Discrete and Continuous Dynamical Systems
Vol. 42, No. 7, July 2022, pp. 3077–3101
doi:10.3934/dcds.2022009

ANALYTIC LINEARIZATION OF A GENERALIZATION OF THE SEMI-STANDARD MAP: RADIUS OF CONVERGENCE AND BRJUNO SUM

Claire Chavaudret*

IMJ-PRG, Université de Paris
Bâtiment Sophie Germain, Boite Courrier 7012
8 Place Aurélie Nemours
75205 Paris Cedex 13, France

Stefano Marmi

Scuola Normale Superiore
Piazza dei Cavalieri, 7
56126 Pisa, Italy

(Communicated by Rafael de la Llave)

Abstract. One considers a system on $\mathbb{C}^2$ close to an invariant curve which can be viewed as a generalization of the semi-standard map to a trigonometric polynomial with many Fourier modes. The radius of convergence of an analytic linearization of the system around the invariant curve is bounded by the exponential of the negative Brjuno sum of $d\alpha$, where $d \in \mathbb{N}^*$ and α is the frequency of the linear part, and the error function is non decreasing with respect to the smallest coefficient of the trigonometric polynomial.

1. Introduction.

Consider the following discrete dynamical system:

$$\left\{ \begin{array}{l} x_{n+1} = x_n + y_n + A(x_n) \\ y_{n+1} = y_n + A(x_n) \end{array} \right. \tag{1}$$

where A is a trigonometric polynomial with positive Fourier modes: $A(x) = \sum_{K=1}^{N} a_K e^{iKx}$, $N \in \mathbb{N}$, $a_K \in \mathbb{C}$. This can be viewed as a generalization of the semi-standard map, which is given by $A(x) = e^{ix}$. The semi-standard map was introduced as a model which is similar to, but easier to study than, the standard map given by $A(x) = \sin x$, where the coexistence of both positive and negative Fourier modes makes the linearization problem more difficult to solve. The standard map, or Taylor-Chirikov map, is the Poincaré section of a kicked rotator and can be used as an approximating model in several areas of physics. It is an example of a simple low-dimensional system displaying chaos and numerical studies (see for

2020 *Mathematics Subject Classification.* Primary: 37F50, 39A45; Secondary: 37E99,39A30, 39A33,37C05.

Key words and phrases. Brjuno sum, Brjuno condition, anaytic linearization, semi-standard map, small divisors.

The second author is supported by the Dynamics and Information Theory Institute at the Scuola Normale Superiore and the MIUR PRIN Project Regular and stochastic behaviour in dynamical systems nr. 2017S35EHN..

*Corresponding author: Claire Chavaudret.

instance [17]) have investigated the boundary between the KAM regime and chaos for this system. However few analytical results are known on the standard map.

Note that when the non-zero coefficients of A are all multiples of d, the ramified change of variable $w = z^d, \mu = \lambda^d$ semi-conjugates our model to the original semistandard map. With the change of variables $z_n = e^{ix_n}, \lambda_n = e^{iy_n}$, system (1) is conjugated to the system

$$\begin{cases} z_{n+1} = \lambda_n z_n \prod_{K=1}^N e^{ia_K z_n^K} \\ \lambda_{n+1} = \lambda_n \prod_{K=1}^N e^{ia_K z_n^K} \end{cases} \tag{2}$$

for which every point of $\{0\} \times S^1$ is a fixed point.

Let $F(z,\lambda) = (\lambda z \prod_{K=1}^N e^{ia_K z^K}, \lambda \prod_{K=1}^N e^{ia_K z^K})$ so the system (2) can be written $(z_{n+1}, \lambda_{n+1}) = F(z_n, \lambda_n)$. One looks for a linearization of the system, that is to say, a map $H(z,\lambda) = (h_1(z,\lambda), h_2(z,\lambda))$, close to the identity, such that $F \circ H = H \circ R$ where $R(z,\lambda) = (\lambda z, \lambda)$. If there is such a linearization and if it is analytic in z, then the invariant curves of the rotation R are smoothly preserved. Then one has a family of invariant closed curves in $\mathbb{C}^2$, corresponding to the numbers λ with modulus 1 and z in a neighbourhood of 0 where the linearization is analytic in z. Now if λ is a root of unity, there is no way of finding a linearization which is analytic in z. Thus we will have to assign a value to the parameter λ, with $|\lambda| = 1$ and $\frac{arg(\lambda)}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$, to construct a linearization which is close to the identity and analytic; its radius of convergence will depend on the arithmetical properties of λ.

Let $\alpha = \frac{arg(\lambda)}{2\pi}$. Davie [13]nd Marmi [19] proved that concerning the semi-standard map, the radius of convergence $\rho(\alpha)$ of the linearization is bounded as follows:

$$\exp(-2B(\alpha) - C) \leq \rho \leq \exp(-2B(\alpha) + C')$$

where $C > 0, C' > 0$ do not depend on the complex argument α of λ and where $B(\alpha)$ is the Brjuno sum of α (see Section 4 for a definition of the Brjuno sum). In particular, if $B(\alpha)$ diverges, then there is no analytic linearization around 0. This can be reformulated as stating that the error function $\alpha \mapsto 2B(\alpha) + \ln \rho(\alpha)$ is bounded. The boundedness of this error function implies that the the divergence of the radius of convergence at rational numbers and at non-Brjuno irrational numbers is exactly compensated by a purely-arithmetical function, thus emphasizing the intimate number-theoretical nature of this stability problem.

After the numerical evidence in [21], a similar result about the standard map, in the perturbative case, was proved in [2] and [3]. However, concerning the semi-standard map and in the present paper, the strong assumption of Fourier modes being only positive makes it possible to remove the perturbative assumption.

The Brjuno sum was first introduced in [5] to give a sufficient condition to the convergence of the linearization for analytic vector fields around a fixed point.

Yoccoz proved in [23] that the Brjuno condition (i.e the convergence of the Brjuno function, or equivalently of the Brjuno sum, although the Brjuno function and the Brjuno sum may differ) is necessary and sufficient to the analytic linearization of the quadratic polynomial and of germs of diffeomorphisms of $(\mathbb{C}, 0)$. This resulted in the study of the error function $B_f + \ln r$, where B_f is the Brjuno function and r is the radius of convergence of the linearization for the quadratic polynomial. It was conjectured in [20] that this function is 1/2-Hölder continuous and Buff and Chéritat showed first that it is bounded in [6], then that it is continuous in [7].

Cheraghi-Chéritat then proved that a restriction of this error function is 1/2-Hölder continuous.

Brjuno's lower bound on the convergence radius for linearization of analytic vector fields was improved in [15]. In [22], Stolovitch replaced the arithmetical condition on the spectrum of the linear part by a condition of algebraic nature. The Brjuno sum was also proved to play a role in other analytic linearization problems, as for instance linearization of vector fields around an invariant torus (see [1] and [10]), or reducibility of quasiperiodic cocycles ([14], [18], [11]).

The optimality of the Brjuno condition was also studied in other linearization problems. Carletti-Marmi proved in [8] that the Brjuno condition is also necessary to linearize analytically a germ of diffeomorphism around a fixed point, and generalize it to Gevrey classes. However the continuity of the analogue of the error function for linearization problems more general than the quadratic polynomial remains open up to now.

Our main result is stated in the following two theorems:

Main Theorem 1. *Let ρ be the radius of convergence of the linearization of the system (2). Let d be the greatest common divisor of the indices of the Fourier modes of A. There exists $C \geq 0$, which is a non decreasing function of $\max_{K=1,\dots,N} |a_K|$, and which only depends on this quantity and on the indices of the Fourier modes of A, such that*

$$\rho \geq \exp(-\frac{2}{d}B(d\alpha) - C) \tag{3}$$

Remark 1. When $A(x)$ is a convergent series with an infinite number of eneric?harmonics, a minor adaptation of our arguments guarantee the convergence of the linearization under the Brjuno condition.

Main Theorem 2. *Assume that the coefficients of A satisfy the following assumption: there exists $\theta \in \mathbb{R}$ such that for all $k = 1, \dots, N$ with $a_k \neq 0$, the complex argument of the number $i^{k-1}a_k$ is $k\theta$.*

Let ρ be the radius of convergence of the linearization of the system (2). Let d be the greatest common divisor of the indices of the Fourier modes of A. There exists $C' \geq 0$, which is a non increasing function of $\min\{|a_K| \neq 0,\ K = 1, \dots, N\}$, an increasing function of $\max_{K=1,\dots,N} |a_K|$, and depends only on these quantities, such that

$$\rho \leq \exp(-\frac{2}{d}B(d\alpha) + C') \tag{4}$$

In particular, if $B(d\alpha)$ diverges, then there is no analytic linearization.

Remark 2. Let κ_0 be the smallest integer such that $a_{\kappa_0} \neq 0$. The assumption in Theorem 2 says that the argument of every non zero a_k is the following function of a_{κ_0}:

$$Arg(a_k) = \frac{k}{\kappa_0} Arg(a_{\kappa_0}) + \frac{\pi}{2}(1 - \frac{k}{\kappa_0})$$

This assumption is in the spirit of Cremer's counterexample of non-linearizable germs, except that the arguments are defined from the beginning instead of recursively.

Remark 3. It is also worth mentioning that $d\alpha$ is a Brjuno number (i.e $B(d\alpha)$ converges) if and only if α is a Brjuno number (i.e $B(\alpha)$ converges; see Appendix, section 8.2 for a proof).

The main result is obtained by a direct analysis of the coefficients of the formal linearization (which always exists, and is unique if one requires it to be formally close to the identity); as in [19] and [13], it appears that there is a link between those coefficients and the Brjuno sum of $d\alpha$. To get an upper bound of the radius of convergence, that is to say, a lower bound on the coefficients of the linearization, one uses a strong assumption on the complex arguments of the coefficients of A, in order to be able to bound the sum from below by just one of its terms. However the lower bound on the radius of convergence does not use this assumption.

2. **Notations.**

Let $x \in \mathbb{R}$, one denotes by $||x||_{\mathbb{Z}}$ the distance between x and the closest integer: $||x||_{\mathbb{Z}} = \min_{p\in\mathbb{Z}} |x-p|$.

Considering the trigonometric polynomial $A(x)$ defined at the beginning, denote by $\kappa_0 < \kappa_1 < \cdots < \kappa_\nu$ all indices of Fourier modes of A: thus for all $0 \leq i \leq \nu$, $a_{\kappa_i} \neq 0$, and if $\forall i = 0, \dots, \nu$, $K \neq \kappa_i$, then $a_K = 0$. Also denote by κ_{min} (resp. κ_{max}) the number κ_i minimizing (resp. maximizing) $\{|a_{\kappa_i}|, i = 0, \dots, \nu\}$.

Denote by $\mathcal{M} \subset \mathbb{N}$ the additive semi-group generated by $\{\kappa_0, \dots, \kappa_\nu\}$:

$$\mathcal{M} = \{p_0\kappa_0 + \cdots + p_\nu\kappa_\nu > 0, p_0 \in \mathbb{N}, \dots, p_\nu \in \mathbb{N}\} \tag{5}$$

For all $0 \leq k \leq n$, we shall denote by $\binom{n}{k}$ the combinatorial number $\frac{n!}{k!(n-k)!}$.

3. **Analysis of the linearization.**

With a reasoning similar to the one in [4] (which is reproduced and adapted to the present model in the appendix), one proves that if H linearizes the system and is close to the identity, that is, its Taylor series starts with (z,λ), then the first component of the linearization is a function $h_1(z,\lambda) = ize^{\Phi_\lambda(z)}$ satisfying (mod $2i\pi$):

$$\sum_{K=1}^{N} ia_K(iz)^K e^{K\Phi_\lambda(z)} = \Phi_\lambda(\lambda^{-1}z) + \Phi_\lambda(\lambda z) - 2\Phi_\lambda(z) \tag{6}$$

(see Lemma 8.2 in the appendix for a proof).

In what follows, λ being fixed with modulus 1 and argument $2\pi\alpha$ such that $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, we will denote for all $l \in \mathbb{N}\setminus\{0\}$,

$$D_l := -4\sin^2(\pi l\alpha) < 0$$

Then $|D_l|^{-1} \geq \frac{1}{4}$ for all $l \geq 1$. Also denote $\Phi_\lambda(z) = \sum_{l\geq 1}\phi_l z^l$ (so the radius of analyticity of Φ_λ is $(\limsup_{l\to+\infty}|\phi_l|^{\frac{1}{l}})^{-1}$). Then

$$\sum_{K\in\{\kappa_0,\dots,\kappa_\nu\}} i^{K+1}a_K z^K \left(1 + \sum_{p\geq 1}\frac{1}{p!}(\sum_{j\geq 1}\phi_j z^j)^p\right)^K = \sum_{l\geq 1}\phi_l z^l D_l \tag{7}$$

Let $l \geq 1$, then $\phi_l = 0$ if $l < \kappa_0$ and if $l \geq \kappa_0$,

$$\phi_l = \frac{1}{D_l} \sum_{\substack{K \in \\ \{\kappa_0, \dots, \\ \kappa_\nu\}, \\ K \le l}} i^{K+1} a_K [\delta_{K,l} + \sum_{\substack{m=1,\dots,K \\ m \le l-K}} \binom{K}{m} \sum_{p_1=1}^{l-K} \cdots \sum_{p_m=1}^{l-K} \frac{1}{p_1! \dots p_m!} \sum_{\substack{j_1^1, \dots, j_{p_1}^1, \\ \vdots \\ j_1^m, \dots, j_{p_m}^m, \\ j_1^1 + \dots + j_{p_1}^1 + \dots \\ + j_1^m + \dots + j_{p_m}^m = l-K}} \phi_{j_1^1} \dots \phi_{j_{p_1}^1} \dots \phi_{j_1^m} \dots \phi_{j_{p_m}^m}] \tag{8}$$

Another recurrence relation is worth mentioning: let $\Psi_\lambda(z) = \sum_{K \in \{\kappa_0, \dots, \kappa_N\}} \Psi_K(z)$, $\Psi_K(z) = i^{K+1} a_K z^K e^{K \Phi_\lambda(z)}$ and denote by $\sum_{k \ge 1} \psi_k z^k$ the Taylor expansion of Ψ_λ and by $\sum_l \psi_{K,l} z^l$ the Taylor expansion of Ψ_K. Then

$$\Psi_\lambda(z) = \Phi_\lambda(\lambda^{-1} z) + \Phi_\lambda(\lambda z) - 2\Phi_\lambda(z) \tag{9}$$

which implies that for all $l \ge \kappa_0$,

$$\psi_l = D_l \phi_l \tag{10}$$

Moreover, differentiating Ψ_K, one sees that $\psi_{K,l} = 0$ for all $l < K$, $\psi_{K,K} = i^{K+1} a_K$, and for all $n \ge K+1$,

$$(n-K)\psi_{K,n} = \sum_{k=1}^{n-1} k \phi_k \psi_{K,n-k} \tag{11}$$

Remark 4. One sees that $\phi_l \ne 0 \Rightarrow l \in \mathcal{M}$. This can be shown by recurrence: for $l = \kappa_0$, the property holds. Assume it holds for all $\kappa_0 \le l' \le l-1$, for a fixed $l > \kappa_0$. Assume that $\phi_l \ne 0$. If $l \in \{\kappa_0, \dots, \kappa_\nu\}$, then $l \in \mathcal{M}$; otherwise, there exists $K \in \{\kappa_0, \dots, \kappa_\nu\}$, $m \le l-K$, $p_1 \le l-K, \dots, p_m \le l-K$ and non vanishing $\phi_{j_1^1}, \dots, \phi_{j_{p_m}^m}$ such that $j_1^1 + \dots + j_{p_m}^m = l-K$. By recurrence assumption, $j_1^1, \dots, j_{p_m}^m \in \mathcal{M}$ therefore $l \in \mathcal{M}$.

A similar fact holds for every Ψ_K: if $\psi_{K,l} \ne 0, l > K$, then $l - K \in \mathcal{M}$. Indeed $\psi_{K,l} = 0$ if $l < K$ and that $\psi_{K,K} = i^{K+1} a_K$ so the recurrence property holds for $l \le K$. Assume the property holds up to a fixed $l \ge K$. If $\psi_{K,l+1} \ne 0$ then by (11) there exist $k \in \{1, \dots, n-1\}$ such that $\phi_k \ne 0$ and $\psi_{K,l+1-k} \ne 0$ so $k \in \mathcal{M}$ and $l+1-k \in \mathcal{M}$, therefore $l+1 \in \mathcal{M}$.

Example: If A is a monomial of order K, then $\Phi_\lambda(z)$ only has coefficients indexed by multiples of K. Then there exists a function $\Xi_\lambda : \mathbb{C} \to \mathbb{C}$ such that for all z, $\Phi_\lambda(z) = \Xi_\lambda(z^K)$.

The following lemma will be used to bound the coefficients of Φ from below by one of the terms of the sum determining them.

Lemma 3.1. *If there exists $\theta \in \mathbb{R}$ such that for all $K \in \{\kappa_0, \dots, \kappa_\nu\}$, the complex number $i^{K-1} a_K$ has argument $K\theta$, then for all $l \in \mathcal{M}$, ϕ_l has argument $l\theta$ and for all $K \in \{\kappa_0, \dots, \kappa_\nu\}$, if $\psi_{K,l} \neq 0$ then $\psi_{K,l}$ has argument $l\theta + \pi$.*

Proof. First one proves the part of the statement concerning ϕ_l:

1. If $l = \kappa_0$, then (8) implies that $\phi_l = \frac{1}{D_{\kappa_0}} i^{\kappa_0+1} a_{\kappa_0}$ which is, by assumption, the product of a number of modulus $\kappa_0\theta$ with $\frac{-1}{D_{\kappa_0}}$, the latter being real and positive.
2. Let $l \geq \kappa_0 + 1$. Assume that for all $l' \in \mathcal{M}$ such that $l' \leq l-1$, $\phi_{l'}$ has argument $l'\theta$. If l differs from all κ_i, then by (8), ϕ_l is the sum of terms which are the product of a number $\frac{i^{K+1} a_K}{D_l}$, which by assumption has argument $K\theta$, with a number which by recurrence assumption has argument $(l-K)\theta$. If l is one of the κ_i, then one has to add the term $\frac{1}{D_l} i^{l+1} a_l$, which has argument $l\theta$.

As for the statement concerning $\psi_{K,l}$, since $\psi_{K,K} = i^{K+1} a_K$ which has argument $K\theta + \pi$, the property holds for $l \leq K$. Assume this property holds up to a fixed $l \geq K$. By equation (11), if $\psi_{K,l+1} \neq 0$ then $\psi_{K,l+1}$ is a sum of terms with argument $k\theta + (l+1-k)\theta + \pi$ for $k \in \{1, \dots, n-1\}$, therefore it has argument $(l+1)\theta + \pi$. □

Remark 5. For instance, $A(x)$ satisfies the assumption of Lemma 3.1 if it only has coefficients $a_K \in i\mathbb{R}^+$ (then $\theta = \frac{\pi}{2}$), or such that $i^{k+1} a_K \in \mathbb{R}^-$ (then $\theta = 0$), or if A is a monomial (without restriction on θ).

3.1. The semigroup $\mathcal{M}$.

We shall also need the following lemma on the set $\mathcal{M}$:

Lemma 3.2. *Let d be the greatest common divisor of $\kappa_0, \dots, \kappa_\nu$. There exists $N_\mathcal{M}$ such that for all integer $m \geq N_\mathcal{M}$, if m is a multiple of d, then $m \in \mathcal{M}$. If $\kappa_0 = 1$ then $N_\mathcal{M} = 1$.*

Proof. Let $I_0 = [\kappa_0, \kappa_0 + \dots + \kappa_\nu[\cap \mathbb{N}$ and for $p \geq 1$, let

$$I_p = [p(\kappa_0 + \dots + \kappa_\nu), (p+1)(\kappa_0 + \dots + \kappa_\nu)[\cap \mathbb{N}$$

Then $(I_p)_{p\geq 0}$ is a partition of $[\kappa_0, +\infty[\cap \mathbb{N}$. Moreover, if $jd \in I_p \cap \mathcal{M}$ for some $p \geq 0$, then $jd + \kappa_0 + \dots + \kappa_\nu$, which is also a multiple of d, belongs to $I_{p+1} \cap \mathcal{M}$; therefore, in order to prove this lemma, it is sufficient to prove that for all $jd \in I_0 \setminus \mathcal{M}$, there exists $p \geq 1$ such that $jd + p(\kappa_0 + \dots + \kappa_\nu) \in \mathcal{M}$. After a finite number of steps, one obtains an integer P such that all multiples of d belonging to I_P are also elements of $\mathcal{M}$. Translating by $\kappa_0 + \dots + \kappa_\nu$, one will deduce that all multiples of d greater than $p(\kappa_0 + \dots + \kappa_\nu)$ are also elements of $\mathcal{M}$.

Now, let $jd \in I_0 \setminus \mathcal{M}$ if it exists (otherwise the proof is finished). By Bezout's theorem, there are relative integers $b_0, \dots, b_\nu$, at least one of which is positive, such that $b_0\kappa_0 + \dots + b_\nu\kappa_\nu = jd$. Sorting the coefficients b_i by sign, one infers that $jd - \sum_{i/\ b_i \leq 0} b_i\kappa_i$ is a linear combination of the κ_i with non negative coefficients, at least one of which is non zero, therefore $jd - \sum_{i/\ b_i\leq 0} b_i\kappa_i \in \mathcal{M}$. Therefore $jd + \max(-b_0, \dots, -b_\nu)(\kappa_0 + \dots + \kappa_\nu) \in \mathcal{M}$. □

Lemma 3.2 has the following corollary:

Corollary 1. *There exists $N_\mathcal{M}$ such that for all $a, b \in \mathcal{M}$, if $a - b \geq N_\mathcal{M}$, then $a - b \in \mathcal{M}$.*

Proof. Let d be the greatest common divisor of $\kappa_0, \dots, \kappa_N$. Since every element of $\mathcal{M}$ is a multiple of d, if $a, b \in \mathcal{M}$, then $a - b$ is a multiple of d. If moreover $a - b \geq N_\mathcal{M}$, where $N_\mathcal{M}$ was defined in Lemma 3.2, then $a - b \in \mathcal{M}$. □

4. **The Brjuno sum and the Brjuno function.** Let d be the greatest common divisor of $\kappa_0, \dots, \kappa_\nu$. Let us consider the continued fraction expansion of $d\alpha$ and let (q_k) be the sequence of the denominators of the approximants of $d\alpha$.

Definition 4.1. The Brjuno sum $B(d\alpha)$ of the number $d\alpha$ is the (non necessarily convergent) sum

$$B(d\alpha) = \sum_{k\geq 0} \frac{\ln q_{k+1}}{q_k}$$

Recall the well-known recurrence relation: for all $j \geq 0$,

$$q_{j+2} = a_{j+2}q_{j+1} + q_j \tag{12}$$

where (a_j) is the sequence of integers given by the continued fraction expansion.

The following lemmas are given in order to relate the Brjuno sum with the small divisors of our linearization problem.

Lemma 4.2. *For all $k \geq 1$, there is*

$$\frac{\pi}{2q_{k+1}} \leq |D_{dq_k}|^{\frac{1}{2}} \leq \frac{3\pi}{q_{k+1}}$$

Proof. For all $l \in \mathbb{Z}$,

$$|D_l| = 4|\sin(\pi l\alpha)|^2$$

and the Taylor formula implies

$$\left| |D_l|^{\frac{1}{2}} - 2\pi||l\alpha||_{\mathbb{Z}} \right| \leq 2\pi||l\alpha||_{\mathbb{Z}}^2$$

whence, if $||l\alpha||_{\mathbb{Z}} \leq \frac{1}{2}$,

$$\pi||l\alpha||_{\mathbb{Z}} \leq |D_l|^{\frac{1}{2}} \leq 3\pi||l\alpha||_{\mathbb{Z}} \tag{13}$$

Now for all $k \geq 1$,

$$||q_k d\alpha||_{\mathbb{Z}} = \min_{p\in\mathbb{Z}} |q_k d\alpha - p| = q_k \min_{p\in\mathbb{Z}} |d\alpha - \frac{p}{q_k}| = q_k|d\alpha - \frac{p_k}{q_k}|$$

Since

$$\frac{1}{2q_kq_{k+1}} \leq |d\alpha - \frac{p_k}{q_k}| \leq \frac{1}{q_kq_{k+1}} \tag{14}$$

(see for instance [20] remark 1.6), there is

$$\frac{\pi}{2q_{k+1}} \leq |D_{dq_k}|^{\frac{1}{2}} \leq \frac{3\pi}{q_{k+1}}. \square$$

□

Remark 6. This cannot be extended to generalized continued fractions (i.e continued fractions where the Gauss map refers to the closest integer, not to the integer part) since the inequality (14) does not hold anymore for generalized continued fractions.

The following lemmas come from [13]. They concern the structure of the set of small divisors and will be used in the lower bound on the radius of convergence. We recall them here and apply the second to our setting.

Lemma 4.3. *([13], lemma 2.2) Let $k \in \mathbb{N}$ and $n \in \mathbb{N}, n \geq 1$, such that d is a divisor of n. If $|D_n|^{\frac{1}{2}} < \frac{1}{q_k}$, then $\frac{n}{d} \geq q_k$ and either q_k divides $\frac{n}{d}$ or $\frac{n}{d} \geq \frac{q_{k+1}}{4}$.*

Lemma 4.4. *([13], lemma 2.3) For all $k \geq 0, n \geq 1$, let $A_k(n) = \{dq_k \leq j \leq n,\ d|j,\ \frac{1}{6q_{k+1}} \leq |D_j|^{\frac{1}{2}} < \frac{1}{6q_k}\}$. Let $E = \max(dq_k, \frac{dq_{k+1}}{4})$. Then there is a function $g_k : \mathbb{N} \to \mathbb{R}^+$ such that:*

- *$g_k(n) \leq (1 + \frac{2dq_k}{E})\frac{n}{dq_k}$;*
- *for all $n_1, n_2 \in \mathbb{N}$, $g_k(n_1) + g_k(n_2) \leq g_k(n_1 + n_2)$;*
- *if $n \in A_k(n)$, then $g_k(n) \geq g_k(n-1) + 1$.*

Remark 7. This application of Davie's lemma is possible because the set A_k satisfies: if $j_1 = dj_1' < j_2 = dj_2' \in A_k$, then either dq_k divides $j_2 - j_1$ or $j_2 - j_1 \geq \frac{dq_{k+1}}{4}$. Indeed, letting p_1 (resp. p_2) be the integer closest to $j_1' d\alpha$ (resp. $j_2' d\alpha$),

$$||(j_2-j_1)\alpha||_{\mathbb{Z}} \leq |p_2-p_1-(j_2'-j_1')d\alpha| \leq ||j_2' d\alpha||_{\mathbb{Z}}+||j_1' d\alpha||_{\mathbb{Z}} \leq |D_{j_2}|^{\frac{1}{2}}+|D_{j_1}|^{\frac{1}{2}} < \frac{1}{3q_k}$$

(the last inequality comes from $j_1, j_2 \in A_k$). Therefore, by (13), $|D_{j_2-j_1}|^{\frac{1}{2}} < \frac{1}{q_k}$. Lemma 4.3 then implies that q_k divides $j_2' - j_1'$ or $j_2' - j_1' \geq \frac{q_{k+1}}{4}$.

Lemma 4.5. *There exists $C_0, C_1 > 0$ such that*

$$\sum_{l \geq 0} \frac{1}{q_l} \leq C_0$$

and

$$\sum_{l \geq 0} \frac{1}{q_l} \ln q_l \leq C_1$$

Proof. Let us give a bound on $\sum_{l\geq 0} \frac{1}{q_l} \ln q_l$. Because of the recurrence relation (12), the sequence (q_l) increases at least as fast as a Fibonacci sequence the first two terms of which are in $\{\kappa_0, \dots, \kappa_\nu\}$: denoting by (f_k) the Fibonacci sequence with $f_0 = f_1 = 1$, one recursively proves that

$$q_k \geq f_k \tag{15}$$

Indeed, $q_0 \geq 1$ and $q_1 \geq 1$. Assume that $q_{k-1} \geq f_{k-1}$ and $q_k \geq f_k$, then

$$q_{k+1} = a_{k+1} q_{k+1-1} + q_{k+1-2} \geq q_k + q_{k-1} \geq (f_k + f_{k-1}) = f_{k+1}.$$

Therefore, since the function $t \mapsto \frac{\ln t}{t}$ decreases on $[e, +\infty[$,

$$\sum_{l \geq 0} \frac{1}{q_l} \ln q_l \leq \ln 2 + \sum_{l \geq 0} \frac{\ln f_l}{f_l} \leq C_1$$

where C_1 is a numerical constant. From the inequality (15), one also infers that

$$\sum_{l \geq 0} \frac{1}{q_l} \leq \sum_{l \geq 0} \frac{1}{f_k} \leq C_0$$

where C_0 is a numerical constant. □ □

4.1. **Subsequence of fast increasing denominators.** Given the number $N_{\mathcal{M}}$ which was defined in Lemma 3.2, let (n_k) the (possibly finite) subsequence containing all indices such that

$$\begin{cases} q_{n_0} \geq \max(N_{\mathcal{M}}+2, 1+\kappa_{max}(2+\frac{1}{d})), \\ q_{n_k+1} \geq q_{n_k}^2 + \zeta(\kappa_{max}, d) q_{n_k} + \eta(\kappa_{max}, d) \end{cases}$$

where ζ, η are given by

$$\zeta(\kappa_{max}, d) = \frac{\kappa_{max}}{d} + 3\kappa_{max} + 2, \quad \eta(\kappa_{max}, d) = (\frac{\kappa_{max}}{d} + 2\kappa_{max})(\kappa_{max}+1)$$

Denote by $\mu + 1 \in \mathbb{N} \cup \{\infty\}$ the cardinal of this subsequence (so μ, if finite, is the greatest k such that n_k is defined).

Lemma 4.6. *For all $0 \leq k \leq \mu$, it holds that $q_{n_k} \geq \max(N_{\mathcal{M}}+2, \kappa_{max})^{2^k}$.*

Proof. By definition, $q_{n_0} \geq \max(N_{\mathcal{M}}+2, \kappa_{max})$.
Assume that $q_{n_k} \geq \max(N_{\mathcal{M}}+2, \kappa_{max})^{2^k}$, then $q_{n_{k+1}} \geq q_{n_k+1} \geq q_{n_k}^2 \geq \max(N_{\mathcal{M}}+2, \kappa_{max})^{2^{k+1}}$. ☐

Corollary 2. *It holds that $\sum_{k=0}^{\mu} \frac{1}{q_{n_k}} \leq \frac{1}{\max(N_{\mathcal{M}}+2, \kappa_{max})-1}$.*

Proof. Indeed

$$\begin{aligned} \sum_{k=0}^{\mu} \frac{1}{q_{n_k}} &\leq \sum_{k\geq 0} \frac{1}{\max(N_{\mathcal{M}}+2, \kappa_{max})^{2^k}} \leq \sum_{k\geq 1} \frac{1}{\max(N_{\mathcal{M}}+2, \kappa_{max})^{k}} \\ &\leq \frac{1}{\max(N_{\mathcal{M}}+2, \kappa_{max})-1}. \end{aligned} \tag{16}$$

☐

Lemma 4.7. *There exists $C_2 > 0$ which is an increasing function of κ_{max} and a non increasing function of d such that*

$$B(d\alpha) - C_2 \leq \sum_{l=0}^{\mu} \frac{|\ln |D_{q_{n_l}}||}{2q_{n_l}} \leq B(d\alpha)$$

Proof. Lemma 4.2 implies that

$$\sum_{l=0}^{\mu} \frac{\ln q_{n_l+1} - \ln(3\pi)}{q_{n_l}} \leq \sum_{l=0}^{\mu} \frac{|\ln |D_{q_{n_l}}||}{2q_{n_l}} \leq \sum_{l=0}^{\mu} \frac{\ln q_{n_l+1}}{q_{n_l}} + \sum_{l=0}^{\mu} \frac{\ln 2 - \ln \pi}{q_{n_l}} \leq B(d\alpha)$$

Now

$$\sum_{l=0}^{\mu} \frac{\ln q_{n_l+1}}{q_{n_l}} = B(d\alpha) - \sum_{l\geq 0, q_l^2+\zeta q_l+\eta > q_{l+1}} \frac{\ln q_{l+1}}{q_l}$$

(where ζ, η were defined at the beginning of the section). Thus

$$\sum_{l=0}^{\mu}\frac{\ln q_{n_l+1}}{q_{n_l}} \geq B(d\alpha) - \sum_{l\geq 0}\frac{\ln(q_l^2+\zeta q_l+\eta)}{q_l} \geq B(d\alpha) - \sum_{l\geq 0}\frac{\ln q_l + \ln(q_l+\zeta+\eta)}{q_l}$$
$$\geq B(d\alpha) - 2\sum_{l\geq 0}\frac{\ln q_l}{q_l} - \sum_{l\geq 0}\frac{\ln(\zeta+\eta+1)}{q_l} \tag{17}$$

By Lemma 4.5,

$$\sum_{l\geq 0}\frac{\ln(\zeta+\eta+1)}{q_l} \leq C_0\ln(\zeta+\eta+1)$$

Therefore one can define $C_2 = 2C_1 + C_0(\ln(3\pi)+\ln(\zeta+\eta+1))$. □

5. **Recursively defined lower bound.** In this section we introduce a function F which will be used in giving a lower bound on the coefficients of the linearization. More precisely, in the case of a subsequence q_{n_k} with infinite support, we shall prove that

$$\limsup_{k\to+\infty}\frac{F(q_{n_k})}{q_{n_k}} - C \leq \limsup_{k\to+\infty}\frac{\ln|\phi_{q_{n_k}}|}{q_{n_k}}$$

where C is a constant not depending on α.
Then we shall bound $\limsup_{k\to+\infty}\left(\frac{F(q_{n_k})}{q_{n_k}}\right)$ by means of the Brjuno sum in the Lemma 5.1 below.

The function F is recursively defined on the set $\{q_{n_k}, 0\leq k\leq\mu\}$ as follows:

$$\begin{aligned}&F(q_{n_0}) = 0 \text{ and } \forall\ 1\leq k\leq\mu,\\ &F(q_{n_k}) = p_0^k|\ln|D_{q_{n_0}}|| + p_0^kF(q_{n_0}) + \cdots + p_{k-1}^k|\ln|D_{q_{n_{k-1}}}|| + p_{k-1}^kF(q_{n_{k-1}})\end{aligned} \tag{18}$$

where the integers p_i^k are given by successive euclidean divisions on $q_{n_k}-\kappa_{max}$:

$$\begin{aligned}&dq_{n_k}-\kappa_{max} = p_{k-1}^k dq_{n_{k-1}} + r_{k-1},\ r_{k-1} < dq_{n_{k-1}},\\ &r_{k-1} = p_{k-2}^k dq_{n_{k-2}} + r_{k-2},\ r_{k-2} < dq_{n_{k-2}},\\ &\dots,\\ &r_1 = p_0^k dq_{n_0} + r_0,\ r_0 < dq_{n_0}.\end{aligned} \tag{19}$$

Remark 8. Let $i\geq 0$. For all $k\geq i+1$,

$$p_i^k \leq \frac{q_{n_{i+1}}}{q_{n_i}} \tag{20}$$

Indeed, the integer p_i^k are recursively given by

$$p_{k-1}^k = E\left(\frac{dq_{n_k}-\kappa_{max}}{dq_{n_{k-1}}}\right)$$

and for $i=0,\dots,k-2$, $p_i^k = E(\frac{r_{i+1}}{dq_{n_i}})$, which satisfies (20) by definition of r_{i+1}.

Lemma 5.1. *There exist $C_4 > 0$ which is an increasing function of κ_{max} and a non increasing function of d such that the function F satisfies for all $0 \leq k \leq \mu$, if $\mu = \infty$,*

$$2B(d\alpha) - C_4 \leq \limsup_{k \to +\infty} \frac{F(q_{n_k})}{q_{n_k}} \tag{21}$$

and if $\mu < +\infty$,

$$2B(d\alpha) - C_4 \leq \frac{F(q_{n_\mu})}{q_{n_\mu}} - \frac{\ln |D_{dq_{n_\mu}}|}{dq_{n_\mu}} \tag{22}$$

Proof. This can be recursively shown. Assume that for a fixed $1 \leq k \leq \mu$ and for all $k' \leq k-1$, one has

$$\frac{F(q_{n_{k'}})}{q_{n_{k'}} + \kappa_{max}/d} \geq \sum_{0 \leq l \leq k'-1} |\ln |D_{dq_{n_l}}||(\frac{1}{q_{n_l}} - \frac{2}{q_{n_{l+1}}})$$

(which holds for $k = 1$). Then

$$\begin{aligned}\frac{F(q_{n_k})}{q_{n_k} + \kappa_{max}/d} &\geq \frac{p_{k-1}}{q_{n_k} + \kappa_{max}/d}(F(q_{n_{k-1}}) + |\ln |D_{q_{n_{k-1}}}||) \\ &\geq \frac{p_{k-1}(q_{n_{k-1}} + \kappa_{max}/d)}{q_{n_k} + \kappa_{max}/d} \sum_{l \leq k-2} |\ln |D_{dq_{n_l}}||(\frac{1}{q_{n_l}} - \frac{2}{q_{n_{l+1}}}) \\ &+ \frac{p_{k-1}}{q_{n_k} + \kappa_{max}/d}|\ln |D_{q_{n_{k-1}}}||\end{aligned} \tag{23}$$

Now by definition of p_{k-1},

$$\begin{aligned}\frac{p_{k-1}(q_{n_{k-1}} + \kappa_{max}/d)}{q_{n_k} + \kappa_{max}/d} &\geq \frac{(\frac{dq_{n_k} - \kappa_{max}}{dq_{n_{k-1}}} - 1)(q_{n_{k-1}} + \kappa_{max}/d)}{q_{n_k} + \kappa_{max}/d} \\ &= \frac{(q_{n_k} - q_{n_{k-1}} - \frac{\kappa_{max}}{d})(q_{n_{k-1}} + \kappa_{max}/d)}{q_{n_{k-1}}(q_{n_k} + \kappa_{max}/d)}\end{aligned} \tag{24}$$

and this quantity is greater than 1 since, by assumption on the subsequence n_k,

$$q_{n_k} \geq q_{n_{k-1}}^2 + \zeta q_{n_{k-1}} + \eta$$

where ζ, η were defined at the beginning of Section 4.1. Therefore

$$\begin{aligned}\frac{F(q_{n_k})}{q_{n_k} + \kappa_{max}/d} &\geq \sum_{l \leq k-2} |\ln |D_{dq_{n_l}}||(\frac{1}{q_{n_l}} - \frac{2}{q_{n_{l+1}}}) \\ &+ |\ln |D_{q_{n_{k-1}}}|| \frac{q_{n_k} - q_{n_{k-1}} - \frac{\kappa_{max}}{d}}{q_{n_{k-1}}(q_{n_k} + \kappa_{max}/d)} \\ &\geq \sum_{l \leq k-2} |\ln |D_{dq_{n_l}}||(\frac{1}{q_{n_l}} - \frac{2}{q_{n_{l+1}}}) \\ &+ |\ln |D_{q_{n_{k-1}}}||(\frac{1}{q_{n_{k-1}}} - \frac{2}{q_{n_k}})\end{aligned} \tag{25}$$

(where the last inequality comes from the fact that $q_{n_0} \geq 1+\kappa_{max}(2+\frac{1}{d})$). Therefore the property holds for all $k \geq 0$. Lemma 4.2 implies that

$$\frac{F(q_{n_k})}{q_{n_k}+\kappa_{max}/d} \geq \sum_{0\leq l\leq k-1} \frac{|\ln|D_{dq_{n_l}}||}{q_{n_l}} - \sum_{0\leq l\leq k-1} \frac{4\ln(\frac{2}{\pi}q_{n_{l+1}})}{q_{n_{l+1}}}$$

which implies

$$\frac{F(q_{n_k})}{q_{n_k}+\kappa_{max}/d} \geq \sum_{0\leq l\leq k-1} \frac{|\ln|D_{dq_{n_l}}||}{q_{n_l}} - 4\sum_{l\geq 0}\frac{\ln q_l}{q_l}$$

Lemma 4.5 then implies

$$\frac{F(q_{n_k})}{q_{n_k}+\kappa_{max}/d} \geq \sum_{0\leq l\leq k-1} \frac{|\ln|D_{dq_{n_l}}||}{q_{n_l}} - 4C_1$$

Finally, by Lemma 4.7, if $\mu=\infty$, then

$$\limsup_{k\to+\infty}\frac{F(q_{n_k})}{q_{n_k}} = \limsup_{k\to+\infty}\frac{F(q_{n_k})}{q_{n_k}+\kappa_{max}/d} \geq 2B(d\alpha)-2C_2-4C_1$$

If $\mu<+\infty$, then

$$\frac{F(q_{n_\mu})}{q_{n_\mu}} - \frac{\ln|D_{dq_{n_\mu}}|}{dq_{n_\mu}} \geq \sum_{0\leq l\leq \mu} \frac{|\ln|D_{dq_{n_l}}||}{q_{n_l}} - 4C_1$$

therefore by Lemma 4.7,

$$\frac{F(q_{n_\mu})}{q_{n_\mu}} - \frac{\ln|D_{dq_{n_\mu}}|}{dq_{n_\mu}} \geq 2B(d\alpha)-2C_2-4C_1$$

Thus one can define $C_4 = 2C_2+4C_1$. □

6. **An upper bound on the radius of convergence.** In this section, one shall assume the following:

Assumption 1. *Assume that there exists $\theta\in\mathbb{R}$ such that for all $k=1,\dots,N$ with $a_k\neq 0$, the complex number $i^{k-1}a_k$ has argument $k\theta$.*

In this case, one can prove a lower bound on the coefficients ϕ_l of the linearization in order to bound the radius of convergence from above.

One needs the following simple lower bound on all coefficients, including those not corresponding to a small divisor.

Lemma 6.1. *For all $r\in\mathbb{N}$, if $|\phi_r|\neq 0$, then $|\phi_r|\geq \min(1,\frac{|a_{\kappa_{min}}|}{4})^{\frac{r}{\kappa_0}}$.*

Proof. For all K such that $a_K\neq 0$, it holds that $|\phi_K|\geq \frac{K|a_K|}{4}\geq\frac{|a_{\kappa_{min}}|}{4}$. If $\frac{|a_{\kappa_{min}}|}{4}<1$, then $|\phi_K|\geq(\frac{|a_{\kappa_{min}}|}{4})^{\frac{K}{\kappa_0}}$. If $\frac{|a_{\kappa_{min}}|}{4}\geq 1$ then $|\phi_K|\geq 1$.

Let $r\in\mathcal{M}, r\geq 2$. Assume that the property holds for all $r'\leq r-1$. Then, either there exists $i\in\{0,\dots,\nu\}$ such that $r=\kappa_i$, and in this case, $|\phi_r|\geq\frac{a_{\kappa_i}}{4}\geq \min(1,\frac{|a_{\kappa_{min}}|}{4})\geq\min(1,\frac{|a_{\kappa_{min}}|}{4})^{\frac{\kappa_i}{\kappa_0}}$, or there exists $i\in\{0,\dots,\nu\}$ such that

$$|\phi_r|\geq\frac{|a_{\kappa_i}|}{4}|\phi_{r-\kappa_i}|\geq\frac{|a_{\kappa_{min}}|}{4}|\phi_{r-\kappa_i}|$$

If $\frac{|a_{\kappa_{min}}|}{4}<1$, then the recurrence assumption implies that

$$|\phi_r| \geq (\frac{|a_{\kappa_{min}}|}{4})^{\frac{\kappa_i}{\kappa_0}} \min(1, \frac{|a_{\kappa_{min}}|}{4})^{\frac{r-\kappa_i}{\kappa_0}} \geq \min(1, \frac{|a_{\kappa_{min}}|}{4})^{\frac{r}{\kappa_0}}.$$

If $\frac{|a_{\kappa_{min}}|}{4} \geq 1$ then $|\phi_r| \geq 1$. □

Lemma 6.2. *For all $j > \kappa_0$ and $p \in \mathbb{N}^*$, there is*

$$|\phi_{pj}| \geq \frac{1}{p|D_{pj}| \dots |D_{2j}|} |\phi_j|^p |D_j|^{p-1}$$

Proof. Notice that

$$|\phi_{pj}| = \frac{1}{|D_{pj}|}|\psi_{pj}| = \frac{1}{|D_{pj}|} \sum_{K \in \{\kappa_0, \dots, \kappa_\nu\}} |\psi_{K,pj}| \tag{26}$$

(one uses the fact that the $\psi_{K,pj}$ have the same argument for every K). Thus by (11),

$$|\phi_{pj}| \geq \frac{1}{|D_{pj}|} \sum_K \frac{(p-1)j}{pj-K} |\phi_{(p-1)j}||\psi_{K,j}| \geq \frac{(p-1)j}{|D_{pj}|pj} |\phi_{(p-1)j}||\psi_j| \tag{27}$$

Iterating this, one obtains

$$|\phi_{pj}| \geq \frac{1}{p|D_{pj}| \dots |D_{2j}|} |\phi_j||\psi_j|^{p-1} = \frac{1}{p|D_{pj}| \dots |D_{2j}|} |\phi_j|^p |D_j|^{p-1}. \tag{28}$$

□

The following lemma states a better lower bound for the coefficients of the linearization corresponding to a small divisor.

Lemma 6.3. *For all $0 \leq k \leq \mu - 1$, there is*

$$\begin{aligned} |\phi_{dq_{n_{k+1}}}| \geq & \frac{1}{|D_{dq_{n_{k+1}}}|} \frac{|a_{\kappa_{max}}|}{(k+2)!} (\frac{1}{4})^{d(p_0^{k+1} + \dots + p_k^{k+1})} \\ & \cdot \frac{|D_{dq_{n_0}}|^{dp_0^{k+1}-1}}{dp_0^{k+1}} \dots \frac{|D_{dq_{n_k}}|^{dp_k^{k+1}-1}}{dp_k^{k+1}} \\ & \cdot |\phi_{dq_{n_0}}|^{p_0^{k+1}} \dots |\phi_{dq_{n_k}}|^{p_k^{k+1}} \min(1, \frac{|a_{\kappa_{min}}|}{4})^{\frac{R_k}{\kappa_0}} \end{aligned} \tag{29}$$

where the integers $p_0^{k+1}, \dots, p_k^{k+1}, R_k$ were defined in equation (18).

Proof. Taking in the recurrence relation (8) the term with $l = q_{n_{k+1}}, K = \kappa_{max}, m = 1, p_1 = k+2, j_1^1 = dp_0^{k+1}(q_{n_0} + \kappa_{max}), \dots, j_{k+1}^1 = dp_k^{k+1}(q_{n_k} + \kappa_{max})$ and $j_{k+2}^1 = R_k$, in order to have $j_1^1 + \dots + j_{k+2}^1 = dq_{n_{k+1}} - \kappa_{max}$, one obtains

$$|\phi_{dq_{n_{k+1}}}| \geq \frac{1}{|D_{dq_{n_{k+1}}}|} \frac{\kappa_{max}|a_{\kappa_{max}}|}{(k+2)!} |\phi_{dp_0^{k+1}(q_{n_0}+\kappa_{max})}| \dots |\phi_{dp_k^{k+1}(q_{n_k}+\kappa_{max})}||\phi_{R_k}|$$

Now apply Lemma 6.2 to every factor in the right hand side, except the last one, and apply Lemma 6.1 to the last factor. One obtains

$$
\begin{aligned}
|\phi_{dq_{n_{k+1}}}| \geq & \frac{1}{|D_{dq_{n_{k+1}}}|}\frac{|a_{\kappa_{max}}|}{(k+2)!}(\frac{1}{4})^{d(p_0^{k+1}+\dots+p_k^{k+1})} \\
& \cdot \frac{|D_{dq_{n_0}}|^{dp_0^{k+1}-1}}{dp_0^{k+1}}\dots\frac{|D_{dq_{n_k}}|^{dp_k^{k+1}-1}}{dp_k^{k+1}} \\
& \cdot |\phi_{dq_{n_0}}|^{p_0^{k+1}}\dots|\phi_{dq_{n_k}}|^{p_k^{k+1}}\min(1,\frac{|a_{\kappa_{min}}|}{4})^{\frac{R_k}{\kappa_0}}.
\end{aligned} \tag{30}
$$

□

The following proposition links the radius of convergence of the linearization to the function F.

Proposition 1. *There exists $C \geq 0$ such that for all $0 \leq k \leq \mu$,*

$$\frac{1}{dq_{n_k}}\ln|D_{dq_{n_k}}\phi_{dq_{n_k}}| \geq \frac{F(q_{n_k})}{dq_{n_k}} - C$$

Moreover, one can have $C = -2\ln(\frac{|a_{\kappa_{min}}|}{4}) + \frac{2+N_{\mathcal{M}}}{1+N_{\mathcal{M}}}(1-\ln|a_{\kappa_{min}}|)$ if $\frac{|a_{\kappa_{min}}|}{4} < 1$ and $C = 1 + \frac{1}{1+N_{\mathcal{M}}}$ otherwise.

Remark 9. Observe that the coefficients of the linearization increase with the nonlinear part of the system.

Proof. Denote $\tilde{C} = -2\ln(\frac{|a_{\kappa_{min}}|}{8})$ if $\frac{|a_{\kappa_{min}}|}{8} < 1$ and $\tilde{C} = 0$ otherwise. Let $S_0 = \frac{\ln 4}{q_{n_0}}$ and for all $1 \leq k \leq \mu$,

$$S_k = \sum_{1\leq j\leq k}\frac{j\ln j + \ln(d^j q_{n_j})}{q_{n_j}} + \sum_{0\leq j\leq k}\frac{\ln 4}{q_{n_j}}(1+p_0^j+\dots+p_{j-1}^j+\frac{R_j}{\kappa_0})$$

Thus, for all $0 \leq k \leq \mu$, $S_k \geq 0$.

Let $0 \leq k \leq \mu$. Let us formulate the following recurrence property:

$$\frac{1}{dq_{n_k}}\ln|D_{dq_{n_k}}\phi_{dq_{n_k}}| \geq \frac{F(q_{n_k})}{dq_{n_k}} - S_k - \tilde{C} \tag{31}$$

First note that this property holds for $k = 0$. Indeed, since by Corollary 1, $q_{n_0} \geq N_{\mathcal{M}}$, which implies that $dq_{n_0} \in \mathcal{M}$, then there exists $b_0, b_1, \dots, b_\nu \geq 0$ such that $dq_{n_0} = \sum_{i=0}^{\nu} b_i\kappa_i$. Let I be such that $b_I > 0$. Then

$$|\phi_{dq_{n_0}}| \geq \frac{1}{|D_{dq_{n_0}}|}\kappa_I|a_{\kappa_I}||\phi_{\kappa_I}|^{b_I-1}\prod_{J\neq I}|\phi_{\kappa_J}|^{b_J}$$

Now on the other side, for all $0 \leq i \leq N$,

$$|\phi_{\kappa_i}| \geq \frac{|a_{\kappa_i}|}{|D_{\kappa_i}|} \geq \frac{|a_{\kappa_{min}}|}{4} \geq \frac{|a_{\kappa_{min}}|}{8}$$

hence, if $\frac{|a_{\kappa_{min}}|}{8} < 1$,

$$|\phi_{dq_{n_0}}| \geq \frac{1}{|D_{dq_{n_0}}|}|a_{\kappa_I}|(\frac{|a_{\kappa_{min}}|}{8})^{dq_{n_0}-1}$$

and if $\frac{|a_{\kappa_{min}}|}{8} \geq 1$,

$$|\phi_{dq_{n_0}}| \geq \frac{1}{|D_{dq_{n_0}}|}|a_{\kappa_I}|$$

Thus if $\frac{|a_{\kappa_{min}}|}{8} < 1$,

$$\frac{1}{dq_{n_0}} \ln|\phi_{dq_{n_0}}| \geq \frac{|\ln|D_{dq_{n_0}}||}{dq_{n_0}} + \frac{dq_{n_0}-1}{dq_{n_0}} \ln(\frac{|a_{\kappa_{min}}|}{8}) + \frac{\ln(|a_{\kappa_I}|)}{dq_{n_0}} \geq \frac{|\ln|D_{dq_{n_0}}||}{dq_{n_0}} - S_0 - \tilde{C}$$

and if $\frac{|a_{\kappa_{min}}|}{8} \geq 1$,

$$\frac{1}{dq_{n_0}} \ln|\phi_{dq_{n_0}}| \geq \frac{|\ln|D_{dq_{n_0}}||}{dq_{n_0}} + \frac{\ln(|a_{\kappa_I}|)}{dq_{n_0}} \geq \frac{|\ln|D_{dq_{n_0}}||}{dq_{n_0}}$$

therefore the property (31) holds for $k=0$.
Now assume that the recurrence property holds for all $0 \leq k' \leq k$, for a fixed $0 \leq k \leq \mu-1$. Lemma 6.3 implies

$$\begin{aligned} \ln|D_{dq_{n_{k+1}}}\phi_{dq_{n_{k+1}}}| \geq \ln\left(\frac{|a_{\kappa_{max}}|}{(k+2)!}\right) &- d((p_0^{k+1}+\dots+p_k^{k+1})+k+1)\ln 4 \\ &- \ln(dp_0^{k+1}\dots dp_k^{k+1}) + \frac{R_k}{\kappa_0}\ln\min(1,\frac{|a_{\kappa_{min}}|}{4}) \\ &+ p_0^{k+1}\ln|D_{dq_{n_0}}\phi_{dq_{n_0}}| + \dots + p_k^{k+1}\ln|D_{dq_{n_k}}\phi_{dq_{n_k}}| \end{aligned} \tag{32}$$

By recurrence assumption, one infers

$$\begin{aligned} \frac{1}{dq_{n_{k+1}}}\ln|D_{dq_{n_{k+1}}}\phi_{dq_{n_{k+1}}}| \geq &-\frac{(k+2)\ln(k+2)}{dq_{n_{k+1}}} + \frac{\ln|a_{\kappa_{max}}|}{dq_{n_{k+1}}} \\ &- \frac{d(p_0^{k+1}+\dots+p_k^{k+1})+k+1}{dq_{n_{k+1}}}\ln 4 \\ &+ \frac{p_0^{k+1}dq_{n_0}}{dq_{n_{k+1}}}(\frac{F(q_{n_0})}{dq_{n_0}} - S_0 - \tilde{C}) + \dots \\ &+ \frac{p_k^{k+1}dq_{n_k}}{dq_{n_{k+1}}}(\frac{F(q_{n_k})}{dq_{n_k}} - S_k - \tilde{C}) \\ &+ \frac{R_k}{d\kappa_0 q_{n_{k+1}}}\ln\min(1,\frac{|a_{\kappa_{min}}|}{4}) \\ &- \frac{\ln(dp_0^{k+1}\dots dp_k^{k+1})}{dq_{n_{k+1}}} \end{aligned} \tag{33}$$

thus by definition of F,

$$\begin{aligned}\frac{1}{dq_{n_{k+1}}}\ln|D_{dq_{n_{k+1}}}\phi_{dq_{n_{k+1}}}| \geq{}& \frac{F(q_{n_{k+1}})}{dq_{n_{k+1}}} - \frac{(k+2)\ln(k+2)}{dq_{n_{k+1}}}\\
&+\frac{\ln|a_{\kappa_{max}}|}{dq_{n_{k+1}}} - \frac{d(p_0^{k+1}+\dots+p_k^{k+1})+k+1}{q_{n_{k+1}}}\ln 4\\
&+\frac{R_k}{d\kappa_0 q_{n_{k+1}}}\ln\min(1,\frac{|a_{\kappa_{min}}|}{4}) - \tilde{C}\\
&-\frac{p_0^{k+1}dq_{n_0}}{dq_{n_{k+1}}}S_0 - \dots - \frac{p_k^{k+1}dq_{n_k}}{dq_{n_{k+1}}}S_k\\
&-\frac{\ln(dp_0^{k+1}\dots dp_k^{k+1})}{dq_{n_{k+1}}}\end{aligned} \tag{34}$$

Now for all $i=0,\dots,k$, $p_i \leq \frac{q_{n_{i+1}}}{q_{n_i}}$ therefore

$$\frac{\ln(dp_0^{k+1}\dots dp_k^{k+1})}{dq_{n_{k+1}}} \leq \frac{\ln(d^{k+1}\frac{q_{n_{k+1}}}{q_{n_0}})}{dq_{n_{k+1}}}$$

and

$$\frac{1}{dq_{n_{k+1}}}\ln|D_{dq_{n_{k+1}}}\phi_{dq_{n_{k+1}}}| \geq \frac{F(q_{n_{k+1}})}{dq_{n_{k+1}}} - \tilde{C} - S_{k+1}$$

Therefore, the property (31) holds for all $0\leq k\leq\mu$.

Finally note that, even if $\mu=+\infty$, the partial sums S_k converge, since from one side, by choice of the subsequence q_{n_j} and corollary 2,

$$\sum_{1\leq j\leq k}\frac{j\ln j+\ln(d^j q_{n_j})}{q_{n_j}} \leq \sum_{1\leq j\leq k}\frac{2}{\sqrt{q_{n_j}}}+C_1 \leq \sum_{1\leq j\leq k}\frac{2}{q_{n_{j-1}}}+C_1 \leq \frac{2}{1+N_{\mathcal{M}}}+C_1$$

and from the other side, by Remark 8,

$$\begin{aligned}&\frac{\ln 4}{q_{n_0}}+\sum_{j=1}^{k}\frac{\ln 4}{q_{n_j}}(1+\frac{R_j}{\kappa_0}+p_0^j+\dots+p_{j-1}^j)\\
&\leq \frac{\ln 4}{q_{n_0}}+\sum_{j=1}^{k}\frac{\ln 4}{q_{n_j}}(1+q_{n_0}+\frac{q_{n_1}+\kappa_{max}}{q_{n_0}+\kappa_{max}}+\dots+\frac{q_{n_{j-1}}+\kappa_{max}}{q_{n_{j-2}}+\kappa_{max}}+\frac{q_{n_j}}{q_{n_{j-1}}})\\
&\leq \ln 4(\frac{1}{q_{n_0}}+\sum_{j=1}^{k}\frac{(j+1)q_{n_{j-1}}}{q_{n_j}}+\frac{1}{q_{n_{j-1}}})\\
&\leq \ln 4(\frac{1}{q_{n_0}}+\sum_{j=1}^{k}\frac{j+2}{q_{n_{j-1}}})\\
&\leq \ln 4(\frac{8}{q_{n_0}}+\sum_{j=2}^{k-1}\frac{j+3}{q_{n_j}}) \leq \ln 4(\frac{8}{q_{n_0}}+\sum_{j=2}^{k-1}\frac{1}{q_{n_{j-1}}})\\
&\leq \frac{9\ln 4}{1+N_{\mathcal{M}}}\end{aligned} \tag{35}$$

(this last inequality used Corollary 2). Thus, let $C = \tilde{C} + \frac{1}{1+N_{\mathcal{M}}}(2 + 9\ln 4) + C_1$, then C satisfies the statement of this proposition. $\square$

The following theorem concludes the proof of our main theorem 2.

Theorem 6.4. *Let $C \geq 0$ be the constant defined in Proposition 1. Let ρ be the radius of convergence of Φ.*

If $\mu = +\infty$, $\rho \leq \exp(-\limsup_{k\to+\infty} \frac{F(q_{n_k})}{dq_{n_k}} + C)$.

If $\mu < +\infty$, $\rho \leq \exp(-\frac{F(q_{n_\mu})}{dq_{n_\mu}} + \frac{\ln|D_{dq_{n_\mu}}|}{dq_{n_\mu}} + C)$.

One also has the bound $\rho \leq \exp(-\frac{2}{d}B(d\alpha) + C_{\mathcal{M}})$ where $C_{\mathcal{M}} = \frac{C_4}{d} + C$, with C_4 defined in Lemma 5.1.

Proof. Recall that $\rho^{-1} = \limsup_{j\to+\infty} |\phi_j|^{\frac{1}{j}}$.
Case 1: $\mu = +\infty$. Then

$$\begin{aligned} -\ln\rho &\geq \limsup_{k\to+\infty} \frac{1}{dq_k}\ln|\phi_{dq_k}| \geq \limsup_{k\to+\infty}\frac{1}{dq_{n_k}}\ln|\phi_{dq_{n_k}}| \\ &\geq \limsup_{k\to+\infty}\frac{|\ln|D_{q_{n_k}}||}{dq_{n_k}} + \frac{F(q_{n_k})}{dq_{n_k}} - C \end{aligned} \tag{36}$$

where C was defined in Proposition 1. By Lemma 5.1,

$$-\ln\rho \geq \frac{2}{d}B(d\alpha) - \frac{C_4}{d} - C.$$

Case 2: $\mu < +\infty$. Then

$$\begin{aligned} -\ln\rho &= \limsup_{j\to+\infty}\frac{1}{j}\ln|\phi_j| \geq \limsup_{j\to+\infty}\frac{1}{jq_{n_\mu}}\ln|\phi_{jq_{n_\mu}}| \\ &\geq \limsup_{j\to+\infty}\frac{1}{jq_{n_\mu}}(j\ln|\phi_{q_{n_\mu}}| - \ln j) \end{aligned} \tag{37}$$

where the last inequality has used Lemma 6.2. Therefore by Proposition 1,

$$\begin{aligned} -\ln\rho &\geq \frac{\ln|\phi_{q_{n_\mu}}|}{qn_\mu} \\ &\geq \frac{F(q_{n_\mu})}{dq_{n_\mu}} - \frac{\ln|D_{dq_{n_\mu}}|}{dq_{n_\mu}} - C \end{aligned} \tag{38}$$

which gives the first part of the statement. Finally, by Lemma 5.1, one also obtains in this case

$$-\ln\rho \geq \frac{2}{d}B(d\alpha) - \frac{C_4}{d} - C.$$

One can then define $C_{\mathcal{M}} = \frac{C_4}{d} + C$. $\square$

7. **A lower bound on the radius of convergence.**

In this section, the second part of the main result is proved. The assumption 1 on the coefficients a_K is relaxed.

The following theorem implies our main theorem 1.

Theorem 7.1. *The radius of convergence is at least* $\exp(-C' - \sum_{l\geq 0} \frac{2\ln q_{l+1}}{dq_l})$, *where* $C' \geq 0$ *is defined by*

$$C' = \ln(|a_{\kappa_{max}}|) + r + \frac{16C_1}{d}$$

if $|a_{\kappa_{max}}| > 1$ *and*

$$C' = r + \frac{16C_1}{d}$$

otherwise, with $r > 0$ *only depending on the Fourier modes* $\kappa_0, \ldots, \kappa_\nu$ *of the trigonometric polynomial* A *and* C_1 *defined in Lemma 4.5.*

Proof. Let w be an analytic solution of the functional equation

$$w(z) = \sum_{k\in\{\kappa_0,\ldots,\kappa_\nu\}} (ze^{w(z)})^k$$

and let R its radius of convergence. Expanding w in its Taylor series, $w(z) = \sum_{n\geq 0} \sigma_n z^n$, one obtains the following relation between the coefficients σ_n (it is the same relation as between the coefficients ϕ_j, only replacing a_k by i^{k+1} and without small divisors):

$$\sigma_l = \sum_{\substack{K\in\\ \kappa_0,\ldots,\\ \kappa_\nu\},\\ K\leq l}} [\delta_{K,l} + \sum_{\substack{m=1,\ldots,K\\ m\leq l-K}} \binom{K}{m} \sum_{p_1=1}^{l-K} \cdots \sum_{p_m=1}^{l-K} \frac{1}{p_1!\ldots p_m!} \cdot \sum_{\substack{j_1^1,\ldots,j_{p_1}^1,\\ \vdots\\ j_1^m,\ldots,j_{p_m}^m,\\ j_1^1+\cdots+j_{p_1}^1+\cdots\\ +j_1^m+\cdots+j_{p_m}^m=l-K}} \sigma_{j_1^1}\ldots\sigma_{j_{p_1}^1}\ldots\sigma_{j_1^m}\ldots\sigma_{j_{p_m}^m}] \tag{39}$$

One can recursively show that the σ_n are non negative real numbers. Moreover the function w is analytic, therefore $\limsup_{n\to+\infty} \frac{-1}{n}\ln\sigma_n$ is equal to $-\ln R$.

The function g giving the upper bound is defined as follows: for all $k \geq 0$, let g_k be the function defined by Davie's lemma 4.4. Then, for all integer $\kappa_0 \leq j < q_{n_0}$, let

$$g(j) = j\ln(|a_{\kappa_{max}}|) + 36j$$

if $|a_{\kappa_{max}}| > 1$, and

$$g(j) = 36j$$

otherwise. Now let $j \geq q_{n_0}$ and assume that k is the greatest index such that $q_k \leq j$; let

$$g(j) = \sum_{l=0}^{k} 2g_l(j)\ln q_{l+1} + j\ln(|a_{\kappa_{max}}|) + 36j$$

if $|a_{\kappa_{max}}| > 1$, and

$$g(j) = \sum_{l=0}^{k} 2g_l(j) \ln q_{l+1} + 36j$$

otherwise.

The function g is increasing. Moreover for all $j_1, j_2 \geq \kappa_0$, as a consequence of Davie's lemma,

$$g(j_1) + g(j_2) \leq g(j_1 + j_2) \tag{40}$$

Now let us prove that $|\phi_j| \leq \sigma_j e^{g(j)}$ for all $j \geq \kappa_0$. First, if $|D_{\kappa_0}|^{-1}|a_{\kappa_{max}}| > 1$, then

$$|\phi_{\kappa_0}| = |D_{\kappa_0}|^{-1}|a_{\kappa_0}| \leq |D_{\kappa_0}|^{-1}|a_{\kappa_{max}}| \leq e^{g(\kappa_0)}$$

and $|\phi_{\kappa_0}| \leq 1 = e^{g(\kappa_0)}$ otherwise. Assume that this holds for all $j' \leq j-1$ and consider $|\phi_j|$. The relation (8) implies

$$\begin{aligned} |\phi_j| \leq |D_j^{-1}| \sum_{\substack{K \in \\ \kappa_0, \ldots, \\ \kappa_\nu\}, \\ K \leq j}} |a_K| [\delta_{K,j} \\ + \sum_{\substack{m=1,\ldots,K \\ m \leq j-K}} \binom{K}{m} \sum_{p_1=1}^{j-K} \cdots \sum_{p_m=1}^{j-K} \frac{1}{p_1! \ldots p_m!} \\ \cdot \sum_{\substack{j_1^1, \ldots, j_{p_1}^1, \\ \vdots \\ j_1^m, \ldots, j_{p_m}^m, \\ j_1^1 + \cdots + j_{p_1}^1 + \ldots \\ + j_1^m + \cdots + j_{p_m}^m = j-K}} |\phi_{j_1^1}| \ldots |\phi_{j_{p_1}^1}| \ldots |\phi_{j_1^m}| \ldots |\phi_{j_{p_m}^m}|] \end{aligned} \tag{41}$$

hence, by recurrence assumption,

$$\begin{aligned} |\phi_j| \leq |D_j^{-1}| \sum_{\substack{K \in \\ \kappa_0, \ldots, \\ \kappa_\nu\}, \\ K \leq j}} |a_K| [\delta_{K,j} + \sum_{\substack{m=1,\ldots,K \\ m \leq j-K}} \binom{K}{m} \sum_{p_1=1}^{j-K} \cdots \sum_{p_m=1}^{j-K} \frac{1}{p_1! \ldots p_m!} \\ \cdot \sum_{\substack{j_1^1, \ldots, j_{p_1}^1, \\ \vdots \\ j_1^m, \ldots, j_{p_m}^m, \\ j_1^1 + \cdots + j_{p_1}^1 + \ldots \\ + j_1^m + \cdots + j_{p_m}^m = j-K}} \sigma_{j_1^1} \ldots \sigma_{j_{p_m}^m} e^{g(j_1^1) + \ldots g(j_{p_m}^m)}] \end{aligned} \tag{42}$$

therefore

$$|\phi_j| \leq |D_j^{-1}| |a_{\kappa_{max}}| e^{g(j-\kappa_0)} \sigma_j$$

We shall distinguish two cases:

- if $j \geq \kappa_0$ and $|D_j|^{\frac{1}{2}} \geq \frac{1}{6}$, then

$$|D_j^{-1}||a_{\kappa_{max}}|e^{g(j-\kappa_0)}\sigma_j \leq 36|a_{\kappa_{max}}|e^{g(j-\kappa_0)}\sigma_j \leq e^{g(j)}\sigma_j$$

- otherwise there exists $k \geq 0$ such that $\frac{1}{6q_{k+1}} \leq |D_j|^{\frac{1}{2}} < \frac{1}{6q_k}$, and in this case, $|D_j^{-\frac{1}{2}}| \leq 6q_{k+1}$. Moreover, $j \in \mathcal{M}$ implies that d divides j; then $j \in A_k(j)$ therefore, by construction of g_k,

$$g_k(j) = g_k(j - \kappa_0) + 1$$

therefore

$$\ln|D_j^{-1}| + \ln|a_{\kappa_{max}}| + g(j-\kappa_0) \leq 2\ln q_{k+1} + \ln 36 + \ln|a_{\kappa_{max}}| + g(j-\kappa_0) \leq g(j)$$

thus

$$|\phi_j| \leq |D_j^{-1}||a_{\kappa_{max}}|e^{g(j-\kappa_0)}\sigma_j \leq e^{g(j)}\sigma_j$$

The recurrence is finished. Thus, for all $j \geq \kappa_0$,

$$\frac{1}{j}\ln|\phi_j| \leq \frac{1}{j}g(j) + \frac{1}{j}\ln\sigma_j \tag{43}$$

Now

$$\frac{1}{j}g(j) \leq \sum_{0\leq l\leq k}\frac{2}{j}g_l(j)\ln q_{l+1} + \delta\ln(|a_{\kappa_{max}}|) + 36$$

therefore by Lemma 4.4,

$$\frac{1}{j}g(j) \leq \sum_{0\leq l\leq k}(\frac{2}{dq_l} + \frac{16}{dq_{l+1}})\ln q_{l+1} + \delta\ln(|a_{\kappa_{max}}|) + 36$$

where $\delta = 1$ if $|a_{\kappa_{max}}| > 1$ and 0 otherwise. Thus,

$$\begin{aligned}\limsup_{j\to+\infty}\frac{1}{j}\ln|\phi_j| &\leq \limsup_{j\to+\infty}\frac{1}{j}g(j) + \limsup_{j\to+\infty}\frac{1}{j}\ln\sigma_j \\ &\leq \sum_{l\geq 0}\frac{2\ln(q_{l+1})}{dq_l} + \sum_{l\geq 0}\frac{16\ln(q_{l+1})}{dq_{l+1}} \\ &\quad + \delta\ln(|a_{\kappa_{max}}|) + 36 + \limsup_{j\to+\infty}\frac{1}{j}\ln\sigma_j\end{aligned} \tag{44}$$

Thus,

$$\limsup_{j\to+\infty}\frac{1}{j}\ln|\phi_j| \leq 2\sum_{l\geq 0}\frac{\ln q_{l+1}}{dq_l} + C'$$

where

$$C' = \ln(|a_{\kappa_{max}}|) - \ln R + \frac{16C_1}{d} + 36$$

if $|a_{\kappa_{max}}| > 1$ and $R < 1$,

$$C' = -\ln R + \frac{16C_1}{d} + 36$$

if $R < 1$ and $|a_{\kappa_{max}}| \leq 1$, and

$$C' = \frac{16C_1}{d} + 36$$

otherwise. It only remains to use the characterization of the radius as a function of $\limsup_{j\to+\infty} \frac{1}{j}\ln|\phi_j|$. □

Remark 10. In the proof of the above theorem, one sees that the constant C' can be made independent of the indices $\kappa_0, \dots, \kappa_\nu$, because the radius of convergence R is bounded away from zero uniformly with respect to these indices.

In fact, the proof can be generalized to an analytic function with infinitely many Fourier coefficients, as long as the sequence of the Fourier coefficients is bounded. Then the integer d is defined as the limit of the greatest common divisor of $\kappa_0, \dots, \kappa_n$ as n tends to ∞.

8. **Appendix.**

8.1. **A proof of equation (6).** Here we give a proof of equation (6) in two lemmas.

Lemma 8.1. *Let $H(z,\lambda) = (h_1(z,\lambda), h_2(z,\lambda))$ be the linearization. Then $h_2(z,\lambda) = \frac{h_1(z,\lambda)}{h_1(\lambda^{-1}z,\lambda)}$.*

Proof. Let $F(z,\lambda) = (\lambda z\prod_{k=1}^N e^{ia_k z^k}, \lambda\prod_{k=1}^N e^{ia_k z^k})$ be the function generating the system, and $R(z,\lambda) = (\lambda z, \lambda)$. Note that if $f(z,\lambda) = \lambda\prod_{k=1}^N e^{ia_k z^k}$ then $F(z,\lambda) = (zf(z,\lambda), f(z,\lambda))$. We shall expand the identity $H = F\circ H\circ R^{-1}$ to infer the desired identity. On one side,

$$\begin{aligned} &F\circ H\circ R^{-1}(z,\lambda) = F\circ H(\lambda^{-1}z,\lambda) = F(h_1(\lambda^{-1}z,\lambda), h_2(\lambda^{-1}z,\lambda)) \\ &= (h_1(\lambda^{-1}z,\lambda)f(h_1(\lambda^{-1}z,\lambda), h_2(\lambda^{-1}z,\lambda)), f(h_1(\lambda^{-1}z,\lambda), h_2(\lambda^{-1}z,\lambda))) \end{aligned} \tag{45}$$

By matching the components of $F\circ H\circ R^{-1}$ with those of H, one infers that

$$h_1(z,\lambda) = h_1(\lambda^{-1}z,\lambda)f(h_1(\lambda^{-1}z,\lambda), h_2(\lambda^{-1}z,\lambda))$$

and

$$h_2(z,\lambda) = f(h_1(\lambda^{-1}z,\lambda), h_2(\lambda^{-1}z,\lambda)))$$

Those two identities imply that for all (z,λ),

$$h_2(z,\lambda) = \frac{h_1(z,\lambda)}{h_1(\lambda^{-1}z,\lambda)}.$$

□

Lemma 8.2. *Let $h_1(z,\lambda) = ize^{\Phi_\lambda(z)}$, then equation (6) holds.*

Proof. Let $F(z,\lambda) = (\lambda z\prod_{k=1}^N e^{ia_k z^k}, \lambda\prod_{k=1}^N e^{ia_k z^k})$ be the function generating the system, and let $R(z,\lambda) = (\lambda z,\lambda)$. By definition, $F\circ H = H\circ R$. Now using Lemma 8.1,

$$F \circ H(z,\lambda) = (\frac{h_1^2(z,\lambda)}{h_1(\lambda^{-1}z,\lambda)} \prod_{k=1}^{N} e^{ia_k h_1(z,\lambda)^k}, \frac{h_1(z,\lambda)}{h_1(\lambda^{-1}z,\lambda)} \prod_{k=1}^{N} e^{ia_k h_1(z,\lambda)^k})$$

and $H \circ R(z,\lambda) = (h_1(\lambda z,\lambda), \frac{h_1(\lambda z,\lambda)}{h_1(z,\lambda)})$. By matching the components and taking the logarithm modulo $2i\pi$, one obtains equation (6). $\square$

8.2. **A number is Brjuno if and only if its multiple by an integer is Brjuno.** In this section, we will give a proof of the following fact:

Proposition 2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $N \in \mathbb{N}, N \geq 2$. Then α is a Brjuno number if and only if $N\alpha$ is a Brjuno number.*

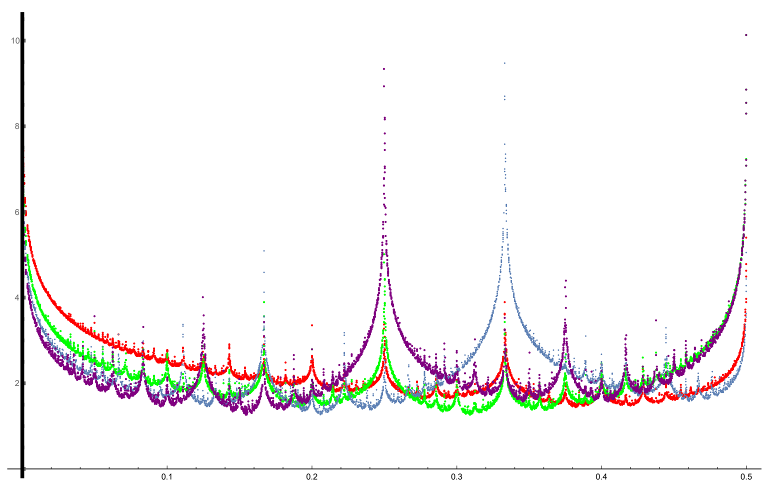

FIGURE 1. The Brjuno function of $N\alpha$ for $\alpha \in (0,1)$. Red: $N = 1$; green: $N = 2$; blue: $N = 3$; purple: $N = 4$.

In order to prove this, it will be useful to recall a few classical facts about continued fractions. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $(\frac{p_n}{q_n})_{n\geq 0}$ be the sequence of best approximants. They have the property that for all rational $\frac{p}{q}$,

$$|\alpha - \frac{p}{q}| \leq \frac{1}{2q^2} \Rightarrow \exists n \geq 0, \frac{p}{q} = \frac{p_n}{q_n}$$

Let (a_n) be the sequence of integers given by the algorithm $a_0 = [\alpha]$, $x_0 = \{\alpha\}$, $x_{n+1} = \{\frac{1}{x_n}\}$, $a_{n+1} = [\frac{1}{x_n}]$. Then

$$q_{n+1} = a_{n+1}q_n + q_{n-1}$$

which implies that

$$a_{n+1} \leq \frac{q_{n+1}}{q_n} \tag{46}$$

Also, there is the following important identity:

$$|\alpha - \frac{p_k}{q_k}| = \frac{1}{q_k} \frac{1}{q_{k+1} + x_{k+1}q_k} \tag{47}$$

As mentioned in [9], there exists a constant $C \geq 0$ such that for all irrational α,

$$|\sum_{n\geq 0} \frac{\ln q_{n+1}}{q_n} - \frac{\ln a_{n+1}}{q_n}| \leq C \tag{48}$$

Therefore α is Brjuno if and only if $\sum_{n\geq 0} \frac{\ln a_{n+1}}{q_n} < +\infty$.

Now let $N \geq 2$; let $a'_0 = [N\alpha], x'_0 = \{N\alpha\}$, $a'_{n+1} = [\frac{1}{x'_n}], x'_{n+1} = \{\frac{1}{x'_n}\}$. Let $(\frac{p'_n}{q'_n})$ be the sequence of best approximants of $N\alpha$.

Lemma 8.3. *If $a'_{n+1} > 2N$, then there exists $k \geq 0$ such that $\frac{p'_n}{Nq'_n} = \frac{p_k}{q_k}$.*

Proof. By (47), one has

$$|\alpha - \frac{p'_n}{Nq'_n}| \leq \frac{1}{N}\frac{1}{q'^2_n a'_{n+1}} < \frac{1}{2(Nq'_n)^2}$$

thus $\frac{p'_n}{Nq'_n}$ is one if the best approximants of α: there exists $k \geq 0$ such that $\frac{p'_n}{Nq'_n} = \frac{p_k}{q_k}$.
□

Now we can prove the Proposition 2. For all $n \geq 0$ such that $a'_{n+1} > 2N$, let k_n be an integer such that $\frac{p'_n}{Nq'_n} = \frac{p_{k_n}}{q_{k_n}}$. Note that $q_{k_n} \leq Nq'_n$. By (46),

$$\frac{\ln a'_{n+1}}{q'_n} \leq \frac{\ln q'_{n+1}}{q'_n} - \frac{\ln q'_n}{q'_n} \tag{49}$$

The identity (47) adapted to $N\alpha$ implies that $q'_n q'_{n+1} \leq \frac{1}{|N\alpha - \frac{p'_n}{q'_n}|}$ therefore

$$\frac{\ln a'_{n+1}}{q'_n} \leq \frac{|\ln|N\alpha - \frac{p'_n}{q'_n}||}{q'_n} - 2\frac{\ln q'_n}{q'_n} \tag{50}$$

If $a'_{n+1} > 2N$, then

$$\frac{\ln a'_{n+1}}{q'_n} \leq \frac{N|\ln|N\alpha - \frac{Np_{k_n}}{q_{k_n}}||}{q_{k_n}} - 2N\frac{\ln q'_n}{q'_n} \tag{51}$$

Again by (47), $|\alpha - \frac{p_{k_n}}{q_{k_n}}| \geq \frac{1}{2q_{k_n}q_{k_n+1}}$ therefore

$$\frac{\ln a'_{n+1}}{q'_n} \leq \frac{N\ln q_{k_n+1}}{q_{k_n}} + \frac{N\ln q_{k_n}}{q_{k_n}} + \frac{N(\ln N + \ln 2)}{q_{k_n}} - 2N\frac{\ln q'_n}{q'_n} \tag{52}$$

By taking the sum over $n \geq 0$,

$$\begin{aligned}
\sum_{n\geq 0} \frac{\ln a'_{n+1}}{q'_n} &= \sum_{n\geq 0, a'_{n+1} > 2N} \frac{\ln a'_{n+1}}{q'_n} + \sum_{n\geq 0, a'_{n+1}\leq 2N} \frac{\ln a'_{n+1}}{q'_n} \\
&\leq \sum_{n\geq 0, a'_{n+1} > 2N} \frac{N\ln q_{k_n+1}}{q_{k_n}} + \frac{N\ln q_{k_n}}{q_{k_n}} \\
&+ \frac{N(\ln N + \ln 2)}{q_{k_n}} + \sum_{n\geq 0} \frac{\ln(2N)}{q'_n} \\
&\leq \sum_{k\geq 0} \frac{N\ln q_{k+1}}{q_k} + \frac{N\ln q_k}{q_k} + \frac{N(\ln N + \ln 2)}{q_k} + \sum_{n\geq 0} \frac{\ln(2N)}{q'_n}
\end{aligned} \tag{53}$$

If α is a Brjuno number, the right hand side is finite, which implies that $N\alpha$ is a Brjuno number.

It remains to prove that if α is a Brjuno number, then $\frac{1}{N}\alpha$ is Brjuno. For an irrational number α, let $B_f(\alpha)$ be the Brjuno function as defined in [20]: the convergence of the Brjuno sum $B(\alpha)$ is equivalent to the convergence of $B_f(\alpha)$, therefore α is a Brjuno number if and only if $B_f(\alpha)$ converges. Now the functional identity from [20], which holds for $\alpha \in (0,1)$:

$$B_f(\alpha) = -\ln\alpha + \alpha B_f(\frac{1}{\alpha})$$

implies that any irrational α is Brjuno if and only if $\frac{1}{\alpha}$ is Brjuno. If α is Brjuno, then $\frac{1}{\alpha}$ is Brjuno, therefore, by the above, $\frac{N}{\alpha}$ is Brjuno, and thus its inverse $\frac{\alpha}{N}$ is a Brjuno number. □

Acknowledgments. S.M. acknowledges the financial support of UniCredit Bank through the Dynamics and Information Theory Institute at the Scuola Normale Superiore and of the MIUR PRIN Project Regular and stochastic behaviour in dynamical systems nr. 2017S35EHN. The authors would like to express thanks to the anonymous referees for comments which helped to improve this article.

REFERENCES

[1] J. Aurouet, *Normalisation de Champs de Vecteurs Holomorphes et Équations Différentielles Implicites*, Ph.D thesis, University of Nice, 2013

[2] A. Berretti and G. Gentile, Bryuno function and the standard map, *Comm. Math. Phys.*, **220** (2001), 623–656.

[3] A. Berretti and G. Gentile, Periodic and quasi-periodic orbits for the standard map, *Comm. Math. Phys.*, **231** (2002), 135–156.

[4] A. Berretti, S. Marmi and D. Sauzin, Limit at resonances of linearizations of some complex analytic dynamical systems, *Ergodic Theory Dynam. Systems*, **20** (2000), 963–990.

[5] A. D. Brjuno, Analytic forms of differential equations, *Trudy Moskov. Mat. Obšč.*, **25** (1971), 119–262; **26** (1972), 199–239.

[6] X. Buff and A. Chéritat, Upper bound for the size of quadratic Siegel disks, *Invent. Math.*, **156** (2004), 1–24.

[7] X. Buff and A. Chéritat, The Brjuno function continuously estimates the size of quadratic Siegel disks, *Ann. of Math.*, **164** (2006), 265–312.

[8] T. Carletti and S. Marmi, Linearization of analytic and non analytic germs of diffeomorphisms of $(\mathbb{C}, 0)$, *Bull. Soc. Math. France*, **128** (2000), 69–85.

[9] C. Carminati and S. Marmi, Linearization of germs: Regular dependence on the multiplier, *Bull. Soc. Math. France*, **136** (2008), 533–564.

[10] C. Chavaudret, Normal form of holomorphic vector fields with an invariant torus under Brjuno's A condition, *Ann. Inst. Fourier*, **66** (2016), 1987–2020.

[11] C. Chavaudret and S. Marmi, Reducibility of quasiperiodic cocycles under a Brjuno-Rüssmann arithmetical condition, *J. Mod. Dyn.*, **6** (2012), 59–78.

[12] D. Cheraghi and A. Chéritat, A proof of the Marmi-Moussa-Yoccoz conjecture for rotation numbers of high type, *Invent. Math.*, **202** (2015), 677–742.

[13] A. M. Davie, The critical function for the semistandard map, *Nonlinearity*, **7** (1994), 219–229.

[14] G. Gentile, Resummation of perturbation series and reducibility for Bryuno skew-product flows, *J. Stat. Phys.*, **125** (2006), 321–361.

[15] A. Giorgilli and S. Marmi, Improved estimates for the convergence radius in the Poincaré-Siegel problem, *Discrete Contin. Dyn. Syst. Ser. S 3*, **4** (2010), 601–621.

[16] A. Giorgilli, U. Locatelli and M. Sansottera, Improved convergence estimates for the Schröder-Siegel problem, *Ann. Mat. Pura Appl.*, **194** (2015), 995–1013.

[17] J. M. Greene and I. C. Percival, Hamiltonian maps in the complex plane, *Phys. D*, **3** (1981), 530–548.

[18] J. Lopes Dias, A normal form theorem for Brjuno skew-systems through renormalization, *J. Differential Equations*, **230** (2006), 1–23.
[19] S. Marmi, Critical functions for complex analytic maps, *J. Phys. A*, **23** (1990), 3447–3474.
[20] S. Marmi, P. Moussa and J.-C.Yoccoz, The Brjuno functions and their regularity properties, *Comm. Math. Phys.*, **186** (1997), 265–293.
[21] S. Marmi and J. Stark, On the standard map critical function, *Nonlinearity*, **5** (1992), 743–761.
[22] L. Stolovitch, Singular complete integrability, *Inst. Hautes Études Sci. Publ. Math.*, **91** (2001), 133–210.
[23] J-C.Yoccoz, Petits diviseurs en dimension 1 *Astérisque*, **231** (1995), 1–88.

Received for publication August 2021; early access February 2022.

E-mail address: claire.chavaudret@imj-prg.fr
E-mail address: stefano.marmi@sns.it