SCUOLA
NORMALE
SUPERIORE

Invariants and Parameter Space Models for Rational Maps

Doctoral Thesis
by

Anastasia Shepelevtseva

Doctoral Program in Mathematics

Supervisor
Stefano Marmi, SNS, Pisa
Supervisor
Vladlen Timorin, HSE, Moscow

Invariants and Parameter Space Models for Rational Maps

by

Anastasia Shepelevtseva

Monday 24th January, 2022 18:42

Submitted to the Scuola Normale Superiore di Pisa
in January 2022, in partial fulfillment of the requirements for the
Doctoral Program in Mathematics

Abstract

This thesis deals with classification of special classes of 1-variable holomorphic rational functions.

In the first part we focus on the class of rational maps with bounded orbits of post-critical points – so-called Thurston maps. These maps can be viewed as a class of topological objects – branching coverings. There exists a natural equivalence relation on the class of Thurston maps, such that different rational functions are almost never equivalent. We provide an algorithm which allows to represent these algebraic (or topological) objects by means of completely combinatorial objects – invariant graphs with marked vertices. We also introduce a computational procedure for finding such graphs.

Then we look at the cubic polynomials with connected Julia sets with a fixed multiplier, thus we get a slice of such polynomials. Such slices can be considered as parameter spaces. We introduce a parametrization of these cubic slices via reglued Julia set of the quadratic polynomial with the same multiplier. We also show that the parametrizing map is continuous.

Acknowledgments

This work would not have been possible without the support and advice on the part of my supervisors Stefano Marmi and Vladlen Timorin, to whom I am very thankful.

I thank ICERM Brown University, Universitat de Barcelona, CIRM Luminy, ICTP Trieste, Jacobs University Bremen, Universita di Pisa and Centro di Giorgi for the opportunities to present my results during my research journey and to have a lot of useful discussions.

I would also like to thank my colleague Francesco Grotto and all my fantastic fellow PhD candidates, who made my stay in Pisa so much more joyful. Enormous thanks go to Gevorg Yeghikyan, for being my greatest inspiration and encouragement. Finally, I thank my family for always supporting me.

Contents

Chapter 1

Introduction

The principal objects of this thesis are holomorphic rational functions of one variable. We classify these functions in some special cases using mainly a topological approach.

The main results are related to two different classes of 1-dimensional holomorphic rational functions: first we deal with *post-critically finite* rational functions and then with cubic polynomials with a fixed multiplier. For the first class we provide an algorithm for finding a purely combinatorial invariant and for the second one we parametrize families of polynomials via some special set on the dynamical plane.

1.1 Invariant trees for Thurston map

The first part is dedicated to the class of rational maps with finite orbits of critical values (images of critical points) – so-called *Thurston maps.* Thurston maps were first introduced by W. Thurston in context of rational maps. His main result, *Thurston's characterization theorem* (see [DH93]), allows to study these algebraic objects by topological tools. More precisely, rational maps can be viewed as a class of topological objects – branched coverings. On the class of Thurston maps there exists a natural equivalence relation, called Thurston equivalence, such that different rational functions are almost never equivalent. Roughly speaking, Thurston theorem enables to understand if a Thurston map is equivalent to a rational function depending on the existence of a purely topological object: *combinatorial obstruction*, which is some special union of simple curves outside the set of *post-critical points.* The *post-*

critical set is defined as the smallest closed f-stable set including the set of critical values. We denote it as $P(f)$ and the set of critical points as $C(T)$. A Thurston map turns out to be equivalent to a rational function if and only if there is no obstruction. Showing "non-existence" of obstructions is a very non-trivial problem, since it deals with checking infinitely many possibilities for sets of curves. Thus, classification of Thurston maps up to equivalence remains an important problem. It has been focus of recent developments, for example [BN06, BD17, CG+15, KL18, Hlu17]. We will be interested in degree 2 Thurston maps.

In our work we introduce combinatorial objects called *invariant spanning trees*, which enable us to "restrict" the dynamics of some Thurston map f to the dynamics only on this combinatorial object. Then we define the invariant spanning tree T for a Thurston map f as a spanning tree, vertices of which map again to vertices of T and $f(T) \subset T$. This concept somehow can be seen as a generalization of one of the first combinatorial invariant – *Hubbard trees* (see [DH85a, BFH92, Poi93]). Moreover, if we know a Hubbard tree of some Thurston map f, we can obtain an invariant spanning tree by connecting it to infinity. Other examples are, for example, *formal matings* (joining two Hubbard trees) or invariant trees obtained from classical *captures* in the in the sense of [Wit88, Ree92]. We use terminology of [MA41], defining ribbon trees as isomorphism classes of embedded trees in $\mathbb{S}^2$. It turns out, that invariant spanning trees completely define the Thurston equivalence class of a Thurston map. More precisely, this result is stated as the following theorem:

Theorem 3.1.5. *Suppose that f, $g : \mathbb{S}^2 \to \mathbb{S}^2$ are two Thurston maps of degree 2. Let T_f and T_g be invariant spanning trees for f and g, respectively. Suppose that there is a cellular homeomorphism $\tau : T_f \to T_g$ with the following properties:*

1. *The map τ is an isomorphism of ribbon graphs.*
2. *We have $\tau \circ f = g \circ \tau$ on $V(T_f) \cup C(T_f)$.*
3. *The critical values of f are mapped to critical values of g by τ.*

Suppose also that τ can be extended to edges of $f^{-1}(T_f)$ incident to points in $C(T_f)$ so that to preserve the cyclic order of edges incident to a given vertex of $C(T_f)$ and so that to satisfy (2). Then f and g are Thurston equivalent.

There exist also algebraic invariants of Thurston maps called *bisets*. They were introduced in [Nek05] (bisets are called *bimodules* there). We consider a set of all homotopy classes of paths from some fixed base point $y \in \mathbb{S}^2 - P(f)$ to its preimages $f^{-1}(y)$ in $\mathbb{S}^2 - P(f)$. We denote it as $\mathcal{X}_f(y)$. We also consider the fundamental group $\pi_f = \pi_1(\mathbb{S}^2 - P(f), y)$. Then $\mathcal{X}_f(y)$ is a *biset over fundamental group*. Roughly speaking, being a biset means that both left and right actions of $\mathcal{X}_f(y)$ on π_f are given. We show, that knowing an invariant spanning tree we can fully describe the biset:

Theorem 3.2.1. *Suppose that f is a Thurston map of degree 2, and T is an invariant spanning tree for f. There is an explicit presentation of the biset of f based only on the data* $(1) - (2)$ *listed above.*

1. *the ribbon graph structure on T,*

2. *the restriction of f to $V(T) \cup C(T)$.*

Other different approaches to the problem of combinatorial encoding of Thurston maps by means of invariant graphs were presented in [CFP01, BM17, Hlu17, LMS15] for specific families of rational maps.

But finding the invariant spanning tree is not always a trivial problem. We show that the statement of the previous Theorem can be generalized. For any (even non-invariant) spanning tree we can find another spanning tree T^* that maps onto T. Then we define a *dynamical tree pair* as a pair of trees T and T^* such that

1. $f(T^*) \subset T$;

2. the vertices of T^* are mapped to vertices of T under f;

3. all critical values of f are vertices of T.

Having some spanning T for $P(f)$ we can introduce a generating set $\mathcal{E}_T$ of the fundamental group $\pi_1(\mathbb{S}^2 - P(f), y)$ consisting of the identity element and the homotopy classes of smooth loops based at y intersecting T only once and transversely. Then we show, that the following data is sufficient for the explicit representation of a biset:

1. the ribbon graph structures on T^*, T;
2. the map $f : V(T^*) \cup C(T^*) \to V(T)$;
3. how elements of $\mathcal{E}_{T^*}$ are expressed through elements of $\mathcal{E}_T$ (or how both $\mathcal{E}_{T^*}$, $\mathcal{E}_T$ are expressed through some other generating set of $\pi_1(\mathbb{S}^2 - P(f), y)$).

It is easy to see, that T^* should be obtained as some subset of the full preimage $f^{-1}(T)$. There are several choices of T^*. We define *ivy object* as a homotopy rel. $P(f)$ class of spanning trees for $P(f)$. Then we introduce the *pullback relation* $[T] \multimap [T^*]$ on the set $Ivy(f)$ of ivy objects. A similar relation on isotopy classes of simple closed curves in $\mathbb{S}^2 - P(f)$ was discussed in [Pil03, KPS16]. We say that there is a pullback relation between trees T and T^* if (T^*, T) is a dynamical tree pair. We can equip the set $Ivy(f)$ with the structure of an abstract directed graph: we connect two vertices corresponding to two ivy objects $[T_1]$ and $[T_2]$ by an oriented arrow from $[T_1]$ to $[T_2]$ if (T_2, T_1) is a dynamical tree pair. We show that all the data corresponding to this graph can be encoded combinatorially (thus it can be inserted into the Wolfram Mathematica program). Moving by each arrow of the graph is the transition from T to T^*. If we want to find an invariant spanning tree, it is natural to consider the iterative process of such transitions. We call this process *ivy iteration*. We define a *pullback invariant* subset $C \subset \mathrm{Ivy}(f)$ as the set of ivy objects such that $[T] \in C$ and $[T] \multimap [T^*]$ imply $[T^*] \in C$. Finding a pullback invariant subset corresponds to finding periodic ivy objects.

Finally, we introduce some examples of the ivy iteration, obtained as the results of the computer program.

1.2 Zakeri slices parametrization

The second class of mappings we are interested in is the class of cubic polynomials $\mathbb{C}_\lambda$ with fixed multiplier $|\lambda| \leqslant 1$ of a fixed point, which we assume to be 0. This space $\mathbb{C}_\lambda$ is called λ-*slice*. We use a following notation: for a cubic polynomial P write $[P]$ for its affine conjugacy class. For a cubic polynomial $P(z) = \lambda z + \ldots$, let $[P]_0$ be its class in $\mathbb{C}_\lambda$. If we suppose that the rotational number of fixed λ is of bounded type, then for the polynomials in this slice the origin is a fixed Siegel point. Such slices as parameter spaces were studied by S. Zakeri, so we call them *Zakeri slices*.

There is classial and powerful method of studying polynomials with fixed or periodic points based upon *linearizations*. A function $f(z)$ is called *linearizable* if there exists a holomorphic change of coordinates h (the linearization of f) such that $h^{-1} \circ f \circ h = \lambda z$, i.e. f is conjugate to λz. The region, where linearization exists is the *Siegel disc* or a *Herman ring*, or a part of *(super)attracting domain*.

The problem of the linearizability of actually any holomorphic germ depending on the multiplier was solved more than 70 years ago, but a lot of connected questions are much better studied only in the case of quadratic polynomial. For example, according to Yoccoz's result for quadratic polynomials the sum of the logarithm of the radius of convergence of a linearizing function and of the Brjuno sum of the rotation number can be extended to a bounded function (see [Yoc95]), but the same result does not hold for general higher degree polynomials.

We are interested in the space $\mathcal{C}_\lambda$ of the cubic polynomials in $\mathbb{C}_\lambda$ with connected Julia set. We will focus on the particular subset of such polynomials called the *principal hyperbolic component*. It is a set of hyperbolic polynomials whose Julia sets are Jordan curves. Denote the closure of this subset as $\mathcal{P}_\lambda$. Then on the parameter plane $\mathcal{P}_\lambda$ is the central part of $\mathcal{C}_\lambda$. Pieces of the boundary of the principal hyperbolic component for $|P'(0)| < 1$ were described via analytic parametrization in [PT09].

We provide a way to parametrize $\mathcal{P}_\lambda$ when the rotation number has bounded type. We introduce a parametrization $\Phi_\lambda : \mathcal{P}_\lambda \to \tilde{K}(Q)$ of these cubic slices via some model of the Julia set of the quadratic polynomial $Q(z) = Q_\lambda(z) = \lambda z(1-z/2)$ with the same multiplier. The model space $\tilde{K}(Q)$ is the set $K(Q)\backslash\Delta(Q)$ where $\Delta(Q)$ is the Siegel disc of Q and the factorization is made by the following relation: two different points z and w on the boundary of the Siegel disc are equivalent if $\mathrm{Re}\left(\bar{\psi}^{-1}(z)\right) = \mathrm{Re}\left(\bar{\psi}^{-1}(w)\right)$, where ψ is a conformal map $\psi : \mathbb{D} \to \Delta(Q)$.

The main result of Chapter 4 is the following theorem:

Theorem 4.2.3. *Suppose that $\theta \in \mathbb{R}/\mathbb{Z}$ is of bounded type, and $\lambda = e^{2\pi i\theta}$. Let*

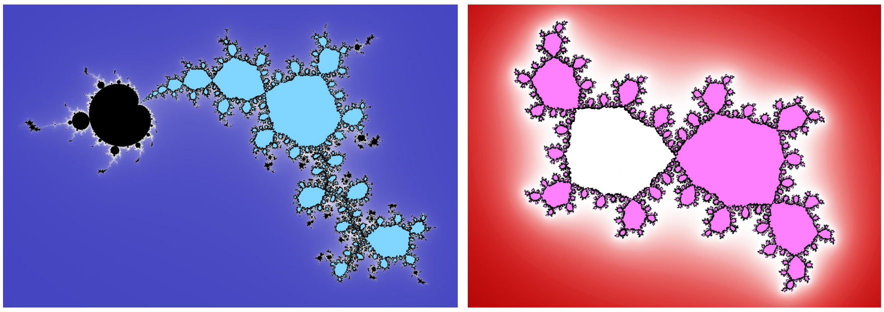


Figure 1-1: Left: the parameter plane $\mathbb{C}_\lambda$ with $\lambda = \exp(\pi i\sqrt{2})$. We used the parameterization, in which every linear conjugacy class from $\mathbb{C}_\lambda$ is represented by a polynomial of the form $f(z) = \lambda z + \sqrt{a}z^2 + z^3$, where a is the parameter (that is, the figure shows the a-plane). The conjugacy class of f is independent on the choice between the two values of the square root. Regions with light uniform shading are interior components of $\mathcal{P}_\lambda$. There are also various "decorations" of $\mathcal{P}_\lambda$ (that is, components of $\mathcal{C}_\lambda - \mathcal{P}_\lambda$) shown in black; these decorations contain copies of the Mandelbrot set. Right: the dynamical plane of $Q = Q_\lambda$. The bounded white region near the center is the Siegel disk $\Delta(Q)$. A conjectural model of $\mathcal{P}_\lambda$ is obtained from $K(Q)$ by removing this white region and gluing its boundary into a simple curve. Our main theorem provides a continuous map from $\mathcal{P}_\lambda$ to this conjectural model.

$Q = Q_\lambda$ be a quadratic polynomial with a fixed point of multiplier λ. Then there is a continuous map $\Phi_\lambda : \mathcal{P}_\lambda \to \tilde{K}(Q)$ taking $[P]_0$ to the η_P-image of some critical point of P.

The map Φ_λ is illustrated in Figure 1-1.

This theorem is a partial extension of [PT09]. The main idea in the construction of the map Φ_λ is parametrization of the polynomials $\mathcal{P}_\lambda$ by the critical point, which does not belong to the closure of the Siegel disc. Then it turns out that it is possible to set up the correspondence between points in the cubic and the quadratic filled Julia sets. The main tools which we are using are *bubbles* and *bubble rays*. We define bubbles as the pullbacks of the Siegel disc. These terms were first introduced for the superattracting instead of the Siegel domains in [Luo95] and the related ideas were further developed in [AY09, Yan17, BBCO10]. Using bubbles we introduce the subset $X(P)$ of the cubic filled Julia set. Informally, in almost every case it is a set of points which can be reached by the chain of bubbles. Thus, we show that Φ_λ satisfy the following property:

Property D. *For any $P \in \mathcal{P}_\lambda$, there exist a full P-invariant continuum $X(P)$ containing a critical point c of P and a continuous monotone map $\eta_P : X(P) \to$*

$K(Q)$ such that η_P semi-conjugates $f|_{X(P)}$ with $Q|_{\eta_P(X(P))}$, and $\Phi_\lambda(P)$ is the image of $\eta_P(c)$ in $\tilde{K}(Q)$.

Here the letter D in *Property D* stands for "Dynamics" (or "Douady").

Chapter 2

Background

In this Chapter we recall the basic notions, required for the further work.

2.1 Basic objects in holomorphic dynamics

We consider a rational mapping R from a sphere S^2 to itself. We denote its n-*th iteration* (meaning map R applied n times) as R^n where $R^1 = R$ and $R^n = R^{n-1} \circ R$, $n \in \mathbb{N}$. For each $R(z)$ we can divide the sphere S^2 into two disjoint invariant subsets, such that the dynamics on the first one (called the *Julia set*) has a chaotic behavior and the dynamics on the second one (called the *Fatou set*) has a regular behavior. Further we will explain what *regular* and *chaotic* mean.

First we need to recall the definition of *the spherical metric*. We will be considering the sphere as the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the the system of coordinates determined in two charts by following relation $z : \overline{\mathbb{C}}\backslash\{\infty\} \to \mathbb{C}, z(0) = 0$ and $w : \overline{\mathbb{C}}\backslash\{0\} \to \mathbb{C}, w(\infty) = 0$ with $zw = 1$. Then we define the spherical metric as follows:

$$ds_{\overline{\mathbb{C}}} = \begin{cases} \frac{2|dz|}{1+|z|^2} & \text{in the } z\text{-chart;} \\ \frac{2|dw|}{1+|w|^2} & \text{in the } w\text{-chart} \end{cases}$$

Now we can define *a normal family* of functions on some open subset:

Definition 2.1.1. *For a family $\mathcal{F}_U$ of all meromorphic functions on some open subset $U \subset \overline{\mathbb{C}}$ a family $\mathcal{F} \subset \mathcal{F}_U$ is* normal *if it is relatively compact in $\mathcal{F}_U$ with resect to the spherical metric.*

This definition enables us to introduce Julia and Fatou sets:

Definition 2.1.2. *For a rational mapping R acting on a sphere $\overline{\mathbb{C}}$ its* Fatou set *$F(R)$ is the set of points $z_0 \in \overline{\mathbb{C}}$ such that $\{R^n\}$ forms a normal family in some neighborhood of z_0. The complement of the Fatou set is the* Julia set *$J(R)$.*

If we consider not the rational functions, but the smaller class of polynomials, then we can also define one other important object, connected to Julia and Fatou sets. Let $P(z)$ be a polynomial. Then its filled Julia set *$K(P)$ is defined as set of points $z_1 \in \overline{\mathbb{C}}$ such that their orbits are bounded. More precisely,*

$$K(P) \stackrel{def}{=} \left\{ z \in \mathbb{C} : P^k(z) \nrightarrow \infty \text{ as } k \to \infty \right\}$$

2.1.1 Parameter plane

Any quadratic polynomial can be conjugate to the following form: $f(z) = z^2 + c$. We denote such polynomial as f_c. Now instead of the complex plane of parameter z we can consider the complex plane of the parameter c. On this plane we can define the following important set:

Definition 2.1.3 (Mandelbrot set)**.** *The set of complex numbers c for which the orbit of point 0 under the function f_c remains bounded in absolute value is called the Mandelbrot set.*

2.2 Thurston maps

This section is a brief introduction to the theory of Thurston maps.

2.2.1 Background on the branched coverings and definition of the Thurston map

First of all we need to recall the fundamental definition of a branched covering:

Definition 2.2.1. *Let X and Y be two oriented compact connected surfaces. Informally a continuous surjective map $f : X \to Y$ is called a* branched covering *if it can be written in orientation preserving homeomorphic coordinates as $z \mapsto z^d$ where d can be different depending on point z. More precisely, if for each point $p \in X$ there exists $d \in \mathbb{N}$, topological disks $U \subset X$ and $V \subset Y$ with $p \in U$ and $q := f(p) \in V$, and orientation-preserving homeomorphisms $\varphi : U \to \mathbb{D}$ and $\psi : V \to \mathbb{D}$ with $\varphi(p) = 0$ and $\psi(q) = 0$ such that the following diagram is commutative*

$$\begin{array}{ccc} p \in U & \xrightarrow{\;f\;} & q \in V \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ 0 \in \mathbb{D} & \xrightarrow[z \mapsto z^d]{} & 0 \in \mathbb{D} \end{array}$$

The integer d depends on z. This number d is called the *local degree* of a branched covering f at x. The *degree* $deg(f)$ of a Thurston map f is a number of preimages of a general point.

Definition 2.2.2. *A point $c \in X$ where a local degree $d \geqslant 2$ is called a* critical point *of a map f. Images $v = f(c)$ of critical points are called* critical values.

We can see that the set of critical points is exactly the set where f fails to be a local homeomorphism.

Denote the set of critical points of f as $C(f)$. Then the following union $P(f) := \bigcup_{n=1}^{\infty} f^n(C(f))$ is called the *post-critical set* of f. If $P(f)$ is a finite set, then f is said to be *post-critically finite*.

Now we can define a key object of this section:

Definition 2.2.3. *A post-critically finite orientation preserving branched covering f with $deg(f) \geqslant 2$ is called a* Thurston map.

The first and very natural examples of the Thurston maps are given by post-critically finite rational functions on the Riemann sphere. This maps are also called *rational Thurston maps*. We will provide more examples in the next Chapter.

2.2.2 Thurston equivalence

There is a natural equivalence relation on Thurston maps. To define it properly we need to recall the following definition of a purely topological construction:

Definition 2.2.4 (Homotopies between topological spaces and homeomorphisms)**.** *Let X and Y be two topological spaces and $f : X \to Y$ and $g : X \to Y$ – two continuous functions from X to Y.*

- *A* homotopy *between f and g is a continuous map $H : X \times \mathbb{I} \to Y$ where* $\mathbb{I} := [0,1] \subset \mathbb{R}$ *such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for any $x \in X$. Denote $H_t(x) := H(x,t)$. We can also see the homotopy as a family of continuous functions $h_t : X \to Y$, $t \in [0,1]$, where $(x,t) \mapsto h_t(x)$ is a continuous map.*
- *Let $A \subset X$ be a subset of X. A homotopy H is called* a homotopy relative to A *if $H_t(a) = H_0(a)$ for all $a \in A$ and $t \in \mathbb{I}$.*
- *Two homeomorphisms $h_0, h_1 : X \to Y$ are called* homotopic *(relative to $A \subset X$) if there exists an homotopy $H : X \times \mathbb{I} \to Y$ (relative to A) with $H_0 = h_0$ and $H_1 = h_1$.*

Suppose $f : S^2 \to S^2$ and $g : \tilde{S}^2 \to \tilde{S}^2$ are two Thurston maps. Let $P(f)$ and $P(g)$ denote their post-critical sets. Here S^2 and $\tilde{S}^2$ are two oriented surfaces homeomorphic to the 2-sphere.

Definition 2.2.5. *Two Thurston maps $f : S^2 \to S^2$ and $g : \tilde{S}^2 \to \tilde{S}^2$ are called (Thurston)* equivalent *if there exist two orientation-preserving homeomorphisms h_0, $h_1 : S^2 \to \tilde{S}^2$ such that $h_0 \circ f = g \circ h_1$ and such that h_0 and h_1 are homotopic relative the post-critical set $P(f)$.*

We can see that *topological conjugacy* (existence of a homeomorphism $h : S^2 \to \tilde{S}^2$ such that $h \circ f = g \circ h$) is a special case of a Thurston equivalence.

It turns out that different rational functions are almost never equivalent. And this fact gives rise to a very natural question of understanding if the given Thurston map is equivalent to the rational function.

2.2.3 Thurston characterization theorem

The comparison of branched coverings with rational functions is one of the central problems in the theory of Thurston mappings. There is a very powerful tool for it called *Thurston characterization theorem*, which actually is one of the most important results in holomorphic dynamics. Here we provide the statement of the theorem, details and proof can be found in [DH93]. To formulate this theorem, we need some additional notion.

Definition 2.2.6 (Stable multicurve). *For a Thurston map $f : S^2 \to S^2$ a Jordan curve $\gamma \subset S^2 \backslash P(f)$ is called* non-peripheral *if each component of $S^2 - \gamma$ contains at least 2 points of $P(f)$. A system $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of simple, closed, disjoint, non-homotopic, non-peripheral curves on $S^2 - P(f)$ is called a* multicurve *on $S^2 - P(f)$. A multicurve Γ is said to be f*-stable *if for any $\gamma \in \Gamma$, all the non-peripheral component of the preimage $f^{-1}(\gamma)$ are homotopic in $S^2 - P(f)$ to elements of Γ.*

Definition 2.2.7 (Thurston linear transformation). *Let $f : S^2 \to S^2$ be a Thurston map and Γ be an f-stable multicurve. To each multicurve Γ from the space $\mathbb{R}^\Gamma$ of all multicurves we can associate the* Thurston linear transformation $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ *defined as follows: for the components $\gamma_{i,j,\alpha}$ of $f^{-1}(\gamma_j)$ homotopic to γ_i in $S^2 - P(f)$*

$$f_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d_{i,j,\alpha}} \gamma_i$$

where

$$d_{i,j,\alpha} = \deg f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j.$$

Now we are ready to provide the statement of the Thurston theorem. The theory of orbifolds is covered in Section 9 of [DH93]. However, for a Thurston map having a *hyperbolic orbifold* is a technical requirement, which we will not need in our work.

Theorem 2.2.8 (Thurston characterization theorem). *A Thurston map $f : S^2 \to S^2$ with hyperbolic orbifold is equivalent to a rational function if and only if there is no f-stable multicurve Γ such that the spectral radius of f_Γ satisfies $\lambda(f_\Gamma) \geqslant 1$. Such curve is called a* combinatorial obstruction.

Remark: All the others Thurston maps (without hyperbolic orbifold) are also fully classified.

2.3 Bisets

In this section we will describe a very important invariant of a Thurston map – a biset. A biset is a algebraic object, which fully encodes the Thurston equivalence class.

For a Thurston map f let us fix some objects and notations for them:

• a fixed basepoint $y \in \mathbb{S}^2 - P(f)$;

• the set $\mathcal{X}_f(y)$ denoting the set of all homotopy classes of paths from y to $f^{-1}(y)$ in $\mathbb{S}^2 - P(f)$;

• the fundamental group $\pi_1(\mathbb{S}^2 - P(f), y)$ denoted as π_f.

It turns out, that there is a structure of a *biset over a fundamental group* π_f on the $\mathcal{X}_f(y)$. The most full definition with algebraic background can be found in [Nek05] (bisets are called *bimodules* there, see Chapter 2). Here we give the formal definition:

Definition 2.3.1 (Biset)**.** *Let π be a group. A set $\mathcal{X}$ is called a* biset over π*, or a* π-biset*, if commuting left and right actions of π on $\mathcal{X}$ are given.*

Let us describe the biset structure on $\mathcal{X}_f(y)$. Thus we need to introduce the left and right actions of π_f on $\mathcal{X}_f(y)$. Left action is just a simple composition (paths are composed from left to right). For a representative $[\gamma] \in \pi_f$ and a representative $[\alpha] \in \mathcal{X}_f(y)$:

$$[\gamma][\alpha] = [\gamma\alpha]$$

It is well defined once we choose y as the basepoint of γ so that the loop γ ends where the path α starts.

In the case of the right action we take the path α first, so it terminates at $f^{-1}(y)$. There are two lifts of the loop γ. To define composition correctly we take the the lift of γ originating at the terminal point of α. Let us denote it as $\tilde{\gamma}$. So we can see this element as:

$$[\alpha][\gamma] = [\alpha\tilde{\gamma}]$$

We want to be able to compare bisets. We need the following definitions to do it:

Definition 2.3.2 (Biset basis and conjugate bisets)**.** *The biset $\mathcal{X}$ is said to be* left free *if there exists a subset $\mathcal{B} \subset \mathcal{X}$ such that every element $a \in \mathcal{X}$ can be uniquely represented as gb, where $g \in \pi$ and $b \in \mathcal{B}$.*

- *the subset $\mathcal{B}$ described above is called a* basis *of $\mathcal{X}$.*

Let π' be another group, and $\mathcal{X}'$ be a π'-biset.

- *a group isomorphism $\rho : \pi \to \pi'$ is said to* conjugate *$\mathcal{X}$ with $\mathcal{X}'$ if there exists a bijection $\sigma : \mathcal{X} \to \mathcal{X}'$ such that*

$$\sigma(g_1 a.g_2) = \rho(g_1)\sigma(a)\rho(g_2)$$

for all g_1, $g_2 \in \pi$ and $a \in \mathcal{X}$.

If ρ and σ with these properties exist, then $\mathcal{X}$ and $\mathcal{X}'$ are said to be conjugate. *If moreover $\pi = \pi'$ and $\rho = id$, we say that $\mathcal{X}$ and $\mathcal{X}'$ are* isomorphic.

The following theorem of Nekrashevich (Theorem 6.5.2 of [Nek05], see also [Kam01, Pil03]) shows why is the biset such an important object. The conjugacy class of the biset determines the Thurston equivalence class of the map.

Theorem 2.3.3. *Let f_1 and f_2 be Thurston maps, and $\mathcal{X}_{f_i}$ be the corresponding π_{f_i}-bisets, $i = 1, 2$. Here π_{f_i} is the fundamental group of $\mathbb{S}^2 - P(f_i)$.*

1. *The maps f_1 and f_2 are Thurston equivalent if and only if there exists an orientation preserving homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $h(P(f_1)) = P(f_2)$ and the induced isomorphism $h_* : \pi_{f_1} \to \pi_{f_2}$ conjugates $\mathcal{X}_{f_1}$ with $\mathcal{X}_{f_2}$.*

2. *Suppose that $P(f_1) = P(f_2) = P$ and the base points chosen for $\mathcal{X}_{f_1}$, $\mathcal{X}_{f_2}$ coincide. The maps f_1 and f_2 are homotopic rel. P if and only if $\mathcal{X}_{f_1}$ and $\mathcal{X}_{f_2}$ are isomorphic.*

Since we work with Thurston maps of degree 2, a basis of a biset $\mathcal{X}_f(y)$ consists of two elements. The base point y has two preimages y_0 and y_1. Then we want to choose the basis of $\mathcal{X}_f(y)$ as the homotopy classes of two paths α_0 and α_1, where they connect base point y with y_0 and y_1 respectively. Then we can associate an *automaton* with the given biset.

Let us give a formal definition of an automaton.

Definition 2.3.4 (Automaton)**.** *Let us assume that we have a two sets A and S with the following notations:*

- *the elements of the set A are considered as* symbols *(or* letters*) and we call the set A an* alphabet*;*
- *the elements of the set A are considered as* states *of automaton.*

An automaton *is a map $\Sigma : A \times S \to S \times A$, acting on the set of finite words in the alphabet A.*

For each state $s \in S$ there is a map $\Sigma(a, s) = (t, b)$ defined on all letters $a \in A$. It changes the letter a to the letter b and the state of automaton from s to t. The automaton starts to read a finite word symbol by symbol ***right to left****. Then if in the state s it reads a letter a it writes down the letter b on the place of the letter a, changes its state to t and moves left on one symbol. Then the procedure repeats.*

Let us go back to our concrete biset $\mathcal{X}_f(y)$. Its basis $[\alpha_0]$, $[\alpha_1]$ is described above. Let us assume that we also have a finite generating set $\mathcal{E} \subset \pi_f$ of a fundamental group π_f. It is possible to choose such $\mathcal{E}$ (this set will be discussed in the next chapter) that for $\varepsilon \in \{0, 1\}$ and any element $a \in \mathcal{E}$, we have $[\alpha_\varepsilon].a = a^*[\alpha_{\varepsilon^*}]$ for some $\varepsilon^* \in \{0, 1\}$ and $a^* \in \mathcal{E}$ depending on a and ε. An automaton $\Sigma : \{0, 1\} \times \mathcal{E} \to \mathcal{E} \times \{0, 1\}$ is defined as automaton which takes (ε, a) to (a^*, ε^*).

2.4 Graphs and embedded trees

2.4.1 Graph terminology

We need to specify our notations for the graphs embedded in the sphere. By the sphere $\mathbb{S}^2$ we mean the oriented topological 2-sphere.

Definition 2.4.1. *A 1-dimensional cell complex embedded into* $\mathbb{S}^2$ *is called a* graph*. 0-cells and 1-cells are called* vertices *and* edges*, respectively. For a graph* G*, we denote the set of its vertices as* $V(G)$ *and the set of its edges as* $E(G)$*.*

A simply connected graph is called a *tree*. A *branch point* is a vertex a of a tree T such that $T - \{a\}$ has more than 2 components. Suppose that $P \subset \mathbb{S}^2$ is some finite subset. A tree T in $\mathbb{S}^2$ such that $P \subset V(T)$ is called a *spanning tree* for P if $V(T) - P$ consists of branch points.

We are also interested in some additional structure on a graph. For a vertex x of a tree $T \subset \mathbb{S}^2$ and an edge e of T if x is in the closure of e, then x is called *incident* to e. In this case e is also *incident* to x.

We want to work not only with embedded graphs, but also with abstract graphs. Then we should define a *ribbon graph*:

Definition 2.4.2 (Ribbon graph (or a fat graph, or a cyclic graph))**.** *An abstract graph in which the edges incident to each particular vertex are cyclically ordered is called a* ribbon graph*.*

We also want to be able to describe graphs as purely combinatorial objects, thus we need the terminology of cyclic sets and pseudoaccesses following [Poi93].

Definition 2.4.3 (Cyclic set)**.** *A set* A *with a fixed cyclic order of its elements is called a* cyclic set*.*

In cyclic set we can define a following pair:

Definition 2.4.4 (Pseudoaccess)**.** *A* pseudoaccess *of a cyclic set* A *is an ordered pair* (a, b) *of its elements of* A *which are successive in the cyclic order.*

We can apply this definition also to a vertex a of a graph $G \in \mathbb{S}^2$. The edges incident to this vertex form a cyclic set $E(G, a)$ with orientation induced by the orientation of the sphere.

Let us assume that we have a Thurston map $f : \mathbb{S}^2 \to \mathbb{S}^2$ with a post-critical set $P(f)$. Then we want to have an invariant graph for it defined in a following way:

Definition 2.4.5 (Invariant spanning tree)**.** *Let* $f : \mathbb{S}^2 \to \mathbb{S}^2$ *be a Thurston map. A spanning tree* T *for* $P(f)$ *is called an* invariant spanning tree *for* f *if:*

1. *we have $f(T) \subset T$;*

2. *vertices of T map to vertices of T.*

This object can be seen in some way as an analogue of a very important object – a *Hubbard tree.*

Now we will describe this construction of invariant graphs introduced by A. Douady and J. Hubbard in [DH85]. First we need a definition of an *angled tree.*

Definition 2.4.6. *An* angled tree *H is a finite connected acyclic m-dimensional (where $m = 0, 1$) simplicial complex together with an* angle *function $\ell, \ell' \mapsto \angle(\ell, \ell') = \angle_v(\ell, \ell') \in \mathbf{Q}/\mathbf{Z}$, assigning to each pair of edges ℓ, ℓ' a rational modulo 1. This function satisfies the following properties: $\angle_v(\ell, \ell')$ is skew-symmetric; $\angle(\ell, \ell') = 0$ if and only if $\ell = \ell'$ and $L_v(\ell, \ell'') = \angle_v(\ell, \ell') + L_v(\ell', \ell'')$ for any three edges incident at a vertex v.*

Then we can define an *abstract Hubbard tree.*

Definition 2.4.7. *An angled tree $\mathbf{H} = ((H, V, \tau, \delta), L)$ is called an* abstract Hubbard tree *if angles at any vertex, which eventually maps to a non-critical cycle (* Julia vertex*), with m incident edges are multiples of $1/m$.*

We know, for example, from Theorem A in [Poi93], that we can associate an abstract Hubbard tree with any postcritically finite polynomial.

2.5 Local dynamics

2.5.1 Basic definitions

Consider the group G of all germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$. Let G_λ denote the set of all germs with derivative $f'(0) = \lambda$:

$G_\lambda = \{f(z) = \sum_{n=1}^{\infty} f_n z^n \in \mathbb{C}\{z\}, f_1 = \lambda\}$, where $\mathbb{C}\{z\}$ denotes the ring of convergent power series.

Definition 2.5.1. *Two germs f_1 and f_2 are* conjugate *if there exists $h \in G_1$ such that $h \circ f_1 = f_2 \circ h$.*

When λ is not equal to 0, the germ $f(z)$ is asymptotic to λz near 0, so the question is "Is the analytic function $f(z)$ conjugate to λz?".

Let R_λ denote the germ $R_\lambda(z) = \lambda z$.

Definition 2.5.2. *A germ $f \in G_\lambda$ is* linearizable *if there exists $h_f \in G_1$ (the linearization of f) such that $h_f^{-1} f h_f = R_\lambda$, i.e. f is conjugate to R_λ.*

It turns out that the answer to the question about existence of linearization depends on the arithmetic properties of λ.

2.5.2 Classification of fixed points

If z_0 is a fixed point of an analytic function f, i.e. $f(z_0) = z_0$, then the number $\lambda = f'(z_0)$ is called the *multiplier* of f at z_0. There is the following classification of fixed points according to λ: a fixed point z_0 is

–*attracting* if $0 < |\lambda| < 1$ (if $\lambda = 0$ then z_0 is *superattracting* fixed point)

–*repelling* if $|\lambda| > 1$

–*rationally neutral or parabolic* if $|\lambda| = 1$ and $\lambda^n = 1$ for some integer n

–*irrationally neutral* if $|\lambda| = 1$ but λ^n is never 1.

Let us assume that f has a periodic point z' of period m. Then we can apply the same classification also to periodic points. If we consider a map f^m, then z' is a fixed point for it. Then the classification is based on the multiplier $\lambda = (f^m)'(z')$.

The first result about linearization was obtained by G. Koenigs and H. Poincare in 1884 in the case of an attracting fixed point:

Theorem 2.5.3. *If a function f has an attracting fixed point with multiplier λ, then it is locally conjugate near 0 to the linear function $g(t) = \lambda t$. The conjugating function is unique up to multiplication by a nonzero complex number.*

This theorem is Theorem 2.1 in [CG] and its proof can be found there.

There exists also a congugating map in the case of a repelling fixed point. It follows immediately from the previous theorem by considering a local inverse f^{-1}.

In the superattracting case it was shown by L. Boettcher that f is conjugate to the map z^d where the degree d is the smallest index i of f_i in the series $f(z) = \sum_{n=1}^{\infty} f_n z^n$ such that $f_i \neq 0$.

When we have a rationally neutral fixed point, i.e., λ is a root of unity, the result about linearization is following:

Theorem 2.5.4. *If a function f has a rationally neutral fixed point and the multiplier λ is a root of unity of order n, then it is conjugate to the linear function $g(t) = \lambda t$ if and only if $f^n = id$.*

In the case of irrationally neutral fixed point, the multiplier λ can be rewritten as $\lambda = e^{2\pi i\alpha}$. As λ is not a root of unity, α is irrational number. Whether $f \in G_\lambda$ is linearizable or not depends crucially on the number theoretic properties of α.

In 1938 H. Cremer showed the following:

Theorem 2.5.5. *For any λ such that $\limsup_{n\to+\infty} |\lambda^n - 1|^{-1/n} = +\infty$ there exists $f \in G_{e^{2\pi i\alpha}}$ that is not linearizable.*

The proofs of Theorem 2.5.4 and Theorem 2.5.5 can be found in [Ma]. The problem of linearizability in the irrationally neutral case was open for a long time (until C.L. Siegel's work [Si] in 1942): the difficulties in this problem were similar to the difficulties in the problems from celestial mechanics connected with number theory.

If we have $f \in G_\lambda$ we want to find a function $\varphi(z) = \sum_{n=1}^{\infty} c_n z^n$ such that $\varphi(\lambda z) = f \circ \varphi(z)$. Formally, we can get the formal series solution of the functional equation by the recurrent formulas for the coefficients c_i:

$$c_1 = 1,\ c_n = \left(\frac{1}{f_1^n - f_1}\right)\left(\sum_{j=2}^{n} f_j \sum_{n_1+\ldots+n_j=n} c_{n_1}\ldots c_{n_j}\right)$$

This formula defines φ explicitly but depending on α in $f_1 = \lambda = e^{2\pi i\alpha}$ the denominators in $\frac{1}{f_1^n - f_1}$ can get very small, so some coefficients in the formal series can get very large. In this case this series will not converge and we have only formal solution. This problem is one of the examples of *small divisor problems* which came from celestial mechanics.

The first example of linearizable functions with irrationally neutral multiplier was introduced by C. Siegel in the work [Si] in 1942. The condition on the α in $\lambda = e^{2\pi i\alpha}$ deals with the idea of approximation of real numbers by rational numbers.

Definition 2.5.6. *A real number α is called* diophantine *if there exist $c > 0, \mu \geqslant 0 \in \mathbb{R}$ such that for all integers p and q, $q \neq 0$ holds $|\alpha - \frac{p}{q}| \geqslant cq^{-2-\mu}$.*

In other words *diophantine* numbers are irrational numbers which are badly approximable by rational numbers. Almost all (w.r.t. Lebsegue measure) irrational numbers are diophantine. The numbers in the complement (in the set of irrationals) of the set of diophantine numbers are called *Liouville numbers*.

The result of Siegel was the following theorem:

Theorem 2.5.7. *If f has a fixed point at 0 with multiplier $\lambda = e^{2\pi i\alpha}$ where α is Diophantine, then f is conformally conjugate on some open neighborhood around 0 to multiplication by λ.*

2.5.3 The Brjuno function and its properties

A. Brjuno developed the ideas of Siegel in 1971, he weakened the number-theoretic conditions on the number α for the convergence and introduced a larger class of numbers α (which includes some but not all Liouville numbers), such that every analytic function f with a fixed point at 0 with multiplier $\lambda = e^{2\pi i\alpha}$ is linearizable.

These conditions are coming from the approximation of real numbers by means of *continued fractions.*

Let $\{x\}$ denote the fractional part of a real number x: $\{x\} = x - [x]$, where $[x]$ is the integer part of x. Then $p_0 = [\alpha]$ and $q_0 = 1$. The continued fraction expansion of $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is the sequence $[a_0, a_1, a_2, ...]$ where $a_j = [1/\alpha_j]$, $\alpha_j = \{1/\alpha_{j-1}\}$, $\alpha_0 = \alpha - [\alpha]$ and $a_0 = [\alpha]$ for all $j \geqslant 1$.

Let us denote $\beta_n = \alpha_0 ... \alpha_n$, where $\beta_{-1} = 1$.

Definition 2.5.8. *A number α is called* Brjuno number *if $B(x) := \sum\limits_{n=0}^{\infty} \beta_{n-1} log\left(\frac{1}{\alpha_n}\right) < +\infty$ The function $B : \mathbb{R}\backslash\mathbb{Q} \to (0; +\infty]$ is called the* Brjuno function.

Theorem 2.5.9 (Brjuno theorem)**.** *Let f have a fixed point at 0 with multiplier $\lambda = e^{2\pi i\alpha}$. If α is a Brjuno number then f is linearizable.*

In 1988 J.-C. Yoccoz proved the converse:

Theorem 2.5.10. *Let $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. All $f \in G_\lambda$ are linearisable if and only if the Brjuno condition on α is verified.*

The proof of this theorem can be found in [Yoc95].

There is also one important subclass of Brjuno numbers:

Definition 2.5.11. *The number θ is* bounded type *if the continued fraction coefficients of θ are bounded.*

Any bounded type irrational number is Brjuno; the converse is not true (cf. [Yoc95]).

The next theorem of J.-C. Yoccoz also shows that considering a linearization problem for a polynomial $P_\lambda(z) = \lambda(z - z^2)$ enables us to generalise the result on all the class G_λ:

Theorem 2.5.12. *If P_λ, where $\lambda = e^{2\pi i\alpha}$, α is irrational, is linearizable then every $f \in G_\lambda$ is also linearizable.*

If P_λ, where $\lambda = e^{2\pi i\alpha}$ is linearisable, let h_λ denote its linearization, i.e. the conformal solution of the functional equation $h_\lambda \circ P_\lambda = \lambda z \circ h_\lambda$, and let $r(\alpha)$ denote the radius of convergence of h_λ. In [Yoc95] Yoccoz showed that $r(\alpha) > e^{-B(\alpha)-C}$ for some constant $C > 0$. This approximation of $r(\alpha)$ leads to considering the error function

$$E(\alpha) : \alpha \to B(\alpha) + log(r(\alpha))$$

For the rational numbers $B(\alpha) = +\infty$ and $log(r(\alpha)) = -\infty$, and it was conjectured in 1990 by S. Marmi [Ma90] that function $E(\alpha)$ can be extended to $\mathbb{R}$ as a continuous function. This conjecture was stated after computer experiments revealed a continuous graph for $E(\alpha)$. It took more than 15 years to prove this conjecture and it was shown in [BC06] only in 2006:

Theorem 2.5.13 (Theorem 1 in [BC06])**.** *The function* $E(\alpha) : \alpha \to B(\alpha) + log(r(\alpha))$ *extends to* $\mathbb{R}$ *as a continuous function.*

Actually, even stronger conjecture was stated in [MMY97]: it was conjectured that function $E(\alpha)$ is Hoelder of exponent $1/2$. It was shown only for some particular class of numbers α in [CC15] in 2015.

2.6 Fatou components

In this section we consider R to be a rational function. Let $F(R)$ be its Fatou set. Then the *Fatou component* is any connected component of $F(R)$.

We can distinguish 4 types of Fatou components depending on their dynamical behavior:

1. if $R(U) = U$, then U is a *fixed* component;
2. if $R^n(U) = U$ for some minimal $n \geqslant 2$, then U is a *periodic* component of *period* n;
3. if U eventually maps to some periodic component U', then U is called *preperiodic*;
4. otherwise if all $\{R^n(U)\}$ are different components, then U is a *wandering domain.*

A famous result of Sullivan says that for a rational function only types $(1)-(3)$ are possible:

Theorem 2.6.1 (Sullivan)**.** *A rational map has no wandering domains.*

This theorem is Theorem 1.3 in [CG], the proof can be found there. It turns out (Theorem 2.1 in [CG]), that the periodic component U can be only one of four following types:

1. U contains an attracting periodic point. In this case U is called a *basin of attraction*;

2. there is a parabolic point on the boundary of U, such that all points in U converge to it. Then U is called *parabolic*;

3. U is a *rotational domain* defined below: it can be either *Siegel disc* or *Herman ring.*

Now we give the precise definitions of the rotational domains. As we have seen in the previous section the property of holomorphic function f with multiplier λ being conjugate to multiplication by λ near 0 is exactly the fact that f is conjugate to a rotation in some area around 0. This set turns out to be a very important object.

Definition 2.6.2 (Siegel disc)**.** *For the holomorphic mapping f the Siegel disk is the biggest open subset containing* 0 *on which f is analytically conjugate to a rotation.*

Then we can also define a *Herman ring*:

Definition 2.6.3 (Herman ring)**.** *For the holomorphic mapping f its Fatou component U is called a* Herman ring *if it is conformally isomorphic to some annulus $A_r = \{z : 1 < |z| < r\}$, and if U is periodic (of period n) under f, and $f^n : U \to U$ is conjugate to a rotation on A_r.*

2.7 Polynomial maps and external rays

Let us consider a polynomial map $P : \mathbb{C} \to \mathbb{C}$. As we already have seen in Section 2.1, for polynomial mappings we can define the filled Julia set $K(P)$ as the set of points with bounded orbits. The reason for existence of this set is the fact that if P polynomial, than ∞ is a supper-attracting fixed point, and thus all points with unbounded orbits belong to its basin of attraction. In fact, the following proposition describes the structure of the filled Julia set:

Proposition 2.7.1. *For a polynomial P its filled Julia set $K(P)$ is a compact set which consists of the Julia set $J(P)$ together with all of the bounded components of the complement $\mathbb{C} \backslash J(P)$. The Julia set $J(P)$ is equal to the topological boundary $\partial K(P)$ of the filled Julia set.*

Proof. Lemma 18.1 in [Mil06]. □

2.7.1 External rays

Let us assume that $K(P)$ is connected. In this case it has to contain all critical points of P. Let us also choose the following normalization: the leading coefficient of P is equal to 1. Then we can introduce a conformal isomorphism $\phi : \mathbb{C}\backslash K(P) \to \mathbb{C}\backslash\bar{\mathbb{D}}$ such that $\phi(z) = z + o(z)$ near infinity between the complement $\mathbb{C}\backslash K(P)$ and the complement to the closed disc $\mathbb{C}\backslash\bar{\mathbb{D}}$. Let ψ be the map, inverse to ϕ.

Definition 2.7.2 (External rays). *The image of a radial line*

$$l_r = \left\{re^{2\pi it} : r > 1, t \in [0,1)\right\}$$

under ψ *is called the* external ray R_t *at* angle t *in* $\mathbb{C}\backslash K(P)$*.*

We say that an external ray $R_t = \psi(re^{2\pi it})$ lands *at a point* z_t*, if there exists a limit point* z_t*, such that points of* R_t *tend to* z_t *as* r *tends to* ∞*.*

According to Corollary 17.4 in [Mil06], if $J(P)$ is locally connected, then every external ray lands.

Definition 2.7.3. *The external ray* R_t *of a map* P *of degree* d *is called* periodic *if* t *is periodic under multiplication by* d*, i.e. if* $d^n t \equiv t (\mathrm{mod} 1)$ *for some* $n \geqslant 1$*.*

2.8 Holomorphic motions

The concept of holomorphic motions was introduced by Mañé, Sad and Sullivan in [MSS83]. Informally, we can understand it as a motion of a subset of a complex plane continuously depending on some parameter. Now we give a formal definition:

Definition 2.8.1. *Let* $A \subset \overline{\mathbb{C}}$ *be any subset and* Λ *be a metric space with a marked base point* τ_0*. A map* $(\tau, z) \mapsto \iota_\tau(z)$ *from* $\Lambda \times A$ *to* $\overline{\mathbb{C}}$ *is an* equicontinuous motion (of A over Λ) *if* $\iota_{\tau_0} = id_A$*, the family of maps* $\tau \mapsto \iota_\tau(z)$ *parameterized by* $z \in A$ *is equicontinuous, and* ι_τ *is injective for every* $\tau \in \Lambda$*.*

An equicontinuous motion is called holomorphic *if* Λ *is a Riemann surface, and each function* $\tau \mapsto \iota_\tau(z)$*, where* $z \in A$*, is holomorphic.*

Let us now consideris a family $F_\tau : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of rational maps such that $F_{\tau_0}(A) \subset A$. An equicontinuous motion $(\tau, z) \mapsto \iota_\tau(z)$ is *equivariant* with respect to the family F_τ if $\iota_\tau(F_{\tau_0}(z)) = F_\tau(\iota_\tau(z))$ for all $z \in A$.

2.9 Cubic polynomials and cubic slices

As we have seen in the definition of the Mandelbrot set, any quadratic polynomial is conjugate to the form $Q_c = z^2 + c$. In the cubic case any cubic polynomial is conjugate to the form:

$$P_{\lambda,b} = \lambda z + bz^2 + z^3, z \in \mathbb{C}$$

.

Then we will denote the space of such polynomials by $\mathcal{F}$.

Now we can define *cubic slices* as families in $\mathcal{F}$ in the following way:

Definition 2.9.1. *We call the* λ-slice $\mathcal{F}_\lambda$ *the space of all cubic polynomials* $g \in \mathcal{F}$ *with* $g'(0) = \lambda$.

2.9.1 The connectedness locus.

We can generalize the Mandelbrot set for higher degrees:

Definition 2.9.2. *The degree* d connectedness locus $\mathcal{C}_d$ *is the set of classes of degree* d *polynomials* f *with connected filled Julia set.*

We are interested in $\mathcal{C}_3$.

2.9.2 The main cubioid and Zakeri curve

We need to describe some special parts of the connectedness locus which play crucial role in the parametrization of $\mathcal{F}_\lambda$.

Definition 2.9.3 ([BOPT14])**.** *The main cubioid* CU *is defined as the set of all classes of cubic polynomials that have a fixed non-repelling point, no repelling peri-*

odic cutpoints and at most one non-repelling periodic point with multiplier different from 1.

It is easy to see that CU $\subset \mathcal{C}_3$

Let us denote the Siegel disc of cubic polynomial P as $\Delta(P)$. Then

Definition 2.9.4. *The* Zakeri curve *is the following set of polynomials:*

$Z_\lambda = \{P_{\lambda,b} \in \mathbf{C}^*_\lambda$; *both critical points of* $P_{\lambda,b}$ *belong to* $\Delta(P_{\lambda,b})\}$.

Chapter 3

Invariant trees for Thurston maps

In this chapter we provide an algorithm for finding invariant spanning trees of degree 2 Thurston maps. This algorithm was introduced by the author and V. Timorin in [ST19] and it was called the *ivy iteration*.

3.1 Invariant spanning trees

3.1.1 Definition and examples

Brief geometric background related to this section was discussed in Section 2.4. Let us recall that we call the following object a *invariant spanning tree*:

Definition 3.1.1 (Invariant spanning tree)**.** *A spanning tree for $P(f)$ is called an* invariant spanning tree *for f if:*

1. *we have $f(T) \subset T$;*

2. *vertices of T map to vertices of T.*

First we want to introduce some examples:

• *Quadratic polynomials.* Let $p(z) = z^2 + c$ be a post-critically finite quadratic polynomial. Let us denote the landing point of the dynamical external ray $R_p(0)$ of p with argument 0 by x_β. We will need the notion of a regulated hull following the terminology of [Mil09, Poi93, Poi10]. Define T as the union of $\{\infty\} \cup R_p(0)$ and the regulated hull of $X \cup \{x_\beta\}$. Then T is an invariant spanning tree for p. Let us note, that the regulated hull of X is exactly the *Hubbard tree* of p (Definition 2.4.7). Thus we can see that T is strictly bigger than the Hubbard tree of p.

- *Matings.* Let p and q be two post-critically finite quadratic polynomials. We may consider them acting on two different copies of a compactification $\copyright$ of $\mathbb{C}$, obtained by adding a circle of infinity. Let us denote this copies as $\copyright_p$ and $\copyright_q$ respectively. For each of these copies circles at infinity can be parameterized by the arguments of external rays (Definition 2.7.1). Let us write $E_p(\theta)$ for the point in the circle at infinity, corresponding to the external ray $R_p(\theta)$ of the argument $\theta \in \mathbb{R}/\mathbb{Z}$.

Now we can define an equivalence relation on the disjoint union $Y = \copyright_p \sqcup \copyright_q$ as follows: $x \sim y$ and $x \neq y$ if and only if one of the two points x has the form $E_p(\theta)$ and the other point y has the form $E_q(-\theta)$. The quotient space $\mathbb{S}^2_{p \amalg q} = Y/\sim$ is called the *formal mating space* of p and q. Since $\mathbb{S}^2_{p \amalg q}$ is homeomorphic to $\mathbb{S}^2$ and the map $F : Y \to Y$ defined as p on $\copyright_p$ and q on $\copyright_q$ descends to the quotient space, we have a naturally defined map $f : \mathbb{S}^2_{p \amalg q} \to \mathbb{S}^2_{p \amalg q}$. We write $f = p \amalg q$ and call f the *formal mating* of p and q.

Then we can construct an invariant spanning tree for f by taking the union of the two invariant spanning trees for p and q constructed as in the previous example.

- *Captures.*

Here we follow the description from [Ree92] with slight changes in phrasing the definition. Let us introduce a smooth structure on $\mathbb{S}^2$ and fix a smooth spherical metric on $\mathbb{S}^2$. Then given a vector v_x at some point $x \in \mathbb{S}^2$ and $\varepsilon > 0$, there is a vector field $D(v_x, \varepsilon)$ such that

1. outside of the ε-neighborhood of x with respect to the spherical metric, $D(v_x, \varepsilon) = 0$;

2. at point x, the vector $D(v_x, \varepsilon)_x$ coincides with v_x.

We may consistently choose vector fields $D(v_x, \varepsilon)$ for all x, v_x and ε so that they depend continuously (or even smoothly) on all parameters. Consider a smooth path $\beta : [0, 1] \to \mathbb{S}^2$ and choose a small $\varepsilon > 0$. Define the map $\sigma_\beta : \mathbb{S}^2 \to \mathbb{S}^2$ as the time $[0, 1]$ flow of the non-autonomous vector field $D(\dot\beta(t), \varepsilon)$. Here $\dot\beta(t)$ is the velocity vector of β at the point $\beta(t)$. The map σ_β is a self-homeomorphism of $\mathbb{S}^2$ with the following properties:

1. we have $\sigma_\beta(\beta(0)) = \beta(1)$;
2. the map σ_β is the identity outside of the ε-neighborhood $U_\varepsilon(\beta)$ of $\beta[0,1]$;
3. the map σ_β is homotopic to the identity modulo $\mathbb{S}^2 - U_\varepsilon(\beta)$.

The homeomorphism σ_β depends on β, ε and on a particular choice of $D(v_x, \varepsilon)$. However, if the path β is fixed, then any two such homeomorphisms σ_β and $\tilde{\sigma}_\beta$ are homotopic relative to $\mathbb{S}^2 - U_\varepsilon(\beta)$.

We can consider a composition $\sigma_\beta \circ p$, where p is a post-critically finite quadratic polynomial, and the choice of β depends on p. Set $\beta(0) = \infty$, and place $\beta(1)$ at some strictly preperiodic point that is not postcritical. If $U_\varepsilon(\beta)$ does not contain finite post-critical points of the map p and iterated images of $\beta(1)$, then all such maps $\sigma_\beta \circ p$ with fixed β are equivalent. In other words, the Thurston equivalence class of $f = \sigma_\beta \circ p$ depends only on β and p. The post-critical set of f is the union of $P(p)$ and the forward orbit of $\beta(1)$, including $\beta(1)$. Note that $\beta(1)$ is a critical value of f, the image of the critical point ∞. In fact, the homotopy class of f does not change if we deform β within the same homotopy class relative to $P(f)$. When talking about $\sigma_\beta \circ p$, we will always assume that the set $\beta[0,1)$ is disjoint from $P(p)$ and from the forward orbit of $\beta(1)$. The path β is called a *capture path* for p.

Definition 3.1.2. *The map $\sigma_\beta \circ p$ defined as above is called the (generalized)* capture *of p associated with β. The capture $\sigma_\beta \circ p$ is said to be* simple *if there is only one $t_0 \in [0,1]$ with $\beta(t_0) \in J(p)$. In the latter case, the corresponding capture path is called a* simple capture path.

Suppose that $\beta(1)$ is eventually mapped to a periodic critical point of p, i.e., to 0 if $p(z) = z^2 + c$. Then a simple capture path $\beta : [0,1] \to \mathbb{S}^2$ looks as follows. There is a parameter $t_0 \in (0,1)$ such that $\beta[0,t_0)$ is in the basin of infinity, $\beta(t_0,1]$ is in the Fatou component eventually mapped to a super-attracting periodic basin, and $\beta(t_0)$ is a point of the Julia set. We may arrange $\beta|_{[0,t_0)}$ to go along an external ray, and $\beta|_{(t_0,1]}$ to go along an internal ray. If $\beta(1) \in J(p)$, then $\beta[0,1]$ can be chosen as the union of an external ray and its landing point. Different simple capture paths lead to at most two different Thurston equivalence classes of captures provided that p and $\beta(1)$ are fixed, cf. [Ree10, Section 2.8].

Generalized captures were first defined by M.Rees in [Ree92]. Simple captures go back to B.Wittner [Wit88]. Both Wittner and Rees used the word "capture" to mean simple capture. We, on the contrary, use the word "capture" to mean a generalized capture. It is worth noting that the original approach of Wittner also used invariant trees. The study of captures is motivated by the following theorem of M.Rees:

Theorem 3.1.3 (Polynomial-and-Path Theorem, Section 1.8 of [Ree92])**.** *Suppose that R is a rational function of degree two with a periodic critical point c_1. Suppose also that the other critical point c_2 of R is not periodic but is eventually mapped to c_1. Then R is equivalent to some capture $\sigma_\beta \circ p$. Moreover, the quadratic polynomial p has a periodic critical point of the same period as c_1.*

Suppose that β is a simple capture path for p, and $f = \sigma_\beta \circ p$ is the corresponding capture. Let T be the minimal subtree of the extended Hubbard tree of p that includes $P(f)$. Then T satisfies the property $p(T) \subset T$. Note that it may happen that $f(T) \not\subset T$, so that T is not an invariant spanning tree for f. For example, let p be the airplane polynomial. Choose $\beta(1)$ to be an iterated p-preimage of 0 on an edge of the Hubbard tree of p. Then T coincides with the Hubbard tree set-theoretically but has more vertices. Some edge e of T maps under p so that the $\beta(1) \in p(e)$ but $\beta(1)$ is not an endpoint of $p(e)$. The latter is a consequence of the fact that there are no vertices of T mapping to $\beta(1)$. The homeomorphism σ_β displaces $p(e)$ so that $\sigma_\beta(p(e))$ no longer contains $\beta(1)$. Thus T is not forward invariant under $f = \sigma_\beta \circ p$.

It may seem plausible that T can be deformed slightly into a genuine invariant spanning tree. Unfortunately, this is not always true. It is known that different simple captures (even those for which $\beta(1)$ is the same) may yield different Thurston equivalence classes, see e.g. [Ree10, Section 2.8]. If T were deformable into an invariant spanning tree, then, by Theorem A, all simple captures with given $\beta(1)$ would be Thurston equivalent, a contradiction.

In the following lemma, by the *support* of a homeomorphism $\sigma : \mathbb{S}^2 \to \mathbb{S}^2$ we mean the closure of the set of points $x \in \mathbb{S}^2$ with $\sigma(x) \neq x$.

Lemma 3.1.4. *Let p, β and T be as above. Assume that the support of σ_β is a sufficiently narrow neighborhood of $\beta[0,1]$, i.e., a subset of the ε-neighborhood of $\beta[0,1]$ for sufficiently small $\varepsilon > 0$. Then T is an invariant spanning tree for $f = \sigma_\beta \circ p$ whenever $\beta[0,1] \cap p(T) = \varnothing$.*

Recall our assumption that the capture path β is simple.

Proof. Suppose that $\beta[0,1] \cap p(T) = \varnothing$. Then the support of σ_β can be made disjoint from $p(T)$. It follows that $\sigma_\beta = id$ on $p(T)$, therefore, $f(T) = \sigma_\beta(p(T)) = p(T) \subset T$. □

3.1.2 Thurston equivalence via invariant spanning trees

The following theorem explains why invariant spanning trees are important objects for Thurston maps. It turnes out that for some Thurston map f it suffices to know the ribbon graph structure of its invariant spanning tree T, the restriction of the map f to the set of vertices and the set of critical points of f in T and the cyclic order, in which pullbacks of certain edges appear around a critical point, to recover its Thurston equivalence class. Now let us formulate what it exactly means. We denote the set of critical points of f in T by $C(T)$.

Theorem 3.1.5. *Let us have two degree 2 Thurston maps f, $g : \mathbb{S}^2 \to \mathbb{S}^2$ with corresponding invariant spanning trees T_f and T_g. Then if there is a cellular homeomorphism $\tau : T_f \to T_g$ with the following properties:*

1. *The map τ is an isomorphism of ribbon graphs.*
2. *We have $\tau \circ f = g \circ \tau$ on $V(T_f) \cup C(T_f)$.*
3. *The critical values of f are mapped to critical values of g by τ.*

which also can be extended to edges of $f^{-1}(T_f)$ incident to points in $C(T_f)$ in the following mode: the cyclic order of edges incident to a given vertex of $C(T_f)$ has to *be preserved and so that to satisfy property* (2) *on edges and vertices. Then these two maps f and g are Thurston equivalent.*

The proof of this theorem is based on another somehow similar theorem, more general version of which is Proposition 3.4.3 of [Hlu17], which is based on the result from [BM17]. Here we will formulate it using our terminology.

Theorem 3.1.6. *Let f and g be two Thurston maps of degree two which have the same invariant spanning tree T. Moreover, suppose that $f^{-1}(T) = g^{-1}(T) = G$, that $f = g$ on $V(G)$, and that the critical values of f coincide with the critical values of g. Then there is an orientation preserving homeomorphism ψ isotopic to the identity relative to $V(T)$ and such that $f = g \circ \psi$.*

Then the proof comes down to the problem of showing that the ribbon graph isomorphism between T_f and T_g can be extended also to an isomorphism between $f^{-1}(T_f)$ and $g^{-1}(T_g)$. Let us denote $f^{-1}(T_f)$ as G_f. Then vertices of G_f are preimages of vertices of T_f and edges of G_f are components of $f^{-1}(T_f - V(T_f))$.

First we describe the topological construction for recovering G_f from T_f. Preimage of T_f under the map of degree 2 consists of two copies of T_f "glued" at the critical points. Thus we also need to know the position of the critical values of f. We will be distinguishing critical values between each other as well as critical points, thus we denote critical points of f as $c_1(f)$ and $c_2(f)$ and the corresponding critical values as $v_1(f)$ and $v_2(f)$. Then if we make a good enough cut between two critical values of f such that it is disjoint from T_f, we can "open" the sphere by this cut and thus have a semispere with one copy of T_f on it. In the full preimages we have two semispheres glued by this cut. To understand the ribbon structure of G_f we should also understand how exactly these semispheres are glued together. It turns out, that we even do not need to know the map f itself, but only the ribbon graph structure of T_f induced by it and the position of critical values of f as it was explained above. As we can see, the essential data are purely combinatorial, so we can describe all the procedure of describing G_f in purely combinatorial language. Further we will do it using the terminology of cyclic sets and pseudoaccesses from [Poi93], which was briefly explained in Section 2.4.1. Informally we can see pseudoaccess of graph at some vertex as a sector between two consecutive edges incident to this point.

This cut is exactly a Jordan arc C_f that connects $v_1(f)$ with $v_2(f)$ and is otherwise disjoint from T_f. Then C_f defines two pseudoaccesses of T_f, one at each of the critical values. From the assumptions of the Theorem 3.1.5 there exists a homeomorphism τ which preserves the cyclic order of edges at every vertex, thus there is a correspondence between pseudoaccesses of T_f and pseudoaccesses of T_g. Then there exists a Jordan arc C_g connecting $v_1(g)$ with $v_2(g)$, otherwise disjoint from T_g, which is again exactly a "good enough cut" now for T_g. The set $U_f = \mathbb{S}^2 - C_f$ is a disk and since both critical values of f are contained in C_f its preimage $f^{-1}(U_f)$ consists of two open disks U_f^0 and U_f^1 on which f is a one-to-one mapping. We are free in the choice of labeling U_f^0 and U_f^1. The same construction can be made for $g^{-1}(U_g)$.

Let us denote the closure of the f-pullback of $T_f - C_f$ in U_f^0 as T_f^0 and in U_f^1 as T_f^1. We write T_f^i for the closures in U_f^i respectively, where $i = 0, 1$. These closures have isomorphic ribbon graph structures. Thus these are exactly two copies of T_f described above. The critical points of f are the vertices of T_f^i that correspond, under the natural isomorphism between T_f^i and T_f, to the critical values $v_1(f)$ and $v_2(f)$. Thus since T_f^0 and T_f^1 are glued at the critical points of f we get an abstract description of the ribbon graph G_f.

We want to separate the edges of G_f in two groups depending on which copy of T_f do they belong. Then we can assign a number to each edge depending on the copy of T_f. We will name this number as *label*. Since we have two copies T_f^0 and T_f^1 the label can take value either 0 or 1. Formally, we say that an $e \in E(\overline{T}_f)$ has *label* $\ell(e)$ if $e \subset T_f^{\ell(e)}$. We want to show that the construction of assigning labels is well-defined, more precisely that it does not depend on the choice of the cut. This statement can be described by the following lemma:

Lemma 3.1.7. *Either* $\tau(T_f \cap T_f^i) \subset T_g \cap T_g^i$ *for every* $i = 0, 1$ *or* $\tau(T_f \cap T_f^i) \subset T_g \cap T_g^{1-i}$ *for every* $i = 0, 1$.

Proof. We can translate the assignment of the levels to purely combinatorial language. First we need to choose an initial edge, with which we will compare all the others. We call this edge the *reference edge* and denote it as e_r. We also can consider labels as label function $\ell(e)$ defined on all edges of $\overline{T}_f$. Now if we know the label of e_r we can start assigning labels to all the edges, starting with the edges sharing a common vertex with e_r by the following rule:

Let us assume that two edges e and e' share a vertex a. Then there are two possibilities:

1. if a is not critical, then $\ell(e') = \ell(e)$;

2. if a is critical, then $\ell(e') \neq \ell(e)$ if and only if e and e' are separated by the *critical pseudoaccesses* at a.

By the critical pseudoaccesses we mean two pseudoaccesses of $\overline{T}_f$ defined by the curve $f^{-1}(C_f)$.

Two edges e ang e' are separated by the critical pseudoaccesses at the common point a if and only if they are separated by $f^{-1}(C_f)$, i.e., lie in different components of $f^{-1}(U_f)$. This explains the assignment rule.

To prove the statement of the Lemma we need to prove that $\ell(\tau(e)) = \ell(e)$. Knowing the assignment rule it is equal to showing that critical pseudoaccesses of $\overline{T}_f$ maps to critical pseudoaccesses of $\overline{T}_g$ under τ. By the properties of τ we know that critical value $v = f(c)$ of f maps to the critical value and that $\tau(v) = \tau \circ f(c) = g(\tau(c))$. It follows that $\tau(c)$ has to be a critical point since g is a map of degree 2. Since τ preserves the cyclic order of edges and maps critical points to the critical points it has to map critical pseudoaccesses to critical pseudoaccesses. $\square$

By proving Lemma 3.1.7 we have shown that τ either preserves all labels or reverses all labels. We had the freedom of choice of the labeling when we were assigning the label to e_r. Thus we can choose the labeling so that labels are preserved by τ.

Now we can introduce a map τ^* with the properties similar to the properties of τ acting on the graph G_f and mapping it to G_g. More precisely we want to prove the following proposition:

Proposition 3.1.8. *There is a ribbon graph isomorphism $\tau_* : G_f \to G_g$ with the following properties:*

1. *We have $\tau_* = \tau$ on $V(\overline{T}_f)$.*

2. *We have $\tau_* \circ f = g \circ \tau_*$ on all vertices of G_f.*

Proof. Since by the previous lemma we know that $\tau : T_f \to T_g$ preserves labels we can "separate" it into two maps τ^0 and τ^1 acting on T_f^0 and T_f^1 respectively. Let us define maps f_i^{-1} and g_i^{-1} as the inverse of $f : T_f^i \to T_f$ and the of $g : T_g^i \to T_g$ respectively. Formally, there are two maps τ^i, where $i = 0, 1$, defined by the following formula: $\tau^i = g_i^{-1} \circ \tau \circ f$. Now we can define τ^* as the map from G_f to G_g, whose restriction to T_f^i is τ^i and then we have to show, that the obtained map satisfies the properties (1) – (2) from the statement of the proposition.

• Property (1) states that $\tau_* = \tau$ on $V(\overline{T}_f)$. Again by the previous lemma we know that τ maps $T_f \cap T_f^i$ to $T_g \cap T_g^i$. Moreover $\tau \circ f = g \circ \tau$ on $V(\overline{T}_f)$. Thus $\tau = g_i^{-1} \circ \tau \circ f$ on $V(\overline{T}_f) \cap T_f^i$, which gives us exactly two "halves" τ^0 and τ^1 of the map τ^*. The cyclic order is obviously preserved at the preimages of non-critical values, since the initial mappings preserve the orientation. In the case of the critical values it is preserved on the each "half-sphere", generated by the critical pseudoaccess and thus for the whole G_f.

• Property (2) states that $\tau_* \circ f = g \circ \tau_*$ on all vertices of G_f. A vertex a_* can be represented in one of two possible ways: $a_* = f_i^{-1}(a)$ where i is either 0 or 1. Then $\tau_* \circ f(a_*) = \tau(a) = g \circ \tau^i \circ f_i^{-1}(a) = g \circ \tau_*(a_*)$.

□

The obtained isomorphism $\tau_* : G_f \to G_g$ in general does not coincide with τ on $\overline{T}_f$. But we have shown that they coincide on $V(\overline{T}_f)$ and that it satisfies assumptions (1)–(3) of Theorem 3.1.5. It means that we can consider τ_* instead of τ and extend it to the entire sphere. To show that this extension is possible we want to apply the following known theorem to our settings:

Theorem 3.1.9 (Corollary 6.6 of [BFH92]). *Let G and G' be two connected graphs embedded into $\mathbb{S}^2$. Consider a homeomorphism $h : G \to G'$ that induces an isomorphism of ribbon graphs. Then there is an orientation preserving homeomorphism $h_* : \mathbb{S}^2 \to \mathbb{S}^2$ whose restriction to G is h.*

And thus we get the following statement:

Corollary 3.1.10. *Suppose that $\tau_* : G_f \to G_g$ satisfies the properties listed in Proposition 3.1.8. Then τ_* extends to an orientation preserving homeomorphism $\tau_* : \mathbb{S}^2 \to \mathbb{S}^2$.*

Now we finally have all the tools to prove Theorem 3.1.5.

Proof of Theorem 3.1.5. Let us recall, that we have formulated Theorem 3.1.6, needed for the further proof. It was stated there that if two Thuston maps f and g share the same invariant spanning tree T, their critical values coincide, $f^{-1}(T) = g^{-1}(T) = G$ and $f = g$ on $V(G)$, then there exists an orientation preserving homeomorphism ψ isotopic to the identity relative to $V(T)$ such that $f = g \circ \psi$.

Now we want to introduce the maps, to which we will apply Theorem 3.1.6.

We have introduced a homeomorphism $\tau_* : G_f \to G_g$ such that

1. we have $\tau_* = \tau$ on $V(T_f)$;

2. we have $g \circ \tau_* = \tau_* \circ f$ on all vertices of G_f.

We have also seen, that we can exchange τ and τ_* since they satisfy the same properties, which are important for our proof. Thus we can extend τ to the whole sphere by Corollary 3.1.10. Set $g_* = \tau^{-1} \circ g \circ \tau$. This map together with f are the desired maps satisfying the conditions of Theorem 3.1.6. Map g_* is also a Thurston map of degree two with an invariant spanning tree T_f. Their critical values coincide since τ maps the critical values of f to the critical values of g. Now we should check the last property $f^{-1}(T_f) = g_*^{-1}(T_f)$. In fact, $g_*^{-1}(T_f) = \tau^{-1} \circ g^{-1} \circ \tau(T_f) = \tau^{-1}(G_g) = G_f = f^{-1}(T_f)$. Applying Theorem 3.1.6 we get that g_* is Thurston equivalent to f, g_* is topologically conjugate to g by the definition of g_* and thus g is Thurston equivalent to f. The statement of Theorem 3.1.5 is proved.

□

3.2 Finding an explicit presentation for the biset

As we have seen in the previous section, the invariant spanning trees are very useful objects. In this section we show how they give rise to some other important invariants and moreover we introduce the generalization of invariant spanning trees that turns out to give rise to the same invariants.

3.2.1 Biset

Let f be a Thurston map with the postritical set $P(f)$. Following our notation from Section 2.3 $\mathcal{X}_f(y)$ is a set of all homotopy classes of paths from some fixed base point $y \in \mathbb{S}^2 - P(f)$ to its preimages $f^{-1}(y)$ in $\mathbb{S}^2 - P(f)$. We also have the fundamantal group $\pi_f = \pi_1(\mathbb{S}^2 - P(f), y)$. Then let us recall that $\mathcal{X}_f(y)$ is a *biset over fundamental group*. The full definition of a biset was also given in the Section 2.3. Here we only recall that being a biset means that both left and right actions of π_f on $\mathcal{X}_f(y)$ are given. Left action is just a simple composition (paths are composed from left to right). For a representative $[\gamma] \in \pi_f$ and a representative $[\alpha] \in \mathcal{X}_f(y)$:

$$[\gamma][\alpha] = [\gamma\alpha]$$

It is well defined once we choose y as the basepoint of γ so that the loop γ ends where the path α starts.

In the case of the right action we take the path α first, so it terminates at $f^{-1}(y)$. Then we take the the lift of γ originating at the terminal point of α. We denote it as $\tilde{\gamma}$ So we can see this element as:

$$[\alpha][\gamma] = [\alpha\tilde{\gamma}]$$

As we have seen in Theorem 2.3.3, the biset is an invariant for the Thurston equivalence class. It turns out that we can extend Theorem 3.1.6 to the following result:

Theorem 3.2.1. *Suppose that f is a Thurston map of degree 2, and T is an invariant spanning tree for f. There is an explicit presentation of the biset of f based only on the data listed above.*

1. *the ribbon graph structure on T,*
2. *the restriction of f to $V(T) \cup C(T)$.*

3.2.2 Generating set for the fundamental group

Let us start with describing the connection between an invariant spanning tree T and the fundamental group $\pi_f = \pi_1(\mathbb{S}^2 - P(f), y)$ with a base point y. We can introduce a generating set for π_f based on T. Informally this set can be described as the loops, which cross each edge in the fixed "direction" and do not have other intersections with T. First we explain what does "direction" of intersection mean. We consider a smooth path γ that crosses some oriented smooth Jordan arc A only once and transversely. We say that γ approaches A *from the left* if, at the intersection point, the velocity vectors to γ and to A (in this order) form a positively oriented basis in the tangent plane to the sphere.

Then the generating set for π consists of the following elements (here we assume that all edges are oriented):

- the neutral element id;
- elements $g_e \in \pi$ corresponding to each edge of T, where homotopy class g_e is represented by a smooth loop γ_e based at y that crosses e just once and transversely, approaches it from the left, and has no other intersection points with T.

We denote this set as $\mathcal{E}_T$.

Proposition 3.2.2. *Let f be a degree 2 Thurston map with an invariant spanning tree T. Then a set $\mathcal{E}_T$ described as above is a stable set under the action of f.*

Proof. If a loop γ belongs to $\mathcal{E}_T$ then there are two possible cases for its preimages:

Case 1: The full preimage consists of two loops, starting at the two preimages y_0 and y_1 of the basepoint y. In this case each of them is a pullback and maps to the loop γ one-to-one, so each preimage can have maximum one point of intersection with the graph T, so the preimages belong to the required set.

Case 2: The full preimage consists of the one loop, which maps to the loop γ two-to-one. Then this loop consists of two paths joining a_0 and a_1, each of them maps to the loop γ one-to-one, so each preimage can have maximum one point of intersection with the graph T. In this case their closure by the path joining the y_0 and y_1 and not crossing the graph Γ is the loop, which has maximum one point of intersection with the graph T, so the preimages belong to the required set. □

3.2.3 Dynamical tree pair

But it is not always trivial to find the invariant spanning tree. It turns out that the statement of Theorem 3.1.6 can be generalized even more than in Theorem 3.2.1. We need to introduce one more object for being able to do it:

Definition 3.2.3. *For a Thurston map $f : \mathbb{S}^2 \to \mathbb{S}^2$ of degree 2 two spanning trees T^* and T for $P(f)$ such that $f(T^*) \subset T$ are called a* dynamical tree pair *for f if*

1. *the vertices of T^* are mapped to vertices of T under f;*

2. *all critical values of f are vertices of T;*

We denote a dynamical tree pair as (T^*, T).

We can see that dynamical tree pairs generalize invariant spanning trees.

3.2.4 Combinatorial description for the biset

Since we are dealing with the mappings of degree 2 a basis of $\mathcal{X}_f(y)$ consists of two elements. We denote them as α_0 and α_1, where they connect base point y with y_0 and y_1 respectively. The description of the biset can be done in the terms of an *automaton* (Definition 2.3.4). If we know the generating set for the fundamental group, it is enough to define the automaton action on the generators.

For the description of the automaton action on any generator $\gamma \in \pi_f$ we should be able to recover its pullbacks γ_i from the combinatorial data only. If γ does not encompass a critical value, than its pullbacks are just two loops, belonging to the different semispheres, containing two different copies of T and glued together, as it was discussed in Section 3.1.2. Since we have shown, that there is correspondence between loops in the generating set and edges, we can assign *labels*, which were introduced in the previous section, also to the loops.

For formal distinguishing of the copies of the sphere in the preimage we need to define a *base edge*.

Let us consider a dynamical tree pair (T^*, T). The preimage of the smallest arc Z in the tree T which contains two critical values is a loop $f^{-1}(Z) = Z^*$ containing two critical points. From the definition of T^* the loop Z^* can not be fully contained in T^*. Thus there is at least one edge e'_b of Z^* which does not belong to T^*. Let us call the edge $e_b = f(e'_b)$ a *base edge*. Here we made the only free choice in the further construction. Since it is done, all the rest of the procedure can be defined uniquely.

Let us consider a Jordan arc C_f that connects critical values $v_1(f)$ and $v_2(f)$ and is otherwise disjoint from T_f. Then we call each of two pseudoaccesses defined by C_f a *post-critical pseudoaccess*. The curve $f^{-1}(C_f)$ defines two pseudoaccesses of $\overline{T}_f$ at the critical point c, not necessarily different. We will call these distinguished pseudoaccesses *critical pseudoaccesses*.

Definition 3.2.4 (The label of an edge)**.** *The* labeling *on a graph $G = f^{-1}(T)$ is a function $\ell : E(G) \to \{0, 1\}$ which satisfies the following properties:*

- *$\ell(e'_b) = 1$;*
- *if two edges e_1 and e_2 share a vertex, then $\ell(e_1) = \ell(e_2)$ if and only if e_1, e_2 are not separated by the critical pseudoaccesses. For $e \in E(G)$, the value of the function $\ell(e)$ is called the* label of the edge *e.*

If γ contains a critical value, than its full preimage consists of just one loop, which maps to γ two-to-one. In this case this loop is going through both preimages y_0 and y_1 of the basepoint. The two pullbacks of the loop γ starting at y_0 and y_1 are the two paths joining y_0 and y_1. Here we have to deal with the orientation of the edges. The following definition enables us to translate the orientation of the sequence of the edges into the combinatorial language. Let us recall that by an *oriented edge* of the graph G we mean its edge $e \in E(G)$ with fixed orientation.

Definition 3.2.5 (Boundary Circuits)**.** *A* boundary circuit *of G (also called a left-turn path in G) is a cyclically ordered sequence $[e_0, \ldots, e_{n-1}]$ of oriented edges of G satisfying the following property: if e_i terminates at a vertex a, then $e_{i-1 \pmod n}$ originates at a, and $e_{i-1 \pmod n}$ is the immediate* ***predecessor*** *of e_i in the cyclic order on $E(G, a)$.*

We can see the $C(T)$ as a walk around the graph T **clockwise**. The critical pseudoaccess divide all oriented edges from $C(T)$ into two groups which we denote

$S^0(T)$ and $S^1(T)$. Then this groups also could be seen as paths with fixed directions. We want to choose the notation in the following way:

- $S^0(T)$ originates at v_1 and terminates at v_2;
- $S^1(T)$ originates at v_2 and terminates at v_1.

For being able to distinguish pullbacks of the edges corresponding to the loop with or without critical value, we need to understand if this loop is changing the orientation when we take its pullback. To be able to do this we need the following definition:

Definition 3.2.6 (Signatures of edges)**.** *An oriented edge e of T is* of signature (i, j) *if e appears in $S^i(T)$, and e^{-1} appears in $S^j(T)$. The notation e^{-1} corresponds to an oriented edge e with the opposite orientation. Thus there exist four possible signatures:* $(0, 0)$*,* $(0, 1)$*,* $(1, 0)$*, and* $(1, 1)$.

Let us recall that we use the following notation $G = f^{-1}(T)$. There exists the boundary circuit $C(T)$. From Section 3.1.2 we remember that the complement $\mathbb{S}^2 - G$ of G consists of two disks Ω^0 and Ω^1. Using the notation from the same section we can define the subsets in $C(T)$ in the following way: we denote the pullback of $S^i(T)$ in T^j as $S^i(T^j)$, where $i, j = 0, 1$. Then the disk Ω^0 is bounded by $S^0(T^0) \cup S^1(T^1)$ and the disk Ω^1 by $S^0(T^1) \cup S^1(T^0)$. We can note that different boundary circuits of G correspond to two different discs Ω^0 and Ω^1. Let us denote as $C^i(G)$ the boundary circuit which corresponds to Ω^i. The full preimage of an edge $e \in E(T)$ in a graph can be represented as a union $e^0 \cup e^1$, where $e^0 \in E(T^0)$ and $e^1 \in E(T^1)$. Then the boundary circuits $C^i(G)$, where $i = 0, 1$, can be described by the following lemma:

Proposition 3.2.7. *Let e be an oriented edge of T of signature (i, j). Then its pullback e^0 belongs to the boundary circuit $C^i(G)$ and the pullback e^1 to the boundary circuit $C^{1-i}(G)$.*

Then we have the following notation: e^δ belongs to $C^{i+\delta}(G)$, where $\delta = 0, 1$ and the addition is mod 2.

Proof. We are assuming that the signature of an edge e is (i, j). It means by definition that $e \in S^i(T)$. The edge e^0 of G is a part of T^0 hence also of $S^i(T^0)$. It follows that $e^0 \in C^i(G)$. The same for $e^1 \in C^{1-i}(G)$. □

3.2.5 The main Theorem

Now we can give the main statement about the computation of the biset in purely combinatorial language. We use here the representation of a biset in the terms of automata, which was briefly described in Section 2.3. Let us recall that for invariant spanning tree T_0 of the degree 2 Thurston map f we were able to write a biset as a map $\Sigma_0 : \{0,1\} \times \mathcal{E}_{T_0} \to \mathcal{E}_{T_0} \times \{0,1\}$ where $\mathcal{E}_{T_0}$ is a generating set of the fundamental group $\pi_f = \pi_1(\mathbb{S}^2 - P(f), y)$ described in Section 2.3. But let us assume that instead of the invariant spanning tree T_0 we have only dynamical tree pair (T^*, T). Then we can describe a map $\Sigma : \{0,1\} \times \mathcal{E}_T \to \mathcal{E}_{T*} \times \{0,1\}$ which defines a presentation of the biset $\mathcal{X}_f(y)$. For a tree T^* we are also able to introduce a generating set $\mathcal{E}_{T*}$ of the fundamental group. Knowing signatures or the edges we can know how any edge $e^* \in T*$ *maps over* an edge $e \in T$: preserving or reversing the orientation. This enables us to define the states of the automaton. Following the introduced terminology now we can formulate the theorem:

Theorem 3.2.8. *We define the automaton in the following way:*

$$\Sigma(\varepsilon, g) = (g^*, \varepsilon^*),$$

where $\varepsilon \in \{0,1\}$ and $g \in \mathcal{E}_T$.

The definition of elements ε^* and g^* is given by the followong rule:

1. *for a neutral element $g = id$ we have $\varepsilon^* = \varepsilon$ and $g^* = id$;*

2. *for an element $g = g_e$ corresponding to an oriented edge e of T of signature $(\varepsilon + \delta, \varepsilon^* + \delta)$ if there exists is an oriented edge e^* of $\overline{T}^*$ with labeling δ such that it maps over e preserving the orientation, then $g^* = g_{e*}$. If there is no such edge, then $g^* = id$.*

Thus we get an automaton

$$\Sigma : \{0,1\} \times \mathcal{E}_T \to \mathcal{E}_{T*} \times \{0,1\}$$

which provides a full description of the biset of the Thurston map f with the dynam-

ical tree pair (T^*, T).

Proof. The state of automata ε corresponds to the choice of the path α_0 or α_1 from the biset basis. Then proving that $\Sigma(\varepsilon, g) = (g^*, \varepsilon^*)$ is equivalent to proving the following statement:

$$[\alpha_\varepsilon].g = g^* \, [\alpha_{\varepsilon *}]$$

Let us recall that we assume that γ_e is a smooth loop based at y, crossing T just once transversely and approaching it from the left. Two preimages of y named y_0 and y_1 belong to Ω_0 and Ω_1 respectfully.

The case where $g = id$ is obvious. If we consider $g = g_e$ corresponding to an oriented edge e of T of signature $(\varepsilon + \delta, \varepsilon^* + \delta)$ we choose g^* as in the statement of the theorem. More precosely: there are two edges mapping to e. In each T^δ there is one. For a labeling δ if this edge belongs to T^* we denote it as e^* and set $g^* = g_{e^*}$. Otherwise if $e^\delta \notin T^*$, then we set $g^* = id$. By $[\alpha_\varepsilon].g$ we understand the concatenation of α_ε and a pullback γ_e^* of γ_e, which start at y_ε, where α_ε ends. The path α_ε ends at Ω^ε. Let us consider the pullback γ_e^*. As a path it approaches some boundary edge e' of Ω^ε, which thus has to belong to $E(G)$ by the definition of Ω^i. We are free to chose the orientation of e' and we want to chose it in such a way, that e' will belong to the boundary circuit $C^\varepsilon(G)$. It is enough to require that γ_e^* approaches e' from the left. The described edge e' must coincide with e^0 or with e^1. As we have shown in Proposition 3.2.7, the edge e^δ belongs to $C^{i+\delta}(G) = C^\varepsilon(G)$. Therefore, we have $e' = e^\delta$ and we know that the edge e is of signature $(\varepsilon+\delta, \varepsilon^*+\delta)$. Then the two sides of the arc e' belong to Ω^ε and Ω^{ε^*}, where we have already chosen orientation of e' such that the left side of e' is in Ω^ε and by the same Proposition 3.2.7 the right side of e' is in Ω^{ε^*}. We do not know if ε and ε^* are different signs. It follows that when we consider the concatenation $\alpha_\varepsilon.\gamma_e$ it must terminate in $\Omega_{\varepsilon *}$. Now we can check the initial equality $[\alpha_\varepsilon].g = g^* \, [\alpha_{\varepsilon *}]$. For γ_e it becomes

$$[\alpha_\varepsilon].[\gamma] = [\alpha_\varepsilon \gamma_e^* \alpha_{\varepsilon *}^{-1}] \, [\alpha_{\varepsilon *}]$$

Now let us show that $[\alpha_\varepsilon \gamma^* \alpha_{\varepsilon *}^{-1}] = g^*$. In the statement of the Theorem there are two cases depending on if $e' \in T^*$:

1. Let us assume that $e' \notin T^*$. Then from the statement of the Theorem we get $g^* = id$. The loop γ_e^* has to be disjoint from T^*. The basis paths α_0 and α_1 are also disjoint from T^*, thus the entire loop $\alpha_\varepsilon \gamma_e^* \alpha_{\varepsilon^*}^{-1}$ is also disjoint from T^*. It means that it is contractible in $\mathbb{S}^2 - T^*$ and in $\mathbb{S}^2 - P(f) \supset \mathbb{S}^2 - T^*$. Then the both sides of equality are equal to id.

2. Now assume that $e' \in T^*$. We know that e^* of $\overline{T}^*$ has label δ, then $e^* \supset e'$. The path γ_e^* intersects G once, and approaches e^* from the left. Therefore, $[\alpha_\varepsilon \gamma_e^* \alpha_{\varepsilon'}^{-1}] = g_{e^*}$, which proves the desired.

□

3.3 Ivy iteration

We introduce a combinatorial method of finding invariant spanning trees. It means that using this method we can insert the combinatorial information about the dynamical tree pair of a Thurston map f into the computer program and thus find invariant spanning trees.

3.3.1 Topological ivy objects, pullback relation and ivy graph

We start with describing the geometric objects we want to represent as combinatorial data.

Definition 3.3.1 (Topological ivy object)**.** *A (topological)* ivy object *is defined as a homotopy rel. $P(f)$ class of spanning trees for $P(f)$. We will write* $\mathrm{Ivy}(f)$ *for the set of all ivy objects for f. The notation $[T]$ will be used for the ivy object corresponding to a spanning tree T for f*

We want to define the following relation on the set $\mathrm{Ivy}(f)$ of ivy objects: if (T^*, T) is a dynamical tree pair (we still need to describe the procedure of choosing the tree T^*) then there is the *pullback relation* $[T] \multimap [T^*]$ between T and T^*.

Now we need to explain how to recover the dynamical tree pair (T^*, T) knowing some spanning tree T. Actually, the choice of T^* is not unique, but we want to make some restrictions which allow us to define T^* and then all the further procedures will be completely combinatorial. We can find the full preimage $G = f^{-1}(T)$ and we know that T^* should be contained in it. A graph G contains a simple loop since we know that it consists of two copies of T glued at the critical points. First we want to remove an edge from it to get a tree. We have already discussed that the only free choice we make is the choice of the base edge e_b. We choose it as an edge separating two critical values. Let us choose a pullback e_b' of e_b such that it maps onto e_b preserving the orientation and also the path from the basepoint y to this edge outside G approaches e_b' from the left. Now let us construct T^*:

1. we remove the edge e_b' from G and get a tree;
2. we choose the smallest subtree of this tree containing $P(f)$ and denote it as $\widehat{T}$;
3. we erase all vertices of $\widehat{T}$ that are not in $P(f)$ and are not branch points of $\widehat{T}$ such that they become points in the edges of T^*.

The tree which we got by these operations is exactly T^*.

Thus we have described how exactly topological ivy objects are connected under the pullback relation. Now we want to equip the set $\mathrm{Ivy}(f)$ with the structure of an abstract directed graph. Each ivy object $[T]$ is a vertex. We connect two vertices corresponding to two ivy objects $[T_1]$ and $[T_2]$ by an oriented arrow from $[T_1]$ to $[T_2]$ if (T_2, T_1) is a dynamical tree pair. Let us recall that we defined this relation as a *pullback relation.* For any tree T there are only finitely many options for the choice of T^*, thus there are only finitely many arrows originating at each element of $\mathrm{Ivy}(f)$.

Definition 3.3.2 (Ivy graph)**.** *Let f be degree 2 Thurston map. Then the* ivy graph *of f is the set* $\mathrm{Ivy}(f)$ *equipped with the graph structure where two elements* $[T]$ *and* $[T^*]$ *are connected by an arrow if there is a* pullback relation $[T] \multimap [T^*]$ *between them.*

Let us come back to the problem of finding periodic spanning trees. We can see that in the ivy graph periodic spanning tree corresponds to some invariant subset. We can precisely define this subset:

Definition 3.3.3. *A subset* $C \subset \mathrm{Ivy}(f)$ *is called* pullback invariant *if for any* $[T] \in C$ *and* $[T] \multimap [T^*]$ *we also have* $[T^*] \in C$. *Then the elements* $\tau \in \mathrm{Ivy}(f)$ *belonging to this invariant cycle* C *are called* periodic objects. *We can define a* period p *of* τ *as the length of any simple cycle containing* τ. *Then the spanning tree* T *corresponding to this element is also periodic of period* p.

3.3.2 Combinatorial terminology: push forwards, vertex structures and vertex words

As we have already mentioned, the main objective of this work is to introduce a purely combinatorial algorithm. We have described the ivy graph and the topological procedure of the pullback relation. Now our goal is to describe all the objects and constructions from the previous subsection using only combinatorial data and to show that the the objects we get by these different descriptions are the same.

Now we have an automaton as a very convenient method of description, which is exactly a purely combinatorial object. We want to see how it acts on a generating set $\mathcal{E}$ of a fundamental group π. We are assuming that $\mathcal{E}$ is symmetric, i.e., that $g^{-1} \in \mathcal{E}$ for every $g \in \mathcal{E}$, and that $1 \in \mathcal{E}$. Let $\mathcal{X}$ be an abstract left free biset over a group π and $\mathcal{B}$ be a basis of $\mathcal{X}$. We can write down the automaton $\Sigma : \mathcal{B} \times \pi \to \pi \times \mathcal{B}$ of $\mathcal{X}$ in the following form:

$$\Sigma = (\sigma, \iota), \text{ where } \Sigma(a, g) = (\sigma(a, g), \iota(a, g)) \text{ for all } a \in \mathcal{B},\ g \in \pi.$$

Definition 3.3.4. *We define the* push forward $P\mathcal{E}$ *of* $\mathcal{E}$ *as the set* $\sigma(\mathcal{B}, \mathcal{E})$.

We have already discussed that there is freedom of choice of a basis $\mathcal{B}$ of $\mathcal{X}$. We want to see how the automaton changes if we apply some basis change. Let us explain the question more precisely: a basis change C_λ can be defined by a function $\lambda : \mathcal{B} \to \pi$ (any such function defines a basis change). The new basis $C_\lambda \mathcal{B}$ consists of $\lambda(a)a$, where $a \in \mathcal{B}$. To lighten the notation denote $\lambda(a)a$ as a_λ. Now we have two automata Σ and $C_\lambda\Sigma$ of $\mathcal{X}$ in basis sets $\mathcal{B}$ and $C_\lambda\mathcal{B}$ respectively. We want to express $C_\lambda\Sigma$ in terms of Σ and C_λ.

Proposition 3.3.5. *The automaton* $C_\lambda\Sigma = (C_\lambda\sigma, C_\lambda\iota)$ *is given by*

$$C_\lambda\sigma(a_\lambda, g) = \lambda(a)g^*\lambda(a^*)^{-1}, \quad C_\lambda\iota(a_\lambda, g) = a^*_\lambda.$$

Proof. We have the automaton Σ which on any $a \in \mathcal{B}$ and any $g \in \pi$ acts as $(g^*, a^*) = \Sigma(a, g)$. Then the statement of the proposition follows from the following equalities in $\mathcal{X}$:

$$a_\lambda.g = \lambda(a)a.g = \lambda(a)g^*a^* = \lambda(a)g^*\lambda(a^*)^{-1}a^*_\lambda.$$

□

Given a symmetric generating set $\mathcal{E}$ of π containing 1, set $P_\lambda\mathcal{E}$ to be the generating set obtained as the push forward of $\mathcal{E}$ under $C_\lambda\Sigma$.

For being able to describe the ivy algorithm in combinatorial terms we need to be able to fully describe spanning tree symbolically.

If we have a finite set $\mathcal{E}$, then consider the free semi-group generated by it. We denote this group as $\mathrm{FS}(\mathcal{E})$. We can see that this group is exactly the set of all finite words in the alphabet $\mathcal{E}$ on which the automaton Σ acts. We denote the product of two elements g, $h \in \mathcal{E}$ in $\mathrm{FS}(\mathcal{E})$ also as in the description of automaton: $g \cdot h$. Now let us assume that $\mathcal{E}$ is a symmetric subset of some group π (if $g \in \mathcal{E}$ than also $g^{-1} \in \mathcal{E}$, where g^{-1} is the inverse of g in the group π) and that the identity element id of π belongs to $\mathcal{E}$. Let us notice that neutral element of $\mathrm{FS}(\mathcal{E})$ is not the same element id but the empty word. We set $\mathcal{E}^\star$ to be the quotient of $\mathrm{FS}(\mathcal{E})$ modulo the relations $id \cdot g = g \cdot id = g$ for all $g \in \mathcal{E}$. We can define the following map on this set:

Definition 3.3.6. *The* evaluation map $\Pi : \mathcal{E}^\star \to \pi$ *is a map that takes every word in the alphabet* $\mathcal{E}$ *to the product of its symbols.*

Moreover, $\mathcal{E}^\star$ enables us to equip $\mathcal{E}$ with additional structure:

Definition 3.3.7. *A* vertex structure *on $\mathcal{E}$ is a subset $\mathcal{V} \subset \mathcal{E}^\star$ with the following property: for every $g \in \mathcal{E}$, there is a unique element of $\mathcal{V}$ of the form $u_1 \cdot g \cdot u_2$ for some u_1, $u_2 \in \mathcal{E}^\star$. Any vertex structure gives rise to an abstract directed graph $G(\mathcal{V})$ as follows. The vertices of $G(\mathcal{V})$ are identified with elements of $\mathcal{V}$. The oriented edges of $G(\mathcal{V})$ are labeled by elements of $\mathcal{E}$. Two vertices v, $w \in \mathcal{V}$ are connected with an oriented edge g (from v to w) if*

$$v = v_1 \cdot g \cdot v_2, \quad w = w_1 \cdot g^{-1} \cdot w_2$$

for some elements v_1, v_2, w_1, w_2 of $\mathcal{E}^\star$. Since $\mathcal{E}$ is symmetric, the edges of $G(\mathcal{V})$ always come in pairs so that paired edges connect the same vertices but go in different directions. These pairs of edges correspond to pairs of the form $\{g, g^{-1}\}$ in $\mathcal{E}$. Thus $G(\mathcal{V})$ can also be regarded as an undirected graph, by identifying each pair of oppositely directed edges with an undirected edge. A vertex structure $\mathcal{V}$ on $\mathcal{E}$ is called a tree structure *if $G(\mathcal{V})$ is a tree. Then we say that $\mathcal{E}$ is* tree-like *if there exists a tree structure $\mathcal{V}$ on $\mathcal{E}$*

Let us recall that in our problem the set $\mathcal{E}$ is generated by a spanning tree T for a finite marked set P. Actually we denoted the corresponding set as $\mathcal{E}_T$, but here to lighten the notation we keep using the notation $\mathcal{E}$. Each element $g_e \in \mathcal{E}$ corresponds to an oriented edge e in T. But we want a full tree structure on $\mathcal{E}$ also to be defined by the combinatorial data, which encodes T. Thus we need to be able to describe the objects, which correspond to vertices. We assume that there is some pseudoaccess fixed at every vertex of T. We can define this objects in the following way:

Definition 3.3.8 (Vertex word)**.** *Let x be a vertex of T. Consider all edges $e_0, \dots,$ e_{k-1} incident to x and oriented outwards. The linear order of these edges is well defined if we impose that*

1. *it follows the natural* ***clockwise*** *order around x;*

2. *the chosen pseudoaccess at x coincides with (e_{k-1}, e_0).*

Then we define the vertex word *of x as the product $g_{e_0} \cdot \dots \cdot g_{e_{k-1}} \in \mathcal{E}^\star$.*

In the introduced construction the vertex structure $\mathcal{V}$ is the set of all vertex words associated with the vertices of T. We can see that $\mathcal{V}$ is a tree structure on $\mathcal{E}$: $G(\mathcal{V})$ is a tree since it is isomorphic to T as a ribbon graph. Then according to Definition 3.3.7 $\mathcal{E}$ is a tree-like set.

Now we want to describe the basis changes for which the push forwards remain tree-like sets. In the previous definitions of push forwards we considered abstract bisets, but now we work again in the assumptions of our problem and assume that the basis $\mathcal{B}$ of the biset consists of two elements. Let us define these push forwards first:

1. we apply a basis change C_λ defined by function $\lambda : \mathcal{B} \to \pi$ to $\mathcal{B}$;
2. we get the push forward of $\mathcal{E}$ under new automaton $C_\lambda\Sigma$, denote this push forward as $P_\lambda\mathcal{E}$;
3. there is a tree structure $P_\lambda\mathcal{V}$ on $P_\lambda\mathcal{E}$.

Let us describe $P_\lambda\mathcal{V}$ precisely. First, we can extend $\Sigma : \mathcal{B} \times \mathcal{E} \to P\mathcal{E} \times \mathcal{B}$ to a map $\Sigma^\star = (\sigma^\star, \iota^\star) : \mathcal{B} \times \mathcal{E}^\star \to (P\mathcal{E})^\star \times \mathcal{B}$, where $\Sigma^\star$ is defined inductively as follows:

- on the empty word $\varnothing \in \mathcal{E}^\star$ we have $\Sigma^\star(a, \varnothing) = (\varnothing, a)$;
- now let us consider an element of $\mathcal{E}^\star$ of the form $g \cdot w$, where $g \in \mathcal{E}$ and $w \in \mathcal{E}^\star$.

Recall that $(g^*, a^*) = \Sigma(a, g)$. Then

$$\sigma^\star(a, g \cdot w) = g^* \sigma^\star(a^*, w), \quad \iota^\star(a, g \cdot w) = \iota^\star(a^*, w).$$

Replacing $\mathcal{B}$ with $C_\lambda\mathcal{B}$, we may assume that $\lambda \equiv 1$. Suppose that $\mathcal{B} = \{a, b\}$. In order to define the new vertex set $P\mathcal{V} \subset (P\mathcal{E})^\star$, we first consider the following three sets:

$$\begin{aligned}
\mathcal{V}(a) &= \{\sigma^\star(a, v) \mid v \in \mathcal{V},\ \iota^\star(a, v) = a\},\\
\mathcal{V}(b) &= \{\sigma^\star(b, v) \mid v \in \mathcal{V},\ \iota^\star(b, v) = b\},\\
\mathcal{V}(a, b) &= \{\sigma^\star(a, v) \cdot \sigma^\star(b, v) \mid v \in \mathcal{V},\ \iota^\star(a, v) = b\}.
\end{aligned}$$

Now take the union of these three sets, remove the trivial element $1 \in \mathcal{E}^\star$ form it as well as all elements of the form $g \cdot g^{-1}$, where $g \in \mathcal{E}$. The remaining set is $P\mathcal{V}$.

3.3.3 The combinatorial ivy iteration

The main objective of this subsection is to show that *geometrical* and *combinatorial* ivy graphs coincide. Now we have all the tools to formally fully define all combinatorial objects connected with combinatorial graph itself.

First we should adapt the basis changes to our purposes. Let us recall that the first basis element of the biset in our assumptions did not have intersections with $G = f^{-1}(T)$, thus all freedom of choice was related to the choice of the second basis element. Set basis $\mathcal{B} = \{a, b\}$. Fixing the first element a unchanged is equivalent to consider only basis changes associated with functions $\lambda : \mathcal{B} \to \pi$ such that $\lambda(a) = 1$. We want to set up the correspondence between elements in $\mathcal{E}$ (which, as we remember, correspond to edges of T) and the basis changes leading to the different choices of the second basis element b. We want to define a *base element* as any element $g \in \mathcal{E}$ such that $\iota(0, g) = 1$. Then the desired basis change associated with base element g is the function $\lambda_g : \mathcal{B} \to \pi$ such that $\lambda_g(b) = \sigma(0, g)$.

We define the *combinatorial ivy iteration* as the process of passing from $\mathcal{E}$ to $P_{\lambda_g}\mathcal{E}$. Informally, we can see parallels between this process and the pullback relations, which were geometrically defined in Section 3.3.1. Now we need to formally define the class of objects, on which the combinatorial ivy iteration acts:

Definition 3.3.9. *A* combinatorial ivy object *is a conjugacy class of tree-like generating sets. In the case of geometrical ivy objects we denoted ivy object, corresponding to a spanning tree T as $[T]$. Following that notation, for any tree-like generating set $\mathcal{E}$ we use the notation $[\mathcal{E}]$ for its conjugacy class. The set of all combinatorial ivy objects with $\pi = \pi_f$ is denoted as* $\mathrm{Ivy}_c(f)$.

We can equip $\mathrm{Ivy}_c(f)$ with oriented graph structure exactly as we did it with the set of geometrical ivy objects $\mathrm{Ivy}(f)$. For a pair of generating sets $\mathcal{E}$ and $\mathcal{E}' = P_{\lambda_g}\mathcal{E}$ we connect $[\mathcal{E}]$ with $[\mathcal{E}']$ by a directed edge. Now our main goal is to show that the combinatorial ivy iteration represents the topological ivy iteration. This result can be formulated as the following theorem:

Theorem 3.3.10. *There is an isomorphic embedding of* $\mathrm{Ivy}(f)$ *into* $\mathrm{Ivy}_c(f)$*. This embedding takes a class* $[T]$ *of a spanning tree* T *to the conjugacy class of the corresponding generating set* $\mathcal{E}_T$*. Let* T *and a base edge* e_b *of* T *define a dynamical tree pair* (T^*, T) *as in Section 3.3.1. If* g *is the element of* $\mathcal{E}_T$ *corresponding to* e_b*, then* T^* *corresponds to* $P_{\lambda_g}\mathcal{E}_T$*. There is a canonical tree structure* $\mathcal{V}$ *on* $\mathcal{E}_T$ *such that* $G(\mathcal{V})$ *is isomorphic to* T*. The tree structure* $\mathcal{V}^*$ *on* $\mathcal{E}_{T^*}$ *corresponding to* T^* *is obtained as* $P_{\lambda_g}\mathcal{V}$*.*

3.3.4 Proof of Theorem 3.3.10

The proof will consist of the three propositions about the images of the biset basis, vertex words and tree structures under the pullback relation. Let us start with the basis.

We know that choosing a base edge e_b in a spanning tree T we define a dynamical tree pair (T^*, T). Let $\mathcal{B} = \{[\alpha_0], [\alpha_1]\}$ be the basis of $\mathcal{X}_f(y_0)$ associated with T. Let us recall, that we choose $[\alpha_0]$ as the class of the constant path and α_1 as the path which connects y_0 with another preimage y_1 of y_0 outside of T and thus $\mathcal{B}$ is defined uniquely by this choice. Now we want to describe the basis $\mathcal{B}^*$ associated with T^*.

Proposition 3.3.11. *Let* $\mathcal{B}^* = \{[\alpha_0], [\alpha_1^*]\}$ *be the basis associated with* T^**. Then* $[\alpha_1^*] = \sigma(0, g)[\alpha_1]$*, where* g *is the element of* $\mathcal{E}_T$ *corresponding to* e_b*, and thus* $\mathcal{B}^* = C_{\lambda_g}\mathcal{B}$*.*

Proof. Set $h = \sigma(0, g)$. Let e_b' be the edge of $f^{-1}(T)$ of label 1 that is not in T^*. Then α_1^* is a path from y_0 to y_1 crossing e_b' and disjoint from $f^{-1}(T)$ otherwise. We have agreed to chose e_b' so that α_1^* approaches it from the left. Then $[\alpha_0].g = [\alpha_1^*]$ in $\mathcal{X}_f(y_0)$. Thus we can define α_1^* as $\alpha_0.\gamma_{e_b}$. On the other hand, we have $\Sigma(0, g) = (h, 1)$, therefore, $[\alpha_1^*] = [\alpha_0].g = h[\alpha_1]$, as desired.

□

Now we want to understand what happens with vertex words.

Proposition 3.3.12. *Let* x *be a vertex of* T*, and* $v \in \mathcal{E}^\star$ *be the corresponding vertex word. The vertex* x *is a critical value of* f *if and only if* $\iota^\star(0, v) = 1$*. In this case, we also have* $\iota^\star(1, v) = 0$*.*

Proof. Let $\Pi : \mathcal{E}^\star \to \pi_f$ be the evaluation map. Then $\iota^\star(\varepsilon, v) = \iota(\varepsilon, \Pi(v))$ for each $\varepsilon \in \{0,1\}$. The element $\Pi(v) \in \pi_f$ is a loop γ around x that crosses T only in a small neighborhood of x, wgich we may assume to be smooth and simple. It bounds a disk D such that $D \cap V(T) = \{x\}$. As we have discussed in Section 3.2.4, the two f-pullbacks of γ are loops or not loops depending on whether x is a critical value or not. These pullbacks are loops if and only if $\iota(\varepsilon, \Pi(v)) = \varepsilon$ for all $\varepsilon = 0, 1$. □

Now let us consider vertex word $v = a_0 \cdots a_{k-1}$ and elements $\sigma^\star(0, v) = b_0 \cdots b_{k-1}$ and $\sigma^\star(1, v) = c_0 \cdots c_{k-1}$ of $(P\mathcal{E})^\star$. Here b_i and c_i are elements of π_f, for $i = 0, \ldots,$ $k-1$. Set $\varepsilon_0 = 0$ and $\varepsilon_i = \iota^\star(0, a_0 \cdots a_{i-1})$ for $i = 1, \ldots, k$ and also set $\delta_0 = 0$ and $\delta_i = \iota^\star(1, a_0 \cdots a_{i-1})$ for $i = 1, \ldots, k$.

If $\varepsilon_k = 1$ (and then also $\delta_k = 0$), then we define the word $w(0, v) = w(1, v) \in (P\mathcal{E})^\star$ as

$$\sigma^\star(0, v)\sigma^\star(1, v) = b_0 \cdots b_{k-1} \cdot c_0 \cdots c_{k-1}.$$

If $\varepsilon_k = 0$ (and then also $\delta_k = 1$), then we define

$$w(0, v) = \sigma^\star(0, v) = b_0 \cdots b_{k-1}, \quad w(1, v) = \sigma^\star(1, v) = c_0 \cdots c_{k-1}.$$

Finally, we want to prove the following statement about the tree structures.

Proposition 3.3.13. *Suppose that the basis $\mathcal{B}$ of $\mathcal{X}_f(y_0)$ corresponds to T^*. Then the tree structure $\mathcal{V}^*$ on $P\mathcal{E}$ corresponding to T^* coincides with the set of $w(\varepsilon, v) \in (P\mathcal{E})^\star$, where ε runs through $\{0,1\}$, and v runs through $\mathcal{V}$, except that we omit $w(\varepsilon, v)$ if it is empty or it has the form $a \cdot a^{-1}$ for some $a \in \mathcal{E}$. In other words, we have $\mathcal{V}^* = P\mathcal{V}$.*

Proof. Let us consider the following elements: a vertex x of T, the corresponding vertex word v, be the oriented edges of T named $A_0, \ldots, A_{k-1}$ such that $a_i = g_{A_i}$ for $i = 0, \ldots, k-1$. Then let β_i be a smooth path disjoint from T, starting at y, and ending in a small neighborhood of x. We can choose β_i in such a way that it approaches x between A_{i-1} and A_i (for $i = 0$ we have $A_{i-1} = A_{k-1}$, $i-1$ is modulo k). After we choose the paths γ_{A_i} with $a_i = [\gamma_{A_i}]$ as $\beta_i \gamma_i \beta_{i+1}^{-1}$, where γ_i is a short path in a small neighborhood of x crossing A_i just once and transversely and disjoint from all other edges of T. The pullback β_i^ε of β_i is the pullback of β_i which starts at y_ε for $\varepsilon \in \{0,1\}$. If we suppose that x is not a critical value of f, then it has two preimages x_0 and x_1. Now we should prove that the vertex words of x_0, x_1 are $w(0,v)$, $w(1,v)$, provided that x_0 and x_1 are vertices of T^*. We may choose the notations such that β_0^0 ends near x_0. Set γ_i^ε to be the pullback of γ_i near x_ε. Since x is not critical, γ_i^ε must indeed stay near just one preimage of x. The by induction on i we get that b_i is represented by $\alpha_{\varepsilon_i}\beta_i^{\varepsilon_i}\gamma_i^0(\beta_{i+1}^{\varepsilon_{i+1}})^{-1}\alpha_{\varepsilon_{i+1}}^{-1}$ and that all $\beta_i^{\varepsilon_i}$ end near x_0. If we take the composition of all γ_i^0, we get a small loop around x_0, which is exactly a pullback of the small loop γ around x, which, in turn, is the composition of all γ_i. Thus we can see, that the oriented edges of T^* coming out of x_0 correspond precisely to non-identity elements b_i. In particular, we get that the vertex word for x_0 is $w(0,v)$ and in the similar way we can get, that the vertex word for x_1 is $w(1,v)$. We have proved the statement for non-critical x.

Now we consider the case, when x is a critical value of f, thus it has only one preimage $x_0 = x_1$. We need to show, that $w(0,v) = w(1,v)$ is the vertex word for $x_0 = x_1$ provided that x_0 is a vertex of T^*. Set γ_i^0 be the pullback of γ_i originating where $\beta_i^{\varepsilon_i}$ ends, and γ_i^1 be the pullback of γ_i originating where $\beta_i^{\delta_i}$ ends. In this case b_i is represented by $\alpha_{\varepsilon_i}\beta_i^{\varepsilon_i}\gamma_i^0(\beta_{i+1}^{\varepsilon_{i+1}})^{-1}\alpha_{\varepsilon_{i+1}}^{-1}$ and c_j is represented by $\alpha_{\delta_j}\beta_j^{\delta_j}\gamma_j^1(\beta_{j+1}^{\delta_{j+1}})^{-1}\alpha_{\delta_{j+1}}^{-1}$. Looking again on the composition of all γ_i^0, we still see, that it is a pullback of γ, but now not a loop, but just a "half" of a loop. And the other half is the composition of all γ_i^1, corresponding to the other pullback of γ. Thus we see that oriented edges of T^* coming out of x_0 correspond precisely to non-identity elements b_i or non-identity elements c_j. This means that $w(0,v) = w(1,v)$ is the vertex word for x_0.

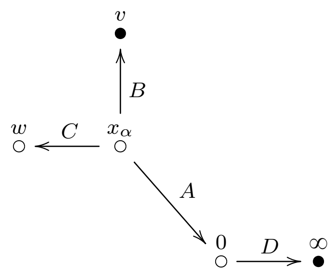
Let us note, that any vertex of T^* is mapped to a vertex of T and thus it follows that any vertex of T^* can be obtained as described above. If $w(\varepsilon,v)$ is empty, then the corresponding point x_ε of $f^{-1}(T)$ does not belong to T^*. If $w(\varepsilon,v)$ has the form $a \cdot a^{-1}$, then x_ε belongs to an edge of T^* corresponding to a. Conversely, if x_ε is not a vertex of T^*, then this may be due to one of the following reasons. Firstly, we may have $x_\varepsilon \notin T^*$, then $w(\varepsilon,v)$ is empty. Secondly, x_ε may belong to some edge of T^*. In this case, $w(\varepsilon,v)$ must have the form $a \cdot a^{-1}$, where $a \in P\mathcal{E}$ corresponds to this edge. □

Now having proved Proposition 3.3.11, Proposition 3.3.12 and Proposition 3.3.13 we get Theorem 3.3.10 from three of them together.

3.3.5 Examples

In this subsection we show some examples of the Ivy graph, obtained with the help of a computer.

• *Rabbit polynomial.* Considering the example of the rabbit polynomial $p(z) = z^2 + c$, where c is chosen such that 0 is periodic of period 3, and $\mathrm{Im}(c) > 0$, we want to explain the usage of the Ivy algorithm. In this case finding invariant tree is trivial even without the Ivy algorithm, but we still can apply it. Let us consider the following tree T, which is invariant:



Let us denote the elements of π_f corresponding to the edges of T as a, b, c, d, where loop e is exactly a loop g_E for any edge $E = A, B, C, D$ constructed as in Subsection 3.2.2 (the choice of the basepoint y of $\mathbb{S}^2 - P(f)$ is free). Thus we get $\mathcal{E}_T = \{id, a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}\}$.

Let us set D as the base edge. Then the labels are

$$\ell(A) = \ell(B) = \ell(C) = 1, \quad \ell(D) = 0.$$

Considering the unique pseudoaccess we also get the following signatures: the edges A and D have signature $(0, 1)$, the edge B has signature $(1, 0)$, the edge C has signature $(1, 1)$.

From Theorem 3.2.8 we get that the map $\Sigma : \{0, 1\} \times \mathcal{E}_T \to \mathcal{E}_T \times \{0, 1\}$ is the

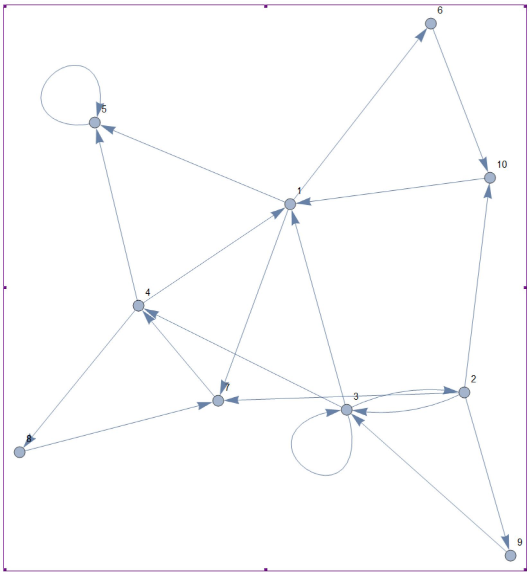


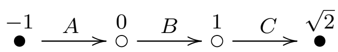
Figure 3-1: The pullback invariant subset of Ivy(f) containing $[T]$, where f is the rabbit polynomial, and T is the invariant spanning tree for f obtained by connecting the Hubbard tree to ∞. This subset consists of 10 elements. Vertex 5 represents an invariant spider in in the sense of Hubbard–Schleicher [HS94], and vertex 3 represents T.

following (since $a = b^{-1}c^{-1}$ we use only b, c and d as generators):

	a	b	c	d	a^{-1}	b^{-1}	c^{-1}	d^{-1}
0	$d\,1$	$a\,1$	$b\,0$	$d\,1$	$c^{-1}1$	$d\,1$	$b^{-1}0$	1
1	$c\,0$	$d^{-1}0$	1	0	$d^{-1}0$	$a^{-1}0$	1	$d^{-1}0$

Applying ivy iteration, we get a pullback invariant subset of Ivy(p) consisting of 10 ivy objects, see Figure 3-1.

•*Simple capture of the basilica at* $\sqrt{2}$. Let us consider the simple capture f of the basilica polynomial $p(z) = z^2 - 1$ at the point $\sqrt{2}$. Then as in the previous case f has an obvious invariant tree T presented below:



Applying the Ivy iteration to it, we get the pullback invariant subset of Ivy(f) of order 40 containing $[T]$. Within this subset, there are three invariant spanning trees for f, up to homotopy, see Figure 3-2.

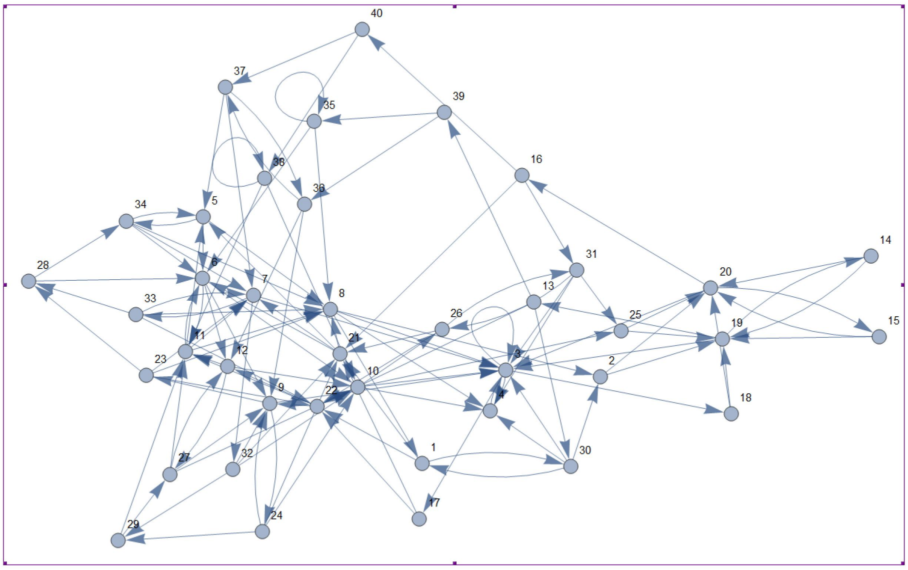
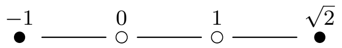
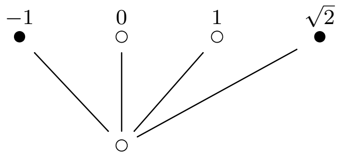


Figure 3-2: The pullback invariant subset of 40 elements in Ivy(f) containing $[T]$. Here f is a simple capture of the basilica at $\sqrt{2}$. Vertices 3, 35 and 38 represent invariant spanning trees for f.

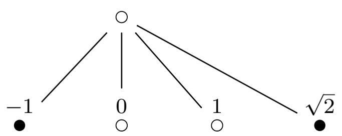
Vertex 3 corresponds to the invariant spanning tree



Vertex 35 corresponds to the invariant spanning tree



Finally, vertex 38 corresponds to the invariant spanning tree



• *A capture of the Chebyshev polynomial.* We consider a simple capture of the Chebyshev polynomial $p(z) = z^2 - 2$ at $\sqrt{2}$. In this example, unlike in the previous two, an invariant spanning tree is not known a priori. We start Ivy iteration from

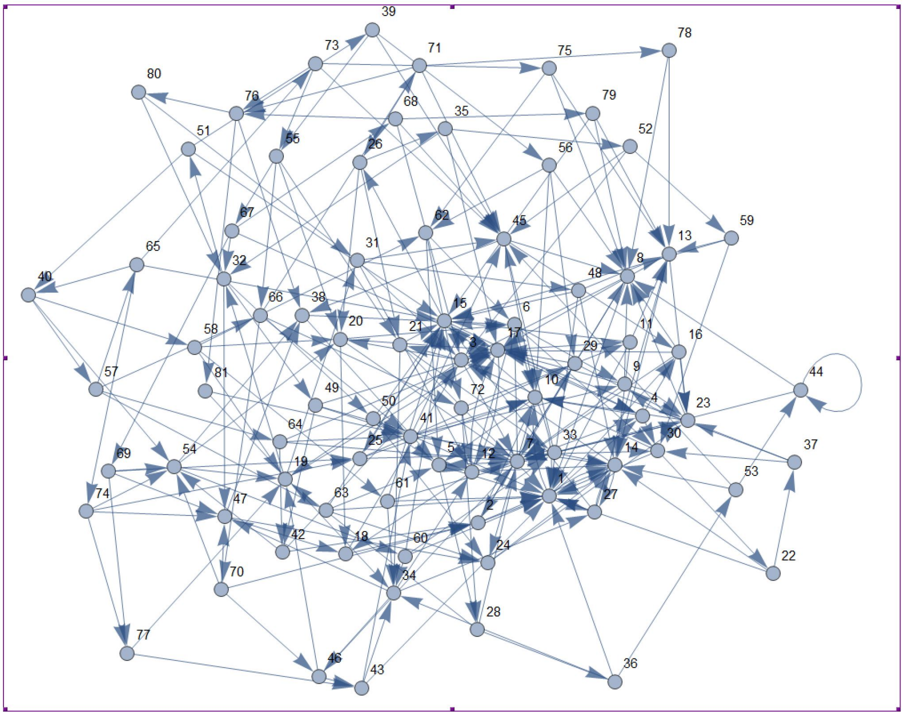
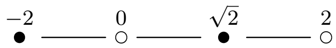


Figure 3-3: The pullback invariant subset of $\mathrm{Ivy}(f_u)$ containing $[T]$. There are 81 objects in this subset. Vertex 44 represents an invariant ivy object.

the extended Hubbard tree T with vertices in $\{\sqrt{2}, 0, -2, 2\}$:



With the help of a computer, we found a pullback invariant subset of size 81 in $\mathrm{Ivy}(f)$ containing $[T]$, see Figure 3-3. This subset contains an invariant ivy object. Thus, we found an invariant (up to homotopy) spanning tree for f.

Chapter 4

Parametrization of Zakeri slices

In this chapter we present a result about the parametrization of *slices* of cubic polynomials with fixed multiplier $|\lambda| \leqslant 1$ with connected Julia set. This result was reported in [BOST22].

4.1 Renormalizable and non-renormalizable polynomials

Let us precisely describe which class of polynomials we are interested in. For a cubic polynomial P write $[P]$ for its affine conjugacy class. We can assume the fixed point of the polynomial to be 0 since we always can find a conjugation that maps a fixed point to 0. Let us assume the multiplier $|\lambda| \leqslant 1$ of P fixed. We denote the space of polynomials with fixed $|\lambda| \leqslant 1$ as $\mathcal{F}_\lambda$. Consider all polynomials P with $P(0) = 0$ and consider all linear conjugacies fixing 0. Then the corresponding conjugacy classes form the space $\mathbb{C}_\lambda$. For a cubic polynomial $P(z) = \lambda z + \ldots$, let $[P]_0$ be its class in $\mathbb{C}_\lambda$.

The dynamical behavior of polynomials in these classes crucially depends on the behavior of its critical points. In the case of quadratic polynomials with an attracting fixed point a critical point necessary had to belong to the *basin of attraction* (defined in Section 2.6) of 0, but now there are 2 critical points and belonging to that basin is guaranteed only for one of them. The condition that none of critical points escapes to infinity is equal to the condition that the Julia set is connected due to Fatou.

Let us consider the case when one critical point escapes to infinity so Julia set is not connected. This case is well studied (can be found in [BH01]). It turns out that in this case the polynomial is *immediately renormalizable*, i.e. for a polynomial P a map $P : U \to V$ is *quadratic-like* (defined below) for some open U and V such that $\overline{U} \subset V$. In [IK12] there was introduced an even more general renormalization scheme, allowing to find copies of $\mathcal{M}_2 \times \mathcal{M}_2$ in the cubic connectedness locus (defined in Section 2.9.2). Let us assume that non-escaping point is w_1 and escaping one is w_2. In the case of non-repelling multiplier at point 0 one critical point should belong to the connected component $K_0(P)$ of the filled Julia set containing 0. Now we have to show that the restriction of P to this component is *quadratic-like*. The following definition is due to Douady-Hubbard [DH85]:

Definition 4.1.1. *A* polynomial-like map of degree d *is a triple* (U, V, f) *where* U *and* V *are open bounded subsets of* $\mathbb{C}$ *isomorphic to discs, with the following properties:*

1)$\overline{U} \subset V$*;*

2)$f : U \to V$ *is proper;*

3) $deg_U f = d$*.*

For $d = 2$ *a triple* (U, V, f) *is called* quadratic-like*.*

Our goal now is to construct subsets U and V in such a way that $1) - 3)$ hold and that $J_0 \subset U$. The proof in details can be found in [BH01].

We will need notion of the *potential function*. This function shows how fast some point escapes to infinity in the logarithmic scale:

Definition 4.1.2. $g_b(z) = \lim_{n\to\infty} \frac{1}{3^n} \log^+ |f_b^{\circ n}(z)|$*, where* $\log^+$ *is the supremum of* $\log$ *and* 0*.*

Then we define Green function $G : \mathbb{C} \to \mathbb{R}^+$:

Definition 4.1.3. $G(b) = \sup_{\{\omega | f'_{\lambda,b}(\omega)=0\}} g_b(\omega)$

Sketch of the proof: We will chose V as the following set $V = \{z \in \mathbb{C} | g_b(z) < 3G(b)\}$ and U as the connected component of its preimage containing 0. Then the restriction of the initial map on it is is a ramified covering of degree 2 (and one-to-one map to all the other components). And from the properties of the Green function it is possible to show, that $K_0(P)$ should be contained in V.

Now we can apply the *Straightening Theorem*([DH85]) to our map :

Theorem 4.1.4. ***(Straightening Theorem)*** *If $f : U \to V$ is a polynomial-like mapping of degree d, then there exist:*

1) a polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree d,

2) a neighborhood W of the filled-in Julia set $K(P)$ such that the mapping $P : P^{-1}(W) = W' \to W$ is a polynomial-like map,

3) a quasiconformal homeomorphism $\varphi : V \to W$ with $\varphi(U) = W'$, such that $\bar{\partial}\varphi = 0$ almost everywhere on $K(f)$ and such that on U $\varphi \circ f = P \circ \varphi$.

This case of renormalizable polynomials under the additional assumption that P is a hyperbolic polynomial (a polynomial is said to be *hyperbolic* if and only if all its critical points are attracted to attracting periodic orbits) was studied in [IK12, SW20]. So we will be mainly interested in the non-renormalizable case. In this case we have cubic polynomials with bounded orbits of critical points (and thereby with connected Julia sets). We will focus on the particular subset of such hyperbolic polynomials whose Julia sets are Jordan curves:

$$\mathcal{P}^0 = \{P \text{ hyperbolic } | P(0) = 0, P'(0) = \lambda, |\lambda| < 1, J(P) \text{ is a Jordan curve}\}$$

and let us denote its closure as $\mathcal{P} = \overline{\mathcal{P}^0}$.

This set is called the *principal hyperbolic component.* It was shown in [BOPT16] that all P in $\mathcal{P}$ satisfy the following properties:

Theorem 4.1.5. *If $P \in \mathcal{P}$ then:*

1) $|\lambda| \leqslant 1$,

2) there are no repelling cutpoints in $J(f)$,

3)all non-repelling periodic points, except possibly 0*, have multiplier* 1.

Write $\mathcal{C}_\lambda \subset \mathbb{C}_\lambda$ for the connectedness locus in the λ-slice $\mathbb{C}_\lambda$. A central part of $\mathcal{C}_\lambda$ is the set $\mathcal{P}_\lambda$ of all polynomials in $\mathcal{C}_\lambda$ that lie in the closure of the principal hyperbolic component.

4.2 Model and the Main Theorem

Consider a quadratic polynomial $Q_\lambda(z) = \lambda z(1 - z/2)$. Then it has multiplier λ of the fixed point 0 and the finite critical point of Q is 1. Suppose that either $|\lambda| < 1$ or $\lambda = e^{2\pi i\theta}$, where $\theta \in \mathbb{R}/\mathbb{Z}$ is of bounded type. We denote the Julia set of Q_λ by $J(Q_\lambda)$, its filled Julia set by $K(Q_\lambda)$, and its Siegel disc (it exists since $\lambda = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R}$ of bounded type) by $\Delta(Q_\lambda)$.

For a subset $X \subset \mathbb{C}$, set $\overline{X}$ to be the closure of X, and ∂X to be the boundary of X. Let $\psi = \psi_{\Delta(Q_\lambda)} : \mathbb{D}(r) \to \Delta(Q_\lambda)$ be a Riemann map. Here $\mathbb{D}(r) = \{z \in \mathbb{C} \mid |z| < r\}$ is the open disk of radius r around 0. The map ψ can be normalized so that $\psi(0) = 0$ and $\psi'(0) = 1$. Then r is uniquely defined and is called the *conformal radius of* $\Delta(Q_\lambda)$. For α is of bounded type $\overline{\Delta}(Q_\lambda)$ is locally connected, this is a classical result of Douady–Ghys–Herman–Shishikura, see [Dou87, Her87, Swi98]. It follows that the Riemann map ψ extends to the boundary. Abusing the notation, we will write $\psi : \overline{\mathbb{D}}(r) \to \overline{\Delta}(Q_\lambda)$ for the continuous map thus obtained. The map ψ conjugates the rigid rotation by angle α with the restriction of Q_λ to $\overline{\Delta}(Q_\lambda)$. The critical point $\omega_Q = 1$ is in $\partial\Delta(Q_\lambda)$ and also the critical value $c_Q = Q_\lambda(\omega_Q)$. Set $u_Q = \psi^{-1}(c_Q)$.

Definition 4.2.1 (The model space $\tilde{K}(Q_\lambda)$)**.** *Based on $K(Q_\lambda)$, we define a new topological space in order to model a certain subset in a parameter slice of cubic polynomials. The new topological space $\tilde{K}(Q_\lambda)$ will be a quotient space of $K(Q_\lambda)\backslash\Delta(Q_\lambda)$ with respect to an equivalence relation $\sim$. By definition, classes of $\sim$ are singletons outside of $\partial\Delta(Q_\lambda)$. Two points on $\partial\Delta(Q_\lambda)$ are $\sim$-equivalent if their ψ-pullbacks are symmetric with respect to the diameter of $\mathbb{D}(r)$ connecting u_Q with $-u_Q$. Now the equivalence relation $\sim$ is well defined. We set*

$$\tilde{K}(Q_\lambda) = (K(Q_\lambda)\backslash\Delta(Q_\lambda)) / \sim .$$

This is a compact connected Hausdorff space.

The following theorem allows to realize $\tilde{K}(Q_\lambda)$ as a plane continuum.

Theorem 4.2.2 (Moore, [Mr25])**.** *Let $\sim$ be a closed equivalence relation on the* 2-

sphere S^2 such that all equivalence classes are connected and non-separating, and not all points are equivalent. Then the quotient space $S^2/\sim$ is homeomorphic to the 2-sphere.

Here, a closed equivalence relation on S^2 is defined as an equivalence relation on S^2, which is a closed subset of $S^2 \times S^2$ A set $A \subset S^2$ is called non-separating if the complement $S^2 \backslash A$ is connected.

Observe that there is no dynamics on $\tilde{K}(Q_\lambda)$.

Informally we can say that there is a conformal mapping from the Siegel disc to a round disc; we know that there is a critical point on the boundary of the Siegel disc; then we are denoting as α the angle between conformal image of any point on the boundary and the image of the critical point. We are identifying points with the angles α and $-\alpha$. We denote this reglued filled Julia set as $\tilde{K}(Q_\lambda)$.

The next Property D shows the existence of a partially defined correspondence between the dynamical planes of P and Q. Here we use the notion of a monotone continuous map: a continuous map $\eta : X \to Y$ between two compacta is said to be *monotone* if, for every connected subset $B \subset Y$, the set $\eta^{-1}(B) \subset X$ is connected.

Property D. *For any cubic polynomial P with $[P]_0 \in \mathcal{P}_\lambda$, there exist a full P-invariant continuum $X(P)$ containing both critical points of P and a continuous map $\eta_P : X(P) \to K(Q)$ that semi-conjugates $f|_{X(P)}$ with $Q|_{\eta_P(X(P))}$. If both critical points of P are in the Julia set, then the map η_P is monotone.*

Now we can formulate the Main Theorem.

Theorem 4.2.3 (Main Theorem)**.** *Suppose that $\theta \in \mathbb{R}/\mathbb{Z}$ is of bounded type, and $\lambda = e^{2\pi i\theta}$. Let $Q = Q_\lambda$ be a quadratic polynomial with a fixed point of multiplier λ. Then there is a continuous map $\Phi_\lambda : \mathcal{P}_\lambda \to \tilde{K}(Q)$ taking $[P]_0$ to the η_P-image of some critical point of P.*

4.2.1 Parametrization

As we have mentioned before, we are free to choose a representing polynomial from the conjugacy class. To lighten the future work with coordinates we choose the polynomial in the following way: let us keep multiplier λ fixed and fixed point to be 0. We want critical points to be 1 and c. As we have mentioned above, one critical point necessary belongs to the boundary of the Siegel disc. We can choose the conjugation such that this point is 1. The described polynomial has a following form:

$$P_{c,\lambda}(z) = P_c(z) = \lambda z\left(1 - \frac{1}{2}\left(1 + \frac{1}{c}\right)z + \frac{1}{3c}z^2\right)$$

4.3 Julia sets structure

We want to understand the connection between cubic and quadratic Julia sets. Let us recall that a *pullback* of a connected set A under a continuous map $f : \mathbb{C} \to \mathbb{C}$ as a connected component of $f^{-1}(A)$.

Let us recall that we are interested in $f : \mathbb{C} \to \mathbb{C}$ in $\mathcal{C}_\lambda$, or $f = Q_\lambda$ with fixed $\lambda = e^{2\pi i\theta}$. Above we have chosen such parametrization that 1 is a critical point of f.

4.3.1 Bubbles

Here we assume that either $f : \mathbb{C} \to \mathbb{C}$ is in $\mathcal{C}_\lambda^c$, or $f = Q_\lambda$, where $\lambda = e^{2\pi i\theta}$ is fixed. *An iterated pullback* of A under f is by definition an f^n-pullback of A for some $n > 0$. We can start constructing Julia set by taking pullbacks of the Siegel disc $\Delta(Q_\lambda)$.

Definition 4.3.1. Bubbles *of f are iterated pullbacks of $\Delta(f)$. Let A be a bubble of f, and let n be the smallest integer with $f^n(A) = \Delta(f)$. Such n is called the* generation *of A and denoted by* Gen(A).

In our assumptions on f bubbles are open Jordan disks. Now let $U \subset \mathbb{C}$ be an open topological disk. It is equipped with a following data:

- a *center* $a \in U$;
- a *radius* $r_U \in (0, \infty)$;
- a *base point* $b \in \partial U$ accessible from U.

Then an open topological disk U together with these data is called a *framed domain* and these data is called a *framing* of U.

We can equip the bubbles with the following structure:

Definition 4.3.2 (Polar coordinates, internal rays). *Let U be a framed domain with center $a \in U$, root point b, and radius r_U. For this domain we can consider the Riemann map $\psi_U : \mathbb{D}(r_U) \to U$ such that $\psi(0) = a$ and $\lim_{u \to r_U} \psi_U(u) = b$ with u converging to r_U radially. Then a point $z \in U$ has the form $\psi_U(\rho_z e^{2\pi i \theta_z})$ for some $\rho_z \in [0, r_U)$ and $\theta_z \in \mathbb{R}/\mathbb{Z}$. The* polar radius function *is by definition the function $z \mapsto \rho_z$ on U. We always extend this function (keeping the notation) to $\overline{U}$ by setting $\rho_z = r_U$ for all $z \in \partial U$. The* polar angle function *is by definition the function $z \mapsto \theta_z$ on $U \backslash \{a\}$. Note that this function is undefined when $\rho_z = 0$. If U is a Jordan disk (and only in this case), we extend the polar angle function to $\overline{U}$ by continuity. Then, for $z \in \partial U$, the angle θ_z is determined by the relation $z = \overline{\psi}_U(r_U e^{2\pi i \theta_z})$. Given any $\alpha \in \mathbb{R}/\mathbb{Z}$ define the* internal ray *$R_U(\alpha)$ as the set $\{z \in U \mid \theta_z = \alpha\}$. Say that $R_U(\alpha)$* lands *at a point $w \in \partial U$ if w is the only point in $\overline{R}_U(\alpha) \backslash U$. If U is a Jordan disk, then every internal ray $R_U(\alpha)$ lands at the point $\overline{\psi}_U(r_U e^{2\pi i \alpha})$.*

Then for a bubble A with $\mathrm{Gen}(A) = n$ we have the following:

- the *center* o_A of A is the only iterated preimage of 0 in A;
- the base point of ∂A – the point b_A such that $f^n(b_A) = f^n(1)$;
- internal rays of A are defined as in the previous Definition 4.3.2.

We can see that if z has polar coordinates ρ and α, then $f(z)$ has polar coordinates ρ and $\alpha + \theta$

4.3.2 Correspondence between cubic and quadratic Julia sets

For a cubic polynomial $P = P_{c,\lambda}$ we want to construct P-invariant continua $X(P)$ as in Property D. Let us first explain informal idea of this set and then provide tools for the formal definition. When we take all pullbacks of the Siegel disc we get chains of bubbles. There is natural homeomorphism between the Siegel disc of $P_{c,\lambda}(z)$ and that of Q_λ. Since we are working with the maps with the same λ, we denote them as P and Q correspondingly. Let us recall that we assume that

$\lambda = e^{2\pi i\alpha}$, where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is of bounded type and thus under our assumption the Siegel discs $\overline{\Delta}(Q)$ and $\overline{\Delta}(P)$ are Jordan disks. The statement about $\overline{\Delta}(P)$ is due to [Zak99]. Recall that there exists a holomorphic isomorphism $\psi_Q : \mathbb{D}(r_Q) \to \Delta(Q)$ with $\psi_Q(0) = 0$ and $\psi'_Q(0) = 1$; then ψ_Q conjugates the map $z \mapsto \lambda z$ on $\mathbb{D}(r_Q)$ with $Q : \Delta(Q) \to \Delta(Q)$. Here r_Q is the conformal radius of $S(Q)$. Similarly, we set ψ_P to be the corresponding Riemann map for the Siegel disk of P. We write r_P for its conformal radius. Since $\overline{\Delta}(Q)$ is locally connected, the map ψ_Q extends to a continuous map $\overline{\psi}_Q : \overline{\mathbb{D}}(r_Q) \to \overline{\Delta}(Q)$. By continuity, we have $\overline{\psi}_Q(\lambda z) = Q(\overline{\psi}_Q(z))$ for all $z \in \mathbb{D}(r_Q)$. Since $\overline{\Delta}(Q)$ is a Jordan disk, the map $\overline{\psi}_Q$ is a homeomorphism. Similarly, we define a homeomorphism $\overline{\psi}_f$ between $\mathbb{D}(r_P)$ and $\overline{\Delta}(P)$ that conjugates the rotation by α with the restriction of P to $\Delta(P)$. Recall that in the definition of the model space $\tilde{K}(Q)$ (Definition 4.2.1) u_Q was defined as a point of the circle $\mathbb{S}^1(r_Q)$ such that $\overline{\psi}_Q(u_Q)$ is the critical value of Q. Similarly, define u_P as a point of the circle $\mathbb{S}^1(r_P)$ such that $\overline{\psi}_P(u_P)$ is a critical value of P. Note that the boundary of $\Delta(P)$ may contain two critical points and thus two critical values. In this case, the choice of u_P is ambiguous. We make an arbitrary choice. We can now define a homeomorphism ϕ between $\overline{\Delta}(P)$ and $\overline{\Delta}(Q)$ as follows:

$$\phi(z) = \psi_Q\left(\frac{u_Q}{u_f}\psi_P^{-1}(z)\right).$$

Observe that the multiplication by u_Q/u_f commutes with rotations about 0. Therefore, ϕ conjugates $Q : \overline{\Delta}(P) \to \overline{\Delta}(P)$ with $f : \overline{\Delta}(Q) \to \overline{\Delta}(Q)$. On the other hand, since multiplication by u_Q/u_f takes u_f to u_Q, the map ϕ takes *some* critical value of f to the only finite critical value of Q.

Let us assume that there is only one critical point of P_λ on the boundary of its Siegel disc (for the parameter plane it means that our polynomial does not belong to the *Zakeri curve* (Definition 2.9.4) Then the homeomorphism can be extended to the bubbles of generation 1. If there is no critical point in these bubbles, we can continue iterating and extend the homeomorphism.

This construction gives rise to the following motivation: since the we can parametrize polynomials from the cubic λ-slice by the critical point, we can assign to them the "image" of the critical point c in $\tilde{K}(Q)$. There is a problem of what we understand as the image since there is no homeomorphism at the critical point. We want to be able to encode the points on bubbles in such a way, that this encoding will be the same in both quadratic and cubic case.

4.4 Encoding points in $K(P)$ and $K(Q)$

We want to define a path from 0 to any point in the bubble. It can be chosen in many ways. But we want this path to consist of the segments of arcs connecting bubble centers with their points of intersection. We will call such paths *legal arcs.*

4.4.1 Legal arcs

Let us give a formal definition. Let A be a bubble for a cubic polynomial $P_c \in \mathcal{C}_\lambda$ or for Q_λ.

Definition 4.4.1 (Legal arcs)**.** *Consider an oriented (directed) topological arc $I \subset K(f)$. Suppose that I° is an open dense subset of I such that the following holds:*

1. *the set $I \backslash I^\circ$ can accumulate only at the terminal point of I;*
2. *each component of I° is contained in one bubble A and coincides with a component of $(A \backslash \bigcup_{n\geqslant 0} f^{-n}(0)) \cap I$.*
3. *the polar angle function is defined and constant on each component of I°;*
4. *$P^n(I)$ is not separated by c for $n \geqslant 0$.*

Then I is called a legal arc *(see Fig. 4-1). Let α_0, ..., α_k, ... be the values of the polar angle on I° taken in the order they appear on I°. A linear order of α_is is well defined since I is oriented. The finite or infinite sequence $(\alpha_0, \dots, \alpha_k, \dots)$ is called the* (polar) multi-angle *of I.*

Let us start with considering legal arcs for quadratic Julia set $K(Q)$. We want to show that there exists a legal arc to any point of $K(Q)$. Moreover, we can describe its multi-angle:

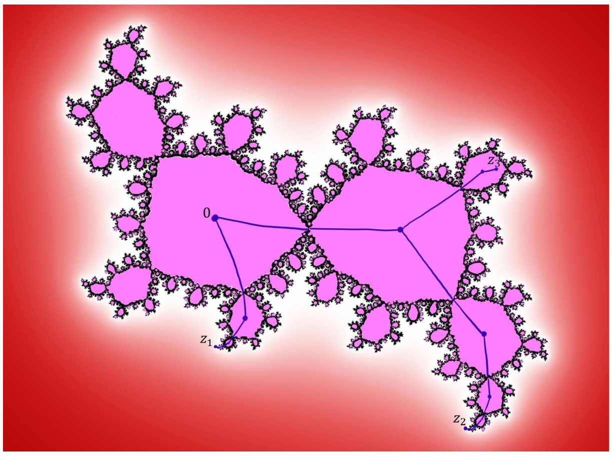


Figure 4-1: A schematic illustration of legal arcs in the dynamical plane of Q_λ. Three legal arcs are shown, connecting 0 with points z_1, z_2, and z_3. Observe that the legal arcs from 0 to z_2 and from 0 to z_3 have an initial segment in common.

Lemma 4.4.2. *For any point $z \in K(Q)$ (we remind that $Q = \lambda z(1 - z/2) = e^{2\pi i\theta} z(1 - z/2)$) there exists a legal arc I_z connecting 0 with this point. Then each term α_i of its multi-angle $(\alpha_0, \ldots, \alpha_k, \ldots)$, except possibly the last term, has the form $\alpha_i = -m_i\theta$, where m_i are nonnegative integers such that $m_{2i+1} = m_{2i}$ and $m_{2i+2} > m_{2i+1}$.*

Following the terminology of the legal arcs we also can define corresponding angles:

Definition 4.4.3. *We say that angle of the form $-m\theta$ ($m \in \mathbb{N}\backslash 0$) is a* legal angle.

Then we call a sequence $(\alpha_0, \ldots, \alpha_k, \ldots)$ with the terms satisfying the properties of the multi-ange listed in Lemma 4.4.2 a legal sequence of angles.

Now we can provide the proof of the lemma:

Proof. There are 2 cases related to the position of the point $z \in K(Q)$. It may belong to the closure of a bubble A of Q or it may not belong. Then in the second case it can be approached by some sequence of bubbles in the Hausdorff metric.

Let us start with the first case: $z \in \overline{A}$. The existence of multi-angle can be proved by induction on $n = \text{Gen}(A)$. For $n = 1$ the point z belongs to the closure of the Siegel disc $\overline{\Delta}(Q)$. Then the legal arc I_z is just a segment or the closure of some internal ray of $\Delta(Q)$, defined by θ_z and ρ_z.

Now let us assume that $n > 1$. It follows that the arc I_z has to intersect the boundary of the Siegel disc $\partial\Delta(Q)$. Denote this point as x. It maps to the critical point 1 after some iterations of Q. Set the number of iterations as m_0 (i.e. $Q^{m_0}(x) = 1$), then the polar angle of x, corresponding to α_0 is $-m_0\theta$. Let us consider the arc $Q^{m_0}(I_z) = I_{Q^{m_0}(z)}$, since its point of intersection with $\partial\Delta(Q)$ is 1 it contains the internal ray $R_{\Delta(Q)}(0)$. Then both initial segments of considered arc map onto $R_{\Delta(Q)}(\theta)$ and thus the multi-angle of $Q^{m_0}(z)$ starts with 0, 0. Assume by induction that the multi-angle of $Q^{m_0+1}(z)$ is $(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_k)$, where $\tilde{\alpha}_i = -\tilde{m}_i\theta$ for $2 \leqslant i < k$. Then the multi-angle of z is $(\alpha_0, \ldots, \alpha_k)$, where $\alpha_0 = \alpha_1 = -m_0\theta$ and $\alpha_i = \tilde{\alpha}_i - (m_0 + 1)\theta = -m_i\theta$ with $m_i = \tilde{m}_i + m_0 + 1$ for $i \geqslant 2$ and $2 \leqslant i < k$. Thus I_z is a legal arc.

Now we consider the second case when $z \notin \overline{A}$ for any bubble. As we have mentioned in the begging, in this case z can be approached by some sequence of pairwise different bubbles $A_0, \ldots, A_k, \ldots$. Set $A_0 = \Delta(Q)$ and the points of bubbles intersection $A_i \cap A_{i+1} = \{z_i\}$. Then we define I_z as the closure of the union of all legal arcs from 0 to z_i, where z_i defined as above. There is an an infinite legal sequence of angles such that the multi-angle z_i is an initial segment of this sequence. We define the multi-angle of z as this infinite sequence. □

4.4.2 Definition of $\eta_P : Y(P) \to K(Q)$

In the proof of Lemma 4.4.2 we have defined sequences of bubbles (both finite an infinite), corresponding to all points $z \in K(Q)$. We will call these sequences *bubble rays*. We have seen that these sequences exist for all z in the filled quadratic Julia set $K(Q)$. Now let us consider the filled Julia sets of cubic polynomials $P = P_c \in \mathcal{C}_\lambda$ in which we are interested.

We can extract the subsets of these Julia sets, whose every point is connected to 0 by a legal arc. Formally, we denote the set of all points $z \in K(P)$ for which there is a legal arc I_z from 0 to z as $Y(P)$. Now we can define the correspondence between $Y(P)$ and $K(Q)$. We define the map $\eta_P : Y(P) \to K(Q)$ as a map which takes $z \in Y(P)$ to a unique point $w = \eta_P(z)$ with the same multi-angle and polar radius. If the multi-angle of z is infinite we set the polar radius ρ_z to be ∞.

We have to show that the map $\eta_P : Y(P) \to K(Q)$ is well-defined, more precisely that the legal arc for any z on the dynamical plane is defined uniquely. To show it we will need some additional concepts and notation corresponding to bubbles.

4.4.3 Bubble rays and bubble chains

We can distinguish the bubbles depending on whether the critical point belongs to it or not: Let us consider a bubble A of $P = P_c \in \mathcal{C}_\lambda^c$ of generation n and a Sigel disc $\Delta(P)$ of P.

Definition 4.4.4 (Root point and multi-angle of a bubble)**.** *The bubble A can be of three types:*

1. *if a map $P^n : A \to \Delta(P)$ is one-to-one, then A is called* off-critical*;*
2. *if critical point c belongs to A, then A is called* critical*;*
3. *if A is a pullback of a critical bubble, it is said to be* precritical.

Moreover, for any bubble A we can define its root point *$r(A)$. When A is off-critical, the root point is uniquely defined by the formula $P^{n-1}(r(A)) = 1$. When A is critical or precritical, there are legal paths from 0 to some points in A. All these paths intersect the boundary of A at the same point; this point is by definition the root point $r(A)$. There are two points z', $z'' \in \partial A$ such that $P^{n-1}(z') = P^{n-1}(z'') = 1$, and the point $r(A)$ is one of them.*

We define the multi-angle *of A as the multi-angle of the root point of A.*

Now we want to set not only the correspondence between points in $K(P)$ connected with 0 by legal arcs and points in $K(Q)$, but also between the bubbles:

Definition 4.4.5 (Legal bubbles and bubble correspondence)**.** *A bubble A of P with $A \cap Y(P) \neq \varnothing$ is called* legal*. Thus, A is legal if and only if $r(A) \in Y(P)$, and $P^i(r(A)) \neq c$ for $i < \mathrm{Gen}(A)$. If a legal bubble A is off-critical, then $\overline{A} \subset Y(P)$. Clearly, $\eta_P(A \cap Y(P))$ lies in a unique bubble A_Q of Q. Say that A and A_Q* correspond *to each other. This correspondence between some bubbles of P and all bubbles of Q is called the* bubble correspondence*. By definition, if A is a legal* *bubble of P, then $P(A)$ is also a legal bubble of P. Moreover, if A corresponds to A_Q, then $P(A)$ corresponds to $Q(A_Q)$.*

Here we give the formal definition of bubble rays, which we have mentioned before in the proof of Lemma 4.4.2. As we have described in that proof, the bubble rays are related to legal arcs. Let A be a legal bubble and z a point in it, different from the root point.

Definition 4.4.6 (Bubble rays, bubble chains, core curves)**.** *Consider a legal arc I_z from 0 to z. Then we can write down the ordered sequence of bubbles $A_0 = \Delta(P)$, ..., $A_n = A$ through which I_z passes exactly in this order. Then the sequence A_0, ..., A_n is called a* bubble chain (to z)*. A* bubble ray *is a sequence $\mathcal{A} = (A_0, A_1, \dots)$ of legal bubbles A_i such that A_0, ..., A_n is a bubble chain, for every finite n. Set $\bigcup \mathcal{A} = \bigcup_{i \geqslant 0} A_i$. Bubble chains and bubble rays for $Q = Q_\lambda$ are defined similarly. The* core curve *of $\mathcal{A}$ is defined as the union of I_{z_i}, where $z_i \in \overline{A}_i \cap \overline{A}_{i+1}$. (Note that $I_{z_i} \subset I_{z_j}$ for $i < j$.)*

Now we can formulate the main proposition of this subsection, which shows that the function $\eta_P : Y(P) \to K(Q)$ from the previous subsection is well-defined:

Proposition 4.4.7. *For z in the dynamical plane of P, there is at most one bubble chain or a bubble ray to z.*

Proof. Suppose that $\mathcal{A}'$ and $\mathcal{A}''$ are different bubble rays or bubble chains to z. If $\mathcal{A}'$ is a bubble ray, then set I' to be its core curve; otherwise set $I' = I_z$. The arc I'' is defined similarly, with $\mathcal{A}'$ replaced by $\mathcal{A}''$. If $\mathcal{A}' \neq \mathcal{A}''$, then there is a bounded open set U in $\mathbb{C}$ whose boundary is contained in $I' \cup I''$. By the Maximum Modulus Principle, the sequence P^n is bounded on U, hence equicontinuous. We conclude that U is in the Fatou set, that is, U is in a single bubble, a contradiction. □

4.4.4 Landing of bubble rays

Further we will extend the map $\eta_P : Y(P) \to K(Q)$ to bigger subsets of $K(P)$. For being able to do so, we should understand the possible limits of bubble rays. Here we introduce some concepts, following the terminology of external rays, defined in Section 2.7.

Definition 4.4.8 (Landing bubble rays)**.** *Let $\mathcal{A} = (A_i)$ be a bubble ray for P or Q. Then we say that $\mathcal{A}$* lands *at a point z or that $\mathcal{A}$ is a bubble ray* to *z if $\{z\}$ is the upper limit of the sequence A_i, that is*

$$\{z\} = \bigcap_i \overline{A_i \cup A_{i+1} \cup \dots} = \lim \mathcal{A}.$$

The right hand side set $\lim \mathcal{A}$ *is called the* limit set *of $\mathcal{A}$.*

Let us explain what we understand by image $P(\mathcal{A})$ of a bubble ray $\mathcal{A} = (A_i)$. If the first bubble $P(A_1) \neq A_0$, then we set $P(\mathcal{A})$ as $(A_0, P(A_1), P(A_2), \dots)$. If $P(A_1) = A_0$ instead, we set $P(\mathcal{A})$ as $(A_0, P(A_2), P(A_3), \dots)$. Now we can define *perodic bubble rays*:

Definition 4.4.9. *We say that a bubble ray $\mathcal{A}$ is* periodic of period *m if $P^m\left(\bigcup \mathcal{A}\right) = \bigcup \mathcal{A}$.*

The main result of this subsection is the following Theorem:

Theorem 4.4.10. *Any periodic bubble ray for P lands at a periodic repelling or parabolic non-rotational point (parabolic point where no invariant periodic external ray lands) of P.*

To prove this Theorem we will need some additional results. The first needed theorem deals with a *topological hull* of a continuum $X \subset \mathbb{C}$: a topological hull is the union of X and all bounded complementary components of X. It is denoted as $\mathrm{Th}(X)$.

Theorem 4.4.11 (Theorem 7.5.2 [BFMOT13])**.** *Let f be a polynomial, let $K(f)$ be connected, and let $X \subset J(f)$ be an invariant continuum. Suppose that X is not a singleton. Then* $\mathrm{Th}(X)$ *contains a rotational fixed point or an invariant parabolic domain.*

Here a *rotational fixed point* means one of the following:

- an attracting fixed point;

- a repelling or parabolic fixed point where no invariant periodic external ray lands;
- a Siegel point;
- a Cremer point.

Moreover, we define a *rotational object* of f as either a rotational fixed point or an invariant parabolic domain.

This Theorem is related to the second needed result:

Theorem 4.4.12 ([GM93]). *Let f be a polynomial of any degree > 1. Consider the union Σ_f of all invariant external f-rays with the set Fix_f of their landing points. Every component of $\mathbb{C}\backslash\Sigma_f$ contains a unique invariant rotational object of f.*

A subset of Σ_f consisting of two rays landing at the same point and their common landing point is called a *cut*.

Now we are ready to prove the main result of this subsection.

Proof of Theorem 4.4.10. Let us denote the limit set of $\mathcal{A}$ as L and its minimal period as m. We need to show that that L is a singleton. If it is not true, then by Theorem 4.4.11, the set L contains a rotational object T which is invariant under the map $f = P^m$. By Theorem 4.4.12 we know that one of the cuts of Σ_f separates T and 0. Evidently, $\mathcal{A}$ cannot intersect this cut which implies that L must be located on one side of the cut while T is located on the other side. A contradiction. Hence L is an f-fixed point. Since it belongs to $J(P)$, it is not attracting. If it is Cremer or Siegel, then, again relying on Theorem 4.4.12, we separate L from 0 with a rational cut, again a contradiction. Hence L is an f-fixed repelling or parabolic point a. If a is rotational, then periodic rays landing at a undergo a nontrivial combinatorial rotation under f. Let $W \supset \bigcup \mathcal{A}$ be a wedge bounded by two consecutive f-rays landing at a. Locally near a, the wedge W is mapped to some other wedge disjoint from W. A contradiction with $\bigcup \mathcal{A} \subset W$.

□

4.5 Stability

All our previous results were about the dynamical map between dynamical planes of cubic and quadratic polynomials. But our main goal is to obtain a continuous map $\Phi_\lambda : \mathcal{P}_\lambda \to \tilde{K}(Q)$ from the parameter plane $\mathcal{P}_\lambda$ to dynamical plane of the reglued quadratic Julia set. We will parametrize polynomials $P_\lambda \in \mathcal{P}_\lambda$ by their critical points. Informally, we want to know that this correspondence is continuous. The main tools in this section are equicontinuous and holomorphic motions, defined in Section 2.8. The precise statement we want to show is the following Theorem:

Theorem 4.5.1. *Choose an arbitrary base point $P_0 \in \mathcal{C}^c_\lambda$ and an arbitrary point $z \in \overline{\Delta}(P_0)$. There is an equivariant equicontinuous motion ι_P of $\overline{\Delta}(P)$ over $\mathcal{C}^c_\lambda$ such that $\iota_{P_0} = id$. Moreover, $\iota_P(z)$ has the same polar coordinates in $\overline{\Delta}(P)$ as z in $\overline{\Delta}(P_0)$. If $P_0 \notin \mathcal{Z}^c_\lambda$, then this equicontinuous motion is holomorphic on $\mathcal{C}^c_\lambda \backslash \mathcal{Z}^c_\lambda$.*

First, we start with introducing some results from [Zak99], which enable us to construct ι_P.

4.5.1 An overview of [Zak99]

The main result of Zakeri in that paper is the fact, that the boundary of a Siegel disc corresponding to the fixed point with bounded type rotation number is a quasicircle. But we will be interested in particular parametrized space of *modified Blaschke products* introduced there.

Zakeri introduced an auxiliary family of degree 5 generalized Blaschke products given by

$$B(z) = e^{2\pi i t} z^3 \left(\frac{z-p}{1-\overline{p}z}\right)\left(\frac{z-q}{1-\overline{q}z}\right), |p| > 1, |q| > 1.$$

The numbers p and q are chosen so that B has a multiple critical point in $\mathbb{S}^1$ and two other critical points c_B, $1/\overline{c}_B$ (which may or may not belong to $\mathbb{S}^1$). It is also possible to choose normalization such that one of the critical points is 1 and the angle $t \in \mathbb{R}/\mathbb{Z}$ such that $B : \mathbb{S}^1 \to \mathbb{S}^1$ has rotation number θ. Then it is possible to introduce the parametrized space $\mathcal{B}_\lambda$ depending on a parameter $\lambda = e^{2\pi i\theta}$. The map $B : \mathbb{S}^1 \to \mathbb{S}^1$ is quasi-symmetrically conjugate to a rigid rotation by some conjugacy

H_B (a theorem of Świcatek and Herman [Swi98]). There exists quasi-conformal extension of this conjugasy to the closed disc, denote it as H (precise extension can be the Douady–Earle extension described in [DE86]).

Now we can finally introduce the *modified Blaschke product* $\tilde{B}$:

$$\tilde{B}(z) = \begin{cases} B(z), & \text{if } |z| \geqslant 1 \\ H^{-1} \circ \mathrm{Rot}_\theta \circ H(z), & \text{if } |z| < 1 \end{cases}$$

where Rot_θ is the rigid rotation about 0 by angle θ. It is shown in the paper of Zakeri, that it is possible to find such a $\tilde{B}$-invariant conformal structure σ on $\overline{\mathbb{C}}$ and straighten it, that $\tilde{B}$ is quasi-conformally conjugate to a cubic polynomial $P \in \mathcal{C}^c_\lambda$. Then the *non-escaping locus* of $\mathcal{B}_\lambda$ is defined as the set of such B's that the orbit of a critical point, different from 1 and from multiple critical point, is bounded. Since B is conjugate to a polynomial P, there exists a map $\mathcal{S} : \mathcal{B}_\lambda \to \mathcal{C}^c_\lambda$ such that $P = \mathcal{S}(B)$. Then $\mathcal{S}$ is called the *surgery map.*

The following proposition is Corollary 10.5 of [Zak99]:

Proposition 4.5.2. *There is an equicontinuous family of qc homeomorphisms* $\varphi_B : \mathbb{C} \to \mathbb{C}$ *parameterized by* B *in the non-escaping locus of* $\mathcal{B}_\lambda$ *such that* $\mathcal{S}(B) = \varphi_B \circ \tilde{B} \circ \varphi_B^{-1}$*, and normalized so that* $\varphi_B(1) = 1$*.*

Let us denote Siegel disk of P as $\Delta(P)$. Then $\Delta(P) = \varphi_B(\mathbb{D})$. Denote the inverse map as $\psi_{\Delta(P)} : \mathbb{D} \to \Delta(P)$, then $\psi_{\Delta(P)} = \varphi_B \circ H_B^{-1}$. Previous proposition together with Theorem 4.4.1 of [Hub06] enable us to formulate the following property of $\psi_{\Delta(P)}$:

Corollary 4.5.3. *The extended Riemann maps* $\overline{\psi}_{\Delta(P)}$ *(where* P *varies through* $\mathcal{C}^c_\lambda$*) form an equicontinuous family.*

Let $C(\overline{\mathbb{D}}, \mathbb{C})$ be the space of all continuous maps from $\overline{\mathbb{D}}$ to $\mathbb{C}$ with the sup-norm. Corollary 4.5.3, in turn, implies the following.

Corollary 4.5.4. *The map from* $\mathcal{C}^c_\lambda$ *to* $C(\overline{\mathbb{D}}, \mathbb{C})$ *taking* P *to* $\overline{\psi}_{\Delta(P)}$ *is continuous.*

Proof. Let us consider a sequence of polynomials $P_n \in \mathcal{C}^c_\lambda$ which converges to $P \in \mathcal{C}^c_\lambda$. Then we have to show that the corresponding sequence $\overline{\psi}_{\Delta(P_n)}$ converges uniformly to $\overline{\psi}_{\Delta(P)}$. By the choice of normalization of φ_B we have $\overline{\psi}_{\Delta(P_n)}(1) = \overline{\psi}_{\Delta(P)(1)} = 1$. We know that each of $\overline{\psi}_{\Delta(P_n)}$ conjugates the rotation by angle θ with P_n on its Siegel disc. Let us consider the point $z = P^k(1) \in \partial\Delta(P)$ for some fixed positive integer k. We can choose n large enough, such that the points $z_n = P_n^k(1) \in \partial\Delta(P_n)$ are close to z. Then $\overline{\psi}_{\Delta(P_n)}(e^{2\pi ik\theta}) = z_n$ and $\overline{\psi}_{\Delta(P)}(e^{2\pi ik\theta}) = z$. Thus $\overline{\psi}_{\Delta(P_n)} \to \overline{\psi}$ point-wise on a dense subset of $\mathbb{S}^1$. By equicontinuity $\overline{\psi}_{\Delta(P_n)}$ converges to $\overline{\psi}_{\Delta(P)}$ uniformly on $\mathbb{S}^1$. And by the Maximum Modulus Principle, $\overline{\psi}_n \to \overline{\psi}$ on $\overline{\mathbb{D}}$. □

Now we are ready to prove Theorem 4.5.1.

4.5.2 Definition of stability

Let us start with the proof of existence of equivariant equicontinuous motion ι_P of the closure of the Siegel disc $\overline{\Delta}(P)$.

Proof of Theorem 4.5.1. We can precisely define $\iota_P(z)$ as $\overline{\psi}_{\Delta(P)} \circ \overline{\psi}^{-1}_{\Delta(P_0)}$. It is an equicontinuous motion Corollary 4.5.4. Since $\overline{\psi}_{\Delta(P)}$ conjugates the rotation by angle θ with P on its Siegel disc, $\iota_P(z)$ is also equivariant. Now let us consider $P_0 \notin \mathcal{Z}^c_\lambda$ and $P \in \mathcal{C}^c_\lambda \backslash \mathcal{Z}^c_\lambda$. There is a holomorphic motion of the P-orbit of 1 with respect to P as above. It possible to extend it to an equivariant holomorphic motion of $\partial\Delta(P)$ (by [MSS83] and [Che20]) and then to extend the obtained holomorphic motion to an equivariant holomorphic motion $\iota_P : \overline{\Delta}(P_0) \to \overline{\Delta}(P)$ (by [Sul] or [Zak16]), such that $\iota_P : \Delta(P_0) \to \Delta(P)$ is a conformal isomorphism taking 0 to 0 and 1 to 1. Since the Riemann map is unique, the obtained map coincides with ι_P defined in the beginning and it preserves the polar coordinates.

□

But if we want to get continuity of the parameter map to the dynamical plane we need not only holomorphic motion of the closure of the Siegel disc, but also of the other parts of the filled Julia set. Further concept of *stability* enables us to extend the equicontinuous motion of $\overline{\Delta}(P)$ on bigger sets.

Definition 4.5.5 (Stability). *Consider $P_0 \in \mathcal{C}^c_\lambda$ and a subset $A \subset Y(P_0)$. Since $Y(P_0)$ is by definition forward invariant, it follows that $P_0^n(A) \subset Y(P_0)$ for all*

$n \geqslant 0$. *Set* $B = \bigcup_{n\geqslant 0} P_0^n(A)$. *Say that* A *is* stable *(or* λ-stable*) if there is an equivariant equicontinuous motion* $\{\iota_P^B\}$ *of* B *over an open neighborhood of* P_0 *in* $\mathcal{C}_\lambda^c$ *such that, for every* $z \in B$*, the point* $z\langle P\rangle = \iota_P^B(z)$ *has the same multi-angle and the same polar radius as* z*. Clearly, if such an equicontinuous motion exists, then it is unique. Write* $A\langle P\rangle$ *for* $\iota_P^B(A)$ *etc. If the equicontinuous motion is in fact holomorphic, then say that* A *is* holomorphically stable.

4.5.3 Stability of legal arcs and Siegel rays

Let us start with finite legal arcs. To show this stability we will need a very general continuity property. Let Rat_d be the space of all degree d rational self-maps of $\overline{\mathbb{C}}$ with the topology of uniform convergence and let Comp be the space of all compact subsets of $\overline{\mathbb{C}}$ with the Hausdorff metric. The the following lemma holds:

Lemma 4.5.6. *Consider the map from* $\mathrm{Rat}_d \times \mathrm{Comp} \to \mathrm{Comp}$ *given by*

$$(f, X) \mapsto f^{-1}(X).$$

This map is continuous.

Proof. Fix $(f, X) \in \mathrm{Rat}_d \times \mathrm{Comp}$. We want to show that for fixed $\varepsilon > 0$ if $\delta = \delta(\varepsilon) > 0$ is sufficiently small and (g, Y) is δ-close to (f, X), then $g^{-1}(Y)$ is ε-close to $f^{-1}(X)$. It means showing that for every point $x \in f^{-1}(X)$, there is $y \in g^{-1}(Y)$ that is ε-close to x, and vice versa: for every y with $g(y) \in Y$, there is $x \in f^{-1}(X)$ that is ε-close to y. For some point $x \in f^{-1}(X)$ we have $g(x)$ is δ-close to $f(x)$ and there is a point in Y which is δ-close to $f(x)$. Denote it as y^*, then y^* is 2δ-close to $g(x)$. Thus there exists point $y \in g^{-1}(y^*)$ that is ε-close to x. By the Open Mapping property, the f-image of the ε-neighborhood of x includes the 4δ-neighborhood of $f(x)$, so δ does not depend on g. We can see that the g-image of the ε-neighborhood of x includes the 2δ-neighborhood of $g(x)$ since it includes the 3δ-neighborhood of $f(x)$. For some point $y \in g^{-1}(Y)$ the similar argument works. Now knowing that $g(y)$ is δ-close to $f(y)$ and there exists a point $x^* \in X$ that is δ-close to $g(y)$, the fact that x^* is 2δ-close to $f(y)$ implies that there exists a point x such that $f(x) = x^*$ and x is ε-close to y. □

More precisely, for the proof of stability of the finite legal arcs we will need the corollary of the previous lemma. Let us state it.

Corollary 4.5.7. *Suppose that* $(f, X) \in \mathrm{Rat}_d \times \mathrm{Comp}$*, that* X *is connected, and that* A *is a component of* $f^{-1}(X)$*. If there are no critical points of* f *in* A*, then for all* (g, Y) *close to* (f, X) *there is a component of* $g^{-1}(Y)$ *close to* A *and not containing critical points of* g*.*

Proof. For some $\varepsilon > 0$, the 5ε-neighborhood of A maps homeomorphically onto a neighborhood of X. By previous Lemma 4.5.6 we can take (g, Y) so close to (f, X) that $g^{-1}(Y)$ is ε-close to $f^{-1}(X)$ and assume g to be injective on the 4ε-neighborhood of A. Let us consider a component of $g^{-1}(Y)$ which intersects the 2ε-neighborhood of A. Denote it as B. Since we know that f is injective on the 5ε-neighborhood of A and that points in B are ε-close to $f^{-1}(X)$, then B has to lie entirely in the 2ε-neighborhood of A. By the injectivity of g we know that B is the unique component of $g^{-1}(Y)$ in the 2ε-neighborhood of A. The closest point in $g^{-1}(Y)$ to any point $a \in A$ has to be in B, thus sets A and B are ε-close. □

Now we can finally proceed to showing stability. As we have already mentioned above, the first result is about stability of finite legal arcs.

Lemma 4.5.8. *Take* $P_0 \in \mathcal{C}^c_\lambda$ *and a point* $z \in Y(P_0)$ *that has a finite multi-angle. If* z *is never mapped to* c*, or if* $c \in \overline{\Delta(P_0)}$*, then the legal arc* I_z *from* 0 *to* z *in* $K(P_0)$ *is stable. It is holomorphically stable if* $P_0 \notin \mathcal{Z}^c_\lambda$*.*

Proof. Set $\alpha = (\alpha_0, \ldots, \alpha_k)$ to be the multi-angle of z. We will prove the lemma by induction on k and the minimal integer m such that $\Pi^m(\alpha)$ has length one. For $k = 0$ the statement is true by Theorem 4.5.1. Let w be the last point of I_z with multi-angle $(\alpha_0, \ldots, \alpha_{k-1})$. Then I_z consists of two legal arcs with smaller k: arc I_w from 0 to w and arc $I_{[w,z]}$ from w to z). By assumption of induction we assume that I_w is stable. In particular, $w\langle P\rangle$ is defined for all P close to P_0, and $w\langle P\rangle$ has the same multi-angle and polar radius as w.

Now let us proceed with induction assumption on m. We assume that $P_0(I_z)$ is stable, then the following set $T = P_0(I_{[w,z]})$ is also stable. Define $I_{[w,z]}\langle P\rangle$ as the

P-pullback of $T\langle P\rangle$ containing $w\langle P\rangle$. From the previous Corollary 4.5.7 we know, that if P is close to P_0, then $I_{[w,z]}\langle P\rangle$ is also close to $I_{[w,z]}$. Thus we can choose an inverse branch of P which defines an equicontinuous motion of $I_{[w,z]}$ on on $T\langle P\rangle$ since $I_{[w,z]}\langle P\rangle$ contains no critical points of P.

Now we know that both parts I_w and $I_{[w,z]}$ of I_z are stable, thus I_z is stable itself.

□

Now we can extend the previous result to infinite periodic legal arcs.

Theorem 4.5.9. *Let $\mathcal{A}$ be a periodic bubble ray for $P_0 \in \mathcal{C}_\lambda^c$ landing at a repelling periodic point x. Suppose that $P_0^i(I_x)$ does not contain c for $i \geqslant 0$. Then I_x is stable; it is holomorphically stable if $P_0 \notin \mathcal{Z}_\lambda^c$.*

Proof. By the definition of the minimal P_0-period m of $\mathcal{A}$ we have $P_0^m(x) = x$. We can choose a small round disk D around x, for which $D \Subset P_0^m(D)$, and the map $P_0^m : D \to P_0^m(D)$ is a homeomorphism since x is a repelling point. We know that $\mathcal{A}$ lands at x, and thus if $\mathcal{A} = (A_n)$, then $\overline{A}_n \subset D$ for n big enough. The map P_0^m shifts bubbles of $\mathcal{A}$, let us denote the integer by which the shift occurs as k. Note that $k \geqslant 1$. Then there is a point $y \in I_x$ such that

$$I_y = I_x \cap \overline{A_0 \cup \cdots \cup A_{N+k}}.$$

We know that I_y is stable by Lemma 4.5.8, thus if P is close to P_0, then there exists a legal arc $I_y\langle P\rangle$ close to I_y with the same multi-angle. Let us denote the final point of $I_y\langle P\rangle$ (after passing through legal bubbles $A_0\langle P\rangle, \ldots, A_{N+k}\langle P\rangle$) as $y\langle P\rangle$. Then the segment $I_{[z,y]}$ from the point $z = P_0^m(y) \in I_y$ to y is stable and the corresponding segment $I_{[z,y]}\langle P\rangle$ for P connects $z\langle P\rangle$ with $y\langle P\rangle$. If P is close to P_0, then $D \Subset P^m(D)$, and $P^m : D \to P^m(D)$ is a homeomorphism. Write P_D^{-m} for the inverse of this homeomorphism. Then P_D^{-m} is a well-defined holomorphic map on D depending analytically on P. Since x is repelling, it is stable, so that there is a nearby repelling point $x\langle P\rangle$ for P of the same period. Set

$$I_x\langle P\rangle = I_y\langle P\rangle \cup \left(\bigcup_{k=1}^{\infty}(P_D^{-m})^k(I_{[z,y]}\langle P\rangle)\right) \cup \{x\langle P\rangle\}.$$

Every term in the right-hand side moves equicontinuously with P as long as P stays close to P_0. The infinite union moves equicontinuously since for P_D^{-m} the point $x\langle P\rangle$ is attracting (the iterates cannot inflate the modulus of continuity). It is also clear that the motion is holomorphic provided that $P_0 \notin \mathcal{Z}_\lambda^c$ and P is close to P_0.

□

As we have seen in Theorem 4.4.10 any periodic bubble ray lands at a repelling or parabolic point.

Let $\mathcal{A}$ be a periodic bubble ray for some P, landing at a repelling or parabolic point a. Then its core curve is a a legal arc I_a from 0 to a. Informally, we want to introduce an "extension" of the external rays, which will coincide with a bubble ray $\mathcal{A}$.

Definition 4.5.10 (Siegel rays)**.** *By the Landing Theorem for polynomials (see e.g. [Mil06, Theorem 18.11]), one or several periodic external rays for P land at a. Let R be an external ray landing at a. Then $I \cup \{a\} \cup R$ is a simple curve connecting 0 with ∞. It is called a* Siegel ray*. The* argument *of the Siegel ray $I \cup \{a\} \cup R$ is defined as the argument of R.*

Let us note that from this definition it follows that every Siegel ray contains precisely one periodic point $a \neq 0$, which can be repelling or parabolic. In the case if the periodic point a is repelling we can show the stability of corresponding Siegel ray.

Theorem 4.5.11. *Let Σ be a Siegel ray for $P_0 \in \mathcal{C}_\lambda^c$. Suppose that the non-zero periodic point in Σ is repelling. Then, for all $P \in \mathcal{C}_\lambda^c$ sufficiently close to P_0, there is a Siegel ray $\Sigma\langle P\rangle$ close to Σ in the spherical metric and having the same argument. Moreover, the periodic point in $\Sigma\langle P\rangle$ depends holomorphically on P provided that $P \notin \mathcal{Z}_\lambda^c$.*

The Siegel ray Σ can be separated into two parts: a legal arc in a periodic bubble ray and an external ray. Let us show that both of them move holomorphically:

1. a legal arc of a bubble ray, which lands at a repelling periodic point, moves holomorphically by Theorem 4.5.9;

2. there is a holomorphic motion of an external ray landing at a repelling periodic by the following Lemma:

   Lemma 4.5.12 ([DH85a], cf. Lemma B.1 [GM93]). *Let P_0 be a polynomial, and z be a repelling periodic point of P_0. If an external ray $R_{P_0}(\theta)$ with rational argument θ lands at z, then, for every polynomial P sufficiently close to P_0, the ray $R_P(\theta)$ lands at a repelling periodic point $z\langle P\rangle$ of P close to z, and $z\langle P\rangle$ depends holomorphically on P.*

Theorem 4.5.11 is proved.

4.5.4 Siegel wedges

Now we want to consider two different Siegel rays Σ and Σ' for a map P. Then they may have some common initial segment $I = \Sigma \cap \Sigma'$ and then branch off either at an iterated preimage of 0 or at a landing point of some bubble ray. Let us assume that they branch off in some iterated preimage of 0, we denote it as b. Then we want to define the area between branched segments of Σ and Σ':

Definition 4.5.13 (Siegel wedge). *A wedge W bounded by segments of Σ and Σ' from b to infinity is called a* Siegel wedge (bounded by Siegel rays Σ and Σ') *with a* root point b*, see Fig. 4-2. There are two Sigma wedges corresponding to rays Σ and Σ': one contains their common segment I and the other one does not contain it. We define the* multi-angle of a wedge*, which does not contain I as the multi-angle of b. The multi-angle of the second one is the empty sequence $()$.*

If Σ and Σ' do not have a common segment and thus $b = 0$, we set multi-angles of both wedges to be $()$.

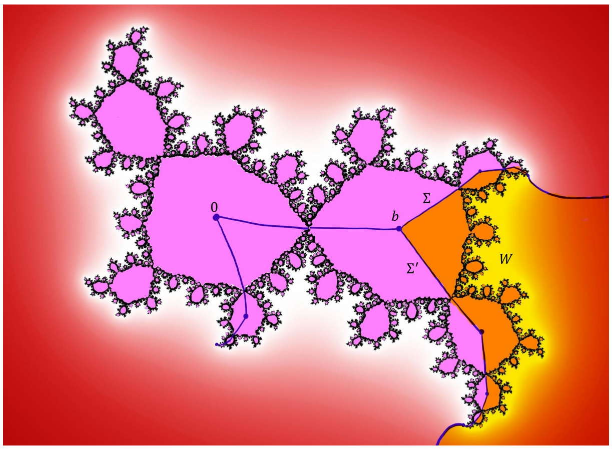


Figure 4-2: A Siegel wedge W in the dynamical plane of Q. Here, W is bounded by Siegel rays Σ and Σ' that have the legal arc I_b from 0 to b as the common initial segment and that branch off at point b.

4.6 Continuous extension of the map η_P

Let us recall that we have defined $\eta_P : Y(P) \to K(Q)$ as a map which takes $z \in Y(P)$ to a unique point $w = \eta_P(z)$ with the same multi-angle and polar radius. In this section we define an extension $Y(P)$ of $X(P)$ such that it satisfies the conditions of **Property D** formulated in Section 4.2 and continuously extend the map $\eta_P : Y(P) \to K(Q)$ to $X(P)$.

4.6.1 Extended set $X(P)$

We define $X(P)$ depending on whether $\overline{Y(P)}$ contains parabolic points of P or not. If it contains some parabolic point a, then there is a parabolic periodic cycle in $\overline{Y(P)}$ containing a, which is unique by the Fatou–Shishikura inequality. Then $X(P)$ is defined in the following way:

1. if $\overline{Y(P)}$ contains no parabolic points of P, then $X(P) = \overline{Y(P)}$;

2. if $\overline{Y(P)}$ contains parabolic point a, then $X(P)$ is the union of $\overline{Y(P)}$ and the closures of all immediate parabolic basins of the points from the cycle of a.

We want to describe the structure of $X(P)$. For being able to do it we need some additional structure on the bubbles. We define a *terminal segment* of an internal ray of A as a segment from some point in the ray to ∂A. Then the structure of $X(P) \cap Fatou(P)$ can be described by the following theorem:

Theorem 4.6.1. *Let A be a legal bubble of P, and A_Q be the corresponding bubble of Q. Then $\eta_P : Y(P) \cap A \to A_Q$ extends to a continuous map $\eta_P : X(P) \cap A \to A_Q$, and one of the following two cases holds:*

1. *The set $X(P) \cap A$ is a terminal segment of the internal ray of A landing at the root point of A. The map $\eta_P : X(P) \cap A \to A_Q$ is one-to-one, and $\eta_P(X(P) \cap A)$ is a terminal segment of the internal ray of A_Q landing at the root point of A_Q.*

2. *The set $X(P) \cap A$ is a "sector" bounded by two "separatrices" in A. It is mapped under η_P onto A_Q, the boundary of the sector mapping two-to-one, and otherwise the map being one-to-one.*

The first step to prove Theorem 4.6.1 is the following lemma:

Lemma 4.6.2. *Let A be a bubble of P. Suppose that a point $z \in \overline{A} \cap X(P)$ is different from the root point $r(A)$ of A. Then A is a legal bubble, and the entire bubble chain to z consists of legal bubbles.*

Proof. If some point $z \neq r(A)$ is in $X(P)$, then $r(A) \in Y(P)$ and the there exists a legal arc I_z from 0 to z non-disjoint from A. Since A is open and $\overline{A} \cap X(P)$, then $A \cap Y(P) \neq \varnothing$. In means that A is legal and the I_z intersects every bubble in the bubble chain to z. It shows that all bubbles in this chain are legal. □

This lemma can be restated as the following Corollary:

Corollary 4.6.3. *Let A be a bubble of P. Either A is legal, or $\overline{A}$ has no points of $X(P)$ except possibly $r(A)$ in which case $r(A)$ is eventually mapped to c, non-strictly before it is mapped to 1 and strictly before A is mapped to $\Delta(P)$.*

Now we can proceed with the proof.

Proof of Theorem 4.6.1. For considering the extension of $Y(P)$ to $X(P)$ we should consider legal bubbles which do not belong to $Y(P)$. Then if A is such a bubble, there is a point $z \in A$ that is eventually mapped to critical point c, assume that $P^n(z) = c$. We can assume $c \in A$, since we know that $P^n : A \to P^n(A)$ is a homeomorphism from $Y(P)$ to itself. There must be at least one point of A which belongs to $Y(P)$ since A is open. Two cases from the statement of the Theorem correspond to two different possibilities for the multi-angle of A. Set $\alpha = (\alpha_0, \ldots, \alpha_{2k})$ to be the multi-angle of A.

(1) In the first case the multi-angle of any point in $A \cap Y(P)$ is $(\alpha.\alpha_{2k})$. Then $\overline{A} \cap X(P)$ is the legal arc from the $r(A)$ to c. We can see that η_P is well-defined and continuous on this legal arc and the image of this legal arc under η_P is again a legal arc in the closure of the bubble of Q corresponding to A.

(2) In the second case only some points in A have multi-angles $(\alpha.\alpha_{2k})$ while others have multi-angles $(\alpha.\alpha_{2k}\alpha_{2k+1})$. Then let us choose a point with multi-angle $\alpha.(\alpha_{2k}, \alpha_{2k+1})$. If we denote the image of the bubble A as $B = P(A)$, then $P(A \cap Y(P))$ includes the center of B and all internal rays of B but the one containing critical value $P(c)$. Denote this ray as R. We define a segment T from $P(c)$ to the boundary of B (which is not contained in $P(A \cap Y(P))$ as the *special segment*. Let us consider the pullback of T. It consists of an arc $T' \subset A$ that contains c and of the two arcs connecting c with two preimages of the point $T \cap \partial B$. Then the bubble A is divided by the arc T' into two disjoint pullbacks of $B \backslash T$, one of which is $A \cap Y(P)$. This pullback contains P-preimage of the center of B and all rays starting from it. All rays but the one passing through c extend to ∂A. The ray containing c branches at this point. Then the set $\partial(A \cap Y(P))$ consists of c and these two branched "*separatrices*". The map η_P extends to these separatrices and the image $\eta_P(A \cap X(P))$ coincides with the entire bubble of Q corresponding to A. Then the radial segment from $\eta_P(c)$ to the boundary of the bubble is exactly the sector the boundary of which is mapping two-to-one. □

4.6.2 A separation property

In this subsection we deal with Siegel wedges, defined in Subsection 4.5.4. We want to set the correspondence between cubic and quadratic wedges and show how quadratic wedges *separate* points in the bubbles.

Definition 4.6.4 (Corresponding Siegel wedges)**.** *Let W be the Siegel wedge of P. Then $\eta_P(\partial W \cap K(P))$ is the union of the core curves of two periodic bubble rays for Q, which land at some repelling periodic points x and y of Q. Then we define the wedge W_Q* corresponding *to the Siegel wedge W of P as the wedge bounded by the unique external rays landing at x and y and containing points of $\eta_P(W \cap Y(P))$.*

We will also call W_Q *P-adapted* wedge and say that a P-adapted wedge W_Q *separates* a point x from a point x' if $x \in W_Q$ and $x' \notin \overline{W}_Q$.

Now let us focus on the structure of entire $\eta_P(Y(P))$. We say that $\eta_P(Y(P)) \subset K(Q)$ is *legal convex*, that is, any two points of this set can be connected by a legal arc lying entirely in this set (since $K(Q)$ is locally connected, any two points of $K(Q)$ can be connected by a legal arc). It is easy to show that the closure of a legal convex is a legal convex: if x_n, $x'_n \in K(Q)$ are two sequences converging to x, x', respectively, then the legal arc from x_n to x'_n converges to the legal arc from x to x'. We get the following Lemma from this statement:

Lemma 4.6.5. *The set $\overline{\eta_P(Y(P))} \subset K(Q)$ is legal convex.*

Now we can formulate the following Lemma about separation of the points in the quadratic bubble:

Lemma 4.6.6. *Let $A_Q \neq \Delta(Q)$ be a bubble of Q, and $x \in \partial A_Q \cap \overline{\eta_P(Y(P))}$ be a point different from the root point of A_Q. Then, for any other point $x' \in \partial A_Q$, there is a P-adapted wedge W_Q with root point in the center of A_Q such that W_Q separates x from x'. Any pair of different points in $\partial\Delta(Q)$ is also separated by a P-adapted wedge unless $c \in \partial\Delta(P)$ and $P^k(c) = 1$ for some $k \geqslant 0$.*

Proof. Let us consider point x as in the statement of the theorem. Then we can connect 0 and x by the legal arc $I_x \in \overline{\eta_P(Y(P))}$ since $\overline{\eta_P(Y(P))}$ is a legal convex by Lemma 4.6.5. Set A to be a legal bubble of P corresponding to the bubble A_Q (it exists by Lemma 4.6.2). Now we are in the setting of the case (2) of Theorem 4.6.1. By this Theorem the root points of other bubbles attached to A_Q on ∂A_Q except possibly one are in $\eta_P(Y(P))$ (they also form a dense subset in ∂A_Q). Then there are exist a pair of points b, b' in ∂A_Q separating x from x'. We can extend the legal arcs from the center a of A_Q to b and b' to periodic Siegel rays. Then we can choose the Siegel rays for P in a way that the wedge W bounded by them corresponds to a wedge W_Q whose boundary intersects ∂A_Q at points b and b'. The obtained wedge W_Q is the desired wedge, separating x from x'.

The case of different points in $\partial\Delta(Q)$ such that $c \in \partial\Delta(P)$ and $P^k(c) = 1$ can be reduced tho the case above by considering two iterated preimages b, b' of 1 separating two paints in $\partial\Delta(Q)$ since these points are again are root points of legal bubbles attached to $\Delta(P)$. □

We can generalize the previous lemma.

Proposition 4.6.7. *A pair of distinct points $x, x' \in J(Q) \cap \overline{\eta_P(Y(P))}$ is separated by a P-adapted wedge, except when both x and x' are in $\partial\Delta(Q)$, and $c \in \partial\Delta(P)$ is eventually mapped to* 1.

Proof. Let us assume that one point x belongs to the boundary of some bubble $\overline{A_Q}$, while x' belongs to another. Then we can consider an intersection of the legal arc I connecting x with x' with the bubble A_Q. Denote the intersection point as x'' and the center of A_Q as a. Now we have points x and x'' which satisfy the conditions of Lemma 4.6.6. Then the same P-adapted wedge W_Q as in Lemma 4.6.6 which separates x from x'' will also separate x from x' since the legal arc from x' to x'' cannot intersect ∂W_Q.

Now let us proceed with the case, when none of the points x and x' belongs to the boundary of a bubble. Then we know that there are bubble rays $\mathcal{A}_Q$ and $\mathcal{A}'_Q$ landing at x and x' respectively. By Lemma 4.6.2 any bubble from the chain $\mathcal{A}_Q$ or $\mathcal{A}'_Q$ should intersect $\eta_P(Y(P))$, then there exist corresponding bubble rays $\mathcal{A}$ and $\mathcal{A}'$ respectively for P. Let us consider some bubble $B_Q \in \mathcal{A}_Q$ which does not belong to $\mathcal{A}'_Q$ and the bubble B of P corresponding to it. Then a P-adapted wedge W_Q should separate x from the root point of B_Q and thus from x'. □

4.6.3 Continuous extension of η_P

Now we are ready to describe the extension of the map η_P to the set $X(P)$. In the case when there is no parabolic cycle in $\overline{Y(P)}$ it is enough to show that η_P extends continuously to $\overline{Y(P)}$. In the case when there is a parabolic cycle, we know that any point $z \in X(P)\backslash\overline{Y(P)}$ belongs to the closure of a parabolic domain at a parabolic point a_z. Then we set $\eta_P(z) = \eta_P(a_z)$ and thus get a continuous map. Actually, we want to prove even stronger statement:

Theorem 4.6.8. *The map $\eta_P : Y(P) \to K(Q)$ extends to a continuous map $\eta_P : X(P) \to K(Q)$. Unless $c \in X(P)\backslash J(P)$, this map is monotone.*

Let us start with the continuity.

Proof of Theorem 4.6.8, continuity. Let us show that η_P extends continuously to $\overline{Y(P)}$. Take $y \in \partial Y(P)\backslash Y(P)$; by Theorem 4.6.1 we can only consider points y which are not in a bubble. For proving continuity it is enough to show, that images of two sequences in $Y(P)$ with the same limit can not converge to the different limits. Let us assume that it is not true and that there exist two sequences y_n, $y'_n \in Y(P)$ with the limit y, but the limits of $\eta_P(y_n)$ and $\eta_P(y'_n)$ are two different points x and x' respectively. Also set $x_n = \eta_P(y_n)$ and $x'_n = \eta_P(y'_n)$. Then we know by Proposition 4.6.7, that there exists a P-adapted wedge W_Q in the dynamical plane of Q separating x from x' so that $x \in W_Q$ and $x' \notin \overline{W}_Q$. For all n big enough $x_n \in W_Q$ since W_Q is open. Then the sequence y_n for all n big enough has to be contained in some compact subset of the legal wedge W corresponding to W_Q for P, then also $y \in W$ and $y'_n \in W$ for n big enough. It follows that $x'_n \in W_Q$ for large n, therefore, $x' \in \overline{W}_Q$, a contradiction. From the definition of $\eta_P(z)$ in the case when there is a parabolic cycle in $\overline{Y(P)}$, we have seen that η_P is continuous also in that case. □

4.6.4 Monotonicity

In this subsection we are dealing with the second half of Theorem 4.6.8 about the monotonicity. Note that to show that η_P is monotone we should check that all point preimages are connected. Let us recall, that preimages of points under η_P are called *fibers* of η_P. The following lemma deals with this connectedness.

Lemma 4.6.9. *A nonempty intersection of finitely many Siegel wedges for P is connected and has a connected intersection with $X(P)$.*

Proof. We want to distinguish two types of Siegel wedges: type 1 wedge is disjoint from the legal arc connecting 0 and its root point and all other Siegel wedges are said to be of type 2. Then finitely many Siegel wedges of type 1 intersect over a Siegel wedge of type 1. Let as denote a set, whose complement is the closure of a finite union of pairwise disjoint type 1 Siegel wedges as B. Then finitely many Siegel wedges of type 2 intersect over such a set B. Moreover, such a set B intersects with a Siegel wedge of type 1 either over exactly this type 1 Siegel wedge or over B minus the closure of a finite union of pairwise disjoint type 1 Siegel wedges contained in B. Thus we can see that a nonempty intersection U of finitely many Siegel wedges for P is connected and also $U \cap Y(P)$ is connected.

Now we have 2 cases: $X(P) = \overline{Y(P)}$ or there are parabolic points in $U \cap X(P)$. In the first case $U \cap X(P)$ is a superset of $U \cap Y(P)$ and a subset of $\overline{U \cap Y(P)}$, thus the lemma statement is true. In the case with parabolic points their parabolic domains should also belong to $U \cap X(P)$, and the statement again holds. □

Now we can finish the proof.

Proof of Theorem 4.6.8, the monotonicity part. Let us consider the point x_Q which belongs to $\eta_P(X(P))$ and the fiber $\eta_P^{-1}(x_Q)$. For x_Q belonging to a bubble of Q we know that the fiber of x_Q is a singleton by Theorem 4.6.1. Assume that $x_Q \in J(Q)$. First suppose that $x_Q \in \partial\Delta(Q)$ and $c \in \partial\Delta(P)$ is eventually mapped to 1. There is a unique point $x \in \partial\Delta(P)$ such that $\eta_P(x) = x_Q$ since $\eta_P : \partial\Delta(P) \to \partial\Delta(Q)$ is a homeomorphism. Then suppose that some point $y \notin \overline{\Delta}(P)$ is mapped to x_Q. There should exist a sequence $y_n \in Y(P)$ converging to y since $y \in X(P)$. It is also possible to choose this sequence in a way such $y_n \in J(P)$ since $y \in X(P)$. Then we also can choose a subsequence of the legal arcs I_{y_n} from 0 to y_n such that they converge to a continuum $C_y \ni y$.

If the point y is on the boundary of a legal bubble A of P, we can choose the bubble A so that $y \neq r(A)$. Then the image $\eta_P(y)$ is on the boundary of the bubble A_Q of Q corresponding to A, and $\eta_P(y) \neq r(A_Q)$. Thus we get a contradiction with $\eta_P(y) = x$. So we know that y is not on the boundary of a bubble.

Let us consider some bubble attached to $\Delta(P)$ and intersecting I_{y_n}, denote it as

A_n. Then there are two cases: there are finitely many different bubbles A_n or there are infinitely many pairwise different A_ns.

In the first case, choosing a suitable subsequence, we may consider the sequence A_n with the same members A. The intersection $I_{y_n} \cap A$ consists of two internal rays landing at $r(A) \in \Delta(P)$ and $b_n \neq r(A)$. In the case when $b_n \to r(A)$ in is possible to replace C_y with $C_y^* = C_y \backslash A$. Then we get a continuum containing y, and $C_y^* \backslash \{x\}$ is disjoint from all bubbles. If $b_n \not\to r(A)$ instead, then, choosing a suitable subsequence, we can get b_n such that $b_n \to b \neq r(A)$. Thus $\eta_P(C_y \backslash \overline{A})$ is attached to the bubble A_Q of Q, corresponding to the bubble A at $\eta_P(b)$. And it does not accumulate on $\partial\Delta(Q)$. This is a contradiction with $x_Q = \eta_P(y) \in \partial\Delta(Q)$.

In the second case when there are infinitely many pairwise different bubbles A_n let us set $C_y^* = C_y$. But the set C_y^* is still a continuum containing y such that $C_y^* \backslash \{x\}$ is disjoint from all bubbles. Thus $\eta_P(C_y^*) = \{x_Q\}$. Since $\eta_P^{-1}(x_Q)$ is the union of $\{x\}$ and C_y^* over all $y \in \eta_P^{-1}(x_Q) \backslash \{x\}$, the fiber $\eta_P^{-1}(x_Q)$ is connected.

Now we can assume that $x_Q \notin \partial\Delta(Q)$ or that $c \notin \partial\Delta(P)$ or that $c \in \partial\Delta(P)$ is never mapped to 1. If we consider the intersection Z_Q of all P-adapted wedges containing x_Q, then $Z_Q \cap J(Q) = \{x_Q\}$ by Proposition 4.6.7. This set Z_Q may also include certain external rays of Q as well as certain internal rays in bubbles of Q appart from x_Q. Thus we can see, that $Z_Q \cap K(Q)$ consists of $\{x_Q\}$ and one or two internal rays in bubbles A_Q such that $x \in \partial A_Q$. The full preimage $Z = \eta_P^{-1}(Z_Q \cap K(Q))$ consists of $\eta_P^{-1}(x_Q)$ and one or two internal rays in legal bubbles A of P. Every such bubble may intersect Z by at most one internal ray. Thus if we have connectedness of Z we get connectedness of the fiber $\eta_P^{-1}(x_Q)$. Now we need to show connectedness of Z. Let us consider any $x \in \eta_P^{-1}(x_Q) \cap J(P)$. Then Z is the intersection of all Siegel wedges of P containing x with $X(P)$. Moreover, it is enough to intersect countably many Siegel wedges $W_1, \ldots, W_n, \ldots$:

$$Z = X(P) \cap \bigcap_{n=1}^{\infty} W_n = X(P) \cap \bigcap_{n=1}^{\infty} U_n, \quad U_n = W_1 \cap \cdots \cap W_n.$$

We know from Lemma 4.6.9 that the sets $X(P) \cap U_n$ are connected and form a nested sequence. And thus their intersection is also connected. □

4.7 The parameter maps Φ_λ^c and Φ_λ

In this section we shift from the maps on the dynamical plane to the maps on the parameter plane and the we prove the Main Theorem 4.2.3. First we need to introduce some additional space of polynomials.

4.7.1 Connectedness locus $\mathcal{C}_\lambda^c$ in the space of polynomials with marked critical point

As we have described in Section 4.2.1, we have chosen the parametrization in such a way that the fixed point is 0 with multiplier λ and the critical points are 1 and c. Then the point c is a free critical point. Let us denote the space of these polynomials as $\mathbb{C}_\lambda^*$ and since we consider λ to be fixed, any polynomial in this space is parametrized by c and we will denote the polynomials in $\mathbb{C}_\lambda^*$ as P_c. Then the *Zakeri curve*, (Definition 2.9.4), in this space is the following set: $\mathcal{Z}_\lambda^c = \{P_c \in \mathbb{C}_\lambda^* \mid \{c, 1\} \subset \partial\Delta\,(P_c)\}$. It divides the punctured plane $\mathbb{C}_\lambda^*$ into two components, $\mathcal{O}_\lambda^*(0)$ and $\mathcal{O}_\lambda^*(\infty)$, each isomorphic to the punctured disk $\mathbb{D}\backslash\{0\}$. The corresponding punctures are $c = 0$ and $c = \infty$, respectively.

We can identify $\mathbb{C}_\lambda^*$ with the space $\mathbb{C}_\lambda$ by a quotient map τ, which identifies P_c with $P_{1/c}$. This map τ is a homeomorphism on the components $\mathcal{O}_\lambda^*(0)$ and $\mathcal{O}_\lambda^*(\infty)$ and it maps Zakeri curve $\mathcal{Z}_\lambda^c$ to a simple arc $\mathcal{Z}_\lambda$. Then the precise identification can be described as following: $\mathbb{C}_\lambda = \tau(\mathcal{O}_\lambda^*(\infty) \cup \mathcal{Z}_\lambda^c)$.

The main object we are interested in is the closure of the principal hyperbolic component $\mathcal{P}_\lambda$ in the the connectedness locus $\mathcal{C}_\lambda$. We can define the connectedness locus $\mathcal{C}_\lambda^c$ in $\mathcal{O}_\lambda^*(\infty) \cup \mathcal{Z}_\lambda^c$ and also the closure of the principal hyperbolic component $\mathcal{P}_\lambda^c$ in it.

It turns out, that it is more convenient to describe the structure of $\mathcal{P}^c$ first and then to obtain $\mathcal{P}$ by applying the factorization τ.

4.7.2 The map Φ_λ^c and its domain $\mathcal{D}_\lambda^c$

Let us denote the set of all $P \in \mathcal{C}_\lambda^c$ such that $c \in X(P)$ as $\mathcal{D}_\lambda^c$. Then on this set we can define a map $\Phi_\lambda^c(P) : \mathcal{D}_\lambda^c \to K(Q)$ which corresponds to each polynomial $P \in \mathcal{D}^c$

the image of its critical point $c \in X(P)$ under η_P. More precisely, $\Phi_\lambda^c(P) = \eta_P(c)$.

The main goal of this section is to show, that the set $\mathcal{P}^c$ we are interested in is a subset of $\mathcal{D}_\lambda^c$. It turns out that polynomials in $\mathcal{C}_\lambda^c$, which do not belong to $\mathcal{D}_\lambda^c$, are immediately renormalizable (this property was discussed in Section 4.1). In addition to the Straightening Theorem 4.1.4, we will need one more theorem about polynomial-like mappings. The following theorem is Theorem B in [BOPT16].

Theorem 4.7.1. *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial, and $Y \subset \mathbb{C}$ be a full P-invariant continuum. The following assertions are equivalent:*

1. *the set Y is the filled Julia set of some polynomial-like map $P : U^* \to V^*$ of degree k,*
2. *Y is a component of the set $P^{-1}(P(Y))$, and, for every attracting or parabolic point y of P in Y, the immediate attracting basin of y or the union of all parabolic domains at y is a subset of Y.*

Now we can formulate and prove the proposition about polynomials, which do not belong to $\mathcal{D}_\lambda^c$

Proposition 4.7.2. *Suppose that $P \in \mathcal{C}_\lambda^c \backslash \mathcal{D}_\lambda^c$. Then P is immediately renormalizable with $X(P)$ being the corresponding quadratic-like Julia set.*

Proof. By the definition of $\mathcal{D}_\lambda^c$ we know that $c \notin X(P)$. We also know that the set $X(P)$ is compact and it is easy to see that $X(P)$ is a component of $P^{-1}(X(P))$. Moreover, $X(P)$ could not contain parabolic periodic points of P since otherwise c should belong to $X(P)$, more precisely to one of the parabolic domains which we add to $Y(P)$. By Theorems 4.1.4 and 4.7.1, there is a Jordan domain $U \supset X(P)$ such that $P : U \to P(U)$ is a quadratic-like map whose filled Julia set coincides with $X(P)$.

□

For any degree d polynomial f belongs to PHD_d if and only if all its critical points are in the immediate attracting basin of the same attracting (or super-attracting) fixed point. Then the set $\mathcal{P}_\lambda^c$ consists of polynomials in $\mathcal{C}_\lambda^c$ that can be approximated by sequences $P_n \in \mathcal{C}_{\lambda_n}^c$ with $|\lambda_n| < 1$ and both critical points of P_n in the immediate basin of 0.

The following lemma shows, that polynomials in the set $\mathcal{P}_\lambda^c$ are always also in the set $\mathcal{D}_\lambda^c$.

Corollary 4.7.3. *The set $\mathcal{P}_\lambda^c$ is a subset of $\mathcal{D}_\lambda^c$.*

Proof. If $P \in \mathcal{C}_\lambda^c \backslash \mathcal{D}_\lambda^c$, then by Proposition 4.7.2, there is a quadratic-like map $P : U \to V$ with filled Julia set $X(P)$, and we may choose U and V so that $c \notin V$. If we take some parameter μ very close to λ but with $|\mu| < 1$; then we can choose any $f \in \mathcal{C}_\mu^c$ very close to P. By Lemma 4.5.6 we know, that $f^{-1}(V)$ is close to $P^{-1}(V)$ and a component U_f of $f^{-1}(V)$ is close to U. It follows that $f : U_f \to V$ is a quadratic-like map, and its filled Julia set contains an attracting point 0. And now we know that the filled Julia set of $f : U_f \to V$ is a Jordan disk, f is two-to-one on it. Thus it is impossible that f is in the principal hyperbolic component, and so $P \notin \mathcal{P}_\lambda^c$. □

4.7.3 Continuity of the parameter map Φ_λ^c

We start with showing continuity of Φ_λ^c at point P_1 – the polynomial in $\mathbb{C}_\lambda^*$ such that $c = 1$ is a multiple critical point.

Lemma 4.7.4. *Suppose that a sequence $P_{c_n} \in \mathcal{D}_\lambda^c$ converges to P_1 (so that $c_n \to 1$). If $\eta_{P_{c_n}}(c_n)$ converges, then the limit is equal to 1.*

Proof. Let us assume that the sequence $c_{Q,n} = \eta_{P_{c_n}}(c_n)$ in the dynamical plane of the polynomial Q converges to some different from 1. Denote it as c_Q. We can choose a sequence of points close to the sequence (c_n) with a limit also 1, more precisely, a sequence (y_n) such that its term y_n is very close to c_n for any n. Then there are two cases: $c_Q \in \partial\Delta(Q)$ or $c_Q \notin \partial\Delta(Q)$. Let us start with the first one.

1) We assume that $c_Q \in \partial\Delta(Q)$, then the multi-angle of c_Q is just some (α_0). We know that 1 and c_Q in $\overline{\Delta}(Q)$ can be separated, moreover here are two internal rays R_Q, L_Q in $\Delta(Q)$ such that $\overline{R_Q \cup L_Q}$ separates these two points. Then there is a simple unbounded curve Γ_Q separating 1 and c_Q in $\mathbb{C}$ and containing R_Q and

L_Q. This curve Γ_Q is the union of two legal arcs from 0 to repelling periodic points of Q and the external rays of Q landing at these repelling points. We want to introduce a curve Γ corresponding to Γ_Q also in the dynamical plane of P_1. The natural requirement for this correspondence is Γ being consisted of internal rays in various bubbles of P_1, centers of those bubbles, landing points of those rays, a couple of repelling periodic points of P_1, and a couple of external rays of P_1 landing at these repelling points as in the case with Γ_Q. Moreover, there should be a bijective correspondence between bubbles A intersecting Γ and bubbles A_Q intersecting Γ_Q so that $A \cap \Gamma$ includes internal rays of the same arguments as $A_Q \cap \Gamma_Q$, adjacent bubbles correspond to adjacent bubbles, and $\Delta(P_1)$ corresponds to $\Delta(Q)$. Such a curve Γ exists (but is not unique) by Theorem 4.4.10. If we now consider some c close to 1 we obtain a set Γ_c for a polynomial P_c, which is close to Γ. But this set may become disconnected, since Γ is not stable. Depending on c, we may choose Γ since it is not unique so that Γ_c stays connected. We need to require that if Γ passes trough bubbles A and A' consecutively, then these two bubble should not be attached by the root point $r(A)$. Thus for every c_n close to c we get two curves Γ and Γ_{c_n}close to each other in the spherical metric and separating 1 from the landing point y of the internal ray in $\Delta(P_1)$ of argument α_0. Then y_n should have multi-angle $(\alpha_{0,n}, \dots)$ with $\alpha_{0,n}$ close to α_0, for n big enough. It follows that y and y_n are on the same side of Γ_{c_n}, and 1 is on the other side. Moreover, Γ_{c_n} cannot accumulate on 1. We get a contradiction with $y_n \to 1$.

2) Now we consider the second case when $c_Q \notin \partial\Delta(Q)$; we may assume that $c_Q \in J(Q)$. Then c_Q has some multi-angle $(\alpha_0, \alpha_1, \alpha_2, \dots)$ and for n big enough, since $\eta_{P_{cn}}$ is continuous, y_n has multi-angle $(\alpha_{0,n}, \alpha_{1,n}, \alpha_{2,n}, \dots)$ with $\alpha_{0,n}, \alpha_{1,n}, \alpha_{2,n}$ close to α_0, α_1, α_2. If $\alpha_0 \neq 0$, then we can apply the same separation argument as in the case 1).

Now if $\alpha_0 = 0$ we also have $\alpha_{0,n} = 0$ for large n. It turns out that we still can get a simple curve Γ_Q with properties as above. Let us consider the bubble of Q with multiangle (0). Denote it as $A_{Q,1}$ and the landing point of the internal ray in $A_{Q,1}$ of argument α_2 as x_Q. Then the desired curve Γ_Q will be the union of two legal arcs from the bubble center $o_{A_{Q,1}}$ to certain repelling periodic points of Q and the external rays of Q landing at these repelling points. It separates x_Q from 1 and differs from the curve Γ_Q in the previous case only by the fact that it is now centered at $o_{A_{Q,1}}$ rather than 0. Thus we can apply the same proof as in the case 1). □

Now we want to reformulate the Main Theorem 4.2.3 for the map Φ^c_λ. The statement becomes the following:

Theorem 4.7.5. *The map $\eta_P : X(P) \to K(Q)$ is monotone for every $P \in \mathcal{C}^c_\lambda$ except when $c \in X(P)\backslash J(P)$. For every $P_c \in \mathcal{P}^c_\lambda$, the critical point c is in $X(P_c)$. The map $\Phi^c_\lambda : P_c \mapsto \eta_{P_c}(c)$ is defined and continuous on $\mathcal{P}^c_\lambda$. It takes values in $K(Q)\backslash\Delta(Q)$.*

The Main Theorem can be obtained from Theorem 4.7.5 by applying the quotient projection τ from $\mathbb{C}^*_\lambda$ to $\mathbb{C}_\lambda$. The last part which we need to complete the proof of Theorem 4.7.5 is the following Theorem:

Theorem 4.7.6. *The map $\Phi^c_\lambda : \mathcal{D}^c_\lambda \to K(Q)$ is continuous.*

Proof. Let us consider some polynomial $P \in \mathcal{D}^c_\lambda$ and a sequence $P_{c_n} \in \mathcal{D}^c_\lambda$ which converge to this polynomial $P = P_c$. We have to show that the sequence of images $c_{Q,n} = \Phi^c_\lambda(P_n)$ converge to $c_Q = \Phi^c_\lambda(P)$. Let us assume that it is not true, then we can choose a subsequence such that $c_{Q,n} \to c'_Q \neq c_Q$. There are three cases depending on the position of c_Q in $J(Q)$.

1) The first case: c_Q belongs to a bubble A_Q of Q. Then there is a legal bubble A of P corresponding to A_Q. This bubble contains c, which we know is a limit of the sequence c_n. Since the bubble $P(A)$ is stable, for each P_{c_n} with n big enough there exists a unique bubble B_n close to $P(A)$, which also contains the critical value $P_{c_n}(c_n)$. Then by Lemma 4.5.6 a component A_n of $P^{-1}_{c_n}(B_n)$ should contain the critical point c_n. It is also is close to A. Each A_n corresponds to A_Q since they all A_n have the same multi-angle. The map $\eta_{\tilde{P}}(\tilde{P}(\tilde{c}))$ depends continuously on $\tilde{P} = P_{\tilde{c}}$ near P. Let us consider two points $Q(c_Q)$ and $Q(c'_Q)$ (they are different since c_Q and c'_Q lie in the same bubble A_Q). We get $Q(c_{Q,n}) = \eta_{P_{c_n}}(P_{c_n}(c_n)) \to \eta_P(P(c)) = Q(c)$, but also $Q(c_{Q,n}) \to Q(c'_Q)$ since $c_{Q,n} \to c'_Q$. Thus c_Q has to coincide with c'_Q.

2) The second case: $c_Q \in J(Q)$ and either $c \notin \partial\Delta(P)$ or $c \in \partial\Delta(P)$ is never mapped to 1 under P. Points c'_Q and c_Q are separated by an adapted wedge W_Q

(Proposition 4.6.7) and the sequence $c_{Q,n}$ has to belong to W_Q for all sufficiently large n (since W_Q is open). We can even choose some compact subset C_Q of W_Q for all large n, such that $c_{Q,n}$ belongs to it. We know that there is some Siegel wedge W of P corresponding to W_Q. Since the boundary of W is stable, there are Siegel wedges W_n for P_{c_n} close to W that correspond to the same W_Q. Then $c_n \in W_n$ for n big enough since $c_{Q,n} \in C_Q$ and thus $c \in \overline{W}$, a contradiction.

3) In the last case we assume that $c \in \partial\Delta(P)$ and $P^k(c) = 1$. For $k = 0$ it is a consequence of Lemma 4.7.4. Case with $k > 0$ can be brought to the previous one by considering the following conjugation: by a linear map that takes c to 1 we conjugate P to a polynomial $\tilde{P} \in \mathcal{Z}^c_\lambda$. Then 1 is mapped to a critical point $\tilde{c}$ of $\tilde{P}$ such that $\tilde{P}^k(1) = \tilde{c}$ and thus we know that $\tilde{c}$ is never mapped to 1 under $\tilde{P}$. □

4.7.4 The final projection

As we have mentioned in the previous section, the Main Theorem can be obtained from Theorem 4.7.5 by applying the quotient projection τ from $\mathbb{C}^*_\lambda$ to $\mathbb{C}_\lambda$. Now we want to show, that we get the same result applying the the quotient map $\pi : K(Q)\backslash\Delta(Q) \to \tilde{K}(Q)$, corresponding to the quotient of $K(Q)\backslash\Delta(Q)$ described in Definition 4.2.1. If we apply π, we get well defined and continuous map $\pi\circ\Phi^c_\lambda : \mathcal{P}^c_\lambda \to \tilde{K}(Q)$. The only thing we need to show to prove that thus we obtain a continuous map Φ_λ from $\mathcal{P}_\lambda$ to $\tilde{K}(Q)$ is the fact, that P_c and $P_{1/c}$ have the same images under $\pi \circ \Phi^c_\lambda$. More precisely, we want to prove the following Lemma:

Lemma 4.7.7. *The points $\Phi^c_\lambda(P_c)$ and $\Phi^c_\lambda(P_{1/c})$ have the same π-images in $\tilde{K}(Q)$.*

We will need the notion of the *angular difference* for the proof.

Definition 4.7.8. *The difference $\alpha - \beta \in \mathbb{R}/\mathbb{Z}$, where $a = \overline{\psi}_{\Delta(P)}(e^{2\pi i\alpha})$ and $b = \overline{\psi}_{\Delta(P)}(e^{2\pi i\beta})$ is called the* angular difference *between two points $a, b \in \partial\Delta(P)$.*

Proof of Lemma 4.7.7. For P_c and $P_{1/c}$ the angular differences between the two critical points in the boundary of the Siegel disk have the same module and different signs since P_c and $P_{1/c}$ are affinely conjugate. Thus $\Phi^c_\lambda(P_c)$ and $\Phi^c_\lambda(P_{1/c})$ have the same angular difference with 1 up to a sign in $\partial\Delta(Q)$. Then two such points are identified in $\tilde{K}(Q)$ by definition of $\tilde{K}(Q)$. □

Thus by proving Lemma 4.7.7 we complete the proof of the Main Theorem 4.2.3.

Chapter 5

Open questions

In this Chapter we focus on some open questions and possible further research directions.

The *Ivy algorithm*, presented in Chapter 3, makes a contribution to the understanding of the combinatorics of invariant of Thurston map. However, we do not make any statement about the structure of the Ivy graph. For example, it is an interesting question, if there could exist an infinite sequence of pairwise different ivy objects $[T_n]$ such that $[T_n] \multimap [T_{n+1}]$. It should be also very useful to understand, if there could be two disjoint pullback invariant subsets of $\mathrm{Ivy}(f)$ for some quadratic rational Thurston map f. Moreover, considering complicated examples of Ivy graphs of Thuston maps with no a priori known invariant spanning trees is a separate non-trivial problem.

In Chapter 4 we have presented the continuous parameter map $\Phi_\lambda : \mathcal{P}_\lambda \to \tilde{K}(Q)$. But the properties of this map are not well-studied yet. For example, we would like to figure out, if Φ_λ is surjective or monotone.

Bibliography

[Ahl66] L.V. Ahlfors, *Lectures on quasiconformal mappings*, D. Van Nostrand Company 1966.

[AB56] L. Ahlfors, A. Beurling, *The boundary correspondence for quasiconformal mappings*, Acta Math. **96** (1956), 125–142.

[AY09] M. Aspenberg, M. Yampolsky, *Mating non-renormalizable quadratic polynomials*, Commun. Math. Phys. **287** (2009), 1–40.

[BC06] X. Buff, A. Cheritat, *The Brjuno Function Continuously Estimates the Size of Siegel Disks*, Annals of Mathematics. **164/1** (2006), 265–312

[BBCO10] A. Blokh, X. Buff, A. Cheritat, L. Oversteegen, *The solar Julia sets of basic quadratic Cremer polynomials*, Ergodic Theory and Dynamical Systems **30** (2010), 51–65

[BCOT19] A. Blokh, A. Chéritat, L. Oversteegen, V. Timorin, *Location of Siegel capture polynomials in parameter spaces*, preprint `arXiv:1809.02586` (2019), to appear in Nonlinearity.

[BD17] L. Bartholdi, D. Dudko, *Algorithmic aspects of branched coverings*, Ann. Fac. Sci. Toulouse **5** (2017), 1219–1296.

[BFH92] B. Bielefeld, Y. Fisher, J. Hubbard, *The classification of critically preperiodic polynomials as dynamical systems*, J. of the AMS, **5**:4 (1992), 721–762.

[BFMOT13] A. Blokh, R. Fokkink, J. Mayer, L. Oversteegen, E. Tymchatyn, *Fixed point theorems for plane continua with applications*, Memoirs of the American Mathematical Society, **224** (2013), no. 1053.

[BH01] X. Buff, C. Henriksen, *Julia Sets in Parameter Spaces*, Commun. Math. Phys. **220** (2001), 333–375.

[BM17] M. Bonk, D. Meyer, *Expanding Thurston maps*, AMS Mathematical Surveys and Monographs **225** (2017).

[BN06] L. Bartholdi, V. Nekrashevych, *Thurston equivalence of topological polynomials*, Acta Math. **197** (2006), 1–51

[BOPT14] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *The main cubioid*, Nonlinearity **27** (2014), 1879–1897

[BOPT16] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *Quadratic-Like Dynamics of Cubic Polynomials*, Commun. Math. Phys. (2016), 341–733.

[BOST22] A. Blokh, L. Oversteegen, A. Shepelevtseva, V. Timorin, *Invariant spanning trees for quadratic rational maps*, to appear in Moscow Mathematical Journal

[BP08] X. Buff, C. Petersen, *On the size of linearization domains*, Math. Proc. Camb. Phil. Soc. **145**, no. 2 (2008), 443–456.

[Br72] A. Brjuno, *Analytic form of differential equations*, Tr. Mosk. Mat. Obs., **26** (1972), 199–239.

[CC15] D. Cheraghi, A. Chéritat, 2015, *A proof of the Marmi-Moussa-Yoccoz conjecture for rotation numbers of high type*, Inventiones Mathematicae, **202** (2015), 677–742

[CFP01] J. Cannon, W. Floyd, W. Parry, *Finite subdivision rules*, Conformal Geometry and Dynamics, **5** (8) (2001), 153–196.

[CG] L. Carleson, T. W. Gamelin, *Complex Dynamics*, Springer-Verlag, (1993)

[CG+15] K. Cordwell, S. Gilbertson, N. Nuechterlein, K. Pilgrim, S. Pinella, *The classification of critically fixed rational maps*, Conform. Geom. Dyn. **19** (2015), 51–94.

[Che20] A. Chéritat, *On the size of Siegel disks with fixed multiplier for cubic polynomials*, preprint `arXiv:2003.13337` (2020).

[Dou87] A. Douady, *Disques de Siegel et anneaux de Herman*, Astérisque, **152-153**, exp. 677 (1987), 151–172.

[DE86] A. Douady, C. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. **157** (1986), 23–48.

[DH85] A. Douady, J. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. Ec. Norm. Sup. 18 (1985), 287–343.

[DH85a] A. Douady, J. H. Hubbard, *Étude dynamique des polynômes complexes* I, II, Publications Mathématiques d'Orsay **84-02** (1984), **85-04** (1985).

[DH93] A. Douady, J. Hubbard, *A proof of Thurston's topological characterization of rational functions*, Acta Math., 171(2):263 297, 1993.

[FM12] B. Farb, D. Margalit, *A Primer on Mapping Class Groups*, Princeton Univ. Press, 2012.

[GM93] L.R. Goldberg, J. Milnor, "Fixed points of polynomial maps. Part II. Fixed point portraits", Annal. sci. de l'École Normale Supérieure, 4^e série, **26**, n° 1 (1993), 51–98

[Her87] M. Herman, *Conjugaison quasi symétrique des homéomorphismes du cercle à des rotations*, preliminary manuscript, https://www.math.kyoto-u.ac.jp/ mitsu/Herman/qsconj1/

[Hlu17] M. Hlushchanka, *Invariant graphs, tilings, and iterated monodromy groups*, Ph.D. Thesis, Jacobs University Bremen (2017).

[HS94] J. Hubbard, D. Schleicher, *The spider algorithm*, in: Proc. Sympos. Appl. Math **49**, AMS (1994), 155–180.

[Hub06] J.H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics.* Volume 1. Matrix Editions 2006.

[IK12] H. Inou, J. Kiwi, *Combinatorics and topology of straightening maps, I: Compactness and bijectivity*, Adv. in Math. **231** (2012), 2666–2733.

[Kam01] A. Kameyama, *The Thurston equivalence for postcritically finite branched coverings*, Osaka J. Math. **38** (2001), no. 3, 565–610.

[Kiw00] J. Kiwi, *Non-accessible critical points of Cremer polynomials*, Ergodic Theory and Dynamical Systems, **20** (2000), no. 5, 1391–1403.

[KL18] G. Kelsey, R. Lodge, *Quadratic Thurston maps with few postcritical points*, Geom. Dedicata (2018). `https://doi.org/10.1007/s10711-018-0387-5`

[KPS16] S. Koch, K. Pilgrim, N. Selinger, *Pullback invariants of Thurston maps.* Trans. Amer. Math. Soc. **368** (2016), no. 7, 4621–4655.

[LMS15] R. Lodge, Y. Mikulich, D. Schleicher, *A classification of postcritically finite Newton maps.* `https://arxiv.org/abs/1510.02771`

[Luo95] J. Luo, *Combinatorics and Holomorphic Dynamics: Captures, Matings and Newton's Method*, PhD Thesis, Cornell University, 1995.

[Lyu83] M. Lyubich, *Some typical properties of the dynamics of rational mappings*, Russian Math. Surveys **38** (1983), no. 5, 154–155.

[Lyu91] M. Lyubich, *On the Lebesgue measure of the Julia set of a quadratic polynomial*, `arXiv:math/9201285` (1991)

[MA41] S. MacLane, V. W. Adkisson, *Extensions of homeomorphisms on the sphere.* Lectures in Topology, University of Michigan Press, Ann Arbor, Mich. 1941, 223–236.

[Ma] S. Marmi, *An Introduction to Small Divisors Problems*, arXiv:math/0009232

[Ma90] S. Marmi, *Critical functions for complex analytic maps*, J. Phys. A: Math. Gen. **23** (1990), 3447-=3474

[Man93] R. Mañé, *On a theorem of Fatou*, Bol. Soc. Brasil. Mat. (N.S.) **24** (1993), 1–11.

[MMY97] S. Marmi, P. Moussa , J.-C. Yoccoz, *The Brjuno functions and their regularity properties*, Comm. Math. Phys. **186-2** (1997), 265–293

[MSS83] R. Mañé, P. Sad, D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 2, 193–217.

[McM94] C. McMullen, *Complex dynamics and renormalization*, Annals of Mathematics Studies **135**, Princeton University Press, Princeton, NJ (1994)

[Mil06] J. Milnor, *Dynamics in One Complex Variable*, Annals of Mathematical Studies **160**, Princeton (2006).

[Mil09] J. Milnor, *Cubic polynomial maps with periodic critical orbit I*, in: "Complex Dynamics: Families and Friends", ed. D. Schleicher, A.K. Peters (2009), 333–411.

[Mr25] R.L. Moore, *Concerning upper-semicontinuous collections of continua*, Transactions of the AMS **27**, Vol. 4 (1925), 416–428

[Nek05] V. Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, Volume: **117** (2005), AMS.

[Nek09] V. Nekrashevych, *Combinatorics of polynomial iterations*, in: "Complex Dynamics: Families and Friends", ed. D. Schleicher, A.K. Peters, Wellesley, MA (2009), 169–214.

[Pet96] C. Petersen, *Local connectivity of some Julia sets containing a circle with an irrational rotation*, Acta Math., 177 (1996), 163-224

[Pil03] K. Pilgrim, *An algebraic formulation of Thurston's combinatorial equivalence*, Proc. Amer. Math. Soc. **131** (2003), no. 11, 3527–3534.

[Poi93] A. Poirier, *On postcritically finite polynomials, part 2: Hubbard trees*, Stony Brook IMS preprint **93/7** (1993). arXiv:math/9307235

[Poi10] A. Poirier, *Hubbard trees*, Fundamenta Mathematicae **208**(3) (2010), 193–248.

[PT09] C. Petersen, Tan Lei, *Analytic coordinates recording cubic dynamics*. In: Complex dynamics, Families and Friends, A K Peters, Wellesley, MA (2009), 413–449.

[Ree92] M. Rees, *A partial description of the Parameter Space of Rational Maps of Degree Two: Part 1* Acta Math. **168** (1992), 11–87

[Ree10] M. Rees, *A Fundamental Domain for V_3*, Mém. Soc. Math. Fr. (N.S) (2010).

[SW20] W. Shen, Y. Wang, *Primitive tuning via quasiconformal surgery*, preprint `arXiv:2010.05199`, to appear in Israel Journal of Mathematics.

[Si] C. Siegel, *Iteration of analytic functions*, Ann. of Math. **43**(2) (1942), 607–612.

[Slo91] Z. Slodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. **111** (1991), 347–355.

[ST19] A. Shepelevtseva, V. Timorin, Invariant spanning trees for quadratic rational maps, Arnold Math J. **5**, (2019), 435–481

[Sul] D. Sullivan, *Quasiconformal homeomorphisms and dynamics III: Topological conjugacy classes of analytic endomorphisms*, manuscript.

[Swi98] G. Świ**c**atek, *On critical circle homeomorphisms*, Bol. Soc. Bras. Mat. **29** (1998), 329–351.

[Wit88] B. Wittner, "On the bifurcation loci of rational maps of degree two", PhD Thesis, Cornell University, 1988

[Yan17] J. Yang, *Applications of Renormalization to Irrationally Indifferent Complex Dynamics*, PhD Thesis, University of Toronto, 2017.

[Yoc95] J.C. Yoccoz, *Petits Diviseurs en Dimension 1*, Astérisque **231** (1995)

[Zak99] S. Zakeri, *Dynamics of cubic siegel polynomials*, Commun. Math. Phys. **206** (1999), 185–233.

[Zak16] S. Zakeri, *Conformal fitness and uniformization of holomorphically moving disks*, Trans. Amer. Math. Soc. **368** (2016), 1023–1049.