SOLVABILITY AND REGULARITY FOR THE ELECTROSTATIC BORN-INFELD EQUATION WITH GENERAL CHARGES

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ABSTRACT. In electrostatic Born-Infeld theory, the electric potential u_ρ generated by a charge distribution ρ in $\mathbb{R}^m$ (typically, a Radon measure) minimizes the action

$$\int_{\mathbb{R}^m}\left(1-\sqrt{1-|D\psi|^2}\right)\mathrm{d}x-\langle\rho,\psi\rangle$$

among functions which decay at infinity and satisfy $|D\psi|\leq 1$. Formally, its Euler-Lagrange equation ($\mathcal{BI}$) prescribes ρ as being the Lorentzian mean curvature of the graph of u_ρ in Minkowski spacetime $\mathbb{L}^{m+1}$. However, because of the lack of regularity of the functional when $|D\psi|=1$, whether or not u_ρ solves ($\mathcal{BI}$) and how regular is u_ρ are subtle issues that were investigated only for few classes of ρ. In this paper, we study both problems for general sources ρ, in a bounded domain with a Dirichlet boundary condition and in the entire $\mathbb{R}^m$. In particular, we give sufficient conditions to guarantee that u_ρ solves ($\mathcal{BI}$) and enjoys improved $W^{2,2}_{\mathrm{loc}}$ estimates, and we construct examples helping to identify sharp thresholds for the regularity of ρ to ensure the validity of ($\mathcal{BI}$). One of the main difficulties is the possible presence of light segments in the graph of u_ρ, which will be discussed in detail.

MSC2020: Primary 35B65, 35B38, 35R06, 53B30; Secondary 35J62, 49J40.
Keywords: Born–Infeld model, Euler–Lagrange equation, Regularity of solutions, Measure data, Prescribed Lorentzian mean curvature.

CONTENTS

Date: January 1, 2022.

1. Introduction

The purpose of this paper is to investigate the existence and regularity properties of spacelike hypersurfaces M with prescribed Lorentzian mean curvature in the Minkowski space

$$\mathbb{L}^{m+1} \doteq \mathbb{R} \times \mathbb{R}^m \qquad \text{with Lorentzian metric} \qquad -\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \sum_{i=1}^{m} \mathrm{d}x^i \otimes \mathrm{d}x^i.$$

The spacelike condition ensures that M is the graph, over some open subset Ω of the totally geodesic slice $\mathbb{R}^m \doteq \{x^0 = 0\}$, of a function u with $|Du| < 1$. We consider both the problem in a bounded domain Ω, and the problem in the entire $\mathbb{R}^m$. In the first case, given $\phi \in C(\partial\Omega)$, a spacelike hypersurface with Lorentzian mean curvature ρ and boundary (the graph of) ϕ is the graph of a solution $u : \overline{\Omega} \to \mathbb{R}$ to

$$(\mathcal{BI}) \qquad \begin{cases} -\mathrm{div}\left(\dfrac{Du}{\sqrt{1-|Du|^2}}\right) = \rho & \text{on } \Omega \subset \mathbb{R}^m, \\[2em] u = \phi & \text{on } \partial\Omega, \end{cases}$$

where D and $|\cdot|$ are the connection and norm in $\mathbb{R}^m$. The source term ρ will be taken to be a Radon measure, or more generally a bounded linear functional on a natural space to which solutions belong. Following the convention in the literature, we say that the graph M of $u \in W^{1,\infty}(\Omega)$ is

- *weakly spacelike* if $|Du| \le 1$ on Ω;
- *spacelike* if $|u(x) - u(y)| < |x - y|$ whenever $x, y \in \Omega$, $x \neq y$ and the line segment $\overline{xy}$ is contained in Ω;
- *strictly spacelike* if $u \in C^1(\Omega)$ and $|Du| < 1$ in Ω.

The equation in ($\mathcal{BI}$) is of interest already in the case of constant ρ, due to the prominent role of spacelike constant mean curvature hypersurfaces in General Relativity (see [36] and the references therein). It was observed in [36, 4, 5, 8] that a variational approach to ($\mathcal{BI}$) by minimizing the functional

$$I_\rho(v) \doteq \int_\Omega \left(1 - \sqrt{1-|Dv|^2}\right) \mathrm{d}x - \langle \rho, v \rangle \tag{1.1}$$

($\langle \cdot, \cdot \rangle$ stands for the duality pairing) may not lead to a solution to ($\mathcal{BI}$), and the core problem is the lack of smoothness of the functional when $|Du| = 1$, in particular, the possible appearance of light segments in the graph of u. To the present, the literature on the existence and regularity problem for solutions to ($\mathcal{BI}$) is still fragmentary, and only a few classes of sources ρ, detailed below, were studied. In this paper, we investigate the problem for more general ρ and develop new tools to grasp the behavior of u both in the case of bounded domains and in the entire $\mathbb{R}^m$. Although we restrict our investigation to Minkowski space, we believe that some of our techniques might be extendable to more general ambient Lorentzian manifolds.

The Born-Infeld model. A further motivation for investigating the functional I_ρ comes from the Born-Infeld model of electromagnetism, proposed by M. Born and L. Infeld in [12, 13]. Concise but informative introductions can be found in [8, 9], see also [47, 30] for a thorough account of the physical literature. One of the main concerns of the theory was to overcome the failure of the principle of finite energy occurring in Maxwell's model, that we shall briefly recall. We remark that the Born-Infeld model also proved to be relevant in the theory of superstrings and membranes, see [27, 47] and the references therein.

In a spacetime (N^4, g) with metric $g = g_{ab}\mathrm{d}y^a \otimes \mathrm{d}y^b$ of signature $(-,+,+,+)$ ($g_{00} < 0$), the electromagnetic field is described as a closed 2-form $F = \frac{1}{2}F_{ab}\mathrm{d}y^a \wedge$

$\mathrm{d}y^b$ which, according to Maxwell's theory and in the absence of charges and currents, is required to be stationary for the action

$$\mathscr{L}_{\mathrm{M}} \doteq \int_{N^4} \mathsf{L}_{\mathrm{M}} \sqrt{-|g|}\mathrm{d}y \qquad \text{with} \quad \mathsf{L}_{\mathrm{M}} \doteq -\frac{F^{ab}F_{ab}}{4},$$

where $|g|$ is the determinant of g and $F^{ab} \doteq g^{ac}g^{bd}F_{cd}$. The presence of a vector field J describing charges and currents is taken into account by adding the Lagrangian

$$\mathscr{L}_J \doteq \int_{N^4} \mathsf{L}_J \sqrt{-|g|}\mathrm{d}y, \qquad \mathsf{L}_J = J^a\Phi_a,$$

where we assumed that F is globally exact and we set $F = \mathrm{d}\Phi$. By its very definition, the energy-impulse tensor T associated to $\mathscr{L}_{\mathrm{M}} + \mathscr{L}_J$ has components

$$T_{ab} = \frac{-2}{\sqrt{-|g|}} \frac{\partial((\mathsf{L}_{\mathrm{M}} + \mathsf{L}_J)\sqrt{-|g|})}{\partial g^{ab}} = F_{ac}F_{bp}g^{cp} - \frac{1}{4}F^{cp}F_{cp}g_{ab} + J^c\Phi_c g_{ab}$$

and in particular T_{00} describes the energy density. In Minkowski space $\mathbb{L}^4$, by writing in Cartesian coordinates $\{x^a\}$ the electromagnetic tensor in terms of the electric and magnetic fields $\mathbf{E} = E_j\mathrm{d}x^j$ and $\mathbf{B} = B_j\mathrm{d}x^j$ as

$$F = \sum_{j=1}^{3} E_j\mathrm{d}x^j \wedge \mathrm{d}x^0 + B_1\mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_2\mathrm{d}x^3 \wedge \mathrm{d}x^1 + B_3\mathrm{d}x^1 \wedge \mathrm{d}x^2,$$

the vector potential as $\Phi = -\varphi\mathrm{d}x^0 + \mathbf{A} = -\varphi\mathrm{d}x^0 + A_j\mathrm{d}x^j$ and $J = \rho\partial_{x^0} + \mathbf{J} = \rho\partial_{x^0} + J^j\partial_{x^j}$, the Maxwell Lagrangian and energy densities become

$$\mathsf{L}_{\mathrm{M}} + \mathsf{L}_J = \frac{1}{2}\big(|\mathbf{E}|^2 - |\mathbf{B}|^2\big) - \rho\varphi + \mathbf{A}(\mathbf{J}), \qquad T_{00} = \frac{1}{2}\big(|\mathbf{E}|^2 + |\mathbf{B}|^2\big) + \rho\varphi - \mathbf{A}(\mathbf{J}).$$

Restricting to the electrostatic case with no current density ($\mathbf{B} = 0$, $\mathbf{E}$ independent of x^0, $\mathbf{J} = 0$), from $\mathbf{E} = -\mathrm{d}\varphi$ the potential φ turns out to be stationary for the reduced action

$$J_\rho(v) \doteq \frac{1}{2}\int_{\mathbb{R}^3} |Dv|^2\mathrm{d}x - \langle \rho, v\rangle,$$

where $\langle \rho, v\rangle$ is the duality pairing given, for smooth ρ, by integration. However, for $\rho = \delta_{x_0}$ the Dirac delta centered at a point x_0, the Newtonian potential $\bar{u}_\rho = \mathrm{const}\cdot|x - x_0|^{2-m}$ solving the Euler-Lagrange equation $-\Delta\bar{u}_\rho = \rho$ for J_ρ has infinite energy on punctured balls centered at x_0:

$$\int_{B_R\setminus B_\varepsilon} T_{00}\mathrm{d}x = \frac{1}{2}\int_{B_R\setminus B_\varepsilon} |D\bar{u}_\rho|^2\mathrm{d}x \to \infty \qquad \text{as } \varepsilon \to 0,$$

a fact of serious physical concern (cf. [13]). The problem also persists for certain sources $\rho \in L^1(\mathbb{R}^m)$, see [22, 8]. To avoid it, Born and Infeld in [12] proposed to replace L_{M} with the Lagrangian density[1]

$$\mathsf{L}_{\mathrm{BI}} = 1 - \sqrt{1 + \frac{1}{2}F^{ab}F_{ab}},$$

an expression first suggested by the parallelism with the relativistic corrections to classical mechanics, and later derived from a general invariance principle [13]. In fact, other choices were also studied in [13]. In Minkowski space with Cartesian coordinates $\{x^a\}$,

$$\mathsf{L}_{\mathrm{BI}} = 1 - \sqrt{1 - |\mathbf{E}|^2 + |\mathbf{B}|^2},$$

so the energy-impulse tensor associated to $\mathscr{L}_{\mathrm{BI}} + \mathscr{L}_J$, and its component T_{00} in Cartesian coordinates, become

$$\begin{aligned} T_{ab} &= \mathsf{L}_{\mathrm{BI}} g_{ab} + \frac{F_{ac}F_{bp}g^{cp}}{\sqrt{1 + F_{cd}F^{cd}/2}} + J^c \Phi_c g_{ab}, \\ T_{00} &= \frac{1 + |\mathbf{B}|^2}{\sqrt{1 - |\mathbf{E}|^2 + |\mathbf{B}|^2}} - 1 + \rho\varphi - \mathbf{A}(\mathbf{J}). \end{aligned}$$

In the electrostatic case, the potential u_ρ generated by a charge ρ is therefore required to minimize the action I_ρ in (1.1) on $\Omega = \mathbb{R}^3$ among weakly spacelike functions with a suitable decay at infinity. It is easy to see that u_ρ exists and is unique (cf. [8] and Subsection 3.1). Formally, ($\mathcal{BI}$) is the Euler-Lagrange equation of I_ρ coupled with the physically meaningful condition $\lim_{x\to\infty}\psi(x) = 0$. The energy density of u_ρ is given by

$$T_{00} = \frac{1}{\sqrt{1 - |Du_\rho|^2}} - 1 + \rho u_\rho.$$

As shown in [13], the explicit solution generated by the distribution $\rho = \delta_{x_0}$ is bounded on $\mathbb{R}^3$ (thus, $\langle \rho, u_\rho\rangle$ is bounded) and satisfies

$$T_{00} - \rho u_\rho \in L^1(\mathbb{R}^3). \tag{1.2}$$

Remarkably, by [8, Proposition 2.7] property (1.2) holds for ρ lying in a large class of distributions including any finite measure on $\mathbb{R}^3$. Among the results proved in the present paper, we show that the same desirable property holds for solutions in bounded domains, that is, $T_{00} - \rho u_\rho \in L^1_{\mathrm{loc}}(\Omega)$ whenever the boundary data ϕ is not too degenerate. Since the local integrability of $T_{00} - \rho u_\rho$ is equivalent to that of

$$w_\rho \doteq \frac{1}{\sqrt{1 - |Du_\rho|^2}},$$

hereafter, with an abuse of notation, we will say that w_ρ is *the energy density* of u_ρ.

Notation and agreements.
Hereafter, we write ω_{m-1} for the volume of the unit sphere $\mathbb{S}^{m-1}$, and indicate with

[1]We followed the convention in [47], which changes signs in L_{BI} with respect to [13]. Also, we set the maximal field strength b to be 1 for convenience.

$\mathbb{1}_A$ the characteristic function of a set A. The subscript δ will denote quantities referred to the Euclidean metric on $\mathbb{R}^m$: d_δ will be the Euclidean distance, $\mathrm{diam}_\delta(E)$ the diameter of a set $E \subset \mathbb{R}^m$ and $|\cdot|_\delta$, $\mathscr{H}^{m-1}_\delta$ the volume and $(m-1)$-dimensional Hausdorff measure in d_δ. Given $x, y \in \mathbb{R}^m$, we let $\overline{xy}$ be the closed segment joining x and y. If $\Omega \subset \mathbb{R}^m$ is an open set, we denote by $\mathcal{M}(\Omega)$ the set of all finite (signed) Borel measures on Ω equipped with the total variation norm $\|\cdot\|_{\mathcal{M}(\Omega)}$. The set $\mathrm{Lip}_c(\Omega)$ will denote the set of Lipschitz functions with compact support in Ω, and we write $\Omega' \Subset \Omega$ when Ω' has compact closure in Ω.

1.1. **Known results for bounded domains.** After work of F. Flaherty [21] for maximal hypersurfaces ($\rho = 0$), solutions to ($\mathcal{BI}$) in bounded domains Ω and for sources $\rho \in L^\infty(\Omega)$ were studied in depth in the influential work by R. Bartnik and L. Simon [4]. To describe the main result therein, for $\phi \in C(\partial\Omega)$, we define

$$\mathcal{Y}_\phi(\Omega) \doteq \Big\{ u \in W^{1,\infty}(\Omega) : u \text{ weakly spacelike}, u = \phi \text{ on } \partial\Omega \Big\}. \tag{1.3}$$

Remark 1.1. We assumed no regularity of $\partial\Omega$, so the boundary condition has to be intended as in [4]: $u = \phi$ on $\partial\Omega$ iff, for each $x \in \partial\Omega$ and any straight line $\gamma : (0,1) \to \Omega$ with $\gamma(0^+) = x$, it holds $u(\gamma(t)) \to \phi(x)$ as $t \to 0^+$. In Proposition 3.5 below, we will prove that this definition suffices to guarantee that functions $u \in \mathcal{Y}_\phi(\Omega)$ can be extended continuously on $\partial\Omega$ with value ϕ.

The class of boundary data for which $\mathcal{Y}_\phi(\Omega) \neq \emptyset$ was characterized in [4, p. 149] in terms of the function

$$\mathrm{d}_{\overline{\Omega}}(x,y) \doteq \inf \Big\{ \mathrm{length}(\gamma) \ : \ \gamma \in \Gamma_{x,y} \Big\} \le +\infty \qquad \forall\, x, y \in \overline{\Omega}, \tag{1.4}$$

where

$$\Gamma_{x,y} = \Big\{ \gamma \in C([0,1], \overline{\Omega}) \ : \ \gamma((0,1)) \subset \Omega, \ \ \gamma \text{ piecewise affine and } \gamma(0) = x, \gamma(1) = y \Big\},$$

the infimum is defined to be $+\infty$ if $\Gamma_{x,y} = \emptyset$, and γ is called piecewise affine if it consists of finitely many intervals where it is affine. In fact, it is showed in [4, p. 149] that

$$\mathcal{Y}_\phi(\Omega) \neq \emptyset \qquad \Longleftrightarrow \qquad |\phi(x) - \phi(y)| \le \mathrm{d}_{\overline{\Omega}}(x,y) \quad \forall\, x, y \in \partial\Omega.$$

Note that the restriction d_Ω of $\mathrm{d}_{\overline{\Omega}}$ to $\Omega \times \Omega$ gives the intrinsic metric on Ω. Remarks on the relation between $\mathrm{d}_{\overline{\Omega}}(x,y)$ for $x, y \in \partial\Omega$ and the distance in the metric completion of $(\Omega, \mathrm{d}_\Omega)$ will be given in Subsection 3.2.

Next, we introduce a class of weak solutions to ($\mathcal{BI}$) in bounded domains.

Definition 1.2. Let Ω be a bounded domain in $\mathbb{R}^m$. For $\rho \in W^{1,\infty}(\Omega)^*$, a *weak solution to* ($\mathcal{BI}$) is a function $u \in \mathcal{Y}_\phi(\Omega)$ such that

$$\text{(i)} \quad w \doteq \frac{1}{\sqrt{1 - |Du|^2}} \in L^1_{\mathrm{loc}}(\Omega) \qquad \text{and}$$

$$\text{(ii)} \quad \int_\Omega \frac{Du \cdot D\eta}{\sqrt{1 - |Du|^2}} \mathrm{d}x = \langle \rho, \eta \rangle \qquad \forall\, \eta \in \mathrm{Lip}_c(\Omega).$$

Given a subdomain $\Omega' \subset \Omega$, we say that u *weakly solves* ($\mathcal{BI}$) on Ω' if $w \in L^1_{\rm loc}(\Omega')$ and (ii) holds for $\eta \in \mathrm{Lip}_c(\Omega')$.

Equation ($\mathcal{BI}$) is formally the Euler-Lagrange equation for the functional

$$I_\rho \ : \ \mathcal{Y}_\phi(\Omega) \to \mathbb{R}, \qquad I_\rho(v) \doteq \int_\Omega \Big(1 - \sqrt{1-|Dv|^2}\Big)\mathrm{d}x - \langle \rho, v\rangle. \tag{1.5}$$

Although, for ρ lying in a large subset of $W^{1,\infty}(\Omega)^*$, the variational problem for I_ρ admits a unique minimizer u_ρ (cf. Subsection 3.1), the example of a hyperplane with slope 1 and $\rho = 0$ indicates that the requirement $\mathcal{Y}_\phi(\Omega) \neq \emptyset$ does not suffice to guarantee that u_ρ solves ($\mathcal{BI}$) (see K. Ecker [17]). In this respect, note that any solution to ($\mathcal{BI}$) is easily seen to coincide with the minimizer u_ρ (cf. Proposition 3.14 below). In [4, Theorem 4.1 and Corollaries 4.2, 4.3], the authors obtained the following striking result:

Theorem 1.3. [4] *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, and let $\phi \in C(\partial\Omega)$. The following properties are equivalent:*

(i) *ϕ admits a spacelike extension on Ω, that is, there exists $\bar\phi \in \mathcal{Y}_\phi(\Omega)$ which is spacelike on Ω;*
(ii) *$|\phi(x) - \phi(y)| < \mathrm{d}_{\overline{\Omega}}(x, y)$ for every $x, y \in \partial\Omega$, $x \neq y$;*
(iii) *for each $\rho \in L^\infty(\Omega)$, there exists $u \in C^1(\Omega) \cap W^{2,2}(\Omega)$, which is strictly spacelike and weakly solves ($\mathcal{BI}$).*

We therefore define the set

$$\mathcal{S}(\partial\Omega) \doteq \Big\{ \phi \in C(\partial\Omega) \ : \ \text{any among (i), (ii), (iii) in Theorem 1.3 holds} \Big\}.$$

Remark 1.4. No regularity of Ω is assumed in Theorem 1.3. This is quite a contrast with the linear problem $-\Delta u = \rho$ in Ω, $u = \phi$ on $\partial\Omega$, for which we need certain regularity properties of $\partial\Omega$, and comes from the strong restriction $u \in W^{1,\infty}(\Omega)$ for ($\mathcal{BI}$).

Remark 1.5. In a broader setting, the equivalence (i) $\Leftrightarrow$ (ii) was studied in [33, Theorem 1].

Theorem 1.3 does not contain the full generality of the statements in [4]. Indeed, under the only assumption $\mathcal{Y}_\phi(\Omega) \neq \emptyset$ the authors showed that the minimizer u_ρ is strictly spacelike on the complement of the set

$$K^\rho_\phi \doteq \overline{\bigcup \Big\{ \overline{xy} \ : \ x, y \in \Omega, \ x \neq y, \ \overline{xy} \subset \Omega, \ |u_\rho(x) - u_\rho(y)| = |x - y| \Big\}},$$

hence it solves ($\mathcal{BI}$) on $\Omega \setminus K^\rho_\phi$. Note that the condition $|Du_\rho| \le 1$ forces u_ρ to be affine with slope 1 on any $\overline{xy} \subset K^\rho_\phi \cap \Omega$, so the graph of u_ρ has a light segment over $\overline{xy}$. With a slight abuse of notation, in such case we call $\overline{xy}$ a *light segment*, and K^ρ_ϕ the set of light segments of u_ρ. A key fact proved in [4, Theorem 3.2] is that when $\rho \in L^\infty(\Omega)$, every light segment has to extend up to $\partial\Omega$, a property called there the *anti-peeling Theorem*. The proof depends on a comparison argument that is not applicable to more general sources ρ, in which case, to our knowledge, the relationship between singularities of ρ and properties of light segments, including their existence, is currently unknown. As we shall see below, its understanding is one of the core issues to obtain sharp regularity results.

For the study of hypersurfaces with $\rho \in L^\infty(\Omega)$ on more general ambient Lorentzian manifolds, we suggest to consult the works of K. Gerhardt [26] and Bartnik [5]. Moving to more singular $\rho \in \mathcal{M}(\Omega)$, juxtaposition of point charges were treated in depth in a series of works by V. Miklyukov and V.A. Klyachin [33, 34, 31]. We quote in particular [34, Theorem 2], that we rephrase as follows:

Theorem 1.6 ([34]). *Let $\Omega \subset \mathbb{R}^m$ be a domain such that $(\Omega, \mathrm{d}_\Omega)$ has compact completion, and let $\phi \in \mathcal{S}(\partial\Omega)$. Fix a k-tuple of points $\mathscr{P} = (x_1, \dots, x_k) \in \Omega \times \ldots \times \Omega$. Then, there exists a constant $M_m(\phi, \mathscr{P})$ such that, for each $a \doteq (a_1, \dots, a_k) \in \mathbb{R}^k$ satisfying $|a| < M_m(\phi, \mathscr{P})$, the minimizer u_ρ with source*

$$\rho = \sum_{j=1}^{k} a_j \delta_{x_j}$$

solves ($\mathcal{BI}$) and it is strictly spacelike (hence, smooth) on $\Omega \backslash \mathscr{P}$. Furthermore, $M_2(\phi, \mathscr{P}) = +\infty$.

The above result also contains a lower bound for $M_m(\phi, \mathscr{P})$ when $m \geq 3$, which depends on the solution to ($\mathcal{BI}$) with $\rho = 0$, on $\{x_1, \dots, x_k\}$ and on the geometry of Ω.

The case $m = 2$ is rather special and, indeed, maximal surfaces with singularities in $\mathbb{L}^3$ were also studied from a different point of view by using complex-analytic tools (cf. [18, 20]). Exploiting Weierstrass data, [35, 43, 23] described in detail classes of maximal surfaces whose singular set is suitably controlled. It should be pointed out that, in the works cited below, the authors consider the equation

$$(1 - |Du|^2)^{3/2} \mathrm{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) = (1 - |Du|^2)^{3/2} H, \qquad H \in \mathbb{R},$$

for which the role of light segments may be different. Examples of maximal surfaces in $\mathbb{L}^3$ whose singular set contains an entire light line were constructed in [24, 45, 2], while an investigation of points at which Du_ρ is light-like can be found in [32, 44, 45]. The behavior near isolated singularities of surfaces with nonconstant, smooth ρ was characterized in [25]. To the best of our knowledge, whether or not the singular sets described in the above mentioned references induce a singular measure in the mean curvature ρ, and which kind of measure, is a problem that is not considered yet.

1.2. **Our contributions for bounded domains.** From a variational point of view, even though the minimizer u_ρ for I_ρ in (1.5) may not solve ($\mathcal{BI}$) weakly, if $\phi \in \mathcal{S}(\partial\Omega)$ then u_ρ enjoys nice properties for each reasonably well-behaved source ρ, including signed Radon measures. Inspired by [8], we prove in Proposition 3.9 that the energy density of u_ρ is locally integrable, namely

$$w_\rho = \frac{1}{\sqrt{1-|Du_\rho|^2}} \in L^1_{\mathrm{loc}}(\Omega),$$

and in particular $|Du_\rho| < 1$ a.e. on Ω; moreover,

$$\int_\Omega \frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1-|Du_\rho|^2}} \mathrm{d}x \le \langle \rho, u_\rho - \psi \rangle \qquad \forall \psi \in \mathcal{Y}_\phi(\Omega), \tag{1.6}$$

where the integrand in the LHS is shown to belong to $L^1(\Omega)$. As we shall see in Proposition 3.14, u_ρ weakly solves ($\mathcal{BI}$) if and only if equality holds in (1.6), a fact that is not obvious in view of the lack of regularity of $\partial\Omega$ and of ϕ.

Next, we investigate the relation between the integrability of ρ and the possible existence of a light segment in the graph of u_ρ. Putting together Propositions 4.3 and A.1, respectively for $\ell = 1$ and for $\ell \in \{2, \dots, m-2\}$, we prove the following

Proposition 1.7. *For each $m \ge 3$ and $\ell \in \{1, \dots, m-2\}$, there exists a function $u \in C^2_c(\mathbb{R}^m)$ with the following properties:*

(i) *the set K of light segments of u is a closed cylinder $\overline{B}^{\ell-1} \times [a, b]$ in a totally geodesic ℓ-plane of $\mathbb{R}^m$ (in particular, if $\ell = 1$ it is a single light segment), and $|Du| < 1$ on $\mathbb{R}^m \backslash K$;*

(ii) *u satisfies*

$$\int_{\mathbb{R}^m} \frac{Du \cdot D\eta}{\sqrt{1-|Du|^2}} \mathrm{d}x = \int_{\mathbb{R}^m} \rho_u \eta \, \mathrm{d}x \qquad \forall \eta \in \mathrm{Lip}_c(\mathbb{R}^m),$$

where $\rho_u \in L^q(\mathbb{R}^m)$ for each $q < m - \ell$. In particular, if $\Omega \subset \mathbb{R}^m$ is a smooth open subset containing the support of u, then u weakly solves ($\mathcal{BI}$) with $\phi \equiv 0$ and $\rho = \rho_u$;

(iii) *for each $q < m - \ell$, it holds*

$$w, \quad w|D^2u|, \quad w^2|D^2u(Du, \cdot)|, \quad w^3 D^2u(Du, Du) \quad \in L^q(\mathbb{R}^m),$$

where $w = (1-|Du|^2)^{-1/2}$ is the energy density of u.

The above construction also allows us to provide examples of minimizers u_ρ that *do not solve* ($\mathcal{BI}$), even though the source ρ is rather mild. In Theorem 5.5, we shall prove the following result:

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^m$ be either a bounded domain or $\Omega = \mathbb{R}^m$. In the first case, let $\phi \in \mathcal{S}(\partial\Omega)$. Let u_ρ be a minimizer for I_ρ and assume that u_ρ has a light segment $\overline{xy} \subset \Omega$ with $u_\rho(y) - u_\rho(x) = |y - x|$. Then, for each $\alpha > 0$, u_ρ also minimizes the functional I_{ρ_α} with*

$$\rho_\alpha = \rho + \alpha(\delta_y - \delta_x)$$

but it does not solve ($\mathcal{BI}$) weakly for ρ_α.

Applying Theorem 1.8 to the example in Proposition 1.7, we have

Corollary 1.9. *There exists a smooth open set $\Omega \Subset \mathbb{R}^m$, a function $u \in C^2_c(\Omega) \cap \mathcal{Y}_0(\Omega)$, points $x, y \in \Omega$ with $x \neq y$ and a function $\rho_{\mathrm{AC}} \in L^q(\Omega)$ for any $q < m-1$, such that the following properties hold:*

(i) *$\overline{xy}$ is a light segment for u, and $|Du| < 1$ on $\Omega \backslash \overline{xy}$;*
(ii) *u minimizes I_ρ with source*

$$\rho = \alpha(\delta_y - \delta_x) + \rho_{\mathrm{AC}}, \qquad \text{for each fixed } \alpha \in \mathbb{R}^+,$$

but it does not solve ($\mathcal{BI}$) weakly.

Observe that Corollary 1.9 makes it impossible to extend Theorem 1.6 (i.e. [34, Theorem 2]) for dimension $m \geq 3$ to more general sources of the type

$$\rho = \sum_{j=1}^{k} a_j \delta_{x_j} + \rho_{\mathrm{AC}} \qquad \text{with } \rho_{\mathrm{AC}} \in L^q(\Omega),\ q < m-1.$$

We next move to results that guarantee the solvability of ($\mathcal{BI}$). To get elliptic estimates, our boundary data shall be restricted to compact subsets $\mathscr{F} \subset \mathcal{S}(\partial\Omega)$ with respect to uniform convergence. Examples of $\mathscr{F}$ include a singleton $\{\phi\}$ and the sets of uniformly bounded c-Lipschitz functions on $\partial\Omega$ with respect to d_δ with $c < 1$. A more general example, $\mathcal{S}_{b,\zeta}(\partial\Omega)$, will be defined for given $b \in \mathbb{R}^+$ and $\zeta : \mathbb{R}^+ \to [0, 1)$ under the assumption that the metric space $(\Omega, \mathrm{d}_\Omega)$ has compact completion, and will be studied in Subsection 3.2.

We first consider the 2-dimensional case.

Theorem 1.10. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain, and let $\Sigma \Subset \Omega$ be a compact subset satisfying $\mathscr{H}^1_\delta(\Sigma) = 0$. Suppose that $\rho \in \mathcal{M}(\Omega)$ decomposes as*

$$\rho = \rho_{\mathrm{S}} + \rho_{\mathrm{AC}}, \qquad \text{with } \begin{cases} \operatorname{supp} \rho_{\mathrm{S}} \subset \Sigma \\ \rho_{\mathrm{AC}} \in L^1(\Omega) \cap L^2_{\mathrm{loc}}(\Omega \backslash \Sigma). \end{cases}$$

Then,

(i) *for each $\phi \in \mathcal{S}(\partial\Omega)$, the minimizer $u_\rho \in \mathcal{Y}_\phi(\Omega)$ weakly solves ($\mathcal{BI}$) in Ω and does not have light segments;*
(ii) *for any given compact set $\mathscr{F} \subset \mathcal{S}(\partial\Omega)$, $\mathcal{I}_1, \mathcal{I}_2, \varepsilon > 0$, $q_0 \geq 0$, and any given open set $\Omega' \Subset \Omega \backslash \Sigma$ satisfying*

$$\|\rho\|_{\mathcal{M}(\Omega)} \leq \mathcal{I}_1, \qquad \|\rho\|_{L^2(\Omega')} \leq \mathcal{I}_2,$$

there exists a constant $\mathcal{C} = \mathcal{C}\big(\Omega, \mathscr{F}, q_0, \mathrm{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, \mathrm{d}_\delta(\Omega', \partial\Omega), \Omega'\big)$ such that, for each $\phi \in \mathscr{F}$, it holds

$$\int_{\Omega'_\varepsilon} (1 + \log w_\rho)^{q_0} \Big\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 \left| D^2 u_\rho \left(Du_\rho, \cdot \right) \right|^2 + w_\rho^5 \left[D^2 u_\rho (Du_\rho, Du_\rho) \right]^2 \Big\} \mathrm{d}x + \int_{\Omega'_\varepsilon} w_\rho (1 + \log w_\rho)^{q_0+1} \mathrm{d}x \leq \mathcal{C},$$

where $\Omega'_\varepsilon \doteq \{x \in \Omega' : \mathrm{d}_\delta(x, \partial\Omega') > \varepsilon\}$;

(iii) *if* $\Omega' \Subset \Omega \setminus \Sigma$ *and* $\rho \in L^\infty(\Omega')$, *then* $u_\rho \in C^{1,\alpha}_{\mathrm{loc}}(\Omega')$ *for some* $\alpha > 0$. *In particular, if* $\rho \in C^\infty(\Omega')$ *so is* u_ρ.

Remark 1.11. If ρ_S is a sum of Dirac deltas and $\rho_{AC} = 0$, we recover the result by Klyachin-Miklyukov (see Theorem 1.6). However, we stress that our proof is completely different. Indeed, the clever proof in [34] is quite specific to Dirac delta singularities, and it seems difficult to extend to sources whose absolutely continuous part is not in L^∞.

Remark 1.12. Regarding the second order regularity of u, for general ρ one cannot expect $u_\rho \in W^{2,q}_{\mathrm{loc}}$ for $q \geq 1$, see the discussion after Example 5.6.

We briefly overview the strategy of the proof, that relies on several steps. We refer to $\Omega, \mathscr{F}, \mathrm{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \mathrm{d}_\delta(\Omega', \partial\Omega)$ in (ii) as being the *data* of our problem, and fix $\varepsilon > 0$. Hereafter, a constant C will be assumed to depend on the data. We proceed by approximating ρ via convolution to get $\rho_j \rightharpoonup \rho$ weakly in $\mathcal{M}(\Omega)$, let $u_j \in \mathcal{Y}_\phi(\Omega)$ minimize I_{ρ_j} and denote by $w_j \doteq (1 - |Du_j|^2)^{-1/2}$ its energy density. First, we show the following two properties:

($\mathscr{P}0_1$) Proposition 5.10 and Corollary 5.11 (**local second fundamental form estimate**): the squared norm of the second fundamental form II_j for the graph of u_j over Ω satisfies

$$\int_{\Omega'_{\varepsilon/2}} \| \mathrm{II}_j \|^2 w_j^{-1} \mathrm{d}x \leq C;$$

($\mathscr{P}0_2$) Lemma 5.4 (**energy estimate**): on Euclidean balls B_r contained in $\Omega'_{\varepsilon/2}$,

$$\int_{B_r} w_j \mathrm{d}x \leq Cr.$$

Properties ($\mathscr{P}0_1$) and ($\mathscr{P}0_2$) hold in any dimension $m \geq 2$. We stress that, writing II_j in terms of u_j as in (2.4), ($\mathscr{P}0_1$) implies bounds on the derivative of the energy density w_j. For the surface case $m = 2$, ($\mathscr{P}0_1$) and ($\mathscr{P}0_2$) imply

($\mathscr{P}1$) Theorem 5.12 (**higher integrability for** $m = 2$):

$$\int_{\Omega'_\varepsilon} w_j \log w_j \leq C.$$

The uniform integrability of $\{w_j\}$ granted by ($\mathscr{P}1$) enables us to show

($\mathscr{P}2$) Step 2 in Proof of Theorem 1.10 (**no-light-segment**): u_ρ has no light segments in Ω' (the statement is quantitative in terms of the data).

With the aid of ($\mathscr{P}2$), we can then refine the integral estimates leading to ($\mathscr{P}0_1$) as follows.

($\mathscr{P}3$) Theorem 5.13 (**higher integrability and second fundamental form estimates**): for each $q_0 \geq 0$,

$$\int_{\Omega'_\varepsilon} \left\{ w_j \log w_j + \| \mathrm{II}_j \|^2 w_j^{-1} \right\} \log^{q_0} w_j \mathrm{d}x \leq C, \tag{1.7}$$

where C also depends on q_0 (and on Ω' in a subtler way). Item (ii) in Theorem 1.10 follows from (1.7), which is technically one of the core parts of the paper. It is important to notice that ($\mathscr{P}3$) holds in a given dimension m provided that so does ($\mathscr{P}2$), and in particular, the higher integrability of w_j does not depend on ($\mathscr{P}1$). To the present, we are able to prove ($\mathscr{P}2$) only in dimension $m = 2$, and the example in Proposition 1.7 shows the possible failure of ($\mathscr{P}2$) in dimension $m \geq 4$ when $\rho \in L^2(\Omega')$.

Also, Item (iii) in Theorem 1.10 follows from ($\mathscr{P}2$) by applying arguments in [4]. To prove Item (i) we need one last piece of information. Clearly, ($\mathscr{P}2$) and the fact that $\mathscr{H}^1_\delta(\Sigma) = 0$ guarantee that u_ρ does not have light segments on the entire Ω. However, the local uniform integrability of $\{w_j\}$ on each $\Omega' \Subset \Omega\backslash\Sigma$ implies

$$\int_\Omega w_\rho Du_\rho \cdot D\eta = \langle \rho, \eta\rangle \qquad \forall\, \eta \in \mathrm{Lip}_c(\Omega\backslash\Sigma).$$

To extend the above identity to test functions $\eta \in \mathrm{Lip}_c(\Omega)$, we shall prove the following removable singularity property.

($\mathscr{P}4$) Theorem 5.2 (**removable singularity**): if $\{w_j\}$ is locally uniformly integrable on $\Omega\backslash\Sigma$ and $\mathscr{H}^1_\delta(\Sigma) = 0$, then u_ρ solves weakly ($\mathcal{BI}$).

In higher dimensions, the possible failure of ($\mathscr{P}2$) makes it necessary to investigate the set of light segments K^ρ_ϕ of u_ρ. With the aid of Theorem 5.13, however, outside of K^ρ_ϕ we can still deduce a few properties of u_ρ:

Theorem 1.13. *Let $m \geq 3$ and $\Omega \subset \mathbb{R}^m$ be a domain, $\Sigma \Subset \Omega$ be compact and $\rho \in \mathcal{M}(\Omega)$ satisfy $\mathscr{H}^1_\delta(\Sigma) = 0$ and*

$$\rho = \rho_{\mathrm{S}} + \rho_{\mathrm{AC}}, \qquad \textit{with} \quad \left\{ \begin{array}{l} \operatorname{supp} \rho_{\mathrm{S}} \subset \Sigma, \\ \rho_{\mathrm{AC}} \in L^1(\Omega) \cap L^2_{\mathrm{loc}}(\Omega\backslash\Sigma). \end{array}\right.$$

Given $\phi \in \mathcal{S}(\partial\Omega)$, consider the set of light segments of the minimizer $u_\rho \in \mathcal{Y}_\phi(\Omega)$:

$$K^\rho_\phi = \overline{\bigcup \left\{ \overline{xy} \ : \ x, y \in \Omega,\ x \neq y,\ \overline{xy} \subset \Omega,\ |u_\rho(x) - u_\rho(y)| = |x-y| \right\}}.$$

Then,

(i) *u_ρ weakly solves ($\mathcal{BI}$) on $\Omega\backslash K^\rho_\phi$.*
Moreover, if $K^\rho_\phi \cap (\partial\Omega \cup \Sigma) = \emptyset$, then u_ρ weakly solves ($\mathcal{BI}$) on the entire Ω.

(ii) *For each $\Omega' \Subset \Omega\backslash(\Sigma \cup K^\rho_\phi)$ and $q_0 \geq 0$,*

$$\int_{\Omega'} (1+\log w_\rho)^{q_0} \left\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 \left| D^2 u_\rho \left(Du_\rho, \cdot \right)\right|^2 + w_\rho^5 \left[D^2 u_\rho(Du_\rho, Du_\rho)\right]^2 \right\} \mathrm{d}x$$
$$+ \int_{\Omega'} w_\rho (1+\log w_\rho)^{q_0+1} \mathrm{d}x < \infty.$$

(iii) *If $\Omega' \Subset \Omega \setminus (\Sigma \cup K^\rho_\phi)$ and $\rho \in L^\infty(\Omega')$, then $u_\rho \in C^{1,\alpha}_{\mathrm{loc}}(\Omega')$ for some $\alpha > 0$.*
In particular, if $\rho \in C^\infty(\Omega')$ so is u_ρ.

Remark 1.14. Corollary 1.9 shows that, in dimension $m \geq 4$, there exists $\rho_{\mathrm{AC}} \in L^2(\Omega)$ and $\rho_{\mathrm{S}} = \delta_y - \delta_x$ such that $u_\rho \in \mathcal{Y}_0(\Omega)$ does not solve ($\mathcal{BI}$) weakly on the entire Ω. Notice that the support $\Sigma = \{x, y\}$ of ρ_{S} satisfies $\Sigma \subset K^\rho_\phi$, and therefore condition $K^\rho_\phi \cap \Sigma = \emptyset$ in (i) of Theorem 1.13 cannot be removed.

1.3. **Known results for $\Omega = \mathbb{R}^m$.** The picture for constant ρ on the entire $\mathbb{R}^m$ is by now well understood. Thanks to E. Calabi [15], S.Y. Cheng and S.T. Yau [16] and Bartnik (Ecker [17, Theorem F]), we know that if $u : \mathbb{R}^m \to \mathbb{R}$ minimizes I_0 (i.e. $\rho = 0$) on each open subset $\Omega \Subset \mathbb{R}^m$ with respect to compactly supported variations in Ω, then u is a hyperplane, possibly with slope 1. Note that no growth conditions on u are imposed a-priori. On the contrary, many examples of smooth spacelike graphs with constant $\rho \neq 0$ were constructed in [41, 42].

In view of applications to Born-Infeld theory, we study I_ρ in $\mathbb{R}^m$ with $m \geq 3$ and for functions decaying at infinity to zero, taking advantage of the different functional settings described by M.K.H. Kiessling in [30] and D. Bonheure, P. d'Avenia and A. Pomponio in [8]. For our purposes, we mildly modify their frameworks and define in Subsection 3.1 a Banach space $\mathcal{Y}(\mathbb{R}^m)$ in such a way that I_ρ is well defined on

$$\mathcal{Y}_0(\mathbb{R}^m) \doteq \Big\{ v \in \mathcal{Y}(\mathbb{R}^m) \ : \ \|Dv\|_\infty \leq 1 \Big\},$$

and so that the latter is closed (and convex) in $\mathcal{Y}(\mathbb{R}^m)$. Our choice does not affect the functional properties of I_ρ showed in [8]: in particular, following [8, Lemma 2.2], I_ρ has a unique minimizer $u_\rho \in \mathcal{Y}_0(\mathbb{R}^m)$ which, by [8, Proposition 2.7] (cf. also Proposition 3.9 herein), satisfies

$$T_{00} - \rho u_\rho = \frac{|Du_\rho|^2}{\sqrt{1 - |Du_\rho|^2}} \in L^1(\mathbb{R}^m) \tag{1.8}$$

and the variational inequality

$$\int_{\mathbb{R}^m} \frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1 - |Du_\rho|^2}} \mathrm{d}x \leq \big\langle \rho, u_\rho - \psi \big\rangle \qquad \forall \psi \in \mathcal{Y}_0(\mathbb{R}^m). \tag{1.9}$$

Note that from (1.8) we deduce $w_\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. We then say that u_ρ *weakly solves* ($\mathcal{BI}$) if

$$\int_{\mathbb{R}^m} \frac{Du_\rho \cdot D\eta}{\sqrt{1 - |Du_\rho|^2}} \mathrm{d}x = \langle \rho, \eta \rangle \qquad \forall \eta \in \mathrm{Lip}_c(\mathbb{R}^m).$$

Even though the literature on the regularity theory for u_ρ in the entire $\mathbb{R}^m$ is more extensive than the one in bounded domains, only a few classes of ρ were investigated in detail. Among them, u_ρ was shown to solve ($\mathcal{BI}$) weakly whenever $\rho \in \mathcal{Y}(\mathbb{R}^m)^*$ satisfies any of the following assumptions:

(i) ρ is radial ([8, Theorem 1.4]);
(ii) $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^m)$ ([8, Theorem 1.5]). In this case, u_ρ is locally strictly spacelike and thus $u_\rho \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^m)$ for some $\alpha > 0$, by the regularity theory for quasilinear equations.

(iii) $\rho \in L^q(\mathbb{R}^m) \cap L^p(\mathbb{R}^m)$ for $q > m$ and $p \in [1, 2_*]$ ([29, Theorem 1.3] and [11, Theorem 1.4 and Corollary 1.5]), see below.

Here and in what follows,

$$2_* \doteq \frac{2m}{m+2}$$

is the conjugate exponent of the Sobolev one 2^*.

The case of point charges.
The problem for

$$\rho = \sum_{i=1}^{k} a_i \delta_{x_i} \tag{1.10}$$

was treated in [7, 8]: in particular, see [7, Theorem 1.2], u_ρ was shown to be locally strictly spacelike (hence, smooth) away from the charges $\{x_i\}$ provided that the points x_i are sufficiently far away depending on the sizes a_i, in the quantitative way recalled in Remark 1.17 below. In this case, u_ρ weakly (indeed, classically) solves ($\mathcal{BI}$) on $\mathbb{R}^m \setminus \{x_1, x_2, \ldots, x_k\}$. However, in [7, 8] the authors did not prove equality in (1.9) for test functions which do not vanish at x_i, see [8, Remark 4.4] for more detailed comments.

In [30] Kiessling claimed that for ρ as in (1.10) u_ρ satisfies ($\mathcal{BI}$) without any restriction on the charges a_i. However, in [8] Bonheure, d'Avenia and Pomponio pointed out a flaw in his subtle argument, and Kiessling later published the erratum [30]. Kiessling's method uses a dual approach, and it would be desirable to have a proof with a direct use of the functional I_ρ.

The case $\rho \in L^q$ for large q.
It is natural to seek a sharp condition on ρ that guarantees the strict spacelikeness of u_ρ and $u_\rho \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^m)$ for some $\alpha \in (0,1)$. The investigation of the radial case in [8, Section 3] suggests that $\rho \in L^q_{\mathrm{loc}}(\mathbb{R}^m)$ with $q > m$ would be sufficient. This evidence, further motivated by the detailed discussion in the Introduction of [10], led Bonheure and A. Iacopetti to formulate the following

Conjecture (Conjecture 1.4 in [10])**.** *If $m \geq 3$ and $\rho \in \mathcal{Y}^* \cap L^q_{\mathrm{loc}}(\mathbb{R}^m)$ with $q > m$, then u_ρ is strictly spacelike on $\mathbb{R}^m$ and $u_\rho \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^m)$ for some $\alpha \in (0,1)$.*

Here, $\mathcal{Y}^*$ is the dual of a functional space $\mathcal{Y}$ where $\mathcal{Y}_0(\mathbb{R}^m)$ embeds as a closed, convex set, and can be taken to be $\mathcal{Y}(\mathbb{R}^m)^*$. In fact, in the stated assumptions on ρ, $C^{1,\alpha}_{\mathrm{loc}}$ regularity easily follows from strict spacelikeness by standard theory of quasilinear equations.

To the present, a complete answer to the conjecture is still unknown. After a first partial result in [10], which is in itself remarkable, an almost exhaustive positive answer was given by the combined efforts of A. Haarala [29] and Bonheure–Iacopetti [11]:

Theorem 1.15 (Theorem 1.3 in [29], Theorems 1.4 and 1.5 in [11])**.** *Assume $m \geq 3$ and $\rho \in L^q(\mathbb{R}^m) \cap L^p(\mathbb{R}^m)$ with $p \in [1, 2_*]$ and $q > m$. Then, u_ρ is strictly spacelike* *and*

$$u_\rho \in C^{1,1-\frac{m}{q}}_{\mathrm{loc}}(\mathbb{R}^m) \cap W^{2,q}_{\mathrm{loc}}(\mathbb{R}^m).$$

Furthermore, u_ρ weakly solves ($\mathcal{BI}$).

Note that the restriction $p \in [1, 2_*]$ is to guarantee that ρ defines a continuous functional. The proof of the theorem is deep, and combines different ingredients that are of independent interest. We emphasize that the global L^q integrability of ρ is fundamental at various stages of the proofs in [29, 11], and hence, the case $\rho \in L^q_{\mathrm{loc}}(\mathbb{R}^m)$ remains an open problem.

1.4. **Our contributions for $\Omega = \mathbb{R}^m$.** We first address the problem with a superposition of point charges. With the aid of Theorem 5.2 (removable singularity) and Theorem 5.13 (higher integrability), we can complement the works in [7, 8] and prove that u_ρ weakly solves ($\mathcal{BI}$) on the entire $\mathbb{R}^m$:

Theorem 1.16. *Let ρ be as in (1.10). If the minimizer u_ρ does not have any light segment, then u_ρ weakly solves ($\mathcal{BI}$). Furthermore, around x_i, u_ρ is asymptotic to a light cone in the sense of [17], where the cone is future (respectively, past) pointing provided that $a_i < 0$ (respectively, $a_i > 0$).*

Remark 1.17. According to [7, Proof of Theorem 1.2], u_ρ has no light segments whenever

$$\left(\frac{m}{\omega_{m-1}}\right)^{\frac{1}{m-1}} \frac{m-1}{m-2}\left[\left(\sum_{i\in I_-}|a_i|\right)^{\frac{1}{m-1}} + \left(\sum_{i\in I_+}|a_i|\right)^{\frac{1}{m-1}}\right] < \min_{i\neq j}|x_i - x_j|, \tag{1.11}$$

where I_+ (I_-) is the set of indices for which $a_i > 0$ ($a_i < 0$).

The last part of Theorem 1.16 needs some comment. In [17], Ecker defined an *isolated singularity* for

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0 \qquad \text{on an open set } B$$

as being a point $x_0 \in B$ such that u minimizes I_0 on any $\Omega' \Subset B\backslash\{x_0\}$ (that is, among functions in $\mathcal{Y}_{u_\rho}(\Omega')$), but not on the entire B. He then proves in [17, Theorem 1.5] that an isolated singularity is asymptotic to a future or past pointing light cone centered at x_0. As a direct application of Ecker's result, in [7, Theorem 3.5] (see also [8, Theorem 1.5]) the authors claim that, for ρ as in (1.10) and $\{x_i\}$, $\{a_i\}$ matching (1.11), near x_i, u_ρ is asymptotic to a light cone which is upward or downward pointing according to whether $a_i < 0$ or $a_i > 0$. However, without knowing the validity of the Euler-Lagrange equation around x_i, it is not clear to us how to exclude the possibility that u_ρ also minimizes I_0 in a neighborhood of x_i. The solvability of ($\mathcal{BI}$) suffices to guarantee that this does not happen, and therefore to fully justify the conclusions in [8, 7].

Next, we consider the behavior of u_ρ for sources $\rho \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$, and obtain the next

Theorem 1.18. *Let $m \geq 3$ and*

$$\rho \in \left(L^1(\mathbb{R}^m) + L^p(\mathbb{R}^m)\right) \cap L^2_{\rm loc}(\mathbb{R}^m), \qquad \textit{for some } \ p \in (1, 2_*].$$

Then, the minimizer u_ρ weakly solves ($\mathcal{BI}$). Moreover, for a given $\mathcal{I} \in \mathbb{R}^+$, there exists a positive constant $\mathcal{I}_0 = \mathcal{I}_0(m, p, \mathcal{I})$ with the following property: if

$$\|\rho\|_{L^1(\mathbb{R}^m)+L^p(\mathbb{R}^m)} \leq \mathcal{I},$$

then for any pair of open sets $\Omega'' \Subset \Omega' \Subset \mathbb{R}^m$ with $\mathrm{d}_\delta(\Omega'', \partial\Omega') \geq \mathcal{I}_0$, any $\mathcal{I}_2 > 0$ with

$$\|\rho\|_{L^2(\Omega')} \leq \mathcal{I}_2,$$

and any $q_0 \geq 0$, there exists a constant $C = C(q_0, m, p, \mathcal{I}, \mathcal{I}_0, \mathcal{I}_2, |\Omega'|_\delta)$ such that
(1.12)

$$\int_{\Omega''} (1 + \log w)^{q_0} \left\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 \left| D^2 u_\rho \left(Du_\rho, \cdot \right) \right|^2 + w_\rho^5 \left[D^2 u_\rho (Du_\rho, Du_\rho) \right]^2 \right\} \mathrm{d}x$$
$$+ \int_{\Omega''} w_\rho (1 + \log w_\rho)^{q_0+1} \mathrm{d}x \leq C.$$

Some comments are in order. First, we stress that u_ρ may have light segments, at least if $m \geq 4$, as the example in Proposition 1.7 shows. The existence/nonexistence of light segments in dimension $m = 3$ is unknown even in the global setting. Second, the enhanced second fundamental form estimate (1.12) holds provided that the inequality

$$\int_{\Omega'} \rho^2 \frac{(1 + \log w_\rho)^{q_0+2}}{w_\rho} \mathrm{d}x \leq \mathcal{I}_1 \tag{1.13}$$

is satisfied, which is trivially implied by $\rho \in L^2(\Omega')$. Whether (1.13) may be satisfied by less regular sources ρ is an open problem.

If ρ contains a singular measure, a few properties still hold.

Theorem 1.19. *Let $m \geq 3$ and let $\Sigma \Subset \mathbb{R}^m$ be a compact set satisfying $\mathscr{H}^1_\delta(\Sigma) = 0$. Assume that ρ decomposes as*

$$\rho = \rho_{\rm S} + \rho_2, \qquad \textit{with} \ \left\{ \begin{array}{l} \rho_{\rm S} \in \mathcal{M}(\mathbb{R}^m), \ \operatorname{supp} \rho_{\rm S} \subset \Sigma, \\ \rho_2 \in \left(L^1(\mathbb{R}^m) + L^p(\mathbb{R}^m)\right) \cap L^2_{\rm loc}(\mathbb{R}^m \backslash \Sigma), \ \ p \in (1, 2_*], \end{array} \right.$$

and let K^ρ be the set of light segments of the minimizer u_ρ:

$$K^\rho \doteq \overline{\bigcup \left\{ \overline{xy} \ : \ x, y \in \mathbb{R}^m, \ x \neq y, \ |u_\rho(x) - u_\rho(y)| = |x - y| \right\}},$$

Then, the following hold.

(i) *u_ρ weakly solves ($\mathcal{BI}$) on $\mathbb{R}^m \backslash K^\rho$.*
Moreover, if $K^\rho \cap \Sigma = \emptyset$, then u_ρ weakly solves ($\mathcal{BI}$) on $\mathbb{R}^m$.

(ii) *For each* $\Omega' \Subset \mathbb{R}^m \backslash (\Sigma \cup K^\rho)$ *and* $q_0 \geq 0$,

$$\int_{\Omega'} (1+\log w_\rho)^{q_0} \left\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 \left| D^2 u_\rho \left(Du_\rho, \cdot \right) \right|^2 + w_\rho^5 \left[D^2 u_\rho (Du_\rho, Du_\rho) \right]^2 \right\} \mathrm{d}x$$

$$+ \int_{\Omega'} w_\rho (1+\log w_\rho)^{q_0+1} \mathrm{d}x < \infty.$$

(iii) *If* $\Omega' \Subset \Omega \setminus (\Sigma \cup K^\rho)$ *and* $\rho \in L^\infty(\Omega')$, *then* $u_\rho \in C^{1,\alpha}_{\mathrm{loc}}(\Omega')$ *for some* $\alpha > 0$. *In particular, if* $\rho \in C^\infty(\Omega')$ *so is* u_ρ.

Adapting Remark 1.14, we see that in (i) of the above theorem u_ρ may not solve ($\mathcal{BI}$) weakly on the entire $\mathbb{R}^m$, at least if $m \geq 4$.

1.5. **Open problems and outline of the paper.** We first address the existence problem for light segments. We think that the regularity of ρ_u in Proposition 1.7 might be sharp, and we are tempted to propose the following

Conjecture 1. *If* $\phi \in \mathcal{S}(\partial\Omega)$ *and* $\rho \in L^q_{\mathrm{loc}}(\Omega)$ *with* $q > m-1$, *then the minimizer* u_ρ *does not have light segments.*

The case $q = m-1$, which includes $\rho \in L^2_{\mathrm{loc}}(\Omega)$ when $m = 3$, is particularly subtle.

Question 2. *If* $\phi \in \mathcal{S}(\partial\Omega)$ *and* $\rho \in L^{m-1}_{\mathrm{loc}}(\Omega)$, *could the minimizer have light segments?*

In view of the techniques developed herein, a negative answer to the above question would be sufficient to extend Theorem 1.10 to dimension $m \geq 3$ and to $\rho_{\mathrm{AC}} \in L^{m-1}_{\mathrm{loc}}(\Omega \backslash \Sigma)$.

Related to the above problems, and in view of Corollary 1.9, we also formulate the following

Question 3. *If* $\phi \in \mathcal{S}(\partial\Omega)$ *and*

$$\rho = \sum_{i=1}^k a_i \delta_{x_i} + \rho_{\mathrm{AC}} \qquad \textit{with} \quad \rho_{\mathrm{AC}} \in L^q(\Omega),\ q > m-1,$$

does the minimizer u_ρ *solve* ($\mathcal{BI}$) *weakly?*

An ambitious goal would be to relate the integrability of ρ to the Hausdorff dimension of the set K^ρ_ϕ of light segments. In view of Proposition 1.7 and of its proof, we may expect that the following holds:

Conjecture 4. *If* $m \geq 3$, $\phi \in \mathcal{S}(\partial\Omega)$ *and* $\rho \in L^q(\Omega)$ *for some* $2 \leq q \leq m$, *then the Hausdorff dimension of* K^ρ_ϕ *satisfies* $\dim_{\mathscr{H}_\delta}(K^\rho_\phi) \leq m-q$.

It might be possible that $\dim_{\mathscr{H}_\delta}(K^\rho_\phi) \leq m-q$ could be strengthened to $\mathscr{H}^{m-q}_\delta(K^\rho_\phi) = 0$. If this were true, notice that it would also imply a negative answer to Question 2. If ρ is more singular, we propose the next

Conjecture 5. *For* $\rho \in \mathcal{M}(\Omega)$, $\mathscr{H}^{m-1}_\delta(K^\rho_\phi) = 0$.

Still about the set of light segments, it would be important to understand the weak limit

$$w_j \mathrm{d}x \rightharpoonup \vartheta \qquad \text{in } \mathcal{M}(\Omega'), \quad \Omega' \Subset \Omega :$$

can one characterize the singular part of ϑ, and relate its support to the set K^ρ_ϕ? Can one characterize the non-negative functional

$$\langle \mathcal{T}, \eta \rangle \doteq \langle \rho, \eta \rangle - \int_\Omega \frac{Du_\rho \cdot D\eta}{\sqrt{1 - |Du_\rho|^2}} \qquad \eta \in C^\infty_c(\Omega),$$

describing the loss in (1.9)?

Regarding the energy density, we first observe that the integrability of w_ρ in Proposition 1.7 is much higher than the one that we can prove in Theorem 5.13. However, the latter is uniform on a sequence of approximated solutions $\{u_{\rho_j}\}$. We can ask the following

Question 6. *Can one prove a local higher integrability $w_\rho \in L^p_{\mathrm{loc}}(\Omega)$, for suitable $p > 1$, under a local higher integrability of ρ, for instance for $\rho \in L^q_{\mathrm{loc}}(\Omega)$ and $q > m - 1$?*

Even the case $\rho \in L^q_{\mathrm{loc}}(\mathbb{R}^m)$ and $q > m$ is currently unknown, cf. [29, 11].

Question 7. *What about the regularity of u_ρ and w_ρ when $\rho \in L^q$ and $q \in (1, 2)$?*

About the higher order regularity for u_ρ, $W^{2,q}$ estimates are unknown apart from the case $q = 2$, considered in the present paper, and $q > m$ treated in [29, 11] for $\Omega = \mathbb{R}^m$. We think that there might be an interpolation result, and therefore propose the following

Question 8. *Can one prove that, for $p \in [2, m]$ and $\rho \in L^p_{\mathrm{loc}}$, the minimizer u_ρ satisfies $u_\rho \in W^{2,p}_{\mathrm{loc}}$?*

The paper is organized as follows. Section 2 contains some background material from Lorentzian Geometry. Section 3 introduces the functional setting, then moves to discuss the basic properties of u_ρ (convergence under approximation of ρ, integrability), together with various equivalent conditions for the solvability of ($\mathcal{BI}$). In particular, we mention Propositions 3.9 and 3.14, which may have an independent interest. Though preparatory, most of the material in this section did not appear elsewhere in the literature. In Section 4, we construct examples of solutions to ($\mathcal{BI}$) with a single light segment, and defer the example with a higher dimensional set of light segments to Appendix A. In Section 5, we develop our main new tools: a removable singularity result, Theorem 1.8, a second fundamental form estimate and a higher integrability result. These are the bulk of the paper, the techniques therein differ from those in the literature and we believe they are applicable beyond the purposes of the present work. The concluding Section 6 contains the proof of our main existence results.

To a certain extent, each of Sections 2 to 5 can be read independently. In particular, the reader acquainted with Lorentzian Geometry and not focusing on the functional analytic setting may directly skip to Section 4.

A note on constants in elliptic estimates

When constants in our theorems are stated to depend on $\mathrm{diam}_\delta(\Omega)$, $|\Omega'|_\delta$, $\mathrm{d}_\delta(\Omega', \partial\Omega)$, in fact they can be bounded uniformly in terms of, respectively, uniform upper bounds for $\mathrm{diam}_\delta(\Omega)$ and $|\Omega'|_\delta$, and lower bounds for $\mathrm{d}_\delta(\Omega', \partial\Omega)$. Regarding the dependence of C in Theorem 1.10 from the domain Ω' and from $\mathrm{d}_\delta(\Omega', \partial\Omega)$, if $\mathrm{d}_\delta(\Omega', \partial(\Omega\backslash\Sigma)) \ge \tau$ and

$$\|\rho\|_{L^2(U_\tau)} \le \mathcal{I}_2 \qquad \text{where } U_\tau = \Big\{ x \in \Omega\backslash\Sigma \ : \ \mathrm{d}_\delta\big(x, \partial(\Omega\backslash\Sigma)\big) \ge \tau \Big\},$$

then C merely depends on τ. On the other hand, anywhere we write $C = C(\Omega, \ldots)$ we mean that we did not investigate the stability of the bounds for sequences of open sets $\{\Omega_j\}$ for which the other data are kept uniformly controlled.

2. Preliminaries from Lorentzian Geometry

In this section, we briefly recall some differential-geometric background that will be used henceforth. Let $\mathbb{L}^{m+1}$ be the Lorentz space with coordinates $(x^0, x^1, \ldots, x^m)$ and metric

$$-\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \sum_{i=1}^m \mathrm{d}x^i \otimes \mathrm{d}x^i, \qquad x \cdot y \doteq -x^0 y^0 + \sum_{i=1}^m x^i y^i, \qquad |x|_{\mathbb{L}} \doteq \sqrt{|x \cdot x|}.$$

Given a smooth function $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}$, consider the graph map

$$F \ : \ \Omega \to \mathbb{L}^{m+1}, \qquad F(x) \doteq (u(x), x),$$

and define M to be the manifold $F(\Omega)$ endowed with the metric induced from $\mathbb{L}^{m+1}$, equivalently, M is Ω endowed with the pull-back metric $g \doteq F^*(\cdot)$. When convenient, g will also be denoted by $\langle\, , \,\rangle$. Let $\|\cdot\|, \nabla, \Delta_M$ be, respectively, the norm, Levi-Civita connection and Laplace-Beltrami operator associated to g. The Hessian of a function u in the metric g will be denoted by $\nabla^2 u$.

We identity $\mathbb{R}^m$ with the slice $\{x^0 = 0\}$, so $\{x^i\}$ are Cartesian coordinates on $\mathbb{R}^m$ with associated vector fields $\{\partial_i\}$. Given an open set $\Omega \subset \mathbb{R}^m$ and $u \in C^\infty(\Omega)$, we let $u_i \doteq \partial_i u$ and $u_{ij} \doteq (D^2u)_{ij} = \partial^2_{ij} u$. By defining

$$X_i \doteq F_* \partial_i = \partial_i + u_i \partial_0,$$

the components of g are written as

$$g_{ij} \doteq X_i \cdot X_j = \delta_{ij} - u_i u_j.$$

Hereafter we assume that g is Riemannian (equivalently, $|Du| < 1$). The inverse metric has components

$$g^{ij} = \delta^{ij} + w^2 u^i u^j, \qquad \text{with} \quad w \doteq \frac{1}{\sqrt{1 - |Du|^2}},$$

where $u^i = \delta^{ij} u_j$ are the components of the gradient Du. Then, the volume measure $\mathrm{d}x_g$ of g relates to the measure $\mathrm{d}x$ on $\mathbb{R}^m$ as follows:

$$\mathrm{d}x_g = w^{-1} \mathrm{d}x. \tag{2.1}$$

The future-pointing, unit normal vector to the graph M is given by $\mathbf{n} \doteq w(\partial_0 + u^i \partial_i)$. Note that $\mathbf{n}\cdot\mathbf{n} = -1$ and $w = -\mathbf{n}\cdot\partial_0$. Let superscripts $\|$ and $\perp$ denote, respectively, the projection onto TM and $TM^\perp$ with respect to the inner product $\cdot$ in $\mathbb{L}^{m+1}$. From the chain of identities

$$\langle \partial_0^{\|}, \partial_j \rangle = \partial_0 \cdot F_* \partial_j = -u_j = -\langle \nabla u, \partial_j \rangle,$$

we deduce that

$$\partial_0^{\|} = -\nabla u. \tag{2.2}$$

Denoting by $\bar{D}$ the Levi-Civita connection of $\mathbb{L}^{m+1}$, we define the second fundamental form of M by

$$\mathrm{II}(\partial_i, \partial_j) \doteq \left(\bar{D}_{X_i} X_j\right)^{\perp} = h_{ij}\mathbf{n}, \qquad \text{thus} \qquad h_{ij} = -\bar{D}_{X_i} X_j \cdot \mathbf{n} = \bar{D}_{X_i}\mathbf{n} \cdot X_j.$$

From the definition of X_i we obtain $h_{ij} = w\, u_{ij}$. The (unnormalized) scalar mean curvature $H \doteq g^{ij} h_{ij}$ in direction $\mathbf{n}$ is therefore

$$H = w\Delta u + w^3 D^2 u(Du, Du) = \operatorname{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right),$$

where Δ is the Laplacian on $\mathbb{R}^m$. Next, since the Christoffel symbols of g are given by $\Gamma_{ij}^k = -w^2 u^k u_{ij}$, we compute the Hessian and Laplacian of a smooth function $\phi : \Omega \to \mathbb{R}$ in the graph metric g:

$$\begin{aligned} &\nabla^2_{ij}\phi = \phi_{ij} + w^2 \phi_k u^k u_{ij}; \\ &\Delta_M \phi = g^{ij}\nabla^2_{ij}\phi = \Delta\phi + w^2 D^2\phi(Du, Du) + H w D\phi \cdot Du. \end{aligned} \tag{2.3}$$

In addition, the norm of the second fundamental form II of the graph u is given by

$$\begin{aligned} \|\,\mathrm{II}\,\|^2 &= g^{ij} g^{kl} h_{ik} h_{jl} = w^2\big(\delta^{ij} + w^2 u^i u^j\big) u_{ik}\big(\delta^{kl} + w^2 u^k u^l\big) u_{jl} \\ &= w^2 |D^2 u|^2 + 2w^4 \left| D^2 u\,(Du, \cdot)\right|^2 + w^6 \big[D^2 u(Du, Du)\big]^2. \end{aligned} \tag{2.4}$$

In particular,

$$\nabla^2_{ij} u = w^2 u_{ij} = w\, h_{ij}, \quad \|\nabla^2 u\|^2 = w^2 \|\,\mathrm{II}\,\|^2, \quad \Delta_M u = Hw \quad \text{on } M. \tag{2.5}$$

Given $o \in \mathbb{R}^m$, we denote by $r_o : \Omega \to \mathbb{R}$ and $\ell_o : \Omega \to \mathbb{R}$, respectively, the Euclidean distance from o and the Lorentzian distance from $(u(o), o)$ restricted to the graph of u, that is, we set

$$\begin{aligned} &r_o(x) \doteq |x - o|, \\ &l_o(s, x) \doteq |(s, x) - (u(o), o)|_{\mathbb{L}} = \sqrt{-(s - u(o))^2 + |x - o|^2}, \\ &\ell_o(x) \doteq l_o\,(u(x), x)\,. \end{aligned} \tag{2.6}$$

We also denote the extrinsic Lorentzian ball centered at o, and more generally the one centered at a subset $A \subset \mathbb{R}^m$, by

$$L_R(o) \doteq \left\{x \in \Omega \ :\ \ell_o(x) < R\right\}, \qquad L_R(A) \doteq \bigcup_{o \in A} L_R(o). \tag{2.7}$$

When it is necessary, we will write ℓ_o^ρ, L_R^ρ to emphasize their dependence on the minimizer $u = u_\rho$ of I_ρ. By (2.3), we get

$$
\begin{aligned}
\bar{D}l_o^2(u(x), x) &= 2\left(x^j - o^j\right)\partial_j + 2\left(u(x) - u(o)\right)\partial_0; \\
\left\|\nabla \ell_o(x)\right\|^2 &= \left|\bar{D}l_o(u(x), x)\right|_{\mathbb{L}}^2 + \left(\bar{D}l_o(u(x), x)\cdot \mathbf{n}\right)^2 \\
&= 1 + \frac{w^2}{\ell_o^2}\left|Du\cdot(x-o) - (u(x)-u(o))\right|^2; \\
\Delta_M \ell_o^2(x) &= 2m + 2wH\left[(x-o)\cdot Du - (u(x)-u(o))\right] \\
&= 2m + H\left(\bar{D}l_o^2(u(x), x)\cdot \mathbf{n}\right).
\end{aligned}
\tag{2.8}
$$

As we shall see in the proof of Theorem 5.13, the construction of cut-off functions based on the Lorentzian distance, instead of those based on the Euclidean one, will be the key to obtain the higher integrability of u_ρ in dimension $m \geq 3$.

3. Basic properties of u_ρ

In this section, we obtain basic properties of the minimizer u_ρ of I_ρ, both for $\Omega \subset \mathbb{R}^m$ a bounded domain ($m \geq 2$) and for $\Omega = \mathbb{R}^m$ ($m \geq 3$).

3.1. **Functional setting.**

We first choose our functional spaces. If $\Omega = \mathbb{R}^m$, our treatment mildly departs from those in [30, 8], and is basically designed to get an explicit description of the sources ρ covered by the method. On the other hand, for bounded Ω, subtleties related to a possibly rough boundary $\partial\Omega$ require extra care in the choice of the functional space, which significantly differs from that in [4].

Definition 3.1. Given $m \geq 2$, we fix $p_1 \in (m, \infty)$ and assume also $p_1 \geq 2^*$ for $m = 3$.

(i) When $m \geq 2$ and $\Omega \subset \mathbb{R}^m$ is a bounded domain, we set

$$
\mathcal{Y}(\Omega) \doteq W^{1,p_1}(\Omega) \cap C(\overline{\Omega}), \qquad \|v\|_{\mathcal{Y}} \doteq \max\left\{\|v\|_{W^{1,p_1}(\Omega)}, \|v\|_{C(\overline{\Omega})}\right\};
$$

(ii) When $\Omega = \mathbb{R}^m$ and $m \geq 3$, we set

$$
\mathcal{Y}(\mathbb{R}^m) \doteq \overline{C_c^\infty(\mathbb{R}^m)}^{\|\cdot\|_{\mathcal{Y}}}, \qquad \|v\|_{\mathcal{Y}} \doteq \max\left\{\|Dv\|_2, \|Dv\|_{p_1}\right\}.
$$

Note that, if Ω is bounded and sufficiently regular (Lipschitz is enough), by Morrey's Embedding Theorem $\mathcal{Y}(\Omega) = W^{1,p_1}(\Omega)$ with the equivalent norm $\|\cdot\|_{W^{1,p_1}(\Omega)}$.

Remark 3.2. The case $\Omega = \mathbb{R}^2$ will not be considered in the present paper. We observe that the radially symmetric solution in [13] with a Dirac delta source (cf. Example 5.6 herein with $H = 0$) has a logarithmic behavior at infinity when $m = 2$, which calls for a different functional setting. For ρ a superposition of point charges, complete classification theorems for entire solutions in $\mathbb{R}^2$ were obtained by A.A. Klyachin [31], and I. Fernández, F.J. López and R. Souam [20].

The following result can be proved in a similar way as [8, Lemma 2.1], but we give full details for the sake of completeness.

Proposition 3.3. *Assume $m \ge 3$ and $\Omega = \mathbb{R}^m$. Then $(\mathcal{Y}(\mathbb{R}^m), \|\cdot\|_{\mathcal{Y}})$ is a reflexive Banach space. Moreover,*

$$\mathcal{Y}(\mathbb{R}^m) \hookrightarrow W^{1,q}(\mathbb{R}^m) \qquad \forall\, q \in [2^*, p_1]. \tag{3.1}$$

In particular, $\|\cdot\|_{\mathcal{Y}}$ is equivalent to $\|D\cdot\|_2 + \|\cdot\|_{W^{1,p_1}}$, and $\mathcal{Y}(\mathbb{R}^m) \hookrightarrow C_0(\mathbb{R}^m) \doteq \{u \in C(\mathbb{R}^m) : \lim_{|x|\to\infty} u(x) = 0\}$ holds.

Proof. First, $\|\cdot\|_{\mathcal{Y}}$ is equivalent to the norm $|u|_{\mathcal{Y}} \doteq \sqrt{\|Du\|_2^2 + \|Du\|_{p_1}^2}$. Hence, to prove the reflexivity of $(\mathcal{Y}(\mathbb{R}^m), \|\cdot\|_{\mathcal{Y}})$ it suffices to show that $(\mathcal{Y}(\mathbb{R}^m), |\cdot|_{\mathcal{Y}})$ is uniformly convex. This easily follows by using the criterion in [14, Exercise 3.29] and the uniform convexity of the norms $\|Du\|_2$ and $\|Du\|_{p_1}$.

To obtain (3.1), let $u \in \mathcal{Y}(\mathbb{R}^m)$. From the choice of p_1 and Hölder's inequality, the next interpolation inequality holds:

$$\|Du\|_q \le \|u\|_{\mathcal{Y}} \quad \text{for all } q \in [2, p_1]. \tag{3.2}$$

Since $m \in [2, p_1)$ and $q^* \to \infty$ as $q \to m^-$, there exists $\hat{q} \in [2, m)$ so that $\hat{q}^* = p_1$. Thus, Sobolev's inequality and (3.2) yield $\|u\|_{p_1} \le C\|Du\|_{\hat{q}^*} \le C\|u\|_{\mathcal{Y}}$. Hence, $\mathcal{Y}(\mathbb{R}^m) \hookrightarrow W^{1,p_1}(\mathbb{R}^m)$ holds. In addition, from $\|u\|_{2^*} \le C\|Du\|_2 \le \|u\|_{\mathcal{Y}}$, $2 < 2^* \le p_1$ and (3.2), we see $\mathcal{Y}(\mathbb{R}^m) \hookrightarrow W^{1,2^*}(\mathbb{R}^m)$. Therefore, by the interpolation, (3.1) holds.

The equivalence between $\|\cdot\|_{\mathcal{Y}}$ and $\|D\cdot\|_2 + \|\cdot\|_{W^{1,p_1}}$ is an immediate consequence of (3.1), while $\mathcal{Y}(\mathbb{R}^m) \hookrightarrow C_0(\mathbb{R}^m)$ follows from Morrey's embedding Theorem once we observe that $u \in L^{2^*}(\mathbb{R}^m) \cap C^{0,\alpha}(\mathbb{R}^m)$ implies that u vanishes at infinity. □

Remark 3.4 (**Dual spaces**)**.** If $q \in (1,\infty)$ and $\Omega \subset \mathbb{R}^m$ is any domain, then it is well-known that elements in the dual space $W^{1,q}(\Omega)^* = W^{-1,q'}(\Omega)$ can be represented as pairs $(v, V) \in L^{q'}(\Omega) \times [L^{q'}(\Omega)]^m$ where $q' \doteq q/(q-1)$, with the action

$$\langle \rho, \psi \rangle \doteq \int_\Omega \psi v \mathrm{d}x + \int_\Omega D\psi \cdot V \mathrm{d}x \qquad \forall\, \psi \in W^{1,q}(\Omega),$$

see for instance [1, Theorem 3.9]. Furthermore, recall that if X_1, X_2 are Banach spaces with $X_1 \cap X_2$ dense in X_1 and X_2, then $(X_1 \cap X_2)^* = X_1^* + Y_2^*$ with the natural norm

$$\|\rho\|_{X_1^*+X_2^*} = \inf\left\{ \|\rho_1\|_{X_1^*} + \|\rho_2\|_{X_2^*} : \rho_j \in X_j^*,\ \rho = \rho_1 + \rho_2 \right\},$$

see [6, Theorem 2.7.1]. Indeed, inspecting the proof in [6], one deduces that every functional $\rho \in (X_1 \cap X_2)^*$ can be represented as

$$\rho = \rho_1 + \rho_2 \in X_1^* + X_2^*, \qquad \text{with} \qquad \|\rho_1\|_{X_1^*} + \|\rho_2\|_{X_2*} \le \|\rho\|_{(X_1\cap X_2)^*},$$

the representation being unique (with equality between norms) when $X_1 \cap X_2$ is dense in both X_1 and X_2. Taking the above observations into account,

(i) if Ω is a bounded domain, every $\rho \in \mathcal{Y}(\Omega)^*$ can be represented as $\rho = \rho_1 + \rho_2 \in W^{-1,p_1'}(\Omega) + \mathcal{M}(\Omega)$, for some ρ_1, ρ_2 satisfying

$$\|\rho_1\|_{W^{-1,p_1'}} + \|\rho_2\|_{\mathcal{M}} \le \|\rho\|_{\mathcal{Y}^*}.$$

The representation is unique when $C(\overline{\Omega}) \cap W^{1,p_1}(\Omega)$ is dense in $W^{1,p_1}(\Omega)$, a fact which entails some mild requirement on $\partial\Omega$ such as the segment condition (cf. [1, Theorem 3.22]). However, uniqueness of the representation will not be used in the present work. Notice the continuous inclusion $\mathcal{M}(\Omega) \hookrightarrow \mathcal{Y}(\Omega)^*$.

(ii) if $\Omega = \mathbb{R}^m$ and $m \geq 3$, then $\mathcal{Y}(\mathbb{R}^m)^* = \mathcal{D}^{1,2}(\mathbb{R}^m)^* + W^{-1,p_1'}(\mathbb{R}^m)$, with $\mathcal{D}^{1,2}(\mathbb{R}^m)$ being the closure of $C_c^\infty(\mathbb{R}^m)$ with respect to the norm $\|v\|_{\mathcal{D}^{1,2}} \doteq \|Dv\|_2$. In particular, because of Proposition 3.3 and Morrey's embedding, $\mathcal{M}(\mathbb{R}^m) \hookrightarrow \mathcal{Y}(\mathbb{R}^m)^*$ and $W^{-1,q'}(\mathbb{R}^m) \hookrightarrow \mathcal{Y}(\mathbb{R}^m)^*$ for each $q \in [2^*, p_1]$. Hence,

$$\mathcal{M}(\mathbb{R}^m) + L^{q'}(\mathbb{R}^m) \hookrightarrow \mathcal{Y}(\mathbb{R}^m)^* \qquad \forall\, q \in [2^*, p_1],$$

where $L^{q'}(\mathbb{R}^m)$ consists of the pairs $(v, 0)$.

Clearly, $\mathcal{Y}_0(\mathbb{R}^m)$ is a closed convex subset of $\mathcal{Y}(\mathbb{R}^m)$. The situation is more subtle for $\mathcal{Y}_\phi(\Omega)$ defined in (1.3), because of the lack of regularity of $\partial\Omega$. However, as the next result shows, the mild sense in which the boundary condition is considered, see Remark 1.1, suffices to guarantee that $\mathcal{Y}_\phi(\Omega) \subset \mathcal{Y}(\Omega)$.

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, let $\mathscr{F} \subset C(\partial\Omega)$ be a relatively compact (resp. compact) subset with respect to uniform convergence, and consider*

$$\mathcal{Y}_{\mathscr{F}}(\Omega) \doteq \left\{ v \ : \ v \in \mathcal{Y}_\phi(\Omega) \text{ for some } \phi \in \mathscr{F} \right\}.$$

Then $\mathcal{Y}_{\mathscr{F}}(\Omega) \subset C(\overline{\Omega})$ as a relatively compact (resp. compact) subset, where we extend each $v \in \mathcal{Y}_{\mathscr{F}}(\Omega)$ onto $\overline{\Omega}$ by setting $v(x) \doteq \phi(x)$ for $x \in \partial\Omega$.

Proof. First, observe that if $x \in \Omega$ and $\widetilde{x} \in \partial\Omega$ is a nearest point to x in the metric d_δ, the boundary condition in Remark 1.1 tested on the segment $tx + (1-t)\widetilde{x} \in \Omega$ for any $t \in (0,1]$ gives, for each $v \in \mathcal{Y}_{\mathscr{F}}(\Omega)$,

$$(3.3) \qquad \left|v(x) - \phi(\widetilde{x})\right| = \left| v(x) - \lim_{t \to 0^+} v(tx + (1-t)\widetilde{x}) \right| \leq \left| x - \widetilde{x} \right|.$$

The inequality trivially holds also if $x \in \partial\Omega$, by the way v is extended. Whence,

$$(3.4) \qquad \|v\|_{L^\infty(\Omega)} \leq \|\phi\|_{C(\partial\Omega)} + \mathrm{diam}_\delta(\Omega) \leq \sup_{\phi \in \mathscr{F}} \|\phi\|_{C(\partial\Omega)} + \mathrm{diam}_\delta(\Omega) < \infty,$$

where the last inequality follows since $\mathscr{F}$ is relatively compact in $C(\partial\Omega)$. This proves the uniform boundedness of $\mathcal{Y}_{\mathscr{F}}(\Omega)$.

Next, we shall show $v \in C(\overline{\Omega})$ for each $v \in \mathcal{Y}_{\mathscr{F}}(\Omega)$, and that $\mathcal{Y}_{\mathscr{F}}(\Omega)$ is uniformly equicontinuous. Let $\varepsilon > 0$ be arbitrary. Since $\mathscr{F}$ is relatively compact in $C(\partial\Omega)$, $\mathscr{F}$ is uniformly equicontinuous on $\partial\Omega$, hence, there exists $\widetilde{\delta}_\varepsilon > 0$ such that

$$\phi \in \mathscr{F},\ x_1, x_2 \in \partial\Omega,\ |x_1 - x_2| < \widetilde{\delta}_\varepsilon \quad \Rightarrow \quad \left|\phi(x_1) - \phi(x_2)\right| < \frac{\varepsilon}{4}.$$

Set

$$\delta_\varepsilon \doteq \frac{1}{4} \min\left\{ \varepsilon, \widetilde{\delta}_\varepsilon \right\} > 0,$$

and pick $x_1, x_2 \in \overline{\Omega}$ with $|x_1 - x_2| < \delta_\varepsilon$. If one among $B_{\delta_\varepsilon}(x_1)$ and $B_{\delta_\varepsilon}(x_2)$ is contained in Ω, property $v \in \mathcal{Y}_\phi(\Omega)$ implies that v is 1-Lipschitz there, whence

$$|x_1 - x_2| < \delta_\varepsilon \quad \Rightarrow \quad \big|v(x_1) - v(x_2)\big| \le \big|x_1 - x_2\big| < \delta_\varepsilon < \varepsilon.$$

We therefore assume that $B_{\delta_\varepsilon}(x_j) \cap \partial\Omega \neq \emptyset$ for $j = 1, 2$, and choose $\widetilde{x}_j \in B_{\delta_\varepsilon}(x_j) \cap \partial\Omega$ satisfying $|x_j - \widetilde{x}_j| = \mathrm{d}_\delta(x_j, \partial\Omega)$. From $|x_1 - x_2| < \delta_\varepsilon$ and $|x_j - \widetilde{x}_j| < \delta_\varepsilon$ for each j, the triangle inequality implies $|\widetilde{x}_1 - \widetilde{x}_2| < 3\delta_\varepsilon < \widetilde{\delta}_\varepsilon$ and therefore, by using (3.3),

$$\begin{aligned} \big|v(x_1) - v(x_2)\big| &\le \big|v(x_1) - \phi(\widetilde{x}_1)\big| + \big|\phi(\widetilde{x}_1) - \phi(\widetilde{x}_2)\big| + \big|\phi(\widetilde{x}_2) - v(x_2)\big| \\ &\le \big|x_1 - \widetilde{x}_1\big| + \frac{\varepsilon}{4} + \big|\widetilde{x}_2 - x_2\big| < 2\delta_\varepsilon + \frac{\varepsilon}{4} \le \varepsilon. \end{aligned}$$

Hence, $v \in C(\overline{\Omega})$ and $\mathcal{Y}_{\mathscr{F}}(\Omega)$ is uniformly equicontinuous on $\overline{\Omega}$. The relative compactness of $\mathcal{Y}_{\mathscr{F}}(\Omega)$ in $C(\overline{\Omega})$ follows by the Arzelá–Ascoli theorem. If $\mathscr{F}$ is compact, then any limit point of a sequence $\{v_j\} \subset \mathcal{Y}_{\mathscr{F}}(\Omega)$ lies in $\mathcal{Y}_{\mathscr{F}}(\Omega)$, thus $\mathcal{Y}_{\mathscr{F}}(\Omega)$ is compact in $C(\overline{\Omega})$. □

Corollary 3.6. *For each bounded domain $\Omega \subset \mathbb{R}^m$ and each $\phi \in C(\partial\Omega)$, $\mathcal{Y}_\phi(\Omega) \subset \mathcal{Y}(\Omega)$ and it is bounded, closed, convex and sequentially weakly compact in $\mathcal{Y}(\Omega)$.*

Proof. By Proposition 3.5, $\mathcal{Y}_\phi(\Omega) \subset C(\overline{\Omega})$ is a compact subset. Since clearly $\mathcal{Y}_\phi(\Omega)$ is contained in $W^{1,p_1}(\Omega)$ as a closed, bounded subset, we deduce that $\mathcal{Y}_\phi(\Omega) \subset \mathcal{Y}(\Omega)$ is closed and bounded. the fact that $\mathcal{Y}_\phi(\Omega)$ is convex is obvious. To prove the sequential weak compactness, let $\{v_j\}$ be sequence in $\mathcal{Y}_\phi(\Omega)$. Then, up to passing to a subsequence, $v_j \to v$ weakly in $W^{1,p_1}(\Omega)$ and strongly in $C(\overline{\Omega})$, for some $v \in \mathcal{Y}(\Omega)$. By Remark 3.4, we can represent a given $\rho \in \mathcal{Y}(\Omega)^*$ as $\rho = \rho_1 + \rho_2$ with $\rho_1 \in W^{-1,p_1'}(\Omega)$ and $\rho_2 \in \mathcal{M}(\Omega)$, whence

$$\big\langle \rho, v_j \big\rangle = \big\langle \rho_1, v_j \big\rangle + \big\langle \rho_2, v_j \big\rangle \to \langle \rho_1, v\rangle + \langle \rho_2, v\rangle = \langle \rho, v\rangle \qquad \text{as } j \to \infty,$$

thus $\{v_j\}$ is weakly convergent. □

Regarding the minimization problem, for the readers' convenience we reproduce the argument in [8] to show the existence and uniqueness of the minimizer u_ρ in our functional setting. For $\rho \in \mathcal{Y}(\Omega)^*$, we recall that $I_\rho : \mathcal{Y}_\phi(\Omega) \to \mathbb{R}$ is defined by

$$I_\rho(v) \doteq \int_\Omega \left(1 - \sqrt{1 - |Dv|^2}\right) \mathrm{d}x - \langle \rho, v\rangle \quad \text{for } v \in \mathcal{Y}_\phi(\Omega).$$

The above discussion guarantees that $\mathcal{Y}_\phi(\Omega)$ is a closed convex subset of $\mathcal{Y}(\Omega)$ (when Ω is bounded, we suppose that $\phi \in C(\partial\Omega)$ is chosen such that $\mathcal{Y}_\phi(\Omega) \neq \emptyset$), and I_ρ is strictly convex since $\overline{B_1(0)} \ni p \mapsto 1 - \sqrt{1 - |p|^2} \in [0, 1]$ is strictly convex. Furthermore, from the inequality $1 - \sqrt{1 - |p|^2} \le |p|^2$ for $|p| \le 1$ and using Lebesgue's dominated convergence theorem, I_ρ is continuous on $\mathcal{Y}_\phi(\Omega)$. Combining convexity and continuity, we deduce that I_ρ is weakly lower-semicontinuous. If Ω is a bounded domain, by Corollary 3.6 the set $\mathcal{Y}_\phi(\Omega)$ is bounded and sequentially weakly compact in $\mathcal{Y}(\Omega)$, so the existence of a minimizer is then obvious by the direct method. On the other hand, if $\Omega = \mathbb{R}^m$, then $\|Dv\|_q^q \le \|Dv\|_2^2$ holds for every $v \in \mathcal{Y}_0(\mathbb{R}^m)$ and $q \in [2,\infty)$ thanks to $\|Dv\|_\infty \le 1$. Thus, in view of the identity

$$1 - \sqrt{1-t} = \sum_{j=1}^{\infty} b_j t^j \qquad \text{with } b_j \doteq \frac{(2j-2)!}{j!(j-1)!2^{2j-1}}, \quad t \in [0,1], \tag{3.5}$$

it follows from $3 \le m < p_1$ that for $v \in \mathcal{Y}_0(\mathbb{R}^m)$,

$$\begin{aligned} \|v\|_{\mathcal{Y}}^2 &\le \left(\|Dv\|_2^2 + \|Dv\|_{p_1}^2 \right) \le \left(\|Dv\|_2^2 + \|Dv\|_2^{4/p_1} \right) \\ &\le 2\left(\|Dv\|_2^2 + 1 \right) \le 2\left[1 + b_1^{-1}\left(I_\rho(v) + \|\rho\|_{\mathcal{Y}^*}\|v\|_{\mathcal{Y}} \right) \right]. \end{aligned} \tag{3.6}$$

Hence, I_ρ is coercive. Since $\mathcal{Y}(\mathbb{R}^m)$ is reflexive, the existence and uniqueness of u_ρ is then a consequence, for instance, of [14, Corollary 3.23].

3.2. **Compact subsets of $\mathcal{S}(\partial\Omega)$: the class $\mathcal{S}_{b,\zeta}(\partial\Omega)$.** To define the compact set $\mathcal{S}_{b,\zeta}(\partial\Omega) \subset \mathcal{S}(\partial\Omega)$ mentioned in the Introduction, we assume that $(\Omega, \mathrm{d}_\Omega)$ has compact metric completion, that following [34] we denote by Ω_{d}. We set $\partial\Omega_{\mathrm{d}} = \Omega_{\mathrm{d}} \backslash \Omega$. *To stress the difference with* $\mathrm{d}_{\overline{\Omega}}$ *in* (1.4)*, we write* d *instead of* d_Ω *for the metric on* Ω_{d}. The identity $i : (\Omega, \mathrm{d}_\Omega) \to (\Omega, \mathrm{d}_\delta)$ extends by density to a distance non-increasing map $\widetilde{i} : (\Omega_{\mathrm{d}}, \mathrm{d}) \to (\overline{\Omega}, \mathrm{d}_\delta)$. Since Ω_{d} is compact and $(\overline{\Omega}, \mathrm{d}_\delta)$ is Hausdorff, $\widetilde{i}$ is a closed map. From $\widetilde{i}(\Omega_{\mathrm{d}}) \supset \Omega$, we deduce that $\widetilde{i}$ is also surjective, hence, $\widetilde{i}$ is a quotient map. Given $\phi \in C(\partial\Omega)$, let $\widetilde{\phi} = \phi \circ \widetilde{i} \in C(\partial\Omega_{\mathrm{d}})$ be its lift. For given $b \in \mathbb{R}^+$ and $\zeta : \mathbb{R}^+ \to [0,1)$, we set

$$\mathcal{S}_{b,\zeta}(\partial\Omega) \doteq \left\{ \phi \in \mathcal{S}(\partial\Omega) \ : \ \|\phi\|_\infty \le b, \quad \sup_{\substack{x,y \in \partial\Omega_{\mathrm{d}}, \\ \mathrm{d}(x,y)=t}} \frac{|\widetilde{\phi}(x) - \widetilde{\phi}(y)|}{\mathrm{d}(x,y)} \le \zeta(t) \ \forall\, t \in \mathbb{R}^+ \right\}, \tag{3.7}$$

where the supremum is defined to be zero if $t > \mathrm{diam}_{\mathrm{d}_\Omega}(\Omega)$. We prove that $\mathcal{S}_{b,\zeta}(\partial\Omega)$ is compact in $C(\partial\Omega)$, so let $\{\phi_j\} \subset \mathcal{S}_{b,\zeta}(\partial\Omega)$. By the Arzelá–Ascoli Theorem, $\{\widetilde{\phi}_j\}$ is relatively compact in $C(\partial\Omega_{\mathrm{d}})$ and thus, up to subsequences, $\widetilde{\phi}_j \to \widetilde{\phi}$ for some $\widetilde{\phi} \in C(\partial\Omega_{\mathrm{d}})$ which is constant on the fibers of $\widetilde{i}$, and therefore factorizes as $\widetilde{\phi} = \phi \circ \widetilde{i}$. Since $\widetilde{i}$ is a quotient map, $\phi \in C(\partial\Omega)$ (see, for instance, [39, Theorem 22.2]). From $\widetilde{\phi}_j \to \widetilde{\phi}$ on $\partial\Omega_d$, we deduce that $\phi_j \to \phi$ on $\partial\Omega$ and ϕ satisfies the last two conditions in (3.7). To show that $\mathcal{S}_{b,\zeta}(\partial\Omega)$ is compact in $C(\partial\Omega)$, it suffices to prove that $\phi \in \mathcal{S}(\partial\Omega)$. Suppose by contradiction that $\phi \notin \mathcal{S}(\partial\Omega)$, and take $x, y \in \partial\Omega$, $x \ne y$ such that $|\phi(x) - \phi(y)| \ge \mathrm{d}_{\overline{\Omega}}(x,y)$. Then, being the left-hand side finite, $\Gamma_{xy} \ne \emptyset$ and we can lift the interior of any path $\gamma \in \Gamma_{xy}$ to a path $\widetilde{\gamma} : (0,1) \to \Omega_{\mathrm{d}}$ of the same length of γ, with $\widetilde{\gamma}((0,1)) \subset \Omega$. Choose paths $\gamma_\varepsilon \in \Gamma_{x,y}$ with $\mathscr{H}^1_\delta(\gamma_\varepsilon) \downarrow \mathrm{d}_{\overline{\Omega}}(x,y)$ as $\varepsilon \downarrow 0$. It is easy to check that $\widetilde{\gamma}_\varepsilon(0^+) \doteq \widetilde{x}_\varepsilon \in \widetilde{i}^{-1}(x)$ and $\widetilde{\gamma}_\varepsilon(1^-) \doteq \widetilde{y}_\varepsilon \in \widetilde{i}^{-1}(y)$. Since the fibers $\widetilde{i}^{-1}(x)$ and $\widetilde{i}^{-1}(y)$ are compact, up to subsequences $\widetilde{x}_{\varepsilon_k} \to \widetilde{x} \in \widetilde{i}^{-1}(x)$ and $\widetilde{y}_{\varepsilon_k} \to \widetilde{y} \in \widetilde{i}^{-1}(y)$. By $x \ne y$, we have $0 < \mathrm{d}(\widetilde{x}, \widetilde{y}) = \lim_{k\to\infty} \mathrm{d}(\widetilde{x}_{\varepsilon_k}, \widetilde{y}_{\varepsilon_k}) \le \mathrm{d}_{\overline{\Omega}}(x,y)$. However, from the last property in

(3.7) for $\widetilde{\phi}_j$, we get the following contradiction:

$$\mathrm{d}(\widetilde{x},\widetilde{y}) \le \mathrm{d}_{\overline{\Omega}}(x,y) \le |\phi(x)-\phi(y)| = \left|\widetilde{\phi}(\widetilde{x})-\widetilde{\phi}(\widetilde{y})\right| = \lim_{j\to\infty}\left|\widetilde{\phi}_j(\widetilde{x})-\widetilde{\phi}_j(\widetilde{y})\right| \le \zeta\left(\mathrm{d}(\widetilde{x},\widetilde{y})\right)\mathrm{d}(\widetilde{x},\widetilde{y}) < \mathrm{d}(\widetilde{x},\widetilde{y}).$$

3.3. **Convergence of minimizers.** Our proof of the solvability of ($\mathcal{BI}$) depends on an approximation procedure, smoothing ρ by convolution. Thus, it entails a convergence result for minimizers.

Proposition 3.7. *Let $\rho_k \in \mathcal{Y}(\Omega)^*$, and consider the following assumptions:*

(i) *$\Omega \subset \mathbb{R}^m$ is a bounded domain with $m \ge 2$, $\{\phi_k\} \subset C(\partial\Omega)$ satisfy $\mathcal{Y}_{\phi_k}(\Omega) \neq \emptyset$ and $\phi_k \to \phi$ strongly in $C(\partial\Omega)$. Assume that $\rho_k = \mu_k + f_k$, where $\mu_k \in \mathcal{M}(\Omega)$, $f_k \in \mathcal{Y}(\Omega)^*$, and that*

$$\mu_k \rightharpoonup \mu \quad \text{weakly in } \mathcal{M}(\Omega), \qquad f_k \to f \quad \text{strongly in } \mathcal{Y}(\Omega)^*. \tag{3.8}$$

(ii) *$\Omega = \mathbb{R}^m$ with $m \ge 3$, $\rho_k = \mu_k + f_k$ where μ_k and f_k satisfy (3.8). Assume also that, for each $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that*

$$|\mu_k|\left(\mathbb{R}^m \setminus B_{R_\varepsilon}\right) < \varepsilon \quad \text{for each } k \ge 1. \tag{3.9}$$

Under either (i) *or* (ii), *$\mathcal{Y}_\phi(\Omega) \neq \emptyset$ and, by setting $\rho \doteq \mu + f$, up to a subsequence, $u_{\rho_k} \to u_\rho$ strongly in $W^{1,q}(\Omega) \cap C(\overline{\Omega})$, respectively, for every $q \in [1,\infty)$ if Ω is a bounded domain, and for every $q \in [2^*,\infty)$ if $\Omega = \mathbb{R}^m$. Furthermore, $\|Du_{\rho_k} - Du_\rho\|_q \to 0$ for every $q \in [2,\infty)$ when $\Omega = \mathbb{R}^m$. In particular,*

$$\left\langle \rho_k, u_{\rho_k}\right\rangle \to \left\langle \rho, u_\rho\right\rangle \qquad \text{as } k \to \infty.$$

Proof. We first suppose that Ω is bounded. Due to Proposition 3.5 and $u_{\rho_k} \in \mathcal{Y}_{\phi_k}(\Omega)$, $\{u_{\rho_k}\}$ is relatively compact in $C(\overline{\Omega})$ and hence it is bounded in $W^{1,q}(\Omega)$ for any $q \in [1,\infty]$. Up to a subsequence, $u_{\rho_k} \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$ for each fixed $q \in (1,\infty)$, and strongly in $C(\overline{\Omega})$. In particular, $u = \phi$ on $\partial\Omega$, and $u_{\rho_k} \rightharpoonup u$ weakly in $\mathcal{Y}(\Omega)$ due to Remark 3.4 (i). From $|u_{\rho_k}(x) - u_{\rho_k}(y)| \le \mathrm{d}_{\overline{\Omega}}(x,y)$ for every $x,y \in \overline{\Omega}$, we deduce $|u(x)-u(y)| \le \mathrm{d}_{\overline{\Omega}}(x,y)$ and $u \in \mathcal{Y}_\phi(\Omega)$. Hence, the minimizer u_ρ does exist.

From (3.5) we get

$$\begin{aligned}\int_\Omega \left(1-\sqrt{1-|Du|^2}\right)\mathrm{d}x = \sum_{j=1}^\infty b_j\|Du\|_{2j}^{2j} &\le \sum_{j=1}^\infty b_j \liminf_{k\to\infty}\|Du_{\rho_k}\|_{2j}^{2j} \\ &\le \lim_{n\to\infty}\liminf_{k\to\infty}\sum_{j=1}^n b_j\|Du_{\rho_k}\|_{2j}^{2j} \\ &\le \liminf_{k\to\infty}\int_\Omega\left(1-\sqrt{1-|Du_{\rho_k}|^2}\right)\mathrm{d}x.\end{aligned} \tag{3.10}$$

From

$$\left\langle \rho_k, u_{\rho_k} \right\rangle = \left\langle \mu_k, u_{\rho_k} \right\rangle + \left\langle f_k, u_{\rho_k} \right\rangle$$

and the facts that $u_{\rho_k} \rightharpoonup u$ weakly in $\mathcal{Y}(\Omega)$ and strongly in $C(\overline{\Omega})$, our assumptions on $\{\mu_k\}$ and $\{f_k\}$ give

$$\lim_{k\to\infty} \left\langle \rho_k, u_{\rho_k} \right\rangle = \langle \mu, u \rangle + \langle f, u \rangle = \langle \rho, u \rangle . \tag{3.11}$$

Hence, by (3.10), we obtain

$$I_\rho(u_\rho) \le I_\rho(u) \le \liminf_{k\to\infty} I_{\rho_k}(u_{\rho_k}) \le \liminf_{k\to\infty} I_{\rho_k}(u_\rho) = I_\rho(u_\rho).$$

Thus, $I_\rho(u) = I_\rho(u_\rho)$, which yields $u = u_\rho$ by the uniqueness of the minimizer, and

$$\int_\Omega \left(1 - \sqrt{1 - |Du_{\rho_k}|^2}\right) \mathrm{d}x \to \int_\Omega \left(1 - \sqrt{1 - |Du_\rho|^2}\right) \mathrm{d}x.$$

If there exists $j_0 > 0$ such that

$$\varepsilon_0 \doteq \liminf_{k\to\infty} \|Du_{\rho_k}\|_{2j_0}^{2j_0} - \|Du_\rho\|_{2j_0}^{2j_0} > 0,$$

then by (3.5) we can choose $h_0 > j_0$ so large that

$$\int_\Omega \left(1 - \sqrt{1 - |Du_\rho|^2}\right) \mathrm{d}x - \sum_{j=1}^{h_0} b_j \|Du_\rho\|_{2j}^{2j} < \frac{b_{j_0}\varepsilon_0}{2},$$

and therefore deduce the following contradiction:

$$\begin{aligned}
\int_\Omega \left(1 - \sqrt{1 - |Du_\rho|^2}\right) \mathrm{d}x &< \frac{b_{j_0}\varepsilon_0}{2} + \sum_{j=1}^{h_0} b_j \|Du_\rho\|_{2j}^{2j} \\
&\le \liminf_{k\to\infty} \sum_{j=1}^{h_0} b_j \|Du_{\rho_k}\|_{2j}^{2j} - \frac{b_{j_0}\varepsilon_0}{2} \\
&\le \liminf_{k\to\infty} \int_\Omega \left(1 - \sqrt{1 - |Du_{\rho_k}|^2}\right) \mathrm{d}x - \frac{b_{j_0}\varepsilon_0}{2} \\
&= \int_\Omega \left(1 - \sqrt{1 - |Du_\rho|^2}\right) \mathrm{d}x - \frac{b_{j_0}\varepsilon_0}{2}.
\end{aligned}$$

Thus, $\|Du_{\rho_k}\|_{2j} \to \|Du_\rho\|_{2j}$ for each $j \ge 1$. The uniform convexity of $L^{2j}(\Omega)$ and $\|u_{\rho_k} - u_\rho\|_\infty \to 0$ imply that $Du_{\rho_k} \to Du_\rho$ in $L^{2j}(\Omega)$, hence $u_{\rho_k} \to u_\rho$ in $W^{1,2j}(\Omega)$ for any $j \ge 1$. By Hölder's inequality, $u_{\rho_k} \to u_\rho$ strongly in $W^{1,q}(\Omega)$ for each $q \in [1, \infty)$ and we complete the proof for the case Ω is a bounded domain.

When $\Omega = \mathbb{R}^m$ with $m \ge 3$, first observe that by our assumptions $\{\rho_k\}$ is uniformly bounded in $\mathcal{Y}(\Omega)^*$. Hence, from $I_{\rho_k}(u_{\rho_k}) \le I_{\rho_k}(0) = 0$ and the coercivity estimate (3.6) for $v = u_{\rho_k}$, we deduce that $\{u_{\rho_k}\}$ is uniformly bounded in $\mathcal{Y}(\mathbb{R}^m)$. By Proposition 3.3 and $\|Du_{\rho_k}\|_\infty \le 1$, $\{u_{\rho_k}\}$ is bounded in $W^{1,q}(\mathbb{R}^m)$ for each $q \in [2^*, \infty)$, hence in $L^\infty(\mathbb{R}^m)$. Up to a subsequence, $u_{\rho_k} \rightharpoonup u$ weakly in $W^{1,q}(\mathbb{R}^m)$ for each $q \in [2^*, \infty)$, $u_{\rho_k} \to u$ in $C_{\mathrm{loc}}(\mathbb{R}^m)$, and $u_{\rho_k} \to u$ weakly in $\mathcal{Y}(\mathbb{R}^m)$ by the reflexivity of $\mathcal{Y}(\mathbb{R}^m)$. Since each u_{ρ_k} is 1-Lipschitz, so is u and $u \in \mathcal{Y}_0(\mathbb{R}^m)$. Coupling condition (3.9) for $\{\mu_k\}$ with the convergence $u_{\rho_k} \to u$ in $C_{\mathrm{loc}}(\mathbb{R}^m)$ and the uniform boundedness of $\{u_{\rho_k}\}$, we deduce that $\langle \mu_k, u_{\rho_k}\rangle \to \langle \mu, u\rangle$, hence (3.11) holds. Then, arguing as above, we may verify $u = u_\rho$ and $Du_{\rho_k} \to Du_\rho$ strongly in $L^q(\mathbb{R}^m)$ for each $q \in [2,\infty)$. Hence, $u_{\rho_k} \to u_\rho$ strongly in $W^{1,q}(\mathbb{R}^m)$ for every $q \in [2^*,\infty)$, concluding the proof. □

3.4. **Local integrability of w and the Euler-Lagrange inequality.** Assuming $\phi \in \mathcal{S}(\partial\Omega)$ if Ω is bounded, in this subsection we show that the minimizer u_ρ is not too degenerate and solves an Euler-Lagrange inequality. We begin with a simple but useful Lemma, which will be repeatedly used.

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^m$ be a domain, let $\mathcal{G} \subset W^{1,\infty}(\Omega)$ be compact in $C(K)$ for each compact set $K \subset \Omega$, and assume that $\|Du\|_\infty \le 1$ on Ω for each $u \in \mathcal{G}$. Fix an open subset $\Omega' \Subset \Omega$ and $\widetilde{\varepsilon} > 0$. Then, the following are equivalent:*

- (a) *For each $\Omega'' \Subset \Omega'$ with $\mathrm{d}_\delta(\Omega'', \partial\Omega') \ge \widetilde{\varepsilon}$, every $u \in \mathcal{G}$ does not have a light segment $\overline{xy} \subset \overline{\Omega'}\backslash\Omega''$ with $x \in \partial\Omega''$, $y \in \partial\Omega'$.*
- (b) *There exists $R = R(\mathcal{G}, \widetilde{\varepsilon}, \Omega') > 0$ such that $L^u_R(\Omega'') \Subset \Omega'$ for each $u \in \mathcal{G}$ and each $\Omega'' \Subset \Omega'$ satisfying $\mathrm{d}_\delta(\Omega'', \partial\Omega') \ge \widetilde{\varepsilon}$, where L^u_R is the Lorentzian ball of radius R associated to the graph of u.*

Furthermore, the following are equivalent:

- (a') *Every $u \in \mathcal{G}$ does not have light segments in Ω'.*
- (b') *For each $\varepsilon > 0$, there exists $R = R(\mathcal{G}, \varepsilon, \Omega') > 0$ such that for each pair of open subsets $\Omega_1 \Subset \Omega_2 \subset \Omega'$ with $\mathrm{d}_\delta(\Omega_1, \partial\Omega_2) \ge \varepsilon$, it holds $L^u_R(\Omega_1) \Subset \Omega_2$ for each $u \in \mathcal{G}$.*

Proof. (b) ⇒ (a) and (b') ⇒ (a') are obvious. The proofs of (a) ⇒ (b) and (a') ⇒ (b') are analogous, so we only prove (a') ⇒ (b'). Assume by contradiction the existence of $\varepsilon > 0$, $\Omega_1^{(j)} \Subset \Omega_2^{(j)}$ with $\mathrm{d}_\delta(\Omega_1^{(j)}, \partial\Omega_2^{(j)}) \ge \varepsilon$, $u_j \in \mathcal{G}$, points $z_j \in \partial\Omega_1^{(j)}$ and $p_j \in \partial\Omega_2^{(j)}$ such that

$$\overline{z_j p_j} \subset \overline{\Omega_2^{(j)}} \subset \overline{\Omega'}, \quad \mathcal{H}^1_\delta\left(\overline{z_j p_j}\right) \ge \varepsilon, \quad \left|z_j - p_j\right| - \left|u_j(z_j) - u_j(p_j)\right| \le \frac{1}{j}. \tag{3.12}$$

Since $\mathcal{G}$ is compact in $C(\overline{\Omega'})$, up to subsequences, $u_j \to u \in \mathcal{G}$ in $C(\overline{\Omega'})$, $z_j \to z \in \overline{\Omega'}$ and $p_j \to p \in \overline{\Omega'}$. Passing to the limit in (3.12), u has a light segment $\overline{zp}$ of length $\ge \varepsilon$. Noticing that $B_\varepsilon(z_j) \subset \Omega$ for each j, we get $B_\varepsilon(z) \subset \Omega'$ and thus part of $\overline{zp}$ lies in Ω', a contradiction. □

We are ready to state our first regularity result. The argument in the proof is inspired by [8, Proposition 2.6], in particular, case (ii) in the following is essentially contained therein.

Proposition 3.9. *Let $\Omega \subset \mathbb{R}^m$ be a domain.*

- (i) *Assume that $m \ge 2$ and that Ω is bounded. For any given compact subset $\mathcal{F} \subset \mathcal{S}(\partial\Omega)$, and any $\varepsilon, \mathcal{I}_1 > 0$, there exists a constant $C = C(\Omega, \mathcal{F}, m, p_1, \mathcal{I}_1, \mathrm{diam}_\delta(\Omega), \varepsilon)$*

such that if

$$\phi \in \mathscr{F}, \qquad \rho \in \mathcal{Y}(\Omega)^* \ \textit{ with } \ \|\rho\|_{\mathcal{Y}^*} \le \mathcal{I}_1,$$

then for each open subset $\Omega' \Subset \Omega$ *with* $\mathrm{d}_\delta(\Omega', \partial\Omega) \ge \varepsilon$ *the minimizer* u_ρ *satisfies*

$$\int_{\Omega'} \frac{\mathrm{d}x}{\sqrt{1-|Du_\rho|^2}} \le C. \tag{3.13}$$

In particular, $|Du_\rho| < 1$ *a.e. on* Ω*. Moreover, for each* $\psi \in \mathcal{Y}_\phi(\Omega)$*,*

$$\frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1-|Du_\rho|^2}} \in L^1(\Omega), \tag{3.14}$$

$$\sqrt{1-|D\psi|^2} - \sqrt{1-|Du_\rho|^2} \le \frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1-|Du_\rho|^2}} \qquad \textit{a.e. on } \Omega \tag{3.15}$$

and

$$\int_\Omega \left(\sqrt{1-|D\psi|^2} - \sqrt{1-|Du_\rho|^2}\right)\mathrm{d}x \le \int_\Omega \frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1-|Du_\rho|^2}}\mathrm{d}x \le \left\langle \rho, u_\rho - \psi \right\rangle. \tag{3.16}$$

(ii) *Assume that* $m \ge 3$ *and that* $\Omega = \mathbb{R}^m$*. For any given* $\mathcal{I}_1 > 0$ *and* $\Omega' \Subset \mathbb{R}^m$*, there exists a constant* $C' = C'(m, p_1, \mathcal{I}_1, |\Omega'|_\delta) > 0$ *such that if* $\|\rho\|_{\mathcal{Y}^*} \le \mathcal{I}_1$*, then* (3.13) *holds with* C'*. Furthermore,* (3.14)–(3.16) *hold for each* $\psi \in \mathcal{Y}_0(\mathbb{R}^m)$*.*

Remark 3.10. Notice that choosing $\Omega = \mathbb{R}^m$ and $\psi = 0$ in (3.14) we infer the integrability condition in (1.8) mentioned in the Introduction.

Proof. (i) We first prove (3.13). Fix $\Omega' \Subset \Omega$ with $\mathrm{d}_\delta(\Omega', \partial\Omega) \ge \varepsilon$. Given $\psi \in \mathcal{Y}_\phi(\Omega)$, observe that $u_t \doteq (1-t)u_\rho + t\psi \in \mathcal{Y}_\phi(\Omega)$ for every $t \in (0,1]$. Thus, $I_\rho(u_\rho) \le I_\rho(u_t)$, and rearranging we get

$$\frac{1}{t}\int_\Omega \left(\sqrt{1-|Du_t|^2} - \sqrt{1-|Du_\rho|^2}\right)\mathrm{d}x \le \left\langle \rho, u_\rho - \psi \right\rangle \quad \forall\, t \in (0,1]. \tag{3.17}$$

Next, the concavity of the map $p \mapsto \sqrt{1-|p|^2}$ on $\overline{B_1(0)}$ implies that

$$\sqrt{1-|Du_t|^2} \ge (1-t)\sqrt{1-\left|Du_\rho\right|^2} + t\sqrt{1-|D\psi|^2} \qquad \text{a.e. on } \Omega,\ \forall\, t \in (0,1],$$

which yields

$$\sqrt{1-|D\psi|^2} - \sqrt{1-|Du_\rho|^2} \le \frac{1}{t}\left\{\sqrt{1-|Du_t|^2} - \sqrt{1-|Du_\rho|^2}\right\} \qquad \text{a.e. on } \Omega. \tag{3.18}$$

Let $\mathscr{G} \subset \mathcal{Y}(\Omega)$ be the set of minimizers of I_0 (i.e. with $\rho = 0$) whose boundary value lies in $\mathscr{F}$. For given $\phi \in \mathscr{F}$ we denote by $\bar{\phi} \in \mathscr{G}$ the corresponding minimizer. The compactness of $\mathscr{F}$ and Propositions 3.5 and 3.7 guarantee that $\mathscr{G}$ is compact in $C(\overline{\Omega})$. By Theorem 1.3, every $u \in \mathscr{G}$ does not have light segments in Ω, thus applying the first part of Lemma 3.8 for $\Omega_\varepsilon \Subset \Omega_{\varepsilon/2}$ we obtain $R = R(\Omega, \mathscr{F}, \varepsilon) > 0$ such that $L^u_R(\Omega_\varepsilon) \Subset \Omega_{\varepsilon/2}$ for each $u \in \mathscr{G}$. From the monotonicity formula [4, Lemma 2.1], we infer the existence of $\theta = \theta(\Omega, \mathscr{F}, \varepsilon)$ such that

$$\sup_{\Omega'} |D\bar{\phi}| \le 1 - 4\theta. \tag{3.19}$$

We take $\psi = \bar{\phi}$, and note that on the set of full measure $V \subset \Omega'$ of points where u_ρ is differentiable it holds $|Du_t| < 1$ for every $t \in (0,1]$. We set

$$K \doteq \Big\{ x \in \Omega \ : \ 1 - \theta < |Du_\rho(x)| \Big\},$$

and split the domain of integration Ω in (3.17) into the sets $\Omega \setminus \Omega'$, $V \cap K$ and $V \cap K^c$. We use (3.18) on $\Omega \setminus \Omega'$ and the identity

$$\begin{aligned} &\frac{1}{t}\Big\{ \sqrt{1 - |Du_t|^2} - \sqrt{1 - |Du_\rho|^2} \Big\} \\ &\qquad = \frac{2Du_\rho \cdot (Du_\rho - D\psi) - t|Du_\rho - D\psi|^2}{\sqrt{1 - |Du_t|^2} + \sqrt{1 - |Du_\rho|^2}} \qquad \text{a.e. on } \Omega \cap \big\{ |D\psi| + |Du_\rho| < 2 \big\} \end{aligned} \tag{3.20}$$

to deduce that

$$\begin{aligned} \big\langle \rho, u_\rho - \bar{\phi} \big\rangle \ge & \int_{\Omega \setminus \Omega'} \Big(\sqrt{1 - |D\bar{\phi}|^2} - \sqrt{1 - |Du_\rho|^2} \Big) \mathrm{d}x \\ & + \int_{V \cap K} \frac{2Du_\rho \cdot (Du_\rho - D\bar{\phi}) - t|Du_\rho - D\bar{\phi}|^2}{\sqrt{1 - |Du_t|^2} + \sqrt{1 - |Du_\rho|^2}} \mathrm{d}x \\ & + \int_{V \cap K^c} \frac{2Du_\rho \cdot (Du_\rho - D\bar{\phi}) - t|Du_\rho - D\bar{\phi}|^2}{\sqrt{1 - |Du_t|^2} + \sqrt{1 - |Du_\rho|^2}} \mathrm{d}x. \end{aligned} \tag{3.21}$$

Recalling (3.19), we restrict to t small enough so that $4t < \theta^2$. By the definition of K, the next inequality holds on $\Omega' \cap K$:

$$2Du_\rho \cdot (Du_\rho - D\bar{\phi}) - t|Du - D\bar{\phi}|^2 \ge 2\left[(1-\theta)^2 - (1 - 4\theta) \right] - 4t > 4\theta > 0. \tag{3.22}$$

Remark also that the last term in the right-hand side of (3.21) is bounded uniformly with respect to $t \in (0,1)$. Thus, letting $t \to 0$ in (3.21) and using (3.22), Fatou's lemma and the dominated convergence theorem, we infer

$$\begin{aligned} \big\langle \rho, u_\rho - \bar{\phi} \big\rangle \ge & \int_{\Omega \setminus \Omega'} \Big(\sqrt{1 - |D\bar{\phi}|^2} - \sqrt{1 - |Du_\rho|^2} \Big) \mathrm{d}x \\ & + \int_{V \cap K} \frac{2\theta}{\sqrt{1 - |Du_\rho|^2}} \mathrm{d}x + \int_{V \cap K^c} \frac{Du_\rho \cdot (Du_\rho - D\bar{\phi})}{\sqrt{1 - |Du_\rho|^2}} \mathrm{d}x. \end{aligned} \tag{3.23}$$

From

$$\left|\int_{\Omega\setminus\Omega'} \sqrt{1-|D\bar{\phi}|^2} - \sqrt{1-|Du_\rho|^2}\,\mathrm{d}x\right| \le |\Omega\setminus\Omega'|_\delta \tag{3.24}$$

and the following straightforward estimate on $\Omega'\cap K^c$:

$$\int_{\Omega'\cap K^c} \left|\frac{Du_\rho\cdot(Du_\rho - D\bar{\phi})}{\sqrt{1-|Du_\rho|^2}}\right| \mathrm{d}x \le \int_{\Omega'\cap K^c} \frac{2\mathrm{d}x}{\sqrt{2\theta-\theta^2}} \le \frac{2|\Omega'|_\delta}{\sqrt{2\theta-\theta^2}},$$

it follows from (3.23) and $|\Omega'\setminus V| = 0$ that

$$\int_{\Omega'\cap K} \frac{2\theta}{\sqrt{1-|Du_\rho|^2}}\mathrm{d}x \le |\Omega\setminus\Omega'|_\delta + \langle \rho, u_\rho - \bar{\phi}\rangle + \frac{2|\Omega'|_\delta}{\sqrt{2\theta-\theta^2}}.$$

Therefore,

$$\begin{aligned}\int_{\Omega'} \frac{\mathrm{d}x}{\sqrt{1-|Du_\rho|^2}} &= \int_{\Omega'\cap K} \frac{\mathrm{d}x}{\sqrt{1-|Du_\rho|^2}} + \int_{\Omega'\cap K^c} \frac{\mathrm{d}x}{\sqrt{1-|Du_\rho|^2}} \\ &\le \frac{1}{2\theta}\left(|\Omega\setminus\Omega'|_\delta + \|\rho\|_{\mathcal{Y}^*}\|u_\rho - \bar{\phi}\|_{\mathcal{Y}} + \frac{2|\Omega'|_\delta}{\sqrt{2\theta-\theta^2}}\right) + \frac{|\Omega'|_\delta}{\sqrt{2\theta-\theta^2}}.\end{aligned} \tag{3.25}$$

For $\psi \in \mathcal{Y}_\phi(\Omega)$, (3.4) and simple estimates for the W^{1,p_1} norm give

$$\|u_\rho - \bar{\phi}\|_{\mathcal{Y}} \le 4\left(\sup_{\phi\in\mathscr{F}} \|\phi\|_{C(\partial\Omega)} + \mathrm{diam}_\delta(\Omega) + |\Omega|_\delta^{\frac{1}{p_1}}\right).$$

Hence, (3.13) holds by (3.25). Notice that, by (3.13) and the arbitrariness of Ω', $|Du_\rho| < 1$ a.e. on Ω.

Next, we shall prove (3.14)–(3.16). Let $\psi \in \mathcal{Y}_\phi(\Omega)$ and consider as above $u_t \doteq (1-t)u_\rho + t\psi \in \mathcal{Y}_\phi(\Omega)$ for $t \in (0,1)$. By combining $|Du_\rho| < 1$ a.e. Ω, (3.20) and (3.18), for each $t \in (0,1)$,

$$\sqrt{1-|D\psi|^2} - \sqrt{1-|Du_\rho|^2} \le \frac{2Du_\rho\cdot(Du_\rho - D\psi) - t|Du_\rho - D\psi|^2}{\sqrt{1-|Du_t|^2} + \sqrt{1-|Du_\rho|^2}} \qquad \text{a.e. on } \Omega. \tag{3.26}$$

Thus letting $t \to 0$ on the set $\{|Du_\rho| < 1\}$, we deduce (3.15).

On the other hand, from (3.17) and (3.20), it follows that

$$\int_\Omega \frac{2Du_\rho\cdot(Du_\rho - D\psi) - t|Du_\rho - D\psi|^2}{\sqrt{1-|Du_t|^2} + \sqrt{1-|Du_\rho|^2}}\mathrm{d}x \le \langle \rho, u_\rho - \psi\rangle.$$

Using a variant of Fatou's lemma as $t \to 0$ and (3.26), we therefore deduce

$$\int_\Omega \left(\sqrt{1-|D\psi|^2} - \sqrt{1-|Du_\rho|^2}\right)\mathrm{d}x \le \int_\Omega \frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1-|Du_\rho|^2}}\mathrm{d}x \le \left\langle \rho, u_\rho - \psi \right\rangle,$$

which proves (3.16). Taking (3.15) into account, both the negative and the positive part of

$$\frac{Du_\rho \cdot (Du_\rho - D\psi)}{\sqrt{1-|Du_\rho|^2}}$$

are integrable, and (3.14) holds.

(ii) We first observe that (3.6), $I_\rho(u_\rho) \le I_\rho(0) = 0$ and $\|\rho\|_{\mathcal{Y}^*} \le \mathcal{I}_1$ imply that $\|u_\rho\|_{\mathcal{Y}} \le C_1(m, \mathcal{I}_1)$. One can therefore perform the same computations in (3.17)–(3.23) with $\Omega = \mathbb{R}^m$, $\bar{\phi} = 0$, $\theta = 1/8$ and replacing (3.24) with

$$0 \le \int_{\mathbb{R}^m \setminus \Omega'} \left(1 - \sqrt{1-|Du_\rho|^2}\right)\mathrm{d}x \le I_\rho(u_\rho) + \left\langle \rho, u_\rho \right\rangle \le \mathcal{I}_1 C_1.$$

Inequality (3.25) becomes

$$\int_{\Omega'} \frac{\mathrm{d}x}{\sqrt{1-|Du_\rho|^2}} \le 4\left(2\mathcal{I}_1 C_1 + C_2|\Omega'|_\delta\right) + C_2|\Omega'|_\delta,$$

for some absolute constant C_2. The rest of the proof follows verbatim, taking into account that $1-\sqrt{1-|p|^2} \le |p|^2$ on $\overline{B_1(0)}$ and thus $\sqrt{1-|D\psi|^2} - \sqrt{1-|Du_\rho|^2} = (1-\sqrt{1-|Du_\rho|^2}) - (1-\sqrt{1-|D\psi|^2}) \in L^1(\mathbb{R}^m)$. This completes the proof. □

Remark 3.11. Inequality (3.15) has a nice geometric interpretation, holding more generally for pairs of Lipschitz functions u, ψ with $|Du| < 1$, $|D\psi| \le 1$ a.e. on Ω. Indeed, if we consider the normal vectors $\mathbf{n}'_u \doteq Du + \partial_0$, $\mathbf{n}'_\psi = D\psi + \partial_0$ (respectively, timelike and causal a.e. on Ω), the reversed Cauchy-Schwarz inequality $-\mathbf{n}'_u \cdot \mathbf{n}'_\psi \ge |\mathbf{n}'_u|_{\mathbb{L}} |\mathbf{n}'_\psi|_{\mathbb{L}}$ is equivalent to

$$\frac{\mathbf{n}'_u}{|\mathbf{n}'_u|_{\mathbb{L}}} \cdot (\mathbf{n}'_u - \mathbf{n}'_\psi) \ge |\mathbf{n}'_\psi|_{\mathbb{L}} - |\mathbf{n}'_u|_{\mathbb{L}},$$

that can be rewritten as (3.15) with u replacing u_ρ.

3.5. **Global minimizers VS solutions to ($\mathcal{BI}$).** In this section, we describe in detail the interplay between solutions of ($\mathcal{BI}$) and global minimizers of I_ρ, stating some useful equivalent characterizations of the solvability of ($\mathcal{BI}$) that, perhaps surprisingly, hold without assuming any regularity of $\partial\Omega$.

Proposition 3.12 (Approximation)**.** *Let $\Omega \subset \mathbb{R}^m$ be an open set, let $u, \psi : \Omega \to \mathbb{R}$ and for $\varepsilon > 0$ define*

$$\psi^u_\varepsilon \doteq \max\{u, \psi - \varepsilon\} + \min\{u, \psi + \varepsilon\} - u = \begin{cases} u & \text{if } |\psi - u| < \varepsilon, \\ \psi + \varepsilon & \text{if } u \ge \psi + \varepsilon, \\ \psi - \varepsilon & \text{if } u \le \psi - \varepsilon. \end{cases}$$

Consider a sequence $\{\varepsilon_j\} \subset \mathbb{R}^+$, $\varepsilon_j \to 0$ and functions $u_j : \Omega \to \mathbb{R}$, and define $\psi_j \doteq \psi^{u_j}_{\varepsilon_j}$.

(i) *If $m \ge 2$, Ω is a bounded domain, $\phi \in \mathcal{S}(\partial\Omega)$ and $u, u_j, \psi \in \mathcal{Y}_\phi(\Omega)$ satisfy $u_j \to u$ in $\mathcal{Y}(\Omega)$, then $\{\psi_j\} \subset \mathcal{Y}_\phi(\Omega)$ and*
   (a) *$\psi_j \equiv u_j$ on $\Omega \setminus \Omega_j$ for some set $\Omega_j \Subset \Omega$. Moreover, if for some $\Omega' \Subset \Omega$ it holds $\psi \equiv u$ and $|u_j - u| < \varepsilon_j$ on $\Omega \setminus \Omega'$, then $\psi_j \equiv u_j$ on $\Omega \setminus \Omega'$;*
   (b) *as $j \to \infty$, $\psi_j \to \psi$ in $W^{1,q}(\Omega) \cap C(\overline{\Omega})$ for each $q \in [1, \infty)$;*

(ii) *If $m \ge 3$, $\Omega = \mathbb{R}^m$ and $u, u_j, \psi \in \mathcal{Y}_0(\mathbb{R}^m)$ satisfy $u_j \to u$ in $\mathcal{Y}(\mathbb{R}^m)$, then $\{\psi_j\} \subset \mathcal{Y}_0(\mathbb{R}^m)$ and* (a) *holds. Furthermore,* (b) *holds with $q \in [2^*, \infty)$, and $\|D\psi_j - D\psi\|_q \to 0$ for all $q \in [2, \infty)$.*

Proof. (i) By $u, u_j, \psi \in \mathcal{Y}_\phi(\Omega)$ and Proposition 3.5, $u, u_j, \psi \in C(\overline{\Omega})$ with $u = u_j = \psi = \phi$ on $\partial\Omega$. Remark that by construction,

$$\psi_j \in C(\overline{\Omega}), \qquad \|\psi_j - \psi\|_\infty \le \varepsilon_j \to 0, \qquad \Omega_j \doteq \{|u_j - \psi| \ge \varepsilon_j\} \Subset \Omega. \tag{3.27}$$

Note also that $\psi_j \equiv u_j$ on $\Omega \setminus \Omega_j$. Furthermore, if $\psi \equiv u$ and $|u_j - u| < \varepsilon_j$ on $\Omega \setminus \Omega'$ for some $\Omega' \Subset \Omega$, then the identity $|u_j - \psi| = |u_j - u| < \varepsilon_j$ holds on $\Omega \setminus \Omega'$ and the definition of ψ_j guarantees that $\psi_j \equiv u_j$ on $\Omega \setminus \Omega'$. Therefore, (a) holds.

Next, the identity

$$D\psi_j = \begin{cases} D\psi & \text{a.e. on } |\psi - u_j| \ge \varepsilon_j, \\ Du_j & \text{a.e. on } |\psi - u_j| < \varepsilon_j \end{cases} \tag{3.28}$$

implies that $|D\psi_j| \le 1$ a.e. on Ω. Since $\psi_j = u_j$ on $\Omega \setminus \Omega_j$ and $u_j \in \mathcal{Y}_\phi(\Omega)$, we infer $\psi_j \in \mathcal{Y}_\phi(\Omega)$. In addition, from $u_j \to u$ in $\mathcal{Y}(\Omega)$, we infer $u_j \to u$ in $C(\overline{\Omega})$. Thus, fix $\{\delta_j\}$ such that $\delta_j \to 0$ and $\|u_j - u\|_\infty < \delta_j$. Taking a subsequence $\{j_k\}$, we have $Du_{j_k}(x) \to Du$ a.e. in Ω. Then as $k \to \infty$, a.e. Ω,

$$\begin{aligned} |D\psi_{j_k} - D\psi| = |Du_{j_k} - D\psi| \cdot \mathbb{1}_{\{|\psi - u_{j_k}| < \varepsilon_{j_k}\}} &\le |Du_{j_k} - D\psi| \cdot \mathbb{1}_{\{|\psi - u| < \varepsilon_{j_k} + \delta_{j_k}\}} \\ &\to |Du - D\psi| \cdot \mathbb{1}_{\{|\psi - u| = 0\}} = 0, \end{aligned} \tag{3.29}$$

where we used Stampacchia's theorem (see [19, Theorem 4.4]). Since the limit is unique, $D\psi_j \to D\psi$ a.e. on Ω. Thus, the dominated convergence theorem with $\|D\psi_j\|_\infty \le 1$ yields $\|D\psi_j - D\psi\|_q \to 0$ for each $q \in [1, \infty)$. From (3.27), (b) also holds.

(ii) When $\Omega = \mathbb{R}^m$, from (3.28) it is easily seen that $\psi_j \in \mathcal{Y}_0(\mathbb{R}^m)$. In addition, by Proposition 3.3, $\|u_j - u\|_\infty \to 0$ and $\mathcal{Y}_0(\mathbb{R}^m) \hookrightarrow C_0(\mathbb{R}^m)$. Hence, we may apply the same argument as above to prove (a) in this case. As for (b), setting

$f_k \doteq |Du_{j_k} - D\psi|$, $g_k \doteq \mathbb{1}_{\{|\psi - u| < \varepsilon_{j_k} + \delta_{j_k}\}}$ and $f = |Du - D\psi|$, we deduce from (3.29) that

$$\|D\psi_{j_k} - D\psi\|_2 \le \|f_k g_k\|_2 \le \|(f_k - f)g_k\|_2 + \|f g_k\|_2 \le \|(f_k - f)\|_2 + \|f g_k\|_2 \to 0$$

as $k \to \infty$, where we used $u_{j_k} \to u$ in $\mathcal{Y}(\mathbb{R}^m)$, $f g_k \to 0$ a.e. $\mathbb{R}^m$ and the dominated convergence theorem. The bound $\|D\psi_{j_k} - D\psi\|_\infty \le 2$ then implies $\|D\psi_{j_k} - D\psi\|_q \to 0$ for all $q \in [2, \infty)$. Since the limit is unique, $\|D\psi_j - D\psi\|_q \to 0$ for all $q \in [2, \infty)$. From $\|\psi_j - \psi\|_\infty \to 0$ and Sobolev's inequality, it follows that $\|\psi_j - \psi\|_{W^{1,q}} \to 0$ for all $q \in [2^*, \infty)$ and (b) also holds. □

Definition 3.13. We say that $u \in \mathcal{Y}_\phi(\Omega)$ is a *local minimizer for* I_ρ if $I_\rho(u) \le I_\rho(\psi)$ for every $\psi \in \mathcal{Y}_\phi(\Omega)$ with $\{u \neq \psi\} \Subset \Omega$. Similarly, for $\Omega = \mathbb{R}^m$, we say that $u \in \mathcal{Y}_0(\mathbb{R}^m)$ is a local minimizer for I_ρ if $I_\rho(u) \le I_\rho(\psi)$ for every $\psi \in \mathcal{Y}_0(\mathbb{R}^m)$ with $\{u \neq \psi\} \Subset \mathbb{R}^m$.

We are ready to state the following

Proposition 3.14 (Minimizers VS solutions to ($\mathcal{BI}$)). *Let $m \ge 2$, Ω be a bounded domain, $\phi \in \mathcal{S}(\partial\Omega)$ and u a local minimizer. Then, $u = u_\rho$. Furthermore, the following are equivalent:*

(i) *u is a weak solution to ($\mathcal{BI}$), that is,*

(3.30)
$$\frac{1}{\sqrt{1 - |Du|^2}} \in L^1_{\mathrm{loc}}(\Omega), \qquad \int_\Omega \frac{Du \cdot D\eta}{\sqrt{1 - |Du|^2}} \mathrm{d}x = \langle \rho, \eta \rangle \quad \forall\, \eta \in \mathrm{Lip}_c(\Omega);$$

(ii) *$u = u_\rho$ and*

$$\int_\Omega \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} \mathrm{d}x = \langle \rho, u - \psi \rangle \qquad \forall\, \psi \in \mathcal{Y}_\phi(\Omega) \ \textit{strictly spacelike};$$

(iii) *$u = u_\rho$ and*

(3.31)
$$\int_\Omega \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} \mathrm{d}x = \langle \rho, u - \psi \rangle \qquad \forall\, \psi \in \mathcal{Y}_\phi(\Omega) \ \textit{with} \ \{\psi \neq u\} \Subset \Omega;$$

(iv) *$u = u_\rho$ and*

$$\int_\Omega \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} \mathrm{d}x = \langle \rho, u - \psi \rangle \qquad \forall\, \psi \in \mathcal{Y}_\phi(\Omega).$$

In particular, if u is a classical solution to ($\mathcal{BI}$), then u satisfies any of (i)–(iv).

The same assertions hold true for $m \ge 3$ and $\Omega = \mathbb{R}^m$.

Proof. Since the case $\Omega = \mathbb{R}^m$ may be proved similarly, we only deal with bounded domains. Let Ω be a bounded domain and u a local minimizer. For $\psi \in \mathcal{Y}_\phi(\Omega)$ and $\varepsilon > 0$, consider the approximation ψ^u_ε constructed in Proposition 3.12, that satisfies $\{\psi^u_\varepsilon \neq u\} \Subset \Omega$. We first notice $I_\rho(u) \le I_\rho(\psi^u_\varepsilon)$. Since $I_\rho \in C(\mathcal{Y}_\phi(\Omega), \mathbb{R})$ as observed in Subsection 3.1, Proposition 3.12 implies $I_\rho(\psi^u_\varepsilon) \to I_\rho(\psi)$ and $I_\rho(u) \le I_\rho(\psi)$ for every $\psi \in \mathcal{Y}_\phi(\Omega)$. Thus, $u = u_\rho$. Also, if u is a classical solution to ($\mathcal{BI}$), then an integration by parts shows that (3.30) holds.

We next prove that (iv) ⇒ (ii) ⇒ (i) ⇒ (iii) ⇒ (iv).

(iv) ⇒ (ii) is obvious.

(ii) ⇒ (i).
Since $u = u_\rho$, the integrability $(1-|Du|^2)^{-1/2} \in L^1_{\mathrm{loc}}(\Omega)$ follows by Proposition 3.9. By density and the dominated convergence theorem, it is enough to prove (i) for $\eta \in C^1_c(\Omega)$. Fix an open set Ω' satisfying $\{\eta \neq 0\} \Subset \Omega' \Subset \Omega$, and choose a strictly spacelike extension $\bar\phi$ of ϕ, for instance the solution to ($\mathcal{BI}$) for $\rho = 0$ as in Theorem 1.3. Since $\sup_{\Omega'} |D\bar\phi| < 1$, for $|t|$ small enough, the function $\psi \doteq \bar\phi + t\eta \in \mathcal{Y}_\phi(\Omega)$ is strictly spacelike and thus

$$\int_\Omega \frac{Du\cdot(Du - D\bar\phi - tD\eta)}{\sqrt{1-|Du|^2}}\mathrm{d}x = \langle \rho, u - \bar\phi - t\eta\rangle.$$

Differentiating at $t = 0$ gives (3.30).

(i) ⇒ (iii).
Identity (3.31) follows immediately from (3.30) since $u - \psi \in \mathrm{Lip}_c(\Omega)$. To show that (3.31) implies $u = u_\rho$, first observe that $|Du| < 1$ a.e on Ω, in view of the first property in (3.30). Let $\psi \in \mathcal{Y}_\phi(\Omega)$ with $\{\psi \neq u\} \Subset \Omega$. Apply Remark 3.11 and (3.31) to deduce

$$\int_\Omega \left(\sqrt{1-|D\psi|^2} - \sqrt{1-|Du|^2}\right)\mathrm{d}x \le \int_\Omega \frac{Du\cdot(Du - D\psi)}{\sqrt{1-|Du|^2}}\mathrm{d}x = \langle \rho, u-\psi\rangle,$$

which can be rewritten as $I_\rho(u) \le I_\rho(\psi)$. Hence, u is a local minimizer and thus it coincides with u_ρ.

(iii) ⇒ (iv).
We recall (3.16), argue by contradiction and suppose that there exist $\psi \in \mathcal{Y}_\phi(\Omega)$ and $\delta > 0$ such that

$$\int_\Omega \frac{Du\cdot(Du - D\psi)}{\sqrt{1-|Du|^2}}\mathrm{d}x \le \langle \rho, u-\psi\rangle - \delta. \tag{3.32}$$

Select $\Omega' \Subset \Omega$ satisfying

$$\int_{\Omega\setminus\Omega'} \left|\frac{Du\cdot(Du - D\psi)}{\sqrt{1-|Du|^2}}\right|\mathrm{d}x < \frac{\delta}{4}, \tag{3.33}$$

which is possible by (3.14). Fix a sequence $\varepsilon_j \downarrow 0$ and consider the approximation ψ_j for ψ constructed in Proposition 3.12 by choosing $u_j = u$ for each j. By construction, $\psi_j \equiv u$ on $\Omega \setminus \Omega_j$ for some $\Omega_j \Subset \Omega$, and, without loss of generality, we can assume that $\Omega' \subset \Omega_j$ as well as $D\psi_j \to D\psi$ a.e. Ω. From $\psi_j \to \psi$ strongly in $\mathcal{Y}(\Omega)$, we get

$$\langle \rho, u - \psi_j\rangle \to \langle \rho, u-\psi\rangle \qquad \text{as } j \to \infty. \tag{3.34}$$

Also, by (3.13) in Proposition 3.9 and the dominated convergence theorem,

$$\int_{\Omega'} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1-|Du|^2}} \mathrm{d}x \to \int_{\Omega'} \frac{Du \cdot (Du - D\psi)}{\sqrt{1-|Du|^2}} \mathrm{d}x \qquad \text{as } j \to \infty. \tag{3.35}$$

By the definition of ψ_j,

$$Du - D\psi_j = (Du - D\psi) \cdot \mathbb{1}_{V_j}, \qquad \text{where} \qquad V_j \doteq \{|u-\psi| \ge \varepsilon_j\}, \tag{3.36}$$

hence from (3.32) and (3.34), we infer

$$\begin{aligned}
\langle \rho, u - \psi_j \rangle - \delta &\ge \int_{\Omega} \frac{Du \cdot (Du - D\psi)}{\sqrt{1-|Du|^2}} \mathrm{d}x - o_j(1) \\
&= \int_{\Omega \setminus \Omega'} \frac{Du \cdot (Du - D\psi)}{\sqrt{1-|Du|^2}} \mathrm{d}x + \int_{\Omega'} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1-|Du|^2}} \mathrm{d}x - o_j(1) \qquad \text{by (3.35)} \\
&\ge -\frac{\delta}{4} + \int_{\Omega'} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1-|Du|^2}} \mathrm{d}x - o_j(1) \qquad \text{by (3.33)} \\
&= -\frac{\delta}{4} + \int_{\Omega_j} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1-|Du|^2}} \mathrm{d}x - \int_{\Omega_j \setminus \Omega'} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1-|Du|^2}} \mathrm{d}x - o_j(1) \\
&= -\frac{\delta}{4} + \langle \rho, u - \psi_j \rangle - \int_{(\Omega_j \setminus \Omega') \cap V_j} \frac{Du \cdot (Du - D\psi)}{\sqrt{1-|Du|^2}} \mathrm{d}x - o_j(1) \\
&\quad \text{by (3.31) and (3.36)} \\
&\ge -\frac{\delta}{2} + \langle \rho, u - \psi_j \rangle - o_j(1) \qquad \text{by (3.33)},
\end{aligned}$$

a contradiction if j is large enough. □

Remark 3.15. For $\Omega = \mathbb{R}^m$, a different proof of (iii) ⇒ (iv) was given in [8, Theorem 6.4].

4. Weak solutions with light segments

In this section, for $m \ge 3$ we give examples of weak solutions u of ($\mathcal{BI}$) with a light segment, and whose mean curvature is of class L^q for suitable q's. The first example is instructive, but the boundary data do not satisfy the strict spacelike condition. The second is slightly complicated, but the solution satisfies the zero Dirichlet boundary condition. The third example, deferred to Appendix A for computational reasons, is similar to the second one but has a higher dimensional set of light segments. Here is our first example:

Proposition 4.1. *Assume $m \ge 3$, $\kappa \in [1, m-1)$ and $\Omega \subset \mathbb{R}^m$ is a bounded domain with $0 \in \Omega$. Then, by setting $x' = (x^1, \dots, x^{m-1})$, for sufficiently small $\varepsilon > 0$ the function $u(x', x_m) \doteq (1 - \varepsilon^{2\kappa}|x'|^{2\kappa})x_m$ satisfies*

$$\int_{\Omega} \frac{Du \cdot D\eta}{\sqrt{1-|Du|^2}} \mathrm{d}x = \int_{\Omega} \rho_u \eta \, \mathrm{d}x \qquad \forall \eta \in \mathrm{Lip}_c(\Omega),$$

where $\rho_u \doteq -\mathrm{div}(wDu)$ *and* $w \doteq (1-|Du|^2)^{-\frac{1}{2}}$. *Moreover,*

(4.1)
$$\rho_u \in L^q(\Omega) \quad \forall\, q < \frac{m-1}{2-\kappa} \quad \text{if } 1 \le \kappa < 2, \qquad \rho_u \in L^\infty(\Omega) \quad \text{if } 2 \le \kappa < m-1.$$

Furthermore, $w \in L^q(\Omega)$ *for* $q < (m-1)/\kappa$ *and the second fundamental form* II_u *corresponding to the graph of* u *satisfies*

$$\left\|\mathrm{II}_u\right\| \in L^q(\Omega) \quad \text{for all } q < m-1. \tag{4.2}$$

In particular, u *is a weak solution to* ($\mathcal{BI}$) *and has a light segment on* $\{x'=0\}\cap\Omega$.

Remark 4.2. In Proposition 4.1, the function $u \in C^2$ is weak solution to ($\mathcal{BI}$) on Ω and $\rho_u \in L^\infty(\Omega)$ for $\kappa \ge 2$. Thus, for $\rho \in L^\infty$ the fact that u does not have light segments is not a necessary condition for the $C^{1,\alpha}$–regularity of u.

Below, we shall use the following formula for functions $u(y,z,x_m) = u(|y|,|z|,x_m)$, where $1 \le \ell \le m-2$, $y \in \mathbb{R}^{m-\ell}$, $z \in \mathbb{R}^{\ell-1}$ and $x = (y,z,x_m) \in \mathbb{R}^m$. By writing $u(r,s,x_m)$ for $r = |y|$ and $s = |z|$, it is readily checked that

$$Du = u_r \frac{y}{|y|} + u_s \frac{z}{|z|} + u_m e_m \tag{4.3}$$

and

(4.4)
$$D^2u = \begin{pmatrix} u_{rr}\frac{y}{|y|}\otimes\frac{y}{|y|} + \frac{u_r}{r}\left(I_{m-\ell} - \frac{y}{|y|}\otimes\frac{y}{|y|}\right) & u_{rs}\frac{y}{|y|}\otimes\frac{z}{|z|} & u_{rm}\frac{y}{|y|} \\ u_{rs}\frac{z}{|z|}\otimes\frac{y}{|y|} & u_{ss}\frac{z}{|z|}\otimes\frac{z}{|z|} + \frac{u_s}{s}\left(I_{\ell-1} - \frac{z}{|z|}\otimes\frac{z}{|z|}\right) & u_{sm}\frac{z}{|z|} \\ u_{rm}\frac{y^T}{|y|} & u_{sm}\frac{z^T}{|z|} & u_{mm} \end{pmatrix},$$

where I_k is the identity matrix of size k. Since the matrix

$$u_{rr}\frac{y}{|y|}\otimes\frac{y}{|y|} + \frac{u_r}{r}\left(I_{m-\ell} - \frac{y}{|y|}\otimes\frac{y}{|y|}\right)$$

has eigenvalues u_{rr} and u_r/r with multiplicities 1 and $m-\ell-1$ respectively, we see that

(4.5)
$$\left|D^2u\right|^2 = u_{rr}^2 + (m-\ell-1)\frac{u_r^2}{r^2} + u_{ss}^2 + (\ell-1)\frac{u_s^2}{s^2} + u_{mm}^2 + 2u_{rs}^2 + 2u_{rm}^2 + 2u_{sm}^2,$$

and

$$\Delta u = u_{rr} + \frac{m-\ell-1}{r}u_r + u_{ss} + \frac{\ell-2}{s}u_s + u_{mm}. \tag{4.6}$$

From (4.3) and (4.4) it follows that

$$\begin{aligned} D^2u\,(Du,\cdot) = {} & \left[u_{rr}u_r + u_{rs}u_s + u_{rm}u_m\right]\frac{y}{|y|} + \left[u_{rs}u_r + u_{ss}u_s + u_{sm}u_m\right]\frac{z}{|z|} \\ & + \left[u_{rm}u_r + u_{sm}u_s + u_{mm}u_m\right]e_m, \end{aligned} \tag{4.7}$$

and

(4.8)
$$D^2u\,(Du,Du) = u_{rr}u_r^2 + 2u_{rs}u_ru_s + 2u_{rm}u_ru_m + u_{ss}u_s^2 + 2u_{sm}u_su_m + u_{mm}u_m^2.$$

We remark that for $u(|y|, x_m)$, where $x = (y, x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$, (4.3)–(4.8) also hold with $\ell = 1$ and $u_s, u_{rs}, u_{ss}, u_{ms} = 0$. Hereafter C will stand for a constant whose value may change from line to line.

Proof of Proposition 4.1. We first prove (4.1). Writing $u(x', x_m) = u(|x'|, x_m) = u(r, x_m)$, we have

$$u_r = -2\kappa\varepsilon^{2\kappa} r^{2\kappa-1} x_m, \quad u_m = 1 - \varepsilon^{2\kappa} r^{2\kappa},$$
$$u_{rr} = -2\kappa(2\kappa-1)\varepsilon^{2\kappa} r^{2\kappa-2} x_m, \quad u_{rm} = -2\kappa\varepsilon^{2\kappa} r^{2\kappa-1}, \quad u_{mm} = 0.$$

Hence,

$$w^{-2} = 1 - |Du|^2 = \varepsilon^{2\kappa} r^{2\kappa} \left[2 - \varepsilon^{2\kappa} r^{2\kappa} - 4\kappa^2 \varepsilon^{2\kappa} r^{2\kappa-2} x_m^2\right].$$

Since $\Omega \subset \mathbb{R}^m$ is bounded and $\kappa \geq 1$, if $\varepsilon > 0$ is sufficiently small, then $2 - \varepsilon^{2\kappa} r^{2\kappa} - 4\kappa^2 \varepsilon^{2\kappa} r^{2\kappa-2} x_m^2 \geq 1$ for each $(x', x_m) \in \Omega$. This yields

$$w(x) = \left(1 - |Du(x)|^2\right)^{-\frac{1}{2}} \leq \varepsilon^{-\kappa} r^{-\kappa} = C r^{-\kappa} \quad \forall\, x \in \Omega \text{ with } r = |x'| > 0. \tag{4.9}$$

Since $r = |x'|$ and $x' \in \mathbb{R}^{m-1}$ with $m \geq 3$, $w \in L^q(\Omega)$ holds for all $q < (m-1)/\kappa$. By (4.6) and (4.8), using (4.9), the assumption $\kappa \geq 1$ and the fact that Ω is bounded, we have

$$\left|\rho_u\right| \leq |w \Delta u| + \left|w^3 D^2 u\,(Du, Du)\right| \leq C\left[r^{\kappa-2} + r^{3\kappa-4}\right] \leq C r^{\kappa-2}, \tag{4.10}$$

which implies (4.1).

Next, since $\kappa \geq 1$ and Ω is bounded, it follows from (4.5) and (4.7) that

$$\left|D^2 u\right| \leq C\left\{\left|u_{rr}\right| + \left|\frac{u_r}{r}\right| + \left|u_{rm}\right| + \left|u_{mm}\right|\right\} \leq C\left\{r^{2\kappa-2} + r^{2\kappa-1}\right\} \leq C r^{2\kappa-2},$$
$$\left|D^2 u(Du, \cdot)\right| \leq \left|u_{rr} u_r + u_{rm} u_m\right| + \left|u_{rm} u_r + u_{mm} u_m\right| \leq C\left\{r^{4\kappa-3} + r^{2\kappa-1} + r^{4\kappa-2}\right\} \leq C r^{2\kappa-1}.$$

With (4.9), we infer from (2.4) that

$$\|\mathrm{II}_u\| \leq w\left|D^2 u\right| + 2w^2\left|D^2 u\,(Du, \cdot)\right| + w^3\left|D^2 u\,(Du, Du)\right| \leq C\left[r^{\kappa-2} + r^{-1}\right] \leq C r^{-1}. \tag{4.11}$$

Therefore, (4.2) holds.

Finally, we prove that u is a weak solution. Let $\eta \in \mathrm{Lip}_c(\Omega)$. From (4.1) and the dominated convergence theorem, it follows that

$$\int_\Omega \rho_u \eta \,\mathrm{d}x = \lim_{\tau\to 0} \int_{\Omega\cap\{|x'|>\tau\}} \rho_u \eta \,\mathrm{d}x = -\lim_{\tau\to 0} \int_{\Omega\cap\{|x'|>\tau\}} \operatorname{div}(w Du)\,\eta \,\mathrm{d}x. \tag{4.12}$$

Since η has compact support in Ω, by the divergence theorem,

$$-\int_{\Omega\cap\{|x'|>\tau\}} \operatorname{div}(w Du)\,\eta \,\mathrm{d}x = \int_{\Omega\cap\{|x'|=\tau\}} w\eta Du \cdot \frac{x'}{|x'|} \,\mathrm{d}\mathscr{H}_\delta^{m-1} + \int_{\Omega\cap\{|x'|>\tau\}} w Du \cdot D\eta \,\mathrm{d}x. \tag{4.13}$$

By

$$\left|Du(x) \cdot \frac{x'}{|x'|}\right| = 2\kappa\varepsilon^{2\kappa} \tau^{2\kappa-1} |x_m| \quad \text{if } |x'| = \tau,$$

it follows from (4.9) and the assumption $m \geq 3$ that

$$\limsup_{\tau\to 0} \int_{\Omega\cap\{|x'|=\tau\}} \left| w\eta Du \cdot \frac{x'}{|x'|} \right| \mathrm{d}\mathscr{H}^{m-1}_{\delta} \leq \lim_{\tau\to 0} C\tau^{-\kappa}\tau^{2\kappa-1}\tau^{m-2} = 0.$$

Finally, since $w \in L^1$ because of (4.9) and $\kappa < m-1$, it follows from (4.12) and (4.13) that

$$\int_{\Omega} \rho_u \eta \,\mathrm{d}x = \int_{\Omega} w Du \cdot D\eta \,\mathrm{d}x,$$

and we complete the proof. □

Next, we modify the function in Proposition 4.1 to make the boundary data satisfy a strictly spacelike condition. To this end, for $\varepsilon > 0$, we first fix $\zeta_\varepsilon \in C_c^\infty(\mathbb{R})$ satisfying

$$\zeta_\varepsilon \equiv 1 \quad \text{on } \left[-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right], \quad \zeta_\varepsilon \equiv 0 \quad \text{on } \mathbb{R} \setminus \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), \quad \|\zeta_\varepsilon'\|_{L^\infty(\mathbb{R})} \leq 4\varepsilon. \tag{4.14}$$

Next, let $a_\varepsilon \in C_c^\infty(\mathbb{R})$ be a function with

$$\begin{aligned} &a_\varepsilon(-t) = a_\varepsilon(t), \quad a_\varepsilon(t) = \begin{cases} 1 & \text{if } t \in [0,\varepsilon], \\ 0 & \text{if } t \in [2\varepsilon, \infty), \end{cases} \\ &a_\varepsilon'(t) < 0 \quad \text{if } t \in (\varepsilon, 2\varepsilon), \quad a_\varepsilon(t) = 1 - d_\varepsilon \exp\left(-\frac{1}{t-\varepsilon}\right) \quad \text{if } t \in \left(\varepsilon, \frac{3\varepsilon}{2}\right], \end{aligned} \tag{4.15}$$

where $d_\varepsilon > 0$ is chosen so that $a_\varepsilon(3\varepsilon/2) = 1/2$. Then we set

$$A_\varepsilon(t) \doteq \int_0^t a_\varepsilon(s)\,\mathrm{d}s \in C^\infty(\mathbb{R}) \tag{4.16}$$

and for $\kappa \geq 1$,

$$u_\varepsilon(x', x_m) \doteq \zeta_\varepsilon(|x'|)\left(1 - \varepsilon^{2\kappa}|x'|^{2\kappa}\right)\zeta_\varepsilon(x_m) A_\varepsilon(x_m) \in C_c^2(\mathbb{R}^m).$$

We remark that u_ε has compact support in $\mathbb{R}^m$ and a light segment:

$$u_\varepsilon(0, x_m) = x_m \quad \text{if } |x_m| \leq \varepsilon.$$

Proposition 4.3. *Let $m \geq 3$, $\kappa \in [1, m-1)$ and assume that $\varepsilon > 0$ is sufficiently small. Write $w_\varepsilon \doteq (1 - |Du_\varepsilon|^2)^{-1/2}$, $\rho_{u_\varepsilon} \doteq -\mathrm{div}(w_\varepsilon Du_\varepsilon)$ and denote by $\mathrm{II}_{u_\varepsilon}$ the second fundamental form corresponding to the graph of u_ε. Then*

$$w_\varepsilon \in L^q_{\mathrm{loc}}(\mathbb{R}^m) \quad \text{and} \quad \rho_{u_\varepsilon}, \left\|\mathrm{II}_{u_\varepsilon}\right\| \in L^q(\mathbb{R}^m) \quad \text{for all } q < \frac{m-1}{\kappa}. \tag{4.17}$$

Moreover, u_ε satisfies

$$\int_{\mathbb{R}^m} \frac{Du_\varepsilon \cdot D\eta}{\sqrt{1 - |Du_\varepsilon|^2}}\,\mathrm{d}x = \int_{\mathbb{R}^m} \rho_{u_\varepsilon}\eta\,\mathrm{d}x \qquad \forall\, \eta \in \mathrm{Lip}_c(\mathbb{R}^m).$$

In particular, if $\Omega \subset \mathbb{R}^m$ satisfies $Q_\varepsilon \doteq [-\varepsilon^{-1}, \varepsilon^{-1}]^m \subset \Omega$, then u_ε is a weak solution to ($\mathcal{BI}$) with zero Dirichlet boundary condition.

Remark 4.4. Between Propositions 4.1 and 4.3, the role of $\kappa \in [1, m-1)$ is different. In fact, in Proposition 4.3, the integrability of ρ_{u_ε} and $\mathrm{II}_{u_\varepsilon}$ becomes worse when we increase κ. However, the integrability of w and w_ε does not change.

Proof of Proposition 4.3. Writing $u_\varepsilon(r,x_m) = \zeta_\varepsilon(r)(1-\varepsilon^{2\kappa}r^{2\kappa})\zeta_\varepsilon(x_m)A_\varepsilon(x_m)$, we first prove (4.17). From (4.15), we see that

$$\left|A_\varepsilon(x_m)\right| \le 2\varepsilon \quad \text{for all } x_m \in \mathbb{R}. \tag{4.18}$$

Moreover, notice that

$$\begin{aligned}(u_\varepsilon)_r &= \left[\zeta_\varepsilon'(r)\left(1-\varepsilon^{2\kappa}r^{2\kappa}\right) - \zeta_\varepsilon(r)2\kappa\varepsilon^{2\kappa}r^{2\kappa-1}\right]\zeta_\varepsilon(x_m)A_\varepsilon(x_m),\\ (u_\varepsilon)_m &= \zeta_\varepsilon(r)(1-\varepsilon^{2\kappa}r^{2\kappa})\left[\zeta_\varepsilon'(x_m)A_\varepsilon(x_m)+\zeta_\varepsilon(x_m)a_\varepsilon(x_m)\right].\end{aligned} \tag{4.19}$$

When $|x_m| \ge \frac{3\varepsilon}{2}$, since $a_\varepsilon(x_m) \le \frac{1}{2}$ and $0 \le \xi_\varepsilon(r)(1-\varepsilon^{2\kappa}r^{2\kappa}) \le 1$ due to (4.14), if $\varepsilon > 0$ is small, then (4.18) and (4.14) give

$$1 - |Du_\varepsilon(x)|^2 \ge 1 - C\varepsilon^2 - \left(a_\varepsilon(x_m)\right)^2 \ge \frac{1}{2}.$$

Since $u_\varepsilon \in C^2(\mathbb{R}^m)$,

$$w_\varepsilon(x) \le \sqrt{2}, \quad \left|\rho_{u_\varepsilon}(x)\right| + \left\|\mathrm{II}_{u_\varepsilon}(x)\right\| \le C \quad \text{for each } x \in \mathbb{R}^m \text{ with } |x_m| \ge \frac{3\varepsilon}{2}. \tag{4.20}$$

When $\frac{1}{2\varepsilon} \le r$ and $|x_m| \le \frac{3\varepsilon}{2}$, remark that for $\delta_\kappa \doteq 2^{-2\kappa} > 0$,

$$0 \le \zeta_\varepsilon(r)\left(1 - \varepsilon^{2\kappa}|x'|^{2\kappa}\right) \le 1 - \delta_\kappa.$$

Thus, by (4.14), (4.18), (4.19) and $0 \le a(x_m) \le 1$, if ε is small enough, then for some constant $\gamma_\kappa > 0$,

$$1 - |Du(x)|^2 \ge 1 - C\varepsilon^2 - \left(1-\delta_\kappa\right)^2\left[C\varepsilon^2 + 1\right] \ge \gamma_\kappa^2 > 0.$$

Hence,

$$w_\varepsilon(x) \le \gamma_\kappa^{-1}, \quad \left|\rho_{u_\varepsilon}(x)\right| + \left\|\mathrm{II}_{u_\varepsilon}(x)\right\| \le C \quad \forall\, x \in \mathbb{R}^m \text{ with } \frac{1}{2\varepsilon} \le r \text{ and } |x_m| \le \frac{3\varepsilon}{2}. \tag{4.21}$$

When $r \le \frac{1}{2\varepsilon}$ and $|x_m| \le \varepsilon$, since $u_\varepsilon(x', x_m) = (1-\varepsilon^{2\kappa}r^{2\kappa})x_m = u(r,x_m)$ where u appears in Proposition 4.1, (4.9) holds for w_ε. Moreover, (4.10) and (4.11) yield

$$w_\varepsilon(x) \le Cr^{-\kappa}, \quad \left|\rho_\varepsilon(x)\right| \le Cr^{\kappa-2}, \quad \left\|\mathrm{II}_{u_\varepsilon}(x)\right\| \le Cr^{-1} \quad \text{for each } r \le \frac{1}{2\varepsilon} \text{ and } |x_m| \le \varepsilon. \tag{4.22}$$

When $r \le \frac{1}{2\varepsilon}$ and $\varepsilon < |x_m| \le \frac{3\varepsilon}{2}$, from $u_\varepsilon(r,x_m) = (1-\varepsilon^{2\kappa}r^{2\kappa})A_\varepsilon(x_m)$, it follows that

$$\begin{aligned}(u_\varepsilon)_r &= -2\kappa\varepsilon^{2\kappa}r^{2\kappa-1}A_\varepsilon(x_m), & (u_\varepsilon)_m &= \left(1-\varepsilon^{2\kappa}r^{2\kappa}\right)a_\varepsilon(x_m),\\ (u_\varepsilon)_{rr} &= -2\kappa(2\kappa-1)\varepsilon^{2\kappa}r^{2\kappa-2}A_\varepsilon(x_m), & (u_\varepsilon)_{rm} &= -2\kappa\varepsilon^{2\kappa}r^{2\kappa-1}a_\varepsilon(x_m),\\ (u_\varepsilon)_{mm} &= \left(1-\varepsilon^{2\kappa}r^{2\kappa}\right)a_\varepsilon'(x_m).\end{aligned} \tag{4.23}$$

By $\frac{1}{2} \leq a_\varepsilon(x_m) \leq 1$ due to (4.15), we see from (4.18) that

$$\begin{aligned} 1 - |Du_\varepsilon(x)|^2 &= 1 - 4\kappa^2\varepsilon^{4\kappa}r^{4\kappa-2}A_\varepsilon^2(x_m) - \left(1 - \varepsilon^{2\kappa}r^{2\kappa}\right)^2 a_\varepsilon^2(x_m) \\ &= \left(1 - a_\varepsilon(x_m)\right)\left(1 + a_\varepsilon(x_m)\right) \\ &+ \varepsilon^{2\kappa}r^{2\kappa}\left[\left(2 - \varepsilon^{2\kappa}r^{2\kappa}\right)a_\varepsilon^2(x_m) - 4\kappa^2\varepsilon^{2\kappa}r^{2\kappa-2}A_\varepsilon^2(x_m)\right] \\ &\geq 1 - a_\varepsilon(x_m) + \varepsilon^{2\kappa}r^{2\kappa}\left[\frac{1}{4} - 16\kappa^2\varepsilon^4\right]. \end{aligned}$$

Thus, for sufficiently small $\varepsilon > 0$,

(4.24)
$$w_\varepsilon(x) \leq C\left(1 - a_\varepsilon(x_m) + r^{2\kappa}\right)^{-\frac{1}{2}} \quad \text{for all } x \in \mathbb{R}^m \text{ with } r \leq \frac{1}{2\varepsilon} \text{ and } \varepsilon < |x_m| \leq \frac{3\varepsilon}{2}.$$

Now, $w_\varepsilon \in L^q_{\rm loc}(\mathbb{R}^m)$ for all $q < (m-1)/\kappa$ easily follow from (4.20), (4.21), (4.22) and (4.24), because of $a_\varepsilon(-x_m) = a_\varepsilon(x_m) \in [0,1]$ and $(1 - a_\varepsilon(x_m) + r^{2k})^{-1/2} \leq r^{-\kappa}$.

Regarding ρ_{u_ε}, recall that

$$\rho_{u_\varepsilon} = -w_\varepsilon \Delta u_\varepsilon - w_\varepsilon^3 D^2u_\varepsilon(Du_\varepsilon, Du_\varepsilon).$$

By (4.24), we get

$$\left|w_\varepsilon \Delta u_\varepsilon\right| \leq Cr^{-\kappa} \quad \text{for all } x \in \mathbb{R}^m \text{ with } r \leq \frac{1}{2\varepsilon} \text{ and } \varepsilon < |x_m| \leq \frac{3\varepsilon}{2}. \tag{4.25}$$

On the other hand, by (4.23) and (4.8),

(4.26)
$$w_\varepsilon^3\left|D^2u_\varepsilon(Du_\varepsilon, Du_\varepsilon)\right| \leq Cw_\varepsilon^3\left[r^{6\kappa-4} + r^{4\kappa-2} + \left|a_\varepsilon'(x_m)\right|\right] \leq Cr^{\kappa-2} + Cw_\varepsilon^3\left|a_\varepsilon'(x_m)\right|.$$

From (4.20), (4.21), (4.22), (4.25), (4.26) and $a_\varepsilon(x_m) = a_\varepsilon(-x_m)$, to show (4.17) for ρ_{u_ε} it suffices to verify

$$w_\varepsilon^3\left|a_\varepsilon'(x_m)\right| \in L^q\left(B'_{1/(2\varepsilon)}(0) \times \left(\varepsilon, \frac{3\varepsilon}{2}\right)\right) \quad \text{for each } q < \frac{m-1}{\kappa}. \tag{4.27}$$

It is enough to check it for $\frac{m-1}{3\kappa} < q < \frac{m-1}{\kappa}$. Due to (4.24) and $m \geq 3$,

$$\begin{aligned} &\int_\varepsilon^{\frac{3}{2}\varepsilon} \mathrm{d}x_m \int_{B'_{1/(2\varepsilon)}(0)} w_\varepsilon^{3q}\left|a_\varepsilon'(x_m)\right|^q \mathrm{d}x' \\ &\leq C\int_\varepsilon^{\frac{3\varepsilon}{2}} \mathrm{d}x_m \int_0^{\frac{1}{2\varepsilon}} \left|a_\varepsilon'(x_m)\right|^q \left(1 - a_\varepsilon(x_m) + r^{2\kappa}\right)^{-\frac{3q}{2}} r^{m-2}\,\mathrm{d}r \\ &\leq C\int_\varepsilon^{\frac{3\varepsilon}{2}} \mathrm{d}x_m \int_0^{(1-a_\varepsilon(x_m))^{1/(2\kappa)}} \left|a_\varepsilon'(x_m)\right|^q \left(1 - a_\varepsilon(x_m)\right)^{-\frac{3q}{2}} r^{m-2}\,\mathrm{d}r \\ &\quad + C\int_\varepsilon^{\frac{3\varepsilon}{2}} \mathrm{d}x_m \int_{(1-a_\varepsilon(x_m))^{1/(2\kappa)}}^{\frac{1}{2\varepsilon}} \left|a_\varepsilon'(x_m)\right|^q r^{-3q\kappa+m-2}\,\mathrm{d}r \\ &\leq C\int_\varepsilon^{\frac{3\varepsilon}{2}} \left|a_\varepsilon'(x_m)\right|^q \left(1 - a_\varepsilon(x_m)\right)^{-\frac{3q}{2}+\frac{m-1}{2\kappa}} \mathrm{d}x_m. \end{aligned} \tag{4.28}$$

Recalling $a_\varepsilon(x_m) = 1 - d_\varepsilon \exp\left(-\frac{1}{x_m-\varepsilon}\right)$ in (4.15), we have

$$\left|a'_\varepsilon(x_m)\right|^q \left(1 - a_\varepsilon(x_m)\right)^{\frac{m-1-3q\kappa}{2\kappa}} \le C_\varepsilon \left(x_m - \varepsilon\right)^{-2q} \exp\left(\frac{\kappa q - (m-1)}{2\kappa(x_m - \varepsilon)}\right).$$

Hence, if $\frac{m-1}{3\kappa} < q < \frac{m-1}{\kappa}$, then

$$\int_\varepsilon^{\frac{3\varepsilon}{2}} \left|a'_\varepsilon(x_m)\right|^q \left(1 - a_\varepsilon(x_m)\right)^{-\frac{3q}{2}+\frac{m-1}{2\kappa}} \mathrm{d}x_m < \infty.$$

Thus, $\rho_{u_\varepsilon} \in L^q(\mathbb{R}^m)$ holds for each $q < (m-1)/\kappa$.

For the assertion that u_ε is a weak solution to ($\mathcal{BI}$), thanks to (4.22) and (4.24), we may obtain it as in the proof of Proposition 4.1 and omit the details.

Finally, we show that $\| \mathrm{II}_{u_\varepsilon} \|$ satisfies (4.17). Due to $u_\varepsilon \in C^2(\mathbb{R}^m)$, (4.20), (4.21), (4.22), it is enough to check that

$$\left\|\mathrm{II}_{u_\varepsilon}\right\| \in L^q\left(B'_{1/(2\varepsilon)}(0) \times \left(\varepsilon, \frac{3\varepsilon}{2}\right)\right) \quad \text{for each } q < \frac{m-1}{\kappa}.$$

Recalling

$$\left\|\mathrm{II}_{u_\varepsilon}\right\| \le w_\varepsilon \left|D^2 u_\varepsilon\right| + 2w_\varepsilon^2 \left|D^2 u_\varepsilon\left(Du_\varepsilon, \cdot\right)\right| + w_\varepsilon^3 \left|D^2 u_\varepsilon\left(Du_\varepsilon, Du_\varepsilon\right)\right|,$$

using (4.26) and (4.27) we only have to check the integrability of $w_\varepsilon |D^2 u_\varepsilon|$ and of $w_\varepsilon^2 |D^2 u_\varepsilon(Du_\varepsilon, \cdot)|$. By (4.5), (4.7) and (4.23),

$$\left|D^2 u_\varepsilon\right| \le C\left(r^{2\kappa-2} + r^{2\kappa-1} + \left|a'_\varepsilon(x_m)\right|\right),$$
$$\left|D^2 u_\varepsilon\left(Du_\varepsilon, \cdot\right)\right| \le C\left(r^{4\kappa-3} + r^{2\kappa-1} + r^{4\kappa-2} + \left|a'_\varepsilon(x_m)\right|\right).$$

Thus, (4.24) and $\kappa \ge 1$ yield

$$w_\varepsilon \left|D^2 u_\varepsilon\right| + w_\varepsilon^2 \left|D^2 u_\varepsilon\left(Du_\varepsilon, \cdot\right)\right| \le Cr^{-1} + Cw_\varepsilon^2 \left|a'_\varepsilon(x_m)\right|.$$

Since $w_\varepsilon \ge 1$ and $w_\varepsilon^2 |a'_\varepsilon(x_m)| \le w_\varepsilon^3 |a'_\varepsilon(x_m)|$, (4.27) implies

$$w_\varepsilon \left|D^2 u_\varepsilon\right| + w_\varepsilon^2 \left|D^2 u_\varepsilon\left(Du_\varepsilon, \cdot\right)\right| \in L^q\left(B'_{1/(2\varepsilon)}(0) \times \left(\varepsilon, \frac{3\varepsilon}{2}\right)\right).$$

Hence, $\mathrm{II}_{u_\varepsilon}$ satisfies (4.17) and we complete the proof. □

5. Main tools

The main results of this section are Theorem 5.2 (Removable singularity), Theorem 5.5 (nonsolvability of ($\mathcal{BI}$)), the L^2-estimate of the second fundamental form II (Proposition 5.10 and Corollary 5.11) and the higher integrability of w_ρ (Theorem 5.13). To prove them, we need to regularize ρ and u_ρ, a device which will also be necessary in Section 6.

5.1. **Setup for our strategy.** According to Remark 3.4, defining $p = q'$ it holds

$$\mathcal{M}(\Omega) + W^{-1,p}(\Omega) \subset \mathcal{Y}(\Omega)^* \qquad \text{for each} \begin{cases} p \in [p_1', \infty) & \text{if } \Omega \text{ is bounded,} \\ p \in [p_1', 2_*] & \text{if } \Omega = \mathbb{R}^m. \end{cases}$$

We shall hereafter restrict to

$$\rho \in \mathcal{M}(\Omega) + L^p(\Omega) \qquad \text{for } p \in (1, 2_*],$$

where $L^p(\Omega) \subset W^{-1,p}(\Omega)$ is the set of pairs $(v, 0)$ as in Remark 3.4.

> *Since $2_* = 1$ when $m = 2$, hereafter the space $L^p(\Omega)$ is tacitly assumed to be empty when $p \in (1, 2_*]$ and $m = 2$.*

Notice that $\mathcal{M}(\Omega) + L^p(\Omega) \hookrightarrow \mathcal{Y}(\Omega)^*$ provided that p_1 is sufficiently large. For instance, we may (and henceforth do) choose

$$p_1 = 3 \quad \text{if } m = 2, \qquad p_1 = \max\{2^*, m\} + p' \quad \text{if } m \geq 3. \tag{5.1}$$

By a standard mollifying argument (see [40, Chapter 2]) and Young's inequality, for given

$$\rho = \mu + f \in \mathcal{M}(\Omega) + L^p(\Omega)$$

we can find sequences of functions $g_j, f_j \in C^\infty(\overline{\Omega})$ such that, setting $\mu_j \doteq g_j \mathrm{d}x$ and recalling $p = q'$,

$$\|\mu_j\|_{\mathcal{M}(\Omega)} \leq \|\mu\|_{\mathcal{M}(\Omega)}, \qquad \|f_j\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}$$
$$\mu_j \rightharpoonup \mu \text{ weakly in } \mathcal{M}(\Omega), \qquad f_j \to f \text{ strongly in } L^p(\Omega) \text{ (hence, in } \mathcal{Y}(\Omega)^*).$$

Define $\rho_j \doteq \mu_j + f_j$. When $\Omega = \mathbb{R}^m$, the construction via convolution also guarantees, for each $\varepsilon > 0$, the existence of $R_\varepsilon > 0$ such that (3.9) holds for $\{\mu_j\}$. Moreover, up to replacing ρ, f by $\rho \mathbb{1}_{B_j}$ and $f \mathbb{1}_{B_j}$ and using a diagonal argument, we can assume that $g_j, f_j \in C_c^\infty(\mathbb{R}^m)$.

Fix $\phi \in \mathcal{S}(\partial\Omega)$ if Ω is bounded, and denote the minimizer of I_{ρ_j} by u_j. Because of Theorem 1.3 or [8, Theorem 1.5 and Remark 3.4], respectively if Ω is bounded or if $\Omega = \mathbb{R}^m$, u_j is a smooth solution to ($\mathcal{BI}$) with Lorentzian mean curvature $H_j \doteq -(g_j + f_j)$ (thus, u_j minimizes I_{ρ_j} with $\rho_j = -H_j \mathrm{d}x$). Write $w_j \doteq (1 - |Du_j|^2)^{-1/2}$. Proposition 3.7 yields $u_j \to u_\rho$ strongly in $W^{1,q}(\Omega) \cap C(\overline{\Omega})$, where $q \in [1, \infty)$ when Ω is bounded, and $q \in [2^*, \infty)$ when $\Omega = \mathbb{R}^m$, and moreover $\langle \rho_j, u_j \rangle \to \langle \rho, u_\rho \rangle$. Therefore, using Proposition 3.14, to show that u_ρ weakly solves ($\mathcal{BI}$) it is enough to prove that

$$\lim_{j \to \infty} \int_\Omega w_j Du_j \cdot D\eta \, \mathrm{d}x = \int_\Omega w_\rho Du_\rho \cdot D\eta \, \mathrm{d}x \qquad \forall \eta \in \mathrm{Lip}_c(\Omega). \tag{5.2}$$

Since $\|Du_j\|_\infty \leq 1$ and we may assume $Du_j \to Du_\rho$ a.e. on Ω, identity (5.2) follows from Vitali's convergence theorem (see [46, Theorem 3.1.9]) provided that $\{w_j\}$ is locally uniformly integrable in the following sense.

Definition 5.1. Let $\Omega \subset \mathbb{R}^m$ be an open subset. We say that a subset $\mathcal{W} \subset L^1_{\rm loc}(\Omega)$ is *locally uniformly integrable on* Ω if, for each $\Omega' \Subset \Omega$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \Omega')$ such that

$$A \subset \Omega' \text{ measurable, } |A| < \delta \qquad \Longrightarrow \qquad \int_A |w| \mathrm{d}x < \varepsilon \quad \forall\, w \in \mathcal{W}.$$

By de la Vallée-Poussin's Theorem (see, for instance, [46, Theorem 3.1.10]), $\mathcal{W}$ is locally uniformly integrable if and only if there exists a compact exhaustion $\{\Omega_k\}_{k=1}^\infty$ of Ω, that is, $\Omega_k \Subset \Omega$, $\Omega_k \uparrow \Omega$, and increasing convex functions $f_k : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that

$$\lim_{t\to\infty} \frac{f_k(t)}{t} = +\infty, \qquad \sup_{w\in\mathcal{W}} \int_{\Omega_k} f_k(|w|) \mathrm{d}x < \infty \quad \forall\, k.$$

The purpose of the next subsections is to obtain a local uniform integrability for $\{w_j\}$. We begin by studying the behavior of u_ρ in regions where ρ is singular.

5.2. **Removable and unremovable singularities.** To our knowledge, the only removable singularity theorem for the prescribed Lorentzian mean curvature equation is the one in [38]. The theorem considers maximal graphs u that are smooth and strictly spacelike in a domain $\Omega' \backslash E$, where $E \Subset \Omega'$ is compact. Under the assumption that the p-capacity of E is zero for some $p \in (1, m]$, and that

$$\int_{\Omega'\backslash E} w^{\frac{p}{p-1}} \mathrm{d}x < \infty, \tag{5.3}$$

then u can be smoothly extended to a spacelike maximal solution on Ω'. In particular, by the known relation between Hausdorff measure and capacity (cf. [19]), compact subsets E with $\mathscr{H}^{m-p}_\delta(E) = 0$ are removable for maximal graphs satisfying (5.3). However, the proof seems not easy to extend to more general measures $\rho \neq 0$, and currently we are unable to prove an a-priori estimate yielding (5.3). Therefore, we take a different approach. Our contribution is the following result, which applies to any measure and only needs a local uniform integrability for the sequence of energy densities $\{w_j\}$.

Theorem 5.2 (**Removable singularity**). *Assume $\Omega \subset \mathbb{R}^m$ is either a bounded domain with $m \geq 2$ or $\mathbb{R}^m$ with $m \geq 3$. Let*

$$\rho \in \mathcal{M}(\Omega) + L^p(\Omega), \qquad p \in (1, 2_*],$$

and, if Ω is bounded, let $\phi \in \mathcal{S}(\partial\Omega)$. Choose $\{p_1, \rho_j, u_j, w_j\}$ as in Subsection 5.1. Suppose that $E \Subset \Omega$ is a compact set with $\mathscr{H}^1_\delta(E) = 0$. Then, for every open subset $\Omega' \subset \Omega$,

$$\begin{array}{c}\{w_j\} \text{ is locally uniformly} \\ \text{integrable on } \Omega'\backslash E\end{array} \quad \Longrightarrow \quad \begin{array}{l} \{w_j\} \text{ is locally uniformly integrable on } \Omega', \text{ and} \\ \displaystyle\int_{\Omega'} \frac{Du_\rho \cdot D\eta}{\sqrt{1-|Du_\rho|^2}} = \langle \rho, \eta \rangle \quad \forall\, \eta \in \mathrm{Lip}_c(\Omega'). \end{array}$$

In particular, if $\{w_j\}$ is locally uniformly integrable on $\Omega\backslash E$, then u_ρ weakly solves ($\mathcal{BI}$).

Remark 5.3. The above requirements on E cannot be weakened to $\mathscr{H}^1_\delta(E) < \infty$. Indeed, consider the example in Corollary 1.9, and set $E = \overline{xy}$. Since $u = u_\rho$ has no light segments in $\Omega \backslash \overline{xy}$, the energies $\{w_j\}$ are locally uniformly integrable there. This can be shown by combining Lemma 3.8 with [4, Lemma 2.1], proceeding as in [4, Proof of Theorem 4.1]. However, u_ρ does not solve ($\mathcal{BI}$), so E is not removable. As a related example, one can see the nice [32, Example 2].

The result is a consequence of the next lemma, which estimates the growth of w on balls centered at a given point.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^m$ be an open set, $H \in C^\infty(\Omega)$ and let u solve*

$$-\mathrm{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = \rho \doteq -H\,\mathrm{d}x \qquad \text{on } \Omega.$$

For any given $y \in \Omega$, define

$$J_y(s) \doteq \int_{B_s(y)} \frac{\mathrm{d}x}{\sqrt{1-|Du|^2}}, \qquad 0 < s < \mathrm{d}_\delta(y, \partial\Omega).$$

Then, for each $0 < s < t < \mathrm{d}_\delta(y, \partial\Omega)$, it holds

$$J_y(s) \le s\left[\frac{J_y(t)}{t} + |\rho|(B_t(y))\right]. \tag{5.4}$$

Proof. Let $\varphi \in \mathrm{Lip}_c(\Omega)$. Up to a translation, we may assume $u(y) = 0$. Let M be the graph of u. Recalling (2.5), we first test $\Delta_M u = Hw$ against $u\varphi$ and integrate by parts to deduce

$$\int \varphi \|\nabla u\|^2\,\mathrm{d}x_g = -\int u\varphi H w\,\mathrm{d}x_g - \int \langle u\nabla u, \nabla\varphi\rangle\,\mathrm{d}x_g.$$

We set $o = y$ in (2.6) and write $\ell(x) = \ell_y(x)$. Multiplying the equation $\Delta_M \ell^2 = 2m + H\bar{D}l^2 \cdot \mathbf{n}$ in (2.8) by φ and integrating by parts we get

$$2m\int \varphi\,\mathrm{d}x_g = -2\int \ell\,\langle\nabla\ell, \nabla\varphi\rangle\,\mathrm{d}x_g - \int \varphi H\bar{D}l^2\cdot\mathbf{n}\,\mathrm{d}x_g.$$

Noting that $\ell^2(x) = r^2(x) - u^2(x)$ and $u(y) = 0$, and using the identities

$$\ell\nabla\ell = r\nabla r - u\nabla u, \quad w^2 = 1 + \|\nabla u\|^2, \quad \bar{D}l^2\cdot\mathbf{n} = 2w\,[r\,(Du, Dr) - u]\,,$$

we infer

$$\begin{aligned} m\int \varphi w^2 \mathrm{d}x_g &= m\int \varphi\,\mathrm{d}x_g + m\int \varphi\|\nabla u\|^2\mathrm{d}x_g \\ &= -\int \ell\,\langle\nabla\ell, \nabla\varphi\rangle\,\mathrm{d}x_g - \int \varphi H w\,[r(Du, Dr) - u]\,\mathrm{d}x_g \\ &\quad -m\int u\varphi H w\,\mathrm{d}x_g - m\int \langle u\nabla u, \nabla\varphi\rangle\,\mathrm{d}x_g \\ &= -\int \langle r\nabla r + (m-1)u\nabla u, \nabla\varphi\rangle\,\mathrm{d}x_g - \int \varphi H w\,[r(Du, Dr) + (m-1)u]\,\mathrm{d}x_g. \end{aligned} \tag{5.5}$$

First, since $\|\nabla\varphi\| \le w|D\varphi|$, $|(Du, Dr)| \le 1$ and $|u| \le r$ due to $\|Du\|_\infty \le 1$, we get

$$\langle r\nabla r + (m-1)u\nabla u, \nabla\varphi\rangle \le \|r\nabla r + (m-1)u\nabla u\|\|\nabla\varphi\| \le mr\max\{\|\nabla r\|, \|\nabla u\|\}\|\nabla\varphi\| \le mr|D\varphi|w^2.$$

Setting

$$T_\rho(\varphi) \doteq -\frac{1}{m}\int \varphi H w\big[r(Du, Dr) + (m-1)u\big]\mathrm{d}x_g,$$

we deduce from (5.5) the following inequality:

$$\int \varphi w^2\,\mathrm{d}x_g \le \int |D\varphi| r w^2\,\mathrm{d}x_g + T_\rho(\varphi). \tag{5.6}$$

Let $0 < s < t < \mathrm{d}_\delta(y, \partial\Omega)$ and consider, for $\varepsilon > 0$ small enough,

$$\varphi(x) \doteq \left(\min\left\{1, \frac{s+\varepsilon - r(x)}{\varepsilon}\right\}\right)_+ \in \mathrm{Lip}_c(B_t(y)) \subset \mathrm{Lip}_c(\Omega).$$

From $|u| \le r$, $|(Du, Dr)| \le 1$ on the support of φ, $|\varphi| \le 1$ and (2.1), and using the coarea formula, we get

$$|T_\rho(\varphi)| \le \int_{B_{s+\varepsilon}(y)} r|H|w\mathrm{d}x_g = \int_0^{s+\varepsilon} \sigma\left[\int_{\partial B_\sigma(y)} |H|\mathrm{d}\mathscr{H}^{m-1}_\delta\right]\mathrm{d}\sigma.$$

Letting $\varepsilon \to 0$ and observing that

$$\int |D\varphi| r w^2 \mathrm{d}x_g = \int |D\varphi| r w\,\mathrm{d}x \to s\int_{\partial B_s(y)} w\,\mathrm{d}\mathscr{H}^{m-1}_\delta$$

for a.e. s, from (5.6), we obtain

$$\int_{B_s(y)} w\,\mathrm{d}x \le s\int_{\partial B_s(y)} w\,\mathrm{d}\mathscr{H}^{m-1}_\delta + \int_0^s \left[\sigma\int_{\partial B_\sigma(y)} |H|\mathrm{d}\mathscr{H}^{m-1}_\delta\right]\mathrm{d}\sigma \qquad \text{for a.e. } s \in [0, t].$$

By the coarea formula, the above inequality can also be rewritten as

$$-\frac{\mathrm{d}}{\mathrm{d}s}\frac{J_y(s)}{s} \le \frac{1}{s^2}\int_0^s \sigma f_y(\sigma)\mathrm{d}\sigma \qquad \text{for a.e. } s \in (0, t],$$

where

$$f_y(\sigma) = \int_{\partial B_\sigma(y)} |H|\mathrm{d}\mathscr{H}^{m-1}_\delta.$$

Integrating on $[s,t]$ and using Tonelli's Theorem, we deduce

$$\begin{aligned}
-\frac{J_y(t)}{t}+\frac{J_y(s)}{s} &\le \int_s^t \frac{1}{\tau^2}\left\{\int_0^\tau \sigma f_y(\sigma)\mathrm{d}\sigma\right\}\mathrm{d}\tau \\
&= \int_0^t \sigma f_y(\sigma)\left\{\int_{\max\{s,\sigma\}}^t \frac{\mathrm{d}\tau}{\tau^2}\right\}\mathrm{d}\sigma \\
&\le \int_0^t \sigma f_y(\sigma)\left[-\frac{1}{\tau}\right]_\sigma^t \mathrm{d}\sigma \le \int_0^t \sigma f_y(\sigma)\frac{1}{\sigma}\mathrm{d}\sigma \\
&= \int_0^t f_y(\sigma)\mathrm{d}\sigma = \int_{B_t(y)} |H|\mathrm{d}x = |\rho|\big(B_t(y)\big),
\end{aligned}$$

which proves (5.4). $\square$

Using Lemma 5.4 and a covering argument, we shall prove Theorem 5.2:

Proof of Theorem 5.2. Write $\rho = \mu + f$ with $\mu \in \mathcal{M}(\Omega)$ and $f \in L^p(\Omega)$. Referring to Subsection 5.1, for $m = 2$ the term f does not appear, and our choice of p_1 imply that $\rho \in \mathcal{Y}(\Omega)^*$. Let μ_j, f_j be as therein, thus $\mu_j \to \mu$ weakly in $\mathcal{M}(\Omega)$ and $f_j \to f$ strongly in $L^p(\Omega)$. Choose $0 < R_0 \le \mathrm{d}_\delta(E, \partial\Omega)/20$. The relative compactness of $B_{10R_0}(E)$ implies that $\rho_j = \mu_j + f_j\mathrm{d}x \rightharpoonup \rho$ weakly in $\mathcal{M}(B_{10R_0}(E))$, so in particular there exists a constant $C_{\mathcal{M}}$ such that

$$\left\|\rho_j\right\|_{\mathcal{M}(B_{10R_0}(E))} \le C_{\mathcal{M}} \quad \text{for each } j \ge 1. \tag{5.7}$$

Write $\rho_j = -H_j\mathrm{d}x$. By Proposition 3.9, there exists a constant $C(R_0)$, depending on ϕ, R_0, $\|\rho\|_{\mathcal{Y}^*}$ such that

$$\int_{B_{4R_0}(E)} w_j \,\mathrm{d}x \le C(R_0). \tag{5.8}$$

For $x \in B_{R_0}(E)$ and $s \in (0, R_0]$, set

$$J_{x,j}(s) \doteq \int_{B_s(x)} w_j \mathrm{d}x.$$

Note that (5.8) implies $J_{x,j}(R_0) \le C(R_0)$ for all $j \ge 1$ and $x \in B_{R_0}(E)$, hence Lemma 5.4 and (5.7), (5.8) ensure that for all $x \in B_{R_0}(E)$, $j \ge 1$ and $s \in (0, R_0)$,

$$J_{x,j}(s) \le s\left[\frac{C(R_0)}{R_0} + |\rho_j|(B_{R_0}(x))\right] \le C_1 s,$$

for some $C_1(R_0, C(R_0), C_{\mathcal{M}})$. By our assumption $\mathscr{H}^1_\delta(E) = 0$ and since E is compact, for given $\tau > 0$ we can cover E with finitely many balls $\{B_k\}_{k=1}^N$, $B_k = B_{r_k}(x_k)$ satisfying $r_k < R_0$ and $\sum_k r_k \le \tau$. We can also assume that $x_k \in B_{R_0}(E)$ for each k. Therefore, for each fixed $\varepsilon > 0$ we can take $\tau > 0$ small enough to satisfy

$$\int_{\bigcup_{k=1}^N B_k} w_j\mathrm{d}x \le \sum_{k=1}^N J_{x_k,j}(r_k) \le C_1 \sum_{k=1}^N r_k \le C_1\tau < \frac{\varepsilon}{2}.$$

Let $\Omega'' \Subset \Omega'$ be a relatively compact subset. By defining $U \doteq \bigcup_{k=1}^{N} B_k$, our assumption yields that $\{w_j\}$ is uniformly integrable on $\Omega''\backslash U$. Thus, there exists $\delta > 0$ such that $A \subset \Omega''\backslash U$ and $|A| < \delta$ imply $\int_A w_j \mathrm{d}x < \varepsilon/2$. Then, for each subset $A \subset \Omega''$ with $|A| < \delta$,

$$\int_A w_j \mathrm{d}x \le \int_{A\cap U} w_j \mathrm{d}x + \int_{A\backslash U} w_j \mathrm{d}x < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that $\{w_j\}$ is uniformly integrable on Ω''. In particular, (5.2) holds for every fixed $\eta \in \mathrm{Lip}_c(\Omega')$ by Vitali's Theorem. □

We next consider singularities which cannot be removed. While the examples in Section 4 show that solutions to ($\mathcal{BI}$) may possess light segments when $\rho \in L^q(\Omega)$ and $q < m-1$, we shall now prove that such solutions exhibit, in a sense, a "borderline" behavior.

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^m$ be either a bounded domain with $m \ge 2$ and $\phi \in \mathcal{S}(\partial\Omega)$, or $\Omega = \mathbb{R}^m$ with $m \ge 3$. Let $\rho \in \mathcal{Y}(\Omega)^*$, and assume that the minimizer u_ρ has a light segment $\overline{xy} \subset \Omega$ with $u_\rho(y) - u_\rho(x) = |y-x|$. Then, for each $\alpha > 0$, u_ρ also minimizes the functional I_{ρ_α} with*

$$\rho_\alpha = \rho + \alpha(\delta_y - \delta_x),$$

but it does not solve ($\mathcal{BI}$) weakly for ρ_α.

Proof. For simplicity, we suppress the index ρ and denote by $I \doteq I_\rho$ and $u \doteq u_\rho$. We also write $I_\alpha \doteq I_{\rho_\alpha}$ and denote its minimizer by u_α. We argue by contradiction and assume that $u_\alpha \ne u$ for some $\alpha > 0$. By uniqueness of the minimizer, we infer

$$I(u) = I_\alpha(u) + \alpha\big[u(y) - u(x)\big] > I_\alpha(u_\alpha) + \alpha\big[u(y) - u(x)\big],$$

which implies

$$u(y) - u(x) < \frac{I(u) - I_\alpha(u_\alpha)}{\alpha}.$$

Similarly,

$$I_\alpha(u_\alpha) = I(u_\alpha) - \alpha\big[u_\alpha(y) - u_\alpha(x)\big] > I(u) - \alpha\big[u_\alpha(y) - u_\alpha(x)\big],$$

thus,

$$u_\alpha(y) - u_\alpha(x) > \frac{I(u) - I_\alpha(u_\alpha)}{\alpha}.$$

Therefore, $u_\alpha(y) - u_\alpha(x) > u(y) - u(x) = |y-x|$, contradicting the fact that $u_\alpha \in \mathcal{Y}_\phi(\Omega)$.

We have therefore proved that $u = u_\alpha$ for each $\alpha > 0$. By Theorem 1.3, pick a strictly spacelike extension $\bar\phi$ of ϕ, so that, in particular, $|y-x| - \bar\phi(y) + \bar\phi(x) > 0$. Since u minimizes I, we see from Proposition 3.9 that

$$\begin{aligned}\int_\Omega \frac{Du\cdot(Du - D\bar\phi)}{\sqrt{1-|Du|^2}}\mathrm{d}x \le \left\langle \rho, u - \bar\phi\right\rangle &= \left\langle \rho_\alpha, u - \bar\phi\right\rangle - \alpha\left\langle (\delta_y - \delta_x), u - \bar\phi\right\rangle \\ &= \left\langle \rho_\alpha, u - \bar\phi\right\rangle - \alpha\left[|y-x| - \bar\phi(y) + \bar\phi(x)\right] \\ &< \left\langle \rho_\alpha, u - \bar\phi\right\rangle.\end{aligned}$$

Therefore, due to Proposition 3.14, u does not solve ($\mathcal{BI}$) for ρ_α. □

5.3. **Local second fundamental form estimate.** The study of $W^{2,q}_{\rm loc}$ regularity for u_ρ leads to investigate the second fundamental form II. We first observe that $W^{2,q}_{\rm loc}$ estimates, for $q \geq 1$, are not to be expected for general ρ. An easy counterexample can be produced building on the expression of u_ρ when $\rho = -H + b\omega_{m-1}\delta_0$, that we now recall.

Example 5.6. Given $H \in \mathbb{R}$, $T > 0$ and $b \in \mathbb{R}^+$, the function

$$u_b(x) = \eta_b(|x|) = \int_{|x|}^{T} \frac{b - m^{-1}Ht^m}{\sqrt{t^{2m-2} + (b - m^{-1}Ht^m)^2}}\mathrm{d}t \qquad \text{on } B_T(0) \subset \mathbb{R}^m$$

solves

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du_b}{\sqrt{1-|Du_b|^2}}\right) = -H + b\omega_{m-1}\delta_0 & \text{on } B_T(0), \\ u_b = 0 & \text{on } \partial B_T(0). \end{cases}$$

Note that u_b in Example 5.6 is strictly spacelike outside of the origin. Take u with the choices $b = T = 1$ and $H = 0$. Fix $R \in (0,1)$ and let $s \in (0, \|u\|_\infty)$, be the constant value of u on $\partial B_R(0)$. Then, the function $u_s = \min\{u, s\}$ solves

$$\begin{cases} \mathrm{div}\left(\dfrac{Du_s}{\sqrt{1-|Du_s|^2}}\right) = -R^{1-m}\mathscr{H}^{m-1}_\delta \llcorner \partial B_R(0) & \text{on } B_1(0), \\ u_s = 0 & \text{on } \partial B_1(0). \end{cases}$$

Clearly, $u_s \notin W^{2,q}_{\rm loc}$ for any $q \geq 1$. Note however that, by explicit computation, $u \in W^{2,q}(B_1(0))$ for each $q \in [1, m)$.

It is reasonable to guess that $u_\rho \in W^{2,2}_{\rm loc}(\Omega)$ provided that $\rho \in L^2(\Omega)$. Indeed, a stronger estimate holds. First, observe that integrating (2.4) on a domain Ω' we get
(5.9)

$$\int_{M'} \|\,\mathrm{II}\,\|^2 \mathrm{d}x_g = \int_{\Omega'} w\left\{ |D^2u|^2 + 2w^2 \left|D^2u\,(Du,\cdot)\right|^2 + w^4\left[D^2u(Du,Du)\right]^2 \right\}\mathrm{d}x,$$

where M' denotes the graph of $u = u_\rho$ over Ω'. In this subsection, we prove local second fundamental form estimates for the graph of u_ρ in regions Ω' where $\rho \in L^2$. Let $\rho = -H\mathrm{d}x$ with $H \in C^\infty(\Omega)$ and u be a smooth solution to ($\mathcal{BI}$). Denote by M' the graph of u over an open subset $\Omega' \Subset \Omega$. First, observe that

$$Dw = w^3 D^2u(Du,\cdot), \qquad |Dw|^2 = w^6|D^2u(Du,\cdot)|^2$$

$$\begin{aligned} \|\nabla w\|^2 &= g^{ij}w_iw_j = |Dw|^2 + w^2(Dw,Du)^2 \\ &= w^6|D^2u(Du,\cdot)|^2 + w^8\left[D^2u(Du,Du)\right]^2 \leq w^2\|\,\mathrm{II}\,\|^2, \end{aligned}$$

hence,

$$\|\nabla \log w\|^2 \leq \|\,\mathrm{II}\,\|^2.$$

Next, we rewrite $\|\nabla^2 u\|^2$ as follows:

Lemma 5.7. *Assume* $\mathrm{d}u(x) \neq 0$ *at* $x \in M$ *and set* $\nu \doteq \nabla u/\|\nabla u\|$ *in a neighborhood of* x*. Denote by* A *the traceless second fundamental form of the level set* $\{u = u(x)\}$ *in the direction* $-\nu$ *and write* $u_{\nu\nu} \doteq \nabla^2 u(\nu,\nu)$*. Then*

$$\|\nabla^2 u\|^2 = \|\nabla u\|^2 \|A\|^2 + \frac{1}{m-1}\left(H^2 w^2 - 2Hw\,u_{\nu\nu}\right) + \frac{m}{m-1}\Big\|\nabla \|\nabla u\|\Big\|^2 + \frac{m-2}{m-1}\Big\|\nabla^\top \|\nabla u\|\Big\|^2, \tag{5.10}$$

where $\nabla^\top$ *stands for the component of* ∇ *tangent to the level set* $\{u = u(x)\}$*.*

Proof. Recall that, by (2.5), $\|\,\mathrm{II}\,\|^2 = w^{-2}\|\nabla^2 u\|^2$. Consider an orthonormal frame $\{\nu, e_\alpha\}$, $2 \le \alpha \le m$ on M. We denote by u_{ij} the components of $\nabla^2 u$ in the above frame. Then,

$$\langle \nabla \|\nabla u\|, e_\alpha\rangle = u_{\alpha\nu}, \qquad \langle \nabla \|\nabla u\|, \nu\rangle = u_{\nu\nu},$$

thus

$$\|\nabla^2 u\|^2 = \sum_{\alpha,\beta=2}^m u_{\alpha\beta}^2 + 2\|\nabla^\top \|\nabla u\|\|^2 + u_{\nu\nu}^2. \tag{5.11}$$

Next, it follows from the definition of A that

$$\|\nabla u\| A_{\alpha\beta} = u_{\alpha\beta} - \frac{\sum_{\gamma=2}^m u_{\gamma\gamma}}{m-1}\delta_{\alpha\beta}.$$

Splitting the norm of the matrix $[u_{\alpha\beta}]$ into its trace and traceless parts, and recalling (2.5), we get

$$\begin{aligned}\sum_{\alpha,\beta=2}^m u_{\alpha\beta}^2 &= \|\nabla u\|^2\|A\|^2 + \frac{1}{m-1}\left(\sum_{\alpha=2}^m u_{\alpha\alpha}\right)^2 = \|\nabla u\|^2\|A\|^2 + \frac{(\Delta_M u - u_{\nu\nu})^2}{m-1} \\ &= \|\nabla u\|^2\|A\|^2 + \frac{1}{m-1}\left(H^2 w^2 - 2Hwu_{\nu\nu} + u_{\nu\nu}^2\right).\end{aligned}$$

Inserting this into (5.11) and noting that $\|\nabla\|\nabla u\|\|^2 = \|\nabla^\top\|\nabla u\|\|^2 + u_{\nu\nu}^2$, we obtain (5.10). □

Remark 5.8. When $H = 0$, we obtain the classical refined Kato inequality for harmonic functions

$$\|\nabla^2 u\|^2 \ge \frac{m}{m-1}\Big\|\nabla\|\nabla u\|\Big\|^2.$$

It is convenient to rewrite the equations in terms of the hyperbolic angle

$$\beta \doteq \operatorname{arcch} w = \log\left(w + \sqrt{w^2-1}\right).$$

Note that $w \mapsto \beta$ is a diffeomorphism on $\{\mathrm{d}u \neq 0\}$. The identities

$$w = \operatorname{ch}\beta, \qquad \|\nabla u\| = \sqrt{w^2-1} = \operatorname{sh}\beta, \qquad u_{\nu\nu} = \langle \nabla\|\nabla u\|, \nu\rangle = \operatorname{ch}\beta\langle\nabla\beta,\nu\rangle,$$

(5.10) and the fact that $\mathrm{II} = w^{-1}\nabla^2 u = 0$ a.e. on the set $\{\mathrm{d}u = 0\}$ due to Stampacchia's theorem allow us to rewrite $\|\,\mathrm{II}\,\|^2 = w^{-2}\|\nabla^2 u\|^2$ as

(5.12)

$$\|\,\mathrm{II}\,\|^2 = \left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 + \frac{H^2}{m-1} - \frac{2H\langle\nabla\beta,\nu\rangle}{m-1} + \frac{m\|\nabla\beta\|^2}{m-1} + \frac{m-2}{m-1}\|\nabla^\top\beta\|^2\right]\cdot\mathbb{1}_{\{\mathrm{d}u\neq 0\}}$$

a.e. on Ω. We therefore deduce that, for some constant $C = C(m) > 0$,

$$\|\,\mathrm{II}\,\|^2 \le C(m)\left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 + \|\nabla\beta\|^2 + H^2\right]\cdot\mathbb{1}_{\{\mathrm{d}u\neq 0\}} \tag{5.13}$$

and that, for every $M' \Subset M$,

$$\int_{M'}\|\,\mathrm{II}\,\|^2\mathrm{d}x_g \le C \quad\Longleftrightarrow\quad \int_{M'\cap\{\mathrm{d}u\neq 0\}}\left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 + \|\nabla\beta\|^2 + H^2\right]\mathrm{d}x_g \le C',$$

where C and C' might be different, but with the same qualitative dependence on the data of our problem ($\mathcal{BI}$).

We next rewrite the Jacobi equation in a way that is more suited to our purposes. We begin with the following

Lemma 5.9. *Define*

$$Y \doteq \frac{\nabla w - H\nabla u}{w} \qquad \text{on } M. \tag{5.14}$$

Then,

$$\mathrm{div}_M Y = \|\,\mathrm{II}\,\|^2 - H^2 - \left\langle Y, \frac{\nabla w}{w}\right\rangle. \tag{5.15}$$

Proof. We shall first prove that

$$\Delta_M w = \left(\|\,\mathrm{II}\,\|^2 - H^2\right)w + \mathrm{div}_M\left(H\nabla u\right) \qquad \text{on } M. \tag{5.16}$$

The identity follows from the Jacobi equation (cf. [3], p. 519) and (2.2):

$$\Delta_M w = -\left\langle\nabla H, \partial_0^{\|}\right\rangle + \|\,\mathrm{II}\,\|^2 w = \langle\nabla H, \nabla u\rangle + \|\,\mathrm{II}\,\|^2 w,$$

once we observe that $\langle\nabla H, \nabla u\rangle = \mathrm{div}_M(H\nabla u) - H\Delta_M u = \mathrm{div}_M(H\nabla u) - H^2 w$. From (5.16) we therefore obtain

$$\Delta_M \log w = \|\,\mathrm{II}\,\|^2 - H^2 - \frac{\|\nabla w\|^2}{w^2} + \mathrm{div}_M\left(\frac{H\nabla u}{w}\right) + H\left\langle\frac{\nabla u}{w}, \frac{\nabla w}{w}\right\rangle,$$

which is (5.15) up to rearranging terms. □

By (5.12), $\nabla u = \mathrm{sh}\,\beta\nu$ and $\nabla w/w = \mathrm{sh}\,\beta\nabla\beta/\,\mathrm{ch}\,\beta$, we rewrite the vector field Y as

$$Y = \frac{\mathrm{sh}\,\beta}{\mathrm{ch}\,\beta}\left(\nabla\beta - H\nu\right) \tag{5.17}$$

and $\mathrm{div}_M Y$ as

$$\mathrm{div}_M Y = \left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 - \frac{m-2}{m-1}H^2 - \frac{2}{m-1}H\langle\nabla\beta,\nu\rangle + \frac{m}{m-1}\|\nabla\beta\|^2 + \frac{m-2}{m-1}\|\nabla^\top\beta\|^2 - \frac{\mathrm{sh}\,\beta}{\mathrm{ch}\,\beta}\langle Y,\nabla\beta\rangle\right]\cdot \mathbb{1}_{\{\mathrm{d}u\neq 0\}}$$

a.e. on Ω. By (5.17) with $0\le \mathrm{sh}\,\beta/\mathrm{ch}\,\beta\le 1$ and Cauchy-Schwarz's and Young's inequalities, we have

$$\left|\frac{\mathrm{sh}\,\beta}{\mathrm{ch}\,\beta}\langle Y,\nabla\beta\rangle\right| \le \|\nabla\beta - H\nu\|\,\|\nabla\beta\| \le \|\nabla\beta\|^2 + |H|\|\nabla\beta\| \le (1+\varepsilon)\|\nabla\beta\|^2 + \frac{4}{\varepsilon}H^2,$$

$$|H\langle\nabla\beta,\nu\rangle| \le |H|\|\nabla\beta\| \le \frac{1}{2\varepsilon}|H|^2 + \frac{\varepsilon}{2}\|\nabla\beta\|^2.$$

Thus there exist constants $C_m, C_{m,\varepsilon}$ such that, a.e. Ω,

$$\mathrm{div}_M Y \ge \left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 - C_{m,\varepsilon}H^2 + \left\{\frac{1}{m-1} - \frac{C_m\varepsilon}{2}\right\}\|\nabla\beta\|^2\right]\cdot\mathbb{1}_{\{\mathrm{d}u\neq 0\}} \tag{5.18}$$

a.e. on Ω. We notice from the smoothness of Y, H and from estimate (5.18) that the function $\|\nabla\beta\|^2\mathbb{1}_{\{\mathrm{d}u\neq 0\}}$ is integrable on the graph of u.

Proposition 5.10. *There exists a constant $C = C_m > 0$ such that, for every $\varphi \in \mathrm{Lip}_c(\Omega)$,*

$$\int_M \varphi^2\|\,\mathrm{II}\,\|^2\mathrm{d}x_g \le C_m\left(\int_M \|\nabla\varphi\|^2\mathrm{d}x_g + \int_M \varphi^2 H^2\mathrm{d}x_g\right). \tag{5.19}$$

Proof. We test (5.18) with the function φ^2 to obtain

$$\begin{aligned}&\int_{\{\mathrm{d}u\neq 0\}}\left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 + \left\{\frac{1}{m-1} - \frac{C_m\varepsilon}{2}\right\}\|\nabla\beta\|^2\right]\varphi^2\mathrm{d}x_g\\ &\le \int \varphi^2\mathrm{div}_M Y\,\mathrm{d}x_g + C_{m,\varepsilon}\int H^2\varphi^2\,\mathrm{d}x_g\\ &= -2\int\varphi\langle\nabla\varphi, Y\rangle\,\mathrm{d}x_g + C_{m,\varepsilon}\int H^2\varphi^2\,\mathrm{d}x_g.\end{aligned} \tag{5.20}$$

Since, from its very definition, $Y = 0$ on $\{\mathrm{d}u = 0\}$, and since $0\le \mathrm{sh}\,\beta/\mathrm{ch}\,\beta\le 1$, using Cauchy-Schwarz's and Young's inequalities we see from (5.17) that

$$\begin{aligned}|\varphi\langle\nabla\varphi, Y\rangle| &\le \{|\varphi\langle\nabla\varphi,\nabla\beta\rangle| + |\varphi H\langle\nabla\varphi,\nu\rangle|\}\,\mathbb{1}_{\{\mathrm{d}u\neq 0\}}\\ &\le \frac{1}{2\varepsilon}\|\nabla\varphi\|^2 + \frac{\varepsilon}{2}\varphi^2\|\nabla\beta\|^2\mathbb{1}_{\{\mathrm{d}u\neq 0\}} + \frac{1}{2}\varphi^2H^2 + \frac{1}{2}\|\nabla\varphi\|^2.\end{aligned}$$

Recalling that $\|\nabla\beta\|^2\mathbb{1}_{\{\mathrm{d}u\neq 0\}}$ is integrable, it follows from (5.20) that

$$\begin{aligned}&\int_{\{\mathrm{d}u\neq 0\}}\left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 + \left\{\frac{1}{m-1} - \frac{C_m\varepsilon}{2} - \varepsilon\right\}\|\nabla\beta\|^2\right]\varphi^2\mathrm{d}x_g\\ &\le C_{m,\varepsilon}\int H^2\varphi^2\,\mathrm{d}x_g + C_\varepsilon\int\|\nabla\varphi\|^2\,\mathrm{d}x_g.\end{aligned}$$

Choosing a small $\varepsilon > 0$ and taking (5.13) into account, we readily deduce (5.19) and complete the proof. $\square$

Using (5.9), (5.19) and the approximation in Subsection 5.1, we prove the following result. We recall that, for $m = 2$, the space $L^p(\Omega)$ below is meant to be empty.

Corollary 5.11. *Let $\Omega \subset \mathbb{R}^m$ be a domain. Assume that either*

- *$m \geq 2$, Ω is bounded, $\mathscr{F} \subset \mathcal{S}(\partial\Omega)$ is a compact subset, and $\phi \in \mathscr{F}$;*
- *$m \geq 3$, $\Omega = \mathbb{R}^m$.*

Fix $\mathcal{I}_1, \mathcal{I}_2 \in \mathbb{R}^+$, $\Omega' \Subset \Omega$ and, for $\varepsilon > 0$, define $\Omega'_\varepsilon \doteq \{x \in \Omega' : \mathrm{d}_\delta(x, \partial\Omega') > \varepsilon\}$. Let $p \in (1, 2_*]$. Then, there exists a constant

$$\mathcal{C} = \begin{cases} \mathcal{C}(\Omega, \mathscr{F}, m, \mathrm{diam}_\delta(\Omega), p, \mathcal{I}_1, \mathcal{I}_2, \varepsilon, \mathrm{d}_\delta(\Omega', \partial\Omega)) & \text{if } \Omega \text{ is bounded,} \\ \mathcal{C}(m, p, \mathcal{I}_1, \mathcal{I}_2, \varepsilon, |\Omega'|_\delta) & \text{if } \Omega = \mathbb{R}^m \end{cases} \tag{5.21}$$

such that for each $\rho \in \mathcal{M}(\Omega) + L^p(\Omega)$ satisfying

$$\|\rho\|_{\mathcal{M}(\Omega)+L^p(\Omega)} \leq \mathcal{I}_1, \qquad \|\rho\|_{L^2(\Omega')} \leq \mathcal{I}_2,$$

it holds

$$\int_{\Omega'_\varepsilon} \left\{ w_\rho \left|D^2 u_\rho\right|^2 + w_\rho^3 \left|D^2 u_\rho\left(Du_\rho, \cdot\right)\right|^2 + w_\rho^5 \left[D^2 u_\rho\left(Du_\rho, Du_\rho\right)\right]^2 \right\} \mathrm{d}x \leq \mathcal{C}. \tag{5.22}$$

In particular,

$$\begin{aligned} \int_{\Omega'_\varepsilon} \frac{1}{w_\rho} \left\{ \left|D \log w_\rho\right|^2 + \left|D w_\rho \cdot D u_\rho\right|^2 \right\} \mathrm{d}x \leq \mathcal{C}, \\ \int_{\Omega'_\varepsilon} \left\{ \left|D \log w_\rho\right| + \left|D w_\rho \cdot D u_\rho\right| \right\} \mathrm{d}x \leq \mathcal{C}. \end{aligned} \tag{5.23}$$

Proof. We choose p_1 as in (5.1) to guarantee that $\rho \in \mathcal{Y}(\Omega)^*$, and referring to Subsection 5.1, we approximate ρ through convolution obtaining $\{\rho_j\}$ with $\rho_j = -H_j \mathrm{d}x$ and $H_j \in C^\infty(\overline{\Omega})$ (resp. $H_j \in_c^\infty (\mathbb{R}^m)$). Let u_j be the smooth solution to ($\mathcal{BI}$) with source ρ_j, and write $w_j \doteq (1 - |Du_j|^2)^{-1/2}$. Proposition 3.7 yields $u_j \to u_\rho$ strongly in $W^{1,q}(\Omega)$, for each $q \in [1, \infty)$ if Ω is bounded and each $q \in [2^*, \infty)$ if $\Omega = \mathbb{R}^m$. We fix $\varphi \in C_c^1(\Omega')$ so that $\varphi \equiv 1$ on Ω'_ε and $|D\varphi(x)| \leq 2/\varepsilon$ for each $x \in \Omega$. From

$$\|\nabla\varphi\|^2 = |D\varphi|^2 + w_j^2 \left(Du_j \cdot D\varphi\right)^2 \leq \left(1 + w_j^2 |Du_j|^2\right) |D\varphi|^2 = w_j^2 |D\varphi|^2,$$

(5.9) and Proposition 5.10 with u_j, it follows that

$$\begin{aligned} &\int_\Omega \varphi^2 w_j \left\{ \left|D^2 u_j\right|^2 + 2w_j^2 \left|D^2 u\left(Du_j, \cdot\right)\right|^2 + w_j^4 \left[D^2 u_j\left(Du_j, Du_j\right)\right]^2 \right\} \mathrm{d}x \\ &\qquad \leq C_m \int_\Omega \left\{ w_j |D\varphi|^2 + \varphi^2 \rho_j^2 w_j^{-1} \right\} \mathrm{d}x. \end{aligned}$$

Combining this estimate with $w_j \geq 1$, the properties of φ and Proposition 3.9, we find a constant C as in (5.21) such that

(5.24)
$$\sup_{j\geq 1} \int_{\Omega'_\varepsilon} w_j \left\{ \left|D^2 u_j\right|^2 + 2w_j^2 \left|D^2 u\left(Du_j, \cdot\right)\right|^2 + w_j^4 \left[D^2 u_j\left(Du_j, Du_j\right)\right]^2 \right\} \mathrm{d}x \leq C.$$

In particular, $\{u_j\}$ is bounded in $W^{2,2}(\Omega'_\varepsilon)$ and we may suppose that $u_j \rightharpoonup u_\rho$ weakly in $W^{2,2}(\Omega'_\varepsilon)$. From the $W^{1,q}$ convergence we may also suppose that $u_j(x) \to u_\rho(x)$, $Du_j(x) \to Du_\rho(x)$ and $w_j(x) \to w_\rho(x)$ for a.e. $x \in \Omega'_\varepsilon$.

Fix $N > 1$ and set

$$w_{N,j}(x) \doteq \min\{w_j(x), N\}, \quad w_{N,\rho}(x) \doteq \min\{w_\rho(x), N\}.$$

By (5.24), we have

(5.25)
$$\sup_{j\geq 1, N>1} \int_{\Omega'_\varepsilon} w_{N,j} \left\{ \left|D^2 u_j\right|^2 + 2w_{N,j}^2 \left|D^2 u_j\left(Du_j, \cdot\right)\right|^2 + w_{N,j}^4 \left[D^2 u_j\left(Du_j, Du_j\right)\right]^2 \right\} \mathrm{d}x \leq C.$$

From $w_j \to w_\rho$, $Du_j \to Du_\rho$ a.e. on Ω, $w_{N,j} \leq N$ and $|Du_j| \leq 1$, it follows that for every $1 \leq i_1, i_2 \leq m$ and $q \in [1, \infty)$,

$$\begin{aligned} &\left\| w_{N,j} - w_{N,\rho} \right\|_{L^q(\Omega'_\varepsilon)} + \left\| w_{N,j}^{3/2} (u_j)_{i_1} - w_{N,\rho}^{3/2} (u_\rho)_{i_1} \right\|_{L^q(\Omega'_\varepsilon)} \\ &\qquad + \left\| w_{N,j}^{5/2} (u_j)_{i_1} (u_j)_{i_2} - w_{N,\rho}^{5/2} (u_\rho)_{i_1} (u_\rho)_{i_2} \right\|_{L^q(\Omega'_\varepsilon)} \to 0. \end{aligned}$$

Since $u_j \rightharpoonup u_\rho$ weakly in $W^{2,2}(\Omega'_\varepsilon)$, for any $\psi \in L^\infty(\Omega'_\varepsilon)$, we see

$$\begin{aligned} \int_{\Omega'_\varepsilon} w_{N,j}^{1/2} (u_j)_{i_1,i_2} \psi \, \mathrm{d}x &\to \int_{\Omega'_\varepsilon} w_{N,\rho}^{1/2} (u_\rho)_{i_1,i_2} \psi \, \mathrm{d}x, \\ \int_{\Omega'_\varepsilon} w_{N,j}^{3/2} (u_j)_{i_1,i_2} (u_j)_{i_3} \psi \, \mathrm{d}x &\to \int_{\Omega'_\varepsilon} w_{N,\rho}^{3/2} (u_\rho)_{i_1,i_2} (u_\rho)_{i_3} \psi \, \mathrm{d}x, \\ \int_{\Omega'_\varepsilon} w_{N,j}^{5/2} (u_j)_{i_1,i_2} (u_j)_{i_3} (u_j)_{i_4} \psi \, \mathrm{d}x &\to \int_{\Omega'_\varepsilon} w_{N,\rho}^{5/2} (u_\rho)_{i_1,i_2} (u_\rho)_{i_3} (u_\rho)_{i_4} \psi \, \mathrm{d}x. \end{aligned}$$

Thus, the density of $L^\infty(\Omega'_\varepsilon)$ in $L^2(\Omega'_\varepsilon)$ yields

$$\begin{aligned} &w_{N,j}^{1/2} D^2 u_j \rightharpoonup w_{N,\rho}^{1/2} D^2 u_\rho, \quad w_{N,j}^{3/2} D^2 u_j\left(Du_j, \cdot\right) \rightharpoonup w_{N,\rho}^{3/2} D^2 u_\rho\left(Du_\rho, \cdot\right), \\ &w_{N,j}^{5/2} D^2 u_j\left(Du_j, Du_j\right) \rightharpoonup w_{N,\rho}^{5/2} D^2 u_\rho\left(Du_\rho, Du_\rho\right) \end{aligned}$$

weakly in $L^2(\Omega'_\varepsilon)$. Hence, by (5.25) and the lower semicontinuity of the norm, we obtain

$$\sup_{N>1} \int_{\Omega'_\varepsilon} w_{N,\rho} \left\{ \left|D^2 u_\rho\right|^2 + 2w_{N,\rho}^2 \left|D^2 u_\rho\left(Du_\rho, \cdot\right)\right|^2 + w_{N,\rho}^4 \left[D^2 u_\rho\left(Du_\rho, Du_\rho\right)\right]^2 \right\} \mathrm{d}x \leq C.$$

By letting $N \to \infty$ and using the monotone convergence theorem, (5.22) holds.

The first in (5.23) readily follows from

$$|D \log w_\rho|^2 = w_\rho^4 \left|D^2 u_\rho(Du_\rho, \cdot)\right|^2, \qquad Dw_\rho \cdot Du_\rho = w_\rho^3 D^2 u_\rho(Du_\rho, Du_\rho)$$

a.e. on Ω. On the other hand, the second in (5.23) is derived from Hölder's inequality and Proposition 3.9:

$$\int_{\Omega'_\varepsilon} \left\{ \left| D\log w_\rho \right| + \left| Dw_\rho \cdot Du_\rho \right| \right\} \mathrm{d}x$$
$$\leq \left(\int_{\Omega'_\varepsilon} w_\rho \mathrm{d}x \right)^{1/2} \left(\int_{\Omega'_\varepsilon} \frac{1}{w_\rho} \left\{ \left| D\log w_\rho \right|^2 + \left| Dw_\rho \cdot Du_\rho \right|^2 \right\} \mathrm{d}x \right)^{1/2}.$$

This concludes the proof. □

5.4. **Higher regularity.** We first examine the case $m = 2$:

Theorem 5.12. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, let $\mathscr{F} \subset \mathcal{S}(\partial\Omega)$ be compact and $\phi \in \mathscr{F}$. Fix $\Omega' \Subset \Omega$ and for $\varepsilon > 0$, define $\Omega'_\varepsilon \doteq \{x \in \Omega' : \mathrm{d}_\delta(x, \partial\Omega') > \varepsilon\}$. Let $\rho \in \mathcal{M}(\Omega)$ satisfy*

$$\|\rho\|_{\mathcal{M}(\Omega)} \leq \mathcal{I}_1, \qquad \|\rho\|_{L^2(\Omega')} \leq \mathcal{I}_2$$

for some constants $\mathcal{I}_1, \mathcal{I}_2$. Then, there exists $C = C(\Omega, \mathscr{F}, \mathrm{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, \mathrm{d}_\delta(\Omega', \partial\Omega))$ such that the energy density $w_\rho = (1 - |Du_\rho|^2)^{-1/2}$ satisfies

$$\int_{\Omega'_\varepsilon} w_\rho \log\left(1 + w_\rho\right) \mathrm{d}x \leq C. \tag{5.26}$$

In particular, u_ρ weakly solves ($\mathcal{BI}$) on Ω'.

Proof. We fix p_1 as in (5.1) and, as in the proof of Corollary 5.11, we find $\rho_j \doteq -H_j \mathrm{d}x$ satisfying $H_j \in C^\infty(\overline{\Omega})$ and

$$\sup_{j\geq 1} \|\rho_j\|_{\mathcal{M}(\Omega)} \leq \mathcal{I}_1, \qquad \sup_{j\geq 1} \|\rho_j\|_{L^2(\Omega')} \leq \mathcal{I}_2.$$

Denote by u_j the minimizer of I_{ρ_j} and by $w_j = (1 - |Du_j|^2)^{-1/2}$. We recall that, for each Radon measure μ on $\mathbb{R}^m$, the following trace inequality holds for some constant $C = C(m)$, see [37, Corollary 1.1.2]:

$$\int \varphi \,\mathrm{d}\mu \leq C \left[\sup_{x\in\mathbb{R}^m, r>0} \frac{\mu(B_r(x))}{r^{m-1}} \right] \int |D\varphi| \,\mathrm{d}x \qquad \forall \varphi \in C^\infty_c(\mathbb{R}^m). \tag{5.27}$$

By Proposition 3.9,

$$\int_{\Omega'} w_j \mathrm{d}x \leq C_1\left(\Omega, \mathscr{F}, \mathrm{diam}_\delta(\Omega), \mathcal{I}_1, \mathrm{d}_\delta(\Omega', \partial\Omega)\right),$$

while, by Corollary 5.11,

$$\int_{\Omega'_{\varepsilon/2}} \left| D\log w_j \right| \mathrm{d}x \leq C_2\left(\Omega, \mathscr{F}, \mathrm{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, \mathrm{d}_\delta(\Omega', \partial\Omega)\right).$$

Hereafter, C_j will denote a constant depending on the same data as C_2. We consider the measure $\mu \doteq w_j \mathrm{d}x \llcorner \Omega'_\varepsilon$ and set $\varphi \doteq \psi \log(1 + w_j)$ for a cut-off function ψ

satisfying $\psi \equiv 1$ on $\Omega'_{3\varepsilon/4}$ and $\operatorname{supp}\psi \subset \Omega'_{\varepsilon/2}$. By (5.4), for each $x \in \Omega'_{\varepsilon/4}$ and $r < \varepsilon/8$,

$$\mu(B_r(x)) = \int_{B_r(x)\cap\Omega'_\varepsilon} w_j \mathrm{d}x \le r\left[\frac{8}{\varepsilon}\int_{B_{\varepsilon/8}(x)} w\,\mathrm{d}x + C(\mathcal{I}_1)\right] \le C_3 r.$$

On the other hand, if $x \in \Omega'_{\varepsilon/4}$ and $r \ge \varepsilon/8$, then

$$\mu(B_r(x)) \le \int_{\Omega'} w_j \mathrm{d}x \le C_1 \le C_4 r.$$

When $x \notin \Omega'_{\varepsilon/4}$ and $r < \varepsilon/8$, we clearly have $\mu(B_r(x)) = 0$. Hence, $\mu(B_r(x)) \le C_5 r$ for each $x \in \mathbb{R}^2$, $r > 0$. Our dimensional restriction, (5.27) and (5.23) imply

$$\begin{aligned}\int_{\Omega'_\varepsilon} w_j \log\left(1+w_j\right)\mathrm{d}x &\le C_6 \int_{\mathbb{R}^2}\left|D\left(\psi\log\left(1+w_j\right)\right)\right|\mathrm{d}x\\ &\le C_6\int_{\Omega'_{\varepsilon/2}}\left[\log\left(1+w_j\right)|D\psi| + \psi\left|D\log w_j\right|\right]\mathrm{d}x \le C_7.\end{aligned}$$

Now (5.26) follows by letting $j \to \infty$ and using Fatou's lemma. Finally, the fact that u_ρ weakly solves ($\mathcal{BI}$) on Ω' follows from (5.26) and the discussion in Subsection 5.1. ☐

We remark that Theorem 5.12 cannot be extended to dimension $m \ge 4$. Otherwise, the entire proof of Theorem 1.10 in Subsection 6.2 would work for dimension $m \ge 4$, which contradicts the example in Remark 1.14 (cf. Theorem 5.5). In dimension $m = 3$, proving that $\{w_j\}$ is locally uniformly integrable on a subdomain where ρ is of class L^2 is an open problem, which seems challenging.

Nevertheless, under a relative compactness assumption on Lorentzian balls we can prove a higher integrability of w_ρ in any dimension. We briefly comment on why cut-off functions based on the Lorentzian distance from o are better behaved than those based on the Euclidean distance r_o. If $u \in \mathcal{Y}_\phi(\Omega)$ and $\phi \in \mathcal{S}(\partial\Omega)$, then from (2.8) we get

$$\|\nabla \ell_o^2\|^2 \le 4\ell_o^2 + 16w^2|x-o|, \qquad |\Delta_M \ell_o^2| \le 2m + 4wH|x-o|. \tag{5.28}$$

By Proposition 3.9, given $\Omega' \Subset \Omega$ and $\mathcal{I}_1$ such that $\rho = -H\mathrm{d}x$ and $\|\rho\|_{\mathcal{M}(\Omega)} \le \mathcal{I}_1$, (2.1) yields

$$\int_M |H|w\,\mathrm{d}x_g \le \mathcal{I}_1, \qquad \int_{M'} w^2\,\mathrm{d}x_g \le C,$$

where M' is the graph over Ω' and C is a constant as in Proposition 3.9. On the other hand, computing the gradient and Laplacian of r_o and using (2.3), we get

$$|\Delta_M r_o^2| \le C(1 + w^2 + |H|w).$$

As we will see in the next proof, the advantage of using ℓ_o instead of r_o is exactly the absence of the addendum w^2 in the upper bound (5.28) for $|\Delta_M \ell_o^2|$.

To state the next result, recall the Lorentzian ball $L_R^\rho(A)$ defined in (2.7).

Theorem 5.13. *Let $\Omega \subset \mathbb{R}^m$ be either*

- *a bounded domain,* $m \geq 2$, $\mathscr{F} \subset \mathcal{S}(\partial\Omega)$ *is compact and* $\phi \in \mathscr{F}$, *or*
- $\Omega = \mathbb{R}^m$ *and* $m \geq 3$.

Let

$$H \in C^\infty(\overline{\Omega}) \quad \text{if } \Omega \text{ is bounded}, \qquad H \in C^\infty_c(\mathbb{R}^m) \quad \text{if } \Omega = \mathbb{R}^m,$$

define the measure $\rho = -H\,\mathrm{d}x$, *and let* $u \in \mathcal{Y}_\phi(\Omega)$ *be the minimizer of* I_ρ. *Assume that*

$$\|u\|_{L^\infty(\Omega)} \leq \mathcal{I}_0, \qquad \|\rho\|_{\mathcal{M}(\Omega)+L^p(\Omega)} \leq \mathcal{I}_1, \tag{5.29}$$

for some constants $\mathcal{I}_0, \mathcal{I}_1 > 0$ *and* $p \in (1, 2_*]$. *Suppose that there exist two open subsets* $\Omega'' \Subset \Omega' \Subset \Omega$ *such that*

$$\int_{\Omega'} H^2 \frac{(1+\log w)^{q_0+2}}{w}\mathrm{d}x \leq \mathcal{I}_{2,q_0}, \tag{5.30}$$

for some $q_0 \in \mathbb{N} \cup \{0\}$ *and* $\mathcal{I}_{2,q_0} \in \mathbb{R}^+$, *and that for some* $R > 0$ *it holds*

$$L^\rho_R(\Omega'') \Subset \Omega'.$$

Then, there exists a constant

$$C = \begin{cases} C(\Omega, \mathscr{F}, m, \mathrm{diam}_\delta(\Omega), \mathcal{I}_0, \mathcal{I}_1, q_0, \mathcal{I}_{2,q_0}, \mathrm{d}_\delta(\Omega', \partial\Omega), R) & \text{if } \Omega \text{ is bounded}, \\ C(m, p, \mathcal{I}_0, \mathcal{I}_1, q_0, \mathcal{I}_{2,q_0}, |\Omega'|_\delta, R) & \text{if } \Omega = \mathbb{R}^m \end{cases} \tag{5.31}$$

such that

$$\int_{\Omega''} \frac{(1+\log w)^{q_0}}{w} \left\{ \| \mathrm{II} \|^2 + w^2 \log w \right\} \mathrm{d}x \leq C. \tag{5.32}$$

Proof. By Theorem 1.3 or [8, Theorem 1.5 and Remark 3.4], we know that u is smooth and strictly spacelike. In particular, $L^\rho_s(\Omega'') \Subset L^\rho_t(\Omega'')$ if $0 \leq s < t$. Define p_1 as in (5.1). We proceed by induction on $q \in \{0, \dots, q_0\}$. Set for convenience

$$\bar{R} \doteq \frac{R}{q_0+1},$$

and define the sequence

$$\Omega'' \doteq \Omega_{q_0+1} \Subset \Omega_{q_0} \Subset \ldots \Subset \Omega_1 \Subset \Omega_0 \Subset \Omega', \qquad \Omega_q \doteq L^\rho_{(q_0+1-q)\bar{R}}(\Omega'') \text{ for } q \geq 0.$$

Let M_q be the graph of u over Ω_q. By rephrasing (5.30) in terms of the graph metric and the hyperbolic angle β, there exists a constant $\bar{\mathcal{I}}_{2,q_0}$ only depending on $\mathcal{I}_{2,q_0}$ such that

$$\int_{M_0} H^2(1+\beta)^{q_0+2} \leq \bar{\mathcal{I}}_{2,q_0},$$

where, hereafter in the proof, integration on subsets of the graph of u will always be performed with respect to the graph measure $\mathrm{d}x_g$, that will be omitted as far as no confusion arises. Hence,

$$\int_{M_0} H^2(1+\beta)^{q+2} \leq \bar{\mathcal{I}}_{2,q_0} \qquad \text{for each } q \in \{0, 1, \dots, q_0\}. \tag{5.33}$$

As a starting point, observe that Proposition 3.9 and (5.29) imply the existence of

$$\bar{\mathcal{I}}_{1,0} = \begin{cases} \bar{\mathcal{I}}_{1,0}\big(\Omega, \mathscr{F}, m, \mathrm{diam}_\delta(\Omega), p, \mathcal{I}_0, \mathcal{I}_1, \mathrm{d}_\delta(\Omega', \partial\Omega)\big) & \text{if } \Omega \text{ is bounded,} \\ \bar{\mathcal{I}}_{1,0}\big(m, p, \mathcal{I}_0, \mathcal{I}_1, |\Omega'|_\delta\big) & \text{if } \Omega = \mathbb{R}^m, \end{cases}$$

such that

$$(\mathscr{A}_0) \qquad \int_{M_0} |H| \,\mathrm{ch}\,\beta + \int_{M_0} \mathrm{ch}^2\,\beta \le \bar{\mathcal{I}}_{1,0}.$$

We shall prove the following inductive step:

if there exists

$$\mathcal{J}_{1,q} = \begin{cases} \mathcal{J}_1\big(\Omega, \mathscr{F}, m, \mathrm{diam}_\delta(\Omega), p, \mathcal{I}_0, \mathcal{I}_1, \mathrm{d}_\delta(\Omega', \partial\Omega), q_0, q, R\big) & \text{if } \Omega \text{ is bounded,} \\ \mathcal{J}_1\big(m, p, \mathcal{I}_0, \mathcal{I}_1, |\Omega'|_\delta, q_0, q, R\big) & \text{if } \Omega = \mathbb{R}^m, \end{cases}$$

such that

$(\mathscr{A}_q)$

$$\int_{M_q} |H|(1+\beta)^q \,\mathrm{ch}\,\beta + \int_{M_q} (1+\beta)^q \,\mathrm{ch}^2\,\beta \le \mathcal{J}_{1,q},$$

then there exists

$$\mathcal{J}_{2,q} = \begin{cases} \mathcal{J}_2\big(\Omega, \mathscr{F}, m, \mathrm{diam}_\delta(\Omega), p, \mathcal{I}_0, \mathcal{I}_1, \mathrm{d}_\delta(\Omega', \partial\Omega), q_0, q, \mathcal{J}_{1,q}, R\big) & \text{if } \Omega \text{ is bounded,} \\ \mathcal{J}_2\big(m, p, \mathcal{I}_0, \mathcal{I}_1, |\Omega'|_\delta, q_0, q, \mathcal{J}_{1,q}, R\big) & \text{if } \Omega = \mathbb{R}^m, \end{cases}$$

such that

$(\mathscr{B}_q)$

$$\int_{M_{q+1}} (1+\beta)^q \|\,\mathrm{II}\,\|^2 + \int_{M_{q+1}} (1+\beta)^{q+1} \,\mathrm{ch}^2\,\beta \le \mathcal{J}_{2,q}.$$

In view of (5.13) and (5.33), to obtain $(\mathscr{B}_q)$ from $(\mathscr{A}_q)$ it is enough to show that

$$\int_{M_{q+1} \cap \{\mathrm{d}u \neq 0\}} (1+\beta)^q \left[\frac{\mathrm{sh}^2\,\beta}{\mathrm{ch}^2\,\beta} \|A\|^2 + \|\nabla\beta\|^2 + \beta\,\mathrm{sh}^2\,\beta \right] \le \mathcal{J}_{2,q},$$

with $\mathcal{J}_{2,q}$ possibly different, but depending on the same data. We first show that $(\mathscr{B}_q) \Rightarrow (\mathscr{A}_{q+1})$ for each $0 \le q \le q_0 - 1$: by (5.33) and Young's inequality,

$$\begin{aligned} \int_{M_{q+1}} |H|(1+\beta)^{q+1} \,\mathrm{ch}\,\beta &\le \int_{M_{q+1}} H^2 (1+\beta)^{q+2} + \int_{M_{q+1}} (1+\beta)^q \,\mathrm{ch}^2\,\beta \\ &\le \bar{\mathcal{I}}_{2,q_0} + \mathcal{J}_{2,q}, \end{aligned}$$

hence $(\mathscr{A}_{q+1})$ holds with $\mathcal{J}_{1,q+1} \doteq \bar{\mathcal{I}}_{2,q_0} + 2\mathcal{J}_{2,q}$.

Since we verified $(\mathscr{A}_0)$, if the implication $(\mathscr{A}_q) \Rightarrow (\mathscr{B}_q)$ is proved, then the induction hypothesis implies $(\mathscr{B}_{q_0})$, which is equivalent to (5.32).

With the above preparation, it suffices to prove that $(\mathscr{A}_q) \Rightarrow (\mathscr{B}_q)$. For small $t > 0$, we consider a smooth approximation $\beta_t \in C^\infty(\Omega)$ of β defined by

$$\mathrm{ch}\,\beta_t \doteq \sqrt{w^2 + t} \quad \Leftrightarrow \quad \beta_t = \log\left(\sqrt{w^2+t} + \sqrt{w^2+t-1}\right).$$

Note that

$$\begin{aligned} &\beta \le \beta_t \le \beta + 1 \ \text{ for small enough } t, && \nabla\beta_t = 0 \quad \text{a.e. on } \{\mathrm{d}u = 0\}, \\ &\beta_t \downarrow \beta, \quad \|\nabla\beta_t\| \uparrow \|\nabla\beta\| \cdot \mathbb{1}_{\{\mathrm{d}u\neq 0\}} \ \text{ as } t \downarrow 0, && \langle \nabla\beta_t, \nabla\beta\rangle\, \mathbb{1}_{\{\mathrm{d}u\neq 0\}} \ge 0. \end{aligned} \tag{5.34}$$

Define also

$$\bar{u} \doteq u - \|u\|_\infty \le 0. \tag{5.35}$$

We consider the smooth vector field $Y + \beta_t \nabla e^{\bar{u}}$, where Y is defined in (5.14), and compute its divergence. For $\varepsilon \in (0,1)$ to be specified later, we use (5.18) to deduce that for some positive constants C_m and $C_{m,\varepsilon}$ depending, respectively, on m and on (m,ε),

(5.36)

$$\begin{aligned} \mathrm{div}_M\left(Y + \beta_t\nabla e^{\bar{u}}\right) \ge &\left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 - C_{m,\varepsilon}H^2 + \left\{\frac{1}{m-1} - C_m\varepsilon\right\}\|\nabla\beta\|^2\right]\cdot \mathbb{1}_{\{\mathrm{d}u\neq 0\}} \\ &+ e^{\bar{u}}\langle\nabla\beta_t, \nabla u\rangle + \beta_t e^{\bar{u}} H\,\mathrm{ch}\,\beta + \beta_t e^{\bar{u}}\,\mathrm{sh}^2\beta. \end{aligned}$$

Hereafter, $C_m, C_{m,\varepsilon}$ as well as the constants $C_q, C_{q,\varepsilon}$, may vary from line to line.

We integrate (5.36) against the test function

$$\psi = \varphi^2(1+\beta_t)^q, \quad \varphi \in \mathrm{Lip}_c(\Omega_q), \quad \varphi^2 \in W^{2,\infty}(\Omega_q). \tag{5.37}$$

By

$$\nabla\psi = (1+\beta_t)^q\nabla\varphi^2 + q\varphi^2(1+\beta_t)^{q-1}\nabla\beta_t,$$

we see that

$$\begin{aligned} &\int_{\{\mathrm{d}u\neq 0\}} \varphi^2(1+\beta_t)^q\left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 - C_{m,\varepsilon}H^2 + \left\{\frac{1}{m-1} - C_m\varepsilon\right\}\|\nabla\beta\|^2\right] \\ &\quad + \int_M \varphi^2(1+\beta_t)^q e^{\bar{u}}\langle\nabla\beta_t, \nabla u\rangle + \int_M \varphi^2(1+\beta_t)^q\beta_t e^{\bar{u}} H\,\mathrm{ch}\,\beta \\ &\quad + \int_M \varphi^2(1+\beta_t)^q e^{\bar{u}}\beta_t\,\mathrm{sh}^2\beta \\ &\le -\int_M (1+\beta_t)^q\left\langle\nabla\varphi^2, Y + \beta_t\nabla e^{\bar{u}}\right\rangle - q\int_M \varphi^2(1+\beta_t)^{q-1}\left\langle\nabla\beta_t, Y + \beta_t\nabla e^{\bar{u}}\right\rangle. \end{aligned}$$

Rearranging the terms and using Cauchy-Schwarz's inequality together with (5.34), we obtain

$$\begin{aligned}
&\int_{\{du\neq 0\}} \varphi^2(1+\beta_t)^q \left[\frac{\operatorname{sh}^2\beta}{\operatorname{ch}^2\beta}\|A\|^2 + \left\{\frac{1}{m-1} - C_m\varepsilon\right\}\|\nabla\beta\|^2\right] \\
&\qquad + \int_M \varphi^2(1+\beta_t)^q e^{\bar u}\beta_t \operatorname{sh}^2\beta \\
\leq\ & -\int_M (1+\beta_t)^q \left\langle \nabla\varphi^2, Y + \beta_t\nabla e^{\bar u}\right\rangle - q\int_M \varphi^2(1+\beta_t)^{q-1}\left\langle \nabla\beta_t, Y + \beta_t\nabla e^{\bar u}\right\rangle \\
&\qquad + \int_{\{du\neq 0\}} \varphi^2(1+\beta_t)^q e^{\bar u}\|\nabla\beta\| \operatorname{sh}\beta + \int_M \varphi^2(1+\beta_t)^{q+1}e^{\bar u}|H|\operatorname{ch}\beta \\
&\qquad + C_{m,\varepsilon}\int_M \varphi^2(1+\beta_t)^q H^2.
\end{aligned}$$

From $\bar u \leq 0$ (see (5.35)) and

$$\varphi^2(1+\beta_t)^q e^{\bar u}\|\nabla\beta\|\operatorname{sh}\beta \leq \varepsilon\varphi^2(1+\beta_t)^q\|\nabla\beta\|^2 + \varepsilon^{-1}\varphi^2(1+\beta_t)^q\operatorname{sh}^2\beta,$$

we infer

(5.38)

$$\begin{aligned}
&\int_{\{du\neq 0\}} \varphi^2(1+\beta_t)^q \left[\frac{\operatorname{sh}^2\beta}{\operatorname{ch}^2\beta}\|A\|^2 + \left\{\frac{1}{m-1} - C_m\varepsilon\right\}\|\nabla\beta\|^2\right] \\
&\qquad + \int_M \varphi^2(1+\beta_t)^q e^{\bar u}\beta_t \operatorname{sh}^2\beta \\
\leq\ & -\int_M (1+\beta_t)^q \left\langle \nabla\varphi^2, Y + \beta_t\nabla e^{\bar u}\right\rangle - q\int_M \varphi^2(1+\beta_t)^{q-1}\left\langle \nabla\beta_t, Y + \beta_t\nabla e^{\bar u}\right\rangle \\
&\qquad + \varepsilon^{-1}\int_M \varphi^2(1+\beta_t)^q \operatorname{sh}^2\beta + \int_M \varphi^2(1+\beta_t)^{q+1}|H|\operatorname{ch}\beta \\
&\qquad + C_{m,\varepsilon}\int_M \varphi^2(1+\beta_t)^q H^2.
\end{aligned}$$

Because of ($\mathscr{A}_q$), (5.33) and the first in (5.34),

(5.39)

$$\begin{aligned}
&\int_M \varphi^2(1+\beta_t)^q\operatorname{sh}^2\beta \leq C_q\|\varphi\|_\infty^2 \mathcal{J}_{1,q}, \\
\int_M \varphi^2(1+\beta_t)^{q+1}|H|\operatorname{ch}\beta &\leq \frac{\|\varphi\|_\infty^2}{2}\left\{\int_{M_q}(1+\beta_t)^{q+2}H^2 + \int_{M_q}(1+\beta_t)^q\operatorname{ch}^2\beta\right\} \\
&\leq C_q\|\varphi\|_\infty^2\left[\bar{\mathcal{I}}_{2,q_0} + \mathcal{J}_{1,q}\right].
\end{aligned}$$

Notice that due to (5.17),

$$\|\nabla\varphi\|^2 \leq w^2|D\varphi|^2 = \operatorname{ch}^2\beta|D\varphi|^2, \qquad \|Y\|^2\cdot\mathbb{1}_{\{du\neq 0\}} \leq 2\left[\|\nabla\beta\|^2 + H^2\right]\cdot\mathbb{1}_{\{du\neq 0\}}.$$

Using $Y = 0$ a.e. on $\{\mathrm{d}u = 0\}$, Young's inequality and assumption ($\mathscr{A}_q$), we infer

$$
\begin{aligned}
& -\int_M (1+\beta_t)^q \langle \nabla\varphi^2, Y\rangle \\
& \le \varepsilon \int_{\{\mathrm{d}u\neq 0\}} \varphi^2(1+\beta_t)^q \left[\|\nabla\beta\|^2 + H^2\right] + \frac{4}{\varepsilon}\int_{\{\mathrm{d}u\neq 0\}} (1+\beta_t)^q \|\nabla\varphi\|^2 \\
& \le \varepsilon \int_{\{\mathrm{d}u\neq 0\}} \varphi^2(1+\beta_t)^q \left[\|\nabla\beta\|^2 + H^2\right] + 4\varepsilon^{-1}\|D\varphi\|_\infty^2 \int_{M_q} (1+\beta_t)^q \operatorname{ch}^2\beta \\
& \le \varepsilon \int_{\{\mathrm{d}u\neq 0\}} \varphi^2(1+\beta_t)^q \left[\|\nabla\beta\|^2 + H^2\right] + C_{q,\varepsilon}\|D\varphi\|_\infty^2 \mathcal{J}_{1,q}.
\end{aligned}
$$

Moreover, from (5.17), $\bar u \le 0$, (5.34) and $Y + \beta_t \nabla e^{\bar u} = 0$ a.e. on $\{\mathrm{d}u = 0\}$ it follows that

$$
\begin{aligned}
& - q\int_M \varphi^2(1+\beta_t)^{q-1}\left\langle \nabla\beta_t, Y + \beta_t\nabla e^{\bar u}\right\rangle \\
& \le -q\int_{\{\mathrm{d}u\neq 0\}} \varphi^2(1+\beta_t)^{q-1}\left\langle \nabla\beta_t, -\frac{\operatorname{sh}\beta}{\operatorname{ch}\beta}H\nu + \beta_t\nabla e^{\bar u}\right\rangle \\
& \le q\int_{\{\mathrm{d}u\neq 0\}} \varphi^2(1+\beta_t)^{q-1}\|\nabla\beta\||H| + q\int_{\{\mathrm{d}u\neq 0\}} \varphi^2(1+\beta_t)^q \operatorname{ch}\beta\|\nabla\beta\| \\
& \le 2\varepsilon\int_{\{\mathrm{d}u\neq 0\}} \varphi^2(1+\beta_t)^q\|\nabla\beta\|^2 + \frac{q^2}{\varepsilon}\int_M \varphi^2(1+\beta_t)^{q-2}H^2 + \frac{q^2}{\varepsilon}\int_M \varphi^2(1+\beta_t)^q \operatorname{ch}^2\beta \\
& \le 2\varepsilon\int_{\{\mathrm{d}u\neq 0\}} \varphi^2(1+\beta_t)^q\|\nabla\beta\|^2 + \varepsilon^{-1}C_q\|\varphi\|_\infty^2\left[\bar{\mathcal{I}}_{2,q_0} + \mathcal{J}_{1,q}\right].
\end{aligned}
$$

Plugging these inequalities into (5.38), we get

$$
\begin{aligned}
& \int_{\{\mathrm{d}u\neq 0\}} \varphi^2(1+\beta_t)^q\left[\frac{\operatorname{sh}^2\beta}{\operatorname{ch}^2\beta}\|A\|^2 + \left\{\frac{1}{m-1} - C_m\varepsilon\right\}\|\nabla\beta\|^2\right] \\
& \quad + \int_M \varphi^2(1+\beta_t)^q e^{\bar u}\beta_t \operatorname{sh}^2\beta \\
& \le -\int_M (1+\beta_t)^q\left\langle \nabla\varphi^2, \beta_t\nabla e^{\bar u}\right\rangle + C_{m,q,\varepsilon}\|\varphi\|_{W^{1,\infty}}^2\left[\bar{\mathcal{I}}_{2,q_0} + \mathcal{J}_{1,q}\right].
\end{aligned}
\tag{5.40}
$$

We next examine the term

$$
K \doteq -\int_M (1+\beta_t)^q\left\langle \nabla\varphi^2, \beta_t\nabla e^{\bar u}\right\rangle.
$$

For $U \Subset \Omega_q$, we choose φ satisfying (5.37) and

$$
\varphi = 0 \quad \text{on } \partial U. \tag{5.41}
$$

Hereafter, we will denote by C_j a constant depending on the same quantities as (5.31). Since $\nabla \beta_t = 0$ a.e. on $\{du = 0\}$, we compute

(5.42)
$$\begin{aligned} K &= -\int_M (1+\beta_t)^q \beta_t \left\langle \nabla \varphi^2, \nabla (e^{\bar{u}} - 1) \right\rangle \\ &= -\int_M \left\langle \nabla \varphi^2, \nabla \left[(1+\beta_t)^q \beta_t (e^{\bar{u}} - 1) \right] \right\rangle + \int_{\{du \neq 0\}} (e^{\bar{u}} - 1) \left\langle \nabla \varphi^2, \nabla \left[(1+\beta_t)^q \beta_t \right] \right\rangle . \end{aligned}$$

The last integral can be easily estimated by using (5.29), (5.34) and the definition of $\bar{u}$:

(5.43)
$$\begin{aligned} &\left| \int_{\{du \neq 0\}} (e^{\bar{u}} - 1) \left\langle \nabla \varphi^2, \nabla \left[(1+\beta_t)^q \beta_t \right] \right\rangle \right| \\ &\le \varepsilon \int_{\{du \neq 0\}} \varphi^2 (1+\beta_t)^q \|\nabla \beta\|^2 + 4\varepsilon^{-1} (1+q)^2 \|e^{\bar{u}} - 1\|^2_{L^\infty(\Omega_q)} \int_M (1+\beta_t)^q \|\nabla \varphi\|^2 \\ &\le \varepsilon \int_{\{du \neq 0\}} \varphi^2 (1+\beta_t)^q \|\nabla \beta\|^2 + \varepsilon^{-1} C_1 \|D\varphi\|^2_\infty \mathcal{J}_{1,q}. \end{aligned}$$

On the other hand, since $\varphi^2 \in W^{2,\infty}(\Omega_q)$ with $\operatorname{supp} \varphi \Subset \Omega_q$, we get

(5.44)
$$\begin{aligned} -\int_M \left\langle \nabla \varphi^2, \nabla \left[(1+\beta_t)^q \beta_t (e^{\bar{u}} - 1) \right] \right\rangle &= \int_M (1+\beta_t)^q \beta_t (e^{\bar{u}} - 1) \Delta_M \varphi^2 \\ &= \int_M (1+\beta_t)^q \beta_t \left(1 - e^{\bar{u}}\right) \left(-\Delta_M \varphi^2\right). \end{aligned}$$

We set $U = L_{\bar{R}}(o)$ where $o \in \Omega_{q+1}$. Then $U \Subset \Omega_q$ and since u is smooth with $\|Du\|_\infty < 1$, $\partial L_{\bar{R}}(o)$ is smooth. We also set

$$\varphi(x) \doteq \left(\bar{R}^2 - \ell_o^2(x) \right)_+ .$$

It is easily seen that (5.37) and (5.41) are satisfied. Moreover, by (2.8) and

$$-\Delta_M \ell_o^4 = -2\|\nabla \ell_o^2\|^2 - 2\ell_o^2 \Delta_M \ell_o^2 \le -2\ell_o^2 \Delta_M \ell_o^2, \tag{5.45}$$

it follows that on U,

$$\begin{aligned} -\Delta_M \varphi^2 &= -\Delta_M \left(\bar{R}^4 - 2\bar{R}^2 \ell_o^2 + \ell_o^4 \right) \le 2 \left(\bar{R}^2 - \ell_o^2 \right) \Delta_M \ell_o^2 \\ &\le 4\bar{R}^2 \left(m + 2\,|H|\operatorname{ch}\beta\, |x - o| \right) \\ &\le C_2 \left(1 + |H| \operatorname{ch}\beta \right). \end{aligned} \tag{5.46}$$

Remark also that

$$\|\varphi\|_{W^{1,\infty}} \le C_3.$$

From ($\mathscr{A}_q$), (5.44), (5.46), $0 \le 1 - e^{\bar{u}} \le 1$, $\beta \le \mathrm{ch}^2\beta$, (5.43) and (5.39), we deduce

$$
\begin{aligned}
K &\le C_2 \int_{M_q} \left(1+\beta_t\right)^q \beta_t \left(1 + |H|\,\mathrm{ch}\,\beta\right) \\
&\quad + C_1 \varepsilon^{-1} \|D\varphi\|_\infty^2 \mathcal{J}_{1,q} + \varepsilon \int_{\{\mathrm{d}u\neq 0\}} \varphi^2 (1+\beta_t)^q \|\nabla\beta\|^2 \\
&\le C_3 \varepsilon^{-1} \left[\bar{\mathcal{I}}_{2,q_0} + \mathcal{J}_{1,q}\right] + \varepsilon \int_{\{\mathrm{d}u\neq 0\}} \varphi^2 (1+\beta_t)^q \|\nabla\beta\|^2 .
\end{aligned}
\tag{5.47}
$$

Since $\varphi \ge \bar{R}^2/2$ on $L_{\bar{R}/2}(o)$, it follows from (5.40) and (5.47) that

$$
\begin{aligned}
&\int_{L_{\bar{R}/2}(o)} (1+\beta_t)^q \left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 + \left\{\frac{1}{m-1} - C_m\varepsilon\right\}\|\nabla\beta\|^2\right] \cdot \mathbb{1}_{\{\mathrm{d}u\neq 0\}} \\
&\qquad + \int_{L_{\bar{R}/2}(o)} e^{\bar{u}} (1+\beta_t)^q \beta_t\, \mathrm{sh}^2\beta \le C_4 C_{m,q,\varepsilon} \left[\mathcal{J}_{1,q} + \bar{\mathcal{I}}_{2,q_0}\right].
\end{aligned}
$$

Choosing $\varepsilon = \left[2C_m(m-1)\right]^{-1}$, noting that $e^{\bar{u}} \ge e^{-2\mathcal{I}_0}$ and letting $t \to 0$, we deduce

$$
\int_{L_{\bar{R}/2}(o)} (1+\beta)^q \left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 + \|\nabla\beta\|^2 + \beta\,\mathrm{sh}^2\beta\right] \cdot \mathbb{1}_{\{\mathrm{d}u\neq 0\}} \le C_5. \tag{5.48}
$$

Consider a maximal set of disjoint Euclidean balls $\{B_{\bar{R}/4}(o_1), \ldots, B_{\bar{R}/4}(o_s)\}$ with $o_i \in \Omega_{q+1}$. Since $B_{\bar{R}/4}(o_i) \subset L_{\bar{R}/4}(o_i) \Subset \Omega_q \Subset \Omega'$, we get

$$
s \le \left\lceil \frac{|\Omega'|_\delta}{\omega_m (\bar{R}/4)^m} \right\rceil \doteq \tau(m, R, q_0, |\Omega'|_\delta).
$$

Using that $\{B_{\bar{R}/2}(o_j)\}$ covers Ω_{q+1} and $B_{\bar{R}/2}(o_j) \subset L_{\bar{R}/2}(o_j) \Subset \Omega_q$, summing up (5.48) we conclude

$$
\int_{M_{q+1}} (1+\beta)^q \left[\frac{\mathrm{sh}^2\beta}{\mathrm{ch}^2\beta}\|A\|^2 + \|\nabla\beta\|^2 + \beta\,\mathrm{sh}^2\beta\right] \cdot \mathbb{1}_{\{\mathrm{d}u\neq 0\}} \le C_5\tau,
$$

which proves ($\mathscr{B}_q$). □

Remark 5.14. We comment on the choice of φ in the above proof. For a general cut-off function φ, in view of (2.3), one could just obtain the bound

$$
\left|\Delta_M \varphi^2\right| \le m\|D^2\varphi^2\|_\infty (1 + \mathrm{ch}^2\beta) + \|D\varphi^2\|_\infty |H|\,\mathrm{ch}\,\beta,
$$

which inserted into (5.44) would make necessary to estimate a term of the type

$$
\int_U (1+\beta_t)^q \beta_t\, \mathrm{ch}^2\beta. \tag{5.49}
$$

Such a term cannot be absorbed into the last addendum on the left-hand side of (5.40). This is the main reason why we use the extrinsic Lorentzian distance. Furthermore, the translation performed in the first line of (5.42) and the choice of $\bar{u}$ in (5.35) are crucial to make sure that the coefficient which multiplies $-\Delta_M\varphi^2$ in (5.44) is non-negative. Hence, an upper estimate for $-\Delta_M\varphi^2$ is sufficient and we can get rid of the term $\|\nabla \ell_o\|$ in (5.45), that would have lead, again, to the appearance of an integral of the type (5.49).

6. Proofs of the main theorems

6.1. **Proof of Theorem 1.16.** Consider the approximation $\{\rho_j, H_j, u_j, w_j\}$ in Subsection 5.1 and fix $\Omega' \Subset \mathbb{R}^m \backslash \{x_1, \ldots, x_k\}$ with smooth boundary. Then

$$\sup_{j\ge 1} \|H_j\|_{L^\infty(\Omega')} < \infty. \tag{6.1}$$

By Proposition 3.7, $u_j \to u_\rho$ in $L^\infty(\mathbb{R}^m)$ and $\mathscr{G} \doteq \{u_\rho\} \cup \{u_j : j \in \mathbb{N}\}$ is compact in $C(\mathbb{R}^m)$. Thus, for given $\Omega'' \Subset \Omega'$, by Lemma 3.8 and the assumption that u_ρ has no light-segments, there exists $R > 0$ independent of j such that the Lorentzian ball $L_R^{\rho_j}(\Omega'') \Subset \Omega'$ for all $j \ge 1$. By (6.1), we can apply Theorem 5.13 to deduce

$$\sup_{j\ge 1} \left\| w_j \log\left(1 + w_j\right)\right\|_{L^1(\Omega'')} < \infty.$$

Thus, the sequence $\{w_j\}$ is locally uniformly integrable on Ω'. By the arbitrariness of Ω', $\{w_j\}$ is locally uniformly integrable on $\Omega\backslash\{x_1, \ldots, x_k\}$; hence, Theorem 5.2 with $E = \{x_i\}_{i=1}^k$ implies

$$\int_{\mathbb{R}^m} w_\rho Du_\rho \cdot D\eta \, \mathrm{d}x = \langle \rho, \eta\rangle = \sum_{i=1}^k a_i \eta(x_i) \qquad \forall\, \eta \in \mathrm{Lip}_c(\mathbb{R}^m). \tag{6.2}$$

Therefore, u_ρ weakly solves ($\mathcal{BI}$).

We next prove that u_ρ has an isolated singularity at each x_i, in the sense of Ecker [17]. Fix $B \doteq B_r(x_i)$ with $x_j \notin \overline{B}$ for $j \neq i$, and choose $\eta \in \mathrm{Lip}_c(B)$ with $\eta = -a_i$ in a neighborhood of x_i. Suppose by contradiction that u_ρ minimizes I_0 in B, that is,

$$I_0(u_\rho) = \inf\left\{ I_0(v) \ : \ v \in \mathcal{Y}_{u_\rho}(B)\right\}, \quad I_0(v) \doteq \int_B \left(1 - \sqrt{1 - |Dv|^2}\right) \mathrm{d}x. \tag{6.3}$$

Since u_ρ does not have light segments, for each ball $\widetilde{B} \Subset B\backslash\{x_i\}$ we have

$$|u_\rho(x) - u_\rho(y)| < |x - y| = d_{\widetilde{B}}(x, y) \quad \forall\, x, y \in \partial\widetilde{B} \text{ with } x \neq y.$$

By (6.3), we may verify that u_ρ is a minimizer of I_0 on $\widetilde{B}$, hence Theorem 1.3 and the arbitrariness of $\widetilde{B}$ guarantee that u_ρ is strictly spacelike on $B\backslash\{x_i\}$. Since $D\eta = 0$ around x_i, we infer the existence of $t > 0$ small enough that $u_\rho + t\eta \in \mathcal{Y}_{u_\rho}(B)$. Using Proposition 3.9 and comparing to (6.2), we get

$$0 \ge \int_B w_\rho Du_\rho \cdot \left(Du_\rho - D(u_\rho + t\eta)\right) \mathrm{d}x = -t \int_B w_\rho Du_\rho \cdot D\eta \, \mathrm{d}x = t|a_i|^2 > 0,$$

which is a contradiction.

To conclude, [17, Theorem 1.5] ensures that u_ρ is asymptotic to a light cone C near x_i, and we can therefore apply the argument in [7, Theorem 3.5] to deduce that C is upward or downward pointing respectively when $a_i < 0$ or $a_i > 0$. □

6.2. **Proof of Theorem 1.10.** Let $\Sigma \Subset \Omega$ and $\rho \in \mathcal{M}(\Omega)$ satisfy the assumptions in Theorem 1.10. Fix $\mathscr{F}, \mathcal{I}_1, \mathcal{I}_2, \Omega'$ and ε as in (ii):

$$\phi \in \mathscr{F}, \qquad \|\rho\|_{\mathcal{M}(\Omega)} \le \mathcal{I}_1, \quad \|\rho\|_{L^2(\Omega')} \le \mathcal{I}_2. \tag{6.4}$$

We also choose $p_1 = 3$ for $\mathcal{Y}(\Omega)$ (any $p_1 > 2$ works). We split the proof into several steps.

Step 1: *for each ϕ, ρ satisfying (6.4), and for each $\varepsilon > 0$, there exists*

$$C_1\left(\Omega, \mathscr{F}, \mathrm{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, \mathrm{d}_\delta(\Omega', \partial\Omega)\right)$$

such that

$$\int_{\Omega'_\varepsilon} w_\rho \log\left(1 + w_\rho\right) \mathrm{d}x \le C_1, \quad \Omega'_\varepsilon \doteq \left\{x \in \Omega' \ : \ \mathrm{d}_\delta(x, \partial\Omega') > \varepsilon\right\}.$$

Proof of Step 1. This directly follows from Theorem 5.12 and (6.4). □

The higher integrability allows to prove the next no-light-segment property.

Step 2: *The minimizer u_ρ does not have light segments in Ω'.*

Proof of Step 2. Assume by contradiction that $\overline{xy} \subset \Omega'$ is a light segment for u_ρ. Up to renaming, $u_\rho(y) - u_\rho(x) = |y - x|$. Define

$$\widetilde{\rho} \doteq \rho + \delta_y - \delta_x.$$

By Theorem 5.5, u_ρ also minimizes $I_{\widetilde{\rho}}$: $u_\rho = u_{\widetilde{\rho}}$. To reach our desired contradiction, we tweak the argument in Theorem 5.5 used to show that u_ρ does not solve ($\mathcal{BI}$). Let $\{\varphi_j\}$ be a mollifier and define $\rho_j = \varphi_j * \rho$ and $\widetilde{\rho}_j = \varphi_j * \widetilde{\rho}$. Call $u_j, \widetilde{u}_j \in \mathcal{Y}_\phi(\Omega)$, respectively, the minimizers of I_{ρ_j} and $I_{\widetilde{\rho}_j}$, and denote by w_j and $\widetilde{w}_j$, respectively, their energy densities. In view of Proposition 3.7 and $u_\rho = u_{\widetilde{\rho}}$, as $j \to \infty$, we have $u_j \to u_\rho$ and $\widetilde{u}_j \to u_\rho$ in $C(\overline{\Omega})$. Notice that, by the properties of convolutions (see [40, Proof of Proposition 2.7]),

$$\|\rho_j\|_{\mathcal{M}(\Omega)} \le \|\rho\|_{\mathcal{M}(\Omega)} \le \mathcal{I}_1, \qquad \|\widetilde{\rho}_j\|_{\mathcal{M}(\Omega)} \le \|\widetilde{\rho}\|_{\mathcal{M}(\Omega)} \le \mathcal{I}_1 + 2$$

and for each $\Omega'' \Subset \Omega' \backslash \{x, y\}$, j large enough and ε small enough,

$$\|\rho_j\|_{L^2(\Omega''_{\varepsilon/4})} + \|\widetilde{\rho}_j\|_{L^2(\Omega''_{\varepsilon/4})} \le \|\rho\|_{L^2(\Omega'')} + \|\widetilde{\rho}\|_{L^2(\Omega'')} \le 2\mathcal{I}_2 + 2.$$

Hence, we can apply Theorem 5.12 on $\Omega'' \Subset \Omega' \setminus \{x, y\}$ to both u_j and to $\widetilde{u}_j$ to deduce that $\{w_j\}$ and $\{\widetilde{w}_j\}$ are locally uniformly integrable on $\Omega' \backslash \{x, y\}$. Then, Theorem 5.2 with $E = \{x, y\}$ guarantees that

$$\int w_\rho D u_\rho \cdot D\eta \,\mathrm{d}x = \langle \rho, \eta \rangle, \qquad \int w_\rho D u_\rho \cdot D\eta \,\mathrm{d}x = \langle \widetilde{\rho}, \eta \rangle \quad \forall \eta \in \mathrm{Lip}_c(\Omega').$$

However, choosing η such that $\eta(y) \neq \eta(x)$, we deduce

$$\langle \widetilde{\rho}, \eta \rangle = \langle \rho, \eta \rangle + \eta(y) - \eta(x) \neq \langle \rho, \eta \rangle,$$

giving the desired contradiction. □

Hereafter, we denote with $\{\rho_j, u_j, w_j\}$ the approximation described in Subsection 5.1. With the aid of Step 2 and $\rho \in L^2(\Omega')$, an application of Lemma 3.8, Corollary 5.11 and Theorem 5.13 gives the next improved higher integrability and second fundamental form estimates for u_ρ, which conclude the proof of Theorem 1.10 (ii).

Step 3: *Higher integrability, Theorem 1.10 (ii): for each* $\varepsilon > 0$, $q_0 > 0$, *there exists a constant*

$$C = C(\Omega, \mathscr{F}, \mathrm{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, \Omega', q_0) > 0$$

such that for each ρ *and* ρ *satisfying* (6.4),

$$\int_{\Omega'_\varepsilon} (1 + \log w_\rho)^{q_0} \left\{ w_\rho |D^2 u_\rho|^2 + w_\rho^3 \left| D^2 u_\rho \left(Du_\rho, \cdot \right) \right|^2 + w_\rho^5 \left[D^2 u_\rho (Du_\rho, Du_\rho) \right]^2 \right\} \mathrm{d}x$$
$$+ \int_{\Omega'_\varepsilon} w_\rho (1 + \log w_\rho)^{q_0 + 1} \mathrm{d}x \le C.$$

Proof of Step 3. Let $\mathscr{G} \subset \mathcal{Y}(\Omega)$ be the set of minimizers u_ρ whose boundary value ϕ and source ρ satisfy (6.4). Because of the compactness of $\mathscr{F}$ and of Propositions 3.5 and 3.7, taking into account the lower semicontinuity of $\|\cdot\|_{L^2(\Omega')}$ and $\|\cdot\|_{\mathcal{M}(\Omega)}$ under weak convergence, we deduce that $\mathscr{G}$ is compact in $C(\overline{\Omega})$. Applying the second part of Lemma 3.8, for $\varepsilon > 0$ we infer the existence of

$$R = R\big(\Omega, \mathscr{F}, \mathrm{diam}_\delta(\Omega), \mathcal{I}_1, \mathcal{I}_2, \varepsilon, \Omega'\big).$$

such that $L_R^{\rho_j}(\Omega'_\varepsilon) \Subset L_R^{\rho_j}(\Omega')$ for each $u \in \mathscr{G}$. Theorem 5.13 with $\Omega'' = \Omega'_\varepsilon$ ensures that (5.32) holds for u_j uniformly in j. The corresponding inequality for the pointwise limit u_ρ, which is a rewriting of our desired estimate, then follows by the same method as that in Corollary 5.11. □

Step 4: *Weak solvability and no light segments, Theorem 1.10 (i).*

Proof of Step 4. Applying Step 1 to the mollified sources ρ_j, we deduce that $\{w_j\}$ are locally uniformly integrable in $\Omega \setminus \Sigma$. Using $\mathscr{H}^1_\delta(\Sigma) = 0$, Theorem 5.2 implies that the limit u_ρ is a weak solution to ($\mathcal{BI}$) on Ω. On the other hand, by Step 2, u_ρ does not have light segments in any set $\Omega'' \Subset \Omega \backslash \Sigma$, hence in $\Omega \backslash \Sigma$. Since $\mathscr{H}^1_\delta(\Sigma) = 0$, there are no light segments on the entire Ω. □

Step 5: *Regularity for* $\rho \in L^\infty$, *Theorem 1.10 (iii).*

Proof of Step 5. Let $\rho \in L^\infty(\Omega')$, and fix a domain $\Omega'' \Subset \Omega'$. Due to Step 2, every point $x \in \Omega''$ has positive Lorentzian distance from $\partial\Omega'$, with a uniform bound depending on the data of our problem. We can therefore use the local gradient estimate in [4, Lemma 2.1] as in [4, Proof of Theorem 4.1] to deduce an L^∞-estimate for w_ρ

and a $W^{2,2}$-estimate for u_ρ in Ω''. From Theorem 1.10 (i) and (ii), $u_\rho \in W^{2,2}_{\mathrm{loc}}(\Omega')$ is a strong solution to

$$-\sum_{i=1}^{m} \partial_i \left(a_i(Du_\rho)\right) = \rho \quad \text{in } \Omega'', \text{ where } \ a_i(p) \doteq \left(1-|p|^2\right)^{-1/2} p_i \,:\, B_1(0) \to \mathbb{R}.$$

By differentiating formally the equation in x_k, we see that $(u_\rho)_k \in W^{1,2}(\Omega'')$ is a weak solution to

$$-\sum_{i=1}^{m} \partial_i \sum_{n=1}^{m} \frac{\partial a_i}{\partial p_n}(Du_\rho)(u_\rho)_{nk} = \sum_{i=1}^{m} \partial_i \left(\rho \delta_{ki}\right) \quad \text{in } \Omega''.$$

Since $(\partial a_i/\partial p_n)$ is bounded and uniformly elliptic on Ω'' due to the L^∞-bound of w_ρ, applying [28, Theorem 8.22 or Corollary 8.24], we see that $(u_\rho)_k \in C^\alpha_{\mathrm{loc}}(\Omega'')$ for some α, hence, $u_\rho \in C^{1,\alpha}_{\mathrm{loc}}(\Omega'')$. By bootstrapping, $u_\rho \in C^\infty(\Omega')$ whenever $\rho \in C^\infty(\Omega')$. □

By Steps 1–5, we complete the proof of Theorem 1.10. □

Remark 6.1. Referring to the approximations $\{u_j\}$ of u_ρ in Subsection 5.1, because of Theorem 5.13, Lemma 3.8 and the argument in Step 2 above, we deduce that the uniform integrability of $\{w_j \log w_j\}$ on a subdomain Ω' where $\rho \in L^2$ is *equivalent* to the nonexistence of light segments for u_ρ on Ω'.

6.3. **Proof of Theorem 1.13.** The proof is similar to the one of Theorem 1.10. We consider the approximation $\{\rho_j, H_j, u_j, w_j\}$ in Subsection 5.1. Fix $\Omega' \Subset \Omega \setminus (\Sigma \cup K^\rho_\phi)$ and a small $\varepsilon > 0$. Then,

$$\|\rho_j\|_{L^2(\Omega'_\varepsilon)} \le \|\rho\|_{L^2(\Omega')} \qquad \text{for } j \text{ large enough.}$$

Let $\Omega'' \Subset \Omega'_\varepsilon$. From the definition of K^ρ_ϕ and Proposition 3.7, the first part of Lemma 3.8 applied to $\mathscr{G} \doteq \{u_j\}_j \cup \{u\}$ guarantees the existence of R such that $L^{\rho_j}_R(\Omega'') \Subset \Omega'$ for each j, and therefore, by Theorem 5.13 we deduce that, for each $q_0 \in \mathbb{R}^+$,

$$\sup_j \int_{\Omega''} \left\{ w_j \left(1+\log w_j\right) + \|\,\mathrm{II}_j\,\|^2 w_j^{-1} \right\} \left(1+\log w_j\right)^{q_0} \mathrm{d}x < \infty.$$

Hence, Theorem 1.13 (ii) holds by the same argument as the one in Corollary 5.11. In the case $\rho \in L^\infty(\Omega')$, from $L^{\rho_j}_R(\Omega'') \Subset \Omega'$ and $\|\rho_j\|_{L^\infty(\Omega'')} \le \|\rho\|_{L^\infty(\Omega')}$ for large enough j we can proceed as in the proof of Step 5 in Theorem 1.10 to get $w_\rho \in L^\infty(\Omega'')$ and then $u_\rho \in C^{1,\alpha}_{\mathrm{loc}}(\Omega')$, which proves Theorem 1.13 (iii).

Summarizing, in our assumptions $\{w_j\}$ is locally uniformly integrable on $\Omega \setminus (\Sigma \cup K^\rho_\phi)$. Theorem 5.2 ensures that u_ρ satisfies ($\mathcal{BI}$) on $\Omega \backslash K^\rho_\phi$. Moreover, if $K^\rho_\phi \cap (\partial\Omega \cup \Sigma) = \emptyset$, then we can choose open sets Ω'', Ω' such that $K^\rho_\phi \subset \Omega'' \Subset \Omega' \Subset \Omega \backslash \Sigma$. By the definition of K^ρ_ϕ and applying Lemma 3.8, we get the existence of R such that $L^{\rho_j}_R(\Omega'') \Subset \Omega'$ for each j, and therefore a uniform integrability of $\{w_j\}$ on Ω'' by Theorem 5.13. Hence, $\{w_j\}$ is locally uniformly integrable on the entire $\Omega\backslash\Sigma$, and u_ρ solves ($\mathcal{BI}$) on Ω by Theorem 5.2. Thus, Theorem 1.13 (i) holds and this completes the proof. □

6.4. **Proof of Theorems 1.18 and 1.19.** We begin with the following proposition:

Proposition 6.2. *Let $m \geq 3$ and $\mathcal{I} > 0$ be given. Then there exists a constant $\mathcal{J} = \mathcal{J}(m, \mathcal{I}, p_1) > 0$ such that for any $\rho \in \mathcal{Y}(\mathbb{R}^m)^*$ with $\|\rho\|_{\mathcal{Y}^*} \leq \mathcal{I}$, the minimizer u_ρ satisfies*

$$\|u_\rho\|_\infty \leq \mathcal{J}. \tag{6.5}$$

Moreover, $L^\rho_\varepsilon(\Omega'') \Subset \Omega'$ holds provided $\varepsilon > 0$ and $\Omega'' \subset \Omega' \subset \mathbb{R}^m$ satisfy

$$\mathrm{d}_\delta(\Omega'', \mathbb{R}^m\backslash\Omega') \geq 2\mathcal{J} + \varepsilon. \tag{6.6}$$

Proof. Remark that the minimizer u_ρ satisfies $I_\rho(u_\rho) \leq I_\rho(0) = 0$. Recalling (3.6) and noting that $b_1 = 1/2$ in (3.5), we see that for each $\rho \in \mathcal{Y}(\mathbb{R}^m)^*$ with $\|\rho\|_{\mathcal{Y}^*} \leq \mathcal{I}$,

$$\|u_\rho\|^2_{\mathcal{Y}} \leq 4\left[1 + 2\|\rho\|_{\mathcal{Y}^*}\|u_\rho\|_{\mathcal{Y}}\right] \leq 4 + 8\mathcal{I}\|u_\rho\|_{\mathcal{Y}}.$$

Hence, minimizers are uniformly bounded in $\mathcal{Y}(\mathbb{R}^m)$ when $\|\rho\|_{\mathcal{Y}^*} \leq \mathcal{I}$ and by virtue of Proposition 3.3, (6.5) holds.

Let $\Omega'' \subset \Omega'$ satisfy (6.6). Notice that (6.5) implies that for each $x, o \in \mathbb{R}^m$ and each $\rho \in \mathcal{Y}(\mathbb{R}^m)^*$ with $\|\rho\|_{\mathcal{Y}^*} \leq \mathcal{I}$,

$$\left(\ell^\rho_o\right)^2(x) = r_o^2(x) - \left|u_\rho(x) - u_\rho(o)\right|^2 \geq r_o^2(x) - 4\mathcal{J}^2.$$

Hence, for any $x \in \mathbb{R}^m \setminus \Omega'$ and $o \in \Omega''$,

$$\left(\ell^\rho_o(x)\right)^2 \geq 4\mathcal{J}\varepsilon + \varepsilon^2,$$

which implies $L^\rho_\varepsilon(\Omega'') \Subset \Omega'$. □

Proof of Theorem 1.18. Define p_1 as in (5.1) for $m \geq 3$, and choose $\{\rho_j, u_j, w_j\}$ as in Subsection 5.1. Under the assumptions of Theorem 1.18, in view of Proposition 6.2, there exists $\mathcal{J} = \mathcal{J}(m, \mathcal{I}, p)$ such that $\|u_j\|_\infty \leq \mathcal{J}$ and $L^{\rho_j}_\varepsilon(\Omega'') \Subset \Omega'$ for any $\varepsilon > 0$ with $\mathrm{d}_\delta(\Omega'', \mathbb{R}^m \setminus \Omega') \geq 2\mathcal{J} + \varepsilon$. Then the local uniform higher integrability of $\{w_j\}$ and the fact that u_ρ solves ($\mathcal{BI}$) directly follow from Theorems 5.2 and 5.13. □

Proof of Theorem 1.19. The proof follow verbatim that of Theorem 1.13, with the help of the L^∞ estimates in Proposition 6.2, and is left to the reader. □

Acknowledgements

The authors wish to thank D. Bonheure, A. Iacopetti and S. Terracini for sharing their insight on the problem. In particular, we thank the first two for detailed discussions about the results in [7, 8, 10, 11]. This work was initiated when J.B. and N.I. visited Scuola Normale Superiore in 2017. They would like to express their gratitude for their hospitality and support during their visit to Scuola Normale Superiore. J.B. was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT)(No. NRF-2019R1A5A1028324) N.I. was supported by JSPS KAKENHI Grant Numbers JP 16K17623, 17H02851, 19H01797 and 19K03590. A.M. has been supported by the project *Geometric problems with loss of compactness* from the Scuola Normale Superiore and by GNAMPA as part of INdAM.

APPENDIX A. WEAK SOLUTIONS WITH HIGHER DIMENSIONAL SET OF LIGHT SEGMENTS

We construct a weak solution to ($\mathcal{BI}$) having a higher dimensional set of light segments. The construction is similar to that of the function in Proposition 4.3, but computations are more involved.

Let $4 \le m$, $2 \le \ell \le m-2$ and write

$$x = (y, z, x_m) \in \mathbb{R}^{m-\ell} \times \mathbb{R}^{\ell-1} \times \mathbb{R} = \mathbb{R}^m.$$

Recall $\zeta_\varepsilon(t)$ and $A_\varepsilon(t)$ in (4.14) and (4.16). Define $U_\varepsilon(y, z, x_m)$ by

$$U_\varepsilon(y, z, x_m) \doteq \zeta_\varepsilon(|y|)\left(1 - \varepsilon^{2\kappa}|y|^{2\kappa}\right)\vartheta_\varepsilon(|z|)\zeta_\varepsilon(x_m)A_\varepsilon(x_m),$$

where $\vartheta_\varepsilon(t)$ is defined by $\vartheta_\varepsilon(t) \doteq \vartheta_1(\varepsilon t)$ and $\vartheta_1(t)$ satisfies

$$\begin{aligned}&\vartheta_1(t) \in C_c^\infty(\mathbb{R}), \quad \vartheta_1'(t) \le 0 \quad \text{for } t \ge 0, \quad \operatorname{supp}\vartheta_1 \subset [-2, 2],\\ &\vartheta_1(t) \equiv 1 \text{ for } 0 \le t \le 1, \quad \vartheta_1(t) = 1 - \frac{e^2}{2}\exp\left(-\frac{1}{t-1}\right) \text{ for } 1 < t \le \frac{3}{2}.\end{aligned} \tag{A.1}$$

Remark that

$$U_\varepsilon(y, z, x_m) = u_\varepsilon(y, x_m)\vartheta_\varepsilon(|z|), \quad U_\varepsilon(0, z, x_m) = x_m \quad \text{if } |z| \le \frac{1}{\varepsilon} \text{ and } |x_m| \le \varepsilon.$$

In particular, the set of light segments of U_ε has dimension ℓ.

Write

$$\begin{aligned}W_\varepsilon(y, z, x_m) &\doteq \left(1 - \left|DU_\varepsilon(y, z, x_m)\right|^2\right)^{-\frac{1}{2}},\\ \rho_{U_\varepsilon}(y, z, x_m) &\doteq -W_\varepsilon \Delta U_\varepsilon - W_\varepsilon^3 D^2 U_\varepsilon\left(DU_\varepsilon, DU_\varepsilon\right),\\ \mathrm{II}_{U_\varepsilon} &\doteq \text{the second fundamental form corresponding to the graph } U_\varepsilon.\end{aligned}$$

Then we shall prove the following result.

Proposition A.1. *Assume $4 \le m$, $2 \le \ell \le m-2$ and $\kappa \in [1, m-\ell)$. Then*

$$W_\varepsilon \in L^q_{\rm loc}(\mathbb{R}^m) \quad \text{and} \quad \rho_{U_\varepsilon}, \left\|\mathrm{II}_{U_\varepsilon}\right\| \in L^q(\mathbb{R}^m) \quad \text{for all } q < \frac{m-\ell}{\kappa}, \tag{A.2}$$

and U_ε satisfies

$$\int_{\mathbb{R}^m} \frac{DU_\varepsilon \cdot D\eta}{\sqrt{1 - |DU_\varepsilon|^2}}\,\mathrm{d}x = \int_{\mathbb{R}^m} \rho_{U_\varepsilon}\eta\,\mathrm{d}x \quad \text{for each } \eta \in C_c^\infty(\mathbb{R}^m).$$

Proof. For (A.2), since ρ_{U_ε} is a mean curvature of the graph U_ε and $|\rho_{U_\varepsilon}| \le C\|\,\mathrm{II}_{U_\varepsilon}\|$, it is enough to treat $\|\,\mathrm{II}_{U_\varepsilon}\|$. By $U_\varepsilon(y, z, x_m) = u_\varepsilon(y, x_m)\vartheta_\varepsilon(|z|) = u_\varepsilon(r, x_m)\vartheta_\varepsilon(s)$, we have

$$\left|DU_\varepsilon\right|^2 = \left[(u_\varepsilon)_r^2 + (u_\varepsilon)_m^2\right]\vartheta_\varepsilon^2 + u_\varepsilon^2(\vartheta_\varepsilon')^2 = \left|Du_\varepsilon\right|^2\vartheta_\varepsilon^2 + u_\varepsilon^2(\vartheta_\varepsilon')^2. \tag{A.3}$$

From (4.18) and (A.1), notice that

$$\left|u_\varepsilon(r,x_m)\right| \le 2\varepsilon, \quad 0 \le \vartheta_\varepsilon(s) \le 1, \quad \left|\vartheta_\varepsilon'(s)\right| \le C\varepsilon.$$

Thus, for sufficiently small ε, thanks to (4.20), (4.21) and (A.3), we infer that

$$\left|W_\varepsilon(y,z,x_m)\right| \le C \quad \text{for each } (y,z,x_m) \in \Omega_1, \tag{A.4}$$

where

$$\Omega_1 \doteq \left\{ (y,z,x_m) \in \mathbb{R}^m \,:\, \text{either } |x_m| \ge \frac{3\varepsilon}{2} \text{ or else } \frac{1}{2\varepsilon} \le r = |y| \text{ and } |x_m| \le \frac{3\varepsilon}{2} \right\}.$$

Hence, it is easy to see that

$$\left\| \mathrm{II}_{U_\varepsilon}(y,z,x_m) \right\| \le C \quad \text{for all } (y,z,x_m) \in \Omega_1.$$

Next, we shall check the integrability of $\mathrm{II}_{U_\varepsilon}$ on

$$\Omega_2 \doteq \left\{ (y,z,x_m) \in \mathbb{R}^m \,:\, r = |y| \le \frac{1}{2\varepsilon},\ s = |z| \le \frac{1}{\varepsilon},\ |x_m| \le \varepsilon \right\},$$

$$\Omega_3 \doteq \left\{ (y,z,x_m) \in \mathbb{R}^m \,:\, r = |y| \le \frac{1}{2\varepsilon},\ s = |z| \le \frac{1}{\varepsilon},\ \varepsilon \le |x_m| \le \frac{3\varepsilon}{2} \right\}.$$

By (A.1), we have $U_\varepsilon(y,z,x_m) = u_\varepsilon(y,x_m)$ on $\Omega_2 \cup \Omega_3$, and we may use the computations in the proof of Proposition 4.3. In particular, by (4.22),

$$W_\varepsilon(y,z,x_m) \le C|y|^{-\kappa}, \quad \left\| \mathrm{II}_{U_\varepsilon}(y,z,x_m) \right\| \le C|y|^{-1} \quad \text{for each } (y,z,x_m) \in \Omega_2, \tag{A.5}$$

hence, from $\kappa \ge 1$, it follows that

$$W_\varepsilon,\ \left\| \mathrm{II}_{U_\varepsilon} \right\| \in L^q(\Omega_2) \quad \text{for all } q < \frac{m-\ell}{\kappa}. \tag{A.6}$$

For Ω_3, by (4.24),

$$W_\varepsilon(y,z,x_m) \le C\left[1 - a_\varepsilon(x_m) + |y|^{2\kappa}\right]^{-1/2} \quad \text{for any } (y,z,x_m) \in \Omega_3 \tag{A.7}$$

and as in the proof of Proposition 4.3, we may verify that

$$W_\varepsilon,\ \left\| \mathrm{II}_{U_\varepsilon} \right\| \in L^q(\Omega_3) \quad \text{for each } q < \frac{m-\ell}{\kappa}. \tag{A.8}$$

Finally, we shall check the integrability of $\| \mathrm{II}_{U_\varepsilon} \|$ on

$$\Omega_4 \doteq \left\{ (y,z,x_m) \in \mathbb{R}^m \,:\, r = |y| \le \frac{1}{2\varepsilon},\ \frac{1}{\varepsilon} < s = |z| \le \frac{3}{2\varepsilon},\ |x_m| \le \frac{3\varepsilon}{2} \right\},$$

$$\Omega_5 \doteq \left\{ (y,z,x_m) \in \mathbb{R}^m \,:\, r = |y| \le \frac{1}{2\varepsilon},\ \frac{3}{2\varepsilon} \le s = |z| \le \frac{2}{\varepsilon},\ |x_m| \le \frac{3\varepsilon}{2} \right\}.$$

We first prove $|DU_\varepsilon| < 1$ on $\Omega_4 \cup \Omega_5$. Since $U_\varepsilon(r,s,x_m) = (1 - \varepsilon^{2\kappa} r^{2\kappa}) \vartheta_\varepsilon(s) A_\varepsilon(x_m)$ on $\Omega_4 \cup \Omega_5$,

$$\begin{aligned}
&(U_\varepsilon)_r = -2\kappa \varepsilon^{2\kappa} r^{2\kappa-1} \vartheta_\varepsilon A_\varepsilon, \quad (U_\varepsilon)_s = \left(1 - \varepsilon^{2\kappa} r^{2\kappa}\right) \vartheta_\varepsilon' A_\varepsilon, \quad (U_\varepsilon)_m = \left(1 - \varepsilon^{2\kappa} r^{2\kappa}\right) \vartheta_\varepsilon a_\varepsilon, \\
&(U_\varepsilon)_{rr} = -2\kappa(2\kappa-1) \varepsilon^{2\kappa} r^{2\kappa-2} \vartheta_\varepsilon A_\varepsilon, \quad (U_\varepsilon)_{rs} = -2\kappa \varepsilon^{2\kappa} r^{2\kappa-1} \vartheta_\varepsilon' A_\varepsilon, \\
&(U_\varepsilon)_{rm} = -2\kappa \varepsilon^{2\kappa} r^{2\kappa-1} \vartheta_\varepsilon a_\varepsilon, \quad (U_\varepsilon)_{ss} = \left(1 - \varepsilon^{2\kappa} r^{2\kappa}\right) \vartheta_\varepsilon'' A_\varepsilon, \\
&(U_\varepsilon)_{sm} = \left(1 - \varepsilon^{2\kappa} r^{2\kappa}\right) \vartheta_\varepsilon' a_\varepsilon, \quad (U_\varepsilon)_{mm} = \left(1 - \varepsilon^{2\kappa} r^{2\kappa}\right) \vartheta_\varepsilon a_\varepsilon'.
\end{aligned} \tag{A.9}$$

Thus,

$$\begin{aligned}
&1-\left|DU_\varepsilon(y,z,x_m)\right|^2\\
&=1-4\kappa^2\varepsilon^{4\kappa}r^{4\kappa-2}\vartheta_\varepsilon^2A_\varepsilon^2-\left(1-2\varepsilon^{2\kappa}r^{2\kappa}+\varepsilon^{4\kappa}r^{4\kappa}\right)\left[(\vartheta_\varepsilon')^2A_\varepsilon^2+\vartheta_\varepsilon^2a_\varepsilon^2\right]\\
&=1-(\vartheta_\varepsilon')^2A_\varepsilon^2-\vartheta_\varepsilon^2a_\varepsilon^2+\varepsilon^{2\kappa}r^{2\kappa}\left[\left(2-\varepsilon^{2\kappa}r^{2\kappa}\right)\left\{(\vartheta_\varepsilon')^2A_\varepsilon^2+\vartheta_\varepsilon^2a_\varepsilon^2\right\}-4\kappa^2\varepsilon^{2\kappa}r^{2\kappa-2}\vartheta_\varepsilon^2A_\varepsilon^2\right].
\end{aligned}$$

By

$$\left|A_\varepsilon(x_m)\right|\le 2\varepsilon,\quad \frac{1}{2}\le a_\varepsilon(x_m)\le 1,\quad \varepsilon r=|\varepsilon y|\le\frac{1}{2}\quad\text{for each }(y,z,x_m)\in\Omega_4\cup\Omega_5,$$

if $\varepsilon>0$ is sufficiently small, then

$$\left(2-\varepsilon^{2\kappa}r^{2\kappa}\right)\vartheta_\varepsilon^2a_\varepsilon^2-4\kappa^2\varepsilon^{2\kappa}r^{2\kappa-2}\vartheta_\varepsilon^2A_\varepsilon^2\ge\frac{1}{8}\vartheta_\varepsilon^2.$$

Therefore, for every $(y,z,x_m)\in\Omega_4\cup\Omega_5$,

(A.10)

$$1-\left|DU_\varepsilon(y,z,x_m)\right|^2\ge 1-(\vartheta_\varepsilon'(|z|))^2A_\varepsilon^2(x_m)-\vartheta_\varepsilon^2(|z|)a_\varepsilon^2(x_m)+\frac{1}{8}\varepsilon^{2\kappa}|y|^{2\kappa}\vartheta_\varepsilon^2(|z|).$$

When $(y,z,x_m)\in\Omega_5$, by $3/2\le\varepsilon|z|\le 2$ and (A.1), we see that

$$\left(\vartheta_\varepsilon'(|z|)\right)^2\le C\varepsilon^2,\quad \vartheta_\varepsilon^2(|z|)\le\vartheta_\varepsilon^2\left(\frac{3}{2\varepsilon}\right)=\frac{1}{4},$$

which implies that if ε is sufficiently small, then for all $(y,z,x_m)\in\Omega_5$,

$$1-\left|DU_\varepsilon(y,z,x_m)\right|^2\ge 1-C\varepsilon^4-\frac{1}{4}\ge\frac{1}{2}>0.$$

Hence, $|DU_\varepsilon|<1$ on Ω_5 and

$$W_\varepsilon,\ \left\|\mathrm{II}_{U_\varepsilon}\right\|\in L^\infty(\Omega_5). \tag{A.11}$$

On the other hand, when $(y,z,x_m)\in\Omega_4$, we have $\vartheta_\varepsilon(|z|)\ge 1/2$, and (A.10) yields

$$1-\left|DU_\varepsilon(y,z,x_m)\right|^2\ge 1-4\varepsilon^2(\vartheta_\varepsilon'(|z|))^2-\vartheta_\varepsilon^2(|z|)a_\varepsilon^2(x_m)+\frac{\varepsilon^{2\kappa}|y|^{2\kappa}}{32}.$$

Thus, to show $|DU_\varepsilon|<1$, it suffices to prove

$$4\varepsilon^2\left(\vartheta_\varepsilon'(s)\right)^2+\vartheta_\varepsilon^2(s)=4\varepsilon^4\left(\vartheta_1'(\varepsilon s)\right)^2+\vartheta_1^2(\varepsilon s)<1\quad\text{for each }\frac{1}{\varepsilon}<s\le\frac{3}{2\varepsilon}. \tag{A.12}$$

To this end, from (A.1) and

$$\vartheta_1'(t)=-\frac{e^2}{2}(t-1)^{-2}\exp\left(-(t-1)^{-1}\right),$$

it follows that for $1<t\le\frac{3}{2}$

$$\begin{aligned}
&4\varepsilon^4\left(\vartheta_1'(t)\right)^2+\vartheta_1^2(t)\\
&=\varepsilon^4e^4(t-1)^{-4}\exp\left(-2\,(t-1)^{-1}\right)+\left[1-\frac{e^2}{2}\exp\left(-(t-1)^{-1}\right)\right]^2\\
&=1-e^2\left[1-\frac{e^2}{4}\exp\left(-(t-1)^{-1}\right)-\varepsilon^4e^2(t-1)^{-4}\exp\left(-(t-1)^{-1}\right)\right]\exp\left(-(t-1)^{-1}\right).
\end{aligned}$$

Since

$$1-\frac{e^2}{4}\exp\left(-(t-1)^{-1}\right)\geq 1-\frac{e^2}{4}e^{-2}=\frac{3}{4}\quad\text{for every } 1<t\leq\frac{3}{2},$$

for sufficiently small $\varepsilon>0$,

$$4\varepsilon^4\left(\vartheta_1'(t)\right)^2+\vartheta_1^2(t)\leq 1-\frac{e^2}{2}\exp\left(-(t-1)^{-1}\right)<1. \tag{A.13}$$

Hence, $|DU_\varepsilon|<1$ on Ω_4. In addition, by $1-4\varepsilon^2(\vartheta_\varepsilon'(|z|))^2-\vartheta_\varepsilon^2(|z|)a_\varepsilon^2(x_m)\geq 0$, we have

$$W_\varepsilon(y,z,x_m)\leq C\left[1-4\varepsilon^2\left(\vartheta_\varepsilon'(|z|)\right)^2-\vartheta_\varepsilon^2(|z|)a_\varepsilon^2(x_m)+|y|^{2\kappa}\right]^{-1/2}\qquad\forall\,(y,z,x_m)\in\Omega_4. \tag{A.14}$$

Thus, $W_\varepsilon\in L^q(\Omega_4)$ follows and $W_\varepsilon\in L^q_{\mathrm{loc}}(\mathbb{R}^m)$ holds in view of (A.4), (A.6), (A.8) and (A.11).

To show $\|\,\mathrm{II}_{U_\varepsilon}\|\in L^q(\Omega_4)$, by $\kappa\geq 1$, (4.5), (4.7), (4.8) and (A.9), for $(y,z,x_m)\in\Omega_4$,

$$\begin{aligned}\left|D^2u\right|&\leq C\left\{|y|^{2\kappa-2}+\left|\vartheta_\varepsilon''(|z|)\right|+\left|\vartheta_\varepsilon'(|z|)\right|+\left|a_\varepsilon'(x_m)\right|\right\},\\ \left|D^2u(Du,\cdot)\right|&\leq C\left\{|y|^{2\kappa-1}+\left|\vartheta_\varepsilon'(|z|)\right|+\left|a_\varepsilon'(x_m)\right|\right\},\\ \left|D^2u(Du,Du)\right|&\leq C\left\{|y|^{4\kappa-2}+\left(\vartheta_\varepsilon'(|z|)\right)^2+\left|a_\varepsilon'(x_m)\right|\right\}.\end{aligned} \tag{A.15}$$

Since $W_\varepsilon(y,z,x_m)\leq Cr^{-\kappa}$ holds due to (A.12) and (A.14), we verify that

$$W_\varepsilon(y,z,x_m)|y|^{2\kappa-2}+W_\varepsilon^2(y,z,x_m)|y|^{2\kappa-1}+W_\varepsilon^3(y,z,x_m)|y|^{4\kappa-2}\leq C|y|^{-1}\in L^q(\Omega_4) \tag{A.16}$$

for all $q<(m-\ell)/\kappa$.

On the other hand, by (A.12) and (A.13), we notice that

$$1-4\varepsilon^2\left(\vartheta_\varepsilon'(|z|)\right)^2-\vartheta_\varepsilon^2(|z|)\geq\frac{e^2}{2}\exp\left(-(\varepsilon|z|-1)^{-1}\right),$$

which yields

$$W_\varepsilon(y,z,x_m)\leq C\exp\left(\frac{1}{2}(\varepsilon|z|-1)^{-1}\right)\quad\text{for all } (y,z,x_m)\in\Omega_4.$$

From (A.1),

$$\left|\vartheta_\varepsilon''(|z|)\right|+\left|\vartheta_\varepsilon'(|z|)\right|\leq C\left(|\varepsilon z|-1\right)^{-4}\exp\left(-(\varepsilon|z|-1)^{-1}\right).$$

Hence,

$$\begin{aligned}&W_\varepsilon(y,z,x_m)\left\{\left|\vartheta_\varepsilon''(|z|)\right|+\left|\vartheta_\varepsilon'(|z|)\right|\right\}+W_\varepsilon^3(y,z,x_m)\left(\vartheta_\varepsilon'(|z|)\right)^2\\ &\leq C(\varepsilon|z|-1)^{-4}\exp\left(-\frac{1}{2}(\varepsilon|z|-1)^{-1}\right)\in L^\infty(\Omega_4).\end{aligned} \tag{A.17}$$

Moreover,
(A.18)
$$\begin{aligned}
&W_\varepsilon^2(y,z,x_m)\left|\vartheta_\varepsilon'(|z|)\right| \\
&= W_\varepsilon^{2-\kappa^{-1}}(y,z,x_m)W_\varepsilon^{\kappa^{-1}}(y,z,x_m)\left|\vartheta_\varepsilon'(|z|)\right| \\
&\le C\exp\left(\frac{2-\kappa^{-1}}{2}(\varepsilon|z|-1)^{-1}\right)(C|y|^{-\kappa})^{\kappa^{-1}}(\varepsilon|z|-1)^{-2}\exp\left(-(\varepsilon|z|-1)^{-1}\right) \\
&= C(\varepsilon|z|-1)^{-2}\exp\left(-\frac{1}{2\kappa}(\varepsilon|z|-1)^{-1}\right)|y|^{-1} \in L^q(\Omega_4) \quad \text{if } q<\frac{m-\ell}{\kappa}.
\end{aligned}$$

By (A.15), (A.16), (A.17), (A.18) and $W_\varepsilon \ge 1$, to show $\| \mathrm{II}_{U_\varepsilon} \| \in L^q(\Omega_4)$ for $q<(m-\ell)/\kappa$, it is enough to prove

$$W_\varepsilon^3(x,y,z_m)\left|a_\varepsilon'(x_m)\right| \in L^q(\Omega_4) \quad \text{for each } q<\frac{m-\ell}{\kappa}. \tag{A.19}$$

To prove (A.19), since $a_\varepsilon'(x_m)=0$ for $|x_m|\le \frac{\varepsilon}{2}$ and a_ε is even, we may suppose $\frac{\varepsilon}{2}<x_m\le\frac{3\varepsilon}{2}$. In this case, from (4.15) and (A.1), notice that

$$\begin{aligned}
&1-4\varepsilon^2\left(\vartheta_\varepsilon'(|z|)\right)^2-\vartheta_\varepsilon^2(|z|)a_\varepsilon^2(x_m) \\
&= \left[1+\vartheta_\varepsilon(|z|)a_\varepsilon(x_m)\right]\left[1-\vartheta_\varepsilon(|z|)a_\varepsilon(x_m)\right]-4\varepsilon^4\left(\vartheta_1'(\varepsilon|z|)\right)^2 \\
&\ge 1-\vartheta_\varepsilon(|z|)a_\varepsilon(x_m)-4\varepsilon^4\left(\vartheta_1'(\varepsilon|z|)\right)^2 \\
&\ge 1-\left[1-\frac{e^2}{2}\exp\left(-(\varepsilon|z|-1)^{-1}\right)\right]\left[1-d_\varepsilon\exp\left(-\left(x_m-\varepsilon\right)^{-1}\right)\right]-4\varepsilon^4\left(\vartheta_1'(\varepsilon|z|)\right)^2 \\
&\ge c_0\left\{\exp\left(-(\varepsilon|z|-1)^{-1}\right)+\exp\left(-\left(x_m-\varepsilon\right)^{-1}\right)\right\} \doteq c_0R^2(|z|,x_m).
\end{aligned}$$

Thus, by (A.14),

$$W_\varepsilon(y,z,x_m)\le C\left\{R^2(|z|,x_m)+|y|^{2\kappa}\right\}^{-\frac12}.$$

Then we proceed as in (4.28) and for $\frac{m-\ell}{3\kappa} < q < \frac{m-\ell}{\kappa}$, we obtain

$$\begin{aligned}
&\int_{\varepsilon}^{\frac{3\varepsilon}{2}} \mathrm{d}x_m \int_{\frac{1}{\varepsilon}<|z|<\frac{3}{2\varepsilon}} \mathrm{d}z \int_{|y|\le\frac{1}{2\varepsilon}} \left(W_\varepsilon^3(y,z,x_m)\left|a_\varepsilon'(x_m)\right|\right)^q \mathrm{d}y \\
&\le C\int_{\varepsilon}^{\frac{3\varepsilon}{2}} \mathrm{d}x_m \int_{\frac{1}{\varepsilon}<|z|<\frac{3}{2\varepsilon}} \mathrm{d}z \int_{|y|\le R^{1/\kappa}(|z|,x_m)} R^{-3q}(|z|,x_m)\left|a_\varepsilon'(x_m)\right|^q \mathrm{d}y \\
&\quad + C\int_{\varepsilon}^{\frac{3\varepsilon}{2}} \mathrm{d}x_m \int_{\frac{1}{\varepsilon}<|z|<\frac{3}{2\varepsilon}} \mathrm{d}z \int_{R^{1/\kappa}(|z|,x_m)\le|y|\le\frac{1}{2\varepsilon}} |y|^{-3\kappa q}\left|a_\varepsilon'(x_m)\right|^q \mathrm{d}y \\
&\le C\int_{\varepsilon}^{\frac{3\varepsilon}{2}} \mathrm{d}x_m \int_{\frac{1}{\varepsilon}<|z|<\frac{3}{2\varepsilon}} R^{-3q+\frac{m-\ell}{\kappa}}(|z|,x_m)\left|a_\varepsilon'(x_m)\right|^q \mathrm{d}z \\
&\le C\int_0^{\frac{\varepsilon}{2}} \mathrm{d}t \int_1^{\frac{3}{2}} \left\{\exp\left(-\frac{1}{s-1}\right)+\exp\left(-\frac{1}{t}\right)\right\}^{\frac{m-\ell-3\kappa q}{2\kappa}} t^{-2q}\exp\left(-\frac{q}{t}\right)\mathrm{d}s \\
&= C\int_0^{\frac{\varepsilon}{2}} \mathrm{d}t \int_0^{\frac{1}{2}} \left\{\exp\left(-\frac{1}{s}\right)+\exp\left(-\frac{1}{t}\right)\right\}^{\frac{m-\ell-3\kappa q}{2\kappa}} t^{-2q}\exp\left(-\frac{q}{t}\right)\mathrm{d}s \\
&\le C\int_0^{\frac{\varepsilon}{2}} \mathrm{d}t \int_0^{t} \exp\left(\frac{3\kappa q-m+\ell}{2\kappa t}\right) t^{-2q}\exp\left(-\frac{q}{t}\right)\mathrm{d}s \\
&\quad + C\int_0^{\frac{\varepsilon}{2}} \mathrm{d}t \int_t^{\frac{1}{2}} \exp\left(\frac{3\kappa q-m+\ell}{2\kappa s}\right) t^{-2q}\exp\left(-\frac{q}{t}\right)\mathrm{d}s \\
&\le C\int_0^{\frac{\varepsilon}{2}} t^{-2q}\exp\left(\frac{\kappa q-m+\ell}{2\kappa t}\right)\mathrm{d}t < \infty.
\end{aligned}$$

Hence, (A.19) holds and (A.2) follows.

For the assertion that U_ε is a weak solution, we notice that by (A.4), (A.5), (A.7), (A.11) and (A.14),

$$W_\varepsilon(y,z,x_m) \le C_R|y|^{-\kappa} \quad \text{for each } (y,z,x_m)\in B_R(0).$$

Hence, $W_\varepsilon \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. By arguing as in the proof of Proposition 4.1, we may verify that U_ε is a weak solution and complete the proof. □

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