Mean-Field Games with Absorption and Singular Controls

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Declaration of Authorship

I, Maddalena GHIO, declare that this thesis entitled, "Mean-Field Games with Absorption and Singular Controls" is original work and the content of Chapters 2 and 3 was previously published respectively in Campi et al. (2019) and in Campi et al. (2020). These papers were written in equal proportions by myself and my coauthors Luciano Campi, Tiziano De Angelis and Giulia Livieri. The remaining material, including the Introduction and Chapter 4, is original and has not been published before. In particular Chapter 4 has been developed in collaboration with my coauthors Luciano Campi, Tiziano De Angelis and Giulia Livieri.
Parts of this thesis are extracts from the papers and pre-prints listed below, with minor changes:

- Campi, L., M. Ghio, and G. Livieri (2019). N-player games and mean-field games with smooth dependence on past absorptions.
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List of Papers

The following works have been submitted in fulfillment of the requirements for the attainment of the doctoral degree *Perfezionamento in Matematica per la Finanza*.

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- Campi, L., T. De Angelis, M. Ghio, and G. Livieri (2020). Mean-field games of finite-fuel capacity expansion with singular controls.
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Abstract

Mean-Field Games with Absorption and Singular Controls

by Maddalena GHIO

This Ph.D. thesis studies two newly developed branches of the theory of *mean-field games*: mean-field games with *absorption* and mean-field games with *singular controls*, with a focus on financial applications and numerical methods.

The first part of the work is devoted to mean-field games with *absorption*, a class of games that can be viewed as natural limits of symmetric stochastic differential games with a large number of players who, interacting through a mean-field, leave the game as soon as their private states hit a given boundary. In most of the literature on mean-field games, all players stay in the game until the end of the period, while in many applications, especially in economics and finance, it is natural to have a mechanism deciding when a player has to leave. Such a mechanism can be modelled by introducing an absorbing boundary for the state space.
The second part of the thesis, deals with mean-field games of *finite-fuel capacity expansion* with *singular controls*. While singular control problems with finite (and infinite) fuel find numerous applications in the economic literature and originated from the engineering literature in the late 60's, many-player game versions of these problems have only very recently been introduced. They are a natural extension of the single agent set-up and allow to model numerous applied situations. In our work in particular, we make assumptions on the structure of the interaction across players that are suitable to model the so-called *goodwill* problem.

Altogether, the original contribution to the mean-field games literature of the present work is threefold. First, it contributes to the development of mean-field games with absorption, continuing the work of Campi and Fischer (2018) and considerably generalizing the original model by relaxing the assumptions and setting it into a more abstract, infinite-dimensional, framework. Second, it introduces a new set of tools to deal with mean-field games with singular controls, extending the well-known connection between singular stochastic control and optimal stopping to mean-field games. Finally, it also contributes to the numerical literature on mean-field games, by proposing a numerical scheme to approximate the solutions of mean-field games with singular controls with a constructive approach.
Overall, this thesis focuses on newly introduced branches of the theory of mean-field games that display a high potential for economic and financial applications, contributing to the literature not only by further developing the existing theory but also by working in directions that make the these models more suitable to applications.

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Contents

List of Abbreviations

MFG	**M**ean-**F**ield **G**ame
NE	**N**ash **E**quilibrium
LLN	**L**aw of **L**arge **N**umbers
SDE	**S**tochastic **D**ifferential **E**quation
BSDE	**B**ackward **S**tochastic **D**ifferential **E**quation
FBSDE	**F**orward-**B**ackward **S**tochastic **D**ifferential **E**quation
PDE	**P**artial **D**ifferential **E**quation
FP	**F**okker-**P**lanck Equation
HJB	**H**amilton-**J**acobi-**B**ellman Equation
SC	**S**ingular **C**ontrol
OS	**O**ptimal **S**topping
ABM	**A**rithmetic **B**rownian **M**otion
GBM	**G**eometric **B**rownian **M**otion

List of Symbols

Let $\mathbb{E}$ be a Euclidean space. Then we define

$\lvert\cdot\rvert$	The Euclidean norm on $\mathbb{E}$

Let E be a Polish space, $T \in \mathbb{R}$, $T > 0$, and $p \in \mathbb{Z}_+$. Then we define

$d_E(\cdot,\cdot)$	A complete metric on E
$\mathcal{P}(E)$	The space of Borel probability measures on E
$\mathcal{M}_f(E)$	The space of finite Borel measures on E
$\mathcal{M}_{\leq 1}(E)$	The space of Borel sub-probability measures on E
$\Upsilon^T_{\mathcal{P}}(E)$	The space of measurable flows on $[0,T]$ of probability measures on E
$\Upsilon^T_{\leq 1}(E)$	The space of measurable flows on $[0,T]$ of sub-probability measures on E
$\mathcal{P}_p(E)$	The space of measures $\pi \in \mathcal{P}(E)$ s.t. $\int_E d_E^p(x,x_0)\pi(\mathrm{d}x) < \infty$ for some $x_0 \in E$
$\mathcal{M}_{\leq 1,1}(E)$	The space of measures $\mu \in \mathcal{M}_{\leq 1}(E)$ s.t. $\int_E d_E(x,x_0)\mu(\mathrm{d}x) < \infty$ for some $x_0 \in E$
$W_1(\cdot,\cdot)$	The 1-Wasserstein distance on $\mathcal{P}_1(E)$
$\Upsilon^T_{\mathcal{P},1}(E)$	The space of measurable flows of probability measures in $\mathcal{P}_1(E)$
$\Upsilon^T_{\leq 1,1}(E)$	The space of measurable flows of measures in $\mathcal{M}_{\leq 1,1}(E)$
$\mathrm{Lip}_1(E;\mathbb{R})$	The set of Lipschitz functions on E with unitary Lipschitz constant

Let E be a compact Hausdorff space and $d \in \mathbb{Z}_+$. Then we define

$C(\mathsf{E})$	The space of continuous functions from E to $\mathbb{R}$ with the supremum norm
$\lVert\cdot\rVert_\infty$	The supremum norm on E
$C([0,T];\mathbb{R}^d)$	The space of continuous functions from $[0,T]$ to $\mathbb{R}^d$ with the supremum norm
$\lVert\cdot\rVert_{\infty,t}$	The supremum norm on $C([0,t];\mathbb{R}^d)$, $t \leq T$
$\lVert\cdot\rVert_{\infty,T}$	The supremum norm on $C([0,T];\mathbb{R}^d)$ ($\lVert\cdot\rVert_\infty$ where no confusion is possible)

Let $k, k' \in \mathbb{Z}$, $k, k' \geq 0$, and $\Omega \subset \mathbb{R}^d$ open. Then we define

$\partial\Omega$	The boundary of Ω
$C(\Omega)$	The continuous functions from Ω to $\mathbb{R}$
$C(\Omega;\mathbb{R}^d)$	The continuous functions from Ω to $\mathbb{R}^d$
$C_b(\Omega)$	The space of continuous and bounded functions from Ω to $\mathbb{R}$
$C_b(\Omega;\mathbb{R}^d)$	The space of continuous and bounded functions from Ω to $\mathbb{R}^d$
$C_c(\Omega)$	The space of functions in $C(\Omega)$ with compact support in Ω
$C_c(\Omega;\mathbb{R}^d)$	The space of functions in $C(\Omega;\mathbb{R}^d)$ with compact support in Ω
$C^k(\Omega)$	The space of k times continuously differentiable functions from Ω to $\mathbb{R}$
$C^{k,k'}((0,T)\times\Omega)$	The space of $C((0,T)\times\Omega)$ functions that are k times continuously differentiable in the first variable and k' times continuously differentiable in the second variable

Dedicated to my family

Chapter 1

Introduction

1.1 Mean-Field Games

Mean-field games (MFGs for short) are, loosely speaking, limits of symmetric stochastic differential games with a large number of players that interact with the distributional behaviour of their competitors. These games were introduced in the seminal papers by Lasry and Lions (2006a,b, 2007a,b) and, simultaneously, by Huang et al. (2007a,b,c, 2006). An increasing stream of research has been flourishing since then, producing theoretical results as well as a wide range of applications in many fields such as economics, finance, crowd dynamics and social sciences in general. Two approaches have been adopted: an analytic approach as in the first works by Lasry and Lions and a probabilistic one, that we follow in this work and which has been developed in a series of papers by Carmona, Delarue and their co-authors (see, e.g., Carmona and Delarue (2013a,b); Carmona et al. (2016); Carmona and Lacker (2015)). For an excellent presentation of the theory we refer in particular to the lecture notes by Cardaliaguet (2013) and the two-volume monograph by Carmona and Delarue (2018).

The term "mean-field" comes from physics, in particular from statistical mechanics and condensed matter physics, where the mean-field (or effective-field) approximation, replacing local interactions between neighbouring particles with an average (non-local) effective interaction, enables to find approximate solutions of problems that would otherwise be analytically intractable, like phase transition in the the Ising model in dimension three or higher. Mean-field interactions are also at the heart of propagation of chaos results for diffusive interacting particle systems, see for instance Sznitman (1991), where law-of-large-numbers (LLN) phenomena allow the recovery of renown partial differential equations (PDEs) from physics as a reduced description of particles' dynamics, in the limit for the number of particles going to infinity.
A game with a large number of players whose private states evolve in time according to a controlled diffusion process, can be understood as a particle system with the additional difficulty that now the particle-like players are rational optimizing agents. Analogously to the aforementioned problems in physics, the curse of dimensionality affects also game theory, as soon as the number of players starts increasing. Indeed, analytically finding the Nash equilibria of games with a large number of interacting players is often unfeasible. However, in the symmetric case and when players interact in a "mean-field" way, i.e. via variables that are function of the distribution of all players, the passage to the limit for the number of players going to infinity, the so called "mean-field limit", produces a substantial simplification of the problem, making the effect of a single player's action negligible on the population as a whole.

The many-player game thus reduces to a single-player game, in which a representative agent interacts with the distribution of all players. The representative player of the MFG responds optimally to the distribution of the population which, at equilibrium, coincides with the distribution of the optimally controlled state variable. Resulting from the limit of finitely many-player games, the MFG can be proved to be a good approximation of the corresponding finite-player game, in the sense that its solution can typically be implemented in the N-player game providing approximate Nash equilibria with vanishing error in the limit for $N \to \infty$.

In economics, the idea of studying Nash equilibria in large finite-player games via the approximation of a continuum of players goes back to the strand of literature on *anonymous games*, i.e. games in which single players, as individuals, have no significative influence on the others' behaviour but they do when aggregated. In this regard, we recall in particular the pioneering paper on existence of equilibria in anonymous games by Schmeidler (1973), the later reformulation with distributional strategies by Mas-Colell (1984) and the work by Jovanovic and Rosenthal (1988) introducing a unified model for sequential discounted anonymous games.
Jumping to more recent times, MFGs are a promising tool to be applied to economics (see Achdou et al., 2014, for a general survey on continuous-time models in economics). They have been introduced in the mathematical formalization of *heterogeneous agent models in continuous time*, finding macroeconomics applications that range from the study of the wealth distribution (Achdou et al., 2014, 2017) to optimal social and monetary policy (Nuño, 2017; Nuño and Thomas, 2020, to cite a few) until a very recent application to the Coronavirus crisis (Kaplan et al., 2020).

1.1.1 An illustrative game

Consider a symmetric stochastic N-player game, where the private states of the players are denoted by $\mathbf{X}^N \doteq (X^{N,1}, \dots, X^{N,N})$. In an economic or financial application, players could be households, firms or banks, just to give an example, and the variables $X^{N,i}$ would represent their private wealth or capital. Throughout the section, to give a more practical sense of the theory and in view of the models of Chapters 2 and 3, we will continue developing the parallel with economic and financial models.

Let the $X^{N,i}$ be stochastic processes with values in $\mathbb{R}^d$, evolving over a finite time horizon $[0, T]$, $T > 0$, according to Brownian driven diffusions of the form

$$X_t^{N,i} = X_0^{N,i} + \int_0^t b\left(s, X_s^{N,i}, \mu_s^N, \alpha_s^{N,i}\right) \mathrm{d}s + \sigma W_t^{N,i}, \;\; t \in [0, T], \tag{1.1}$$

for $i \in \{1, \dots, N\}$, where $W^{N,1}, \dots, W^{N,N}$ are independent d-dimensional Wiener processes defined on some filtered probability space, $\boldsymbol{\alpha}^N \doteq (\alpha^{N,1}, \dots, \alpha^{N,N})$ is a vector of adapted strategies, σ is a non-degenerate diffusion matrix and b is a given drift functional.
The symmetric interaction of players is modelled via μ^N, the random flow of empirical probability measures representing the empirical distribution of all players

$$\mu_t^N(\cdot) \doteq \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{N,i}}(\cdot), \;\; t \in [0, T].$$

The simplest form of interaction can happen via average variables, e.g. the empirical average $\overline{O}_t$ of an observable quantity of the system $O : \mathbb{R}^d \to \mathbb{R}$ at time $t \in [0, T]$

$$\overline{O}_t = \frac{1}{N} \sum_{i=1}^{N} O(X_t^{N,i}) = \int_{\mathbb{R}^d} O(x) \mu_t^N(\mathrm{d}x), \quad t \in [0, T].$$

In an economic or financial model, $\overline{O}$ could represent an aggregate variable like aggregate wealth of the economy or average capital reserve of the players. However, the measure dependence of the drift on the empirical measure μ^N, can account for more complex, also non-local, interaction forms, meaning that interaction among the players can take place also via quantities that depend on the entire distribution of the players, not only averages but also tails and in general the full distribution.

Players evaluate their strategy vector $\boldsymbol{\alpha}^N$ according to their own expected costs and depending on the other players' states, via cost functionals of the form

$$J^{N,i}(\boldsymbol{\alpha}^N) \doteq \mathsf{E}\left[\int_0^T f\left(s, X_s^{N,i}, \mu_s^N, \alpha_s^{N,i}\right) \mathrm{d}s + F\left(T, X_T^{N,i}\right)\right]$$

where f and F are respectively a given running cost and terminal cost. The cost functionals $J^{N,i}(\boldsymbol{\alpha}^N)$ take into account all possible costs that players can incur into during the game, from consumption to fees, but also more figurative and indirect costs, like the cost of being "far from the crowd". The players' goal is then to implement a strategy that allows them to minimize their costs at each instant of time of the game. A strategy could be a consumption policy for a household or an investing/borrowing plan for a firm or a bank.
Equilibrium situations in many-player non-cooperative games, where players have no incentive to change their currently employed strategy deviating from equilibrium because they would only experience further losses, are well described by the concept of *Nash equilibrium* (Nash, 1950, 1951). In our framework, a strategy vector $\boldsymbol{\alpha}^N$ is a Nash equilibrium for the N-player game if for every $i \in \{1, \dots, N\}$ and for any alternative admissible single-player strategy β we have

$$J^{N,i}(\boldsymbol{\alpha}^N) \leq J^{N,i}\left(\left[\boldsymbol{\alpha}^{N,-i}, \beta\right]\right)$$

where the standard notation $[\boldsymbol{\alpha}^{N,-i}, \beta]$ denotes a strategy vector equal to $\boldsymbol{\alpha}^N$ for all players but the i-th one, who deviates by playing β instead.

The interaction via the empirical measure μ^N introduces in the N-player dynamics, i.e. in the N-dimensional system of stochastic differential equations describing the time evolution of the players' private states, a strong coupling. Besides coupling and high dimensionality, minimization of the cost functional by the players brings in addition an optimization problem, more precisely a problem of stochastic optimal control. As soon as the number of players starts increasing, analytically finding the exact Nash equilibria of the N-player game becomes often unfeasible. It is at this point that the *mean-field games* theory comes to aid.
First, we relax the notion of Nash equilibrium to that of ε-Nash equilibrium, where deviating from the equilibrium strategy can only lead to limited (at most ε) additional gain. More precisely, a strategy vector $\boldsymbol{\alpha}^N$ is an ε-Nash equilibrium for the

N-player game if for every $i \in \{1, \dots, N\}$ and for any alternative admissible single-player strategy β we have

$$J^{N,i}(\boldsymbol{\alpha}^N) \leq J^{N,i}\left(\left[\boldsymbol{\alpha}^{N,-i}, \beta\right]\right) + \varepsilon.$$

Then, we introduce the associated MFG, as an auxiliary single-player game representing the limit of the N-player games for N going to infinity. In the so-called "mean-field limit", the effect of single players on the population as a whole becomes negligible. Then, the description of the system reduces to that of a representative player interacting with the limit distribution of all other players.
The MFG, with the mathematical simplification produced by the passage to the limit, can be exploited as a tool to find approximate solutions of the original N-player game. In practice, one solves the MFG problem (in a sense that will be defined below) instead of the N-player one, finding an optimal control for the representative player. With this optimal control at hand, it is possible in many situations to construct a vector of strategies for the corresponding N-player game, that proves to be an ε-Nash equilibrium with vanishing ε in the limit for N-going to infinity.
In this sense, we can say that the MFG is a "good" approximation of the original N-player game. Moreover, despite modelling only a representative player, the MFG keeps track of the initial N-players heterogeneity by deriving at the same time also the evolution of the entire distribution of all players, not as an exogenously given background dynamics but as an endogenous product arising from within the model.

The MFG associated to our illustrative N-player game, will be of the following form.
Given a (deterministic) flow of probability measures μ and an admissible control process α, the representative player's state evolves according to the equation

$$X_t = X_0 + \int_0^t b\left(s, X_s, \mu_s, \alpha_s\right) \mathrm{d}s + \sigma W_t, \quad t \in [0, T], \tag{1.2}$$

where X is a d-dimensional stochastic process, W is a d-dimensional Wiener process on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathsf{P})$ and b is the drift defined as above in the N-player game. The flow of probability measures μ is to be understood as the limit, in a sense to be defined, for N going to infinity of the random flow of empirical probability measures μ^N.
The representative agent evaluates the control α according to a cost functional of the form

$$J^{\mu}(\alpha) \doteq \mathsf{E}\left[\int_0^T f\left(s, X_s, \mu_s, \alpha_s\right) \mathrm{d}s + F\left(T, X_T\right)\right] \tag{1.3}$$

where f and F are respectively the running and terminal costs defined as above.
A *solution* of the MFG is then, loosely speaking, a pair (α, μ) where α is optimal for the control problem, i.e. it is a minimizer of the cost functional J^{μ}, while μ is a flow of probability measures, satisfying the fixed-point condition $\mu_t = \mathcal{L}(X_t)$, i.e. μ_t is the law of X_t for all $t \in [0, T]$.

1.1.2 A game with absorption and a game with singular controls

A MFG with *absorption* (Campi and Fischer, 2018; Campi et al., 2019) is a MFG, as introduced is Section 1.1.1, where the private state of the representative player X in Equation (1.2) evolves in a subset $\mathcal{O} \subset \mathbb{R}^d$ until it hits the boundary $\partial\mathcal{O}$, then leaves the game at $\tau \doteq \inf\{t \in [0,T] : X_t \notin \mathcal{O}\}$, equivalently we say that it is *absorbed* by the boundary. The dynamics of the representative player now depends on a flow μ of sub-probability measures instead of the more common flow of probability measures, starting as a normalized probability but then losing mass with the passing of time to represent the fraction of players leaving the game. The cost functional of Equation (1.3) writes

$$J^{\mu}(\alpha) \doteq \mathsf{E}\left[\int_0^{\tau\wedge T} f(s, X_s, \mu_s, \alpha_s)\,\mathrm{d}s + F(\tau\wedge T, X_{\tau\wedge T})\right].$$

A natural application of MFGs with absorption is to model the interbank market and its interconnectedness via bilateral credit exposures, as in Carmona et al. (2015). Here, players are interpreted as banks, whose monetary reserves evolve according to stochastic dynamics as in Equation (1.1) and where the drift depends on both the rate of interbank borrowing/lending and on a controlled borrowing/lending rate to a central bank. However, in Carmona et al. (2015) no absorbing boundary conditions are considered, while it would be natural to include the possibility of default when the capital of a bank hits from above a barrier, ideally associated to minimum regulatory capital requirements. Our research then aims to fill this gap in the existing literature. We refer to Chapter 2 for a complete review of the related literature.

A MFG with *singular* controls (Campi et al., 2020) is a MFG, as introduced is Section 1.1.1, where the control process $\xi = (\xi_t)_{t\in[0,T]}$ is not necessarily absolutely continuous with respect to the Lebesgue measure on $[0,T]$, thus entering the representative player's dynamics as

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s, \xi_s)\ \mathrm{d}s + \int_0^t \mathrm{d}\xi_s + \sigma W_t, \quad t \in [0,T]$$

and satisfying good enough assumptions to give sense to the equation above (adapted, non-negative, non-decreasing, right-continuous and bounded in our model in Chapter 3). A similar dependence can be included also in the cost functional, i.e.

$$J^{\mu}(\alpha) \doteq \mathsf{E}\left[\int_0^T f(s, X_s, \mu_s, \xi_s)\,\mathrm{d}s + \int_0^T \mathrm{d}\xi_s + F(T, X_T)\right]$$

Generally speaking, the singular control ξ, being a monotonically increasing process, can represent the cumulative investment in production of a company producing a good of value X. Investing in production comes with a cost that depends on the total investment in a proportional way, i.e. on the integral of $\mathrm{d}\xi$ over the total investment horizon T. Notice that singularity of the investment process ξ, allowing for jumps, enables to include into the model possibly discontinuous realistic investment decisions. In Chapter 3, we apply this class of games to model the so-called *goodwill* problem. We refer to the introduction of Chapter 3 for a detailed description of the problem and a complete review of the related literature.

1.2 Our original contribution to the mean-field games literature

Our original contribution to the MFGs literature is threefold. First, we contribute to the development of MFGs with absorption, continuing the work of Campi and Fischer (2018) and further generalizing their original model by considerably relaxing the assumptions and setting it into a more abstract, infinite-dimensional, framework. Second, we introduce a new set of tools to deal with MFGs with singular controls, extending the well-known connection between singular stochastic control and optimal stopping to MFGs. Finally, we also contribute to the numerical literature on MFGs, by proposing a numerical scheme to approximate solutions of MFGs of finite-fuel capacity expansion with singular controls and underlying GBM dynamics.
Overall, our work focuses on newly introduced branches of the MFG theory that display a high potential for economic and financial applications. We contribute not only by further developing the existing theory but also by working in directions that make the these models more apt to practical applications.

1.2.1 Our contribution to MFGs with absorption

Regarding MFGs with absorption, our best efforts focus on leveraging them to an infinite-dimensional framework and stretching the assumptions of Campi and Fischer (2018) to include more general and realistic dynamics that are common in standard models. Specifically,

(i) We recast MFGs with absorption in a more general setting, most common to the MFG literature, where the dependence of the dynamics and costs on the empirical measure is infinite-dimensional.

(ii) We introduce a direct dependence on past absorptions in the drift of the Stochastic Differential Equations (SDEs) describing the evolution of the players' states by letting the initial distribution of players lose mass over time. Such a loss of mass corresponds to the exit of the absorbed players from the game, so that the proportion of the absorbed players has an effect on the future evolution of the survivors. This feature was not present in Campi and Fischer (2018), where the empirical measure of the survivors was re-normalized at each time. Such a dependence on past absorptions is also included in the costs.

(iii) We allow both the drift and the cost functional of the players to grow at most linearly with the state, hence they are not necessarily bounded unlike in Campi and Fischer (2018). Moreover, the set of non-absorbing states $\mathcal{O}$ can also be unbounded. Dropping the boundedness of the game data increases the flexibility of our setting, which can include more realistic dynamics from the viewpoint of applications.

Our technical results can be summarized as follows while for more details and a broader overview of the related literature we refer to Chapter 2.

- We prove existence of a relaxed feedback MFG solution and, under an additional convexity assumption, we show that there are optimal feedback strategies in strict form; see Theorem 2.3.1, Proposition 2.3.4 and Proposition 2.3.5. Additionally, we show that there exist relaxed and strict feedback solutions that are Markovian up to the exit time; see Proposition 2.3.6.

- We prove uniqueness of the MFG solution under standard monotonicity conditions of the Lasry-Lions type formulated for sub-probability measures; see Theorem 2.4.1.

- We study approximate Nash equilibria for the N-player game in a setting where the dependence on the measure variable is finite-dimensional. Precisely, we show that if we have a feedback solution of the MFG (either relaxed or strict), we can construct a sequence of approximate Nash equilibria for the corresponding N-player games with a vanishing approximation error as $N \to \infty$; see Theorem 2.5.1 and Corollary 2.5.2. It is worth stressing that the construction produces approximate N-player equilibria in feedback strategies (instead of the more common open-loop strategies).

1.2.2 Our contribution to MFGs with singular controls

For what concerns MFGs of finite-fuel capacity expansion with singular controls, our main contribution to the literature is the following.

(i) We introduce the MFG of finite-fuel capacity expansion with singular controls, with an interaction structure among the players that is particularly apt to modelling the so-called *goodwill* problem.

(ii) We prove that the renown and well-established connection between singular control problems of capacity expansion and problems of optimal stopping also holds in our MFG setting.

(iii) We prove existence of a solution of the MFG of finite-fuel capacity expansion with singular controls via an iterative approximation procedure, naturally suggesting a numerical method that we also formalize and implement (see Chapter 4).

Our technical results can be summarized as follows while for more details and a broader overview of the related literature we refer to Chapter 3.

- We formulate the MFG of capacity expansion as the limit model for a sequence of N-player games of capacity expansion (Section 3.2). Then, under mild assumptions on the problem's data we construct a solution in feedback form of the MFG of capacity expansion (Section 3.3). Our constructive approach, based on an intuitive iterative scheme, allows us to determine the optimal control for the MFG in terms of an optimal boundary $(t, x) \mapsto c(t, x)$ that splits the state space $[0, T] \times \mathbb{R} \times [0, 1]$ into an *action* region and an *inaction* region; see Theorem 3.2.1 in Section 3.2. The optimal strategy prescribes to keep the controlled dynamics underlying the MFG inside the closure of the inaction region by Skorokhod reflection.

- Whenever the optimal boundary in the MFG is Lipschitz continuous in its second variable we can show that it induces a sequence of approximate ε_N-Nash equilibria for the N-player games with vanishing approximation error at rate $O(1/\sqrt{N})$ as N tends to infinity; see Theorem 3.4.1 in Section 3.4. While Lipschitz regularity of optimal boundaries is in general a delicate issue, we provide sufficient conditions on our problem data that guarantee such regularity.

- Based on the iterative procedure in the proof of Theorem 3.2.1, Section 3.3, we construct a numerical scheme to approximate the solution of the MFG of finite-fuel capacity expansion with underlying Geometric Brownian motion dynamics, introduced in Section 4.1. We implement the numerical method in Section 4.2.2.

Chapter 2

Mean-Field Games with Absorption

2.1 Introduction to mean-field games with absorption

Mean-field games with *absorption* is a class of games that has been introduced in Campi and Fischer (2018) and that can be viewed as natural limits of symmetric stochastic differential games with a large number of players who, interacting through a mean-field, leave the game as soon as their private states hit some given boundary. In most of the literature on MFGs, all players stay in the game until the end of the period, while in many applications, especially in economics and finance, it is natural to have a mechanism deciding when some player has to leave. Such a mechanism can be modelled by introducing an absorbing boundary for the state space as in Campi and Fischer (2018), which is the starting point of our study.

Related literature, featuring some form of absorption or exit mechanism for the players and introducing possible applications for this kind of models, is vast.
First, we cite the works of Giesecke et al. (2013) and Giesecke et al. (2015) where a model based on point processes for correlated defaults timing in a portfolio of firms is introduced and analysed. Giesecke et al. (2013) prove a LLN for the default rate as the number N of firms goes to infinity. Motivated by modelling the contagion effect are the works of Hambly and Ledger (2017), Hambly et al. (2019) and Hambly and Søjmark (2019) too. The first work provides a LLN for the empirical measure of a system of finitely many (uncontrolled) diffusions on the half-line, absorbed when they hit zero and correlated through the proportion of absorbed processes. In Hambly et al. (2019) the model is extended to include a positive feedback mechanism when the particles hit the barrier, thus modelling contagious blow-ups. A mathematical complement to the previous work is provided in Ledger and Søjmark (2020). More recently, Hambly and Søjmark (2019) have proposed a general model for systemic (or macroscopic) events. By working on a set-up similar to Hambly and Ledger (2017), they interpret the diffusions as distances-to-default of financial institutions and model the correlation effect through a common source of noise and a form of mean-reversion in the drift. A form of endogenous contagion mechanism is also considered.
On the side of applications to economics, Chan and Sircar (2015) and Chan and Sircar (2017) study oligopolistic models with exhaustible resources formulated as MFGs with absorption at zero. Their model keeps track of the fraction of active players at each time. However, this fraction appears in the objective functions but not in the state variable.
Two more papers to mention are those by Delarue et al. (2015a) and Delarue et al.

(2015b), where a particle system approach is used to study the mathematical properties of an integrate-and-fire model from neurology. The particles' dynamics have some resetting mechanism which activates as soon as some particle hits a given boundary. Besides, we cite two recent papers by Nadtochiy and Shkolnikov (2019, 2020). The first one focuses on the cascade effect in an interbank mean-field model with defaults and a contagion effect modelled via a singular interaction through hitting times. The second one investigates the associated mean-field game also including more general dynamics and connection structures.
Finally, we mention a class of MFGs that has been considered quite recently especially in relation to bank run models, that is MFGs of optimal stopping or timing; see, for instance, Bertucci (2018), Bouveret et al. (2020), Carmona et al. (2017) and Nutz (2018). Therein, the agents solve an optimal stopping problem so that the terminal time is directly chosen by them instead of being determined by the evolution of the controlled state as in our setting. In both settings the terminal time is in fact a random time and the state evolution might be affected by the fraction of players leaving the game and the empirical measure of those that remain.

2.1.1 Model description

The purpose of this chapter is to study N-player games and related MFGs in the presence of an absorbing set (players are eliminated from the game once their private states leave a given open set $\mathcal{O} \in \mathbb{R}^d$), and where the vector of private states $\mathbf{X}^N \doteq (X^{N,1}, \dots, X^{N,N})$ evolves according to

$$X_t^{N,i} = X_0^{N,i} \quad + \quad \int_0^t \bar{b}\left(s, X_s^{N,i}, \mu_s^N, u^{N,i}\left(s, \mathbf{X}^N\right)\right) \mathrm{d}s + \sigma W_t^{N,i}, \quad t \in [0,T], \tag{2.1}$$

for $i \in \{1, \dots, N\}$, where $\boldsymbol{u}^N \doteq (u^{N,1}, \dots, u^{N,N})$ is a vector of feedback strategies, $W^{N,1}, \dots, W^{N,N}$ are independent d-dimensional Wiener processes defined on some filtered probability space, σ is the (non-degenerate) diffusion matrix and $\bar{b}$ is a given drift functional. Finally, μ^N is the random flow of empirical sub-probability measures representing the empirical distribution of the survivors

$$\mu_t^N(\cdot) \doteq \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{N,i}}(\cdot)\, \mathbf{1}_{[0,\tau^{X^{N,i}})}(t).$$

Each player evaluates a strategy vector $\boldsymbol{u}^N$ according to their own expected costs

$$J^{N,i}\left(\boldsymbol{u}^N\right) \doteq \mathsf{E}\left[\int_0^{\tau^{N,i}} \bar{f}\left(s, X_s^{N,i}, \mu_s^N, u^{N,i}\left(s, \mathbf{X}^N\right)\right) \mathrm{d}s + F\left(\tau^{N,i}, X_{\tau^{N,i}}^{N,i}\right)\right] \tag{2.2}$$

over a random time horizon. In Equation (2.2), $\mathbf{X}^N$ is the N-player dynamics under $\boldsymbol{u}^N$ and $\tau^{N,i} \doteq \tau^{X^{N,i}} \wedge T$. In the present work, we are interested in drifts $\bar{b}$ and costs $\bar{f}$ with sub-linear growth, hence possibly unbounded. Further details on the setting with all the technical assumptions will be given in Section 2.2.
The dynamics above is also motivated by economic models for corporate finance, systemic risk, and asset allocation. For instance, we can interpret players as firms whose values are represented by the state variables $X^{N,i}$ for $i \in \{1, \dots, N\}$. Each company is affected by the fraction of both defaulted and non-defaulted firms and takes strategic decisions accordingly. Moreover, sub-linearity of the drift allows to include a mean-reversion term representing some herding behaviour. A possible

application is the pricing of portfolio credit derivatives where the pricing depends upon the so called distance-to-default of the assets in the portfolio (Hambly and Ledger (2017)). Alternatively, each player can be interpreted as a bank, whose monetary reserve evolves according to the stochastic dynamics in Equation (2.1) where the drift depends on both the rate of interbank borrowing/lending and on a controlled borrowing/lending rate to a central bank, as in Carmona et al. (2015). However, in Carmona et al. (2015) no absorbing boundary conditions are considered. The latter features could be incorporated in the model by introducing absorbing boundary conditions at the default level, similarly to Hambly and Ledger (2017). This would enable to study the impact of defaults on systemic risk and stability of the financial system described by the game. Last but not least, the proposed set-up allows for a Brownian motion with an Ornstein–Uhlenbeck type drift modelling for the private state, a model that has been used (for instance) for the notion of flocking to default in the financial literature (Fouque and Sun (2013)). However, in the present work we focus on the mathematical properties of the proposed family of games and we leave the applications for future research.

2.1.2 Methodology and original contribution

The first paper introducing mean-field games with absorption is Campi and Fischer (2018). Therein, existence of solutions of the MFG and construction of approximate Nash equilibria for the N-player games were provided under some boundedness assumptions on the coefficients and without including the effect of past absorption on the survivors' behaviour. The present work continues the investigation of this kind of games, with the following main extensions.

(i) We recast MFGs with absorption in a more general setting, most common to the MFG literature, where the dependence of the dynamics and costs on the empirical measure is infinite-dimensional.

(ii) We introduce a direct dependence on past absorptions in the drift of the Stochastic Differential Equations (SDEs) describing the evolution of the players' states by letting the initial distribution of players lose mass over time. Such a loss of mass corresponds to the exit of the absorbed players from the game, so that the proportion of the absorbed players has an effect on the future evolution of the survivors. This feature was not present in Campi and Fischer (2018), where the empirical measure of the survivors was re-normalized at each time. Such a dependence on past absorptions is also included in the costs.

(iii) We allow both the drift and the cost functional of the players to grow at most linearly with the state, hence they are not necessarily bounded unlike in Campi and Fischer (2018). Moreover, the set of non-absorbing states $\mathcal{O}$ can also be unbounded. Dropping the boundedness of the game data increases the flexibility of our setting, which can include more realistic dynamics from the viewpoint of applications (for more details, see later in this introduction).

The main contributions of the chapter can be summarized as follows:

- We introduce the MFG with smooth dependence on past absorptions, i.e. the limit model corresponding to the above N-player games as N tends to infinity. For a solution of the MFG, the empirical sub-probability measures $(\mu_t^N)_{t\in[0,T]}$ are replaced by flows of sub-probability measures on $\mathbb{R}^d$; see Definition 2.2.1.

- We prove existence of a relaxed feedback MFG solution and, under an additional convexity assumption, we show that there are optimal feedback strategies in strict form; see Theorem 2.3.1, Proposition 2.3.4 and Proposition 2.3.5. Additionally, we show that there exist relaxed and strict feedback solutions that are Markovian up to the exit time; see Proposition 2.3.6.
- We prove uniqueness of the MFG solution under standard monotonicity conditions of the Lasry-Lions type formulated for sub-probability measures; see Theorem 2.4.1.
- We study approximate Nash equilibria for the N-player game in a setting where the dependence on the measure variable is finite-dimensional. Precisely, we show that if we have a feedback solution of the MFG (either relaxed or strict), we can construct a sequence of approximate Nash equilibria for the corresponding N-player games with a vanishing approximation error as $N \to \infty$; see Theorem 2.5.1 and Corollary 2.5.2. It is worth stressing that the construction produces approximate N-player equilibria in feedback strategies (instead of the more common open-loop strategies).

The proof of the existence of feedback solutions of the MFG is inspired by the truncation procedure introduced by Lacker (2015). We construct a sequence of approximating MFGs, each one with bounded drift and cost functional, to which we can apply the results of Campi and Fischer (2018). Then, we prove convergence of the solutions of these approximating MFGs to a solution of the original one. Nonetheless, the procedure in Lacker (2015) cannot be applied directly to our case mainly due to the history dependency and the discontinuities induced by past absorptions. In particular, a different instance of the mimicking result of Brunick and Shreve (2013) applies to our framework.
To establish the uniqueness result we follow standard monotonicity arguments, with some adjustments due to the dependence of the coefficients on a flow of sub-probability measures instead of probability measures. In particular, the uniqueness result relies on an additional (standard) monotonicity assumption on the running cost of the Lasry-Lions type.
The proof of the construction of approximate Nash equilibria for the N-player game is based on weak convergence arguments and controlled martingale problems. The use of martingale problems in proving convergence to the McKean-Vlasov limit and propagation of chaos for weakly interacting systems goes back to Funaki (1984), Oelschläger (1984) and Méléard (1996). We observe that, whereas standard results prove convergence in law of the empirical measures, in the present work we follow the approach of Lacker (2018) to obtain a strong form of propagation of chaos with possibly unbounded and path-dependent drift. We show that the empirical measures converge in a stronger topology (the τ-topology), a result that enables us to take the limit as $N \to \infty$ without assuming any regularity of the feedback strategies with respect to the state process. In our framework, unlike Campi and Fischer (2018), the continuity of the MFG optimal control for almost every path of the state variable with respect of the Wiener measure is no longer feasible. Indeed, the PDE-based estimates that were used in Campi and Fischer (2018) to get such a regularity are not available anymore due to the possible unboundedness of the drift and the running cost.

2.1.3 Preliminaries and notation for mean-field games with absorption

In this section, we provide the definitions of the different spaces of trajectories and measures used in the chapter along with the corresponding topologies, distances and notions of convergence.

Spaces of trajectories. Let $d \in \mathbb{Z}_+$. We denote by $\mathcal{O} \subset \mathbb{R}^d$ an open subset of $\mathbb{R}^d$ representing the space of the players' private states and by $\mathcal{X} \doteq C([0,T];\mathbb{R}^d)$ the space of $\mathbb{R}^d$-valued continuous trajectories on the time interval $[0,T]$, $T < \infty$. The space $\mathbb{R}^d$ is equipped with the standard Euclidean norm, always indicated by $|\cdot|$, while $\mathcal{X}$ with the sup-norm, denoted by $\|\cdot\|_\infty$, which makes $\mathcal{X}$ separable and complete. We use the notation $\|\cdot\|_{\infty,t}$ whenever the sup-norm is computed over the time interval $[0,t]$, $t < T$. Besides, we denote with $\mathcal{X}^N \doteq C([0,T];\mathbb{R}^{d\times N})$ the space of N-dimensional vectors of continuous trajectories and identify it with $\mathcal{X}^{\times N}$.

Spaces of measures. We use flows of probability and sub-probability measures to describe the distribution of players and its time evolution in $\mathcal{O}$. For E a Polish space, let $\mathcal{M}_f(E)$ denote the space of finite Borel measures on E, $\mathcal{P}(E)$ the space of Borel probability measures on E and $\mathcal{M}_{\leq 1}(E)$ the space of Borel sub-probability measures on E, i.e. measures $\mu \in \mathcal{M}_f(E)$ such that $\mu(E) \leq 1$. These spaces are endowed with the weak convergence of measures (Billingsley (1999)). We will often write $\mu^n \overset{w}{\rightharpoonup} \mu$ to indicate weak convergence of μ^n towards μ as $n \to \infty$ and $\xi_n \xrightarrow{\mathcal{L}} \xi$ to denote convergence in law of a sequence of random variables $(\xi_n)_{n\in\mathbb{N}}$ (defined on possibly different probability spaces) to a limit random variable ξ.

We define by $\mathrm{Y}^T_{\mathcal{P}}(E)$ (resp. by $\mathrm{Y}^T_{\leq 1}(E)$) the spaces of measurable flows of probability (resp. sub-probability) measures on E, i.e. the space of Borel measurable maps π (resp. μ) from the time interval $[0,T]$ to $\mathcal{P}(E)$ (resp. $\mathcal{M}_{\leq 1}(E)$). Wherever possible without confusion, we use $\mathrm{Y}^T_{\mathcal{P}}$ (resp. $\mathrm{Y}^T_{\leq 1}$) when $E = \mathbb{R}^d$. We denote by $\mathcal{P}_1(E)$ and by $\mathcal{M}_{\leq 1,1}(E)$ the following subsets of $\mathcal{P}(E)$ and $\mathcal{M}_{\leq 1}(E)$:

$$\begin{aligned}
\mathcal{P}_1(E) &\doteq \left\{ \pi \in \mathcal{P}(E) : \int_E d_E(x,x_0)\pi(\mathrm{d}x) < \infty \text{ for some } x_0 \in E \right\}, \\
\mathcal{M}_{\leq 1,1}(E) &\doteq \left\{ \mu \in \mathcal{M}_{\leq 1}(E) : \int_E d_E(x,x_0)\mu(\mathrm{d}x) < \infty \text{ for some } x_0 \in E \right\}.
\end{aligned}$$

We endow $\mathcal{P}_1(E)$ with the 1-Wasserstein distance W_1

$$W_1(\mu,\nu) \doteq \inf_{\pi\in\Pi(\mu,\nu)} \int_{E\times E} d_E(x,y)\,\mathrm{d}\pi(x,y) = \sup_{f\in \mathrm{Lip}_1(E;\mathbb{R})} \int_E f(x)\mathrm{d}(\mu-\nu)(x) \tag{2.3}$$

where $\Pi(\mu,\nu) \subset \mathcal{P}_1(E\times E)$ represents the set of probability measures with given marginals μ and ν, and $\mathrm{Lip}_1(E;\mathbb{R})$ the set of Lipschitz functions on E with unitary Lipschitz constant. The second equality in Equation (2.3) is due to the Kantorovich-Rubinstein Theorem (see, for instance, Theorem 6.1.1 in Ambrosio et al. (2008)). Notice that $(\mathcal{P}_1(E), W_1)$ is a separable and complete metric space whenever (E, d_E) is separable and complete. Finally, let $\mathrm{Y}^T_{\mathcal{P},1}(E)$ (resp. $\mathrm{Y}^T_{\leq 1,1}(E)$) denote the space of measurable flows of probability measures in $\mathcal{P}_1(E)$ (resp. in $\mathcal{M}_{\leq 1,1}(E)$). Again, wherever possible without confusion, we use $\mathrm{Y}^T_{\mathcal{P},1}$ and $\mathrm{Y}^T_{\leq 1,1}$ when $E = \mathbb{R}^d$.

The canonical space. We will often work on the canonical filtered probability space,

denoted by $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ and defined as follows. Set $\Omega \doteq \mathcal{X}$, let ξ be an $\mathbb{R}^d$-valued random variable with law $\nu \in \mathcal{P}(\mathbb{R}^d)$ and let W be a d-dimensional Wiener process on $\mathcal{X}$ independent of ξ. Define $\mathcal{W}^\nu \in \mathcal{P}(\mathcal{X})$ as the law of $\xi + \sigma W$. Set $\mathcal{F}$ as the $\mathcal{W}^\nu$-completion of the Borel σ-algebra $\mathcal{B}(\mathcal{X})$ and $\mathbb{F} = (\mathcal{F})_{t\in[0,T]}$ as the $\mathcal{W}^\nu$-augmentation of the filtration generated by the canonical process $\hat{X}$ on $\mathcal{X}$, i.e. $\hat{X}_t(\varphi) \doteq \varphi(t)$ for all $(t,\varphi) \in [0,T] \times \mathcal{X}$. In particular, $\mathbb{F}$ satisfies the usual conditions. Finally set $\mathsf{P} \doteq \mathcal{W}^\nu$ and $W \doteq \sigma^{-1}(\xi - \hat{X})$, which is a Wiener process on $\mathcal{X}$. Where no confusion is possible, we will write X for $\hat{X}$.
The extended canonical probability space. When dealing with relaxed controls we will work on the following extension of the canonical probability space $\mathcal{X}$. Set $\widetilde{\Omega} \doteq \mathcal{X} \times \mathcal{V}$, let $\mathcal{F}$ and $\mathbb{F}$ be the canonical σ-algebra and the canonical filtration on $\mathcal{X}$, respectively, whereas $\mathcal{G}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ denote the Borel σ-algebra and the filtration generated by the canonical process $\hat{\Lambda}$ on $\mathcal{V}$, respectively. Finally, we set $\widetilde{\mathcal{F}}_t \doteq \mathcal{F}_t \otimes \mathcal{G}_t$ for all $t \in [0,T]$, and $\widetilde{\mathcal{F}} \doteq \mathcal{F} \otimes \mathcal{G}$.

2.2 The mean-field game with absorption: setting and assumptions

In this section, we describe the MFG with smooth dependence on past absorptions and give the definition of solution of the MFG. We also introduce the MFGs with truncated coefficients, which will be used in the proof of existence of MFG solutions.

Now, let $\mathcal{O} \subset \mathbb{R}^d$ be a non-empty open set, the set of non-absorbing states, and let $\Gamma \subset \mathbb{R}^d$ be the set of control actions. For each $\varphi \in \mathcal{X}$ we set $\tau^\varphi \doteq \inf\{t \in [0,T] : \varphi(t) \notin \mathcal{O}\}$, with the convention $\inf \varnothing = \infty$, and $\tau(\varphi) \doteq \tau^\varphi \wedge T$. In order to set up the dynamics of the players' states, we need to introduce the following functions:

$$\bar{b} : [0,T] \times \mathbb{R}^d \times \mathcal{M}_{\leq 1,1}(\mathbb{R}^d) \times \Gamma \to \mathbb{R}^d, \qquad \sigma \in \mathbb{R}^{d\times d},$$
$$\bar{f} : [0,T] \times \mathbb{R}^d \times \mathcal{M}_{\leq 1,1}(\mathbb{R}^d) \times \Gamma \to [0,\infty), \qquad F : [0,T] \times \mathbb{R}^d \to [0,\infty).$$

Since we will have to impose some joint continuity property for the functions above, in particular with respect to the μ-variable, and there is no natural metrizable topology over the set of sub-probability measures $\mathcal{M}_{\leq 1,1}(\mathbb{R}^d)$, it will be convenient to work with the following reparameterization of a suitable restriction of $\bar{b}$ and $\bar{f}$:

$$\begin{aligned} b(t,\varphi,\theta,u) &\doteq \bar{b}(t,\varphi(t),g(t,\theta),u), \\ f(t,\varphi,\theta,u) &\doteq \bar{f}(t,\varphi(t),g(t,\theta),u) \end{aligned}$$

where b and f are progressively measurable functionals such that

$$b : [0,T] \times \mathcal{X} \times \mathcal{P}_1(\mathcal{X}) \times \Gamma \to \mathbb{R}^d,$$
$$f : [0,T] \times \mathcal{X} \times \mathcal{P}_1(\mathcal{X}) \times \Gamma \to [0,\infty)$$

while $g : [0,T] \times \mathcal{P}_1(\mathcal{X}) \to \mathcal{M}_{\leq 1,1}(\mathbb{R}^d)$ is defined by its action on the test functions of the 1-Wasserstein convergence, i.e., on the functions $\psi \in C(\mathbb{R}^d)$ with sub-linear growth, as

$$\int_{\mathbb{R}^d} \psi(x) g(t,\theta)(\mathrm{d}x) \doteq \int_{\mathcal{X}} \psi(\varphi(t)) \mathbf{1}_{[0,\tau^\varphi)}(t) \theta(\mathrm{d}\varphi). \tag{2.4}$$

In words, the functions b and f above are reparameterizatons of the *restrictions* of $\bar{b}$ and $\bar{f}$, respectively, to the range of the map

$$(t, \varphi, \theta, u) \mapsto (t, \varphi(t), g(t, \theta), u).$$

Moreover, for each $\mu \in \mathcal{M}_{\leq 1,1}(\mathbb{R}^d)$ and $\theta \in \mathcal{P}_1(\mathcal{X})$ we introduce the notation

$$m(\mu) \doteq \int_{\mathbb{R}^d} |x| \mu(\mathrm{d}x) \quad \text{and} \quad m(t; \theta) \doteq \int_{\mathcal{X}} |\varphi(t)| \mathbf{1}_{[0,\tau^\varphi)}(t) \theta(\mathrm{d}\varphi).$$

Now, we collect the necessary assumptions on all initial data in order to state our main results. Some further assumptions will be given later in the chapter when necessary.

(H1) The drift $\bar{b}$ satisfies the following uniform Lipschitz continuity:

$$\left|\bar{b}(t, x, \mu, u) - \bar{b}(t, x', \mu, u)\right| \leq L|x - x'|, \quad x, x' \in \mathbb{R}^d$$

for any $(t, \mu, u) \in [0, T] \times \mathcal{M}_{\leq 1,1}(\mathbb{R}^d) \times \Gamma$. Moreover it has sub-linear growth, i.e.

$$\left|\bar{b}(t, x, \mu, u)\right| \leq C\,(1 + |x| + m(\mu))$$

for all $(t, x, \mu, u) \in [0, T] \times \mathbb{R}^d \times \mathcal{M}_{\leq 1,1}(\mathbb{R}^d) \times \Gamma$ and for a positive constant $C > 0$.

(H2) The running costs $\bar{f}$ and the terminal cost F have sub-linear growth, i.e.

$$\begin{aligned} \bar{f}(t, x, \mu, u) &\leq C(1 + |x| + m(\mu)), \\ F(t, x) &\leq C(1 + |x|), \end{aligned}$$

for all $(t, x, \mu, u) \in [0, T] \times \mathbb{R}^d \times \mathcal{M}_{\leq 1,1}(\mathbb{R}^d) \times \Gamma$, $(t, x) \in [0, T] \times \mathbb{R}^d$ and for a positive constant $C > 0$.

(H3) $\bar{b}$ and $\bar{f}$ are such that their reparametrizations b and f are jointly continuous at points $(t, \varphi, \theta, u) \in [0, T] \times \mathcal{X} \times \mathcal{P}_1(\mathcal{X}) \times \Gamma$ such that $\theta \ll \mathcal{W}^\nu$. Moreover, F is jointly continuous on $[0, T] \times \mathbb{R}^d$.

(H4) The set $\mathcal{O}$ is open, convex and strictly included in $\mathbb{R}^d$ with $\mathcal{C}^2$-boundary, i.e. $\partial\mathcal{O}$ is the graph of a $\mathcal{C}^2$ function. Alternatively, $\mathcal{O} = (0, \infty)^{\times d}$ is also allowed.

(H5) The set $\Gamma \subset \mathbb{R}^d$ is compact.

(H6) The diffusion matrix $\sigma \in \mathbb{R}^{d \times d}$ has full rank.

(H7) The initial distribution $\nu \in \mathcal{P}(\mathbb{R}^d)$ has support in $\mathcal{O}$ and satisfies $\int_{\mathcal{O}} \mathrm{e}^{\lambda |x|^2} \nu(\mathrm{d}x) < \infty$ for some $\lambda > 0$.

(H8) The initial conditions of the N-player game $X_0^{N,i}$, $i \in \{1, \ldots, N\}$, are i.i.d. and with the initial condition of the MFG X_0, they are all distributed as $\nu \in \mathcal{P}(\mathbb{R}^d)$.

Before turning to the MFG dynamics, some remarks on the assumptions above are in order.

Remark 2.2.1. The growth assumptions in (H1) and (H2) could be further refined. For instance, one could assume sub-linear and sub-polynomial growth of the drift and diffusion matrix with suitable exponents as, e.g., in Lacker (2015). Moreover, the running cost f could certainly take real values; however, without loss of generality and given the interpretation as a cost term, we have assumed $f \geq 0$.

Remark 2.2.2. The continuity properties in (H3) are crucial in the passage to the limit performed in Proposition 2.3.2. Since the laws of the processes that we consider are absolutely continuous with respect to the Wiener measure $\mathcal{W}^{\nu}$ (they belong to the set $\mathsf{Q} \subset \mathcal{P}(\mathcal{X})$ of laws of Brownian-driven processes with sub-linear drift that we introduce and characterize in the Appendix A, cfr. Lemma A.0.3), it is sufficient to require continuity at points $\theta \ll \mathcal{W}^{\nu}$. The passage to the limit in the measure argument can then be performed by Lemma A.0.4 together with Lemma A.0.5.

Remark 2.2.3. Admittedly, compactness of Γ is a strong assumption, but it will play an important role in order to obtain existence and uniqueness of weak solutions of the SDEs for the player state's dynamics in both the MFG and the N-player games. In particular, it enables a line of arguments based on Beněs' condition – ensured by the boundedness of the coefficient in the control variable – and Girsanov's theorem (see Remark 2.2.5 for more precise references), which is one of the main tools of our approach.

Remark 2.2.4. The nondegeneracy of σ as in (H6) is justified by the counter-example in Campi and Fischer (2018), Section 7, where it was shown that a feedback MFG solution does not necessarily induce a sequence of approximate Nash equilibria with vanishing error. A careful inspection of such a counter-example reveals that it can be easily adapted to our setting since, in that particular context, dividing by the initial number of players N (as in our setting) or renormalizing each time by the current number of players (as in the counter-example) turn out to be equivalent for N large. Finally, even though state dependency of the diffusion matrix can be handled using very similar techniques, we have decided to leave it out and focus on other more interesting aspects of the model. For the same reason we leave aside a possible dependence of σ on the control, as it would just increase the level of technicality of the proofs due to the use of martingale measures (see Lacker (2015)).

The mean-field dynamics. Given a flow of sub-probability measures $\mu \in \mathrm{Y}^{T}_{\leq 1,1}$ and a feedback progressively measurable control $u : [0,T] \times \mathcal{X} \to \Gamma$, the representative player's state evolves according to the equation

$$X_t = X_0 + \int_0^t \bar{b}\left(s, X_s, \mu_s, u\left(s, X\right)\right) \mathrm{d}s + \sigma W_t, \quad t \in [0,T], \tag{2.5}$$

where X is a d-dimensional stochastic process starting at $X_0 \overset{d}{\sim} \nu \in \mathcal{P}(\mathbb{R}^d)$ and W is a d-dimensional Wiener process on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$. Solutions of Equation (2.5) are understood to be in the weak sense (see Remark 2.2.5 below).
Let $\mathcal{U}_{fb}$ denote the set of all *feedback controls* defined as

$$\mathcal{U}_{fb} \doteq \{u : [0,T] \times \mathcal{X} \to \Gamma : \ u \text{ is progressively measurable}\}.$$

The cost associated with a strategy $u \in \mathcal{U}_{fb}$, a flow of sub-probability measures $\mu \in \mathrm{Y}^{T}_{\leq 1,1}$ and an initial distribution $\nu \in \mathcal{P}(\mathbb{R}^d)$ is given by (we omit, for the sake of simplicity, the explicit dependence on ν)

$$J^{\mu}\left(u\right) \doteq \mathsf{E}\left[\int_0^{\tau} \bar{f}\left(s, X_s, \mu_s, u\left(s, X\right)\right) \mathrm{d}s + F\left(\tau, X_{\tau}\right)\right] \tag{2.6}$$

where $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P}, W, X)$ is a solution of Equation (2.5) under u with initial distribution ν, and $\tau \doteq \tau^X \wedge T$ the random time horizon. Finally we set

$$V^\mu \doteq \inf_{u\in\mathcal{U}_{fb}} J^\mu(u).$$

Remark 2.2.5. For a given flow of sub-probability measures μ, thanks to the linear growth of $\bar{b}$ in the state variable φ and to the boundedness of the action space Γ, we have that both existence and uniqueness in law of a weak solution of Equation (2.5) is guaranteed by Lemma A.0.1, and by Proposition 5.3.6, Remark 5.3.8 and Proposition 5.3.10 in Karatzas and Shreve (1987) (see our Lemma A.0.2). Precisely, this can be proved by means of Girsanov's theorem and Beneš' condition (Beneš, 1971).

The notion of solution we consider for the MFG is the following.

Definition 2.2.1 (*Feedback MFG solution*)**.** A feedback solution of the MFG is a pair $(u, \mu) \in \mathcal{U}_{fb} \times \mathrm{Y}^T_{\leq 1,1}$ such that:

(i) Strategy u is optimal for μ, i.e. $V^\mu = J^\mu(u)$.

(ii) Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P}, X, W)$ is a weak solution of Equation (2.5) with flow of sub-probability measures μ, strategy u and initial condition ν. Then

$$\mu_t(\cdot) = \mathsf{P}(\{X_t \in \cdot\} \cap \{\tau^X > t\}), \quad t \in [0, T].$$

Relaxed controls. It will be very convenient to use relaxed controls (see El Karoui et al. (1987) for a precise definition), which allow us to view progressively measurable controls with values on a compact set Γ as elements of the space of probability measures on Γ. The latter space is compact when endowed with the weak convergence of measures. The space $\mathcal{V}$ of relaxed controls is given by

$$\mathcal{V} \doteq \{q \in \mathcal{M}_f([0,T] \times \Gamma) : q(\mathrm{d}t, \mathrm{d}\gamma) = \mathrm{d}t q_t(\mathrm{d}\gamma),\, t \mapsto q_t \in \mathcal{P}(\Gamma) \text{ Borel measurable}\}$$

i.e. it is the set of all finite positive measures on $[0, T] \times \Gamma$ with Lebesgue time marginal. With a slight abuse of notation, we denote with $\hat{\Lambda}$ both the identity map and the canonical process on $\mathcal{V}$ (where no confusion is possible, we drop the hat and write Λ in place of $\hat{\Lambda}$). Precisely, a single-player relaxed control is a $\mathcal{V}$-valued random variable Λ such that $(\Lambda_t)_{t\in[0,T]}$ is a progressively measurable $\mathcal{P}(\Gamma)$-valued stochastic process. We say that Λ is a feedback control if there exists a progressively measurable functional $\lambda : [0, T] \times \mathcal{X} \to \mathcal{P}(\mathcal{X})$ such that $\Lambda_t = \lambda(t, X)$ for all $t \in [0, T]$, with X denoting the player's dynamics. Moreover, we say that Λ is a strict and feedback control if there exists $u \in \mathcal{U}_{fb}$ such that $\lambda(t, X) = \delta_{u(t,X)}$ for all $t \in [0, T]$.
Let $\widetilde{\mathcal{U}}_{fb}$ be the set of relaxed feedback controls for the MFG. We rewrite the dynamics and the cost functional of the MFG (Equation (2.5)) and Equation (2.6)) using relaxed controls:

$$\begin{aligned} X_t &= X_0 + \int_{[0,t]\times\Gamma} \bar{b}\,(s, X_s, \mu_s, u)\, \lambda\,(s, X)\,(\mathrm{d}u)\mathrm{d}s + \sigma W_t, \qquad (2.7) \\ J^\mu\,(\lambda) &= \mathsf{E}\left[\int_{[0,\tau]\times\Gamma} \bar{f}\,(s, X_s, \mu_s, u)\, \lambda\,(s, X)\,(\mathrm{d}u)\mathrm{d}s + F\,(\tau, X_\tau)\right] \end{aligned}$$

where $t \in [0, T]$ and $\lambda \in \widetilde{\mathcal{U}}_{fb}$. Moreover, we extend accordingly the notion of feedback solutions of the MFG.

Definition 2.2.2 (*Relaxed feedback MFG solution*)**.** A relaxed feedback solution of the MFG is a pair $(\lambda, \mu) \in \widetilde{\mathcal{U}}_{fb} \times \mathrm{Y}^T_{\leq 1,1}$ such that:

(i) λ is optimal, i.e. $V^\mu = J^\mu(\lambda)$.

(ii) Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathsf{Q}, X, W)$ be a weak solution of Equation (2.7) with flow of sub-probability measures μ, control λ and initial condition ν. Then

$$\mu_t(\cdot) = \mathsf{Q}(\{X_t \in \cdot\} \cap \{\tau^X > t\}), \quad t \in [0, T].$$

Feedback and open-loop controls. Feedback controls induce stochastic open-loop controls, i.e. tuples $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P}, X, u, W)$ that are weak solutions of

$$X_t = X_0 + \int_0^t \bar{b}\left(s, X_s, \mu_s, u_s\right) \mathrm{d}s + \sigma W_t, \quad t \in [0, T] \tag{2.8}$$

where u is a progressively measurable Γ-valued stochastic process. As a consequence, the computation of the infimum of $J^\mu(\cdot)$ over the class of stochastic open-loop controls would imply a lower value for V^μ. However, thanks to Proposition 2.6 in El Karoui et al. (1987), the two minimization problems are equivalent from the point of view of the value function.
A similar argument holds also in the case of feedback relaxed controls, that induce relaxed stochastic open-loop controls, tuples $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathsf{Q}, X, \Lambda, W)$ that are weak solutions of

$$X_t = X_0 + \int_{[0,t]\times\Gamma} \bar{b}\left(s, X_s, \mu_s, u\right) \Lambda_s(\mathrm{d}u)\mathrm{d}s + \sigma W_t, \quad t \in [0, T] \tag{2.9}$$

where Λ is a progressively measurable $\mathcal{P}(\Gamma)$-valued stochastic process.
In the rest of the chapter we will call $\mathbb{U}$ the set of open-loop controls and, for the sake of brevity and where no confusion is possible, denote with u an element of $\mathbb{U}$ implying the whole tuple $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P}, X, u, W)$. Similarly, we will call $\widetilde{\mathbb{U}}$ the set of open-loop relaxed controls and denote with Λ an element of $\widetilde{\mathbb{U}}$ implying the whole tuple $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathsf{Q}, X, \Lambda, W)$.

Approximating MFGs. We conclude this preliminary section by introducing a suitable sequence of approximating MFGs, which is obtained by truncation of the coefficients of the original MFG similarly as in Lacker (2015). Such a sequence will be useful in the proof of existence of a MFG solution along the following lines: we will prove existence of feedback MFG solutions of the approximating MFGs in the sequence by extending the existence result of Campi and Fischer (2018). Then, by letting the truncation threshold go to infinity, we will obtain a solution of the original MFG. This approach relies on two additional assumptions (Assumptions (C1) and (C2) below) that will be introduced later in this part.
Let $(K_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $K_n \nearrow +\infty$. The n^{th} approximating MFG model, denoted by MFG(n), is obtained as follows.

($\mathbf{T}_n$) $\bar{b}^n(x) = \bar{b}(x)$ when $|\bar{b}(x)| \leq K_n$, while it is continuously truncated at level K_n, i.e. $|\bar{b}^n(x)| = K_n$, otherwise. Similarly for the costs $\bar{f}^n$ and F^n and for the associated functions b^n and f^n.

Notice that we do not truncate the possibly unbounded set $\mathcal{O}$ of non-absorbing states. In each MFG(n) the representative player's state evolves as in Equation (2.5) with $\bar{b}$ replaced by $\bar{b}^n$, i.e.

$$X_t = X_0 + \int_0^t \bar{b}^n \left(s, X_s, \mu_s, u(s, X)\right) \mathrm{d}s + \sigma W_t, \quad t \in [0, T] \tag{2.10}$$

when the player is using the strict control u, and similarly when using a relaxed control. Moreover, in the cost functional $\bar{f}$ and F are replaced by their truncated counterpart $\bar{f}^n$ and F^n. The associated cost functional is denoted by $J^{n,\mu}(u)$ or $J^{n,\mu}(\lambda)$ depending on whether the player is implementing a strict strategy u or a relaxed one λ. The optimal values are defined, accordingly, by

$$V^{n,\mu} \doteq \inf_{u \in \mathcal{U}_{fb}} J^{n,\mu}(u).$$

The definitions of strict and relaxed MFG solutions given above for the (un-truncated) MFG can clearly be applied to the approximating MFG(n)s with the obvious modifications. We associate to the MFG(n)s the following Hamiltonians:

$$\begin{aligned} h^n(t,x,\theta,z,u) &\doteq f^n(t,x,\theta,u) + z\sigma^{-1} b^n(t,x,\theta,u), \\ H^n(t,x,\theta,z) &\doteq \inf_{u \in \Gamma} h^n(t,x,\theta,z,u) \end{aligned}$$

and the set of minimizers

$$A^n(t,x,\theta,z) \doteq \{u \in \Gamma : h^n(t,x,\theta,z,u) = H^n(t,x,\theta,z)\}$$

for $(t,x,\theta,z) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathcal{X}) \times \mathbb{R}^d$. In the next section on existence of MFG solutions we will rely on the following additional convexity assumptions:

(C1) For each $n \in \mathbb{N}$, $A^n(t,x,\theta,z)$ is convex for all $(t,x,\theta,z) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathcal{X}) \times \mathbb{R}^d$.

(C2) The running cost f is convex in the control variable $u \in \Gamma$.

Remark 2.2.6. Assumption (C1) is common in control theory and it is crucial in order to apply fixed point theorems. In our case it is satisfied if, for instance, the running cost f is bounded and convex in the control variable $u \in \Gamma$. Indeed in this case, due to the flexibility in the choice of the truncation thresholds, choosing $K^n \geq \|f\|_\infty$ for all $n \in \mathbb{N}$ we have $f^n = f$ for all $n \in \mathbb{N}$. Then convexity is preserved by adding any sub-linear term. Finally, we observe that Assumption (C2) will be used in Section 2.3.4 for obtaining the existence of strict MFG solutions.

2.3 Existence of solutions of the mean-field game

Throughout this section Assumptions (H1)-(H8) are in force. Under these and the additional convexity Assumptions (C1) and (C2) we show that both a relaxed and a strict feedback solution of the MFG exist; see Theorem 2.3.1 below together with Proposition 2.3.4 and Proposition 2.3.5. In addition, we guarantee the existence of a feedback solution of the MFG with Markovian feedback strategy up to the exit time; see Proposition 2.3.6. Our main existence result can be stated as follows.

Theorem 2.3.1 (*Existence of relaxed and strict feedback MFG solutions*)**.** *Under Assumptions (H1)-(H8) and (C1), there exists a relaxed feedback MFG solution (λ, μ). Moreover, under the additional Assumption (C2) , there exists a strict feedback MFG solution (u, μ).*

To prove Theorem 2.3.1, we proceed by approximation in the sense that, first, we prove that each MFG(n) introduced in the previous section has a feedback (strict) solution by extending the results in Campi and Fischer (2018); see Subsection 2.3.1. Then, we prove the convergence of such approximating solutions to a feedback (relaxed) solution of the original MFG by passing to the limit with the truncation thresholds; see Subsection 2.3.2.

Before proceeding, we ensure the well-posedness of the game in the sense that we show that the private state X of the representative agent remains in $\mathcal{O}$ up to time T with some positive probability. This is the content of the following lemma.

Lemma 2.3.1. *Grant Assumptions (H1)-(H8). Let* $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P}, X, W)$ *be a weak solution of Equation* (2.5). *Then* $\mathsf{P}(\tau^X > t) > 0$ *for all* $t \in [0,T]$.

Proof. Set $b_t \doteq \bar{b}(t, X_t, \mu_t, u(t,X))$ for $t \in [0,T]$, and define $Z \doteq (Z_t)_{t\in[0,T]}$ as

$$Z_t \doteq \mathcal{E}_t\left(-\int_0^{\cdot} \sigma^{-1} b_s \mathrm{d}W_s\right), \quad t \in [0,T],$$

where $\mathcal{E}_t(\cdot)$ denotes the Doléans-Dade stochastic exponential. By Lemma A.0.1, Z is a true martingale. Define Q by $\frac{d\mathsf{Q}}{d\mathsf{P}} \doteq Z_T$. By Girsanov's theorem $\widetilde{W}_t \doteq W_t + \int_0^t \sigma^{-1} b_s \mathrm{d}s$, $t \in [0,T]$, is a Q-Wiener process, and under Q the process X has law $\mathcal{W}^{\nu}$. As a consequence of the law of iterated logarithms, any Wiener process remains in an open set, hence in $\mathcal{O} \subset \mathbb{R}^d$, for a finite time with strictly positive probability. Therefore $\mathsf{Q}(\tau^X > T) > 0$ and thus $\mathsf{P}(\tau^X > T) > 0$. □

2.3.1 Approximating MFGs

In this subsection we prove existence of solutions of the approximating MFG(n)s.

Theorem 2.3.2 (*Existence of solutions of MFG(n)*)**.** *Let* $n \in \mathbb{N}$. *Under Assumptions (H1)-(H8) and (C1) there exists a feedback solution* (u^n, μ^n) *of MFG(n).*

Proof. The proof follows similar steps to those in Section 6 of Campi and Fischer (2018): we only sketch here the main steps. The main difference with Campi and Fischer (2018) is that, due to Assumption (C1), we have to deal with set-valued maps, hence to apply a version of Kakutani's fixed point theorem instead of Brouwer's. We use the version proposed by Carmona and Lacker (2015), Proposition 7.4, which is in turn based on the results of Cellina (1969). Other adjustments are due to the fact that μ is a flow of sub-probability measures (instead of probability measures) and that $\mathcal{O}$ can be unbounded.
Fix $n \in \mathbb{N}$. The proof is based on the construction of a suitable map $\Psi : \mathcal{P}(\mathcal{X}) \times \mathbb{U} \to \mathcal{P}(\mathcal{X})$ on an appropriate compact and convex subset of $\mathcal{P}(\mathcal{X})$, where $\mathbb{U}$ is the space of progressively measurable Γ-valued stochastic processes. The fixed points of Ψ will provide MFG(n) solutions. More in detail, define $\mathsf{Q}_{\nu,K}$ as the set of laws $\theta \in \mathcal{P}(\mathcal{X})$ of any process of the type

$$\xi + \int_0^t b_s \mathrm{d}s + \sigma W_t, \quad t \in [0,T]$$

defined on some filtered probability space with a Wiener process W, $\xi \overset{d}{\sim} \nu$, drift $(b_t)_{t\in[0,T]}$ adapted and bounded by $K > 0$. Let us consider

$$\Psi : \mathsf{Q}_{\nu,K_n} \times \mathbb{U} \ni (\theta, u) \mapsto \mathsf{P}^{\theta,u} \circ X^{-1} \in \mathsf{Q}_{\nu,K_n},$$

where X is the canonical process on $\mathcal{X}$ and the probability measure $\mathsf{P}^{\theta,u}$ is defined as follows. Let $(\theta, u) \in \mathsf{Q}_{\nu,K_n} \times \mathbb{U}$ and let $\mu^\theta \in \mathsf{Y}^T_{\leq 1}$ be defined as $\mu^\theta_t(\cdot) \doteq \theta(\{X_t \in \cdot\} \cap \{\tau^X > t\})$ for all $t \in [0,T]$. Let $(\Omega, \mathcal{F}^u, \mathbb{F}^u = (\mathcal{F}^u_t)_{t\in[0,T]}, \mathsf{P}^{\theta,u}, X, W^u)$ be the weak solution of

$$X_t = X_0 + \int_0^t \bar{b}^n(s, X_s, \mu^\theta_s, u_s)\,\mathrm{d}s + \sigma W^u_t, \quad t \in [0,T]$$

on the canonical space $(\Omega \doteq \mathcal{X}, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$. Moreover, for $\theta \in \mathsf{Q}_{\nu,K_n}$ we call u^θ an optimal control for the cost

$$J^{n,\mu^\theta}(u) \doteq \mathsf{E}^{\mathsf{P}^{\theta,u}}\left[\int_0^\tau \bar{f}^n(s, X_s, \mu^\theta_s, u_s)\mathrm{d}s + F^n(\tau, X_\tau)\right].$$

Such optimal controls u^θ can be constructed by standard BSDE techniques as in Campi and Fischer (2018), Section 6.1, by means of Darling and Pardoux (1997), Theorem 3.4, due to the random terminal times. Under Assumption (C1) optimal controls u^θ are in general not unique. Indeed

$$A^n(\theta) \doteq \left\{u^\theta \in \mathbb{U} : u^\theta \in A^n(\cdot, X_\cdot, \theta, Z^\theta_\cdot),\ \mathcal{L}_T \otimes \mathsf{P} - a.e.\right\}$$

provides an entire set of optimal controls, where Z^θ is part of the the solution of the associated adjoint BSDE and $\mathcal{L}_T$ denotes the Lebesgue measure on $[0,T]$. Moreover, by measurable selection there exists a measurable function $\hat{u}^{n,\theta} : [0,T] \times \mathbb{R}^d \times \mathsf{Q}_{\nu,K_n} \times \mathbb{R}^d \to \Gamma$ such that

$$\hat{u}^{n,\theta}(\cdot, X_\cdot, \theta, Z^\theta_\cdot) \in A^n(\theta), \quad \mathcal{L}_T \otimes \mathsf{P} - \text{a.e.}$$

Additionally, $\hat{u}^{n,\theta}(t, X_t, \theta, Z^\theta_t)$, for $t \in [0,T]$, is a progressively measurable control process that can be written in feedback form. Indeed, since Z^θ is progressively measurable for the canonical filtration, it can expressed as $Z^\theta_t = \zeta^\theta(t, X)$ for some progressively measurable functional $\zeta^\theta : [0,T] \times \mathcal{X} \to \mathbb{R}^d$ and for any $t \in [0,T]$.
Now, a fixed point for the map Ψ is a probability measure $\theta \in \mathsf{Q}_{\nu,K_n}$ such that $\theta \in \Psi(\theta, A(\theta))$. Existence is provided by Proposition 7.4 in Carmona and Lacker (2015), so to conclude the proof it suffices to check that all the required assumptions are satisfied in our case. The set $\mathsf{Q}_{\nu,K_n} \subset \mathcal{P}(\mathcal{X})$ is a (weakly) compact, convex and metrizable subset of $C^*_b(\mathcal{X})$, the dual of the space of bounded and continuous functions on $\mathcal{X}$, which is a locally convex topological vector space with the weak* topology (that induces the weak convergence of measures on $\mathcal{P}(\mathcal{X})$). We endow the vector space $\mathbb{U}$ with the norm $\|\cdot\|_{\mathbb{U}}$ defined as $\|u\|_{\mathbb{U}} \doteq \mathsf{E}[\int_0^T |u_t|\mathrm{d}t]$. As a consequence of Berge's maximum theorem (Aliprantis and Border, 1994, Theorem 17.31) and of Assumption (C1) the set-valued map $A^n : \mathsf{Q}_{\nu,K_n} \to \mathbb{U}$ is upper hemicontinuous and has non-empty convex and closed values (see the proof of Lemma 7.11 in Carmona and Lacker (2015)). Therefore, Proposition 7.4 in Carmona and Lacker (2015) applies, yielding the existence of a feedback solution of MFG(n). □

A-priori estimates. Here, we show that the moments up to any order $\alpha \geq 1$ of the state process remain bounded uniformly in n. Such estimates will be very useful when we will relax the truncation in the next section.

Lemma 2.3.2 (*A-priori estimates*). *Grant Assumptions (H1)-(H8) and (C1). Consider feedback solutions $(u^n, \mu^n)_{n\in\mathbb{N}}$ and (u, μ) of the MFG(n)'s and of the MFG, respectively. Let*

$(\Omega^n, \mathcal{F}^n, \mathbb{F}^n = (\mathcal{F}^n_t)_{t\in[0,T]}, \mathsf{P}^n, X^n, W^n)_{n\in\mathbb{N}}$ be a sequence of weak solutions of the SDEs in Equation (2.10) and $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P}, X, W)$ a weak solution of the SDE in Equation (2.5). Then for any $\alpha \geq 1$

$$\sup_{n\in\mathbb{N}} \mathsf{E}^{\mathsf{P}^n}\left[\|X^n\|^\alpha_\infty\right] \leq K(\alpha) \quad \textit{and} \quad \mathsf{E}^{\mathsf{P}}\left[\|X\|^\alpha_\infty\right] \leq K(\alpha)$$

where $K(\alpha) < \infty$ is a positive constant independent of n.

Proof. This follows from standard estimates that rely on the drift's sub-linear growth and on Grönwall's lemma. □

2.3.2 Convergence of the approximating MFGs

Let $(u^n, \mu^n)_{n\in\mathbb{N}}$ be a sequence of feedback solutions of the approximating MFGs introduced in the previous Subsection 2.3.1, whose existence is guaranteed by Theorem 2.3.2. In addition, let $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n = (\mathcal{F}^n_t)_{t\in[0,T]}, \mathsf{P}^n, X^n, W^n)_{n\in\mathbb{N}}$ be a sequence of weak solutions of the SDEs in Equation (2.10) associated to $(u^n, \mu^n)_{n\in\mathbb{N}}$. Let θ^n be defined as $\theta^n \doteq \mathsf{P}^n \circ (X^n)^{-1}$ for each $n \in \mathbb{N}$.

To prove the convergence of the approximating MFGs we proceed in the following way. First, we show that there exists a subsequence of $(\theta^n)_{n\in\mathbb{N}}$, say $(\theta^{n_k})_{n_k\in\mathbb{N}}$, that converges in $\mathcal{P}_1(\mathcal{X})$ to some limit $\theta \in \mathcal{P}_1(\mathcal{X})$. To prove this, we interpret $(u^n, \mu^n)_{n\in\mathbb{N}}$ as relaxed feedback solutions, $(\lambda^n, \mu^n)_{n\in\mathbb{N}}$. Second, we show that also the sequence of the corresponding extended laws $(\Theta^n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathcal{X} \times \mathcal{V})$ converges in $\mathcal{P}_1(\mathcal{X} \times \mathcal{V})$ to some limit $\Theta \in \mathcal{P}_1(\mathcal{X} \times \mathcal{V})$. Finally, we characterize the limit points by means of the martingale problem of Stroock and Varadhan (see Stroock and Varadhan (1969, 2007)).

Lemma 2.3.3 (*Relative compactness*)**.** *$(\theta^n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{P}(\mathcal{X})$.*

Proof. First, we prove tightness by applying Aldous' criterion (see, e.g., Jacod and Shiryaev (2013), Condition VI.4.4), that is

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{\tau\leq\sigma\leq\tau+\delta} \mathsf{P}^n\left(|X^n_\sigma - X^n_\tau| \geq r\right) = 0$$

for all $r > 0$ and where τ and σ are stopping times bounded by T. Indeed, we have

$$\mathsf{P}^n\left(|X^n_\sigma - X^n_\tau| \geq r\right) \leq \frac{\mathsf{E}^{\mathsf{P}^n}\left[|X^n_\sigma - X^n_\tau|\right]}{r}$$

and

$$\begin{aligned}
\mathsf{E}^{\mathsf{P}^n}\left[|X^n_\sigma - X^n_\tau|\right] &\leq \mathsf{E}^{\mathsf{P}^n}\left[\int_\tau^{(\tau+\delta)\wedge T} \left|\bar{b}^n(t, X^n_t, \mu^n_t, u^n(t, X^n))\right| \mathrm{d}t\right] + |\sigma|((\tau+\delta)\wedge T - \tau)^{\frac{1}{2}} C^W_T \\
&\leq \mathsf{E}^{\mathsf{P}^n}\left[C\int_\tau^{(\tau+\delta)\wedge T} (1 + \|X^n\|_{\infty,t} + \sup_{n\in\mathbb{N}} \mathsf{E}^{\mathsf{P}^n}\|X^n\|_{\infty,t} + |u^n(t, X^n)|)\mathrm{d}t\right] \\
&\quad + |\sigma|((\tau+\delta)\wedge T - \tau)^{\frac{1}{2}} C^W_T \\
&\leq \mathsf{E}^{\mathsf{P}^n}\left[C\int_\tau^{(\tau+\delta)\wedge T} (1 + \|X^n\|_\infty + K + |u^n(t, X^n)|)\mathrm{d}t\right] \\
&\quad + |\sigma|((\tau+\delta)\wedge T - \tau)^{\frac{1}{2}} C^W_T
\end{aligned}$$

for some constants $C^W_T, K > 0$ independent of $n \in \mathbb{N}$. Then we conclude by Lemma 2.3.2. Relative compactness then follows from Prohorov's Theorem. □

Now, let $\theta \in \mathcal{P}(\mathcal{X})$ be a limit point for $(\theta^n)_{n\in\mathbb{N}}$ and let $(\theta^{n_k})_{n_k\in\mathbb{N}}$ be a subsequence of $(\theta^n)_{n\in\mathbb{N}}$ such that $\theta^{n_k} \xrightarrow{w} \theta$ as $n_k \to \infty$. With a slight abuse of notation, in what follows we identify $(\theta^{n_k})_{n_k\in\mathbb{N}}$ with $(\theta^n)_{n\in\mathbb{N}}$. We now show that the latter convergence is actually stronger by proving that $(\theta^n)_{n\in\mathbb{N}}$ converges to θ in the 1-Wasserstein distance.

Lemma 2.3.4 (*Convergence in the 1-Wasserstein distance*)**.** *Let $(\theta^n)_{n\in\mathbb{N}}$ be as above. Then $W_1(\theta^n, \theta) \to 0$ and $\theta \in \mathcal{P}_1(\mathcal{X})$.*

Proof. Notice that by Lemma 2.3.2 we have $(\theta^n)_{n\in\mathbb{N}} \subset \mathcal{P}_1(\mathcal{X})$. To prove convergence in the 1-Wasserstein distance, we have to show that (see, for instance, Theorem 7.12.ii in Villani (2003))

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \mathsf{E}^{\mathsf{P}^n}\left[\|X^n\|_\infty \mathbf{1}_{\{\|X^n\|_\infty \geq R\}}\right] = 0.$$

Set $\alpha, \beta > 1$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then, for any $\varepsilon > 0$ by Young's and Markov's inequalities, and by Lemma 2.3.2 we have

$$\begin{aligned} \mathsf{E}^{\mathsf{P}^n}\left[\|X^n\|_\infty \mathbf{1}_{\{\|X^n\|_\infty \geq R\}}\right] &\leq \varepsilon^\alpha \frac{\mathsf{E}^{\mathsf{P}^n}[\|X^n\|_\infty^\alpha]}{\alpha} + \frac{\mathsf{P}^n(\|X^n\|_\infty \geq R)}{\varepsilon^\beta \beta} \\ &\leq \varepsilon^\alpha \frac{K(\alpha)}{\alpha} + \frac{K}{\varepsilon^\beta \beta R} \end{aligned}$$

for some positive constants $K(\alpha)$ and K independent of $n \in \mathbb{N}$. The conclusion immediately follows thanks to the fact that convergence in the 1-Wasserstein distance preserves the finiteness of the first moment. □

Proposition 2.3.1 (*Absolute continuity of limit measures*)**.** *Let $\theta, (\theta^n)_{n\in\mathbb{N}} \subset \mathcal{P}_1(\mathcal{X})$ be as in Lemma 2.3.4. Then $\theta \ll \mathcal{W}^\nu$, i.e. θ is absolutely continuous with respect to $\mathcal{W}^\nu$.*

Proof. By construction $\theta^n \ll \mathcal{W}^\nu$ for all $n \in \mathbb{N}$, hence we have to make sure that the absolute continuity is also preserved in the limit. For doing so, we apply Theorem X.3.3 in Jacod and Shiryaev (2013). In particular, we have to verify that all assumptions therein are fulfilled, which in our setting are reduced to the following properties:

(i) The contiguity of the sequence of θ^n with respect to the Wiener measure $\mathcal{W}^\nu$, i.e. for any sequence of measurable sets B_n with $\mathcal{W}^\nu(B_n) \to 0$ we have $\theta^n(B_n) \to 0$ as $n \to \infty$ (see, e.g., Definition V.1.1 in Jacod and Shiryaev (2013)).

(ii) The tightness of the sequence of $\mathcal{W}^\nu$-martingales $(M^n)_{n\in\mathbb{N}}$, where each $M^n = (M^n_t)_{t\in[0,T]}$ is defined as

$$M^n_t \doteq \mathcal{E}_t\left(\int_0^\cdot \sigma^{-1}\bar{b}^n(s, X_s, \mu^n_s, u^n(s, X))\mathrm{d}W_s\right), \quad t \in [0, T].$$

In order to check property (i), we first show that the sequence of Radon-Nikodym derivatives $(\frac{d\theta^n}{d\mathcal{W}^\nu})_{n\in\mathbb{N}}$ is uniformly integrable under $\mathcal{W}^\nu$. This is a consequence of the following bound:

$$\sup_{n\in\mathbb{N}} \mathsf{E}^{\mathcal{W}^\nu}\left[\left(\frac{d\theta^n}{d\mathcal{W}^\nu}\right)^p\right] < \infty, \quad p \in [1, \infty) \tag{2.11}$$

which follows from Corollary A.0.1 and by fact that, by inspection of the proofs of Lemma A.0.1 and Corollary A.0.1, all bounds are uniform in $n \in \mathbb{N}$.
Now, property (i) can be obtained as follows: for all sequences of measurable sets B_n with $\mathcal{W}^\nu(B_n) \to 0$, we have

$$\theta^n(B_n) = \mathbb{E}^{\mathcal{W}^\nu}\left[\frac{d\theta^n}{d\mathcal{W}^\nu}\mathbf{1}_{B_n}\right] \to 0, \quad n \to \infty,$$

by an application of dominated convergence theorem due to the bound in Equation (2.11). Hence the sequence of measures θ^n is contiguous to $\mathcal{W}^\nu$.
Property (ii) follows from Aldous criterion (Jacod and Shiryaev, 2013, Condition VI.4.4), that is

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{\tau \le \sigma \le \tau + \delta} \mathcal{W}^\nu\left(|M^n_\sigma - M^n_\tau| \ge r\right) = 0 \tag{2.12}$$

for all $r > 0$ and where τ and σ are stopping times bounded by T. As a consequence, we will also have the tightness property for the pair $(X, M^n)_{n\in\mathbb{N}}$ under the measure $\mathcal{W}^\nu$. By Theorem VI.4.13 in Jacod and Shiryaev (2013) it is sufficient to check the tightness property for the corresponding quadratic variation processes

$$\langle M^n \rangle_t = \int_0^t \left|\sigma^{-1}\bar{b}^n(s, X_s, \mu^n_s, u^n(s, X))M^n_s\right|^2 \mathrm{d}s, \quad t \in [0, T].$$

First, by Markov's inequality $\mathcal{W}^\nu(|\langle M^n\rangle_\sigma - \langle M^n\rangle_\tau| \ge r) \le \frac{1}{r}\mathsf{E}^{\mathcal{W}^\nu}[|M^n_\sigma - M^n_\tau|]$. Then, by Young's inequality for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned}
\mathsf{E}^{\mathcal{W}^\nu}\left[|\langle M^n\rangle_\sigma - \langle M^n\rangle_\tau|\right] &\le \mathsf{E}^{\mathcal{W}^\nu}\left[\int_\tau^{(\tau+\delta)\wedge T}\left|\sigma^{-1}\right|^2\left|\bar{b}^n(s, X_s, \mu^n_s, u^n(s, X))\right|^2|M^n_s|^2\,\mathrm{d}s\right] \\
&\le \frac{1}{p}\left|\sigma^{-1}\right|^2\int_\tau^{(\tau+\delta)\wedge T}\mathsf{E}^{\mathcal{W}^\nu}\left[\left|\bar{b}^n(s, X_s, \mu^n_s, u^n(s, X))\right|^{2p}\right]\mathrm{d}s \\
&\quad + \frac{1}{q}\left|\sigma^{-1}\right|^2\int_\tau^{(\tau+\delta)\wedge T}\mathsf{E}^{\mathcal{W}^\nu}\left[|M^n_s|^{2q}\right]\mathrm{d}s \\
&\le \left(\frac{K(p)}{p} + \frac{K(q)}{q}\right)\left|\sigma^{-1}\right|^2((\tau+\delta)\wedge T - \tau)
\end{aligned}$$

for some positive constants $K(p)$ and $K(q) > 0$ independent of $n \in \mathbb{N}$. Notice that the last inequality is a consequence of Lemma 2.3.2 and Property (i). Therefore, Aldous' criterion in Equation (2.12) is satisfied.

After checking properties (i) and (ii) above, we can at last apply Theorem X.3.3 in Jacod and Shiryaev (2013), yielding that the tightness of $(\mathcal{W}^\nu \circ (X, M^n)^{-1})_{n\in\mathbb{N}}$ implies the tightness of $(\theta^n \circ (X, M^n)^{-1})_{n\in\mathbb{N}}$. In particular, if $(\mathcal{W}^\nu \circ (X, M^n)^{-1})_{n\in\mathbb{N}}$ weakly converges to some Θ' in $\mathcal{P}(\mathcal{X} \times \mathcal{X})$ then $(\theta^n \circ (X, M^n)^{-1})_{n\in\mathbb{N}}$ weakly converges to some other $\Theta'' \ll \Theta'$ in $\mathcal{P}(\mathcal{X} \times \mathcal{X})$, and the same holds true for their first marginals on $\mathcal{X}$. Therefore, we can conclude that $\theta \ll \mathcal{W}^\nu$. □

Compactification method. So far we have established the convergence of the laws $(\theta^n)_{n\in\mathbb{N}}$ to some limit law θ in the 1-Wasserstein distance. Now, in order to prove the convergence of the approximating feedback solutions $(u^n, \mu^n)_{n\in\mathbb{N}}$ to some feedback MFG solution (u, μ), we need to show that the sequence of optimal controls $(u^n)_{n\in\mathbb{N}}$ converges to a control u, which is optimal for the limit game.

To do this, we interpret the sequence of strict feedback solutions $(u^n, \mu^n)_{n\in\mathbb{N}}$ as a sequence of relaxed feedback solutions $(\lambda^n, \mu^n)_{n\in\mathbb{N}}$, by defining $\lambda^n : [0,T] \times \mathcal{X} \to \mathcal{P}(\Gamma)$ as $\lambda^n(t,\varphi) \doteq \delta_{u^n(t,\varphi)}$ for all $(t,\varphi) \in [0,T] \times \mathcal{X}$ and for all $n \in \mathbb{N}$. Furthermore, we identify each λ^n with a stochastic relaxed control Λ^n. We then fix a sequence of associated weak solutions $(\widetilde{\Omega}^n, \widetilde{\mathcal{F}}^n, \widetilde{\mathbb{F}}^n = (\widetilde{\mathcal{F}}^n_t)_{t\in[0,T]}, \mathrm{Q}^n, X^n, W^n)$ of Equation (2.7) and set $\Theta^n \doteq \mathrm{Q}^n \circ (X^n, \Lambda^n)^{-1} \in \mathcal{P}(\mathcal{X} \times \mathcal{V})$ for all $n \in \mathbb{N}$. Finally, we associate to each MFG(n) and to the limit MFG a martingale problem (Stroock and Varadhan (1969, 2007)) and show that the limit points $\Theta \in \mathcal{P}(\mathcal{X} \times \mathcal{V})$ of $(\Theta^n)_{n\in\mathbb{N}}$ solve the limit relaxed martingale problem. We start with the following lemma.

Lemma 2.3.5 (*Tightness in the 1-Wasserstein distance and absolute continuity*). *Let $(\Theta^n)_{n\in\mathbb{N}}$ be as above. Then the following two properties hold:*

(i) $(\Theta^n)_{n\in\mathbb{N}}$ *is tight in* $\mathcal{P}_1(\mathcal{X} \times \mathcal{V})$*;*

(ii) Any limit point Θ of the sequence $(\Theta^n)_{n\in\mathbb{N}}$ in $\mathcal{P}_1(\mathcal{X} \times \mathcal{V})$ satisfies $\Theta \circ X^{-1} \ll \mathcal{W}^\nu$.

Proof. (i). It follows from Lemma 2.3.4 and the compactness of Γ.

(ii). This is a consequence of Proposition 2.3.1, the fact that by construction $\theta^n = \Theta^n \circ X^{-1}$ for all $n \in \mathbb{N}$, and the fact that weak convergence of the joint laws implies weak convergence of the marginals. □

By the previous lemma, we can assume without loss of generality that the original sequence $(\Theta^n)_{n\in\mathbb{N}}$ converges to some limit measure Θ in $\mathcal{P}_1(\mathcal{X} \times \mathcal{V})$. In order to characterize the limit point Θ, we associate to each approximating MFG(n) and to the limit MFG a (relaxed) martingale problem, henceforth RM(n) and RM, respectively. Then, we show that Θ is also a solution of RM. We will use the notation Dg and D^2g for the gradient and the Hessian of a smooth function $g : \mathbb{R}^d \to \mathbb{R}$, while $\mathrm{Tr}[A]$ denote the trace of a square matrix A. Notice that in the following definition we have used the repameterization b of the drift $\bar{b}$.

Definition 2.3.1 (*The approximating martingale problems (RM(n))*). We say that $\widehat{\Theta} \in \mathcal{P}(\mathcal{X} \times \mathcal{V})$ is a solution of RM(n) if for all $g \in C^2_c(\mathbb{R}^d)$ the process

$$
\begin{aligned}
M^{n,g}_t(\varphi, q; \widehat{\Theta}) \doteq g(\varphi(t)) - g(\varphi(0)) &- \int_{[0,t]\times\Gamma} b^n(s, \varphi, \hat{\theta}, u)^\top Dg(\varphi(s)) q(\mathrm{d}s, \mathrm{d}u) \\
&- \frac{1}{2} \int_0^t \mathrm{Tr}\left[\sigma\sigma^\top D^2 g(\varphi(s))\right] \mathrm{d}s, \quad t \in [0,T]
\end{aligned}
$$

is a $\widehat{\Theta}$-martingale, where $\hat{\theta} \doteq \widehat{\Theta} \circ X^{-1}$ and X is the canonical process on $\mathcal{X}$.

Observe that, by construction, each Θ^n solves RM(n). In Proposition 2.3.2 below we will characterize the limit points as solutions of the following (relaxed) martingale problem.

Definition 2.3.2 (*The limit martingale problem (RM)*). We say that $\widehat{\Theta} \in \mathcal{P}(\mathcal{X} \times \mathcal{V})$ is a solution of RM if for all $g \in C^2_c(\mathbb{R}^d)$ the process

$$
\begin{aligned}
M^{g}_t(\varphi, q; \widehat{\Theta}) \doteq g(\varphi(t)) - g(\varphi(0)) &- \int_{[0,t]\times\Gamma} b(s, \varphi, \hat{\theta}, u)^\top Dg(\varphi(s)) q(\mathrm{d}s, \mathrm{d}u) \\
&- \frac{1}{2} \int_0^t \mathrm{Tr}\left[\sigma\sigma^\top D^2 g(\varphi(s))\right] \mathrm{d}s, \quad t \in [0,T]
\end{aligned}
$$

is a $\widehat{\Theta}$-martingale, where $\hat{\theta} \doteq \widehat{\Theta} \circ X^{-1}$.

Remark 2.3.1. The martingale property in both RM(n) and in RM is understood to hold on $(\mathcal{X} \times \mathcal{V}, \mathcal{B}(\mathcal{X} \times \mathcal{V}))$ with respect to the Θ-augmentation of the canonical filtration made right continuous by a standard procedure. Nonetheless, to conclude it is sufficient to check that the martingale property holds with respect to the canonical filtration on $\mathcal{X} \times \mathcal{V}$ (see, for instance, Problem 5.4.13 in Karatzas and Shreve (1987)).

Now, we can characterize the limit points via the martingale problems.

Proposition 2.3.2 (*Characterization of limit points via martingale problems*). *Θ solves RM as in Definition 2.3.2.*

Proof. Fix $t_1, t_2 \in [0, T]$, $t_1 < t_2$, $g \in C_c^2(\mathbb{R}^d)$ and $\psi \in C_b(\mathcal{X} \times \mathcal{V})$ measurable with respect to $\mathcal{B}_{t_1}(\mathcal{X} \times \mathcal{V})$. Define $\Psi, \Psi^n : \mathcal{P}(\mathcal{X} \times \mathcal{V}) \to \mathbb{R}$ as

$$\begin{aligned}\Psi\left(\Theta';\Theta\right) &\doteq \mathrm{E}^{\Theta'}\left[\psi\left(M_{t_2}^g(\cdot\,;\Theta) - M_{t_1}^g(\cdot\,;\Theta)\right)\right],\\ \Psi^n\left(\Theta';\Theta\right) &\doteq \mathrm{E}^{\Theta'}\left[\psi\left(M_{t_2}^{n,g}(\cdot\,;\Theta) - M_{t_1}^{n,g}(\cdot\,;\Theta)\right)\right]\end{aligned}$$

for $\Theta', \Theta \in \mathcal{P}(\mathcal{X} \times \mathcal{V})$ and for all $n \in \mathbb{N}$. Since $\Psi^n(\Theta^n; \Theta^n) = 0$ for all $n \in \mathbb{N}$, it suffices to prove that $\Psi^n(\Theta^n; \Theta^n) \to \Psi(\Theta; \Theta)$ as $n \to \infty$.
First, we observe that $\Psi^n(\Theta^n; \Theta^n)$ and $\Psi(\Theta; \Theta)$ can be written as

$$\begin{aligned}\Psi^n(\Theta^n;\Theta^n) &= \int_{\mathcal{X}\times\mathcal{V}} \psi(\varphi,q) \int_{[t_1,t_2]\times\Gamma} b^n(s,\varphi,\theta^n,u)^\top Dg(\varphi(s))q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q)\\ &\quad+\int_{\mathcal{X}\times\mathcal{V}} \psi(\varphi,q)\int_{t_1}^{t_2}\frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^\top D^2 g(\varphi(s))\right]\mathrm{d}s\,\Theta^n(\mathrm{d}\varphi,\mathrm{d}q)\end{aligned}$$

and

$$\begin{aligned}\Psi(\Theta;\Theta) &= \int_{\mathcal{X}\times\mathcal{V}} \psi(\varphi,q) \int_{[t_1,t_2]\times\Gamma} b(s,\varphi,\theta,u)^\top Dg(\varphi(s))q(\mathrm{d}s,\mathrm{d}u)\Theta(\mathrm{d}\varphi,\mathrm{d}q)\\ &\quad+\int_{\mathcal{X}\times\mathcal{V}} \psi(\varphi,q)\int_{t_1}^{t_2}\frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^\top D^2 g(\varphi(s))\right]\mathrm{d}s\,\theta(\mathrm{d}\varphi,\mathrm{d}q).\end{aligned}$$

The convergence of the diffusion terms is a straightforward consequence of the weak convergence $\Theta^n \xrightarrow{w} \Theta$ and the fact that the map

$$(\varphi,q) \mapsto \psi(\varphi,q)\int_{t_1}^{t_2}\frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^\top D^2 g(\varphi(s))\right]\mathrm{d}s$$

is in $C_b(\mathcal{X} \times \mathcal{V})$, leading to

$$\begin{aligned}&\int_{\mathcal{X}\times\mathcal{V}} \psi(\varphi,q)\int_{t_1}^{t_2}\frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^\top D^2 g(\varphi(s))\right]\mathrm{d}s\,\Theta^n(\mathrm{d}\varphi,\mathrm{d}q)\\ \underset{n\to\infty}{\longrightarrow}&\int_{\mathcal{X}\times\mathcal{V}} \psi(\varphi,q)\int_{t_1}^{t_2}\frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^\top D^2 g(\varphi(s))\right]\mathrm{d}s\,\Theta(\mathrm{d}\varphi,\mathrm{d}q).\end{aligned}$$

Hence, we only need to study the convergence of the drift terms. We split the rest of the proof in two steps.
Step 1. We prove that

$$\int_{\mathcal{X}\times\mathcal{V}} \psi(\varphi,q) \int_{[t_1,t_2]\times\Gamma} \left(b^n(s,\varphi,\theta^n,u) - b(s,\varphi,\theta^n,u)\right)^\top Dg(\varphi(s))q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q) \underset{n\to\infty}{\longrightarrow} 0.$$

Indeed,

$$
\begin{aligned}
&\left|\int_{\mathcal{X}\times\mathcal{V}} \psi(\varphi,q) \int_{[t_1,t_2]\times\Gamma} (b^n(s,\varphi,\theta^n,u) - b(s,\varphi,\theta^n,u))^\top Dg(\varphi(s))q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q)\right| \\
&\leq C_{Dg}C_\psi \int_{\mathcal{X}\times\mathcal{V}} \int_{[t_1,t_2]\times\Gamma} |b^n(s,\varphi,\theta^n,u) - b(s,\varphi,\theta^n,u)|\, q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q) \\
&\leq C_{Dg}C_\psi \int_{\mathcal{X}\times\mathcal{V}} \int_{[t_1,t_2]\times\Gamma} |b(s,\varphi,\theta^n,u)|\, \mathbf{1}_{\{|b|\geq K_n\}} q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q) \\
&\leq C_{Dg}C_\psi \frac{\varepsilon^\alpha \int_{\mathcal{X}\times\mathcal{V}} \int_{[t_1,t_2]\times\Gamma} |b(s,\varphi,\theta^n,u)|^\alpha\, q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q)}{2\alpha} \\
&+ C_{Dg}C_\psi \frac{\int_{\mathcal{X}\times\mathcal{V}} \int_{[t_1,t_2]\times\Gamma} \mathbf{1}_{\{|b|\geq K_n\}} q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q)}{2\beta\varepsilon^\beta} \\
&\leq C_{Dg}C_\psi \frac{\varepsilon^\alpha \sup_{n\in\mathbb{N}} \int_{\mathcal{X}\times\mathcal{V}} \int_{[t_1,t_2]\times\Gamma} |b(s,\varphi,\theta^n,u)|^\alpha\, q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q)}{2\alpha} \\
&+ C_{Dg}C_\psi \frac{\sup_{n\in\mathbb{N}} \int_{\mathcal{X}\times\mathcal{V}} \int_{[t_1,t_2]\times\Gamma} |b(s,\varphi,\theta^n,u)|\, q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q)}{2K_n\beta\varepsilon^\beta}
\end{aligned}
$$

for all $\varepsilon > 0$, where C_{Dg} and C_ψ are uniform bounds on Dg and ψ, respectively. We applied Young's inequality with exponents $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ for the third inequality, while for the last one we used the Markov's inequality with respect to the measure $\pi(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\varphi,\mathrm{d}q) \doteq q(\mathrm{d}s,\mathrm{d}u)\Theta^n(ud\varphi,\mathrm{d}q)$ on $\mathcal{X}\times\mathcal{V}\times[0,T]\times\Gamma$:

$$
\int_{\mathcal{X}\times\mathcal{V}} \int_{[t_1,t_2]\times\Gamma} \mathbf{1}_{\{|b|\geq K_n\}} q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q) \leq \frac{\int_{\mathcal{X}\times\mathcal{V}} \int_{[t_1,t_2]\times\Gamma} |b(s,\varphi,\theta^n,u)|\, q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q)}{K_n}.
$$

The suprema over $n \in \mathbb{N}$ are bounded due to Lemma 2.3.2. We conclude this step by letting first $n \to \infty$ (so that $K_n \nearrow \infty$) then $\varepsilon \to 0$.

Step 2. We prove that

$$
\begin{aligned}
&\int_{\mathcal{X}\times\mathcal{V}} \psi(\varphi,q) \int_{[t_1,t_2]\times\Gamma} b(s,\varphi,\theta^n,u)^\top Dg(\varphi(s))q(\mathrm{d}s,\mathrm{d}u)\Theta^n(\mathrm{d}\varphi,\mathrm{d}q) \\
\underset{n\to\infty}{\longrightarrow}\ &\int_{\mathcal{X}\times\mathcal{V}} \psi(\varphi,q) \int_{[t_1,t_2]\times\Gamma} b(s,\varphi,\theta,u)^\top Dg(\varphi(s))q(\mathrm{d}s,\mathrm{d}u)\Theta(\mathrm{d}\varphi,\mathrm{d}q).
\end{aligned}
$$

To this aim we show that:

$$
(\theta,\varphi,q) \mapsto \psi(\varphi,q) \int_{[t_1,t_2]\times\Gamma} b(s,\varphi,\theta,u)^\top Dg(\varphi(s))q(\mathrm{d}s,\mathrm{d}u)
$$

is continuous on $\mathcal{P}_1(\mathcal{X})\times\mathcal{X}\times\mathcal{V}$ at points such that $\theta \ll \mathcal{W}^\nu$ and that it has sublinear growth in $(\varphi,q) \in \mathcal{X}\times\mathcal{V}$ so that we can conclude by using the property $W_1(\Theta^n,\Theta) \to 0$ together with Theorem 7.12.iv in Villani (2003). Since $\psi \in C(\mathcal{X}\times\mathcal{V})$, we only need to show the continuity of the second (integral) term. Let $(\theta^n,\varphi^n,q^n,u^n)_{n\in\mathbb{N}} \subset \mathcal{P}_1(\mathcal{X})\times\mathcal{X}\times\mathcal{V}\times\Gamma$ converge to some point $(\theta,\varphi,q,u) \in \mathcal{P}_1(\mathcal{X})\times\mathcal{X}\times\mathcal{V}\times\Gamma$ where $\theta \ll \mathcal{W}^\nu$. Then

$$
b(t,\varphi^n,\theta^n,u^n)^\top Dg(\varphi^n(t)) \underset{n\to\infty}{\longrightarrow} b(t,\varphi,\theta,u)^\top Dg(\varphi(t))
$$

for all $t \in [t_1, t_2]$ by the continuity assumptions on b and Dg, i.e. $b(t,\cdot)^\top Dg(\cdot)$ is jointly continuous for each $t \in [t_1, t_2]$ at points (θ, φ, q, u) with $\theta \ll \mathcal{W}^\nu$. Moreover

$$\begin{aligned} \left| b(t,\varphi,\theta,u)^\top Dg(\varphi(t)) \right| &\leq C_{Dg} C \left(1 + \|\varphi\|_{\infty,t} + m(t;\theta) + |u|\right) \\ &\leq C_{Dg} C \left(1 + K + \|\varphi\|_{\infty,t} + |u|\right) \end{aligned}$$

for some constants $C_{Dg}, C, K > 0$ (this replaces Assumption (2) of Corollary A.5 in Lacker (2015)). We conclude by means of Corollary A.5 in Lacker (2015).

□

We conclude this subsection by characterizing any limit measure Θ as the joint law of state and (relaxed) control for a weak solution of the limit SDE in Equation (2.9) with drift $\bar{b}$. The next corollary is a fairly standard result establishing a well-known connection between solutions of RM and weak solutions of SDEs:

Corollary 2.3.1 (*Representation of limit points*)**.** *Let Θ be a solution of RM, as in Definition 2.3.2. Then there exists a weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathrm{Q}, X, \Lambda, W)$ of*

$$X_t = X_0 + \int_{[0,t]\times\Gamma} \bar{b}\,(s, X_s, \mu_s, u)\, \Lambda_s(\mathrm{d}u)\mathrm{d}s + \sigma W_t, \quad t \in [0,T]$$

such that $\Theta = \mathrm{Q} \circ (X,\Lambda)^{-1}$, $\theta = \Theta \circ X^{-1}$ and $\mu_t = g(t,\theta)$ with $g : [0,T] \times \mathcal{P}_1(\mathcal{X}) \to \mathcal{M}_{\leq 1,1}(\mathbb{R}^d)$ as in Equation (2.4).

Proof. Arguing analogously as in the proofs of Proposition 5.4.6 and Corollary 5.4.8 in Karatzas and Shreve (1987) gives the existence of a weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathrm{Q}, X, \Lambda, W)$ of the SDE

$$X_t = X_0 + \int_{[0,t]\times\Gamma} b\,(s, X, \theta, u)\, \Lambda_s(\mathrm{d}u)\mathrm{d}s + \sigma W_t, \quad t \in [0,T] \tag{2.13}$$

such that Θ is the law of (X,Λ) under Q and $\theta = \Theta \circ X^{-1}$. The conclusion is obtained by going back to the original drift $\bar{b}$, that we recall is given by

$$\bar{b}(t, \varphi(t), g(t,\theta), u) = b(t,\varphi,\theta,u), \quad (t,\varphi,\theta,u) \in [0,T] \times \mathcal{X} \times \mathcal{P}_1(\mathcal{X}) \times \Gamma,$$

and $g(t,\theta) = \mu_t$ as in Equation (2.4). □

2.3.3 Optimality of the limit points

In this subsection, we show that any limit point $\Theta \in \mathcal{P}(\mathcal{X} \times \mathcal{V})$ of $(\Theta^n)_{n\in\mathbb{N}}$ is optimal according to the cost functional of the MFG. In order to do that, we will extend the notion of relaxed MFG solution to controls that are not necessarily in feedback form. In this case we evaluate optimality according to the following cost functional:

$$J^\mu(\Lambda) \doteq \mathsf{E}\left[\int_{[0,\tau]\times\Gamma} \bar{f}\,(s, X_s, \mu_s, u)\, \Lambda_s(\mathrm{d}u)\mathrm{d}s + F\,(\tau, X_\tau)\right],$$

where Λ is any relaxed stochastic control and $\tau \doteq \tau^X \wedge T$, subject to the dynamics

$$X_t = X_0 + \int_{[0,t]\times\Gamma} \bar{b}\,(s, X_s, \mu_s, u)\, \Lambda_s(\mathrm{d}u)\mathrm{d}s + \sigma W_t, \quad t \in [0,T]. \tag{2.14}$$

We set $V^\mu = \inf_\Lambda J^\mu(\Lambda)$, where the minimization is actually performed over the set of relaxed stochastic open-loop controls, i.e. over the tuples $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathsf{Q}, X, \Lambda, W)$ that are weak solutions of Equation (2.14) and where Λ is a progressively measurable $\mathcal{P}(\Gamma)$-valued stochastic process. To simplify the notation, we will just write Λ to refer to the whole tuple. Moreover, when working on the canonical space $\mathcal{X} \times \mathcal{V}$, where the canonical process (X, Λ) is completely characterized by its law Θ, we will simply write $J^\mu(\Theta)$ in place of $J^\mu(\Lambda)$.

Definition 2.3.3 (*Relaxed MFG solution*)**.** A relaxed solution of the MFG is a pair (Λ, μ), where Λ is a relaxed stochastic control and $\mu \in \mathrm{Y}^T_{\leq 1,1}$, such that:

(i) Λ is optimal, i.e. $V^\mu = J^\mu(\Lambda)$.

(ii) Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathsf{Q}, X, \Lambda, W)$ be a weak solution of Equation (2.14) with flow of sub-probability measures μ, stochastic control Λ and initial condition ν. Then

$$\mu_t(\cdot) = \mathsf{Q}(\{X_t \in \cdot\} \cap \{\tau^X > t\}), \quad t \in [0, T].$$

Proposition 2.3.3 (*Existence of relaxed MFG solutions*)**.** *Grant Assumptions (H1)-(H8) and (C1). Let Θ be a limit point of $(\Theta^n)_{n\in\mathbb{N}}$ in $\mathcal{P}_1(\mathcal{X} \times \mathcal{V})$. Set $\mu \in \mathrm{Y}^T_{\leq 1,1}$ as*

$$\mu_t(\cdot) \doteq \Theta\left(\{X_t \in \cdot\} \cap \left\{\tau^X > t\right\}\right) \quad t \in [0, T].$$

Then (Θ, μ) is a relaxed MFG solution according to Definition 2.3.3.

Proof. By construction we immediately have that Λ is a relaxed stochastic control and $\mu \in \mathrm{Y}^T_{\leq 1,1}$. Moreover, property (ii) is a consequence of the fact that Θ is a solution of RM as in Definition 2.3.2. To prove property (i), we proceed through the following steps:

(j) Let $\widetilde{\Theta} \in \mathcal{P}(\mathcal{X} \times \mathcal{V})$ be a solution of RM. Then there exists a sequence of solutions $(\widetilde{\Theta}^n)_{n\in\mathbb{N}}$ of RM(n) such that $\lim_{n\to\infty} J^{n,\mu^n}(\widetilde{\Theta}^n) = J^\mu(\widetilde{\Theta})$.

(jj) $\lim_{n\to\infty} J^{n,\mu^n}(\Theta^n) = J^\mu(\Theta)$.

(jjj) $J^\mu(\Theta) \leq J^\mu(\widetilde{\Theta})$ for any $\widetilde{\Theta} \in \mathcal{P}(\mathcal{X} \times \mathcal{V})$ solution of RM.

The proof of (j)-(jjj) largely follows that of Theorem 3.6 in Lacker (2015). Therefore, we highlight only the main differences with respect to our setting, which are due to the sub-linear growth of the drift and the cost functional and to the path dependency induced by the exit time from $\mathcal{O}$.

Proof of (j). Let $\widetilde{\Theta} \in \mathcal{P}(\mathcal{X} \times \mathcal{V})$ be a solution of RM and let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \widetilde{\Theta}, X, \Lambda, W)$ be a weak solution of Equation (2.14) on the canonical space $\widetilde{\Omega} = \mathcal{X} \times \mathcal{V}$. The existence of this solution is guaranteed by Corollary 2.3.1. Now fix Λ and let X^n be a sequence of strong solutions of:

$$X^n_t = \xi + \int_{[0,t]\times\Gamma} \bar{b}^n\left(s, X^n_s, \mu^n_s, u\right) \Lambda_s(\mathrm{d}u)\mathrm{d}s + \sigma W_t, \quad t \in [0, T]$$

on the filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \widetilde{\Theta})$. Set $\widetilde{\Theta}^n \doteq \widetilde{\Theta} \circ (X^n, \Lambda)^{-1}$ for each $n \in \mathbb{N}$. Notice that $(\widetilde{\Theta}^n)_{n\in\mathbb{N}} \subset \mathcal{P}_1(\mathcal{X} \times \mathcal{V})$. Moreover each $\widetilde{\Theta}^n$ solves RM(n) as in Definition 2.3.1. We now show that:

$$\mathsf{E}^{\widetilde{\Theta}}\left[\|X^n - X\|_\infty\right] \xrightarrow[n\to\infty]{} 0 \quad \text{and} \quad W_1(\widetilde{\Theta}^n, \widetilde{\Theta}) \xrightarrow[n\to\infty]{} 0. \tag{2.15}$$

Regarding the first limit, it is sufficient to note that:

$$\mathsf{E}^{\widetilde{\Theta}}\left[\|X^n - X\|_{\infty,t}\right] \leq L\int_0^t \mathsf{E}^{\widetilde{\Theta}}\left[\|X^n - X\|_{\infty,s}\right] \mathrm{d}s + \mathsf{E}^{\widetilde{\Theta}}\left[\int_{[0,t]\times\Gamma} \Delta b^n(s,u)\Lambda_s(\mathrm{d}u)\mathrm{d}s\right]$$

where we set

$$\Delta b^n(t,u) \doteq |\bar{b}^n(t,X_t,\mu_t,u) - \bar{b}(t,X_t,\mu_t,u)|.$$

The first term can be handled with Grönwall's Lemma, whereas the second one by applying a similar argument as in the first step of the proof of Proposition 2.3.2. Regarding the second limit in Equation (2.15) we can proceed as follows. First, notice that the first limit in Equation (2.15) implies convergence in probability, hence in law, of X^n to X. Thus, by an argument similar to that of Lemma 2.3.5, we can prove the convergence in the 1-Wasserstein distance. At this point, the convergence of the costs is a consequence of the convergence in the 1-Wasserstein distance and the sub-linear growth of the running cost (combined with Theorem 7.12.iv in Villani (2003)), as in the second step of the proof of Proposition 2.3.2.

Proof of (jj). This follows from an argument similar to the second part of (j).

Proof of (jjj). Let $\widetilde{\Theta} \in \mathcal{P}(\mathcal{X}\times\mathcal{V})$ be a solution of RM and let $(\widetilde{\Theta}^n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathcal{X}\times\mathcal{V})$ be an approximating sequence as in (j). By the optimality of Θ^n we have

$$J^{n,\mu^n}(\Theta^n) \leq J^{n,\mu^n}(\Theta^n)$$

for all $n \in \mathbb{N}$. The optimality of Θ follows by taking the limit for $n \to \infty$ on both sides of the inequality above and using the previous properties (j) and (jj). ☐

2.3.4 Existence of solutions

In this subsection we finally conclude the proof of Theorem 2.3.1 by proving the existence of a relaxed feedback MFG solution and, under additional convexity assumptions, the existence of a strict feedback MFG solution. In addition, we also prove existence of solutions that are Markovian up to the exit time.

Relaxed feedback MFG solutions. The main mathematical tool here is the mimicking result of Brunick and Shreve (2013). We follow the procedure in Lacker (2015) but with modifications due to the peculiarities of our model induced mainly by the presence of absorptions. We give more details in the proof below.

Proposition 2.3.4 (*Existence of relaxed feedback MFG solutions*)**.** *Grant Assumptions (H1)-(H8) and (C1). Let (Θ,μ) be a relaxed MFG solution as in Definition 2.3.3.*

Then there exists another relaxed MFG solution (Θ',μ) and a progressively measurable functional $\lambda : [0,T]\times\mathcal{X} \to \mathcal{P}(\Gamma)$ such that $\Theta'((\varphi,q) \in \mathcal{X}\times\mathcal{V} : q_t = \lambda(t,\varphi)) = 1$ for $\mathcal{L}_T$-a.e. $t \in [0,T]$ and $J^\mu(\Theta') = J^\mu(\Theta) = V^\mu$, i.e. (λ,μ) is a relaxed feedback solution of the MFG as in Definition 2.2.2.

Proof. We adapt the proof of Theorem 3.7 in Lacker (2015) to our setting, by exploiting the mimicking result in Corollary 3.11 of Brunick and Shreve (2013) instead of Corollary 3.7 as in Lacker (2015). As a consequence, the mimicking process that we get is not Markovian as in Lacker. However, it has the same law as the original process and not only the same marginals. This is important in our setting due to the path dependency induced by the exit time τ.

We start with the construction of λ by disintegration. Precisely, define $\eta \in \mathcal{P}([0,T]\times\mathcal{X}\times\Gamma)$ as:

$$\eta\left(I\times B\times G\right) \doteq \frac{1}{T}\mathrm{E}^{\Theta}\left[\int_{[0,T]\times\Gamma}\mathbf{1}_{(I\times B\times G)}\left(t,X,u\right)\Lambda\left(\mathrm{d}t,\mathrm{d}u\right)\right]$$

and disintegrate it as $\eta(\mathrm{d}t,\mathrm{d}\varphi,\mathrm{d}u)=\tilde{\eta}(\mathrm{d}t,\mathrm{d}\varphi)\lambda_{t,\varphi}(\mathrm{d}u)$. Then:

$$\eta\left(I\times B\times G\right) = \int_{[0,T]\times\mathcal{X}}\int_{\Gamma}\mathbf{1}_{(I\times B\times G)}\left(t,\varphi,u\right)\lambda_{t,\varphi}\left(\mathrm{d}u\right)\tilde{\eta}\left(\mathrm{d}t,\mathrm{d}\varphi\right)$$

for all $I\in\mathcal{B}([0,T])$, $B\in\mathcal{B}(\mathcal{X})$ and $G\in\mathcal{B}(\Gamma)$. By the disintegration theorem, $(t,\varphi)\mapsto\lambda_{t,\varphi}(\cdot)\in\mathcal{P}(\Gamma)$ is Borel-measurable. Now set $\widetilde{\mathcal{F}}^X_t\doteq\sigma(X_s,s\in[0,t])$ for each $t\in[0,T]$. We claim that:

$$\lambda_{t,X}\left(\cdot\right) = \mathrm{E}^{\Theta}\left[\Lambda_t\left(\cdot\right)|\widetilde{\mathcal{F}}^X_t\right] \quad \Theta\text{-a.s. and for } \mathcal{L}_T\text{-a.e. } t\in[0,T] \tag{2.16}$$

which is measurable and adapted, hence it has a progressively measurable modification λ. We show that for any bounded measurable functional $g:[0,T]\times\mathcal{X}\times\Gamma\to\mathbb{R}$ such that $g(t,\cdot,u)$ is $\widetilde{\mathcal{F}}^X_t$-measurable for all $t\in[0,T]$ and $u\in\Gamma$

$$\int_{\Gamma}g\left(t,X,u\right)\lambda_{t,X}\left(\mathrm{d}u\right) = \int_{\Gamma}g\left(t,X,u\right)\mathrm{E}^{\Theta}\left[\Lambda_t\left(\mathrm{d}u\right)|\widetilde{\mathcal{F}}^X_t\right]$$

Θ-a.s. and for $\mathcal{L}_T$-a.e. $t\in[0,T]$. Indeed, for any other bounded measurable functional $h:[0,T]\times\mathcal{X}\to\mathbb{R}$ such that $h(t,\cdot)$ is $\widetilde{\mathcal{F}}^X_t$-measurable for all $t\in[0,T]$, we have

$$\begin{aligned}
&\frac{1}{T}\mathrm{E}^{\Theta}\left[\int_0^T h\left(t,X\right)\int_{\Gamma}g\left(t,X,u\right)\lambda_{t,X}\left(\mathrm{d}u\right)\mathrm{d}t\right] \qquad (2.17)\\
&= \int_{[0,T]\times\mathcal{X}}h\left(t,\varphi\right)\int_{\Gamma}g\left(t,\varphi,u\right)\lambda_{t,\varphi}\left(\mathrm{d}u\right)\tilde{\eta}\left(\mathrm{d}t,\mathrm{d}\varphi\right)\\
&= \int_{[0,T]\times\mathcal{X}\times\Gamma}h\left(t,\varphi\right)g\left(t,\varphi,u\right)\eta\left(\mathrm{d}t,\mathrm{d}\varphi,\mathrm{d}u\right)\\
&= \frac{1}{T}\mathrm{E}^{\Theta}\left[\int_0^T h\left(t,X\right)\int_{\Gamma}g\left(t,X,u\right)\Lambda_t\left(\mathrm{d}u\right)\mathrm{d}t\right]
\end{aligned}$$

where the first equality comes from the definition of $\tilde{\eta}$, the second one is due to the disintegration of η and the third one holds by definition of η.
Now, let $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{F}}=(\widetilde{\mathcal{F}}_t)_{t\in[0,T]},\mathrm{Q},W,X,\Lambda)$ be a weak solution of Equation (2.14) with relaxed control $\Theta=\mathrm{Q}\circ(X,\Lambda)^{-1}$. By Corollary 3.11 in Brunick and Shreve (2013) there exists a weak solution $(\widetilde{\Omega}',\widetilde{\mathcal{F}}',\widetilde{\mathbb{F}}'=(\widetilde{\mathcal{F}}'_t)_{t\in[0,T]},\mathrm{Q}',W',X')$ of

$$X'_t = \xi+\int_0^t\int_{\Gamma}\bar{b}\left(s,X'_s,\mu_s,u\right)\lambda_{s,X'}(\mathrm{d}u)\mathrm{d}s+\sigma W'_t, \quad t\in[0,T]$$

such that $\mathrm{Q}'\circ(X')^{-1}=\mathrm{Q}\circ X^{-1}$. Define $\Theta'\doteq\mathrm{Q}'\circ(X',\Lambda')^{-1}$ where $\Lambda'(\mathrm{d}t,\mathrm{d}u)\doteq dt\lambda_{t,X'}(\mathrm{d}u)$. Notice that if μ' is the flow of sub-probability measures associated to Θ' then $\mu'=\mu$. Finally, Θ' solves the same relaxed martingale problem as Θ, and it has the same cost as Θ as required:

$$\begin{aligned} J^{\mu}\left(\Theta'\right) &= \mathsf{E}^{\mathsf{Q}'}\left[\int_0^{\tau'}\int_\Gamma \bar{f}\left(t, X'_t, \mu_t, u\right)\lambda_{t,X'}\left(\mathrm{d}u\right)\mathrm{d}t + F\left(\tau', X'_{\tau'}\right)\right] \\ &= \mathsf{E}^{\mathsf{Q}}\left[\int_0^{\tau}\int_\Gamma \bar{f}\left(t, X_t, \mu_t, u\right)\lambda_{t,X}\left(\mathrm{d}u\right)\mathrm{d}t + F\left(\tau, X_\tau\right)\right] \\ &= \mathsf{E}^{\mathsf{Q}}\left[\int_0^{\tau}\int_\Gamma \bar{f}\left(t, X_t, \mu_t, u\right)\mathsf{E}^{\mathsf{Q}}\left[\Lambda_t\left(\mathrm{d}u\right)|\widetilde{\mathcal{F}}^X_t\right]\mathrm{d}t + F\left(\tau, X_\tau\right)\right] \\ &= \mathsf{E}^{\mathsf{Q}}\left[\int_0^{\tau}\int_\Gamma \mathsf{E}^{\mathsf{Q}}\left[\bar{f}\left(t, X_t, \mu_t, u\right)\Lambda_t\left(\mathrm{d}u\right)|\widetilde{\mathcal{F}}^X_t\right]\mathrm{d}t + F\left(\tau, X_\tau\right)\right] \\ &= \mathsf{E}^{\mathsf{Q}}\left[\int_{[0,\tau]\times\Gamma} \bar{f}\left(t, X_t, \mu_t, u\right)\Lambda\left(\mathrm{d}t, \mathrm{d}u\right) + F\left(\tau, X_\tau\right)\right] \\ &= J^{\mu}\left(\Theta\right). \end{aligned}$$

□

Remark 2.3.2. We observe that, due to the discontinuity induced by the exit time τ, it is not possible in general to apply Theorem 3.6 of Brunick and Shreve (2013) to $Z_t = (X_t, \mathbb{I}_{[0,\tau)}(t))$, $t \in [0,T]$, to obtain a control which is Markovian in Z. Moreover the few mimicking results available in the literature for discontinuous processes hold under very restrictive or hardly verifiable assumptions. Nonetheless, Theorem 3.6 of Brunick and Shreve (2013) could still be applied in some particular cases when, for instance, $\mathcal{O} = (0,\infty)$ and $Z_t = (X_t, \inf_{s\in[0,t]} X_s)$.

Strict feedback MFG solutions. Under additional convexity assumptions (Filippov (1962); Haussmann and Lepeltier (1990)), we prove existence of feedback MFG solutions in strict form. Let (Θ,μ) be a relaxed MFG solution according to Definition 2.3.3 and for each $(t,\varphi) \in [0,T] \times \mathcal{X}$ define $K(t,\varphi,\mu)$ as:

$$K\left(t,\varphi,\mu\right) \doteq \left\{\left(\bar{b}\left(t,\varphi(t),\mu_t,u\right), z\right) \,:\, z \geq \bar{f}\left(t,\varphi(t),\mu_t,u\right) \quad \text{and} \quad u \in \Gamma\right\}.$$

Existence of strict MFG solutions is established under the additional Assumption (C2).

Remark 2.3.3. Assumption (C2) is equivalent to requiring that the set $K(t,\varphi,\mu)$ is convex. This assumption is crucial to apply the measurable selection arguments in Dufour and Stockbridge (2012); Haussmann and Lepeltier (1990).

Proposition 2.3.5 (*Existence of strict feedback MFG solutions*)**.** *Grant Assumptions (H1)-(H8), (C1) and Assumption (C2). Let (Θ,μ) be a relaxed MFG solution as in Definition 2.3.3.*

Then there exists another relaxed MFG solution (Θ',μ) and a progressively measurable functional $u \in \mathcal{U}_{fb}$ such that $\Theta'((\varphi,q) \in \mathcal{X}\times\mathcal{V} : q_t = \delta_{u(t,\varphi)}) = 1$ for $\mathcal{L}_T$-a.e. $t \in [0,T]$ and $J^{\mu}(\Theta') = J^{\mu}(\Theta) = V^{\mu}$, i.e. (u,μ) is a strict and feedback solution of the MFG as in Definition 2.2.1.

Proof. We follow once more the proof of Theorem 3.7 in Lacker (2015), highlighting the main differences with respect to our setting. The first part of the proof proceeds as in Proposition 2.3.4. Since for all $(t,\varphi) \in [0,T]\times\mathcal{X}$ the pair $(\bar{b}(t,\varphi(t),\mu_t,u), \bar{f}(t,\varphi(t),\mu_t,u))$ belongs to $K(t,\varphi,\mu)$ for all $u \in \Gamma$ and $K(t,\varphi,\mu)$ is convex, we have

$$\int_\Gamma \left(\bar{b}\left(t,\varphi(t),\mu_t,u\right), \bar{f}\left(t,\varphi(t),\mu_t,u\right)\right)\lambda_{t,\varphi}(\mathrm{d}u) \in K\left(t,\varphi,\mu\right).$$

By applying the measurable selection argument in Dufour and Stockbridge (2012); Haussmann and Lepeltier (1990) (with respect to the progressive σ-algebra, i.e. the σ-algebra generated by progressively measurable processes), we find a progressively measurable functional $u : [0, T] \times \mathcal{X} \to \Gamma$ such that

$$\int_\Gamma \bar{b}\left(t, \varphi(t), \mu_t, u\right) \lambda_{t,\varphi}(\mathrm{d}u) = \bar{b}\left(t, \varphi(t), \mu_t, u(t, \varphi)\right)$$

and

$$\int_\Gamma \bar{f}\left(t, \varphi(t), \mu_t, u\right) \lambda_{t,\varphi}(\mathrm{d}u) \geq \bar{f}\left(t, \varphi(t), \mu_t, u(t, \varphi)\right) \tag{2.18}$$

for all $(t, \varphi) \in [0, T] \times \mathcal{X}$. Define $\Theta' \doteq \mathrm{Q}' \circ (X', \Lambda')^{-1}$ where Q' is as in the proof of Proposition 2.3.4 and $\Lambda'(\varphi, q)(\mathrm{d}t, \mathrm{d}u) \doteq dt\delta_{u(t,\varphi)}(\mathrm{d}u)$. Θ' solves the same relaxed martingale problem as Θ. As for the costs, we have

$$\begin{aligned}
J^\mu\left(\Theta'\right) &= \mathsf{E}^{\mathrm{Q}'}\left[\int_0^{\tau'} \int_\Gamma \bar{f}\left(t, X'_t, \mu_t, u\right) \delta_{u(t,X')}(\mathrm{d}u)\mathrm{d}t + F\left(\tau, X'_\tau\right)\right] \\
&= \mathsf{E}^{\mathrm{Q}'}\left[\int_0^{\tau'} \bar{f}\left(t, X'_t, \mu_t, u(t, X')\right) \mathrm{d}t + F\left(\tau, X'_\tau\right)\right] \\
&\leq \mathsf{E}^{\mathrm{Q}'}\left[\int_0^{\tau'} \int_\Gamma \bar{f}\left(t, X'_t, \mu_t, u\right) \lambda_{t,X'}(\mathrm{d}u)\mathrm{d}t + F\left(\tau, X'_\tau\right)\right] \\
&= \mathsf{E}^{\mathrm{Q}}\left[\int_0^{\tau} \int_\Gamma \bar{f}\left(t, X_t, \mu_t, u\right) \lambda_{t,X}\left(\mathrm{d}u\right) \mathrm{d}t + F\left(\tau, X_\tau\right)\right] \\
&= \mathsf{E}^{\mathrm{Q}}\left[\int_{[0,\tau]\times\Gamma} \bar{f}\left(t, X_t, \mu_t, u\right) \Lambda\left(\mathrm{d}t, \mathrm{d}u\right) + F\left(\tau, X_\tau\right)\right] \\
&= J^\mu\left(\Theta\right)
\end{aligned}$$

where the inequality above is due to Equation (2.18). Given the optimality of (Θ, μ) we already have the converse inequality, i.e. $J^\mu(\Theta) \leq J^\mu(\Theta')$. Hence $J^\mu(\Theta) = J^\mu(\Theta')$. □

We can finally give the proof of Theorem 2.3.1.

Proof of Theorem 2.3.1. Grant Assumptions (H1)-(H8) and (C1). Proposition 2.3.3 guarantees existence of a relaxed MFG solution (Θ, μ) as in Definition 2.3.3. By Proposition 2.3.4 there exists another relaxed MFG solution (Θ', μ) together with a progressively measurable functional $\lambda : [0, T] \times \mathcal{X} \to \mathcal{P}(\Gamma)$ such that $\Theta'((\varphi, q) \in \mathcal{X} \times \mathcal{V} : q_t = \lambda(t, \varphi)) = 1$ for $\mathcal{L}_T$-a.e. t and $J^\mu(\Theta') = J^\mu(\Theta) = V^\mu$. Then (λ, μ) is a relaxed and feedback solution of the MFG as in Definition 2.2.2.

Additionally grant Assumption (C2). By Proposition 2.3.5 there exists another relaxed MFG solution (Θ', μ) and a progressively measurable functional $u \in \mathcal{U}_{fb}$ such that $\Theta'((\varphi, q) \in \mathcal{X} \times \mathcal{V} : q_t = \delta_{u(t,\varphi)}) = 1$ for $\mathcal{L}_T$-a.e. $t \in [0, T]$, and $J^\mu(\Theta') = J^\mu(\Theta) = V^\mu$. Then (u, μ) is a strict and feedback solution of the MFG as in Definition 2.2.1. □

Markovian MFG solutions. We conclude this part with showing that there exist relaxed and strict feedback solutions that are Markovian up to the exit time.

Proposition 2.3.6 (*Markovian MFG solutions*)**.** *Grant Assumptions (H1)-(H8) and (C1). Let (Θ, μ) be a relaxed MFG solution as in Definition 2.3.3. Then there exists another* *relaxed MFG solution* (Θ', μ) *and a function* $\lambda : [0,T] \times \mathbb{R}^d \to \mathcal{P}(\Gamma)$ *such that*

$$\mathcal{L}_T \otimes \Theta'(\{(t,\varphi,q) : q_t = \lambda(t,\varphi(t)), t \le \tau^X(\varphi)\}) = 1$$

and $J^\mu(\Theta') = J^\mu(\Theta) = V^\mu$. *Additionally, grant Assumption (C2). Then there exists a function* $u : [0,T] \times \mathbb{R}^d \to \Gamma$ *such that*

$$\mathcal{L}_T \otimes \Theta'(\{(t,\varphi,q) : q_t = \delta_{u(t,\varphi(t))}, t \le \tau^X(\varphi)\}) = 1$$

and $J^\mu(\Theta') = J^\mu(\Theta) = V^\mu$.

Proof. Let us define the following processes

$$Y_t \doteq (t, X_t), \quad X_t^{\tau^X} \doteq X_{t\wedge\tau^X}, \quad Y_t^{\tau^X} \doteq Y_{t\wedge\tau^X}$$

for $t \in [0,T]$. If X satisfies Equation (2.14) with flow of sub-probability measures μ and relaxed control Λ then the SDE satisfied by X^{τ^X} is (on the same probability space)

$$X_t^{\tau^X} = \xi + \int_{[0,t]\times\Gamma} \bar{b}\left(s, X_s^{\tau^X}, \mu_s, u\right) \mathbf{1}_{[0,\tau^X)}(s)\Lambda_s(du)ds + \sigma\int_0^t \mathbf{1}_{[0,\tau^X)}(s)dW_s$$

for $t \in [0,T]$. Notice that until $t \le \tau^X$ the stopped process X^{τ^X} coincides pathwise with the original process X. We now apply the mimicking result in Corollary 3.7 of Brunick and Shreve (2013), to the stopped process Y^{τ^X}. To this end, we follow the proof of Theorem 3.7 in Lacker (2015) and the proofs of Propositions 2.3.4 and 2.3.5 in the present chapter.
First, we claim that there exists a measurable function $\lambda : [0,T] \times \mathbb{R}^{d+1} \to \mathcal{P}(\Gamma)$ such that

$$\lambda_{t,Y_t^{\tau^X}}(\cdot) = \mathsf{E}^\Theta\left[\Lambda_t(\cdot)\middle|Y_t^{\tau^X}\right], \quad \Theta\text{-a.s. and for } \mathcal{L}_T\text{-a.e. } t \in [0,T].$$

Such a function can be constructed by disintegration as follows. Let $\eta \in \mathcal{P}([0,T] \times \mathbb{R}^{d+1} \times \Gamma)$ be given by

$$\eta(B) \doteq \frac{1}{T}\mathsf{E}^\Theta\left[\int_{[0,T]\times\Gamma} \mathbf{1}_C\left(t, Y_t^{\tau^X}, u\right)\Lambda(dt,du)\right].$$

We define λ through $\eta(dt,dy,du) \doteq \tilde{\eta}(dt,dy)\lambda_{t,y}(du)$. By Corollary 3.7 in Brunick and Shreve (2013) applied to $\lambda_{t,Y_t^{\tau^X}}$ there exists a weak solution $(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{F}}' = (\widetilde{\mathcal{F}}'_t)_{t\in[0,T]}, \mathsf{Q}', W', X')$ of

$$X'_t = \xi + \int_0^t \int_\Gamma \bar{b}\left(s, X'_s, \mu_s, u\right) \mathbf{1}_{[0,\tau^{X'})}(s)\lambda_{s,Y_t^{\tau^{X'}}}(du)ds + \sigma\int_0^t \mathbf{1}_{[0,\tau^{X'})}(s)dW'_s$$

for $t \in [0,T]$, where $Y_t^{\tau^{X'}} \doteq (t\wedge\tau^{X'}, X'_t)$ and $\mathsf{Q}' \circ (t\wedge\tau^{X'}, X'_t)^{-1} = \mathsf{Q} \circ (t\wedge\tau^X, X_t^{\tau^X})^{-1}$ for all $t \in [0,T]$, i.e. $Y^{\tau^{X'}}$ and Y^{τ^X} have the same time marginals. Now set $\tau' \doteq \tau^{X'} \wedge T$. Recall that $\Theta = \mathsf{Q} \circ (X,\Lambda)^{-1}$ and define $\Theta' \doteq \mathsf{Q}' \circ (X',\Lambda')^{-1}$ where $\Lambda'(dt,du) \doteq dt\lambda_{t,Y_t^{\tau^{X'}}}(du)$. Equality of the costs can be shown just as in the proof of Proposition

2.3.4:

$$\begin{aligned}
J^{\mu}\left(\Theta'\right) &= \mathsf{E}^{\mathsf{Q}'}\left[\int_0^{\tau'}\int_\Gamma \bar{f}(t,X'_t,\mu_t,u)\lambda_{t,t\wedge\tau^{X'},X'_t}(\mathrm{d}u)\mathrm{d}t + F\left(\tau',X'_{\tau'}\right)\right] \\
&= \mathsf{E}^{\mathsf{Q}}\left[\int_0^{\tau}\int_\Gamma \bar{f}(t,X^{\tau^X}_t,\mu_t,u)\lambda_{t,t\wedge\tau^X,X^{\tau^X}_t}(\mathrm{d}u)\mathrm{d}t + F\left(\tau,X^{\tau^X}_{\tau}\right)\right] \\
&= \mathsf{E}^{\mathsf{Q}}\left[\int_{[0,\tau]\times\Gamma} \bar{f}(t,X^{\tau^X}_t,\mu_t,u)\Lambda(\mathrm{d}t,\mathrm{d}u) + F\left(\tau,X^{\tau^X}_{\tau}\right)\right] \\
&= J^{\mu}\left(\Theta\right).
\end{aligned}$$

Therefore, $\lambda : [0,T]\times[0,T]\times\mathbb{R}^d \to \mathcal{P}(\Gamma)$ satisfies $\Theta'(q\in\mathcal{V} : q_t = \lambda(t,t\wedge\tau^{\hat{X}},\hat{X}^{\tau^{\hat{X}}}_t)) = 1$ for $\mathcal{L}_T$-a.e. $t\in[0,T]$ and $J^{\mu}(\Theta') = J^{\mu}(\Theta) = V^{\mu}$.

Consider now a weak solution $(\widetilde{\Omega}'',\widetilde{\mathcal{F}}'',\widetilde{\mathbb{F}}'' = (\widetilde{\mathcal{F}}''_t)_{t\in[0,T]},\mathsf{Q}'',W'',X'')$ of

$$X''_t = \xi + \int_0^t\int_\Gamma \bar{b}\left(s,X''_s,\mu_s,u\right)\lambda_{s,Y^{\tau^{X''}}_t}(\mathrm{d}u)\mathrm{d}s + \sigma W''_t, \quad t\in[0,T]$$

where $Y^{\tau^{X''}}_t = (t\wedge\tau^{X''},X''_t)$. Set $\Theta'' \doteq \mathsf{Q}''\circ(X'',\Lambda'')^{-1}$ where $\Lambda''(\mathrm{d}t,\mathrm{d}u) \doteq \mathrm{d}t\lambda_{t,Y^{\tau^{X''}}_t}(du)$. To avoid confusion between specific solutions, here $(\hat{X},\hat{\Lambda})$ denotes the canonical process on $\mathcal{X}\times\mathcal{V}$. First, Θ' solves the martingale problem associated to

$$\begin{aligned}
\widehat{M}^g_t(\varphi,q) \doteq g(\varphi(t)) - g(\varphi(0)) - \int_{[0,t]\times\Gamma} &\bar{b}(s,\varphi(s),\mu_s,u)^\top Dg(\varphi(s))\mathbf{1}_{[0,\tau^{\hat{X}})}(s)q(\mathrm{d}s,\mathrm{d}u) \\
&+\frac{1}{2}\int_0^t \mathrm{Tr}\left[\sigma\sigma^\top D^2 g(\varphi(s))\right]\mathbf{1}_{[0,\tau^{\hat{X}})}(s)\mathrm{d}s, \quad t\in[0,T].
\end{aligned}$$

as well as the one associated to

$$\begin{aligned}
M^g_t(\varphi,q) \doteq g(\varphi(t)) - g(\varphi(0)) - \int_{[0,t]\times\Gamma} &\bar{b}(s,\varphi(s),\mu_s,u)^\top Dg(\varphi(s))q(\mathrm{d}s,\mathrm{d}u) \\
&+\frac{1}{2}\int_0^t \mathrm{Tr}\left[\sigma\sigma^\top D^2 g(\varphi(s))\right]\mathrm{d}s
\end{aligned}$$

up to time $\tau^{\hat{X}}\wedge T$, i.e. the martingale property is satisfied by the processes above stopped at time $\tau^{\hat{X}}\wedge T$. Second, Θ'' solves the latter martingale problem up to time T. Then Θ' and Θ'' solve the same martingale problem up to time $\tau^{\hat{X}}\wedge T$. Moreover, we have $\Theta''(q\in\mathcal{V} : q_t = \lambda(t,t\wedge\tau^{\hat{X}},\hat{X}_t)) = 1$ for $\mathcal{L}_T$-a.e. $t\in[0,T]$. If we set $\Theta_t \doteq \Theta\circ(\hat{X},\hat{\Lambda})^{-1}_{\cdot\wedge t}$ for all $\Theta\in\mathcal{P}(\mathcal{X}\times\mathcal{V})$ and $t\in[0,T]$, then by uniqueness of the solution of the martingale problem up to time $\tau^{\hat{X}}\wedge T$ we have

$$\Theta'_t(\cdot\cap\{t\le\tau^{\hat{X}}\wedge T\}) = \Theta''_t(\cdot\cap\{t\le\tau^{\hat{X}}\wedge T\}).$$

Hence $J^{\mu}(\Theta') = J^{\mu}(\Theta'')$. Now Θ'' satisfies item (ii) of Definition 2.3.3.

To conclude notice that the process $Y^{\tau^{X''}}_t = (t\wedge\tau^{X''},X''_t)$ reduces to (t,X''_t) before time $\tau^{X''}\wedge T$. Hence, also $\lambda_{t,Y^{\tau^{X''}}_t}$, with a slight abuse of notation, reduces to λ_{t,X''_t}. With the additional Assumption (C2), the second part of this lemma follows from the proof of Proposition 2.3.5 applied to the stopped process Y^{τ^X}. □

2.4 Uniqueness of solutions of the mean-field game

In this section we address the problem of uniqueness of MFG solutions. Precisely, under Assumptions (H1)-(H8) and with the additional Assumptions (U1)-(U4) given below, where the second one guarantees monotonicity of the running cost in the same spirit as Lasry and Lions (2007b) (see also Theorem 3.29 in Carmona and Delarue (2018)), we show uniqueness of the MFG solution also in the presence of smooth dependence on past absorptions. The extra assumptions can be formulated as follows.

(U1) The running cost can be split in two terms:

$$\bar{f}(t,x,\mu,u)=\bar{f}_0(t,x,u)+\bar{f}_1(t,x,\mu)$$

for some measurable functions $\bar{f}_0 : [0,T]\times\mathbb{R}^d\times\Gamma\to[0,\infty)$ and $\bar{f}_1 : [0,T]\times\mathbb{R}^d\times\mathcal{M}_{\leq 1,1}(\mathbb{R}^d)\to[0,\infty)$.

(U2) Lasry-Lions monotonicity assumption: Let $\mu,\tilde{\mu}\in\mathcal{M}_{\leq 1,1}(\mathbb{R}^d)$, $\mu\neq\tilde{\mu}$. Then

$$\int_{\mathbb{R}^d}\left(\bar{f}_1(t,x,\mu)-\bar{f}_1(t,x,\tilde{\mu})\right)(\mu-\tilde{\mu})(\mathrm{d}x)\geq 0,\quad t\in[0,T].$$

(U3) The drift b does not depend on the measure variable.

(U4) Let $\bar{\mu}\in\mathrm{Y}^T_{\leq 1,1}$ be fixed. Then the following optimization problem

$$\inf_{\Lambda\in\widetilde{\mathrm{U}}} J^{\bar{\mu}}(\Lambda)\doteq\mathsf{E}\left[\int_{[0,\tau]\times\Gamma}\bar{f}\left(s,X_s,\bar{\mu}_s,u\right)\Lambda_s(\mathrm{d}u)\mathrm{d}s+F\left(\tau,X_\tau\right)\right]\tag{2.19}$$

has a unique solution $\Lambda^{\bar{\mu}}$, where $(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]},\mathsf{P},W,X)$ is a solution of Equation (2.9) under $\Lambda^{\bar{\mu}}$ with initial distribution ν and drift b satisfying (U3).

Theorem 2.4.1 (*Uniqueness*). *Under Assumptions (H1)-(H8) and (U1)-(U4), if there exists a feedback solution of the MFG (λ,μ) (as in Definition 2.2.2) then it is unique.*

Proof. By contradiction, let (λ,μ) and $(\tilde{\lambda},\tilde{\mu})$ be two different feedback MFG solutions (as in Definition 2.2.2). Then

$$J^{\tilde{\mu}}(\lambda)-J^{\tilde{\mu}}(\tilde{\lambda})>0\quad\text{and}\quad J^{\mu}(\tilde{\lambda})-J^{\mu}(\lambda)>0$$

where the inequality is strict by uniqueness of the minimizer in Assumption (U4), and in particular

$$\Delta(\mu,\tilde{\mu},\lambda,\tilde{\lambda})\doteq J^{\tilde{\mu}}(\lambda)-J^{\tilde{\mu}}(\tilde{\lambda})+J^{\mu}(\tilde{\lambda})-J^{\mu}(\lambda)>0.$$

However, thanks to Assumption (U3) that grants independence of the dynamics of the state processes from the flows of measures μ and $\tilde{\mu}$

$$\begin{aligned}\Delta(\mu,\tilde{\mu},\lambda,\tilde{\lambda}) &= \mathsf{E}^{\mathsf{P}}\left[\int_0^T\mathbf{1}_{[0,\tau)}(t)\left(\bar{f}_1(t,X_t,\tilde{\mu}_t)-\bar{f}_1(t,X_t,\mu_t)\right)\mathrm{d}t\right]\\ &\quad+\mathsf{E}^{\widetilde{\mathsf{P}}}\left[\int_0^T\mathbf{1}_{[0,\tilde{\tau})}(t)\left(\bar{f}_1(t,\widetilde{X}_t,\mu_t)-\bar{f}_1(t,\widetilde{X}_t,\tilde{\mu}_t)\right)\mathrm{d}t\right]\end{aligned}$$

where $(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]},\mathsf{P},W,X)$ and $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{F}}=(\widetilde{\mathcal{F}}_t)_{t\in[0,T]},\widetilde{\mathsf{P}},\widetilde{W},\widetilde{X})$ are weak solutions of Equation (2.7) respectively with controls λ and $\tilde{\lambda}$. Set $\theta\doteq\mathsf{P}\circ X^{-1}$ and

$\tilde{\theta} \doteq \widetilde{\mathsf{P}} \circ \widetilde{X}^{-1}$. Then

$$\begin{aligned}
\Delta(\mu,\tilde{\mu},\lambda,\tilde{\lambda}) &= \int_{\mathcal{X}} \int_0^T \mathbf{1}_{[0,\tau(\varphi))}(t) \left[\bar{f}_1(t,\varphi(t),\mu_t) - \bar{f}_1(t,\varphi(t),\tilde{\mu}_t)\right] \mathrm{d}t\tilde{\theta}(\mathrm{d}\varphi) \\
&\quad - \int_{\mathcal{X}} \int_0^T \mathbf{1}_{[0,\tau(\varphi))}(t) \left[\bar{f}_1(t,\varphi(t),\mu_t) - \bar{f}_1(t,\varphi(t),\tilde{\mu}_t)\right] \mathrm{d}t\theta(\mathrm{d}\varphi) \\
&= \int_0^T \int_{\mathcal{X}} \left[\bar{f}_1(t,\varphi(t),\mu_t) - \bar{f}_1(t,\varphi(t),\tilde{\mu}_t)\right] \mathbf{1}_{[0,\tau(\varphi))}(t)\tilde{\theta}(\mathrm{d}\varphi)\mathrm{d}t \\
&\quad - \int_0^T \int_{\mathcal{X}} \left[\bar{f}_1(t,\varphi(t),\mu_t) - \bar{f}_1(t,\varphi(t),\tilde{\mu}_t)\right] \mathbf{1}_{[0,\tau(\varphi))}(t)\theta(\mathrm{d}\varphi)\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^d} \left[\bar{f}_1(t,x,\mu_t) - \bar{f}_1(t,x,\tilde{\mu}_t)\right] \tilde{\mu}_t(\mathrm{d}x)\mathrm{d}t \\
&\quad - \int_0^T \int_{\mathbb{R}^d} \left[\bar{f}_1(t,x,\mu_t) - \bar{f}_1(t,x,\tilde{\mu}_t)\right] \mu_t(\mathrm{d}x)\mathrm{d}t \\
&= - \int_0^T \int_{\mathbb{R}^d} \left[\bar{f}_1(t,x,\mu_t) - \bar{f}_1(t,x,\tilde{\mu}_t)\right] (\mu_t - \tilde{\mu}_t)(\mathrm{d}x)\mathrm{d}t
\end{aligned}$$

which is lower than or equal to zero by Assumption (U2). In the second equality we have used Fubini-Tonelli theorem, while the third one comes from the definitions of μ and $\tilde{\mu}$, i.e.

$$\begin{aligned}
\mu_t(B) &\doteq \theta\left(\{X_t \in B\} \cap \{t < \tau\}\right) \\
&= \int_{\mathcal{X}} \mathbf{1}_B(\varphi(t))\mathbf{1}_{[0,\tau(\varphi))}(t)\theta(\mathrm{d}\varphi) \\
&= \int_{\mathbb{R}^d} \mathbf{1}_B(x)\mu_t(\mathrm{d}x), \quad t \in [0,T]
\end{aligned}$$

for all $B \in \mathcal{B}(\mathbb{R}^d)$ and similarly for $\tilde{\mu}$. □

Example 2.4.1 (Non-local dependence on the measure through a weighted average). We provide and example of running cost $\bar{f}$ satisfying the monotonicity condition (U2), which is an assumption on the measure-dependent term $\bar{f}_1$ only. Let $w : \mathbb{R}^d \to [0,\infty)$ be some measurable function with sub-linear growth so that

$$m_w(\mu) \doteq \int_{\mathbb{R}^d} w(x)\mu(\mathrm{d}x) < \infty, \quad \text{for all } \mu \in \mathcal{M}_{\leq 1,1}(\mathbb{R}^d)$$

and set

$$\bar{f}_1(t,x,\mu) \doteq w(x) \int_{\mathbb{R}^d} w(y)\mu(\mathrm{d}y) = w(x)m_w(\mu), \quad (t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{M}_{\leq 1,1}(\mathbb{R}^d).$$

Since

$$\bar{f}_1(t,x,\mu) - \bar{f}_1(t,x,\tilde{\mu}) = w(x) \int_{\mathbb{R}^d} w(y)(\mu - \tilde{\mu})(\mathrm{d}y)$$

we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(\bar{f}_1(t,x,\mu) - \bar{f}_1(t,x,\tilde{\mu})\right)(\mu - \tilde{\mu})(dx) &= \int_{\mathbb{R}^d} w(x) \int_{\mathbb{R}^d} w(y)(\mu - \tilde{\mu})(\mathrm{d}y)(\mu - \tilde{\mu})(\mathrm{d}x), \\
&= \int_{\mathbb{R}^d} w(x)(\mu - \tilde{\mu})(\mathrm{d}x) \int_{\mathbb{R}^d} w(y)(\mu - \tilde{\mu})(\mathrm{d}y), \\
&= \left(\int_{\mathbb{R}^d} w(x)(\mu - \tilde{\mu})(\mathrm{d}x)\right)^2 \geq 0.
\end{aligned}$$

2.5 Approximate Nash equilibria for the N-player game with finite-dimensional interaction

In this section, we consider an important particular case of our MFG with absorption, where the mean-field interaction is finite-dimensional. This is inspired by the original model of Campi and Fischer (2018). We show that any feedback solution of the MFG can be used to construct a sequence of approximate Nash equilibria for the corresponding N-player game. To this end, we will need two additional assumptions (Assumptions (N1) and (N2) below). We focus on a finite-dimensional example first for technical reasons: this setting is very suitable to the propagation of chaos result that we use in the proofs without being too technical. Second, we think that this case is also particularly relevant for the applications as mentioned in the introduction. Overall, we believe that the finite-dimensional setting enables us to keep a good balance between abstract technicalities and modelling needs.

The approximation result is the content of Theorem 2.5.1 and Corollary 2.5.2. In order to prove this, we interpret the N-player system as a system of N interacting diffusions (as in, e.g., Gärtner (1988); McKean (1966); Sznitman (1991)). While the usual mode of convergence of an N-particle system is the convergence in law of the empirical measures, here we obtain a stronger form of propagation of chaos as in Lacker (2018) but with possibly unbounded drift in the state variable. We prove that the empirical measures converge in the stronger τ-topology, which is widely used in the large deviations literature (see, for instance, Chapter 6.2 in Dembo and Zeitouni (2010)); see Subsection 2.5.3.

2.5.1 The setting with finite-dimensional interaction

Here, we describe the MFG and the corresponding N-player game with smooth dependence on past absorptions, specializing them to the finite-dimensional interaction setting. In particular, we give the definition of ε-Nash equilibrium for the N-player game. Then, we give the assumptions that are specific to this model. We conclude by checking that the MFG with finite-dimensional interactions satisfies the hypotheses of Theorem 2.3.1, granting the existence of relaxed and strict solutions of the MFG.

The mean-field dynamics. Given a feedback control $u \in \mathcal{U}_{fb}$ and a flow of sub-probability measures $\mu \in \mathrm{Y}^T_{\leq 1,1}$, the representative player's state evolves according to the equation

$$X_t = X_0 + \int_0^t \tilde{b}\left(s, X_s, L\left(\mu_s\right), m_w\left(\mu_s\right), u\left(s, X\right)\right) \mathrm{d}s + \sigma W_t, \quad t \in [0, T] \tag{2.20}$$

where X is a d-dimensional stochastic process starting at $X_0 \overset{d}{\sim} \nu \in \mathcal{P}(\mathbb{R}^d)$, W is a d-dimensional Wiener process on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$, $\tilde{b}$ and σ are as in the assumptions below. In addition, $m_w(\mu)$ and $L(\mu)$ are functions $m_w : \mathcal{M}_{\leq 1,1}(\mathbb{R}^d) \to \mathbb{R}^{d_0}$ and $L : \mathcal{M}_{\leq 1,1}(\mathbb{R}^d) \to [0, 1]$ defined as

$$m_w(\mu) \doteq \int_{\mathbb{R}^d} w(x)\, \mu(\mathrm{d}x) \quad \text{and} \quad L(\mu) \doteq 1 - \int_{\mathbb{R}^d} \mu(\mathrm{d}x)$$

where $w : \mathbb{R}^d \to \mathbb{R}^{d_0}$, $d_0 \in \mathbb{N}$, is a fixed weight function with sub-linear growth. Again, solutions of Equation (2.20) are understood in the weak sense (see Remark 2.2.5). The cost associated to a strategy $u \in \mathcal{U}_{fb}$ and a flow of sub-probability measures $\mu \in \mathrm{Y}^T_{\le 1,1}$ is given by

$$J^{\mu}(u) \doteq \mathsf{E}\left[\int_0^{\tau} \tilde{f}(s, X_s, L(\mu_s), m_w(\mu_s), u(s, X))\, ds + F(\tau, X_\tau)\right] \tag{2.21}$$

where $\tau \doteq \tau^X \wedge T$ is the random time horizon as in the previous sections.

The N-player dynamics. Let $N \in \mathbb{N}$ be the number of players. We assume that the players' private states evolve according to the following system of N d-dimensional SDEs: for $i \in \{1, \ldots, N\}$,

$$X_t^{N,i} = X_0^{N,i} + \int_0^t \tilde{b}\left(s, X_s^{N,i}, L\left(\mu_s^N\right), m_w\left(\mu_s^N\right), u^{N,i}\left(s, \mathbf{X}^N\right)\right) ds + \sigma W_t^{N,i} \tag{2.22}$$

for $t \in [0, T]$, where $X_0^{N,i} \overset{d}{\sim} \nu$ i.i.d., $W^{N,1}, \ldots, W^{N,N}$ is an N-dimensional vector of independent d-dimensional Wiener processes, $\mathbf{X}^N$ denotes the vector of all players' private states, $\boldsymbol{u}^N$ the vector of feedback strategies, $\tilde{b}$ and σ are as in the assumptions below. We remind that $\mu^N \in \mathrm{Y}^T_{\le 1,1}$ is the random empirical *sub-probability measures* defined as

$$\mu_t^N(\cdot) \doteq \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}(\cdot)\, \mathbf{1}_{[0,\tau^{X^{N,i}})}(t), \quad t \in [0, T]. \tag{2.23}$$

Solutions of the SDEs in Equation (2.22) are understood to be in the weak sense on some filtered probability space $(\Omega^N, \mathcal{F}^N, \mathbb{F}^N = (\mathcal{F}_t^N)_{t\in[0,T]}, \mathsf{P}^N)$ satisfying the usual conditions (see Remark 2.2.5).
Let $\mathcal{U}_1^N$ be the set of all progressively measurable functionals $u : [0, T] \times \mathcal{X}^N \to \Gamma$, and let $\mathcal{U}_N^N$, the set of all vectors $\boldsymbol{u}^N$ such that $u^{N,i} \in \mathcal{U}_1^N$, $i \in \{1, \ldots, N\}$. Each element of $\mathcal{U}_N^N$ is called *feedback strategy vector*. In this game, player i evaluates a strategy vector $\boldsymbol{u}^N \in \mathcal{U}_N^N$ according to the expected costs

$$J^{N,i}\left(\boldsymbol{u}^N\right) \doteq \mathsf{E}\Bigg[\int_0^{\tau^{N,i}} \tilde{f}\left(s, X_s^{N,i}, L\left(\mu_s^N\right), m_w\left(\mu_s^N\right), u^{N,i}(s, \mathbf{X}^N)\right) ds + F\left(\tau^{N,i}, X^{N,i}_{\tau^{N,i}}\right)\Bigg] \tag{2.24}$$

over a random time horizon, where $\mathbf{X}^N$ is the N-player dynamics under $\boldsymbol{u}^N$ and $\tau^{N,i} \doteq \tau^{X^{N,i}} \wedge T$. Our aim is the construction of approximate Nash equilibria for the N-player game from a solution of the limit problem. In the next definition, we use the standard notation $[u^{N,-i}, v]$ to indicate a strategy vector equal to $\boldsymbol{u}^N$ for all players but the i-th, who deviates by playing $v \in \mathcal{U}_1^N$ instead.

Definition 2.5.1 (*ε-Nash equilibrium*). Let $\varepsilon \ge 0$. A strategy vector $\boldsymbol{u}^N \in \mathcal{U}_N^N$ is called ε-Nash equilibrium for the N-player game if for every $i \in \{1, \ldots, N\}$ and for any deviation $v \in \mathcal{U}_1^N$ we have:

$$J^{N,i}(\boldsymbol{u}^N) \le J^{N,i}\left(\left[u^{N,-i}, v\right]\right) + \varepsilon.$$

Relaxed controls. It will be very convenient to use relaxed controls also in the N-player case. Let $\widetilde{\mathcal{U}}_1^N$ be the set of all single-player relaxed strategies for the N-player game, and let $\widetilde{\mathcal{U}}_N^N$ be the set of N-player relaxed strategy vectors, i.e. vectors $\boldsymbol{\lambda}^N = (\lambda^{N,1}, \dots, \lambda^{N,N})$ with $\lambda^{N,i} \in \widetilde{\mathcal{U}}_1^N$, $i \in \{1, \dots, N\}$. At this point, we can rewrite the dynamics and the cost functional of the N-player game (Equation (2.22) and Equation (2.24)) by using relaxed controls as

$$X_t^{N,i} = X_0^{N,i} + \int_{[0,t]\times\Gamma} \tilde{b}\left(s, X_s^{N,i}, L\left(\mu_s^N\right), m_w\left(\mu_s^N\right), u\right) \lambda^{N,i}\left(s, \mathbf{X}^N\right)(\mathrm{d}u)\mathrm{d}s + \sigma W_t^{N,i} \tag{2.25}$$

with associated cost

$$J^{N,i}\left(\boldsymbol{\lambda}^N\right) = \mathsf{E}\left[\int_{[0,\tau^{N,i}]\times\Gamma} \tilde{f}\left(s, X_s^{N,i}, L\left(\mu_s^N\right), m_w\left(\mu_s^N\right), u\right) \lambda^{N,i}\left(s, \mathbf{X}^N\right)(\mathrm{d}u)\mathrm{d}s \right.$$
$$\left. + F\left(\tau^{N,i}, X_{\tau^{N,i}}^{N,i}\right)\right] \tag{2.26}$$

for $t \in [0,T]$, $i \in \{1, \dots, N\}$, $\boldsymbol{\lambda}^N \in \widetilde{\mathcal{U}}_N^N$ and $\lambda^{N,i} \in \widetilde{\mathcal{U}}_1^N$ for all $i \in \{1, \dots, N\}$. Moreover, we extend accordingly the notion of ε-Nash equilibrium.

Definition 2.5.2 (*Relaxed ε-Nash equilibrium*)**.** A strategy vector $\boldsymbol{\lambda}^N \in \widetilde{\mathcal{U}}_N^N$ is an ε-Nash equilibrium for the N-player game if for every $i \in \{1, \dots, N\}$ and for any single-player strategy $\beta \in \widetilde{\mathcal{U}}_1^N$

$$J^{N,i}(\boldsymbol{\lambda}^N) \leq J^{N,i}\left(\left[\boldsymbol{\lambda}^{N,-i}, \beta\right]\right) + \varepsilon.$$

The drift $\tilde{b}$, the function w, the running cost $\tilde{f}$ and the terminal cost F now satisfy the following assumptions, replacing Assumptions (H1)-(H3):

(H1′) The drift $\tilde{b} : [0,T] \times \mathbb{R}^d \times [0,1] \times \mathbb{R}^{d_0} \times \Gamma \to \mathbb{R}^d$ is jointly continuous and satisfies the following uniform Lipschitz continuity: there exists $L > 0$ such that

$$\left|\tilde{b}\left(t, x, \ell, m, u\right) - \tilde{b}\left(t, x', \ell, m, u\right)\right| \leq L\left|x - x'\right|$$

for all $x, x' \in \mathbb{R}^d$ and all $(t, \ell, m, u) \in [0,T] \times [0,1] \times \mathbb{R}^{d_0} \times \Gamma$. Moreover it has sub-linear growth in (x, m) uniformly in the other variables, i.e. there exists a constant $C > 0$ such that

$$\left|\tilde{b}\left(t, x, \ell, m, u\right)\right| \leq C\left(1 + |x| + |m|\right)$$

for all $(t, x, \ell, m, u) \in [0,T] \times \mathbb{R}^d \times [0,1] \times \mathbb{R}^{d_0} \times \Gamma$.

(H2′) $w : \mathbb{R}^d \to \mathbb{R}^{d_0}$ is continuous and has sub-linear growth: $|w(x)| \leq C(1 + |x|)$ for all $x \in \mathbb{R}^d$.

(H3′) The costs $\tilde{f} : [0,T] \times \mathbb{R}^d \times [0,1] \times \mathbb{R}^{d_0} \times \Gamma \to [0, \infty)$ and $F : [0,T] \times \mathbb{R}^d \to [0, \infty)$ are jointly continuous. Moreover, they have sub-linear growth:

$$\begin{aligned} \left|\tilde{f}(t, x, \ell, m, u)\right| &\leq C\left(1 + |x| + |m|\right), \\ \left|F(t, x)\right| &\leq C\left(1 + |x|\right), \end{aligned}$$

for all $(t, x, \ell, m, u) \in [0,T] \times \mathbb{R}^d \times [0,1] \times \mathbb{R}^{d_0} \times \Gamma$.

We conclude the presentation of the finite-dimensional model by introducing the coefficients' reparametrization on $\mathcal{P}_1(\mathcal{X})$, by checking their joint continuity (as in Assumption (H3)), where continuity in the measure variable is in the 1-Wasserstein distance and at points $\theta \ll \mathcal{W}^\nu$. We set $(\bar{b}, \bar{f})(t, x, \mu, u) \doteq (\tilde{b}, \tilde{f})(t, \varphi(t), L(\mu), m_w(\mu), u)$ for all $(t, x, \mu, u) \in [0, T] \times \mathbb{R}^d \times \mathcal{M}_{\leq 1,1}(\mathbb{R}^d) \times \Gamma$ and define the reparametrization (b, f) as in Section 2.2. Then

$$(b, f)(t, \varphi, \theta, u) \quad = \quad (\tilde{b}, \tilde{f})(t, \varphi(t), L(t; \theta), m_w(t; \theta), u)$$

where

$$\begin{aligned} m_w(t; \theta) &\doteq \int_{\mathcal{X}} w\left(\varphi(t)\right) \mathbf{1}_{[0,\tau(\varphi))}(t) \theta(\mathrm{d}\varphi), \\ L(t; \theta) &\doteq 1 - \int_{\mathcal{X}} \mathbf{1}_{[0,\tau(\varphi))}(t) \theta(\mathrm{d}\varphi) \end{aligned}$$

are called the average and loss process and they equal $m_w(\mu_t)$ and $L(\mu_t)$ in case $\mu_t = g(t, \theta)$ where g is defined as in Equation (2.4).
Joint continuity of b and f follows from joint continuity of $\tilde{b}$ and $\tilde{f}$ and from the following lemma.

Lemma 2.5.1 (*Continuity of the average and loss processes*)**.** *Grant Assumptions (H1')-(H3') and (H4)-(H8). Let $(\theta_n)_{n \in \mathbb{N}} \subset \mathcal{P}_1(\mathcal{X})$ converge to $\theta \in \mathcal{P}_1(\mathcal{X})$, $\theta \ll \mathcal{W}^\nu$, in the 1-Wasserstein distance, then*

(i) $L(t; \theta^n) \to L(t; \theta)$ *as* $n \to \infty$.

(ii) $m_w(t; \theta^n) \to m_w(t; \theta)$ *as* $n \to \infty$.

Proof. (i). Denote by $\mathbb{D}_\tau(t)$ the set of discontinuity points of the map $\varphi \mapsto \mathbf{1}_{[0,\tau(\varphi))}(t)$ for $t \in [0, T]$. In particular $\theta^n \overset{w}{\rightharpoonup} \theta$. Then:

$$L(t; \theta^n) - L(t; \theta) = -\int_{\mathcal{X}} \mathbf{1}_{[0,\tau(\varphi))}(t) \left(\theta^n - \theta\right)(\mathrm{d}\varphi) \underset{n \to \infty}{\longrightarrow} 0$$

for all $t \in [0, T]$. This follows from the definition of weak convergence of measures, the fact that $\theta(\mathbb{D}_\tau(t)) = 0$ for all $t \in [0, T]$ (due to $\theta \ll \mathcal{W}^\nu$) and by Lemma A.0.4.(d).

(ii). Now we have:

$$|m_w(t; \theta^n) - m_w(t; \theta)| \quad \leq \quad \left| \int_{\mathcal{X}} w(\varphi(t)) \mathbf{1}_{[0,\tau(\varphi))}(t) \left(\theta^n - \theta\right)(\mathrm{d}\varphi) \right| \underset{n \to \infty}{\longrightarrow} 0$$

for all $t \in [0, T]$ as a consequence of the convergence in the 1-Wasserstein distance, the fact that $\theta(\mathbb{D}_\tau(t)) = 0$ for all $t \in [0, T]$ and by Lemma A.0.4.(d) together with Lemma A.0.5. □

We conclude by proving that we can use Theorem 2.3.1 and get existence of a feedback relaxed and strict solutions of the MFG with smooth dependence on past absorptions and finite-dimensional dependence on the measure.

Corollary 2.5.1 (*Existence of relaxed and strict feedback MFG solutions*)**.** *Under Assumptions (H1')-(H3'), (H4)-(H8) and (C1) , there exists a relaxed feedback solution (λ, μ) of the MFG with finite dimensional interaction. Moreover, under the additional Assumption (C2) , there exists a strict feedback MFG solution (u, μ).*

Proof. Assumptions (H1′)-(H3′) imply Assumptions (H1)-(H3) of Theorem 2.3.1. Indeed, (H1)-(H2) follow from the definition of the coefficients $\tilde{b}$ and $\tilde{f}$. Assumption (H3), i.e. joint continuity of the reparametrized coefficients, is a consequence of joint continuity of $\tilde{b}$ and $\tilde{f}$ and Lemma 2.5.1. □

2.5.2 The N-player approximation theorem

In order to state the N-player approximation results, we need the following two additional assumptions (N1)-(N2), whose formulation requires some more terminology.
We set

$$d_t^{TV}(\theta,\tilde{\theta}) \doteq \sup_{B\in\mathcal{F}_t}|\theta(B)-\tilde{\theta}(B)|,$$

for all $\theta,\tilde{\theta}\in\mathcal{P}(\mathcal{X})$ and we note that for $t\in[0,T)$, d_t is only a pseudo-metric, whereas for $t=T$ it is a proper metric; d_T^{TV} is called the total variation distance. However, with a slight abuse of terminology, we will often refer to d_t^{TV} as the total variation distance for each $t\in[0,T]$.

(N1) The function $w:\mathbb{R}^d\to\mathbb{R}^{d_0}$ is bounded.

(N2) The drift $\tilde{b}$ satisfies the following Lipschitz continuity:

$$\left|\tilde{b}\left(t,x,\ell,m,u\right)-\tilde{b}\left(t,x',\ell',m',u\right)\right|\leq L\left(\left|x-x'\right|+\left|\ell-\ell'\right|+\left|m-m'\right|\right)$$

for all $(x,\ell,m),(x',\ell',m')\in\mathbb{R}^d\times[0,1]\times\mathbb{R}^{d_0}$ and all $(t,u)\in[0,T]\times\Gamma$, with Lipschitz constant $L>0$. The running cost $\tilde{f}$ can be decomposed as

$$\tilde{f}(t,x,\ell,m,u)=\tilde{f}_0(t,x,u)+\tilde{f}_1(t,x,\ell,m),$$

where

$$|\tilde{f}_0(t,x,u)|\leq K \quad \text{and} \quad |\tilde{f}_1(t,x,\ell,m)|\leq C(1+|x|),$$

for all $(t,x,\ell,m,u)\in[0,T]\times\mathbb{R}^d\times[0,1]\times\mathbb{R}^{d_0}\times\Gamma$ and some constants $C,K>0$.

From Assumptions (N1)-(N2), the reparametrizations b and f inherit a series of properties that are fundamental in the proof of the approximation result. First, being $w:\mathbb{R}^d\to\mathbb{R}^{d_0}$ bounded, the drift b is Lipschitz continuous with respect to the total variation distance, which is a key assumption in Lemma 2.5.2. Indeed

$$\begin{aligned}\left|b(t,\varphi,\theta,u)-b(t,\varphi,\theta',u)\right| &\leq L\left(\left|L(t;\theta)-L(t;\theta')\right|+\left|m_w(t;\theta)-m_w(t;\theta')\right|\right)\\ &\leq L(1+\|w\|_\infty)d_T^{TV}(\theta,\theta')\doteq L_b^{TV}d_T^{TV}(\theta,\theta')\end{aligned}$$

because

$$\begin{aligned}\left|L(t;\theta)-L(t;\theta')\right| &= \left|\int_{\mathcal{X}}\mathbf{1}_{[0,\tau(\varphi))}(t)(\theta'-\theta)(\mathrm{d}\varphi)\right|\leq d_T^{TV}(\theta,\theta') \quad \text{and}\\ \left|m_w(t;\theta)-m_w(t;\theta')\right| &= \left|\int_{\mathcal{X}}w(\varphi(t))\mathbf{1}_{[0,\tau(\varphi))}(t)(\theta-\theta')(\mathrm{d}\varphi)\right|\leq\|w\|_\infty d_T^{TV}(\theta,\theta').\end{aligned}$$

Second, the sub-linear growth property

$$|b(t,\varphi,\theta,u)|\leq C(1+\|w\|_\infty+\|\varphi\|_{\infty,t}),\quad (t,\varphi)\in[0,T]\times\mathcal{X}$$

is uniform in $\theta \in \mathcal{P}(\mathcal{X})$ and in $u \in \Gamma$, implying that b is bounded in the measure and control variables (and analogously f). This means that b and f are well defined on all $\mathcal{P}(\mathcal{X})$ not only on $\mathcal{P}_1(\mathcal{X})$, which is fundamental to apply the fixed point theorem in Lemma 2.5.2. Finally, the running cost f can be decomposed as

$$f(t,\varphi,\theta,u) = f_0(t,\varphi,u) + f_1(t,\varphi,\theta)$$

where its components are

$$f_0(t,\varphi,u) \doteq \tilde{f}_0(t,\varphi(t),u) \quad \text{and} \quad f_1(t,\varphi,\theta) \doteq \tilde{f}_1(t,\varphi(t),L(t;\theta),m_w(t;\theta))$$

which inherit from $\tilde{f}_0$ and $\tilde{f}_1$ the properties

$$|f_0(t,\varphi,u)| \leq K \quad \text{and} \quad |f_1(t,\varphi,\theta)| \leq C(1+\|\varphi\|_{\infty,t})$$

for all $(t,\varphi,\theta,u) \in [0,T] \times \mathcal{X} \times \mathcal{P}(\mathcal{X}) \times \Gamma$. This is a key assumption to perform the passage to the many-player limit in Theorem 2.5.1. Indeed, boundedness in the control of f_0 enables us to exploit convergence in the τ-topology while sub-linearity in the state variable φ uniformly in the measure variable θ makes f_1 a good test function for the convergence in the 1-Wasserstein distance.

Theorem 2.5.1 (*Approximate Nash equilibria - relaxed*)**.** *Let (λ,μ) be a relaxed feedback MFG solution. For all $N \geq 2$, define $\boldsymbol{\lambda}^N = (\lambda^{N,1},\dots,\lambda^{N,N}) \in \widetilde{\mathcal{U}}_N^N$ where $\lambda^{N,i}(t,\varphi^N) \doteq \lambda(t,\varphi^{N,i})$ for all $i \in \{1,\dots,N\}$, $t \in [0,T]$ and $\varphi^N \in \mathcal{X}^N$.*
Then under Assumptions (H1')-(H3'), (H4)-(H8) and (N1)-(N2), for every $\varepsilon > 0$ there exists $N^\varepsilon \in \mathbb{N}$ such that $\boldsymbol{\lambda}^N$ is an ε-Nash equilibrium for the N-player game whenever $N \geq N^\varepsilon$, i.e. for every $i \in \{1,\dots,N\}$ and for any deviation $\beta \in \widetilde{\mathcal{U}}_1^N$

$$J^{N,i}\left(\boldsymbol{\lambda}^N\right) \leq J^{N,i}\left(\left[\lambda^{N,-i},\beta\right]\right) + \varepsilon$$

for all $N \geq N^\varepsilon$.

Corollary 2.5.2 (*Approximate Nash equilibria - strict*)**.** *Let (u,μ) be a strict feedback MFG solution. For all $N \geq 2$, define $\boldsymbol{u}^N = (u^{N,1},\dots,u^{N,N}) \in \mathcal{U}_N^N$ where $u^{N,i}(t,\varphi^N) \doteq u(t,\varphi^{N,i})$ for all $i \in \{1,\dots,N\}$, $t \in [0,T]$ and $\varphi^N \in \mathcal{X}^N$.*
Then under Assumptions (H1')-(H3'), (H4)-(H8) and (N1)-(N2), for every $\varepsilon > 0$ there exists a $N^\varepsilon \in \mathbb{N}$ such that $\boldsymbol{u}^N$ is an ε-Nash equilibrium for the N-player game whenever $N \geq N^\varepsilon$, i.e. for every $i \in \{1,\dots,N\}$ and for any deviation $v \in \mathcal{U}_1^N$

$$J^{N,i}\left(\boldsymbol{u}^N\right) \leq J^{N,i}\left(\left[u^{N,-i},v\right]\right) + \varepsilon$$

for all $N \geq N^\varepsilon$.

Before proceeding, we define the empirical measure ζ^N of the N-player system (Equation (2.25)) as

$$\zeta^N(\cdot) \doteq \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{N,i}}(\cdot) \tag{2.27}$$

which is a $\mathcal{P}(\mathcal{X})$-valued random variable. Moreover, we fix a relaxed feedback MFG solution (λ,μ) and define (cfr. Theorem 2.5.1 and Corollary 2.5.2) $\boldsymbol{\lambda}^N \in \widetilde{\mathcal{U}}_N^N$ as $\boldsymbol{\lambda}^N \doteq (\lambda^{N,i})_{i=1,\dots,N}$ where $\lambda^{N,i}(t,\varphi^N) \doteq \lambda(t,\varphi^{N,i})$ for all $i = 1,\dots,N$, $t \in [0,T]$ and $\varphi^N \in$

$\mathcal{X}^N$. In the next two subsections we consider the following N-particle system:

$$X_t^{N,1} = X_0^{N,1} + \int_{[0,t]\times\Gamma} b\left(s, X^{N,1}, \zeta^N, u\right) \beta\left(s, \mathbf{X}^N\right)(\mathrm{d}u)\mathrm{d}s + \sigma W_t^{N,1}, \quad (2.28)$$

$$X_t^{N,i} = X_0^{N,i} + \int_{[0,t]\times\Gamma} b\left(s, X^{N,i}, \zeta^N, u\right) \lambda\left(s, X^{N,i}\right)(\mathrm{d}u)\mathrm{d}s + \sigma W_t^{N,i} \quad (2.29)$$

for $i = 2, \ldots, N$, $t \in [0,T]$ and where $\beta \in \widetilde{\mathcal{U}}_1^N$ is a generic single-player control. Precisely, in Subsection 2.5.3 we set $\beta(t, \varphi^N) \doteq \lambda(t, \varphi^{N,1})$ for $t \in [0,T]$ and $\varphi^N \in \mathcal{X}^N$ (we say that $\beta = \lambda$ for short); whereas, in Subsection 2.5.4 we let β be generic (unless differently specified), which means that we allow the first player to deviate from the MFG solution λ.

2.5.3 Propagation of chaos

In this subsection we consider the system of N interacting symmetric diffusions given by Equations (2.28) and (2.29) with $\beta = \lambda$. We associate to this system a suitable McKean-Vlasov equation (Equation (2.30) below) and show a propagation of chaos result, that we will need in the proofs of Theorem 2.5.1 and Corollary 2.5.2.

Definition 2.5.3 (*McKean-Vlasov solution*)**.** A law $\theta^* \in \mathcal{P}(\mathcal{X})$ is a McKean-Vlasov solution of equation

$$X_t = X_0 + \int_{[0,t]\times\Gamma} b\left(s, X, \theta^*, u\right) \lambda\left(s, X\right)(\mathrm{d}u)\mathrm{d}s + \sigma W_t, \quad t \in [0,T], \quad X_0 \overset{d}{\sim} \nu \quad (2.30)$$

if there exists a weak solution $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,t]}, \mathsf{P}, X, W)$ with $\mathsf{P}\circ X^{-1} = \theta^*$ and $\mathsf{P}\circ X_0^{-1} = \nu$.

The following lemma ensures the well-posedness of Equation (2.30).

Lemma 2.5.2 (*Existence and uniqueness of McKean-Vlasov solutions*)**.** *Grant Assumptions (H1')-(H3'), (H4)-(H8) and (N1)-(N2). Then, there exists a unique McKean-Vlasov solution for Equation* (2.30)*.*

Proof. We follow Lacker (2018), proof of Theorem 2.4. Precisely, we apply Banach fixed point theorem on the complete metric space $(\mathcal{P}(\mathcal{X}), d_T)$ together with Picard iterations. To this end, we start by defining, for any $\alpha > 0$, the following distance:

$$d^\alpha(\theta, \theta')^2 \doteq \int_0^T e^{-\alpha t} d_t(\theta, \theta')^2 \,\mathrm{d}t, \quad \theta, \theta' \in \mathcal{P}(\mathcal{X}).$$

We note that $d^\alpha(\cdot,\cdot)$ is a complete metric on $\mathcal{P}(\mathcal{X})$. We now define $\Psi : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X}) \subset \mathcal{P}(\mathcal{X})$ as the map $\theta \mapsto \Psi(\theta) \doteq \mathsf{P}^\theta \circ (X^\theta)^{-1}$ where $(\Omega^\theta, \mathcal{F}^\theta, \mathsf{P}^\theta, X^\theta, W^\theta)$ is a weak solution of Equation (2.30) with θ in the drift, which is well defined (see Remark 2.2.5).

We show that Ψ is a contraction on $\mathcal{P}(\mathcal{X})$ with respect to the distance d^α for a sufficiently large $\alpha > 0$. Let $\mathcal{H}(\theta|\theta')$ denote the relative entropy of θ with respect to θ' for $\theta, \theta' \in \mathcal{P}(\mathcal{X})$, and let $\mathcal{H}_t(\theta|\theta') = \mathcal{H}(\theta_t|\theta'_t)$, $\theta_t \doteq \mathsf{P}^\theta \circ (X^\theta_{\cdot\wedge t})^{-1}$. By Pinsker's inequality, there exists a constant $C_H > 0$ such that

$$\begin{aligned} d_t(\Psi(\theta), \Psi(\theta'))^2 &\le C_H \mathcal{H}_t(\Psi(\theta), \Psi(\theta')) \\ &\le \frac{1}{2} C_H |\sigma^{-1}|^2 \widetilde{L}^2 \int_0^t d_s(\theta, \theta')^2 \mathrm{d}s \end{aligned}$$

where we set $\widetilde{L} \doteq L_b^{TV}$. Therefore, we have

$$\begin{aligned} d^\alpha(\Psi(\theta),\Psi(\theta'))^2 &= \int_0^T e^{-\alpha t} d_t(\Psi(\theta),\Psi(\theta'))^2 \mathrm{d}t \\ &\leq \frac{1}{2} C_H |\sigma^{-1}|^2 \widetilde{L}^2 \int_0^T e^{-\alpha t} \int_0^t d_s(\theta,\theta')^2 \mathrm{d}s\, \mathrm{d}t \\ &= \frac{1}{2} C_H |\sigma^{-1}|^2 \widetilde{L}^2 \int_0^T d_t(\theta,\theta')^2 \int_t^T e^{-\alpha s} \mathrm{d}s\, \mathrm{d}t \\ &\leq \frac{1}{2}\frac{C_H}{\alpha} |\sigma^{-1}|^2 \widetilde{L}^2 \int_0^T e^{-\alpha t} d_t(\theta,\theta')^2 \mathrm{d}t = \frac{1}{2}\frac{C_H}{\alpha} |\sigma^{-1}|^2 \widetilde{L}^2 d^\alpha(\theta,\theta')^2 \end{aligned}$$

which shows that Ψ is a contraction whenever $\frac{1}{2}\frac{C_H}{\alpha}|\sigma^{-1}|^2\widetilde{L}^2 < 1$. Thanks to the arbitrariness of $\alpha > 0$, we conclude that Ψ has a unique fixed-point in $\mathcal{P}(\mathcal{X})$. □

We consider the sequence of empirical measures $(\zeta^N)_{N\in\mathbb{N}}$ in Equation (2.27) associated to the N-particle systems in Equations (2.28) and (2.29) (with $\beta = \lambda$). We follow Lacker (2018) and we prove the convergence, both in law and in probability in the τ-topology, of $(\zeta^N)_{N\in\mathbb{N}}$ to the McKean-Vlasov solution $\theta^* \in \mathcal{P}(\mathcal{X})$ of Equation (2.30). We remind that the τ-topology on $\mathcal{P}(\mathcal{X})$, denoted with $\tau(\mathcal{P}(\mathcal{X}))$, is the topology generated by the sets

$$B_{f,x,\delta} \doteq \left\{ \pi \in \mathcal{P}(\mathcal{X}) : \left| \int_{\mathcal{X}} f(y)\pi(\mathrm{d}y) - x \right| < \delta \right\}$$

where $f : \mathcal{X} \to \mathbb{R}$ is any measurable bounded function, $x \in \mathbb{R}$ and δ is any strictly positive constant. In particular, the τ-topology is the coarsest topology that makes the maps $\pi \mapsto \int_{\mathcal{X}} f(y)\pi(\mathrm{d}y)$ continuous for all measurable bounded functions $f : \mathcal{X} \to \mathbb{R}$ (see, for instance, Chapter 6.2 in Dembo and Zeitouni (2010)).
Moreover, we denote by $w(\mathcal{P}(\mathcal{X}))$ the weak topology on $\mathcal{P}(\mathcal{X})$ and with $\mathcal{B}(\mathcal{P}(\mathcal{X}))$ the Borel σ-algebra on $\mathcal{X}$ generated by the open sets of the weak topology. The following lemma adapts Theorem 2.6.1-2 in Lacker (2018) to our framework, in particular to the case of diffusions with possibly unbounded drift.

Lemma 2.5.3 (*Propagation of chaos*)**.** *Grant Assumptions (H1')-(H3'), (H4)-(H8) and (N1)-(N2). Let $\theta^* \in \mathcal{P}(\mathcal{X})$ be the unique McKean-Vlasov solution of Equation (2.30). Then the sequence $(\zeta^N)_{N\in\mathbb{N}}$ converges in law to θ^*, i.e. $\zeta^N \xrightarrow{\mathcal{L}} \theta^*$, as $N \to \infty$. Moreover*

$$\lim_{N\to\infty} \mathsf{P}^N \left(\zeta^N \notin B \right) = 0$$

for all open neighbourhoods B of θ^* in the τ-topology that are in $\mathcal{B}(\mathcal{P}(\mathcal{X}))$.

Proof. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space that supports an i.i.d. sequence of $\mathcal{X}$-valued random variables with law θ^*. For each $N \in \mathbb{N}$, set $(\mathcal{F}_t^N)_{t\in[0,T]}$ to be the filtration generated by $X^1, \dots, X^N$. Define

$$W_t^i \doteq \sigma^{-1} \left(X_t^i - \xi - \int_{[0,t]\times\Gamma} b(s, X^i, \theta^*, u)\lambda(s, X^i)(\mathrm{d}u)\mathrm{d}s \right),\ t \in [0,T],\ i \in \{1,\dots,N\}.$$

In particular, $W^1, \dots, W^N$ are independent Wiener processes on $(\Omega, \mathcal{F}, \mathbb{F}^N = (\mathcal{F}_t^N)_{t\in[0,T]}, \mathsf{P})$.
Fix $N \in \mathbb{N}$, and consider the tuple $(\Omega, \mathcal{F}, \mathbb{F}^N = (\mathcal{F}_t^N)_{t\in[0,T]}, \mathsf{P}, (X^{N,1}, \dots, X^{N,N}), (W^1, \dots, W^N))$,

with $X^{N,i} \doteq X^i$, for all $i \in \{1,\dots,N\}$. This is a weak solution of

$$X^{N,i} = \xi + \int_{[0,t]\times\Gamma} b(s, X^{N,i}, \theta^*, u)\lambda(s, X^{N,i})(\mathrm{d}u)\mathrm{d}s + \sigma W^i_t, \quad t \in [0,T], \quad i \in \{1,\dots,N\}.$$

Now, define the probability P^N via its density with respect to P, $\frac{d\mathrm{P}^N}{d\mathrm{P}} \doteq Z^N_T$, where, for all $t \in [0,T]$

$$Z^N_t \doteq \mathcal{E}_t \left(\int_0^{\cdot} \sum_{i=1}^N \int_\Gamma \sigma^{-1} \left(b(s, X^{N,i}, \zeta^N, u) - b(s, X^{N,i}, \theta^*, u) \right) \lambda(s, X^{N,i})(\mathrm{d}u)\mathrm{d}W^i_s \right).$$

A standard application of Girsanov's theorem gives

$$X^{N,i}_t = \xi + \int_{[0,t]\times\Gamma} b(s, X^{N,i}, \zeta^N, u)\lambda(s, X^{N,i})(\mathrm{d}u)\mathrm{d}s + \sigma W^{N,i}_t, \quad t \in [0,T], \; i \in \{1,\dots,N\}$$

for some P^N-Wiener process W^N. Notice that $(\Omega, \mathcal{F}, \mathbb{F}^N = (\mathcal{F}^N_t)_{t\in[0,T]}, \mathrm{P}^N, X^N, W^N)$ is a weak solution of the N-particle system in Equations (2.28) and (2.29), with $\beta(t, \varphi^N) \doteq \lambda(t, \varphi^{N,1})$ for $t \in [0,T]$ and $\varphi^N \in \mathcal{X}^N$.

At this point, the rest of the proof can be performed as in Lacker (2018), Theorem 2.6.1-2, along the following steps:

(i) Show that $F_{t_1,t_2} : \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ defined as

$$F_{t_1,t_2}(\theta) \doteq \int_{\mathcal{X}} \int_{t_1}^{t_2} \left| \int_\Gamma \sigma^{-1} \left(b(s,\varphi,\theta,u) - b(s,\varphi,\theta^*,u) \right) \lambda(s,\varphi)(\mathrm{d}u) \right|^2 \mathrm{d}s\,\theta(\mathrm{d}\varphi) \tag{2.31}$$

is τ-continuous for all $t_1, t_2 \in [0,T]$, $t_1 < t_2$ and $\mathcal{B}(\mathcal{P}(\mathcal{X}))$-measurable, which is done aside at the end of this proof. Moreover $F_{t_1,t_2}(\theta) \le \widetilde{L}(t_2 - t_1)\mathcal{H}(\theta|\theta^*)$ for all $t_1, t_2 \in [0,T]$, $t_1 < t_2$ and for all $\theta \in \mathcal{P}(\mathcal{X})$, which is a straightforward consequence of the Lipschitz continuity in the total variation distance.

(ii) Since $X^{N,1}, X^{N,2}, \dots X^{N,N}$ are i.i.d. under P, Sanov's Theorem (e.g. Theorem 6.2.10 in Dembo and Zeitouni (2010)) can be applied to $\mathrm{P} \circ (\zeta^N)^{-1}$.

(iii) Derive a large deviation principle for $\mathrm{P}^N \circ (\zeta^N)^{-1}$, precisely

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathrm{P}^N \left(\zeta^N \notin B \right) \le -\mathrm{e}^{-\widetilde{L}T} \inf_{\theta \notin B} \mathcal{H}\left(\theta|\theta^*\right)$$

for all open neighbourhoods B of θ in the τ-topology that are in $\mathcal{B}(\mathcal{P}(\mathcal{X}))$, for some constant $\widetilde{L} > 0$.
To this aim, we stress that we can proceed just as in Lacker (2018)[1]. Indeed, regardless of the sub-linear growth of the drift, we can adapt Lacker's estimates thanks to

$$\left| b\left(t,\varphi,\theta,u\right) - b\left(t,\varphi,\theta',u\right) \right| \le 2\widetilde{L}.$$

Moreover we can apply Varadhan's integral lemma (Dembo and Zeitouni, 2010, Theorem 4.3.1) thanks to the continuity of F_{t_1,t_2}.

[1]Precisely we can show by induction that Equation (4.1) in Lacker (2018) holds also in this case, then conclude observing that P^N and P agree on $\mathcal{F}_0$.

(iv) Conclude by showing that $\inf_{\theta \notin B} \mathcal{H}(\theta|\theta^*) > 0$ so that

$$\lim_{N\to\infty} \mathrm{P}^N \left(\zeta^N \notin B \right) = 0$$

which can be performed as in Lacker (2018).

Proof of the continuity of F_{t_1,t_2} in the τ-topology. We actually prove the stronger claim that the functional F_{t_1,t_2} in Equation (2.31) is continuous in the weak topology (w-topology for short). First, we can write $F_{t_1,t_2}(\theta) = \int_{\mathcal{X}} f_{t_1,t_2}(\varphi,\theta)\theta(\mathrm{d}\varphi)$ for $\theta \in \mathcal{P}(\mathcal{X})$, where

$$f_{t_1,t_2}(\varphi,\theta) \doteq \int_{t_1}^{t_2} \left| \int_{\Gamma} \sigma^{-1} \left(b(s,\varphi,\theta,u) - b(s,\varphi,\theta^*,u) \right) \lambda(s,\varphi)(\mathrm{d}u) \right|^2 \mathrm{d}s$$

which is a real-valued bounded measurable function defined on $\mathcal{X} \times \mathcal{P}(\mathcal{X})$. Let $(\theta^n)_{n\in\mathbb{N}}, \theta \in \mathcal{P}(\mathcal{X})$ be such that $\theta^n \xrightarrow{w} \theta$. We want to show that $F_{t_1,t_2}(\theta^n) \to F_{t_1,t_2}(\theta)$ as $n \to \infty$.
Set $f_n(\varphi) \doteq f_{t_1,t_2}(\varphi,\theta^n)$ and $f(\varphi) \doteq f_{t_1,t_2}(\varphi,\theta)$. They are all in $C_b(\mathcal{X})$ with uniform bound in $n \in \mathbb{N}$. Moreover, $f_n \to f$ in the sup-norm. Indeed

$$\sup_{\varphi\in\mathcal{X}} |f_n(\varphi) - f(\varphi)| \leq 4L_b^{TV} L \int_{t_1}^{t_2} |L(s;\theta^n) - L(s;\theta)| + |m_w(s;\theta^n) - m_w(s;\theta)| \,\mathrm{d}s$$

which vanishes in the limit for $n \to \infty$ due to Lemma 2.5.1. As a consequence, we obtain

$$F_{t_1,t_2}(\theta^n) = \int_{\mathcal{X}} f_n(\varphi)\theta^n(\mathrm{d}\varphi) \underset{n\to\infty}{\longrightarrow} \int_{\mathcal{X}} f(\varphi)\theta(\mathrm{d}\varphi) = F_{t_1,t_2}(\theta).$$

□

2.5.4 Proof of the The N-player approximation theorem

This section is devoted to the construction of approximate Nash equilibria for the N-player game from a solution of the limit problem, in the particular case of finite-dimensional interaction as described before. The results of previous Subsection 2.5.3 allow us to pass to the many-player limit even if feedback MFG strategies are discontinuous in the state variable. We have observed in the introduction that the construction of approximated Nash equilibria for the N-player games in Campi and Fischer (2018) was crucially based on the continuity of the limit optimal control for almost every paths of the state variable with respect to the Wiener measure. In our setting, such a regularity property is no longer feasible due to the possible unboundedness of the coefficients, which makes it difficult to apply PDE-based estimates as in Campi and Fischer (2018) to get the needed continuity. Therefore, in order to overcome this obstacle, we will use the strong form of propagation of chaos in Lemma 2.5.3, which allows to pass to the limit even through possibly discontinuous MFG optimal controls.
In this part, we consider the dynamics in Equation (2.28) and Equation (2.29) without necessarily taking $\beta = \lambda$, unless differently specified. We start with some preliminary estimates ensuring that the costs remain bounded in the mean-field limit despite the sub-linear growth.

Lemma 2.5.4 (*A-priori estimates*)**.** *Grant Assumptions (H1')-(H3'), (H4)-(H8) and (N1)-(N2). Consider the dynamics in Equations* (2.28) *and* (2.29)*. Then for any* $\alpha \geq 1$

$$\sup_{N\in\mathbb{N}} \mathsf{E}^{\mathsf{P}^N}\left[\|X^{N,i}\|_\infty^\alpha\right] \quad \leq \quad K(\alpha)$$

for $i \in \{1,\dots,N\}$ *and where* $K(\alpha) < \infty$ *is a positive constant independent of* N*.*

Proof. This is a consequence of Grönwall's lemma together with uniform boundedness of the drift in the measure and control variables. □

Now, we prove the tightness of the sequence of laws $(\mathsf{P}^N \circ (\zeta^N)^{-1})_{N\in\mathbb{N}}$ when $\beta = \lambda$ in Equation (2.28), i.e. when the dynamics are symmetric. Then, thanks to Lemma 2.5.3, we characterize the limit points of $(\mathsf{P}^N \circ (\zeta^N)^{-1})_{N\in\mathbb{N}}$ as McKean-Vlasov solutions of Equation (2.30); see Lemma 2.5.6.

Lemma 2.5.5 (*Tightness*)**.** *Grant Assumptions (H1')-(H3'), (H4)-(H8) and (N1)-(N2). Let* ζ^N *be the empirical measure of the system given by Equations* (2.28) *and* (2.29) *with* $\beta = \lambda$*. Then the sequence* $(\mathsf{P}^N \circ (\zeta^N)^{-1})_{N\in\mathbb{N}}$ *is tight in* $\mathcal{P}(\mathcal{P}(\mathcal{X}))$*.*

Proof. The tightness of such a sequence follows from Sznitman (1991), Proposition 2.2, combined with Kolmogorov-Chentsov criterion (see, for instance, Corollary 14.9 in Kallenberg (2006)). □

Lemma 2.5.6 (*Characterization of limit points*)**.** *Grant Assumptions (H1')-(H3'), (H4)-(H8) and (N1)-(N2). Let* ζ^N *be the empirical measure of the system given by Equations* (2.28) *and* (2.29) *with* $\beta = \lambda$*. Let* $(\mathsf{P}^{N_k} \circ (\zeta^{N_k})^{-1})_{k\in\mathbb{N}}$ *be a convergent subsequence of* $(\mathsf{P}^N \circ (\zeta^N)^{-1})_{N\in\mathbb{N}}$*. Let* ζ *be a random variable defined on some probability space* $(\Omega, \mathcal{F}, \mathsf{P})$ *with values in* $\mathcal{P}(\mathcal{X})$ *such that* $\zeta^{N_k} \xrightarrow{\mathcal{L}} \zeta$*. Then*

(i) ζ *coincides* P*-a.s. with the unique McKean-Vlasov solution* θ^* *of Equation* (2.30)*.*

(ii) The sequence $(\zeta^N)_{N\in\mathbb{N}}$ *converges in probability (hence also in law) to* θ^* *when* $\mathcal{P}(\mathcal{X})$ *is equipped with the* τ*-topology.*

Proof. By Lemma 2.5.5 there exists a subsequence $(\mathsf{P}^{N_k} \circ (\zeta^{N_k})^{-1})_{k\in\mathbb{N}} \subset \mathcal{P}(\mathcal{P}(\mathcal{X}))$ converging to $\mathsf{P} \circ \zeta^{-1} \in \mathcal{P}(\mathcal{P}(\mathcal{X}))$. Lemma 2.5.3 guarantees the convergence in law of the whole sequence $(\zeta^N)_{N\in\mathbb{N}}$ to the deterministic limit θ^*, which is the unique McKean-Vlasov solution of Equation (2.30). By uniqueness in law of the weak limit we have $\mathsf{P} \circ \zeta^{-1} = \delta_{\theta^*}$, yielding $\zeta = \theta^*$ P-a.s.. Lemma 2.5.3 also gives convergence in probability in the τ-topology of $(\zeta^N)_{N\in\mathbb{N}}$ to θ^*. □

Corollary 2.5.3 (*Characterization of the convergence*)**.** *Under the assumptions of Lemma 2.5.6, the following properties hold:*

(i) For all Borel-measurable bounded function $f : \mathcal{X} \to \mathbb{R}$ *such that* $\theta \mapsto \int_{\mathcal{X}} f(\varphi)\theta(\mathrm{d}\varphi)$ *is* $\tau(\mathcal{P}(\mathcal{X}))$*-continuous*

$$\mathsf{E}^{\mathsf{P}^N}\left[\int_{\mathcal{X}} f(\varphi)\zeta^N(\mathrm{d}\varphi)\right] \underset{N\to\infty}{\longrightarrow} \mathsf{E}^{\mathsf{P}}\left[\int_{\mathcal{X}} f(\varphi)\zeta(\mathrm{d}\varphi)\right] \equiv \mathsf{E}^{\mathsf{P}}\left[\int_{\mathcal{X}} f(\varphi)\theta^*(\mathrm{d}\varphi)\right].$$

(ii) $\mathsf{P}^N \circ (X^{N,1}, \zeta^N)^{-1} \xrightarrow{w} \theta^* \otimes \delta_{\theta^*}$*. Moreover,* $\mathsf{P}^N \circ (X^{N,1})^{-1} \xrightarrow{w} \theta^*$ *and* $\mathsf{P}^N \circ (\zeta^N)^{-1} \xrightarrow{w} \delta_{\theta^*}$*.*

(iii) For all $f \in C(\mathcal{X})$ with sub-linear growth, i.e. $|f(\varphi)| \leq C_f(1+\|\varphi\|_\infty)$ for some $C_f > 0$ and all $\varphi \in \mathcal{X}$, we have

$$\mathsf{E}^{\mathsf{P}^N}\left[\int_{\mathcal{X}} f(\varphi)\zeta^N(\mathrm{d}\varphi)\right] \underset{N\to\infty}{\longrightarrow} \mathsf{E}^{\mathsf{P}}\left[\int_{\mathcal{X}} f(\varphi)\zeta(\mathrm{d}\varphi)\right] \equiv \mathsf{E}^{\mathsf{P}}\left[\int_{\mathcal{X}} f(\varphi)\theta^*(\mathrm{d}\varphi)\right].$$

Proof. (i) This is a consequence of Lemma 2.5.3, Lemma 2.5.6 and of the almost sure equality $\zeta = \theta^*$.
(ii) We already know that $\mathsf{P}^N \circ (\zeta^N)^{-1} \xrightarrow{w} \delta_{\theta^*}$ from Lemma 2.5.6. Therefore, the convergence of $\mathsf{P}^N \circ (X^{N,1})^{-1}$ to θ^* follows from Sznitman (1991), Proposition 2.2, and the symmetry of the system.
(iii) Let $f \in C(\mathcal{X})$ with sub-linear growth. It is enough to show that

$$\mathsf{E}^{\mathsf{P}^N}\left[\int_{\mathcal{X}} \|\varphi\|_\infty\zeta^N(\mathrm{d}\varphi)\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathcal{X}} \|\varphi\|_\infty\theta^*(\mathrm{d}\varphi).$$

To this aim, for fixed $R > 0$, we consider the decomposition

$$\begin{aligned}\mathsf{E}^{\mathsf{P}^N}\left[\int_{\mathcal{X}} \|\varphi\|_\infty(\zeta^N-\theta^*)(\mathrm{d}\varphi)\right] \leq\ & \mathsf{E}^{\mathsf{P}^N}\left[\int_{\mathcal{X}} (\|\varphi\|_\infty \wedge R)(\zeta^N-\theta^*)(\mathrm{d}\varphi)\right] \\ & + \mathsf{E}^{\mathsf{P}^N}\left[\int_{\mathcal{X}} \|\varphi\|_\infty \mathbf{1}_{\{\|\varphi\|_\infty \geq R\}}(\zeta^N+\theta^*)(\mathrm{d}\varphi)\right].\end{aligned}$$

By property (i), for any fixed $R > 0$, we have

$$\lim_{N\to\infty} \mathsf{E}^{\mathsf{P}^N}\left[\int_{\mathcal{X}} (\|\varphi\|_\infty \wedge R)(\zeta^N-\theta^*)(\mathrm{d}\varphi)\right] = 0$$

so that

$$\limsup_{N\to\infty} \mathsf{E}^{\mathsf{P}^N}\left[\int_{\mathcal{X}} \|\varphi\|_\infty(\zeta^N-\theta^*)(\mathrm{d}\varphi)\right] \leq \limsup_{N\to\infty} \mathsf{E}^{\mathsf{P}^N}\left[\int_{\mathcal{X}} \|\varphi\|_\infty \mathbf{1}_{\{\|\varphi\|_\infty \geq R\}}(\zeta^N+\theta^*)(\mathrm{d}\varphi)\right].$$

Now, we let $R \to \infty$ and we show that the RHS vanishes in the limit. To do so, recall that, due to Lemma 2.5.4, there exist constants $K(\alpha), K > 0$ such that

$$\sup_{N\in\mathbb{N}} E^{\mathsf{P}^N}\left[\|X^{N,i}\|_\infty^\alpha\right] \leq K(\alpha) \quad \text{and} \quad \sup_{N\in\mathbb{N}} E^{\mathsf{P}^N}\left[\|X^{N,i}\|_\infty\right] \leq K$$

independently of $i \in \{1,\dots,N\}$. Then, set $\alpha, \beta > 1$ such that $\frac{1}{\alpha}+\frac{1}{\beta} = 1$ and let $\varepsilon > 0$. By definition of ζ^N and by Young's and Markov's inequalities, we have

$$\begin{aligned}\limsup_{N\to\infty} \mathsf{E}^{\mathsf{P}^N}\left[\int_{\mathcal{X}} \|\varphi\|_\infty \mathbf{1}_{\{\|\varphi\|_\infty \geq R\}}\zeta^N(\mathrm{d}\varphi)\right] &= \limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \mathsf{E}^{\mathsf{P}^N}\left[\|X^{N,i}\|_\infty \mathbf{1}_{\{\|X^{N,i}\|_\infty \geq R\}}\right] \\ &\leq \left(\varepsilon^\alpha \frac{K(\alpha)}{\alpha} + \frac{K}{\varepsilon^\beta \beta R}\right) \qquad (2.32)\end{aligned}$$

which converges to zero by letting $R \to \infty$ and then $\varepsilon \to 0$. A similar reasoning applies to the same expectation with θ^* instead of ζ^N. □

Remark 2.5.1. Let $\mathbb{D} \doteq \{\varphi \in \mathcal{X} : \tau(\varphi)$ is discontinuous at $\varphi\}$. Since $\zeta \overset{a.s.}{=} \theta^* \in \mathsf{Q}$, Lemma A.0.4 implies $\theta^*(\mathbb{D}) = 0$ and the statement of Corollary 2.5.3 holds for $f =$

$\mathbf{1}_{\mathbb{D}}$ as well.

Finally, we conclude this section with the proof of Theorem 2.5.1, which leads immediately to Corollary 2.5.2.

Proof of Theorem 2.5.1. The proof is structured in three steps.

(j) $\lim_{N\to\infty} J^{N,1}(\boldsymbol{\lambda}^N) = J^{\mu}(\lambda)$.

(jj) Let $\beta^{N,1} \in \mathcal{U}_1^N$ be such that

$$J^{N,1}([\boldsymbol{\lambda}^{N,-1}, \beta^{N,1}]) \leq \inf_{\beta \in \mathcal{U}_1^N} J^{N,1}([\boldsymbol{\lambda}^{N,-1}, \beta]) + \frac{\varepsilon}{2}.$$

Then

$$\liminf_{N\to\infty} J^{N,1}\left(\left[\boldsymbol{\lambda}^{N,-1}, \beta^{N,1}\right]\right) \geq J^{\mu}(\lambda).$$

(jjj) $J^{N,1}(\boldsymbol{\lambda}^N) \leq \inf_{\beta\in\mathcal{U}_1^N} J^{N,1}([\boldsymbol{\lambda}^{N,-1}, \beta]) + \varepsilon$.

We consider the dynamics in Equation (2.25). In (j) we set $\lambda^{N,1}(t, \varphi^N) = \lambda(t, \varphi^{N,i})$ for all $(t, \varphi^N) \in [0, T] \times \mathcal{X}^N$ and prove convergence of the first-player cost functional to the cost functional of the MFG. In (jj) instead we allow the first player to deviate and choose $\lambda^{N,1}(t, \varphi^N) = \beta^{N,1}(t, \varphi^N)$ for all $(t, \varphi^N) \in [0, T] \times \mathcal{X}^N$ where $\beta^{N,1} \in \widetilde{\mathcal{U}}_1^N$ is a generic single-player relaxed control. We conclude the proof in (jjj) by combining the results in (j) and (jj).

Proof of (j). To prove that $J^{N,1}(\boldsymbol{\lambda}^N) \to J^{\mu}(\lambda)$, as $N \to \infty$, we split each cost functional in the sum of two terms:

$$\begin{aligned} J^{N,1}(\boldsymbol{\lambda}^N) &= \mathsf{E}^{\mathsf{P}^N}\left[\int_{[0,T]\times\Gamma}\int_{\mathcal{X}} \mathbf{1}_{[0,\tau(\varphi))}(t) f_0(t,\varphi,u)\lambda(t,\varphi)(\mathrm{d}u)\zeta^N(d\varphi)\mathrm{d}t\right] \\ &\quad + \mathsf{E}^{\mathsf{P}^N}\left[\int_0^T \mathbf{1}_{[0,\tau^{N,1})}(t) f_1(t, X^{N,1}, \zeta^N)\mathrm{d}t + F(\tau^{N,1}, X^{N,1}_{\tau^{N,1}})\right] \end{aligned}$$

and

$$\begin{aligned} J^{\mu}(\lambda) &= \mathsf{E}^{\mathsf{P}}\left[\int_{[0,T]\times\Gamma}\int_{\mathcal{X}} \mathbf{1}_{[0,\tau(\varphi))}(t) f_0(t,\varphi,u)\lambda(t,\varphi)(\mathrm{d}u)\zeta(\mathrm{d}\varphi)\mathrm{d}t\right] \\ &\quad + \mathsf{E}^{\mathsf{P}}\left[\int_0^T \mathbf{1}_{[0,\tau)}(t) f_1(t, X, \zeta)\mathrm{d}t + F(\tau, X_\tau)\right]. \end{aligned}$$

Since f_0 is bounded, the convergence of the first summand in the decomposition of $J^{N,1}(\boldsymbol{\lambda}^N)$ to the corresponding term in $J^{\mu}(\lambda)$ is a consequence of Corollary 2.5.3(i) and of Lemma 2.5.6. On the other hand, since both f_1 and F have sub-linear growth, the convergence of the second summand in $J^{N,1}(\boldsymbol{\lambda}^N)$ follows from Corollary 2.5.3(iii), Lemma 2.5.6 and the fact that $\theta^* \in \mathsf{Q}$ together with Lemma A.0.5.

Proof of (jj). We follow the proof of Theorem 3.10 in Lacker (2020) with suitable modifications due to the possibly unbounded drift and the dependence on the first exit time from the set $\mathcal{O}$.
Let $(\Omega^N, \mathcal{F}^N, \mathbb{F}^N = (\mathcal{F}_t^N)_{t\in[0,T]}, \mathsf{Q}^N, Y^N, W^N)_{N\in\mathbb{N}}$ be a weak solutions of the N-player system. Let $(\zeta^N)_{N\in\mathbb{N}}$ be the associated empirical measures. Under Q^N the

first player's dynamics is

$$Y_t^{N,1} = Y_0^{N,1} + \int_{[0,t]\times\Gamma} b(s, Y^{N,1}, \zeta_Y^N, u)\beta^{N,1}(s, \mathbf{Y}^N)(du)ds + \sigma W_t^{N,1}, \quad t \in [0,T].$$

Now, let P^N be the probability measure under which the first player's dynamics becomes

$$Y_t^{N,1} = Y_0^{N,1} + \int_{[0,t]\times\Gamma} b(s, Y^{N,1}, \zeta_Y^N, u)\lambda(s, Y^{N,1})(du)ds + \sigma \widetilde{W}_t^{N,1}, \quad t \in [0,T]$$

where $\widetilde{W}^{N,1}$ is a P^N-Wiener process. In other terms, P^N satisfies $\frac{d\mathsf{Q}^N}{d\mathsf{P}^N} = Z_T^N$ where

$$Z_t^N = \mathcal{E}_t\left(\int_0^{\cdot}\int_\Gamma b(s, Y^{N,1}, \zeta_Y^N, u)(\beta^{N,1}(s, \mathbf{Y}^N) - \lambda(s, Y^{N,1}))(du)d\widetilde{W}_s\right), \quad t \in [0,T].$$

By inspection of the proofs of Lemma A.0.1 and Corollary A.0.1, all bounds are uniform in $N \in \mathbb{N}$, hence Corollary A.0.1 gives the uniform integrability of the sequence of exponential martingales $(Z^N)_{N\in\mathbb{N}}$. More in detail, we apply Corollary A.0.1 to the drift

$$b(t, \varphi^N) \doteq \int_\Gamma b(t, \varphi^{N,1}, \zeta_{\varphi^N}, u)(\beta^{N,1}(t, \varphi^N) - \lambda(t, \varphi^{N,1}))(du)$$

for $(t, \varphi^N) \in [0,T] \times \mathcal{X}^N$. Notice that this drift is sublinear in φ^N. Therefore convergence of the empirical measures to θ^* in probability in the τ-topology under P^N implies convergence of the empirical measures to the same limit in probability in the τ-topology under Q^N. Hence $\zeta_Y^N \xrightarrow{\mathcal{L}} \theta^*$ under Q^N and

$$\lim_{N\to\infty} \mathsf{Q}^N\left(\zeta_Y^N \notin B\right) = 0$$

for all neighbourhoods B of θ in the τ-topology which belong to $\mathcal{B}(\mathcal{P}(\mathcal{X}))$. The tightness of $(Y^{N,1})_{N\in\mathbb{N}}$ under Q^N still follows from their tightness under P^N. Consider $(\beta^{N,1}(t, \mathbf{Y}^N))_{t\in[0,T]}$ as a single-player relaxed stochastic open-loop control and denote it simply by $(\beta_t^{N,1})_{t\in[0,T]}$. Interpret $(Y^{N,1}, \beta^{N,1}, \zeta_Y^N)_{N\in\mathbb{N}}$ as a sequence of random variables with values in $\mathcal{X} \times \mathcal{V} \times \mathcal{P}(\mathcal{X})$. Compactness of $\mathcal{V}$ and tightness of $(Y^{N,1}, \zeta_Y^N)_{N\in\mathbb{N}}$ imply the tightness of $(Y^{N,1}, \beta^{N,1}, \zeta_Y^N)_{N\in\mathbb{N}}$ under Q^N.

Let (Y, β, θ^*) be a limit point of the sequence $(Y^{N,1}, \beta^{N,1}, \zeta_Y^N)_{N\in\mathbb{N}}$, defined on some probability space with probability measure Q. Then by a standard martingale argument it can be shown to satisfy

$$Y_t = \xi + \int_{[0,t]\times\Gamma} b(s, Y, \theta^*, u)\beta_t(du)ds + \sigma W_t, \quad t \in [0,T] \tag{2.33}$$

where W is a Q-Wiener process. As in (j) we split $J^{N,1}([\lambda^{N,-1}, \beta^{N,1}])$ in two terms as

$$\begin{aligned} J^{N,1}([\lambda^{N,-1}, \beta^{N,1}]) &= \mathsf{E}^{\mathsf{Q}^N}\left[\int_{[0,T]\times\Gamma} \mathbf{1}_{[0,\tau^{N,1})}(t) f_0(t, Y^{N,1}, u)\beta_t^{N,1}(du)dt\right] \\ &+ \mathsf{E}^{\mathsf{Q}^N}\left[\int_0^T \mathbf{1}_{[0,\tau^{N,1})}(t) f_1(t, Y^{N,1}, \zeta_Y^N)dt + F(\tau^{N,1}, Y_{\tau^{N,1}}^{N,1})\right]. \end{aligned}$$

We move along a weakly converging subsequence of $(Y^{N,1}, \beta^{N,1}, W^{N,1})_{N\in\mathbb{N}}$ under

Q^N to the limit point (Y, β, W) in Equation (2.33). Convergence of the first and second summands above now works as in the proof of (j). Considering again the whole sequence, we obtain

$$\begin{aligned}\liminf_{N\to\infty} J^{N,1}([\lambda^{N,-1}, \beta^{N,1}]) &\geq \inf_{\beta} \mathrm{E}^{Q^N}\left[\int_{[0,T]\times\Gamma} \mathbf{1}_{[0,\tau)}(t) f(t, Y, \theta^*, u)\beta_t(\mathrm{d}u)\mathrm{d}t + F(\tau, Y_\tau)\right] \\ &= V^\mu\end{aligned}$$

where the infimum on the RHS above is taken over all relaxed stochastic open-loop controls and the last equality follows from embedding the set of strict controls into the set of relaxed controls combined with the chattering lemma (Bahlali et al., 2006; El Karoui et al., 1987; Fleming and Rishel, 2012).

Proof of (jjj). This is a consequence of steps (j) and (jj). Indeed

$$J^{N,1}(\boldsymbol{\lambda}^N) - \inf_{\beta\in\mathcal{U}_1^N} J^{N,1}([\boldsymbol{\lambda}^{N,-1}, \beta]) \leq J^{N,1}(\boldsymbol{\lambda}^N) - J^\mu(\lambda) + J^\mu(\lambda) - J^{N,1}([\boldsymbol{\lambda}^{N,-1}, \beta^{N,1}]) + \frac{\varepsilon}{2}.$$

Now by steps (j) and (jj) there exists $N^\varepsilon \in \mathbb{N}$ such that for all $N \geq N^\varepsilon$

$$J^{N,1}(\boldsymbol{\lambda}^N) - J^\mu(\lambda) \leq \frac{\varepsilon}{4} \quad \text{and} \quad J^\mu(\lambda) - J^{N,1}([\boldsymbol{\lambda}^{N,-1}, \beta^{N,1}]) \leq \frac{\varepsilon}{4}.$$

Therefore, we can conclude that $J^{N,1}(\boldsymbol{\lambda}^N) \leq \inf_{\beta\in\mathcal{U}_1^N} J^{N,1}([\boldsymbol{\lambda}^{N,-1}, \beta]) + \varepsilon$ for all $N \geq N^\varepsilon$, which establishes the statement of Theorem 2.5.1. □

Chapter 3

Mean-Field Games of Finite-Fuel Capacity Expansion with Singular Controls

3.1 Introduction to mean-field games with singular controls

Singular control problems with finite (and infinite) fuel find numerous applications in the economic literature and originated from the engineering literature in the late 60's (see Bather and Chernoff (1967) for a seminal paper and, for example, Beneš et al. (1980); El Karoui and Karatzas (1988); Karatzas (1985) for early contributions to the finite fuel case). Game versions of these problems are a natural extension of the single agent set-up and allow to model numerous applied situations.

Here in particular we make assumptions on the structure of the interaction across players that are suitable to model the so-called *goodwill* problem (see, e.g., Jack et al. (2008); Marinelli (2007) in a stochastic environment and Buratto and Viscolani (2002) in a deterministic one). Specifically, players can be interpreted as firms that produce the same good (e.g., mobile phones) and must decide how to advertise it over a finite time horizon. The i-th firm's product has a market price that depends on the particular type/brand (e.g., Apple, Huawei, etc.) and we model that by the process $X^{N,i}$. Each firm can invest in marketing strategies in order to raise the profile of their product and its popularity. The i-th firm's cumulative amount of investment that goes towards advertising is modelled by the process $Y^{N,i}$, where the finite-fuel feature naturally incorporates the idea that firms set a maximum budget for advertising over the period $[0, T]$. All firms measure their performance in terms of future discounted revenues: they use a running profit function $(x, y) \mapsto f(x, y)$ and deduct the (proportional) cost of advertising $c_0 \mathrm{d}\xi$. A typical example is $f(x, y) = x \cdot y^\alpha$, $\alpha \in (0, 1)$, where profits are linear in the product's price and increasing and concave as function of the total investment made towards advertising.
From the point of view of the i-th firm, investing $\Delta\xi^{N,i} > 0$ has a cost $c_0\Delta\xi^{N,i}$ and produces two effects. First of all it increases the popularity of the i-th firm's product, hence increasing the running profit to the level $f(x, y + \Delta\xi^{N,i})$ (we are tacitly assuming $y \mapsto f(x, y)$ increasing). Secondly, it has a broader impact on the visibility of the type of product (e.g., mobile phones) and will stimulate the public's demand for that good. This has a knock-on effect on the trend of the prices of all the firms that produce the same good. We model this fact via the interaction term m_t^N in the price dynamics and we assume that the drift function increases with the average spending in advertising across all companies, i.e. $m \mapsto a(x, m)$ is non-decreasing.

The literature on MFGs is rapidly growing. Most of the papers deal with games where players use "regular controls" in order to optimise their payoffs. Here by regular controls we mean those having a bounded impact on the velocity of the underlying dynamics. Only few papers have studied the case of MFGs with *singular* controls, which is a larger class of controls allowing for unbounded changes in the velocity of the underlying process and possible discontinuities in the state trajectories. More specifically, Fu and Horst (2017) established an abstract existence result for solutions of a general MFG with singular control, using the notion of relaxed solutions. The same approach was also applied in Fu (2019) to extend the previous results to MFG with interaction through the controls as well. In both papers, the issue of finding approximate Nash equilibria in the N-player games is left aside. To the best of our knowledge, only the works of Cao et al. (2017) and of Guo and Xu (2019) tackle simultaneously MFGs and N-player games with singular controls. Their analysis is based on verification theorems and quasi-variational inequalities specifically designed for their settings and not amenable to simple extensions. For completeness, we also mention the two papers Hu et al. (2014), Zhang (2012), which use a maximum principle approach to solve singular control problems with mean-field dynamics for the state variables. A class of controls closely related to singular controls is that of impulses, which has also received attention recently within MFG theory. We mention the two papers Basei et al. (2019) and Zhou and Huang (2017), where MFGs with impulse controls are considered and solved using an approach based on quasi-variational inequalities and exploiting the stationarity properties of their settings. Finally, the article Bertucci (2020) provides a variational characterization of the density of jumping particles coming from an impulse control problem.

3.1.1 Model description

In this chapter, we study Nash equilibria for a class of symmetric N-player stochastic differential games, for large N, and we characterize the solutions of the associated MFG. Specifically, we consider a class of *finite-fuel capacity expansion* games with *singular controls*.

In order to set out our main results, we provide here a short description of the N-player game of capacity expansion (see Section 3.4.1 for a full account). The game is set over a finite-time horizon T given and fixed. We consider a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ equipped with a right-continuous filtration $\mathbb{F} \doteq (\mathcal{F}_t)_{t\in[0,T]}$ which is augmented with all the P-null sets. There are N players in the game and the i-th player $i = 1, \dots, N$ chooses a strategy $\xi^{N,i} = (\xi^{N,i}_t)_{t\in[0,T]}$ from the set of all right-continuous non-decreasing adapted processes, affecting their own private state variables $(X^{N,i}, Y^{N,i})$. Given a drift $a : \mathbb{R} \times [0,1] \to \mathbb{R}$ and a volatility $\sigma : \mathbb{R} \to \mathbb{R}^+$, the private states have dynamics

$$\begin{aligned} X^{N,i}_t &= X^i_0 + \int_0^t a(X^{N,i}_s, m^N_s)\, \mathrm{d}s + \int_0^t \sigma(X^{N,i}_s)\, \mathrm{d}W^i_s, \\ Y^{N,i}_t &= Y^i_{0-} + \xi^{N,i}_t, \qquad t \in [0,T], \end{aligned} \tag{3.1}$$

where $(W^1, \dots, W^N)$ is a N-dimensional Brownian motion. The initial conditions (X^i_0, Y^i_{0-}) are i.i.d. random variables with common distribution $\nu \in \mathcal{P}(\Sigma)$, where $\mathcal{P}(\Sigma)$ is the space of all probability measures on $\Sigma \doteq \mathbb{R} \times [0,1]$. The players interact through the mean-field term m_t^N appearing in the drift and given by

$$m_t^N = \frac{1}{N}\sum_{i=1}^{N} Y_t^{N,i} = \int_\Sigma y\,\mu_t^N(\mathrm{d}x,\mathrm{d}y), \quad t \in [0,T], \tag{3.2}$$

where $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^{N,i},Y_t^{N,i})}$ denotes the empirical probability measure of the players' states with δ_z the Dirac delta mass at $z \in \Sigma$. In Equation (3.1), each $\xi^{N,i}$ represents the investment in additional capacity made by the i-th player. Each player aims at maximizing an expected payoff of the form

$$J^{N,i} \doteq \mathsf{E}\left[\int_0^T e^{-rt} f(X_t^{N,i}, Y_t^{N,i})\,\mathrm{d}t - \int_{[0,T]} e^{-rt}\,c_0\,\mathrm{d}\xi_t^{N,i}\right], \tag{3.3}$$

for a fixed discount rate $r > 0$, some cost $c_0 > 0$ and some running payoff f (the same for all players). The optimisation is subject to the so-called *finite-fuel constraint*: $Y_{0-} + \xi_t^{N,i} \in [0,1]$ for all $t \in [0,T]$ and all $i = 1,\dots,N$. We are interested in computing (approximate) Nash equilibria of this N-player game via the MFG approach. This requires to pass to the limit as $N \to \infty$ and to identify the limiting MFG. The latter must be solved (as explicitly as possible) and the associated optimal control is then implemented in the N-player game for sufficiently large N, as a proxy for the equilibrium strategy.

3.1.2 Methodology and original contribution

We focus on the construction of approximate Nash equilibria for the N-player game through solutions of the corresponding MFG.
First, we formulate the MFG of capacity expansion, i.e. the limit model corresponding to the above N-player games as $N \to \infty$ (Section 3.2). Then, under mild assumptions on the problem's data we construct a solution in feedback form of the MFG of capacity expansion (Section 3.3). Our constructive approach, based on an intuitive iterative scheme, allows us to determine the optimal control in the MFG in terms of an optimal boundary $(t,x) \mapsto c(t,x)$ that splits the state space $[0,T] \times \mathbb{R} \times [0,1]$ into an *action* region and an *inaction* region; see Theorem 3.2.1 in Section 3.2. The optimal strategy prescribes to keep the controlled dynamics underlying the MFG inside the closure of the inaction region by Skorokhod reflection. Finally, whenever the optimal boundary in the MFG is Lipschitz continuous in its second variable we can show that it induces a sequence of approximate ε_N-Nash equilibria for the N-player games with vanishing approximation error at rate $O(1/\sqrt{N})$ as N tends to infinity; see Theorem 3.4.1 in Section 3.2.
While Lipschitz regularity of optimal boundaries is in general a delicate issue, we provide sufficient conditions on our problem data that guarantee such regularity. Since our conditions are far from being necessary, in Section 4.1 we also illustrate an example with a clear economic interpretation which violates these conditions and yet yields a Lipschitz boundary.

The proof of Theorem 3.2.1 on the existence of a feedback solution for the limit MFG hinges on a well-known connection between singular stochastic control and optimal stopping (e.g., Baldursson and Karatzas (1996); Karatzas and Shreve (1984, 1985)), which we apply in the analysis of our iterative scheme. This approach allows us to overcome the usual difficulties arising from fixed-point arguments often employed in the literature on MFGs. Moreover, as a byproduct we also obtain that a connection between singular control problems of capacity expansion and problems of optimal stopping holds in the setting of our MFG. The finite-fuel condition is not a structural condition and it is clear that our choice of $y \in [0,1]$ is not restrictive: indeed, we could equally consider $y \in [0,\bar{y}]$ for any $\bar{y} > 0$ (see Remark 3.2.1). This is suggestive that our results may, in future work, be extended to the infinite-fuel setting by considering sequences of problems with increasing fuel and limiting arguments.

3.1.3 Preliminaries and notation for mean-field games of finite-fuel capacity expansion with singular controls

Let $\Sigma \doteq \mathbb{R} \times [0,1]$ and let $\mathcal{P}(\Sigma)$ denote the set of all probability measures on Σ equipped with the Borel σ-field $\mathcal{B}(\Sigma)$. Let $\mathcal{P}_2(\Sigma)$ be the subset of $\mathcal{P}(\Sigma)$ of probability measures with finite second moment. The set Σ and the N-fold product space Σ^N are the state spaces for the controlled processes (X,Y) and (X^N, Y^N) that are underlying the MFG and the N-player game, respectively. Since our problems are set on a finite-time horizon, we also consider time as a state variable and use the state space $[0,T] \times \Sigma$. Given a set $A \subset [0,T] \times \Sigma$ we denote its closure by $\overline{A}$.
Given a filtered probability space $\Pi \doteq (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ satisfying the usual conditions and a $\mathcal{F}_0$-measurable random variable $Z \in [0,1]$, we denote

$$\Xi^{\Pi}(Z) \doteq \big\{\xi : (\xi_t)_{t\geq 0} \text{ is } \mathbb{F}\text{-adapted, non-decreasing, right-continuous,} \\ \text{with } \xi_{0-} = 0 \text{ and } Z + \xi_t \in [0,1] \text{ for all } t \in [0,T],\ \mathsf{P}\text{-a.s.}\big\}.$$

The set $\Xi^{\Pi}(Z)$ is the set of admissible strategies for the players in our games. The random variable Z will be replaced by the initial value of the process Y (for the MFG) or $Y^{N,i}$ (for the i-th player in the N-player game). Often, we drop the dependence of Ξ on the probability space Π and the random variable Z, as no confusion shall arise. Finally, the parameters $c_0 > 0$ and $r \geq 0$ are fixed throughout the chapter and describe the cost of exerting control and the discount factor, respectively.

3.2 The mean-field game of finite-fuel capacity expansion with singular controls: setting and main results

In this section we set-up the MFG associated with the N-player game described above and we state the main result concerning the existence and structure of the optimal control for this game; see Theorem 3.2.1. Later, in Section 3.4 we link the MFG to the N-player game. Below we use the notation introduced in Section 3.1.3.

3.2.1 The MFG of finite-fuel capacity expansion with singular controls

Let $\Pi = (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space satisfying the usual conditions and supporting a one-dimensional $\mathbb{F}$-Brownian motion W. Notice that the initial σ-field $\mathcal{F}_0$ is not necessarily trivial.
Let (X_0, Y_{0-}) be a two-dimensional $\mathcal{F}_0$-measurable random variable with joint law $\nu \in \mathcal{P}(\Sigma)$ and let $\xi \in \Xi^{\Pi}(Y_{0-})$ be an admissible strategy. Then, given a bounded

Borel measurable function $m : [0,T] \to [0,1]$, for all $t \in [0,T]$ we define the 2-dimensional, degenerate, controlled dynamics

$$
\begin{aligned}
X_t &= X_0 + \int_0^t a(X_s, m(s))\,\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s, \\
Y_t^{\xi} &= Y_{0-} + \xi_t.
\end{aligned} \tag{3.4}
$$

The goal of the representative player consists in maximizing over the set of all admissible strategies $\xi \in \Xi^{\Pi}(Y_{0-})$ the following objective functional

$$
J(\xi) = \mathsf{E}\left[\int_0^T e^{-rt} f(X_t, Y_t^{\xi})\mathrm{d}t - \int_{[0,T]} e^{-rt} c_0 \mathrm{d}\xi_t\right], \tag{3.5}
$$

where f is some running payoff function and we recall that $c_0 > 0$ is some cost and $r \geq 0$. Assumptions on all the coefficients appearing in the state variables' dynamics and in the objective functional are given below. The integral with respect to the positive random measure $\mathrm{d}\xi$ includes possible atoms at the initial and terminal time (corresponding to possible jumps of ξ).

Remark 3.2.1. The choice $Y \in [0,1]$ in the definition of the set Ξ of admissible strategies is with no loss of generality and we could equally consider $Y \in [0, \bar{y}]$ for $\bar{y} > 0$. The assumption of finite fuel is consistent with real-world applications, where a firm would set aside a certain budget to be spent over a given period $[0,T]$.

Since we are interested in the MFG that arises from the N-player game, Equations (3.1)-(3.3), in the limit as $N \to \infty$, it is natural to seek for an admissible optimal strategy ξ (given m) such that the following consistency condition holds

$$
m(t) = \mathsf{E}[Y_t^{\xi}], \quad t \in [0,T]. \tag{3.6}
$$

The precise definition of MFG solution is given in Definition 3.2.1 below. In order to develop our methodology, it is convenient to state a version of the MFG starting from any time $t \in [0,T]$ and any realization $(x,y) \in \Sigma$ of the states (X_t, Y_{t-}). Therefore, let us consider the dynamics in Equation (3.4) conditional on the initial data $(t,x,y) \in [0,T] \times \Sigma$, i.e.

$$
\begin{aligned}
X_{t+s}^{t,x} &= x + \int_0^s a(X_{t+u}^{t,x}, m(t+u))\,\mathrm{d}u + \int_0^s \sigma(X_{t+u}^{t,x})\mathrm{d}W_{t+u}, \\
Y_{t+s}^{t,x,y;\xi} &= y + (\xi_{t+s} - \xi_{t-}), \quad s \in [0, T-t],
\end{aligned} \tag{3.7}
$$

where $\mathrm{d}W_{t+u} = \mathrm{d}(W_{t+u} - W_t)$. Since the increments of the control $\xi \in \Xi^{\Pi}(Y_{0-})$, after time t, may in general depend on (t,x,y), we account for that dependence by denoting $Y^{t,x,y;\xi}$ (and $\xi^{t,x,y}$ if necessary). Instead, given a bounded measurable function m, the dynamics of X only depends on the initial condition $X_t = x$, which motivates the use of the notation $X^{t,x}$. For the original case of the process started at time zero (i.e. $t = 0$), we use the simpler notation $(X_s^x, Y_s^{x,y;\xi})_{s\in[0,T]}$.

The notation introduced above is somewhat cumbersome, so we often use $\mathsf{P}_{t,x,y}(\,\cdot\,) = \mathsf{P}(\,\cdot\,|X_t = x, Y_{t-} = y)$ for simplicity. So for any bounded measurable function g and any stopping time $\tau \in [0, T-t]$ we have

$$
\mathsf{E}\left[g(t+\tau, X_{t+\tau}^{t,x}, Y_{t+\tau}^{t,x,y;\xi})\right] = \mathsf{E}_{t,x,y}\left[g(t+\tau, X_{t+\tau}, Y_{t+\tau}^{\xi})\right],
$$

and, moreover, we use $\mathsf{P}_{x,y} = \mathsf{P}_{0,x,y}$ for the special case $t = 0$.
It is clear that given $\xi \in \Xi^{\Pi}(Y_{0-})$ the process $\hat{\xi}_s \doteq \xi_{t+s} - \xi_{t-}$ is right continuous, non-decreasing and adapted with $\hat{\xi}_{0-} = 0$. Moreover, $y + \hat{\xi} \in [0,1]$, $\mathsf{P}_{t,x,y}$ a.s. (i.e. conditionally on $(X_t, Y^{\xi}_{t-}) = (x,y)$) because $\xi \in \Xi^{\Pi}(Y_{0-})$. Then, it is useful to introduce the set

$$\Xi^{\Pi}_{t,x}(y) \doteq \{\xi : (\xi_s)_{s\geq 0} \text{ is } (\mathcal{F}_{t+s})_{s\geq 0}\text{-adapted, non-decreasing, right-continuous,} \\ \text{with } \xi_{0-} = 0 \text{ and } y + \xi_s \in [0,1] \text{ for all } s \in [0, T-t],\ \mathsf{P}_{t,x,y}\text{-a.s.}\}.$$

Clearly $\Xi^{\Pi}_{0,x}(y) = \Xi^{\Pi}(y)$. Here Π is fixed, so we can drop the superscript in the definition of the set of admissible controls. We sometimes drop also the subscript x and just write $\Xi_t(y) = \Xi_{t,x}(y)$. Furthermore, when no confusion shall arise we write $\xi \in \Xi_t(y)$ although we refer to $\hat{\xi} \in \Xi_t(y)$ with $\hat{\xi}_s = \xi_{t+s} - \xi_{t-}$.
Assuming that the mapping $(x,y) \mapsto \mathsf{E}_{x,y}[Y^{\xi}_t]$ is measurable for any admissible ξ, we can express the consistency condition in Equation (3.6) as

$$m(t) = \int_{\Sigma} \mathsf{E}_{x,y}[Y^{\xi}_t]\nu(dx,dy) = \int_{\Sigma}\int_{\Sigma} y' \mu^{x,y;\xi}_t(dx',dy')\nu(dx,dy),$$

where $\mu^{x,y;\xi}_t \doteq \mathcal{L}(X^x_t, Y^{x,y;\xi}_t) \in \mathcal{P}(\Sigma)$ is the law of the pair $(X^x_t, Y^{x,y;\xi}_t)$ and the integral with respect to $\nu(dx,dy)$ accounts for the fact that $(X_0, Y_{0-}) \overset{d}{\sim} \nu$.

Turning our attention to the optimisation problem, we have that the maximal expected payoff associated with a condition $(t,x,y) \in [0,T] \times \Sigma$ is given by

$$\begin{aligned} v(t,x,y) &\doteq \sup_{\xi\in\Xi_{t,x}(y)} J(t,x,y;\xi) \quad \text{with} \\ J(t,x,y;\xi) &\doteq \mathsf{E}_{t,x,y}\left[\int_0^{T-t} e^{-rs} f(X_{t+s}, Y^{\xi}_{t+s})\, ds - \int_{[0,T-t]} e^{-rs} c_0 d\xi_s\right]. \end{aligned} \tag{3.8}$$

The initial objective function in Equation (3.5) and the optimisation problem in Equation (3.8) are easily linked by averaging the latter over the initial condition $(X_0, Y_{0-}) \overset{d}{\sim} \nu \in \mathcal{P}(\Sigma)$. That is

$$V^{\nu} \doteq \sup_{\xi\in\Xi} J(\xi) \quad \text{with} \quad J(\xi) \doteq \int_{\Sigma} J(0,x,y;\xi)\nu(dx,dy), \tag{3.9}$$

where we write $\Xi = \Xi(Y_{0-})$ for simplicity.

Now we define solutions of the MFG of capacity expansion.

Definition 3.2.1 (Solution of the MFG of capacity expansion)**.** A solution of the MFG of capacity expansion with initial condition $\nu \in \mathcal{P}_2(\Sigma)$ is a pair (m^*, ξ^*) with $m^* : [0,T] \to [0,1]$ a measurable function and $\xi^* \in \Xi$ such that:

(i) *(Optimality property).* ξ^* is optimal, i.e.

$$J(\xi^*) = V^{\nu} = \sup_{\xi\in\Xi} \mathsf{E}\left[\int_0^T e^{-rt} f(X^*_t, Y^{\xi}_t)dt - \int_{[0,T]} e^{-rt} c_0 d\xi_t\right],$$

where (X^*, Y^{ξ}) is a solution of Equation (3.4) associated to (m^*, ξ).

(ii) *(Mean-field property)*. Letting (X^*, Y^*) be the solution of Equation (3.4) associated to (m^*, ξ^*), the consistency condition holds, i.e.

$$m^*(t) = \int_\Sigma \mathsf{E}_{x,y}[Y_t^*]\nu(\mathrm{d}x, \mathrm{d}y),$$

for each $t \in [0, T]$.

We say that a solution ξ^* of the MFG is in *feedback form* if we have $\xi_t^* = \eta(t, X, Y_{0-})$, $t \in [0, T]$, for some non-anticipative mapping

$$\eta : [0, T] \times C([0, T]; \mathbb{R}) \times [0, 1] \to [0, 1]$$

(i.e. such that $\eta(t, X, Y_{0-}) = \eta(t, (X_{s\wedge t})_{s\in[0,T]}, Y_{0-})$).

We observe that the definition of MFG solution above mimics the structure of a Nash equilibrium (NE) in classical game theory. Indeed, for a NE we first need to compute the best response of each player while keeping the strategies of the competitors fixed, and then we obtain the equilibrium as a fixed point of the best response map. Likewise, the optimality condition (i) corresponds to computing the best response against a given behaviour of the population described by m^*; condition (ii) is a fixed point condition, stating that m^* has to be consistent with the best response of the representative player.

3.2.2 Assumptions and main result

Before stating our main result regarding the existence and structure of the solution to the MFG, we list below the assumptions needed in our approach.

Assumption 3.2.1 (Coefficients of the SDE)**.** For the functions $a : \Sigma \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$ the following hold:

(i) a and σ are Lipschitz continuous with constant $L > 0$, i.e. for all $x, x' \in \mathbb{R}$ and $m, m' \in [0, 1]$ we have

$$|a(x, m) - a(x', m')| + |\sigma(x) - \sigma(x')| \leq L(|x - x'| + |m - m'|).$$

(ii) The mapping $m \mapsto a(x, m)$ is non-decreasing on $[0, 1]$ for all $x \in \mathbb{R}$.

Part (i) of the assumption guarantees that given any Borel measurable function $m : [0, T] \to [0, 1]$ the first equation in System (3.7) admits a unique strong solution (see, e.g., Karatzas and Shreve (1987), Theorem 5.2.9). Moreover, by a well-known application of Kolmogorov-Chentsov's continuity theorem, there exists a modification $\tilde{X}$ of X which is continuous as a random field, i.e. $(t, x, s) \mapsto \tilde{X}_{t+s}^{t,x}$ is continuous P-a.s. (see, e.g., Karatzas and Shreve (1987), pp. 397-398, or Baldi (2017), Theorem 9.9). From now on we tacitly assume that we always work with such modification and we denote it again by X.
Part (i) of the assumption could be relaxed but at the cost of additional technicalities in the proofs. In principle we only need sufficient regularity on the coefficients to guarantee existence of a unique strong solution for X which is also continuous with respect to its initial datum (t, x). Part (ii) instead is instrumental in our construction of the optimal control in the MFG and will be used later for a comparison result (Lemma 3.3.1). Notice that (ii) is well-suited for the application to the *goodwill* problem described in Section 3.1.1 in the Introduction. Typical examples that we have in

mind for the drift are $a(x,m) = (m-x)$ (mean-reverting), $a(x,m) = mx$ (geometric Brownian motion) and $a(x,m) = m$ (arithmetic Brownian motion).

Next we give assumptions on the running profit appearing in the optimisation problem.

Assumption 3.2.2 (Profit function)**.** The running profit $f : \Sigma \to [0,\infty)$ is continuous and the partial derivatives $\partial_y f$ and $\partial_{xy} f$ exist and are continuous on $\mathbb{R} \times (0,1)$. Furthermore, we have

(i) Monotonicity: $x \mapsto f(x,y)$, $y \mapsto f(x,y)$ and $x \mapsto \partial_y f(x,y)$ are increasing, with

$$\lim_{x \to -\infty} \partial_y f(x,y) < rc_0 < \lim_{x \to +\infty} \partial_y f(x,y); \tag{3.10}$$

(ii) Concavity: $y \mapsto f(x,y)$ is strictly concave for all $x \in \mathbb{R}$.

(iii) The mixed derivative is strictly positive, i.e. $\partial_{xy} f > 0$ on $\mathbb{R} \times (0,1)$.

The set of assumptions above is in line with the literature on irreversible investment and is fulfilled for example by profit functions of Cobb-Douglas type (i.e. $f(x,y) = x^\alpha y^\beta$ with $\alpha \in [0,1]$, $\beta \in (0,1)$ and $x > 0$).

We conclude with some standard integrability conditions that guarantee that the problem is well-posed and allows us to use dominated convergence theorem in some of the technical steps in the proofs.

Assumption 3.2.3 (Integrability)**.** There exists $p > 1$ such that, given any Borel measurable $m : [0,T] \to [0,1]$ and letting X be the associated solution of the SDE in Equation (3.7), we have

$$\mathsf{E}_{t,x,y}\left[\int_0^{T-t} e^{-rs}\Big(\,|f(X_{t+s},y)|^p + |\partial_y f(X_{t+s},y)|^p\,\Big)\mathrm{d}s\right] < \infty,$$

for all $(t,x,y) \in [0,T] \times \mathbb{R} \times [0,1]$. Finally, $\nu \in \mathcal{P}_2(\Sigma)$.

Remark 3.2.2 (State space)**.** For specific applications it may be convenient to restrict the state space of the process X to the positive half-line $[0,\infty)$ or to a generic (possibly unbounded) interval $(\underline{x},\overline{x})$. In those cases the assumptions above and the further ones in the next sections can be adapted in a straightforward manner. In particular the limits in Equation (3.10) are amended by letting x tend to the endpoints of the relevant domain. If the end-points of the domain are inaccessible to the process X all our arguments of proof continue to hold up to trivial changes in the notation. For a more general boundary behaviour of the process some tweaks may be needed on a case by cases basis.

We are now ready to state the main results concerning the MFG described above. The proof requires a number of technical steps and hinges on a iterative method whose details are provided in Section 3.3.

Theorem 3.2.1 (Solution of the MFG of capacity expansion)**.** *Suppose Assumptions 3.2.1, 3.2.2 and 3.2.3 hold. Then, there exists a upper-semi continuous function $c : [0,T] \times \mathbb{R} \to [0,1]$, with $t \mapsto c(t,x)$ and $x \mapsto c(t,x)$ both non-decreasing, such that the pair (m^*,ξ^*) with*

$$\xi^*_t \doteq \sup_{0 \le s \le t} (c(s,X^*_s) - Y_{0-})^+, \quad m^*(t) \doteq \int_\Sigma \mathsf{E}_{x,y}\left[Y^*_t\right]\nu(\mathrm{d}x,\mathrm{d}y), \quad t \in [0,T],$$

is a solution of the MFG as in Definition 3.2.1.

Differently from the vast majority of papers that analyse MFGs here we are able not only to prove existence of a solution but also to characterise the optimal control in terms of a upper semi-continuous, monotone surface in the state space $[0, T] \times \Sigma$. Moreover, the iterative scheme that we devise for the proof of the theorem suggests a procedure to actually construct the optimal boundary numerically.
The second key result in this chapter shows that the optimal control ξ^* solution of the MFG can be used (under mild additional assumptions) to construct an ε-Nash equilibrium in the N-player game. The statement and proof of this fact are given in Section 3.4 below, whereas in the next section we prove Theorem 3.2.1.

3.3 Construction of the solutions of the mean-field game

In this section, we provide the complete proof of Theorem 3.2.1 together with an intuitive description of the iterative scheme that underpins it. Some of the auxiliary results used along the way can be found in the Appendix as indicated.

3.3.1 Description of the iterative scheme

The idea is to start an iterative scheme based on singular control problems that are analogue to the one in the MFG but without consistency condition in the mean-field interaction.

We initialise the scheme by setting $m^{[-1]}(t) \equiv 1$, for $t \in [0, T]$. At the n-th step, $n \geq 0$, assume a non-decreasing, right-continuous function $m^{[n-1]} : [0, T] \to [0, 1]$ is given and fixed and consider the dynamics

$$X^{[n];t,x}_{t+s} = x + \int_0^s a(X^{[n];t,x}_{t+u}, m^{[n-1]}(t+u))\mathrm{d}u + \int_0^s \sigma(X^{[n];t,x}_{t+u})\mathrm{d}W_{t+u}, \tag{3.11}$$

$$Y^{[n];t,x,y}_{t+s} = y + (\xi_{t+s} - \xi_{t-}), \tag{3.12}$$

for $(x, y) \in \Sigma$, $s \in [0, T-t]$, $t \in [0, T]$ and where $\xi \in \Xi(Y_{0-})$. As already noticed we have $\xi_{t+\cdot} - \xi_{t-} \in \Xi_t(y)$ and we define the singular control problem $\mathbf{SC}^{[n]}_{t,x,y}$ as:

$$v_n(t, x, y) \doteq \sup_{\xi \in \Xi_t(y)} J_n(t, x, y; \xi) \qquad \text{with} \tag{3.13}$$

$$J_n(t, x, y; \xi) \doteq \mathsf{E}_{t,x,y}\left[\int_0^{T-t} e^{-rs} f(X^{[n]}_{t+s}, y + \xi_s)\mathrm{d}s - \int_{[0,T-t]} e^{-rs} c_0 \mathrm{d}\xi_s\right]. \tag{3.14}$$

Now, in order to define the $(n+1)$-th step of the algorithm, let us assume that we can find an optimal control $\xi^{[n]*}$ for problem $\mathbf{SC}^{[n]}_{0,x,y}$ for each $(x, y) \in \Sigma$. Set $Y^{[n]*} \doteq y + \xi^{[n]*}$ and assume that $(x, y) \mapsto \mathsf{E}_{x,y}[Y^{[n]*}_t]$ is measurable for all $t \in [0, T]$. Then, we define

$$m^{[n]}(t) \doteq \int_\Sigma \mathsf{E}_{x,y}\left[Y^{[n]*}_t\right] \nu(\mathrm{d}x, \mathrm{d}y).$$

The map $t \mapsto m^{[n]}(t)$ is non-decreasing and right-continuous (by dominated convergence) with values in $[0, 1]$, so we can use it to define $(X^{[n+1]}, Y^{[n+1]})$ and v_{n+1} by iterating the above construction.

It is well-known in singular control theory that since $y \mapsto f(x,y)$ is concave and the dynamics of $X^{[n]}$ is independent of the control ξ, then the y-derivative of $v_n(t,x,y)$ corresponds to the value function of an optimal stopping problem. While we re-derive this fact in Proposition 3.3.3 for completeness, here we state the optimal stopping problem that should be associated to $\mathbf{SC}^{[n]}_{t,x,y}$ above.
For $(t,x,y) \in [0,T] \times \Sigma$ we define the stopping problem $\mathbf{OS}^{[n]}_{t,x,y}$ as

$$u_n(t,x,y) \doteq \inf_{\tau \in \mathcal{T}_t} U_n(t,x,y;\tau) \qquad \text{with} \tag{3.15}$$

$$U_n(t,x,y;\tau) \doteq \mathsf{E}_{t,x}\left[\int_0^\tau e^{-rs}\partial_y f(X^{[n]}_{t+s},y)\mathrm{d}s + c_0 e^{-r\tau}\right], \quad \text{for } \tau \in \mathcal{T}_t \tag{3.16}$$

and where $\mathcal{T}_t$ is the set of stopping times for the filtration generated by the Brownian motion in Equation (3.11), with values in $[0, T-t]$. Since $W_{t+u} - W_t = W_u$ in law, it is convenient for the analysis of the stopping problems (and there is no loss of generality) to use always the same Brownian motion in the dynamics of the process $X^{[n];t,x}$, irrespectively of $t \in [0,T]$. With this convention we have the useful fact that $\mathcal{T}_{t_2} \subset \mathcal{T}_{t_1}$ for $t_1 < t_2$. This stopping problem is standard (see, e.g., Peskir and Shiryaev (2006), Chapter I, Section 2, Theorem 2.2): thanks to Assumption 3.2.3 and continuity of the gain process

$$u \mapsto \int_0^u e^{-rs}\partial_y f(X^{[n]}_{t+s},y)\mathrm{d}s + c_0 e^{-ru}$$

we know that the smallest optimal stopping time is

$$\tau^{[n]}_*(t,x,y) = \inf\{s \in [0,T-t] : u_n(t+s, X^{[n];t,x}_{t+s}, y) = c_0\}. \tag{3.17}$$

Letting

$$Z^{[n]}_s \doteq e^{-rs}u_n(t+s, X^{[n]}_{t+s}, y) + \int_0^s e^{-ru}\partial_y f(X^{[n]}_{t+u},y)\mathrm{d}u \tag{3.18}$$

we have that, under $\mathsf{P}_{t,x,y}$

$$(Z^{[n]}_s)_{s\in[0,T-t]} \text{ is a submartingale and } (Z^{[n]}_{s\wedge\tau^{[n]}_*})_{s\in[0,T-t]} \text{ is a martingale.} \tag{3.19}$$

Accordingly, we define the continuation region $\mathcal{C}^{[n]}$ and the stopping region $\mathcal{S}^{[n]}$ of the optimal stopping problem as

$$\mathcal{C}^{[n]} \doteq \{(t,x,y) \in [0,T] \times \Sigma : u_n(t,x,y) < c_0\},$$
$$\mathcal{S}^{[n]} \doteq \{(t,x,y) \in [0,T] \times \Sigma : u_n(t,x,y) = c_0\}.$$

Finally, we introduce an auxiliary set, which is useful for our analysis

$$\mathcal{H} \doteq \{(x,y) \in \mathbb{R} \times [0,1] : \partial_y f(x,y) - rc_0 < 0\}. \tag{3.20}$$

Notice that condition in Equation (3.10), Assumption 3.2.2, implies that $\mathcal{H}$ is not empty. This is needed to prove that the continuation and stopping regions are not empty either.

The rest of our algorithm of proof for Theorem 3.2.1 goes as follows:

Step 1. Using a probabilistic approach we study in detail continuity and monotonicity of the value function u_n, for a generic $n \geq 0$.

Step 2. Thanks to the results in step 1 we construct a solution to $\mathbf{OS}^{[n]}_{t,x,y}$ by determining the geometry of the stopping region $\mathcal{S}^{[n]}$. In particular we need to prove regularity properties of the optimal stopping boundary $\partial\mathcal{C}^{[n]}$ that guarantee that we can construct a process $Y^{[n]*}$ so that the couple $(X^{[n]}, Y^{[n]*})$ is bound to evolve in the closure $\overline{\mathcal{C}}^{[n]}$ of the continuation set, by Skorokhod reflection.

Step 3. We confirm that $Y^{[n]*}$ is optimal in the singular control problem $\mathbf{SC}^{[n]}_{t,x,y}$ and that v_n can be constructed by integrating u_n with respect to y (as already shown in the existing literature).

Step 4. We prove that the sequence $(u_n)_{n\geq 0}$ is decreasing and use this fact to prove that the iterative scheme converges to the MFG, in the sense that $(X^{[n]}, Y^{[n]*}, m^{[n]})$ converges to (X^*, Y^*, m^*) from Definition 3.2.1 and that (Y^*, m^*) are expressed as in Theorem 3.2.1.

3.3.2 Solution of the n-th stopping problem

Here we construct the solution to problem $\mathbf{OS}^{[n]}_{t,x,y}$ for a generic $n \geq 0$. In particular, $t \mapsto m^{[n-1]}(t)$ is a given right-continuous, non-decreasing function bounded between zero and one.

First we state a simple but useful comparison result. In order to prove it, we introduce an auxiliary state process on a filtered probability space $\overline{\Pi} = (\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\geq 0}, \overline{\mathsf{P}})$ with filtration generated by the Brownian motion $\overline{W}$

$$\overline{X}^{[n];t,x}_{t+s} = x + \int_0^s a(\overline{X}^{[n];t,x}_{t+u}, m^{[n-1]}(t+u))\mathrm{d}u + \int_0^s \sigma(\overline{X}^{[n];t,x}_{t+u})\mathrm{d}\overline{W}_u, \tag{3.21}$$

for $x \in \mathbb{R}$, $s \in [0, T-t]$ and $t \in [0, T]$. So that

$$\left(\overline{X}^{[n];t,x}_{t+\cdot}, \overline{W}_\cdot\right) = \left(X^{[n];t,x}_{t+\cdot}, W_{t+\cdot} - W_t\right) \quad \text{in law.}$$

Then, for $(t, x, y) \in [0, T] \times \Sigma$, we introduce an auxiliary stopping problem:

$$\overline{u}_n(t,x,y) \doteq \inf_{\tau\in\overline{\mathcal{T}}_t} \overline{U}_n(t,x,y;\tau) \qquad \text{with} \tag{3.22}$$

$$\overline{U}_n(t,x,y;\tau) \doteq \mathsf{E}_{t,x}\left[\int_0^\tau e^{-rs}\partial_y f(\overline{X}^{[n]}_{t+s}, y)\mathrm{d}s + c_0 e^{-r\tau}\right], \quad \text{for } \tau \in \overline{\mathcal{T}}_t \tag{3.23}$$

where $\overline{\mathcal{T}}_t$ is the set of stopping times for the filtration generated by the Brownian motion $\overline{W}$. We observe that $\overline{u}_n(t,x,y) = u_n(t,x,y)$ for all $(t,x,y) \in [0,T] \times \Sigma$, which will be key to the proof of item (iii) of Proposition 3.3.1.

Lemma 3.3.1 (Comparison)**.** *Let Assumption 3.2.1 hold and recall that $m^{[n-1]} : [0,T] \to [0,1]$ is non-decreasing. Then, for any $t \leq t'$ we have*

$$\mathsf{P}\left(\overline{X}^{[n];t,x}_{t+s} \leq \overline{X}^{[n];t',x}_{t'+s}, \ \forall s \in [0, T-t']\right) = 1, \tag{3.24}$$

under the dynamics in Equation (3.11).

Proof. It suffices to compare the drift coefficients of the auxiliary processes $\overline{X}^{[n];t,x}$ and $\overline{X}^{[n];t',x}$ and then apply the comparison result in (Karatzas and Shreve, 1987, Proposition 5.2.18) (that proof does not use time-continuity of the drift and it is the same as the proof of Proposition 5.2.13 therein). Set $A(x,s) \doteq a(x, m(t+s))$ and $A'(x,s) \doteq a(x, m(t'+s))$. Since both $t \mapsto m(t)$ and $m \mapsto a(x,m)$ are non-decreasing (Assumption 3.2.1-(ii)), we have $A(x,s) \leq A'(x,s)$ for all $(x,s) \in \mathbb{R} \times [0, T-t']$. Therefore, applying (Karatzas and Shreve, 1987, Proposition 5.2.18) we obtain Equation (3.24). □

Next we prove continuity and monotonicity of the value function.

Proposition 3.3.1 (Value function of $\mathbf{OS}^{[n]}$). *Let Assumptions 3.2.1–3.2.3 hold. Then the value function of the optimal stopping problem $\mathbf{OS}^{[n]}_{t,x,y}$ has the following properties:*

(i) $0 \leq u_n(t,x,y) \leq c_0$*;*

(ii) the map $x \mapsto u_n(t,x,y)$ *is non-decreasing for each fixed* $(t,y) \in [0,T] \times [0,1]$ *and* $y \mapsto u_n(t,x,y)$ *is non-increasing for each* $(t,x) \in [0,T] \times \mathbb{R}$*;*

(iii) the map $t \mapsto u_n(t,x,y)$ *is non-decreasing for each fixed* $(x,y) \in \Sigma$*;*

(iv) the value function is continuous, i.e. $u_n \in C([0,T] \times \Sigma; \mathbb{R})$.

Proof. (i). The upper bound is due to $u_n(t,x,y) \leq U_n(t,x,y;0) = c_0$. For the lower bound it is enough to recall that $\partial_y f \geq 0$ by Assumption 3.2.2-(i).

(ii). Fix $(t,y) \in [0,T] \times [0,1]$. Let $x_2 > x_1$ and set $\tau_2 \doteq \tau_*^{[n]}(t,x_2,y)$ as in Equation (3.17), which is optimal in $u_n(t,x_2,y)$. Then

$$u_n(t,x_2,y) - u_n(t,x_1,y) \geq \mathsf{E}\left[\int_0^{\tau_2} e^{-rs}\left(\partial_y f(X^{[n];t,x_2}_{t+s}, y) - \partial_y f(X^{[n];t,x_1}_{t+s}, y)\right) \mathrm{d}s\right] \geq 0$$

because $X^{[n];t,x_2}_{t+s} \geq X^{[n];t,x_1}_{t+s}$ by uniqueness of the solution of Equation (3.11) and $x \mapsto \partial_y f(x,y)$ is increasing by Assumption 3.2.2-(i). By a similar argument we also obtain monotonicity in y, since $y \mapsto \partial_y f(x,y)$ is decreasing by Assumption 3.2.2-(ii).

(iii). For this part of the proof we use Lemma 3.3.1. Fix $(x,y) \in \Sigma$ and take $t_2 > t_1$ in $[0,T]$. Consider $\mathbf{SC}^{[n]}_{t_1,x,y}$ and $\mathbf{SC}^{[n]}_{t_2,x,y}$ with underlying dynamics given by the auxiliary processes $\overline{X}^{[n];t_1,x}$ and $\overline{X}^{[n];t_2,x}$. Let $\overline{\mathcal{T}}_t$ be the set of optimal stopping times adapted to the filtration generated by the Brownian motion $\overline{W}$ and with values in $[0, T-t]$. Then let $\tau_2 = \tau_*^{[n]}(t_2,x,y)$ be optimal in $\overline{u}_n(t_2,x,y)$ and notice that the stopping time is also admissible for $\overline{u}_n(t_1,x,y)$ because $\overline{\mathcal{T}}_{t_2} \subset \overline{\mathcal{T}}_{t_1}$. Then

$$\overline{u}_n(t_2,x,y) - \overline{u}_n(t_1,x,y) \geq \mathsf{E}\left[\int_0^{\tau_2} e^{-rs}\left(\partial_y f(\overline{X}^{[n];t_2,x}_{t_2+s}, y) - \partial_y f(\overline{X}^{[n];t_1,x}_{t_1+s}, y)\right) \mathrm{d}s\right] \geq 0,$$

where the final inequality uses that $\overline{X}^{[n];t_2,x}_{t_2+s} \geq \overline{X}^{[n];t_1,x}_{t_1+s}$ for $s \in [0, T-t_2]$, P-a.s. by Lemma 3.3.1 and $x \mapsto \partial_y f(x,y)$ is non-decreasing by Assumption 3.2.2-(i). We conclude by observing that $\overline{u}_n(t,x,y) = u(t,x,y)$ for all $(t,x,y) \in [0,T] \times \Sigma$.

(iv). Joint continuity of the value function can be deduced by separate continuity in each variable and monotonicity (see, e.g., Kruse and Deely (1969)). Thanks to (ii)

and (iii), it suffices to show that u_n is continuous separately in each variable.
Fix $(t,x,y) \in [0,T] \times \Sigma$. Let $x_k \to x$ as $k \to \infty$ and let $\tau_* = \tau_*^{[n]}(t,x,y)$ be optimal for $u_n(t,x,y)$. First we show right-continuity of $u_n(t,\cdot,y)$ and assume that $x_k \downarrow x$. For each k, using monotonicity proven in (ii) we have

$$
\begin{aligned}
0 \leq & u_n(t,x_k,y) - u_n(t,x,y) \\
\leq & \mathsf{E}\left[\int_0^{\tau_*} e^{-rs}\left(\partial_y f(X_{t+s}^{[n];t,x_k},y) - \partial_y f(X_{t+s}^{[n];t,x},y)\right)\mathrm{d}s\right] \\
\leq & \mathsf{E}\left[\int_0^{T-t} e^{-rs}\left|\partial_y f(X_{t+s}^{[n];t,x_k},y) - \partial_y f(X_{t+s}^{[n];t,x},y)\right|\mathrm{d}s\right].
\end{aligned} \tag{3.25}
$$

Taking limits as $k \to \infty$, Assumption 3.2.3 allow us to use dominated convergence so that we only need

$$
\lim_{k\to\infty}\left|\partial_y f(X_{t+s}^{[n];t,x_k},y) - \partial_y f(X_{t+s}^{[n];t,x},y)\right| = 0, \quad \mathsf{P}-a.s.
$$

The latter holds by continuity of $\partial_y f$ and continuity of the flow $x \mapsto X^{[n];t,x}$ (which is guaranteed by Assumption 3.2.1).
We can prove left-continuity by analogous arguments. Letting $x_k \uparrow x$ and, for each k, selecting the stopping time $\tau_k = \tau_*^{[n]}(t,x_k,y)$ which is optimal for $u_n(t,x_k,y)$ we get

$$
\begin{aligned}
0 \leq & u_n(t,x,y) - u_n(t,x_k,y) \\
\leq & \mathsf{E}\left[\int_0^{\tau_k} e^{-rs}\left(\partial_y f(X_{t+s}^{[n];t,x},y) - \partial_y f(X_{t+s}^{[n];t,x_k},y)\right)\mathrm{d}s\right].
\end{aligned}
$$

Then we can conclude as in Equation (3.25). Completely analogous arguments allow to prove continuity of the value function with respect to y and we omit them here for brevity.
Continuity in time only requires a small adjustment to the argument above. Let $t_k \to t$ as $k \to \infty$, with $(t,x,y) \in [0,T] \times \Sigma$ fixed. First let us consider $t_k \downarrow t$ and set $\tau_* = \tau_*^{[n]}(t,x,y)$, which is optimal for $u_n(t,x,y)$. Then $\tau_* \wedge (T-t_k)$ is admissible for $u_n(t_k,x,y)$ and, by the monotonicity proven in (iii), we have

$$
\begin{aligned}
0 \leq & u(t_k,x,y) - u(t,x,y) \\
\leq & \mathsf{E}\left[\int_0^{\tau_*\wedge(T-t_k)} e^{-rs}\left(\partial_y f(X_{t_k+s}^{[n];t_k,x},y) - \partial_y f(X_{t+s}^{[n];t,x},y)\right)\mathrm{d}s\right] \\
& + \mathsf{E}\left[\int_{\tau_*\wedge(T-t_k)}^{\tau_*} e^{-rs}\partial_y f(X_{t+s}^{[n];t,x},y)\mathrm{d}s\right] \\
\leq & \mathsf{E}\left[\int_0^{T-t_k} e^{-rs}\left|\partial_y f(X_{t_k+s}^{[n];t_k,x},y) - \partial_y f(X_{t+s}^{[n];t,x},y)\right|\mathrm{d}s\right] \\
& + \mathsf{E}\left[\int_{T-t_k}^{T-t} e^{-rs}\left|\partial_y f(X_{t+s}^{[n];t,x},y)\right|\mathrm{d}s\right].
\end{aligned}
$$

Now we can let $k \to \infty$ and use dominated convergence (thanks to Assumption 3.2.3), continuity of the stochastic flow $t \mapsto X_{t+\cdot}^{t,x}$ and continuity of $\partial_y f$ (Assumption 3.2.2) to obtain right-continuity of $u_n(\cdot,x,y)$. An analogous argument allows to prove left-continuity as well. □

Thanks to the properties of the value function we can easily determine the shape of the continuation region $\mathcal{C}^{[n]}$, whose boundary $\partial\mathcal{C}^{[n]}$ turns out to be a surface with "nice" monotonicity properties, that we subsequently use to obtain a solution of the singular control problem $\mathbf{SC}^{[n]}$. Part of the proof is based on the following equivalent representation of the value function:

$$u_n(t,x,y) = c_0 + \inf_{\tau\in\mathcal{T}_t} \mathsf{E}_{t,x}\left[\int_0^\tau e^{-rs}\left(\partial_y f(X^{[n]}_{t+s},y) - rc_0\right) \mathrm{d}s\right]. \tag{3.26}$$

Proposition 3.3.2 (Optimal boundary). *Under Assumptions 3.2.1–3.2.3, the continuation and stopping regions, $\mathcal{C}^{[n]}$ and $\mathcal{S}^{[n]}$, are non-empty. The boundary of $\mathcal{C}^{[n]}$ can be expressed as a function $c_n : [0,T]\times\mathbb{R}\to[0,1]$, such that*

$$\mathcal{C}^{[n]} = \{(t,x,y)\in[0,T]\times\Sigma : y > c_n(t,x)\}, \quad \mathcal{S}^{[n]} = \{(t,x,y)\in[0,T]\times\Sigma : y \le c_n(t,x)\}.$$

The map $(t,x)\mapsto c_n(t,x)$ is upper semi-continuous with $t\mapsto c_n(t,x)$ and $x\mapsto c_n(t,x)$ non-decreasing (hence $c_n(\,\cdot\,,x)$ and $c_n(t,\,\cdot\,)$ are right-continuous).

Proof. Thanks to (ii) in Proposition 3.3.1, for any $(t,x)\in[0,T]\times\mathbb{R}$ we can define

$$c_n(t,x) \doteq \inf\{y\in[0,1] : u_n(t,x,y) < c_0\} = \inf\{y\in[0,1] : (t,x,y)\in\mathcal{C}^{[n]}\} \tag{3.27}$$

with the convention that $\inf\varnothing = 1$. Since $x\mapsto u_n(t,x,y)$ and $t\mapsto u_n(t,x,y)$ are non-decreasing we have, for any $\varepsilon>0$

$$(t,x,y)\in\mathcal{S}^{[n]} \implies (t,x+\varepsilon,y)\in\mathcal{S}^{[n]}$$

and

$$(t,x,y)\in\mathcal{S}^{[n]} \implies (t+\varepsilon,x,y)\in\mathcal{S}^{[n]}.$$

Then, c_n is non-decreasing in both t and x.

To show upper semi-continuity we fix (t,x) and take a sequence $(t_k,x_k)_{k\ge1}$ that converges to (t,x). Then $(t_k,x_k,c_n(t_k,x_k))\in\mathcal{S}^{[n]}$ for all k's and, since the stopping region is closed, in the limit we get

$$\limsup_{k\to\infty}(t_k,x_k,c_n(t_k,x_k)) = (t,x,\limsup_{k\to\infty} c_n(t_k,x_k))\in\mathcal{S}^{[n]}.$$

Then, by definition of c_n it must be

$$\limsup_{k\to\infty} c_n(t_k,x_k) \le c_n(t,x).$$

It only remains to show that $\mathcal{C}^{[n]}$ and $\mathcal{S}^{[n]}$ are both non-empty. A standard argument implies that $[0,T)\times\mathcal{H}\subset\mathcal{C}^{[n]}$ with $\mathcal{H}$ the open set in Equation (3.20). Indeed, starting from $(t,x,y)\in[0,T)\times\mathcal{H}$ and taking the suboptimal strategy

$$\tau_{\mathcal{H}} \doteq \inf\{s\in[0,T-t] : (X^{[n];t,x}_{t+s},y)\notin\mathcal{H}\}$$

we easily obtain $u_n(t,x,y)\le U_n(t,x,y;\tau_{\mathcal{H}}) < c_0$ by continuity of paths of $X^{[n]}$ and since $\mathsf{P}_{t,x,y}(\tau_{\mathcal{H}}>0)=1$. So $\mathcal{C}^{[n]}\neq\varnothing$ because $\mathcal{H}\neq\varnothing$ thanks to Equation (3.10) in Assumption 3.2.2. We conclude with an argument by contradiction. Assume that $\mathcal{S}^{[n]}=\varnothing$. Then, given any $(t,x,y)\in[0,T)\times\Sigma$ we have

$$u_n(t,x,y) = c_0 + \mathsf{E}\left[\int_0^{T-t} e^{-rs}\left(\partial_y f(X^{[n];t,x}_{t+s},y) - rc_0\right)\mathrm{d}s\right],$$

thanks to Equation (3.26). Taking limits as $x \to \infty$ and using monotone convergence to pass it under the expectation and the integral (Assumption 3.2.2-(i)) we get

$$\lim_{x\to\infty} u_n(t,x,y) - c_0 = \mathsf{E}\left[\int_0^{T-t} e^{-rs}\left(\lim_{x\to\infty} \partial_y f(X^{[n];t,x}_{t+s}, y) - rc_0\right) \mathrm{d}s\right] > 0$$

thanks to Equation (3.10). This contradicts $u_n(t,x,y) \le c_0$, hence $\mathcal{S} \neq \varnothing$. □

3.3.3 Solution of the n-th singular control problem

Here we follow a well-trodden path to show that the boundary c_n obtained in the section above is actually all we need to construct the optimal control in the singular control problem $\mathbf{SC}^{[n]}$.

First we provide the candidate optimal control in the next lemma.

Lemma 3.3.2. *Fix $(t,x,y) \in [0,T] \times \Sigma$ and let $\xi^{[n]*}$ be defined $\mathsf{P}_{t,x,y}$-almost surely as*

$$\xi^{[n]*}_{t+s} \doteq \sup_{0\le u\le s}\left(c_n(t+u, X^{[n]}_{t+u}) - y\right)^+ \quad \textit{with} \quad \xi^{[n]*}_{t-} = 0.$$

Then, $\xi^{[n]*} \in \Xi_{t,x}(y)$ and realises $\mathsf{P}_{t,x,y}$-almost surely the Skorokhod reflection of the process $(X^{[n]}, Y^{[n]*})$ inside the continuation region $\mathcal{C}^{[n]}$, where $Y^{[n]*} = y + \xi^{[n]*}$. That is, $\mathsf{P}_{t,x,y}$-almost surely we have

(i) $(X^{[n]}_{t+s}, Y^{[n]*}_{t+s}) \in \overline{\mathcal{C}}^{[n]}$ for all $s \in [0, T-t]$ (recall that $\overline{\mathcal{C}}^{[n]}$ is the closure of $\mathcal{C}^{[n]}$);

(ii) Minimality condition:

$$\int_{[t,T]} \mathbf{1}_{\{Y^{[n]*}_{s-} > c_n(s, X^{[n]}_s)\}} \mathrm{d}\xi^{[n]*}_s = \sum_{t<s\le T} \int_{Y^{[n]*}_{s-}}^{Y^{[n]*}_s} \mathbf{1}_{\{Y^{[n]*}_{s-} + z > c_n(s, X^{[n]}_s)\}} \mathrm{d}z = 0. \tag{3.28}$$

Proof. Clearly $\xi^{[n]*}$ is non-decreasing, adapted and bounded by $1-y$. So if we prove that it is also right-continuous we have shown that it belongs to $\Xi_{t,x}(y)$. The proof of right-continuity uses ideas as in De Angelis et al. (2017). For any $\varepsilon > 0$ we have

$$\xi^{[n]*}_{t+s} \le \xi^{[n]*}_{t+s+\varepsilon} = \xi^{[n]*}_{t+s} \vee \sup_{0<u\le\varepsilon}\left(c_n(t+s+u, X^{[n]}_{t+s+u}) - y\right)^+.$$

By upper semi-continuity of the boundary and continuity of the trajectories of $X^{[n]}$ we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sup_{0<u\le\varepsilon}\left(c_n(t+s+u, X^{[n]}_{t+s+u}) - y\right)^+ \\ &= \limsup_{u\to 0}\left(c_n(t+s+u, X^{[n]}_{t+s+u}) - y\right)^+ \le \left(c_n(t+s, X^{[n]}_{t+s}) - y\right)^+ \le \xi^{[n]*}_{t+s}\end{aligned}$$

Then, combining the above expressions we get $\xi^{[n]*}_{t+s} = \lim_{\varepsilon\to 0} \xi^{[n]*}_{t+s+\varepsilon}$ as needed. Next we show the Skorokhod reflection property. By construction we have

$$Y^{[n]*}_{t+s} = y + \xi^{[n]*}_{t+s} \ge c_n(t+s, X^{[n]}_{t+s})$$

so that $(X^{[n]}_{t+s}, Y^{[n]*}_{t+s}) \in \overline{\mathcal{C}}^{[n]}$ for all $s \in [0, T-t]$ as claimed in (i). For the minimality condition (ii) fix $\omega \in \Omega$ and let $s \in [t, T]$ be such that $Y^{[n]*}_{s-}(\omega) > c_n(s, X^{[n]}_s(\omega))$. Then by definition of $Y^{[n]*}$ and by upper semi-continuity of c_n we have

$$\sup_{t \le u < s} \Big(c_n(u, X^{[n]}_u(\omega)) - y \Big)^+ > c_n(s, X^{[n]}_s(\omega)) - y, \tag{3.29}$$

which implies $Y^{[n]*}_{s-}(\omega) = Y^{[n]*}_s(\omega)$. The latter and Equation (3.29) imply that there exists $\delta > 0$ such that

$$\Big(c_n(s, X^{[n]}_s(\omega)) - y \Big)^+ \le \sup_{t \le u \le s} \Big(c_n(u, X^{[n]}_u(\omega)) - y \Big)^+ - \delta. \tag{3.30}$$

By upper semi-continuity of $s \mapsto c_n(s, X^{[n]}_s(\omega))$ there must exist $s' > s$ such that

$$\Big(c_n(u, X^{[n]}_u(\omega)) - y \Big)^+ \le \Big(c_n(s, X^{[n]}_s(\omega)) - y \Big)^+ + \frac{\delta}{2}$$

for all $u \in [s, s')$. The latter and Equation (3.30) imply $Y^{[n]*}_{s-}(\omega) = Y^{[n]*}_u(\omega)$ for all $u \in [s, s')$. Hence $\mathrm{d}\xi^{[n]*}(\omega) = 0$ on $[s, s')$ as needed to show that the first term in Equation (3.28) is zero. For the second term, it is enough to notice that by the explicit form of $\xi^{[n]*}$ we easily derive $\{\Delta\xi^{[n]*}_s > 0\} = \{Y^{[n]*}_{s-} < c(s, X^{[n]}_s)\}$ for any $s \in [t, T]$. Therefore

$$\begin{aligned} Y^{[n]*}_{s-} + \Delta\xi^{[n]*}_s =& Y^{[n]*}_{s-} + \xi^{[n]*}_{s-} \vee \Big(c_n(s, X^{[n]}_s) - y \Big)^+ - \xi^{[n]*}_{s-} \\ =& Y^{[n]*}_{s-} + \Big(c_n(s, X^{[n]}_s) - Y^{[n]*}_{s-} \Big)^+ = Y^{[n]*}_{s-} \vee c_n(s, X^{[n]}_s), \end{aligned}$$

as needed (i.e. any jump of the control $\xi^{[n]*}$ brings the controlled process to the boundary of the continuation set). □

Using the lemma we can now establish optimality of $\xi^{[n]*}$ and obtain v_n as the integral of u_n. The proof of the next proposition follows very closely the proof of Theorem 5.1 in De Angelis et al. (2017), except that here we have a finite-fuel problem (see also Baldursson and Karatzas (1996); El Karoui and Karatzas (1991) for earlier similar proofs). So we move it to the appendix for completeness.

Proposition 3.3.3 (Value function of **SC**$^{[n]}$)**.** *Let Assumptions 3.2.1–3.2.3 hold. For any $(t, x, y) \in [0, T] \times \Sigma$ we have*

$$v_n(t, x, y) = \Phi_n(t, x) - \int_y^1 u_n(t, x, z)\mathrm{d}z, \tag{3.31}$$

with

$$\Phi_n(t, x) \doteq \mathsf{E}_{t,x}\left[\int_0^{T-t} e^{-rs} f(X^{[n]}_{t+s}, 1)\mathrm{d}s \right].$$

Moreover, $\xi^{[n]*}$ as in Lemma 3.3.2 is optimal, i.e. $v_n(t, x, y) = J_n(t, x, y; \xi^{[n]*})$.

3.3.4 Limit of the iterative scheme

Now that we have characterised the solution of the n-th singular control problem, we turn to the study of convergence of the iterative scheme.

First we show monotonicity of the scheme in terms of the sequence of value functions $(u_n)_{n\geq 0}$ of the stopping problems.

Proposition 3.3.4 (Monotonicity of the iterative scheme)**.** *Under Assumptions 3.2.1–3.2.3 we have* $u_n \geq u_{n+1}$ *on* $[0,T]\times\Sigma$ *and* $c_n \geq c_{n+1}$ *on* $[0,T]\times\mathbb{R}$*. Moreover, for any* $(t,x,y)\in[0,T]\times\Sigma$ *we also have*

$$X^{[n]}_{t+s} \geq X^{[n+1]}_{t+s} \text{ and } Y^{[n]*}_{t+s} \geq Y^{[n+1]*}_{t+s} \text{ for } s\in[0,T-t],\ \mathsf{P}_{t,x,y}\text{-a.s.} \tag{3.32}$$

Finally, $m^{[n]} \geq m^{[n+1]}$ *on* $[0,T]$*.*

Proof. We argue by induction and assume that for some $n\geq 0$ we have $m^{[n-1]} \geq m^{[n]}$ on $[0,T]$. Then, by monotonicity of the drift coefficient (Assumption 3.2.1-(ii), we have $a(x,m^{[n]}(t)) \leq a(x,m^{[n-1]}(t))$ for all $(t,x)\in[0,T]\times\mathbb{R}$. It follows from comparison results for SDEs (see, e.g., Karatzas and Shreve, 1987, Proposition 5.2.18) and Equation (3.11) that $X^{[n]}_{t+s} \geq X^{[n+1]}_{t+s}$ for all $s\in[0,T-t]$, $\mathsf{P}_{t,x}$-a.s., for all $(t,x)\in[0,T]\times\mathbb{R}$. By monotonicity of the profit function (Assumption 3.2.2-(i)) we have $\partial_y f(X^{[n+1]}_{t+s},y) \leq \partial_y f(X^{[n]}_{t+s},y)$ and therefore Equations (3.15) and (3.16) imply $u_{n+1}\leq u_n$ on $[0,T]\times\Sigma$. The latter and the definition of the optimal boundary in Equation (3.27) give us $c_{n+1}\leq c_n$ on $[0,T]\times\mathbb{R}$. Now, using the definition of the optimal control in Lemma 3.3.2 we have $\mathsf{P}_{t,x,y}$-a.s.

$$\begin{aligned}\xi^{[n+1]*}_{t+s} &= \sup_{0\leq u\leq s}\Big(c_{n+1}(t+u,X^{[n+1]}_{t+u})-y\Big)^+ \leq \sup_{0\leq u\leq s}\Big(c_n(t+u,X^{[n+1]}_{t+u})-y\Big)^+\\ &\leq \sup_{0\leq u\leq s}\Big(c_n(t+u,X^{[n]}_{t+u})-y\Big)^+ = \xi^{[n]*}_{t+s},\end{aligned}$$

where the first inequality is due to $c_n\geq c_{n+1}$ and the second one to $X^{[n]}\geq X^{[n+1]}$, since $x\mapsto c_n(t,x)$ is non-decreasing (Proposition 3.3.2). Monotonicity of the optimal controls implies monotonicity of the optimally controlled processes $Y^{[n]*}_{t+s}\geq Y^{[n+1]*}_{t+s}$ for all $s\in[0,T-t]$ and from the latter we obtain

$$m^{[n+1]}(t) = \int_\Sigma \mathsf{E}_{x,y}\left[Y^{[n+1]*}_t\right]\nu(\mathrm{d}x,\mathrm{d}y) \geq \int_\Sigma \mathsf{E}_{x,y}\left[Y^{[n]*}_t\right]\nu(\mathrm{d}x,\mathrm{d}y) = m^{[n]}(t).$$

So the argument is complete once we show that we can find $n\geq 0$ such that $m^{[n-1]}\geq m^{[n]}$ on $[0,T]$. The latter is true in particular for $n=0$ since $m^{[-1]}\equiv 1$ and $m^{[0]}\leq 1$ on $[0,T]$. □

It is clear that by construction $0\leq c_n(t,x)\leq 1$ and $0\leq m^{[n]}(t)\leq 1$ for all $(t,x)\in[0,T]\times\mathbb{R}$ and all $n\geq 0$. Moreover, $a(x,0)\leq a(x,m^{[n]}(t))\leq a(x,1)$ for all $(t,x)\in[0,T]\times\mathbb{R}$ and all $n\geq 0$, so that by the comparison principle $\bar{X}^0_{t+s}\leq X^{[n]}_{t+s}\leq X^{[0]}_{t+s}$, for all $s\in[0,T-t]$, $\mathsf{P}_{t,x,y}$-a.s. for all $n\geq 0$ and with $\bar{X}^0$ the solution of Equation (3.11) associated to $a(x,0)$.
By monotonicity of the sequences $(u_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$ and $(m^{[n]})_{n\geq 0}$ we can define the functions

$$u(t,x,y) \doteq \lim_{n\to\infty} u_n(t,x,y), \quad c(t,x)\doteq\lim_{n\to\infty} c_n(t,x) \tag{3.33}$$
$$\text{and } \widetilde{m}(t)\doteq\lim_{n\to\infty} m^{[n]}(t),$$

for all $(t,x,y) \in [0,T] \times \Sigma$. Pointwise limit preserves the monotonicity of $\widetilde{m}$, c and u with respect to (t,x,y). Moreover, since u_n is continuous and c_n, $m^{[n]}$ are upper semi-continuous for all $n \geq 0$ we have that

$$\text{the functions } u,\, \widetilde{m} \text{ and } c \text{ are upper semi-continuous} \tag{3.34}$$

on their respective domains as decreasing limit of upper semi-continuous functions. Since $\widetilde{m}$ is also non-decreasing, then it must be right-continuous.
Notice that for each $n \geq 0$ the null set in Equation (3.32) depends on n and (t,x,y) so we denote it by $N^n_{t,x,y}$. Then we can define a universal null set $N_{t,x,y} \doteq \cup_{n\geq 0} N^n_{t,x,y}$ and for any $(t,x,y) \in [0,T] \times \Sigma$ and all $\omega \in \Omega \setminus N_{t,x,y}$ we define the processes $\widetilde{X}$ and $\widetilde{\xi}$ as

$$\widetilde{X}_{t+s}(\omega) \doteq \lim_{n\to\infty} X^{[n]}_{t+s}(\omega) \quad \text{and} \quad \widetilde{\xi}_{t+s}(\omega) \doteq \lim_{n\to\infty} \xi^{[n]*}_{t+s}(\omega), \tag{3.35}$$

for all $s \in [0, T-t]$. We can then set $\widetilde{X} \equiv 0$ and $\widetilde{\xi} \equiv 0$ on $N_{t,x,y}$ and recall that the filtration is completed with $\mathsf{P}_{t,x,y}$-null sets, so that the limit processes are adapted. Of course we also have

$$\widetilde{Y}_t \doteq y + \widetilde{\xi}_t = \lim_{n\to\infty} Y^{[n]*}_t$$

and thanks to monotone convergence we can immediately establish

$$\widetilde{m}(t) = \lim_{n\to\infty} \int_\Sigma \mathsf{E}_{x,y}\big[Y^{[n]*}_t\big]\nu(\mathrm{d}x,\mathrm{d}y) = \int_\Sigma \mathsf{E}_{x,y}\big[\,\widetilde{Y}_t\,\big]\nu(\mathrm{d}x,\mathrm{d}y). \tag{3.36}$$

Notice that here we are using that $(x,y) \mapsto \mathsf{E}_{x,y}[\xi^{[n]*}_t]$ is measurable, thanks to the explicit expression of $\xi^{[n]*}$ and measurability of c_n. Therefore $(x,y) \mapsto \mathsf{E}_{x,y}[\,\widetilde{\xi}_t\,]$ is measurable too as pointwise limit of measurable functions.

We now derive the dynamics of $\widetilde{X}$ and show that $\widetilde{\xi} \in \Xi$.

Lemma 3.3.3 (Limit state processes)**.** *Suppose Assumptions 3.2.1–3.2.3 hold. For any $(t,x,y) \in [0,T] \times \Sigma$ the process $\widetilde{X}$ is the unique strong solution of*

$$\widetilde{X}_{t+s} = x + \int_0^s a\big(\widetilde{X}_{t+u}, \widetilde{m}(t+u)\big)\mathrm{d}u + \int_0^s \sigma\big(\widetilde{X}_{t+u}\big)\mathrm{d}W_{t+u}, \quad s \in [0, T-t], \tag{3.37}$$

and the process $\widetilde{\xi}$ belongs to $\Xi_{t,x}(y)$.

Proof. Fix $(t,x,y) \in [0,T] \times \Sigma$. The first observation is that $\widetilde{X}$ and $\widetilde{\xi}$ are $(\mathcal{F}_{t+s})_{s\geq 0}$-adapted processes as pointwise limit of adapted processes on $\Omega \setminus N_{t,x,y}$ and by $\mathsf{P}_{t,x,y}$-completeness of the filtration. Since $\widetilde{\xi}$ is decreasing limit of right-continuous non-decreasing processes (hence upper semi-continuous), then it is also non-decreasing and upper-semi continuous. The latter two properties imply right-continuity of the limit process $\widetilde{\xi}$ as well. Since $\xi^{[n]*}_{t-} = 0$ and $\xi^{[n]*}_T \leq 1-y$ for all $n \geq 0$ we also have $\widetilde{\xi}_{t-} = 0$ and $\widetilde{\xi}_T \leq 1-y$. Hence $\widetilde{\xi} \in \Xi_{t,x}(y)$.
Let us now prove Equation (3.37). Denote by X' the unique strong solution of Equation (3.37) and let us show that $\widetilde{X} = X'$. By standard estimates and using Lipschitz

continuity of the drift $a(\,\cdot\,)$ (Assumption 3.2.1-(i))we have

$$\begin{aligned}
&\mathsf{E}_{t,x}\left[\sup_{0\le s\le T-t}\left|X^{[n]}_{t+s}-X'_{t+s}\right|^2\right]\\
&\le 2\,\mathsf{E}_{t,x}\left[L\cdot T\int_0^{T-t}\left(\left|X^{[n]}_{t+s}-X'_{t+s}\right|^2+\left|m^{[n]}(t+s)-\widetilde{m}(t+s)\right|^2\right)\mathrm{d}s\right]\\
&\quad+2\,\mathsf{E}_{t,x}\left[\sup_{0\le s\le T-t}\left|\int_0^s\left(\sigma(X^{[n]}_{t+s})-\sigma(X'_{t+s})\right)\mathrm{d}W_{t+s}\right|^2\right].
\end{aligned}$$

Since σ enjoys linear growth and $X^{[n]}$ and X' are solutions of SDEs with Lipschitz coefficients, then

$$s\mapsto\int_0^s\left(\sigma(X^{[n]}_{t+s})-\sigma(X'_{t+s})\right)\mathrm{d}W_{t+s}$$

is a martingale on $[0,T-t]$ and we can use Doob's inequality to get

$$\begin{aligned}
&\mathsf{E}_{t,x}\left[\sup_{0\le s\le T-t}\left|\int_0^s\left(\sigma(X^{[n]}_{t+s})-\sigma(X'_{t+s})\right)\mathrm{d}W_{t+s}\right|^2\right]\\
&\le 4\,\mathsf{E}_{t,x}\left[\int_0^{T-t}\left(\sigma(X^{[n]}_{t+s})-\sigma(X'_{t+s})\right)^2\mathrm{d}s\right]\le 4L^2\mathsf{E}_{t,x}\left[\int_0^{T-t}\left|X^{[n]}_{t+s}-X'_{t+s}\right|^2\mathrm{d}s\right],
\end{aligned}$$

Combining the estimates above and using Gronwall's inequality we obtain

$$\mathsf{E}_{t,x}\left[\sup_{0\le s\le T-t}\left|X^{[n]}_{t+s}-X'_{t+s}\right|^2\right]\le c\int_0^{T-t}\left|m^{[n]}(t+s)-\widetilde{m}(t+s)\right|^2\mathrm{d}s,$$

for some constant $c>0$. Letting $n\to\infty$ and using bounded convergence and the definition of $\widetilde{m}$ we conclude. □

Next we connect $u(\,\cdot\,)$ and $c(\,\cdot\,)$ with an optimal stopping problem for $\widetilde{X}$. Recall that u and c are upper semi-continuous by Equation (3.34) and enjoy the same monotonicity properties of u_n and c_n.

Lemma 3.3.4 (Limit optimal stopping problem)**.** *Suppose Assumptions 3.2.1–3.2.3 hold. Then, for all $(t,x,y)\in[0,T]\times\Sigma$ we have*

$$\begin{aligned}
u(t,x,y)&=\inf_{\tau\in\mathcal{T}_t}U(t,x,y;\tau)\qquad\textit{with}\\
U(t,x,y;\tau)&\doteq\mathsf{E}_{t,x}\left[\int_0^\tau e^{-rs}\partial_y f(\widetilde{X}_{t+s},y)\mathrm{d}s+c_0e^{-r\tau}\right]
\end{aligned}\tag{3.38}$$

and

$$c(t,x)=\inf\{y\in[0,1]:u(t,x,y)<c_0\}\quad\textit{with}\inf\varnothing=1.$$

In particular c is the boundary of the set

$$\mathcal{C}\doteq\{(t,x,y)\in[0,T]\times\Sigma:u(t,x,y)<c_0\}\tag{3.39}$$

and, moreover, both $\mathcal{C}$ and $\mathcal{S}\doteq([0,T]\times\Sigma)\setminus\mathcal{C}$ are not empty.

Proof. Since $X^{[n]}\ge\widetilde{X}$ for all $n\ge 0$ and $x\mapsto\partial_y f(x,y)$ is non-decreasing, for any $\tau\in\mathcal{T}_t$ we have $U_n(t,x,y;\tau)\ge U(t,x,y;\tau)$ and therefore

$$u(t,x,y)=\lim_{n\to\infty}\inf_{\tau\in\mathcal{T}_t}U_n(t,x,y;\tau)\ge\inf_{\tau\in\mathcal{T}_t}U(t,x,y;\tau).$$

Now, given $\varepsilon > 0$ we can find a stopping time $\tau_\varepsilon \in \mathcal{T}_t$ such that

$$\inf_{\tau\in\mathcal{T}_t} U(t,x,y;\tau) + \varepsilon \geq U(t,x,y;\tau_\varepsilon).$$

Moreover, by dominated convergence (Assumption 3.2.3) and continuity of $\partial_y f$ we have

$$U(t,x,y;\tau_\varepsilon) = \mathsf{E}_{t,x}\left[\int_0^{\tau_\varepsilon} e^{-rs} \lim_{n\to\infty} \partial_y f(X^{[n]}_{t+s}, y) ds + c_0 e^{-r\tau_\varepsilon}\right] = \lim_{n\to\infty} U_n(t,x,y;\tau_\varepsilon).$$

So combining the above we get

$$\inf_{\tau\in\mathcal{T}_t} U(t,x,y;\tau) + \varepsilon \geq \lim_{n\to\infty} U_n(t,x,y;\tau_\varepsilon) \geq \lim_{n\to\infty} u_n(t,x,y) = u(t,x,y)$$

and since $\varepsilon > 0$ was arbitrary we conclude

$$u(t,x,y) \leq \inf_{\tau\in\mathcal{T}_t} U(t,x,y;\tau)$$

as needed for the first claim.
Let us next prove that c coincides with the optimal stopping boundary for the limit problem. Since $u \leq u_n$ for all $n \geq 0$ we have

$$c_n(t,x) = \inf\{y \in [0,1] : u_n(t,x,y) < c_0\} \geq \inf\{y \in [0,1] : u(t,x,y) < c_0\}$$

so that

$$c(t,x) \geq \inf\{y \in [0,1] : u(t,x,y) < c_0\}.$$

For the reverse inequality, let us fix $(t,x) \in [0,T] \times \mathbb{R}$, take $\eta \in [0,1]$ such that

$$\eta > \inf\{y \in [0,1] : u(t,x,y) < c_0\}. \tag{3.40}$$

Then there must be $\delta > 0$ such that $u(t,x,\eta) \leq c_0 - \delta$. By pointwise convergence, there exists $n_\delta \geq 0$ such that $u_n(t,x,\eta) \leq u(t,x,\eta) + \delta/2$ for all $n \geq n_\delta$ and therefore, $u_n(t,x,\eta) \leq c_0 - \delta/2$ for all $n \geq n_\delta$. Hence, $\eta > c_n(t,x)$ for all $n \geq n_\delta$ and $\eta > c(t,x)$ too. The result holds for any $\eta \in [0,1]$ such that Equation (3.40) is true and therefore

$$c(t,x) \leq \inf\{y \in [0,1] : u(t,x,y) < c_0\}.$$

Since $y \mapsto u(t,x,y)$ is decreasing it is clear that c is the boundary of the set $\mathcal{C}$ defined in Equation (3.39).
The exact same arguments as in the proof of Proposition 3.3.2 apply to the stopping problem with value u and allow us to show that $\mathcal{C} \neq \varnothing$ and $\mathcal{S} \neq \varnothing$ thanks to Equation (3.10) in Assumption 3.2.2. □

Thanks to the probabilistic representation of u we can use the same arguments as in the proof of Proposition 3.3.1 to show that u indeed fulfils the same properties as u_n.

Corollary 3.3.1. *Under Assumptions 3.2.1-3.2.3 the function u satisfies (i)–(iv) in Proposition 3.3.1.*

In what follows we let

$$\tau_*(t,x,y) = \inf\{s \in [0,T-t] : u(t+s, \widetilde{X}^{t,x}_{t+s}, y) = c_0\}, \tag{3.41}$$

which is optimal for the limit problem with value $u(t,x,y)$. Continuity of the value function allows a simple proof of convergence of optimal stopping times. The result is of independent interest and might be used for numerical approximation of the optimal stopping rule τ_*. We state the result here but put its proof in the appendix as it is not be needed in the rest of the chapter.

Lemma 3.3.5. *For all $(t,x,y) \in [0,T] \times \Sigma$ we have $\tau_*^{[n]} \uparrow \tau_*$, $\mathsf{P}_{t,x,y}$-a.s., as $n \to \infty$.*

Since the dynamics of $(\widetilde{X}_t)_{t\in[0,T]}$ is fully specified and we have obtained a solution of the optimal stopping problem with value u (Lemma 3.3.4), we can state a result similar to Proposition 3.3.3.

Proposition 3.3.5. *Let Assumptions 3.2.1–3.2.3 hold and let $\widetilde{X}$ be specified as in Lemma 3.3.3. Define*

$$\begin{aligned} \hat{v}(t,x,y) &\doteq \sup_{\xi\in\Xi_{t,x}(y)} \hat{J}(t,x,y;\xi) \qquad \textit{with} \\ \hat{J}(t,x,y;\xi) &\doteq \mathsf{E}_{t,x}\left[\int_0^{T-t} e^{-rs} f(\widetilde{X}_{t+s}, y+\xi_s)\mathrm{d}s - \int_{[0,T-t]} e^{-rs} c_0 \mathrm{d}\xi_s\right]. \end{aligned} \tag{3.42}$$

Then, for any $(t,x,y) \in [0,T] \times \Sigma$ we have

$$\hat{v}(t,x,y) = \Phi(t,x) - \int_y^1 u(t,x,z)\mathrm{d}z \quad \textit{with} \quad \Phi(t,x) \doteq \mathsf{E}_{t,x}\left[\int_0^{T-t} e^{-rs} f(\widetilde{X}_{t+s},1)\mathrm{d}s\right]. \tag{3.43}$$

Moreover,

$$\xi^*_{t+s} \doteq \sup_{0\le u\le s} \Big(c(t+u,\widetilde{X}_{t+u}) - y\Big)^+ \quad \textit{with} \quad \xi^*_{t-} = 0, \tag{3.44}$$

is the unique optimal control in Equation (3.42), up to indistinguishability.

Remark 3.3.1. Before passing to the proof we would like to emphasise that at this stage we are not claiming that $(\widetilde{m},\xi^*)$ is a solution of the MFG because $\widetilde{m}$ is specified in Equation (3.36) and a priori the consistency condition may not hold. Hence, a priori $\hat{v}$ is not the function v defined in Equation (3.8). Of course uniqueness of the optimal control also holds in Proposition 3.3.3 albeit not stated there.

Proof. We only need to prove uniqueness of the optimal control as all remaining claims are obtained by repeating verbatim the proof of Proposition 3.3.3. As usual, uniqueness follows by strict concavity of the map $y \mapsto f(x,y)$, convexity of the set $\Xi_{t,x}(y)$ of admissible controls and an argument by contradiction.
For notational simplicity and with no loss of generality we take $t=0$. Assume that $\eta \in \Xi_{0,x}(y)$ is another optimal control. Since, η and ξ^* are both right-continuous they are indistinguishable as soon as they are modifications, i.e. $\mathsf{P}_{x,y}(\xi^*_s = \eta_s) = 1$ for all $s \in [0,T]$. Arguing by contradiction assume there exists $s_0 \in [0,T)$ such that $3p \doteq \mathsf{P}_{x,y}(\xi^*_{s_0} \neq \eta_{s_0}) > 0$. Then, there also exists $\varepsilon > 0$ such that $\mathsf{P}_{x,y}(|\xi^*_{s_0} - \eta_{s_0}| \ge \varepsilon) \ge 2p$ and, by right-continuity of the paths, there exists $s_1 > s_0$ such that

$$\mathsf{P}_{x,y}\left(\inf_{s_0\le u\le s_1} |\xi^*_u - \eta_u| \ge \varepsilon\right) \ge p > 0.$$

Let us denote

$$A_0 \doteq \left\{\omega : \inf_{s_0 \le u \le s_1} |\xi^*_u(\omega) - \eta_u(\omega)| \ge \varepsilon\right\}. \tag{3.45}$$

For any $\lambda \in (0,1)$, since η and ξ^* are both optimal, we have

$$\begin{aligned}\hat{v}(0,x,y) =& \lambda \hat{J}(0,x,y;\eta) + (1-\lambda)\hat{J}(0,x,y;\xi^*) \\ =& \mathsf{E}_x\left[\int_0^T e^{-rs}\left[\lambda f(\widetilde{X}_s, y+\eta_s) + (1-\lambda) f(\widetilde{X}_s, y+\xi^*_s)\right] \mathrm{d}s\right] \\ &- \mathsf{E}_x\left[\int_{[0,T]} e^{-rs} c_0 (\lambda \mathrm{d}\eta_s + (1-\lambda)\mathrm{d}\xi^*_s)\right].\end{aligned}$$

Now, letting $\zeta^\lambda \doteq \lambda\eta + (1-\lambda)\xi^*$ it is immediate to check that $\zeta^\lambda \in \Xi_{0,x}(y)$ and, by strict concavity of $y \mapsto f(x,y)$ (and joint continuity of f), we have

$$\mathbf{1}_{A_0}\left[\lambda f(\widetilde{X}_s, y+\eta_s) + (1-\lambda) f(\widetilde{X}_s, y+\xi^*_s)\right] < \mathbf{1}_{A_0} f(\widetilde{X}_s, y+\zeta^\lambda_s), \quad \text{for } s \in [s_0, s_1]$$

with $\mathbf{1}_{A_0}$ the indicator of the event in Equation (3.45). Since $\mathsf{P}_{0,x}(A_0) > 0$ and $s_0 < s_1$, the strict inequality holds for the expected values as well. Hence we reach the contradiction

$$\hat{v}(0,x,y) < \hat{J}(0,x,y;\zeta^\lambda),$$

which concludes the proof. □

3.3.5 Solution of the MFG of finite-fuel capacity expansion with singular controls

In this section we first show that $\widetilde{\xi}$ obtained in the previous section (see Equation (3.35)) is optimal for the control problem in Proposition 3.3.5 and then conclude that $(\widetilde{m}, \widetilde{\xi})$ solves the MFG.

Proposition 3.3.6. *Let Assumptions 3.2.1–3.2.3 hold, take $\widetilde{\xi}$ as in Equation (3.35), $\widetilde{m}$ as in Equation (3.36) and $\widetilde{X}$ as in Lemma 3.3.3. Then*

$$\hat{v}(t,x,y) = \hat{J}(t,x,y;\widetilde{\xi}), \quad \textit{for any } (t,x,y) \in [0,T]\times\Sigma$$

and $\widetilde{\xi}$ is indistinguishable from ξ^* as in Equation (3.44).

Proof. We only need to prove optimality of $\widetilde{\xi}$ as the rest follows by uniqueness of the optimal control (Proposition 3.3.5).
Recall the value function v_n of $\mathbf{SC}^{[n]}$ (see Equations (3.13)–(3.14)) and its expression from Proposition 3.3.3. Using dominated convergence we obtain

$$\begin{aligned}&\lim_{n\to\infty}\left(\Phi_n(t,x) - \int_y^1 u_n(t,x,z)\mathrm{d}z\right) \\ &= \mathsf{E}_{t,x}\left[\int_0^{T-t} e^{-rs}\lim_{n\to\infty} f(X^{[n]}_{t+s}, 1)\mathrm{d}s\right] - \int_y^1 \lim_{n\to\infty} u_n(t,x,z)\mathrm{d}z = \hat{v}(t,x,y),\end{aligned}$$

where the final equality is due to Equation (3.33), Equation (3.35) and Proposition 3.3.5. Therefore we have

$$\lim_{n\to\infty} v_n(t,x,y) = \hat{v}(t,x,y).$$

Since $v_n(t,x,y) = J_n(t,x,y;\xi^{[n]*})$, if we can show that

$$\lim_{n\to\infty} J_n(t,x,y;\xi^{[n]*}) = \hat{J}(t,x,y;\widetilde{\xi}),$$

the proof is complete. The latter is not difficult, indeed by integration by parts and dominated convergence we have

$$\begin{aligned}
&\lim_{n\to\infty} J_n(t,x,y;\xi^{[n]*}) \\
&= \lim_{n\to\infty} \mathsf{E}_{t,x}\left[\int_0^{T-t} e^{-rs} f(X^{[n]}_{t+s}, y+\xi^{[n]*}_{t+s})\mathrm{d}s - c_0 e^{-r(T-t)}\xi^{[n]*}_T - rc_0\int_0^{T-t} e^{-rs}\xi^{[n]*}_{t+s}\mathrm{d}s\right] \\
&= \mathsf{E}_{t,x}\bigg[\int_0^{T-t} e^{-rs}\lim_{n\to\infty} f(X^{[n]}_{t+s}, y+\xi^{[n]*}_{t+s})\mathrm{d}s \\
&\qquad\qquad - c_0 e^{-r(T-t)}\lim_{n\to\infty}\xi^{[n]*}_T - rc_0\int_0^{T-t} e^{-rs}\lim_{n\to\infty}\xi^{[n]*}_{t+s}\mathrm{d}s\bigg] \\
&= \mathsf{E}_{t,x}\left[\int_0^{T-t} e^{-rs} f(\widetilde{X}_{t+s}, y+\widetilde{\xi}_{t+s})\mathrm{d}s - c_0 e^{-r(T-t)}\widetilde{\xi}_T - rc_0\int_0^{T-t} e^{-rs}\widetilde{\xi}_{t+s}\mathrm{d}s\right] \\
&= \hat{J}(t,x,y;\widetilde{\xi}),
\end{aligned}$$

where the penultimate equality comes from Equation (3.35) and the final one is obtained by undoing the integration by parts. □

By construction $\widetilde{Y}$ and $\widetilde{m}$ fulfill the consistency condition in Equation (3.36) hence we have a simple corollary.

Corollary 3.3.2. *The pair $(\widetilde{m},\widetilde{\xi})$ is a solution of the MFG as in Definition 3.2.1. Since $\widetilde{\xi}$ is indistinguishable from ξ^* in Equation (3.44) then Theorem 3.2.1 holds with*

$$X^* = \widetilde{X}, \quad Y^* = Y_{0-} + \widetilde{\xi} = Y_{0-} + \xi^* \quad \text{and} \quad m^* = \widetilde{m}.$$

As a byproduct of this result and of Proposition 3.3.5 we also have that the classical connection between singular stochastic control and optimal stopping still holds in our specific MFG.

3.4 Approximate Nash equilibria for the N-player game of finite-fuel capacity expansion with singular controls

3.4.1 The N-player game: setting and assumptions

Here we start with a formal description of the N-player game sketched in the introduction.

Let $\Pi \doteq (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space satisfying the usual conditions, supporting an infinite sequence of independent one-dimensional $\mathbb{F}$-Brownian motions $(W^i)_{i=1}^{\infty}$, as well as i.i.d. $\mathcal{F}_0$-measurable initial states $(X_0^i, Y_0^i)_{i=1}^{\infty}$ with common distribution $\nu \in \mathcal{P}(\Sigma)$, independent of the Brownian motions. For each $N \geq 1$, define $\mathbb{F}^N = (\mathcal{F}_0 \vee \mathcal{F}_t^N)_{t\geq 0}$, where $(\mathcal{F}_t^N)_{t\geq 0}$ is the augmented filtration generated by the Brownian motions $(W^i)_{i=1}^N$. Then the filtered probability spaces $\Pi^N \doteq (\Omega, \mathcal{F}, \mathbb{F}^N, \mathsf{P})$ satisfy the usual conditions. These are the spaces on which we define strong solutions of the N-player systems.

Each player $i = 1, \dots, N$ observes/controls their own private state process $(X^{N,i}, Y^{N,i})$, whose dynamics is

$$
\begin{aligned}
X_t^{N,i} &= X_0^i + \int_0^t a(X_s^{N,i}, m_s^N)\, ds + \int_0^t \sigma(X_s^{N,i})\, \mathrm{d}W_s^i, \\
Y_t^{N,i} &= Y_{0-}^i + \xi_t^{N,i}, \qquad t \in [0,T],
\end{aligned} \tag{3.46}
$$

where $\xi^{N,i} \in \Xi^{\Pi^N}(Y_{0-}^i)$ is the strategy chosen by the i-th player, while m^N is the mean-field interaction term given by

$$
m_t^N = \frac{1}{N}\sum_{i=1}^N Y_t^{N,i} = \int_\Sigma y\, \mu_t^N(\mathrm{d}x, \mathrm{d}y), \quad \mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^{N,i}, Y_t^{N,i})}. \tag{3.47}
$$

The process μ_t^N above denotes the empirical distribution of the players' states, with δ_z the Dirac delta mass at $z \in \Sigma$.

In the rest of this section we use the notation

$$
\Xi^N(Y_{0-}) = \{(\xi^i)_{i=1}^N : \xi^i \in \Xi^{\Pi^N}(Y_{0-}^i) \text{ for all } i = 1, \dots N\}.
$$

We also consider the dynamics of (X^N, Y^N) conditionally on specific initial conditions $(\mathbf{x}, \mathbf{y}) \doteq (x^i, y^i)_{i=1}^N$ drawn independently from the common initial distribution ν. Therefore, the dynamics of the state variables under $\mathsf{P}_{\mathbf{x},\mathbf{y}}$ reads

$$
\begin{aligned}
X_t^{N,i} &= x^i + \int_0^t a(X_s^{N,i}, m_s^N)\, \mathrm{d}s + \int_0^t \sigma(X_s^{N,i})\, \mathrm{d}W_s^i, \\
Y_t^{N,i} &= y^i + \xi_t^{N,i}, \qquad t \in [0,T].
\end{aligned}
$$

Accordingly, since the initial conditions $(\mathbf{x}, \mathbf{y}) = (x^i, y^i)_{i=1}^N$ are drawn from the N-fold product of the measure ν, denoted ν^N, the expected payoff of the i-th player is given by

$$
J^{N,i}(\xi^N) \doteq \int_{\Sigma^N} J^{N,i}(\mathbf{x}, \mathbf{y}; \xi^N)\nu^N(\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{y}),
$$

where

$$
J^{N,i}(\mathbf{x}, \mathbf{y}; \xi^N) \doteq \mathsf{E}_{\mathbf{x},\mathbf{y}}\left[\int_0^T e^{-rt} f\left(X_t^{N,i}, Y_t^{N,i}\right) \mathrm{d}t - \int_{[0,T]} e^{-rt} c_0 \mathrm{d}\xi_t^{N,i}\right].
$$

Definition 3.4.1 (ε-Nash equilibrium for the N-player game)**.** Given $\varepsilon \geq 0$, an admissible strategy vector $\xi^\varepsilon \in \Xi^N(Y_{0-})$ is called *ε-Nash equilibrium* for the N-player game of capacity expansion if for every $i = 1, \dots, N$ and for every admissible individual strategy $\xi^i \in \Xi^{\Pi^N}(Y_{0-}^i)$, we have

$$
J^{N,i}(\xi^\varepsilon) \geq J^{N,i}([\xi^{\varepsilon,-i}, \xi^i]) - \varepsilon,
$$

where $[\xi^{\varepsilon,-i}, \xi^i]$ denotes the N-player strategy vector that is obtained from ξ^ε by replacing the i-th entry with ξ^i.

In order to construct ε-Nash equilibria using the optimal control obtained in the MFG it is convenient to make an additional set of assumptions on the profit function.

Assumption 3.4.1. The running payoff f is locally Lipschitz, i.e.

$$|f(x,y) - f(x',y')| \leq \Lambda(x,x')(|x-x'| + |y-y'|), \quad (x,y),(x',y') \in \Sigma.$$

Moreover, there exists $q > 1$ such that the function $\Lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ satisfies the integrability condition

$$C(\Lambda, q) \doteq \sup_{N \in \mathbb{N}} \sup_{\xi^{N,1}} \mathsf{E}\left[\int_0^T \Lambda^q(X_t^{N,1}, X_t^*)\mathrm{d}t\right] < \infty \tag{3.48}$$

where $X^{N,1}$ is the solution of Equation (3.46), $X^* = \widetilde{X}$ is the solution of Equation (3.4) obtained in the MFG (see also Equation (3.37)) and the supremum is taken over all admissible controls $\xi^{N,1} \in \Xi^{\Pi^N}(Y^1_{0-})$ and all $N \in \mathbb{N}$.

The integrability condition in Equation (3.48) is redundant if f is Lipschitz continuous. Since $Y^{N,1} \in [0,1]$ there is no loss of generality in taking Λ independent of y and the supremum over $\xi^{N,1}$ is not restrictive either. If Λ has polynomial growth of order $p \geq 1$, then Equation (3.48) holds thanks to the Lipschitz continuity of the coefficients $a(x,m)$ and $\sigma(x)$ as soon as $\mathsf{E}[(X_0)^{p \cdot q}] < \infty$. Later in Section 4.1 we consider and example where Λ has exponential growth and Equation (3.48) holds.

The next is an assumption on the optimal boundary found in Theorem 3.2.1.

Assumption 3.4.2. The optimal boundary $(t,x) \mapsto c(t,x)$ of Theorem 3.2.1 is uniformly Lipschitz continuous in x with constant $\theta_c > 0$, i.e.

$$\sup_{0 \leq t \leq T} |c(t,x) - c(t,x')| \leq \theta_c |x - x'|, \quad x, x' \in \mathbb{R}.$$

Lipschitz regularity of optimal stopping/control boundaries is a delicate issue in general. However, the question can be addressed by analytical methods (see, e.g., Soner and Shreve (1991)) or by probabilistic methods as in De Angelis and Stabile (2019a). In Section 3.4.3 we show how ideas from the latter paper can be used in our context to prove that Assumption 3.4.2 indeed holds in a large class of examples.

3.4.2 Approximate Nash equilibria

Here we prove that the MFG solution constructed in Theorem 3.2.1 induces approximate Nash equilibria in the N-player game of capacity expansion, when N is large enough.

Theorem 3.4.1 (Approximate Nash equilibiria for the N-player game)**.** *Suppose Assumptions 3.2.1–3.4.2 hold. Recall the feedback solution (m^*, ξ^*) of the MFG of capacity expansion constructed in Theorem 3.2.1. Recall also that*

$$\xi_t^* = \eta^*(t, X^*, Y_{0-}), \quad t \in [0,T],$$

with $\eta^* : [0,T] \times C([0,T];\mathbb{R}) \times [0,1] \to [0,1]$ the non-anticipative mapping defined by

$$\eta^*(t, \varphi, y) \doteq \sup_{0 \leq s \leq t} \Big(c(s, \varphi(s)) - y\Big)^+$$

and with X^* the dynamics in Equation (3.4) associated to m^*. Setting $\hat{\xi}^{N,i}_t \doteq \eta^*(t, X^{N,i}, Y^i_{0-})$, the vector $\hat{\xi}^N$ is a ε_N-Nash equilibrium for the N-player game of capacity expansion according to Definition 3.4.1 with $\varepsilon_N \to 0$ as $N \to \infty$. Further, if $q = 2$ in Equation (3.48) of Assumption 3.4.1, then the rate of convergence is of order $N^{-1/2}$.

Proof. For each Brownian motion W^i we introduce the following auxiliary dynamics, which are the analogues of the solution (X^*, Y^*) of Equation (3.4) with (ξ^*, m^*) as in Theorem 3.2.1:

$$\begin{aligned} X^i_t &= X^i_0 + \int_0^t a(X^i_s, m^*(s))ds + \int_0^t \sigma(X^i_s)\mathrm{d}W^i_s, \\ Y^i_t &= Y^i_{0-} + \zeta^i_t \doteq Y_{0-} + \eta^*(t, X^i, Y^i_{0-}), \quad t \in [0,T], \quad i \in \{1, \dots, N\}. \end{aligned} \tag{3.49}$$

Notice that the initial conditions above are the same as in the dynamics of $(X^{N,i}, Y^{N,i})$. Moreover, $(X^i_t, Y^i_t)_{i=1}^\infty$ is a sequence of i.i.d. random variables with values in $\mathbb{R} \times [0,1]$, so that in particular the law of large numbers (LLN) holds. The rest of the proof is structured in three steps:

(i) We prove that $J^{N,1}(\hat{\xi}^N) \to J(\xi^*)$ as $N \to \infty$.

(ii) Recalling the notation $[\hat{\xi}^{N,-1}, \xi] = (\xi, \hat{\xi}^{N,2}, \dots, \hat{\xi}^{N,N})$ introduced in Definition 3.4.1 we prove

$$\limsup_{N\to\infty} \sup_{\xi \in \Xi^{\Pi^N}(Y^1_{0-})} J^{N,1}([\hat{\xi}^{N,-1}, \xi]) \le J(\xi^*) = V^\nu.$$

(iii) Combining (i) and (ii), for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$J^{N,1}(\hat{\xi}^N) \ge \sup_{\xi \in \Xi^{\Pi^N}(Y^1_{0-})} J^{N,1}([\hat{\xi}^{N,-1}, \xi]) - \varepsilon$$

for all $N \ge N_\varepsilon$.

In the three steps above we singled out the first player with no loss of generality since the N-player game is symmetric.

(i) Let us start by observing that (X^*, Y^*, ξ^*) from Theorem 3.2.1 and (X^1, Y^1, ζ^1) defined above have the same law, so that

$$J(\xi^*) = \mathsf{E}\left[\int_0^T e^{-rs} f(X^1_t, Y^1_t)\mathrm{d}t - c_0 \int_{[0,T]} e^{-rt}\mathrm{d}\zeta^1_t\right].$$

By triangular inequality we get

$$\begin{aligned} |J^{N,1}(\hat{\xi}^N) - J(\xi^*)| \le \mathsf{E}\left[\int_0^T e^{-rt}\left|f(\hat{X}^{N,1}_t, \hat{Y}^{N,1}_t) - f(X^1_t, Y^1_t)\right|\mathrm{d}t\right] \\ + c_0\mathsf{E}\left[\left|\int_{[0,T]} e^{-rt}\mathrm{d}(\hat{\xi}^{N,1}_t - \zeta^1_t)\right|\right], \end{aligned} \tag{3.50}$$

where we use $(\hat{X}^{N,i}, \hat{Y}^{N,i})$ for the state process of the i-th player when all players use the control vector $\hat{\xi}^N$. Similarly we denote by $\hat{m}^N$ the empirical average of the processes $\hat{Y}^{N,i}$.

We estimate the first term on the right-hand side using Assumption 3.4.1 and obtain

$$\begin{aligned}
&\mathsf{E}\left[\int_0^T e^{-rt}\left|f(\hat{X}_t^{N,1},\hat{Y}_t^{N,1}) - f(X_t^1,Y_t^1)\right|\mathrm{d}t\right] \\
&\le \mathsf{E}\left[\int_0^T e^{-rt}\Lambda(\hat{X}_t^{N,1},X_t^1)\big(|\hat{X}_t^{N,1}-X_t^1| + |\hat{Y}_t^{N,1}-Y_t^1|\big)\mathrm{d}t\right] \\
&\le C_1\mathsf{E}\left[\int_0^T \Lambda^q(\hat{X}_t^{N,1},X_t^1)\mathrm{d}t\right]^{\frac{1}{q}}\mathsf{E}\left[\int_0^T \big(|\hat{X}_t^{N,1}-X_t^1|^p + |\hat{Y}_t^{N,1}-Y_t^1|^p\big)\mathrm{d}t\right]^{\frac{1}{p}} \\
&\le C_1 C(\Lambda,q)\mathsf{E}\left[\int_0^T \big(|\hat{X}_t^{N,1}-X_t^1|^p + |\hat{Y}_t^{N,1}-Y_t^1|^p\big)\mathrm{d}t\right]^{\frac{1}{p}},
\end{aligned}$$

for some positive constant $C_1 = C_1(T,q)$, using Hölder's inequality with $p = q/(q-1)$ and $q > 1$ as in Assumption 3.4.1. For the remaining term in Equation (3.50) we use integration by parts and $\hat{\xi}_{0-}^{N,1} = \zeta_{0-}^1 = 0$ to obtain

$$\mathsf{E}\left[\left|\int_{[0,T]} e^{-rt}\mathrm{d}(\hat{\xi}_t^{N,1}-\zeta_t^1)\right|\right] = \mathsf{E}\left[\left|\int_0^T e^{-rt}(\hat{\xi}_t^{N,1}-\zeta_t^1)\mathrm{d}t\right|\right] \le \mathsf{E}\left[\int_0^T e^{-rt}\left|\hat{\xi}_t^{N,1}-\zeta_t^1\right|\mathrm{d}t\right].$$

Recall that $\hat{\xi}_t^{N,1} = \eta^*(t,\hat{X}^{N,1},Y_{0-}^1)$ and $\zeta_t^1 = \eta^*(t,X^1,Y_{0-}^1)$. Then using Assumption 3.4.2 we obtain

$$\left|\hat{\xi}_t^{N,1}-\zeta_t^1\right| \le \sup_{0\le s\le t}\left|c(s,\hat{X}_s^{N,1}) - c(s,X_s^1)\right| \le \theta_c \sup_{0\le s\le t}\left|\hat{X}_s^{N,1}-X_s^1\right| \tag{3.51}$$

and the same bound also holds for $|\hat{Y}_t^{N,1} - Y_t^1|$. Then, combining the above estimates we arrive at

$$\begin{aligned}
|J^{N,1}(\hat{\xi}^N) - J(\xi^*)| \le{}& C_1 C(\Lambda,q)T(1+\theta_c)\mathsf{E}\Big[\sup_{0\le t\le T}|\hat{X}_t^{N,1}-X_t^1|^p\Big]^{\frac{1}{p}} \\
&+ c_0\theta_c T\,\mathsf{E}\left[\sup_{0\le s\le t}|\hat{X}_s^{N,1}-X_s^1|\right].
\end{aligned} \tag{3.52}$$

Since $p > 1$ it remains to show that

$$\lim_{N\to\infty}\mathsf{E}\Big[\sup_{0\le t\le T}|\hat{X}_t^{N,1}-X_t^1|^p\Big] = 0. \tag{3.53}$$

Repeating the same estimates as those in the proof of Lemma 3.3.3 but with $(\hat{X}^{N,1},X^1)$ instead of $(X^{[n]},X')$ and with $(\hat{m}^N,m^*)$ instead of $(m^{[n]},\widetilde{m})$ we obtain

$$\mathsf{E}\Big[\sup_{0\le t\le T}|\hat{X}_t^{N,1}-X_t^1|^p\Big] \le C\,\mathsf{E}\left[\int_0^T\left|\hat{m}_t^N - m^*(t)\right|^p\mathrm{d}t\right], \tag{3.54}$$

for some constant $C > 0$ depending on p, T and the Lipschitz constant of the coefficients $a(\,\cdot\,)$ and $\sigma(\,\cdot\,)$.
Now, observe that

$$\varepsilon_{p,N} \doteq \int_0^T \mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^N Y_t^i - m^*(t)\right|^p\right]\mathrm{d}t \to 0, \quad N\to\infty, \tag{3.55}$$

by the LLN and the bounded convergence theorem, since $(Y_t^i)_{i=1}^N$ are i.i.d. with mean

$m^*(t)$ (recall that η^* is the feedback map of the optimal control in the MFG). Hence, we have

$$\begin{aligned}\mathsf{E}\left[\int_0^T |\hat{m}_t^N - m^*(t)|^p \mathrm{d}t\right] \leq & 2^{p-1} \int_0^T \mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^N (\hat{Y}_t^{N,i} - Y_t^i)\right|^p\right] \mathrm{d}t + 2^{p-1}\varepsilon_{p,N} \\ \leq & 2^{p-1} \int_0^T \frac{1}{N}\sum_{i=1}^N \mathsf{E}\left[|\hat{Y}_t^{N,i} - Y_t^i|^p\right] \mathrm{d}t + 2^{p-1}\varepsilon_{p,N} \\ = & 2^{p-1} \int_0^T \mathsf{E}\left[\left|\hat{Y}_t^{N,1} - Y_t^1\right|^p\right] \mathrm{d}t + 2^{p-1}\varepsilon_{p,N} \\ \leq & 2^{p-1}\theta_c^p \int_0^T \mathsf{E}\left[\sup_{0\leq s\leq t}\left|\hat{X}_s^{N,1} - X_s^1\right|^p\right] \mathrm{d}t + 2^{p-1}\varepsilon_{p,N}\end{aligned} \tag{3.56}$$

where the first inequality uses $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, the second inequality follows by Jensen's inequality ($p > 1$), the equality by the fact that the processes $(\hat{Y}^{N,i} - Y^i)_{i=1}^N$ are exchangeable and the final inequality uses Equation (3.51) applied to $|\hat{Y}_t^{N,1} - Y_t^1|$.
Plugging the latter estimate back into Equation (3.54) and applying Gronwall's lemma once again we obtain

$$\mathsf{E}\Big[\sup_{0\leq t\leq T} |\hat{X}_t^{N,1} - X_t^1|^p\Big] \leq C'\varepsilon_{p,N},$$

for a suitable constant $C' > 0$ depending on T and the other constants above. Thanks to Equation (3.55) we obtain Equation (3.53).

(ii). This part of the proof is similar to the above but now the first player deviates by choosing a generic admissible control ξ while all remaining players pick $\hat{\xi}^{N,i}$, $i = 2, \dots, N$; we denote this strategy vector $\beta^N = [\hat{\xi}^{N,-1}, \xi]$. In particular we notice that the empirical average associated to this strategy reads

$$\frac{1}{N}\left(Y_{0-}^1 + \xi_t + \sum_{i=2}^N (Y_{0-}^i + \hat{\xi}_t^{N,i})\right) = \bar{m}_t^N + \frac{1}{N}(\xi_t - \hat{\xi}_t^{N,1}),$$

where $\bar{m}_t^N \doteq N^{-1}\sum_{i=1}^N (Y_{0-}^i + \hat{\xi}_t^{N,i})$. One should be careful here that $\bar{m}^N$ is different to $\hat{m}^N$ used in the proof of (i) above, because the deviation of player 1 from the strategy vector $\hat{\xi}^N$ causes a knock-on effect on the dynamics of $\hat{\xi}^{N,i}$ for all i's through the non-anticipative mapping $\eta^*(t, X^{N,i;\beta}, Y_{0-}^i)$. To keep track of this subtle aspect we use the notations $\hat{\xi}_t^{N,i;\beta} = \eta^*(t, X^{N,i;\beta}, Y_{0-}^i)$ and $\bar{Y}_t^{N,i;\beta} = Y_{0-}^i + \hat{\xi}_t^{N,i;\beta}$, for $i = 1, \dots, N$, in the calculations below. Accordingly, the state process of the first player reads

$$\begin{aligned} X_t^{N,1;\beta} &= X_0^1 + \int_0^t a\big(X_s^{N,1;\beta}, \bar{m}_s^N + N^{-1}(\xi_s - \hat{\xi}_s^{N,1;\beta})\big)\mathrm{d}s + \int_0^t \sigma(X_s^{N,1;\beta})\mathrm{d}W_s^1 \\ Y_t^{N,1;\beta} &= Y_{0-}^1 + \xi_t, \qquad t \in [0,T]. \end{aligned}$$

Using the above expression for $X^{N,1;\beta}$ and the same arguments as in Equation (3.54) we obtain

$$\begin{aligned}
&\mathsf{E}\Big[\sup_{0\le t\le T}|X_t^{N,1;\beta}-X_t^1|^p\Big]\\
&\le C\,\mathsf{E}\left[\int_0^T\left|\bar m_t^N+N^{-1}(\xi_t-\hat\xi_t^{N,1;\beta})-m^*(t)\right|^p\mathrm{d}t\right]\\
&\le 2^{p-1}C\,\mathsf{E}\left[\int_0^T\left|\bar m_t^N-m^*(t)\right|^p\mathrm{d}t\right]+2^p2^{p-1}CTN^{-p},
\end{aligned}$$

where the final inequality uses $|a+b|^p\le 2^{p-1}(|a|^p+|b|^p)$ and $|\xi_t-\hat\xi_t^{N,1;\beta}|\le 2$ (by the finite-fuel condition), and $C>0$ is a suitable constant. Repeating the same steps as in Equation (3.56) we have

$$\begin{aligned}
\mathsf{E}\left[\int_0^T|\bar m_t^N-m^*(t)|^p\mathrm{d}t\right]\le&2^{p-1}\int_0^T\mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^N(\bar Y_t^{N,i;\beta}-Y_t^i)\right|^p\right]\mathrm{d}t+2^{p-1}\varepsilon_{p,N}\\
\le&2^{p-1}\int_0^T\mathsf{E}\left[\left|\bar Y_t^{N,1;\beta}-Y_t^1\right|^p\right]\mathrm{d}t+2^{p-1}\varepsilon_{p,N}\\
\le&2^{p-1}\theta_c^p\int_0^T\mathsf{E}\left[\sup_{0\le s\le t}\left|X_s^{N,1;\beta}-X_s^1\right|^p\right]\mathrm{d}t+2^{p-1}\varepsilon_{p,N}
\end{aligned}$$

where we have used that $(\bar Y^{N,i;\beta}-Y^i)_{i=1}^N$ are exchangeable by construction. Combining the two estimates above and using Gronwall's inequality we obtain a bound which is uniform with respect to $\xi\in\Xi^{\Pi^N}(Y_{0-}^1)$. In particular we have

$$\lim_{N\to\infty}\sup_{\xi\in\Xi^{\Pi^N}(Y_{0-}^1)}\mathsf{E}\Big[\sup_{0\le t\le T}|X_t^{N,1;\beta}-X_t^1|^p\Big]\le C'\lim_{N\to\infty}\varepsilon_{p,N}=0,\tag{3.57}$$

where $C'>0$ is the constant appearing from Gronwall's inequality. Since any $\xi\in\Xi^{\Pi^N}(Y_{0-}^1)$ is admissible but suboptimal in the MFG with state process X^1 as in Equation (3.49) we get

$$\begin{aligned}
&\sup_{\xi\in\Xi^{\Pi^N}(Y_{0-}^1)}J^{N,1}([\hat\xi^{N,-1},\xi])-V^\nu\\
&\le\sup_{\xi\in\Xi^{\Pi^N}(Y_{0-}^1)}\Big(J^{N,1}([\hat\xi^{N,-1},\xi])-J(\xi)\Big)\\
&\le\sup_{\xi\in\Xi^{\Pi^N}(Y_{0-}^1)}\mathsf{E}\left[\int_0^Te^{-rs}\Big(f(X_t^{N,1;\beta},Y_{0-}^1+\xi_t)-f(X_t^1,Y_{0-}^1+\xi_t)\Big)\mathrm{d}t\right]\\
&\le\sup_{\xi\in\Xi^{\Pi^N}(Y_{0-}^1)}\mathsf{E}\left[\int_0^Te^{-rt}\Lambda(X_t^{N,1;\beta},X_t^1)|X_t^{N,1;\beta}-X_t^1|\mathrm{d}t\right]
\end{aligned}$$

where in the final inequality we used Assumption 3.4.1. Now, arguing as in (i) and using Equations (3.57) and (3.48) we obtain

$$\limsup_{N\to\infty}\sup_{\xi\in\Xi^{\Pi^N}(Y_{0-}^1)}J^{N,1}([\hat\xi^{N,-1},\xi])\le V^\nu=J(\xi^*),$$

where the final equality holds by optimality of ξ^* in the MFG.

(iii). This step follows from the previous two. With no loss of generality we consider only the first player as the game is symmetric. Given $\varepsilon > 0$, thanks to (ii) there exists $N_\varepsilon > 0$ sufficiently large that for any $\xi \in \Xi^{\Pi^N}(Y^1_{0-})$

$$J^{N,1}([\hat{\xi}^{N,-1}, \xi]) \leq V^\nu + \frac{\varepsilon}{2} \quad \text{for all } N > N_\varepsilon.$$

From (i), with no loss of generality we can also assume $N_\varepsilon > 0$ sufficiently such large that

$$J^{N,1}(\hat{\xi}^N) \geq V^\nu - \frac{\varepsilon}{2} \quad \text{for all } N > N_\varepsilon.$$

Combining the two inequalities above we obtain that for all $\xi \in \Xi^{\Pi^N}(Y^1_{0-})$ holds

$$J^{N,1}(\hat{\xi}^N) \geq J^{N,1}([\hat{\xi}^{N,-1}, \xi]) - \varepsilon \quad \text{for all } N > N_\varepsilon.$$

The final claim on the speed of convergence can be verified by taking $q = p = 2$ in the above estimates. The leading term in the convergence of Equation (3.52) is $\sqrt{\varepsilon_{2,N}}$ (see Equation (3.57)). Since Equation (3.55) reads

$$\varepsilon_{2,N} = \int_0^T \operatorname{Var}\left(\frac{1}{N}\sum_{i=1}^N Y^i_t\right) \mathrm{d}t = \frac{1}{N}\int_0^T \operatorname{Var}(Y^1_t)\,\mathrm{d}t$$

upon noticing that $\mathsf{E}\left[N^{-1}\sum_{i=1}^N Y^i_t\right] = \mathsf{E}[Y^1_t] = m^*(t)$ since $(Y^i)_{i=1}^N$ are i.i.d., the claim follows. □

3.4.3 Conditions for a Lipschitz continuous optimal boundary

Here we complement results from De Angelis and Stabile (2019a) to provide sufficient conditions under which Assumption 3.4.2 holds. Notice that our problem is parabolic and degenerate as there is no diffusive dynamics in the y-direction. Therefore classical PDE results cannot be applied. Moreover, we extend De Angelis and Stabile (2019a) by considering non-constant diffusion coefficients in the the dynamics of X^*. Thanks to Lemma 3.3.4, the question reduces to finding sufficient conditions on the data of the optimal stopping problem in Equation (3.38) that guarantee a Lipschitz stopping boundary. In Equation (3.38) the dynamics of $\widetilde{X}$ was obtained from Lemma 3.3.3 and corresponds to the dynamics of X^* in the MFG. In the rest of this section we always use such X^*.

We make some additional assumptions on the coefficients of the SDE.

Assumption 3.4.3. We have $x \mapsto a(x,m)$ and $x \mapsto \sigma(x)$ continuously differentiable with $\partial_x\sigma(x) \geq 0$ and $\partial_x a(x,m) \leq \bar{a}$ for some $\bar{a} > 0$.

Thanks to this assumption we have that the stochastic flow $x \mapsto X^{*;t,x}(\omega)$ is continuously differentiable. The dynamics of $Z^{t,x} \doteq \partial_x X^{*;t,x}$ is given by (see Protter, 2005, Chapter V.7)

$$Z^{t,x}_{t+s} = 1 + \int_0^s \partial_x a(X^{*;t,x}_{t+u}, m^*(t+u))Z^{t,x}_{t+u}\mathrm{d}u + \int_0^s \partial_x\sigma(X^{*;t,x}_{t+u})Z^{t,x}_{t+u}\,\mathrm{d}W_{t+u}, \quad (3.58)$$

for all $(t,x) \in [0,T] \times \mathbb{R}$ and $s \in [0, T-t]$. The solution of Equation (3.58) is explicit in terms of X^* and it reads

$$Z^{t,x}_{t+s} = \exp\Big[\int_0^s \Big(\partial_x a(X^{*;t,x}_{t+u}, m^*(t+u)) - \frac{1}{2}\partial_x \sigma(X^{*;t,x}_{t+u})^2\Big)\,\mathrm{d}u + \int_0^s \partial_x \sigma(X^{t,x}_{t+u})\,\mathrm{d}W_{t+u}\Big], \tag{3.59}$$

for $(t,x) \in [0,T] \times \mathbb{R}$ and $s \in [0,T-t]$. Thanks to this explicit formula we can deduce that $(t,x) \mapsto Z^{t,x}$ is a continuous flow, by continuity of the flow $(t,x) \mapsto X^{*;t,x}$.

Later on we will perform a change of measure using Z and for that we also require the following assumption.

Assumption 3.4.4. For all $(t,x) \in [0,T] \times \mathbb{R}$ we have

$$\mathsf{E}_{t,x}\left[\int_0^{T-t} \Big(\partial_x \sigma(X^*_{t+u}) Z_{t+u}\Big)^2 \mathrm{d}u\right] < +\infty. \tag{3.60}$$

Then

$$\frac{\mathrm{d}\mathsf{Q}}{\mathrm{d}\mathsf{P}}\Big|_{\mathcal{F}_T} \doteq Z_T \exp\Big(-\int_0^T \partial_x a(X^*_t, m^*(t))\mathrm{d}t\Big) \tag{3.61}$$

defines the Radon-Nikodym derivative of the absolutely continuous change of measure from P to Q.

Next we assume some extra conditions on the profit function.

Assumption 3.4.5. We have $f \in C^2(\mathbb{R} \times (0,1))$ and either $\sigma(x) = \sigma$ is constant or we have $x \mapsto \partial_{xy} f(x,y)$ non-increasing. Moreover, the integrability condition below holds:

$$\sup_{(t,x,y)\in K} \mathsf{E}_{t,x,y}\left[\int_0^{T-t} e^{-rs}\Big(\big|\partial_{yy} f(X^*_{t+s}, y)\big| + (1 + Z_{t+s})\,\big|\partial_{xy} f(X^*_{t+s}, y)\big|\Big)\mathrm{d}s\right] < \infty,$$

for any compact $K \subset [0,T] \times \Sigma$.

Notice that $f(x,y) = x^\alpha y^\beta$ with $\alpha \in (0,1]$ and $\beta \in (0,1)$ satisfies Assumption 3.4.5 combined with Assumption 3.2.1. The next proposition provides sufficient conditions for Lipschitz continuity of the optimal boundary.

Proposition 3.4.1. *Let Assumptions 3.2.1–3.2.3 and Assumptions 3.4.3–3.4.5 hold. If either of the two conditions below holds:*

(i) there exist $\alpha, \gamma > 0$ such that

$$|\partial_{yy} f| \geq \alpha > 0 \quad \textit{and} \quad |\partial_{xy} f| \leq \gamma(1 + |\partial_{yy} f|) \quad \textit{on } \Sigma;$$

(ii) there exists $\gamma > 0$ such that $|\partial_{xy} f| \leq \gamma |\partial_{yy} f|$ on Σ;

then Assumption 3.4.2 holds.

The proof of the proposition uses the next lemma concerning the optimal stopping time defined in Equation (3.41), whose slightly technical proof we move to the appendix.

Lemma 3.4.1. *The mapping* $(t,x,y) \mapsto \tau_*(t,x,y)$ *is* P*-almost surely continuous on* $[0,T] \times \Sigma$ *with* $\tau_*(t,x,y) = 0$, P*-a.s. for* $(t,x,y) \in \partial\mathcal{C}$.

Proof of Proposition 3.4.1. The idea of the proof combines ideas from De Angelis and Stabile (2019a) and De Angelis and Peskir (2020). First, for $\delta > 0$ we define

$$c_\delta(t,x) \doteq \inf\{y \in [0,1] : u(t,x,y) < c_0 - \delta\}$$

with $\inf \varnothing = 1$. Then it is clear that $c_\delta(\,\cdot\,) > c_{\delta'}(\,\cdot\,) > c(\,\cdot\,)$ for all $0 < \delta' < \delta$ by monotonicity of $y \mapsto u(t,x,y)$. Since u is continuous then

$$\lim_{\delta \downarrow 0} c_\delta(t,x) = c(t,x) \quad (t,x) \in [0,T] \times \mathbb{R}$$

and if we can prove that $x \mapsto c_\delta(t,x)$ is Lipschitz with a constant independent of δ we can conclude. By continuity of u we know that

$$u\big(t,x,c_\delta(t,x)\big) = c_0 - \delta$$

so that by the implicit function theorem, whose use is justified in step 2 below, we have

$$\partial_x c_\delta(t,x) = -\frac{\partial_x u\big(t,x,c_\delta(t,x)\big)}{\partial_y u\big(t,x,c_\delta(t,x)\big)}, \quad (t,x) \in [0,T] \times \mathbb{R}. \tag{3.62}$$

Thanks to Corollary 3.3.1 we have $\partial_x c_\delta(t,x) \geq 0$. In step 2 below wefind an upper bound so that $|\partial_x c_\delta| \leq \theta_c$ on $[0,T] \times \mathbb{R}$, for a suitable constant $\theta_c > 0$. This concludes the proof.

Step 1: (Gradient estimates). We fix an arbitrary $(t,x,y) \in [0,T] \times \Sigma$ and let $\tau_* = \tau_*(t,x,y)$. Then for any $\varepsilon > 0$ we have

$$\begin{aligned}
&u(t,x,y+\varepsilon) - u(t,x,y) \\
&\leq \mathsf{E}\left[\int_0^{\tau_*} e^{-rs}\Big(\partial_y f(X^{*;t,x}_{t+s}, y+\varepsilon) - \partial_y f(X^{*;t,x}_{t+s}, y)\Big)\mathrm{d}s\right] \\
&= \int_0^{\varepsilon} \mathsf{E}\left[\int_0^{\tau_*} e^{-rs}\partial_{yy} f(X^{*;t,x}_{t+s}, y+z)\mathrm{d}s\right]\mathrm{d}z,
\end{aligned}$$

where we used Fubini's theorem in the final equality. Dividing by ε, letting $\varepsilon \to 0$ and using the integrability condition from Assumption 3.4.5 we conclude

$$\limsup_{\varepsilon \to 0} \frac{u(t,x,y+\varepsilon) - u(t,x,y)}{\varepsilon} \leq \mathsf{E}\left[\int_0^{\tau_*} e^{-rs}\partial_{yy} f(X^{*;t,x}_{t+s}, y)\mathrm{d}s\right].$$

Taking $\tau_*^\varepsilon \doteq \tau_*(t,x,y+\varepsilon)$ in the first expression above we have

$$\begin{aligned}
&u(t,x,y+\varepsilon) - u(t,x,y) \\
&\geq \mathsf{E}\left[\int_0^{\tau_*^\varepsilon} e^{-rs}\Big(\partial_y f(X^{*;t,x}_{t+s}, y+\varepsilon) - \partial_y f(X^{*;t,x}_{t+s}, y)\Big)\mathrm{d}s\right] \\
&= \int_0^{\varepsilon} \mathsf{E}\left[\int_0^{\tau_*^\varepsilon} e^{-rs}\partial_{yy} f(X^{*;t,x}_{t+s}, y+z)\mathrm{d}s\right]\mathrm{d}z.
\end{aligned}$$

Dividing again by $\varepsilon > 0$ and letting $\varepsilon \to 0$, we can now invoke Lemma 3.4.1 to justify that $\tau_*^\varepsilon \to \tau_*$ and obtain

$$\liminf_{\varepsilon \to 0} \frac{u(t,x,y+\varepsilon) - u(t,x,y)}{\varepsilon} \geq \mathsf{E}\left[\int_0^{\tau_*} e^{-rs} \partial_{yy} f(X_{t+s}^{*;t,x}, y) \mathrm{d}s\right].$$

So, in conclusion we have shown that $\partial_y u$ exists in $[0,T] \times \Sigma$ and it reads

$$\partial_y u(t,x,y) = \mathsf{E}\left[\int_0^{\tau_*} e^{-rs} \partial_{yy} f(X_{t+s}^{*;t,x}, y) \mathrm{d}s\right].$$

Further, in light of the fact that $(t,x,y) \mapsto \tau_*(t,x,y)$ and $(t,x,y) \mapsto (X_{t+s}^{*;t,x}, y)$ are P-a.s. continuous, we deduce that $\partial_y u$ is also continuous on $[0,T] \times \Sigma$, by dominated convergence and Assumption 3.4.5. Finally, since $\partial_{yy} f < 0$ (Assumption 3.2.2-(ii)), we have that

$$\partial_y u\big(t,x,c_\delta(t,x)\big) < 0, \quad \text{for all } (t,x) \in [0,T] \times \Sigma, \tag{3.63}$$

because $\big(t,x,c_\delta(t,x)\big) \in \mathcal{C}$ and $\tau_* > 0$ at those points.
Next we obtain a similar result for $\partial_x u$. With the same notation as above we have

$$\begin{aligned}
&u(t,x+\varepsilon,y) - u(t,x,y) \\
&\leq \mathsf{E}\left[\int_0^{\tau_*} e^{-rs}\Big(\partial_y f(X_{t+s}^{*;t,x+\varepsilon}, y) - \partial_y f(X_{t+s}^{*;t,x}, y)\Big)\mathrm{d}s\right] \\
&= \int_x^{x+\varepsilon} \mathsf{E}\left[\int_0^{\tau_*} e^{-rs} \partial_{xy} f(X_{t+s}^{*;t,\eta}, y) Z_{t+s}^{t,\eta} \mathrm{d}s\right] \mathrm{d}\eta.
\end{aligned}$$

Dividing by ε and letting $\varepsilon \to 0$, we use dominated convergence (Assumption 3.4.5) and continuity of the flows $x \mapsto \big(X^{*;t,x}, Z^{t,x}\big)$ to conclude

$$\limsup_{\varepsilon \to 0} \frac{u(t,x+\varepsilon,y) - u(t,x,y)}{\varepsilon} \leq \mathsf{E}\left[\int_0^{\tau_*} e^{-rs} \partial_{xy} f(X_{t+s}^{*;t,x}, y) Z_{t+s}^{t,x} \mathrm{d}s\right].$$

By a symmetric argument and continuity of the optimal stopping time we also obtain the reverse inequality and therefore conclude

$$\partial_x u(t,x,y) = \mathsf{E}\left[\int_0^{\tau_*} e^{-rs} \partial_{xy} f(X_{t+s}^{*;t,x}, y) Z_{t+s}^{t,x} \mathrm{d}s\right].$$

Also in this case continuity of $(t,x,y) \mapsto \tau_*(t,x,y)$, due to Lemma 3.4.1, and $(t,x) \mapsto \big(X^{*;t,x}, Z^{t,x}\big)$, combined with dominated convergence, imply that $\partial_x u$ is continuous on $[0,T] \times \Sigma$.

Step 2: (Bound on $\partial_x c_\delta$). Since $\partial_y u$ and $\partial_x u$ are continuous and Equation (3.63) holds, the equation in Equation (3.62) is fully justified as an application of the implicit function theorem. In this step we use the probabilistic representations of $\partial_x u$ and $\partial_y u$ to obtain an upper bound on $\partial_x c_\delta$. First we recall the change of measure induced by Z (see Equation (3.61)) and we use it to write

$$\partial_x u(t,x,y) = \mathsf{E}_{t,x}^{\mathsf{Q}}\left[\int_0^{\tau_*} e^{-rs+\int_0^s a(X_{t+u}^*, m^*(t+u))\mathrm{d}u} \partial_{xy} f(X_{t+s}^*, y) \mathrm{d}s\right].$$

We want to find an upper bound for $\partial_x u$ in terms of the process under the original measure P. Under the measure Q we have, by Girsanov theorem, that X^* evolves according to

$$\mathrm{d}X^*_{t+s} = \big[a(X^*_{t+s}, m^*(t+s)) + \sigma(X^*_{t+s})\partial_x\sigma(X^*_{t+s})\big]\mathrm{d}s + \sigma(X^*_{t+s})\mathrm{d}W^{\mathsf{Q}}_{t+s},$$

where $W^{\mathsf{Q}}_{t+s} = W_{t+s} - \int_0^s \partial_x\sigma(X^*_{t+u})\mathrm{d}u$ defines a Brownian motion under Q. Analogously, under the original measure P we can define a process $\bar{X}$ with the same dynamics, i.e.,

$$\mathrm{d}\bar{X}_{t+s} = \big[a(\bar{X}_{t+s}, m^*(t+s)) + \sigma(\bar{X}_{t+s})\partial_x\sigma(\bar{X}_{t+s})\big]\mathrm{d}s + \sigma(\bar{X}_{t+s})\mathrm{d}W_{t+s},$$

and denote

$$\bar{\tau}_* \doteq \inf\{s \in [0, T-t] : c(t+s, \bar{X}_{t+s}) \geq y\}.$$

Then we have that the processes and stopping times are equal in law, i.e.

$$\mathrm{Law}^{\mathsf{Q}}(X^*, \tau_*) = \mathrm{Law}^{\mathsf{P}}(\bar{X}, \bar{\tau}_*)$$

and we can express $\partial_x u$ in terms of the original measure as

$$\partial_x u(t,x,y) = \mathsf{E}_{t,x}\left[\int_0^{\bar{\tau}_*} e^{-rs+\int_0^s a(\bar{X}_{t+u}, m^*(t+u))\mathrm{d}u}\partial_{xy} f(\bar{X}_{t+s}, y)\mathrm{d}s\right]. \tag{3.64}$$

Let us first consider $x \mapsto \sigma(x)$ not constant. By comparison principles we have $\bar{X} \geq X^*$ since $\partial_x\sigma \geq 0$ (Assumption 3.4.3), therefore $\partial_{xy} f(\bar{X}, y) \leq \partial_{xy} f(X^*, y)$ by Assumption 3.4.5. Since $x \mapsto c(t,x)$ is non-decreasing as pointwise limit of non-decreasing functions (recall Proposition 3.3.2), then $c(t+s, \bar{X}_{t+s}) \geq c(t+s, X^*_{t+s})$, hence implying $\bar{\tau}_* \leq \tau_*$, P-a.s. Recalling that $\partial_{xy} f > 0$ from Assumption 3.2.2 and combining these few facts we have

$$\partial_x u(t,x,y) \leq e^{\bar{a}(T-t)}\mathsf{E}_{t,x}\left[\int_0^{\tau_*} e^{-rs}\partial_{xy} f(X^*_{t+s}, y)\mathrm{d}s\right],$$

where we also used $\partial_x a \leq \bar{a}$ (Assumption 3.4.3). If instead $\sigma(x) = \sigma$ is constant then $X^* = \bar{X}$ by uniqueness of the SDE and therefore the above estimate follows directly from Equation (3.64). Plugging this bound in Equation (3.62) and recalling that $\partial_{yy} f < 0$ we obtain

$$0 \leq \partial_x c_\delta(t,x) \leq e^{\bar{a}(T-t)} \frac{\mathsf{E}_{t,x}\left[\int_0^{\tau_*} e^{-rs}\partial_{xy} f\big(X^*_{t+s}, c_\delta(t,x)\big)\mathrm{d}s\right]}{\mathsf{E}_{t,x}\left[\int_0^{\tau_*} e^{-rs}\big|\partial_{yy} f\big(X^*_{t+s}, c_\delta(t,x)\big)\big|\mathrm{d}s\right]}.$$

Now, if condition (i) holds we obtain

$$0 \leq \partial_x c_\delta(t,x) \leq e^{\bar{a}(T-t)}\left(\frac{\gamma}{\alpha} + \gamma\right), \quad \text{for all } (t,x) \in [0,T]\times\mathbb{R} \text{ and any } \delta > 0,$$

whereas if condition (ii) holds we obtain

$$0 \leq \partial_x c_\delta(t,x) \leq e^{\bar{a}(T-t)}\gamma, \quad \text{for all } (t,x) \in [0,T]\times\mathbb{R} \text{ and any } \delta > 0.$$

So in the first case Assumption 3.4.2 holds with

$$\theta_c = e^{\bar{a}T}\left(\frac{\gamma}{\alpha} + \gamma\right),$$

and in the second case with $\theta_c = \gamma \exp(\bar{a}T)$. □

Next we provide a couple of examples meeting the requirements of Proposition 3.4.1.

Example 3.4.1 (Ornstein-Uhlenbeck dynamics with exponential-Cobb-Douglas profit)**.** Let $a(x,m) \doteq \alpha(m-x)$ for some $\alpha > 0$ and $\sigma(x) \equiv \sigma$ for some $\sigma > 0$. Given a Borel function $m : [0,T] \to [0,1]$ the dynamics of X from Equation (3.4) reads

$$X_t = X_0 + \int_0^t \alpha(m(s) - X_s)\mathrm{d}s + \sigma W_t, \quad t \in [0,T]. \tag{3.65}$$

Let $f(x,y) \doteq e^x y^\beta$ for some $\beta \in (0,1)$ and for all $(x,y) \in \Sigma$. Finally assume that $\mathsf{E}[\exp(qX_0)] < \infty$ for some $q \geq 1$.

We check the assumptions of Proposition 3.4.1. Assumptions 3.2.1 and Assumption 3.4.3 on the dynamics' coefficients are trivially satisfied. The profit function f has the monotonicity required by Assumption 3.2.2-(i) and it is strictly concave (Assumption 3.2.2-(ii)). Also, $\partial_{xy} f(x,y) = \beta e^x y^{\beta-1} > 0$ (Assumption 3.2.2-(iii)) and Equation (3.10) is satisfied since

$$\lim_{x\to-\infty} \frac{\beta e^x}{y^{1-\beta}} = 0 < rc_0 < \lim_{x\to\infty} \frac{\beta e^x}{y^{1-\beta}} = +\infty$$

for any $y \in (0,1)$ fixed. The integrability Assumption 3.2.3 is satisfied by the Ornstein-Uhlenbeck dynamics with initial condition as above. Assumption 3.4.4 is another integrability assumption that reduces to finiteness of the second moment of the exponential martingale Z (which is satisfied by boundedness of $\partial_x a$ and $\partial_x \sigma$). Assumption 3.4.5 holds because σ is constant. Finally Assumption (ii) in Proposition 3.4.1 is satisfied with any $\gamma \geq \frac{1}{1-\beta}$ since

$$|\partial_{xy} f(x,y)| = \frac{\beta e^x}{y^{1-\beta}} \quad \text{and} \quad |\partial_{yy} f(x,y)| = \frac{\beta(1-\beta)e^x}{y^{2-\beta}}.$$

We also notice that Assumption 3.4.1 holds so that Theorem 3.4.1 can be applied.

Example 3.4.2 (GBM dynamics with linear-Cobb-Douglas profit)**.** Let $a(x,m) \doteq \alpha m x$ for some $\alpha > 0$ and $\sigma(x) \doteq \sigma x$ for some $\sigma > 0$. Given a Borel function $m : [0,T] \to [0,1]$, the dynamics of X from Equation (3.4) reads

$$X_t = X_0 + \int_0^t \alpha X_s m(s)\mathrm{d}s + \int_0^t \sigma X_s \mathrm{d}W_s, \quad t \in [0,T]. \tag{3.66}$$

Non-negativity of the trajectories reduces the state space Σ to $[0,\infty) \times [0,1]$ (see Remark 3.2.2). Let $f(x,y) \doteq (1+x)(1+y)^\beta$ for some $\beta \in (0,1)$ and for all $(x,y) \in \Sigma$. Finally assume that $\nu \in \mathcal{P}_2(\Sigma)$ and that $rc_0 > \beta$.
Let us check the assumptions of Proposition 3.4.1. Assumptions 3.2.1 and Assumption 3.4.3 on the dynamics' coefficients are trivially satisfied. The profit function f has the monotonicity required by Assumption 3.2.2-(i) and is strictly concave (Assumption 3.2.2-(ii)). Also, $\partial_{xy} f(x,y) = \beta(1+y)^{\beta-1} > 0$ (Assumption 3.2.2-(iii)) and Equation(3.10) is satisfied since

$$\lim_{x\to 0} \frac{\beta(1+x)}{(1+y)^{1-\beta}} = \frac{\beta}{(1+y)^{1-\beta}} < \beta < rc_0 < \lim_{x\to\infty} \frac{\beta(1+x)}{(1+y)^{1-\beta}} = +\infty$$

for any $y \in (0,1)$ fixed. The integrability Assumption 3.2.3 is satisfied with $p = 2$ (or higher provided the initial condition has finite p-th moment) thanks to sub-linearity of the logarithm and standard estimates on the GBM dynamics. Assumption 3.4.4 is another integrability assumption that reduces to finiteness of the second moment of the exponential martingale Z (which is satisfied by boundedness of $\partial_x a$ and $\partial_x \sigma$). Assumption 3.4.5 holds because $x \mapsto \partial_{xy} f(x,y)$ is decreasing. Finally Assumption (ii) in Proposition 3.4.1 is satisfied with any $\gamma \geq \frac{2}{1-\beta}$ since

$$|\partial_{xy} f(x,y)| = \frac{\beta}{(1+y)^{1-\beta}} \quad \text{and} \quad |\partial_{yy} f(x,y)| = \frac{\beta(1-\beta)(1+x)}{(1+y)^{2-\beta}}.$$

We also notice that Assumption 3.4.1 holds so that Theorem 3.4.1 can be applied.

We would like to emphasise that the conditions of Proposition 3.4.1 are far from being necessary. While it would be overly complicated to state a general theorem in this sense, we provide below an example with a clear economic interpretation for which Proposition 3.4.1 is not directly applicable.

Chapter 4

Numerical Methods for Mean-Field Games of Finite-Fuel Capacity Expansion with Singular Controls

Based on the constructive procedure in the proof of Theorem 3.2.1, Section 3.3, we propose a numerical scheme to approximate the solution of a MFG of finite-fuel capacity expansion with underlying Geometric Brownian motion (GBM hereafter) dynamics that we introduce in Section 4.1. The numerical method is an iterative Picard scheme, based on an integral equation that we derive in Section 4.2.1 for the optimal stopping boundary of an optimal stopping problem of the form of problems $\mathbf{OC}^{[n]}$.
We then apply the iterative procedure at each step n of the proof of Theorem 3.2.1. While convergence of the approximating optimal stopping problems to the MFG solution is granted by Theorem 3.2.1, we here prove convergence of the iterative numerical method. This, at each step n, grants convergence of the numerical approximating procedure to the theoretical limit represented at step n by the optimal stopping boundary b_n (or equivalently its generalized inverse c_n), from which we can construct the optimal control $\xi^{[n]*}$ of problem $\mathbf{OC}^{[n]}$.
Theorem 3.2.1 combined with the numerical Picard-like iterative scheme proposed in the present Chapter, produces a numerical method to approximate the solution of a MFG of finite-fuel capacity expansion with underlying GBM dynamics.

4.1 A model of mean-field game of finite-fuel capacity expansion with singular controls and Geometric Brownian motion

In this section, we describe the MFG with singular controls and GBM underlying dynamics to which we tailor the numerical method.

Let us assume that $a(x,m) = (\mu + m)x$ and $\sigma(x) = \sigma x$ for some $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$. Let us also assume that $f(x,y) = xg(y)$ with $g \in C^2([0,1])$, $g > 0$, strictly concave and strictly increasing. This specification corresponds to the classical model of the goodwill problem in which firms produce a good whose price evolves as a GBM and revenues are linear in the price of the good and increasing and concave in the amount of investment that goes towards advertising.

On the one hand, Assumptions 3.2.1–3.2.3 are easily verified and Theorem 3.2.1 holds (i.e., our construction of the solution to the MFG holds). On the other hand, neither (i) nor (ii) in Proposition 3.4.1 hold, so we cannot apply directly the result on Lipschitz continuity of the boundary which is needed for the approximation result in Theorem 3.4.1. However, we shall now see how an alternative argument of proof can be applied to prove that Assumption 3.4.2 remains valid.

First of all we change our coordinates by letting $\psi \doteq \ln x$, so that the value function of the optimal stopping problem can be written as

$$\widetilde{u}(t,\psi,y) \doteq u(t,e^{\psi},y) = \inf_{\tau\in\mathcal{T}_t} \mathsf{E}_{t,\psi}\left[\int_0^{\tau} e^{-rs} g'(y) e^{\Psi_{t+s}} \mathrm{d}s + e^{-r\tau} c_0\right],$$

where $\Psi_{t+s} \doteq \ln X^*_{t+s}$ is just a Brownian motion with drift, i.e.,

$$\Psi^{t,\psi}_{t+s} = \psi + \int_0^s \left(\mu - \sigma^2/2 + m^*(t+s)\right)\mathrm{d}s + \sigma \mathrm{d}(W_{t+s} - W_t).$$

The optimal boundary can also be expressed in terms of (t,ψ) by putting $\widetilde{c}(t,\psi) = c(t,e^{\psi})$. Then the mean-field optimal control reads

$$\xi^*_t = \sup_{0\le s\le t}\left(\widetilde{c}(s,\Psi_s) - y\right)^+, \quad t\in[0,T]$$

whereas the optimal stopping time for the value $\widetilde{u}(t,\psi,y)$ reads

$$\tau_* = \inf\{s\in[0,T-t] : \widetilde{c}(t+s,\Psi_{t+s}) \ge y\}.$$

Now we show that the optimal boundary $\widetilde{c}(\,\cdot\,)$ is indeed Lipschitz with respect to ψ and therefore the proof of Theorem 3.4.1 can be repeated with Ψ instead of X^* so that the theorem holds as stated. Since $\partial_\psi \Psi^{t,\psi}_{t+s} \equiv 1$ for $s\in[0,T-t]$ and Assumption 3.4.5 holds, we can use the same arguments as in step 1 of the proof of Proposition 3.4.1 to obtain

$$\partial_y \widetilde{u}(t,\psi,y) = g''(y)\mathsf{E}_{t,\psi}\left[\int_0^{\tau_*} e^{-rs+\Psi_{t+s}}\mathrm{d}s\right]$$

and

$$\partial_\psi \widetilde{u}(t,\psi,y) = g'(y)\mathsf{E}_{t,\psi}\left[\int_0^{\tau_*} e^{-rs+\Psi_{t+s}}\mathrm{d}s\right].$$

Then, by the same arguments as in step 2 of the proof of Proposition 3.4.1 we obtain

$$\partial_\psi \widetilde{c}_\delta(t,\psi) = -\frac{\partial_x \widetilde{w}(t,\psi,\widetilde{c}_\delta(t,\psi))}{\partial_y \widetilde{w}(t,\psi,\widetilde{c}_\delta(t,\psi))} = \frac{g'(\widetilde{c}_\delta(t,\psi))}{|g''(\widetilde{c}_\delta(t,\psi))|} \le \kappa,$$

for some $\kappa > 0$, where the final inequality holds because $g \in C^2([0,1])$ and strictly concave. Therefore for the optimal boundary we have

$$\sup_{0\le t\le T} |\widetilde{c}(t,\psi_1) - \widetilde{c}(t,\psi_2)| \le \kappa|\psi_1 - \psi_2|, \quad \psi_1,\psi_2\in\mathbb{R},$$

as needed. In conclusion, the result of Theorem 3.4.1 remains valid, even though the optimal boundary in the original parametrisation of the problem is not uniformly Lipschitz.

4.2 A numerical method for mean-field games of finite-fuel capacity expansion with singular controls and Geometric Brownian motion

4.2.1 The integral equation for the optimal boundary

Let $f(x,y) \doteq e^x g(y)$ with $g \in C^2([0,1])$, positive, strictly concave and strictly increasing. In this section, we derive the integral equation, Equation (4.6), for the optimal stopping boundary b of an optimal stopping problem of the form

$$\begin{aligned} u(t,x,y) &\doteq \inf_{\tau\in\mathcal{T}_t} U(t,x,y;\tau) \qquad \text{with} \\ U(t,x,y;\tau) &\doteq \mathsf{E}_{t,x}\left[\int_0^\tau e^{-rs}\partial_y f(X_{t+s},y)\mathrm{d}s + c_0 e^{-r\tau}\right], \quad \text{for } \tau \in \mathcal{T}_t \end{aligned} \tag{4.1}$$

with underlying dynamics

$$X^{t,x}_{t+s} = x + \int_0^s m(t+u)\,\mathrm{d}u + \sigma(W_{t+s} - W_t),$$

where $m : [0,T] \to [0,1]$ is a Borel measurable function, i.e. we consider the dynamics described in Section 4.1 after the logarithmic change of coordinates. In particular, we have that $X^{t,x}_{t+s} \overset{d}{\sim} \mathcal{N}(x + \int_0^s m(t+u)\,du, \sigma^2 s)$, where $\mathcal{N}(\mu,\sigma^2)$ denotes a Gaussian random variable with mean μ and variance σ^2.
The numerical scheme we propose is based on the following Proposition.

Proposition 4.2.1. *Assume that, for all $y \in [0,1]$, $u(\cdot,y) \in C^{1,1}((0,T)\times\mathbb{R})$, that the partial derivative $\partial_x u(\cdot,y)$ is such that*

$$\mathsf{E}\left[\int_0^{T-t} \left|\partial_x u\left(X^{t,x}_{t+s},y\right)\right|^2 \mathrm{d}s\right] < \infty \tag{4.2}$$

for all $t \in [0,T]$ and that there exists $\partial^2_{xx}u(\cdot,y)$, bounded and continuous on $(0,T)\times\mathbb{R}\setminus\partial\mathcal{C}_y$, where $\mathcal{C}_y$ is the y-section of $\mathcal{C}$. Assume also that $u(\cdot,y) \in C^{1,2}((0,T)\times\mathbb{R}\setminus\partial\mathcal{C}_y)$. Then, the value function of the optimal stopping problem $u(t,x,y)$ has the representation

$$\begin{aligned} u(t,x,y) &= e^{-r(T-t)}c_0 + \mathsf{E}\left[\int_0^{T-t} e^{-rs}\partial_y f(X^{t,x}_{t+s},y)\,\mathrm{d}s\right] \\ &\quad + \mathsf{E}\left[\int_0^{T-t} e^{-rs}\left(rc_0 - \partial_y f(X^{t,x}_{t+s},y)\right)1_{\{X^{t,x}_{t+s}\geq b(t+s,y)\}}\right] \end{aligned} \tag{4.3}$$

for all $(t,x,y) \in \mathcal{C}$. Assume also that $b(T,y) = \log(rc_0) - \log g'(y)$ for all $y \in [0,1]$. Then the optimal stopping boundary b is a solution of the following integral equation

$$\begin{cases} c_0(1-e^{-r(T-t)}) = \int_0^{T-t}\left(\int_0^\infty e^{-rs}\partial_y f(z,y)p(t,b(t,y),t+s,z)\,\mathrm{d}z\right)\mathrm{d}s \\ \qquad\qquad + \int_0^{T-t}\left(\int_{b(t+s,y)}^\infty e^{-rs}\left(rc_0 - \partial_y f(z,y)\right)p(t,b(t,y),t+s,z)\mathrm{d}z\right)\mathrm{d}s \\ b(T,y) = \log(rc_0) - \log g'(y) \end{cases} \tag{4.4}$$

for all $(t,y) \in [0,T]\times[0,1]$, where $p(t,x,t+s,z) \doteq \partial_z \mathsf{P}\left[X^{t,x}_{t+s} \leq z\right]$ is the transition density of the arithmetic Brownian motion dynamics.

Remark 4.2.1. The continuity assumptions of Proposition 4.2.1, i.e. that the value function u is continuously differentiable and that its second derivative exists bounded and continuous outside the stopping boundary, are standard to be verified in the optimal stopping literature. The assumption on the terminal condition for the optimal stopping boundary, i.e. that $b(T,y) = \log(rc_0) - \log g'(y)$, is less trivial, however it can be verified in many situations (see e.g. De Angelis and Milazzo, 2020; De Angelis and Stabile, 2019b). We here hence give these assumptions for granted and leave their proof for a later work.

Proof. Fix $y \in [0,1]$. Let $(u^{(k)})_{k\in\mathbb{N}}$, $u^{(k)} : [0,T]\times\Sigma \to \mathbb{R}$, $u^{(k)}(\cdot,y) \in C^{1,2}((0,T)\times\mathbb{R})$ be an approximating sequence for $u(\cdot,y)$ constructed via mollification, so that

$$\left(u^{(k)}(\cdot,y), \partial_x u^{(k)}(\cdot,y), \partial_t u^{(k)}(\cdot,y)\right) \underset{k\to\infty}{\longrightarrow} (u(\cdot,y), \partial_x u(\cdot,y), \partial_t u(\cdot,y))$$

uniformly on compact sets and

$$\partial^2_{xx} u^{(k)}(t,x,y) \underset{k\to\infty}{\longrightarrow} \partial^2_{xx} u(t,x,y)$$

for all $(t,x) \notin \partial\mathcal{C}_y$. Let $(K^{(\ell)})_{\ell\in\mathbb{N}}$ be an increasing sequence of compact sets such that $K^{(\ell)} \uparrow \mathcal{C}_y$ and

$$\tau^{(\ell)}(t,x) \doteq \inf\{s \geq 0 \,:\, (t+s, X^{t,x}_{t+s}) \in \partial K^{(\ell)}\} \wedge (T-t).$$

Then, for each fixed $y \in [0,1]$ and for each $\ell \in \mathbb{N}$, we can apply Itô's lemma up to any $s \leq \tau^{(\ell)}$ to each $u^{(k)}$ getting

$$\begin{aligned} e^{-rs}u^{(k)}(t+s, X^{t,x}_{t+s}, y) &= u^{(k)}(t,x,y) + \int_0^s e^{-ru}(\mathcal{L}_{t,x} - r)u^{(k)}(t+u, X^{t,x}_{t+u}, y)du \\ &+ \int_0^s \sigma e^{-ru}\partial_x u^{(k)}(t+u, X^{t,x}_{t+u}, y)dW_{t+u} \end{aligned}$$

for all $(t,x,y) \in [0,T]\times\Sigma$ and $s \in [0, \tau^{(\ell)}(t,x)]$, where

$$\mathcal{L}_{t,x} \doteq \partial_t + \frac{\sigma^2(\,\cdot\,)}{2}\partial_{xx} + a(\,\cdot\,, m(\,\cdot\,))\partial_x - r.$$

Taking expectations on both sides, exploiting boundedness of $\partial_x u$ and choosing $s = \tau^{(\ell)}(t,x)$ we obtain

$$u^{(k)}(t,x,y) = c_0 e^{-r\tau^{(\ell)}(t,x)} + \mathsf{E}\left[\int_0^{\tau^{(\ell)}(t,x)} (r - \mathcal{L}_{t,x})u^{(k)}(t+s, X^{t,x}_{t+s}, y)e^{-rs}ds\right].$$

Since the process $X^{t,x}_{t+\cdot\wedge\tau^{(\ell)}}$ is confined to $K^{(\ell)}$, we can apply the uniform convergence on compact sets of the sequence $(u^{(k)}(\cdot,y))_{k\in\mathbb{N}}$ and its first derivatives to $u(\cdot,y)$ and its first derivatives. Thanks to the boundedness assumption on $\partial^2_{xx}u$ and the fact that $\mathsf{P}((t+s, X^{t,x}_{t+s}) \in \partial\mathcal{C}_y) = 0$ we can pass to the limit also in the second-order terms by the dominated convergence theorem. We find the representation in Equation (4.3) by letting $\ell \uparrow \infty$ and observing that $\tau^{(\ell)} \uparrow (T-t)$.
In particular, if we evaluate this equation at $x = b(t,y)$ and if we set $p(t,x,t+s,z) \doteq \partial_z \mathsf{P}\left[X^{t,x}_{t+s} \leq z\right]$ the transition density of the arithmetic Brownian motion defined

above, we obtain the integral equation for the optimal stopping boundary $b(t,y)$:

$$\begin{aligned} c_0(1-e^{-r(T-t)}) &= \int_0^{T-t}\left(\int_0^{\infty} e^{-rs}\partial_y f(z,y)p(t,b(t,y),t+s,z)\,\mathrm{d}z\right)\mathrm{d}s \\ &+ \int_0^{T-t}\left(\int_{b(t+s,y)}^{\infty} e^{-rs}\left(rc_0-\partial_y f(z,y)\right)p(t,b(t,y),t+s,z)\mathrm{d}z\right)\mathrm{d}s. \end{aligned}$$

□

Further manipulation of the integral equation, as described below, leads to an implicit equation for the boundary $b(t,y)$ of the form:

$$b(t,y) = F(t,b(\cdot,y);T,r,\sigma) \tag{4.5}$$

that naturally lends itself to the initialization of a Picard iterative scheme.

Indeed, the inner integrals can be computed explicitly thanks to the properties of the Gaussian distribution. In particular the first integral reads:

$$\int_0^{\infty} e^{-rs}\partial_y f(z,y)p(t,b(t,y),t+s,z)\mathrm{d}z = g^{'}(y)e^{b(t,y)}\exp\left(\int_0^s m(t+u)\mathrm{d}u+\frac{\sigma^2 s}{2}-rs\right).$$

At this point, we split the second integral in two parts (one involving the function f and one not) and we compute them separately. For the first part we have:

$$\begin{aligned} &\int_{b(t+s,y)}^{\infty} e^{-rs}\,r\,c_0\,p(t,b(t,y),t+s,z)\mathrm{d}z = r\,c_0\,e^{-rs}\left[1-\Phi_{(t,b(t,y),t+s)}(b(t+s,y))\right] \\ &= r\,c_0\,e^{-rs}\left[1-\Phi(\beta_0(s))\right] \end{aligned}$$

where $\Phi_{(t,x,t+s)}(z)$ is the cumulative distribution function of the $X_{t+s}^{t,b(t,y)}$ computed at z, $\Phi(z)$ denotes the cumulative distribution function of a standard Gaussian random variable at z and $\beta_0(s) \doteq \frac{b(t+s,y)-b(t,y)-\int_0^s m(t+u)\mathrm{d}u}{\sigma\sqrt{s}}$. For the second part we have:

$$\begin{aligned} &\int_{b(t+s,y)}^{\infty} e^{-rs}g^{'}(s)e^{z}p(t,b(t,y),t+s,z)\mathrm{d}z \\ &= g^{'}(y)e^{b(t,y)}\exp\left(\int_0^s m(t+u)\mathrm{d}u\frac{\sigma^2 s}{2}-rs\right)\left[1-\Phi(\beta_1(s))\right], \end{aligned}$$

where $\beta_1(s) \doteq \frac{b(t+s,y)-b(t,y)-\int_0^s m(t+u)\mathrm{d}u}{\sigma\sqrt{s}} - \sigma\sqrt{s}$. Therefore, the initial integral equation becomes:

$$\begin{aligned} c_0(1-e^{-r(T-t)}) &= rc_0\int^{T-t} e^{-rs}\left[1-\Phi(\beta_0(s))\right]\mathrm{d}s \\ &+ g^{'}(y)e^{b(t,y)}\int_0^{T-t}\exp\left(\int_0^s m(t+u)\mathrm{d}u+\frac{\sigma^2 s}{2}+rs\right)\Phi(\beta_1(s))\mathrm{d}s \\ &\doteq rc_0 I^{(1)}(t,b(t,y);T,r,\sigma) + g'(y)e^{b(t,y)}I^{(2)}(t,b(t,y);T,r,\sigma). \end{aligned}$$

Solving for $b(t,y)$ we obtain:

$$\begin{aligned} b(t,y) &= \log c_0 + \log\left((1-e^{-r(T-t)}) - rI^{(1)}(t,b(t,y);T,r,\sigma)\right) \\ &- \log g'(y) - \log I^{(2)}(t,b(t,y);T,r,\sigma), \end{aligned} \tag{4.6}$$

where we observe that $(1-e^{-r(T-t)}) - rI^{(1)}(t,b(t,y);T,r,\sigma) \geq 0$.

Equation (4.6) is the starting point for our Picard iterative numerical scheme. In order for the numerical method to be *well posed*, we conclude this section by proving that, for each $y \in [0,1]$, Equation (4.6) has solution $b(\cdot,y)$ which is unique in the following class:

$$\begin{aligned} B_y \quad &\doteq \quad \{\beta \in C([0,T]) \,:\, \beta(T) = \log(rc_0) - \log g'(y) \\ \text{and} \quad &\quad \beta(t) > \log(rc_0) - \log g'(y) \quad \forall t < T\} \quad y \in [0,1]. \end{aligned} \tag{4.7}$$

Theorem 4.2.1. *Assume that $t \mapsto b(t,y)$ is continuous on $[0,T]$ for each $y \in [0,1]$. Then, for all $y \in [0,1]$, the integral equation (4.6) admits a unique solution in B_y equal to $b(\cdot,y)$.*

Remark 4.2.2. The continuity assumption of Theorem 4.2.1, i.e. that the optimal stopping boundary b is continuous on $[0,T]$, is standard to be verified in the optimal stopping literature (see e.g. De Angelis and Milazzo, 2020; De Angelis and Stabile, 2019b). Together with the continuity assumptions of Proposition 4.2.1, we her give it for granted and leave its proof for a later work.

Proof. We follow the approach in troduced by Peskir (2005).
First, we trivially extend the auxiliary set $\mathcal{H}$ defined in Equation (3.20) to the whole space $[0,T] \times \Sigma$ by setting

$$\widehat{\mathcal{H}} \doteq \{(t,x,y) \in [0,T] \times \Sigma : \partial_y f(x,y) - rc_0 < 0\}.$$

Then $\widehat{\mathcal{H}} \subset \mathcal{C}$ and the boundary of its y-section $\widehat{\mathcal{H}}_y \subset \mathcal{C}_y$ is constant and equal to $\beta(y) \doteq \log(rc_0) - \log g'(y)$. Now, assume that there exists $\gamma : [0,T] \times [0,1] \to \mathbb{R}$ such that $\gamma(\cdot,y) \in B_y$ and it solves the integral equation

$$\begin{aligned} c_0 \quad &= \quad c_0 e^{-r(T-t)} + \mathsf{E}\left[\int_0^{T-t} e^{-rs} \partial_y f(X_{t+s}^{t,\gamma(t,y)}, y) \mathrm{d}s\right] + \\ &+ \quad \mathsf{E}\left[\int_0^{T-t} e^{-rs} \left(rc_0 - \partial_y f(X_{t+s}^{t,\gamma(t,y)}, y)\right) 1_{\{X_{t+s}^{t,\gamma(t,y)} \geq \gamma(t+s,y)\}} \mathrm{d}s\right] \end{aligned}$$

for each $y \in [0,1]$. If we set

$$\begin{aligned} U^\gamma(t,x,y) \quad &= \quad c_0 e^{-r(T-t)} + \mathsf{E}\left[\int_0^{T-t} e^{-rs} \partial_y f(X_{t+s}^{t,x}, y) \mathrm{d}s\right] + \\ &+ \quad \mathsf{E}\left[\int_0^{T-t} e^{-rs} \left(rc_0 - \partial_y f(X_{t+s}^{t,x}, y)\right) 1_{\{X_{t+s}^{t,x} \geq \gamma(t+s,y)\}} \mathrm{d}s\right] \end{aligned}$$

for all $(t,x,y) \in [0,T] \times \Sigma$, then trivially $U^\gamma(T,x,y) = c_0$ and $U^\gamma(t,\gamma(t,y),y) = c_0$. Moreover, we claim that the following processes are martingales:

$$\begin{aligned} \widehat{U}_s^{\gamma,t} \quad &\doteq \quad e^{-rs} U^\gamma(t+s, X_{t+s}^{t,x}, y) + \int_0^s e^{-ru} \partial_y f(X_{t+u}^{t,x}, y) \mathrm{d}u + \\ &+ \quad \int_0^s e^{-ru} \left(rc_0 - \partial_y f(X_{t+u}^{t,x}, y)\right) 1_{\{X_{t+u}^{t,x} \geq \gamma(t+u,y)\}} \mathrm{d}u \end{aligned}$$

and

$$\begin{aligned}\widehat{Z}^t_s &\doteq e^{-rs}u(t+s, X^{t,x}_{t+s}, y) + \int_0^s e^{-ru}\partial_y f(X^{t,x}_{t+u}, y)\mathrm{d}u + \\ &+ \int_0^s e^{-ru}\left(rc_0 - \partial_y f(X^{t,x}_{t+u}, y)\right) 1_{\{X^{t,x}_{t+u} \geq b(t+u,y)\}}\mathrm{d}u.\end{aligned}$$

In the following steps, we consider $y \in [0,1]$ fixed.

Step 1: $U^\gamma(t,x,y) = c_0$ for all $x \geq \gamma(t,y)$. Equality is trivial when $x = \gamma(t,y)$ or when $t = T$. Let then $t < T$ and $x > \gamma(t,y)$. Define

$$\tau_\gamma \doteq \inf\{s \geq 0 \,:\, X^{t,x}_{t+s} \leq \gamma(t+s,y)\} \wedge (T-t).$$

Then by the martingale property

$$U^\gamma(t,x,y) = \mathsf{E}\left[e^{-r\tau_\gamma}U^\gamma(t+\tau_\gamma, X^{t,x}_{t+\tau_\gamma}, y)\right] + \mathsf{E}\left[c_0(1-e^{-r\tau_\gamma})\right]$$

and we conclude by observing that $U^\gamma(t+\tau_\gamma, X^{t,x}_{t+\tau_\gamma}, y) = c_0$.

Step 2: $U^\gamma(t,x,y) \geq c_0$. This inequality is trivial for $x \geq \gamma(t,y)$ and for $t = T$. Let then $t < T$ and $x < \gamma(t,y)$. Define

$$\tau^\gamma \doteq \inf\{s \geq 0 \,:\, X^{t,x}_{t+s} \geq \gamma(t+s,y)\} \wedge (T-t).$$

Then by the martingale property

$$\begin{aligned}U^\gamma(t,x,y) &\doteq \mathsf{E}\left[e^{-r\tau^\gamma}U^\gamma(t+\tau^\gamma, X^{t,x}_{t+\tau^\gamma}, y)\right] + \mathsf{E}\left[\int_0^{\tau^\gamma} e^{-ru}\partial_y f(X^{t,x}_{t+u}, y)\mathrm{d}u\right] \\ &= \mathsf{E}\left[e^{-r\tau^\gamma}c_0\right] + \mathsf{E}\left[\int_0^{\tau^\gamma} e^{-ru}\partial_y f(X^{t,x}_{t+u}, y)\mathrm{d}u\right] \\ &\geq u(t,x,y)\end{aligned}$$

where the last inequality is a consequence of the definition of u.

Step 3: $\gamma(t,y) \leq b(t,y)$. By contradiction assume that there exists $t \in [0,T)$ such that $\gamma(t,y) > b(t,y)$ and let $x \geq \gamma(t,y)$. Define

$$\tau_b \doteq \inf\{s \geq 0 \,:\, X^{t,x}_{t+s} \leq b(t+s,y)\} \wedge (T-t).$$

By the martingale property, the fact that $U^\gamma(t,x,y) = u(t,x,y)$ when $x \geq \gamma(t,y)$ and that $U^\gamma \geq u$ we obtain

$$\mathsf{E}\left[\int_0^{\tau_b} e^{-rs}\left(rc_0 - \partial_y f(X^{t,x}_{t+s}, y)\right) 1_{\{X^{t,x}_{t+s} < \gamma(t+s,y)\}}\mathrm{d}s\right] \geq 0.$$

However $rc_0 - \partial_y f(x,y) < 0$ when $x > b(t,y)$ hence

$$\mathsf{E}\left[\int_0^{\tau_b} e^{-rs}\left(rc_0 - \partial_y f(X^{t,x}_{t+s}, y)\right)\mathrm{d}s\right] < 0.$$

We conclude by showing that $\mathsf{P}(X^{t,x}_{t+s} < \gamma(t+s,y)) > 0$ for $s > 0$ sufficiently small. Indeed, by continuity of γ and b there exists $s_0 > 0$ such that $\gamma(t+s,y) > b(t+s,y)$

for all $s \in [0, s_0]$. Then by continuity of the trajectories there is a positive probability that $X^{t,x}_{t+\cdot}$ will visit the region $[b(t+s), \gamma(t+s))$ for $s \in [s_1, s_2] \subset [0, s_0]$, for some $s_1, s_2 > 0$, $s_2 > s_1$.

Step 4: $\gamma(t,y) \geq b(t,y)$. By contradiction assume that there exists $t \in [0,T)$ such that $\gamma(t,y) < b(t,y)$ and let $x \in (\gamma(t,y), b(t,y))$. Define

$$\tau^b \doteq \inf\{s \geq 0 \,:\, X^{t,x}_{t+s} \geq b(t+s,y)\} \wedge (T-t).$$

By a similar reasoning as in Step 3 we find

$$\mathsf{E}\left[\int_0^{\tau^b} e^{-rs}\left(rc_0 - \partial_y f(X^{t,x}_{t+s}, y)\right) 1_{\{X^{t,x}_{t+s} \geq \gamma(t+s,y)\}} \mathrm{d}s\right] \geq 0.$$

However $rc_0 - \partial_y f(x,y) < 0$ when $x \in (\gamma(t,y), b(t,y))$ so we conclude as in Step 3.

Proof of the claim. We show that $(\widehat{U}^{\gamma,t}_s)_{s\in[0,T-t]}$ and $(\widehat{Z}^t_s)_{s\in[0,T-t]}$ are martingales. Indeed, let τ be a stopping time with values in $[0, T-t]$, then

$$\begin{aligned}\mathsf{E}\left[\widehat{U}^{\gamma,t}_\tau\right] &= \mathsf{E}\left[e^{-r\tau} U^\gamma(t+\tau, X^{t,x}_{t+\tau}, y)\right] \\ &+ \mathsf{E}\left[\int_0^\tau e^{-ru}\partial_y f(X^{t,x}_{t+u}, y)\mathrm{d}u\right] + \\ &+ \mathsf{E}\left[\int_0^\tau e^{-ru}\left(rc_0 - \partial_y f(X^{t,x}_{t+u}, y)\right) 1_{\{X^{t,x}_{t+u} \geq \gamma(t+u,y)\}}\mathrm{d}u\right].\end{aligned}$$

We compute the first term on the right-hand-side by means of the definition of U^γ as

$$\begin{aligned}&\mathsf{E}\left[e^{-r\tau} U^\gamma(t+\tau, X^{t,x}_{t+\tau}, y)\right] = c_0 e^{-r(T-t)} + \\ &+ \mathsf{E}\left[e^{-r\tau}\mathsf{E}\left[\int_0^{T-t-\tau} e^{-rs}\partial_y f(X_{t+\tau+s}, y)\mathrm{d}s\Big|\mathcal{F}_{t+\tau}\right]\right] + \\ &+ \mathsf{E}\left[e^{-r\tau}\mathsf{E}\left[\int_0^{T-t-\tau} e^{-rs}\left(rc_0 - \partial_y f(X_{t+\tau+s}, y)\right) 1_{\{X_{t+\tau+s} \geq \gamma(t+\tau+s,y)\}}\mathrm{d}s\Big|\mathcal{F}_{t+\tau}\right]\right] \\ &= c_0 e^{-r(T-t)} + \\ &+ \mathsf{E}\left[\int_\tau^{T-t} e^{-rs}\partial_y f(X^{t,x}_{t+s}, y)\mathrm{d}s\right] + \\ &+ \mathsf{E}\left[\int_\tau^{T-t} e^{-rs}\left(rc_0 - \partial_y f(X^{t,x}_{t+s}, y)\right) 1_{\{X^{t,x}_{t+s} \geq \gamma(t+s,y)\}}\mathrm{d}s\right]\end{aligned}$$

so that

$$\begin{aligned}\mathsf{E}\left[\widehat{U}^{\gamma,t}_\tau\right] &= c_0 e^{-r(T-t)} + \mathsf{E}\left[\int_0^{T-t} e^{-rs}\partial_y f(X^{t,x}_{t+s}, y)\mathrm{d}s\right] + \\ &+ \mathsf{E}\left[\int_0^{T-t} e^{-rs}\left(rc_0 - \partial_y f(X^{t,x}_{t+s}, y)\right) 1_{\{X^{t,x}_{t+s} \geq \gamma(t+s,y)\}}\mathrm{d}s\right] \\ &= U^\gamma(t,x,y)\end{aligned}$$

for any stopping time with values in $[0, T-t]$. The computation for $(\widehat{Z}^t_s)_{s\in[0,T-t]}$ is analogous so we conclude. □

4.2.2 The numerical method

In this section, we describe the numerical scheme for MFGs with singular controls. Precisely, we develop a numerical Picard-like iterative scheme based on Equation (4.6) that we then apply to each step n of the proof of Theorem 3.2.1.

Let $n \in \mathbb{N}$ be fixed. In order to clearly distinguish the n-th iteration of the iterative procedure from the iterations of the numerical method, we denote with $b^{[n]}$ and $c^{[n]}$ the optimal stopping boundaries of problem $\mathbf{OS}^{[n]}$ and introduce a subscript to denote the k-step of the numerical scheme.
Let then $b_k^{[n]}(t, y)$ be the stopping boundary obtained after the k-th iteration of the Picard scheme. In the following, we describe how we initialize the scheme and how we derive $b_{k+1}^{[n]}(t, y)$ from $b_k^{[n]}(t, y)$ for each $n, k \in \mathbb{N}$.
First, we start by discretizing the state space, i.e. we introduce equispaced partitions $\Pi_t \doteq \{0 \doteq t_0 < t_1 < \ldots < t_n \doteq T\}$, $\Pi_y \doteq \{0 \doteq y_0 < y_1 < \ldots < y_m \doteq 1\}$ and $\Pi_x \doteq \{x_1 < x_2 < \ldots < x_\ell\}$ with mesh $\Delta_t = \frac{T}{n}$, $\Delta_y = \frac{1}{m}$ and $\Delta_x = \frac{(x_\ell - x_1)}{\ell}$ and where the interval $[x_1, x_\ell] \subset \mathbb{R}$ is such that it includes the support of the initial distribution ν. Partitions will remain constant throughout all steps.

Initialization. At step $n = 0$ of the iterative scheme, we have $m^{[-1]*} \equiv 1$. In order to compute $b^{[0]*}(t, y)$, we initialize the algorithm by setting $b_0^{[0]}(t_j, y_i) \doteq \log(rc_0) - \log(g^{'}(y_i))$, for each fixed $i \in \{1, \ldots, m\}$ and for all $j = 1, \ldots, n$, which is the solution of $\partial_y f(x, y) = r\, c_0$ (see Figure 4.1). We then apply the procedure described below (see Figure 4.2).

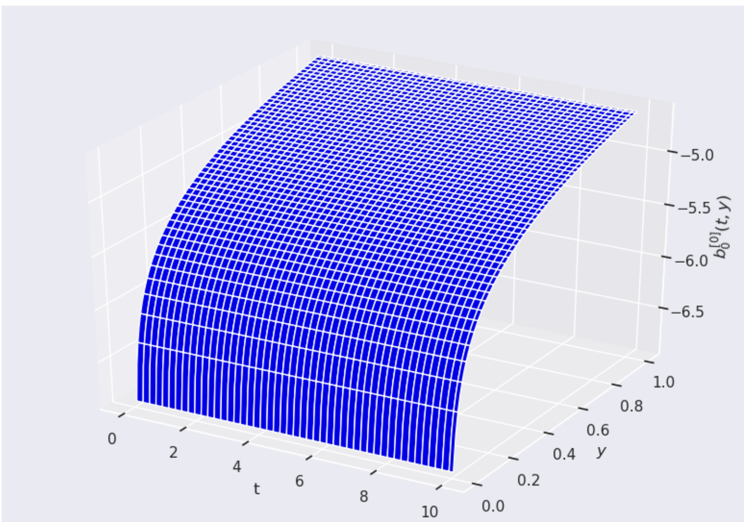

FIGURE 4.1: The figure displays the initialized boundary $b_0^{[0]}(t, y)$. Simulation parameters are listed in Table 4.1.

From iteration k to iteration $k+1$. Let $n \in \mathbb{N}$ be fixed. For each fixed $i \in \{1, \ldots, m\}$ and for all $j = 0, \ldots, n$, let $b_k^{[n]}(t_j, y_i)$ denote the values of the boundary

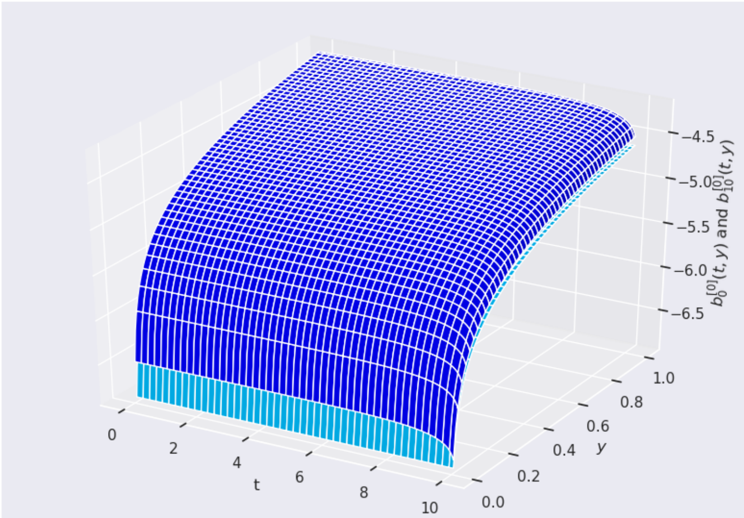

FIGURE 4.2: The figure displays the numerical boundary $b_{10}^{[0]}(t,y)$ at step $n = 0$ and at a iteration $k = 10$ of the Picard scheme superimposed on the initialized boundary $b_0^{[0]}(t,y)$. Simulation parameters are listed in Table 4.1.

obtained after the k-th iteration. Then, the values of the $(k+1)$-th iteration are computed as:

$$\begin{aligned} b_{k+1}^{[n]}(t_j, y_i) = \log c_0 + \log \Big((1 - e^{-r(T-t_j)}) - rI^{(1)}(t_j, b_k^{[n]}(t_j, y_i); T, r, \sigma) \Big) \\ - \log g'(y_i) - \log I^{(2)}(t_j, b_k^{[n]}(t_j, y_i); T, r, \sigma), \end{aligned} \tag{4.8}$$

for each fixed $i \in \{1, \dots, m\}$ and for all $j = 0, \dots, n$. See Figure 4.3. Here, the integral with respect to the time variable is computed by a standard quadrature method.

From step n to step $n+1$. Once that we have approximated the optimal boundary $b^{[n]*}(t,y)$, in order to compute $b^{[n+1]*}(t,y)$, we first need to compute $\xi_t^{[n]*} = \sup_{0 \le s \le t}(c^{[n]*}(s, X_s) - y)^+$, then $Y_t^{[n]*} \doteq y + \xi^{[n]*}$ and finally $m^{[n]*}(t)$, by taking the average over all possible initial positions (x,y) for the conditional expectation of $Y_t^{[n]*}$ (see Figure 4.4).

The expected value of $Y_t^{[n]*}$ is computed via a Monte-Carlo method, by simulating N trajectories of the process for all possible initial conditions $(x,y) \in \Pi_x \times \Pi_y$, $N \in \mathbb{N}$ large enough to the desired precision. Then integration with respect to the initial measure ν is performed via standard numerical integration techniques.

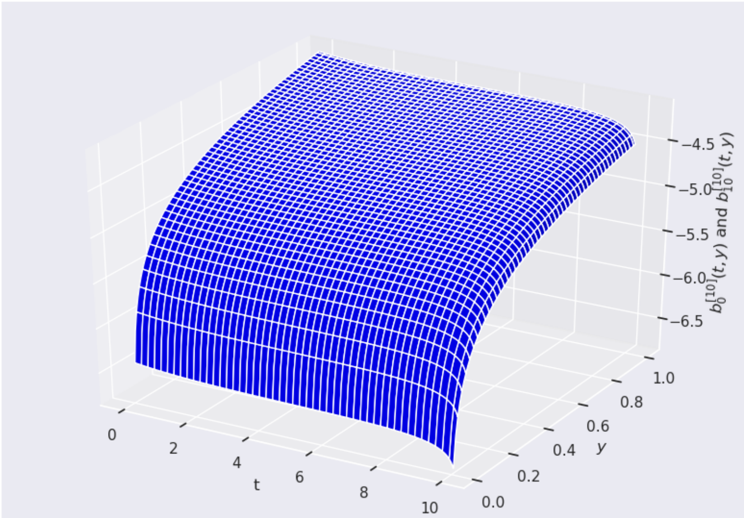

FIGURE 4.3: The figure displays the numerical boundary $b_{10}^{[10]}(t,y)$ at step $n = 10$ and at iteration $k = 10$ of the Picard scheme superimposed on $b_0^{[10]}(t,y) = b_{k_{max}}^{[9]}(t,y)$ where k_{max} is the maximum number of iterations performed at the previous step $n - 1 = 9$. The two boundaries almost coincide (this is supported by the scaling of the error depicted in Figure 4.5). Simulation parameters are listed in Table 4.1.

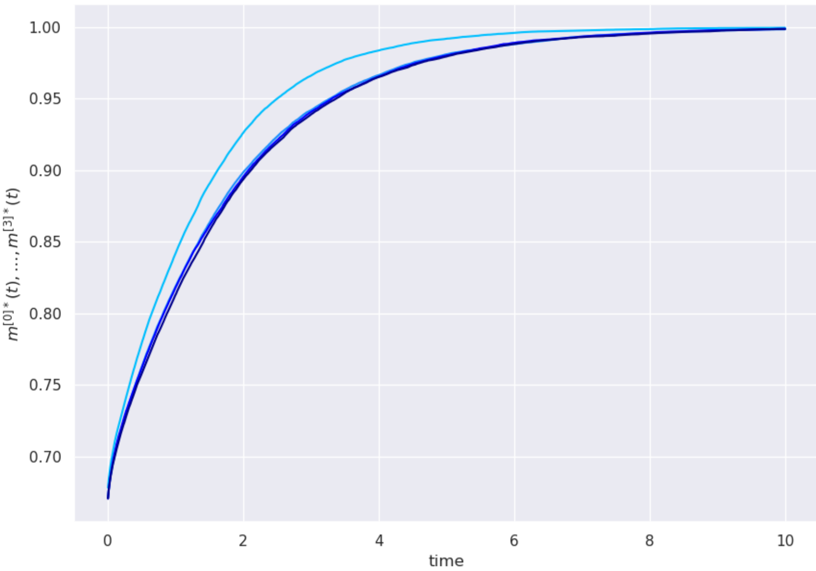

FIGURE 4.4: The figure displays $m^{[n]*}(t)$ for $n = 0, \ldots, 3$. Colors move from light-blue to dark-blue with n increasing. Simulation parameters are listed in Table 4.1.

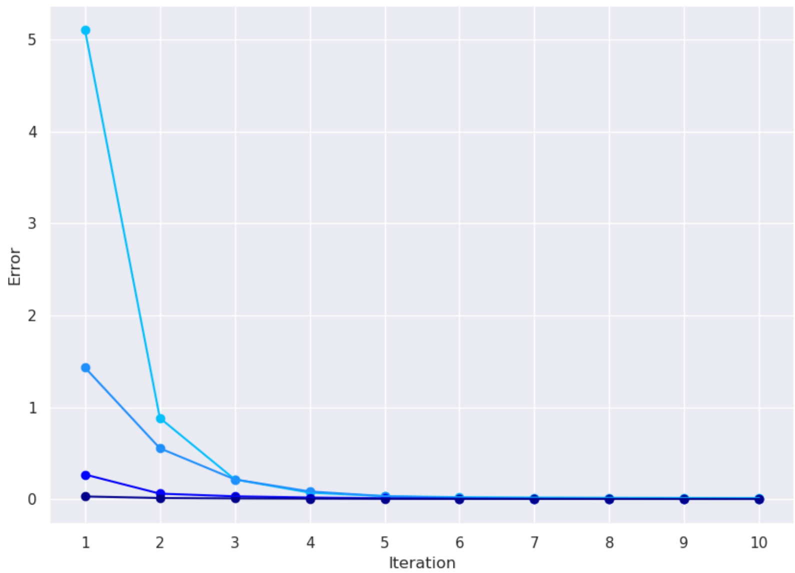

FIGURE 4.5: The figure displays how the computational error scales with the number of iterations (k) for different steps (n) of the iterative procedure. Color moves from light-blue to dark-blue with n increasing. Simulation parameters are listed in Table 4.1.

Parameter	Value
r	0.01
σ	1
μ	$0.5\sigma^2$
c_0	0.5
T	10
x_1	-7
x_ℓ	-3
ν	Uniform($[x_1, x_\ell]$)

TABLE 4.1: Simulation parameters.

Appendix A

Technical results for mean-field games with absorption

This appendix provides some of the technical results used in Chapter 2. More in detail, we state existence and uniqueness of weak solutions of SDEs with sub-linear drift. We characterize the space of laws of processes with sub-linear drift and initial condition ν (Q defined below). We prove some regularity results on the exit time τ^X with respect to measures in Q. Finally, we discuss the convergence of measures in the 1-Wasserstein distance along test functions with sub-linear growth and possibly discontinuous over a set of limit measure zero.

A.0.1 Existence and uniqueness of solution of SDEs with sub-linear drift

In this subsection we prove a slight variation of the well-known Beneš' condition (Beneš (1971)), leading to an existence and uniqueness result for weak solutions of SDEs with a sub-linear drift. More precisely, we allow the drift to depend on a rescaled Wiener process with a independent random initial condition. We recall that $\mathcal{E}_t(\cdot)$ denotes the Doléans-Dade stochastic exponential. Moreover, given a function $f : E \to \mathbb{R}$ where E is a Polish space, we denote by $\mathbb{D}_f$ the set of its discontinuity points.

As a preliminary, we introduce the set Q of laws of stochastic processes with sub-linear drift in the sense of Beneš to which these results apply.

Laws of processes with sub-linear drift. Let $\beta : [0, T] \times \mathcal{X} \to \mathbb{R}^d$ be a progressively measurable functional such that

$$|\beta(t, \varphi)| \leq C(1 + \|\varphi\|_\infty), \quad (t, \varphi) \in [0, T] \times \mathcal{X}$$

for some constant $C > 0$. Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathsf{P}, X)$ be a weak solution of the following SDE

$$X_t = \xi + \int_0^t \beta(s, X)\mathrm{d}s + \sigma W_t, \quad \xi \overset{d}{\sim} \nu, \quad t \in [0, T]$$

where W is a Wiener process independent of ξ. Existence and uniqueness of a weak solution follows from an application of Girsanov's theorem and Beneš' condition (see Lemma A.0.1 and Lemma A.0.2). Moreover such laws turn out to be absolutely continuous with respect to the Wiener measure $\mathcal{W}^\nu$ (Lemma A.0.3). Then, we denote by Q the set of laws $\theta \in \mathcal{P}(\mathcal{X})$ of all continuous processes X solving the SDE above.

Lemma A.0.1 (*Beneš' condition*)**.** *Let $b : [0,T] \times \mathcal{X} \to \mathbb{R}^d$ be a progressively measurable functional such that*

$$|b(t,\varphi)| \leq C\left(1 + \|\varphi\|_\infty\right), \quad (t,\varphi) \in [0,T] \times \mathcal{X}.$$

Let $\sigma \in \mathbb{R}^{d\times d}$ be a full rank matrix. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space satisfying usual conditions, supporting a random variable $\xi \overset{d}{\sim} \nu$ and a Wiener process W independent of ξ. Set

$$X_t \doteq \xi + \sigma W_t, \quad t \in [0,T].$$

Then

$$Z_t \doteq \mathcal{E}_t\left(\int_0^\cdot \sigma^{-1} b(s,X) dW_s\right), \quad t \in [0,T]$$

is a martingale.

Proof. We follow the proof of Corollary 3.5.16 in Karatzas and Shreve (1987). Precisely let $t_0 = 0 < t_1 < \ldots < t_{n-1} < t_n = T$ be a partition of the interval $[0,T]$. Then thanks to the sub-linearity of the drift

$$\int_{t_{n-1}}^{t_n} |b(s,X)|^2 ds \leq (t_n - t_{n-1}) C^2 \left(1 + \|X\|_\infty\right)^2.$$

Let $Y^n \doteq (Y^n_t)_{t\in[0,T]}$ be defined by

$$Y^n_t \doteq \mathrm{e}^{\frac{1}{4}(t_n - t_{n-1})C^2(1+|X_t|)^2}.$$

Notice that Y^n is a sub-martingale and that by Doob's maximal inequality (Karatzas and Shreve, 1987, Theorem 1.3.8.iv) we have $\mathsf{E}[\|Y^n\|^2_\infty] \leq 4\mathsf{E}[(Y^n_T)^2]$. Moreover

$$\begin{aligned}
\mathsf{E}\left[(Y^n_T)^2\right] &\leq \mathsf{E}\left[\mathrm{e}^{\frac{1}{2}(t_n-t_{n-1})C^2(1+2|\xi|^2+2|\sigma|^2|W_T|^2)}\right] \\
&= \mathsf{E}\left[\mathrm{e}^{(t_n-t_{n-1})C^2|\sigma|^2|W_T|^2}\right] \mathsf{E}\left[\mathrm{e}^{\frac{1}{2}(t_n-t_{n-1})C^2(1+2|\xi|^2)}\right]
\end{aligned}$$

where in the equality we have used the independence between ξ and W. To conclude, it is sufficient to choose $(t_k - t_{k-1}), k = 1, \ldots, n$, sufficiently small, for instance $(t_k - t_{k-1}) < \min\{\frac{1}{2C^2|\sigma|^2}, \frac{\lambda}{C^2}\}$, and to apply Corollary 3.5.14 in Karatzas and Shreve (1987). □

Corollary A.0.1 (*Moments of the stochastic exponential*)**.** *Under the assumptions of Lemma A.0.1, the process $Z = (Z_t)_{t\in[0,T]}$ has finite moments of any order $p \in [1,\infty)$, i.e. $\mathbb{E}\left[Z^p_T\right] < \infty$ for all $p \in [1,\infty)$.*

Proof. The proof follows directly from Lemma A.0.1 combined with Corollary 2 in Grigelionis and Mackevičius (2003). □

Lemma A.0.2 (*Existence and uniqueness of weak solutions*)**.** *Let $b : [0,T] \times \mathcal{X} \to \mathbb{R}^d$ be a progressively measurable functional such that*

$$|b(t,\varphi)| \leq C\left(1 + \|\varphi\|_\infty\right), \quad (t,\varphi) \in [0,T] \times \mathcal{X}.$$

Let $\sigma \in \mathbb{R}^{d\times d}$ a full rank matrix. Then there exists a weak solution $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P}, X, W)$ of

$$X_t = \xi + \int_0^t b(s, X)ds + \sigma dW_t, \quad \xi \overset{d}{\sim} \nu, \quad t \in [0, T].$$

Additionally, this solution is unique in law.

Proof. The proof follows directly from Lemma A.0.1 and Girsanov's theorem (see Karatzas and Shreve, 1987, Propositions 5.3.6 and 5.3.10). □

A.0.2 Characterization of the set Q

Lemma A.0.3 (*Laws of processes with sub-linear drift*)**.** *Let $\theta \in \mathsf{Q}$. Then $\theta \sim \mathcal{W}^\nu$, i.e. θ is equivalent to the Wiener measure $\mathcal{W}^\nu$.*

Proof. The proof follows directly from Lemma A.0.1, Girsanov's theorem and Bayes' rule to ensure that Z^{-1} given by Lemma A.0.1 is still a martingale. □

Before proceeding further, we recall that τ^X is the first exit time from $\mathcal{O}$ in the path space, i.e.

$$\tau^X(\varphi) = \inf\{t \geq 0 : \varphi(t) \notin \mathcal{O}\}, \quad \varphi \in \mathcal{X},$$

where $\mathcal{O} \subset \mathbb{R}^d$ satisfies Assumption (H4).

Lemma A.0.4 (*Regularity results*)**.** *Let $\theta \in \mathsf{Q}$. Let $\mathcal{O} \subset \mathbb{R}^d$ satisfy Assumption (H4) and let X be the identity process on $\mathcal{X}$. Then*

(a) $\tau^X < \infty$, *θ-almost surely.*

(b) The mapping $\varphi \mapsto \tau^X(\varphi)$, from $\mathcal{X}$ to $[0, \infty]$, is θ-a.s. continuous.

(c) $\theta(\tau^X = t) = 0$ *for all $t \in [0, T]$.*

(d) The mapping $\varphi \mapsto \mathbf{1}_{[0,\tau^X(\varphi))}(t)$, from $\mathcal{X}$ to $\mathbb{R}$, is θ-a.s. continuous for all $t \in [0, T]$.

(e) Properties (a)-(d) hold for $\mathcal{O} = (0, \infty)^{\times d}$ as well.

Proof. The proof is similar to the one of Lemma D.3 in Campi and Fischer (2018). Notice that by Lemma A.0.3 each $\theta \in \mathsf{Q}$ is equivalent to $\mathcal{W}^\nu$. So, it is sufficient to check properties (a)-(d) for $\mathcal{W}^\nu$.

(a) This is a consequence of the law of iterated logarithms (as time tends to infinity) and the fact that $\mathcal{O}$ is strictly included in $\mathbb{R}^d$.

(b) This, again, is a consequence of the law of iterated logarithms (as time tends to zero), the smoothness of $\mathcal{O}$'s boundary, the non-degeneracy of σ and the fact that $\mathcal{O}$ is strictly included in $\mathbb{R}^d$ (Kushner and Dupuis (2013), pp. 260-261).

(c) This is a consequence of the following relations

$$\mathcal{W}^\nu(\tau^X = t) \leq \mathcal{W}^\nu(X_t \in \partial\mathcal{O}) = 0 \quad \text{for all } t \in [0, T]$$

where in the last equality we use the fact that the Lebesgue measure of the boundary of a convex subset of $\mathbb{R}^d$ is identically zero (Lang (1986)), and that $\mathcal{W}^\nu \circ X_t^{-1}$ is absolutely continuous with respect to the Lebesgue measure for all $t \in [0, T]$.

(d) This is a consequence of properties (b) and (c) above.
(e) When $\mathcal{O} = (0,\infty)^{\times d}$ it turns out that

$$\tau^X(\varphi) = \min_{i=1,\dots,d} \tau^i(\varphi), \quad \varphi \in \mathcal{X}$$

where $\tau^i(\varphi) \doteq \inf\{t \in [0,T] : \varphi_i(t) \leq 0\}$, for $i \in \{1,\dots,d\}$ and $\varphi \in \mathcal{X}$. Then the conclusion follows from the continuity result in dimension $d = 1$ (Kushner and Dupuis (2013), pp. 260-261) applied to each τ^i. □

A.0.3 Additional convergence results

Lemma A.0.5 (*Convergence in the 1-Wasserstein distance*)**.** *Let E be a Polish space with a complete metric d_E. Let $\theta, (\theta^n)_{n\in\mathbb{N}} \subset \mathcal{P}_1(E)$ such that $W_1(\theta^n, \theta) \to 0$ as $n \to \infty$. Let $f : E \to \mathbb{R}$ be a measurable function such that $|f(x)| \leq C(1 + d_E(x, x_0))$ for all $x \in E$, for some $x_0 \in E$ and for some constant $C > 0$. Let $\mathbb{D}_f$ be the set of its discontinuity points and assume $\theta(\mathbb{D}_f) = 0$. Then*

$$\int_E f(x)\theta^n(dx) \underset{n\to\infty}{\longrightarrow} \int_E f(x)\theta(dx).$$

Proof. The proof works as in Villani (2003), proof of Theorem 7.12.iv, the only difference being that here f can have discontinuities with $\theta(\mathbb{D}_f) = 0$. In particular, we perform the same decomposition as in Villani (2003), i.e. $f(x) = f_R^1(x) + f_R^2(x)$ with $f_R^1(x) \doteq f(x) \wedge (C(1+R))$ and $f_R^2(x) \doteq f(x) - f_R^1(x)$ for all $x \in E$ and for some $R > 0$. We have that $|f_R^1|$ is bounded by $C(1+R)$ and $\theta(\mathbb{D}_{f_R^1}) = 0$ since $\mathbb{D}_{f_R^1} \subset \mathbb{D}_f$. Then all limits can be performed just as in Villani (2003), proof of Theorem 7.12.iv. □

Appendix B

Technical results for mean-field games with singular controls

In this appendix we collect a number of technical results for mean-field games with singular controls.

Proof of Proposition 3.3.3. Fix $(t,x,y) \in [0,T] \times \Sigma$. Take any admissible control $\zeta \in \Xi_{t,x}(y)$ and define, for $q \geq 0$, its right-continuous inverse (see, e.g., (Revuz and Yor, 2013, Ch. 0, Sec. 4)) $\tau^{\zeta}(q) \doteq \inf\{s \in [t,T) : \zeta_{s-t} > q\} \wedge T$. The process $\tau^{\zeta} \doteq (\tau^{\zeta}(q))_{q\geq 0}$ has increasing right-continuous sample paths, hence it admits left limits $\tau^{\zeta}_{-}(q) \doteq \inf\{s \in [t,T) : \zeta_{s-t} \geq q\} \wedge T$, for $q \geq 0$. It can be shown that both $\tau^{\zeta}(q)$ and $\tau^{\zeta}_{-}(q)$ are $(\mathcal{F}_{t+s})$-stopping times for any $q \geq 0$.

Let now $q = z - y$ for $z \geq y$ and consider the function w defined as

$$w(t,x,y) \doteq \Phi_n(t,x) - \int_y^1 u_n(t,x,z)\,\mathrm{d}z. \tag{B.1}$$

Since $\tau^{\zeta}(z-y)$ is admissible for $u_n(t,x,z)$ we have

$$w(t,x,y) - \Phi_n(t,x) \quad \geq - \int_y^1 \mathsf{E}_{t,x}\left[c_0 e^{-r\tau^{\zeta}(z-y)} + \int_t^{\tau^{\zeta}(z-y)} e^{-rs}\partial_y f(X^{[n]}_s, z)\mathrm{d}s\right]\mathrm{d}z.$$

In order to compute the integral with respect to $\mathrm{d}z$ we observe that for $t \leq s < T$ we have

$$\{\zeta_{s-t} < z-y\} \subseteq \{s < \tau^{\zeta}(z-y)\} \subseteq \{\zeta_{s-t} \leq z-y\}$$

by right-continuity and monotonicity of the process $s \mapsto \zeta_{s-t}$. The left-most and right-most events above are the same up to $\mathrm{d}z$-null sets. Then, applying Fubini's theorem more than once we obtain

$$\begin{aligned}
&w(t,x,y) - \Phi_n(t,x)\\
&\geq \mathsf{E}_{t,x}\left[-\int_y^1 e^{-r\tau^{\zeta}(z-y)}c_0\,\mathrm{d}z - \int_t^T e^{-rs}\int_y^1 \partial_y f(X^{[n]}_s, z)\mathbf{1}_{\{s<\tau^{\zeta}(z-y)\}}\,\mathrm{d}z\,\mathrm{d}s\right]\\
&= \mathsf{E}_{t,x}\left[-\int_y^1 e^{-r\tau^{\zeta}(z-y)}c_0\,\mathrm{d}z - \int_t^T e^{-rs}\int_y^1 \partial_y f(X^{[n]}_s, z)\mathbf{1}_{\{\zeta_{s-t}<z-y\}}\mathrm{d}z\,\mathrm{d}s\right]\\
&= \mathsf{E}_{t,x}\left[-\int_y^1 e^{-r\tau^{\zeta}(z-y)}c_0\,\mathrm{d}z - \int_t^T e^{-rs}[f(X^{[n]}_s, 1) - f(X^{[n]}_s, y+\zeta_{s-t})]\,\mathrm{d}s\right]\\
&= J_n(t,x,y;\zeta) - \Phi_n(t,x),
\end{aligned}$$

where the final equality uses the well-known change of variable formula (see, e.g., Revuz and Yor, 2013, Ch. 0, Proposition 4.9)

$$\int_y^1 e^{-r\tau^{\zeta}(z-y)}\mathrm{d}z = \int_{[t,T]} e^{-rs}\mathrm{d}\zeta_{s-t}.$$

By the arbitrariness of $\zeta \in \Xi_{t,x}(y)$ we conclude $w_n(t,x,y) \geq v_n(t,x,y)$.

For the reverse inequality we take $\zeta_s = \xi^{[n]*}_{t+s}$ as defined in Lemma 3.3.2. Recall that

$$\tau^{[n]}_*(t,x,z) = \inf\left\{s \in [0,T-T] : z \leq c_n(t+s, X^{[n];t,x}_{t+s})\right\}.$$

and since $s \mapsto c_n(s, X^{[n];t,x}_s) - z$ is upper semi-continuous, it attains a maximum over any compact interval in $[t,T)$. In particular, for $s \in [t,T)$

$$\tau^{[n]}_*(t,x,z) \leq s \iff \text{there exists } \theta \in [t,t+s] \text{ such that } c_n(\theta, X^{[n];t,x}_\theta) \geq z.$$

For any $y < z$, the latter is also equivalent to

$$\xi^{[n]*}_{t+s} = \sup_{0\leq u\leq s}\left(c_n(t+u, X^{[n];t,x}_{t+u}) - y\right)^+ \geq z - y$$

and, therefore, it is also equivalent to $\tau^{\xi^{[n]*}}_{-}(z-y) \leq s$. Since $s \in [t,T]$ was arbitrary the chain of equivalences implies that $\tau^{\xi^{[n]*}}_{-}(z-y) = \tau^{[n]}_*(t,x,z)$, P-a.s. for any $z > y$. However, we have already observed that for a.e. $z > y$ it must be $\tau^{\xi^{[n]*}}_{-}(z-y) = \tau^{\xi^{[n]*}}(z-y)$, P-a.s., hence $\tau^{\xi^{[n]*}}(z-y) = \tau^{[n]}_*(t,x,z)$ as well. The latter, in particular implies

$$w(t,x,y) - \Phi_n(t,x) = -\int_y^1 \mathsf{E}_{t,x}\Big[c_0 e^{-r\tau^{\xi^{[n]*}}(z-y)} + \int_t^{\tau^{\xi^{[n]*}}(z-y)} e^{-rs}\partial_y f(X^{[n]}_s, z)\mathrm{d}s\Big]\mathrm{d}z,$$

by optimality of $\tau^{[n]}_*(t,x,z)$ in $u_n(t,x,z)$. Repeating the same steps as above we then find

$$w(t,x,y) = J_n(t,x,y;\xi^{[n]*}),$$

which combined with $v_n \leq w$ concludes the proof. □

Proof of Lemma 3.3.5. We have $\mathcal{C}^{[n]} \subset \mathcal{C}^{[n+1]} \subset \mathcal{C}$ because the sequence $(c_n)_{n\geq 0}$ is decreasing. Then, the sequence $(\tau^{[n]}_*)_{n\geq 0}$ is increasing and $\lim_{n\to\infty}\tau^{[n]}_* \leq \tau_*$, $\mathsf{P}_{t,x,y}$-a.s. for any $(t,x,y) \in [0,T]\times\Sigma$. In order to prove the reverse inequality, first we observe that $t \mapsto X^{[k]}_t(\omega)$ is continuous for all $\omega \in \Omega \setminus N_k$ with $\mathsf{P}(N_k) = 0$, for all $k \geq 0$. Moreover, $t \mapsto X_t(\omega)$ is continuous for all $\omega \in \Omega\setminus N$ with $\mathsf{P}(N) = 0$. Let us set $N_0 \doteq (\cup_k N_k)\cup N$ and $\Omega_0 \doteq \Omega\setminus N_0$ so that $\mathsf{P}(\Omega_0) = 1$. Fix $(t,x,y) \in [0,T]\times\Sigma$ and $\omega \in \Omega_0$. Let $\delta > 0$ be such that $\tau_*(\omega) > \delta$ (if no such δ exists, then $\tau_*(\omega) = 0$ and $\tau^{[n]}_*(\omega) \geq \tau_*(\omega)$ for all $n \geq 0$). Then, since $s \mapsto u(t+s, \widetilde{X}_{t+s}(\omega), y)$ is continuous, there exists $\varepsilon > 0$ such that

$$\sup_{0\leq s\leq\delta}\Big(u(t+s, \widetilde{X}_{t+s}(\omega), y) - c_0\Big) \leq -\varepsilon. \tag{B.2}$$

At the same time we also notice that $s \mapsto u_n(t+s, X^{[n]}_{t+s}(\omega), y)$ is continuous and moreover

$$u_n(t+s, X^{[n]}_{t+s}(\omega), y) \geq u_{n+1}(t+s, X^{[n]}_{t+s}(\omega), y) \geq u_{n+1}(t+s, X^{[n+1]}_{t+s}(\omega), y)$$

by monotonicity of the sequences $(u_n)_{n\geq 0}$ and $(X^{[n]})_{n\geq 0}$ and of the map $x \mapsto u_n(t,x,y)$. So we have that $u_n(t+\cdot, X^{[n]}_{\cdot}(\omega), y)$ is a decreasing sequence of continuous functions of time and since the limit is also continuous, the convergence is uniform on $[0,\delta]$. Then, there exists $n_0 \geq 0$ sufficiently large that

$$\sup_{0\leq s\leq \delta} \left| u(t+s, \widetilde{X}_{t+s}(\omega), y) - u_n(t+s, X^{[n]}_{t+s}(\omega), y) \right| \leq -\frac{\varepsilon}{2}, \quad \text{for } n \geq n_0.$$

Using this fact and Equation (B.2) we have

$$\sup_{0\leq s\leq \delta} \left(u_n(t+s, X^{[n]}_{t+s}(\omega), y) - c_0 \right) \leq -\frac{\varepsilon}{2}$$

and $\tau^{[n]}_*(\omega) > \delta$, for all $n \geq n_0$. Since $\delta > 0$ was arbitrary, we obtain

$$\lim_{n\to\infty} \tau^{[n]}_*(\omega) \geq \tau_*(\omega).$$

Recalling that $\omega \in \Omega_0$ was also arbitrary we obtain the desired result. □

Proof of Lemma 3.4.1. The proof is divided in two steps: first we show that $(t,x,y) \mapsto \tau_*(t,x,y)$ is lower semi-continuous and then that it is upper semi-continuous. Both parts of the proof rely on continuity of the flow $(t,x,s) \mapsto X^{*;t,x}_{t+s}(\omega)$. The latter holds for all $\omega \in \Omega \setminus N$ where N is a universal set with $\mathsf{P}(N) = 0$. For simplicity, in the rest of the proof we just write X instead of X^*.

Step 1. (Lower semi-continuity). This part of the proof is similar to that of Lemma 3.3.5. Fix $(t,x,y) \in [0,T] \times \Sigma$ and take a sequence $(t_n, x_n, y_n)_{n\geq 1}$ that converges to (t,x,y) as $n \to \infty$. Denote $\tau_* = \tau_*(t,x,y)$ and $\tau_n \doteq \tau_*(t_n,x_n,y_n)$ and fix an arbitrary $\omega \in \Omega \setminus N$. If $(t,x,y) \in \mathcal{S}$ then $\tau_*(\omega) = 0$ and $\liminf_n \tau_n(\omega) \geq \tau_*(\omega)$ trivially. Let $\delta > 0$ be such that $\tau_*(\omega) > \delta$. Then by continuity of the value function u and of the trajectory $s \mapsto X^{t,x}_{t+s}(\omega)$ there exists $\varepsilon > 0$ such that

$$\sup_{0\leq s\leq \delta} \left(u(t+s, X^{t,x}_{t+s}(\omega), y) - c_0 \right) \leq -\varepsilon.$$

Thanks to continuity of the stochastic flow there is no loss of generality in assuming that $(t_n + s, X^{t_n,x_n}_{t_n+s}(\omega), y_n)$ lies in a compact K for all $n \geq 1$ and $s \leq \delta$. Then there exists $n_\varepsilon > 0$ such that

$$\sup_{0\leq s\leq \delta} \left| u(t+s, X^{t,x}_{t+s}(\omega), y) - u(t_n+s, X^{t_n,x_n}_{t_n+s}(\omega), y) \right| \leq \varepsilon/2$$

for all $n \geq n_\varepsilon$ (by uniform continuity). Combining the above we get

$$\sup_{n\geq n_\varepsilon} \sup_{0\leq s\leq \delta} \left(u(t_n+s, X^{t_n,x_n}_{t_n+s}(\omega), y_n) - c_0 \right) \leq -\varepsilon/2,$$

which implies $\tau_n(\omega) > \delta$ for all $n \geq n_\varepsilon$. Hence $\liminf_n \tau_n(\omega) > \delta$ and since δ and ω were arbitrary we conclude $\liminf_n \tau_n(\omega) > \tau_*(\omega)$, for all $\omega \in \Omega \setminus N$.

Step 2. (Upper semi-continuity). For this part of the proof we need to introduce the *hitting time* σ_*° to the interior of the stopping set $\mathcal{S}^\circ \doteq \mathrm{int}(\mathcal{S})$ (which is not empty thanks to the argument of proof of Lemma 3.3.4), i.e.,

$$\sigma_*^\circ(t,x,y) \doteq \inf\{s \in (0, T-t] \,:\, (t+s, X_{t+s}^{t,x}, y) \in \mathcal{S}^\circ\}.$$

Assume for a moment that

$$\mathsf{P}_{t,x,y}(\tau_* = \sigma_*^\circ) = 1 \quad \text{for all } (t,x,y) \in [0,T] \times \Sigma. \tag{B.3}$$

Then we can invoke Lemma 4 in De Angelis and Peskir (2020) (see Eq. (3.7) therein) to conclude that $(t,x,y) \mapsto \sigma_*^\circ(t,x,y)$ is upper semi-continuous. Hence, given $(t,x,y) \in [0,T] \times \Sigma$ and any sequence $(t_n, x_n, y_n)_{n\geq 1}$ converging to (t,x,y) as $n \to \infty$, setting $\tau_n = \tau_*(t_n, x_n, y_n)$ and $\sigma_n^\circ = \sigma_*^\circ(t_n, x_n, y_n)$, we have $\tau_n(\omega) = \sigma_n^\circ(\omega)$ for all $\omega \in \Omega_n$ with $\mathsf{P}(\Omega_n) = 1$ for each $n \geq 1$; therefore letting $\bar{\Omega} \doteq \cap_{n\geq 1}\Omega_n$ we have $\mathsf{P}(\bar{\Omega}) = 1$ and

$$\limsup_{n\to\infty} \tau_n(\omega) = \limsup_{n\to\infty} \sigma_n^\circ(\omega) \leq \sigma_*^\circ(\omega) = \tau_*(\omega),$$

with $\tau_* = \tau_*(t,x,y)$ and $\sigma_*^\circ = \sigma_*^\circ(t,x,y)$, for all $\omega \in \bar{\Omega} \cap \{\tau_* = \sigma_*^\circ\}$ where $\mathsf{P}(\bar{\Omega} \cap \{\tau_* = \sigma_*^\circ\}) = 1$.

Let us now prove Equation (B.3). We introduce the generalised left-continuous inverse of $x \mapsto c(t,x)$, i.e.

$$b(t,y) = \sup\{x \in \mathbb{R} : c(t,x) < y\}.$$

Then it is easy to check that $t \mapsto b(t,y)$ is non-increasing. This implies that $\mathsf{P}_{t,x,y}(\tau_* = \sigma_*^\circ) = 1$ for all $(t,x,y) \in \mathcal{S}^\circ$ by continuity of the paths of X. Moreover, for $(t,x,y) \in \mathcal{C}$ we have $\sigma_*^\circ = \tau_* + \sigma_*^\circ \circ \theta_{\tau_*}$, where $\{\theta_t,\, t \geq 0\}$ is the shift operator, i.e., $(t, X_t(\omega)) \circ \theta_s = (t+s, X_{t+s}(\omega))$. Then, $\tau_* = \sigma_*^\circ$ if and only if $\sigma_*^\circ \circ \theta_{\tau_*} = 0$. Since $\sigma_*^\circ \circ \theta_{\tau_*}$ is the hitting time to $\mathcal{S}^\circ$ after the process (t, X, y) has reached the boundary $\partial\mathcal{C}$ of the continuation set, the previous condition is implied by $\mathsf{P}_{t,x,y}(\sigma_*^\circ = 0) = 1$ for $(t,x,y) \in \partial\mathcal{C}$. So we now focus on proving the latter.

We claim that

$$\mathcal{S}^\circ = \{(t,x,y) \,:\, x > b(t,y)\}$$

and will give a proof of this fact in Lemma B.0.1 below. Then by the law of iterated logarithm and non-increasing $t \mapsto b(t,y)$ we immediately obtain $\mathsf{P}_{t,x,y}(\sigma_*^\circ = 0) = 1$ for $(t,x,y) \in \partial\mathcal{C}$ because $(t,x,y) \in \partial\mathcal{C}$ if and only if $x \geq b(t,y)$. □

Lemma B.0.1. *We have*

$$\mathcal{S}^\circ = \{(t,x,y) \,:\, x > b(t,y)\}. \tag{B.4}$$

Proof. While the claim may seem obvious, since $y \mapsto b(t,y)$ is non-decreasing, one should notice that for it to hold we must rule out the case $b(t,y) < b(t,y+)$ for all $(t,y) \in [0,T] \times [0,1)$. Indeed, if the latter occurs for some (t_0, y_0) we have $\{t_0\} \times (b(t_0,y_0), b(t_0,y_0+)) \times \{y_0\} \in \partial\mathcal{C}$ and Equation (B.4) fails.

Here we use an argument by contradiction inspired to De Angelis (2015). Assume that there exists $(t_0, y_0) \in [0,T] \times [0,1]$ such that $x_1^0 \doteq b(t_0,y_0) < b(t_0,y_0+) =: x_2^0$. Then we proceed in two steps.

Step 1. (A PDE for the value function). Since $(t,x) \mapsto a(x, m^*(t))$ is not continuous in general we cannot immediately apply standard PDE arguments that guarantee that

$$\partial_t u + \frac{\sigma^2(\,\cdot\,)}{2}\partial_{xx} u + a(\,\cdot, m^*(\,\cdot\,))\partial_x u - ru = -\partial_y f, \quad \text{for } (t,x,y) \in \mathcal{C}$$

(see Peskir and Shiryaev, 2006, Chapter III). However, given $\delta > 0$ and letting $\mathcal{O}_\delta \doteq (t_0 - \delta, t_0) \times (x_1^0, x_2^0)$ we have $\mathcal{O}_\delta \times (y_0, y_0 + \delta) \subset \mathcal{C}$. Moreover, with no loss of generality we can assume $\delta > 0$ sufficiently small and such that $m^*_-(t) \doteq \lim_{\varepsilon \downarrow 0} m^*(t - \varepsilon)$ is continuous on $(t_0 - \delta, t_0]$ (recall that m^* is non-decreasing and right-continuous). Since m^* is non-decreasing it has at most countably many jumps on any compact and therefore replacing m^* with m^*_- in the dynamics of X Equation (3.37) (recall that $m^* = \widetilde{m}$) we obtain a new process X' which is indistinguishable from the original one. Then, starting from $(t,x,y) \in \mathcal{O}_\delta \times (y_0, y_0 + \delta)$ and letting $\tau_{\mathcal{O}}$ be the first exit time of $(t + s, X'_{t+s})$ from $\mathcal{O}_\delta$ we have that

$$s \mapsto e^{-r(s \wedge \tau_{\mathcal{O}})} u(t + s \wedge \tau_{\mathcal{O}}, X'_{s \wedge \tau_{\mathcal{O}}}, y) + \int_0^{s \wedge \tau_{\mathcal{O}}} e^{-ru} \partial_y f(X'_{t+u}, y) \mathrm{d}u$$

is a continuous martingale. By standard arguments (see Peskir and Shiryaev, 2006, Chapter III) this translates to the fact that *for each* $y \in (y_0, y_0 + \delta)$ the value function $u(\,\cdot\,, y)$ is the unique solution in $C^{1,2}(\mathcal{O}_\delta) \cap C(\overline{\mathcal{O}}_\delta)$ of the boundary value problem

$$\partial_t w + \frac{\sigma^2(\,\cdot\,)}{2}\partial_{xx} w + a(\,\cdot, m^*_-(\,\cdot\,))\partial_x w - rw = -\partial_y f(\,\cdot\,, y), \quad \text{on } \mathcal{O}_\delta \tag{B.5}$$

with $w(\,\cdot\,) = u(\,\cdot\,, y)$ at $\partial_P \mathcal{O}_\delta$, where $\partial_P \mathcal{O}_\delta$ is the parabolic boundary of $\mathcal{O}_\delta$ (notice that now the claim is correct because $(t,x) \mapsto a(x, m^*_-(t))$ is continuous in $\mathcal{O}_\delta$).

Step 2. (Contradiction). Thanks to the result in step 1 we can now find a contradiction. Pick an arbitrary $\varphi \in C_c^\infty((x_1^0, x_2^0))$, $\varphi \geq 0$ and multiply Equation (B.5) by φ (with $w(\,\cdot\,) = u(\,\cdot\,, y)$). Since $t \mapsto u(t,x,y)$ is non-decreasing we have $\partial_t u(\,\cdot\,, y) \geq 0$ on $\mathcal{O}_\delta$ and therefore, for each $y \in (y_0, y_0 + \delta)$ we have

$$\varphi(\,\cdot\,)\left[\frac{\sigma^2(\,\cdot\,)}{2}\partial_{xx} u(\,\cdot\,, y) + a(\,\cdot\,, m^*_-(\,\cdot\,))\partial_x u(\,\cdot\,, y) - ru(\,\cdot\,, y)\right] \leq -\varphi(\,\cdot\,)\partial_y f(\,\cdot\,, y), \quad \text{on } \mathcal{O}_\delta.$$

In the inequality above we fix t_0 and integrate over (x_1^0, x_2^0). By using integration-by-parts formula we obtain[1]

$$\begin{aligned} &\int_{x_1^0}^{x_2^0} \left(\tfrac{1}{2}\partial_{xx}\left[\sigma^2(x)\varphi(x)\right] - \partial_x\left[a(x, m(t_0))\varphi(x)\right] - r\varphi(x)\right) u(t_0, x, y)\mathrm{d}x \\ &\leq -\int_{x_1^0}^{x_2^0} \partial_y f(x,y)\varphi(x)\mathrm{d}x. \end{aligned}$$

[1] Notice that the argument seems to require $\sigma \in C^2(\mathbb{R})$. However, given that the final estimate in Equation (B.6) does not depend on σ we can apply a smoothing of σ, if necessary, and then pass to the limit at the end.

Now, letting $y \downarrow y_0$, using dominated convergence and $u(t_0, x, y_0) = c_0$ for $x \in (x_1^0, x_2^0)$, and undoing the integration by parts we obtain

$$\int_{x_1^0}^{x_2^0} (\partial_y f(x, y_0) - rc_0)\varphi(x)\mathrm{d}x \leq 0. \tag{B.6}$$

Hence $\partial_y f(x, y_0) - rc_0 \leq 0$ for all $x \in (x_1^0, x_1^0)$ by arbitrariness of $\varphi \geq 0$ and continuity of $x \mapsto \partial_y f(x, y_0) - rc_0$. However, since $\mathcal{S} \subseteq [0, T] \times (\Sigma \setminus \mathcal{H})$ (recall Equation (3.20)), then it must be $\partial_y f(x, y_0) - rc_0 = 0$ for all $x \in (x_1^0, x_1^0)$, which contradicts $\partial_{xy} f > 0$ (Assumption 3.2.2-(iii)). □

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