Scuola Normale Superiore

Classe di Scienze Matematiche, Fisiche e Naturali

Invariant Measures and Stationary Solutions of 2-dimensional Euler Equations and Related Models

Tesi di Perfezionamento in Matematica

Candidato: Francesco Grotto

Relatore: Prof. Franco Flandoli

Pisa, June 20, 2020

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A mia madre.

Ringraziamenti

La presente tesi conclude un lungo percorso alla Scuola Normale Superiore, prima come Studente Ordinario e poi come Perfezionando. È stato un onore far parte di questa istituzione, e sarò sempre grato alla Scuola, ed alla Repubblica Italiana che la sostiene, per le opportunità che mi sono state offerte.

Tra i miei mentori spicca naturalmente il Professor Franco Flandoli, mio relatore prima come Professore all'Università di Pisa durante i miei studi universitari e quindi alla Scuola Normale. Negli ultimi sei anni, oltre ad una eccellente formazione in Matematica, ho ricevuto da Franco innumerevoli consigli e insegnamenti sulla vita accademica, nonchè un supporto costante, attento e disinteressato. Non posso che ritenermi privilegiato per aver potuto conoscerlo ed essere stato suo allievo.

Un ruolo altrettanto importante lo ha avuto per me il Professor Marco Romito, che parallelamente al nostro comune maestro ha guidato la mia formazione in modo determinante.

Sono molte poi le persone con cui ho lavorato e diviso momenti importanti: il Professor Stefano Marmi, i miei collaboratori Professor Dejun Luo e Umberto Pappalettera, i miei colleghi ed amici Elia Bruè, Federico Cesare Giorgio Conti, Julian Lawrence Demeio, Marta Leocata, Valentino Liu, Alessandra Occelli, Cristiano Ricci, José Danuso Rocha de Oliveira, Daniele Semola, Anastasia Shepelevtseva, Giorgio Stefani, Matteo Verzobio, Gevorg Yeghikyan. Posso inoltre contare in questi anni un gran numero di conversazioni scientifiche che hanno indirizzato il mio lavoro, e per queste ringrazio i Professori Sergio Albeverio, Benjamin Gess, Massimiliano Gubinelli, i Dottori Luigi Amedeo Bianchi, Michele Coghi, Mario Maurelli, Dario Trevisan, Giovanni Zanco, Margherita Zanella, ed infine Clara Antonucci.

Ringraziamenti speciali vanno ovviamente ai miei familiari per il loro supporto, mio fratello Giovanni, mio padre, mio nonno, quest'ultimo nel ricordo a un anno dalla scomparsa. Su tutti ringrazio mia madre, per l'attenzione, l'amore e il lavoro infaticabili che mi ha dedicato.

Contents

CHAPTER 1

Introduction

Invariant measures and their associated stationary flows play a very important and distinguished role in the study of incompressible fluid dynamics. Although such relevance is due by a large extent to a particular goal of the theory, that is to obtain a statistical mechanics description of phenomena associated to *turbulence*, many mathematical aspects of the topic have also proved to be of independent interest.

What follows is a collection of original contributions by the author and his collaborators to the study of stationary flows in 2-dimensional, incompressible fluid dynamics models. Regularity regimes prescribed by physically motivated invariant measures turn out to be quite singular for the models under consideration because of their nonlinear structure. The unifying scope of our treatment is the application of ideas and techniques belonging to Probability Theory, or rather Stochastic Analysis, and Statistical Mechanics, to mathematical problems arising in such context.

1.1. PDE Models in Incompressible Fluid Dynamics

The three chief open research directions in deterministic incompressible fluid mechanics deal with: (a) well-posedness results, (b) inviscid limits, (c) turbulence. We refer to [**9, 69, 115, 117**] for general surveys. Probability has obvious relations with turbulence, while it is less clear how much it can be related to the former two. Inviscid limits have been essentially left untouched by stochastic methods. As for problem (a), a huge effort has been devoted to the attempt at extending and improving the deterministic theory by means of probability and stochastic models, and most of what follows fits into this framework.

The most important open problems in class (a) concern basic deterministic equations, the outstanding example being the 3-dimensional Navier-Stokes equations (3dNS), [**66**]. Striking well-posedness results for SDEs with very irregular drift and additive noise such as [**162, 114**] lead to the general belief that suitably non-degenerate additive noise may regularize several classes of differential equations, providing for instance uniqueness results in cases where the deterministic equation may not have a unique solution. Relevant infinite-dimensional examples of such phenomena are described for instance in [**52, 53, 55**], in which, however, the drift terms is still far from the irregularity and unboundedness of the inertial term of 3dNS, and requirements on the noise restrict applications to parabolic 1-dimensional equations. The strategy of those works consisted in directly solve the infinite dimensional Kolmogorov equation associated to the SPDE. In the case of 3dNS the corresponding Kolmogorov equation has been solved in [**57**], but regularity of solutions is not sufficient to deduce uniqueness results of weak solutions to the stochastic 3dNS. However, [**78**] established existence of global in time Markov selections satisfying strong Feller property — a striking continuous dependence on initial conditions — which has no deterministic counterpart in the theory of 3dNS.

In the inviscid case, the main open problems concern the 3-dimensional Euler equations: only local results are known, except for special notions of solutions, see [**127, 128, 60**]. Such equations represent a too difficult task for a first stage understanding of regularization by noise. We will instead focus on the 2-dimensional Euler equations (2dE), expressed in *vorticity formulation*, on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \operatorname{div} u = 0, \\ \omega = \nabla^\perp u, \end{cases} \tag{1.1.1}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. When the initial condition $\omega|_{t=0}$ is bounded measurable, a celebrated result of Judovič [105] establishes the existence of a unique solution. The result has been extended to stochastic versions forced by regular additive noise in [21], and by multiplicative transport noise in [35].

When the regularity of the initial condition $\omega|_{t=0}$ is decreased, say to $L^p(\mathbb{T}^2)$, $p \in (1, \infty)$, global existence can still be proved with arguments based on the formal conservation of the L^p-norm of ω. Uniqueness, however, is an open problem: see [127] for a discussion. It is therefore natural to explore stochastic approaches to restore uniqueness below the class L^∞: unfortunately we still do not know whether there exists a noise, either additive or multiplicative, producing such an effect.

This and other closely related open problems originated a considerable amount of research: several attempts have been made to prove that suitable multiplicative transport type noises — a natural choice in inviscid problems due to its conservation properties — regularize first order, transport type PDEs. The case of linear transport equations has been understood quite well, see for instance [16, 67, 69, 77, 137].

The nonlinear case is much harder to treat, and only fragmentary results are available: point vortex solutions to 2dE, which we will extensively discuss, are regularized [68]. For dyadic models and their generalizations on trees [15] uniqueness holds thanks to multiplicative noise [13, 23], and a variant of the same technique applied also to a 3D Leray α-model [14]. Hamilton-Jacobi equations [84] and scalar conservation laws [85] are also regularized by suitable multiplicative noise, although not of transport type.

Well-posedness problem (a) has another relevant aspect, that is to extend existence theory to distributional classes of vorticity fields ω in 2dE. The motivation is twofold: to understand the limits of PDE theory in terms of roughness of solutions, and to establish a rigorous setup for investigation of explicit Gibbsian invariant measures. Indeed, the latter is one of the main topics we will discuss in the following, and it is of potential interest also for turbulence theory. Early results in this direction are reviewed in [9], including the basic existence result of [8] for stationary solutions of (1.1.1) in negative order Sobolev spaces, with fixed time marginal being the 2-dimensional space white noise, also known as Enstrophy measure in this context, or the Energy-Enstrophy Gibbs measure. This theory was recently revised by means of an alternative approach based on point vortex approximation, introduced in [71]. These works, devoted to the deterministic equation (1.1.1) with random initial conditions, have been also generalized to stochastic cases. Multiplicative transport noise was covered by [73, 74, 75], whereas the additive space-time white noise forcing was considered in [94, 72]. In the additive case, friction is needed to allow stationary solutions, while multiplicative noise is conservative.

The 2dE with additive noise, possibly including friction, their corresponding stationary solutions and invariant measures had already been considered before. However, in earlier studies the space regularity of noise was such that solutions were function-valued, and invariant measures were supported on spaces of functions: we refer for instance to [21, 36, 18, 22, 49, 90, 19, 88, 20], and also to other related results in [116, 117, 69]. Many of those models and results are inspired by the open problem of turbulence (c): in connection with this question and the previous references we also mention [24, 76, 98].

1.2. The 2-dimensional Euler Equations

We now proceed to introduce in detail the content of the forthcoming Chapters, the obvious starting point being Euler equations (1.1.1), which are commonly written in terms of the velocity vector field u as

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0, \\ \operatorname{div} u = 0. \end{cases}$$

Here p is the scalar pressure field, which vanishes when passing to vorticity form. We conveniently refer to the monographies [126, 128, 135] for the deterministic theory of these equations.

The stationary solutions we are interested in are *not* deterministic, constant in time solutions of Euler equations, but random velocity (equivalently, vorticity) fields whose distribution is preserved by Euler dynamics, that is invariant measures. All candidate invariant measures obtained by formal arguments are too singular to give meaning to the PDE, unless it is considered in a suitable, non trivial, weak formulation. This situation is quite commmon in the theory of dispersive PDEs and more generally of Hamiltonian PDEs: we refer to the recent survey [161] and references therein regarding the dispersive setting. The next two paragraphs will detail the two main examples of invariant measures we will deal with in our fluid-dynamics setting.

Before that, let us introduce an essential tool that allows to solve the problem of defining the nonlinear term for irregular vorticity fields, in fact highlighting the peculiarities of fluid-dynamics PDEs we consider.

To fix ideas, let us consider here the torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ as space domain: generalisations of topics below to bounded domains or other compact surfaces will often be possible, and they will be detailed separately in their respective Chapters.

We denote $d(x, y)$ the distance between two points $x, y \in \mathbb{T}^2$. Since vorticity is the curl of velocity, $\omega = \nabla^\perp u$, it has zero space average: in what follows we will thus consider only functions (or distributions) having zero average on $\mathbb{T}^2$, and denote by $\dot{L}^p(\mathbb{T}^2), \dot{H}^\alpha(\mathbb{T}^2)$ Lebesgue and Sobolev spaces of zero averaged functions.

It will often be convenient to work with Fourier series: let $e_k(x) = e^{2\pi \mathrm{i}\, k\cdot x}$, for $k \in \mathbb{Z}^2_0 = \mathbb{Z}^2 \setminus \{0\}$, $x \in \mathbb{T}^2$, be the orthonormal basis of $\dot{L}^2(\mathbb{T}^2)$ diagonalising the Laplace operator, and recall that Sobolev spaces (of zero average distributions) are characterised as follows:

$$\forall \alpha \in \mathbb{R}, \quad \dot{H}^\alpha(\mathbb{T}^2) = \left\{ u \in C^\infty(\mathbb{T}^2)' : \|u\|^2_{\dot{H}^\alpha} = \sum_{k \in \mathbb{Z}^2_0} |k|^{-2\alpha} |\hat{u}_k|^2 < \infty \right\},$$

where $\hat{u}_k = \langle u, e_k \rangle$, the brackets denoting (complex) L^2-based duality couplings from now on. We will also denote by $\mathcal{M}(\mathbb{T}^2)$ the linear space of finite signed measures on $\mathbb{T}^2$, which is continuously embedded in $H^\alpha(\mathbb{T}^2)$ for any $\alpha < -1$, since Fourier coefficients of measures are uniformly bounded by 1.

The Green function of the Laplace operator with zero average, $G = (-\Delta)^{-1}$, is the unique solution of

$$\forall x, y \in \mathbb{T}^2 \quad -\Delta_x G(x, y) = \delta_y(x) - 1, \quad \int_{\mathbb{T}^2} G(x, y) dx = 0;$$

we recall that G is a symmetric function, and moreover it is translation invariant. It has the explicit representation in Fourier series

$$G(x,y) = G(x-y) = \sum_{k\in\mathbb{Z}^2_0} \frac{e_k(x-y)}{4\pi^2|k|^2},$$

and moreover it can be expressed as the sum of Green's function on the whole plane and a bounded function,

$$G(x,y) = -\frac{1}{2\pi}\log d(x,y) + g(x,y), \tag{1.2.1}$$

with $g(x,y) \in C^0_{sym}(\mathbb{T}^{2\times 2})$. The latter representation holds more generally on any compact Riemannian surface without boundary (see [**11**]), and it can be recovered comparing the $G(x,y)$ to the solution of $-\Delta_x u(x) = \delta_y(x)$ on a small ball centred in y with Dirichlet boundary conditions.

As already mentioned, we will focus on solutions to (1.1.1) of low space regularity: in order to give meaning to the PDE the starting point is thus the weak formulation against a smooth test $\phi \in C^\infty(\mathbb{T}^2)$,

$$\begin{aligned}\langle \phi, \omega_t\rangle - \langle \phi, \omega_0\rangle &= \int_0^t \int_{\mathbb{T}^{2\times 2}} K(x-y)\omega_s(y)\omega_s(x)\nabla\phi(x)dxdyds\\ &= \int_0^t \langle (K * \omega_s)\omega_s, \nabla\phi\rangle\, ds,\end{aligned}$$

where we introduced the notation $K = \nabla^\perp G$. The convolution kernel K is in fact the Biot-Savart kernel in dimension 2, since it allows to express the velocity field in terms of vorticity:

$$\omega = \nabla^\perp \cdot u \quad \Rightarrow u = K * \omega.$$

Of course, K is a singular kernel, and this makes the weak formulation above unsuitable to treat even measure-valued solutions. Here comes into play the fundamental symmetrisation introduced in the works of Delort and Schochet [**62, 154, 155**].

For smooth solutions of Euler equations, by symmetrising the variables x, y in the integral expressing the right-hand side of weak formulation, and using the fact that $K(x-y)$ is skew-symmetric (since $G(x-y)$ is symmetric), one obtains

$$\begin{aligned}\langle \phi, \omega_t\rangle - \langle \phi, \omega_0\rangle &= \int_0^t \int_{\mathbb{T}^{2\times 2}} H_\phi(x,y)\omega_s(x)\omega_s(y)dxdyds \qquad (1.2.2)\\ &= \int_0^t \langle H_\phi, \omega_s \otimes \omega_s\rangle\, ds,\end{aligned}$$

$$H_\phi(x,y) = \frac{1}{2}(\nabla\phi(x) - \nabla\phi(y)) \cdot K(x-y), \quad x,y \in \mathbb{T}^2, \tag{1.2.3}$$

where $H_\phi(x,y)$ is a bounded symmetric function with zero average in both variables and smooth outside the diagonal set $\triangle^2 = \{(x,x) : x \in \mathbb{T}^2\}$, where it has a jump discontinuity. Because of this, by interpreting brackets $\langle\cdot,\cdot\rangle$ as suitable duality couplings, one can give meaning to Euler equations when vorticity ω has low space regularity.

1.3. Euler Point Vortices

We now introduce a dynamics of point measures on $\mathbb{T}^2$ satisfying (1.1.1) in weak sense, providing in a natural way a class of invariant random point measures. We consider a set of N *point vortices* described by their positions $x_1, \dots x_N \in \mathbb{T}^2$

and *intensities* $\xi_1, \dots \xi_N \in \mathbb{R}$, whose dynamics is given by

$$\dot{x}_i(t) = \sum_{j \neq i}^{N} \xi_j K(x_i(t), x_j(t)), \tag{1.3.1}$$

with K as defined above. The system is Hamiltonian with respect to conjugate coordinates $(x_{i,1}, \xi_i x_{i,2})$, and Hamiltonian function

$$H(\underline{x}) = \sum_{i<j}^{N} \xi_i \xi_j G(x_i, x_j)$$

(the interaction energy of vortices). The product area on phase space $\mathbb{T}^{2\times N}$ is preserved, at least formally, thanks to Liouville's theorem. The point vortices system is a classical model. We refer to [**10, 135**] for a general introduction and most of the notions we are going to rely on, and to [**127**] for an overview of the statistical mechanics point of view.

The first natural observation is that the vector field of this (system of) ODE is singular when two positions coincide, that is when two vortices *collide*. More generally speaking, in classical, finite-dimensional Hamiltonian systems whose Hamiltonian function involves singular interaction, there may exist singular trajectories in which, at finite time, the driving vector field diverges. When this happens only for a negligible set of initial conditions with respect to an invariant measure, thus a physically relevant measure on phase space, the motion is said to be *almost complete*. A relevant example is the so called *improbability of collisions* in N-body systems, a problem that has received attention both in classical [**2, 151, 150**] and more recent [**79**] works.

The point vortices system perfectly fits the setting we just outlined, and its almost completeness is a classical result.

Theorem 1.3.1 (Dürr-Pulvirenti). *Let $\xi_1, \dots \xi_N \in \mathbb{R}$ be fixed. There exists a full-measure set $M \subset \mathbb{T}^{2\times N}$ and a one-parameter group of maps $T_t : M \to M$ such that $\underline{x}(t) = T_t(\underline{x}) \in \mathbb{T}^{2\times N}$ is the unique, smooth solution of (1.3.1) with initial positions $(x_1(0), \dots, x_N(0)) = \underline{x} \in M$. For all $t \in \mathbb{R}$, T_t defines a measurable, measure preserving, dx^N-almost everywhere invertible transformation of $(\mathbb{T}^2)^N$.*

Define moreover, for $t > 0$ and $\underline{x} \in (\mathbb{T}^2)^N$,

$$d_t(\underline{x}) = \inf_{s\in[0,t]} \min_{i\neq j} |(T_s\underline{x})_i - (T_s\underline{x})_j|.$$

Then there exists a constant $C > 0$ independent of $c \in (0,1)$ such that

$$|\{d_t(\underline{x}) < c\}| \leq \frac{C(t+1)}{-\log c}. \tag{1.3.2}$$

Another natural question is whether the Liouville operator, that is the time evolution generator for the dynamics of observables, is essentially self-adjoint on a class of observables smooth in a dense set obtained by removing singular points from the phase space, [**149**, Section X.14]. To be specific, let us consider the one-parameter group of Koopman unitaries U_t associated to such flow,

$$U_t f = f \circ T_t, \quad f \in L^2(\mathbb{T}^{2\times N}).$$

By Stone's theorem, $U_t = e^{\mathrm{i}\,tA}$ is generated by a self-adjoint operator A, the latter being defined on a dense subset of $L^2(\mathbb{T}^{2\times N})$. An explicit expression of L can be given only on certain sets of observables, and in [**95**] the following result was established.

Theorem 1.3.2. *Consider the linear space X of functions $f \in L^\infty(\mathbb{T}^{2\times N})$ such that:*

- *there exists a version of f and a full-measure open set $M \subset \mathbb{T}^{2\times N}$ on which $f|_M \in C^\infty(M)$, and moreover $\nabla f|_M \in L^\infty(M)$;*
- *there is a neighbourhood of $\triangle^N = \{x_i = x_j \text{ for some } i \neq j\}$, the the collision set, on which f vanishes.*

It is dense in $L^2(\mathbb{T}^{2\times N})$, and for any $\underline{\xi} \in \mathbb{R}^N$, $f \in X$ the following expression is well defined as a function in $L^\infty(\mathbb{T}^{2\times N})$:

$$Lf(\underline{x}) = -\mathrm{i}\sum_{i=1}^{N}\sum_{j\neq i}\nabla_i f(\underline{x})\cdot\xi_j K(x_i - x_j). \tag{1.3.3}$$

Moreover, (L, X) is a symmetric operator and if A is the generator of U_t, then X is a subset of the domain of A, and $L = A|_X$.

The latter Theorem (to be discussed in Chapter 3), was inspired by [**3**], in which the Liouville operator was defined on a set of cylinder functions of Fourier modes, and the question of essential self-adjointness was raised on such domain. We shall discuss the setting of [**3**] in comparison to ours in subsection 3.2.4.

Let us now go back to the link with Euler equations. As detailed in [**155**], the empirical measure $\omega = \sum_{i=1}^N \xi_i\delta_{x_i}$ with x_i evolving as in (1.3.1) satisfies (1.2.2) if we assume that $H_\phi(x,x) = 0$, thas is if we neglect self-interactions of vortices. More precisely, brackets $\langle H_\phi, \cdot\rangle$ are to be interpreted as duality couplings between continuous functions and measures on $\mathbb{T}^{2\times 2} \setminus \triangle^2$.

This should not be surprising: the vector field acting on vortex x_i is in fact given by the convolution of $K = \nabla^\perp G$ with the empirical measure of the other vortices $x_j \neq x_i$. Indeed, it is possible to obtain the point vortices system as a limit of solutions to Euler equations made of *vorticity patches*, thus providing a rigorous motivation for the model: such approximation arguments are the object of a rich literature, among which we mention [**132, 133, 134, 42, 38**]. We also mention the recent [**39**] on similar arguments in dimension 3.

Thanks to the Hamiltonian structure, point vortices also preserve the *canonical Gibbs ensemble* at inverse temperature $\beta \geq 0$,

$$\nu_{\beta,N}(dx_1,\dots,dx_n) = \frac{1}{Z_{\beta,N}}\exp\left(-\beta H(x_1,\dots,x_n)\right)dx_1,\dots,dx_n.$$

This measure was first introduced by Onsager in this context, [**143**]. Equilibrium ensembles at high kinetic energy, which exhibit the tendency to cluster vortices of same sign intensities expected in a turbulent regime, were proposed by Onsager allowing negative values of β. Unfortunately, we will not be able to treat the case $\beta < 0$ in our arguments.

For a fixed choice of intensities, considering point vortices whose positions are distributed as the Gibbs ensemble, we obtain a stationary solution of Euler equations in the sense above. A classical scaling limit of these solutions is the Mean Field limit, in which intensities are rescaled as $\xi_i \mapsto \frac{1}{N}$, inverse temperature as $\beta \mapsto N\beta$ and the number of vortices is sent to infinity, $N \to \infty$: in this limit, one obtains deterministic stationary solutions of Euler's equation, the correlations of vortices vanishing in the limit. Indeed the precise rate of such decay has been obtained in [**99**], see Chapter 5 and in particular Theorem 5.1.2 below.

Looking at Mean Field Limit as a Law of Large Numbers, we will then consider a different scaling corresponding to Central Limit Theorem, the latter producing relevant Gaussian invariant measures. We will provide a comparison between the two settings, together with adequate references to classical Mean Field theory, in Chapter 4 and Chapter 5.

1.4. Gaussian Invariant Measures

Smooth solutions of Euler equations on a 2-dimensional domain D preserve the quadratic first integrals *energy* and *enstrophy*,

$$E = \int_D |u|^2 dx, \qquad S = \int_D \omega^2 dx. \tag{1.4.1}$$

Let us stick to $D = \mathbb{T}^2$ for the sake of exposition, and leave other choices to subsequent Chapters. The Gaussian field associated to the quadratic form $\beta E + \gamma S$ on $\mathbb{T}^2$, known as *energy-enstrophy measure*, formally defined as

$$d\mu_{\beta,\gamma}(\omega) = \frac{1}{Z_{\beta,\gamma}} e^{-\beta E(\omega) - \gamma S(\omega)} d\omega, \tag{1.4.2}$$

is thus a natural candidate as an invariant measure of the flow. However, the field is only supported on spaces of quite rough distributions –not even measures– so that making sense of Euler equations in this setting is not trivial: this problem has been effectively tackled by means of Fourier analysis in [**7, 8**].

Energy-Enstrophy measure is rigorously defined as follows: for $\gamma > 0$ and $\beta \geq 0$, let $\omega_{\beta,\gamma}$ be the centred, zero averaged, Gaussian random field on $\mathbb{T}^2$ with covariance

$$\forall f, g \in \dot{L}^2(\mathbb{T}^2), \quad \mathbb{E}\left[\langle \omega_{\beta,\gamma}, f \rangle \langle \omega_{\beta,\gamma}, g \rangle\right] = \langle f, Q_{\beta,\gamma} g \rangle, \quad Q_{\beta,\gamma} = (\gamma - \beta\Delta)^{-1}.$$

Equivalently, $\omega_{\beta,\gamma}$ is a centred Gaussian stochastic process indexed by $\dot{L}^2(\mathbb{T}^2)$ with the specified covariance. Since the embedding of $Q^{1/2}_{\beta,\gamma} \dot{L}^2(\mathbb{T}^2)$ into $\dot{H}^s(\mathbb{T}^2)$ is Hilbert-Schmidt for all $s < -1$, $\omega_{\beta,\gamma}$ can be identified with a random distribution taking values in the latter spaces (see [**59**]). We will denote by $\mu_{\beta,\gamma}$ the law of $\omega_{\beta,\gamma}$ on $\dot{H}^s(\mathbb{T}^2)$, any $s < -1$: a rigorous interpretation of (1.4.2) will be provided in Section 4.1 below.

Let us notice that the special case of the *enstrophy measure* $\mu_{0,1} = \mu$ is the *white noise* on $\mathbb{T}^2$, the unique invariant measure of (infinite dimensional) Ornstein-Uhlenbeck equation

$$dZ = -\alpha Z\, dt + \sqrt{2\alpha}\, dW, \quad \alpha > 0, \tag{1.4.3}$$

with W a cylindrical Wiener process on $L^2(\mathbb{T}^2)$. The triple

$$(E = H^{-1-\delta}(\mathbb{T}^2), L^2(\mathbb{T}^2), \mu), \quad \delta > 0, \tag{1.4.4}$$

is an abstract Wiener space with identity covariance operator.

The definition of the nonlinear term in (1.1.1) when the law of ω_t is μ, (or more generally when it is absolutely continuous with respect to μ) is not immediate, and it has been thoroughly discussed in [**71**] and related works, [**58, 94**]. We will rely upon the arguments of Subsection 2.5 of [**71**], which we now review.

We consider the weak vorticity formulation of Euler equations, so that the problem is to make sense of the coupling $\langle H_\phi, \omega \otimes \omega \rangle$ with ϕ some smooth function of $\mathbb{T}^2$ and $\omega \sim \mu$ a realization of white noise. A lengthy but elementary computation in Fourier series reveals that the Sobolev regularity of H_ϕ is at best $H^{2-}(\mathbb{T}^2 \times \mathbb{T}^2)$, thus the above symmetrized formulation, allows us to give a proper meaning to (1.2.2) in the case when $\omega_t \in H^{-1+}(\mathbb{T}^2)$, which is not the case if $\omega_t \sim \mu$. Here comes into play the essential role of Probability: the following statement is proved in [**71**, Section 2.5], and it will be discussed again in the forthcoming Chapters.

Proposition 1.4.1. *Let $\phi \in C^\infty(\mathbb{T}^2)$ and ω be a random distribution on $\mathbb{T}^2$ with law $\rho\, d\mu$, $\rho \in L^p(E, \mu)$ for some $p > 1$. For any sequence $(H^n_\phi)_{n\in\mathbb{N}} \subset C^\infty(\mathbb{T}^2 \times$*

$\mathbb{T}^2$) *of symmetric functions such that*

$$L^2(\mathbb{T}^2\times\mathbb{T}^2)-\lim_{n\to\infty}H^n_\phi=H_\phi, \tag{1.4.5}$$

$$\lim_{n\to\infty}\int_{\mathbb{T}^2}H^n_\phi(x,x)dx=0, \tag{1.4.6}$$

the limit

$$\langle\omega\diamond\omega,H_\phi\rangle:=\lim_{n\to\infty}\langle\omega\otimes\omega,H^n_\phi\rangle \tag{1.4.7}$$

exists in $L^1(\mu)$ *and it does not depend on the approximating sequence* H^n_ϕ *among the ones satisfying the above properties. Moreover,*

$$\mathbb{E}\left[\left|\left\langle\omega\diamond\omega,H^n_\phi-H_\phi\right\rangle\right|\right]\le C_p\left\|H^n_\phi-H_\phi\right\|^{1/p'}_{L^2(\mathbb{T}^2\times\mathbb{T}^2)}+\left|\int_{\mathbb{T}^2}H^n_\phi(x,x)dx\right|, \tag{1.4.8}$$

with $\frac{1}{p}+\frac{1}{p'}=1$, *and for any* $q\in[1,\infty)$ *it holds*

$$\mathbb{E}\left[\left|\left\langle\omega\diamond\omega,H_\phi\right\rangle\right|^q\right]\le C_q\left\|\rho\right\|_{L^p(E,\mu)}\left\|\phi\right\|^q_{C^2(\mathbb{T}^2)}, \tag{1.4.9}$$

with C_q *a constant depending only on* q.

If $\rho_t\in L^\infty([0,T],L^p(E,\mu))$ *and* ω_t *is a process with trajectories in* $C([0,T],E)$ *and marginals* $\omega_t\sim\rho_t\,d\mu$ *(in particular we are assuming* $\int_E\rho_t\,d\mu=1$ *for all* t*), the sequence of real processes* $\left\langle\omega_t\otimes\omega_t,H^n_\phi\right\rangle$ *converges in* $L^1\left(E,\mu;L^1([0,T])\right)$ *to a process* $\langle\omega_t\otimes\omega_t,H_\phi\rangle$ *which does not depend on the approximations* H^n_ϕ *as above.*

It is worth noticing that the approximations H^n_ϕ as in (1.4.5) can always be obtained by regularizing the kernel K in the definition of H_ϕ. We also remark that if $\omega\sim\mu$, the limit (1.4.7) coincides with the double Wiener-Itô integral of the kernel H_ϕ on the Gaussian Hilbert space (E,μ) (see [**94**] and Chapter 6 for a discussion).

We are now able to give meaning to Euler equations with marginals (absolutely continuous with respect to) μ; let us do so for the following stochastic generalisation of (1.1.1),

$$\begin{cases}d\omega+u\cdot\nabla\omega dt=-\alpha\omega dt+\sqrt{2\alpha}dW,\\ \nabla^\perp u=\omega.\end{cases} \tag{1.4.10}$$

with $\alpha\ge 0$ and W the cylindrical Wiener process on $L^2(\mathbb{T}^2)$. Equation (1.4.10) will be the the object of Chapter 2 and Chapter 6, the latter providing in a sense a physical motivation, see Section 1.5 below. It has been widely investigated especially as inviscid limit of driven and damped Navier-Stokes equation, see for instance [**19**], [**48**] and references therein. Aside from the fact that we are dealing directly with the inviscid case, the substantial difference with respect to those works is of course the space regularity of solutions. Let us conclude this Section with an overview of the results of [**72**] and Chapter 2, providing definitions of solutions to (1.4.10) and its associated (infinite dimensional) Fokker-Planck equation.

Consider the orthonormal Fourier basis $e_k(x)=e^{i\,k\cdot x}$ of $L^2(\mathbb{T}^2,dx)$, and denote as usual $\hat\omega_k=\langle\omega,e_k\rangle$. In fact, we will only deal with real-valued objects: Fourier coefficients of opposite modes will henceforth be complex conjugated. Let $\mathcal{FC}_b$ be the linear space of *cylinder functions* of the form

$$\varphi(\omega)=f(\hat\omega_{k_1},\dots,\hat\omega_{k_n}),\quad k_1,\dots,k_n\in\mathbb{Z}^2_0,$$

with $n\ge 1$ and $f\in C^\infty_b(\mathbb{R}^n)$. The infinitesimal generator of the linear part of (1.4.10) is $\alpha\mathcal{L}$, with $\mathcal{L}$ the generator of the *Ornstein-Uhlenbeck semigroup* acting on cylinder functions as

$$\mathcal{L}\varphi(\omega)=\sum_{i=1}^n\partial_{ii}f(\hat\omega_{k_1},\dots,\hat\omega_{k_n})-\sum_{i=1}^n\partial_i f(\hat\omega_{k_1},\dots,\hat\omega_{k_n})\hat\omega_{k_i}.$$

The generator associated to (1.4.10) can be written formally as

$$\mathcal{A}\varphi(\omega) = \mathcal{B}\varphi(\omega) + \alpha\mathcal{L}\varphi(\omega), \quad \mathcal{B}\varphi(\omega) = -\langle (K*\omega)\cdot\nabla\omega, D\varphi(\omega)\rangle, \tag{1.4.11}$$

whose action on cylinder functions $\varphi \in \mathcal{FC}_b$ is given in terms of $\mathcal{L}$ and

$$D\varphi(\omega) = \sum_{i=1}^n \partial_i f(\hat\omega_{k_1},\dots,\hat\omega_{k_n}) e_{k_i}.$$

To give a rigorous definition of $\mathcal{B}$, we make use of Proposition 1.4.1 (see also the discussion in [**58**]). First, we combine the latter two expressions with (1.2.2) to obtain, say first for smooth ω,

$$\begin{aligned}\mathcal{B}\varphi(\omega) &= -\sum_{i=1}^n \partial_i f(\hat\omega_{k_1},\dots,\hat\omega_{k_n}) \langle (K*\omega)\cdot\nabla\omega, e_{k_i}\rangle, \\ &= \sum_{i=1}^n \partial_i f(\hat\omega_{k_1},\dots,\hat\omega_{k_n}) \left\langle \omega\otimes\omega, H_{e_{k_i}}\right\rangle.\end{aligned}$$

By Proposition 1.4.1, we can define the real random variable

$$\mathcal{B}\varphi(\eta) = \sum_{i=1}^n \partial_i f(\hat\omega_{k_1},\dots,\hat\omega_{k_n}) \left\langle \omega\diamond\omega, H_{e_{k_i}}\right\rangle \in L^1(\mu),$$

for all cylinder functions $\varphi\in\mathcal{FC}_b$. As already observed above, $\langle \eta\diamond\eta, H_\phi\rangle$ is in fact an element of the second Wiener chaos of the Gaussian process η, since it coincides with the double Itô-Wiener integral. As a consequence, $\mathcal{B}$ is exponentially integrable when acting on cylinder functions:

$$\mathbb{E}\left[\exp\left(\varepsilon|\mathcal{B}\varphi(\eta)|\right)\right] < \infty \quad \text{for all small } \varepsilon > 0 \tag{1.4.12}$$

(see [**58**, Theorem 8] for an explicit computation).

The singularity of the nonlinear term is such that the operator $\mathcal{B}$, regarded as a vector field acting as a derivation on the Gaussian space (1.4.4), does not take values in the Cameron-Martin space $L^2(\mathbb{T}^2)$, or even in $H^{-\delta}(\mathbb{T}^2)$, see [**8**]. Nonetheless, it formally holds $\operatorname{div}_\mu \mathcal{B} = 0$, in agreement with the fact that the SPDE under consideration formally preserves μ.

Let us consider the Fokker-Planck equation associated to (1.4.10):

$$\begin{cases}\partial_t\rho = \mathcal{A}^*\rho = -\mathcal{B}\rho + \alpha\mathcal{L}\rho, \\ \rho|_{t=0} = \rho_0.\end{cases} \tag{1.4.13}$$

Definition 1.4.2. *Given $\rho_0\in L^1(\mu)$, we say that $\rho \in L^1_{loc}\big(\mathbb{R}_+, L^1(E,\mu)\big)$, for $\alpha\ge 0$, is a weak solution of the Fokker-Planck equation (1.4.13) if*

(a) *for any $\varphi\in\mathcal{FC}_b$ and $T>0$,*

$$\int_0^T\int_E |\rho_t\mathcal{A}\varphi| d\mu dt < \infty;$$

(b) *for any $f\in C^1_c(\mathbb{R}_+)$ and $\varphi\in\mathcal{FC}_b$ it holds*

$$f(0)\int_E \rho_0\varphi d\mu + \int_0^\infty\int_E f'(t)\rho_t\varphi\, d\mu dt + \int_0^\infty\int_E f(t)\rho_t\mathcal{A}\varphi\, d\mu dt = 0. \tag{1.4.14}$$

Identity (1.4.14) implies that, in the distributional sense,

$$\frac{d}{dt}\int_E \rho_t\varphi\, d\mu = \int_E \rho_t\mathcal{A}\varphi\, d\mu \quad \text{for a.e. } t\in(0,\infty).$$

Since the right-hand side is locally integrable in $t\in(0,\infty)$, the map $[0,\infty)\ni t\mapsto \int_E \rho_t\varphi\, d\mu$ is absolutely continuous, thus ρ_t is weakly continuous in time. This also gives meaning to the initial condition specification $\rho|_{t=0} = \rho_0$. Moreover, taking $\varphi \equiv 1$ yields $\int_E \rho_t \, d\mu = \int_E \rho_0 \, d\mu$ for all $t > 0$.

Let us state the existence results for (1.4.10) and its Fokker-Planck equations we will prove in Chapter 2 by means of Galerkin approximations.

THEOREM 1.4.3. *Let $\rho_0 \in L \log L(E, \mu; \mathbb{R}_+)$ and $\alpha \geq 0$. Then,*

(i) *there exists a weak solution $(\rho_t)_{t \in \mathbb{R}_+}$ of the Fokker-Planck equation (1.4.13) in the sense of Definition 1.4.2;*

(ii) *for almost every $t > 0$ it holds*

$$\int_E \rho_t \log \rho_t \, d\mu \leq e^{-2\alpha t} \int_E \rho_0 \log \rho_0 \, d\mu + \left(1 - e^{-2\alpha t}\right) \|\rho_0\|_{L^1} \log \|\rho_0\|_{L^1}.$$

In particular, if ρ_0 is a probability density and $\alpha > 0$, then the relative entropy of the weak solution ρ_t decreases exponentially fast, which in turn implies the convergence to equilibrium of ρ_t: for almost every $t > 0$ it holds

$$\|\rho_t - 1\|_{L^1} \leq e^{-\alpha t} \sqrt{2 \int_E \rho_0 \log \rho_0 d\mu}.$$

The last assertion is an immediate consequence of the exponential decay of entropy and Kullback's inequality, see [**118**, (11)]. We will also deduce an existence result for L^p $(p > 1)$ initial densities: let us state it explicitly since it will play an important role in building solutions to the stochastic equation (1.4.10).

THEOREM 1.4.4. *Let $\rho_0 \in L^p(E, \mu)$ with $p > 1$ and $\alpha \geq 0$. Then,*

(i) *there exists a weak solution $\rho \in L^\infty\left(\mathbb{R}_+, L^p(E, \mu)\right)$ to Fokker-Planck equation (1.4.13) in the sense of Definition 1.4.2;*

(ii) *if $p = 2$, then, denoting by $\bar{\rho}_0 = \int_E \rho_0 \, d\mu$, we have, for a.e. $t > 0$,*

$$\|\rho_t - \bar{\rho}_0\|_{L^2} \leq e^{-\alpha t} \|\rho_0 - \bar{\rho}_0\|_{L^2}.$$

Finally, we have existence of weak (both in probabilistic and analytical sense) solutions to the Euler equation (2.0.1) in the setting of Theorem 1.4.4.

THEOREM 1.4.5. *Let $p > 1$, $\alpha \geq 0$, $T > 0$. Assume that $\rho_0 \in L^p(E, \mu; \mathbb{R}_+)$ is a probability density, and let $\rho \in L^\infty\left(0, T; L^p(E, \mu)\right)$ be a weak solution obtained in Theorem 1.4.4 to Fokker-Planck equation (1.4.13) with initial datum ρ_0. There exist a filtered probability space on which a cylindrical Wiener process W on $L^2(\mathbb{T}^2)$ and an adapted process ω_t are defined such that*

(i) *$\omega \in C([0, T], E)$ with probability one;*

(ii) *for almost every $t \in [0, T]$, ω_t has law $\rho_t \, d\mu$;*

(iii) *for any $\phi \in C^\infty(\mathbb{T}^2)$ and $t \in [0, T]$,*

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \diamond \omega_s, H_\phi \rangle \, ds - \alpha \int_0^t \langle \omega_s, \phi \rangle \, ds + \sqrt{2\alpha} \, \langle W_t, \phi \rangle,$$

the nonlinear term being defined as in Proposition 1.4.1.

For all $\alpha \geq 0$, if the initial datum has white noise distribution μ, the solution to (1.4.10) we build is stationary. Indeed, this is true for all Energy-Enstrophy measures $\mu_{\beta,\gamma}$, $\beta \geq 0, \gamma > 0$ for $\alpha = 0$, that is Euler equations, but when $\alpha > 0$ the result above on Fokker-Planck equation implies that the solution we build converges to white noise for large times. We shall discuss it further in Section 2.4. It is important to remark that uniqueness of (1.4.10), including deterministic Euler equations as a particular case, in the stationary regime with white noise marginals, remains an important open problem.

Let us conclude this Section by mentioning that (1.4.5) actually generalises the results of [**8**]: their notion of solution was given in terms of Fourier series, but a close inspection reveals that the above definition of the nonlinear term (as a limit of certain approximations) actually include theirs (as limit of Fourier truncated objects).

1.5. Invariant Measures and Scaling Limits

The close resemblance between Onsager's point vortices ensembles and Energy-Enstrophy Gaussian invariant measures for the two dimensional Euler flow is known since the works of Kraichnan on two-dimensional turbulence, [**113**], and it will be thoroughly discussed in Chapter 4. The main result of the latter is to obtain the Gaussian Energy-Enstrophy measure as a limit of Gibbsian point vortices ensembles, in a sort of Central Limit Theorem.

We will consider increasingly many vortices sending $N \to \infty$, while decreasing their intensities $\xi_i = \frac{\sigma_i}{\sqrt{\gamma N}}$, with $\gamma > 0$ and $\sigma_i = \pm 1$, as in the familiar central limit scaling. We will prove that, if positions of vortices $x_1, \dots, x_N$ have joint distribution $\nu_{\beta,N}$ on $\mathbb{T}^{2N}$ with intensities scaling as above, the random measure $\mu^N_{\beta,\gamma}$, which is the law of the $\mathcal{M}(\mathbb{T}^2)$-valued random variable

$$\omega^N_{\beta,\gamma} = \sum_{i=1}^N \xi_i(\delta_{x_i} - 1) \xrightarrow{N\to\infty} \mu_{\beta,\gamma}$$

converges in law to the energy-enstrophy measure. On $\mathbb{T}^2$, the result does not depend on the choice of signs σ_i: to each Dirac delta representing a vortex we are subtracting its space average, so that the global average vanishes and we are thus looking at fluctuations around a null profile. In fact, the result can be regarded as an investigation of Gaussian fluctuations around the well-known mean-field limit, in the case where the latter vanishes, see Section 4.4 below. This is the reason why we will need to impose (asymptotic) neutrality of the global intensity on bounded domains D, that is, to ensure that the limit in the law of large numbers scaling is naught, since in that case it is not possible to renormalise Dirac deltas because of the boundary condition.

Let us provide some further insight on the analogy between those random measures, first pointed out by Kraichnan ([**113**]). The Hamiltonian function H can be seen as a renormalised energy to the extent that it includes all mutual interactions save the ones of vortices with themselves. To make this intuition more precise, let us first recall that in the Gaussian case $\omega \sim \mu_{0,1}$ (white noise), the double Itō-Wiener integral of a smooth function $h \in C^\infty(\mathbb{T}^{2\times 2})$ is given by

$$\langle h, \omega \diamond \omega \rangle = \langle h, \omega \otimes \omega \rangle - \int_{\mathbb{T}^2} h(x,x)dx, \tag{1.5.1}$$

where: on the left-hand side we used the notation introduced in Proposition 1.4.1, coupling against $\omega \otimes \omega$ on the right-hand side is understood as the (almost surely defined) integral against the tensor product of the random distribution ω with itself. One can directly verify the above formula by the definitions: we refer to [**104**, Chapter 7], which includes a discussion on how Wick ordering in double stochastic integrals can be seen as removing singular self-interactions, *cf.* Remark 7.27. As observed in Section 2.4, the renormalised energy can be expressed as

$$2 :E: (\omega) = \langle G, \omega \diamond \omega \rangle = \lim_{n\to\infty} \int_{\mathbb{T}^{2\times 2}} G_n(x,y) d\omega(x) d\omega(y), \tag{1.5.2}$$

where $G_n \in C^\infty(\mathbb{T}^{2\times 2})$ are symmetric and vanish on the diagonal, G_n converge to G in $L^2(\mathbb{T}^{2\times 2})$, and the limit holds in $L^2(\mu_\gamma)$.

In the case of a point vortices cluster $\omega^N \sim \mu^N_{0,\gamma}$, one can define renormalised double integrals in an analogous way. Considering centred distributions (as it is

$\mu_{0,1}$) is essential in the forthcoming Lemma, and in the case of point vortices on $\mathbb{T}^2$ the condition is ensured if we consider the zero average setting.

LEMMA 1.5.1. *Let $\omega^N \sim \mu^N_{0,\gamma}$. On continuous functions $h \in C(\mathbb{T}^{2\times 2})$ with zero average in both variables and vanishing on the diagonal,* i.e. *$h(x,x) = 0$ for all x, define the map*

$$h \mapsto \int_{\mathbb{T}^{2\times 2}} h(x,y) d\omega^N(x) d\omega^N(y) = \sum_{i\neq j} \xi_i \xi_j h(x_i, x_j).$$

Since it holds

$$\mathbb{E}\left[\left(\sum_{i\neq j} \xi_i\xi_j h(x_i,x_j)\right)^2\right] \leq C_\gamma \|h\|^2_{L^2(\mathbb{T}^{2\times 2})}$$

with C_γ a constant independent of N, the map takes values in $L^2(\mu^N_{0,1})$, and it extends by density to a bounded linear map which we will denote

$$\dot{L}^2(\mathbb{T}^{2\times 2}) \ni f \mapsto \int_{\mathbb{T}^{2\times 2}} f(x,y) :d\omega^N(x) d\omega^N(y): \in L^2(\mu^N_{0,1}).$$

PROOF. For any function h as above it holds

$$\mathbb{E}\left[\left(2\sum_{i<j}^N \xi_i\xi_j h(x_i,x_j)\right)^2\right] = 4\sum_{i<j}^N \sum_{\ell<k}^N \xi_i\xi_j\xi_\ell\xi_k \mathbb{E}\left[h(x_i,x_j)h(x_\ell,x_k)\right]$$
$$= \frac{4}{\gamma^2 N^2}\sum_{i<j}^N \int_{\mathbb{T}^{2\times 2}} h(x,y)^2 dxdy = 2\frac{N-1}{\gamma^2 N}\int_{\mathbb{T}^{2\times 2}} h(x,y)^2 dxdy,$$

where the middle passage makes essential use of the zero average condition: all summands except the ones with $i=\ell, j=k$ vanish. □

This construction is analogous to the one of double stochastic integrals with respect to Gaussian measures (Itō-Wiener integrals) and Poisson point process; the above computation is also an important tool in [**71**]. Define, in analogy with (1.5.2), the renormalised energy in the vortices ensemble μ^N_γ case as the renormalised double integral of the potential G with respect to μ^N_γ, that is as a random variable in $L^2(\mu^N_\gamma)$: by Lemma 1.5.1, considering approximations G_n of G as above, we actually recover the Hamiltonian:

$$2 :E: (\omega^N) = \sum_{i\neq j} \xi_i\xi_j G(x_i,x_j) = 2H(x_1,\dots x_n).$$

The convergence of Hamiltonian functions of point vortices to the renormalised Gaussian energy in the case $\beta = 0$ is an important part in the proof of the forthcoming main result of Chapter 4.

THEOREM 1.5.2. *Let $\beta/\gamma \geq 0$. It holds:*

(1) $\lim_{N\to\infty} Z_{\beta,\gamma,N} = Z_{\beta,\gamma}$, where $Z_{\beta,\gamma,N}$ is the partition function of $\nu_{\beta,\gamma,N}$ with intensities $\xi_i = \frac{\sigma_i}{\sqrt{\gamma N}}$ and

$$Z_{\beta,\gamma} = \int e^{-\beta :E:(\omega)} d\mu_{0,\gamma}(\omega).$$

(2) the sequence of $\mathcal{M}$-valued random variables $\omega^N \sim \mu^N_{\beta,\gamma}$ converges in law on $\dot{H}^s(\mathbb{T}^2)$, any $s < -1$, to a random distribution $\omega \sim \mu_{\beta,\gamma}$, as $N \to \infty$;

(3) the sequence of real random variables $H(\omega^N)$ converges in law to $:E:(\omega)$ as $N \to \infty$, with ω^N, ω as in point (2).

The core argument is a uniform bound for partition functions of canonical Gibbs measures, the strategy being the following:

- we split the interaction potential, the Laplacian Green function G, into a regular, *long range* part and a singular, *short range* part, the latter being the Green function of the operator $m^2 - \Delta$ (*2-dimensional Yukawa potential*);
- the contribution of the regular part can be interpreted as an exponential integral of a regular Gaussian field: since the covariance kernel corresponds to a fourth order operator, no normal ordering is required;
- on the other hand, the contribution of the (pointwise vanishing) singular part is controlled by estimating the partition function of vortices interacting by Yukawa potential with diverging mass $m \to \infty$.

Besides the Central Limit Theorem for correlated, Gibbs distributed vortices, we will consider in Chapter 6 a somewhat simpler case. We will consider a system of vortices, whose positions are distributed uniformly and independently on $\mathbb{T}^2$, in which at random times new vortices are created, and the intensities are overall exponentially damped. These two effect compensate to produce a stationary regime. In the Central Limit scaling for vortices of above, and increasing the rate of generation of new vortices, the system will be shown to converge to a stationary solution of (1.4.10), the cylindrical noise W emerging as the limit of the generation process.

Equation (1.4.10) can be regarded as an inviscid version of the one considered in [**27**], which aimed to describe the energy cascades phenomena in stationary, energy-dissipated, 2-dimensional turbulence. Even if our point vortices model is not able to capture turbulence phenomena such as the celebrated energy spectrum decay law of inverse cascade predicted by Kolmogorov, the mechanism of creation and damping of point vortices we describe might contribute to provide a description of experimental behaviours of models such as the ones in [**27**].

1.6. Stationary Solutions of Barotropic Quasi-Geostrophic Equations

Barotropic quasi-geostrophic equations in channel domains constitute a physically relevant partial differential equation in oceanography and atmospheric modeling, with applications including for instance the Antarctic circumpolar current. Significance of the model is discussed for instance in [**92, 45, 130, 64**] and references therein, to which we refer.

The presence of conserved quantities and their associated equilibrium statistical mechanics constitute an important feature of the model; although numerical reasons naturally lead to consider Fourier truncated or other approximations of the stationary flow, as for instance in [**129**, Section 6], the full infinite-dimensional setting is of great interest because of its geophysical relevance and mathematical difficulty, as discussed in [**130**]. The latter monography thoroughly discusses in its Chapter 8 equations for fluctuations around the mean state for the truncated model, and then considers a continuum limit by scaling parameters of invariant measures so to neglect fluctuations, obtaining a mean state description for the PDE model.

The contribution of [**96**], to be reviewed in Chapter 7, in a sense furthers their study: we will show how fluctuations can be included in the continuum limit by defining a suitably weak notion of solution mimicking the one discussed above for Euler equations, so to include the distributional regimes dictated by the full infinite-dimensional invariant measure, under which fluctuations of comparable order are observed at all scales.

The model under consideration, for the derivation of whom we refer to [**130**, Chapter 1], is the following. We consider the rectangle $R = [-\pi, \pi] \times [0, \pi]$ as a space domain, and denote $z = (x, y) \in R$ its points; we also fix a finite interval for time $t \in [0, T]$. The governing dynamics is the inviscid quasi-geostrophic equation for the scalar *potential vorticity* $q(t, z)$,

$$\partial_t q + \nabla^\perp \psi \cdot \nabla q = 0, \tag{1.6.1}$$

where $\nabla^\perp = (-\partial_y, \partial_x)$, and $\psi(t, z)$ is the *stream function* determining the divergenceless velocity field $\nabla^\perp \psi$. The channel geometry prescribes that velocity $\nabla^\perp \psi$ be tangent to the top and bottom boundaries of R, and we further assume the flow to be periodic in the x coordinate. Such boundary conditions are encoded in terms of ψ as follows:

$$\partial_x \psi(t, x, \pi) = \partial_x \psi(t, x, 0) = 0, \tag{1.6.2}$$

$$\nabla^\perp \psi(t, x + 2\pi, y) = \nabla^\perp \psi(t, x, y). \tag{1.6.3}$$

As a consequence, at fixed t the stream function ψ is constant on the impermeable boundaries $y = 0, \pi$. Using the fact that ψ is defined up to an additive constant, possibly depending on time, we will set $\psi(t, x, 0) \equiv 0$, from which it is easily seen that ψ takes the form

$$\psi = -Vy + \psi',$$

with $V(t)$ a function of time only describing a large-scale mean flow, and $\psi'(t, z)$ the scalar *small-scale stream function*, periodic in x and null at $y = 0, \pi$ at all times. Potential vorticity is then linked to ψ' by

$$q = \Delta \psi' + h + \beta y, \tag{1.6.4}$$

where $h(z)$ is a smooth scalar function modelling the effect of the underlying topography on the fluid, and βy, $\beta \in \mathbb{R}$, is the beta-plane approximation of Coriolis' force.

Dynamics of $V(t)$ is derived by imposing conservation of *total energy*,

$$E = \frac{1}{2} \fint_R |\nabla^\perp \psi|^2 dxdy = \frac{1}{2} V^2 + \frac{1}{2} \fint_R |\nabla^\perp \psi'|^2 dxdy, \tag{1.6.5}$$

from which one obtains an equation for time evolution of the mean flow,

$$\frac{dV}{dt} = -\fint_R \partial_x h(z) \psi'(z) dz,$$

the right-hand side being usually referred to as *topographic stress*. This last relation completes our set of equations,

$$\begin{cases} \partial_t q + \nabla^\perp \psi \cdot \nabla q = 0, \\ q = \Delta \psi' + h + \beta y, \\ \psi = -Vy + \psi', \\ \frac{dV}{dt} = -\fint_R \partial_x h(z) \psi'(z) dz. \end{cases} \tag{BQG}$$

Since both ψ and ψ' can be recovered from V and q, taking into account the boundary conditions (1.6.2), (1.6.3) in solving Poisson's equation (1.6.4), we will consider (V, q) as the state variables of the system. This particular choice has the advantage of retaining the active scalar form for the dynamics (1.6.1) of q.

Besides the total energy E, (BQG) preserve the *large-scale enstrophy*

$$Q(V, q) = \beta V + \frac{1}{2} \fint_R (q - \beta y)^2. \tag{1.6.6}$$

Due to the Hamiltonian nature of the fluid dynamics, it is thus expected that the Gibbsian ensembles

$$d\nu_{\alpha,\mu}(V, q) = \frac{1}{Z_{\alpha,\mu}} e^{-\alpha(\mu E(V,q) + Q(V,q))} dV dq, \quad \alpha, \mu > 0, \tag{1.6.7}$$

are invariant measures for (BQG). Since Boltzmann's exponents are quadratic functionals of the state variables (V, q), these are Gaussian measures. Just as Energy-Enstrophy measures described above, they are only supported on spaces of distributions –they give null mass to any space of functions– so some effort is required to give meaning to the dynamics (BQG) in the low-regularity regime dictated by $\nu_{\alpha,\mu}$.

We will describe in Chapter 7 a notion of solution to (BQG) completely analogous to the Delort-Schochet formulation described above for Euler equations, robust enough to admit samples of $\nu_{\alpha,\mu}$ as fixed-time distributions, and then produce by means of a Galerkin approximation scheme such a solution.

THEOREM 1.6.1. *Let $\beta \neq 0$ and h as above. For any $\alpha, \mu > 0$ there exists a stationary stochastic process $(V_t, q_t)_{t\in[0,T]}$ with fixed-time marginals $\nu_{\alpha,\mu}$, whose trajectories solve* (BQG) *in the weak vorticity formulation of Definition 7.2.8.*

As in the case of 2-dimensional Euler's equations in the Energy-Enstrophy stationary regime, or more generally when fixed time marginals are absolutely continuous with respect to space white noise, uniqueness remains an important open problem. We will not discuss uniqueness of solutions of (BQG) in the above stationary regime; thus, in particular, we are not able to state that the solutions we produce form a flow, *i.e.* a one-parameter group of transformations of phase space indexed by time.

1.7. 2-Dimensional Primitive Equations: a More Singular Geophysical Model

Primitive Equations constitute a fundamental model in geophysical fluid dynamics. The work [**97**], to be reviewed in Chapter 8, is devoted to the study of Gaussian invariant measures in the stochastically forced 2-dimensional case: the model under analysis is thus a stochastic PDE of the form:

$$\begin{cases} \partial_t v + v\partial_x v + w\partial_z v + \partial_x p = \mathcal{D}(\Delta)v + \eta, \\ \partial_z p = 0, \\ \partial_x v + \partial_z w = 0, \end{cases} \tag{1.7.1}$$

where (x, z) are coordinates of the bounded domain $D = [0, 2\pi]^2$ on which suitable boundary conditions are imposed, (v, w) are the components of the *velocity* vector field, p is the *pressure*, the term $\mathcal{D}(\Delta)$ describes a dissipation mechanism and η is a Gaussian stochastic process.

It is in fact the case with $\mathcal{D}(\Delta) = \nu\Delta$ and $\eta = 0$ to be usually referred to as *2-dimensional Primitive Equations* (2dPE), together with its variants including effects such as density and temperature variations, and other geophysical effects. When those physical phenomena are neglected, equations (1.7.1) have many aspects in common with the 2-dimensional Navier-Stokes equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \nu\Delta u, \\ \operatorname{div} u = 0. \end{cases}$$

This familiarity naturally leads to look for applications of concepts and techniques developed in the extensive theory of Navier-Stokes equations, especially in the 2-dimensional setting. However, the nonlinearity of 2-dimensional Primitive Equations is in fact harder to treat.

When considering the stochastically forced case, it is well-known that stochastic Navier-Stokes equations (SNS) in dimension 2, in their vorticity form

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = \nu \Delta \omega + \sqrt{2\nu}\xi, \\ \operatorname{div} u = 0,\ \omega = \operatorname{curl} u, \end{cases}$$

ω being the scalar *vorticity* field, preserve the enstrophy measure $\mu = \mu_{0,\gamma}$ introduced above, when driven by space-time white noise ξ. Indeed, enstrophy is a first integral of motion in the case $\nu = 0$, the 2-dimensional Euler equations, and enstrophy measure is the unique, ergodic invariant measure of the linear part of the dynamics when $\nu > 0$. Notwithstanding the low space regularity under enstrophy measure, existence and pathwise uniqueness of stationary solutions for SNS in this setting are by now classical results due to [**56, 5**].

We have mentioned another quadratic invariant for Euler equations: the energy $\|u\|_{L^2}^2$. When SNS is driven by space-time white noise at the level of velocity, the *energy measure*, a white noise at the level of u, is *formally* preserved. The cursive is here in order, because the energy measure regime is so singular that no solution theory is yet available in this case. Nonetheless, the existing stochastic analysis techniques allow to deal with such regime in *hyperviscous* cases, that is replacing the viscous term Δu with $-(-\Delta)^\theta u$, $\theta > 1$. Indeed, a procedure known as *Itō trick* in the literature related to regularisation by noise is employed in [**100**] to give meaning and solve SNS under energy measure with sufficiently strong hyperviscosity. We also mention the recent development [**101**], in which Kolmogorov equations are solved by means of Gaussian analysis tools, broadening the result to solutions absolutely continuous with respect to energy measure.

The analogue of vorticity field for 2-dimensional Primitive Equations is $\omega = \partial_z v$, as the quadratic observable $\|\partial_z v\|_{L^2}^2$ is a first integral of the 2-dimensional *hydrostatic Euler equations*:

$$\begin{cases} \partial_t v + v\partial_x v + w\partial_z v + \partial_x p = 0, \\ \partial_z p = 0, \\ \partial_x v + \partial_z w = 0. \end{cases} \tag{1.7.2}$$

Prescribing the correct additive Gaussian noise η, the linear part of (1.7.1) with $\mathcal{D}(\Delta) = \Delta$ preserves the Gibbsian measure associated to $\|\partial_z v\|_{L^2}^2 = \|\omega\|_{L^2}^2$, formally defined by

$$d\mu(\omega) = \frac{1}{Z} e^{-\frac{1}{2}\|\omega\|_{L^2}^2} d\omega,$$

that is, white noise distribution for ω. However, as we will detail below, the stationary regime with μ-distributed marginals for (1.7.1) is not comparable to the enstrophy measure stationary regime of SNS, because of the more singular nonlinearity. Indeed, unlike in [**56, 5**], the nonlinear terms of (1.7.1) can not be defined as distributions when $\partial_z v$ has law μ. Still, as in the case of energy measure SNS, hyperviscosity allows to apply the techniques of [**100**].

In [**97**] it was presented a solution theory of 2-dimensional Primitive Equations in the hyperviscous setting $\mathcal{D}(\Delta) = -(-\Delta)^\theta$, for large enough θ and a suitable stochastic forcing. The regularising effect of hyperviscosity for Navier-Stokes and Primitive Equations is well-understood in the deterministic setting, and it is often used in numerical simulations [**120**]; we refer to [**119, 123, 124**] and, more recently, [**103**] for a thorough discussion. The main contribution of [**97**] is thus to introduce a Gaussian invariant measure in the context of 2-dimensional Primitive Equations, and then to exploit the techniques of [**100**] to provide a first well-posedness result for this singular SPDE in a hyperviscous setting.

Although stochastic versions of Primitive Equations both in two and three dimensions have already been considered, to the best of our knowledge the existing literature is limited to more regular regimes. To mention a few relevant previous works, in [**89**, **87**, **159**], 2-dimensional Primitive Equations are considered with a multiplicative noise taking values in function spaces, the same is done in the 3-dimensional case in [**61**, **83**], and in [**86**] the authors prove the existence of an invariant measure in this setting. In the 2-dimensional cases, large deviation principles are studied in [**82**, **158**]. Let us also mention the works [**146**, **32**] on deterministic 2-dimensional Primitive Equations, whose study began with [**124**, **125**, **122**], and [**136**] on their inviscid version, by which the vorticity formulation we present below for our model is inspired.

CHAPTER 2

Gaussian Solutions by Fokker-Planck Equation

This Chapter covers the results of [**72**]. It is devoted to existence of stationary solutions of the 2-dimensional stochastic Euler equation on the torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, including a friction term and space-time additive white noise forcing,

$$\begin{cases} d\omega + u\cdot\nabla\omega dt = -\alpha\omega\, dt + \sqrt{2\alpha}dW, \\ \nabla^\perp u = \omega. \end{cases} \tag{2.0.1}$$

As discussed above, we assume that ω has zero space average on $\mathbb{T}^2$: all function spaces on $\mathbb{T}^2$ are tacitly assumed to have zero averaged elements. This Chapter follows the introductory discussion of Section 1.4, and in particular we make use of the notation introduced there: on $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ we consider the normalized Haar measure dx such that $\int_{\mathbb{T}^2} dx = 1$, and the orthonormal Fourier basis $e_k(x) = e^{\mathrm{i}\, k\cdot x}$ of $L^2(\mathbb{T}^2, dx)$. We only deal with real-valued objects: Fourier coefficients of opposite modes will henceforth be complex conjugated. In order to lighten notation, we fix $\delta > 0$ and denote $E = H^{-1-\delta}(\mathbb{T}^2)$. Moreover, we denote η the space white noise on $\mathbb{T}^2$, μ is its law.

As a preliminary, we prove an existence result for the associated Fokker-Planck equation: this becomes necessary since solutions to (2.0.1) under cylindrical white noise forcing exist only in distributional spaces where classical energy or enstrophy estimates are not available. Such estimates are thus replaced by probabilistic estimates, taking averages with respect to the solution of the Fokker-Planck equation.

Using the method of Galerkin approximation, we shall prove existence of solutions ρ_t to the Fokker-Planck equation with initial data ρ_0 which are $L\log L$-integrable with respect to white noise measure μ, that is $\int \rho_0 \log^+ \rho_0 d\mu < \infty$. In the case $\alpha > 0$, the relative entropy of these solutions decrease exponentially fast as t grows to ∞; this together with an inequality of Kullback [**118**] implies the convergence to equilibrium of the solutions we constructed. In the case $\rho_0 \in L^2(\mu)$, we also have exponential convergence of ρ_t in L^2-norm.

These results put forward a difficult question that we will not treat here, namely the search for a notion of uniqueness and ergodicity of the invariant measure μ, and convergence to equilibrium of the non-stationary solutions.

Before moving on, we briefly recall some recent works on the Fokker–Planck equations in infinite dimensional settings, mainly due to Bogachev, Da Prato, Röckner and their coauthors. The work [**28**] considered Fokker–Planck equations associated to the stochastic evolution equations in a Hilbert space: under suitable conditions on the nonlinear term, they established existence and uniqueness of measure valued solution to the Fokker–Planck equation with Dirac initial condition, the solution satisfying Chapman–Kolmogorov equation. This method was further developed in [**29**] under weaker conditions on the coefficients. Existence of solutions of Fokker–Planck equations on Hilbert spaces with non-trace class second order coefficients was established in [**54**], with applications to stochastic 2D and 3D Navier–Stokes equations with non-trace class additive noise. In the more recent paper [**30**], the authors considered cases with non-constant coefficients in the second order part, and also degenerate cases where these coefficients can even vanish.

Assuming that the infinite-dimensional drifts admit certain finite-dimensional approximations, they proved a new uniqueness result for solutions to Fokker–Planck–Kolmogorov (FPK) equations for probability measures, and presented some applications for FPK equations associated to SPDEs. We refer to the last chapter of the monograph [**31**] for some general discussions on infinite dimensional FPK equations.

2.1. Galerkin Approximation and $L\log L$ Initial Data

Let us define the finite-dimensional projection of $H = L^2(\mathbb{T}^2, dx)$ onto the finite set of modes $\Lambda_N = \left\{k \in \mathbb{Z}_0^2 : |k|_\infty \le N\right\}$,

$$\Pi_N : H \ni f \mapsto \Pi_N f = \sum_{k\in\Lambda_N} \langle f, e_k\rangle_H \, e_k \in H_N, \tag{2.1.1}$$

where we can identify the finite dimensional codomain with

$$H_N = \left\{\xi \in \mathbb{C}^{\Lambda_N} : \bar{\xi}_k = \xi_{-k}\right\} \tag{2.1.2}$$

(whose dimension is $|\Lambda_N|$). On H_N we consider the Euclidean inner product induced by $\mathbb{C}^{\Lambda_N}$, and the Gaussian measure μ_N having Fourier coefficients $\hat{\mu}_N(k) = \overline{\hat{\mu}_N(-k)}$ with the law of independent standard complex Gaussian distributions.

We consider the following Galerkin approximation of (2.0.1):

$$d\Pi_N\omega + \Pi_N((K * \Pi_N\omega)\cdot\nabla\Pi_N\omega)dt = -\alpha\Pi_N\omega dt + \sqrt{2\alpha}d\Pi_N W. \tag{2.1.3}$$

This equation is in fact an SDE in $\omega^N \in H_N$, and it can be rewritten as

$$d\omega^N + b_N(\omega^N)dt = -\alpha\omega^N dt + \sqrt{2\alpha}dW^N, \quad W^N = \sum_{k\in\Lambda_N} W^k e_k, \tag{2.1.4}$$

where the W^k's are independent standard complex Brownian motions such that $\overline{W^k} = W^{-k}$, and the drift is given by

$$b_N(\xi) = -\sum_{n\in\Lambda_N} e_n \sum_{k\in\Lambda_N} \frac{k^\perp\cdot n}{|k|^2}\xi_k\xi_{n-k}, \quad \xi \in H_N,$$

as one can prove by a straightforward computation in Fourier series using that $K(x) = \sum_{k\in\mathbb{Z}_0^2} \frac{\mathrm{i}\, k^\perp}{|k|^2} e_k(x)$. By means of the above expression, it is easy to check that, for all $\xi \in H_N$,

$$\langle b_N(\xi), \xi\rangle_{H_N} = 0, \quad \operatorname{div}_{\mu_N} b_N(\xi) = \operatorname{div} b_N(\xi) - \langle b_N(\xi), \xi\rangle_{H_N} = 0. \tag{2.1.5}$$

The SDE (2.1.4) has smooth coefficients, so there exists a unique strong local solution ω_t^N given an initial datum $\omega_0^N \in H_N$; the forthcoming estimate shows that it is also global in time.

Lemma 2.1.1. *If ω_t^N is a solution of (2.1.4), then, for any $t \ge 0$,*

$$\mathbb{E}\left[\left|\omega_t^N\right|_{H_N}^2\right] \le \left|\omega_0^N\right|_{H_N}^2 e^{-2\alpha t} + |\Lambda_N|\,(1 - e^{-2\alpha t}).$$

Proof. By the Itô formula and (2.1.5), and omitting all subscripts H_N,

$$\begin{aligned} d\left|\omega_t^N\right|^2 &= -2\left\langle \omega_t^N, b_N\left(\omega_t^N\right) + \alpha\omega_t^N\right\rangle dt + 2\sqrt{2\alpha}\left\langle \omega_t^N, dW_t^N\right\rangle + 2\alpha\left\langle dW_t^N, dW_t^N\right\rangle \\ &= -2\alpha\left|\omega_t^N\right|^2 dt + 2\sqrt{2\alpha}\left\langle \omega_t^N, dW_t^N\right\rangle + 2\alpha\,|\Lambda_N|\,dt, \end{aligned}$$

and therefore

$$d\left(e^{2\alpha t}\left|\omega_t^N\right|^2\right) = 2\sqrt{2\alpha}\,e^{2\alpha t}\left\langle \omega_t^N, dW_t^N\right\rangle + 2\alpha e^{2\alpha t}\,|\Lambda_N|\,dt.$$

If we define, for $R > 0$, the stopping time

$$\tau_R = \inf\left\{t > 0 : \left|\omega_t^N\right| \ge R\right\},$$

then we have

$$\begin{aligned}\mathbb{E}\left[e^{2\alpha(t\wedge\tau_R)}\left|\omega_t^N\right|^2\right] &= \left|\omega_0^N\right|^2 + 2\sqrt{2\alpha}\,\mathbb{E}\left[\int_0^{t\wedge\tau_R} e^{2\alpha s}\left\langle\omega_s^N, dW_s^N\right\rangle\right] \\ &\quad + |\Lambda_N|\,\mathbb{E}\left[e^{2\alpha(t\wedge\tau_R)}-1\right] \\ &\le \left|\omega_0^N\right|^2 + |\Lambda_N|\left(e^{2\alpha t}-1\right),\end{aligned}$$

which concludes the proof if we let $R\uparrow\infty$ by Fatou's lemma. □

2.1.1. Finite dimensional Fokker-Planck equation. Let $\mathcal{L}_N$ be the Ornstein-Uhlenbeck operator on H_N; then $\alpha\mathcal{L}_N$ is the infinitesimal generator of the linear part of (2.1.4). We can introduce the Galerkin approximation $\mathcal{A}_N$ of $\mathcal{A}$, acting on smooth functions $F\in C_b^2(H_N)$ as

$$\mathcal{A}_N F(\xi) = -\left\langle b_N(\xi), \nabla F(\xi)\right\rangle_{H_N} + \alpha\mathcal{L}_N F(\xi). \tag{2.1.6}$$

We can thus write the Fokker-Planck equation corresponding to (2.1.4): if the law of ω_0^N has a smooth probability density ρ_0^N (with respect to μ_N), so does ω_t^N for any later time, and the density ρ_t^N satisfies

$$\begin{cases}\partial_t\rho_t^N = \mathcal{A}_N^*\rho_t^N,\\ \rho^N|_{t=0} = \rho_0^N.\end{cases} \tag{2.1.7}$$

REMARK 2.1.2. Simple heuristic arguments immediately give rise to an a priori estimate on the entropy of ρ_t^N. Indeed, if ρ_t^N is a smooth solution of (2.1.7), for any $t\ge 0$,

$$\begin{aligned}\partial_t\left(\rho_t^N\log\rho_t^N\right) &= \left(1+\log\rho_t^N\right)\partial_t\rho_t^N \\ &= \left(1+\log\rho_t^N\right)\left\langle b_N, \nabla\rho_t^N\right\rangle_{H_N} + \alpha\left(1+\log\rho_t^N\right)\mathcal{L}_N\rho_t^N.\end{aligned}$$

Integrating on H_N with respect to μ_N and using (2.1.5) we get

$$\int_{H_N}\rho_t^N\log\rho_t^N\,d\mu_N + \alpha\int_0^t\int_{H_N}\frac{\left|\nabla\rho_s^N\right|^2}{\rho_s^N}d\mu_N ds = \int_{H_N}\rho_0^N\log\rho_0^N\,d\mu_N.$$

However, the above computation is somewhat formal, since the drift b_N has quadratic growth. In the following we give a more rigorous proof of the a priori estimate, and at the same time give a meaning to the equation (2.1.7).

In the remainder of this subsection, we fix $N\in\mathbb{N}$ and assume that the initial condition of (2.1.7) belongs to

$$\rho_0^N \in L^\infty(H_N, \mathbb{R}_+). \tag{2.1.8}$$

One can extend the result below to more general initial data, but since the study of (2.1.7) is only an intermediate step, we do not pursue such generality here. Consider cut-off functions $\chi_n(\xi)=\chi(\xi/n)$, $n\ge 1$, where $\chi\in C_c^\infty(H_N,[0,1])$ is a radial function (i.e., $\chi(\xi)=\chi(|\xi|_{H_N})$ by a slight abuse of notation) such that $\chi|_{B_N(1)}\equiv 1$ and $\chi|_{B_N(2)^c}\equiv 0$, $B_N(r)$ being the ball in H_N centered at the origin with radius $r>0$. Define

$$b_N^{(n)}(\xi) = \chi_n(\xi)b_N(\xi), \quad \xi\in H_N, n\in\mathbb{N};$$

then $b_N^{(n)}$ is a smooth vector field on H_N with compact support for any $n\in\mathbb{N}$. Notice that b_N^n is still divergence-free since by (2.1.5) and $\nabla\chi_n(\xi)=\chi'\left(\frac{|\xi|}{n}\right)\frac{\xi}{n|\xi|}$ one has

$$\operatorname{div}_{\mu_N}\left(b_N^{(n)}\right) = \operatorname{div}_{\mu_N}\left(\chi_n b_N\right) = \chi_n\operatorname{div}_{\mu_N}(b_N) - \left\langle b_N, \nabla\chi_n\right\rangle_{H_N} = 0. \tag{2.1.9}$$

Now we consider the approximating operators

$$\mathcal{A}_N^{(n)}F(\xi) = -\left\langle b_N^{(n)}(\xi), \nabla F(\xi)\right\rangle_{H_N} + \alpha\mathcal{L}_N F(\xi)$$

and the corresponding Fokker-Planck equations

$$\begin{cases} \partial_t \rho_t^{(n)} = \left(\mathcal{A}_N^{(n)}\right)^* \rho_t^{(n)}, \\ \rho^{(n)}|_{t=0} = \rho_0^{(n)} = P_{1/n}^N \rho_0^N, \end{cases} \tag{2.1.10}$$

where the initial datum is regularized by means of the Ornstein-Uhlenbeck semigroup $P_t^N = e^{t\mathcal{L}_N}$ on H_N: for $t \geq 0$ the latter is explicitly given by

$$P_t^N \rho_0^N(\xi) = \int_{H_N} \rho_0^N(\eta)\left[2\pi\left(1-e^{-2t}\right)\right]^{-|\Lambda_N|/2} \exp\left(-\frac{|\eta - e^{-t}\xi|^2}{2(1-e^{-2t})}\right) d\eta. \tag{2.1.11}$$

LEMMA 2.1.3. *For any $n \geq 1$, $\rho_0^{(n)} \in C_b^\infty(H_N, \mathbb{R}_+)$ and*

$$\int_{H_N} \rho_0^{(n)} \log \rho_0^{(n)} d\mu_N \leq \int_{H_N} \rho_0^N \log \rho_0^N d\mu_N. \tag{2.1.12}$$

Moreover, the solutions $\rho_t^{(n)}$ of the equations (2.1.10) satisfy

$$\sup_{t\geq 0} \left\|\rho_t^{(n)}\right\|_\infty \leq \left\|\rho_0^N\right\|_\infty, \tag{2.1.13}$$

$$\begin{aligned} \int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} d\mu_N \leq e^{-2\alpha t} & \int_{H_N} \rho_0^N \log \rho_0^N d\mu_N \\ & + \left(1-e^{-2\alpha t}\right)\left\|\rho_0^N\right\|_{L^1(\mu_N)} \log \left\|\rho_0^N\right\|_{L^1(\mu_N)} \quad \forall\, t \geq 0. \end{aligned} \tag{2.1.14}$$

PROOF. The first assertion follows from (2.1.8) and (2.1.11); the estimate (2.1.12) is a consequence of Jensen's inequality and the invariance of μ_N for the semigroup $\left(P_t^N\right)_{t\geq 0}$.

Inequality (2.1.13) follows from (2.1.8) and the representation

$$\rho_t^{(n)}(\xi) = \mathbb{E}\left[\rho_0^{(n)}\left(X_t^{(n)}\right)\right],$$

where $X_t^{(n)}$ is the solution to the SDE

$$dX_t^{(n)} = b_N^{(n)}\left(X_t^{(n)}\right)dt - \alpha X_t^{(n)} dt + \sqrt{2\alpha}\, dW_t^N, \quad X_0^{(n)} = \xi.$$

Thanks to (2.1.9), the arguments in Remark 2.1.2 are now rigorous and we have

$$\frac{d}{dt}\int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} d\mu_N = -\alpha \int_{H_N} \frac{\left|\nabla \rho_t^{(n)}\right|^2}{\rho_t^{(n)}} d\mu_N. \tag{2.1.15}$$

Recall the log-Sobolev inequality on the finite-dimensional Gaussian space (H_N, μ_N):

$$\int_{H_N} \varphi^2 \log \frac{\varphi^2}{\|\varphi\|_{L^2(\mu_N)}^2} d\mu_N \leq 2\int_{H_N} |\nabla\varphi|^2 d\mu_N, \quad \forall\, \varphi \in W^{1,2}(H_N, \mu_N).$$

Taking $\varphi = \left(\rho_t^{(n)}\right)^{1/2}$ yields

$$\int_{H_N} \rho_t^{(n)} \log \frac{\rho_t^{(n)}}{\left\|\rho_t^{(n)}\right\|_{L^1(\mu_N)}} d\mu_N \leq \frac{1}{2}\int_{H_N} \frac{\left|\nabla \rho_t^{(n)}\right|^2}{\rho_t^{(n)}} d\mu_N.$$

Combining the latter inequality with (2.1.15) we obtain

$$\begin{aligned} \frac{d}{dt}\int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} d\mu_N \leq -2\alpha & \int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} d\mu_N \\ & + 2\alpha\left\|\rho_0^N\right\|_{L^1(\mu_N)} \log \left\|\rho_0^N\right\|_{L^1(\mu_N)}, \end{aligned}$$

where we have used the fact that

$$\|\rho_t^{(n)}\|_{L^1(\mu_N)} = \|\rho_0^{(n)}\|_{L^1(\mu_N)} = \|\rho_0^N\|_{L^1(\mu_N)} \quad \forall\, t > 0.$$

Integrating in time, we conclude that

$$\int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} d\mu_N \le e^{-2\alpha t} \int_{H_N} \rho_0^{(n)} \log \rho_0^{(n)} d\mu_N \\ + \big(1 - e^{-2\alpha t}\big) \|\rho_0^N\|_{L^1(\mu_N)} \log \|\rho_0^N\|_{L^1(\mu_N)},$$

which, together with (2.1.12), leads to the final result. □

Corollary 2.1.4. *Let $\rho_0^N \in L^\infty(H_N, \mathbb{R}_+)$. There exists a nonnegative function $\rho^N \in L^\infty\big(\mathbb{R}_+, L^\infty(H_N, \mu_N)\big)$ satisfying*

$$\sup_{t \in [0,\infty)} \|\rho_t^N\|_{L^\infty(\mu_N)} \le \|\rho_0^N\|_{L^\infty(\mu_N)}, \tag{2.1.16}$$

$$\int_{H_N} \rho_t^N \log \rho_t^N d\mu_N \le e^{-2\alpha t} \int_{H_N} \rho_0^N \log \rho_0^N d\mu_N \\ + \big(1 - e^{-2\alpha t}\big) \|\rho_0^N\|_{L^1(\mu_N)} \log \|\rho_0^N\|_{L^1(\mu_N)} \tag{2.1.17}$$

for almost every $t > 0$; moreover, for any $f \in C_c^1(\mathbb{R}_+)$ and $\psi \in C_b^\infty(H_N)$,

$$0 = f(0) \int_{H_N} \psi \rho_0^N d\mu_N \\ + \int_0^\infty \int_{H_N} \rho_t^N \Big[f'(t)\psi + f(t) \langle b_N, \nabla \psi \rangle_{H_N} + \alpha f(t) \mathcal{L}_N \psi \Big] d\mu_N dt. \tag{2.1.18}$$

In particular, the above equation shows that ρ^N satisfies (2.1.7) in a weak sense.

Remark 2.1.5. It might be possible to give a strong (i.e. pointwise) meaning to (2.1.7), but the weak formulation, combined with the estimate (2.1.17), is enough to prove existence of solutions to Fokker-Planck equations in the infinite dimensional case.

Proof. Thanks to (2.1.13), we can find a subsequence $\{\rho^{(n_i)}\}_{i\in\mathbb{N}}$ weakly-$*$ converging in $L^\infty\big(\mathbb{R}_+, L^\infty(H_N, \mu_N)\big)$ to some ρ^N satisfying (2.1.16).

Now we fix any $T > 0$. We know that $\rho^{(n_i)}$ also converges weakly in $L^1\big([0,T] \times H_N\big)$ to ρ^N. The sequence $\{\rho^{(n_i)}\}_{i\in\mathbb{N}}$ is contained in the set

$$\mathcal{S} = \left\{ u \in L^1\big([0,T] \times H_N\big) : u_t \ge 0,\ \int_{H_N} u_t \log u_t \, d\mu_N \le \Lambda(t) \text{ for all } t \in [0,T] \right\},$$

where we write $\Lambda(t)$ for the right hand side of (2.1.17). The convexity of the function $s \mapsto s \log s$ implies that $\mathcal{S}$ is a convex subset of $L^1\big([0,T] \times H_N\big)$. Since the weak closure of $\mathcal{S}$ coincides with the strong one, there exists a sequence of functions $u^{(n)} \in \mathcal{S}$ which converge strongly to ρ^N in $L^1\big([0,T] \times H_N\big)$. Up to a subsequence, $u^{(n)}$ converge to ρ^N almost everywhere, thus Fatou's lemma and (2.1.14) implies that (2.1.17) holds for a.e. $t \in (0,T)$. The arbitrariness of $T > 0$ implies that it holds for a.e. $t \in (0,\infty)$.

Finally, multiplying both sides of (2.1.10) (with n replaced by n_i) by $f \in C_c^1(\mathbb{R}_+)$ and $\psi \in C_b^\infty(H_N)$, and integrating by parts leads to

$$0 = f(0) \int_{H_N} \psi \rho_0^{(n_i)} d\mu_N \\ + \int_0^\infty \int_{H_N} \rho_t^{(n_i)} \Big[f'(t)\psi + f(t) \Big\langle b_N^{(n_i)}, \nabla \psi \Big\rangle_{H_N} + \alpha f(t) \mathcal{L}_N \psi \Big] d\mu_N dt.$$

Recall that $b_N^{(n)} = \chi_n b_N$; it is clear that $\left\langle b_N^{(n)}, \nabla\psi \right\rangle_{H_N}$ converges strongly to $\langle b_N, \nabla\psi\rangle_{H_N}$ in $L^2(\mu_N)$. By the weak-$*$ convergence of $\rho^{(n_i)}$, letting $i \to \infty$ yields (2.1.18). □

2.1.2. Proof of Theorem 1.4.3. We assume that $\rho_0 \in L\log L(E,\mu;\mathbb{R}_+)$. Define

$$\rho_0^N = P_{1/N}^N \mathbb{E}\left[\rho_0 \wedge N | \Pi_N\right], \quad N \in \mathbb{N}, \tag{2.1.19}$$

where $\mathbb{E}\left[\cdot|\Pi_N\right]$ is the conditional expectation with respect to the sub-σ-algebra generated by coordinates in H_N. Note that, for any $f \in L^1(\mu)$ and all $N \geq 1$, we can regard $\mathbb{E}\left[f|\Pi_N\right]$ as a function on E. By the invariance of μ_N for the Ornstein-Uhlenbeck semigroup $\left(P_t^N\right)_{t\geq 0}$ and Jensen's inequality,

$$\begin{aligned}\int_{H_N} \rho_0^N \log \rho_0^N \, d\mu_N &\leq \int_{H_N} \mathbb{E}\left[\rho_0 \wedge N|\Pi_N\right] \log \mathbb{E}\left[\rho_0 \wedge N|\Pi_N\right] d\mu_N \\ &= \int_E \mathbb{E}\left[\rho_0 \wedge N|\Pi_N\right] \log \mathbb{E}\left[\rho_0 \wedge N|\Pi_N\right] d\mu.\end{aligned}$$

Using again Jensen's inequality, for all $N \in \mathbb{N}$,

$$\int_{H_N} \rho_0^N \log \rho_0^N \, d\mu_N \leq \int_E (\rho_0 \wedge N) \log(\rho_0 \wedge N) d\mu \leq \int_E \rho_0 \log \rho_0 \, d\mu. \tag{2.1.20}$$

Moreover, it is easy to see that

$$\|\rho_0^N\|_{L^1(\mu_N)} \leq \|\rho_0\|_{L^1(\mu)}. \tag{2.1.21}$$

For any $N \geq 1$, taking ρ_0^N as the initial value, by the arguments in the last subsection, we have a nonnegative solution ρ^N to the finite dimensional Fokker-Planck equation (2.1.18) which verifies (2.1.17). We shall regard the solutions as functions on $E = H^{-1-\delta}(\mathbb{T}^2)$, i.e. $\rho_t^N(\omega) = \rho_t^N(\Pi_N \omega), (t,\omega) \in \mathbb{R}_+ \times E$. Then, combining (2.1.17) with (2.1.20) and (2.1.21), for a.e. $t > 0$,

$$\int_E \rho_t^N \log \rho_t^N \, d\mu \leq e^{-2\alpha t} \int_E \rho_0 \log \rho_0 \, d\mu + \left(1 - e^{-2\alpha t}\right) \|\rho_0\|_{L^1(\mu)} \log \|\rho_0\|_{L^1(\mu)}. \tag{2.1.22}$$

From this estimate and a diagonal argument, there exist a subsequence $\left\{\rho^{N_i}\right\}_{i\geq 1}$ and some function $\rho : \mathbb{R}_+ \times E \to \mathbb{R}_+$ such that, for any $T > 0$, ρ^{N_i} converges weakly in $L^1\big(0,T; L^1(E,\mu)\big)$ to ρ, and for a.e. $t > 0$,

$$\int_E \rho_t \log \rho_t \, d\mu \leq e^{-2\alpha t} \int_E \rho_0 \log \rho_0 \, d\mu + \left(1 - e^{-2\alpha t}\right) \|\rho_0\|_{L^1(\mu)} \log \|\rho_0\|_{L^1(\mu)}.$$

The proof is similar to that of Corollary 2.1.4. Moreover, by the duality of Orlicz spaces, one has, for any $T > 0$,

$$\lim_{i\to\infty} \int_0^T \int_E G(t,\omega) \rho_t^{N_i}(\omega) d\mu dt = \int_0^T \int_E G(t,\omega) \rho_t(\omega) d\mu dt$$

for any G such that, for some small $\varepsilon > 0$,

$$\sup_{t\in[0,T]} \int_E e^{\varepsilon |G(t,\omega)|} d\mu dt < +\infty. \tag{2.1.23}$$

Fixing any cylindrical function ψ and $f \in C_c^1(\mathbb{R}_+)$, for N big enough we always have the equation (2.1.18); replacing N by N_i, it can be rewritten as

$$0 = f(0) \int_E \psi \rho_0^{N_i} d\mu + \int_0^\infty \int_E \rho_t^{N_i} \Big[f'(t)\psi + f(t) \langle b_{N_i}, D\psi\rangle + \alpha f(t) \mathcal{L}\psi\Big] d\mu dt.$$

By the definition (2.1.19), it is not difficult to show that, for any cylindrical ψ,

$$\lim_{i\to\infty}\int_E \psi\rho_0^{N_i}d\mu = \int_E \psi\rho_0\,d\mu.$$

Moreover, the first and the third terms in the second integral also converge to the corresponding limits. The only term that requires our attention is the nonlinear part. We have

$$\begin{aligned}\Big|\int_0^\infty\int_E \rho_t^{N_i} f(t)\,\langle b_{N_i}, D\psi\rangle\,d\mu dt - \int_0^\infty\int_E \rho_t f(t)\,\langle \mathcal{B}, D\psi\rangle\,d\mu dt\Big| \\ \le \Big|\int_0^\infty\int_E \rho_t^{N_i} f(t)\big(\,\langle b_{N_i}, D\psi\rangle - \langle \mathcal{B}, D\psi\rangle\,\big)d\mu dt\Big| \\ + \Big|\int_0^\infty\int_E \big(\rho_t^{N_i}-\rho_t\big)f(t)\,\langle \mathcal{B}, D\psi\rangle\,d\mu dt\Big|.\end{aligned}$$

By (1.4.12), $G(t,\omega) := f(t)\,\langle \mathcal{B}, D\psi\rangle$ satisfies (2.1.23). Thus, the second term on the right hand side tends to 0 as $i\to\infty$. Next, one can prove that $\langle b_{N_i}, D\psi\rangle$ converges strongly in $L^1(E,\mu)$ to $\langle \mathcal{B}, D\psi\rangle$ as $i\to\infty$, see for instance [**58**, Section 3.3.1]. Combining the convergence with the uniform exponential integrability of these quantities, we deduce that the sequence $\langle b_{N_i}, D\psi\rangle$ actually converges to $\langle \mathcal{B}, D\psi\rangle$ in the Orlicz norm. Therefore, by (2.1.22), the first term also vanishes as $i\to\infty$. Thus, we can let $i\to\infty$ in the above equality to get the equation

$$0 = f(0)\int_E \psi\rho_0 d\mu + \int_0^\infty\int_E \rho_t\Big[f'(t)\psi + f(t)\,\langle \mathcal{B}, D\psi\rangle + \alpha f(t)\mathcal{L}\psi\Big]d\mu dt. \tag{2.1.24}$$

Therefore, ρ_t solves the Fokker-Planck equation (1.4.13) for $L\log L$ initial condition. The proof of Theorem 1.4.3 is complete.

2.2. L^p-initial data

In this section we assume the initial data of the Fokker-Planck equation (1.4.13) to be integrable of order $p>1$. In this case, we can follow the arguments in the last section to prove the existence of weak solutions to the Fokker-Planck equations (1.4.13). Here we only prove new a priori estimates on the Galerkin approximations and the exponential convergence in $L^2(\mu)$ norm in the case $p=2$.

2.2.1. A priori estimates for $p>1$. Assume first $\rho_0^N\in L^\infty(H_N,\mu_N)$ and consider as above the Fokker-Planck equation (2.1.10):

$$\begin{cases}\partial_t\rho_t^{(n)} = \big(\mathcal{A}_N^{(n)}\big)^*\rho_t^{(n)},\\ \rho^{(n)}|_{t=0} = \rho_0^{(n)} = P^N_{1/n}\rho_0^N.\end{cases}$$

Jensen's inequality implies

$$\int_{H_N}\big|\rho_0^{(n)}\big|^p d\mu_N \le \int_{H_N}\big|\rho_0^N\big|^p d\mu_N \quad \text{for all } n\ge 1, \tag{2.2.1}$$

and we can extend this bound for all subsequent times.

LEMMA 2.2.1. *For any $n\in\mathbb{N}$, it holds that*

$$\int_{H_N}\big|\rho_t^{(n)}\big|^p d\mu_N \le \int_{H_N}\big|\rho_0^N\big|^p d\mu_N \quad \textit{for all } t>0.$$

PROOF. Using equation (2.1.10),

$$\begin{aligned}\partial_t\Big[\big|\rho_t^{(n)}\big|^p\Big] &= p\Big[\big(\rho_t^{(n)}\big)^2\Big]^{\frac{p}{2}-1}\rho_t^{(n)}\partial_t\rho_t^{(n)}\\ &= b_N^{(n)}\cdot\nabla\Big[\big|\rho_t^{(n)}\big|^p\Big] + p\alpha\Big[\big(\rho_t^{(n)}\big)^2\Big]^{\frac{p-1}{2}}\mathcal{L}_N\rho_t^{(n)}.\end{aligned}$$

Integrating by parts on H_N with respect to μ_N gives us

$$\frac{d}{dt}\int_{H_N}\Big[\big(\rho_t^{(n)}\big)^p\Big]d\mu_N = -p\alpha\int_{H_N}\big(\rho_t^{(n)}\big)^{p-2}\Big|\nabla\rho_t^{(n)}\Big|^2 d\mu_N.$$

Next, integrating in time between 0 and t leads to

$$\int_{H_N}\Big[\big(\rho_t^{(n)}\big)^p\Big]d\mu_N \le \int_{H_N}\Big[\big(\rho_0^{(n)}\big)^p\Big]d\mu_N,$$

which, together with (2.2.1), yields the desired estimate. □

As a consequence, $\{\rho^{(n)}\}_{n\ge 1}$ is bounded in $L^\infty\big(\mathbb{R}_+, L^p(H_N,\mu_N)\big)$. Thus we can find a subsequence which converges weakly-$*$ to some limit

$$\rho^N \in L^\infty\big(\mathbb{R}_+, L^p(H_N,\mu_N)\big),$$

satisfying the estimate

$$\sup_{t\in\mathbb{R}_+}\int_{H_N}\big|\rho_t^N\big|^p d\mu_N \le \int_{H_N}\big|\rho_0^N\big|^p d\mu_N \tag{2.2.2}$$

and the finite dimensional Fokker-Planck equation

$$\begin{aligned} 0 = f(0)&\int_{H_N}\psi\rho_0^N d\mu_N \\ &+\int_0^\infty\int_{H_N}\rho_t^N\Big[f'(t)\psi + f(t)\left\langle b_N,\nabla\psi\right\rangle_{H_N} + \alpha f(t)\mathcal{L}_N\psi\Big]d\mu_N dt \end{aligned} \tag{2.2.3}$$

for any $\psi\in C_b^\infty(H_N)$ and $f\in C_c^1(\mathbb{R}_+)$.

Next, if $\rho_0\in L^p(E,\mu)$, we define, for $N\in\mathbb{N}$,

$$\rho_0^N = P_{1/N}^N\mathbb{E}\left[(-N)\vee(\rho_0\wedge N)\middle|\Pi_N\right], \tag{2.2.4}$$

which, by Jensen's inequality, satisfies

$$\sup_{N\ge 1}\int_{H_N}\big|\rho_0^N\big|^p d\mu_N \le \int_E|\rho_0|^p d\mu. \tag{2.2.5}$$

Consider the finite dimensional Fokker-Planck equations (2.2.3) with initial data ρ_0^N, and regard the solutions ρ_t^N as functions on E. From estimate (2.2.2) and inequality (2.2.5) we deduce

$$\sup_{N\ge 1}\sup_{t\in\mathbb{R}_+}\int_E\big|\rho_t^N\big|^p d\mu \le \int_E|\rho_0|^p d\mu. \tag{2.2.6}$$

Hence, we can find a subsequence ρ^{N_i} converging weakly-$*$ in $L^\infty\big(\mathbb{R}_+, L^p(E,\mu)\big)$ to some ρ, which can be shown to satisfy the Fokker-Planck equation (1.4.13), thus completing the proof of point (i) of Theorem 1.4.4. We omit the details.

2.2.2. The case $p = 2$. We want to show the exponential decay of the energy, proving point (ii) of Theorem 1.4.4. We start again from equation (2.1.10) with the initial condition $\rho_0^{(n)} = P_{1/n}^N\rho_0^N$, where $\rho_0^N\in L^\infty(H_N)$. It is clear that for all $n\ge 1$,

$$\bar\rho_0^{(n)} := \int_{H_N}\rho_0^{(n)}d\mu_N = \int_{H_N}\rho_0^N d\mu_N =: \bar\rho_0^N.$$

Lemma 2.2.2. *It holds that*

$$\int_{H_N}\big(\rho_t^{(n)} - \bar\rho_0^N\big)^2 d\mu_N \le e^{-2\alpha t}\int_{H_N}\big(\rho_0^N - \bar\rho_0^N\big)^2 d\mu_N \quad \textit{for all } t>0.$$

PROOF. According to equation (2.1.10), we have

$$\partial_t\Big[\big(\rho_t^{(n)}-\bar\rho_0^N\big)^2\Big]=2\big(\rho_t^{(n)}-\bar\rho_0^N\big)b_N^{(n)}\cdot\nabla\rho_t^{(n)}+2\alpha\big(\rho_t^{(n)}-\bar\rho_0^N\big)\mathcal{L}_N\rho_t^{(n)}.$$

By (2.1.9), integrating by parts with respect to μ_N yields

$$\frac{d}{dt}\int\big(\rho_t^{(n)}-\bar\rho_0^N\big)^2d\mu_N=-2\alpha\int\Big|\nabla\rho_t^{(n)}\Big|^2d\mu_N.$$

Recall that μ_N satisfies the Poincaré inequality on H_N: for any $\varphi\in W^{1,2}(H_N,\mu_N)$,

$$\int(\varphi-\bar\varphi)^2d\mu_N\le\int|\nabla\varphi|^2d\mu_N,$$

where $\bar\varphi=\int\varphi\,d\mu_N$. Therefore,

$$\frac{d}{dt}\int\big(\rho_t^{(n)}-\bar\rho_0^N\big)^2d\mu_N\le-2\alpha\int\big(\rho_t^{(n)}-\bar\rho_0^N\big)^2d\mu_N,$$

where we used the fact that $\bar\rho_t^{(n)}:=\int\rho_t^{(n)}d\mu_N=\bar\rho_0^{(n)}=\bar\rho_0^N$ for all $t>0$. As a result,

$$\int\big(\rho_t^{(n)}-\bar\rho_0^N\big)^2d\mu_N\le e^{-2\alpha t}\int\big(\rho_0^{(n)}-\bar\rho_0^N\big)^2d\mu_N\quad\text{for all } t>0.$$

Finally, we complete the proof by noting that

$$\begin{aligned}\int\big(\rho_0^{(n)}-\bar\rho_0^N\big)^2d\mu_N&=\int\big(\rho_0^{(n)}\big)^2d\mu_N-\big(\bar\rho_0^N\big)^2\\&\le\int\big(\rho_0^N\big)^2d\mu_N-\big(\bar\rho_0^N\big)^2=\int\big(\rho_0^N-\bar\rho_0^N\big)^2d\mu_N,\end{aligned}$$

where we have used Jensen's inequality in the second step. □

Repeating the arguments below Lemma 2.2.1, there exists a subsequence $\rho^{(n_i)}$ converging weakly-$*$ to some $\rho^N\in L^\infty\big(\mathbb{R}_+,L^2(H_N,\mu_N)\big)$, which is a weak solution to the finite dimensional Fokker-Planck equations (2.2.3) with the initial datum ρ_0^N. Moreover, replacing the set $\mathcal{S}$ in the proof of Corollary 2.1.4 by

$$\tilde{\mathcal{S}}=\Big\{u\in L^2\big([0,T]\times H_N\big):\big\|u_t-\bar\rho_0^N\big\|_{L^2(\mu_N)}\le e^{-\alpha t}\big\|\rho_0^N-\bar\rho_0^N\big\|_{L^2(\mu_N)}\ \forall\,t\in[0,T]\Big\},$$

similar discussions imply that for a.e. $t\in(0,T)$, one has

$$\big\|\rho_t^N-\bar\rho_0^N\big\|_{L^2(\mu_N)}\le e^{-\alpha t}\big\|\rho_0^N-\bar\rho_0^N\big\|_{L^2(\mu_N)}.$$

The arbitrariness of $T>0$ yields that the above inequality holds for a.e. $t>0$.

Next, for $\rho_0\in L^2(E,\mu)$ and $N\in\mathbb{N}$, we define ρ_0^N as in (2.2.4). We have

$$\bar\rho_0^N=\int_{H_N}\rho_0^N\,d\mu_N=\int_{H_N}\mathbb{E}\left[(-N)\vee(\rho_0\wedge N)\big|\Pi_N\right]d\mu_N=\int_E(-N)\vee(\rho_0\wedge N)d\mu,$$

therefore,

$$\lim_{N\to\infty}\bar\rho_0^N=\int_E\rho_0\,d\mu=\bar\rho_0.$$

This together with (2.2.5) (taking $p=2$) implies

$$\limsup_{N\to\infty}\int_{H_N}\big(\rho_0^N-\bar\rho_0^N\big)^2d\mu_N\le\int_E(\rho_0-\bar\rho_0)^2d\mu.\tag{2.2.7}$$

For any $N\ge1$, there exists a weak solution $\big(\rho_t^N\big)_{t\in\mathbb{R}_+}$ to the equation (2.2.3) with the initial condition ρ_0^N, satisfying

$$\big\|\rho_t^N-\bar\rho_0^N\big\|_{L^2(\mu_N)}\le e^{-\alpha t}\big\|\rho_0^N-\bar\rho_0^N\big\|_{L^2(\mu_N)}\quad\text{for a.e. } t\in(0,\infty).\tag{2.2.8}$$

As usual, we view $\rho_t^N (N \geq 1)$ as functionals on E. As in Section 2.2.1, there is a subsequence ρ^{N_i} converging weakly-$*$ to some $\rho \in L^\infty\big(\mathbb{R}_+, L^2(E,\mu)\big)$. By (2.2.7) and (2.2.8), we can show the exponential decay of the energy of ρ_t for a.e. $t > 0$.

2.3. Existence of Weak solutions

Thanks to the control on densities we have gained in the last Section, we are now in the position to prove Theorem 1.4.5. Let us thus take $\rho_0 \in L^p(E,\mu;\mathbb{R}_+)$ for some $p > 1$, satisfying $\bar\rho_0 = \int_E \rho_0 \, d\mu = 1$. We define ρ_0^N similarly to (2.2.4):

$$\rho_0^N = c_N^{-1} P_{1/N}^N \mathbb{E}\left[(\rho_0 \wedge N)\big|\Pi_N\right], \tag{2.3.1}$$

where c_N is the normalizing constant such that $\bar\rho_0^N = \int_{H_N} \rho_0^N d\mu_N = 1$. Clearly,

$$\lim_{N\to\infty} c_N = 1.$$

Let ρ_t^N be the solution of the finite dimensional Fokker-Planck equations (2.2.3) with initial data ρ_0^N. Combining the above fact with (2.2.6), we see that

$$\sup_{N\geq 1} \sup_{t\in[0,T]} \big\|\rho_t^N\big\|_{L^p(\mu)} \leq c_0 \|\rho_0\|_{L^p(\mu)}. \tag{2.3.2}$$

Consider the solution ω_t^N of the SDEs (2.1.4), for which the initial values ω_0^N is distributed as $\rho_0^N \mu_N$; then ρ_t^N is the probability density function (with respect to μ_N) of ω_t^N. In this part we regard ω_t^N and ρ_t^N as objects defined on $E = H^{-1-}$, i.e. $\omega_t^N(\omega) = \omega_t^N(\Pi_N\omega)$, $\rho_t^N(\omega) = \rho_t^N(\Pi_N\omega)$. We want to show that the laws Q^N of $\omega_\cdot^N$ on $C\big([0,T],E\big)$ are tight. To this end we will use the compactness criterion proved in [**156**, Corollary 9, p. 90]. The arguments here follow those of [**75**, Section 3].

Take $\delta \in (0,1)$, $\kappa > 5$ (this choice is due to estimates below) and consider the spaces

$$X = H^{-1-\delta/2}(\mathbb{T}^2), \quad B = H^{-1-\delta}(\mathbb{T}^2), \quad Y = H^{-\kappa}(\mathbb{T}^2).$$

Then $X \subset B \subset Y$ with compact embeddings and we also have, for a suitable constant $C > 0$ and for

$$\theta = \frac{\delta/2}{\kappa - 1 - \delta/2}, \tag{2.3.3}$$

the interpolation inequality

$$\|\omega\|_B \leq C\|\omega\|_X^{1-\theta}\|\omega\|_Y^\theta, \quad \omega \in X.$$

These are the preliminary assumptions of [**156**, Corollary 9, p. 90]. We consider here a particular case:

$$\mathcal{S} = L^{p_0}(0,T;X) \cap W^{1/3,4}(0,T;Y),$$

where for $0 < \alpha < 1$ and $p \geq 1$,

$$W^{\alpha,p}(0,T;Y) = \left\{ f : f \in L^p(0,T;Y) \text{ and } \int_0^T\int_0^T \frac{\|f(t)-f(s)\|_Y^p}{|t-s|^{\alpha p+1}} dtds < \infty \right\}.$$

Lemma 2.3.1. *Let $\delta \in (0,1)$ and $\kappa > 5$ be given. If*

$$p_0 > \frac{12(\kappa - 1 - 3\delta/2)}{\delta},$$

then $\mathcal{S}$ is compactly embedded into $C\big([0,T], H^{-1-\delta}(\mathbb{T}^2)\big)$.

Proof. Recall that θ is defined in (2.3.3). In our case, we have $s_0 = 0, r_0 = p_0$ and $s_1 = 1/3, r_1 = 4$. Hence $s_\theta = (1-\theta)s_0 + \theta s_1 = \theta/3$ and

$$\frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} = \frac{1-\theta}{p_0} + \frac{\theta}{4}.$$

It is clear that for p_0 given above, it holds $s_\theta > 1/r_\theta$, thus the desired result follows from the second assertion of [**156**, Corollary 9]. ☐

For $N \geq 1$, let Q^N be the law of $\omega_\cdot^N$ on $\mathcal{X} := C\big([0,T], H^{-1-}(\mathbb{T}^2)\big)$. We want to prove that the family $\{Q^N\}_{N\geq 1}$ is tight in $\mathcal{X}$. The next result follows from the definition of the topology in $\mathcal{X}$.

LEMMA 2.3.2. *The family $\{Q^N\}_{N\geq 1}$ is tight in $\mathcal{X}$ if and only if it is tight in the space $C\big([0,T], H^{-1-\delta}(\mathbb{T}^2)\big)$ for any $\delta > 0$.*

In view of the above two lemmas, it is sufficient to prove that $\{Q^N\}_{N\geq 1}$ is bounded in probability in $W^{1/3,4}\big(0,T; H^{-\kappa}(\mathbb{T}^2)\big)$ and in each $L^{p_0}\big(0,T; H^{-1-\delta}(\mathbb{T}^2)\big)$ for any $p_0 > 0$ and $\delta > 0$.

We show first that the family $\{Q^N\}_{N\geq 1}$ is bounded in probability on the space $L^{p_0}\big(0,T; H^{-1-\delta}(\mathbb{T}^2)\big)$. Let us recall that, for any $q > 1$ and $\delta > 0$, there exists $C_{q,\delta} > 0$ such that

$$\int \|\omega\|_{H^{-1-\delta}}^q \, d\mu \leq C_{q,\delta}.$$

We have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \|\omega_t^N\|_{H^{-1-\delta}}^{p_0} dt\right] &= \int_0^T \mathbb{E}\left[\|\omega_t^N\|_{H^{-1-\delta}}^{p_0}\right] dt \\
&= \int_0^T \int \|\omega\|_{H^{-1-\delta}}^{p_0} \rho_t^N(\omega) d\mu dt \\
&\leq \int_0^T \left[\int \|\omega\|_{H^{-1-\delta}}^{p_0 q} d\mu\right]^{1/q} \left[\int \big(\rho_t^N(\omega)\big)^p d\mu\right]^{1/p} dt \\
&\leq C_{p_0 q,\delta} T \sup_{t\in[0,T]} \|\rho_t^N\|_{L^p(\mu)} \leq C_{p_0 q,\delta} T \|\rho_0\|_{L^p(\mu)},
\end{aligned} \tag{2.3.4}$$

where q is the conjugate number of p and we have used the above estimate and (2.3.2) in the last two steps. By Chebyshev's inequality, the family $\{Q^N\}_{N\geq 1}$ is bounded in probability in $L^{p_0}\big(0,T; H^{-1-\delta}(\mathbb{T}^2)\big)$.

Next, we prove boundedness in probability of $\{Q^N\}_{N\geq 1}$ in $W^{1/3,4}\big(0,T; H^{-\kappa}(\mathbb{T}^2)\big)$ where $\kappa > 5$. Again by Chebyshev's inequality, it suffices to show that

$$\sup_{N\geq 1} \mathbb{E}\left[\int_0^T \|\omega_t^N\|_{H^{-\kappa}}^4 dt + \int_0^T \int_0^T \frac{\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4}{|t-s|^{7/3}} dt ds\right] < \infty.$$

In view of (2.3.4), we see that it is sufficient to establish a uniform estimate on the expectation $\mathbb{E}\left[\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4\right]$. We write $\langle \cdot, \cdot \rangle$ for the inner product in $L^2(\mathbb{T}^2)$.

LEMMA 2.3.3. *There exists $C > 0$ depending on α, δ and $\|\rho_0\|_{L^p(\mu)}$ such that for any $k \in \Lambda_N$, we have*

$$\mathbb{E}\left[\langle \omega_t^N - \omega_s^N, e_k \rangle^4\right] \leq C(t-s)^2\big(|k|^8 + 1\big).$$

PROOF. By equation (2.1.4),

$$\begin{aligned}
\langle \omega_t^N, e_k \rangle &= \langle \omega_0^N, e_k \rangle + \int_0^t \langle \omega_s^N, u(\omega_s^N) \cdot \nabla e_k \rangle ds \\
&\quad - \alpha \int_0^t \langle \omega_s^N, e_k \rangle ds + \sqrt{2\alpha} \int_0^t \left\langle dW_s^{(N)}, e_k \right\rangle \\
&= \langle \omega_0^N, e_k \rangle + \int_0^t \langle \omega_s^N \otimes \omega_s^N, H_{e_k} \rangle ds - \alpha \int_0^t \langle \omega_s^N, e_k \rangle ds + \sqrt{2\alpha}\, W_t^k.
\end{aligned}$$

Therefore, for $0 \le s < t \le T$,

(2.3.5)
$$\langle \omega_t^N - \omega_s^N, e_k \rangle = \int_s^t \langle \omega_r^N \otimes \omega_r^N, H_{e_k} \rangle \, dr - \alpha \int_s^t \langle \omega_r^N, e_k \rangle \, dr + \sqrt{2\alpha}(W_t^k - W_s^k).$$

First, we control by Hölder's inequality:

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\int_s^t \langle \omega_r^N \otimes \omega_r^N, H_{e_k} \rangle \, dr\bigg)^4\bigg] \\
&\le (t-s)^3 \, \mathbb{E}\bigg[\int_s^t \langle \omega_r^N \otimes \omega_r^N, H_{e_k} \rangle^4 \, dr\bigg] \\
&= (t-s)^3 \int_s^t \int \langle \omega \otimes \omega, H_{e_k} \rangle^4 \, \rho_r^N \, d\mu dr \\
&\le (t-s)^3 \int_s^t \bigg[\int \langle \omega \otimes \omega, H_{e_k} \rangle^{4q} \, d\mu\bigg]^{1/q} \bigg[\int (\rho_r^N)^p d\mu\bigg]^{1/p} dr.
\end{aligned}$$

By (1.4.9) and the uniform density estimate (2.3.2),

$$\begin{aligned}
\mathbb{E}\bigg[\bigg(\int_s^t \langle \omega_r^N \otimes \omega_r^N, H_{e_k} \rangle \, dr\bigg)^4\bigg] &\le C_q \|e_k\|_{C^2(\mathbb{T}^2)}^4 (t-s)^4 \sup_{t\in[0,T]} \big\|\rho_r^N\big\|_{L^p(\mu)} \\
&\le C_q (t-s)^4 |k|^8 \|\rho_0\|_{L^p(\mu)}.
\end{aligned} \tag{2.3.6}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\bigg[\bigg(\int_s^t \langle \omega_r^N, e_k \rangle \, dr\bigg)^4\bigg] &\le (t-s)^3 \, \mathbb{E} \int_s^t \langle \omega_r^N, e_k \rangle^4 \, dr \\
&= (t-s)^3 \int_s^t \int \langle \omega, e_k \rangle^4 \, \rho_r^N \, d\mu dr \\
&\le C_q (t-s)^4 \|\rho_0\|_{L^p(\mu)}.
\end{aligned} \tag{2.3.7}$$

Finally,

$$\mathbb{E}\big[(W_t^k - W_s^k)^4\big] \le C(t-s)^2.$$

Combining this estimate with (2.3.5)–(2.3.7) yields the result. □

As a result of Lemma 2.3.3, by Cauchy's inequality,

$$\begin{aligned}
\mathbb{E}\big(\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4\big) &= \mathbb{E}\bigg[\bigg(\sum_{k\in\mathbb{Z}_0^2} |k|^{-2\kappa} \langle \omega_t^N - \omega_s^N, e_k \rangle^2\bigg)^2\bigg] \\
&\le \bigg(\sum_{k\in\mathbb{Z}_0^2} |k|^{-2\kappa}\bigg) \sum_{k\in\mathbb{Z}_0^2} |k|^{-2\kappa} \, \mathbb{E}\Big[\langle \omega_t^N - \omega_s^N, e_k \rangle^4\Big] \\
&\le \tilde{C}(t-s)^2 \sum_{k\in\mathbb{Z}_0^2} |k|^{-2\kappa} |k|^8 \le \hat{C}(t-s)^2,
\end{aligned}$$

since $2\kappa - 8 > 2$ due to the choice of κ. Consequently,

$$\mathbb{E}\bigg[\int_0^T \int_0^T \frac{\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4}{|t-s|^{7/3}} dtds\bigg] \le \hat{C} \int_0^T \int_0^T \frac{|t-s|^2}{|t-s|^{7/3}} dtds < \infty.$$

The proof of the boundedness in probability of $\{Q^N\}_{N\ge1}$ in $W^{1/3,4}\big(0,T; H^{-\kappa}(\mathbb{T}^2)\big)$ is complete.

To summarize, we have shown that the family $\{Q^N\}_{N\ge1}$ of laws of $\{\omega_\cdot^N\}_{N\ge1}$ is tight on $\mathcal{X} = C([0,T], E)$. Since we are dealing with the SDEs (2.1.4), it is necessary to consider the laws of $\omega_\cdot^N$ together with the law $\mathcal{W}$ on $\mathcal{Y} = C\big([0,T], \mathbb{R}^{\mathbb{Z}_0^2}\big)$ of the family of Brownian motions $W := \{W_\cdot^k\}_{k\in\mathbb{Z}_0^2}$. For any $N \in \mathbb{N}$, we denote $Q^N \otimes \mathcal{W}$ the joint law (not the product measure) of $(\omega_\cdot^N, W)$ on

$$\mathcal{X} \times \mathcal{Y} = C([0,T],E) \times C\Big([0,T],\mathbb{R}^{\mathbb{Z}_0^2}\Big).$$

Then, it is easy to see that the family $\big\{Q^N \otimes \mathcal{W}\big\}_{N\geq 1}$ of joint laws is tight on $\mathcal{X}\times\mathcal{Y}$, cf. the arguments above [**75**, Lemma 3.4]. Thus, by Prohorov's theorem (see [**25**, Theorem 5.1, p. 59]), we can find a subsequence $\big\{Q^{N_i} \otimes \mathcal{W}\big\}_{i\geq 1}$ which converges weakly to some $Q \otimes \mathcal{W}$, a probability measure on $\mathcal{X} \times \mathcal{Y}$. Next, the Skorokhod theorem (see [**25**, Theorem 6.7, p. 70] implies that there exist a probability space $\big(\tilde\Theta, \tilde{\mathcal{F}}, \tilde{\P}\big)$, a sequence of processes $\big\{\big(\tilde\omega_\cdot^{N_i}, \tilde W^{N_i}\big)\big\}_{i\in\mathbb{N}}$ and a limit process $\big(\tilde\omega_\cdot, \tilde W\big)$ defined on this probability space such that, for all $i \in \mathbb{N}$, the law of $\big(\tilde\omega_\cdot^{N_i}, \tilde W^{N_i}\big)$ is $Q^{N_i} \otimes \mathcal{W}$, and $\tilde{\P}$-a.s., $\big(\tilde\omega_\cdot^{N_i}, \tilde W^{N_i}\big)$ converges in $\mathcal{X}\times\mathcal{Y}$ to $\big(\tilde\omega_\cdot, \tilde W\big)$ as $i \to \infty$. Note that $\tilde W^{N_i}$ and $\tilde W$ are families of Brownian motions indexed by $\mathbb{Z}_0^2$.

We need one last result before proving the existence of solutions to (2.0.1).

LEMMA 2.3.4. *For a.e. $t \in [0,T]$, the law of $\tilde\omega_t$ on E has a density ρ_t with respect to μ, where ρ_t is a weak solution to the Fokker–Planck equation* (1.4.13).

PROOF. Fix any $F \in C_b(E,\mathbb{R})$ and $f \in C([0,T])$. By the $\tilde{\P}$-a.s. convergence of $\tilde\omega_\cdot^{N_i}$ to $\tilde\omega_\cdot$ in $\mathcal{X} = C([0,T],E)$, we have

$$\begin{aligned}\mathbb{E}_{\tilde{\P}} \int_0^T f(t)F(\tilde\omega_t)dt &= \lim_{i\to\infty} \mathbb{E}_{\tilde{\P}} \int_0^T f(t)F\big(\tilde\omega_t^{N_i}\big)dt = \lim_{i\to\infty} \mathbb{E}_{\P} \int_0^T f(t)F\big(\omega_t^{N_i}\big)dt \\ &= \lim_{i\to\infty} \int_0^T f(t) \int_E F(\omega)\rho_t^{N_i}(\omega)\, d\mu(\omega)dt.\end{aligned}$$

The densities $\rho_\cdot^{N_i} (i \in \mathbb{N})$ satisfy the estimates (2.3.2), thus, taking a further subsequence if necessary, we can assume that $\rho_\cdot^{N_i}$ converge weakly to some limit $\rho_\cdot$, which by the first half of Theorem 1.4.4, is a weak solution of the Fokker–Planck equation (1.4.13). Next, we have

$$\int_0^T f(t)\mathbb{E}_{\tilde{\P}} F(\tilde\omega_t)dt = \int_0^T f(t) \int_E F(\omega)\rho_t(\omega)d\mu(\omega)dt.$$

The arbitrariness of $f \in C([0,T])$ implies that, for almost every $t \in [0,T]$,

$$\mathbb{E}_{\tilde{\P}} F(\tilde\omega_t) = \int_E F(\omega)\rho_t(\omega)d\mu(\omega).$$

We can take a countable dense subset $\mathcal{C} \subset C_b(E,\mathbb{R})$ of functionals F such that, for almost every $t \in [0,T]$, the above equality holds for all $F \in \mathcal{C}$. Thus the law of $\tilde\omega_t$ is ρ_t. □

Up to now, we have indeed obtained the assertions (i) and (ii) of Theorem 1.4.5. Finally, we can prove the existence of weak solutions to the stochastic Euler equation (2.0.1).

PROOF OF THEOREM 1.4.5, ITEM (III). Recall that ω^{N_i} solves the finite dimensional equation (2.1.4) with N_i in place of N, and $\Big(\tilde\omega^{N_i}, \tilde W^{N_i}\Big)$ has the same law as $\big(\omega^{N_i}, W\big)$, where we write W for the family of Brownian motions $\{W_\cdot^k\}_{k\in\mathbb{Z}_0^2}$, similarly for $\tilde W^{N_i}$. Thus, for any $\phi \in C^\infty(\mathbb{T}^2)$,

$$\begin{aligned}\Big\langle \tilde\omega_t^{N_i}, \phi\Big\rangle = \Big\langle \tilde\omega_0^{N_i}, \phi\Big\rangle &+ \int_0^t \big\langle \tilde\omega_s^{N_i}, \big(K * \tilde\omega_s^{N_i}\big)\cdot\nabla\phi_{N_i}\big\rangle\, ds \\ &- \alpha \int_0^t \big\langle \tilde\omega_s^{N_i}, \phi\big\rangle\, ds + \sqrt{2\alpha}\Big\langle \tilde W_t^{N_i}, \phi\Big\rangle,\end{aligned}$$

where $\phi_{N_i} = \Pi_{N_i}\phi = \sum_{k\in\Lambda_{N_i}} \langle \phi, e_k\rangle e_k$.

By the $\tilde{\mathbb{P}}$-a.s. convergence of $\left(\tilde\omega^{N_i}, \tilde W^{N_i}\right)$ to $\left(\tilde\omega_\cdot, \tilde W\right)$ in $\mathcal{X}\times\mathcal{Y}$ as $i\to\infty$, it is clear that all the terms but the nonlinear part converge in $L^1\big(\tilde\Theta, \tilde{\mathbb{P}}, C([0,T],\mathbb{R})\big)$ to the corresponding one in the limit. As for the nonlinear part,

$$\int_0^t \left\langle \tilde\omega_s^{N_i}, \left(K * \tilde\omega_s^{N_i}\right)\cdot\nabla\phi_{N_i}\right\rangle ds = \int_0^t \left\langle \tilde\omega_s^{N_i}\otimes\tilde\omega_s^{N_i}, H_{\phi_{N_i}}\right\rangle ds.$$

and we can bound

$$\begin{aligned}
&\mathbb{E}_{\tilde{\mathbb{P}}}\left[1\wedge \sup_{0\le t\le T}\left|\int_0^t \left\langle \tilde\omega_s^{N_i}\otimes\tilde\omega_s^{N_i}, H_{\phi_{N_i}}\right\rangle ds - \int_0^t \langle\tilde\omega_s\otimes\tilde\omega_s, H_\phi\rangle\, ds\right|\right]\\
&\quad\le \mathbb{E}_{\tilde{\mathbb{P}}}\left[1\wedge\int_0^T \left|\left\langle \tilde\omega_s^{N_i}\otimes\tilde\omega_s^{N_i}, H_{\phi_{N_i}}\right\rangle - \langle\tilde\omega_s\otimes\tilde\omega_s, H_\phi\rangle\right| ds\right]\\
&\quad\le \mathbb{E}_{\tilde{\mathbb{P}}}\left[1\wedge\int_0^T \left|\left\langle \tilde\omega_s^{N_i}\otimes\tilde\omega_s^{N_i}, H_{\phi_{N_i}} - H_\phi\right\rangle\right| ds\right]\\
&\qquad + \mathbb{E}_{\tilde{\mathbb{P}}}\left[1\wedge\int_0^T \left|\left\langle \tilde\omega_s^{N_i}\otimes\tilde\omega_s^{N_i} - \tilde\omega_s\otimes\tilde\omega_s, H_\phi\right\rangle\right| ds\right].
\end{aligned}$$

We denote the two terms on the right hand side by $I_1^{N_i}$ and $I_2^{N_i}$, respectively. By the definition of H_ϕ, we have $H_{\phi_{N_i}} - H_\phi = H_{\phi_{N_i}-\phi}$, and therefore, by (1.4.9),

$$\begin{aligned}
I_1^{N_i} &\le \mathbb{E}_{\tilde{\mathbb{P}}}\left[\int_0^T \left|\left\langle \tilde\omega_s^{N_i}\otimes\tilde\omega_s^{N_i}, H_{\phi_{N_i}} - H_\phi\right\rangle\right| ds\right]\\
&\le CT\|\phi_{N_i}-\phi\|_{C^2(\mathbb{T}^2)} \sup_{0\le s\le T}\left\|\rho_s^{N_i}\right\|_{L^p(\mu)} \le C'T\|\rho_0\|_{L^p(\mu)}\|\phi_{N_i}-\phi\|_{C^2(\mathbb{T}^2)},
\end{aligned}$$

where the last step follows from (2.3.2). Since $\phi\in C^\infty(\mathbb{T}^2)$, the Fourier series $\phi_N = \Pi_N\phi$ converge to ϕ in $C^\infty(\mathbb{T}^2)$. Thus we deduce

$$\lim_{i\to\infty} I_1^{N_i} = 0. \tag{2.3.8}$$

Next, let $H_\phi^n\in C^\infty(\mathbb{T}^2\times\mathbb{T}^2)$ be an approximating sequence of H_ϕ as in (1.4.5) and (1.4.6). By the triangle inequality,

$$\begin{aligned}
I_2^{N_i} &\le \mathbb{E}_{\tilde{\mathbb{P}}}\left[1\wedge\int_0^T \left|\left\langle \tilde\omega_s^{N_i}\otimes\tilde\omega_s^{N_i}, H_\phi^n - H_\phi\right\rangle\right| ds\right]\\
&\quad + \mathbb{E}_{\tilde{\mathbb{P}}}\left[1\wedge\int_0^T \left|\left\langle \tilde\omega_s\otimes\tilde\omega_s, H_\phi^n - H_\phi\right\rangle\right| ds\right]\\
&\quad + \mathbb{E}_{\tilde{\mathbb{P}}}\left[1\wedge\int_0^T \left|\left\langle \tilde\omega_s^{N_i}\otimes\tilde\omega_s^{N_i} - \tilde\omega_s\otimes\tilde\omega_s, H_\phi^n\right\rangle\right| ds\right]\\
&= J_{1,n}^{N_i} + J_{2,n} + J_{3,n}^{N_i}.
\end{aligned} \tag{2.3.9}$$

Recall that, by Lemma 2.3.4, $\tilde\omega_s$ has the density ρ_s for almost every $s\in(0,T)$ and the estimate below holds:

$$\sup_{0\le s\le T}\|\rho_s\|_{L^p(\mu)} \le \liminf_{i\to\infty}\sup_{0\le s\le T}\left\|\rho_s^{N_i}\right\|_{L^p(\mu)} \le c_0\|\rho_0\|_{L^p(\mu)}.$$

Therefore, by (1.4.8),

$$J_{2,n} \le \mathbb{E}_{\tilde{\mathbb{P}}}\left[\int_0^T \left|\left\langle \tilde\omega_s\otimes\tilde\omega_s, H^n_\phi - H_\phi\right\rangle\right| ds\right]$$
$$\le T\left[C_p\left\|H^n_\phi - H_\phi\right\|^{1/p'}_{L^2(\mathbb{T}^2\times\mathbb{T}^2)} + \left|\int_{\mathbb{T}^2} H^n_\phi(x,x)dx\right|\right],$$

which tends to 0 as $n\to\infty$. Next, thanks to the uniform estimates (2.3.2) on the densities $\rho^{N_i}_s$ of $\tilde\omega^{N_i}_s$, the same arguments of above yield

$$\lim_{n\to\infty} J^{N_i}_{1,n} = 0 \quad \text{uniformly in } i\in\mathbb{N}.$$

Finally, fix $n\in\mathbb{N}$; $\tilde{\mathbb{P}}$-almost surely, $\tilde\omega^{N_i}$ converges in $C([0,T],E)$ to $\tilde\omega$ as $i\to\infty$, thus

$$\lim_{i\to\infty}\int_0^T \left|\left\langle \tilde\omega^{N_i}_s\otimes\tilde\omega^{N_i}_s - \tilde\omega_s\otimes\tilde\omega_s, H^n_\phi\right\rangle\right| ds = 0.$$

As a result, for any fixed n, the dominated convergence theorem implies

$$\lim_{i\to\infty} J^{N_i}_{3,n} = 0,$$

from which, first letting $i\to\infty$ and then $n\to\infty$ in (2.3.9), we obtain

$$\lim_{i\to\infty} I^{N_i}_2 = 0.$$

Combining this limit with (2.3.8) we conclude the proof. □

2.4. Gibbsian Energy-Enstrophy Measures

We conclude this Chapter with a relevant example of an absolutely continuous measure with respect to the white noise measure μ, that is the Energy-Enstrophy measure introduced in Chapter 1. The Gaussian random distributions $\mu_{\beta,\gamma}$ are best understood in terms of Fourier series: we can write

$$\omega_{\beta,\gamma} = \sum_{k\in\mathbb{Z}^2_0}\hat\omega_{\beta,\gamma,k}e_k, \qquad \text{where } \hat\omega_{\beta,\gamma,k} = \langle\omega_{\beta,\gamma}, e_k\rangle \sim N_{\mathbb{C}}\left(0, \frac{4\pi^2|k|^2}{\beta+4\pi^2|k|^2\gamma}\right)$$

are independent $\mathbb{C}$-valued Gaussian variables, and the Fourier expansion thus converges in $L^2\left(H^s(\mathbb{T}^2),\mu_{\beta,\gamma}\right)$ for $s<-1$. The measure $\mu_{\beta,\gamma}$ is also characterised by its Fourier transform (characteristic function) on $\dot H^s(\mathbb{T}^2)$: for any $f\in\dot H^{-s}(\mathbb{T}^2)$,

$$\int e^{\mathrm{i}\langle\omega,f\rangle}d\mu_{\beta,\gamma}(\omega) = \exp\left(-\frac12\sum_{k\in\mathbb{Z}^2_0}\frac{4\pi^2|k|^2|\hat f_k|^2}{\beta+4\pi^2|k|^2\gamma}\right). \tag{2.4.1}$$

Let us recall an equivalent definition of $\mu_{\beta,\gamma}$: for a smooth vorticity distribution ω, energy is given by

$$2E(\omega) = -\left\langle\omega,\Delta^{-1}\omega\right\rangle = \sum_{k\in\mathbb{Z}^2_0}\frac{|\hat\omega_k|^2}{4\pi^2|k^2|},$$

which does not make sense as a random variable if instead ω has white noise law $\mu_{0,\gamma}=\mu_\gamma$, since in that case $\hat\omega_k$'s are i.i.d. Gaussian variables, and the series diverges almost surely. However, one can define a *renormalised energy* by means of normal ordering:

$$2:E:= \lim_{K\to\infty}\sum_{|k|\le K}\frac{:\hat\omega_k\hat\omega^*_k:}{4\pi^2|k^2|} = \lim_{K\to\infty}\sum_{|k|\le K}\left(\frac{|\hat\omega_k|^2}{4\pi^2|k^2|} - \int\frac{|\hat\omega_k|^2}{4\pi^2|k^2|}d\mu_\gamma(\omega)\right) \tag{2.4.2}$$

where the limit holds in $L^2(\mu_\gamma)$ (see [7] and Theorem 1.5.2 below), and it defines an element of the second Wiener chaos $H^{:2:}(\mu_\gamma)$. As a consequence, $:E:$ can be expressed as a double Itō-Wiener stochastic integral with respect to the white noise μ_γ, the kernel being naturally Green's function G: in the notation introduced in Chapter 1,

$$2 :E: (\omega) = \langle G, \omega \diamond \omega \rangle .$$

LEMMA 2.4.1. *The probability measure on $\dot{H}^s(\mathbb{T}^2)$, any $s < -1$, defined by density as*

$$d\tilde{\mu}_{\beta,\gamma} = \frac{1}{Z_{\beta,\gamma}} e^{-\beta :E:(\omega)} d\mu_\gamma(\omega), \quad Z_{\beta,\gamma} = \int e^{-\beta :E:(\omega)} d\mu_\gamma(\omega), \tag{2.4.3}$$

is well-posed. It coincides with the energy-enstrophy measure, $\tilde{\mu}_{\beta,\gamma} = \mu_{\beta,\gamma}$.

The computations we perform in the forthcoming proof find analogues in the infinite product representations of energy-enstrophy measures given for instance in [**8, 17**].

PROOF. The variable $:E:$ has exponential moments because it belongs to the second Wiener chaos, so the partition function is finite and the measure well-defined. If characteristic functionals $\mathbb{E}\left[e^{\mathrm{i}\langle f,\omega\rangle}\right]$ coincide for all $f \in \dot{H}^{-s}(\mathbb{T}^2)$, the two measures coincide. Since under $\mu_{0,\gamma}$ the Fourier modes $\hat{\omega}_k$ are independent centred $\mathbb{C}$-valued Gaussian variables with variance γ^{-1}, we can compute

$$\begin{aligned}
\int e^{\mathrm{i}\langle f,\omega\rangle - \beta :E:(\omega)} d\mu_\gamma &= \int \exp\left(\sum_{k \in \mathbb{Z}^2_0} \mathrm{i}\, \hat{f}_k \hat{\omega}_k^* - \beta \frac{|\hat{\omega}_k|^2 - \gamma^{-1}}{8\pi^2 |k|^2} \right) d\mu_\gamma \\
&= \prod_{k \in \mathbb{Z}^2_0} \int_{\mathbb{C}} \frac{\gamma}{2\pi} \exp\left(\mathrm{i}\, \hat{f}_k z^* - \beta \frac{|z|^2 - \gamma^{-1}}{8\pi^2|k|^2} - \frac{\gamma |z|^2}{2} \right) dz \\
&= \prod_{k \in \mathbb{Z}^2_0} \frac{4\pi^2 \gamma |k|^2}{\beta + 4\pi^2\gamma|k^2|} e^{\frac{\beta}{8\pi^2|k^2|}} \exp\left(-\frac{|\hat{f}_k|^2}{2} \cdot \frac{4\pi^2|k|^2}{4\pi^2\gamma|k^2| + \beta} \right),
\end{aligned}$$

and since the partition function $Z_{\beta,\gamma}$ can be evaluated setting $f \equiv 0$ in the latter formula,

$$Z^{-1}_{\beta,\gamma} \int e^{\mathrm{i}\langle f,\omega\rangle - \beta :E:(\omega)} d\mu_\gamma = \prod_{k \in \mathbb{Z}^2_0} \exp\left(-\frac{|\hat{f}_k|^2}{2} \cdot \frac{4\pi^2|k|^2}{4\pi^2\gamma|k^2| + \beta} \right),$$

where the right-hand side is the characteristic function of $\mu_{\beta,\gamma}$, (2.4.1). □

Intuition suggests that the renormalized energy is invariant for Euler's equation, and we can express this fact rigorously by means of the above discussion. The idea is to exhibit a solution of the Fokker-Planck equation (1.4.13) — in the case where friction and forcing are absent, $\alpha = 0$ — such that $\rho_t \equiv :E:$. In fact, since no uniqueness results are available, this is the best notion of invariance we can produce, and as we see below it is a consequence of the infinitesimal invariance already observed in the literature.

PROPOSITION 2.4.2. *For any cylinder function $\phi \in \mathcal{FC}_b$ and $\beta > -1$ it holds*

$$\mathbb{E}\left[:E: (\eta)\, \mathcal{B}\phi(\eta) \right] = \mathbb{E}\left[\frac{1}{Z_\beta} e^{-\beta :E:(\eta)} \mathcal{B}\phi(\eta) \right] = 0. \tag{2.4.4}$$

As a consequence, for $\alpha = 0$, there exist constant solutions of (1.4.13) (in the sense specified in Section 1.4) such that $\rho_t \equiv :E:$ or $\frac{1}{Z_\beta} e^{-\beta :E:}$. Moreover, there exists a weak solution of (2.0.1), again in the sense of Section 1.4, whose fixed time marginals are constant in time, and coincide with $\frac{1}{Z_\beta} e^{-\beta :E:}$.

PROOF. The fact that $\mathbb{E}\left[:E:(\eta)\,\mathcal{B}\phi(\eta)\right]=0$ is detailed in [**46**, Theorem 3.1], and infinitesimal invariance of Gibbs density can be obtained by a completely analogous computation. By means of (2.4.4), one can straightforwardly check that the constant densities $\rho_t \equiv :E:$ or $\frac{1}{Z_\beta}e^{-\beta :E:}$ solve the Fokker-Planck equation (1.4.13) for $\alpha = 0$ in the sense of Definition 1.4.2. In order to apply Theorem 1.4.5 and deduce existence of a stationary solution to Euler equation, we are only left to verify suitable integrability conditions.

Since $:E:$ belongs to the second Wiener chaos of μ, it has finite moments of all orders, as well as exponential moments: we already mentioned that $e^{-\beta :E:}$ is integrable as soon as $\beta > -1$. This threshold can be deduced from the explicit Gaussian expression (2.4.2) and the standard result [**104**, Theorem 6.1]. □

Thanks to the integrability properties of $:E:$ and $\frac{1}{Z_\beta}e^{-\beta :E:}$ we just recalled, Theorem 1.4.4 and Theorem 1.4.5 provide existence of solutions to the stochastic Euler equation (2.0.1) and the associated Fokker-Planck equation with initial data μ_β also for $\alpha > 0$.

However, $:E:$ *is not invariant for the Ornstein-Uhlenbeck generator* $\mathcal{L}$, as one can verify with an elementary computation in Fourier series in the same fashion of the above ones. The resulting flow is thus not stationary. When $\beta > -\frac{1}{2}$, by the decay estimate in Theorem 1.4.4 for the case $p = 2$ we know that the solutions we have built converge for large time to the space white noise. Since uniqueness results are not available, we cannot rule out existence of "anomalous" solutions with a different behavior. As already remarked in the Introduction, just like uniqueness of weak solutions, convergence to equilibrium in this setting remains a fascinating open problem.

CHAPTER 3

Liouville Operator of the Point Vortices System

This Chapter covers the arguments of [**95**]. As outlined in the Introduction, the main result is the identification of a core for the infinitesimal generator of vortex dynamics. We first consider a fixed number N of vortices on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ as space domain: we will discuss other geometries in Section 3.2 below.

By the aforementioned result of Dürr and Pulvirenti [**65**], Theorem 1.3.1, the point vortex dynamics is well-posed for initial data in a full measure set of the phase space, thus defining a measurable flow $T_t : \mathbb{T}^{2\times N} \to \mathbb{T}^{2\times N}$ and giving positive answer to the problem of almost completeness. We will consider the one-parameter group of Koopman unitaries U_t associated to such flow,

$$U_t f = f \circ T_t, \quad f \in L^2(\mathbb{T}^{2\times N}),$$

and a set of smooth functions on full-measure open sets, vanishing in a neighbourhood of singular points of the driving vector field, on which we are able to explicitly write the generator A of the Koopman group $U_t = e^{\mathrm{i}\, tA}$, and show that such observables form a core for A.

Even if we will achieve our aim by means of an approximation noticeably differing from the one of [**65, 135**], much of their understanding of the point vortices system will be crucial to our efforts. Our method also draws ideas from the work [**131**], which discusses essential self-adjointness of Liouville's operator for an infinite particle system with regular interactions. Literature regarding the evolution of infinite particles is extensive (let us only quote the recent work [**37**], and refer to its references), but Liouville operators in that context are a somewhat uncommon topic, and may thus be a source of interesting problems

Besides references on point vortices given above, it is worth mentioning that integrable and non integrable behaviours in point vortices systems are also the subject of a considerable literature. We refer to [**157, 112**] for vortices on $\mathbb{T}^2$, [**109**] for vortices on $\mathcal{S}^2$ and to [**10**] for a survey on the topic: complete references can be found in those works, including a large number of studies on vortices on $\mathbb{R}^2$.

Remark. In the present Chapter, the symbol D is used to denote domains of operators, as it is customary in Functional Analysis. We will use instead $\mathcal{D}$ for bounded domains of $\mathbb{R}^2$, in subsection 3.2.2. In later Chapters D will reprise its original role.

3.1. The Liouville Operator for Point Vortices Systems

In this section, $\underline{x} = (x_1, \dots x_N) \in (\mathbb{T}^2)^N$ are the positions of point vortices on $\mathbb{T}^2$, and $\underline{\xi} = (\xi_1, \dots \xi_N) \in \mathbb{R}^N$ their intensities. We denote by dx^N the uniform (Haar's) measure on $(\mathbb{T}^2)^N$, and by $d(x, y)$ the distance of points $x, y \in \mathbb{T}^2$. Also, we will denote by $|B|$ the measure of measurable subsets $B \subset (\mathbb{T}^2)^N = \mathbb{T}^{2\times N}$. All observables are intended as complex valued, and we will denote $L^2(\mathbb{T}^{2\times N}) = L^2(\mathbb{T}^{2\times N}; \mathbb{C})$. We distinguish the imaginary unit $\mathrm{i} \in \mathbb{C}$ and the index $i \in \mathbb{N}$ (in italics). Time $t \in \mathbb{R}$ ranges the whole real line, but for simplicity we will often consider positive times $t > 0$, the other case being completely analogous.

3.1.1. Classical Results on Improbability of Collisions. The point vortices system (1.3.1) is an ordinary differential equation in finite dimension, whose vector field is given by

$$B_i(\underline{x}) = \sum_{j\neq i}^{N} \xi_j K(x_i, x_j), \quad \underline{x} \in (\mathbb{T}^2)^N, \tag{3.1.1}$$

where $K = \nabla^\perp G$. As already remarked, the vector field is singular on the generalised diagonal

$$\triangle^N = \left\{ \underline{x} \in (\mathbb{T}^2)^N : x_i = x_j \text{ some } i \neq j \right\}.$$

Although B is smooth outside $\triangle^N$, classical well-posedness theorems can only provide existence and uniqueness of solutions only locally in time. Indeed, if some vortices collapse, that is if a solution reaches $\triangle^N$, the vector field diverges. However, by exploiting the peculiar structure of B, it is possible to prove that in fact, for any fixed –but arbitrary– choice of intensities ξ, the system (1.3.1) has a global (in time), smooth solution for almost every initial condition with respect to dx^N.

The case in which all intensities ξ_i have the same sign is easier, since the minimum distance between vortices along a trajectory in phase space can be controlled by means of the Hamiltonian H. Indeed, from the definition of H and (1.2.1) (using in particular the fact that g is uniformly bounded), we have

$$\begin{aligned} H(\underline{x}) &\geq \left(\min_i |\xi_i|\right)^2 \sum_{i<j}^{N} \left(-\frac{1}{2\pi} \log d(x_i, x_j) + \min_{\mathbb{T}^2} g \right) \\ &\geq -C \log \min_{i\neq j} d(x_i, x_j) - C', \end{aligned}$$

where $C, C' > 0$ are constants depending on $\underline{\xi}, N$, and thus

$$\min_{i\neq j} d(x_i, x_j) \geq e^{-CH(\underline{x})-C'}. \tag{3.1.2}$$

Since the right-hand side is a first integral of the motion, we can extend local-in-time solutions of (1.3.1) starting from $\underline{x} \in \mathbb{T}^{2\times N} \setminus \triangle^N$ to global solutions which are also smooth in time.

When vortices intensities $\underline{\xi} \in \mathbb{R}^N$ take both positive and negative values, there might exist initial conditions leading to collapse, see [**135**, Section 4.2] and the references above on integrable motion of vortices. Indeed, the energy $H(\underline{x})$ gives us no control whatsoever on the vortices distances along the trajectory of $\underline{x}$, since H include now both positive and negative terms which can cancel out large contributions of close couples of vortices.

We already mentioned the classical result of Dürr and Pulvirenti, [**65**], establishing almost completeness in the general case of intensities with positive and negative signs, Theorem 1.3.1. We also refer to [**135**] for the case of vortices on the whole plane (see Section 3.2 below).

The core idea in the proof of Theorem 1.3.1 is to consider a smooth vector field on $\mathbb{T}^{2\times N}$ given by

$$B_i^\varepsilon(\underline{x}) = \sum_{j\neq i}^{N} \xi_j K_\varepsilon(x_i, x_j), \quad K_\varepsilon = \nabla^\perp G_\varepsilon, \tag{3.1.3}$$

that is the point vortices vector field (3.1.1) with a smoothed interaction obtained by $G_\varepsilon \in C^\infty(\mathbb{T}^2)$ such that:

$$G_\varepsilon|_{B(0,\varepsilon)^c} = G|_{B(0,\varepsilon)^c}, \quad |\nabla G^\varepsilon(x)| \leq |\nabla G(x)| \leq \frac{C}{|x|} \quad \forall x \in \mathbb{T}^2, \tag{3.1.4}$$

where $B(0,\varepsilon) \subset \mathbb{T}^2$ is the ball of radius ε centred in 0 and $C > 0$ is a universal constant. We denote by $T_t^\varepsilon \underline{x} = T^\varepsilon(t, \underline{x})$ the flow of the ordinary differential equation

$$\begin{cases} \dot{\underline{x}}(t) = B^\varepsilon(\underline{x}(t)) \\ \underline{x}(0) = \underline{x} \end{cases},$$

which is globally well-posed since its driving vector field is smooth. Moreover, we define for $\varepsilon > 0, t > 0$ and $\underline{x} \in \mathbb{T}^{2\times N}$,

$$d_t^\varepsilon(\underline{x}) = \inf_{s\in[0,t]} \min_{i\neq j} |(T_s^\varepsilon \underline{x})_i - (T_s^\varepsilon \underline{x})_j|. \tag{3.1.5}$$

In order to control the minimum distance between vortices, instead of the Hamiltonian one can resort to the Lyapunov function

$$\mathcal{L}^\varepsilon(\underline{x}) = \sum_{i\neq j} G_\varepsilon(x_i, x_j) = \sum_{i\neq j} G_\varepsilon(x_i - x_j),$$

for which the following analogue of (3.1.2) holds:

$$d_t^\varepsilon(\underline{x}) \geq \exp\left(-C \sup_{s\in[0,t]} |\mathcal{L}^\varepsilon(T_s^\varepsilon \underline{x})| - C' \right), \tag{3.1.6}$$

with $C, C' > 0$ only depending on N. By differentiating in time $\mathcal{L}^\varepsilon(T_t^\varepsilon \underline{x})$, and exploiting the Hamiltonian structure of the vector field B^ε –in particular volume conservation on phase space– [**65**] established the uniform bound

$$\int_{\mathbb{T}^{2\times N}} \sup_{s\in[0,t]} |\mathcal{L}^\varepsilon(T_s^\varepsilon \underline{x})| dx^N \leq C(t+1),$$

with $C > 0$ independent of $\varepsilon > 0$. This, in combination with Markov inequality and (3.1.6) produces the crucial estimate, for $C > 0$ independent of $c \in (0,1)$,

$$|\{d_t^\varepsilon(\underline{x}) < c\}| \leq \frac{C(t+1)}{-\log c}, \tag{3.1.7}$$

from which (1.3.2) follows, since $\{d_t < c\}$ is the almost sure limit of $\{d_t^\varepsilon < c\}$.

In fact, on the set $\{d_t^\varepsilon(\underline{x}) > \varepsilon\}$ the flow $T_s^\varepsilon(\underline{x})$ of B^ε and $T_t(\underline{x})$ of B coincide: for all $t, \varepsilon > 0$,

$$T_s^\varepsilon(\underline{x}) = T_s(\underline{x}) \quad \forall s \in [0,t], \underline{x} \in \{d_t > \varepsilon\}. \tag{3.1.8}$$

Sending $\varepsilon \to 0$ we obtain the full-measure set $\{d_t(\underline{x}) > 0\}$ on which the flow $T_t(\underline{x})$ of B is well-defined, and intersecting the sets $\{d_{t_n}(\underline{x}) > 0\}$ over a sequence of times $t_n \uparrow \infty$ (and one $t_n \downarrow -\infty$) Theorem 1.3.1 is completed.

3.1.2. Functional Analytic Setting. This paragraph collects abstract definitions and results we are going to apply to point vortices systems. We assume knowledge of basic notions in the theory of groups of unitary operators on Hilbert spaces, for which we refer the reader to [**149**, Chapter VIII].

Let $(X, \mathcal{F}, \mu)$ a standard Borel probability space, *i.e.* X is a Polish space and $\mathcal{F}$ the associated Borel σ-algebra. The following results establishes a relation between groups of maps on X and groups of operators on $L^2(\mu) = L^2(X, \mathcal{F}, \mu)$. Its first part, the easier one, is well known as Koopman's Lemma, whereas the second part, a converse implication, is a relevant result in Ergodic Theory, for the proof of which we refer to [**91**].

THEOREM 3.1.1. *Let the mapping*

$$\mathbb{R} \times X \ni (t, x) \mapsto T_t(x) \in X$$

be such that: for μ-almost every $x \in X$, $t \mapsto T_t(x)$ is a continuous map; for all $t \in \mathbb{R}$, $x \mapsto T_t(x)$ is a μ-almost surely invertible, measurable and measure preserving map and for all $t, s \in \mathbb{R}$

$$T_t \circ T_s(x) = T_{t+s}(x)$$

(that is, $(T_t)_{t\in\mathbb{R}}$ is a group). Then

$$L^2(X, \mathcal{F}, \mu) \ni f \mapsto U_t f = f \circ T_t \in L^2(X, \mathcal{F}, \mu)$$

*defines a strongly continuous group of unitary, positive and unit-preserving operators on $L^2(X, \mathcal{F}, \mu)$ (*Koopman group*).*

Conversely, let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous group of of unitary, positive and unit-preserving operators on $L^2(X, \mathcal{F}, \mu)$ with generator A; then there exists a group of μ-almost surely invertible, measurable and measure preserving maps $T_t : X \to X$, $t \in \mathbb{R}$, such that $U_t f = f \circ T_t$ for all $f \in L^2(X, \mathcal{F}, \mu)$; moreover, $t \mapsto T_t(x)$ is weakly measurable for all $t \in \mathbb{R}$.

Remark 3.1.2. It is worth mentioning that the characterisation of Koopman groups is an important problem in Ergodic Theory. We refer to [**121**] for a review on the topic, and to the works [**81, 160**] for a characterisation of Koopman groups in terms of properties of their generators.

Our aim is to identify a core for the generator of vortex dynamics. This problem is intimately linked to the one of uniquely extending densely defined symmetric operators and essential self-adjointness. We recall the following terminology.

Proposition 3.1.3. *Consider a symmetric linear operator (L, D) on $L^2(\mu)$; each of the following statements implies the next one:*

- Essential self-adjointness*: the closure of (L, D) is self-adjoint;*
- $L^2(\mu)$ uniqueness*: there exists a unique one-parameter strongly continuous group of unitaries whose generator extends (L, D).*
- Markov uniqueness*: there exists a unique one-parameter strongly continuous group of unitaries preserving positivity and unit whose generator extends (L, D).*

While the second implication is trivial, the first one is due to Stone's theorem: any one-parameter strongly continuous groups of unitaries on a Hilbert space H is generated by a self-adjoint operator. We also recall that the first two definitions coincide if (L, D) is semi-bounded; however this will not be the case in our discussion.

The basic self-adjointness criterion is the following (see [**149**, Theorem VIII.1]).

Proposition 3.1.4. *Let H be a complex Hilbert space, $U_t = e^{\mathrm{i}\,tA}$ a strongly continuous unitary group on H and A its generator. If $D \subset D(A)$ is a dense linear subset such that $U_t(D) \subseteq D$, then $(A|_D, D)$ is essentially self-adjoint and D is a core for A, $\overline{A|_D} = A$.*

We will in fact use a modified version of this criterion: the proof is a standard argument, but we report it for the sake of completeness.

Proposition 3.1.5. *Let H be a complex Hilbert space, $U_t = e^{\mathrm{i}\,tA}$ a strongly continuous unitary group on H and A its generator. If $D \subset D(A)$ is a dense subset, $L = A|_D$ and*

$$\forall t \in \mathbb{R}, \forall f \in D, \quad U_t f \in D\left(\overline{L}\right), \tag{3.1.9}$$

then (L, D) is essentially self-adjoint and D is a core for A, $\overline{L} = A$.

PROOF. By [149, Theorem VIII.3], if $\ker(L^* \pm \mathrm{i}) = \{0\}$, then $\overline{L}$ is self-adjoint. Assume by contradiction that there exists $f \in D(L^*)$ such that $L^* f = \mathrm{i}\, f$ (the case of $L^* f = -\,\mathrm{i}\, f$ is analogous). Then, for all $g \in D = D(L)$ it holds

$$\frac{d}{dt} \langle U_t g, f\rangle_H = \langle \mathrm{i}\, A U_t g, f\rangle_H = \left\langle \mathrm{i}\, \overline{L} U_t g, f\right\rangle_H = \langle U_t g, f\rangle_H \,, \tag{3.1.10}$$

where the second passage makes use of the hypothesis $U_t g \in D(\overline{L})$, and the last one of $L^* = (\overline{L})^*$. The operator U_t is unitary, so the only solution to the above differential equation for $\langle U_t g, f\rangle$ in t is the constant 0, and thus, since g varies on the dense set D, $f = 0$.

We are left to show that $\overline{L} = A$: this follows easily by differentiating in time $e^{\mathrm{i}\, t\overline{L}}$ on D and noting that the result coincides by definition with the derivative of U_t, so that $U_t = e^{\mathrm{i}\, t\overline{L}}$. □

Let us note that condition (3.1.9) can be rephrased as: for all $t \in \mathbb{R}$ and $f \in D$, there exists a sequence $g_n \in D$ such that

$$g_n \xrightarrow{n\to\infty} U_t f, \quad L g_n \xrightarrow{n\to\infty} \overline{L} U_t f \tag{3.1.11}$$

in the strong topology of H.

3.1.3. The Koopman Group for Point Vortices Systems. We denote by T_t the group of transformations of $\mathbb{T}^{2\times N}$ defined in Theorem 1.3.1 –that is the point vortices flow– and U_t its associated Koopman group for the remainder of this section. We now define a first set of observables on which we are able to write explicitly the generator of U_t, and which will turn out to be a core for the generator in the simple case where vortices all have positive (or negative) intensity.

PROPOSITION 3.1.6. *The linear subspace*

$$D = \left\{ f \in C^\infty(\mathbb{T}^{2\times N}) : \operatorname{supp} f \cap \triangle^N = \emptyset \right\}.$$

is dense in $L^2(\mathbb{T}^{2\times N})$.

Fix $\underline{\xi} \in \mathbb{R}^N$. For any $f \in D$ the following expression is well defined as a function in $L^\infty(\mathbb{T}^{2\times N})$:

$$Lf(\underline{x}) = -\,\mathrm{i} \sum_{i=1}^N \sum_{j\neq i} \nabla_i f(\underline{x}) \cdot \xi_j K(x_i - x_j), \tag{3.1.12}$$

where $\nabla_i f$ denotes the gradient in the i-th coordinate of $\mathbb{T}^{2\times N} = (\mathbb{T}^2)^N$.

The operator (L, D) is symmetric; moreover, if A is the generator of U_t, then $D \subset D(A)$ and $L = A|_D$.

For the sake of clarity, we recall that $\operatorname{supp} f$, the *support* of f, is the closure of the set of points on which $f \neq 0$. Let us also introduce the useful notation

$$\triangle^N_\varepsilon = \left\{ \underline{x} \in \mathbb{T}^{2\times N} : d(x_i, x_j) \leq \varepsilon \text{ for some } i \neq j \right\}, \tag{3.1.13}$$

and notice that the support of any $f \in D$ and $\triangle^N_\varepsilon$ are disjoint for any small enough $\varepsilon > 0$.

PROOF. The density statement is straightforward: smooth functions $C^\infty(\mathbb{T}^{2\times N})$ are dense in $L^2(\mathbb{T}^{2\times N})$, so we only need to show that we can approximate in L^2-norm the elements of $C^\infty(\mathbb{T}^{2\times N})$ with the ones of D. This is readily done by means of Urysohn's lemma —or rather its C^∞ version on smooth manifolds, see [47, Theorem 3.5.1])— which ensures existence of smooth functions g_ε vanishing on $\triangle^N_\varepsilon$ and coinciding with a given $g \in C^\infty(\mathbb{T}^{2\times N})$ outside $\triangle^N_{\varepsilon'}$ for $0 < \varepsilon < \varepsilon'$.

The expression (3.1.12) is well-defined for $f \in D$ since ∇f vanishes in a neighbourhood of $\triangle^N_\varepsilon$, and moreover

$$\|Lf\|_\infty \leq C_{\xi,N} \|f\|_{C^1(\mathbb{T}^{2\times N})} \left(\min_{\underline{x}\in \operatorname{supp} f} \min_{i\neq j} |x_i - x_j| \right)^{-1} < \infty,$$

because $K(x,y) \sim |x-y|^{-1}$ for $x \to y$ in $\mathbb{T}^2$.

As for the symmetry part: first one replaces $K = \nabla^\perp G$ with the cut off kernel $K_\varepsilon = \nabla^\perp G_\varepsilon$ as in (3.1.3). Integration by parts and the fact that K_ε is divergence free readily show that

$$\int_{\mathbb{T}^{2\times N}} \nabla_i f(\underline{x}) \cdot K_\varepsilon(x_i - x_j) g(\underline{x}) dx^N = - \int_{\mathbb{T}^{2\times N}} \nabla_i g(\underline{x}) \cdot K_\varepsilon(x_i - x_j) f(\underline{x}) dx^N,$$

in which we can send $\varepsilon \to 0$ by bounded convergence. Summing up all contributions, multiplying by i and taking into account complex conjugation in the scalar product of $L^2(\mathbb{T}^{2\times N})$ we conclude that (L, D) is symmetric.

It remains to show the following limit in $L^2(\mathbb{T}^{2\times N})$,

$$\lim_{t\to 0} \frac{U_t f - f}{t} = Lf, \quad \forall f \in D.$$

But thanks to Theorem 1.3.1, for almost every $\underline{x} \in \mathbb{T}^{2\times N}$ we have that $U_t f(\underline{x}) = f(T_t \underline{x})$ is a smooth function of t and

$$\left.\frac{d}{dt} f(T_t\underline{x})\right|_{t=0} = \sum_{i=1}^N \nabla_i f(\underline{x}) \cdot \dot{x}_i(0) = Lf(\underline{x}), \tag{3.1.14}$$

so that we can conclude by bounded convergence. □

Uniqueness of the flow in the almost-everywhere sense of Theorem 1.3.1 already gives us, by means of Theorem 3.1.1, the following uniqueness result.

Proposition 3.1.7. *For any fixed $\underline{\xi} \in \mathbb{R}^N$, (L, D) is Markov unique, and it extends to the self-adjoint generator A of $U_t = e^{\mathrm{i}\, tA}$, the Koopman group of T_t.*

Before we move on, in the next section, to identify a core for the generator of the Koopman group in the general case $\underline{\xi}$, let us analyse the simpler case of vortices with positive (equivalently, negative) intensities, $\underline{\xi} \in (\mathbb{R}^+)^N$. This indeed is a simpler case because the energy $H(\underline{x})$ controls the minimum distance of vortices as noted above in (3.1.2)

Theorem 3.1.8. *Let $\underline{\xi} \in (\mathbb{R}^+)^N$. Then the operator (L, D) is essentially self-adjoint, and its closure coincides with the generator of U_t.*

Proof. We apply the classical criterion of Proposition 3.1.4 by showing that D is left invariant by U_t, that is for any $f \in D$ and $t \in \mathbb{R}$ it holds $f \circ T_t \in D$. By (3.1.2) and invariance of H, it holds

$$\forall t \in \mathbb{R}, \quad \min_{\underline{x}\in \operatorname{supp} f} \min_{i\neq j} d\left((T_t\underline{x})_i, (T_t\underline{x})_j\right) \geq \exp\left(-C \max_{\underline{x}\in \operatorname{supp} f} H(\underline{x}) - C' \right) > 0,$$

with $C, C' > 0$ constants depending only on $\underline{\xi}$ and N, so that for any $\varepsilon > 0$ smaller than the right-hand side of the latter inequality, $f \circ T_t = f \circ T^\varepsilon_t$, with T^ε being the flow of (3.1.3) as above. This implies that $f \circ T_t$ is still a smooth function and that its support is disjoint from $\triangle^N$, which concludes the proof. □

3.1.4. A Core for the Liouville Operator: The General Case. As we mentioned above, when vortices intensities $\underline{\xi} \in \mathbb{R}^N$ take both positive and negative values, there might exist initial conditions leading to collapse. More generally, the minimum distance of vortices along a globally defined trajectory of the flow might be 0, that is the configuration might pass arbitrarily close to $\triangle^N$.

As a consequence, even if for $f \in D$ the support of f has a positive distance from the diagonal $\triangle^N$, trajectories starting from $\operatorname{supp} f$ can travel arbitrarily close to $\triangle^N$ in finite time, and D is thus not invariant for U_t. Moreover, there is no clue that U_t should preserve C^∞ regularity.

Instead of Proposition 3.1.4, we rely in this case on Proposition 3.1.5, which allows us to check a sort of "approximate invariance" of the candidate core. The key is in choosing the correct approximation of U_t, and a natural choice might be to consider the Koopman group of the flow T^ε of the smoothed vector field B^ε: unfortunately this choice is inadequate to our purposes, see subsection 3.1.5 below.

We now define a new set of observables, which we will prove to be a core for A, and a truncated flow that will serve us to check conditions of Proposition 3.1.5. In fact, the result was already stated in the Introduction:

DEFINITION 3.1.9. *We denote by $\tilde{D}$ the linear space of functions $f \in L^\infty(\mathbb{T}^{2\times N})$ such that:*

- *there exists a version of f and a full-measure open set $M \subset \mathbb{T}^{2\times N}$ on which $f|_M \in C^\infty(M)$, and moreover $\nabla f|_M \in L^\infty(M)$;*
- *f vanishes in a neighbourhood of $\triangle^N$.*

PROPOSITION 3.1.10. *The linear subspace $\tilde{D}$ is dense in $L^2(\mathbb{T}^{2\times N})$, and for any $\underline{\xi} \in \mathbb{R}^N$, $f \in \tilde{D}$ the following expression is well defined as a function in $L^\infty(\mathbb{T}^{2\times N})$:*

$$Lf(\underline{x}) = -\mathrm{i} \sum_{i=1}^{N} \sum_{j\neq i} \nabla_i f(\underline{x}) \cdot \xi_j K(x_i - x_j). \tag{3.1.15}$$

Moreover, $(L, \tilde{D})$ is a symmetric operator and if A is the generator of U_t, then $\tilde{D} \subset D(A)$ and $L = A|_{\tilde{D}}$.

The proof of the latter Proposition is completely analogous to the one for D above. The following is the main result of the present paper.

THEOREM 3.1.11. *Let $\underline{\xi} \in \mathbb{R}^N$. Then the operator $(L, \tilde{D})$ is essentially self-adjoint, and its closure coincides with the generator A of U_t.*

Instead of smoothing the driving vector field, we simply stop trajectories of the flow drawing too close to $\triangle^N$. Since $T^\varepsilon : [0,t] \times \mathbb{T}^{2\times N} \to \mathbb{T}^{2\times N}$ is a smooth function on a compact set, by definition $d_t^\varepsilon : \mathbb{T}^{2\times N} \to \mathbb{R}$, defined in (3.1.5), is a continuous function. In particular, the sets $\{d_t^\varepsilon < c\}$, $\{d_t^\varepsilon > c\}$ are open subsets of $\mathbb{T}^{2\times N}$. Moreover, since

$$\left| \bigcup_{c\geq 0} \{\underline{x} : d_t^\varepsilon(\underline{x}) = c\} \right| = 1,$$

the closed sets $\{d_t^\varepsilon = c\}$ are negligible for almost all $c \geq 0$. Let us stress that

$$\forall \underline{x} \in \{d_t^\varepsilon > \varepsilon\} = \{d_t > \varepsilon\}, \quad \forall s \in [0,t], \quad T_s^\varepsilon \underline{x} = T_s \underline{x}.$$

We define the (lower semicontinuous) function

$$\tau_{t,\varepsilon}(\underline{x}) = \begin{cases} t & \underline{x} \in \{d_t^\varepsilon > \varepsilon\} = \{d_t > \varepsilon\} \\ 0 & \underline{x} \in \{d_t^\varepsilon < \varepsilon\} \end{cases}, \tag{3.1.16}$$

and the arrested flow

$$R_t^\varepsilon \underline{x} = T_{\tau_{t,\varepsilon}(\underline{x})}\underline{x} = \begin{cases} T_t\underline{x} & \underline{x} \in \{d_t^\varepsilon > \varepsilon\} = \{d_t > \varepsilon\} \\ \underline{x} & \underline{x} \in \{d_t^\varepsilon < \varepsilon\} \end{cases}. \tag{3.1.17}$$

We can assume without loss of generality that $|\{d_t^\varepsilon = \varepsilon\}| = 0$, so that (3.1.16) and (3.1.17) define $\tau_{t,\varepsilon}, R_t^\varepsilon$ on a full-measure open set. Indeed, if $\{d_t^\varepsilon = \varepsilon\}$ has positive measure, we can redefine $R_t^\varepsilon = T_t = T^\varepsilon = T^{\varepsilon'}$ on $\{d_t^\varepsilon > \varepsilon'\}$ with a slightly larger $\varepsilon' > \varepsilon$ such that $\{d_t^\varepsilon = \varepsilon'\}$ is negligible, and the identity outside $\{d_t^\varepsilon > \varepsilon'\}$: this does not influence any of the forthcoming arguments. That being said, we see that R_t^ε has the following properties: for any $\varepsilon > 0, t \in \mathbb{R}$,

- it is a diffeomorphism on the full-measure open set $\{d_t^\varepsilon \neq \varepsilon\}$,
- it is a discontinuous but measurable transformation of the whole $\mathbb{T}^{2\times N}$,
- it is a measure preserving map.

Finally, we define the approximating Koopman operators

$$V_t^\varepsilon f(\underline{x}) = f(R_t^\varepsilon \underline{x}) \quad f \in L^2(\mathbb{T}^{2\times N}), \tag{3.1.18}$$

which are positivity and unit preserving maps taking values in $L^2(\mathbb{T}^{2\times N})$.

PROPOSITION 3.1.12. *Fix $f \in \tilde{D}$ and $t \in \mathbb{R}$. Then:*

(i) $V_t^\varepsilon f \in \tilde{D}$*;*
(ii) $V_t^\varepsilon f$ *converges to* $U_t f$ *in* $L^2(\mathbb{T}^{2\times N})$ *as* $\varepsilon \to 0$*;*
(iii) $AV_t^\varepsilon f = LV_t^\varepsilon f$ *is well-defined since* $V_t^\varepsilon f \in \tilde{D}$*, and it converges to* $AU_t f$ *in* $L^2(\mathbb{T}^{2\times N})$ *as* $\varepsilon \to 0$.

PROOF. Starting from item (i), first of all we notice that $V_t^\varepsilon f = f \circ R_t^\varepsilon \in L^\infty(\mathbb{T}^{2\times N})$ because $f \in L^\infty(\mathbb{T}^{2\times N})$. Let M be, as above, the open set on which (a version of) f is smooth, then

$$f \circ R_t^\varepsilon(\underline{x}) = \begin{cases} f(T_t\underline{x}) & \underline{x} \in \{d_t^\varepsilon > \varepsilon\} \cap (T_t^\varepsilon)^{-1}M \\ f(\underline{x}) & \underline{x} \in \{d_t^\varepsilon < \varepsilon\} \cap M \end{cases}. \tag{3.1.19}$$

The sets on the right-hand side are disjoint since $\{d_t^\varepsilon > \varepsilon\} \cap \{d_t^\varepsilon < \varepsilon\} = \emptyset$, and open because intersection of open sets. Moreover, since T_t^ε is measure-preserving,

$$\left|(T_t^\varepsilon)^{-1}M\right| = |M| = 1 \quad \Rightarrow \left|\{d_t^\varepsilon > \varepsilon\} \cap (T_t^\varepsilon)^{-1}M\right| = |\{d_t^\varepsilon > \varepsilon\}|$$

and also $|\{d_t^\varepsilon < \varepsilon\} \cap M| = |\{d_t^\varepsilon < \varepsilon\}|$. This shows that $f \circ R_t^\varepsilon$ coincides with a smooth function on a full-measure open set. As for its gradient,

$$\nabla\left(f \circ R_t^\varepsilon\right)(\underline{x}) = \begin{cases} \nabla f(T_t^\varepsilon\underline{x})DT_t^\varepsilon(\underline{x}) & \underline{x} \in \{d_t^\varepsilon > \varepsilon\} \cap (T_t^\varepsilon)^{-1}M \\ \nabla f(\underline{x}) & \underline{x} \in \{d_t^\varepsilon < \varepsilon\} \cap M \end{cases},$$

where $\|DT_t^\varepsilon(\underline{x})\|_\infty < \infty$, and thus $\nabla(V_t^\varepsilon f) \in L^\infty(\mathbb{T}^{2\times N})$ since $\nabla f \in L^\infty(M)$. By definition, $R_t^\varepsilon(\underline{x}) = \underline{x}$ on $\{d_t^\varepsilon < \varepsilon\}$, which is a neighbourhood of $\triangle^N$, since it contains all $\triangle^N_{\varepsilon'}$ for $\varepsilon' < \varepsilon$; thus on the intersection of $\{d_t^\varepsilon < \varepsilon\}$ and the neighbourhood of $\triangle^N$ on which f vanishes, so must vanish also $V_t^\varepsilon f$, concluding item (i).

Item (ii) follows directly from (3.1.7) and the fact that U_t is unit preserving:

$$\begin{aligned}\|U_t f - V_t^\varepsilon f\|_{L^2}^2 &= \int_{\{d_t^\varepsilon < \varepsilon\}} |U_t f(\underline{x}) - f(\underline{x})|\, dx^N \\ &\le 2\|f\|_\infty^2 |\{d_t^\varepsilon < \varepsilon\}| \le \frac{Ct\|f\|_\infty^2}{-\log\varepsilon}.\end{aligned}$$

Let us now consider how the generator A acts on $V_t^\varepsilon f$. By definition,

$$AV_t^\varepsilon f(\underline{x}) = \left.\frac{d}{ds}\right|_{s=0} U_s V_t^\varepsilon f(\underline{x}) = \left.\frac{d}{ds}\right|_{s=0} f(R_t^\varepsilon T_s \underline{x}).$$

For a fixed $\underline{x}$ in the open set $\{d_t^\varepsilon > \varepsilon\}$, $T_s\underline{x}$ is well-defined for s in a neighbourhood of 0, and it is a smooth function in such time interval. Thus, for small enough s depending on the $\underline{x}$ we are fixing, $T_s\underline{x} \in \{d_t^\varepsilon > \varepsilon\}$, and the same is true if $\underline{x} \in \{d_t^\varepsilon < \varepsilon\} \setminus \triangle^N$ (we are removing the closed negligible diagonal $\triangle^N$). As a consequence, for all $\underline{x} \in \{d_t^\varepsilon > \varepsilon\}$,

$$\left.\frac{d}{ds}\right|_{s=0} f(R_t^\varepsilon T_s\underline{x}) = \left.\frac{d}{ds}\right|_{s=0} f(T_tT_s\underline{x}) = \left.\frac{d}{ds}\right|_{s=0} f(T_sT_t\underline{x}) = Lf(T_t\underline{x}) = U_tLf(\underline{x}),$$

and analogously for $\underline{x} \in \{d_t^\varepsilon < \varepsilon\} \setminus \triangle^N$,

$$\left.\frac{d}{ds}\right|_{s=0} f(R_t^\varepsilon T_s\underline{x}) = \left.\frac{d}{ds}\right|_{s=0} f(T_s\underline{x}) = Lf(\underline{x}).$$

We thus see that, for $\underline{x}$ in a full-measure set,

$$LV_t^\varepsilon f(\underline{x}) = V_t^\varepsilon Lf(\underline{x}).$$

Since a strongly continuous unitary group always commutes with its generator on the domain of the latter, and since $Af = Lf$ for $f \in \tilde{D}$,

$$\begin{aligned}\|AU_tf - AV_t^\varepsilon f\|_{L^2}^2 = \|U_tLf - V_t^\varepsilon Lf\|_{L^2}^2 &= \int_{\{d_t^\varepsilon<\varepsilon\}} |U_tLf(\underline{x}) - Lf(\underline{x})|\, dx^N \\ &\leq 2\,\|Lf\|_\infty^2\, |\{d_t^\varepsilon < \varepsilon\}| \leq \frac{Ct\,\|f\|_{C^1}^2}{-\log\varepsilon},\end{aligned}$$

where C is a constant depending on N and $\underline{\xi}$. This concludes the proof of (iii). ☐

Theorem 3.1.11 is a direct consequence of Proposition 3.1.5 and Proposition 3.1.12. Indeed, for fixed $f \in \tilde{D}$ and $t \in \mathbb{R}$, the we have shown that $V_t^\varepsilon f$ satisfies condition (3.1.11), and thus $\tilde{D}$ is a core for A.

3.1.5. Considerations on unsuccessful approaches. In the proof of Theorem 3.1.11 we use in an essential way the peculiar structure of our approximating flow R_t^ε in items (i) and (iii), while (ii) still holds true if we replace U_tf with $U_t^\varepsilon f = f \circ T_t^\varepsilon$, the approximating flow of [**65**], for any smooth $f \in D$. There are two reasons why we are not able to treat the latter setting.

Using the fact that T_t^ε is smooth one can show with some care that U_t^ε preserves D. This and estimate (3.1.7) would show that D is a core for A provided that we can also show that $AU_t^\varepsilon f$ strongly converges to AU_tf for fixed $f \in D, t \in \mathbb{R}$ (*cf.* Proposition 3.1.12). Since U_tf and $U_t^\varepsilon f$ coincide on $\{d_t^\varepsilon > \varepsilon\}$, we only need to evaluate their difference on $\{d_t^\varepsilon \leq \varepsilon\}$. The set over which we integrate has small measure $t\log\left(\frac{1}{\varepsilon}\right)$, but if we try to bound $LU_t^\varepsilon f$ uniformly in $\underline{x}$ ($LU_tf = U_tLf$ is uniformly bounded since Lf is), we are led to control terms including $\|DT_t^\varepsilon\|_\infty$: since the vector field $\|B^\varepsilon\|_\infty \simeq \varepsilon^{-2}$, we get $\|DT_t^\varepsilon\|_\infty \simeq e^{C\varepsilon^{-2}}$, which is way too large to be compared with the measure of the integration set. Considering estimates in L^2 or L^p norms does not seem to solve the issue, either.

We have seen above how an abrupt truncation of the flow allows us to show that $\tilde{D}$ is a core for A, and it is clear that allowing functions of lower regularity was necessary to employ this kind of approximation. We further mention only one more smooth approach. One might define the vector field

$$B^\delta(\underline{x}) = M^\delta(\underline{x})B(\underline{x}), \tag{3.1.20}$$

with $M^\delta \in C^\infty(\mathbb{T}^{2\times N})$ vanishing on a δ-neighbourhood of $\triangle^N$ and taking value 1 far from it. The Koopman operators V_t^δ of its associated flow would preserve D and strongly converge to U_t; however, L and V_t^δ would not commute unless M^δ is a first integral of the motion. As there can not be invariants of the vortex motions controlling the minimum distance of vortices (as M^δ would do) in the case of coexisting positive and negative vortices, we would not be able to continue the proof as we did for Theorem 3.1.11, and thus have to face explicit computations, in which the difficulties of the same kind of the ones outlined above appear.

3.2. Generalisations

3.2.1. Point Vortices on the Sphere. All the arguments above still work with almost no modifications when the torus $\mathbb{T}^2$ is replaced with a smooth compact surface with no boundary, such as the sphere $\mathcal{S}^2 = \left\{x \in \mathbb{R}^3 : |x| = 1\right\}$ (to be regarded as an embedded surface). On $\mathcal{S}^2$, $d\sigma$ is the Riemannian volume, so that $\int_{\mathcal{S}^2} d\sigma = 1$, and $x \cdot y, x \times y$ respectively denote scalar and vector products in $\mathbb{R}^3$. Euler equations on $\mathcal{S}^2$ are given by, for $x \in \mathcal{S}^2$,

$$\begin{cases} \partial_t \omega(x,t) = x \cdot (\nabla\psi(x,t) \times \nabla\omega(x,t)), \\ -\Delta\psi(x,t) = \omega(x,t). \end{cases}$$

Here Δ is the Laplace-Beltrami operator, and we have to supplement the Poisson equation for the *stream function* ψ with the zero average condition. The Green function of $-\Delta$ is simply given by

$$-\Delta G(x,y) = \delta_y(x) - 1, \quad G(x,y) = -\frac{1}{2\pi}\log|x-y| + c,$$

$c \in \mathbb{R}$ a universal constant making G zero-averaged. To satisfy in weak sense Euler equations, the point vortices vorticity distribution $\omega = \sum_1^N \xi_i \delta_{x_i}$ must evolve according to

$$\dot{x}_i = \frac{1}{2\pi}\sum_{i \neq j}^N \xi_j \frac{x_j \times x_i}{|x_i - x_j|^2}, \tag{3.2.1}$$

which is still a Hamiltonian system with

$$H(x_1, \dots x_N) = \sum_{i<j}^N \xi_i \xi_j G(x_i, x_j).$$

In fact, setting $K(x,y) = \frac{1}{2\pi}\frac{x \times y}{|x-y|^2}$, (3.2.1) takes the same form of (1.3.1), and K is still a skew-symmetric, divergence free function on $\mathcal{S}^2$ (divergence being the adjoint of the gradient operator on functions of $\mathcal{S}^2$). We refer to [**147**] for a more complete discussion of this setting. It is easy to see that all the features we relied on in Section 3.1 are still present:

(i) the flow is locally well posed when positions of vortices do not coincide, and it is measure-preserving because of the Hamiltonian nature of the equations;
(ii) the crucial cancellation leading to integrability of the Lyapunov function $\mathcal{L}(x_1, \dots x_N) = \sum_{i \neq j} G(x_i, x_j)$ and thus required for the proof of Theorem 1.3.1 to work still takes place, in this case because $(x \times y) \perp (x - y)$ for any $x, y \in \mathcal{S}^2$;
(iii) as a consequence, the almost-surely well defined point vortices flow T_t coincide with a smooth one on open sets of large measures, so that we can implement again the strategy of subsection 3.1.4.

3.2.2. Point Vortices on Bounded Domains. Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and Lebesgue measure $|\mathcal{D}| = 1$, $G(x,y)$ the Green function of $-\Delta$ on $\mathcal{D}$ with Dirichlet boundary conditions, which can be represented as the sum of its free version $G_{\mathbb{R}^2}(x,y) = -\frac{1}{2\pi}\log|x-y|$ and the harmonic extension in $\mathcal{D}$ of the values of $G_{\mathbb{R}^2}$ on $\partial\mathcal{D}$,

$$G(x,y) = -\frac{1}{2\pi}\log|x-y| + g(x,y), \quad \begin{cases} \Delta g(x,y) = 0 & x \in \mathcal{D} \\ g(x,y) = \frac{1}{2\pi}\log|x-y| & x \in \partial D \end{cases} \tag{3.2.2}$$

for all $y \in \mathcal{D}$. Both G and g are symmetric, and maximum principle implies that

$$-\frac{1}{2\pi}\log(d(x) \vee d(y)) \le g(x,y) \le \frac{1}{2\pi}\log\operatorname{diam}(\mathcal{D}), \tag{3.2.3}$$

with $d(x)$ the distance of $x \in \mathcal{D}$ from the boundary $\partial\mathcal{D}$.

The motion of a system of N vortices with intensities $\xi_1,\dots,\xi_N \in \mathbb{R}$ and positions $x_1,\dots,x_N \in \mathcal{D}$ is governed by the Hamiltonian function

$$H(x_1,\dots,x_n) = \sum_{i<j}^{N} \xi_i\xi_j G(x_i,x_j) + \sum_{i=1}^{N} \xi_i^2 g(x_i,x_i),$$

leading to the system of equations

$$\dot{x}_i(t) = \sum_{j\neq i}^{N} \xi_j \nabla^\perp G(x_i(t),x_j(t)) + \xi_i^2\nabla^\perp g(x_i,x_i).$$

The additional (with respect to the cases with no boundary) self-interaction terms involving g are due to the presence of an impermeable boundary: it is thanks to these terms that the system satisfies (in weak sense) Euler's equations. We refer to [**135**, Section 4.1] for a thorough motivation of this fact.

In this setting, the relevant features (i)–(iii) we individuated in the last paragraph are still present, but the boundary enters as an additional singular set of the vector field, and thus our arguments must take it into account. Without going into details, we just mention the relevant required modifications:

- the smoothed vector field B^ε and its associated flow T^ε of subsection 3.1.1 must be defined by smoothing both $\log|\cdot|$ and g in (3.2.2): B^ε will coincide with the original vortices vector field B whenever vortices are at least $\varepsilon > 0$ apart from each other and from the boundary;
- functions of D must now satisfy $\operatorname{supp} f \subset \mathcal{D}^N \setminus \triangle^N$, where $\triangle^N$ is the diagonal set of $\mathcal{D}^N$ (the definition being the same as in the torus case), whereas function of $\tilde{D}$ must vanish not only around $\triangle^N$, but also in a neighbourhood of $\left\{\underline{x} \in \overline{\mathcal{D}}^N : \exists i : x_i \in \partial\mathcal{D}\right\}$.

3.2.3. Gibbsian Ensembles and Vortices on the Whole Plane. We now return to point vortices on $\mathbb{T}^2$. We have already mentioned that besides the uniform measure dx^N on $\mathbb{T}^{2\times N}$, the point vortices flow T_t also preserves all *(Canonical) Gibbs measures*

$$d\mu_{\beta,N}(\underline{x}) = \frac{1}{Z_{\beta,N}} e^{-\beta H(\underline{x})} dx^N, \quad Z_{\beta,N} = \int_{\mathbb{T}^{2\times N}} e^{-\beta H(\underline{x})} dx^N. \tag{3.2.4}$$

When $\beta > 0$, $\mu_{\beta,N}$ gives more weight to configurations where vortices of the same sign are far from each other, but positive and negative vortices are close. Vice-versa, if $\beta < 0$, vortices of the same sign tend to cluster. Invariance of $\mu_{\beta,N}$ is an easy consequence of the one of $H(\underline{x})$ and dx^N, and it can be achieved by considering the smoothed vortices interaction B^ε with Hamiltonian H^ε and sending $\varepsilon \to 0$.

Whatever β is, since $\mu_{\beta,N}$ is absolutely continuous with respect to dx^N, the flow T_t is still globally well-defined on a full-measure set. However, the density of $\mu_{\beta,N}$ is singular in $\triangle^N$ (save for trivial cases), so uniform integrability of the Lyapunov functions $\mathcal{L}^\varepsilon$ in Theorem 1.3.1 is spoiled. As a consequence, the arguments in subsection 3.1.4 also fail.

Let us now spend a few words on point vortices on $\mathbb{R}^2$. The system is given by (1.3.1) with $G(x) = -\frac{1}{2\pi}\log|x|$, and it is well-posed for almost all initial conditions with respect to the product Lebesgue measure provided that no subset of the intensities $\{\xi_1, \dots \xi_N\}$ sums to zero, see [**135**]. The latter condition ensures that vortices can not travel to infinity in finite time.

The product Lebesgue measure on $\mathbb{R}^{2\times N}$ is not a probability measure, so we are led to look for an integrable density on $\mathbb{R}^2$ left invariant by the dynamics. To the best of our knowledge, this is only achieved by the Gaussian measure

$$d\mu_{\alpha,\eta,N}(\underline{x}) = \frac{1}{Z_{\alpha,\eta,N}} e^{-\eta\cdot M(\underline{x}) - \alpha I(\underline{x})} dx^N, \quad \eta \in \mathbb{R}^2, \alpha \in \mathbb{R}^+,$$
$$M(\underline{x}) = \sum_{i=1}^N \xi_i x_i, \qquad I(\underline{x}) = \sum_{i=1}^N \xi_i |x_i|^2,$$

when *all vortices are positive*, I and M being first integrals of vortices motion, the *moment of inertia* and *centre of vorticity* (see [**127**, Section 5.3]). The interaction energy H can be also added to the Gibbs exponential, but this is not a substantial modification. As we have seen above, the case of positive vortices can be dealt with by exploiting conservation of energy, so we shall not discuss it further. Unfortunately, the more interesting case of arbitrary signs seems to be impossible to include in our discussion.

3.2.4. The Configuration Space and Non-Uniqueness. In the point vortices time evolution, the number and intensities of vortices are constant — at least when no vortices collide, as we will see. As a consequence, everything we said still applies if instead of fixing $N, \underline{\xi}$ we choose them at random, provided that all objects are well defined. In order to discuss an arbitrary number of vortices, one can consider the phase space

$$\bigcup_{N\geq 0} (\mathbb{T}^2 \times \mathbb{R})^N,$$

on which, conditioned to the random choice of N, to be made for instance with a sample of a Poisson distribution, we consider the product measures $dx^N \otimes \nu^{\otimes N}$, with ν the probability law of a single intensity $\xi_i \in \mathbb{R}$.

An equivalent (up to symmetrisation of products) point of view is the *configuration space* setting, in which one looks at the law of the vorticity distribution $\omega = \sum_{i=1}^N \xi_i \delta_{x_i}$ (the empirical measure of vortices) under the law of the aforementioned ensemble of vortices. This is the approach of [**3**]. Let us define

$$\Gamma = \bigcup_{N\geq 0} \Gamma_N, \quad \Gamma_N = \left\{ \gamma = \sum_{i=1}^N \xi_i \delta_{x_i} : \xi_i \in \mathbb{R}, x_i \in \mathbb{T}^2, x_i \neq x_j \text{ if } i \neq j \right\},$$

to be regarded as a subset of finite signed measures $\mathcal{M}(\mathbb{T}^2)$. There is a one-to-one correspondence between elements of Γ_N and classes of equivalence of $(\mathbb{T}^2 \times \mathbb{R})^N$ up to permutations. Let ν be a probability measure on $\mathbb{R}$ with finite second moment and $\lambda > 0$; we define the measure μ_N on Γ_N as the image of $dx^N \otimes \nu^{\otimes N}$ on $(\mathbb{T}^2 \times \mathbb{R})^N$ under the aforementioned correspondence, and then we define μ on Γ as

$$\mu = e^{-\lambda} \sum_{N\geq 0} \frac{\lambda^N}{N!} \mu_N.$$

Equivalently, μ can be realised by considering a Poisson point process on $\mathbb{T}^2 \times \mathbb{R}$ with intensity measure $\lambda dx \otimes d\nu$, the samples of which are vectors $(x_1, \xi_1, \dots x_N, \xi_N)$, and setting μ to be the image law under the map $\gamma = \sum_{i=1}^N \xi_i \delta_{x_i}$. We refer to [**6**] for a complete discussion of Poisson processes and the configuration space.

By Theorem 1.3.1, for μ-almost every $\gamma \in \sum_{i=1}^N \xi_i \delta_{x_i} \Gamma$ the point vortices flow with initial positions x_i and intensities ξ_i is globally well-defined. Moreover, the flow defines a group of invertible measurable maps $\mathcal{T}_t : \Gamma \to \Gamma$, the cursive to distinguish it from the flow T_t on $L^2(\mathbb{T}^{2\times N})$ in Section 3.1. The map $\mathcal{T}_t$ preserves μ since it leaves each Γ_N invariant, and for fixed N the point vortices evolution does not change intensities and preserves the product measure on the torus.

The main contribution of [**3**] is an explicit expression of the generator of the Koopman group U_t on $L^2(\Gamma,\mu)$ associated to $\mathcal{T}_t$, on the set of cylinder functions of Fourier modes. In order to comment the problem of essential self-adjointness in this setting we now repeat their result: we do so perhaps in a more concise way, by means of the aforementioned Delort-Schochet symmetrisation. Let us thus consider Euler equations in the weak vorticity form

$$\begin{aligned}
\langle \phi, \omega_t \rangle - \langle \phi, \omega_0 \rangle &= \int_0^t \int_{\mathbb{T}^{2\times 2}} H_\phi(x,y)\omega_s(x)\omega_s(y)dxdyds \\
&= \int_0^t \langle H_\phi, \omega_s \otimes \omega_s \rangle \, ds, \\
H_\phi(x,y) &= \frac{1}{2}(\nabla\phi(x) - \nabla\phi(y)) \cdot K(x-y), \quad x,y \in \mathbb{T}^2,
\end{aligned}$$

where $H_\phi(x,y)$ is a symmetric function with zero average in both variables and smooth outside the diagonal set $\triangle^2$, where it has a jump discontinuity. We already noticed that the empirical measure $\omega = \sum_{i=1}^N \xi_i \delta_{x_i}$ satisfies such equations when x_i are positions of point vortices.

We define *local observables* on Γ as the family $\mathcal{F}$ of functions of the form

$$F(\gamma) = f(\langle \phi_1, \gamma \rangle, \dots \langle \phi_n, \gamma \rangle), \tag{3.2.5}$$

where $f \in C_c^\infty(\mathbb{C}^n, \mathbb{C})$ and $\phi_1, \dots \phi_n \in C^\infty(\mathbb{T}^2, \mathbb{C})$, the brackets $\langle \cdot, \cdot \rangle$ denoting coupling of continuous functions and measures. In [**3**] the functions ϕ_k were chosen in the Fourier orthonormal basis, but this would not change anything in our discussion.

PROPOSITION 3.2.1. *Let $\mathcal{U}_t$ be the Koopman group on $L^2(\mu)$ associated with $\mathcal{T}_t$, and $\mathcal{A}$ be its generator. For any $F \in \mathcal{F}$ of the form (3.2.5), the following expression defines an observable in $L^2(\mu)$,*

$$\mathcal{L}F(\gamma) = -\mathrm{i} \sum_{k=1}^n \partial_k f(\langle \phi_1, \gamma \rangle, \dots \langle \phi_n, \gamma \rangle) \langle H_{\phi_k}, \gamma \otimes \gamma \rangle . \tag{3.2.6}$$

The operator $(\mathcal{L}, \mathcal{F})$ is symmetric, $\mathcal{F} \subseteq D(\mathcal{A})$, and $\mathcal{A}|_{\mathcal{F}} = \mathcal{L}$. Moreover, $(\mathcal{L}, \mathcal{F})$ is Markov unique, that is $\mathcal{A}$ is the unique self-adjoint extension generating a strongly continuous, positivity and unit preserving group of unitaries, which is $\mathcal{U} = e^{\mathrm{i}\, t\mathcal{A}}$.

PROOF. To show that $\mathcal{L}F \in L^2(\mu)$, since $\partial_k f$ is uniformly bounded, we just need to compute for $\phi \in C^\infty(\mathbb{T}^2, \mathbb{C})$,

$$\begin{aligned}
\int \langle H_\phi, \gamma \otimes \gamma \rangle^2 d\mu_N(\gamma) &= \int_{\mathbb{T}^{2\times N}} \int_{\mathbb{R}^N} \left(\sum_{i\neq j} \xi_i \xi_j H_\phi(x_i, x_j) \right)^2 dx^N d\nu^N(\underline{\xi}) \\
&= \int_{\mathbb{T}^{2\times N}} \int_{\mathbb{R}^N} \sum_{i\neq j, \ell \neq k} \xi_i \xi_j \xi_\ell \xi_k H_\phi(x_i,x_j) H_\phi(x_\ell, x_k) dx^N d\nu^N(\underline{\xi}) \\
&= 2 \sum_{i \neq j} \int_{\mathbb{R}^2} \xi_i^2 \xi_j^2 d\nu(\xi_i) d\nu(\xi_j) \int_{\mathbb{T}^{2\times 2}} H_\phi(x,y)^2 dxdy \leq C_{\phi,\nu} N^2,
\end{aligned}$$

where we made essential use of the fact that H_ϕ is zero-averaged in both variables, so the only non vanishing terms in the double sum are the ones with $i = \ell, j = k$

(or vice-versa). We also recall that the ξ_i's are independent with finite second moments. From here,

$$\int \langle H_\phi, \gamma \otimes \gamma \rangle^2 \, d\mu(\gamma) \leq e^{-\lambda} \sum_{N \geq 0} \frac{\lambda^N}{N!} C_{\phi,\nu} N^2 < \infty,$$

from which we easily conclude $\mathcal{L}F \in L^2(\mu)$.

We are left to prove that $\mathcal{U}_t$ is differentiable on $\mathcal{F}$ and that its derivative at time $t = 0$ is given by $\mathcal{L}$. However, this is equivalent to show that $\omega_t = \mathcal{T}_t \gamma$ solves (1.2.2), which we already know. □

Local observables $\mathcal{F}$ are not invariant for $\mathcal{U}_t$: this is due to the nonlinearity of the dynamics, not to singularity of the interaction. Our techniques thus does not seem to be suited to this setting.

We conclude by mentioning an idea of [**155**], from which we quote: *"Considering point vortices to be solutions of the weak vorticity formulation allows us to extend their dynamics beyond collisions simply by merging vortices that collide into a single vortex whose strength is the algebraic sum of the colliding vortices. Clearly this defines a solution for times less than and for times greater than the collision time, and the resulting vorticity is continuous in time in the weak-* topology of measures, so that there is no contribution [...] from the "jump" at the collision time. Of course, this extended notion of point-vortex dynamics is horribly nonunique since the time-reversibility of the Euler equations implies that a single vortex can split equally well into several vortices at any time."* Non uniqueness for the weak formulation of Euler equation in the point vortices case might be a clue that $(\mathcal{L}, \mathcal{F})$ is not essentially self-adjoint or even $L^2(\mu)$ unique. However, producing counterexamples with collisions or splitting of vortices is a difficult problem: explicit examples of collisions rely on integrability properties of the Hamiltonian dynamics. Whether $(\mathcal{L}, \mathcal{F})$ is essentially self-adjoint thus remains an interesting open question.

CHAPTER 4

A Central Limit Theorem for Gibbs Ensembles of Vortices

This Chapter follows [98], and it is devoted to the proof of Theorem 1.5.2 and its generalisations on other space domains: we have given a general introduction to the result in Section 1.5 above. The result in a sense completes the one of [17], in which the same scaling limit of point vortices was performed, but with a smoothed interaction potential. We also mention that a Central Limit Theorem for fluctuations of point vortices in the case where D is a disk was derived at the end of [26]: that result is unfortunately incomplete, since it proves convergence of integrals of the fluctuation field against a restricted set of test functions. Both [17, 26] emphasise the relevance of a good control of partition functions, which in fact is crucial in the present work. Most of the underlying physical understanding of the topic goes back to classical works: we mainly refer to the ones of Kraichnan and Onsager, see respectively [113, 143] and references therein.

In Section 4.1 we discuss in detail the result in the case where $D = \mathbb{T}^2$ is the 2-dimensional torus: the main result is Theorem 1.5.2, which we presented above in Chapter 1. In principle, the result could be extended to compact Riemannian surfaces D: we do not pursue such generality, and we only consider two other physically relevant geometries, namely the 2-dimensional sphere $\mathcal{S}^2$ and bounded domains of $\mathbb{R}^2$. The former, being a compact surface without boundary, is completely analogous to the case on $\mathbb{T}^2$, and it is briefly discussed in Section 4.2. In Section 4.3 we show how to adapt the previous arguments to the case of a bounded domain, the main issue being the self-interaction terms in the Hamiltonian due to the presence of a boundary. Finally in Section 4.4, as concluding remarks, we outline how our result compares to the well established literature on mean field limits for point vortices.

4.1. The Periodic Case

Let $N \in \mathbb{N}$ (the number of vortices), $\gamma > 0$, $\beta \geq 0$ (the *inverse temperature*), $\xi_1, \dots, \xi_N \in \mathbb{R}$ (the intensities of vortices), $x_1, \dots, x_N \in \mathbb{T}^2$ (the positions of vortices) and the Hamiltonian

$$H(x_1, \dots, x_N) = \sum_{i<j}^{N} \xi_i \xi_j G(x_i, x_j)$$

on the phase space $\mathbb{T}^{2\times N}$. In what follows, intensities will always be given as $\xi_i = \frac{\sigma_i}{\sqrt{\gamma N}}$, with signs $\sigma_i = \pm 1$, according to the central limit scaling. The arguments of the present Section work for any choice of the sequence of signs $\sigma_1^N, \dots \sigma_N^N = \pm 1$ for $N \geq 1$: we assume that such a choice is performed once and for all, and drop the apex N to ease notation. The main result of this section is convergence of Gibbs ensemble of vortices $\mu^N_{\beta,\gamma}$ defined above to the energy-enstrophy measure $\mu_{\beta,\gamma}$, which we introduced in Chapter 1 and rigorously discussed in Section 2.4.

4.1.1. On Gibbs Measures for Point Vortices. Let us consider the measure on $\mathbb{T}^{2\times N}$ defined by

$$\nu_{\beta,\gamma,N}(dx_1,\dots,dx_N)=\frac{1}{Z_{\beta,\gamma,N}}\exp\left(-\beta H(x_1,\dots,x_N)\right)dx_1,\dots,dx_N, \tag{4.1.1}$$

with $Z_{\beta,\gamma,N}$, the *partition function*, being the constant such that $\nu_{\beta,\gamma,N}$ is a probability measure. Notice that, even if it is not made explicit, the partition function depends also on the choice of signs σ_i. The measure $\nu_{\beta,\gamma,N}$ is usually referred to as the *canonical Gibbs' measure*. Since the potential G has a logarithmic singularity, the existence of such measure, or equivalently the finiteness of $Z_{\beta,\gamma,N}$, is not completely trivial. The issue is addressed in [**127**] on bounded domains of $\mathbb{R}^2$ for vortices with equal intensities. The technique we apply was first introduced in [**63**] in the similar case of a log-gas: a more refined computation deriving the asymptotics in N in the latter setting can be found in [**102**].

PROPOSITION 4.1.1. *For any choice of $\gamma>0$, $\beta\geq 0$, and signs $\sigma_i=\pm 1$ as above, if $N>\frac{\beta}{\pi\gamma}$ then $Z_{\beta,\gamma,N}<\infty$, and the measure $\nu_{\beta,\gamma,N}$ is thus well-defined.*

PROOF. By (1.2.1) and Hölder's inequality,

$$Z_{\beta,\gamma,N}\leq\left(\int_{\mathbb{T}^{2N}}\prod_{i<j}d(x_i,x_j)^{\frac{\beta\xi_i\xi_j}{\pi}}\right)^{1/2}\left(\int_{\mathbb{T}^{2N}}\prod_{i<j}e^{-2\beta\xi_i\xi_j g(x_i,x_j)}\right)^{1/2},$$

where the second factor on the right-hand side is bounded (by a constant depending on all parameters including N) since g is. Let us now turn to the first term. We relabel the variables as follows: $y_1,\dots y_k$ are the ones with positive intensities, and $z_1,\dots z_{n-k}$ the negative ones; moreover, y_i and z_i are couples of closest positive-negative neighbours, so that

$$d(y_i,z_i)\leq d(y_i,z_j)\wedge d(y_j,z_i)\quad\forall j\geq i. \tag{4.1.2}$$

We accordingly split

$$\prod_{i<j}d(x_i,x_j)^{\frac{\beta\sigma_i\sigma_j}{\pi\gamma N}}=\left(\frac{\prod_{i<j}d(y_i,y_j)\prod_{i<j}d(z_i,z_j)}{\prod_{i,j}d(y_i,z_j)}\right)^{\frac{\beta}{\pi\gamma N}},$$

the indices running over all admissible values. By definition and the triangular inequality,

$$\begin{aligned}d(y_i,y_j)&\leq d(y_i,z_i)+d(y_j,z_i)\leq 2d(y_j,z_i),\\ d(z_i,z_j)&\leq d(y_i,z_i)+d(y_i,z_j)\leq 2d(y_j,z_i),\end{aligned}$$

so that we can use the terms in the numerator to cancel all terms in the denominator save for the ones corresponding to closest neighbours (if $k\neq N/2$ some terms in the numerator are left over, and we bound them with constants):

$$\prod_{i<j}d(x_i,x_j)^{\frac{\beta\sigma_i\sigma_j}{\pi\gamma N}}\leq C\left(\prod_{1\leq i\leq k\wedge n-k}d(y_i,z_i)\right)^{-\frac{\beta}{\pi\gamma N}},$$

where C is again a constant depending on all parameters. As soon as $N>\frac{\beta}{2\pi\gamma}$, factors of the latter product are integrable, thus concluding the proof. □

In dealing with limits as N goes to infinity, Gibbs measure will always be (ultimately) defined, so we will ignore the issue henceforth in this section. Finally, let us note that $\omega^N_{\beta,\gamma}$ can be regarded as random variables in $\dot H^s(\mathbb{T}^2)$ for all $s<-1$, since signed measures have uniformly bounded Fourier coefficients.

4.1.2. Potential Splitting and the Sine-Gordon transformation. In this paragraph we introduce the key tools in the proof of Theorem 1.5.2. The main issue is the logarithmic singularity of the Green function G. To deal with it we will decompose G in two parts, a smooth approximation of G and a remainder retaining logarithmic singularity: for $m > 0$,

$$G = -\Delta^{-1} = \left(-\Delta^{-1} - (m^2 - \Delta)^{-1}\right) + (m^2 - \Delta)^{-1} := V_m + W_m. \tag{4.1.3}$$

Physically, the smooth part V_m corresponds to the long-range part of the potential, and the singular part W_m to short-range interactions. We will also denote

$$H = H_{V_m} + H_{W_m} = \sum_{i<j}^{N} \xi_i \xi_j V_m(x_i, x_j) + \sum_{i<j}^{N} \xi_i \xi_j W_m(x_i, x_j),$$

the relative splitting of the Hamiltonian. In terms of Fourier series,

$$W_m(x, y) = \sum_{k \in \mathbb{Z}_0^2} \frac{e_k(x - y)}{m^2 + 4\pi^2 |k|^2}, \quad V_m(x, y) = \sum_{k \in \mathbb{Z}_0^2} \frac{m^2 e_k(x - y)}{4\pi^2 |k|^2 (m^2 + 4\pi^2 |k|^2)}.$$

The Green function W_m is called the 2-dimensional *Yukawa potential* or *screened Coulomb potential* with mass m (as opposed to the *Coulomb potential* G).

We will regard the regular part of the Hamiltonian corresponding to V_m as the covariance of a Gaussian field. The idea, dating back to [**152**], originated as a connection between the classical Coulomb gas theory and sine-Gordon field theory (hence the name): it will allow us to analyse the convergences in Theorem 1.5.2 by standard Gaussian computations, up to a remainder term involving the Yukawa potential W_m (whose associated partition function we bound in subsection 4.1.3). We thus define F_m as the centred Gaussian field on $\mathbb{T}^2$ with covariance kernel V_m, that is

$$\forall f, g \in \dot{L}^2(\mathbb{T}^2), \quad \mathbb{E}\left[\langle F_m, f\rangle \langle F_m, g\rangle\right] = \left\langle f, \left(-\Delta^{-1} - (m^2 - \Delta)^{-1}\right) g \right\rangle. \tag{4.1.4}$$

The remainder of this paragraph deals with properties of F_m. The reproducing kernel Hilbert space is

$$\sqrt{-\Delta^{-1} - (m^2 - \Delta)^{-1}}\, \dot{L}^2(\mathbb{T}^2) \subseteq \dot{H}^2(\mathbb{T}^2),$$

so that F_m has a $\dot{H}^s(\mathbb{T}^2)$-valued version for all $s < 1$, into which $\dot{H}^2(\mathbb{T}^2)$ has Hilbert-Schmidt embedding. As a consequence, by Sobolev embedding, F_m has a version taking values in $\dot{L}^p(\mathbb{T}^2)$ for all $p \geq 1$.

The field F_m can also be evaluated at points $x \in \mathbb{T}^2$: the coupling $F_m(x) := \langle \delta_x, F_m\rangle$ is defined as the series, converging in $L^2(F_m)$ *uniformly in* $x \in \mathbb{T}^2$,

$$\langle \delta_x, F_m\rangle = \sum_{k \in \mathbb{Z}_0^2} e^{2\pi\,\mathrm{i}\, x\cdot k} \hat{F}_{m,k}, \quad \hat{F}_{m,k} = \langle e_k, F_m\rangle \sim N_{\mathbb{C}}\left(0, \frac{m^2}{4\pi^2 |k|^2 \left(m^2 + 4\pi^2 |k|^2\right)}\right).$$

In other terms, $x \mapsto F_m(x)$ is a measurable random field, and $F_m(x)$ are centred Gaussian variables of variance $V_m(x, x) = V_m(0, 0)$. A straightforward application of Kolmogorov continuity theorem shows that there exists a version of $F_m(x)$ which is α-Hölder for all $\alpha < 1/2$.

Lemma 4.1.2. *For any $\alpha > 0$, $p \geq 1$ and $m \to \infty$,*

$$\mathbb{E}\left[\|F_m\|_p^p\right] \simeq_p (\log m)^{p/2} \tag{4.1.5}$$

$$\mathbb{E}\left[\exp\left(-\alpha \|F_m\|_2^2\right)\right] \simeq m^{-\frac{\alpha}{2\pi}}. \tag{4.1.6}$$

PROOF. Let us begin with moments: by Fubini-Tonelli theorem,

$$\mathbb{E}\left[\|F_m\|_p^p\right] = \int_{\mathbb{T}^2} \mathbb{E}\left[|F_m(x)|^p\right] dx = c_p \int_{\mathbb{T}^2} V_m(x,x)^{p/2} dx = c_p V_m(0,0)^{p/2},$$

where $V_m(0,0) = \frac{1}{2\pi}\log m + o(\log m)$ can be checked by explicit computation in Fourier series. As for exponential moments, a standard Gaussian computation (see [59, Proposizion 2.17]) gives

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-\alpha\|F_m\|_2^2\right)\right] &= \exp\left\{-\frac{1}{2}\operatorname{Tr}\left(\log\left(1+2\alpha\left(-\Delta^{-1}-(m^2-\Delta)^{-1}\right)\right)\right)\right\}\\
&= \exp\left(-\frac{1}{2}\sum_{k\in\mathbb{Z}_0^2}\log\left(1+\frac{2\alpha m^2}{4\pi^2|k|^2(m^2+4\pi^2|k|^2)}\right)\right)\\
&> \exp\left(-\sum_{k\in\mathbb{Z}_0^2}\frac{\alpha m^2}{4\pi^2|k|^2(m^2+4\pi^2|k|^2)}\right)\\
&= \exp\left(-\alpha V_m(0,0)\right) \simeq m^{-\frac{\alpha}{2\pi}},
\end{aligned}$$

the other inequality descending from analogous computations using $\log(1+x) > x - \frac{x^2}{2}$, $x > 0$, instead of the inequality $\log(1+x) < x$ we just applied. □

Since it holds, for $s,t\in\mathbb{R}$,

$$\mathbb{E}\left[e^{\mathrm{i}\,sF_m(x)}e^{\mathrm{i}\,tF_m(y)}\right] = e^{-\frac{s^2+t^2}{2}V_m(0,0)}e^{-stV_m(x,y)},$$

(and analogous expressions for n-fold products) we can transform the partition function relative to the regular part of the Hamiltonian H_{V_m}:

$$\begin{aligned}
(4.1.7)\quad &\int_{\mathbb{T}^{2N}} e^{-\beta H_{V_m}} dx_1\cdots dx_n\\
&= \int_{\mathbb{T}^{2N}} \exp\left(-\beta\sum_{i\neq j}^{N}\frac{\sigma_i\sigma_j}{2\gamma N}V_m(x_i,x_j)\right) dx_1\cdots dx_n\\
&= e^{\frac{\beta}{2\gamma}V_m(0,0)}\mathbb{E}\left[\int_{\mathbb{T}^{2N}}\exp\left(-\mathrm{i}\sqrt{\frac{\beta}{\gamma N}}\sum_{i=1}^{N}\sigma_i F_m(x_i)\right)dx_1\cdots dx_n\right].
\end{aligned}$$

Rewriting the partition function in these terms is the first step in the analysis of $Z_{\beta,\gamma,N}$, the next one being a control of the singular part of the potential, which we could not transform. We deal with W_m in the next paragraph: let us conclude the present one with the estimate we will use on complex exponentials of F_m. It relies essentially on:

LEMMA 4.1.3. *If $f\in\dot{L}^4(\mathbb{T}^2)$, then*

$$\left|\int_{\mathbb{T}^2} e^{\mathrm{i}\,f(x)}dx - e^{-\frac{1}{2}\|f\|_2^2}\right| \leq \frac{\|f\|_3^3}{6} + \frac{\|f\|_2^4}{8}.$$

PROOF. Thanks to the zero average condition, we can expand

$$\begin{aligned}
&\int_{\mathbb{T}^2} e^{\mathrm{i}\,f(x)}dx - e^{-\frac{1}{2}\|f\|_2^2}\\
&\quad= \int_{\mathbb{T}^2}\left(e^{if(x)} - 1 - \mathrm{i}\,f(x) + \frac{f(x)^2}{2}\right)dx - \left(e^{-\frac{1}{2}\|f\|_2^2} - 1 + \frac{\|f\|_2^2}{2}\right)
\end{aligned}$$

and then apply Taylor expansions

$$\left|e^{it}-1-it+\frac{t^2}{2}\right|\le\frac{t^3}{6},\qquad \left|e^{-t}-1+t\right|\le\frac{t^2}{2}.$$

□

PROPOSITION 4.1.4. *For any $\beta,\gamma>0$ and integer $p\ge 1$, if $m=m(N)$ grows at most polynomially in N, then it holds*

$$\int_{\mathbb{T}^{2N}} e^{-\beta H_{V_m}}\,dx_1\cdots dx_n \le C_{\beta,\gamma,p}\left(1+\frac{m^{\frac{\beta}{4\pi\gamma}}(\log m)^{2p}}{N^{p/2}}\right)$$

uniformly in N.

To ease notation, in the following argument we will denote

$$E_j=\int_{\mathbb{T}_2} e^{\mathrm{i}\,\xi_j\sqrt{\beta}F_m(x_j)}\,dx_j,\qquad \mathcal{E}=e^{-\frac{\beta}{2N\gamma}\|F_m\|_{L^2}^2},$$

(notice that both depend on N,m) and thus write (4.1.7) as

$$\int_{\mathbb{T}^{2N}} e^{-\beta H_{V_m}}\,dx_1\cdots dx_n = e^{\frac{\beta}{2\gamma}V_m(0,0)}\mathbb{E}\left[\prod_{j=1}^N E_j\right]$$

In sight of Lemma 4.1.3, we expect the 0-th order term (in $1/N$) to be $e^{\frac{\beta}{2\gamma}V_m(0,0)}\mathbb{E}\left[\mathcal{E}^N\right]$, which is $O(1)$ as shown above in Lemma 4.1.2. The forthcoming proof applies the Taylor expansion of Lemma 4.1.3 to further and further orders.

PROOF. For $p=1$, we expand the product $\prod_{j=1}^N E_j$ by means of the algebraic identity

$$\prod_{j=1}^N E_j=\mathcal{E}^N+\sum_{k=1}^N(E_k-\mathcal{E})\mathcal{E}^{N-k}\left(\prod_{j=1}^{k-1}E_j\right), \tag{4.1.8}$$

from which we can estimate

$$\begin{aligned}
\mathbb{E}\left[\prod_{j=1}^N E_j\right] &= \mathbb{E}\left[\mathcal{E}^N\right]+\sum_{k=1}^N\mathbb{E}\left[\left(\prod_{j=1}^{k-1}E_j\right)(E_k-\mathcal{E})\mathcal{E}^{N-k}\right]\\
&\le \mathbb{E}\left[\mathcal{E}^N\right]+\sum_{k=1}^N\mathbb{E}\left[|E_k-\mathcal{E}|\right]\\
&\le \mathbb{E}\left[\mathcal{E}^N\right]+N\cdot\mathbb{E}\left[\frac{1}{6}\left(\frac{\beta}{\gamma N}\right)^{3/2}\|F_m\|_3^3+\frac{1}{8}\left(\frac{\beta}{\gamma N}\right)^2\|F_m\|_2^4\right]\\
&\le C_{\beta,\gamma}\left(m^{-\frac{\beta}{4\pi\gamma}}+\frac{(\log m)^2}{\sqrt{N}}\right).
\end{aligned}$$

The higher order terms (in $1/N$) have been dealt with in the following way: exponential factors have been bounded with $|E_j|,|\mathcal{E}|\le 1$, only leaving differences $E_k-\mathcal{E}$ from which smallness is obtained. The third step is the crucial application of Lemma 4.1.3, and the last one is Lemma 4.1.2 and Hölder inequality. The thesis now follows recalling once again that $V_m(0,0)\simeq\frac{1}{2\pi}\log m$.

For $p=2$, by iterating (4.1.8) we get the identity

$$\prod_{j=1}^N E_j=\mathcal{E}^N+\mathcal{E}^{N-1}\sum_{k=1}^N(E_k-\mathcal{E})+\sum_{k=2}^N\sum_{\ell=1}^{k-1}(E_\ell-\mathcal{E})(E_k-\mathcal{E})\mathcal{E}^{N-\ell-1}\prod_{j=1}^{\ell-1}E_j.$$

Taking expectations and controlling separately the summands,

$$\begin{aligned}\mathbb{E}\left[\prod_{j=1}^N E_j\right] &\leq \mathbb{E}\left[\mathcal{E}^N\right] + \sum_{k=1}^N \mathbb{E}\left[|E_k-\mathcal{E}|\mathcal{E}^{N-k}\right] + \sum_{k=2}^N\sum_{\ell=1}^{k-1}\mathbb{E}\left[|E_\ell-\mathcal{E}|\,|E_k-\mathcal{E}|\right]\\ &\leq \mathbb{E}\left[\mathcal{E}^N\right] + N\mathbb{E}\left[\mathcal{E}^{2(N-k)}\right]^{1/2}\mathbb{E}\left[|E_1-\mathcal{E}|^2\right]^{1/2}\\ &\quad+\frac{1}{2}N(N-1)\mathbb{E}\left[|E_1-\mathcal{E}|^2\right]\\ &\lesssim m^{-\frac{\beta}{4\pi\gamma}} + m^{-\frac{\beta(N-k)}{4\pi\gamma N}}N^{-1/2}(\log m)^{3/2} + N^{-1}(\log m)^3.\end{aligned}$$

In the latter computation, the second step is Cauchy-Schwarz inequality, while the third combines Hölder inequality and Lemma 4.1.2 to control

$$\mathbb{E}\left[|E_k-\mathcal{E}|^2\right] \lesssim \mathbb{E}\left[\left(N^{-3/2}\left\|F_m\right\|_3^3 + N^{-2}\left\|F_m\right\|_2^4\right)^2\right] \lesssim N^{-3}(\log m)^3.$$

The thesis for $p=2$ is obtained, since we have shown that

$$\int_{\mathbb{T}^{2N}} e^{-\beta H_{V_m}}dx_1\cdots dx_n \lesssim 1 + N^{-1/2}(\log m)^{3/2} + m^{\frac{\beta}{4\pi\gamma}}N^{-1}(\log m)^3,$$

where the middle term is always $o(1)$ because we are assuming that $m(N)$ grows at most polynomially.

Further iterations of (4.1.8) to expand products of E_j produce in a completely analogous manner the required estimate for arbitrary $p\geq 1$. Let us only report, as an example, the third order iteration of (4.1.8):

$$\begin{aligned}\prod_{j=1}^N E_j &= \mathcal{E}^N + \mathcal{E}^{N-1}\sum_{k=1}^N(E_k-\mathcal{E}) + \mathcal{E}^{N-2}\sum_{k=2}^N\sum_{\ell=1}^{k-1}(E_\ell-\mathcal{E})(E_k-\mathcal{E})\\ &\quad+\sum_{k=3}^N\sum_{\ell=2}^{k-1}\sum_{m=1}^{\ell-1}(E_k-\mathcal{E})(E_\ell-\mathcal{E})(E_m-\mathcal{E})\mathcal{E}^{N-m-2}\Big(\prod_{j=1}^{m-1}E_j\Big).\end{aligned}$$ □

4.1.3. Controlling Partition Functions. We want to analyse separately the contributions of regular and singular parts of the potential to the partition function

$$Z_{\beta,\gamma,N} = \int_{\mathbb{T}^{2N}} e^{-\beta H_{V_m}}e^{-\beta H_{W_m}}dx^N.$$

The core idea is that if we send $m(N)\to\infty$ along $N\to\infty$ with a suitable rate, the contribution of the Yukawa part of the potential, W_m, becomes irrelevant, and we can bound $Z_{\beta,\gamma,N}$ uniformly in N. With a uniform bound at hand, identifying the limit becomes quite simple: we will do so in the next Section, reducing ourselves to the case $\beta=0$.

Let us thus focus on W_m. Its free version $W_{m,\mathbb{R}^2}$, that is the Green function of $m^2-\Delta$ on the whole plane, can be expressed in term of the modified Bessel function of the second kind K_0 as

$$W_{m,\mathbb{R}^2}(x,y) = W_{m,\mathbb{R}^2}(|x-y|) = \frac{1}{2\pi}K_0(m|x-y|), \quad x,y\in\mathbb{R}^2, \tag{4.1.9}$$

where K_0 is the positive solution of

$$r^2K_0''(r) + rK_0'(r) - r^2K_0(r) = 0, \quad r\geq 0,$$

with logarithmic divergence in $r = 0$ and exponential decay for large r,

$$K_0(r) = -\log(r) + O(1), \qquad r \to 0, \tag{4.1.10}$$

$$K_0(r) \le \frac{\sqrt{\pi}e^{-r}}{\sqrt{2r}}, \qquad \forall r > 0 \tag{4.1.11}$$

(see [1]). Unlike $G_{\mathbb{R}^2}(x) = -\frac{1}{2\pi}\log|x|$, $W_{m,\mathbb{R}^2} \in L^1(\mathbb{R}^2)$, hence by Poisson summation formula it holds, for any distinct $x, y \in \mathbb{T}^2$,

$$W_m(x,y) = \sum_{k\in\mathbb{Z}^2} W_{m,\mathbb{R}^2}(|x+k-y|) - \int_{\mathbb{R}^2} W_{m,\mathbb{R}^2}(|x|)dx, \tag{4.1.12}$$

the integral on right-hand side taking care of the space average. Notice that, since K_0 is positive, so is the first summand in (4.1.12). This representation allows for a quite precise control of W_m, which we now use to control the rate at which the partition function relative to Yukawa potential goes to 1 as $m \to \infty$.

Proposition 4.1.5. *Let $N \ge 1$, $\beta/\gamma > -8\pi$ and $m > 0$. There exists a constant $C_{\beta,\gamma} > 0$ such that*

$$\int_{\mathbb{T}^{2N}} e^{-\beta H_{W_m}} dx_1 \cdots dx_n \le \left(1 + C_{\beta,\gamma}\frac{(\log m)^2}{m^2}\right)^N$$

(uniformly with respect to the choice of signs σ_i).

Proof. As a first step we produce an estimate on $W_m(x) = W_m(x,0)$ which separates the short-range, relevant part and a long range remainder. We do so by means of the representation (4.1.12), so first we have to take a closer look at $W_{m,\mathbb{R}^2}$. We choose a small radius $\frac{1}{m} \ll r_m = \frac{2\log m}{m} \ll 1$, below which we control $W_{m,\mathbb{R}^2}$ with logarithm: by (4.1.11), and since K_0 is decreasing, $W_{m,\mathbb{R}^2}(x) \le \frac{C}{m^2}$ when $|x| \ge r_m$ (C will denote possibly different positive constants throughout this proof). Inside the ball $B(0, r_m)$, by comparison principle,

$$\forall x \in \bar{B}(0,r_m) \quad W_{m,\mathbb{R}^2}(x) \le -\frac{1}{2\pi}\log\left(\frac{|x|}{r_m}\right) + \frac{C}{m^2} \tag{4.1.13}$$

since the right-hand side is the solution to the problem

$$\begin{cases} -\Delta u = \delta_0 & \text{in } B(0,r_m) \\ u = \frac{C}{m^2} & \text{in } \partial B(0,r_m) \end{cases}.$$

Applying (4.1.11) we can bound

$$\sum_{k\in\mathbb{Z}^2_0} W_{m,\mathbb{R}^2}(|x+k|) \le C \sum_{k\in\mathbb{Z}^2_0} e^{-m|k|} \le \frac{C}{m^2},$$

so going back to (4.1.12), we control separately the summand $k = 0$ with (4.1.13) and the others as above, to get

$$0 < \sum_{k\in\mathbb{Z}^2} W_{m,\mathbb{R}^2}(|x+k|) =\le -\frac{1}{2\pi}\log\left(\frac{d(x,0)}{r_m}\right)\chi_{B(0,r_m)}(x) + \frac{C}{m^2} \tag{4.1.14}$$

(compare with the expansion (1.2.1)). Change of variables and (4.1.9) show that also

$$0 < \int_{\mathbb{R}^2} W_{m,\mathbb{R}^2}(|x|)dx \le \frac{C}{m^2}. \tag{4.1.15}$$

We now apply Hölder's inequality to obtain the thesis in the regime $|\beta/\gamma| < 8\pi$. Keeping in mind that W_m is translation invariant,

$$\int_{\mathbb{T}^{2N}} e^{-\beta H_{W_m}} dx_1 \cdots dx_n = \int_{\mathbb{T}^{2N}} \prod_{i=1}^{N} \prod_{j\neq i, j=1}^{N} \exp\left(-\frac{\beta\sigma_i\sigma_j}{2\gamma N} W_m(x_i,x_j)\right) dx_1 \cdots dx_n$$
$$\leq \prod_{i=1}^{N} \left(\prod_{j\neq i, j=1}^{N} \int_{\mathbb{T}^2} \exp\left(-\frac{\beta\sigma_i\sigma_j}{2\gamma} W_m(x_j,0)\right) dx_j \right)^{1/N},$$

so we can restrict ourselves to the case of two particles. Since we are already neglecting possible cancellations due to signs (and allowing for negative inverse temperatures β), they are irrelevant: let us say they are opposite to fix ideas. Applying the above pointwise estimates then leads to

$$\int_{\mathbb{T}^2} \exp\left(\frac{\beta}{2\gamma} W_m(x)\right) dx \leq \left(1 + \int_{d(x,0)\leq r_m} \left(\frac{d(x,0)}{r_m}\right)^{\frac{\beta}{4\pi\gamma}} dx\right) e^{C/m^2}$$
$$\leq \left(1 + C r_m^2\right) e^{C/m^2} = 1 + O\left(\frac{(\log m)^2}{m^2}\right) \tag{4.1.16}$$

as soon as $\frac{\beta}{\gamma} < 8\pi$ for integrability, from which the thesis follows.

To cover all positive temperatures $\beta/\gamma \geq 0$, we resort instead to the technique employed in Proposition 4.1.1. Assume first that positive and negative vortices are in equal number, and relabel them by minimal distance dipoles as in Proposition 4.1.1 (see (4.1.2), whose notation we employ in the following). Then we can group the summands of the Hamiltonian function as

$$H_{W_m} = \frac{1}{\gamma N} \sum_{i<j} (W_m(y_i - y_j) - W_m(z_i, y_j))$$
$$+ \frac{1}{\gamma N} \sum_{i<j} (W_m(z_i - z_j) - W_m(y_i, z_j)) - \frac{1}{2\gamma N} \sum_i W_m(y_i, z_i). \tag{4.1.17}$$

The first and second term in the formula above are similar, so we only look at the first one. There are two possible cases to consider. For $i < j$,

- if $d(z_i, y_j) > \frac{r_m}{2}$, by (4.1.14) and (4.1.15) it holds
$$W(z_i, y_j) - W_m(y_i, y_j) \leq -\frac{1}{2\pi} \log\left(\frac{d(z_i,y_j)}{r_m}\right) + \frac{C}{m^2} \lesssim \frac{1}{m^2};$$
- if $d(z_i, y_j) \leq \frac{r_m}{2}$, then it must be $d(y_i, z_i) \leq \frac{r_m}{2}$, and thus
$$d(y_i, y_j) \leq d(y_i, z_i) + d(z_i, y_j) \leq 2d(z_i, y_j) \leq r_m,$$
so that we can bound, again by (4.1.14) and (4.1.15),

$$W(z_i, y_j) - W_m(y_i, y_j) \leq -\frac{1}{2\pi} \log\left(\frac{d(z_i,y_j)}{r_m}\right) + \frac{1}{2\pi} \log\left(\frac{d(y_i,y_j)}{r_m}\right) + \frac{C}{m^2}$$
$$\leq \frac{1}{2\pi} \log\left(\frac{d(y_i,y_j)}{d(z_i,y_j)}\right) + \frac{C}{m^2} \leq C.$$

We conclude that, in either case,

$$W(z_i, y_j) - W_m(y_i, y_j) \leq C\left(\chi_{d(y_i,z_i)\leq r_m/2} + \frac{1}{m^2}\right)$$

Applying these estimates to the first and second sums in (4.1.17), we can control the Gibbsian exponential density by

$$e^{-\beta H_{W_m}} \leq \prod_{i=1}^{N} e^{\frac{\beta}{2\gamma N} W_m(y_i,z_i)} e^{C_{\beta,\gamma}\left(\chi_{d(y_i,z_i)\leq r_m/2}+\frac{1}{m^2}\right)},$$

so that, integrating over all variables,

$$\int_{\mathbb{T}^{2N}} e^{-\beta H_{W_m}} dx^N \leq \left(\int_{\mathbb{T}^4} e^{\frac{\beta}{2\gamma N} W_m(y,z)} e^{C\left(\chi_{d(y,z)\leq r_m/2}+\frac{1}{m^2}\right)} dydz\right)^N.$$

We are now able to control the two exponentials separately by Cauchy-Schwarz inequality and (4.1.14), (4.1.15). If $0<\delta<\frac{4\pi\gamma}{\beta}$, the same computation of (4.1.16) leads to

$$\int_{\mathbb{T}^4} e^{\frac{\beta}{\gamma N} W_m(y,z)}\, dy\, dz \leq \left(\int_{\mathbb{T}^4} e^{\frac{\delta\beta}{\gamma} W_m(y,z)}\, dy\, dz\right)^{\frac{1}{\delta N}} \leq \left(1+Cr_m^2\right)^{\frac{1}{\delta N}} e^{\frac{C}{Nm^2}},$$

while the second factor to control is

$$\int_{\mathbb{T}^{2N}} e^{C\left(\chi_{d(y,z)\leq r_m/2}+\frac{1}{m^2}\right)} dydz \leq (1+Cr_m^2) e^{\frac{C}{m^2}}.$$

The thesis now follows collecting all estimates. The case in which there are more positive than negative vortices, or vice-versa, is readily settled as follows. Let P_N and Q_N be the numbers of positive and negative vortices, say $Q_N < P_N$. Then (4.1.17) becomes

$$\begin{aligned} H_{W_m} &= \frac{1}{\gamma N}\sum_{i=1}^{Q_N}\sum_{j=i+1}^{P_N}\left(W_m(y_i-y_j)-W_m(z_i,y_j)\right) \\ &+ \frac{1}{\gamma N}\sum_{i=1}^{Q_N}\sum_{j=i+1}^{Q_N}\left(W_m(z_i-z_j)-W_m(y_i,z_j)\right) - \frac{1}{2\gamma N}\sum_{i=1}^{Q_N} W_m(y_i,z_i) \\ &+ \frac{1}{\gamma N}\sum_{i=Q_N+1}^{P_N}\sum_{j=i+1}^{P_N} W_m(y_i-y_j). \end{aligned} \tag{4.1.18}$$

Since it is always $W_m \gtrsim -\frac{1}{m^2}$, the new term appearing in (4.1.18) –the fourth one in the right-hand side– contributes at most with a factor $\exp\left(C_{\beta,\gamma}N/m^2\right)$ to the exponential integral, so the proof carries on as before. □

Corollary 4.1.6. *If $N\geq 1$, $\beta/\gamma \geq 0$, $Z_{\beta,\gamma,N}$ is uniformly bounded in N by a constant depending only on β,γ.*

Proof. Let $a>0$, $m(N)=N^a$ and $p\geq 1$ an integer, then by Proposition 4.1.4 and Proposition 4.1.5 we have

$$\begin{aligned} Z_{\beta,\gamma,N} &= \int_{\mathbb{T}^{2N}} e^{-\beta H_{V_m}} e^{-\beta H_{W_m}} dx^N \\ &\leq \left(\int_{\mathbb{T}^{2N}} e^{-2\beta H_{V_m}} dx^N\right)^{1/2} \left(\int_{\mathbb{T}^{2N}} e^{-2\beta H_{W_m}} dx^N\right)^{1/2} \\ &\leq C_{\beta,\gamma,p}\left(1+N^{\frac{a\beta}{4\pi\gamma}-\frac{p}{2}} a^{2p}\right)\left(1+C_{\beta,\gamma}\frac{a^2}{N^{2a}}\right)^N. \end{aligned}$$

The partition function $Z_{\beta,\gamma,N}$ is then uniformly bounded in N as soon as

$$\frac{\beta}{4\pi\gamma} a < \frac{p}{2}, \qquad 1-2a<0.$$

Since, for any given $\beta/\gamma > 0$, we can choose $p \geq 1$ in Proposition 4.1.4 large enough for the interval $\frac{1}{2} < a < \frac{2\pi\gamma}{\beta} p$ not to be empty, the thesis follows. □

REMARK 4.1.7. The separation of long-range relevant interaction and singular short range ones in $G = V_m + W_m$ may in fact be obtained in a variety of ways: a notable mention is the decomposition $G = V_\varepsilon + W_\varepsilon$, with

$$V_\varepsilon = e^{-\varepsilon\Delta} * G = \int_\varepsilon^\infty e^{-t\Delta} dt, \quad W_\varepsilon = G - V_\varepsilon = \int_0^\varepsilon e^{-t\Delta} dt$$

(in fact, V_ε is the smoothed potential considered in [**17**]). The singular part W_ε admits the representation (4.1.12), with Bessel's function K_0 replaced by the exponential integral function E_1. The latter behaves very similarly to K_0: it diverges logarithmically in the origin and decays exponentially for large arguments. Indeed, this decomposition is completely equivalent to the one we chose for our purposes.

REMARK 4.1.8. Bounds on partition functions of point vortices -or the closely related 2-dimensional Coulomb gas ensembles- are a central part in many works on the topic. We refer for instance to the ones obtained in [**63, 102, 26**]. However, the uniform bound we obtain with our particular scaling of intensities does not seem to be obtainable from their estimates.

4.1.4. Proof of Central Limit Theorem. We are now able to conclude the proof of Theorem 1.5.2. The first step is the case $\beta = 0$, which in fact does not rely on the above arguments, and is essentially due to [**71**].

PROOF OF THEOREM 1.5.2, $\beta = 0$. The statement on partition functions is trivial in this case. Convergence in law of $\omega^N \sim \mu_\gamma^N$ to $\omega \sim \mu_\gamma$ on $\dot H^s(\mathbb{T}^2)$, any $s < -1$, is ensured by a straightforward application of the Central Limit Theorem for sums of independent variables on Hilbert spaces. As for the convergence of the Hamiltonian: let G_n converge to G in $L^2(\mathbb{T}^{2\times 2})$, with G_n vanishing on the diagonal, and split

$$\begin{aligned}
\int G(x,y) :d\omega^N(x) d\omega^N(y): &- \int G(x,y) :d\omega(x) d\omega(y): \\
&= \int G(x,y) :d\omega^N(x) d\omega^N(y): - \int G_n(x,y) d\omega^N(x) d\omega^N(y) \\
&\quad + \int G_n(x,y) d\omega^N(x) d\omega^N(y) - \int G_n(x,y) d\omega(x) d\omega(y) \\
&\quad + \int G_n(x,y) d\omega(x) d\omega(y) - \int G(x,y) :d\omega(x) d\omega(y): .
\end{aligned}$$

The $L^2(\Omega, \mathbb{P})$-norms of the differences on the right-hand side vanish in the limit. Indeed, thanks to Lemma 1.5.1, the first one is controlled uniformly in N by

$$\mathbb{E}\left[\left|\int G(x,y) :d\omega^N(x) d\omega^N(y): - \int G_n(x,y) d\omega^N(x) d\omega^N(y)\right|^2\right] \lesssim_\gamma \|G - G_n\|^2_{L^2(\mathbb{T}^{2\times 2})},$$

and the very same estimate holds for the third summand by Gaussian Itō isometry, *cf.* (1.5.2). The second moment of the middle term vanishes as $N \to \infty$ since we have already proved that ω^N converges in law on $H^s(\mathbb{T}^2)$ for $s < -1$, so that $\omega^N \otimes \omega^N$ converges in law on $H^{2s}(\mathbb{T}^{2\times 2})$ (uniform integrability descends again by the above estimate and Itō isometry). □

PROOF OF THEOREM 1.5.2, $\beta > 0$. Consider variables $\omega_\gamma^N \sim \mu_\gamma^N$ converging to $\omega_\gamma \sim \mu_\gamma$ as above. We have just seen that if $\beta = 0$ the Hamiltonian $H(\omega_\gamma^N)$ converges to $:E:(\omega_\gamma)$ in $L^2(\Omega, \mathbb{P})$. Since $x \mapsto e^{-\beta x}$ is a continuous function on $\mathbb{R}$, this implies that $e^{-\beta H(\omega_\gamma^N)}$ converges in probability to $e^{-\beta :E:(\omega_\gamma)}$ for all $\beta \in \mathbb{R}$. If

$e^{-\beta H(\omega_\gamma^N)}$ is uniformly integrable in N, then its expected value $Z_{\beta,\gamma,N}$ converges to $Z_{\beta,\gamma} = \mathbb{E}\left[e^{-\beta :E:(\omega_\gamma)}\right]$. By Corollary 4.1.6,

$$\mathbb{E}\left[\left(e^{-\beta H(\omega_\gamma^N)}\right)^p\right] = Z_{p\beta,\gamma,N}$$

is uniformly bounded in N for all $p\beta/\gamma \geq 0$. As a consequence, $e^{-\beta H(\omega_\gamma^N)}$ is uniformly integrable if $\beta/\gamma \geq 0$, thus proving point (1).

Since $(e^{-\beta H(\omega_\gamma^N)}, \omega_\gamma^N)$ converges in law to $(e^{-\beta :E:(\omega_\gamma)}, \omega_\gamma)$ on the Polish space $\mathbb{R} \times \dot{H}^s(\mathbb{T}^2)$, any $s < -1$, we deduce the convergence on $\dot{H}^s(\mathbb{T}^2)$ of the probability distributions

$$d\mu_{\beta,\gamma}^N(\omega) = e^{-\beta H(\omega)} d\mu_\gamma^N(\omega) \to e^{-\beta :E:(\omega)} d\mu_\gamma(\omega) = d\mu_{\beta,\gamma}(\omega)$$

for all $\beta \geq 0$. We are only left to prove convergence of the Hamiltonian $H(\omega_{\beta,\gamma}^N)$ for $\omega_{\beta,\gamma}^N \sim \mu_{\beta,\gamma}^N$. Since its Laplace transform is given by

$$\mathbb{E}\left[e^{\alpha H(\omega_{\beta,\gamma}^N)}\right] = \int e^{\alpha H(\omega)} \frac{e^{-\beta H(\omega)}}{Z_{\beta,\gamma,N}} d\mu_\gamma^N = \frac{Z_{\beta-\alpha,\gamma,N}}{Z_{\beta,\gamma,N}},$$

convergence of partition functions and Lemma 2.4.1 show that

$$\mathbb{E}\left[e^{\alpha H(\omega_{\beta,\gamma}^N)}\right] \xrightarrow{N\to\infty} \mathbb{E}_{\mu_{\beta,\gamma}}\left[e^{\alpha :E:(\omega)}\right]$$

with $\omega_{\beta,\gamma} \sim \mu_{\beta,\gamma}$, for any α in a neighbourhood of 0 (β/γ as above), and we can conclude by Lévy continuity theorem (see [**106**, Theorem 4.3]). $\square$

4.2. The Case of the 2-dimensional Sphere

Consider the 2-dimensional sphere $\mathcal{S}^2 = \left\{x \in \mathbb{R}^3 : |x| = 1\right\}$ as an embedded surface in $\mathbb{R}^3$, its tangent spaces as subsets of $\mathbb{R}^3$ and gradients of scalar functions as vectors of $\mathbb{R}^3$. On $\mathcal{S}^2$ we consider the uniform measure $d\sigma$ such that $\int_{\mathcal{S}^2} d\sigma = 1$. The expressions $x \cdot y, x \times y$ respectively denote in this section the scalar and vector products in $\mathbb{R}^3$.

Euler equations on $\mathcal{S}^2$ are given by, for $x \in \mathcal{S}^2$,

$$\begin{cases} \partial_t \omega(x,t) = x \cdot (\nabla\psi(x,t) \times \nabla\omega(x,t)), \\ -\Delta\psi(x,t) = \omega(x,t). \end{cases}$$

Here Δ denotes the Laplace-Beltrami operator, and we have to supplement the Poisson equation for the *stream function* ψ with the zero average condition (just as we did on $\mathbb{T}^2$). The Green function of $-\Delta$,

$$-\Delta G(x,y) = \delta_y(x) - 1$$

has the simple form

$$G(x,y) = -\frac{1}{2\pi} \log|x-y| + c,$$

with $|\cdot|$ the Euclidean distance of $\mathbb{R}^3$ between $x, y \in \mathcal{S}^2$ and c a constant. Just like in the case of flat geometries, smooth solutions preserve energy and enstrophy (1.4.1). The definition of point vortices dynamics is also completely analogous to the case on $\mathbb{T}^2$: the vorticity distribution $\omega = \sum_1^N \xi_i \delta_{x_i}$ evolves according to the Hamiltonian dynamics (*Helmholtz law*)

$$\dot{x}_i = \frac{1}{2\pi} \sum_{i<j}^N \xi_j \frac{x_j \times x_i}{|x_i - x_j|^2},$$

with Hamiltonian function corresponding to the (renormalised) energy of the configuration,

$$H(x_1, \dots x_N) = \sum_{i<j}^{N} \xi_i \xi_j G(x_i, x_j).$$

We refer to [**147**] for a more complete discussion of this setting.

The similarity with the periodic case is such that almost the whole Section 4.1 applies to $\mathcal{S}^2$: the very same statement of Theorem 1.5.2 holds on $\mathcal{S}^2$, with all the involved objects defined as in that case. The proof proceeds analogously, splitting $G = V_m + W_m$ as in (4.1.3). The content of subsections 4.1.2 and 4.1.4 only needs the replacement of Fourier basis e_k (which we used in Gaussian computations) with spherical harmonics. In fact, the only argument in the proof of Theorem 1.5.2 which needs to be adapted to the case on $\mathcal{S}^2$ is the control on Yukawa partition function of subsection 4.1.3. A careful analysis of the proof of Proposition 4.1.5 reveals that it is sufficient to prove the following bound on $W_m = (m^2 - \Delta)^{-1}$.

REMARK 4.2.1. The distance between $x, y \in \mathcal{S}^2$ on the surface is given by the angle $\theta \in [0, \pi]$ formed by the vectors $x, y \in \mathbb{R}^3$; therefore, by rotation invariance, $G(x, y) = G(\theta)$ and $W_m(x, y) = W_m(\theta)$.

PROPOSITION 4.2.2. *Let $r_m = c\frac{\log m}{m}$ with $c \geq 0$ large enough. It holds, as $m \to \infty$, uniformly in $\theta \in [0, \pi]$,*

$$W_m(\theta) = \left(-\frac{1}{2\pi} \log \frac{\theta}{r_m} + O(1) \right) \chi_{\theta \leq r_m} + O(m^{-2}).$$

On $\mathbb{T}^2$, we relied on an explicit representation of W_m. Here, we seize the opportunity to present a more robust argument, based on the well-known representation

$$W_m(x, y) = \int_0^\infty e^{-m^2 t} p(t, x, y) dt. \tag{4.2.1}$$

in terms of the heat kernel $p(t, x, y)$. Indeed, the following arguments work more generally on compact Riemannian surfaces without boundary. We nevertheless prefer to keep using the terminology of $\mathcal{S}^2$, for the sake of simplicity. We will make use of the following properties of the heat kernel $p(t, x, y) = p(t, \theta)$, for which we refer to [**139, 142**].

LEMMA 4.2.3. *It holds, for any $\theta \in [0, 2\pi]$,*

$$p(t, \theta) \leq C, \qquad t \geq 1, \tag{4.2.2}$$

$$p(t, \theta) \leq \frac{C}{t\sqrt{\pi - \theta + t}} e^{-\frac{\theta^2}{4t}}, \qquad t \leq 1, \tag{4.2.3}$$

with $C > 0$ independent from t. Moreover, for small t, uniformly on θ on compact sets of $[0, \pi)$,

$$p(t, \theta) = q_t(\theta) H(\theta) + O(1), \quad q_t(\theta) = \frac{1}{4\pi t} e^{-\frac{\theta^2}{4t}}, \quad H(\theta) = \frac{\theta}{\sin \theta}. \tag{4.2.4}$$

PROOF OF PROPOSITION 4.2.2. It is not difficult to see, using the estimates (4.2.2) and (4.2.3), that

$$\int_{r_m^2}^\infty e^{-m^2 t} p(t, \theta) dt + \chi_{\{\theta \geq r_m\}} \int_0^{r_m^2} e^{-m^2 t} p(t, \theta) dt = O(m^{-2}),$$

so we focus on the main term, $\chi_{\theta \leq r_m} \int_0^{r_m^2} e^{-m^2 t} p(t, \theta) dt$. Thanks to (4.2.4), we have

$$\int_0^{r_m^2} e^{-m^2 t} p(t, \theta) dt = H(\theta) \int_0^{r_m^2} e^{-m^2 t} q_t(\theta) dt + O(1).$$

Integrating by parts, straightforward computations show that

$$\int_0^{r_m^2} e^{-m^2 t} q_t(\theta) dt = \frac{1}{4\pi} \int_0^1 \exp\left(-c^2 \log^2 m - \frac{\theta^2}{r_m^2}\right) \frac{ds}{s} = -\frac{1}{2\pi} \log \frac{\theta}{r_m} + O(1),$$

and since $H(0) = 1$ and H is differentiable in 0, the thesis follows. □

4.3. The Case of a Bounded Domain

In this Section, $D \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, $G(x, y)$ is the Green function of $-\Delta$ on D with Dirichlet boundary conditions. The naught subscript refers to boundary conditions: $H_0^\alpha(D)$, $\alpha > 0$, are the (fractional) $L^2(D)$-based Sobolev spaces defined as the closure of compactly supported functions $C_c^\infty(D)$ with respect to the norm

$$\|u\|_{H_0^\alpha(D)} = \left\|(1 - \Delta)^{\alpha/2} u\right\|_{L^2(D)},$$

whereas $H^{-\alpha}(D) = H_0^\alpha(D)'$. We recall the representation for the Green function G given in (3.2.2), and the estimate (3.2.3) deriving from it.

4.3.1. Gibbs Ensembles and Gaussian Measures. The motion of a system of N vortices with intensities $\xi_1, \dots, \xi_N \in \mathbb{R}$ and positions $x_1, \dots, x_N \in D$ is governed by the Hamiltonian function

$$H(x_1, \dots, x_n) = \sum_{i<j}^N \xi_i \xi_j G(x_i, x_j) + \frac{1}{2} \sum_{i=1}^N \xi_i^2 g(x_i, x_i).$$

The additional (with respect to the cases with no boundary) self-interaction terms involving g are due to the presence of an impermeable boundary: it is thanks to these terms that the system satisfies (in weak sense) Euler's equations. We refer again to [**135**, Section 4.1] for further details. We will consider intensities $\xi_i = \frac{\sigma_i}{\sqrt{\gamma N}}$ with signs $\sigma_i = \pm 1$ as in the previous section. We denote by dx the normalized Lebesgue measure on D, and for $\gamma > 0$, $\beta \geq 0$ we define

$$\nu_{\beta,\gamma,N}(dx_1, \dots, dx_n) = \frac{1}{Z_{\beta,\gamma,N}} \exp\left(-\beta H(x_1, \dots, x_n)\right) dx_1, \dots, dx_n. \tag{4.3.1}$$

PROPOSITION 4.3.1. *For any choice of $\gamma > 0$, $\beta \in \mathbb{R}$, and signs $\sigma_i = \pm 1$, if*

$$-8\pi \frac{N}{\max(n_+, n_-)} < \frac{\beta}{\gamma} < 4\pi \frac{N}{1 + \min(n_+, n_-)},$$

then $Z_{\beta,\gamma,N} < \infty$, and the measure $\nu_{\beta,\gamma,N}$ is thus well-defined, where n_+, n_- are, respectively, the number of vortices with positive and negative intensity.

PROOF. Let us denote by H_i the interaction part and by H_s the self-interaction part of the Hamiltonian H,

$$H_i = \sum_{i<j}^N \xi_i \xi_j G(x_i, x_j), \quad H_s = \frac{1}{2} \sum_{i=1}^N \xi_i^2 g(x_i, x_i).$$

If $\beta < 0$, $-\beta H_s$ is bounded from above by (3.2.3). Since $G \geq 0$, we can neglect in H_i the contribution of vortices with different sign and

$$\beta H_i \leq -\frac{\beta}{2\gamma N} \sum_{\sigma_i, \sigma_j > 0} G(x_i, x_j) - \frac{\beta}{2\gamma N} \sum_{\sigma_i, \sigma_j < 0} G(x_i, x_j) := -\beta H_i^+ - \beta H_i^-$$

The terms H_i^+, H_i^- are functions on disjoint sets of variables, so the integral of their exponential factorizes in the product of two integrals. We analyse the first integral, the estimate of the second will follow likewise. Let $I_+ = \{i : \sigma_i > 0\}$. Again by (3.2.3), the self-interaction terms is bounded, therefore

$$\int D^{i_+} e^{-\beta H_i^+} \lesssim \int_{D^{i_+}} \prod_{i\in I_+} \prod_{j\in I_+, j\neq i} |x_i - x_j|^{\frac{\beta}{4\pi\gamma N}} \leq \prod_{i\in I_+} \Big(\int_D dx_i \prod_{j\in I_+, j\neq i} \int_D |x_i - x_j|^{\frac{\beta}{4\pi\gamma N} n_+} dx_j \Big)^{\frac{1}{n_+}}$$

The integrals above are finite if $\frac{\beta}{4\pi\gamma N} n_+ > -2$. Likewise, for H_i^- we obtain $\frac{\beta}{4\pi\gamma N} n_- > -2$.

We turn to the case $\beta > 0$. By the Hölder inequality with conjugate exponents p and q, we can bound separately the contributions of H_i and H_s

Thanks to (3.2.3), it holds

$$\int_{D^N} e^{-\beta q H_s(x_1,\dots,x_N)} dx_1 \dots dx_N \leq \Big(\int_D d(x)^{-\frac{\beta q}{4\pi\gamma N}} dx \Big)^N < \infty$$

as soon as $\frac{\beta}{4\pi\gamma} < \frac{N}{q}$. As for the interaction term, since G is positive and g is uniformly bounded from above,

$$-p\beta H_i \leq -\frac{\beta p}{2\pi\gamma N} \sum_{\sigma_i\cdot\sigma_j<0}^N \log|x_i - x_j| + CN.$$

Assume without loss of generality that $n_- \leq n_+$, then by the Hölder inequality,

$$\begin{aligned}
\int_{D^N} e^{-\beta H_i} &\lesssim \int_{D^{n_-}} \prod_{i\in I_+} \Big(\int_D \prod_{j\in I_-} |x_i - x_j|^{-\frac{p\beta}{2\pi\gamma N}} dx_i \Big) \\
&= \int_{D^{n_-}} \Big(\int_D \prod_{j\in I_-} |y - x_j|^{-\frac{p\beta}{2\pi\gamma N}} dy \Big)^{n_+} \\
&\leq \int_{D^{n_-}} \prod_{j\in I_-} \Big(\int_D |y - x_j|^{-\frac{p\beta}{2\pi\gamma N} n_-} dy \Big)^{\frac{n_+}{n_-}}.
\end{aligned}$$

The right-hand side is finite if $\frac{p\beta}{2\pi\gamma} n_- < 2$. Combining the two conditions on p, q we get the announced restriction on β/γ. □

The reader will notice that, unlike in Proposition 4.1.1, when $N \to \infty$ we still have a restriction on the values of β/γ. See Remark 4.3.7 for more details.

We define the probability $\mu_{\beta,\gamma}^N$ on finite signed measures $\mathcal{M}(D)$ as the law of

$$\omega_{\beta,\gamma}^N = \sum_{i=1}^N \xi_i \delta_{x_i},$$

with $x_1, \dots x_n$ sampled under $\nu_{\beta,\gamma,N}$. In the case of a bounded domain we will assume the *neutrality condition*

$$\sum_{i=1}^N \sigma_i = 0, \tag{4.3.2}$$

so that $\omega_{\beta,\gamma}^N$ has zero average.

The limiting Gaussian random field should also have zero space average. Since the constant function 1 does not belong to the spaces in which we set the problem (it does not satisfy the Dirichlet b.c.), the definition is somewhat more involved than it was on $\mathbb{T}^2$. Define the bounded linear operator

$$M : L^2(D) \to L^2(D), \quad Mf(x) = f(x) - \int_D f(y)dy.$$

For $\gamma > 0$ and $\beta \geq 0$, let $\omega_{\beta,\gamma}$ be the centred Gaussian random field on D with covariance

$$\forall f, g \in L^2(D), \quad \mathbb{E}\left[\langle \omega_{\beta,\gamma}, f\rangle \langle \omega_{\beta,\gamma}, g\rangle\right] = \langle f, Q_{\beta,\gamma} g\rangle, \quad Q_{\beta,\gamma} = M^*(\gamma - \beta\Delta)^{-1} M.$$

Equivalently, $\omega_{\beta,\gamma}$ is a centred Gaussian stochastic process indexed by $L^2(D)$ with the specified covariance. Analogously to the torus case, $\omega_{\beta,\gamma}$ can be identified with a random distribution taking values in $H^s(D)$ for all $s < -1$.

Renormalised energy of the vorticity distribution $\mu_{\beta,\gamma}$ is defined just as in (2.4.2), and the equivalent definition of $\mu_{\beta,\gamma}$ provided by Lemma 2.4.1 still applies in this context. In fact, all Gaussian computations in Fourier series of the last Section still work on domains $D \subset \mathbb{R}^2$ if one considers an orthonormal basis of $L^2(D)$ diagonalising the Laplace operator: for $n \in \mathbb{N}$,

$$-\Delta e_n = \lambda_n e_n, \quad \lambda_n \sim n,$$

the latter being the well known Weyl's law. The main difference is that explicit expression in Fourier series on D are complicated by the presence of the zero-averaging operator M in the covariance. We are now able to state the main result of the Section, a perfect analogue of the Central Limit Theorem we proved above on $\mathbb{T}^2$.

THEOREM 4.3.2. *Let $\beta/\gamma \in [0, 8\pi)$, assume the neutrality condition (4.3.2), and set $\bar g = \int_D g(y,y)dy$. It holds:*

(1) $\lim_{N\to\infty} Z_{\beta,\gamma,N} = e^{\beta \bar g} Z_{\beta,\gamma}$;
(2) the sequence of $\mathcal{M}$-valued random variables $\omega^N \sim \mu^N_{\beta,\gamma}$ converges in law on $H^s(D)$, any $s < -1$, to a random distribution $\omega \sim \mu_{\beta,\gamma}$, as $N \to \infty$;
(3) the sequence of real random variables $H(\omega^N) - \bar g$ converges in law to $:E:(\omega)$ as $N \to \infty$, with ω^N, ω as in point (2).

REMARK 4.3.3. Minor modifications of our arguments allow to replace the neutrality condition on intensities with the hypothesis $\sum_{i=1}^N \xi_i = o((\log N)^{-1/2})$. Moreover, it is possible to consider random signs σ_i taking values ± 1 with probability $1/2$, or more generally i.i.d. bounded signs with zero expected value. Such generalisations are in fact inessential from the physical point of view, namely we are still dealing with fluctuations around a null profile (see Section 4.4): we omit details.

We conclude this paragraph proving the case $\beta = 0$ (and $\gamma = 1$, for notational simplicity): if we can then provide a uniform bound for partition functions $Z_{\beta,\gamma,N}$, the content of subsection 4.1.4 completely carries on to the domain case. In the remainder of this Section we show out how to adapt the strategy we used in the torus case to control partition functions.

The expression (1.5.1) of double stochastic integrals with respect to white noise still holds, and so does Lemma 1.5.1 in the following form:

LEMMA 4.3.4. *Let $\omega^N \sim \mu^N_{0,\gamma}$. On continuous functions $h \in C(D^2)$ vanishing on the diagonal,* i.e. *$h(x,x) = 0$ for all x, define the map*

$$h \mapsto \int_{D^2} h(x,y) d\omega^N(x) d\omega^N(y) = \sum_{i \neq j} \xi_i \xi_j h(x_i, x_j).$$

Since it holds

$$\mathbb{E}\left[\left(\sum_{i\neq j} \xi_i \xi_j h(x_i, x_j)\right)^2\right] \leq C_\gamma \left\|h\right\|^2_{L^2(D^2)}$$

with C_γ a constant independent of N, the map takes values in $L^2(\mu^N_{0,1})$, and it extends by density to a bounded linear map from $\dot{L}^2(D^2)$ to $L^2(\mu^N_{0,1})$ which we will denote by

$$f \mapsto \int_{D^2} f(x,y) :d\omega^N(x) d\omega^N(y): .$$

The proof only differs from the one on $\mathbb{T}^2$ in that is uses neutrality of total intensity in place of the zero average condition. In considering the relation between the Hamiltonian and renormalised energy, another relevant difference with respect to the torus case appears: defining the renormalised energy of point vortices as in Section 4.1,

$$\begin{aligned} 2:E: &= \int_{D^2} G(x,y) :d\omega^N \otimes d\omega^N := \sum_{i\neq j} \xi_i \xi_j G(x_i,x_j) \\ &= 2H - \sum_{i=1}^N \xi_i^2 g(x_i,x_i). \end{aligned}$$

This is why we need corrections depending on $\bar{g} = \int_D g(y,y)dy$ in points (1) and (3) of Theorem 4.3.2: the Hamiltonian H alone is not a centred variable, and its mean value is

$$\sum_{i=1}^N \xi_i^2 g(x_i,x_i) = \frac{1}{N}\sum_{i=1}^N g(x_i,x_i),$$

which converges by the law of large numbers to $\bar{g}$. That being said, proceeding as in subsection 4.1.4 straightforwardly concludes the proof of the case $\beta = 0$.

4.3.2. Potential Splitting on Bounded Domains. We want to decompose $G = V_m + W_m$ as in Section 4.1, with V_m a regular (*long range*) potential converging to G as $m \to \infty$, and W_m a singular but vanishing remainder. In order for our strategy to work we need to rewrite the part of H corresponding to V_m as sum of covariances (in particular, positive terms) of a regular Gaussian field with zero space average. At the same time, we will need a quite precise description of W_m. We thus choose W_m as the Green function of $m^2 - \Delta$ on D with Dirichlet boundary conditions, that is

(4.3.3)
$$W_m(x,y) = \frac{1}{2\pi} K_0(m|x-y|) + w_m(x,y), \qquad \begin{cases} (m^2-\Delta) w_m(x,y) = 0 & x \in D \\ w_m(x,y) = -\frac{1}{2\pi} K_0(m|x-y|) & x \in \partial D \end{cases}$$

for all $y \in D$, and where we notice that $\frac{1}{2\pi}K_0(m|x-y|) = W_{m,\mathbb{R}^2}(x,y)$ is the Green function of $m^2 - \Delta$ on the whole plane. We then set

$$V_m = G - W_m, \quad v_m = g - w_m.$$

Unfortunately, V_m is not zero averaged, so we need to further define the potential

$$V_m^0(x,y) = V_m(x,y) - \int_D V_m(x,y)dy - \int_D V_m(x,y)dx + \int_{D^2} V_m(x,y)dxdy, \tag{4.3.4}$$

which we will use as covariance kernel for the Gaussian field F_m: indeed, notice that, as an integral kernel,

$$V_m^0 = M^* m^2 (-\Delta(m^2-\Delta))^{-1} M,$$

thus V_m^0 is positive definite and zero averaged.

Looking now at the corresponding decomposition of the Hamiltonian,

$$H = \sum_{i<j}^N \xi_i\xi_j W_m(x_i,x_j) + \frac{1}{2}\sum_{i=1}^N \xi_i^2 w_m(x_i,x_i) + \sum_{i<j}^N \xi_i\xi_j V_m(x_i,x_j) + \frac{1}{2}\sum_{i=1}^N \xi_i^2 v_m(x_i,x_i)$$
$$:= H_{W_m} + H_{V_m},$$

a simple computation exploiting the neutrality condition yields

$$\sum_{i,j}^N \xi_i\xi_j V_m(x_i,x_j) = \sum_{i,j}^N \xi_i\xi_j V_m^0(x_i,x_j) - \sum_{i=1}^N \xi_i^2 V_m(x_i,x_i),$$

so that, since $V_m + v_m = V_{m,\mathbb{R}^2}$ (the Green function of $-m^{-2}\Delta(m^2-\Delta)$), we can rewrite

$$H_{V_m} = \frac{1}{2}\sum_{i,j}^N \xi_i\xi_j V_m^0(x_i,x_j) - \frac{1}{2}\sum_{i=1}^N \xi_i^2 V_{m,\mathbb{R}^2}(x_i,x_i).$$

One can easily show that $V_{m,\mathbb{R}^2}$ is a regular, symmetric, translation invariant function; moreover, it has a global maximum in $V_{m,\mathbb{R}^2}(0,0) = \frac{1}{2\pi}\log m + o(\log m)$, as it is shown by taking the difference of

$$G_{\mathbb{R}^2}(x,y) = -\frac{1}{2\pi}\log|x-y|, \quad W_{m,\mathbb{R}^2}(x,y) = \frac{1}{2\pi}K_0(m|x-y|) \sim -\frac{1}{2\pi}\log(m|x-y|),$$

for close $x,y\in\mathbb{R}^2$. This, together with (4.3.4), implies that for all $x\in D$ we also have $V_m^0(x,x) = \frac{1}{2\pi}\log m + o(\log m)$.

Lemma 4.3.5. *Let F_m be the centred Gaussian field on D with covariance kernel V_m^0. There exists a version of $F_m(x)$ which is α-Hölder for all $\alpha < 1/2$, and moreover for any $\alpha>0$, $p\geq 1$ and $m\to\infty$, it holds*

$$\mathbb{E}\left[\|F_m\|_p^p\right] \simeq_p (\log m)^{p/2} \tag{4.3.5}$$

$$\mathbb{E}\left[\exp\left(-\alpha\|F_m\|_2^2\right)\right] \lesssim m^{-\frac{\alpha}{2\pi}}. \tag{4.3.6}$$

Proof. Hölder property descends from Kolmogorov continuity theorem since V_m is continuously differentiable (and so is V_m^0). The estimate of p-moments is the same as in the periodic case, so let us turn to exponential moments. Identifying kernels and their associated integral operators, it holds

$$\mathbb{E}\left[\exp\left(-\alpha\|F_m\|_2^2\right)\right] = \exp\left\{-\frac{1}{2}\operatorname{Tr}\left(\log\left(1+2\alpha V_m^0\right)\right)\right\}.$$

Hence, we only need to compute the asymptotic behaviour in m of $\operatorname{Tr} V_m^0$, since then we can apply the inequalities $x - \frac{x^2}{2} < \log(1+x) < x$ and conclude as in Lemma 4.1.2. We resort again to Fourier series: by definition of the kernel V_m^0 we have

$$\begin{aligned}
\operatorname{Tr} V_m^0 &= \sum_{n=1}^\infty \int_{D^2} V_m^0(x,y)e_n(x)e_n(y)dxdy \\
&= \operatorname{Tr} V_m - 2\sum_{n=1}^\infty \bar e_n \int_{D^2} V_m(x,y)e_n(x)dxdy + \int_{D^2} V_m(x,y)dxdy \sum_{n=1}^\infty \bar e_n^2 \\
&= \operatorname{Tr} V_m - \sum_{n=1}^\infty \frac{m^2\bar e_n^2}{\lambda_n(m^2+\lambda_n)} = \operatorname{Tr} V_m + O(1), \quad m\to\infty,
\end{aligned}$$

where we denoted $\bar{e}_n$ the space averages of $e_n(x)$ (that is, the Fourier coefficients of the constant function 1). The last passage is a consequence of

$$0 \leq \sum_{n=1}^{\infty} \frac{m^2 \bar{e}_n^2}{\lambda_n(m^2+\lambda_n)} \leq \left(\sum_{n=1}^{\infty} \frac{m^4}{\lambda_n^2 (m^2+\lambda_n)^2} \right)^{1/2} \lesssim \left(\int_1^{\infty} \frac{m^4}{x^2(m^2+x)^2} dx \right)^{1/2}$$
$$= \left(\frac{m^2+2}{m^2+1} - \frac{2\log(m^2+1)}{m^2} \right)^{1/2} = O(1), \quad m \to \infty,$$

where we used $\left(\sum_{n=1}^{\infty} \bar{e}_n^4\right)^{1/2} \leq \sum_{n=1}^{\infty} \bar{e}_n^2 = \|1\|_{L^2(D)}^2 = 1$ and Cauchy-Schwarz inequality. We conclude by noting that

$$\operatorname{Tr} V_m = \sum_{n=1}^{\infty} \frac{m^2}{\lambda_n(m^2+\lambda_n)} = V_m(0,0). \qquad \square$$

We can now apply the transformation

$$e^{-\beta H_{V_m}} = e^{\frac{\beta}{2\gamma} V_{m,\mathbb{R}^2}(0,0)} \mathbb{E}\left[e^{\mathrm{i}\sqrt{\beta} \sum_{i=1}^N \xi_i F_m(x_i)} \right]$$

and proceed as in the previous Section. The proof of Proposition 4.1.4 in the bounded domain setting is just the same, thanks to Lemma 4.3.5. We are only left to prove the analogue of Proposition 4.1.5, from which a uniform bound on partition functions is derived as in Corollary 4.1.6.

Proposition 4.3.6. *Let $N \geq 1$, $|\beta/\gamma| \leq 8\pi$ and $m > 0$. There exists a constant $C_{\beta,\gamma} > 0$ such that*

$$\int_{\mathbb{T}^{2N}} e^{-\beta H_{W_m}} dx_1 \cdots dx_n \leq \left(1 + C_{\beta,\gamma} \frac{(\log m)^2}{m^2} \right)^N.$$

Proof. As in the first part of the proof of Proposition 4.1.5, we reduce by means of Hölder inequality to bound the integral

$$I = \int_{D^2} e^{\frac{\beta}{2\gamma} W_m(x,d)} dx dy.$$

We thus proceed to bound pointwise the interaction potential $W_m(x,y) = W_{m,\mathbb{R}^2}(x,y) + w_m(x,y)$. Let us first fix x, and consider the small radius $r_m = \frac{2\log m}{m}$, as we did in Proposition 4.1.5. For m large enough, $B(x,r_m) \subseteq D$, and we have showed in Section 4.1 that for all $x,y \in \mathbb{R}^2$,

$$W_{m,\mathbb{R}^2}(x) \leq -\frac{1}{2\pi} \log\left(\frac{|x-y|}{r_m} \right) \chi_{B(x,r_m)}(y) + \frac{C}{m^2}.$$

We are thus left to bound $w_m(x,y)$: by definition (4.3.3) and the maximum principle, it holds, for all x uniformly in y,

$$w_m(x,y) \leq \frac{1}{2\pi} K_0(md(x)) \leq -\frac{1}{2\pi} \log\left(\frac{d(x)}{r_m} \right) \chi_{d(x)<r_m} + \frac{C}{m^2}.$$

Going back to I, we get

$$I \leq e^{C/m^2} \int_{B(x,r_m)} \left(1 + \left(\frac{|x-y|}{r_m} \right)^{-\frac{\beta}{4\pi\gamma}} \right) dy \cdot \int_D \left(1 + \left(\frac{d(x)}{r_m} \right)^{-\frac{\beta}{4\pi\gamma}} \right) dx$$
$$\leq e^{C/m^2} \left(1 + C r_m^2 \right)^2,$$

which concludes just as in Proposition 4.1.5. $\square$

REMARK 4.3.7. The technical reason behind the parameter restriction in Proposition 4.3.6 and Proposition 4.3.1 above could be avoided if a local decomposition of the Yukawa potential as in Proposition 4.2.2 is available for a general domain D with smooth enough boundary. Indeed, in that case, one could deduce that $Z_{\beta,\gamma,N} < \infty$, and thus that the meaure $\nu_{\beta,\gamma,N}$ is well defined for all values of $\beta > 0$, $\gamma > 0$. Likewise, Proposition 4.3.6 and in turns Proposition 4.3.1 would hold woithout restrictions.

A way to prove a local decomposition for the Yukawa potential is to use the same strategy of Section 4.2, namely the general representation (4.2.1), that holds beyond the geometry of the sphere. Through the point of view of the heat kernel, the role of the geometry of the domain and of its boundary becomes apparent in terms of the divergence in time of the heat kernel, whose behaviour depends on the number of geodetics and their intersection with the boundary. We refer to the fundamental [**139**] for further details. We notice in particular that if the intrinsic geometry of the domain is geodesically convex, that in the flat metric means that the domain is convex, the same estimates, in particular [**139**, Theorem 2.1], of the case without boundary such as the sphere or the torus, hold. This justify the following corollary, that fully generalizes the central limit theorem of [**26**] from the sphere to general convex domains.

COROLLARY 4.3.8. *Assume the neutrality condition* (4.3.2)*. If D is a convex domain, then the conclusions of Theorem 4.3.2 hold for all $\beta > 0$ and $\gamma > 0$.*

4.4. A Comparison with Mean Field Theory

Let us spend a few words about how our results compare with the mean field limit studied by [**40, 41, 110**], on which we will focus in the next Chapter. Those works cover the case of vortices with identical intensities, while [**26, 141**] consider vortices with (random) intensities of different signs. Vortices with random intensities on $\mathbb{S}^2$ have been analyzed in [**111**].

The scaling of intensities $|\xi| \sim N^{-1}$, is dictated by energy considerations, in order for the dominant (infinite) self-interaction term to vanish. It is *not* the scaling we assumed in the previous Sections, as it corresponds to the law of large number scaling. The scaling of inverse temperature $\beta \sim N$ is chosen so that the limit is non-trivial, see [**135**]. The resulting Hamiltonian on a bounded domain $D \subset \mathbb{R}^2$, with parameters of order one up to rescaling, is

$$\frac{1}{N}\sum_{i<j}\sigma_i\sigma_j G(x_i,x_j) + \frac{1}{2N}\sum_{i=1}^{N}\sigma_i^2 g(x_i,x_i),$$

with σ_i uniformly bounded. The corresponding Gibbs measure coincides with our $\nu_\beta^N = \nu_{\beta,1}^N$.

In the case of a bounded domain, for vortices with the same intensity, [**40**] proved that the single vortex distribution, that is the one dimensional marginal of ν_β^N, converges to a superposition of solutions to the Mean Field Equation,

$$\omega = \frac{e^{-\beta\psi}}{\int_D e^{-\beta\psi}dx}, \quad -\Delta\psi = \omega, \tag{4.4.1}$$

with the Poisson equation for the stream function ψ being complemented with Dirichlet boundary conditions. Solutions to (4.4.1) are particular steady solutions of the Euler equations that minimize the energy-entropy functional $\beta E + S$, defined in (1.4.1). A unique minimum exists when $\beta > 0$ (and for $\beta \leq 0$ close enough to 0), so that ν_β^N converges, in the sense of finite dimensional distributions, to an infinite product measure (*propagation of chaos*). Connections of the mean field equation with the microcanonical ensemble and equivalence with the canonical ensemble are considered in [**41**].

The case of intensities with different signs is studied in [**26**] through a large deviations approach. Under the assumption that the empirical measure of intensities converges to a probability distribution μ, the joint empirical measure of intensities and positions satisfies a large deviation principle with speed N^{-1}, and the extended energy-entropy functional as rate function:

$$H(\nu) + \frac{\beta}{2} \int_{\mathbb{R}^2 \times D^2} \sigma\sigma' G(x, x') \nu(d\sigma, dx) \nu(d\sigma', dx'), \tag{4.4.2}$$

where H is the relative entropy of ν with respect to the product of μ and the normalized Lebesgue measure on D. The mean field equation satisfied by the density (corresponding to the Euler-Lagrange equation for the minimisation problem of the rate function) is

$$\rho(\sigma, x) = \frac{1}{Z} e^{-\beta\sigma\psi},$$

with Z a normalising constant and ψ is the averaged stream function,

$$\psi(x) = \int \sigma G(x, y) \rho(\sigma, y) \mu(d\sigma) dy. \tag{4.4.3}$$

Similar statement also hold in the periodic case.

Looking back to our setting, in both the case of zero average vortices on $\mathbb{T}^2, \mathcal{S}^2$, and the one of vortices in a bounded domain D with neutral global intensity, for $\beta \geq 0$, the free energy (4.4.2) is non-negative and attains the value zero on the N-fold product uniform measure. Moreover, the stream function (4.4.3) is null. The large deviations principle of [**26**] implies a law of large numbers, while our Theorem 1.5.2 and Theorem 4.3.2 provide the convergence of fluctuations with respect to the null average. We mention again the central limit theorem derived in [**26**], which is however restricted to a disk domain and to a small class of test function.

CHAPTER 5

Decay of Correlations in the Mean Field Limit

Mean Field scaling limits of 2-dimensional Euler point vortices, or the equivalent 2-dimensional Coulomb gas, are a classical topic in Statistical Mechanics, and a well established literature is devoted to them. The contribution to such theory of [99], to which this Chapter is devoted, consists in determining the rate at which correlations of vortices, *i.e.* charges, decay in the Mean Field limit.

Once again we focus on $\mathbb{T}^2$ as space domain: other 2-dimensional compact manifolds without boundary, or bounded domains of $\mathbb{R}^2$ with smooth boundaries can be covered by minor modifications of our arguments. On $\mathbb{T}^2$, in the Mean Field scaling limit, that is in the limit $N \to \infty$, $\beta \to 0$, $N\beta = 1$, the k-particle correlation function of the Gibbsian enseble converge to 1. In other words, in such limit the positions of vortices completely decorrelates. To evaluate the rate at which this happens we will resort to the technique developed in the previous Chapter, in fact exploiting techniques dating back to classical works on statistical mechanics of the Coulomb Gas, such as the aforementioned [80, 152, 33, 34, 107, 108].

5.1. Mean Field Theory and Previous Results

Our discussion begins with a brief review of the Mean Field theory for point vortices on the torus $\mathbb{T}^2$. We consider a system of an even number N of vortices with positions

$$(x_1, \dots x_N) = \left(y_1, \dots y_{N/2}, z_1, \dots z_{N/2}\right);$$

the first $N/2$ vortices have intensity $+1$, the others -1. For brevity, we will denote $\underline{x} = (\underline{y}, \underline{z}) \in \mathbb{T}^{2\times N}$ the array of all positions. We consider the Canonical Gibbs measure at inverse temperature β associated to the Hamiltonian

$$H_N(\underline{x}) = \frac{1}{2}\sum_{i\neq j}^{N/2} \left(G(y_i, y_j) + G(z_i, z_j)\right) - \sum_{i=1}^{N/2}\sum_{j=1}^{N/2} G(y_i, z_j). \tag{5.1.1}$$

In order to avoid redundant notation, we already introduce in the definition of Gibbs' measures the Mean Field Limit scaling, $\beta \mapsto \frac{\beta}{N}$.

We have seen above that for any $0 \leq \beta < 4\pi N$,

$$Z_{\beta,N} = \int_{\mathbb{T}^{2\times N}} e^{-\frac{\beta}{N}H_N(\underline{x})} dx^N < \infty, \quad d\mu_{\beta,N}(\underline{x}) = \frac{1}{Z_{\beta,N}} e^{-\frac{\beta}{N}H_N(\underline{x})} dx^N,$$

defines a probability measure on $\mathbb{T}^{2\times N}$, symmetric in its first $N/2$ variables y_i and in the second $N/2$ variables z_i.

The central object of our discussion is the k-point correlation function, the aim being understanding its asymptotic behaviour in the limit $N \to \infty$. We fix a finite number of vortices: by symmetry, there is no loss in considering $(y_1, \dots, y_h, z_1, \dots z_\ell)$ for $N \geq h + \ell$. To ease notation, we will write

$$\underline{x} = (\hat{x}, \check{x}), \quad \hat{x} = (\hat{y}, \hat{z}) = (y_1, \dots, y_h, z_1, \dots z_\ell),$$

and analogously $\check{x}$ the array of vortices we are not fixing. We define

$$\rho^N_{h,\ell}(y_1,\dots,y_h,z_1,\dots z_\ell)=\rho^N_{h,\ell}(\hat{x})=\frac{1}{Z_{\beta,N}}\int_{\mathbb{T}^{N-h-\ell}}e^{-\frac{\beta}{N}H_N(\underline{x})}d\check{x}.$$

Here and from now on $d\check{x}$ (respectively $d\hat{x}$) indicates integration with respect to the $N-h-\ell$ 2-dimensional variables $\check{x}$ (resp. the $h+\ell$ variables $\hat{x}$).

THEOREM 5.1.1. *Let $\beta>0$; the free energy functional*

$$\mathcal{F}(\rho_+,\rho_-)=\frac{1}{\beta}\int_{\mathbb{T}^2}(\rho_+\log\rho_++\rho_-\log\rho_-)+\int_{\mathbb{T}^2}(\rho_+-\rho_-)G*(\rho_+-\rho_-),\tag{5.1.2}$$
$$\rho_+,\rho_-\ \textit{probability densities on}\ \mathbb{T}^2\ \textit{such that}\ \rho_\pm\log\rho_\pm\in L^1(\mathbb{T}^2),$$

admits the unique minimiser $\rho_+=\rho_-\equiv 1$. For any $1\leq h+\ell\leq N$ and $1\leq p<\infty$, the $(h+\ell)$-point correlation function $\rho^N_{h,\ell}$ converges to $\rho_+^{\otimes h}\otimes\rho_-^{\otimes\ell}\equiv 1$ in L^p topology,

$$\lim_{N\to\infty}\left\|\rho^N_{h,\ell}-1\right\|_{L^p(\mathbb{T}^{2\times N})}=0.\tag{5.1.3}$$

The latter is a classical result, valid for more general geometries of the space domain and for small negative temperatures regimes, although in such generality the minimiser of the functional (maximiser for $\beta<0$) might not be unique and limit points of the sequence $(\rho^N_{k,h})_{N\in\mathbb{N}}$ can thus be superpositions of minima (resp. maxima) of $\mathcal{F}$. We refer to [**40, 41**] and the monography [**127**] for a complete discussion.

Stationary points of the free energy can be characterised as solutions of the Mean Field equation for the potential $\phi=G*(\rho_+-\rho_-)$,

$$-\Delta\phi=\frac{e^{-\beta\phi}}{Z_+}-\frac{e^{\beta\phi}}{Z_-},\quad Z_\pm=\int_{\mathbb{T}^2}e^{\mp\beta\phi}dx,$$

which, up to a suitable choice of the average $\psi=\phi+c$, is equivalent to the sinh-Poisson equation,

$$\Delta\psi=\frac{1}{\alpha}\sinh(\beta\psi),\quad 4\alpha^2=\int_{\mathbb{T}^2}e^{-\beta\psi}dx\int_{\mathbb{T}^2}e^{\beta\psi}dx,\tag{5.1.4}$$

see [**135**, section 7.5]. Since on the torus there is a unique and trivial solution $\rho\equiv 1$, such equivalence is trivial in our setting: it is nonetheless a more general fact.

The main result of the present paper is the following refinement of Theorem 5.1.1, concerning the rate at which the convergence (5.1.3) takes place.

THEOREM 5.1.2. *For any $\beta>0$, $1\leq k+h\leq N$ and $1\leq p<\infty$,*

$$\|\rho^N_{h,\ell}-1\|_{L^p(\mathbb{T}^{2\times N})}\leq\frac{C_{\beta,p,h,\ell}}{\sqrt{N}}(\log N)^{\frac{3}{2}}.$$

The core idea behind our computations is the correspondence, provided by Gaussian integration, between functionals of the vortex ensemble and certain Euclidean field theoretic integrals. We are able to exploit such link, to be outlined in the forthcoming section, only for positive temperatures, $\beta>0$. This unfortunately rules out a relevant regime, $\beta<0$, in which the Mean Field equation on $\mathbb{T}^2$ admits nontrivial solutions, see [**127**].

5.2. The Coulomb Gas and Sine-Gordon Field Theory

The 2-dimensional, Coulomb gas is a classical mechanics system consisting of point charges: we will consider the case in which there are two species of charges of opposite signs, but with same intensity. For a system of N charges, say half positive and half negative, their dynamics is described by the Hamiltonian function

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j} \sigma_i\sigma_j G(x_i, x_j),$$

where G is the Green function of the Laplacian, as above, x_i are the positions and p_i the momenta of the charges, $\sigma_i = \pm 1$ the signs of the charges. In Gibbsian ensembles of the system, momenta have Maxwellian (Gaussian) independent distributions; when dealing with correlation functions or analogous functionals –which is ultimately the aim of the present work– we can always integrate them out: it is thus convenient to only consider the configurational (interaction) part of the Hamiltonian.

We consider the system of charges in a bounded domain $D \subseteq \mathbb{R}^2$, so boundary conditions have to be supplemented to define G: for the sake of this discussion there is no difference in considering free boundary conditions, Dirichlet boundary conditions (physically interpreted as considering the system in a cavity inside a conductor) or the periodic case $\mathbb{T}^2$. It is immediate to observe that the (configurational) Canonical Gibbs ensemble for the 2D Coulomb gas actually coincides with the vortices ensemble defined above, provided that the same boundary conditions are taken into account, since the configurational part of the Hamiltonian $\mathcal{H}$ is in fact the same as (5.1.1).

5.2.1. The Sine-Gordon representation. It is a classical and well-known fact that Gaussian integration provides a correspondence between two-dimensional Coulomb gas and the Sine-Gordon field theory, as described in [**152**]. This equivalence has been instrumental in the study of both systems, see for instance [**80, 33, 34**], since it allowed to employ techniques from both statistical mechanics and field theory. The remainder of this section is dedicated to review such correspondence, which we will exploit in the proof of Theorem 5.1.2. The following arguments are mostly formal and not rigorous: indeed we only aim to provide a heuristic motivation of the techniques we are going to use.

The equivalence with Sine-Gordon theory is exact only when the Coulomb gas is considered in the Grand Canonical ensemble. Let us then consider the (configurational part of the) Grand Canonical partition function,

$$\mathfrak{Z}_{z,\beta} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{D^n} \exp\left(-\beta H_n(x_1, \sigma_1, \dots x_n, \sigma_n)\right) dx^n d\nu^n, \tag{5.2.1}$$

where the *activity* $z > 0$ controls the arbitrary (Poisson distributed) number n of charges and ν is the law of a $\frac{1}{2}$-Bernoulli variable on $\{\pm 1\}$; the positions x_i and signs σ_i are thus independent variables with law, respectively, dx on $\mathbb{T}^2$ and ν. Notice that the neutrality condition has been replaced with an average neutrality, $\int \sigma d\nu(\sigma) = 0$; this is only for the sake of simplicity of exposition, different and more general choices can be made.

The corresponding (Euclidean) Sine-Gordon field theory has Lagrangian

$$\mathcal{L}(\phi) = \beta\left|\nabla\phi\right|^2 - 2z\cos\left(\beta\phi\right),$$

so that the vacuum expectation value is

$$\mathcal{V}_{z,\beta} = \int e^{-\int \mathcal{L}(\phi)dx}\mathcal{D}\phi = \int \exp\left(-\beta\int_D |\nabla\phi|^2 dx + 2z\int_D \cos(\beta\phi)dx\right)\mathcal{D}\phi.$$

The equivalence with Grand Canonical Coulomb gas is most immediately seen by observing that the partition function $\mathfrak{Z}_{z,\beta}$ actually coincides with the Sine-Gordon vacuum expectation, up to a normalising factor given by the vacuum expectation of the free field,

$$\mathfrak{Z}_{z,\beta} = \mathcal{V}_{z,\beta}/\mathcal{V}_{0,\beta}. \tag{5.2.2}$$

This can be shown with the following formal computation. If X, Y are two real standard Gaussian variables, it holds

$$e^{\frac{s^2+t^2}{2}}\mathbb{E}\left[e^{\mathrm{i}\,sX}e^{\mathrm{i}\,tY}\right] = e^{-st\mathbb{E}[XY]}.$$

By means of this Fourier transform, we can thus formally see any exponential function $e^{-G(x_i,x_j)}$ as the field theoretic correlation function of the field operators $e^{\mathrm{i}\,\chi(x_i)}, e^{\mathrm{i}\,\chi(x_j)}$ with respect to the free (Gaussian) theory with action $\int |\nabla\chi|^2 dx$ (the 2-dimensional *Gaussian free field*). More explicitly, we write

$$\frac{\int e^{\mathrm{i}\,\beta\sum_{i=1}^n \sigma_i\chi(x_i)}e^{-\beta\int_D |\nabla\phi|^2 dx}\mathcal{D}\phi}{\int e^{-\beta\int_D |\nabla\phi|^2 dx}\mathcal{D}\phi} = \exp\left(-\frac{\beta}{2}\sum_{i\neq j}^n \sigma_i\sigma_j G(x_i,x_j)\right).$$

The computation is only formal since the random field χ has singular covariance: its samples are not functions (χ can be realised as a random distribution), and thus the above complex exponentials need renormalisation to be rigorously defined. Proceeding with the formal computation (in which for a moment we omit the infinite renormalisation term $\mathcal{V}_{0,\beta} = \int e^{-\beta\int|\nabla\phi|^2}\mathcal{D}\phi$),

$$\begin{aligned}
&\sum_{n=0}^\infty \frac{z^n}{n!}\int dx_1\cdots dx_n d\nu(\sigma_1)\cdots d\nu(\sigma_n)\int e^{-\beta\int_D |\nabla\phi|^2 dx}\mathcal{D}\phi e^{\mathrm{i}\,\beta\sum_{i=1}^n\sigma_i\chi(x_i)}\\
&= \sum_{n=0}^\infty \frac{z^n}{n!}\int e^{-\beta\int_D |\nabla\phi|^2 dx}\mathcal{D}\phi\left(\int dx d\nu(\sigma)e^{\mathrm{i}\,\beta\sigma\chi(x)}\right)^n\\
&= \int e^{-\beta\int_D |\nabla\phi|^2 dx}\mathcal{D}\phi\sum_{n=0}^\infty \frac{z^n}{n!}\left(\int dx 2\cos(\beta\chi(x))\right)^n\\
&= \int e^{2z\int\cos(\beta\phi(x))dx}e^{-\beta\int_D |\nabla\phi|^2 dx}\mathcal{D}\phi = \mathcal{V}_{z,\beta},
\end{aligned}$$

from which (5.2.2).

5.2.2. Mean Field Scaling and Correlation Functions. The Mean Field scaling of Coulomb charges in the Canonical ensemble is

$$\beta \mapsto \varepsilon\beta, \quad N \mapsto \frac{N}{\varepsilon}, \quad \varepsilon \to 0,$$

and it corresponds in the Grand Canonical Ensemble to

$$\beta \mapsto \varepsilon\beta, \quad z \mapsto \frac{z}{\varepsilon}, \quad \varepsilon \to 0$$

(ε sometimes referred to as the *plasma parameter*). Applying the Mean Field scaling to the Sine-Gordon theory one recovers the Klein-Gordon field theory: looking at vacuum expectations,

$$\mathcal{V}_{z/\varepsilon,\varepsilon\beta} \xrightarrow{\varepsilon\to 0} \int \exp\left(\int_D |\nabla\phi|^2 dx + z\beta\int_D \phi^2 dx\right)\mathcal{D}\phi,$$

the right-hand side being the vacuum expectation of the theory with Lagrangian

$$\mathcal{L}(\phi) = |\nabla\phi|^2 - z\beta\phi^2,$$

This is because in such a scaling every term in the power expansion of the interaction term $\cos(\xi\sqrt{\beta}\phi)$ is negligible save for the quadratic one. A straightforward computation –using for instance Fourier series on $\mathbb{T}^2$– reveals that the

Mean Field scaling limit of $\mathfrak{Z}_{z,\beta}$ in fact coincides with the partition function of the Energy-Enstrophy invariant measure of the 2-dimensional Euler equations,

$$\mathfrak{Z}_{z/\varepsilon,\varepsilon\beta} = \mathcal{V}_{z/\varepsilon,\varepsilon\beta}/\mathcal{V}_{0,0} \xrightarrow{\varepsilon\to 0} Z_\beta = \int \exp\left(-\beta \int_D \omega\Delta^{-1}\omega dx\right) d\mu(\omega),$$
$$d\mu(\omega) = \frac{1}{Z}\int e^{-\int_D \omega^2 dx}\mathcal{D}\omega,$$

where μ –the Enstrophy measure– is actually the space white noise on $\mathbb{T}^2$. The following result of [**98**] (to which we refer for a complete discussion of the involved Gaussian measures), rigorously establishes such convergence for the Canonical ensemble of charges on the torus $\mathbb{T}^2$.

THEOREM 5.2.1. *For any $\beta \geq 0$,*

$$\lim_{N\to\infty} Z_{\beta,N} = Z_\beta.$$

Let us now fix the first k charges, with positions $x_1, \dots x_k \in D$ and intensities $\xi_i = \sigma_i$, $\sigma_i \in \{\pm 1\}$, $i = 1, \dots k$. Their Grand Canonical correlation function is obtained considering the ensemble composed of those and other n charges with random position and intensities, n being also randomly distributed as before,

$$\rho(x_1, \xi_1, \dots x_k, \xi_k) = \frac{1}{\mathfrak{Z}_{z,\beta}} \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{D^n} e^{-\beta H_{n+k}(x_i,\sigma_i)} \prod_{i=k+1}^{n+k} dx_i d\nu(\sigma_i)$$

In the Sine-Gordon correspondence, these statistical mechanics correlation functions transform into the correlation (Green function) of the field operators $e^{\mathrm{i}\,\xi_i \chi(x_i)}$,

$$\rho(x_1, \xi_1, \dots x_k, \xi_k) = \frac{\int \prod_{i=1}^k e^{\mathrm{i}\sqrt{\beta}\sigma_i\phi(x_i)} e^{-\frac{1}{\varepsilon}\int_D \mathcal{L}(\phi)dx}\mathcal{D}\phi}{\int e^{-\frac{1}{\varepsilon}\int_D \mathcal{L}(\phi)dx}\mathcal{D}\phi} \tag{5.2.3}$$

The latter expression follows from the same formal computations of the previous paragraph: we applied the Gaussian integration formula with respect to the free field with Lagrangian $\frac{1}{\varepsilon}\int |\nabla\phi|^2 dx$, so that the dependence on ε is factored out from the action.

As ε goes to zero, the dominant contribution of the functional integrals in (5.2.3) comes from the stationary points of the action $\mathcal{S}(\phi) = \int_D \mathcal{L}(\phi)dx$, which are given by

$$\frac{\delta\mathcal{S}}{\delta\phi} = \Delta\phi - 2z\sin(\sqrt{\beta}\phi) = 0,$$

which is equivalent, setting $\psi = -\mathrm{i}\,\phi$, to the Debye-Hückel Mean Field equation,

$$\Delta\psi = 2z\sinh(\sqrt{\beta}\psi),$$

which is a sinh-Poisson equation in agreement with the one in (5.1.4). In the particular case of the torus, $D = \mathbb{T}^2$, this equation only admits the trivial solution $\psi \equiv 0$. The limit of correlation functions can thus be obtained by evaluating the field operator $\prod_{i=1}^k e^{\mathrm{i}\sqrt{\beta}\sigma_i\phi(x_i)}$ at the stationary point,

$$\rho(x_1, \xi_1, \dots x_k, \xi_k) \sim \prod_{i=1}^k e^{-\sqrt{\beta}\xi_i\psi(x_i)} = 1.$$

Formal computations involving power expansion of the cosine interaction term leads to further orders behaviour of the correlation function in ε, see [**107**].

Our work actually finds an analogue in [**107**], with some important differences: they consider Coulomb charges in dimension 3 (while we exclusively focus on the 2-dimensional case), and their charges are smeared, the cutoff parameter going to zero in a suitable rate with respect to the Mean Field scaling, while we retain the whole singularity of the interaction. The latter difference is analogous to the one between the two works [17] and [98].

5.3. Decay of Correlations

Let us now proceed to the proof of our main result, Theorem 5.1.2. The main difficulty is due to the logarithmic singularity of the Green function G, which we solve splitting the potential as in the previous Chapter. We thus make use of notation and results of Section 4.1: for $m > 0$,

$$G = -\Delta^{-1} = \left(-\Delta^{-1} - (m^2 - \Delta)^{-1}\right) + (m^2 - \Delta)^{-1} := V_m + W_m. \tag{5.3.1}$$

and accordingly

$$H = H_{V_m} + H_{W_m} = \sum_{i<j}^{N} \xi_i \xi_j V_m(x_i, x_j) + \sum_{i<j}^{N} \xi_i \xi_j W_m(x_i, x_j).$$

We will regard the regular part of the Hamiltonian corresponding to V_m as the covariance of a Gaussian field as we formally did in Section 5.2 for the full Hamiltonian. Let F_m be the centred Gaussian field on $\mathbb{T}^2$ with covariance kernel V_m: we have shown above that F_m has a version taking values in $\dot{L}^p(\mathbb{T}^2)$ for all $p \geq 1$, and a version which is α-Hölder for all $\alpha < 1/2$. Moreover, we have the following estimates:

Lemma 5.3.1. *For any $\alpha > 0$, $p \geq 1$ and $m \to \infty$,*

$$\mathbb{E}\left[\|F_m\|_p^p\right] \simeq_p (\log m)^{p/2}, \tag{5.3.2}$$

$$\mathbb{E}\left[\exp\left(-\alpha \|F_m\|_2^2\right)\right] \simeq m^{-\frac{\alpha}{2\pi}}, \tag{5.3.3}$$

and, moreover, for $0 < \alpha \leq \alpha'$,

$$\mathbb{E}\left[\exp(-\alpha \|F_m\|_{L^2}^2)\right] - \mathbb{E}\left[\exp(-\alpha' \|F_m\|_{L^2}^2)\right] \lesssim (\alpha' - \alpha) m^{-\frac{\alpha}{2\pi}} \log m. \tag{5.3.4}$$

The first two estimates were proved in Section 4.1; (5.3.4) is obtained considering the first order Taylor expansion of the exponential and controlling the remainder by means of Gaussian computations analogous to the ones above.

Let us now consider the Sine-Gordon transformation applied to H_{V_m}, that is we recall (4.1.7):

$$\begin{aligned}
&\int_{\mathbb{T}^{2N}} e^{-\beta H_{V_m}} dx_1 \cdots dx_n \\
&\quad = \int_{\mathbb{T}^{2N}} \exp\left(-\frac{\beta}{2N} \sum_{i \neq j}^{N} \sigma_i \sigma_j V_m(x_i, x_j)\right) dx_1 \cdots dx_n \\
&\quad = e^{\frac{\beta}{2} V_m(0,0)} \mathbb{E}\left[\int_{\mathbb{T}^{2N}} \exp\left(-\mathrm{i}\sqrt{\frac{\beta}{N}} \sum_{i=1}^{N} \sigma_i F_m(x_i)\right) dx_1 \cdots dx_n\right].
\end{aligned}$$

In both expressions, $\mathbb{E}$ denotes expectation with respect to the law of the Gaussian field F_m. We obtained above the following estimate on the regular Gibbs partition function (see also Proposition 5.3.4 below):

Proposition 5.3.2. *For any $\beta > 0$ and integer $n \geq 1$, if $m = m(N)$ grows at most polynomially in N, then it holds*

$$\int_{\mathbb{T}^{2N}} e^{-\beta H_{V_m}} dx_1 \cdots dx_n \leq C_{\beta,n} \left(1 + \frac{m^{\frac{\beta}{4\pi}} (\log m)^{2n}}{N^{n/2}}\right)$$

uniformly in N.

We will also need the control on (the partition function associated to) the singular part of the potential W_m provided by Proposition 4.1.5. Finally, we will need some elementary properties of real and complex exponential integrals, which we isolate here for the reader's convenience.

LEMMA 5.3.3. *Let (X,μ) be a probability space and $f \in L^1(X,\mu)$ with $\int f d\mu = 0$ and $\int e^{-\alpha f} d\mu < \infty$ for $\alpha > 0$. Then for all $n \geq 1$,*

$$\int \left(e^{-f} - 1\right)^{2n} d\mu \leq 2^{2n-2} \int (e^{-2nf} - 1) d\mu.$$

Moreover, if additionally $f \in L^4(X,\mu)$, then

$$\left| \int e^{\mathrm{i}\, f} d\mu - e^{-\frac{1}{2}\|f\|_2^2} \right| \leq \frac{\|f\|_3^3}{6} + \frac{\|f\|_2^4}{8}.$$

PROOF. Expanding the product,

$$\int \left(e^{-f} - 1\right)^{2n} d\mu = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \int e^{-kf} d\mu,$$

and controlling positive and negative terms respectively with Young's and Jensen's inequalities,

$$1 \leq e^{-k \int f d\mu} \leq \int e^{-kf} d\mu \leq \frac{k}{2n} \int e^{-2nf} d\mu + \frac{2n-k}{2n},$$

we get

$$\begin{aligned}
\int \left(e^{-f} - 1\right)^{2n} d\mu &\leq \left(\sum_{k=0}^{n} \frac{k}{n} \binom{2n}{2k} \right) \int e^{-2nf} d\mu \\
&\qquad + \sum_{k=0}^{n} \binom{2n}{2k} \frac{n-k}{n} - \sum_{k=0}^{n-1} \binom{2n}{2k+1} \\
&= 2^{2n-2} \int (e^{-2nf} - 1) d\mu,
\end{aligned}$$

which proves the first statement. As for the second one, thanks to the zero average condition, we can expand

$$\begin{aligned}
&\int_{\mathbb{T}^2} e^{\mathrm{i}\, f(x)} dx - e^{-\frac{1}{2}\|f\|_2^2} \\
&\quad = \int_{\mathbb{T}^2} \left(e^{if(x)} - 1 - \mathrm{i}\, f(x) + \frac{f(x)^2}{2} \right) dx - \left(e^{-\frac{1}{2}\|f\|_2^2} - 1 + \frac{\|f\|_2^2}{2} \right)
\end{aligned}$$

and then apply Taylor expansions

$$\left| e^{it} - 1 - it + \frac{t^2}{2} \right| \leq \frac{t^3}{6}, \qquad \left| e^{-t} - 1 + t \right| \leq \frac{t^2}{2}. \qquad \square$$

5.3.1. Proof of Theorem 5.1.2. To ease notation, in the following argument we will denote

$$E_j = \int_{\mathbb{T}_2} e^{\mathrm{i}\, \xi_j \sqrt{\beta} F_m(x_j)} dx_j, \qquad \mathcal{E} = e^{-\frac{\beta}{2N\gamma} \|F_m\|_{L^2}^2},$$

(notice that both depend on $N, m = m(N)$) and thus write (4.1.7) as

$$\int_{\mathbb{T}^{2N}} e^{-\beta H_{V_m}} dx_1 \cdots dx_n = e^{\frac{\beta}{2\gamma} V_m(0,0)} \mathbb{E} \left[\prod_{j=1}^{N} E_j \right]$$

In sight of Lemma 5.3.3, we expect the 0-th order term (in $1/N$) to be given by $e^{\frac{\beta}{2\gamma}V_m(0,0)}\mathbb{E}\left[\mathcal{E}^N\right]$, which is $O(1)$ as shown above in Lemma 5.3.1. The forthcoming proof applies the Taylor expansion of Lemma 5.3.3 to further and further orders.

PROPOSITION 5.3.4. *For any $\beta \geq 0$ and integer $k \geq 0$, let*

$$\mathscr{R}_k = \Big(\prod_{j=k+1}^{N} E_j\Big) - \mathcal{E}^{N-k}.$$

If $m = m(N)$ grows at most polynomially in N, for every integer $n \geq 1$

$$\mathbb{E}[|\mathscr{R}_k|] \leq \frac{C_{\beta,k,n}}{\sqrt{N}} m^{-\frac{\beta}{4\pi}}(\log m)^{\frac{3}{2}} + \frac{C_{\beta,k,n}}{N^{\frac{n}{2}}}(\log m)^{3n/2}.$$

PROOF. For $n = 1$, we expand the product $\prod_{j=k+1}^{N} E_j$ by means of the algebraic identity introduced in (4.1.8),

$$\prod_{j=k+1}^{N} E_j = \mathcal{E}^{N-k} + \sum_{\ell=k+1}^{N}(E_\ell - \mathcal{E})\mathcal{E}^{N-\ell}\left(\prod_{j=k+1}^{\ell-1} E_j\right). \tag{5.3.5}$$

For $n = 2$, by iterating (5.3.5) we get the identity

$$\begin{aligned}\mathscr{R}_k = \mathcal{E}^{N-k-1} &\sum_{\ell=k+1}^{N}(E_\ell - \mathcal{E}) \\ &+ \sum_{k+1\leq \ell_1<\ell_2\leq N}^{N} \mathcal{E}^{N-k-\ell_1+1}(E_{\ell_1} - \mathcal{E})(E_{\ell_2} - \mathcal{E})\left(\prod_{j=k+1}^{\ell_1-1} E_j\right).\end{aligned}$$

For general n, the iteration of (5.3.5) yields,

$$\begin{aligned}\mathscr{R}_k = &\sum_{\ell=1}^{n-1}\mathcal{E}^{N-k-\ell} \sum_{k+1\leq k_1<\cdots<k_\ell\leq N}\prod_{j=1}^{\ell}(E_{k_j} - \mathcal{E}) \\ &+ \sum_{k+1\leq k_1<\cdots<k_n\leq N} \mathcal{E}^{N-n-k_1+1}\left(\prod_{j=1}^{n}(E_{k_j} - \mathcal{E})\right)\left(\prod_{j=k+1}^{k_1-1} E_j\right).\end{aligned}$$

To estimate the expectation of $\mathscr{R}_k$, everything boils down to estimate expectations of terms $\mathcal{E}^a\|F_m\|_{L^3}^{3b}$ for $a, b > 0$. Indeed, we notice that $|E_j| \leq 1$ and $\mathcal{E} \leq 1$, and that by Taylor expansion, and since F_m has zero average on the torus, $|E_j - \mathcal{E}| \leq N^{-3/2}\|F_m\|_{L^3}^3$. By Lemma 5.3.1 and Cauchy-Schwarz,

$$\mathbb{E}[\mathcal{E}^a\|F_m\|_{L^3}^{3b}] \leq \mathbb{E}[\mathcal{E}^{2a}]^{\frac{1}{2}}\|F_m\|_{L^3}^{6b}]^{\frac{1}{2}} \leq m^{-\frac{a}{4\pi N}\beta}(\log m)^{3b}.$$

Thus,

$$\begin{aligned}\mathbb{E}[|\mathscr{R}_k|] &\lesssim \sum_{\ell=1}^{n-1} N^{-\ell/2}\mathbb{E}\big[\mathcal{E}^{N-k-\ell}\|F_m\|_{L^3}^{3\ell}\big] \\ &\quad + \frac{1}{N}\sum_{k_1=k+1}^{N-n+1} N^{-n/2}\mathbb{E}\big[\mathcal{E}^{N-n-k_1+1}\|F_m\|_{L^3}^{3n}\big] \\ &\lesssim \sum_{\ell=1}^{n-1} N^{-\ell/2}m^{-\frac{N-k-\ell}{4\pi N}\beta}(\log m)^{3\ell/2} + N^{-n/2}(\log m)^{3n/2} \\ &\lesssim \frac{1}{\sqrt{N}}m^{-\frac{\beta}{4\pi}}(\log m)^{\frac{3}{2}} + N^{-n/2}(\log m)^{3n/2},\end{aligned}$$

since m is polynomial in N, therefore $N^{-1/2}(\log m)^{3/2}m^{\beta/4\pi N}$ is smaller than 1 for N large enough. $\square$

REMARK 5.3.5. In fact, Proposition 5.3.4 reprises the argument used in [98] to prove Proposition 5.3.2: indeed, the latter can be deduced from the former.

PROOF OF THEOREM 5.1.2. Fix an even integer $N \geq 1$ large enough, an exponent $p \in [1, \infty)$, and denote by $q \in (1, \infty]$ the Hölder conjugate exponent, so that $1/p + 1/q = 1$. Let $f \in L^q(\mathbb{T}^{2\times k})$ be a test function such that $\|f\|_{L^q} \leq 1$. We use the potential splitting (4.1.3), with m polynomial in N, to decompose the integral of f,

$$\begin{aligned}\int_{\mathbb{T}^{2k}} f(\hat{x})\rho^N_{h,\ell}(\hat{x})\,d\hat{x} &= \frac{1}{Z_{\beta,N}}\int_{\mathbb{T}^{2k}} f(\hat{x})(e^{-\frac{\beta}{N}H_{W_m}} - 1)e^{-\frac{\beta}{N}H_{V_m}}\,d\hat{x}\,d\check{x} \\ &\quad + \frac{1}{Z_{\beta,N}}\int_{\mathbb{T}^{2k}} f(\hat{x})e^{-\frac{\beta}{N}H_{V_m}}\,d\hat{x}\,d\check{x} \\ &:= [S] + [R].\end{aligned}$$

We first consider $[S]$. Let $r, s \geq 1$ be such that $1/r + 1/s = 1/p$, then by the Hölder inequality,

$$[S] \leq \frac{1}{Z_{\beta,N}}\|e^{-\frac{\beta}{N}H_{W_m}} - 1\|_{L^r}\|e^{-\frac{\beta}{N}H_{V_m}}\|_{L^s}.$$

By Jensen's inequality, $Z_{\beta,N} \geq 1$, moreover, by Proposition 5.3.2, $\|e^{-\frac{\beta}{N}H_{V_m}}\|_{L^s}$ is uniformly bounded in N by our choice of m. If n is the smallest integer such that $2n \geq r$ (thus $2n \leq r+2$), by Proposition 4.1.5 and Lemma 5.3.3,

$$\|e^{-\frac{\beta}{N}H_{W_m}} - 1\|_{L^r} \leq \Big(\int_{\mathbb{T}^{2N}} e^{-\frac{2n\beta}{N}H_{W_m}} - 1\Big)^{\frac{1}{2n}} \lesssim \Big(\frac{N}{m^2}(\log m)^2\Big)^{\frac{1}{r+2}}, \tag{5.3.6}$$

since by our choice of m, N/m^2 converges to 0 polynomially in $1/N$.

We turn to the estimate of $[R]$. Set

$$\delta(\hat{x}) = \left(\prod_{j=1}^{h} e^{\mathrm{i}\sqrt{\beta}F_m(y_j)}\right)\left(\prod_{j=1}^{\ell} e^{-\mathrm{i}\sqrt{\beta}F_m(z_j)}\right),$$

then as in (4.1.7),

$$[R] = \frac{1}{Z_{\beta,N}}e^{\frac{1}{2}\beta V_m(0,0)}\mathbb{E}\left[\Big(\prod_{j=k+1}^{N} E_j\Big)\int_{\mathbb{T}^{2k}} f(\hat{x})\delta(\hat{x})\,d\hat{x}\right].$$

Consider the two terms that originate from the decomposition of the product in $\mathcal{E}^{N-k} + \mathscr{R}_k$. First, by Proposition 5.3.4,

$$\begin{aligned}\frac{e^{\frac{1}{2}\beta V_m(0,0)}}{Z_{\beta,N}}\mathbb{E}\Big[\mathscr{R}_k\int_{\mathbb{T}^{2k}} f(\hat{x})\delta(\hat{x})\,d\hat{x}\Big] &\leq \frac{1}{Z_{\beta,N}}e^{\frac{1}{2}\beta V_m(0,0)}\mathbb{E}[|\mathscr{R}_k|] \\ &\leq \frac{(\log m)^{\frac{3}{2}}}{\sqrt{N}} + \frac{m^{\frac{\beta}{4\pi}}}{N^{n/2}}(\log m)^{3n/2}.\end{aligned} \tag{5.3.7}$$

By a Taylor expansion,

$$|\delta(\hat{x}) - 1| \lesssim \frac{1}{\sqrt{N}}\sum_{j=1}^{h} F_m(y_j) + \frac{1}{\sqrt{N}}\sum_{j=1}^{\ell} F_m(z_j),$$

therefore, by Lemma 5.3.1,

$$\frac{e^{\frac{1}{2}\beta V_m(0,0)}}{Z_{\beta,N}}\Big|\mathbb{E}\Big[\mathcal{E}^{N-k}\int_{\mathbb{T}^{2k}} f(\hat{x})(\delta(\hat{x})\,d\hat{x} - 1)\Big]\Big| \lesssim \frac{1}{\sqrt{N}}(\log m)^{1/2}. \tag{5.3.8}$$

It remains to consider only the term,

$$\frac{1}{Z_{\beta,N}}\left(e^{\frac{1}{2}\beta V_m(0,0)}(\mathbb{E}[\mathcal{E}^{N-k}] - Z_{\beta,N}\right)\int_{\mathbb{T}^{2k}} f(\hat{x})\,d\hat{x} + \int_{\mathbb{T}^{2k}} f(\hat{x})\,d\hat{x}$$

and we wish to estimate the contribution to the rate of convergence of the term in brackets in the formula above. Applying the same estimates of above to $f \equiv 1$, we see that the term in brackets is, up to error terms of the same order of those in (5.3.6) and (5.3.7), controlled by

$$e^{\frac{1}{2}\beta V_m(0,0)}(\mathbb{E}[\mathcal{E}^{N-k}] - \mathbb{E}[\mathcal{E}^N]) \lesssim \frac{1}{N}\log m \tag{5.3.9}$$

The last inequality follows from Lemma 5.3.1. We finally choose $m = N^a$. With $a = 1 + \frac{r}{4}$, (5.3.6) is controlled by $N^{-1/2}(\log N)^{3/2}$, as well as (5.3.8) and (5.3.9). Likewise for (5.3.7) if we choose the integer $n > 1 + \frac{\beta}{2\pi}a$. $\square$

CHAPTER 6

Stationary Solutions by Point Vortices Approximations

In this Chapter, following [94], we discuss again solutions to the 2-dimensional incompressible Euler's equation with frictional damping, on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$$\partial_t u_t + u_t \cdot \nabla u_t + \nabla p_t = -\theta u_t + F_t, \quad \nabla \cdot u_t = 0, \tag{6.0.1}$$

where $\theta > 0$ and F_t is a stochastic forcing term. We will consider weak solutions in the form of point vortices systems, and their scaling limit to Gaussian solutions, in the spirit of [71] and Chapter 4.

We will treat our model equation in vorticity form,

$$\partial_t \omega_t = -\theta \omega_t + u_t \cdot \nabla \omega_t + \Pi_t, \quad \omega_t = \nabla^\perp \cdot u_t, \tag{6.0.2}$$

and exhibit solutions by adapting the point vortices model for Euler's equation. Inclusion of the damping term in our model will amount to an exponential quenching of the vortex intensities, with rate θ. Because of dissipation due to friction (which physically results from the 3-dimensional environment in which the 2-dimensional flow is embedded), a forcing term is necessary in order for the model to exhibit stationary behaviour. We will choose as Π_t a Poisson point process, so to add new vortices and rekindle the system. The linear part of (6.0.2) suggests that stationary distributions are made of countable vortices with exponentially decreasing intensity, but in fact dealing with solutions of (6.0.2) having such marginals seems to be as hard as the white noise marginals case. The latter will be also addressed, considering a central limit scaling of the vortices model, resulting in solutions of (6.0.2) with space white noise marginal, and space-time white noise as forcing term.

The main result of the Chapter is the existence of solutions to (6.0.2) in these two cases: infinite vortices marginals and Poisson point process forcing; white noise marginals and space-time white noise forcing. The latter one draws us closer to the models in [27], where the forcing term was Gaussian with delta time-correlations. We will apply a compactness method: our approximant processes will not be approximated solutions (as in Faedo-Galerkin methods), but true point vortices solutions with finitely many vortices.

6.1. Preliminaries and Main Result

Our space domain is the torus $\mathbb{T}^2$: we adopt notation and conventions established in previous Chapters. We denote $H^\alpha = H^\alpha(\mathbb{T}^2) = W^{\alpha,2}(\mathbb{T}^2)$ for $\alpha \in \mathbb{R}$, and $L^2 = L^2(\mathbb{T}^2)$. We recall that Sobolev spaces enjoy the compact embeddings $H^\alpha \hookrightarrow H^\beta$ whenever $\beta < \alpha$, the injections being furthermore Hilbert-Schmitd if $\alpha > \beta + 1$. We will often consider functions of two space variables, *i.e.* functions on $\mathbb{T}^2 \times \mathbb{T}^2 = \mathbb{T}^{2\times 2}$, and denote by $L^2_{sym}(2\times 2)$ the space of symmetric square integrable functions,

$$L^2_{sym}(2\times 2) = \left\{ f \in L^2(2\times 2) : f(x,y) = f(y,x)\ \forall x, y \in \mathbb{T}^2 \right\}.$$

Analogously, $H^\alpha_{sym}(\mathbb{T}^{2\times 2})$, $\alpha \in \mathbb{R}$ will be the $L^2_{sym}(2\times 2)$-based Sobolev space of symmetric functions. The space of finite signed measures on the torus is denoted by $\mathcal{M} = \mathcal{M}(\mathbb{T}^2)$.

The capital letter C will denote (possibly different) constants, and subscripts will point out occasional dependences of C on other parameters. Lastly, we write $X \sim Y$ when the random variables X, Y have the same law.

6.1.1. Random Variables. In order to lighten notation, in this paragraph we denote random variables (or stochastic processes) and their laws with the same symbols. Let us also fix $H := H^{-1-\delta}$, with $\delta > 0$, the Sobolev space in which we embed our random measures and distributions. We will deal with stochastic objects of Gaussian and Poissonian nature: we refer to [**145, 153**] for a complete discussion of the underlying classical theory.

We denote W_t be the cylindrical Wiener process on $L^2(\mathbb{T}^2)$, that is $\langle W_t, f\rangle$ is a real-valued centred Gaussian process indexed by $t \in [0,\infty)$ and $f \in L^2(\mathbb{T}^2)$ with covariance

$$\mathbb{E}\left[\langle W_t, f\rangle, \langle W_s, g\rangle\right] = t \wedge s \,\langle f, g\rangle_{L^2(\mathbb{T}^2)} \tag{6.1.1}$$

for any $t, s \in [0,\infty)$ and $f, g \in L^2(\mathbb{T}^2)$. Since the embedding $L^2(\mathbb{T}^2) \hookrightarrow H^{-1-\delta}(\mathbb{T}^2)$ is Hilbert-Schmidt, W_t defines a $H^{-1-\delta}$-valued Wiener process. The law η of W_1 is the white noise on $\mathbb{T}^2$, and it can thus be regarded as a Gaussian probability measure on $H^{-1-\delta}$. Analogously, the law ζ of the (distributional) time derivative of W is the space-time white noise and it can be identified both with a centred Gaussian process indexed by $L^2([0,\infty)\times\mathbb{T}^2)$ and identity covariance operator or with a centred Gaussian probability measure on $H^{-3/2-\delta}([0,\infty)\times\mathbb{T}^2)$.

Besides those Gaussian distributions, we will be interested in a number of Poissonian variables, which we now define in the framework of [**145**]. For $\lambda > 0$, let π^λ be the Poisson random measure on $[0,\infty)\times H^{-1-\delta}$ with intensity measure ν given by the product of the measure λdt on $[0,\infty)$ and the image of $\sigma\delta_x$ where $\sigma = \pm 1$ and $x \in \mathbb{T}^2$ are chosen uniformly at random. In other terms, one can define the compound Poisson process on $H^{-1-\delta}$ (in fact on $\mathcal{M}$),

$$\Sigma^\lambda_t = \sum_{i: t_i \le t} \sigma_i \delta_{x_i} = \int_0^t d\pi^\lambda, \tag{6.1.2}$$

starting from the jump times t_i of a Poisson process of parameter λ, a sequence σ_i of i.i.d. ± 1-valued Bernoulli variable of parameter $1/2$ and a sequence x_i of i.i.d uniform variables on $\mathbb{T}^2$. Notice that, since its intensity measure has 0 mean, π^λ is a compensated Poisson measure, or equivalently Σ^λ_t is a $H^{-1-\delta}$-valued martingale. Moreover, Σ^λ_t has the same covariance of W_t (up to the factor λ):

$$\mathbb{E}\left[\left\langle \Sigma^\lambda_t, f\right\rangle \left\langle \Sigma^\lambda_s, g\right\rangle\right] = \lambda(t\wedge s)\,\langle f, g\rangle^2_{L^2}, \tag{6.1.3}$$

and also the same quadratic variation,

$$\left[\left\langle \Sigma^\lambda, f\right\rangle\right]_t = \lambda t \,\|f\|^2_{L^2}. \tag{6.1.4}$$

We will need a symbol for another Poissonian integral, the $H^{-1-\delta}$-valued (in fact $\mathcal{M}$-valued) variable

$$\Xi^{\lambda,\theta}_M = \sum_{i: t_i \le M} \sigma_i e^{-\theta t_i}\delta_{x_i} = \int_0^M e^{-\theta t} d\pi^\lambda, \tag{6.1.5}$$

where $M, \theta > 0$. Thanks to the negative exponential, the above integrals converge also when $M = \infty$, defining a random measure: we will call it $\Xi^{\lambda,\theta} = \Xi^{\lambda,\theta}_\infty$.

REMARK 6.1.1. By (6.1.5), a sample of the random measure $\Xi_M^{\lambda,\theta}$ is a finite sum of *point vortices* $\xi_i\delta_{x_i}$ with $\xi_i \in \mathbb{R}, x_i \in \mathbb{T}^2$. We will say that the random vector $(\xi_i, x_i)_{i=1\dots N} \in (\mathbb{R}\times\mathbb{T}^2)^N$ (with random length N) is sampled under $\Xi_M^{\lambda,\theta}$ if $\sum_{i=1}^N \xi_i\delta_{x_i}$ has the law of $\Xi_M^{\lambda,\theta}$. Analogously (and in a sense more generally speaking), the sequence $(t_i,\sigma_i,x_i)_{i\in\mathbb{N}}$ is sampled under π^λ if the sum of $\sigma_i\delta_{t_i}\delta_{x_i}$ has the law of the Poisson point process π^λ.

These Poissonian measures are characterised by their Laplace transforms: for any measurable and bounded $f:\mathbb{T}^2\to\mathbb{R}$,

$$\mathbb{E}\left[\exp\left(\alpha\left\langle f,\Sigma_t^\lambda\right\rangle\right)\right] = \exp\left(\lambda t\int_{\{\pm1\}\times\mathbb{T}^2}(e^{\alpha\sigma f(x)}-1)d\sigma dx\right), \tag{6.1.6}$$

$$\mathbb{E}\left[\exp\left(\alpha\left\langle f,\Xi_M^{\lambda,\theta}\right\rangle\right)\right] = \exp\left(\lambda\int_{[0,M]\times\{\pm1\}\times\mathbb{T}^2}(e^{\alpha\sigma e^{-\theta t}f(x)}-1)dtd\sigma dx\right), \tag{6.1.7}$$

where $d\sigma$ denotes the uniform measure on ±1. By the isometry property of Poissonian integrals, the second moments of Σ_t^λ and $\Xi_M^{\lambda,\theta}$ are given by

$$\mathbb{E}\left[\left\|\Sigma_t^\lambda\right\|_{H^{-1-\delta}}^2\right] = C\lambda t, \quad \mathbb{E}\left[\left\|\Xi_M^{\lambda,\theta}\right\|_{H^{-1-\delta}}^2\right] = C\frac{\lambda}{\theta}(1-e^{-\theta M}),$$

where $C=\|\delta\|_{H^{-1-\delta}}^2$ is the Sobolev norm of Dirac's delta.

An important link between the objects we have defined so far is the following:

PROPOSITION 6.1.2 (Ornstein-Uhlenbeck process). *Consider the $H^{-1-\delta}$-valued linear stochastic differential equation*

$$du_t = -\theta u_t dt + d\Pi_t. \tag{6.1.8}$$

If $\Pi_t = \sqrt{\lambda}W_t$, there exists a unique stationary solution with invariant measure $\sqrt{\frac{\lambda}{2\theta}}\eta$, and if $u_0\sim C\eta$ ($C>0$), the invariant measure is approached exponentially fast, $u_t \sim \sqrt{\frac{\lambda}{2\theta}(1-e^{-2\theta t}(1-C^2))}\eta$.

Analogously, if $\Pi_t=\Sigma_t^\lambda$, there exists a unique stationary solution with invariant measure $\Xi_\infty^{\theta,\lambda}$, and if $u_0\sim\Xi_M^{\theta,\lambda}$, then u_t will have law $\Xi_{M+t}^{\theta,\lambda}$ for any later time $t>0$.

The linear equation (6.1.8), in both the outlined cases, has a unique $H^{-1-\delta}$-valued strong solution, with continuous trajectories in the Gaussian case, and *cadlag* trajectories in the Poissonian one. Well-posedness of the linear equation and uniqueness of the invariant measure are part of the classical theory, and they descend from the explicit solution by stochastic convolution:

$$u_t = e^{-\theta t}u_0 + \int_0^t e^{-\theta(t-s)}d\Pi_s, \tag{6.1.9}$$

from which it is not difficult to derive also the last statement of the Proposition.

6.1.2. Stochastic Double Integrals. Let η be the space white noise on $\mathbb{T}^2$ as in the previous section. Considering η as a random distribution in $H^{-1-\delta}$, the tensor product $\eta\otimes\eta$ is defined as a distribution in $H^{-2-2\delta}(\mathbb{T}^{2\times2})$, so for $h\in H^{2+\delta}(\mathbb{T}^{2\times2})$ we can couple $\langle h,\eta\otimes\eta\rangle$.

The couplings of η against $L^2(\mathbb{T}^2)$ functions are only defined as Ito-Wiener integrals: *double* Ito-Wiener integrals play a crucial role in our discussion, so let us reprise and expand the arguments of Section 1.4. We have denoted above the double stochastic integral with respect to η as

$$L^2_{sym}(\mathbb{T}^{2\times2}) \ni h \mapsto \langle h,\eta\diamond\eta\rangle \in L^2(\eta)$$

the map being an isometry of Hilbert spaces which is not onto: its image is the second Wiener chaos. The following Lemma provides a rigorous definition expanding the one given above.

LEMMA 6.1.3. *The following provide equivalent definitions of the map* $h \mapsto \langle h, \eta \diamond \eta \rangle$*:*

- *the extension by density in* $L^2(\mathbb{T}^{2\times 2})$ *of*
  $$L^2_{sym}(\mathbb{T}^{2\times 2}) \ni f \odot g \mapsto : \langle \eta, f\rangle \langle \eta, g\rangle := \langle \eta, f\rangle \langle \eta, g\rangle - \langle f, g\rangle \in L^2(\eta),$$
  for all $f, g \in L^2(\mathbb{T}^2)$ *and with* $f \odot g(x,y) = \frac{f(x)g(y)+f(y)g(x)}{2}$*;*
- *the extension by density in* $L^2(\mathbb{T}^{2\times 2})$ *of the map*

$$\sum_{\substack{i_1,i_2=1,\dots,n\\ i_1\neq i_2}} a_{i_1,i_2}\mathbf{1}_{A_{i_1}\times A_{i_2}}(x,y) \mapsto \sum_{\substack{i_1,i_2=1,\dots,n\\ i_1\neq i_2}} a_{i_1,i_2}\eta(A_{i_1})\eta(A_{i_2}) \in L^2(\eta), \tag{6.1.10}$$

  where $n \geq 0$*,* $A_1, \dots, A_n \subset \mathbb{T}^2$ *are disjoint Borel sets and* $a_{i,j} \in \mathbb{R}$*, so that functions of the form above vanish on the diagonal* $x = y \in \mathbb{T}^2$*;*
- *the extension by density in* $L^2(\mathbb{T}^{2\times 2})$ *of the map*

$$\forall h \in C^\infty(\mathbb{T}^{2\times 2}) : \forall x \in \mathbb{T}^2\ h(x,x) = 0, \qquad h \mapsto \langle h, \eta \otimes \eta\rangle . \tag{6.1.11}$$

For any $h \in H^{2+\delta}_{sym}(\mathbb{T}^{2\times 2})$ it holds as an equality between $L^2(\eta)$ variables

$$\langle h, \eta\otimes\eta\rangle = \langle h, \eta\diamond\eta\rangle + \int_{\mathbb{T}^2} h(x,x)dx \tag{6.1.12}$$

(since it is true for the dense subset of symmetric products) where we remark that $\int_{\mathbb{T}^2} h(x,x)dx$ makes sense since h has a continuous version by Sobolev embedding. We thus see that Ito-Wiener integration corresponds to "subtract the diagonal contribution" to the tensor product.

Let $\lambda, \theta, M > 0$. In the Poissonian case, we can define double integrals against continuous functions $h \in C(\mathbb{T}^{2\times 2})$ $\mathbb{P}$-almost surely as

$$\left\langle h, \Xi^{\lambda,\theta}_M \otimes \Xi^{\lambda,\theta}_M \right\rangle = \sum_{i,j: t_i, t_j \leq M} \sigma_i \sigma_j e^{-\theta(t_i+t_j)} h(x_i, x_j),$$

where x_i, σ_i, t_i are distributed as in the definition of $\Xi^{\lambda,\theta}_M$, (6.1.5). If we consider in the Poissonian case the third approximation procedure of Lemma 6.1.3, we obtain a different, "renormalised" Poissonian double integral.

LEMMA 6.1.4. *Let* $\mathcal{A} \subset C(\mathbb{T}^{2\times 2})$ *be the family of continuous functions vanishing on the diagonal,* $h(x,x) = 0$ *for all* $x \in \mathbb{T}^2$*. Then for all* $h \in \mathcal{A}$

$$\mathbb{E}\left[\left|\left\langle h, \Xi^{\lambda,\theta}_M \otimes \Xi^{\lambda,\theta}_M\right\rangle\right|^2\right] = \frac{\lambda^2}{\theta}(1-e^{-\theta M})^2 \left\|h\right\|^2_{L^2(\mathbb{T}^{2\times 2})} . \tag{6.1.13}$$

As a consequence, the map $\mathcal{A} \ni h \mapsto \left\langle h, \Xi^{\lambda,\theta}_M \otimes \Xi^{\lambda,\theta}_M\right\rangle \in L^2(\Xi^{\lambda,\theta}_M)$ *extends by continuity to a map*

$$L^2(\mathbb{T}^{2\times 2}) \ni h \mapsto \left\langle h, \Xi^{\lambda,\theta}_M \diamond \Xi^{\lambda,\theta}_M\right\rangle \in L^2(\Xi^{\lambda,\theta}_M)$$

which satisfies (6.1.13)*, and which is given, for functions* $h \in L^2(\mathbb{T}^{2\times 2})$ *continuous outside the diagonal set* $\left\{(x,x) : x \in \mathbb{T}^2\right\} \subset \mathbb{T}^{2\times 2}$*, but possibly discontinuous or singular on it, by*

$$\left\langle h, \Xi^{\lambda,\theta}_M \diamond \Xi^{\lambda,\theta}_M\right\rangle = \sum_{\substack{i,j: t_i, t_j \leq M\\ i\neq j}} \sigma_i \sigma_j e^{-\theta(t_i+t_j)} h(x_i, x_j). \tag{6.1.14}$$

The proof of the latter (as well as the one of Lemma 6.1.3) is a straightforward computation. In a sense, in the Poissonian case the "subtraction of diagonal contributions" is made even more evident then in the Gaussian case by (6.1.14), where the sum runs over distinct indices.

6.1.3. Main Results. Fix $\lambda, \theta > 0$. Our model is the stochastic differential equation

$$d\omega = -\theta\omega dt + (K * \omega) \cdot \nabla\omega dt + d\Pi_t, \tag{6.1.15}$$

where $d\Pi_t$ is either the Poisson process $d\Sigma_t^\lambda$ or the space-time white noise dW_t. We have seen in Proposition 6.1.2 how the linear part of the equation behaves; the intuition provided by the point vortices system suggests that, thanks to the Hamiltonian form of the nonlinearity, the latter only contributes to "shuffle" the vorticity without changes to the fixed time statistics. This intuition can be motivated as follows. Since the point vortices system preserves the product Lebesgue measure, the system must preserve the Poissonian random measures $\Xi_M^{\lambda,\theta}$ we introduced in subsection 6.1.1, because the positions of vortices under those measures are uniformly, independently scattered (this fact will be rigorously proved in Section 6.2 for $M < \infty$). Building Gaussian solutions by approximation with Poissonian ones thus must produce the same phenomenon. In other words, with an eye towards stationary solutions, we expect to be able to build a Poissonian stationary solution with $\omega_t \sim \Xi_\infty^{\theta,\lambda}$ in the case $\Pi_t = \Sigma_t^\lambda$, and a stationary Gaussian solution with $\omega_t \sim \sqrt{\frac{\lambda}{2\theta}}\eta$ in the case $\Pi_t = \sqrt{\lambda}W_t$.

Remark 6.1.5. These claims are deeply related with the fact that 2D Euler's equation preserves *enstrophy*, $\int_{\mathbb{T}^2} \omega(x)^2 dx$, when smooth solutions are considered. The quadratic form associated to enstrophy, that is the $L^2(\mathbb{T}^2)$ product, is (up to multiplicative constants) the covariance of random fields $\Xi_M^{\lambda,\theta}$ and η: as already remarked in [**4**], one should expect all random fields with such covariance to be invariant for Euler's equation, even if the very meaning of the latter sentence has to be clarified.

First and foremost, we need to specify a suitable concept of solution: inspired by the discussion of the last paragraph, we give the following one.

Definition 6.1.6. *Fix $T, \delta > 0$, and let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a probability space with a filtration $\mathcal{F}_t$ satisfying the usual hypothesis, and let $(\omega_t)_{t\in[0,T]}$ be a $H^{-1-\delta}$-valued $\mathcal{F}_t$-predictable process, with trajectories of class*

$$L^2([0,T], H^{-1-\delta}) \cap \mathcal{D}([0,T], H^{-3-\delta}) \tag{6.1.16}$$

($\mathcal{D}([0,T], S)$ denotes the space of S-valued cadlag *functions into a metric space S). On $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ we also consider a $H^{-1-\delta}$-valued $\mathcal{F}_t$-martingale $(\Pi_t)_{t\in[0,T]}$. We consider the following cases: for $\theta, \lambda > 0$,*

- *(P) $\Pi_t = \Sigma_t^\lambda$ and $\omega_t \sim \Xi_{M+t}^{\lambda,\theta}$ (defined respectively in (6.1.2) and (6.1.5)) for all $t \in [0,T]$, with $0 \le M < \infty$;*
- *(Ps) $\Pi_t = \Sigma_t^\lambda$ and $\omega_t \sim \Xi_\infty^{\lambda,\theta}$ for all $t \in [0,T]$;*
- *(G) $\Pi_t = \sqrt{\lambda}W_t$ and $\omega_t \sim \sqrt{\frac{\lambda}{2\theta}(1-e^{-2\theta(M+t)})}\eta$ for all $t \in [0,T]$, with $0 \le M < \infty$;*
- *(Gs) $\Pi_t = \sqrt{\lambda}W_t$ and $\omega_t \sim \sqrt{\frac{\lambda}{2\theta}}\eta$ for all $t \in [0,T]$.*

We say that $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, \Pi_t, \omega_0, (\omega_t)_{t\in[0,T]})$ is a weak solution *of (6.1.15) if for any $f \in C^\infty(\mathbb{T}^2)$ it holds $\mathbb{P}$-almost surely for any $t \in [0,T]$:*

$$\langle f, \omega_t\rangle = e^{-\theta t}\langle f, \omega_0\rangle + \int_0^t e^{-\theta(t-s)}\langle H_f, \omega_s \diamond \omega_s\rangle ds + \int_0^t e^{-\theta(t-s)}\langle f, d\Pi_s\rangle, \tag{6.1.17}$$

where $\langle H_f, \omega_s \diamond \omega_s \rangle$ is defined as in Lemma 6.1.4 in cases (P), (Ps) and as in Lemma 6.1.3 in (G), (Gs). If instead, given $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, W_t)$ there exists a process ω_t as above, we call it a strong solution*.*

REMARK 6.1.7. The "variation of constants" expression in the above definition is equivalent to the "integral" one

$$\langle f, \omega_t \rangle = \langle f, \omega_0 \rangle - \theta \int_0^t \langle f, \omega_s \rangle \, ds + \int_0^t \langle H_f, \omega_s \diamond \omega_s \rangle \, ds + \langle f, \Pi_t \rangle \,, \tag{6.1.18}$$

as one can verify integrating by parts in time. Both versions will be useful in what follows, but we deem (6.1.17) more suggestive.

REMARK 6.1.8. The nonlinear term of (6.1.17) is well-defined thanks to the isometry properties of Gaussian and Poissonian double integral (see Section 6.1): indeed, the integrand is bounded in $L^2(\mathbb{P})$ uniformly in time, so that, in particular, $\int_0^t \langle H_f, \omega_s \diamond \omega_s \rangle \, ds$ is a continuous function of time.

We are now able to state our main result.

THEOREM 6.1.9. *There exist weak solutions of (6.1.15) in all the outlined cases, stationary (as $H^{-1-\delta}$-valued stochastic processes) in the cases (Ps) and (Gs).*

As already remarked, equation (6.1.15) is difficult to deal with directly in the Gaussian (or even the stationary Poisson) case: for instance it does not seem possible to treat it with fixed point or semigroup techniques. We prove existence of stationary solutions by taking limits of point vortices solutions, corresponding to the case (P). We begin with a solution ω_M of the equation (6.1.15) with noise Σ_t^λ starting from finitely many vortices distributed as $\Xi_M^{\theta,\lambda}$. Well-posedness in this case is ensured by a generalisation of Theorem 1.3.1, whose proof is the content of Section 6.2. The first limit we consider is $M \to \infty$, so to build a stationary solution with invariant measure $\Xi^{\theta,\lambda}$ and thus obtain existence in case (Ps). Scaling intensities $\sigma \to \frac{\sigma}{\sqrt{N}}$ and generation rate $\lambda \to N\lambda$, we prove that as $N \to \infty$ the limit points are stationary solutions of (6.1.15) driven by space-time white noise and with invariant measure the space white noise. The nonstationary Gaussian case (G) will be derived analogously, in this sort of central limit theorem.

We are applying a *compactness method*: first, we prove probabilistic bounds on the involved distribution, in order to -second step- apply a compactness criterion ensuring tightness of the approximating processes; finally, we pass to the limit the equation satisfied by the approximants.

REMARK 6.1.10. Consider the case when no damping or forcing are present: we noted above that the classical finite vortices system Equation 1.3.1 preserves the product Lebesgue's measure, so in particular the distributions $\Xi_M^{\theta,\lambda}$ with $M < \infty$ and $\theta, \lambda > 0$ are also invariant. The very same limiting procedure we are going to use, as $M \to \infty$, proves existence of stationary solutions to Euler's equation in its weak formulation with invariant measure $\Xi_\infty^{\theta,\lambda}$ (or η, the case of [**71**]). More generally, Poissonian and Gaussian stationary solutions, as suggested in [**4**], should be particular cases of stationary solutions with independently scattered random distributions.

6.2. Solutions with finitely many vortices

Even in the case of initial data distributed as $\Xi_M^{\lambda,\theta}$, that is with almost surely finitely many initial vortices, solving the nonlinear equation

$$d\omega = -\theta \omega dt + (K * \omega) \cdot \nabla \omega dt + d\Sigma_t^\lambda \tag{6.2.1}$$

is not a trivial task. We will build a solution describing explicitly how the initial vortices and the ones added by the noise term evolve, as a system of increasingly numerous differential equations for the positions of vortices x_i. Intuitively, the process $\omega_{M,t}$ is defined as follows: from the initial datum $\omega_M(0)$, which is sampled under $\Xi_M^{\theta,\lambda}$, we let the system evolve according to the deterministic dynamics

$$\dot{x}_i = \sum_{j\neq i} \xi_j e^{-\theta t} K(x_i, x_j)$$

until the first jump time t_1 of the driving noise Σ_t^λ, when we add the vortex corresponding to the jump, and so on. To treat the model rigorously, let us introduce the following notation: let $x_{1,0}, \dots, x_{n,0}$ and $\xi_{1,0}, \dots, \xi_{n,0}$ be the (random) positions and signs of vortices of the initial datum, and set for notational convenience $t_1 = \dots = t_n = 0$ their birth time; at time t_i it is added a vortex with intensity $\xi_{i,t_i} = \pm 1$ in the position x_{i,t_i}, but we can pretend it to actually have existed since time 0, and just come into play at the time t_i. Thus, our equations are

$$x_{i,t} = x_{i,t_i} + \mathbf{1}_{t_i \leq t} \int_{t_i}^{t} \sum_{j\neq i: t_j \leq s} \xi_{j,s} K(x_{i,s}, x_{j,s}) ds, \tag{6.2.2}$$

$$\xi_{i,t} = \begin{cases} \xi_{i,0} & t < t_i \\ e^{-\theta(t-t_i)} \xi_{i,0} & t \geq t_i \end{cases}. \tag{6.2.3}$$

In this formulation of the problem, part of the randomness consists in the positions and intensities of the initial vortices and the ones to be: the random jump times t_i then determine when the latter ones become part of the system. Let us thus fix the t_i's (that is, condition the process given the distribution of the t_i's) so to reduce us to a deterministic problem with random initial data. The existence of a solution for almost every initial condition is ensured by the following generalisation of Theorem 1.3.1.

Proposition 6.2.1. *Let $(x_{i,0})_{i\in\mathbb{N}}$ be a sequence of i.i.d uniform variables on $\mathbb{T}^2$. For every locally finite sequence of jump times $0 \leq t_1 \leq \dots \leq t_i \leq \dots \leq \infty$ and initial intensities $(\xi_{i,0}) \in [-1,1]$ the system of equations (6.2.2) and (6.2.3) possesses a unique, piecewise smooth and continuous, global in time solution, for a full probability set which does not depend on the choice of $t_i, \xi_{i,0}$. At any time, the joint law of positions x_i is the infinite product of Lebesgue measure on $\mathbb{T}^2$.*

We use the hypothesis that the jump times t_i are locally finite (there are only finitely many of them in every compact $[0,T]$) so to reduce ourselves to a system of finitely many vortices. In fact, we repeat the proof of [**65, 135**] adapting it to our context. The issue is the possibility of collapsing vortices, which is ruled out as follows. We define an approximating system with interaction kernel smoothed in a ball around 0: the smooth interaction readily gives well-posedness of the approximants, on which we evaluate a Lyapunov functional measuring how close the vortices can get. Bounding the Lyapunov function then ensures that as the regularisation parameter goes to 0, the approximant vortices in fact perform the same motion prescribed by the non-smoothed equation.

Proof. Let $\delta > 0$, and consider smooth functions G_δ coinciding with G outside the fattened diagonal $\{(x,y) \in \mathbb{T}^{2\times 2} : d(x,y) < \delta\}$ (d being the distance on the torus $\mathbb{T}^2$), and such that

$$|G_\delta(x,y)| \leq C|G(x,y)|, \quad |\nabla G_\delta(x,y)| \leq \frac{C}{d(x,y)} \quad \forall x,y \in \mathbb{T}^2. \tag{6.2.4}$$

Note in particular that the latter inequality was already true for G. Let us first restrict ourselves to a time interval $[0,T]$: in particular, we can consider only the finitely many vortices with $t_i \leq T$, let them be $x_1, \dots, x_n$. Notice that smoothing K does not effect the evolution of the intensities $\xi_{i,t}$.

Thanks to Cauchy-Lipschitz theorem, the system with smoothed interaction kernel $K_\delta = \nabla^\perp G_\delta$ has a unique smooth solution $x^\delta_{i,t}$ for $t \in [0, t_1]$. The time derivative $\dot{x}^\delta_{i,t}$ is not right-continuous at $t = t_1$, but on $(t_1, t_2]$ is again smooth, so we can extend the unique solution applying Cauchy-Lipschitz in $[t_1, t_2]$ starting from x^δ_{i,t_1}; notice that the resulting solution on $[0, t_2]$ is continuous, although not differentiable at t_1. Proceeding as such we extend well-posedness to all $t \geq 0$.

Because of the Hamiltonian structure of the equations, that is, since $K_\delta = \nabla^\perp G_\delta$, it holds $\operatorname{div} \dot{x}^\delta_{i,t} = 0$ for any $t \neq t_1, \dots t_n$. As a consequence, by Liouville's theorem (see for instance [**50**, Section 2.2]) the flow is measure preserving on intervals $(t_i, t_{i+1}]$, where it is smooth. But we have seen that the solution $x^\delta_{i,t}$ is given by a composition of such transformations, so that the product Lebesgue measure is preserved at all times.

Let us now introduce a Lyapunov function measuring how close the existing vortices are by means of G_δ:

$$L_\delta(t) = L_\delta(t, x^\delta_{1,t}, \dots, x^\delta_{n,t}) = -\sum_{i \neq j : t_i, t_j \leq t} G_\delta(x^\delta_{i,t}, x^\delta_{j,t}).$$

By replacing G_δ with $G_\delta - k$ for a large enough $k > 0$ in the definition of L_δ we can assume that L_δ is nonnegative. Observe that, because of (6.2.4), $\int_{\mathbb{T}^{2\times n}} L_\delta(0) dx_1, \dots dx_n \leq C$ for a constant C independent of δ. Upon differentiating, and keeping in mind that

$$\dot{x}^\delta_{i,t} = \sum_{j \neq i : t_j < t} \xi_{j,t} \nabla^\perp G_\delta(x^\delta_{i,t}, x^\delta_{j,t}), \quad \forall t > t_i, t \neq t_1, \dots t_n,$$

(again, the flow is continuous but only differentiable away from jump times), we obtain, for all $t \neq t_1, \dots t_n$,

$$\begin{aligned} \frac{d}{dt} L_\delta(t) &= -\sum_{i \neq j : t_i, t_j \leq t} \nabla G_\delta(x^\delta_{i,t}, x^\delta_{j,t}) \cdot (\dot{x}^\delta_{i,t} + \dot{x}^\delta_{j,t}) \\ &= \sum_{i,j,k \leq n} \tilde{a}_{ijk}(t) \nabla G_\delta(x^\delta_{i,t}, x^\delta_{j,t}) \cdot \nabla^\perp G_\delta(x^\delta_{i,t}, x^\delta_{k,t}), \end{aligned}$$

where $\tilde{a}_{ijk}(t)$ depend on time t as functions of the intensities $\xi_{i,t}$, $\tilde{a}_{ijk} = 0$ whenever two indices are equal, since $\nabla G_\delta(x^\delta_{i,t} - x^\delta_{j,t}) \cdot \nabla^\perp G_\delta(x^\delta_{i,t} - x^\delta_{j,t}) = 0$ and it always holds $|\tilde{a}_{ijk}(t)| \leq 1$. As a consequence, and using the fact that the solution $x^\delta_{i,t}$ is continuous, we have

$$L_\delta(t) = L_\delta(0) + \sum_{i,j,k \leq n} \int_0^t \tilde{a}_{ijk}(t) \nabla G_\delta(x^\delta_{i,s}, x^\delta_{j,s}) \cdot \nabla^\perp G_\delta(x^\delta_{i,s}, x^\delta_{k,s}) ds.$$

We can use this to prove the following integral bound on L_δ: denoting by dx^n the n-fold Lebesgue measure of the distribution of initial position,

$$\begin{aligned} \int_{\mathbb{T}^{2\times n}} \sup_{t \in [0,T]} L_\delta(t) dx^n &\leq \int_{\mathbb{T}^{2\times n}} L_\delta(0) dx^n \\ &\quad + {\sum_{i,j,k}}' \int_0^T \int_{\mathbb{T}^{2\times n}} \left| \nabla G_\delta(x^\delta_{i,s}, x^\delta_{j,s}) \cdot \nabla^\perp G_\delta(x^\delta_{i,s}, x^\delta_{k,s}) \right| dx^n ds \\ &\leq \int_{\mathbb{T}^{2\times n}} L_\delta(0) dx^n + T {\sum_{i,j,k}}' \int_{\mathbb{T}^{2\times n}} \left| \nabla G_\delta(x_i, x_j) \cdot \nabla^\perp G_\delta(x_i, x_k) \right| dx^n \\ &\leq \int_{\mathbb{T}^{2\times n}} L_\delta(0) dx^n + T C_n \int_{\mathbb{T}^{2\times 3}} \left| \nabla G_\delta(x, y) \cdot \nabla^\perp G_\delta(x, z) \right| dx dy dz \leq C_T, \end{aligned}$$

C_T being a constant depending only on T (n depends on T). In the second and third lines, $\sum'$ denotes summation over indices $i, j, k = 1, \dots n$ such that no pair of them coincide. In the second inequality we have used the invariance of Lebesgue's measure, in the third one the fact that summation is over distinct indices and in the last step the aforementioned integrability of $L_\delta(0)$ and the fact that, because of (6.2.4), the integrands in the second term are bounded by

$$\left|\nabla G_\delta(x-y) \cdot \nabla^\perp G_\delta(x-z)\right| \leq \frac{C}{|x-y||x-z|}.$$

With these estimates at hand, we can now pass to the limit as $\delta \to 0$: let

$$d_{\delta,T}(x^n) = \min_{t\in[0,T]} \min_{i\neq j} d(x^\delta_{i,t} - x^\delta_{j,t}),$$

so that

$$d_{\delta,T}(x^n) < \delta \Rightarrow \sup_{t\in[0,T]} L_\delta(t) > -C\log(\delta),$$

since when two points x, y are closer than δ, $G_\delta(x,y) \geq C\log(\delta)$ for some universal constant C. As a consequence, by Čebyšëv's inequality,

$$\mathbb{P}(\Omega_{\delta,T}) := \mathbb{P}(d_{\delta,T}(x^n) < \delta) \leq C'(-\log\delta)^{-1}.$$

By construction, in the event $\Omega^c_{\delta,T}$ the solution $x^\delta_{i,t}$ is in fact a solution of the original system in $[0,T]$. Hence, the thesis holds if the event

$$\bar\Omega = \bigcup_{T>0} \bigcap_{\delta>0} \Omega_{\delta,T}$$

is negligible. But this is true: $\Omega_{\delta,T}$ is monotone in its arguments, so that the intersection in δ is negligible because of the above estimates, hence the increasing union in T must be negligible too. □

The forthcoming Corollary is a direct consequence of Proposition 6.2.1: indeed to complete our construction we only need to randomise the jump times and intensities so that the initial conditions and driving noise have the correct distribution. Assume that

- $(x_{1,0}, \xi_{1,0}), \dots (x_{n,0}, \xi_{n,0})$ are positions and intensities of vortices sampled under $\Xi^{\theta,\lambda}_M$,
- $(t_{n+m}, x_{n+m,0}, \xi_{n+m,0} = \sigma_{n+m})_{m\geq 1}$ is sampled under π^λ,

both in the sense of Remark 6.1.1, with variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a piecewise smooth, *cadlag* solution of the system of equations (6.2.2) and (6.2.3) for all $t \in [0,\infty)$, $\mathbb{P}$-almost surely. Moreover, the positions of vortices at any time t, $x_{i,t}$, are i.i.d. uniform variables on the torus $\mathbb{T}^2$.

Corollary 6.2.2. *In the outlined setting, the process* $\omega_{M,t} = \sum_{i:t_i\leq t} \xi_{i,t}\delta_{x_{i,t}}$ *is a* $\mathcal{M}$*-valued* cadlag *Markov process with fixed time marginals* $\omega_{M,t} \sim \Xi^{\theta,\lambda}_{M+t}$ *for all* $t \geq 0$. *It is a strong solution of*

$$d\omega_M = -\theta\omega_M dt + (K * \omega_M)\cdot\nabla\omega_M dt + d\Sigma^\lambda_t,$$

in the sense of Definition 6.1.6

Proof. Fix $s < t$: by construction, given the positions $x_{i,0}$, the initial intensities $\xi_{i,0}$ and the jump times t_i (in a $\mathbb{P}$-full measure event), $\omega_{M,t}$ is given by a deterministic function of $(x_{i,s}, \xi_{i,s})_{i:t_i<s}$ and $(t_i, x_{i,0}, \xi_{i,0})_{i:s\leq t_i<t}$. As a consequence, $\omega_{M,t}$ is a function of $\omega_{M,s}$ and of the driving noise $(\Sigma^\lambda_r)_{s\leq r<t}$, which is independent from $\omega_{M,s}$: this implies the Markov property. Since the trajectories of positions $x_{i,t}$ and the evolution of intensities $\xi_{i,t}$ are smooth in time, $\omega_{M,t}$ is also smooth in time, save for the jump times t_i when a new Dirac's delta is added.

As for the marginal distributions, let us first evaluate:

$$
\begin{aligned}
\mathbb{E}\left[e^{\mathrm{i}\,\alpha\langle\omega_{M,t},f\rangle}\right] &= \mathbb{E}\left[\exp\left(\mathrm{i}\,\alpha\sum_{i:t_i\le t}\xi_{i,t}f(x_{i,t})\right)\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\exp\left(\mathrm{i}\,\alpha\sum_{i:t_i\le t}\xi_{i,t}f(x_{i,t})\right)\middle|(t_i)_{i\ge 0}\right]\right]\\
&= \mathbb{E}\left[\prod_{i:t_i\le t}\int_{\mathbb{T}^2}e^{\mathrm{i}\,\alpha\xi_{i,t}f(x)}dx\right]=:\mathbb{E}\left[\prod_{i:t_i\le t}F(\xi_{i,t})\right].
\end{aligned}
$$

Using the definition of $\xi_{i,t}$, and distinguishing the cases $i\le n$ and $i>n$ (which correspond to two independent groups of random variables), we can write

$$
\begin{aligned}
\mathbb{E}\left[e^{\mathrm{i}\,\alpha\langle\omega_{M,t},f\rangle}\right] &= \mathbb{E}_N\left[\prod_{s_i\in[0,M]}F(e^{-\theta s_i})\right]\cdot\mathbb{E}_N\left[\prod_{s_i\in[0,t]}F(e^{-\theta(t-s_i)})\right]\\
&= \mathbb{E}_N\left[\prod_{s_i\in[0,M+t]}F(e^{-\theta s_i})\right]
\end{aligned}
$$

where N is a Poisson point process of parameter λ on $\mathbb{R}$ whose points are denoted by s_i, and the second passage follows from the fact that the points N in disjoint intervals are independent and their distribution does not change if we reverse the parametrisation of the interval. Comparing to the characteristic function of Ξ_{M+t} given in (6.1.7), we conclude that $\omega_{M,t}\sim\Xi^{\theta,\lambda}_{M+t}$.

Observe now that in this case it holds, for any $f\in C^\infty(\mathbb{T}^2)$, $\mathbb{P}$-almost surely for all $t\ge 0$,

$$
\langle H_f,\omega_{M,t}\diamond\omega_{M,t}\rangle=\sum_{\substack{i,j:t_i,t_j\le t\\ i\ne j}}\xi_{i,t}\xi_{j,t}H_f(x_{i,t},x_{j,t}),
$$

(*cf.* with subsection 6.1.1). Given this, it is straightforward to show that we do have built solutions of (6.1.17): for $f\in C^\infty(\mathbb{T}^2)$, by (6.2.2) and (6.2.3),

$$
\begin{aligned}
\langle f,\omega_{M,t}\rangle &= \sum_{i:t_i\le t}\xi_{i,t}f(x_{i,t})\\
&= \sum_{i:t_i\le t}\xi_{i,t}\left(f(x_{i,t_i})+\int_{t_i}^t\sum_{j\ne i:t_j\le s}\xi_{j,s}\nabla f(x_{i,s})\cdot K(x_{i,s},x_{j,s})ds\right)\\
&= \left(\sum_{i=1}^n+\sum_{i>n:t_i\le t}\right)\xi_{i,t}f(x_{i,t_i})+\sum_{i:t_i\le t}\xi_{i,t}\int_{t_i}^t\sum_{j\ne i:t_j\le s}\xi_{j,s}\nabla f(x_{i,s})\cdot K(x_{i,s},x_{j,s})ds\\
&= \sum_{i=1}^n e^{-\theta t}f(x_{i,0})+\sum_{i>n:t_i\le t}e^{-\theta(t-t_i)}f(x_{i,t_i})\\
&\quad+\int_0^t\sum_{\substack{i,j:t_i,t_j\le s\\ i\ne j}}e^{-\theta(t-s)}\xi_{i,s}\xi_{j,s}\nabla f(x_{i,s})\cdot K(x_{i,s},x_{j,s})ds\\
&= e^{-\theta t}\langle f,\omega_{M,0}\rangle+\int_0^t e^{-\theta(t-s)}\langle f,d\Sigma_s\rangle+\int_0^t\langle H_f,\omega_{M,s}\diamond\omega_{M,s}\rangle ds.
\end{aligned}
$$

The latter equation holds regardless of the choice of initial positions, intensities and jump times (as soon as the dynamics is defined) so in particular it holds $\mathbb{P}$-almost surely uniformly in t, and this concludes the proof. $\square$

The method of [**65**] thus provides, quite remarkably, existence and pathwise uniqueness of measure-valued strong solutions. Unfortunately, it only seems to apply to systems of finitely many vortices, since it relies on the very particular, discrete nature of the measures involved to control the "diagonal collapse" issue. We refer to [**95**] for further uniqueness results for point vortices systems obtained by means of refinements of the above techniques.

Let us conclude this section noting that we have obtained the first piece of Theorem 6.1.9, namely we have built solutions in the case (P) for all $M<\infty$.

6.3. Proof of the Main Result

In Section 6.2 we built the point vortices processes $\omega_{M,t}=\sum_{i:t_i\leq t}\xi_{i,t}\delta_{x_{i,t}}$. Let us introduce the scaling in $N\geq 1$: we will denote $\omega_{M,N,t}=\sum_{i:t_i\leq t}\frac{\xi_{i,t}}{\sqrt{N}}\delta_{x_{i,t}}$ where $x_{i,t},\xi_{i,t}$ solve equations (6.2.2) and (6.2.3), and where the t_i's are the jump times of a real valued Poisson process of intensity $N\lambda$. In other words, by Corollary 6.2.2, $\omega_{M,N,t}$ is a strong solution of

$$d\omega_{M,N}=-\theta\omega_{M,N}dt+(K*\omega_{M,N})\cdot\nabla\omega_{M,N}dt+\frac{1}{\sqrt{N}}d\Sigma^{N\lambda}_t, \tag{6.3.1}$$

(in the sense of Definition 6.1.6) with fixed time marginals $\omega_{M,N,t}\sim\frac{1}{\sqrt{N}}\Xi^{\theta,N\lambda}_{M+t}$. It is worth to note here that, by construction of $\omega_{M,N,t}$, its natural filtration $\mathcal{F}_t$ coincides with the one generated by the driving noise $\Sigma^{N\lambda}_t$ and the initial datum.

The forthcoming paragraphs deal with, respectively: a recollection of some compactness criterions, the bounds proving that the laws of $\omega_{M,N}$ are tight, the proof of the fact that limit points of our family of processes are indeed solutions in the sense of Definition 6.1.6, that is, the main result.

6.3.1. Compactness Results. Let us first review a deterministic compactness criterion due to Simon (we refer to [**156**] for the result and the required generalities on Banach-valued Sobolev spaces).

PROPOSITION 6.3.1 (Simon). *Assume that*

- $X\hookrightarrow B\hookrightarrow Y$ *are Banach spaces such that the embedding* $X\hookrightarrow Y$ *is compact and there exists* $0<\theta<1$ *such that for all* $v\in X\cap Y$
$$\|v\|_B\leq M\,\|v\|_X^{1-\theta}\,\|v\|_Y^{\theta}\,;$$
- $s_0,s_1\in\mathbb{R}$ *are such that* $s_\theta=(1-\theta)s_0+\theta s>0$.

If $\mathcal{F}\subset W$ *is a bounded family in*

$$W=W^{s_0,r_0}([0,T],X)\cap W^{s_1,r_1}([0,T],Y)$$

with $r_0,r_1\in[0,\infty]$*, and we define*

$$\frac{1}{r_\theta}=\frac{1-\theta}{r_0}+\frac{\theta}{r_1},\qquad s_*=s_\theta-\frac{1}{r_\theta},$$

then if $s_*\leq 0$*,* $\mathcal{F}$ *is relatively compact in* $L^p([0,T],B)$ *for all* $p<-\frac{1}{s_*}$*. In the case* $s_*>0$*,* $\mathcal{F}$ *is moreover relatively compact in* $C([0,T],B)$*.*

Let us specialise this result to our framework. Take

$$X=H^{-1-\delta/2}(\mathbb{T}^2),\quad B=H^{-1-\delta}(\mathbb{T}^2),\quad Y=H^{-3-\delta}(\mathbb{T}^2),$$

with $\delta > 0$: by Gagliardo-Niremberg estimates the interpolation inequality is satisfied with $\theta = \delta/2$. Let us take moreover $s_0 = 0$, $s_1 = 1/2 - \gamma$ with $\gamma > 0$, $r_1 = 2$ and $r_0 = q \geq 1$, so that the discriminating parameter is

$$s_* = -\gamma\theta - \frac{1-\theta}{q}.$$

Note that as we take δ smaller and smaller, and q bigger and bigger, we can get $s_* < 0$ arbitrarily close to 0, but not 0. We have thus derived:

COROLLARY 6.3.2. *If the sequence*

$$\{v_n\} \subset L^p([0,T], H^{-1-\delta}) \cap W^{1/2-\gamma,2}([0,T], H^{-3-\delta})$$

is bounded for any choice of $\delta > 0$ and $p \geq 1$, and for some $\gamma > 0$, then it is relatively compact in $L^q([0,T], H^{-1-\delta})$ for any $1 \leq q < \infty$. As a consequence, if a sequence of stochastic processes $u^n : [0,T] \to H^{-1-\delta}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is such that, for any $\delta > 0$, $p \geq 1$ and some $\gamma > 0$, there exists a constant $C_{\delta,\gamma,q}$ for which

$$\sup_n \mathbb{E}\left[\|u^n(t)\|_{L^p([0,T],H^{-1-\delta})} + \|u^n\|_{W^{1/2-\gamma,1}([0,T],H^{-3-\delta})}\right] \leq C_{\delta,\gamma,p}, \tag{6.3.2}$$

then the laws of u_n on $L^q([0,T], H^{-1-\delta})$ are tight for any $1 \leq q < \infty$.

The processes we will consider are discontinuous in time: this is why we consider only fractional Sobolev regularity in time. However, as we have just observed, this prevents us to use Simon's criterion to prove any time regularity beyond L^q. This is why we will combine the latter result with a compactness criterion for *cadlag* functions. We refer to [138] for both the forthcoming result and the necessary preliminaries on the space $\mathcal{D}([0,T], S)$ of *cadlag* functions taking values in a complete separable metric space S.

THEOREM 6.3.3 (Aldous' Criterion). *Consider a sequence of stochastic processes $u^n : [0,T] \to S$ defined on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ and adapted to filtrations $\mathcal{F}^n_t$. The laws of u^n are tight on $\mathcal{D}([0,T], S)$ if:*

(1) for any $t \in [0,T]$ (a dense subset suffices) the laws of the variables u^n_t are tight;
(2) for all $\varepsilon, \varepsilon' > 0$ there exists $R > 0$ such that for any sequence of $\mathcal{F}^n$-stopping times $\tau_n \leq T$ it holds

$$\sup_n \sup_{0 \leq r \leq R} \mathbb{P}^n\left(d(u^n_{\tau_n}, u^n_{\tau_n+r}) \geq \varepsilon'\right) \leq \varepsilon.$$

6.3.2. Tightness of Point Vortices Processes. The following estimate on our Poissonian random measures is the crux in all the forthcoming bounds; it is essentially a Poissonian analogue of the ones in Section 3 of [71].

PROPOSITION 6.3.4. *Let $\omega_{M,N} \sim \frac{1}{\sqrt{N}} \Xi^{\theta,N\lambda}_M$. For any $1 \leq p < \infty$ there exists a constant $C_p > 0$ such that for any measurable bounded functions $h : \mathbb{T}^2 \to \mathbb{R}$ and $f : \mathbb{T}^{2\times 2} \to \mathbb{R}$ it holds*

$$\mathbb{E}\left[\langle h, \omega_{M,N}\rangle^{2p}\right] \leq C_p \|h\|^{2p}_\infty, \qquad \mathbb{E}\left[\langle f, \omega_{M,N} \otimes \omega_{M,N}\rangle^p\right] \leq C_p \|f\|^p_\infty, \tag{6.3.3}$$

uniformly in $N \geq 0$ and $M \in [0,\infty]$. As a consequence, since for $\delta > 0$ the Green function $\Delta^{-1-\delta}$ is smooth,

$$\mathbb{E}\left[\|\omega_{M,N}\|^{2p}_{H^{-1-\delta}}\right] = \mathbb{E}\left[\left\langle \Delta^{-1-\delta}, \omega_{M,N} \otimes \omega_{M,N}\right\rangle^p\right] \leq C_{p,\delta}, \tag{6.3.4}$$

uniformly in M, N.

PROOF. Since

$$\langle f, \omega_{M,N} \otimes \omega_{M,N} \rangle = \left\langle \tilde{f}, \omega_{M,N} \otimes \omega_{M,N} \right\rangle, \quad \tilde{f}(x,y) = \frac{1}{2}(f(x,y) + f(y,x)),$$

we reduce ourselves to symmetric functions. Moreover, without loss of generality we can check (6.3.3) for functions with separate variables $f(x,y) = h(x)h(y)$, $h : \mathbb{T}^2 \to \mathbb{R}$ measurable and bounded, for which it holds

$$\mathbb{E}\left[\langle f, \omega_{M,N} \otimes \omega_{M,N} \rangle^p\right] = \mathbb{E}\left[\langle h, \omega_{M,N} \rangle^{2p}\right].$$

Moments of the random variable $\langle h, \omega_{M,N} \rangle$ can be evaluated by differentiating the moment generating function (6.1.7): using Faà di Bruno's formula to take $2p$ derivatives we get

$$\begin{aligned}
&\mathbb{E}\left[\langle h, \omega_{M,N} \rangle^{2p}\right] = \\
&= (2p!) \sum_{\substack{r_1,\dots,r_{2p} \geq 0 \\ r_1 + 2r_2 + \dots + 2pr_{2p} = 2p}} \prod_{k=1}^{2p} \frac{1}{(k!)^{r_k} r_k!} \left(N\lambda \int_{[0,M]\times\{\pm 1\}\times\mathbb{T}^2} \frac{\sigma^k}{N^{k/2}} e^{-\theta t k} h(x) d\sigma dx dt \right)^{r_k} \\
&\leq (2p!) \sum_{\substack{r_1,\dots,r_{2p} \geq 0 \\ r_1 + 2r_2 + \dots + 2pr_{2p} = 2p}} \prod_{k=1}^{2p} \frac{(N\lambda)^{r_k} \|h\|_k^{kr_k} \mathbf{1}_{2|k}^{r_k}}{(\theta k)^{r_k} N^{kr_k/2} (k!)^{r_k} r_k!} \\
&= \frac{(2p!) \|h\|_\infty^{2p}}{N^p} \sum_{\substack{r_1,\dots,r_{2p} \geq 0 \\ r_1 + 2r_2 + \dots + 2pr_{2p} = 2p}} \prod_{k=1}^{2p} \frac{(N\lambda)^{r_k} \mathbf{1}_{2|k}^{r_k}}{(\theta k)^{r_k} (k!)^{r_k} r_k!}
\end{aligned}$$

(see [**145, 148**] for similar classical computations). Let us stress that when an integral in the latter formula is null, its 0-th power is to be interpreted as $0^0 = 1$. The contribution of $\mathbf{1}_{2|k} = \int \sigma^k d\sigma$ is crucial: when k is odd, $\mathbf{1}_{2|k}$ is null, so only terms with $m_k = 0$ survive in the sum (again, $0^0 = 1$). Thus, the highest power of N appearing is $N^{r_2} \leq N^{2p/2} = N^p$, which is compensated by the N^{-p} we factored out, and this concludes the proof. □

We can now discuss convergence at fixed times.

PROPOSITION 6.3.5. *The laws of a family of variables $\omega_{M,N} \sim \frac{1}{\sqrt{N}} \Xi_M^{\theta, N\lambda}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in on $H^{-1-\delta}$ are tight, for any fixed $\delta > 0$. Moreover,*

- *the limit as $M \to \infty$ at fixed N, say $N = 1$, is the law of $\Xi_\infty^{\theta,\lambda}$;*
- *the limit as $N \to \infty$ at fixed M (any $M \in (0, \infty]$) is the law of $\sqrt{\frac{(1-e^{-2\theta M})\lambda}{2\theta}}\eta$;*

and if the variables converge almost surely, they do so also in $L^p(\Omega, H^{-1-\delta})$ for any $1 \leq p < \infty, \delta > 0$.

PROOF. The embedding $H^\alpha \hookrightarrow H^\beta$ is compact as soon as $\alpha > \beta$, and we know that the variables are uniformly bounded elements of $L^p(\Omega, H^{-1-\delta})$ for any $p \geq 1$ by (6.3.3), so by Čebyšëv's inequality their laws are tight.

Identification of limit laws is yet another consequence of (6.1.7): by Theorem 2 of [**93**] (an infinite-dimensional Lévy theorem) we only need to check that characteristic functions $\mathbb{E}\left[e^{\mathrm{i}\langle \omega_{M,N}, h \rangle}\right]$ converge to the ones of the announced limits for any $h \in H^{1+\delta}$. Since (6.1.7) is valid for all $M \in [0, \infty]$, the limit for $M \to \infty$ poses no problem. As for the limit $N \to \infty$, for any test function $h \in H^{1+\delta}$,

$$\begin{aligned}\mathbb{E}\left[\exp\left(\mathrm{i}\left\langle h, \omega_{M,N}\right\rangle\right)\right] &= e^{-N\lambda} \exp\left(N\lambda \int_{[0,M]\times\{\pm 1\}\times\mathbb{T}^2} \exp\left(\frac{\mathrm{i}\,\sigma}{\sqrt{N}} h(x) e^{-\theta t}\right) dx d\sigma dt\right) \\ &= e^{-N\lambda} \exp\left(N\lambda \int_0^M \frac{1}{N} \|h\|_2^2 e^{-2\theta t} dt + O_h\left(\frac{1}{N}\right)\right) \\ &\xrightarrow{N\to\infty} \exp\left(\frac{\lambda}{2\theta} \|h\|_2^2 (1 - e^{-2\theta M})\right),\end{aligned}$$

where in the second step we used the following elementary expansion: for $\phi \in C(\mathbb{T}^2)$,

$$\left|\frac{1}{2}\int_{\mathbb{T}^2}\left(\exp\left(\frac{\phi(x)}{\sqrt{N}}\right) + \exp\left(-\frac{\phi(x)}{\sqrt{N}}\right)\right) dx - 1 - \frac{\|\phi\|_2^2}{2N}\right| \leq \frac{\|\phi\|_4^4}{24N^2}. \tag{6.3.5}$$

Since $\mathbb{E}\left[\exp\left(\mathrm{i}\left\langle h, \eta\right\rangle\right)\right] = \exp(-\|h\|_2^2)$, this concludes the proof. □

The latter result provides compactness "in space" ("equi-boundedness"): in order to apply Corollary 6.3.2 and Theorem 6.3.3, we also need to obtain a control on the regularity "in time" ("equi-continuity"). We will obtain it by exploiting the equation satisfied by $\omega_{M,N}$, which we derived in Corollary 6.2.2, which allows us to prove the forthcoming estimate on increments.

Proposition 6.3.6. *Let $\omega_{M,N} : [0,T] \to H^{-1-\delta}$ be the stochastic process defined at the beginning of this Section. For any $\mathcal{F}_t$-stopping time $\tau \leq T$ (possibly constant), $r, \delta > 0$, there exists a constant $C_{\delta,T}$ independent of M, N, τ, r such that*

$$\mathbb{E}\left[\|\omega_{M,N,\tau+r} - \omega_{M,N,\tau}\|_{H^{-3-\delta}}^2\right] \leq C_{\delta,T} \cdot r. \tag{6.3.6}$$

Proof. In order to lighten notation, and since the final result must not depend on M, N, let us drop them when writing $\omega_{M,N,t} = \omega_t$. By its definition in 6.3.1 and Remark 6.1.7 we know that the process satisfies the integral equation

$$\langle f, \omega_{t+r}\rangle - \langle f, \omega_t\rangle = -\theta \int_t^{t+r} \langle f, \omega_s\rangle\, ds + \int_t^{t+r} \langle H_f, \omega_s \diamond \omega_s\rangle\, ds + \left\langle f, \frac{1}{\sqrt{N}}(\Sigma_{t+r}^{N\lambda} - \Sigma_t^{N\lambda})\right\rangle, \tag{6.3.7}$$

for any smooth $f \in C^\infty(\mathbb{T}^2)$. Since this equation holds $\mathbb{P}$-almost surely uniformly in $s, t \in [0,T]$, it is also true when we replace t with the stopping time τ. It is convenient to recall that

$$\|u\|_{H^{-3-\delta}}^2 = \sum_{k\in\mathbb{Z}^2} (1 + |k|^2)^{-3-\delta} |\hat{u}_k|^2,$$

so we can use the weak integral equation against the orthonormal functions e_k to control the full norm:

$$\mathbb{E}\left[\|\omega_{\tau+r} - \omega_\tau\|_{H^{-3-\delta}}^2\right] = \sum_{k\in\mathbb{Z}^2} (1 + |k|^2)^{-3-\delta} \mathbb{E}\left[|\langle \omega_{\tau+r} - \omega_\tau, e_k\rangle|^2\right]. \tag{6.3.8}$$

We estimate increments by bounding separately the terms in the equation, let us start from the linear one:

$$\mathbb{E}\left[\left|\int_\tau^{\tau+r} \langle f, \omega_s\rangle\, ds\right|^2\right] \leq r\mathbb{E}\left[\int_0^T |\langle f, \omega_s\rangle|^2\, ds\right] = r\int_0^T \mathbb{E}\left[|\langle f, \omega_s\rangle|^2\right] ds \leq CTr\,\|f\|_\infty^2, \tag{6.3.9}$$

where the last passage makes use of the uniform estimate (6.3.3). The nonlinearity is the harder one, and its singularity is the reason why we can not obtain space regularity beyond $H^{-3-\delta}$,

$$\mathbb{E}\left[\left|\int_\tau^{\tau+r}\langle H_f,\omega_s\diamond\omega_s\rangle\,ds\right|^2\right]\le r\int_0^T\mathbb{E}\left[|\langle H_f,\omega_s\diamond\omega_s\rangle|^2\right]ds \tag{6.3.10}$$

$$\le CTr\,\|H_f\|_\infty^2\le CTr\,\|f\|^2_{C^2(\mathbb{T}^2)}\,, \tag{6.3.11}$$

where the second passage uses (6.3.3), and the third is due to the fact that by Taylor expansion

$$|H_f(x,y)|=\frac{1}{2}\,|K(x,y)(\nabla f(x)-\nabla f(y))|\le C\frac{|\nabla f(x)-\nabla f(y)|}{d(x,y)}\le C\,\|f\|_{C^2(\mathbb{T}^2)}\,.$$

By (6.1.4), the martingale $(\langle f,N^{-1/2}(\Sigma^{N\lambda}_{t+r}-\Sigma^{N\lambda}_t)\rangle)_{t\in[0,T]}$ has constant quadratic variation $\lambda r\,\|f\|^2_{L^2}$, so Burkholder-Davis-Gundy inequality gives

(6.3.12)

$$\mathbb{E}\left[\left|\left\langle f,N^{-1/2}(\Sigma^{N\lambda}_{\tau+r}-\Sigma^{N\lambda}_\tau)\right\rangle\right|^2\right]\le\mathbb{E}\left[\sup_{t\in[0,T]}\left|\left\langle f,N^{-1/2}(\Sigma^{N\lambda}_{t+r}-\Sigma^{N\lambda}_t)\right\rangle\right|^2\right]\le C\lambda r\,\|f\|^2_{L^2}\,.$$

Applying estimates (6.3.9,6.3.10,6.3.12) to the functions e_k, from (6.3.7) and Cauchy-Schwarz inequality we get

$$\mathbb{E}\left[|\langle\omega_{\tau+r}-\omega_\tau,e_k\rangle|^2\right]\le C_{\theta,\lambda,T}\,r|k|^4,$$

so that (6.3.8) gives us

$$\mathbb{E}\left[\|\omega_{\tau+r}-\omega_\tau\|^2_{H^{-3-\delta}}\right]\le\sum_{k\in\mathbb{Z}^2}(1+|k|^2)^{-3-\delta}Cr\left(T+|k|^4T+\lambda\right)\le C_{\theta,\lambda,T,\delta}\,r,$$

which concludes the proof. □

PROPOSITION 6.3.7. *The laws of the processes $\omega_{M,N}:[0,T]\to H^{-1-\delta}$ are tight in*

$$L^q([0,T],H^{-1-\delta})\cap\mathcal{D}([0,T],H^{-3-\delta})$$

for any $\delta>0,1\le q<\infty$.

PROOF. Since $\omega_{M,N,t}\sim\frac{1}{\sqrt{N}}\Xi^{\theta,N\lambda}_{M+t}$, they are bounded in $L^p(\Omega,H^{-1-\delta})$ for any $\delta>0,1\le p<\infty$ uniformly in M,N,t as shown in Proposition 6.3.5, and as a consequence the processes $\omega_{M,N}$ are uniformly bounded in $L^p(\Omega\times[0,T],H^{-1-\delta})$, for any $\delta>0,1\le p<\infty$. Moreover, we have proved fixed-time tightness. We are thus left to prove Aldous' condition in $H^{-3-\delta}$ and to control a fractional Sobolev norm in time in order to apply Corollary 6.3.2 and Theorem 6.3.3, concluding the proof. As in the previous proof, we denote $\omega_{M,N,t}=\omega_t$.

We only need to apply the uniform bound on increments (6.3.6). Starting from the fractional Sobolev norm, we evaluate

$$\begin{aligned}\mathbb{E}\left[\|\omega\|_{W^{\alpha,1}([0,T],H^{-3-\delta})}\right]&=\mathbb{E}\left[\int_0^T\int_0^T\frac{\|\omega_t-\omega_s\|_{H^{-3-\delta}}}{|t-s|^{1+\alpha}}dtds\right]\\&\le\int_0^T\int_0^T\frac{\mathbb{E}\left[\|\omega_t-\omega_s\|_{H^{-3-\delta}}\right]}{|t-s|^{1+\alpha}}dtds\\&\le C\int_0^T\int_0^T|t-s|^{-1/2-\alpha},\end{aligned}$$

which converges as soon as $\alpha<1/2$. Aldous's condition follows from Čebyšëv's inequality: if τ is a stopping time for ω_t, then

$$\sup_{0\le r\le R}\mathbb{P}\left(\|\omega_{\tau+r}-\omega_\tau\|_{H^{-3-\delta}}\ge\varepsilon\right)\le\varepsilon^{-1}\sup_{0\le r\le R}\mathbb{E}\left[\|\omega_{\tau+r}-\omega_\tau\|_{H^{-3-\delta}}\right]\le C\varepsilon^{-1}R^{1/2},$$

where the right-hand side is smaller than $\varepsilon' > 0$ as soon as R, which we can choose, is small enough. □

Let us conclude this paragraph with a martingale central limit theorem concerning the driving noise of our approximant processes.

PROPOSITION 6.3.8. *Let $(\Pi_t^N)_{t\in[0,T],N\in\mathbb{N}}$ be a sequence of $H^{-1-\delta}$-valued martingale with laws $\Pi^N \sim \frac{1}{\sqrt{N}}\Sigma^{N\lambda}$ (fix $\delta > 0$). The laws of Π^N are tight in*

$$L^q([0,T], H^{-1-\delta}) \cap \mathcal{D}([0,T], H^{-1-\delta}) \tag{6.3.13}$$

for any $\delta > 0, 1 \leq q < \infty$, and limit points have the law of the Wiener process $\sqrt{\lambda}W_t$ on $H^{-1-\delta}$ with covariance

$$\mathbb{E}\left[\langle W_t, f\rangle, \langle W_s, g\rangle\right] = t \wedge s \langle f, g\rangle_{L^2(\mathbb{T}^2)}.$$

PROOF. By (6.3.12) we readily get

$$\mathbb{E}\left[\left\|\Pi_{\tau+r}^N - \Pi_\tau^N\right\|_{H^{-1-\delta}}^2\right] \leq C_{\delta,\lambda} r$$

for any $N \in \mathbb{N}, \delta, r > 0$ and any τ stopping time for Π^N, uniformly in N. The very same argument of the last proposition (here with a better space regularity) proves then the claimed tightness. The martingale property (with respect to the processes own filtrations) carries on to limit points since it can be expressed by means of the following integral formulation: for any $s, t \in [0,T]$,

$$\mathbb{E}\left[(\Pi_t^N - \Pi_s^N)\Phi(\Pi^N |_{[0,s]})\right] = 0$$

for all the real bounded measurable functions Φ on $(H^{-1-\delta})^{[0,s]}$. Limit points are Gaussian processes, since at any fixed time

$$\frac{1}{\sqrt{N}}\Sigma_t^{N\lambda} \sim \frac{1}{\sqrt{N}}\Xi_t^{\theta=0,N\lambda} \xrightarrow{N\to\infty} \sqrt{\lambda t}\eta \sim \sqrt{\lambda}W_t,$$

as one can show by repeating the computations on characteristic functions in Proposition 6.3.5 with $\theta = 0, M = t$. It now suffices to recall the covariance formulas (6.1.1) and (6.1.3),

$$\mathbb{E}\left[\left\langle \frac{1}{\sqrt{N}}\Sigma_t^{N\lambda}, f\right\rangle \left\langle \frac{1}{\sqrt{N}}\Sigma_s^{N\lambda}, g\right\rangle\right] = \lambda(t\wedge s)\langle f, g\rangle_{L^2}^2 = \mathbb{E}\left[\left\langle \sqrt{\lambda}W_t, f\right\rangle, \left\langle \sqrt{\lambda}W_s, g\right\rangle\right],$$

to conclude that any limit point has the law of $\sqrt{\lambda}W$. □

6.3.3. Identifying Limits. The last step is to prove that limit points of the family of processes $\omega_{M,N}$ satisfy Definition 6.1.6. First, let us recall once again our setup for the sake of clarity:

- $\lambda, \theta > 0$ are fixed throughout;
- there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the stochastic processes $\Sigma_t^{N\lambda}$ and the random variables $\Xi_M^{\theta,N\lambda}$ are defined, for $M \geq 0, N \in \mathbb{N}$, their laws being as in Section 6.1;
- the processes $(\omega_{M,N,t})_{t\in[0,T]}$ are defined as at the beginning of this section: strong solutions of (6.3.1) with initial datum $\frac{1}{\sqrt{N}}\Xi_M^{\theta,N\lambda}$ and driving noise $\frac{1}{\sqrt{N}}\Sigma_t^{N\lambda}$, built as in Corollary 6.2.2.

To fix notation, let us consider separately the following three cases: by Proposition 6.3.7, we can consider converging sequences

(Ps) $(\omega_{M_n,N=1})_{n\in\mathbb{N}}$, with $M_n \to \infty$ as $n \to \infty$, the limit being ω_t^P;
(G) $(\omega_{M,N_n})_{n\in\mathbb{N}}$, with $N_n \to \infty$ as $n \to \infty$ and fixed $M < \infty$, the limit being $\omega_{M,t}^G$;
(Gs) $(\omega_{M_n,N_n})_{n\in\mathbb{N}}$, with $M_n, N_n \to \infty$ as $n \to \infty$, the limit being ω_t^G;

the convergence in law takes place in $L^q([0,T],H^{-1-\delta})\cap\mathcal{D}([0,T],H^{-3-\delta})$, for any fixed $\delta>0, 1\le q<\infty$. By Proposition 6.3.5, the Poissonian limit (Ps) has marginals $\omega_t^P\sim\Xi_\infty^{\theta,\lambda}$, and the Gaussian ones $\omega_{M,t}^G\sim\sqrt{\frac{\lambda}{2\theta}(1-e^{-2\theta(M+t)})}\eta$ for all $t\in[0,T]$, $M\in[0,\infty)$, and $\omega_t^G\sim\sqrt{\frac{\lambda}{2\theta}}\eta$ (the labels are given so to match the ones in Definition 6.1.6). Notice that $(\omega_m^P)_{m\in\mathbb{N}}$ have all the same driving noise Σ_t^λ, but different initial data, while in the Gaussian limiting sequences the driving noise also varies. Let us show that the limit laws in the cases where $M\to\infty$ are stationary.

Proposition 6.3.9. *The processes ω_t^P and ω_t^G are stationary.*

Proof. As the intuition suggests, the key is the fact that M is a time-like parameter, and taking $M\to\infty$ corresponds to the infinite time limit. Formally, we observe that for all $r>0$, $0\le t_1\le\cdots\le t_k<\infty$, and M,N,

$$(\omega_{M,N,t_1+r},\dots,\omega_{M,N,t_k+r})\sim(\omega_{M+r,N,t_1},\dots,\omega_{M+r,N,t_k}). \tag{6.3.14}$$

Indeed, by construction (see Section 6.2), for all $s<t$, $\omega_{M,N,t}$ is given as a measurable function of $\omega_{M,N,s}$ and the driving noise,

$$\omega_{M,N,t}=F_{s,t}(\omega_{M,N,s},\Sigma^{N\lambda}\,|_{[s,t]}) \tag{6.3.15}$$

this, combined with the fact that $\omega_{M,N,t}\sim\omega_{M+t,N,0}$ and the invariance of $\Sigma^{N\lambda}$ by time shifts proves (6.3.14). Passing (6.3.14) to the limits (Ps) and (Gs) concludes the proof, since the dependence on r of the right-hand side disappears. □

Remark 6.3.10. Equation (6.3.15) is equivalent to the Markov property, *cf.* the beginning of the proof to Corollary 6.2.2. Equation (6.3.14) is the time omogeneity property. The Markov property is a consequence of uniqueness for the system (6.2.2), (6.2.3). Since uniqueness result in cases (Ps), (G) and (Gs) of Definition 6.1.6 seem to be out of reach by now, we can not hope to derive the Markov property as well.

We are only left to show that our limits do produce the sought solutions of Theorem 6.1.9. First, we apply Skorokhod's theorem to obtain almost sure convergence.

Proposition 6.3.11. *There exist stochastic processes $(\tilde\omega_n^P)_{n\in\mathbb{N}},\tilde\Sigma_t^\lambda$, defined on a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, such that their joint distribution coincides with the one of the original objects and with $\tilde\omega_m^P$ converging to a limit $\tilde\omega^P$ almost surely in $L^q([0,T],H^{-1-\delta})\cap\mathcal{D}([0,T],H^{-3-\delta})$ for any fixed $\delta>0,1\le q<\infty$.*

Analogously, there exist $(\tilde\omega_{M,n}^G,\tilde\omega_n^G,\tilde\Sigma_t^{N_n\lambda})_{n\in\mathbb{N}}$, defined on $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, such that their joint distribution coincides with the one of the original objects and with $\tilde\omega_{M,n}^G,\tilde\omega_n^G$ converging respectively to limits $\tilde\omega_M^G,\tilde\omega^G$ almost surely in $L^q([0,T],H^{-1-\delta})\cap\mathcal{D}([0,T],H^{-3-\delta})$ for any fixed $\delta>0,1\le q<\infty$.

The proof is a straightforward application of the following version of Skorokhod's theorem, which we borrow from [**140**] (see references therein). The required tightness is provided by Proposition 6.3.7 and Proposition 6.3.8.

Theorem 6.3.12 (Skorokhod Representation). *Let $X_1\times X_2$ be the product of two Polish spaces, $\chi^n=(\chi_n^1,\chi_n^2)$ be a sequence of $X_1\times X_2$-valued random variables, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, converging in law and such that χ_n^1 have all the same law ρ. Then there exist a sequence $\tilde\chi^n=(\tilde\chi_n^1,\tilde\chi_n^2)$ of $X_1\times X_2$-valued random variables, defined on a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, such that*

- *χ^n and $\tilde\chi^n$ have the same law for all n;*
- *$\tilde\chi^n$ converge almost surely to a $X_1\times X_2$-valued random variable $\tilde\chi=(\tilde\chi_1^n,\tilde\chi_2^n)$ on $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$;*

- *the variable* $\tilde{\chi}_1^n$ *and* $\tilde{\chi}_1$ *coincide almost surely.*

Proof of Proposition 6.3.11. In the case (P) we apply the above result with $X_1 = X_2 = X = L^q([0,T], H^{-1-\delta}) \cap \mathcal{D}([0,T], H^{-3-\delta})$ and $\chi_1^m = \Sigma_t^\lambda, \chi_2^m = \omega_m^P$, while for the case (G) we take $X_1 = \{0\}$ and $X_2 = X \times X$, with $\chi_2^n = (\omega_n^G, \Sigma_t^{N_n\lambda})$. □

The new processes still are weak solutions of (6.3.1) in the sense of Definition 6.1.6. Consider for instance the $\tilde{\omega}_n^G$ (the other case being identical): clearly their trajectories have the same regularity as ω_n^G, and they have the same fixed time distributions. As for the equation, it holds, for any $f \in C^\infty(\mathbb{T}^2)$ and $t \in [0,T]$, $\mathbb{P}$-almost surely,

$$\langle f, \tilde{\omega}_{n,t}^G \rangle - \langle f, \tilde{\omega}_{n,0}^G \rangle + \theta \int_0^t \langle f, \tilde{\omega}_{n,s}^G \rangle \, ds - \int_0^t \langle H_f, \tilde{\omega}_{n,s}^G \diamond \tilde{\omega}_{n,s}^G \rangle \, ds - \left\langle f, \frac{1}{\sqrt{N_n}} \Sigma_t^{N_n\lambda} \right\rangle = 0,$$

since taking the expectation of the absolute value (capped by 1) of the right-hand side gives a functional of the law of $\tilde{\omega}_n^G, \tilde{\Sigma}_t^{N_n\lambda}$, which is the same of the original ones. Moreover, since all the terms in the last equation are *cadlag* functions in time (in fact they are all continuous but the noise term), one can choose the $\tilde{\mathbb{P}}$-full set on which the equation holds uniformly in $t \in [0,T]$.

Remark 6.3.13. In fact, one can prove more. Following the proof of Lemma 28 in [**71**], it is possible to show that the new Skorokhod process have in fact the same point vortices structure of $\omega_{M,N}$, namely it is possible to represent $\tilde{\omega}_{m,t}^P$ and $\tilde{\omega}_{M,n,t}^G, \tilde{\omega}_{n,t}^G$ as sums of vortices satisfying equations (6.2.2) and (6.2.3) of Section 6.2. The argument would be quite long, and we feel that it would not add much to our discussion, so we refrain to go into details, contenting us with our analytically weak notion of solution.

To ease notation, from now on we will drop all tilde symbols, implying that we are going to work only with the new processes and noise terms. We are finally ready to pass to the limit the stochastic equations satisfied by our approximating processes, thus concluding the proof of our main result.

Proof of Theorem 6.1.9. The limits of ω_n^P, $\omega_{M,n}^G$ and ω_n^G provide respectively the sought solutions in the cases (Ps), (G) and (Gs) of Definition 6.1.6. We focus again our attention on ω_n^G, case (Gs), the other ones being analogous.

Since ω_n^G converges almost surely in the spaces (6.3.13), we immediately deduce that, for any $f \in C^\infty(\mathbb{T}^2)$ and $t \in [0,T]$, $\mathbb{P}$-almost surely,

$$\langle f, \omega_{n,t}^G \rangle \to \langle f, \omega_t^G \rangle, \tag{6.3.16}$$

$$\int_0^t \langle f, \omega_{n,s}^G \rangle \, ds \to \int_0^t \langle f, \omega_s^G \rangle \, ds. \tag{6.3.17}$$

The nonlinear term is only slightly more difficult. Let $H_k \in C^\infty(\mathbb{T}^{2\times 2})$, $k \in \mathbb{N}$, be symmetric functions vanishing on the diagonal converging to H_f as $k \to \infty$ (it is yet another equivalent of the approximation procedure (6.1.10)). Then

$$\langle H_k, \omega_{n,t}^G \diamond \omega_{n,t}^G \rangle = \langle H_k, \omega_{n,t}^G \otimes \omega_{n,t}^G \rangle \to \langle H_k, \omega_t^G \otimes \omega_t^G \rangle = \langle H_k, \omega_t^G \diamond \omega_t^G \rangle$$

in $L^2(\Omega \times [0,T])$ (the last passage is due to (6.1.12)). Almost sure convergence of the noise terms is ensured by Proposition 6.3.11, and the limiting law has been determined in Proposition 6.3.8, hence, summing up, it holds $\mathbb{P}$-almost surely

$$\langle f, \omega_t^G \rangle - \langle f, \omega_0^G \rangle + \theta \int_0^t \langle f, \omega_s^G \rangle \, ds - \int_0^t \langle H_f, \omega_s^G \diamond \omega_s^G \rangle \, ds - \left\langle f, \sqrt{\lambda} W_t \right\rangle = 0.$$

As already noted above, quantifiers in $\mathbb{P}$ and $t \in [0, T]$ can be exchanged thanks to the fact that we are dealing with *cadlag* processes in time. Stationarity of ω_t^P and ω_t^G follows from Proposition 6.3.9. This concludes the proof of Theorem 6.1.9. $\square$

CHAPTER 7

Gaussian Invariant Measures of Barotropic Quasi-Geostrophic Equations

This Chapters review the arguments of [**96**], which we introduced in Chapter 1. It is structured as follows. In Section 7.1 we collect some preliminary material, including a short discussion on regularity regimes in which (BQG) are well-posed. In Section 7.2 we thoroughly discuss the formulation of weak solution required by our low-regularity setting, and finally in Section 7.3 we will prove Theorem 1.6.1 by approximating the infinite-dimensional stationary solution with finite-dimensional, stationary Galerkin truncations of (BQG).

7.1. Definitions and Preliminary Results

We consider mixed boundary conditions on R for the small scale stream function ψ', that is periodicity in the x variable and Dirichlet boundary at $y = 0, \pi$. In order to simplify Fourier analysis, let us extend the space domain to the 2-dimensional torus $D = [-\pi, \pi]^2$ with periodic boundary conditions on both x, y variables, extending ψ' to D so that it becomes an odd function of y. We still denote points $z = (x, y) \in D$.

To study (BQG) on D we also extend q, h in the same way; the extension of h might be discontinuous at $y = 0$, but this will not be relevant in the following. Indeed, it is not difficult to see that equations (BQG) preserve such condition. We also remark that the beta-plane term βy of (1.6.4) is coherent with the domain extension.

Due to the (skew-)symmetry in y variable, it will be convenient to introduce the following set of orthonormal functions of $L^2(D, \mathbb{C})$,

$$(e_j)_{j\in\mathbb{Z}}, (e_j s_k, e_j c_k)_{(j,k)\in\Lambda} \quad \Lambda = \{(j,k) : j \in \mathbb{Z}, k \in \mathbb{N} \setminus \{0\},\}$$
$$e_j(x) = \frac{1}{2\pi} e^{\mathrm{i}\, jx}, \;\; s_k(y) = \sin(ky), \;\; c_k(y) = \cos(ky).$$

Since we work with real valued objects, Fourier coefficients relative to modes (j, k) and $(-j, k)$ will always be complex conjugated. With this relation between Fourier coefficients, $\{e_j, e_j s_k, e_j c_k\}_{(j,k)\in\Lambda}$ is a Hilbert basis of $L^2 = L^2(D, \mathbb{R})$.

Odd functions of y only have non null Fourier coefficients relative to $(e_j s_k)_{(j,k)\in\Lambda}$: we will denote those coefficients, say of ψ', by

$$\mathcal{F}_{j,k}(\psi') = \hat{\psi}'_{j,k} = \int_D \psi'(x, y) e_{-j}(x) s_k(y) dx dy.$$

so that

$$\psi'(x, y) = \sum_{(j,k)\in\Lambda} \hat{\psi}'_{j,k} e_j(x) s_k(y), \quad \hat{\psi}'_{j,k} = \overline{\hat{\psi}'_{-j,k}}.$$

For $\alpha \in \mathbb{R}$, we denote by $H^\alpha = W^{\alpha,2}(D, \mathbb{R})$ the $L^2(D, \mathbb{R})$-based Sobolev spaces, which enjoy the compact embeddings $H^\alpha \hookrightarrow H^\beta$ whenever $\beta < \alpha$, the injections being furthermore Hilbert-Schmidt if $\alpha > \beta + 1$. The scale of Sobolev spaces of odd distributions in y,

$$\mathcal{H}^\alpha = \left\{ u = \sum_{(j,k)\in\Lambda} \hat{u}_{j,k} e_j s_k : \|u\|^2_{\mathcal{H}^\alpha} = \sum_{(j,k)\in\Lambda} |\hat{u}_{j,k}|^2 (j^2+k^2)^{2\alpha} < \infty \right\},$$

clearly share the same properties. We denote with $\mathcal{H}^0$ the subspace of odd functions of y in $L^2(D)$, and more generally each $\mathcal{H}^\alpha$ is a closed subspace of H^α. Brackets $\langle\cdot,\cdot\rangle$ will denote $\mathcal{H}^0$-based duality couplings.

As a convention, C will denote a positive constant, possibly changing in every occurrence even in the same formula and depending only on its eventual subscripts.

7.1.1. Well-posedness regimes. Our main aim is to give meaning to (BQG) in distributional regimes dictated by the formally invariant Gibbs measures. Before we undertake that task, we briefly discuss, for the sake of completeness, more regular regimes in which our equations are actually well-posed. Let us begin by introducing the notion of weak solution.

DEFINITION 7.1.1. *Given $(V_0, q_0) \in \mathbb{R}\times L^\infty(D)$, we say that*

$$(V(t), q(t))_{t\in[0,T]} \in L^\infty([0,T], \mathbb{R}\times D)$$

is a weak solution to (BQG) *with initial datum (V_0,q_0) if for any $\varphi\in C^1([0,T]\times D)$ it holds*

$$\int_D \varphi(T,z)q(T,z)dz - \int_D \varphi(0,z)q_0(z)dz = \int_0^T \int_D (\partial_t\varphi(s,z) + \nabla^\perp\psi(s,z)\cdot\nabla\varphi(s,z))q(s,z)dzds, \tag{7.1.1}$$

$$V(t) = V_0 + \int_0^t \fint_D h(z)\partial_x\psi'(z,s)dzds, \tag{7.1.2}$$

$$\psi = -Vy + \psi', \quad q = \Delta\psi' + h + \beta y. \tag{7.1.3}$$

Thanks to the fact that the equation for q is in the active scalar form, the method of characteristics produces an existence result: a minor modification of the proof of [**135**, Ch.2,Theorem 3.1] leads to the following:

PROPOSITION 7.1.2. *Let $(V_0,q_0)\in\mathbb{R}\times L^\infty(D)$, and consider the Lagrangian formulation of* (BQG) *given by*

$$\begin{cases} \frac{d}{dt}\phi_t(z) = \nabla^\perp\psi(t,\phi_t(z)) \\ \phi_0(z) = z \end{cases}, \quad q(t,z) = q_0(\phi_{-t}(z)), \tag{7.1.4}$$

together with equations (7.1.2),(7.1.3)*. There exists a unique solution (ϕ, V, q) of such system, and moreover (V,q) is a weak solution of* (BQG) *in the sense of Definition 7.1.1.*

The argument ultimately relies on the fact that $\nabla\nabla^\perp\Delta^{-1}$ is a singular kernel of Calderón-Zygmund type, so that its associated convolution operator is a bounded linear map from $L^\infty(D)$ to the Bounded Mean Oscillation (BMO) space. This implies that the vector field

$$\nabla^\perp\psi = V\begin{pmatrix}0\\1\end{pmatrix} + \nabla^\perp\Delta^{-1}(q-h-\beta y)$$

has gradient in BMO, and thus it is log-Lipschitz (*cfr.* [**135**, Ch.2,Lemma 3.1]). The vector field $\nabla^\perp\psi$ then satisfies the Osgood condition ([**144**]) for the associated Cauchy problem (7.1.4), which is thus well-posed; it is not difficult to check that $q(t,z) = q_0(\phi_{-t}(z))$ satisfies the weak formulation (7.1.1). All these ideas date back to the celebrated work of Judovič, [**105**], concerning well-posedness of Euler equations for initial vorticity in L^∞.

PROPOSITION 7.1.3. *For any* $(V_0, q_0) \in \mathbb{R}\times L^\infty(D)$*, the weak solution of* (BQG) *in the sense of Definition 7.1.1 is unique.*

Uniqueness can be obtained by energy estimates at the level of the velocity vector field $v = \nabla^\perp \psi$. Such estimates are performed for instance in [**128**, Theorem 8.2] for the 2D Euler equations ($h = 0$, $\beta = 0$), and again they rely on the fact that $\nabla\nabla^\perp\psi$ is in BMO to arrive at Gronwall-type inequalities, something which is not influenced by the addition of regular terms such as $h + \beta y$ to q. We refer to [**12**] for a thorough discussion of uniqueness for a large class of active scalar equations sharing similar features. We also mention the recent work [**43**], where the arguments we just sketched are applied to a barotropic quasi-geostrophic model closely related to ours: the difference consists in impermeable boundary conditions on the whole boundary and the presence of a free surface effect instead of the fixed topography h we consider. The paper [**44**], moreover, is devoted to multi-layered barotropic quasi-geostrophic equations.

7.1.2. Conserved Quantities and Gibbsian Measures. Smooth solutions of (BQG) preserve the first integrals energy and enstrophy,

$$E = \frac{1}{2}V^2 + \frac{1}{2}\fint \left|\nabla^\perp \psi'\right|^2, \quad Q = \beta V + \frac{1}{2}\fint (q-\beta y)^2.$$

We refer again to [**130**, Section 1.4] for a detailed discussion of conserved quantities. As already remarked, energy E can be seen as a functional of variables (V, q) by solving the Poisson equation (1.6.4).

In (1.6.7) above, we have formally introduced the Gibbsian measures

$$d\nu_{\alpha,\mu}(V,q) = \frac{1}{Z_{\alpha,\mu}} e^{-\alpha(\mu E + Q)} dV dq, \quad \alpha, \mu > 0,$$

the expression meaning that we consider the Gaussian measure whose inverse covariance operator is given by the quadratic functional $\alpha(\mu E + Q)$ of (V, q).

Let us now provide a rigorous framework: we define $\nu_{\alpha,\mu}$ as the joint law of the Gaussian variable $V \sim N\left(-\frac{\beta}{\mu}, \frac{1}{\alpha\mu}\right)$ and the Gaussian random field q indexed by $\mathcal{H}^0$ with mean and covariance given by, for $f, g \in \mathcal{H}^0$,

$$\begin{aligned}\mathbb{E}\left[\langle q, f\rangle\right] = \langle \bar q, f\rangle &= \left\langle \frac{\mu}{\mu - \Delta} h + \beta y, f \right\rangle \\ \mathbb{E}\left[\langle q, f\rangle \langle q, g\rangle\right] - \langle \bar q, f\rangle \langle \bar q, g\rangle &= \left\langle f, \frac{1}{\alpha(1-\mu\frac{1}{\Delta})} g \right\rangle,\end{aligned}$$

V and q being independent. Notice that α only plays a role in the variance. The link between the latter and the formal definition (1.6.7) is perhaps clearer thinking of the formal reference measure $dV dq$ as the infinite product of uniform measures on the infinite product space $\mathbb{R} \times \mathbb{C}^\Lambda$ of Fourier modes (modulo the relation $\hat q_{j,k} = \overline{\hat q_{-j,k}}$), and considering the Boltzmann exponent $e^{-\alpha(\mu E + Q)}$ as the infinite product of densities given by the Parseval expansion of the quadratic form $\alpha(\mu E + Q)$.

In order to deal with centred variables we set

$$U = V + \frac{\beta}{\mu}, \quad \omega = q - \bar q, \tag{7.1.5}$$

the new variables satisfying equations of motion

$$\begin{cases} \partial_t \omega + \nabla^\perp \Delta^{-1}\omega \cdot \nabla\omega + L\omega = 0 \\ \frac{dU}{dt} = \fint_D h \partial_x \Delta^{-1}\omega \end{cases}, \tag{7.1.6}$$

where $L\omega$ collects all affine terms in ω,

$$L\omega = \left(U - \frac{\beta}{\mu}\right)\partial_x\omega + U\frac{\mu\partial_x}{\mu-\Delta}h + \frac{\nabla^\perp}{\mu-\Delta}h\cdot\nabla\omega + \nabla^\perp\Delta^{-1}\omega\cdot\frac{\mu\nabla}{\mu-\Delta}h + \beta\partial_x\Delta^{-1}\omega.$$

The equivalence of (7.1.6) and (BQG) is intended for smooth solutions.

We now define the purely quadratic *pseudoenergy*: for $\mu>0$,

$$S_\mu(U,\omega) = \frac{\mu}{2}U^2 + \frac{1}{2}\int_D(\omega - \mu\Delta^{-1}\omega)\omega dxdy \tag{7.1.7}$$

so that the law of (V,ω) under $\nu_{\alpha,\mu}$ is given by

$$d\eta_{\alpha,\mu}(U,\omega) = \frac{1}{\tilde{Z}_{\alpha,\mu}}e^{-\alpha S_\mu(U,\omega)}dUd\omega, \tag{7.1.8}$$

the latter to be interpreted analogously to the definition of $\nu_{\alpha,\mu}$ above, (1.6.7): it is the joint law of the real Gaussian variable $U\sim N\left(0,\frac{1}{\alpha\mu}\right)$ and the independent centred Gaussian field ω with covariance operator $\alpha^{-1}(1-\mu\Delta^{-1})^{-1}$. In order to lighten the exposition, we will abuse notation denoting by $\eta_{\alpha,\mu}(d\omega)$ the law of ω under $\eta_{\alpha,\mu}$, and analogously for U. We will also denote

$$\sigma^2 := \int U^2 d\nu_{\alpha,\mu}(U,\omega) = \frac{1}{\alpha\mu}, \quad \sigma^2_{j,k} := \int |\hat{\omega}_{j,k}|^2 d\eta_{\alpha,\mu}(U,\omega) = \frac{j^2+k^2}{\alpha(\mu+j^2+k^2)}.$$

Indeed, under $\eta_{\alpha,\mu}$ the Fourier modes $\hat{\omega}_{j,k}$ are independent centred Gaussian variables with the above covariances; notice that they are complex valued, but subject to the condition $\bar{\hat{\omega}}_{j,k} = \hat{\omega}_{-j,k}$.

We have considered ω under $\eta_{\alpha,\mu}$ as a Gaussian random field indexed by $\mathcal{H}^0$ (its *reproducing kernel Hilbert space*): it is well known that it can also be identified with a random distribution in a larger Hilbert space into which $\mathcal{H}^0$ has an Hilbert-Schmidt embedding, such as $\mathcal{H}^{-1-\delta}$ for any $\delta>0$. In other terms, since all $\sigma^2_{j,k}$, (j,k) varying in Λ, are of order 1, the random Fourier series $\omega = \sum_{(j,k)\in\Lambda}\hat{\omega}_{j,k}e_js_k$ converges in $L^2(\eta_{\alpha,\mu})$ in $\mathcal{H}^{-1-\delta}$ for any $\delta>0$, but not for $\delta\geq 0$.

Lemma 7.1.4. *For any $\delta>0$, $(\mathbb{R}\times\mathcal{H}^{-1-\delta},\mathbb{R}\times\mathcal{H}^0,\eta_{\alpha,\mu})$ is a (complex) abstract Wiener space; equivalently, under $\eta_{\alpha,\mu}$, ω can be identified with a $\mathcal{H}^{-1-\delta}$-valued Gaussian random variable.*

7.2. Weak Solutions for Low-Regularity Marginals

We now discuss how to interpret (7.1.6) in the case when, at a fixed time, (U,ω) is a sample of $\eta_{\alpha,\mu}$. Indeed, as we remarked above, in that case ω can be identified at best as a distribution in $\mathcal{H}^{-1-\delta}$, $\delta>0$, and thus the main concern is the nonlinear term of the evolution equation, the affine term $L\omega$ being easily defined pathwise as a distribution of class $\mathcal{H}^{-2-\delta}$.

7.2.1. Fourier Expansion of the Nonlinear Term. Let us fix $\delta>0$, and consider the coupling between the nonlinear term $\nabla^\perp\Delta^{-1}\omega\cdot\nabla\omega$ and a smooth test function $\phi\in C^\infty(D)$. If $\omega\in\mathcal{H}^{-1-\delta}$, we can define the tensor product $\omega\otimes\omega$ as a distribution on $D\times D$ via

$$\langle\omega\otimes\omega,\varphi\otimes\psi\rangle := \langle\omega,\varphi\rangle\langle\omega,\psi\rangle, \qquad \varphi,\psi\in C^\infty(D), \tag{7.2.1}$$

where $\varphi\otimes\psi(z,z') := \varphi(z)\psi(z')$; it is easily observed that the resulting distribution $\omega\otimes\omega$ is of class $\mathcal{H}^{-2-2\delta}(D\times D)$ (with $\mathcal{H}^\alpha(D\times D)$ we denote the closed subspace of $H^\alpha(D\times D)$ generated by vectors $e_j s_k\otimes e_{j'}s_{k'}$), so the expression

$$\int_{D\times D} H(z,z')\omega(dz)\omega(dz') = \langle \omega\otimes\omega, H\rangle \tag{7.2.2}$$

is well defined via duality for every $H\in\mathcal{H}^{2+2\delta}(D\times D)$. Now, given any smooth test function $\phi\in C^\infty(D)$, we look for a suitable function H_ϕ such that

$$\int_D \nabla^\perp\Delta^{-1}\omega(z)\cdot\nabla\omega(z)\phi(z)dz = \int_{D\times D} H_\phi(z,z')\omega(z)\omega(z')dzdz'. \tag{7.2.3}$$

We perform the computation in Fourier series. Let us also recall that we denote points of D by $z=(x,y), z'=(x',y')$. We thus have

$$\omega(z) = \sum_{(j,k)\in\Lambda}\hat\omega_{j,k}e_j(x)s_k(y),\quad \Delta^{-1}\omega(z) = -\sum_{(j,k)\in\Lambda}\frac{\hat\omega_{j,k}}{j^2+k^2}e_j(x)s_k(y),$$

$$\nabla\omega(z) = \sum_{(j,k)\in\Lambda}\begin{pmatrix}\mathrm{i}\,js_k(y)\\ kc_k(y)\end{pmatrix}\hat\omega_{j,k}e_j(x),$$

$$\nabla^\perp\Delta^{-1}\omega(z) = \sum_{(j,k)\in\Lambda}\begin{pmatrix}kc_k(y)\\ -\mathrm{i}\,js_k(y)\end{pmatrix}\frac{\hat\omega_{j,k}}{j^2+k^2}e_j(x),$$

$$\begin{aligned}\nabla^\perp\Delta^{-1}\omega(z)\cdot\nabla\omega(z) = \sum_{(j,k)\in\Lambda}\sum_{(j',k')\in\Lambda}&((jk'-j'k)s_{k+k'}(y)+(j'k+jk')s_{k-k'}(y))\\ &\times\frac{\hat\omega_{j,k}\hat\omega_{j',k'}}{2\,\mathrm{i}(j^2+k^2)}e_{j+j'}(x),\end{aligned}$$

so that equation (7.2.3) becomes

$$\begin{aligned}&\int_D \nabla^\perp\Delta^{-1}\omega(z)\cdot\nabla\omega(z)\phi(z)dz\\ &=\sum_{(j,k)\in\Lambda}\sum_{(j',k')\in\Lambda}\Big((jk'-j'k)\hat\phi_{-j-j',k+k'}+(j'k+jk')\hat\phi_{-j-j',k-k'}\Big)\frac{\hat\omega_{j,k}\hat\omega_{j',k'}}{2\,\mathrm{i}(j^2+k^2)}\\ &=\sum_{(j,k)\in\Lambda}\sum_{(j',k')\in\Lambda}\Big((jk'-j'k)\hat\phi_{-j-j',k+k'}+(j'k+jk')\hat\phi_{-j-j',k-k'}\Big)\\ &\quad\times\left(\frac{1}{j^2+k^2}-\frac{1}{j'^2+k'^2}\right)\frac{\hat\omega_{j,k}\hat\omega_{j',k'}}{4\,\mathrm{i}} = \sum_{\substack{(j,k)\in\Lambda\\(j',k')\in\Lambda}}\mathcal{F}_{-j,k}\mathcal{F}_{-j',k'}(H_\phi)\hat\omega_{j,k}\hat\omega_{j',k'},\end{aligned}$$

the second step consisting in a symmetrisation with respect to indices (j,k) and (j',k'). The last equality is the Fourier expansion of the right-hand side of (7.2.3), and becomes our definition of H_ϕ:

$$\begin{aligned}\mathcal{F}_{j,k}\mathcal{F}_{j',k'}H_\phi \coloneqq &\Big((j'k-jk')\hat\phi_{j+j',k+k'}-(j'k+jk')\hat\phi_{j+j',k-k'}\Big)\\ &\times\left(\frac{1}{j^2+k^2}-\frac{1}{j'^2+k'^2}\right)\frac{1}{4\,\mathrm{i}},\end{aligned}$$

where $\mathcal{F}_{j,k}\mathcal{F}_{j',k'}$ is an abbreviation for the more rigorous notation $\mathcal{F}_{j,k}\otimes\mathcal{F}_{j',k'}$, the Fourier projector on $e_js_k\otimes e_{j'}s_{k'}$. We also adopt the convention

$$\hat\phi_{j+j',k-k'} \coloneqq -\hat\phi_{j+j',k'-k}\quad\text{whenever}\quad k-k'<0.$$

So far, H_ϕ is defined only as a formal Fourier series: the forthcoming Lemma discusses the convergence of the latter, *i.e.* the regularity of H_ϕ.

Lemma 7.2.1. *For every $\phi\in\mathcal{H}^2$, $H_\phi\in\mathcal{H}^0(D\times D)$.*

PROOF. To ease notation, we denote $l = (j,k)$ and $l' = (j',k')$. We have that $H_\phi \in \mathcal{H}^0(D \times D)$ if and only if

$$\sum_{l,l' \in \Lambda} |\mathcal{F}_l \mathcal{F}_{l'}(H_\phi)|^2 < \infty. \tag{7.2.4}$$

The Fourier coefficients of H_ϕ are given by two summands which we estimate separately. The first one is

$$\mathcal{F}_{j+j',k+k'}(\phi)(j'k - jk')\left(\frac{1}{j^2+k^2} - \frac{1}{j'^2+k'^2}\right) = \mathcal{F}_{l+l'}(\phi)\left(-l^\perp \cdot l'\right)\left(\frac{1}{|l|^2} - \frac{1}{|l'|^2}\right),$$

where $|l|^2 = j^2 + k^2$, and similarly for l'; taking squares and summing over $l + l' = m \in \Lambda$ gives us

$$\sum_{m \in \Lambda} |\mathcal{F}_m(\phi)|^2 \sum_{\substack{l \in \Lambda \\ l \neq m}} \left(l^\perp \cdot (m-l)\left(\frac{1}{|l|^2} - \frac{1}{|m-l|^2}\right)\right)^2. \tag{7.2.5}$$

We now resort to the following inequalities:

$$l^\perp \cdot (m-l) = l^\perp \cdot m \leq |l||m|, \tag{7.2.6}$$

$$|m-l|^2 - |l|^2 = m \cdot (m-2l) \leq |m||m-2l| \leq |m|\left(|m-l|^2 + |l|^2\right)^{1/2}, \tag{7.2.7}$$

so that the inner summation in (7.2.5) can be estimated with

$$\begin{aligned}\sum_{\substack{l \in \Lambda \\ l \neq m}} \left(\frac{|l|^2|m|^4|m-l|^2}{|l|^4|m-l|^4} + \frac{|l|^4|m|^4}{|l|^4|m-l|^4}\right) &= |m|^4 \sum_{\substack{l \in \Lambda \\ l \neq m}} \left(\frac{1}{|l|^2|m-l|^2} + \frac{1}{|m-l|^4}\right) \\ &\leq 2|m|^4 \sum_{l \in \Lambda} \frac{1}{|l|^4}.\end{aligned}$$

Modulo a multiplicative constant, (7.2.5) is therefore smaller or equal to

$$\sum_{m \in \Lambda} |\mathcal{F}_m(\phi)|^2 |m|^4,$$

which is finite as soon as $\phi \in \mathcal{H}^2$. The other contribution is given by the terms of the form

$$\mathcal{F}_{j+j',k-k'}(\phi)(j'k + jk')\left(\frac{1}{j^2+k^2} - \frac{1}{j'^2+k'^2}\right),$$

which after the change of variables $(j,k,j',k') \mapsto (j,k,-j',k')$ becomes

$$\mathcal{F}_{l-l'}(\phi)\left(l^\perp \cdot l'\right)\left(\frac{1}{|l|^2} - \frac{1}{|l'|^2}\right),$$

which can be estimated in a similar fashion taking the modulo square and summing over $l - l' = m \in \Lambda$. Thus (7.2.4) is proved. ☐

REMARK 7.2.2. Even though we will not need it in the following, for every $\delta < 1$ the above computation actually yields $H_\phi \in \mathcal{H}^\delta(D \times D)$ if $\phi \in \mathcal{H}^{2+\delta}$. This in fact is the optimal Sobolev regularity, since in general $H_\phi \notin \mathcal{H}^\delta(D \times D)$ for $\delta \geq 1$, even for more regular ϕ. Indeed, for $\phi(x,y) = \sin(y)$ the Fourier coefficients of H_ϕ are given by

$$\mathcal{F}_{j,k}\mathcal{F}_{j',k'}(H_\phi) = \frac{\mathbf{1}_{\{j+j'=0\}}\mathbf{1}_{\{k-k'=1\}}}{4\,\mathrm{i}} \frac{j(1-2k)}{(j^2+k^2)(j^2+(k-1)^2)},$$

therefore $H_\phi \in \mathcal{H}^\delta(D \times D)$ if and only if

$$\sum_{(j,k) \in \Lambda} \left(1 + 2j^2 + k^2 + (k-1)^2\right)^\delta \frac{j^2(1-2k)^2}{(j^2+k^2)^2(j^2+(k-1)^2)^2} < \infty,$$

but the sum above can be estimated from below (modulo a positive multiplicative constant) by

$$\sum_{(j,k)\in\Lambda} \frac{j^2k^2}{(j^2+k^2)^{4-\delta}},$$

the latter converging if and only if $\delta < 1$.

Unfortunately, since H_ϕ does not belong to $\mathcal{H}^{2+2\delta}(D\times D)$, it is not possible to define the nonlinear term of (7.1.6) *pathwise*, that is fixing a realisation of ω under $\eta_{\alpha,\mu}$ and taking products of distributions. It is at this point that we make essential use of the probabilistic approach to invariant measures.

7.2.2. The Nonlinear Term as a Stochastic Integral. Thanks to the peculiar form of the fluid-dynamic nonlinearity, which in our setting is reflected by the coefficients of H_ϕ, when ω is sampled from the Gaussian measure $\eta_{\alpha,\mu}$ it is possible to define the nonlinear term as a (double) stochastic integral, that is, as an $L^2(\eta_{\alpha,\mu})$-limit of suitable approximations.

The following result finds analogues in [**8**, Lemma 1.3.2], see also [**7**], and in [**71**, Theorem 8] or the related [**58, 73, 94, 72**], all dealing with stationary solutions of 2-dimensional Euler equations.

PROPOSITION 7.2.3. *Let $H\in\mathcal{H}^0(D\times D)$ be a symmetric function. Consider functions $(H^n)_{n\in\mathbb{N}}\subset\mathcal{H}^{2+2\delta}(D\times D)$ such that, for every $(j,k),(j',k')\in\Lambda$,*

$$\lim_{n\to\infty}\sum_{(j,k)\in\Lambda}\mathcal{F}_{j,k}\mathcal{F}_{j,k}(H^n)\sigma_{j,k}^2=0,\quad \mathcal{F}_{j,k}\mathcal{F}_{j',k'}(H^n)=\mathcal{F}_{j',k'}\mathcal{F}_{j,k}(H^n), \tag{7.2.8}$$

and suppose that the sequence H^n approximates H in the following sense:

$$\lim_{n\to\infty}\sum_{\substack{(j,k)\in\Lambda\\(j'k')\in\Lambda}}\left(\mathcal{F}_{j,k}\mathcal{F}_{j',k'}(H^n-H)\right)^2\sigma_{j,k}^2\sigma_{j',k'}^2=0. \tag{7.2.9}$$

Under $\eta_{\alpha,\mu}$, the sequence of random variables $\langle\omega\otimes\omega,H^n\rangle$ defined by (7.2.1), converges in mean square. Moreover, the limit does not depend on the approximating sequence H^n.

PROOF. To ease notation we denote $l=(j,k)$ and $l'=(j',k')$. For any function $H\in\mathcal{H}^0(D\times D)$, we compute

$$\begin{aligned}\mathbb{E}\left[\langle\omega\otimes\omega,H\rangle^2\right]&=\mathbb{E}\left[\sum_{\substack{l,l'\in\Lambda\\m,m'\in\Lambda}}\mathcal{F}_l\mathcal{F}_{l'}(H)\mathcal{F}_m\mathcal{F}_{m'}(H)\overline{\hat\omega_l\hat\omega_{l'}\hat\omega_m\hat\omega_{m'}}\right]\\&=\sum_{\substack{l,l'\in\Lambda\\m,m'\in\Lambda}}\mathcal{F}_l\mathcal{F}_{l'}(H)\mathcal{F}_m\mathcal{F}_{m'}(H)\mathbb{E}\left[\overline{\hat\omega_l\hat\omega_{l'}\hat\omega_m\hat\omega_{m'}}\right].\end{aligned}$$

By Wick-Isserlis formula the expected value in the last summand is given by

$$\mathbb{E}\left[\overline{\hat\omega_l\hat\omega_{l'}\hat\omega_m\hat\omega_{m'}}\right]=\sigma_l^2\sigma_m^2\delta_{l,l'}\delta_{m,m'}+\sigma_l^2\sigma_{l'}^2\delta_{l,m}\delta_{l',m'}+\sigma_l^2\sigma_{l'}^2\delta_{l,m'}\delta_{l',m}.$$

Substituting and using the relations (7.2.8) one gets

$$\mathbb{E}\left[\langle\omega\otimes\omega,H\rangle^2\right]=\left(\sum_{l\in\Lambda}\mathcal{F}_l(H)\sigma_l^2\right)^2+2\sum_{l,l'\in\Lambda}\mathcal{F}_l\mathcal{F}_{l'}(H)^2\sigma_l^2\sigma_{l'}^2.$$

If conditions (7.2.8) and (7.2.9) hold, applying the latter equation to differences H^m-H^n we obtain that the sequence of random variables $\langle\omega\otimes\omega,H^n\rangle$ is a Cauchy sequence in $L^2(\Omega)$. The independence of limit from the sequence (H^n) follows from triangular inequality and (7.2.9). □

REMARK 7.2.4. In [71], conditions (7.2.8) and (7.2.9) are replaced by

$$H^n \text{ symmetric}, \quad \lim_{n\to\infty} \int H^n(z,z)dz = 0,$$
$$\lim_{n\to\infty} \int\int (H^n(z,z') - H(z,z'))^2 dz dz' = 0,$$

where integration is performed over the 2-dimensional torus. These conditions are simpler than ours since we deal with coloured noise $\eta_{\alpha,\mu}$ rather then space white noise.

Consider now a test function $\phi \in C^\infty(D)$: Proposition 7.2.3 allows us to define the nonlinearity (7.2.3) as the $L^2(\eta_{\alpha,\mu})$-limit of $\left\langle \omega\otimes\omega, H^n_\phi \right\rangle$ for any sequence H^n_ϕ approximating H_ϕ in the above sense (for instance, progressive truncations of Fourier series). To emphasize the peculiarity of its definition, we adopt a special notation for this object.

DEFINITION 7.2.5. *For any $H \in \mathcal{H}^0(D\times D)$, and H^n as in Proposition 7.2.3,*

$$\langle \omega \diamond \omega, H\rangle := L^2(\eta_{\alpha,\mu}) - \lim_{n\to\infty} \langle \omega\otimes\omega, H^n\rangle . \tag{7.2.10}$$

We chose a distinct symbol because if we consider a smooth H and confront the new object we define and coupling with tensor products (7.2.1), a straightforward computation reveals that

$$\langle \omega \diamond \omega, H\rangle = \langle \omega\otimes\omega, H\rangle - \sum_{(j,k)\in\Lambda} \mathcal{F}_{j,k}\mathcal{F}_{j,k}(H)\sigma^2_{j,k}.$$

Indeed, let H^n be the following approximation of H:

$$\mathcal{F}_{j,k}\mathcal{F}_{j',k'}(H^n) := \mathcal{F}_{j,k}\mathcal{F}_{j',k'}(H) - \frac{S}{n\sigma^2_{0,k}} \mathbf{1}_{\{j=j'=0,k=k'=1,\dots,n\}},$$

where $S := \sum_{(j,k)\in\Lambda} \mathcal{F}_{j,k}\mathcal{F}_{j,k}(H)\sigma^2_{j,k} < \infty$. Hence

$$\langle \omega \diamond \omega, H\rangle = \langle \omega\otimes\omega, H\rangle - \lim_{n\to\infty}\sum_{k=1}^n \hat\omega^2_{0,k}\frac{S}{n\sigma^2_{0,k}} = \langle \omega\otimes\omega, H\rangle - S.$$

as an equality between random variables in $L^2(\eta_{\alpha,\mu})$. Notice that the last summand in the latter expression diverges for a generic $H \in \mathcal{H}^0(D\times D)$, according to the fact that the coupling with tensor product $\omega\otimes\omega$ can not be defined in that case.

REMARK 7.2.6. The present paragraph takes its name because the coupling $\langle \omega\diamond\omega, H\rangle$ we defined in fact corresponds to the double Itō-Wiener integral of H with respect to the Gaussian measure $\eta_{\alpha,\mu}$.

We now extend Proposition 7.2.3 to manage stochastic processes, rather than just random variables.

PROPOSITION 7.2.7. *On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider a stochastic process $(\omega_t)_{t\in[0,T]}$ with trajectories in $C([0,T], \mathcal{H}^{-1-\delta})$ such that the law of ω_t is $\eta_{\alpha,\mu}(d\omega)$ for every $t\in[0,T]$. Let $(H^n_\phi)_{n\in\mathbb{N}} \subseteq \mathcal{H}^{2+2\delta}(D\times D)$ be an approximation of H_ϕ in the sense of Proposition 7.2.3. Then the sequence of processes $t\mapsto \left\langle \omega_t\otimes\omega_t, H^n_\phi\right\rangle$ converges in $L^2([0,T], L^2(\mathbb{P}))$. Moreover, the limit does not depend on the approximating functions H^n_ϕ.*

The proof is a direct consequence of Proposition 7.2.3 and stationarity of the process ω. We are now ready to give the definition of solution we mentioned in Theorem 1.6.1.

DEFINITION 7.2.8. *A stochastic process $(U_t,\omega_t)_{t\in[0,T]}$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, with trajectories in $C([0,T];\mathbb{R}\times\mathcal{H}^{-1-\delta})$, solves the reduced form (7.1.6) of (BQG) in the* weak vorticity formulation *if, for every test function $\phi\in C^\infty(D)$, $\mathbb{P}$-almost surely, for every $t\in[0,T]$,*

$$\langle\omega_t,\phi\rangle=\langle\omega_0,\phi\rangle+\int_0^t\langle\omega_s\diamond\omega_s,H_\phi\rangle\,ds+\int_0^t\langle L\omega_s,\phi\rangle\,ds, \tag{7.2.11}$$

$$U_t-U_0=\int_0^t\fint_D h\left(\partial_x\Delta^{-1}\omega_s+\frac{\partial_x}{\mu-\Delta}h\right)dzds, \tag{7.2.12}$$

where the process $s\mapsto\langle\omega_s\diamond\omega_s,H_\phi\rangle$ is defined by Proposition 7.2.7.

In the remainder of the paper we will focus on equations for centred variables (U,ω), and thus prove the following corresponding version of Theorem 1.6.1, from which the latter is straightforwardly recovered.

THEOREM 7.2.9. *Let $\beta\neq0$ and h as above. For any $\alpha,\mu>0$ there exists a stationary stochastic process $(U_t,\omega_t)_{t\in[0,T]}$ with trajectories in $C([0,T],\mathbb{R}\times\mathcal{H}^{-1-\delta})$ and fixed-time marginals $\eta_{\alpha,\mu}$, whose trajectories solve (7.1.6) in the weak vorticity formulation of Definition 7.2.8.*

7.3. A Galerkin Approximation Scheme

Let us define the finite-dimensional projection of $L^2(D)$ onto the finite set of modes $\Lambda_N=\left\{(j,k)\in\Lambda:j^2+k^2\leq N\right\}$,

$$\Pi_N:L^2(D)\ni f\mapsto\Pi_Nf:=\sum_{(j,k)\in\Lambda_N}\mathcal{F}_{j,k}(f)e_js_k\in\mathcal{H}_N, \tag{7.3.1}$$

where we can identify the finite dimensional codomain with

$$\mathcal{H}_N=\left\{\sum_{(j,k)\in\Lambda_N}\xi_{j,k}e_js_k\ :\ \xi_{j,k}=\overline{\xi_{-j,k}}\right\}\simeq\left\{\xi\in\mathbb{C}^{\Lambda_N}:\xi_{j,k}=\overline{\xi_{-j,k}}\right\}\simeq\mathbb{C}^{\tilde\Lambda_N},$$
$$\tilde\Lambda_N=\left\{(j,k)\in\Lambda:j\geq0,j^2+k^2\leq N\right\}.$$

7.3.1. Truncated Barotropic Quasi-Geostrophic Equations. Let $h^N=\Pi_Nh$: we consider the following truncated version of (7.1.6),

$$\begin{cases}\partial_t\omega^N+\Pi_N\left(\nabla^\perp\Delta^{-1}\omega^N\cdot\nabla\omega^N\right)+L_N\omega^N=0\\ \frac{dU^N}{dt}=\fint_D h^N\partial_x\Delta^{-1}\omega^N,\end{cases} \tag{7.3.2}$$

with $L_N\omega^N$ collecting affine terms in ω^N:

$$\begin{aligned}L_N\omega^N=&\left(U^N-\frac{\beta}{\mu}\right)\partial_x\omega^N+U^N\frac{\mu\partial_x}{\mu-\Delta}h^N+\Pi_N\left(\frac{\nabla^\perp}{\mu-\Delta}h^N\cdot\nabla\omega^N\right)\\&+\Pi_N\left(\nabla^\perp\Delta^{-1}\omega^N\cdot\frac{\mu\nabla}{\mu-\Delta}h^N\right)+\beta\partial_x\Delta^{-1}\omega^N.\end{aligned}$$

For the sake of simplicity, equations (7.3.2) can be rewritten in the compact form

$$\partial_t(U^N,\omega^N)=B^N(U^N,\omega^N), \tag{7.3.3}$$

where ω^N is the vector with components $(\hat\omega^N_{j,k})_{(j,k)\in\tilde\Lambda_N}$, and $B^N:\mathbb{R}\times\mathcal{H}_N\to\mathbb{R}\times\mathcal{H}_N$. Let us stress the fact that we can reduce ourselves to consider Fourier modes in $\tilde\Lambda_N$ thanks to $\hat\omega^N_{j,k}=\overline{\hat\omega^N_{-j,k}}$.

Galerkin approximants (7.3.2) are globally well-posed, and truncation is such that they preserve the following projection of $\eta_{\alpha,\mu}$,

$$\eta^N_{\alpha,\mu}=(\mathrm{Id}_\mathbb{R},\Pi_N)_\#\eta_{\alpha,\mu}.$$

In other words, under $\eta^N_{\alpha,\mu}$, U^N has the same Gaussian distribution of U under $\eta_{\alpha,\mu}$, while ω^N is the projection of ω under $\eta_{\alpha,\mu}$. More explicitly, we can define $\eta^N_{\alpha,\mu}$ by density with respect to the product Lebesgue measure on $\mathcal{H}_N \simeq \tilde{\Lambda}_N$,

$$
\begin{aligned}
d\eta^N_{\alpha,\mu}(U^N,\omega^N) = \frac{1}{Z^N_{\alpha,\mu}} e^{-\frac{\alpha\mu}{2}(U^N)^2} dU^N \\
\times \prod_{(j,k)\in\tilde{\Lambda}_N} \exp\left(-\frac{\alpha}{2}|\hat{\omega}^N_{j,k}|^2\left(1+\frac{\mu}{j^2+k^2}\right)\right) d\hat{\omega}^N_{j,k}.
\end{aligned}
$$

PROPOSITION 7.3.1. *For $\eta^N_{\alpha,\mu}$-almost every initial datum (U^N_0,ω^N_0), there exists a unique solution $(U^N_t,\omega^N_t) \in C^\infty([0,\infty),\mathbb{R}\times\mathcal{H}_N)$ to the ordinary differential equation (7.3.2). Moreover, the global flow preserves $\eta^N_{\alpha,\mu}$.*

PROOF. The components of vector field B^N are polynomials of $U^N, \hat{\omega}^N_{j,k}$, $(j,k)\in\tilde{\Lambda}_N$, and thus B^N and its derivatives have finite moments of all orders under $\eta^N_{\alpha,\mu}$. The thesis then follows from non-explosion results in [**51**, Section 3], as soon as we check that B^N has null divergence with respect to $\eta^N_{\alpha,\mu}$, *i.e.*

$$
\begin{aligned}
0 = \operatorname{div}_{\eta^N_{\alpha,\mu}} B^N = \partial_{U_N} B^N_{U_N} + \sum_{(j,k)\in\tilde{\Lambda}_N} \partial_{j,k} B^N_{j,k} \\
- \alpha\mu U_N B^N_{U_N} - \alpha \sum_{(j,k)\in\tilde{\Lambda}_N}\left(1+\frac{\mu}{j^2+k^2}\right)\hat{\omega}^N_{j,k} B^N_{j,k},
\end{aligned}
$$

subscripts denoting components (and derivatives) relative to U_N or $\hat{\omega}_{j,k}$. In fact, [**51**] treats the case of a standard Gaussian measure on $\mathbb{R}^n$, but their results are easily extended to our case. Showing that B^N is divergence-free with respect to $\eta^N_{\alpha,\mu}$ can be done by direct computation: the full computation in the case of completely periodic geometry can be found in [**129**, Section 6.2], to which we refer, the differences with our case being minimal. □

7.3.2. The Truncated Nonlinear Term. In the finite-dimensional Galerkin truncation (7.3.2) we can repeat the arguments of subsection 7.2.1 to cast couplings of the nonlinear term into a double integral formulation. For any $\phi \in C^\infty(D)$, expanding in Fourier series the equality

$$
\begin{aligned}
\left\langle \Pi_N\left(\nabla^\perp\Delta^{-1}\omega^N\cdot\nabla\omega^N\right),\phi\right\rangle_{\mathcal{H}^0} &= \left\langle \nabla^\perp\Delta^{-1}\omega^N\cdot\nabla\omega^N, \Pi_N\phi\right\rangle_{\mathcal{H}^0} \\
&= \left\langle \omega^N\otimes\omega^N, H^N_\phi\right\rangle_{\mathcal{H}^0(D\times D)},
\end{aligned}
$$

we deduce a Fourier expansion of H^N_ϕ,

$$
\begin{aligned}
\mathcal{F}_{j,k}\mathcal{F}_{j',k'} H^N_\phi = (j'k-jk')\hat{\phi}_{j+j',k+k'} \frac{\mathbf{1}_{\{(j+j',k+k')\in\Lambda_N\}}}{4\,\mathrm{i}}\left(\frac{1}{j^2+k^2}-\frac{1}{j'^2+k'^2}\right) \\
-(j'k+jk')\hat{\phi}_{j+j',k-k'} \frac{\mathbf{1}_{\{(j+j',k-k')\in\Lambda_N\}}}{4\,\mathrm{i}}\left(\frac{1}{j^2+k^2}-\frac{1}{j'^2+k'^2}\right),
\end{aligned}
$$

the computation being completely analogous to the one in subsection 7.2.1.

7.3.3. Compactness Results. The first step towards taking the limit of Galerkin approximants as $N\to\infty$ is to provide estimates from which we can deduce relative compactness of approximations.

We begin by reviewing a deterministic compactness criterion due to Simon, which allows us to control separately time and space regularity, in the spirit of Aubin-Lions compactness Lemma. We refer to [**156**] for the result and the required generalities on Banach-valued Sobolev spaces.

Proposition 7.3.2 (Simon). *Assume that*

- $X \hookrightarrow B \hookrightarrow Y$ *are Banach spaces such that the embedding* $X \hookrightarrow Y$ *is compact and there exists* $0 < \theta < 1$ *such that for all* $v \in X \cap Y$
$$\|v\|_B \le M \|v\|_X^{1-\theta} \|v\|_Y^{\theta};$$
- $s_0, s_1 \in \mathbb{R}$ *are such that* $s_\theta = (1-\theta)s_0 + \theta s_1 > 0$.

If $\mathcal{F} \subset W$ *is a bounded family in*
$$W = W^{s_0,r_0}([0,T],X) \cap W^{s_1,r_1}([0,T],Y)$$
with $r_0, r_1 \in [0,\infty]$*, and moreover*
$$s^* = s_\theta - \frac{1-\theta}{r_0} - \frac{\theta}{r_1} > 0,$$
then if $\mathcal{F}$ *is relatively compact in* $C([0,T],B)$.

Let us specialise this result to our framework. Take
$$X = \mathbb{R} \times \mathcal{H}^{-1-\delta/2}, \quad B = \mathbb{R} \times \mathcal{H}^{-1-\delta}, \quad Y = \mathbb{R} \times \mathcal{H}^{-3-\delta},$$
with $\delta > 0$: by Gagliardo-Niremberg estimates the interpolation inequality is satisfied with $\theta = \delta/2$. Let us take moreover $s_0 = 0$, $s_1 = 1$, $r_1 = 2$ and $r_0 = q \ge 1$; if we can take q large such that
$$s^* = \frac{\delta}{4} - \frac{2-\delta}{2q} > 0,$$
then the hypothesis are satisfied and obtain:

Corollary 7.3.3. *Let* $\delta > 0$. *If a family of functions*
$$\{v_n\} \subset L^q([0,T], \mathbb{R} \times \mathcal{H}^{-1-\delta/2}) \cap W^{1,2}([0,T], \mathbb{R} \times \mathcal{H}^{-3-\delta})$$
is bounded for any $q \ge 1$*, then it is relatively compact in* $C([0,T], \mathbb{R} \times \mathcal{H}^{-1-\delta})$.

As a consequence, if a sequence of stochastic processes $u^n : [0,T] \to \mathbb{R} \times \mathcal{H}^{-1-\delta}$, $n \in \mathbb{N}$*, defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is such that, for any* $q \ge 1$*, there exists a constant* $C_{T,\delta,q}$ *for which*
$$\sup_n \mathbb{E}\left[\|u^n(t)\|^p_{L^q([0,T],\mathbb{R}\times\mathcal{H}^{-1-\delta/2})} + \|u^n\|_{W^{1,2}([0,T],\mathcal{H}^{-3-\delta})}\right] \le C_{T,\delta,q}, \tag{7.3.4}$$
then the laws of u^n *on* $C([0,T], \mathbb{R} \times \mathcal{H}^{-1-\delta})$ *are tight.*

For the sake of completeness we remark that the second, probabilistic part of the latter statement follows from the deterministic one and a simple application of Chebyshev inequality.

We want to apply Corollary 7.3.3 to the sequence of finite dimensional Galerkin approximations we built in Proposition 7.3.1. To obtain the uniform bound (7.3.4), let us begin with the "space regularity" part: by stationarity of the process (U^N, ω^N) we can swap expectations and time integrals, so that
$$\begin{aligned}\mathbb{E}\left[\|U^N\|^p_{L^q([0,T])} + \|\omega^N\|^p_{L^q([0,T],\mathcal{H}^{-1-\delta/2})}\right] &\le T\mathbb{E}\left[|U^N|^p + \|\omega^N\|^p_{\mathcal{H}^{-1-\delta/2}}\right]\\ &\le T\int \left(|U|^p + \|\omega\|^p_{\mathcal{H}^{-1-\delta/2}}\right) d\eta_{\alpha,\mu}(dU,d\omega) \le C_{T,p,\alpha,\mu}.\end{aligned}$$
As for bounds on time regularity: starting with U^N, by the evolution equation
$$\begin{aligned}\|U^N\|^2_{W^{1,2}([0,T])} &= \|U^N\|^2_{L^2([0,T])} + \left\|\frac{dU^N}{dt}\right\|^2_{L^2([0,T])}\\ &\le \int_0^T \left(|U^N_t|^2 + \left|\fint_D h^N \partial_x \Delta^{-1}\omega^N_t\right|^2\right) dt,\end{aligned}$$

from which we deduce, using that ω^N has marginals $\eta^N_{\alpha,\mu}(d\omega^N)$ for every fixed time t, and that $h \in C^\infty(D)$,

$$\begin{aligned}\mathbb{E}\left[\|U^N\|^2_{W^{1,2}([0,T])}\right] &\leq C_{T,h}\left(1+\mathbb{E}\left[\|\omega^N\|^2_{\mathcal{H}^{-1-\delta}}\right]\right)\\ &\leq C_{T,h}\left(1+\mathbb{E}\left[\|\omega\|^2_{\mathcal{H}^{-1-\delta}}\right]\right) \leq C_{T,\alpha,\mu,h}.\end{aligned}$$

Let us now focus on time regularity of ω^N: we have

$$\begin{aligned}&\|\omega^N\|^2_{W^{1,2}([0,T],\mathcal{H}^{-3-\delta})} = \|\omega^N\|^2_{L^2([0,T],\mathcal{H}^{-3-\delta})} + \|\partial_t\omega^N\|^2_{L^2([0,T],\mathcal{H}^{-3-\delta})}\\ &\leq 2\int_0^T \left(\|\omega^N_t\|^2_{\mathcal{H}^{-3-\delta}} + \|\Pi_N\left(\nabla^\perp\Delta^{-1}\omega^N_t\cdot\nabla\omega^N_t\right)\|^2_{\mathcal{H}^{-3-\delta}} + \|L_N\omega^N_t\|^2_{\mathcal{H}^{-3-\delta}}\right)dt.\end{aligned}$$

The affine term is controlled at any fixed time t by

$$\begin{aligned}\mathbb{E}\left[\|L_N\omega^N_t\|^2_{\mathcal{H}^{-3-\delta}}\right] &\leq C_{\alpha,\mu,h}\left(1+\mathbb{E}\left[\|\omega^N\|^2_{\mathcal{H}^{-1-\delta}}\right]\right)\\ &\leq C_{\alpha,\mu,h}\left(1+\mathbb{E}\left[\|\omega\|^2_{\mathcal{H}^{-1-\delta}}\right]\right) \leq C_{\alpha,\mu,h}.\end{aligned}$$

The quadratic term is the one forcing us to consider a large Hilbert space such as $\mathcal{H}^{-3-\delta}$. As above, we denote $m=(j,k)\in\Lambda_N$. We set $\phi_m = e_j s_k$ and consider

$$\mathbb{E}\left[\langle\omega^N\otimes\omega^N, H^N_{\phi_m}\rangle^2\right] = \mathbb{E}\left[\left(\sum_{l,l'\in\Lambda_N}\mathcal{F}_l\mathcal{F}_{l'}(H_{\phi_m})\overline{\hat\omega^N_l\hat\omega^N_{l'}}\right)^2\right],$$

where, by the expansion we derived in subsection 7.3.2,

$$\mathcal{F}_l\mathcal{F}_{l'}(H^N_{\phi_m}) = -l^\perp\cdot l'\,\frac{\mathbf{1}_{\{l+l'=m\}}}{4\,\mathrm{i}}\left(\frac{1}{|l|^2}-\frac{1}{|l'|^2}\right) + l^\perp\cdot l'\,\frac{\mathbf{1}_{\{l-l'=m\}}}{4\,\mathrm{i}}\left(\frac{1}{|l|^2}-\frac{1}{|l'|^2}\right).$$

We can consider only the first contribution of the latter sum, since, up to a constant, we can bound the contribution of the sum with the contributions of the sole first term, similarly to what we did in the proof of Lemma 7.2.1. We obtain:

$$\begin{aligned}\mathbb{E}\left[\langle\omega^N\otimes\omega^N, H^N_{\phi_m}\rangle^2\right] \leq C &\sum_{\substack{l,h\in\Lambda_N\\ l,h\neq m}} l^\perp\cdot(m-l)\left(\frac{1}{|l|^2}-\frac{1}{|m-l|^2}\right)\\ &\times h^\perp\cdot(m-h)\left(\frac{1}{|h|^2}-\frac{1}{|m-h|^2}\right)\mathbb{E}\left[\overline{\hat\omega^N_l\hat\omega^N_{m-l}\hat\omega^N_h\hat\omega^N_{m-h}}\right].\end{aligned} \tag{7.3.5}$$

By Wick-Isserlis Formula the expected value on the right-hand side is given by

$$\begin{aligned}&\mathbb{E}\left[\overline{\hat\omega^N_l\hat\omega^N_{m-l}\hat\omega^N_h\hat\omega^N_{m-h}}\right]\\ &\qquad= \sigma_l^2\sigma_h^2\delta_{l,m-l}\delta_{h,m-h} + \sigma_l^2\sigma_{m-l}^2\delta_{l,h}\delta_{m-l,m-h} + \sigma_l^2\sigma_h^2\delta_{l,m-h}\delta_{m-l,h}\\ &\qquad= \sigma_l^2\sigma_h^2\delta_{l,m-l}\delta_{h,m-h} + \sigma_l^2\sigma_{m-l}^2\delta_{l,h} + \sigma_l^2\sigma_h^2\delta_{l,m-h}.\end{aligned} \tag{7.3.6}$$

Notice that if $l=m-l$ we have $l^\perp(m-l)=0$, hence the first summand in (7.3.6) does not play any role in the computation of (7.3.5). Moreover, it is easy to check that the second and third terms give the same contribution, since $l^\perp\cdot h = -h^\perp\cdot l$.

Therefore, applying inequalities (7.2.6),(7.2.7),

$$\begin{aligned}
\mathbb{E}\left[\left\langle \omega^N \otimes \omega^N, H^N_{\phi_m}\right\rangle^2\right] &\le C \sum_{\substack{l\in\Lambda_N\\ l\neq m}} \sigma_l^2\sigma_{m-l}^2 \left(l^\perp\cdot(m-l)\left(\frac{1}{|l|^2}-\frac{1}{|m-l|^2}\right)\right)^2\\
&\le C \sum_{\substack{l\in\Lambda_N\\ l\neq m}} \sigma_l^2\sigma_{m-l}^2 \left(\frac{|l|^2|m|^4|m-l|^2}{|l|^4|m-l|^4}+\frac{|l|^4|m|^4}{|l|^4|m-l|^4}\right)\\
&= C|m|^4 \sum_{\substack{l\in\Lambda_N\\ l\neq m}} \sigma_l^2\sigma_{m-l}^2 \left(\frac{1}{|l|^2|m-l|^2}+\frac{1}{|m-l|^4}\right)\\
&\le C|m|^4 \sum_{l\in\Lambda_N} \frac{\sigma_l^2\sigma_{m-l}^2}{|l|^4}.
\end{aligned}$$

Recall now the expression for σ_l^2:

$$\sigma_l^2 = \frac{|l|^2}{\alpha(\mu+|l|^2)},$$

which is smaller than α^{-1} for every l. Therefore $\sum_{l\in\Lambda_N}\frac{\sigma_l^2\sigma_{m-l}^2}{|l|^4}$ is bounded form above uniformly in $m\in\Lambda_N, N\in\mathbb{N}$. Hence

$$\begin{aligned}
\mathbb{E}\left[\|\Pi_N\left(\nabla^\perp\Delta^{-1}\omega^N\cdot\nabla\omega^N\right)\|^2_{\mathcal{H}^{-3-\delta}}\right] &\le C\sum_{m\in\Lambda_N}\frac{1}{(1+|m|^2)^{3+\delta}}\mathbb{E}\left[\left\langle \omega^N \otimes \omega^N, H^N_{\phi_m}\right\rangle^2\right]\\
&\le c_\delta \sum_{m\in\Lambda_N}\frac{|m|^4}{(1+|m|^2)^{3+\delta}} \le C_\delta,
\end{aligned}$$

where C_δ is a finite constant which does not depend on N. All in all, we arrive to

$$\mathbb{E}\left[\|\omega^N\|^2_{W^{1,2}([0,T],\mathcal{H}^{-3-\delta})}\right] \le C_{T,\alpha,\mu,h}\left(1+\mathbb{E}\left[\|\omega\|^2_{\mathcal{H}^{-1-\delta}}\right]\right).$$

The estimates made so far, combined with Corollary 7.3.3, lead us finally to:

LEMMA 7.3.4. *The laws $\Theta^N_{\alpha,\mu}$ of the sequence of processes $u^N=(U^N_t,\omega^N_t)_{t\in T}$ defined by Proposition 7.3.1 are tight on $C([0,T],\mathbb{R}\times\mathcal{H}^{-1-\delta})$.*

7.3.4. The Continuous Limit. By Prokhorov theorem there exists a subsequence of $\Theta^N_{\alpha,\mu}$ –with a slight abuse of notation we will denote it with the same symbol– weakly converging to a probability measure $\Theta_{\alpha,\mu}$ on $C([0,T],\mathbb{R}\times\mathcal{H}^{-1-\delta})$. By Skorokhod theorem, there exists a new probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and random variables $\tilde u^N$, $\tilde u$ with values in $C([0,T],\mathbb{R}\times\mathcal{H}^{-1-\delta})$ such that:

- the law of $\tilde u^N$ (resp. $\tilde u$) is $\Theta^N_{\alpha,\mu}$ (resp. $\Theta_{\alpha,\mu}$);
- $\tilde u^N$ converges to $\tilde u$ $\tilde{\mathbb{P}}$-almost surely.

In order to lighten notation, we will drop tilde superscripts in the following.

The aim of this final paragraph is to prove that the stochastic process u is a weak solution of (BQG) in the sense of Definition 7.2.8, thus concluding the proof of Theorem 7.2.9. First of all, we make the following fundamental observation.

LEMMA 7.3.5. *The Galerkin approximations $u^N=(U^N,\omega^N)$ solve (7.3.2) in the sense of Definition 7.2.8. More precisely, given any test function $\phi\in C^\infty(D)$,*

$$\left\langle\omega^N_t,\phi\right\rangle = \left\langle\omega^N_0,\phi\right\rangle + \int_0^t\left\langle\omega^N_s\otimes\omega^N_s,H_\phi\right\rangle ds + \int_0^t\left\langle L_N\omega^N_s,\phi\right\rangle ds, \tag{7.3.7}$$

PROOF. This follows from the discussion made in subsection 7.2.1. □

Proof of Theorem 7.2.9. All but the bilinear term in (7.3.7) converge almost surely because of the convergence of $\omega^N \to \omega$ in $C([0,T],\mathcal{H}^{-1-\delta})$ and continuity of duality coupling with ϕ. The almost sure convergence of U^N to U solving (7.2.12) follows similarly. Let us thus focus on convergence of the nonlinearity. For any given $\phi \in C^\infty(D)$ and $M \in \mathbb{N}$ it holds

$$\begin{aligned}\int_0^t \left\langle \omega_s^N \otimes \omega_s^N, H_\phi \right\rangle ds = &\int_0^t \left\langle \omega_s^N \otimes \omega_s^N, H_\phi - H_\phi^M \right\rangle ds \\ &+ \int_0^t \left\langle \omega_s^N \otimes \omega_s^N - \omega_s \otimes \omega_s, H_\phi^M \right\rangle ds \\ &+ \int_0^t \left\langle \omega_s \otimes \omega_s, H_\phi^M \right\rangle ds.\end{aligned}$$

For the first term on the right-hand side we have the following L^1 estimate:

$$\begin{aligned}&\mathbb{E}\left[\left| \left\langle \omega^N \otimes \omega^N, H_\phi - H_\phi^M \right\rangle \right| \right] \leq \mathbb{E}\left[\sum_{m\in\Lambda} |\hat{\phi}_m| \left| \left\langle \omega^N \otimes \omega^N, H_{\phi_m} - H_{\phi_m}^M \right\rangle \right| \right] \\ &\leq \left(\sum_{m\in\Lambda} |\hat{\phi}_m|^2 (1+|m|^2)^\beta \right)^{1/2} \left(\sum_{m\in\Lambda} \frac{\mathbb{E}\left[\left| \left\langle \omega^N \otimes \omega^N, H_{\phi_m} - H_{\phi_m}^M \right\rangle \right| \right]^2}{(1+|m|^2)^\beta} \right)^{1/2} \\ &\leq \left(\sum_{m\in\Lambda} |\hat{\phi}_m|^2 (1+|m|^2)^\beta \right)^{1/2} \left(\sum_{\substack{m\in\Lambda \\ m\notin\Lambda_M}} \frac{\mathbb{E}\left[\left\langle \omega^N \otimes \omega^N, H_{\phi_m} \right\rangle^2 \right]}{(1+|m|^2)^\beta} \right)^{1/2} \\ &\leq C\|\phi\|_{H^\beta} \left(\sum_{\substack{m\in\Lambda \\ m\notin\Lambda_M}} \frac{|m|^4}{(1+|m|^2)^\beta} \right)^{1/2} \to 0 \text{ as } M \to \infty \text{ for } \beta > 3.\end{aligned}$$

For the last term, Proposition 7.2.7 implies the convergence in $L^2([0,T],L^2(\Omega))$

$$\int_0^t \left\langle \omega_s \otimes \omega_s, H_\phi^M \right\rangle ds \to \int_0^t \langle \omega_s \diamond \omega_s, H_\phi \rangle \, ds$$

as long as we check that H_ϕ^M is an approximation of H_ϕ in the sense of Proposition 7.2.3. But this last property is easily implied by the definition of H_ϕ^M and Lemma 7.2.1. The second term in the right-hand side goes to zero as $N \to \infty$ for every fixed M, since $\omega^N \otimes \omega^N$ converges almost surely to $\omega \otimes \omega$ in $C([0,T],\mathcal{H}^{-2-2\delta}(D\times D))$, and H_ϕ^M belongs to $C^\infty(D\times D)$. Thus, up to subsequences, we have the almost sure convergence

$$\int_0^t \left\langle \omega_s^N \otimes \omega_s^N, H_\phi \right\rangle ds \to \int_0^t \langle \omega_s \diamond \omega_s, H_\phi \rangle \, ds.$$

Therefore, taking the almost sure limit in (7.3.7) we get

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \diamond \omega_s, H_\phi \rangle \, ds. + \int_0^t \langle L\omega_s, \phi \rangle \, ds. \qquad \square$$

CHAPTER 8

Gaussian Invariant Measures of 2-dimensional Stochastic Primitive Equations

This Chapter contains the results obtained in [**97**], which we outlined in Chapter 1, and it is structured as follows: in Section 8.1 we rigorously introduce a stochastic version of 2-dimensional Primitive Equations in terms of vorticity $\omega = \partial_z v$ and a Gaussian measure formally preserved by the dynamics. We then summarize how the theory of [**100**] applies and state a well-posedness result for martingale solutions in sufficiently hyperviscous cases. Finally, in Section 8.3 we collect details and computations completing the proof of our main results.

8.1. Vorticity Formulation and Conservation Laws

The model under consideration in the remainder of the paper is the following stochastic PDE in the space domain $D = [0, 2\pi]^2 \ni (x, z)$,

$$\begin{cases} \partial_t v + v\partial_x v + w\partial_z v + \partial_x p = -(-\Delta)^\theta v + \eta, \\ \partial_z p = 0, \\ \partial_x v + \partial_z w = 0. \end{cases} \tag{8.1.1}$$

Here, $v = v(t, x, z)$ is the horizontal velocity, $w = w(t, x, z)$ is the vertical velocity, $p = p(t, x)$ is the pressure. The parameter θ and the additive Gaussian noise η will be specified below. The unknown fields v, w are subject to the following boundary conditions:

$$\begin{cases} w = 0, & \text{if } z = 0, 2\pi, \\ v = 0, & \text{if } x = 0, 2\pi, \\ \partial_z v = 0, & \text{if } z = 0, 2\pi. \end{cases} \tag{8.1.2}$$

The first two lines impose *impermeability* of the boundary; the third one is called a *free boundary condition* for the surface and the bottom of D. Before moving on, we discuss another possible choice in the next paragraph.

8.1.1. On Physically Realistic Boundary Conditions. While free boundary conditions are suited to describe interfaces between fluids such as the ocean surface, they can not be used to model a solid boundary such as the ocean bottom. Instead, one should consider a no-slip boundary condition, leading to a different set of conditions:

$$\begin{cases} w = 0, & \text{if } z = 0, 2\pi, \\ v = 0, & \text{if } x = 0, 2\pi, \\ v = 0, & \text{if } z = 0, \\ \partial_z v = 0, & \text{if } z = 2\pi. \end{cases} \tag{8.1.3}$$

In other words, we are assuming that the full velocity field (v, w) vanishes on the bottom side.

We prefer the choice (8.1.2) since Laplace operator can be diagonalised on functions satisfying that set of boundary conditions. This is not true when we consider

Dirichlet boundary at the bottom, since the eigenvalue problem is overdetermined. In that case, Fourier analysis can still be carried through with an orthonormal basis differing from usual trigonometric functions, see [**32**, Section 6]. We also refer to [**87**] for further discussion on boundary condition, and conclude the paragraph observing that one can reduce conditions (8.1.3) to the ones (8.1.2).

Assume that (v, w) is a smooth solution of (8.1.1) on D satisfying (8.1.3), for simplicity in the case $\eta = 0$. Then, if we extend the solution to the doubled domain $\tilde{D} = [0, 2\pi] \times [-2\pi, 2\pi]$ so that v, w are odd functions in the z direction, we have obtained a solution of (8.1.1) on $\tilde{D}$ satisfying (8.1.2). The size and aspect ratio of the domain is in fact irrelevant in our discussion.

8.1.2. Vorticity Formulation. Let us first assume to be dealing with smooth solutions of (8.1.1), driven by a smooth deterministic η. The aim is to derive an equivalent formulation of the model in terms of the only scalar field vorticity $\omega = \partial_z v$, on which we will focus the remainder of our discussion.

First of all, let us notice that v must always have zero average in the z direction, since the incompressibility equation $\partial_x v + \partial_z w = 0$ and boundary conditions imply, for all $x \in [0, 2\pi]$:

$$\partial_x \int_0^{2\pi} v(x, z')dz' = \int_0^{2\pi} \partial_x v(x, z')dz' = -\int_0^{2\pi} \partial_z w(x, z')dz' = 0,$$

from which it follows

$$\int_0^{2\pi} v(x, z')dz' = \int_0^{2\pi} v(0, z')dz' = 0. \tag{8.1.4}$$

Because of this, the solution $A(v)$ of the linear problem

$$\begin{cases} -\partial_z^2 A(v)(x, z) = v(x, z), & (x, z) \in [0, 2\pi] \times (0, 2\pi), \\ A(v)(x, z) = 0, & z = 0, 2\pi, \end{cases}$$

is well defined for all v satisfying our hypothesis. Another property of solutions (v, w) holding independently of time is that w is a *diagnostic variable*, *i.e.* it is completely determined by v:

$$w(x, z) = w(x, 0) - \int_0^z \partial_x v(x, z')dz' = \partial_x \partial_z A(v)(x, z).$$

Neglecting for a moment boundary conditions, equations (8.1.1) can thus be rewritten in terms of only v, p by

$$\begin{cases} \partial_t v + v\partial_x v + \partial_x \partial_z A(v) \partial_z v + \partial_x p = -(-\Delta)^\theta v + \eta, \\ \partial_z p = 0. \end{cases}$$

The system is then further simplified by considering the equation for vorticity $\omega = \partial_z v$, which does not involve the pressure p:

$$\partial_t \omega + \nabla^\perp A(\omega) \cdot \nabla \omega = -(-\Delta)^\theta \omega + \partial_z \eta, \tag{8.1.5}$$

where $\nabla^\perp = (-\partial_z, \partial_x)$. Notice that v is completely determined by its partial derivative $\partial_z v$ and the zero average condition (8.1.4), so (8.1.5) is equivalent to (8.1.1). Let us also remark that $A(\omega)$ is well-defined since ω has zero average in the z direction, and that A –to be rigorously defined below as an operator on function spaces– commutes with derivatives.

Let us briefly discuss boundary conditions for ω. Conditions on v immediately prescribe $\omega(x, 0) = \omega(x, 2\pi) = 0$ for $x \in [0, 2\pi]$ and, moreover, since v is constant along the z direction at $x = 0, 2\pi$, we also have $\partial_z v(0, z) = \partial_z v(2\pi, z) = 0$ for all $z \in [0, 2\pi]$: overall ω must vanish on ∂D. The condition $w = 0$ on $z = 0, 2\pi$ is not as easy to translate into a condition for ω, but we will bypass the issue with our

Fourier series approach below. It is worth noticing, however, that it is because of the boundary condition on w that $A(\omega)$ is well defined.

REMARK 8.1.1. The relation between boundary conditions for (v, w) and ω is thoroughly discussed in [**32**] in the setting of subsection 8.1.1.

To conclude the paragraph, let us observe that thanks to the driving vector field $\nabla^\perp A(\omega)$ being Hamiltonian, smooth solutions of the hydrostatic Euler equation (1.7.2) in vorticity form,

$$\partial_t \omega + \nabla^\perp A(\omega) \cdot \nabla \omega = 0, \tag{8.1.6}$$

with $(v, w) = \nabla^\perp A(\omega)$ satisfying boundary conditions, preserve the quadratic observable $\int_D \omega^2 dxdz$. Quite remarkably, this feature is peculiar to the two-dimensional case, since the quantity ω does not seem to have a counterpart in higher dimensions.

8.1.3. Functional Analytic Setting. As we described above, we are not interested in regular solutions of (8.1.1), but rather to singular, distributional regimes. It is thus convenient to encode in Fourier series the boundary conditions, and then set up our results in distribution spaces defined by means of Fourier expansions.

The general Fourier series expansion of a smooth function ω on D such that $A(\omega)$ is well-defined and $(v, w) = \nabla^\perp A(\omega)$ satisfy boundary conditions (8.1.2) is

$$\omega(x, z) = \sum_{k \in \mathbb{N}_0^2} \hat{\omega}_k e_k(x, z), \quad e_k(x, z) = \frac{1}{\pi} \sin(k_1 x) \sin(k_2 z),$$

where the e_k's form an orthonormal set in $L^2(D)$, $\hat{\omega}_k$ are the Fourier coefficients of ω and $k = (k_1, k_2) \in \mathbb{N}_0^2 = (\mathbb{N} \setminus \{0\})^2$. We will denote

$$\mathcal{S} = \left\{ \omega = \sum_{k \in \mathbb{N}_0^2} \hat{\omega}_k e_k : \forall p \in \mathbb{R} \sum_{k \in \mathbb{N}_0^2} |k|^p |\hat{\omega}_k| < \infty \right\}.$$

Equivalently, $\mathcal{S}$ is the space of smooth functions ω on D belonging to the domain of A and such that $(v, w) = \nabla^\perp A(\omega)$ satisfies the boundary conditions (8.1.2). We then denote by $\mathcal{S}'$ its dual space, represented by Fourier series whose coefficients grow at most polynomially. Brackets $\langle \cdot, \cdot \rangle$ will denote duality couplings between functions and distributions

$$\langle f, g \rangle = \sum_{k \in \mathbb{N}_0^2} \hat{f}_k \hat{g}_k,$$

defined whenever the right-hand side converges. Let us also introduce, for $m \in \mathbb{N}$, the projection onto the linear space of functions generated by e_k with $|k|^2 = k_1^2 + k_2^2 \leq m^2$,

$$\pi_n : \mathcal{S}' \to \mathcal{S}, \quad \omega \mapsto \pi_m \omega = \sum_{\substack{k \in \mathbb{N}_0^2, \\ |k| \leq m}} \hat{\omega}_k e_k.$$

Following [**100**], we set up our analysis on the Banach spaces

$$\mathcal{F}L^{p,\alpha} = \left\{ \omega \in \mathcal{S}' : \|\omega\|_{\mathcal{F}L^{p,\alpha}}^p = \sum_{k \in \mathbb{N}_0^2} |k|^{\alpha p} |\hat{\omega}_k|^p < \infty \right\}, \quad \alpha \in \mathbb{R}, p \geq 1,$$

and their $p = \infty$ version with $\|\omega\|_{\mathcal{F}L^{\infty,\alpha}} = \sup_{k \in \mathbb{N}_0^2} |k|^\alpha |\hat{\omega}_k|$.

Moving to the Fourier expression of the dynamics (8.1.1), the crux is clearly the nonlinear term, whose Fourier expansion is given by

$$\nabla^\perp A(\omega) \cdot \nabla \omega = B(\omega) = \sum_{k \in \mathbb{N}_0^2} B_k(\omega) e_k, \quad B_k(\omega) = \sum_{h \in \mathbb{Z}_0^2} \hat{\omega}_h \hat{\omega}_{k-h} \frac{k \cdot h^\perp}{h_2^2}, \tag{8.1.7}$$

where $\mathbb{Z}_0^2 = (\mathbb{Z} \setminus \{0\})^2$ and, for $h = (h_1, h_2) \in \mathbb{Z}_0^2$, $\hat{\omega}_h = \operatorname{sign}(h_1 h_2)\hat{\omega}_{(|h_1|,|h_2|)}$.

With vorticity formulation at hand, the difficulty inherent to the nonlinear term is now apparent: looking at the z component of the divergence-less vector field $\nabla^\perp A(\omega)$, the loss of one ∂_x derivative is not compensated by the gain of one ∂_z derivative. Indeed, such unbalance marks the difference between (8.1.1) and 2-dimensional SNS, which is especially evident in the Fourier series expansion (8.1.7).

8.1.4. Gaussian Invariant Measures and Driving Noise. Referring to [**59**], we now introduce the stochastic analytic tools we will employ below.

Invariance of $S(\omega) = \frac{1}{2}\int_D \omega(x,z)^2 dxdz$ for (1.7.2) suggests that existence of an invariant *Gibbs measure* formally defined by

$$d\mu(\omega) = \frac{1}{Z} e^{-S(\omega)} d\omega. \tag{8.1.8}$$

Since S is quadratic, (8.1.8) can be understood as a Gaussian measure on $\mathcal{S}'$ with covariance operator Id, a multiple of *space white noise* on D. In other words, μ is the law of the centred Gaussian process χ indexed by $\mathcal{F}L^{2,0}$ with covariance

$$\mathbb{E}\left[\chi(f)\chi(g)\right] = \langle f, g\rangle, \quad f, g \in \mathcal{F}L^{2,0}.$$

Such μ can be interpreted as the law of a random distribution supported on all $\mathcal{F}L^{2,\alpha}$ with $\alpha < -1$, the spaces into which the reproducing kernel Hilbert space $\mathcal{F}L^{2,0}$ has Hilbert-Schmidt embedding. Although a fixed realisation of the random field χ is only a distribution, couplings $\langle f, \chi\rangle = \chi(f)$ for $f \in \mathcal{F}L^{2,0}$ are defined as random variables in $L^2(\mu)$ (Itō integrals).

Another equivalent formulation is in terms of infinite products: formally expanding S by Parseval formula, we can write

$$d\mu(\omega) = \prod_{k\in\mathbb{N}_0^2} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}|\hat{\omega}_k|^2} d\hat{\omega}_k\right),$$

that is, under μ the Fourier coefficients $\hat{\omega}_k$ are independent identically distributed standard Gaussian variables. As a consequence, for all $\alpha < 0$, μ is supported by $\mathcal{F}L^{\infty,\alpha}$.

Looking at the laws of Fourier components under μ it is also clear why under this measure equations (1.7.2) and (8.1.12) are *singular*: the series defining a single coefficient $B_k(\omega)$ of the vector field diverges almost surely under μ. On the other hand, the expected value under μ of each summand in the series defining $B_k(\omega)$ vanishes, which is a formal but suggestive argument supporting the invariance of μ. In fact, the argument becomes rigorous when considering Galerkin truncations of B, and we will make essential use of this in the following.

The *space-time* analogue of μ, which we will use to define the stochastic forcing for (8.1.1), can be defined in two equivalent ways. First, we can consider the centred Gaussian field ξ indexed by $L^2([0,T], \mathcal{F}L^{2,0})$, with $T \in [0,\infty]$, whose covariance is given by

$$\mathbb{E}\left[\xi(\Phi)\xi(\Phi')\right] = \langle \Phi, \Phi'\rangle_{L^2([0,T],\mathcal{F}L^{2,0})}.$$

When coupled with test functions of the form $\mathbf{1}_{[0,t]}(s)\phi(x,z)$, $\phi \in \mathcal{S}$, ξ can be regarded as the cylindrical Wiener process W_t on $\mathcal{F}L^{2,0}$:

$$\left\langle \xi, \mathbf{1}_{[0,t]}\phi\right\rangle = \langle W_t, \phi\rangle = \sum_{k\in\mathbb{N}_0^2} \hat{\phi}_k \beta_t^k,$$

the latter part being the usual Karhunen-Loève decomposition with $(\beta_t^k)_{k\in\mathbb{N}_0^2}$ independent standard Wiener processes.

For all $\theta > 0, \nu > 0$, the Gaussian measure μ is the unique, ergodic invariant measure of the infinite-dimensional Langevin's dynamics

$$\partial_t X = -\nu(-\Delta)^\theta X + \sqrt{2\nu}(-\Delta)^{\theta/2}\xi, \tag{8.1.9}$$

which can be interpreted, by means of Fourier decomposition, as the system of independent one-dimensional SDEs

$$d\hat{X}_k = -\nu|k|^{2\theta}\hat{X}_k dt + \sqrt{2\nu}|k|^\theta d\beta_t^k, \quad k \in \mathbb{N}_0^2.$$

For the sake of simplicity, and without loss of generality, we will set $\nu = 1$ in the following. Let us conveniently introduce a symbol for the Generator of the dynamics (8.1.9): first we define cylinder functionals on $\mathcal{S}'$ by

$$\mathcal{C} = \left\{F \in L^2(\mu) : F(\omega) = f(\hat{\omega}_{k_1}, \dots \hat{\omega}_{k_r}),\, f \in C^\infty(\mathbb{R}^r),\, k_1, \dots k_r \in \mathbb{N}_0^2, r \in \mathbb{N}\right\},$$

and for $F \in \mathcal{C}$ we denote

$$\mathcal{L}_\theta F(\omega) = \sum_{i=1}^r |k_i|^{2\theta}\left(-\hat{\omega}_{k_i}\partial_i f + \partial_i^2 f\right). \tag{8.1.10}$$

Let us also introduce the *carré du champ* of the diffusion operator $\mathcal{L}_\theta$: for $F, G \in \mathcal{C}$,

$$\mathcal{E}_\theta(F,G)(\omega) = \sum_{i=1}^n |k_i|^{2\theta}\partial_i f \partial_i g, \tag{8.1.11}$$

which satisfies the Gaussian integration by parts formula

$$\mathbb{E}_\mu\left[F\mathcal{L}_\theta G\right] = -\mathbb{E}_\mu\left[\mathcal{E}_\theta(F,G)\right].$$

The above arguments finally lead us to consider the combination of dynamics (1.7.2) and (8.1.9) as a SPDE preserving μ:

$$\partial_t\omega + \nabla^\perp A(\omega)\cdot\nabla\omega = -(-\Delta)^\theta\omega + \sqrt{2}(-\Delta)^{\theta/2}\xi. \tag{8.1.12}$$

As already noticed, the nonlinear part of the dynamics is not well defined for functions ω in the regularity regime dictated by μ, or rather, it can be given a rigorous meaning only by exploiting cancellations due to the structure of the stochastic equation as a whole.

Remark 8.1.2. In terms of v, the latter equation reads

$$\begin{cases}\partial_t v + v\partial_x v + \partial_x\partial_y A(v)\partial_y v + \partial_x p = -(-\Delta)^\theta v + \partial_z\sqrt{2}(-\Delta)^{\theta/2}\xi,\\ \partial_z p = 0.\end{cases}$$

The forcing term should have white noise regularity in x, and Brownian regularity in y, although the covariance structure is a nontrivial copula of the two.

Remark 8.1.3. Just as in the case of 2D Euler or stochastic Navier-Stokes equations, the invariant measure associated to enstrophy is not able to describe peculiar features of the fluid-dynamic model, such as turbulence phenomena. In fact, such measures are preserved by any flow of measure-preserving diffeomorphisms of the domain, among which the Euler flow is a very distinguished case. Energy ensembles should be in fact more relevant, but they are supported on quite larger distribution spaces.

8.2. Regularisation by Noise in Hyperviscous Regimes

In this section we outline how the solution theory of [**100**] (known as *Energy Solutions* theory in the context of stochastic Burgers and KPZ equations) applies to our model in a sufficiently hyperviscous regime. Computations differ from that work only by small details: we collect them in the last section for the sake of completeness, and in the present one we only recall the core ideas.

8.2.1. Controlled Processes and Martingale Solutions. We recall the notion of controlled process from [**100**].

DEFINITION 8.2.1. *For $\theta \geq 0$ and $T > 0$ we define the space $\mathcal{R}_{\theta,T}$ of stochastic processes with trajectories of class $C([0,T],\mathcal{S}')$ such that any $\omega \in \mathcal{R}_{\theta,T}$ satisfies:*

(1) ω is stationary and for any $t \in [0,T]$, $\omega_t \sim \eta$;
(2) there exists a stochastic process $\mathcal{A}$ with trajectories $C([0,T],\mathcal{S}')$ starting from $\mathcal{A}_0 = 0$ and with null quadratic variation such that, for any $\phi \in \mathcal{S}$,

$$\langle \phi, \omega_t \rangle - \langle \phi, \omega_0 \rangle + \int_0^t \left\langle (-\Delta)^\theta \phi, \omega_s \right\rangle ds - \langle \phi, \mathcal{A}_t \rangle = M_t(\phi)$$

is a martingale with respect to the filtration of ω, and it has quadratic variation $[M(\phi)]_t = 2t \left\| (-\Delta)^{\theta/2} \phi \right\|^2_{\mathcal{F}L^{2,0}}$;
(3) the reversed process $\tilde{\omega}_t = \omega_{T-t}$ satisfies condition (2) with $\tilde{\mathcal{A}}_t = -\mathcal{A}_{T-t}$.

Notice that in fact elements of $\mathcal{R}_{\theta,T}$ are the couples $(\omega, \mathcal{A})$. The forward and backward martingale equations defining the class $\mathcal{R}_{\theta,T}$ allow to obtain good *a priori* estimates for nonlinear functionals of controlled process, in a procedure by now commonly known as *Itō trick*, especially in literature related to regularisation by noise techniques, see [**70, 67, 16**].

In the next paragraph we detail how the Itō trick produces good estimates on Galerkin approximations of (8.1.12): the idea behind Definition 8.2.1 is to collect the features of those approximants allowing such estimates, to form a class of processes on which the nonlinear term of (8.1.12) is defined. In Section 8.3 we will prove the following:

LEMMA 8.2.2. *Let $\theta > 2$, $T > 0$ and $\omega \in \mathcal{R}_{\theta,T}$. Then for every $\zeta < -1$*

$$\lim_{m \to \infty} \int_0^t B(\pi_m \omega_s) ds$$

exists as a limit in $C([0,T],\mathcal{F}L^{\infty,\zeta})$. We denote by $\int_0^t B(\omega_s)ds$ the limiting process.

The latter lemma shows that the nonlinear functional $B(\omega)$ can be defined for $\omega \in \mathcal{R}_{\theta,T}$ as a distribution in both space *and* time. Let us observe that Fourier truncation π_m in Lemma 8.2.2 can in fact be replaced with a large class of mollifiers, the limit being independent of such choice: for the sake of keeping the exposition simple, we refrain from going into details.

We can now give a notion of martingale solution to (8.1.12).

DEFINITION 8.2.3. *Let $\theta > 2$, $T > 0$ and $\omega \in \mathcal{R}_{\theta,T}$. We say that ω is a* martingale solution *to (8.1.12) if it holds almost surely, for any $t \in [0,T]$,*

$$\mathcal{A}_t = \int_0^t B(\omega_s) ds.$$

The solution is pathwise unique *if, for any two controlled processes $\omega, \tilde{\omega} \in \mathcal{R}_{\theta,T}$ defined on the same probability space, satisfying conditions (2) and (3) of Definition 8.2.3 with the same martingales and with $\omega_0 = \tilde{\omega}_0$ almost surely, then almost surely, for all $t \in [0,T]$, $\omega_t = \tilde{\omega}_t$.*

The following is the main result of the paper: its proof will be given in Section 8.3.

THEOREM 8.2.4. *Let $T > 0$. For any $\theta > 2$ there exists a solution to (8.1.12) in the sense of Definition 8.2.3. Moreover, for $\theta > 3$ the solution is pathwise unique.*

8.2.2. Galerkin Approximation and the Itō Trick. Let us introduce approximating processes $(\omega^m)_{m\in\mathbb{N}}$ by their Fourier coefficients dynamics: for $k \in \mathbb{N}_0^2$,

$$d\hat{\omega}_k^m = B_k^m(\omega^m)dt - |k|^{2\theta}\hat{\omega}_k^m dt + \sqrt{2}|k|^\theta d\beta_t^k, \tag{8.2.1}$$

where $B^m(\omega) = \pi_m B(\pi_m \omega)$, and $\omega_0^m \sim \mu$. The vector field B^m satisfies

$$\begin{aligned}\operatorname{div}_\mu B^m(\omega) &= \operatorname{div}_\mu \sum_{\substack{k\in\mathbb{N}_0^2,\\ |k|\leq m}} \sum_{\substack{h\in\mathbb{Z}_0^2,\\ |h|\leq m}} \hat{\omega}_h \hat{\omega}_{k-h} \frac{k\cdot h^\perp}{h_2^2} e_k \\ &= \sum_{\substack{k\in\mathbb{N}_0^2,\\ |k|\leq m}} \sum_{\substack{h\in\mathbb{Z}_0^2,\\ |h|\leq m}} \left(\partial_{\hat{\omega}_k}(\hat{\omega}_h\hat{\omega}_{k-h}) - \hat{\omega}_h\hat{\omega}_{k-h}\hat{\omega}_k\right)\frac{k\cdot h^\perp}{h_2^2} = 0.\end{aligned} \tag{8.2.2}$$

As a consequence, (8.2.1) has a unique, (probabilistically) strong, global in time solution since μ is preserved by the linear part of the dynamics, and thus [**51**, Theorem 3.2] applies. In the following, we denote by $\mathbb{P}_\mu^m$ the law of ω^m in $C(\mathbb{R}_+, \mathcal{S}')$.

By Itō formula, for any cylinder function $F \in \mathcal{C}$, $F(\omega) = f(\hat{\omega}_{k_1}, \dots, \hat{\omega}_{k_n})$, it holds

$$dF(\omega^m) = \mathcal{L}_\theta F(\omega^m)dt + \mathcal{G}^m F(\omega^m)dt + \sum_{i=1}^n \partial_i f(\hat{\omega}_{k_1}^m, \dots, \hat{\omega}_{k_n}^m)\sqrt{2}|k_i|^\theta d\beta_t^{k_i},$$

where $\mathcal{L}_\theta$ is defined in (8.1.9) and

$$\mathcal{G}^m F(\omega) = \sum_{i=1}^n \partial_i f(\hat{\omega}_{k_1}, \dots, \hat{\omega}_{k_n}) B_{k_i}^m(\omega)dt.$$

In other words, the process

$$M_t^{F,m} = F(\omega_t^m) - F(\omega_0^m) - \int_0^t \mathcal{L}_\theta F(\omega_s^m)ds - \int_0^t \mathcal{G}^m F(\omega_s^m)ds \tag{8.2.3}$$

is a martingale with quadratic variation

$$[M^{F,m}]_t = 2\int_0^t \sum_{i=1}^n |k_i|^{2\theta}\left(\partial_i f(\hat{\omega}_{k_1}^m, \dots, \hat{\omega}_{k_n}^m)\right)^2 ds = 2\int_0^t \mathcal{E}_\theta(F)(\omega_s^m)ds.$$

Let us point out that, thanks to the hydrodynamic form of the nonlinearity, $\mathcal{G}^m$ is a skew-symmetric operator with respect to μ: indeed, since

$$\langle \omega, B^m(\omega)\rangle = \left\langle \omega, \pi_m(\nabla^\perp A(\pi_m\omega)\cdot\nabla\pi_m\omega)\right\rangle = 0,$$

Gaussian integration shows that

$$\mathbb{E}_\mu\left[F\mathcal{G}^m G\right] = -\mathbb{E}_\mu\left[G\mathcal{G}^m F\right], \quad \forall F, G \in \mathcal{C}.$$

Let us then consider the reversed process $\tilde{\omega}_t^m = \omega_{T-t}^m$, for a fixed time horizon $T > 0$: $\tilde{\omega}^m$ is a Markov process whose generator is the adjoint of the one of ω^m, that is $\mathcal{L}_\theta - \mathcal{G}^m$. The process

$$\tilde{M}_t^{F,m} = F(\tilde{\omega}_t^m) - F(\tilde{\omega}_0^m) - \int_0^t \mathcal{L}_\theta F(\tilde{\omega}_s^m)ds - \int_0^t \mathcal{G}^m F(\tilde{\omega}_s^m)ds \tag{8.2.4}$$

is thus another martingale with quadratic variation $2\int_0^t \mathcal{E}_\theta(F)(\omega_s^m)ds$. To sum up, we have shown that ω^m is a controlled process in the sense of Definition 8.2.1.

The trick is now to sum the martingale identities (8.2.3), (8.2.4) for ω^m and $\tilde{\omega}^m$: in doing so the nonlinear skew symmetric part, together with boundary terms, is canceled, leaving us with martingales term and the symmetric Ornstein-Uhlenbeck generator,

$$\tilde{M}^{F,m}_{T-t} - \tilde{M}^{F,m}_{T} - M^{F,m}_{t} = 2\int_0^t \mathcal{L}_\theta F(\omega^m_s) ds.$$

Burkholder-Davis-Gundy inequality thus provides, together with stationarity of ω^m, the following powerful estimate: for $p \geq 1$ there exists a constant $C_p > 0$ only depending on p such that for all $F \in \mathcal{C}$

$$\mathbb{E}_{\mathbb{P}^m_\mu}\left[\sup_{t\in[0,T]}\left|\int_0^t \mathcal{L}_\theta F(\omega^m_s) ds\right|^p\right] \leq C_p\sqrt{T}\mathbb{E}_\mu\left[|\mathcal{E}_\theta F|^{p/2}\right]. \tag{8.2.5}$$

REMARK 8.2.5. As already observed, Definition 8.2.1 actually collects the elements we used to establish (8.2.5); indeed, the latter holds more generally for any controlled process $\omega \in \mathcal{R}_{\theta,T}$.

Inequality (8.2.5) provides good estimates on time integrals of observables for ω^m, provided that we are able to solve a Poisson equation in Gaussian space. The main aim are clearly bounds to establish the limit in Lemma 8.2.2, which can be obtained by means of (8.2.5) by solving

$$\mathcal{L}_\theta H^m_k(\omega) = B^m_k(\omega).$$

Since $B^m_k(\omega)$ belongs to the second chaos in the Wiener chaos decomposition of $L^2(\mu)$, and since $\mathcal{L}_\theta$ is diagonalised by such decomposition, it is easy to obtain the explicit solution

$$H^m_k(\omega) = -\chi_{\{|k|\leq m\}} \sum_{\substack{h,\ell\in\mathbb{Z}^2_0,\\ h+\ell=k\\ |h|,|\ell|\leq m}} \hat{\omega}_h\hat{\omega}_\ell \frac{\ell\cdot h^\perp}{h_2^2(|h|^2+|\ell|^2)^\theta}. \tag{8.2.6}$$

The computation is completely analogous to [**100**, Section 3], to which we refer. With the latter expression at hand, one only needs to estimate moments of $\mathcal{E}_\theta(H^m_k)$: we report such computation in the next section, together with some variants from which Theorem 8.2.4 follows.

8.3. Proof of Main Result

We complete in this Section the proof of Theorem 8.2.4. First, by means of the Itō trick estimate (8.2.5) we obtain bounds on Galerkin approximations: the last two paragraphs are then devoted to existence and uniqueness of martingale solutions. In this section, the symbol $\lesssim$ denotes inequality up to a positive multiplicative constant uniform in the involved parameters.

8.3.1. Controlling the Nonlinear Term. We start from the expression (8.2.6) for H^m_k to obtain estimates on the nonlinear term in (8.1.12). By definition of $\mathcal{E}_\theta$, (8.1.11), one has

$$\mathcal{E}_\theta(H^m_k)(\omega) = \chi_{\{|k|\leq m\}} \sum_{\substack{h\in\mathbb{Z}^2_0,\\ |h|\leq m}} |h|^{2\theta}\left|\frac{2(k-h)\cdot h^\perp}{h_2^2(|k-h|^2+|h|^2)^\theta}\hat{\omega}_{k-h}\right|^2,$$

therefore, taking expectation with respect to μ, for $|k| \le m$

$$\begin{aligned}\mathbb{E}_\mu\left[\mathcal{E}_\theta(H_k^m)\right] &= \sum_{\substack{h\in\mathbb{Z}_0^2,\\ |h|\le m}} |h|^{2\theta}\left|\frac{2(k-h)\cdot h^\perp}{h_2^2(|k-h|^2+|h|^2)^\theta}\right|^2 \\ &\lesssim \sum_{\substack{h\in\mathbb{Z}_0^2,\\ |h|\le m}} \frac{|k|^2|h|^{2+2\theta}}{|k-h|^{4\theta}+|h|^{4\theta}} \lesssim \sum_{\substack{h\in\mathbb{Z}_0^2,\\ |h|\le m}} \frac{|k|^2|h|^2}{|k-h|^{2\theta}+|h|^{2\theta}}.\end{aligned}$$

Now we use the fact that, for $\theta > 2$,

$$\sum_{h\in\mathbb{Z}_0^2} \frac{|h|^2}{|k-h|^{2\theta}+|h|^{2\theta}} \lesssim |k|^{4-2\theta}$$

(see [**101**, Lemma 16]) to deduce the following estimate uniformly in m:

$$\mathbb{E}_\mu\left[\mathcal{E}_\theta(H_k^m)\right] \lesssim |k|^{6-2\theta}.$$

Similarly, increments are controlled by

$$\sup_{n>m} \mathbb{E}_\mu[\mathcal{E}_\theta(H_k^n - H_k^m)] \lesssim |k|^2 m^{4-2\theta}.$$

With these estimates at hand, by means of (8.2.5) and Gaussian hypercontractivity, one can prove the following estimates on the nonlinear term of (8.1.12).

LEMMA 8.3.1. *Let $G_t^m := \int_0^t B(\pi_m\omega_s)ds$ and $\mathbb{P}_\mu^m$ be the distribution of the stationary solution of (8.2.1) described above. For any $n > m$ we have the following estimates:*

$$\left\|\sup_{t\in[0,T]} (G_t^m)_k\right\|_{L^p(\mathbb{P}_\mu^m)} \lesssim |k|^{3-\theta}T^{1/2}, \tag{8.3.1}$$

$$\left\|\sup_{t\in[0,T]} (G_t^n)_k - (G_t^m)_k\right\|_{L^p(\mathbb{P}_\mu^m)} \lesssim |k|T^{1/2}m^{2-\theta}. \tag{8.3.2}$$

LEMMA 8.3.2. *Let $\tilde G_t^m := \int_0^t e^{-(t-s)(-\Delta)^\theta}B(\pi_m\omega_s)ds$ and $\mathbb{P}_\mu^m$ as above. For any m fixed, $n > m$, $s, t \in [0,T]$, $s < t$, we have the following estimates:*

$$\left\|\sup_{t\in[0,T]} \left(\tilde G_t^m\right)_k\right\|_{L^p(\mathbb{P}_\mu^m)} \lesssim |k|^{3-2\theta}, \tag{8.3.3}$$

$$\left\|\sup_{t\in[0,T]} \left(\tilde G_t^n\right)_k - \left(\tilde G_t^m\right)_k\right\|_{L^p(\mathbb{P}_\mu^m)} \lesssim |k|^{-1}m^{4-2\theta}, \tag{8.3.4}$$

$$\sup_m \left\|\left(\tilde G_t^m\right)_k - \left(\tilde G_s^m\right)_k\right\|_{L^p(\mathbb{P}_\mu^m)} \lesssim |k|^{3-2\theta+2\varepsilon\theta}(t-s)^\varepsilon, \tag{8.3.5}$$

where the last inequality is meant to hold for $\varepsilon > 0$ small enough.

Proofs of the previous estimates follow along the lines of Lemma 5, Lemma 6 and Corollary 1 of [**100**], so we refrain from repeating them here.

8.3.2. Existence for $\theta > 2$. We first prove Lemma 8.2.2, which gives a meaning to the nonlinear term of (8.1.12). The result easily follows from Lemma 8.3.1.

Proof of Lemma 8.2.2. Let $G_t^m := \int_0^t B(\pi_m\omega_s)ds$. It is clear that G^m is a random process with values in $C([0,T],\mathcal{F}L^{\infty,\zeta})$ for every m and $\zeta\in\mathbb{R}$. Since $\theta>2$, (8.3.2) gives for any p and $n>m$

$$\mathbb{E}_{\mathbb{P}_\mu^m}\left[\sum_k |k|^{\zeta p}\left|\sup_{t\in[0,T]}(G_t^n)_k-(G_t^m)_k\right|^p\right]\to 0$$

as $m\to\infty$ whenever $\zeta<-2/p-1$. Taking p sufficiently large, for any $\zeta<-1$ we obtain the almost sure uniform convergence of G^m in the space $C([0,T],\mathcal{F}L^{\infty,\zeta})$. □

We are now ready to prove the first part of Theorem 8.2.4. The proof relies on subsection 8.3.1 and Skorokhod Theorem.

Proof of Theorem 8.2.4, existence. Let us consider the mild formulation of (8.2.1):

$$\begin{aligned}\omega_t^m &= e^{-t(-\Delta)^\theta}\omega_0+\int_0^t e^{-(t-s)(-\Delta)^\theta}B^m(\omega_s^m)ds\\ &\quad+\sqrt{2}(-\Delta)^{\theta/2}\int_0^t e^{-(t-s)(-\Delta)^\theta}d\beta_s,\end{aligned}\tag{8.3.6}$$

where $\omega_0\sim\mu$ and β is a cylindrical Wiener process on $\mathcal{F}L^{2,0}$. Define

$$\mathcal{A}_t^m := \int_0^t B^m(\omega_s^m)ds,\quad \tilde{\mathcal{A}}_t^m := \int_0^t e^{-(t-s)(-\Delta)^\theta}B^m(\omega_s^m)ds.$$

We prove that, for $\varepsilon>0$ sufficiently small and $\zeta<-1$, the laws of the processes $\left(\omega^m,\mathcal{A}^m,\tilde{\mathcal{A}}^m,\beta\right)_m$ are tight in $C([0,T],\mathcal{X})$, where

$$\mathcal{X} := \mathcal{F}L^{\infty,\zeta}\times\mathcal{F}L^{\infty,\theta-3-\varepsilon}\times\mathcal{F}L^{\infty,2\theta-3-\varepsilon}\times\mathcal{F}L^{\infty,-\varepsilon}.$$

By Borel-Cantelli theorem applied to Fourier expansions, the law μ is concentrated on $\mathcal{F}L^{\infty,-\varepsilon}$, and the stochastic convolution takes values in $C([0,T],\mathcal{F}L^{\infty,\theta-\varepsilon})$ for every $\varepsilon>0$. Tightness in this space is given by Fernique Theorem. Tightness of $(\tilde{\mathcal{A}}^m)_m$ descends from Equation 8.3.5 and tightness of $(\mathcal{A}^m)_m$ descends from Equation 8.3.1. Hence, by a standard application of Prokhorov Theorem and Skorokhod Theorem, we deduce the a.s. convergence, up to a subsequence and a change of the underlying probability space, of $\left(\omega^m,\mathcal{A}^m,\tilde{\mathcal{A}}^m,\beta\right)$ towards some random variable $\left(\omega,\mathcal{A},\tilde{\mathcal{A}},\beta\right)$ in $C([0,T],\mathcal{X})$ which satifies

$$\begin{aligned}\omega_t &= e^{-t(-\Delta)^\theta}\omega_0+\tilde{\mathcal{A}}_t+\sqrt{2}(-\Delta)^{\theta/2}\int_0^t e^{-(t-s)(-\Delta)^\theta}d\beta_s\\ &=\omega_0+\int_0^t(-\Delta)^\theta\omega_s ds+\mathcal{A}_t+\sqrt{2}(-\Delta)^{\theta/2}\beta_t.\end{aligned}$$

Now it is easy to check that $\omega\in\mathcal{R}_{\theta,T}$, see [**100**] for details. □

8.3.3. Uniqueness for $\theta>3$.

Proof of Theorem 8.2.4, uniqueness. We have constructed a sequence of ω^m converging a.s. to a solution ω as random variables in $C([0,T],\mathcal{F}L^{\infty,\zeta})$ for every $\zeta<-1$. Here we prove uniqueness, which comes from an estimate on the quantity $\pi_m(\omega^m-\omega)$ in a suitable space, where $\omega\in\mathcal{R}_{\theta,T}$ is a controlled solution to (8.1.12) and ω^m is its Galerkin approximation defined by (8.3.6). In particular, we prove that $\pi_m(\omega^m-\omega)$ converges a.s. to zero in the space $C([0,T],\mathcal{F}L^{\infty,\xi})$, for suitable $\xi>\zeta$. This would conclude the proof by uniqueness at the level of the Galerkin truncations.

It is easy to see that

$$\begin{aligned}\pi_m(\omega_t^m-\omega_t) \coloneqq \delta_t^m &= \int_0^t e^{-(t-s)(-\Delta)^\theta}(B^m(\omega_s^m)-\pi_m B(\omega_s))ds\\ &= \int_0^t e^{-(t-s)(-\Delta)^\theta}(B^m(\omega_s^m)-B^m(\omega_s))ds\\ &+ \int_0^t e^{-(t-s)(-\Delta)^\theta}(B^m(\omega_s)-\pi_m B(\omega_s))ds\\ &= \alpha_t^m+\gamma_t^m.\end{aligned}$$

for every m, and thus for every ξ

$$\sup_k \sup_{t\in[0,T]} |k|^\xi |(\delta_t^m)_k| \le \sup_k \sup_{t\in[0,T]} |k|^\xi |(\alpha_t^m)_k| + \sup_k \sup_{t\in[0,T]} |k|^\xi |(\gamma_t^m)_k|.$$

By (8.3.3), (8.3.4) and interpolation, γ^m satisfies

$$\left\|\sup_{t\in[0,T]} |(\gamma_t^m)_k|\right\|_{L^p(\mathbb{P}_\mu^m)} \lesssim |k|^{3-2\theta+\varepsilon} m^{-\varepsilon},$$

and therefore for every $\xi<2\theta-3$ we have

$$\sup_k \sup_{t\in[0,T]} |k|^\xi |(\gamma_t^m)_k| \to 0 \text{ a.s. for } m\to\infty.$$

On the other hand, since

$$|B^m(\omega_s^m)-B^m(\omega_s)| \lesssim \sum_{\substack{h\in\mathbb{Z}_0^2,\\ |h|\le m}} |k||h||(\omega_s^m+\omega_s)_h||(\omega_s^m-\omega_s)_{k-h}|,$$

we obtain the following bound on α^m:

$$\begin{aligned}\sup_{t\in[0,T]} |k|^\xi |(\alpha_t^m)_k| \lesssim & |k|^\xi \sup_h \sup_{t\in[0,T]} |h|^\xi |(\delta_t^m)_h|\\ &\times \sup_{t\in[0,T]} \left| \int_0^t e^{-(t-s)|k|^{2\theta}} \sum_{\substack{h\in\mathbb{Z}_0^2,\\ |h|\le m}} |h|^{1-\xi}|k||(\omega_s^m+\omega_s)_{k-h}|ds\right|.\end{aligned}$$

If $\xi>3$ the series $\sum_{h\in\mathbb{Z}_0^2}|h|^{1-\xi}$ converges, therefore by Hölder inequality

$$\begin{aligned}&\left|\int_0^t e^{-(t-s)|k|^{2\theta}} \sum_{|h|\le m} |h|^{1-\xi}|(\omega_s^m+\omega_s)_{k-h}|ds\right|\\ \lesssim& \left|\int_0^t \sum_{|h|\le m} |h|^{1-\xi} e^{-p'(t-s)|k|^{2\theta}}ds\right|^{1/p'} \left|\int_0^t \sum_{|h|\le m} |h|^{1-\xi}|(\omega_s^m+\omega_s)_{k-h}|^p ds\right|^{1/p}\\ \lesssim& |k|^{-2\theta/p'} \left|\int_0^t \sum_{|h|\le m} |h|^{1-\xi}|(\omega_s^m+\omega_s)_{k-h}|^p ds\right|^{1/p}.\end{aligned}$$

Taking $p'\to 1$ such that $\xi+1-2\theta/p'<0$ and using the fact that ω^m and ω have marginals $\sim\mu$, we finally get

$$\sup_k \sup_{t\in[0,T]} |k|^\xi |(\delta_t^m)_k| \to 0 \text{ a.s. for } m\to\infty,$$

for every $3<\xi<2\theta-3$, which corresponds to the additional contraint $\theta>3$. The proof is complete. $\square$

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