TRUTH FROM COMPARISON

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Introduction

Truth from comparison is the idea that the truth of the sentences in a given formal language can be evaluated by means of binary comparisons between the sentences themselves, such as for example

"the sentence ϕ is *less (more) true than* the sentence ψ".

This is presented as an alternative to the standard approach, which consists rather in evaluating sentences by assigning them a certain value, called *truth value*. The research reported in this doctoral dissertation was aimed at laying down the formal foundations and exploring the philosophical implications of truth from comparison.

The shift in focus from *pointwise valuations*, which typically feature in standard truth-value semantics, to *pairwise valuations* based on comparative judgements, is motivated by the need of providing a satisfactory philosophical account of truth values. Truth values are posited as arguments (or values) of functions that associate sentences with their semantic status. The functional approach turns out to be a powerful tool that easily allows for generalizations, starting with the cardinality of the set of truth values. It also poses philosophical problems concerning nature and interpretation of truth values and truth ascriptions, especially when non-classical values are involved, up to the extreme case of infinitely many truth values, or degrees of truth. Philosophical difficulties we encounter in dealing with truth values can be overcome by embracing a relational perspective. A way for doing this is to adopt a structuralist position, according to which the nature of truth values does not count, what really counts is the structure defined over the set of truth values, which is usually both operational (truth functions) and relational (truth ordering). However, any structuralist approach to truth values must postulate at least a set of objects on which to impose the desired structure. Truth from comparison is a more radical step in this direction: we start just with the set of sentences and consider an ordering relation among them which expresses comparisons in respect of truth, without introducing objects whose nature and interpretation one has the onus to clarify.

The main thesis I defend is the following: the comparative perspective I propose is a viable way of dealing with logical valuedness while avoiding explicit recourse to a given set of truth values, and it ultimately provides a philosophical justification thereof.

To this aim, I build on some concepts and techniques from measurement theory. Measurement theory distinguishes qualitative and quantitative approaches to measurement, i.e. pairwise comparisons and pointwise numerical assignments, respectively. A fundamental position in measurement theory holds that quantitative structures are based on qualitative ones, which act as mathematical and conceptual foundations. The task is then to look at which kind of mathematical properties should qualitative data have in order to constrain a quantitative representation. The key step in my argumentation is to bring this methodological lesson to bear on the notion of comparative truth. Let $\mathcal{L}$ be a propositional language and $\mathcal{SL}$ the set of sentences built recursively by means of a set of connectives. I take as primitive a binary relation $\preceq \subseteq \mathcal{SL} \times \mathcal{SL}$ interpreted as *no more true than* and I investigate which conditions this relation should satisfy in order to guarantee the existence of a valuation function v from $\mathcal{SL}$ to a suitable set of truth values representing it, namely such that for all $\phi, \psi \in \mathcal{SL}$

$$\phi \preceq \psi \Rightarrow v(\phi) \leq v(\psi).$$

This investigation involves mathematical methods, more precisely algebraic ones. An algebra is a set with certain operations defined over it. Since the set of sentences can be seen as an algebra with the connectives acting as operations and the same holds for the set of truth values, algebraic logic lends itself as a valuable tool for this discussion. In Chapter 1 I explain in more details motivations, aims and methods.

Chapter 2 is devoted to the main mathematical results. In particular I start, in Section 2.1, by considering an abstract framework in which a propositional language and an arbitrary set of connectives are given. In this setting I prove a first extremely general representation theorem for pairwise valuations. Then, in Section 2.2, I restrict my attention to a specific class of logics, which is wide enough to include all the logics of interest for the discussion. As a corollary of the general result, I prove that, given an underlying logic, a suitable semantics induced by pairwise valuations for it yields the standard intended semantics for the logic itself, to the extent that some axiomatic conditions defining the relation *more* or *less true than* are sufficient for it to be representable by a pointwise valuation of that logic. I also discuss the desirability of the axioms, some remarkable cases, and some implications on the notions of truth, truth values and many-valuedness.

This investigation provides the theoretical framework and the formal results necessary to put forward a novel philosophical account of the notion of degrees of truth, as arising in infinite-valued logics, which is the topic of Chapter 3. In spite of the extensive research on the relevant mathematics of degrees of truth, and in spite of many attempts to link them to the formalisation of vagueness or to probability theory, the very notion of degrees of truth remains somewhat clouded in conceptual mystery and philosophically motivated diffidence. In Section 3.1, I apply the results of the second chapter to the case of infinite-valued logics. One of the contributions of this dissertation is to show that, as a consequence of this, degrees of truth can be thought of as possible measures (or cardinalisations) of a comparative notion of truth, which is taken as primitive and is governed by non-numerical principles. This will allow to clarify issues related to the nature and the role of degrees of truth (Section 3.2). This philosophical account of what degrees of truth are and what role they play in the model triggers a positive feedback also on the much-criticised project of modelling vagueness by degrees of truth. It has been argued that a semantics based on functions from sentences to degrees of truth coded by real numbers misrepresents the phenomenon of vagueness. In Section 3.3 I show that the alternative semantics based on the notion *more or less true than* can provide a satisfactory theory of vagueness, and, furthermore, it is immune to some of the traditional objections raised against the standard one, such as the artificial precision objection and the linearity objection. Moreover, thanks to the representation result, the mathematical convenience and the instrumental value typical of the numerical model can be retained, while having as philosophical support the plausibility of a qualitative model.

The qualitative foundation for degrees of truth I put forward also sheds new light on their debated relation with probabilities. Recall that, under certain conditions, subjective probability measures arise (uniquely) from qualitative comparisons. Since the axiomatizations of *more* or *less probable than* and *more* or *less true than* can be fruitfully compared, a new level of analysis for investigating the relation between probabilities and degrees of truth is available. This will be the topic of Chapter 4. Moreover, I argue that a deeper understanding of the distinction between degrees of truth and probabilities can shed new light on their possible interactions. In the second part of the chapter I shall propose a new framework for connecting them, whose core consists in providing a probabilistic interpretation for the notion of graded truth.

Chapter 1

Truth from comparison

Truth has been one of the central topics of discussion throughout the history of philosophy. The modern reflection about truth takes place on several levels and involves various philosophical disciplines. For the purpose of this work we are mainly interested in how the notion of truth is treated within the scope of philosophical logic. In particular, we consider fully formalised languages, whose constitutive elements are propositional sentences, and we look at how the truth of these sentences is evaluated.[1]

1.1 Truth values

The standard way of providing a formal semantics for a given language consists in evaluating sentences by assigning them a certain value, called *truth value*, representing their semantic status. In classical contexts there are two such values: *the True* and *the False*. The introduction of truth values dates back to Frege who introduced them as a special kind of objects representing denotations for sentences (see the survey article on the history of truth values Béziau, 2012). In spite of their relatively recent history, truth values have been considered a central logical notion. Łukasiewicz for example wrote: "Logic is the science of objects of a special kind, namely a science of *logical values*." (Łukasiewicz, 1970, p. 90). And, indeed, truth values play a crucial role in the formal semantics for logical systems.

In order to have a better grasp of this, consider a propositional language given be a set $\mathcal{L} = \{p_1, p_2, \dots\}$ of propositional variables and a set $\mathcal{C}_{\mathcal{L}} = \{c_n \mid c \colon \mathcal{L}^n \to \mathcal{L} \text{ with } n \in \mathbb{N}\}$ of propositional connectives, with a certain arity n. Let $\mathcal{SL}$ denote the set of compound sentences built by recursion starting from $\mathcal{L}$ and $\mathcal{C}_{\mathcal{L}}$. The classical propositional semantics is given by the set of propositional valuations on $\mathcal{L}$, namely functions $v \colon \mathcal{L} \to \{0, 1\}$. The intended interpretation of '$v(p) = 1(0)$' is 'the

[1] I follow the practice, widespread in works on truth employing a fully formalised language, of taking sentences as truth bearers.

propositional variable p is *true (false)* under the valuation v'. Classical propositional logic is truth-functional. This means that valuations extend uniquely to $\mathcal{SL}$, namely there are fixed functions $f_{c_n}\colon \{0,1\}^n \to \{0,1\}$ from the set of truth values into itself which establish how the truth value of a complex formula is computed given the truth values of the components:

$$v(c_n(\phi_1, \dots, \phi_n)) = f_{c_n}(v(\phi_1), \dots, v(\phi_n)).$$

Connectives are semantically determined by these functions, usually called *truth functions.*

The formulation in terms of truth values helps to clarify a point which is often left unexpressed: in order to establish whether a sentence is true it is necessary to specify a structure, or an interpretation, in relation to which the truth assignment is made. In propositional logics, sentences are uninterpreted objects and the role of providing an interpretation is played by valuations themselves. Any valuation is a possible interpretation for the sentences, i.e., it is a model: it declares what holds and what does not. In this respect, any valuation can be thought as a possible world. That is why, in order to establish what is logically true or what logically follows from what, *all* possible valuations (assignment of values) are considered. Consider, as an example, the standard definition of logical consequence for classical logic: a conclusion logically follows from a set of premisses if and only if, for all the possible valuations, whenever all the premisses are true (receive the True, or 1, as truth value) the conclusion is also true. Formally, let $\mathcal{V}$ be the set of classical logical valuations and $\mathcal{P}(\mathcal{SL})$ the power set of $\mathcal{SL}$, we introduce the relation $\models_{CL} \subseteq \mathcal{P}(\mathcal{SL}) \times \mathcal{SL}$ defined as follows: for all $\Gamma \subseteq \mathcal{SL}$ and $\phi \in \mathcal{SL}$

$$\Gamma \models_{CL} \phi \;:\Leftrightarrow\; \forall v \in \mathcal{V} \;\text{ if }\; \forall \gamma \in \Gamma \;\; v(\gamma) = 1 \;\text{ then }\; v(\phi) = 1.$$[2]

Logical truths or tautologies are sentences following from an empty set of premisses, intuitively they are the sentences which are true in all possible valuations. Formally, for all $\phi \in \mathcal{SL}$:

$$\models_{CL} \phi \;:\Leftrightarrow\; \forall v \in \mathcal{V} \;\; v(\phi) = 1.$$

The classical consequence relation is reflexive, monotone, transitive (it is a Tarskian consequence relation) and moreover it satisfies a requirement of *structurality*, namely it is preserved by uniform substitutions.

Definition 1.1. *A* uniform substitution *is a mapping $\sigma\colon \mathcal{SL} \to \mathcal{SL}$, such that for every $\phi_1, \dots, \phi_n$ and each $c_n \in \mathcal{C}_{\mathcal{L}}$ the following holds:*

$$\sigma(c_n(\phi_1, \dots, \phi_n)) = c_n(\sigma(\phi_1), \dots, \sigma(\phi_n)).$$

[2] I use the symbol $:\Leftrightarrow$ when a definition is in place. Analogously, $:=$ stands for 'is defined to be equal to'.

Substitutions are uniform reinterpretations of the components of a sentence (other than its logical constants). Structurality, or substitution invariance, is the following property: for all $\Gamma \subseteq \mathcal{SL}$ and $\phi \in \mathcal{SL}$ and for any uniform substitution σ of $\mathcal{SL}$:

$$\text{if } \Gamma \models_{CL} \phi \text{ then } \sigma(\Gamma) \models_{CL} \sigma(\phi),$$

where $\sigma(\Gamma)$ is to be interpreted as $\sigma(\gamma)$ for all γ in Γ.

Truth-value semantics for a given logic is based on the assumption that the semantic status of sentences can be represented by values assigned to them. Moreover, the classical account just outlined is characterized by additional modelling hypothesis concerning the set of truth values.

Bivalence: there are exactly two truth values, the True and the False.

Non-contradiction: each sentence is given exactly one truth value by each valuation.

Truth-functionality: the truth value of a compound sentence is determined by the truth values of its components.

Moreover, there is an hypotheses on logical consequence:

Truth preservation: a conclusion logically follows from a set of premisses if, for all the possible valuations, whenever the premisses are true the conclusion is also true.

One of the virtues of the formalisation lies in the disclosure of new theoretical possibilities whose consequences can be explored formally. Once isolated as principles, these requirements can be put under discussion, modified or even abandoned. Different routes can be taken for going *non-classical*, according to which principle is questioned. We are interested in non-classical extension of the classical semantics obtained by dropping the principle of Bivalence. The assumption that there are only two truth values can be, and has been, dropped. This leads to consider values other than true and false, ranging from three values to infinitely many.

The idea of dropping the principle of bivalence was fully exploited from a logical point of view by Łukasiewicz who developed in the late 20's logical systems including further values beyond true and false. This choice was philosophically motivated by the need of modelling some phenomena which are not captured by the traditional true versus false dichotomy. In his paper On determinism (Łukasiewicz, 1970, pp. 110-128), Łukasiewicz advanced the idea of treating sentences concerning the contingent future such as 'I shall be in Warsaw at noon on 21 December of the next year', as having a semantic value different from true and false. Independently and during the same years, Emil Post put forward a many-valued system as a natural generalization of the two-valued case:

> One class of such systems seems to have the same relation to ordinary logic that geometry in a space of an arbitrary number of dimensions has to the geometry of Euclid. [...] In these systems instead of the two truth-values + and −, we have m distinct 'truth-values' t_l, t_2, ..., t_m where m is any positive integer. (Post, 1921)

These two remarkable examples correspond to two different attitudes, philosophical and mathematical, towards many-valued logics. On the one hand, it can be argued that properties other than *being true* and *being false*, such as *being undetermined*, deserve philosophical attention as possible semantic status for sentences and therefore matching truth values are introduced. On the other hand, the functional approach (functions associating sentences with truth values) turns out to be a powerful mathematical tool which easily allows for generalizations, starting with the cardinality of the set of truth values. Once the functional formulation is considered, it becomes natural to enlarge the codomain of the valuation functions and consider other values beyond true and falses and then investigate the possible philosophical interpretation of these further values. Many-valued extensions of classical logic are obtained by extending the codomain of valuation functions and providing new truth functions for defining the meaning of connectives. We are interested in many-valued logics that reject Bivalence and keep all the other principles, Non-contradiction, Truth-functionality and Truth preservation. I refer the interested reader to Gottwald (2001) for a detailed survey on the history, theoretical foundations, mathematical development and applications of many-valued logics.

In the light of the mathematical depth of their properties and philosophical richness of their applications, among the many-valued logical systems it is worth considering *infinite-valued logics* (Hájek, 1998). In these logics the real unit interval is taken as set of truth values and, accordingly, logical valuations are functions $v\colon \mathcal{SL} \to [0, 1]$:

> We shall assume that the truth degrees are linearly ordered, with 1 as maximum and 0 as minimum. Thus truth degrees will be coded by (some) reals. And even if logics of finitely many truth degrees can be developed we choose not to exclude any real number from the set of truth degrees. We shall always take the set $[0, 1]$ with its natural (standard) linear order. (Hájek, 1998, p. 3.)

The values 0 and 1 are interpreted as the traditional truth values (absolutely) *false* and (absolutely) *true*, respectively, and the intermediate values as *truth-degrees* or *degrees of truth*.

1.1.1 Philosophical status of truth values

On the standard semantic account just outlined, there is the implicit assumption that truth values have something to do with a general concept of truth. Indeed, there is a tight correlation between the truth-as-property and truth-as-object perspectives: a sentence has the property of being true if and only if its truth value is *the True*. Similarly, sentences have the property of being false if and only if their truth value is *the False*.[3] Even granted that the interpretation of the classical truth values is clear, introducing truth values as objects is not a philosophically innocuous move, it actually poses ontological problems related to their nature: what kind of entities are truth values?

> In this context, what we are being asked is, "What are these truth values of which you speak?" Now often, amongst philosophers, talk of the truth values of sentences is simply a stylistic variant of talk of whether sentences are true. This is fine in itself, but it should not blind us to the fact that truth values, properly so-called, are not mere *façons de parler*: they are objects. There is a particular point to positing these objects: we wish to bring certain useful mathematical machinery – most notably the machinery of functions – to bear on the analysis of phenomena such as truth and validity. Using this machinery, we can achieve a very elegant and useful picture of language and its relationship to the world, and hence we do not baulk at positing the objects required to get the picture off the ground: we need objects to serve as the arguments and values of functions, and in particular, we need certain objects called truth values. Depending upon our antecedent ideas about the phenomena we wish to model – for example, whether properties may be possessed to intermediate degrees – we will posit different sets of truth values, with different structural properties. (Smith, 2008, p. 212.)

Problems concerning the nature of truth values are independent from the cardinality of the set of truth values, and they arise already at the classical level when Bivalence is still in place. Additional truth values beyond the True and the False inherit these problems related to their ontological status and, in addition, pose new problems of interpretation. Since truth values lie at the heart of logic, the idea – be it philosophically or mathematically motivated – of considering values beyond true and false had a revolutionary character and a great philosophical impact. Łukasiewicz wrote:

> It is not easy to foresee what influence the discovery of non-Chrysippean systems of logic [i.e. many-valued systems] will exercise on philosophical

[3] See Cook (2009) for a discussion on the possible relations between language and the world expressed by truth values.

> speculation. However, it seems to me that the philosophical significance of the systems of logic treated here might be at least as great as the significance of non-Euclidean systems of geometry. (Łukasiewicz, 1970, p. 176.)

Indeed, many-valued logics pose new interesting philosophical challenges. In particular, there are questions concerning the status of the additional values, like how to interpret them and whether it makes sense at all to call them *truth* values. We have seen that classical truth values are deeply interwoven with truth taken as a property: a sentence is true (false) if and only if its truth value is the True (the False). When values beyond true and false are considered the association with a corresponding property or semantic status (or a lack thereof?) is less immediate and it is not clear how additional truth values should be interpreted. Problems of interpretation become particularly pressing when infinitely many truth values are considered, in what follows I shall explain why this is the case.

Many objections have been raised against degrees of truth, addressing their motivation, namely the very fact that there is such a thing like graded truth, or their aim, questioning that they are of any use in philosophy. I shall not deal with those objections here. Neither shall I talk about degrees of truth as a model for vagueness (see Section 3.3). I rather focus here on the alleged intrinsic philosophical implausibility of degrees of truth. I distinguish three kinds of objections about the *nature* of degrees of truth:

1. what *are* degrees of truth?
2. how do we *interpret* the fact that a sentence is true to a certain degree?
3. how can truth be *measured*?

We know that degrees of truth, such as truth values in general, are objects assigned to sentences in order to represent their semantic status. But, again, which kind of objects are they? Moreover: what does it mean for a sentence to be 0.7 true, or $1/\pi$ true? A possible answer, from a logical point of view, is that this objection is ill-posed since one does not assign specific values to sentences, rather one considers all possible assignments and focuses on valid inferences:

> Let us comment that mathematical fuzzy logic concerns the possibility of sound inference, surely not techniques of ascribing concrete truth degrees to concrete propositions. (Hájek, 2009, p. 368.)

This is certainly true, but Objection 2 is actually less naive than that. It aims at underlining the implausibility for a sentence of having an exact real number as truth value. A point that logicians and philosophers defending degrees of truth and fuzzy logics cannot ignore. In the classical case, the fact that a sentence has value 1 (0) is interpreted as the sentence being true (false). How about degrees of truth? Sentences receive a number from the real unit interval $[0, 1]$, but what does that number stand for? The standard interpretation resorts to the idea that truth itself is graded, and, accordingly, sentences can be true to a greater or lesser degree. However, this is far from justifying the fact that sentences receive a unique, exact real number as truth value. This leads to the third objection, which concerns the measurability of truth. Even if we accept that truth is graded, or that it comes in degrees, how can an abstract attribute like the truth of sentences be measured? Again, the problem is not how to measure exactly or concretely the amount of truth of a specific sentence, it is rather an issue of measurability in principle. The assignment of truth values is unique and extremely precise (being a real number) and any choice of a value seems arbitrary, for instance, how can we justify the choice of the truth value 0.24 over 0.23?

Smith (2008) convincingly points out that issues related to the ontological status of degrees of truth are in no respect more problematic than the case of classical truth values. The solution in both cases has a structuralist flavour: there is no issue as to what truth values are, all we should know is the structural properties they satisfy.

> Now, to return to our question as to what these truth values *are*: this is not a special question for the fuzzy theorist. If we want to ask: "What *are* these fuzzy truth values – these degrees of truth?", then we should also ask, "What *are* these classical truth values, 0 and 1?". In both cases, the answer is that the truth values are elements in a particular sort of algebraic structure – and what matters is the structural properties of the latter, not the intrinsic nature of its elements. (Smith, 2008, p. 212.)

The set of truth values, regardless to its cardinality, is a structured set (an algebra, as we will call it later): in particular it is a set endowed with operations, the truth functions, mirroring in number and arity the operations over the set of sentences (connectives). Moreover, the set of truth values is usually endowed with an ordering relation. In classical logic the set of truth values, $\{0, 1\}$ or $\{\perp, \top\}$, forms a Boolean algebra, and, as such, it is endowed with the ordering $0 \leq_T 1$, which is sometimes called the *truth ordering*, because it expresses the fact that truth is increasing: the True is more true than the False. Also truth values of three-valued logic can be ordered to form a lattice, usually with the third value being more true than the False, but less true than the True. In infinite-valued logic, truth values form a chain whose upper and lower bound are the True and the False, respectively (see Shramko and Wansing, 2011, Chapter 3). Building on this, a possible way to avoid issues concerning the nature of truth values is to embrace a structuralist position toward truth values, according to which truth values are elements of a certain structure, their nature is irrelevant, and all that matters are the structural properties of the set. The idea is that any set endowed with that specific structure is a possible set of truth values.

However, this answer is not completely satisfactory. First of all, it may be the case that the nature of truth values *determines* the structural properties of the set, for example when numbers are chosen as truth values. In those cases, it seems arbitrary to determine which arithmetical properties of the chosen set are properties of truth values and which are not. The structuralist perspective over truth values is particularly unsatisfactory for degrees of truth, because they are usually identified with real numbers, and the structure of the real numbers is rich and sophisticated. Which of the arithmetical properties of the real unit interval should be taken as structural properties of degrees of truth? Some of the structural properties of the chosen set of truth values might be essentially related to their numerical nature, for example linearity or density of the reals. It is, therefore, not clear how much of this structure is needed, which properties of the set of numbers chosen as truth values match up with the desired properties of the truth assignment. So, the structuralist position is only a partial solution, because in some cases the structural properties depend on which objects are chosen as truth value.

Moreover, we can doubt of the adequacy of a model that introduces objects, assigns them a crucial role and, still, in order to make philosophical sense of it, we are asked to ignore exactly those objects. Truth values are objects with a major logical role and questions related to their philosophical status (their nature, their interpretation) cannot be ignored. The problem of the interpretation of the semantic fact that a given sentence has value x does not vanish even if a structuralist position is adopted. Until valuation functions remain the single most important element of the formal semantics, both conceptually and formally, the philosophical discourse about logic (especially about infinite-valued logics) cannot leave issues related to the philosophical status of truth values out of consideration. Hence the need of an account that questions the role of valuation functions as constitutive elements of the semantics, like the one I am introducing in Section 1.2.

1.1.2 On the role of formal methods

I argue that the problem of the philosophical status of truth values is actually a problem of formalisation. Justifying this claim requires a general account of the role of formal methods. To this aim, I build on Cook (2002) and the *logic-as-modelling view*. Cook distinguishes three different ways in which formal systems or formal semantics relate to the piece of everyday discourse they are meant to model or explicate:

1. **Descriptive View:** the formalism aims at describing what is really going on and "every aspect of the formalism corresponds (at least roughly) to something

actually occurring in the phenomenon being formalized." (Cook, 2002, p. 234.)

2. **Instrumentalist View:** there is a matching up between the formalism and the phenomenon, however the former does not represent something really occurring in the latter nor provides real explanation of the language being studied.

3. **Modelling View:** formal systems are not descriptions, nor are they completely instrumental, they are rather good models, ways for representing phenomena. Some aspects of the model are representative of the phenomenon, whereas some others are merely artefacts.

The Modelling View implies the acceptance of a theory of formal methods which contemplates that some aspects of the model are meant to correspond to real aspects of the phenomenon being modelled (called *representors*) and some aspects are introduced for strictly formal, mathematical reasons or application-driven reasons (called *artefacts*). Typically, interpretative problems arise for parts of the model which do not find a counterpart in the phenomenon. However, the distinction between representors and artefacts cannot be traced arbitrarily by simply blaming of artificiality problematic aspects of the model, in particular we cannot consider as artefacts elements which play a crucial role in the model (see Keefe, 2012, about what can we ignore in a model).

Applying this framework to our discussion on truth values imposes a reflection on two main questions: (i) what kind of phenomenon have we as target system of the formalisation? (ii) can truth values as objects be considered merely artefacts of the model?

The informal concept we seek to model by introducing truth values is surely multifaceted, nonetheless I argue that two aspects can be isolated. First, there is the practice of natural language to ascribe a semantic status to statements, by saying for example that they are true; and, again at the level of language, there are pieces of everyday discourse which escape classical true/false ascriptions, like for example statements about the contingent future or statements containing vague predicates. Secondly, deeply related to the linguistic aspect, there is a naive theoretical aspect (or pre-theoretical aspect) given by intuitions about the phenomenon itself, like for example that those statements have an intermediate status with respect to truth. This cluster of linguistic and naive theoretical aspects can be made philosophically precise and then formalised. I call *logical valuedness* the idea that the truth of sentences can be evaluated within a formal system by ascribing them a certain semantic status, like truth and falsehood, and *many-valuedness* the idea of considering nonclassical ascriptions, which escape the true versus false dichotomy. We have seen that these ideas can be formalised by introducing truth values, two or more, as objects assigned to sentences by valuation functions.

The just outlined two-fold nature of the informal concept, I argue, is typical of semantic notions.[4] In such a context, the target of a formalisation is a fragmented collection of ideas, in which at least two dimensions can be isolated:

linguistic aspect: concepts we use are linguistically expressed and thus they are subjected to syntactic and grammatical rules, and intuitions related to linguistic competence and practice;

naive theoretical aspect: intuitive concepts are already "theory-laden", they come with theoretical presuppositions. I call this theoretical aspect 'naive' because is implicit, related to the linguistic use, largely unsystematic, unstable, lacking of internal coherence and of intersubjective agreement.

Philosophical conceptions are a first refinement of our intuitions and they take both aspects into account:

> Many, probably most, of the crucial concepts in philosophical discourse originate in idealizations of non-philosophical language. [...] [P]hilosophers (tacitly) assume that there is, or can be constructed, a more fundamental and more straight-lined concept behind the embellished meanings of words and phrases in non-regimented natural language. (Hansson, 2000, p. 163.)

Being the upshot of an explicit and conscious theoretical elaboration, philosophical conceptions systematise the subject matter, operate idealisations and constitute the grounding of the formalisation. A reflection on these aspects leads to a refinement of the artefact/representors distinction. Representative are aspects of the model meant to correspond to real aspects of the phenomenon, either taken in its linguistic dimension or in its naive theoretical dimension; whereas artefactual are aspects related exclusively with the formalisation process which are not present in the pre-formal notion.

The philosophical difficulties related to truth values pointed out in the previous section can be explained by noticing that truth values as objects actually do not mirror any aspect really present in the informal concept, since they are not present either in the natural language or in the naive theoretical aspect. As Béziau (2012, p. 235) notices "It seems that 'truth-value' is exclusively used by logicians, philosophers of logic and analytic philosophers". Truth values as objects are artificial but, nevertheless, and this answers to Question (ii) in the negative, they are not treated as such: they are not dispensable. Truth values are constitutive elements of the formal semantics since sentences are interpreted and evaluated by assigning them a unique object in a given domain. Logicians interested in philosophical issues cannot help but providing

[4]As a further example of this see my discussion on graded truth later in Subsection 3.2.2.

an answer to the pressing interpretative questions outlined in the previous section and they do not have the easy way out of invoking artificiality.

Among the risks of the formalisation against which Hansson (2000, p. 169) warned there was the "introduction of *ad hoc constructions* with no sensible informal interpretation" and, related to this, the "undue focus on problems that are mere artefacts of the formal model, rather than on more general philosophical problems that the model can be helpful in elucidating". This seems exactly our case. Difficulties apparently related to the notions of logical valuedness and many-valuedness *per se* depend, instead, on the chosen formalisation. It is a problem of formalisation in a precise sense: the formal model assigns an essential role to aspects that would better be considered artefacts, namely truth values as objects, including for example the truth value 0.732.

A possible way for addressing this problem is to reduce down to the bear minimum the introduction of artificial elements in the process of philosophical precisification and formalisation. That is why, in what follows, I propose an alternative model, called *truth from comparison*, in which problematic aspects of the present model are treated as artefacts. Truth from comparison improves on the structuralist position, according to which truth values are structured objects whose nature is irrelevant, but goes far beyond it. More precisely, truth from comparison is a model based on the idea that all we need to know for evaluating sentences of a formal language are the relative positions of sentences compared in respect of their truth. Instead of a semantics based on functions into a structured set of truth values (pointwise semantics), I consider a semantics based on comparative judgements among sentences (pairwise semantics). This provides a model for logical valuedness and many-valuedness which does not presuppose recourse to objects other than sentences like truth values and valuation functions, whose nature has to be justified. Moreover, I shall show that truth from comparison is to all effects a way for evaluating sentences, alternative to the functional approach and yet compatible with it.

1.2 Truth from comparison

Truth from comparison is the idea that the truth of the sentences in a given formal language can be evaluated by means of binary comparisons between the sentences themselves, such as for example

"the sentence ϕ is *less (more) true than* the sentence ψ".

I present this idea as an alternative to the standard approach, which consists, as we have seen, in evaluating sentences by assigning them a certain object, the truth value. The key shift in focus is from pointwise valuations, which typically feature in standard truth-value semantics, to pairwise valuations based on comparative judgements.

The motivation for this investigation lies on doubts and concerns surrounding the notion of truth values as objects described in the previous section on the one hand, and, on the greater philosophical plausibility of looking at comparative judgements instead, on the other. I argue that comparative judgements of the form "the sentence ϕ is more (or less) true than the sentence ψ" are a more plausible model for logical valuedness and many-valuedness than absolute judgements of the form "the truth value of the sentence ϕ is x" and "the truth value of the sentence ψ is y", which come almost always together with the information "x is greater (or less) than y". Comparative judgements involve less mathematical structure than absolute ones, are governed by non-numerical principle and thus they guarantee a certain independence from the numerical apparatus. Also, comparative judgements act at a more fundamental level because numerical values are assigned in a way that respects the comparative relations among objects (see next section).[5]

We have seen that in defence of truth values one can advocate a structuralist position. However, we can notice that the pairwise perspective goes much further in the direction of dissolving the issue as to what truth values are than the structuralist approach does. In a pairwise perspective the fact that the nature of truth values as objects does not matter is taken seriously up to the point that there are no objects any more, just structure. More precisely, any structuralist approach to truth values must postulate at least a set of objects on which to impose the (algebraic) structure that is deemed desirable. By contrast, the pairwise approach gets started just with the set of sentences themselves, along with the mere additional structure of a binary relation. In a slogan, whereas pointwise valuedness is a set of objects with a structure, pairwise valuedness is itself just a structure.

One might notice that what we are doing here is nothing but transferring the ordering structure already assumed among truth values (the truth ordering) directly over the set of sentences. Indeed, this is the case. However, the truth ranking over sentences is a structure that can be weaker than the ordering structure among truth values, and, as already said, it does not presuppose the existence of truth values as objects. As a consequence, one of the main virtues of the pairwise perspective emerges here. Transferring the ordering structure among truth values directly over

[5]Another point in favour of comparative judgements could be that they have a greater cognitive plausibility over absolute judgements. In psychological jargon, comparative judgements, namely judgements about whether there is a difference between two or more stimuli, are contrasted with absolute judgements, namely judgements about a single stimulus, e.g. about the value of one of its properties or about whether it is present or absent. The former are taken to be easier to perform than the latter. There is some experimental evidence going in this direction for probability ascriptions (Fontanari et al., 2014). Nonetheless, to the best of my knowledge we have no evidence that this is the case also for truth ascriptions. Although symmetry considerations may strongly suggest that this is the case, especially if degrees of truth are considered.

the set of sentences allows for more freedom in the choice of properties, because the properties of the structure are fixed axiomatically and can be manipulated, questioned and eventually dropped. These reasons guarantee that artificiality problems posed by a functional model of logical valuedness and many-valuedness are avoided, since a pairwise perspective helps in taking a step back in the formalisation by taking off some of the mathematical structure and thus reducing the artificial aspects of the model.

Nevertheless, also the instrumental aspect of models is relevant and the functional perspective has, indeed, a great logical value and is rich in mathematical applications. That is why the present proposal is two-fold: on the one hand, truth from comparison is proposed as a new model for logical valuedness, on the other, it is defined in such a way that pointwise attributions of values arise as mathematical consequence. The ultimate intent is to gain both philosophical plausibility and mathematical convenience, which is the best trade-off for a process of formalisation which aims to be deeply interwoven with philosophical considerations.

This analysis triggers a philosophical feedback on the notions involved and is rich in philosophical implications. To start with, it allows a philosophical analysis of the (informal) notion of *more* or *less true* obtained by investigating (and playing with) the defining conditions of its formal counterpart. The possibilities unclosed by considering axioms defining a formal object as properties of its informal counterpart are one of the value of the pairwise perspective. But also, thanks to bridge results between pairwise and pointwise side, a philosophical analysis of the notion of truth values becomes possible. Indeed, truth values are introduced as pointwise counterparts, or possible cardinalisations, of comparative judgements. Therefore, the comparative perspective I propose is a viable way of dealing with logical valuedness while avoiding explicit recourse to a given set of truth values, and it ultimately also provides a philosophical justification thereof. Moreover, a substantial philosophical account of what truth values are and of which role they play can in turn shed light on related debated philosophical problems. For instance, we will see, this is the case for degrees of truth in relation with vagueness and probability.

The formal counterpart of the relation *more or less true than* will be a binary relation on the set of sentences satisfying certain axiomatic properties. This will lead to the definition of the notion of *pairwise valuation* as a basis for a formal semantics called *pairwise semantics*, where the semantic notions of tautology and logical consequence are defined pairwise. Moreover, the set of defining conditions for pairwise valuations suffices to establish its compatibility with pointwise valuations, so that the latter will be introduced as pointwise reformulation of the former. A crucial role in this investigation is played by bridge results connecting pairwise valuations and pointwise valuations. In order to make this connection formally precise, I bring concepts and techniques from measurement theory to bear on this analysis. Moreover, in order to materially establish results I make extensively use of algebraic logic and semantics. In what follows I shall explain this two-fold methodology in details.

1.2.1 Qualitative foundations of measurement

Measurement theory distinguishes qualitative and quantitative approaches to measurement (Krantz et al., 1971). Qualitative approaches have a comparative nature: measuring objects with respect to an attribute P amounts to performing comparisons among objects in the domain of P in respects related to P. Quantitative approaches imply instead assigning numbers to the objects in the domain of the attribute at stake. Clearly, we associate numbers with objects in such a way that the properties of the attribute are faithfully represented. In particular, in assigning numbers we should respect the more-or-less relations among the objects. This can be pushed further by saying that quantitative structures are based on qualitative ones. Indeed, a fundamental position in measurement theory maintains that the qualitative aspect of measure acts as mathematical and conceptual foundation:

> Measurement elevates the qualitative to quantitative. It is possible only when qualitative data have enough structure to sufficiently constrain a numerical representation. Thus, measurement theory seeks to answer questions like, "When are we justified in representing phenomena with properties X by numerical structure Y?" and "How much are we allowed to read into the numbers that results?". (Furnas, 1991, p. 103.)

The idea is, therefore, to start with a relational structure, and in particular with a set endowed with a binary relation expressing comparisons.

Definition 1.2. *Let X be a set and $\preceq$ be a binary relation on X, i.e., $\preceq$ is a subset of $X \times X$. The relation $\preceq$ is a*

preorder: *if and only if for all $x, y, z \in X$ the following are satisfied*

- Reflexivity: $x \preceq x$,
- Transitivity: *if $x \preceq y$ and $y \preceq z$ then $x \preceq z$.*

Partial order: *if and only if for all $x, y, z \in X$ the following are satisfied*

- Reflexivity,
- Transitivity,
- Antisymmetry: *if $x \preceq y$ and $y \preceq x$ then $x = y$.*

Weak order: *if and only if for all $x, y, z \in X$ the following are satisfied*

- Linearity: *Either $x \preceq y$ or $y \preceq x$,*

- Transitivity.

Total order: *if and only if for all $x, y, z \in X$ the following are satisfied*

- Linearity,
- Transitivity,
- Antisymmetry.

Notice that a weak order is always reflexive.

Definition 1.3. *If $\preceq$ is a binary relation on X, two new relations are defined as follows:*

$$x \prec y \text{ iff } x \preceq y \text{ and not } y \preceq x,$$

and

$$x \sim y \text{ iff } x \preceq y \text{ and } y \preceq x.$$

These are referred to as the strict *and* symmetric *parts of $\preceq$, respectively.*

If $\preceq$ is a preorder then $\sim$ is an equivalence relation on X (reflexive, symmetric and transitive). The distinction between preorders and partial orders (and also between weak orders and total orders) is antisymmetry or the lack thereof: for the former it is possible that $x \preceq y$ and $y \preceq x$ for distinct elements of x, y of X, for the latter there cannot be elements which are equivalent with respect to the relation and nonetheless distinct. However, every preorder (weak order) is associated with a partial order (total order) in a natural way by considering the quotient *modulo* the symmetric part $\sim$, namely the set $X/\sim$ of equivalence classes $[x] = \{y \mid y \in X, y \sim x\}$. Equivalence classes form a partition of A, i.e. a family of pairwise non-empty disjoint subsets whose union is A. The relation induced by the preorder (weak order) on the quotient set is a partial order (total order).

Relational structures of this sort are meant to represent rankings or comparative judgements. Given a relational structure, the task is to look at which kind of mathematical properties they are to have in order to constrain a quantitative representation. The aim is to prove *representation theorems*, i.e., theorems establishing the conditions under which a relation that compares objects with respect to a certain attribute can be represented by numerical assignments. To this respect, representation theorems act as bridge results between the qualitative and quantitative side.

> A *representation theorem* asserts that if a given relational structure satisfies certain axioms, then a homomorphism into a certain numerical relational structure can be constructed. [...] From this standpoint, measurement may be regarded as the construction of homomorphisms from empirical relational structures of interest into numerical relational

structures that are useful. Foundational analysis consists, in part, of clarifying (in the sense of axiomatizing) assumptions of such constructions. (Krantz et al., 1971, p. 9.)

Recall that homomorphisms are structure-preserving maps between structures. Representation theorems consist in mapping a relational structure, i.e. $\langle X, \preceq \rangle$, which expresses comparative judgements, to a numerical one, usually the set of the real numbers endowed with their natural linear order, $\langle \mathbb{R}, \leq \rangle$, by means of a structure-preserving map $\Phi\colon X \to \mathbb{R}$, which acts as a measure. Then, given $x, y \in X$, we say that the measure $\Phi(\cdot)$ on X

- *weakly represents* $\preceq$: if and only if

$$x \preceq y \Rightarrow \Phi(x) \leq \Phi(y),$$

- *strongly represents* $\preceq$: if and only if

$$x \preceq y \Leftrightarrow \Phi(x) \leq \Phi(y).$$

Weak representation ensures that given an order there is a function compatible with it. Intuitively, there is no way back: if you know the function $\Phi(\cdot)$ then you have just a partial knowledge of the corresponding order (it might be that $\Phi(x) = \Phi(y)$ while $x > y$). The general form of representation theorem goes as follows: given a relational structure $\langle X, \preceq \rangle$, if $\preceq$ satisfies a certain set of axiomatic conditions then there exists a function $\Phi\colon X \to \mathbb{R}$ such that Φ weakly (or strongly) represents $\preceq$.

These existence results go hand in hand with uniqueness results:

> [A]n analysis into the foundations of measurement involves, for any particular empirical relational structure, the formulation of a set of axioms that is sufficient to establish two types of theorems: a representation theorem, which asserts the existence of a homomorphism Φ into a particular numerical relational structure, and a uniqueness theorem, which sets forth the permissible transformations $\Phi \to \Phi'$ that also yield homomorphisms into the same numerical relational structure. A measurement procedure corresponds to the construction of a Φ in the representation theorem. (Krantz et al., 1971, p. 12.)

In other words, we wonder whether the function whose existence is stated in representation theorems is unique. It would be better to say that we wonder *to what extent* it is unique, since usually the function is unique up to a certain class of transformations determined by the axioms governing the ordering. Such a class is called a class of *permissible* or *admissible* transformations. This means that, given a function representing an ordering, each permissible transformation of the function continues to represent the ordering. Different *scales of measurement* can be distinguished according to the classes of permissible transformations (Stevens, 1946). In more details, writing $f\colon \mathbb{R} \to \mathbb{R}$ for a generic permissible transformation and $\mathbb{R}_{++}$ for the set of strictly positive reals:

Ordinal scale. *Monotone* transformations are permissible:

$$f \text{ is such that } \Phi(x) \leq \Phi(y) \text{ implies } f(\Phi(x)) \leq f(\Phi(y)).$$

Interval scale. *Linear* transformations are permissible:

$$f(\Phi(x)) = \alpha\Phi(x) + b \text{ with } \alpha \in \mathbb{R}_{++} \text{ and } \beta \in \mathbb{R}.$$

Ratio scale. *Similarity* transformations are permissible:

$$f(\Phi(x)) = \alpha\Phi(x) \text{ with } \alpha \in \mathbb{R}_{++}.$$

Absolute scale. *Identity* transformation is the only permissible transformation:

$$f(\Phi(x)) = \Phi(x).$$

The idea of qualitative foundation has proved itself to be philosophically deep and methodologically relevant. An example of this is given by the so called *ordinal revolution* in the history of utility. In Bentham's tradition, the utility of a certain good or bundle was the quantity which measures the extent to which the good satisfies the individual's desires, that is the value of a pleasure or pain considered by itself. However, the obvious difficulty in measuring the "utiles" of a specific good has led to the misfortune of the notion (see Kauder, 1965), until Pareto argued that cardinal notions of utility should be replaced by ordinal comparisons:

> [I]n order to examine general economic equilibrium, this measurement [of the degrees of utility] is unnecessary. It is sufficient to ascertain if one pleasure is larger or smaller than another. This is the only fact we need to build a theory. (Pareto, 1906)

The comparative counterpart of the notion of cardinal utility is given by the preference relation, a binary relation among goods establishing which of any two goods is preferred over the other. It is now taken for granted that cardinal utility makes sense only in relation to its ordinal representation (see Kreps, 1988). The central result that paved the way for this approach is the Von Neumann and Morgenstern representation theorem (Von Neumann and Morgenstern, 1953) which isolates the conditions under which the preference ordering of a rational agent can be represented by a real-valued utility function according to the general form of representation theorems previously described. The case of utility is emblematic because shows the main value of bridge results like representation theorems: they allow to retain both the philosophical plausibility typical of easily interpretable relations and the mathematical convenience of numerical assignments.

1.2.2 Algebraic methods

The key step in my argumentation is to bring the methodological lesson learned from measurement theory to bear on the notions of logical valuedness and many-valuedness. I take as primitive a binary relation over the set of sentences, $\preceq \subseteq \mathcal{SL} \times \mathcal{SL}$, interpreted as *no more true than*, and I investigate which conditions this relation should satisfy in order to guarantee the existence of a valuation function v from $\mathcal{SL}$ to a suitable set of truth values representing it, namely such that for all $\phi, \psi \in \mathcal{SL}$

$$\phi \preceq \psi \Rightarrow v(\phi) \leq v(\psi).$$

This investigation involves mathematical methods, more precisely algebraic ones. An algebra (or algebraic structure) is a set A together with a collection of operations on A. An n-ary operation on A is a function that takes n elements of A and returns a single element of A. The number of operations and their arity define the so called *type* of the algebra. A *homomorphism* is a structure-preserving map between two algebraic structures of the same type, namely it is a function from the set A to the set B such that for every operation f_A of A and f_B of B of arity n:

$$h(f_A(a_1, \ldots, a_n)) = f_B(h(a_1), \ldots, h(a_n)).$$

A formal language can be seen as an algebra with the connectives acting as operations, called *algebra of terms*. This algebra has the set $\mathcal{SL}$ of sentences as universe and an operation of arity n for each connective c_n in $\mathcal{C}_\mathcal{L}$. For the sake of readability I denote the operations by using the same symbols of the language, namely $(\mathcal{SL}, \mathcal{C}_\mathcal{L})$. This algebra has type $\mathcal{C}_\mathcal{L}$ (we say that it is a $\mathcal{C}_\mathcal{L}$-algebra) and it is sometimes referred to as the *absolutely free algebra* in language $\mathcal{SL}$ with generators $\mathcal{L}$. Truth values form an algebra too: the set of truth values together with the operations establishing the truth-functionality clauses for the connectives, namely $(A, \{f_{c_n} \mid c_n \in \mathcal{C}_\mathcal{L}\})$. This algebra is called *algebra of truth values* and it is of the same type as the algebra of terms, namely it is a $\mathcal{C}_\mathcal{L}$-algebra. Logical valuations are functions mapping sentences into the set of truth values and behaving truth-functionally. Given that the sets of sentences and truth values are algebras, logical valuations can be seen as homomorphisms between the two structures, namely operation-preserving maps, i.e. maps $v\colon (\mathcal{SL}, \mathcal{C}_\mathcal{L}) \to (A, \{f_{c_n} \mid c_n \in \mathcal{C}_\mathcal{L}\})$ such that

$$v(c_n(\phi_1, \ldots, \phi_n)) = f_{c_n}(v(\phi_1), \ldots, v(\phi_n)).$$

.

The algebra of truth values is usually also a relational structure, that is to say, an ordering is defined over its elements. Truth values are either numbers, and thus they are endowed with the natural order *greater than* defined among numbers, or they are generic objects whose nature is unknown with a structure defined over them. In both cases, as we have seen, we can imagine a ranking of some kind, the truth ordering. The core of my proposal consists in taking the algebra of terms as a relational structure as well by defining a preorder over it representing truth comparisons among sentences (this ordering will be in general weaker than the truth ordering among truth values). The aim is to isolate conditions, expressed by axioms over the ordering, that correspond to plausible properties of the informal relation *more* or *less true than* and, at the same time, guarantee that the ranking is representable in the sense of measurement theory. The representing functions should be logical valuations, because I construe the ordering among sentences as a pairwise counterpart of logical valuations. These results will be genuinely measure-theoretic or quantitative when truth values are numbers, and especially so when they are real numbers as in the case of infinite-valued logics (see Chapter 3).

Since logical valuations are homomorphism, in an algebraic context establishing representation results amounts to proving the existence of (classes of) homomorphisms between ordered algebras, in particular homomorphisms from the algebra of terms endowed with the formal counterpart of the relation *no more true than* and the algebra of truth values endowed with its natural ordering. Homomorphisms are order preserving, as representation theorems require. That is why algebraic logic lends itself as a valuable tool for this discussion.

1.3 Preliminaries

1.3.1 Algebraic semantics

We have seen that the formal pointwise semantics can be described in algebraic terms: on the one hand we have the algebra of terms, an absolutely free algebra generated from the set of propositional variables and the set of connectives as operations, on the other hand we have the algebra of truth values, an algebra of the similar type. Logical valuations are homomorphisms between the two structures. Given a logic, an algebraic semantics for it is given by a class of compatible algebras (usually forming a variety, i.e., a class of algebras that is axiomatizable by equational laws. See Burris and Sankappanavar (2000)). For example classical logic is characterised by the class of Boolean algebras, and *any* Boolean algebra, finite or not, is an admissible set of truth values for it. This is because given our language $\mathcal{SL}$, any algebra of type $\mathcal{C}_{\mathcal{L}}$ can act as codomain of the logical valuations, also the algebra of terms itself (in this case valuations would be endomorphisms)! We need a way for discriminating among the admissible algebras, and, indeed, in a class of algebras there are some algebras playing a distinguished role: subdirectly irreducible algebras. Recall that a *product* of some set of algebraic structures of a certain type is the cartesian product of the sets with the operations defined coordinatewise. Given a set I of algebras B_i of the same type, I denote by $\prod_{i\in I} B_i$ the direct product and by $\pi_i\colon \prod_{i\in I} B_i \to B_i$ the projection function, that sends a tuple indexed by I – that is, an element of $\prod_{i\in I} B_i$ – to its i^{th} coordinate.

Definition 1.4. *A $\mathcal{C}_{\mathcal{L}}$-algebra A is* subdirectly irreducible *if and only if for each set I and each injective homomorphism $h\colon A \to \prod_{i\in I} B_i$, with B_i a $\mathcal{C}_{\mathcal{L}}$-algebra, if for all $i \in I$ the composition $\pi_i \circ h$ is onto then there exists $i \in I$ and an isomorphism $\iota\colon A \to B_i$.*

Informally, a subdirectly irreducible algebra is an algebra that cannot be factored as a subdirect product of simpler algebras, since it includes (as a factor) an algebra isomorphic to itself. The notion of subdirectly irreducible algebra helps to isolate, among the possible candidates, an algebraic semantics, whose elements can be justifiably considered truth values. For example, in the case of classical logics, the only subdirectly irreducible Boolean algebra has cardinality two, and that is why its elements can be taken to represent the True and the False.

Given the conceptual and formal importance of subdirectly irreducible algebras for discourses concerning truth values, the following theorem plays a crucial role (Burris and Sankappanavar, 2000, Theorem 8.6, p. 64):

Theorem 1.5 (Birkhoff's Subdirect Representation Theorem)**.** *Every $\mathcal{C}_{\mathcal{L}}$-algebra A is isomorphic to a subdirect product of subdirectly irreducible algebras, which are homomorphic images of A.*

Irreducible algebras factorising a given algebra are obtained by quotienting the universe of the algebra at stake with respect to congruences over the algebra. A congruence over a given $\mathcal{C}_{\mathcal{L}}$-algebra A is a binary relation $\theta \subseteq A \times A$ which is reflexive, symmetric and transitive (equivalence relation), and, in addition it is compatible with the structure of A: for each $c_n \in \mathcal{C}_{\mathcal{L}}$

$$\{(\phi_i, \psi_i)\}_{i\leq n} \in \theta \;\Rightarrow\; (c_n(\phi_1, \dots, \phi_n), c_n(\psi_1, \dots, \psi_n)) \in \theta.$$

This property is a necessary condition for introducing an algebraic structure on the quotient algebra A/θ whose elements are the equivalence classes of the equivalence relation θ. Another way for saying this is that a congruence is a subalgebra of A. The set $ConA$ of all congruence relations on an algebra A is ordered by the inclusion relation and it forms a lattice with respect to set-theoretic union and intersection. In this set a special role is played by meet-irreducible congruences:

Definition 1.6. *A congruence θ on A is* completely meet-irreducible (c.m.i.) *if whenever $\theta = \bigcap_{i \in I} \theta_i$, with $\theta_i \in ConA$, we have $\theta = \theta_i$ for some $i \in I$.*

Given these definitions, the proof of Theorem 1.5 rests on the following facts.

Lemma 1.7. *Given a $\mathcal{C}_{\mathcal{L}}$-algebra A and a congruence θ over A, if θ is completely meet-irreducible then A/θ is subdirectly irreducible.*

Intuitively, we consider the "largest" congruences, the ones that cannot be further extended, in order to obtain by quotienting the "simplest" algebras, the ones that cannot be further factorised.

Lemma 1.8. *Given a $\mathcal{C}_{\mathcal{L}}$-algebra A and $I \subseteq ConA$, the natural map $h \colon A \to \prod_{\theta \in I}(A/\theta)$, that sends $a \in A$ to the tuple $(a/\theta)_{\theta \in I}$ indexed by I of the equivalence classes of a under each $\theta \in I$, is injective if and only if $\bigcap I = Id_A$, where Id_A is the identity relation over A.*

Lemma 1.9 (Lindenbaum lemma)**.** *Given a $\mathcal{C}_{\mathcal{L}}$-algebra A, for each $a, b \in A$ with $a \neq b$, there exists $\theta \in ConA$ such that*

(i) $(a, b) \notin \theta$,

(ii) θ *is maximal with respect to (i) in the order of the lattice $ConA$.*

The proof of this lemma is non-constructive, being based on the Axiom of Choice. We are ready now to prove Theorem 1.5.

Proof. Let A be an algebra. Consider the set $I = \{\theta \in ConA \mid \theta \text{ is c.m.i.}\}$, i.e. the set of congruences over A which are completely meet-irreducible. Then the natural map $e \colon A \to \prod_{\theta \in I} A/\theta$ is a subdirect product representation of A, as stated in the Theorem. Indeed, each A/θ is subdirectly irreducible by Lemma 1.7. Moreover, in order to show that e is injective it suffices to prove that $\bigcap I = Id$ (Lemma 1.8). Identity is clearly included in the intersection of I, namely $Id \subseteq \bigcap I$. We prove $\bigcap I \subseteq Id$ by showing the contrapositive statement. Suppose therefore $(a, b) \notin Id$, or equivalently, $a \neq b$. Then, by Lemma 1.9, there is $\theta \in ConA$ such that $(a, b) \notin \theta$, and θ is completely meet-irreducible. It follows that $\theta \in I$, and therefore $(a, b) \notin \bigcap I$, as was to be shown.

□

1.3.2 Preordered algebras

I consider here algebras with a preorder defined over their elements, and I call them *preordered algebras*. Their formal treatment requires an adjustment of the previously stated definitions and results. Setting up this new formal framework is the aim of this subsection.

Recall that we treat the set of sentences $\mathcal{SL}$ as an algebra whose operations are the logical connectives $c_n \in \mathcal{C}_\mathcal{L}$. $\mathcal{C}_\mathcal{L}$-algebras are algebras of type $\mathcal{C}_\mathcal{L}$, namely algebras similar to $(\mathcal{SL}, \mathcal{C}_\mathcal{L})$. A congruent preorder is a binary relation on A which is reflexive, transitive and such that its symmetric part is a congruence on A. This being in place, the definitions of preordered $\mathcal{C}_\mathcal{L}$-algebra and preorder morphism of $\mathcal{C}_\mathcal{L}$-algebras are straightforward:

Definition 1.10. *A* preordered $\mathcal{C}_\mathcal{L}$-algebra *is a pair $(A, \preceq_A)$ where A is a $\mathcal{C}_\mathcal{L}$-algebra and $\preceq_A \subseteq A^2$ is a congruent preorder over A.*

Definition 1.11. *A* preorder homomorphism (isomorphism) $h\colon (A, \preceq_A) \to (B, \preceq_B)$ *is a homomorphism (isomorphism) of algebras which preserves the preorder, i.e. for all $a, b \in A$*

$$a \preceq_A b \Rightarrow h(a) \preceq_B h(b).$$

Notice that homomorphisms induce in a natural way a preorder over the elements of the codomain algebra, as follows:

Definition 1.12. *If $(A, \preceq_A)$ is a preordered $\mathcal{C}_\mathcal{L}$-algebra and $h\colon A \to B$ is a homomorphism, then the* congruent preorder $\preceq_B$ induced by h and $\preceq_A$ over B *is the transitive and congruent closure of the relation R induced by h and $\preceq_A$, i.e. for all $a, b \in B$*

$$(a, b) \in R \;:\Leftrightarrow\; \exists a' \in \{h^{-1}(a)\}, \exists b' \in \{h^{-1}(b)\} \text{ such that } a' \preceq_A b'.$$

Then $(B, \preceq_B)$ is a preordered $\mathcal{C}_\mathcal{L}$-algebra.

When there is no risk of confusion I drop the subscript of preorders for the sake of readability.

Along these lines, we can define the important notions of quotient and product of preordered algebras:

Definition 1.13. *Let $(A, \preceq)$ be a preordered $\mathcal{C}_\mathcal{L}$-algebra and $\theta \in ConA$. The* quotient with respect to θ *is the preordered $\mathcal{C}_\mathcal{L}$-algebra $(A/\theta, \preceq_\theta)$, where A/θ is the quotient of A and $\preceq_\theta$ is the preorder induced by θ and $\preceq$, namely the transitive and congruent closure of the relation R_θ defined as follows*

$$([a]_\theta, [b]_\theta) \in R_\theta \;:\Leftrightarrow\; \exists a_i \in [a]_\theta, \exists b_i \in [b]_\theta \text{ such that } a_i \preceq b_i.$$

Definition 1.14. *Let I be a set such that for each $i \in I$ a preordered $\mathcal{C}_\mathcal{L}$-algebra $(B_i, \preceq_i)$ is given. The* product over I, $\prod_{i\in I}(B_i, \preceq_i)$, *is defined as the preordered $\mathcal{C}_\mathcal{L}$-algebra whose underlying algebra is $\prod_{i\in I} B_i$ with the preorder $\preceq$ defined pointwise as*

$$(a_i)_{i\in I} \preceq (b_i)_{i\in I} \;:\Leftrightarrow\; \forall i \in I \quad a_i \preceq_i b_i.$$

For each $i \in I$, the function $\pi_i \colon \prod_{i\in I}(B_i, \preceq_i) \to B_i$ is a preorder homomorphism.

The main results stated in Section 1.3.1 must be restated with some modifications. In particular, I prove that preordered algebras, too, enjoy a variant of the Subdirect Representation Theorem (Theorem 1.5).

Theorem 1.15 (Preordered variant of Birkhoff's Theorem)**.** *Every preordered $\mathcal{C}_{\mathcal{L}}$-algebra $(A, \preceq)$ is isomorphic to a subdirect product of subdirectly irreducible preordered $\mathcal{C}_{\mathcal{L}}$-algebras, which are preordered homomorphic images of A.*

Proof. Let $(A, \preceq)$ be a preordered $\mathcal{C}_{\mathcal{L}}$-algebra. Consider the quotient algebra $(A/\sim, \leq_\sim)$. For the sake of readability we denote $A/\sim = B$. By applying Theorem 1.5 we know that there exists a family I of congruences on B together with a homomorphism $e \colon B \to \prod_{\theta\in I} B/\theta$ such that the composition with each projection function is onto, as pictured in Figure 1.1, and such that each B/θ is subdirectly irreducible. Moreover, each B/θ carries a natural structure of preordered $\mathcal{C}_{\mathcal{L}}$-algebra since $\preceq$ induces a congruent preorder through the map e, according to the Definition 1.12. We can conclude that e is a preorder homomorphism.

Figure 1.1: Birkhoff's Theorem Construction.

$$\begin{tikzcd}
(A,\preceq) \arrow[r, two heads, "q_\sim"] \arrow[rr, bend right, "h = e\circ q_\sim"'] & (B,\leq_\sim) \arrow[r, hook, "e"] \arrow[rr, bend right, "\pi_i \circ e"'] & \prod_{\theta\in I} B/\theta \arrow[r, "\pi_i"] & (B/\theta,\preceq)
\end{tikzcd}$$

$\square$

1.3.3 Infinite-valued logics

As already said, part of this work will be devoted to the notion of degrees of truth as arising in infinite-valued logics. In particular, I shall consider Łukasiewicz and Gödel logic. All results cited in this section for which no reference is given may be found either in Hájek (1998) or in Cignoli et al. (2000).

Let $\mathcal{L}$ be a countable propositional language. The set of sentences $\mathcal{SL}$ is built as before by using as primitive connectives a 0-ary connective, $\bot$, and implication, $\to$. Negation can be defined as $\neg\phi \coloneqq \phi \to \bot$, and $\top \coloneqq \neg\bot$. Łukasiewicz logic $\text{Ł} = (\mathcal{SL}, \vdash_{\text{Ł}})$ is given by the following system of axiom schemata:

(Ł1) $\bot \to \phi$

(Ł2) $\phi \to (\psi \to \phi)$

(Ł3) $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$

(Ł4) $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$

(Ł5) $((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)$

and Modus Ponens as inference rule. Łukasiewicz logic is characterized by an infinite-valued semantics. A *Łukasiewicz valuation* is a map $v\colon \mathcal{SL} \to [0,1]$ such that for any sentences $\phi, \psi \in \mathcal{SL}$:

1. $v(\bot) = 0$,
2. $v(\phi \to \psi) = \min\{1, 1 - v(\phi) + v(\psi)\}$.

From this it follows that $v(\neg\phi) = 1 - v(\phi)$ and $v(\top) = 1$.

It is customary to define further connectives:

$$\begin{aligned}
\phi \vee \psi &:= (\phi \to \psi) \to \psi && \text{(lattice) disjunction}\\
\phi \wedge \psi &:= \neg(\neg\phi \vee \neg\psi) && \text{(lattice) conjunction}\\
\phi \oplus \psi &:= \neg\phi \to \psi && \text{strong disjunction}\\
\phi \odot \psi &:= \neg(\neg\phi \oplus \neg\psi) && \text{strong conjunction}
\end{aligned}$$

whose corresponding semantics is

$$\begin{aligned}
v(\phi \vee \psi) &= \max\{v(\phi), v(\psi)\}\\
v(\phi \wedge \psi) &= \min\{v(\phi), v(\psi)\}\\
v(\phi \oplus \psi) &= \min\{1, v(\phi) + v(\psi)\}\\
v(\phi \odot \psi) &= \max\{0, v(\phi) + v(\psi) - 1\}.
\end{aligned}$$

The semantic consequence relation $\models_{\text{Ł}} \subseteq \mathcal{P}(\mathcal{SL}) \times \mathcal{SL}$ is defined classically as absolute truth preservation, namely as preservation of the value 1. Recall also that for Łukasiewicz logic only a local deduction theorem holds:

Theorem 1.16. *For any $\phi, \psi \in \mathcal{SL}$,*
$\phi \vdash_{Ł} \psi$ if and only if $\exists n > 1$ such that $\vdash_{Ł} \underbrace{\phi \odot \cdots \odot \phi}_{n\ times} \to \psi$.

A complete algebraic semantics for Łukasiewicz logic is given by the class of MV-algebras (Cignoli et al., 2000).

Definition 1.17. *An MV-algebra $(A, \neg, \oplus, 0)$ is a set A equipped with a binary operation $\oplus$, a unary operation $\neg$ and a distinguished constant 0 satisfying the following equations:*

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

(MV2) $x \oplus y = y \oplus x$

(MV3) $x \oplus 0 = x$

(MV4) $\neg\neg x = x$

(MV5) $x \oplus \neg 0 = \neg 0$

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Every MV-algebra M is naturally endowed with a preorder relation defined by setting for each $x, y \in M$:

$$x \leq y \;:\Leftrightarrow\; \neg x \oplus y = 1.$$

The real unit interval $[0,1]$ equipped with the operations $\neg x := 1 - x$ and $x \oplus y := \min\{1, x + y\}$, that is $([0,1], \neg, \oplus, 0)$, is an MV-algebra, usually referred to as the *standard MV-algebra.* This algebra is the standard truth-value semantics for Łukasiewicz logic. The MV-algebra $[0,1]$ plays a special role: non-trivial MV-algebras, those with more than one element, are isomorphic to subalgebras of $[0,1]$.

Theorem 1.18. *Let A be a non-trivial MV-algebra. The following hold:*

(i) there exists at least one homomorphism $m\colon A \to [0,1]_{MV}$.

(ii) if A is linearly ordered then m is unique.

(iii) if A is linearly ordered and Archimedean then m is one-to-one. And if m is one-to-one then A is linearly ordered and Archimedean.

This result is obtained by considering quotients with respect to special kind of congruences, the ones corresponding to maximal ideals.

Definition 1.19. *An* ideal *of an MV-algebra A is a subset I of A satisfying the following*

(i) $0 \in I$,

(ii) if $x \in I$, $y \in A$ and $y \leq x$ then $y \in I$,

(iii) if $x \in I$ and $y \in I$ then $x \oplus y \in I$.

Ideals contain the bottom elements, are closed downward and closed under disjunctions. Filters are the dual notion: they contain the top element, are closed upward and closed under conjunction.

Definition 1.20. *An ideal I of A is said to be*

- proper *if $I \neq A$. Let $\mathcal{I}(A)$ the set of proper ideals of A.*
- prime *if I is proper and $\forall x, y \in A$ either $(x \ominus y) \in I$ or $(y \ominus x) \in I$. ($\mathcal{P}(A)$)*

- maximal *if I is proper and no proper ideal of A strictly contains I, i.e. for each ideal $J \neq I$ if $I \subseteq J$ then $J = A$. ($\mathcal{M}(A)$)*

Lemma 1.21. *If M is a proper ideal of A then the following are equivalent:*

1. *M is maximal,*
2. *for any $a \in A$, either $a \notin M$ or there exists $n \in \mathbb{N}$ such that $\neg n \cdot a \in M$.*

Lemma 1.22 (Lindenbaum's Lemma)**.** *Let A be an MV-algebra and $I, F \subseteq A$ be an ideal and a filter. If $I \cap F = \emptyset$, then there exists a maximal ideal M of A such that*

(i) $I \subseteq M$,

(ii) $F \cap M = \emptyset$.

Recall that this is a non-constructive principle, equivalent to a weaker version of the Axiom of Choice.

I sketch the proof of Theorem 1.18:

Proof. Since A is non-trivial there are two elements $0, 1 \in A$ such that $0 \neq 1$. Notice that $\{0\}$ and $\{1\}$ are, respectively, an ideal and a filter of A. Moreover, $\{0\} \cap \{1\} = \emptyset$. Lemma 1.22 guarantees that there exists at least one maximal ideal M of A such that $\{0\} \subseteq M$ and $M \cap \{1\} = \emptyset$. For each of those M we can consider the equivalence relation

$$a \equiv_M b \;:\Leftrightarrow\; a \leftrightarrow b \in M,$$

and the quotient algebra $A/\equiv_M$. Notice than each $A/\equiv_M$ is

- non-trivial, because M is proper;
- linearly ordered, because M is maximal;
- Archimedean, because M is maximal (Lemma 1.21).

Now, $A/\equiv_M$ satisfies premisses that are analogous to the ones in Hölder's Theorem stating that any linearly ordered Archimedean group is isomorphic to a subgroup of the reals.[6] Due to this, it can be proved that there is a unique injective homomorphism $h\colon A/\equiv_M \to [0, 1]$. By composing h with the canonical map $q_M\colon A \to A/\equiv_M$ we obtain a homomorphism $m\colon A \to [0, 1]$. This sketches the proof of (i).

Moreover, notice that if A is linearly ordered then q_M is unique, and so is m and if A is Archimedean then q_M is one-to-one, and so is m. As stated in (ii) and (iii) of Theorem 1.18, respectively. □

[6] For a proof see Bigard et al. (1977), Chapter 2.2.6.

The axiomatization of Gödel logic will not play a relevant role here (see Hájek, 1998, p. 97). It suffices to recall that Gödel valuations are functions $v\colon \mathcal{SL} \to [0,1]$ such that

1. $v(\bot) = 0$,
2. $v(\neg\phi) = \begin{cases} 1, & \text{if } v(\phi) = 0; \\ 0, & \text{if } v(\phi) > 0. \end{cases}$
3. $v(\phi \to \psi) = \begin{cases} 1, & \text{if } v(\phi) \leq v(\psi); \\ v(\psi), & \text{otherwise.} \end{cases}$

Moreover, Gödel logic is characterised by the class of Gödel algebra, which are Heyting algebra satisfying the *prelinearity* condition, stating that for each two elements x, y of the algebra the following holds:

$$(x \to y) \vee (y \to x) = 1.$$

Chapter 2

From pairwise to pointwise valuations

The standard semantics for propositional logics is given by a set of logical valuations, namely functions that map each propositional variable to one of the truth values and, in most cases, behave truth-functionally with respect to the connectives. In Chapter 1, I motivated the interest of considering, as an alternative approach, a semantics based on binary comparisons of sentences with respect to their truth. The key step consists in shifting the focus from pointwise valuations, which typically feature in standard truth-value semantics, to pairwise valuations based on comparative judgements of the form

"the sentence ϕ is *less (more) true than* the sentence ψ".

This chapter develops this idea, in particular, it provides grounds for (i) axiomatically defining pairwise valuations, and (ii) investigating the relation with pointwise valuations. Throughout the discussion, I shall point out some interesting implications on the notions of truth values and many-valuedness.

2.1 Abstract representation theorem

Having set the framework needed for the discussion in Subsection 1.3.2, I move now to the core of the investigation. I make precise what I mean by *pairwise valuation* and I prove a first abstract representation theorem, stating that pairwise valuations can be represented by pointwise valuations. Recall that throughout this section, $\mathcal{L}$ is a countable set of propositional variables, $\mathcal{C}_{\mathcal{L}}$ is a set of connectives defined over $\mathcal{L}$ and $\mathcal{SL}$ is the set of sentences recursively generated from $\mathcal{L}$ and $\mathcal{C}_{\mathcal{L}}$.

2.1.1 Pairwise valuations

To evaluate sentences *pairwise* means to perform comparative judgements of the form "ϕ is more/less true than ψ". As already said, in order to formalise this idea we take as primitive a binary relation over the set of sentences. This relation expresses comparisons, so we assume it to be an order relation and we take the weak relation $\preceq \subseteq \mathcal{SL}^2$, interpreted as *no more true than.* The strict and the symmetric part of $\preceq$, respectively $\prec$ and $\sim$, are interpreted as *(strictly) less true than* and *as true as*, respectively. The relation $\sim$ will have a prominent role throughout the discussion because it allows us to talk about equivalence classes with respect to truth, containing sentences which are evaluated to be *true in the same way* or *true to the same extent.*

I start by requiring some minimal, structural properties as stated in the following:

Definition 2.1. *A relation $\preceq \subset \mathcal{SL}^2$ is a* pairwise valuation *if and only if*

- $\preceq$ *is a preorder, namely it is reflexive and transitive;*
- $\sim$ *is a congruence with respect to the connectives, namely for all $c_n \in \mathcal{C}_\mathcal{L}$*

$$\{\phi_i \sim \psi_i\}_{i\leq n} \Rightarrow c_n(\phi_1, \ldots, \phi_n) \sim c_n(\psi_1, \ldots, \psi_n).$$

The relation *no more true than* expresses a ranking, accordingly it is modelled by an ordering relation that is taken to be weak, so reflexive and transitive. Pairwise valuations are not assumed to be antisymmetric since it should be allowed for two distinct sentences to be one as true as the other. Also, in general, linearity is not required since not each and every pair of sentences are comparable with respect to truth (I shall discuss some problematic aspects related to linearity in the next chapter in connection with vagueness). The derived relation *as true as* is an equivalence relation. I also assume it to be compatible with the underlying algebraic structure given by the connectives. Sentential connectives are operations over the algebra of terms, so the condition of being a congruence with respect to them them reflects the compositional nature of truth, namely the fact that truth values of complex sentences are determined by the truth values of the component. In conclusion, the structural constraints of reflexivity, transitivity and congruence are extremely plausible considered the relation's interpretation in terms of weak comparative truth.

Moreover, it is worth stressing that these requirements are in no respect restrictive because we can always start with a partial specification of a truth order among sentences which is neither reflexive, nor transitive, nor congruent and then consider the closure with respect to those properties. Let $R \subseteq \mathcal{SL}^2$ be a set of ordered pairs of sentences representing some instances of comparative judgement of the form "ϕ is no more true than ψ". We can enlarge the extension of R by adding new comparisons and thus consider the congruent preorder generated by R, namely its reflexive, transitive and congruent closure:

$$\begin{aligned}\preceq_R = R &\cup \{(\phi,\phi) \mid \phi \in \mathcal{SL}\}\\ &\cup \{(\phi,\psi) \mid \exists\chi \in \mathcal{SL} \text{ such that } (\phi,\chi) \in R \text{ and } (\chi,\psi) \in R\}\\ &\cup \{(c_n(\phi_1,\dots,\phi_n), c_n(\psi_1,\dots,\psi_n)) \mid \{(\phi_i,\psi_i)\}_{i\le n} \in R\}.\end{aligned}$$

We say that $\preceq_R$ is the congruent preorder generated by R. Notice that this closure is possible without inconsistencies since R only contains positive instances of a weak truth comparison.

Since truth comparisons are by all means semantic valuations, we consider *families* of congruent preorders, in the same way as we consider all possible assignments in the standard functional setting (see Section 1.1). Let $\mathcal{P}$ be the set of all possible pairwise valuations. For any family of pairwise valuations $\mathcal{F} \subseteq \mathcal{P}$ we can consider the intersection of $\mathcal{F}$, namely

$$\bigcap \mathcal{F} = \{(\phi,\psi) \subseteq \mathcal{SL}^2 \mid \text{ for all } \preceq \text{ in } \mathcal{F} \;\; \phi \preceq \psi\},$$

and the corresponding preorder

$$\phi \preceq_{\mathcal{F}} \psi \;:\Leftrightarrow\; (\phi,\psi) \in \bigcap \mathcal{F}. \tag{2.1}$$

From Proposition 2.2 we know that $\preceq_{\mathcal{F}}$ is a pairwise valuation.

Proposition 2.2. *The properties of being a preorder, an equivalence relation, a congruent preorder and a fully invariant congruent preorder are preserved under arbitrary intersections.*

The *intersection* of a family of preorders plays an important role. The pairwise valuation resulting from the intersection intuitively represents the set of comparisons which hold under all possible interpretations, or in all possible worlds. In the standard truth-value semantics for propositional logic, sentences that are true under all possible assignments are called *tautologies* or *logical truths*. Those have the property of remaining true under all uniform substitutions, that is to say under all reinterpretations of its components (see Definition 1.1). Substitution invariance corresponds to the property of being analytical. From a comparative perspective, we deal with families of possible pairwise valuations. Analogously, the preorder resulting from the intersection of all pairwise valuations expresses the comparative judgements which hold *analytically* or *logically*. For the reasons just stated, substitution invariance is a necessary requirement and I take it as a criterion of *admissibility* of a family of pairwise valuation. In our algebraic setting, a *uniform substitution* is simply an endomorphism $\sigma\colon \mathcal{SL} \to \mathcal{SL}$.

Definition 2.3. *A family of pairwise valuations $\mathcal{A} \subseteq \mathcal{P}$ is* admissible *if and only if $\preceq_{\mathcal{A}}$ is substitution-invariant, i.e. for all endomorphism $\sigma\colon \mathcal{SL} \to \mathcal{SL}$*

$$\phi \preceq_{\mathcal{A}} \psi \Rightarrow \sigma(\phi) \preceq_{\mathcal{A}} \sigma(\psi).$$

2.1.2 Representation theorem

Pairwise valuations in an admissible family are the formal counterpart of the idea of truth from comparison, whose philosophical interest has been defended in Chapter 1. It is worth recalling that part of this interest lies in the possibility of going back from pairwise valuations to pointwise valuations, namely the possibility of starting with binary relations over the set of sentences and eventually using them to derive compatible truth-value assignments in a suitable set of truth values.

Definition 2.4. *A* pointwise valuation of $\mathcal{SL}$ *is a homomorphism* $h\colon \mathcal{SL} \to A$*, where* A *is a* $\mathcal{C}_{\mathcal{L}}$*-algebra.*

Given this we say that

Definition 2.5. *A pointwise valuation* $h\colon \mathcal{SL} \to A$ represents *a pairwise valuation* $\preceq \subseteq \mathcal{SL}^2$ *if and only if for all* $\phi, \psi \in \mathcal{SL}$

$$\phi \preceq \psi \Rightarrow h(\phi) \preceq h(\psi).$$

A pointwise valuation strongly represents *a pairwise valuation if and only if*

$$\phi \preceq \psi \Leftrightarrow h(\phi) \preceq h(\psi).$$

The existence of a pointwise valuation representing a pairwise valuation guarantees that the comparisons between sentences can also be expressed as compositional functions from the set of sentences to a suitable structure (standard functional valuations). The structure of codomain of these functions ought to be compatible with the structure of the set of sentences and with the preorder, in other words ought to be an algebra of the same type of $\mathcal{SL}$, whose preorder reflects the pairwise valuation under all possible interpretations of the sentences. To this purpose I introduce the notion of *model of a pairwise valuation.*

Definition 2.6.

1. *A preordered* $\mathcal{C}_{\mathcal{L}}$*-algebra* $(A, \preceq)$ *is a* model of a pairwise valuation $\preceq \subseteq \mathcal{SL}^2$ *if and only if each homomorphism* $h\colon \mathcal{SL} \to A$ *is a preorder homomorphism, namely is such that*

$$\phi \preceq \psi \Rightarrow h(\phi) \preceq h(\psi).$$

   Let $Mod\{\preceq\}$ *denote the class of models of* $\preceq$*.*

2. *A preordered* $\mathcal{C}_{\mathcal{L}}$*-algebra* $(A, \preceq)$ *is a* model of a family $\mathcal{F} \subseteq \mathcal{P}$ *if and only if for all homomorphism* $h\colon \mathcal{SL} \to A$

$$\phi \preceq_{\mathcal{F}} \psi \Rightarrow h(\phi) \preceq h(\psi),$$

   where $\preceq_{\mathcal{F}}$ *is as in Equation* (2.1)*. Let* $Mod\mathcal{F}$ *denote the class of models of* $\mathcal{F}$*.*

Notice that for all $\preceq \in \mathcal{F}$

$$Mod\{\preceq\} \subseteq Mod \bigcap_{\preceq \in \mathcal{F}} \preceq = Mod\{\preceq_{\mathcal{F}}\} = Mod\mathcal{F}.$$

Indeed, if an algebra is a model of a given $\preceq \in \mathcal{F}$ then it is also a model of any subset of $\preceq$, in particular a model of $\preceq_F$ since by definition for all $\preceq \in \mathcal{F}$ we have $\preceq_F \subseteq \preceq$.

Following this definition, $Mod\mathcal{A}$ is the class of models of $\mathcal{A}$, namely the class of algebras which are compatible with all the pairwise valuations in a given admissible family. The closure properties of this class can be investigated, starting with the following.

Proposition 2.7. *$Mod\mathcal{A}$ is closed under quotients, in the sense of Definition 1.13.*

Proof. Let $(A, \preceq)$ be a preordered $\mathcal{C}_{\mathcal{L}}$-algebra in $Mod\mathcal{A}$. We want to show that given $\theta \in ConA$, we have $A/\theta \in Mod\mathcal{A}$. Let q_θ the canonical map from A to A/θ. This is a homomorphism. Following Definition 1.13, we can define a congruent preorder $\preceq_\theta$ over A/θ induced by q_θ. We prove that the preordered $\mathcal{C}_{\mathcal{L}}$-algebra $(A/\theta, \preceq_\theta)$ is still a model of $\mathcal{A}$, i.e. for all $h_\theta \colon \mathcal{SL} \to A/\theta$ we have $\phi \preceq_{\mathcal{A}} \psi \Rightarrow h_\theta(\phi) \preceq_\theta h_\theta(\psi)$. Notice that every homomorphism h_θ can be written as the composition of $h \colon \mathcal{SL} \to A$ and q_θ. Both are preorder homomorphisms because, by hypothesis, h preserves $\preceq_{\mathcal{A}}$ and, by construction, $\preceq_\theta$ preserves $\preceq$. We conclude that h_θ preserves $\preceq_{\mathcal{A}}$, which implies $A/\theta \in Mod\mathcal{A}$. $\square$

Among the models of $\mathcal{A}$, the quotient structure induced by the congruence relation *as true as* plays a crucial role. Suppose we start with the set of sentences and a congruent preorder $\preceq$ in an admissible family $\mathcal{A}$ which establishes a truth ranking. The quotient with respect to the equivalence relation is obtained by taking as universe the quotient set $\mathcal{SL}/\sim = \{[\phi]_\sim \mid \phi \in \mathcal{SL}\}$, where $[\phi]_\sim = \{\psi \in \mathcal{SL} \mid \psi \sim \phi\}$, and by defining for each $c_n \in \mathcal{C}_{\mathcal{L}}$ the following operation:

$$\tilde{c}_n([\phi_1]_\sim, \dots, [\phi_n]_\sim) \coloneqq [c_n(\phi_1, \dots, \phi_2)]_\sim.$$

Since $\sim$ is a congruence (see Definition 2.1), we have that:

Lemma 2.8. *Each $\tilde{c}_n$ is well defined, namely*

$$\{\phi_i \sim \psi_i\}_{i \leq n} \Rightarrow \tilde{c}_n([\phi_1]_\sim, \dots, [\phi_n]_\sim) = \tilde{c}_n([\psi_1]_\sim, \dots, [\psi_n]_\sim).$$

Call $q_\sim$ the canonical map from $\mathcal{SL}$ to $\mathcal{SL}/\sim$ and notice that it induces a partial order on $\mathcal{SL}/\sim$ defined as follows:

$$[\phi]_\sim \leq_\sim [\psi]_\sim \;:\Leftrightarrow\; \exists \phi_i \in [\phi]_\sim, \exists \psi_i \in [\psi]_\sim \text{ such that } \phi_i \preceq \psi_i.$$

Notice that $(\mathcal{SL}/\sim, \leq_\sim)$ is a preordered $\mathcal{C}_{\mathcal{L}}$-algebra and it is a model of $\mathcal{A}$.

Lemma 2.9. $(\mathcal{SL}/\sim, \leq_\sim) \in Mod\mathcal{A}$.

Proof. Consider the algebra $(\mathcal{SL}, \preceq_\mathcal{A})$. We know that $\preceq_\mathcal{A}$ is substitution-invariant (see Definition 2.3), that is for all $h\colon \mathcal{SL} \to \mathcal{SL}$ it holds $\phi \preceq_\mathcal{A} \psi \Rightarrow h(\phi) \preceq_\mathcal{A} h(\psi)$. We conclude that $(\mathcal{SL}, \preceq_\mathcal{A}) \in Mod\mathcal{A}$. Since $\preceq_\mathcal{A} \subseteq \preceq$, for all $h\colon \mathcal{SL} \to \mathcal{SL}$ we have $h(\phi) \preceq_\mathcal{A} h(\psi) \Rightarrow h(\phi) \preceq h(\psi)$. We conclude that $(\mathcal{SL}, \preceq) \in Mod\mathcal{A}$. Since $Mod\mathcal{A}$ is closed under quotients by Proposition 2.7, the statement follows. □

The canonical map from the set of sentences to the quotient structure, mapping each sentence in the corresponding equivalence class, is already a pointwise valuation representing the pairwise valuation. Indeed, it is a preorder homomorphism from the algebra of terms to an algebra of the same type.[1] Accordingly, the equivalence classes (or congruence classes) can be considered as possible truth values to be used in order to *cardinalise* the original preorder. However, for our philosophical discourse, we need valuations and sets of truth values which are more significant from a semantic point of view. To this aim, I introduce the notion of irreducibility of an algebra in $Mod\mathcal{A}$ as useful condition to discriminate among the possible models of the preorder and to isolate some algebras that count as significant algebraic semantics.

Definition 2.10. *A preordered $\mathcal{C}_\mathcal{L}$-algebra $(A, \preceq)$ is* $Mod\mathcal{A}$-irreducible *if and only if $(A, \preceq) \in Mod\mathcal{A}$, and $(A, \preceq)$ is subdirectly irreducible, namely for each set I and each homomorphism $h\colon A \to \prod_{i\in I} B_i$, with B_i a $\mathcal{C}_\mathcal{L}$-algebra, if for all $i \in I$ the composition $\pi_i \circ h$ is onto then there exists $i \in I$ and an isomorphism $\iota\colon A \to B_i$.*

As already explained in Chapter 1, an algebra A is subdirectly irreducible if any subdirect representation of A includes (as a factor) an algebra isomorphic to A, with the isomorphism given by the projection map. Intuitively, $Mod\mathcal{A}$-irreducible algebras are, among the models of the admissible family $\mathcal{A}$, the "simplest" algebras, the ones that cannot be further factorised. At the same time by virtue of Birkhoff's Theorem we know that such algebras are suitable factors for constructing all other algebras. Indeed, also $Mod\mathcal{A}$ is determined by its subdirectly irreducible members, since every algebra A in the class can be constructed as a subalgebra of a suitable direct product of the subdirectly irreducible quotients of A, all of which belong to the class of models of $\mathcal{A}$ because A does and because of Lemma 2.7.

The main result that can be now proved is the following abstract representation theorem for pairwise valuations:

Theorem 2.11. *Given an admissible family of pairwise valuations $\mathcal{A} \subseteq \mathcal{P}$, for every $\preceq \in \mathcal{A}$ there exists at least one pointwise valuation $v_\preceq\colon \mathcal{SL} \to A$ representing it, where A is a $Mod\mathcal{A}$-irreducible $\mathcal{C}_\mathcal{L}$-algebra.*

[1] Also preorder-preserving endomorphisms of the algebra of terms are pointwise valuations, but those are purely syntactical constructions and thus are not necessarily significant as valuations.

Proof. Consider the quotient structure induced by $\preceq$ and recall that $(\mathcal{SL}/{\sim}, \leq_\sim)$ is a model of $\mathcal{A}$. By Theorem 1.15 we know that there exists at least one homomorphism h from $\mathcal{SL}/{\sim}$ onto A, where A is a subdirectly irreducible preordered $\mathcal{C}_\mathcal{L}$-algebra. Also, $A \in Mod\mathcal{A}$ since A by construction is a preordered quotient of $\mathcal{SL}$ and Lemma 2.7 guarantees that the class $Mod\mathcal{A}$ is closed under quotient.

Define $v_\preceq \colon \mathcal{SL} \to A$ by letting $v_\preceq = h \circ q_\sim$ and observe that

- $v_\preceq$ is a pointwise valuation,
- $v_\preceq$ preserves $\preceq$. Notice that we are not assuming so far that A has a natural order. However, it has a partial order $\leq$ defined over it induced by $\leq_\sim$ (see Definition 1.12). By construction $\leq$ preserves $\preceq$, because $\leq_\sim$ does.
- $v_\preceq$ is onto because it is the composition of two maps which are onto.

We conclude that $v_\preceq$ is the desired pointwise valuation representing the pairwise valuation $\preceq$.

Figure 2.1: From pairwise to pointwise valuation.

$$\begin{tikzcd}
(\mathcal{SL}, \preceq) \arrow[r, two heads, "q_\sim"] \arrow[rrr, dashed, two heads, bend right, "v_\preceq = h \circ q_\sim"'] & (\mathcal{SL}/{\sim}, \leq_\sim) \arrow[r, hook, "e"] \arrow[rr, two heads, bend right, "h"'] & \prod_{i\in I} A_i \arrow[r, "\pi_i"] & (A_i, \leq)
\end{tikzcd}$$

$$\text{where } A_i = \frac{\mathcal{SL}/{\sim}}{\theta} \text{ with } \theta \in Con\mathcal{SL}/{\sim}$$

$\square$

I put forward an alternative method for evaluating the truth of sentences, truth from comparison, modelled by pairwise valuations, which express comparative judgements. The standard approach consists rather in evaluating sentences by assigning a certain value, called truth value, by means of operation-preserving functions, or pointwise valuations. Theorem 2.11 states that if a set of truth comparisons satisfies minimal structural properties then values can be assigned to sentences in a way that is compatible with the truth ranking.

The desired pointwise valuation is obtained by composition of homomorphisms. The canonical map $q_\sim$ is already a representing pointwise valuation. However, we have seen that considerably more can be done, i.e. we can construct pointwise valuations evaluating sentences in more fundamental algebras, the irreducible ones. Irreducible algebras in the class of models of $\mathcal{A}$ are obtained by refining the quotient, namely by composing the congruence $\sim$ with other congruences θ. It is worth stressing that the representing pointwise valuation obtained in this way is far from unique, indeed there is a suitable map for each completely meet-irreducible congruence over the quotient algebra (see Definition 1.6 and Lemma 1.7). Also, the construction is not constructive, since Lemma 1.9 essentially uses Zorn's Lemma.

Pairwise valuations are interpreted as ranking of sentences with respect to their truth, so the functions representing them are truth assignments. The irreducible algebra A_i plays the role of the algebra of truth values: its universe is the set of truth values and its operations (which are of the same signature as sentential connectives) are truth functions assigning truth values to complex sentences on the basis of the truth values of their components. This set of truth values is partially ordered by an order induced by the pairwise valuation itself. Furthermore, the fact that the final pointwise valuation is onto amounts to saying that all the truth values in the algebra are actually needed in order to represent the truth ranking as a functional assignment. It is worth noting that this is not yet a quantitative representation of pairwise valuations since the universe of A_i, the set of truth values, is not numerical and, at this stage, nothing can be said on how to embed it into a numerical structure. Moving to a less abstract framework is the best way to have a better grasp of the significance of the preceding results.

2.2 Starting with a logic

In this section the framework is less abstract to the extent that I specify the structure of the language and introduce some distinguished connectives. Let $\mathcal{L}$ be a propositional language and $\mathcal{SL}$ the set of sentences built recursively by means of a binary connective $\rightarrow$ for implication, along with the constant $\bot$ for *falsum*. As usual negation, constant for *verum*, disjunction, conjunction and biimplication are defined as $\neg\phi := \phi \rightarrow \bot$, $\top := \neg\bot$, $\phi \vee \psi := \neg\phi \rightarrow \psi$, $\phi \wedge \psi := \neg(\neg\phi \vee \neg\psi)$ and $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, respectively.

Moreover, at this stage I assume a logical system, or a logic, to be given. In the spirit of abstract algebraic logic (Font and Pigozzi, 2003), a logic here is pair: language (algebra of terms) and deducibility relation over the language. Let $\mathbf{L} = \langle \mathcal{SL}, \vdash \rangle$, with $\vdash \subseteq \mathcal{SL}^2 \times \mathcal{SL}$, be an arbitrary logic over the language $\mathcal{SL}$ satisfying the following Assumptions.

1. $\mathbf{L}$ is a *Tarskian logic*, i.e the deducibility relation $\vdash$ satisfies for every $\Gamma \subseteq \mathcal{SL}$ and every $\phi, \psi \in \mathcal{SL}$:

   (REF) $\phi \in \Gamma$ implies $\Gamma \vdash \phi$,

   (MON) $\Gamma \subseteq \Delta$ and $\Gamma \vdash \phi$ imply $\Delta \vdash \phi$,

(TRA) $\Gamma \vdash \phi$ and $\Gamma, \phi \vdash \psi$ imply $\Gamma \vdash \psi$.

(STR) for every endomorphism σ of $\mathcal{SL}$, $\Gamma \vdash \phi$ implies $\sigma(\Gamma) \vdash \sigma(\phi)$.

(FIN) $\Gamma \vdash \phi$ implies that there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \phi$.

That is to say that the deducibility relation is reflexive, monotone, transitive, structural and finitary. Also, **L** enjoys a deduction theorem in one of the following forms:

Theorem (Ordinary Deduction Theorem – DT). *If* $\Gamma, \phi \vdash \psi$ *then* $\Gamma \vdash \phi \to \psi$.

Theorem (Local Deduction Theorem – LDT). *If* $\Gamma, \phi \vdash \psi$ *then there is* $n \in \omega$ *such that* $\Gamma \vdash \phi^n \to \psi$, *where* ϕ^n *is an abbreviation for* $\underbrace{\phi \wedge \cdots \wedge \phi}_{n \text{ times}}$.

2. **L** is algebraizable in the sense of Blok and Pigozzi (1989), in particular strongly and regularly algebraizable (see Czelakowski, 2001, Definition 5.1.1, p. 352). This means that the class of algebras characterising the logic forms a variety (strongly algebraizable) and that each algebra A has a distinguished element, $\top$, acting as designated value (regularly algebraizable). We also assume $\top$ to be the interpretation of the logical symbol $\top$ of the algebra of terms. Let $Var_\vdash$ denote the variety characterising **L**. For the sake of readability we denote the operations of the algebras in the variety characterising the logic **L** by using the same symbols of the connectives of $\mathcal{SL}$, for example $\to$ and $\bot$.

3. All $A \in Var_\vdash$ are partially ordered as follows:

$$\forall a, b \in A \;\; a \leq_{\mathcal{V}} b \;:\Leftrightarrow\; a \to b = \top.$$

Moreover, $\leq_{\mathcal{V}}$ is bounded, namely

$$\bot \to a = \top,$$
$$a \to \top = \top.$$

4. Let $\mathcal{SL}_0$ be the set of sentences in which just connectives and logical constants occur. For all $\Gamma \subseteq \mathcal{SL}_0$ and all $\phi \in \mathcal{SL}$

$$\Gamma \vdash \phi \Leftrightarrow \Gamma \vdash_{CL} \phi,$$

where $\vdash_{CL}$ is the deducibility relation of classical propositional logic. In other words, the logic of $\bot$ and $\top$ is classical. This is a conservativity assumption which guarantees that also in many-valued logics there are a *True* and a *False* which are *the True* and *the False* of the classical bivalent semantics.

Assumption 3 can also be expressed syntactically. Building on Cintula (2006), we could equivalently require:

3$'$. **L** is a *weakly implicative logic*, namely the connective $\to$ satisfies

- reflexivity: $\vdash \phi \to \phi$
- transitivity: $\vdash \phi \to \psi, \psi \to \chi \vdash \phi \to \chi$
- congruence: $\phi_1 \leftrightarrow \phi_2, \psi_1 \leftrightarrow \psi_2 \vdash (\phi_1 \to \psi_1) \leftrightarrow (\phi_2 \to \psi_2)$
- MP: $\phi, \phi \to \psi \vdash \psi$.

These conditions on implication guarantee that for any algebra in the algebraic semantics of these logics one can define an order relation from the implication as in Assumption 3. The fact that the order of truth values is bounded can be expressed syntactically as

- $\vdash \bot \to \phi$,
- $\vdash \phi \to \top$.

I prefer a semantic formulation for Assumption 3 because it reflects an assumption about the set of truth values instead of one about a connective of the language. The set of truth values irrespectively of its cardinality usually comes with an ordering structure, called truth ordering. I wish to restrict attention to logics which have as intended semantics a set of truth values partially ordered by the implication and admitting a greatest and smallest element, representing absolute truth and absolute falsity, respectively, which behave classically. Such logics include for example classical logic, fuzzy logics, intuitionistic logic and certain families of substructural logics.

2.2.1 Pairwise L-valuations

In this new framework the definitions of pointwise valuation, pairwise valuation, admissible family of pairwise valuations and representation we stated in the previous section (Definitions 2.4, 2.1, 2.3 and 2.5) need some adjustments.

Definition 2.12. *A* pointwise **L**-valuation *is a homomorphism* $h\colon \mathcal{SL} \to A$, *where* A *is a* $\mathcal{C}_{\mathcal{L}}$*-algebra* $A \in Var_{\vdash}$.

Sentences are evaluated in an algebra of the variety $Var_{\vdash}$ that constitutes the algebraic semantics of the logic, and, as usual, valuations are truth-functional maps, such that the following hold for all sentences $\phi, \psi \in \mathcal{SL}$:

1. $h(\phi \to \psi) = h(\phi) \to h(\psi)$,
2. $h(\top) = \top$.

Let $\mathcal{V}$ be the set of all pointwise **L**-valuations containing for all $A \in Var_{\vdash}$ all the homomorphisms $h\colon \mathcal{SL} \to A$. We say that a sentence $\phi \in \mathcal{SL}$ is a *semantic consequence* of a set of sentences $\Gamma \subseteq \mathcal{SL}$ with respect to the class $Var_{\vdash}$ if for each $\mathcal{C}_{\mathcal{L}}$-algebra A in the variety and each pointwise **L**-valuation $h\colon \mathcal{SL} \to A$ we have $h(\phi) = \top$ whenever $h(\gamma) = \top$ for all $\gamma \in \Gamma$. We denote this circumstance by $\Gamma \models_{\mathcal{V}} \phi$. For future references I state the following

Definition 2.13. *For all* $\Gamma \subseteq \mathcal{SL}$ *and* $\phi \in \mathcal{SL}$

$$\Gamma \models_{\mathcal{V}} \phi \;:\Leftrightarrow\; \forall h \in \mathcal{V} \;\; \textit{if}\, \forall \gamma \in \Gamma \;\; h(\gamma) = \top \;\textit{then}\; h(\phi) = \top.$$

Semantic consequence is defined in terms of preservation of absolute truth ($\top$) under all the possible valuations. It is known that the relation thus defined is reflexive, monotone, transitive and structural, that is:

Proposition 2.14. $\models_{\mathcal{V}}$ *is a Tarskian consequence relation over* $\mathcal{SL}$.

Furthermore, being $\models_{\mathcal{V}}$ defined in terms of the algebras in the variety characterizing **L**, it is sound and complete with respect to the logic, even in the strong sense with respect to arbitrary theories.

Proposition 2.15. *For all* $\Gamma \subseteq \mathcal{SL}$ *and* $\phi \in \mathcal{SL}$

$$\Gamma \models_{\mathcal{V}} \phi \;\Leftrightarrow\; \Gamma \vdash \phi.$$

As before, instead of mapping sentences to an ordered structure (pointwise valuations), we endow the set of sentences with an order (pairwise valuations). I started the formalisation process by requiring minimal structural proprieties: pairwise valuations, defined over pairs of sentences, are congruent preorders, namely binary reflexive and transitive relations whose symmetric part is a congruence with respect to the connectives. In the present less abstract framework, it is immediate to notice that $\sim$ is congruent also with respect to the defined operations:

Proposition 2.16. *1.* $\phi \sim \psi \Rightarrow \neg\phi \sim \neg\psi$,

2. $\phi_1 \sim \psi_2, \psi_1 \sim \psi_2 \Rightarrow \phi_1 \star \psi_1 \sim \phi_2 \star \psi_2 \quad \star \in \{\vee, \wedge, \leftrightarrow\}$

However, now there is also a logical system in the picture, which can be taken into account when it comes to formulating assumptions about pairwise valuations. Pairwise valuations for a specific logic **L** in the class of logics under consideration are defined in the following:

Definition 2.17. *A relation* $\preceq \subset \mathcal{SL}^2$ *is a* pairwise **L**-valuation *if and only if it is a pairwise valuation and it satisfies for all sentences* $\phi, \psi \in \mathcal{SL}$

(A.1) $\vdash \phi \Rightarrow \phi \sim \top$,

(A.2) $(\phi \to \psi) \sim \top \Leftrightarrow \phi \preceq \psi$.

The relation is assumed to be *sound* with respect to the underlying logic, namely if a sentence is provable in the logical system (if it is a theorem) then it should be evaluated as maximally true (Axiom (A.1)). This axiom explicitly anchors the relation which expresses truth comparisons to the underlying logic. This may be considered an unwelcome feature for a semantic notion to possess, however at this level of generality it guarantees neutrality with respect to the chosen logic. When a specific logic is fixed, as far as the logic is finitely axiomatizable, then the axiom at stake can be removed in favour of a list of suitable conditions. We shall see an example of this in the next chapter. Thus Axiom (A.1) might be considered an 'abbreviation' for such conditions. In addition, we require the truth order to coincide with the order of truth values given by the implication. Axiom (A.2) also states the truth condition for the implication: an implicative sentence is true if and only if the truth value of the antecedent is less than or equal to the truth value of the consequent. It follows immediately from the Definition that pairwise valuations are *bounded* preorders:

$$\bot \preceq \phi \preceq \top,$$

meaning that every sentence is no more true than tautologies and no less true than contradictions.

As in the previous section, a family of pairwise **L**-valuations $\mathcal{A} \subseteq \mathcal{P}$ is *admissible* if and only if $\preceq_{\mathcal{A}}$, the intersection of the preorders in $\mathcal{A}$, is substitution-invariant, i.e. for all endomorphism $\sigma \colon \mathcal{SL} \to \mathcal{SL}$

$$\phi \preceq_{\mathcal{A}} \psi \Rightarrow \sigma(\phi) \preceq_{\mathcal{A}} \sigma(\psi).$$

The following propositions give us more details about the relation between pairwise **L**-valuations and the underlying logic.

Proposition 2.18. *If $\preceq$ is a pairwise* **L***-valuation then the following hold:*

1. $\vdash \phi \to \psi \Rightarrow \phi \preceq \psi$,
2. $\vdash \phi \leftrightarrow \psi \Rightarrow \phi \sim \psi$.

Proof. 1. $\vdash \phi \to \psi$ implies $(\phi \to \psi) \sim \top$ by axiom (A.1). $(\phi \to \psi) \sim \top$, in turn, implies $\phi \preceq \psi$ by axiom (A.2).

2. It follows directly from the previous item and the definition of '$\leftrightarrow$'.

□

Proposition 2.19. *1. If $\phi \vdash \psi$ and $\phi \sim \top$ then $\psi \sim \top$,*

2. If $\Gamma \vdash \phi$ and $\forall \gamma \in \Gamma\ \gamma \sim \top$ then $\phi \sim \top$.

Proof. 1. We distinguish two cases according to the kind of deduction theorem the logic satisfies:

(DT) $\phi \vdash \psi \Rightarrow \vdash \phi \to \psi \Rightarrow \phi \preceq \psi$. Since $\phi \sim \top$, we have $\top \preceq \psi$. From this and Axiom (A.1) we conclude $\phi \sim \top$.

(LDT) $\phi \vdash \psi \Rightarrow \vdash \phi^n \to \psi \Rightarrow \phi^n \preceq \psi$. We want to show that $\phi \sim \top \Rightarrow \phi^n \sim \top$ in order to conclude $\psi \sim \top$. Assume $\phi \sim \top$. Since $\sim$ is a congruence (see Proposition 3.10), $\phi \sim \top, \phi \sim \top \Rightarrow \phi \wedge \phi \sim \top \wedge \top$. Since by assumption the logic of $\top$ is classical, $\top$ is idempotent with respect to $\wedge$. So we have $\vdash \top \wedge \top \leftrightarrow \top$. From this and the previous proposition we conclude $\top \wedge \top \sim \top$. Thus by transitivity $\phi \wedge \phi \sim \top$. This can be iterated n times. We conclude $\phi^n \sim \top$.

2. Under the assumption that $\vdash$ is finitary, this is a straightforward generalisation of the previous cases.

□

Proposition 2.18 establishes a crucial relation between the congruence *as true as* and the relation of logical equivalence defined as

$$\phi \equiv \psi \;:\Leftrightarrow\; \vdash \phi \leftrightarrow \psi,$$

which is a congruence over $\mathcal{SL}$ by virtue of the Assumptions. Proposition 2.19 states that a sort of strong soundness holds: if a sentence ϕ is a syntactical consequence of a set of sentences Γ, then ϕ is absolutely true whenever each sentence in Γ is such according to the pairwise valuation.

2.2.2 New version of the representation theorem

We are interested in proving that any pairwise **L**-valuation in a given admissible family can be represented by a pointwise **L**-valuation. This is the focus of this subsection. However before that, it is of interest to point out that, since we are assuming that a partial order is defined over the set of truth values, for any pointwise **L**-valuation there exists a corresponding pairwise **L**-valuation. In other words, each logical valuation of the logic **L** can be expressed by means of a set of comparisons between sentences.

Theorem 2.20 (From pointwise to pairwise)**.** *Each pointwise* **L***-valuation induces a pairwise* **L***-valuation. The family $\mathcal{V}$ of pointwise* **L***-valuations induces an admissible family of pairwise* **L***-valuations.*

Proof. For all h in $S\mathcal{V}$ define

$$\phi \preceq_h \psi \;:\Leftrightarrow\; h(\phi) \leq_{\mathcal{V}} h(\psi),$$

and let $\mathcal{F}$ be the set of pairwise valuations thus generated. We prove that

(i) $\preceq_h$ is a pairwise **L**-valuation. $\preceq_h$ is a congruent preorder because $\leq_\mathcal{V}$ is such.

(A.1) $\vdash \phi \Rightarrow \models_\mathcal{V} \phi \Rightarrow h(\phi) =_\mathcal{V} \top \Rightarrow \phi \sim_h \top$.

(A.2) $(\phi \to \psi) \sim_h \top \Leftrightarrow h(\phi \to \psi) =_\mathcal{V} \top \Leftrightarrow h(\phi) \leq_\mathcal{V} h(\psi) \Leftrightarrow \phi \preceq_h \psi$.

(ii) The set $\mathcal{F}$ is an admissible family of pairwise **L**-valuations, namely the intersection preorder $\bigcap \mathcal{F}$ is substitution-invariant. This rests on the fact that $\models_\mathcal{V}$ is such (see Proposition 2.14).

$$
\begin{aligned}
(\phi, \psi) \in \bigcap \mathcal{F} &\Leftrightarrow \forall \preceq_h \in \mathcal{V} \;\; \phi \preceq_h \psi \\
&\Leftrightarrow \forall h \in \mathcal{V} \;\; h(\phi) \leq_\mathcal{V} h(\psi) \\
&\Leftrightarrow \forall h \in \mathcal{V} \;\; h(\phi) \to h(\psi) = \top \\
&\Leftrightarrow \forall h \in \mathcal{V} \;\; h(\phi \to \psi) = \top \\
&\Leftrightarrow \models_\mathcal{V} \phi \to \psi \\
&\Rightarrow \forall \sigma \colon \mathcal{SL} \to \mathcal{SL} \quad \models_\mathcal{V} \sigma(\phi \to \psi) \\
&\Leftrightarrow \models_\mathcal{V} \sigma(\phi) \to \sigma(\psi) \\
&\Leftrightarrow \forall h \in \mathcal{V} \;\; h(\sigma(\phi) \to \sigma(\psi)) = \top \\
&\Leftrightarrow \forall h \in \mathcal{V} \;\; h(\sigma(\phi)) \to h(\sigma(\psi)) = \top \\
&\Leftrightarrow \forall h \in \mathcal{V} \;\; h(\sigma(\phi)) \leq_\mathcal{V} h(\sigma(\psi)) \\
&\Leftrightarrow (\sigma(\phi), \sigma(\psi)) \in \bigcap \mathcal{F}.
\end{aligned}
$$

□

This is to some extent the easy direction of representation results. Recall that the set of truth values, irrespectively of its cardinality, usually comes with an ordering structure, called *truth ordering*. For instance, in the classical case, truth values constitute a lattice in which *the False* is less true than *the True*, or in the infinite-valued case degrees of truth form a bounded chain. In general, we take this order to be the natural order of the variety characterizing the logic, so it is determined from the implication. This natural order induces a truth ranking over the set of sentences, as Theorem 2.20 states. Notice that the induced pairwise valuation is more than a preorder: it is a partial order. This gives us the chance to make an important point in favour of the comparative perspective. Defining the truth ranking directly over the set of sentences, instead of considering the ranking endowed in the set of truth values, allows us to take some of the mathematical structure off and to get rid of non-essential mathematical properties. This might be a move rich in philosophical and methodological relevance as I shall argue in the next chapter.

The main focus is still to show that it is possible to move from pairwise to pointwise valuations, that is to say to show that if a set of comparative judgements satisfies certain requirements, then it can be represented by a function assigning truth values that are ordered in a compatible way. More precisely, we want the natural order of the set of truth values, namely the order of the variety characterizing the logic, to preserve the original pairwise valuation.

Definition 2.21. *A pointwise* $\mathbf{L}$*-valuation* $h\colon \mathcal{SL} \to A$ represents *a pairwise* $\mathbf{L}$*-valuation* $\preceq \subseteq \mathcal{SL}^2$ *if and only if for all* $\phi, \psi \in \mathcal{SL}$

$$\phi \preceq \psi \Rightarrow h(\phi) \leq_{\mathcal{V}} h(\psi).$$

The new version of the representation theorem is the following:

Theorem 2.22 (From pairwise to pointwise)**.** *Given an admissible family of pairwise* $\mathbf{L}$*-valuations* $\mathcal{A} \subseteq \mathcal{P}$*, for every* $\preceq$ *in* $\mathcal{A}$ *there exists at least one pointwise* $\mathbf{L}$*-valuation representing it.*

This follows as a corollary of Theorem 2.11. We already proved that for every $\preceq$ in an admissible family $\mathcal{A} \subseteq \mathcal{P}$ there exists at least one pointwise valuation $v_{\preceq}\colon \mathcal{SL} \to A$ representing it, where A is a $Mod\mathcal{A}$-irreducible $\mathcal{C}_{\mathcal{L}}$-algebra with an induced partial order defined over it. The following Lemma completes the proof of Theorem 2.22:

Lemma 2.23. *Let* $\preceq$ *be a pairwise valuation. Consider the pointwise valuation* $v_{\preceq}\colon \mathcal{SL} \to A$ *representing it, with* A *a* $Mod\mathcal{A}$*-irreducible* $\mathcal{C}_{\mathcal{L}}$*-algebra. If* $\preceq$ *is a pairwise valuation for a logic* $\mathbf{L}$ *then*

1. $A \in Var_{\vdash}$*,*
2. $\leq_{\mathcal{V}}$*, the natural order of* A*, preserves* $\preceq$*.*

Proof. We prove this by focusing on the quotient algebra $\mathcal{SL}/\sim$. The fact that the truth order refines the order given by the logical equivalence relation, i.e. the fact that

$$\phi \equiv \psi \Rightarrow \phi \sim \psi,$$

allows us to relate the quotient algebra modulo $\sim$ with the algebra $(\mathcal{SL}/\equiv, \to, 1)$, known as the Lindenbaum algebra for $\mathbf{L}$. This algebra exists by virtue of the assumptions over $\mathbf{L}$ and, also, we know that it is in $Var_{\vdash}$.

Recall that the quotient algebra modulo $\sim$ is obtained by taking as universe the quotient set $\mathcal{SL}/\sim = \{[\phi]_{\sim} \mid \phi \in \mathcal{SL}\}$, where $[\phi]_{\sim} = \{\psi \in \mathcal{SL} \mid \psi \sim \phi\}$. The operations

$$[\phi]_{\sim} \tilde{\to} [\psi]_{\sim} := [\phi \to \psi]_{\sim}$$

$$\widetilde{\top} := [\top]_{\sim} := \top$$

are well defined because $\sim$ is a congruence over $\mathcal{SL}$. Let $q_{\equiv}$ and $q_{\sim}$ be the canonical maps from $\mathcal{SL}$ to the quotients $\mathcal{SL}/\equiv$ and $\mathcal{SL}/\sim$, respectively. Notice that these functions are onto. This being in place, the relation between the structures is sketched in Figure 2.2.

Figure 2.2: Connection with the Lindenbaum algebra.

$$\begin{tikzcd}
(\mathcal{SL}, \to, \top, \preceq) \arrow[rr, two heads, "q_{\equiv}"] \arrow[d, "q_{\sim}"] & & (\mathcal{SL}/\equiv, \to, 1) \arrow[dll, dashed, "f"'] \\
(\mathcal{SL}/\sim, \widetilde{\to}, \widetilde{\top}) & &
\end{tikzcd}$$

In order to make the diagram commute, we define a function $f\colon \mathcal{SL}/\equiv \to \mathcal{SL}/\sim$ as follows:

$$\forall \phi \; f([\phi]_{\equiv}) = [\phi]_{\sim}.$$

We can then verify that:

Proposition 2.24. *1. f is well defined,*

2. f is onto,

3. $f(q_{\equiv}(\phi)) = q_{\sim}(\phi)$.

The proof is straightforward. A crucial role is played by the fact that the truth order refines the order given by logical equivalence. Moreover:

Proposition 2.25. *$f\colon \mathcal{SL}/\equiv \to \mathcal{SL}/\sim$ is a homomorphism, namely*

1. $f([\phi]_{\equiv} \to [\psi]_{\equiv}) = f([\phi]_{\equiv}) \,\widetilde{\to}\, f([\psi]_{\equiv})$,

2. $f(1) = \widetilde{\top}$.

Since varieties of algebras are closed under homomorphic images, the following holds:

Lemma 2.26. *$(\mathcal{SL}/\sim, \widetilde{\to}, \widetilde{\top})$ is in the same variety of the Lindenbaum algebra of* **L***, i.e. in the same variety of algebras characterizing the logic* **L***.*

Analogously to the abstract case, in which we have seen that the quotient algebra was among the models of $\mathcal{A}$ (Lemma 2.9), here we showed that $(\mathcal{SL}/\sim, \widetilde{\to}, \widetilde{\top}) \in Var_{\vdash}$. Since varieties are closed under homomorphisms, the algebra A, obtained by applying theorem 2.11, is in the variety characterizing the logic too, as was to be shown.

We have to prove that $\leq_{\mathcal{V}}$, the natural order of A, preserves $\preceq$. Since $(\mathcal{SL}/{\sim}, \tilde{\to}, \tilde{\top}) \in Var_{\vdash}$, it has its natural partial order defined by

$$[\phi]_{\sim} \leq_{\mathcal{V}} [\psi]_{\sim} \;:\Leftrightarrow\; [\phi]_{\sim} \tilde{\to} [\psi]_{\sim} = [\top]_{\sim}.$$

Moreover, $\preceq$ induces in a natural way a partial order $\leq_{\sim}$ on the quotient set defined as

$$[\phi]_{\sim} \leq_{\sim} [\psi]_{\sim} \;:\Leftrightarrow\; \exists \phi_i \in [\phi]_{\sim}, \psi_i \in [\psi]_{\sim} \text{ such that } \phi_i \preceq \psi_i,$$

which is the induced preorder whose preservation is guaranteed from Theorem 2.11. However, Axiom (A.3) guarantees that the two orderings coincide:

$$\begin{aligned}
[\phi]_{\sim} \leq_{\mathcal{V}} [\psi]_{\sim} &\Leftrightarrow [\phi]_{\sim} \tilde{\to} [\psi]_{\sim} = [\top]_{\sim} \\
&\Leftrightarrow [\phi \to \psi]_{\sim} = [\top]_{\sim} \\
&\Leftrightarrow (\phi \to \psi) \sim \top \\
&\Leftrightarrow \phi \preceq \psi \\
&\Leftrightarrow [\phi]_{\sim} \leq_{\sim} [\psi]_{\sim}.
\end{aligned}$$

We conclude that $\preceq$ is preserved.

Figure 2.3: From **L**-pairwise to **L**-pointwise valuation .

$$\begin{tikzcd}
(\mathcal{SL}, \preceq) \arrow[r, two heads, "q_{\sim}"] \arrow[rrr, dashed, bend right, two heads, "v_{\preceq} = h \circ q_{\sim}"'] & (\mathcal{SL}/{\sim}, \leq_{\sim}) \arrow[r, hook, "e"] \arrow[rr, bend right, two heads, "h"'] & \prod_{\theta \in I} A_{\theta} \arrow[r, "\pi_{\theta}"] & A_{\theta}
\end{tikzcd}$$

$$\text{where } A_{\theta} = \frac{\mathcal{SL}/{\sim}}{\theta} \text{ with } \theta \in Con\mathcal{SL}/{\sim}$$

$\square$

The central notion here is the notion of pairwise valuation for a given logic, which consists in a preorder ranging over the set of sentences, and satisfying certain axiomatic conditions. I defended the plausibility of these conditions given the interpretation of $\preceq$ and the Assumptions on the underlying logic. Theorem 2.22 shows that this small set of conditions is sufficient to guarantee the existence of a matching pointwise valuation function, namely a logical valuation for that logic evaluating sentences in the intended truth-value semantic which preserves the original truth ranking.

The underlying logic is syntactically given as a deducibility system satisfying some general Assumptions. It is worth looking at what happens when the underlying logic is classical. Let $\mathbf{CL} = \langle \mathcal{SL}, \vdash_{CL} \rangle$ be a deductive system of classical propositional logic. Recall that the algebraic semantics for classical logic is given by the variety $\mathcal{B}$ of Boolean algebras. It is known that the two-element Boolean algebra $\mathcal{B}_2$ is the only subdirectly irreducible Boolean algebra (Stone's Representation Theorem). This is a peculiar situation: in general a variety can have arbitrarily many non-isomorphic subdirectly irreducible members of arbitrary size. Birkhoff's Theorem for the classical case can then be stated as follows: every Boolean algebra is isomorphic to a subalgebra of a direct product of copies of the two-element Boolean algebra.

A congruent preorder $\preceq$ is a pairwise valuation for **CL** if satisfies axiom (A.2) and

(A.1′) $\vdash_{CL} \phi \Rightarrow \phi \sim \top$.

As one would expect, the algebra $(\mathcal{SL}/{\sim}, \tilde{\rightarrow}, \tilde{\top})$ turns out to be a Boolean algebra (see Lemma 2.23). Accordingly, it can be embedded in the two-element Boolean algebra. So, the representation result can be stated as follows:

Corollary 2.27. *For every pairwise* **CL**-*valuation in a given admissible family, there exists a homomorphism in* $\mathcal{B}_2$ *representing it.*

Homomorphisms from the algebra of terms in $\mathcal{B}_2$ are bivalent, classical, valuation functions evaluating complex sentences according to the classical truth tables.

It makes sense to talk of *more true than* also in a classical setting since the standard semantics for classical logic also determines a truth ordering. This is a binary lattice with two equivalence classes with respect to truth, basically stating that true sentences are more true than false sentences. Interestingly, the axioms do not rule out the possibility of having *genuinely intermediate sentences*, namely sentences strictly less true than $\top$ and strictly more true than $\top$. Those sentences, if any, end up being either true or false according to the corresponding pointwise valuation. This is possible because it is a weak representation result: the representing function can give the same value to sentences that are not considered equivalent with respect to the order. So, in the classical case, starting with pairwise valuations and talking about more or less true amounts to adding some information which is then lost in the quantitative formulation since genuinely intermediate sentences with respect to truth end up being flat-out.

2.2.3 Pairwise semantics

An admissible family of pairwise valuations induces in a natural way a semantics for the logic at stake. The notions of tautology and semantic consequence can be defined as follows:

Definition 2.28.

$$\Gamma \models_{\preceq} \phi :\Leftrightarrow \textit{ for all pairwise } \boldsymbol{L}\textit{-valuations } \preceq \in \mathcal{A}, \textit{ if } \forall \gamma \in \Gamma\ \gamma \sim \top \textit{ then } \phi \sim \top$$

$$\Leftrightarrow \textit{if } \forall\gamma \in \Gamma\ \gamma \sim_{\mathcal{A}} \top \textit{ then } \phi \sim_{\mathcal{A}} \top.$$

$$\models_{\preceq} \phi :\Leftrightarrow \textit{ for all pairwise } \boldsymbol{L}\textit{-valuations } \preceq\in \mathcal{A}\ \phi \sim \top$$

$$\Leftrightarrow \phi \sim_{\mathcal{A}} \top.$$

These are classical definitions in which validity is accounted for in terms of (absolute) truth preservation: whenever all the premisses are absolutely true, the conclusion should be absolutely true as well.[2]

The semantics thus defined can be proved to be strongly sound and complete with respect to the logic **L**. Soundness follows directly from Proposition 2.19 and the definition of $\models_{\preceq}$. In what follows we prove strong completeness.

Theorem 2.30 (Strong completeness)**.** $\forall\Gamma \subseteq \mathcal{SL}, \phi \in \mathcal{SL}$

$$\Gamma \models_{\preceq} \phi \Rightarrow \Gamma \vdash \phi.$$

Direct proof. We prove that if $\Gamma \nvdash \phi$ then there exists an admissible family and a pairwise **L**-valuation such that $\forall\gamma \in \Gamma\ \gamma \sim \top$ and $\phi \nsim \top$. Let

$$\phi \preceq \psi \ :\Leftrightarrow\ \Gamma \vdash \phi \to \psi.$$

We can verify that

(i) $\preceq$ is a pairwise **L**-valuation. $\preceq$ is a congruent preorder because $\to$ by assumption is reflexive, transitive and congruent (see item $3'$ of the Assumptions). Moreover, it satisfies Axioms (A.1) and (A.2):

(A.1) $\vdash \phi \Rightarrow \vdash \top \leftrightarrow \phi \Rightarrow \Gamma \vdash \top \leftrightarrow \phi \Rightarrow \phi \sim \top$.

(A.2) $\phi \to \psi \sim \top \Leftrightarrow \Gamma \vdash (\phi \to \psi) \leftrightarrow \top \Leftrightarrow \Gamma \vdash \phi \to \psi \Leftrightarrow \phi \preceq \psi$.

[2] Pairwise valuations are also particularly suitable for expressing a different notion of consequence relation, alternative to the truth-preserving scheme, which has been proposed for many-valued logics: logical consequence as "preservations of degrees of truth" (see e.g. Font, 2003). If there is some (partial) ordering among truth values then consequence can be understood as follows: whenever all premisses attain at least a certain degree of truth, the conclusion should have at least that degree of truth too. Intuitively, there should be no drop in truth value from the premisses to the conclusion. In terms of pairwise valuation this would be stated as in the following

Definition 2.29. *If $\preceq$ is a partial order:*

$$\Gamma \models_{\preceq} \phi :\Leftrightarrow \textit{ for all pairwise } \boldsymbol{L}\textit{-valuations, } \forall\chi \in \mathcal{SL} \textit{ if } \forall\gamma \in \Gamma\ \gamma \succeq \chi \textit{ then } \phi \succeq \chi.$$

If $\preceq$ is a total order:

$$\Gamma \models_{\preceq} \phi :\Leftrightarrow \textit{ for all pairwise } \boldsymbol{L}\textit{-valuations, } \phi \succeq \textit{ inf } \{\gamma \in \mathcal{SL} \mid \gamma \in \Gamma\}.$$

(ii) $\Gamma \nvDash_{\preceq} \phi$, namely $\forall\gamma \in \Gamma\ \gamma \sim \top$ and $\phi \nsim \top$. Notice that $\Gamma \vdash \phi \Leftrightarrow \Gamma \vdash \top \leftrightarrow \phi \Leftrightarrow \top \sim \phi$, that is to say the defined pairwise valuation gives value $\top$ to all and only the sentences derivable from Γ. We can prove that $\Gamma \vdash \phi \Leftrightarrow \Gamma \vdash \top \leftrightarrow \phi$ by using **(MON)** and Deduction Theorem for the left-to-right direction, and the inverse of Deduction Theorem and **(TRA)** for the right-to-left direction.

(iii) There exists an admissible family $\mathcal{F}$ such that $\preceq \in \mathcal{F}$. $\forall\Gamma \subseteq \mathcal{SL}$ define $\preceq_\Gamma$ such that

$$\phi \preceq_\Gamma \psi :\Leftrightarrow \Gamma \vdash \phi \rightarrow \psi.$$

Consider the family $\mathcal{F} \subseteq \mathcal{P}$ of pairwise **L**-valuations thus defined. We prove that $\mathcal{F}$ is admissible. Let $\preceq_\mathcal{F}$ be the intersection preorder over $\mathcal{F}$. Notice that $\phi \preceq_\mathcal{F} \psi \Leftrightarrow \forall\Gamma \subseteq \mathcal{SL}\quad \Gamma \vdash \phi \rightarrow \psi$. Furthermore, $\forall\Gamma \subseteq \mathcal{SL}\quad \Gamma \vdash \phi \rightarrow \psi$ implies and is implied by $\vdash \phi \rightarrow \psi$. And $\vdash$ is substitution-invariant because it is equivalent to $\models_\mathcal{V}$ that satisfies substitution invariance. Then, we have

$$\begin{aligned}
\phi \preceq_\mathcal{F} \psi &\Leftrightarrow \forall\Gamma \subseteq \mathcal{SL}\ \ \Gamma \vdash \phi \rightarrow \psi \\
&\Leftrightarrow \vdash \phi \rightarrow \psi \\
&\Rightarrow \vdash \sigma(\phi) \rightarrow \sigma(\psi) \\
&\Leftrightarrow \forall\Gamma \subseteq \mathcal{SL}\ \ \Gamma \vdash \sigma(\phi) \rightarrow \sigma(\psi) \\
&\Leftrightarrow \sigma(\phi) \preceq_\mathcal{F} \sigma(\psi).
\end{aligned}$$

□

Indirect proof. Recall that $\models_\mathcal{V}$ as defined in Definition 2.13 enjoys strong completeness with respect to $\vdash$, as stated in Theorem 2.15. It suffices then to prove that $\Gamma \models_\preceq \phi \Rightarrow \Gamma \models_\mathcal{V} \phi$. We prove $\Gamma \nvDash_\mathcal{V} \phi \Rightarrow \Gamma \nvDash_\preceq \phi$, namely that if there exists a pointwise **L**-valuation $h \in \mathcal{V}$ such that $h(\gamma) = \top$ for all $\gamma \in \Gamma$ and $v(\phi) \neq \top$ then there exists a pointwise **L**-valuation $\preceq$ in an admissible family such that $\gamma \sim \top$ for all $\gamma \in \Gamma$ and $\phi \nsim \top$.

We know from Theorem 2.20 that each pointwise **L**-valuation induces a pairwise **L**-valuation in an admissible family defined as $\phi \preceq_h \psi :\Leftrightarrow h(\phi) \leq_\mathcal{V} h(\psi)$. In addition, notice that for all $\phi \in \mathcal{SL}$, $h(\phi) = \top$ if and only if $\phi \sim_h \top$. □

Soundness and completeness results guarantee that pairwise valuations supply **L** with an adequate semantics, alternative to the standard truth-value semantics, though still defined in terms of absolute-truth preservation. The fact that this semantics is sound comes as no surprise given that by assumption each pairwise valuation evaluates theorems of the logic as absolutely true (Axiom (A.1)). I am not assuming the right-to-left direction of Axiom (A.1) so that also sentences which are not tautologies can be considered absolutely true under a certain valuation. However, a form of completeness hold: if a sentence is absolutely true under all possible pairwise valuations in an admissible family then it is also provable in the logic. Also, a strong completeness holds, namely completeness with respect to a set of premisses: whenever a sentence is a semantic consequence of a set of sentences, there is a proof of this sentence in the system. Recall that, for example, the standard pointwise semantics with valuations into $[0,1]$ for Łukasiewicz logic is not strongly complete.

We have seen that the above can be proved directly, or, alternatively, indirectly by building on the completeness of the algebraic semantics. Both strategies used for proving completeness bring out the centrality of theories and the possibility of defining pairwise valuations in terms of the implicative relations holding in a set of sentences. This can be made explicit by noticing that there is a correspondence between the pairwise **L**-valuations as defined in Definition 2.1 and the deductively closed sets or theories of the logic **L**. Recall that a set of sentences $\Gamma \subseteq \mathcal{SL}$ is *deductively closed* if and only if for all $\gamma \in \mathcal{SL}$

$$\Gamma \vdash \gamma \;\Leftrightarrow\; \gamma \in \Gamma.$$

A set of sentences induces in a natural way an ordering between sentences and, vice versa, given an ordering we can always isolate a set of sentences as showed in the following constructions:

(i) Given a relation $\preceq$ define

$$\Gamma_{\preceq} := \{\gamma \in \mathcal{SL} \mid \top \sim \gamma\}.$$

(ii) Given a set Γ define

$$\phi \preceq_{\Gamma} \psi \;:\Leftrightarrow\; \Gamma \vdash \phi \to \psi.$$

This being in place, the following holds.

Proposition 2.31. *1. If $\preceq$ is a pairwise* **L***-valuation then $\Gamma_{\preceq}$ is deductively closed.*

2. If Γ is deductively closed then $\preceq_{\Gamma}$ is a pairwise **L***-valuation.*

Proof. 1. Assume $\Gamma_{\preceq} \vdash \phi$. By definition $\gamma \sim \top$ for all the sentences $\gamma \in \Gamma$, then by Proposition 2.19 we can conclude that $\phi \sim \top$ and thus $\phi \in \Gamma$. Recall that this holds under the hypothesis that **L** is finitary and enjoys a deduction theorem.

2. See item (ii) in the proof of Theorem 2.30.

□

The correspondence can be stressed further by noticing that a Galois connection can be established between pairwise valuations and deductively closed theories.

$$\Lambda \xrightarrow{\text{(ii)}} \preceq_\Lambda \xrightarrow{\text{(i)}} \Theta_{\preceq_\Lambda} \qquad\qquad \Lambda = \Theta_{\preceq_\Lambda}$$

$$\begin{aligned} \phi \in \Theta_{\preceq_\Lambda} &\Leftrightarrow \top \preceq_\Lambda \phi \\ &\Leftrightarrow \Lambda \vdash \top \to \phi \\ &\Leftrightarrow \Lambda \vdash \phi \\ &\Leftrightarrow \phi \in \Lambda. \end{aligned}$$

$$\sqsubseteq \xrightarrow{\text{(i)}} \Theta_\sqsubseteq \xrightarrow{\text{(ii)}} \preceq_{\Theta_\sqsubseteq} \qquad\qquad \sqsubseteq = \preceq_{\Theta_\sqsubseteq}$$

$$\begin{aligned} \phi \preceq_{\Theta_\sqsubseteq} \psi &\Leftrightarrow \Theta_\sqsubseteq \vdash \phi \to \psi \\ &\Leftrightarrow \phi \to \psi \in \Theta_\sqsubseteq \\ &\Leftrightarrow \top \sqsubseteq \phi \to \psi \\ &\Leftrightarrow \phi \sqsubseteq \psi. \end{aligned}$$

The same one-one correspondence holds between totally ordered pairwise **L**-valuations and *prime theories* of **L**, namely sets of sentences Γ such that

$$\text{either } \Gamma \vdash \phi \to \psi \text{ or } \Gamma \vdash \psi \to \phi.$$

The formal relation between pairwise valuations and deductively closed theories does not undermine the interest of investigating truth rankings as primitive objects. On the contrary, it adds depth and weight to the analysis. The idea of truth from comparisons was in the first place called for in terms of the intuitive and informal notion of *more* or *less true than*. Then, formal treatment and representation results contributed in sharpening our intuitions and provided us with a precise definition. So, on the one hand, formal interactions with other logical notions constitute a testing ground for the definitions. On the other hand, these structural similarities confirm that we are bringing to the foreground notions which are deeply interwoven with the logical structure of our systems. Re-elaborating them in a form that allows a philosophical interpretation in terms of the relation *more* or *less true than* is one of the contributions of this investigation.

2.3 Conclusions

Pointwise and pairwise valuations are alternative methods for evaluating the truth of sentences of a formal language. The former method evaluates sentences by assigning them a truth value, whereas the latter by means of binary comparisons. This chapter is concerned with the relationships between these two notions, the central problem being to establish representation results, i.e. to isolate a set of conditions over pairwise valuations sufficient to guarantee the existence of pointwise valuations representing them.

After having introduced the formal notion of pairwise valuations, I proved a first extremely general representation theorem stating under which conditions comparative assessments can be represented by pointwise assignments of truth values (Theorem 2.11). The set of truth values at this stage is the universe of an algebra similar to the algebra of terms and compatible with the original pairwise valuation. This result allows us to isolate the minimal requirements a set of comparisons should meet in order to be considered the comparative counterpart of a valuation: being a congruent preorder. It is, therefore, a first fundamental step towards a measure-theoretical treatment of comparative truth.

Next in this chapter I showed that more significant representation results can be obtained by moving to a less abstract framework. In order to do this I restricted my attention to a specific language with a fixed set of connectives and to a specific class of logics, which is wide enough to include all the logics of interest for the discussion, ranging from classical logic to finite-valued and infinite-valued logics. As a corollary of the general result, I proved that, given an underlying logic, a suitable semantics induced by pairwise valuations for it yields the standard intended semantics for the logic itself, to the extent that some axiomatic conditions defining the relation *more* or *less true than* are sufficient for it to be representable by a valuation function of that logic (Theorem 2.22). The set of truth values in this case is the universe of an algebra in the algebraic semantics of the logic. Moreover, the pairwise semantics can be proved to be strongly sound and complete with respect to the logic (Theorem 2.30).

Representation results like Theorem 2.11 and Theorem 2.22 assure that if sentences can be compared 'well enough' with respect to their truth, where 'well enough' is given by a set of definitory axiomatic conditions, then it is as if we attach them a specific truth value. The existence of a representation allows us to gain the mathematical convenience typical of the functional approach, in which the idea of being true to different extents in modelled by using special, usually numerical, objects, namely truth values. Furthermore, there is a gain in philosophical plausibility in the idea of truth from comparison as defended in Chapter 1. It provides a viable way for dealing with logical valuedness without resorting to a set of objects whose nature and philosophical status are questionable.

Chapter 3

Degrees of truth explained away

The investigation carried out in Chapter 2 provides the theoretical framework and the formal results necessary to put forward a novel philosophical account of the notion of degrees of truth, as arising in infinite-valued logics. In spite of the extensive research on the relevant mathematics of degrees of truth, which turned out to be deep and rich in applications, the very notion of degrees of truth remains somewhat clouded in conceptual mystery and philosophically motivated diffidence. The first aim of this chapter is to bring representation theorems to bear on the analysis of degrees of truth, in order to provide a philosophical account of what degrees of truth are and of their role in the formalisation. In particular, I shall defend the idea that degrees of truth can be thought of as possible measures (or cardinalisations) of a comparative notion of truth, which is taken as primitive and is governed by non-numerical principles.

The second aim of this chapter is to shed new light on the much-criticised project of modelling vagueness by using degrees of truth. It has been argued that a semantics based on functions from sentences to degrees of truth coded by real numbers misrepresents the phenomenon of vagueness. This objection is known as *artificial precision objection* and is particularly compelling. I shall show that this and related difficulties can be overcome by adopting a comparative perspective on degrees of truth. This investigation may therefore contribute to rehabilitate degrees of truth and infinite-valued logics as a competitive model for vagueness.

I start my investigation about degrees of truth by applying to infinite-valued logics the qualitative foundation described in Chapter 2.

3.1 Qualitative perspective on degrees of truth

In Section 1.3.3 I introduced Łukasiewicz and Gödel infinite-valued logics. In the spirit of abstract algebraic logic I take these logics to be defined as pairs, $\mathbf{Ł} = \langle \mathcal{SL}, \vdash_{Ł} \rangle$ and $\mathbf{G} = \langle \mathcal{SL}, \vdash_G \rangle$, respectively, where $\mathcal{SL}$ is the set of sentences built recursively from a set of propositional variables $\mathcal{L}$ and a set of connectives $\mathcal{C}_{\mathcal{L}} = \{(\rightarrow, 2), (\perp, 0)\}$. The standard semantics for these logics is given by homomorphisms in an algebra of truth values that has as support set the real unit interval $[0, 1]$. The elements of this set are called *degrees of truth*.

Recall that the qualitative foundation consists in taking as primitive objects pairwise valuations, that is to say preorders over the set of sentences, and proving that under certain conditions these relations can be represented by pointwise valuations, that is to say by truth-value assignments.

Representation results for infinite-valued logics, stating that pairwise valuations can be represented as functions evaluating sentences in the algebraic variety characterising the logic, follow as corollary from Theorem 2.22, stating that given an admissible family of pairwise **L**-valuations $\mathcal{A} \subseteq \mathcal{P}$, for every $\preceq$ in $\mathcal{A}$ there exists at least one pointwise **L**-valuation representing it. Recall that given a logic $\mathbf{L} = \langle \mathcal{SL}, \vdash \rangle$ a pointwise **L**-valuation is a homomorphism $h \colon \mathcal{SL} \rightarrow A$, where A is an algebra $A \in Var_{\vdash}$, which is the variety characterising the logic **L**. As we have seen in Subsection 1.3.3, Łukasiewicz logic is characterised by the variety of MV-algebras, whereas Gödel logic by the variety of Gödel algebras. A relation $\preceq \subseteq \mathcal{SL}^2$ is a *pairwise* **L**-*valuation* if and only if it is a congruent preorder and it satisfies for all sentences $\phi, \psi \in \mathcal{SL}$

(A.1) $\vdash \phi \Rightarrow \phi \sim \top$,

(A.2) $(\phi \rightarrow \psi) \sim \top \Leftrightarrow \phi \preceq \psi$.

Pairwise valuations for $\mathbf{Ł} = \langle \mathcal{SL}, \vdash_{Ł} \rangle$ and $\mathbf{G} = \langle \mathcal{SL}, \vdash_G \rangle$ are obtained by replacing (A.1) in the previous definition with

(A.1′) $\vdash_{Ł} \phi \Rightarrow \phi \sim \top$,

(A.1″) $\vdash_G \phi \Rightarrow \phi \sim \top$,

respectively.

Given this, through Theorem 2.22, we can derive that any Łukasiewicz (Gödel) pairwise valuation can be represented by a pointwise valuation for Łukasiewicz (Gödel) logic, namely a truth-functional assignment in the support set of an algebra in the variety characterising the Łukasiewicz (Gödel) logic:

Corollary 3.1. *For every pairwise* **Ł***-valuation there exists at least a pointwise valuation $h \colon \mathcal{SL} \rightarrow A$, where A is an MV-algebra, such that for all $\phi, \psi \in \mathcal{SL}$*

$$\phi \preceq \psi \Rightarrow h(\phi) \leq h(\psi).$$

Corollary 3.2. *For every pairwise* **G***-valuation there exists at least a pointwise valuation $h \colon \mathcal{SL} \rightarrow A$, where A is a Gödel algebra, such that for all $\phi, \psi \in \mathcal{SL}$*

$$\phi \preceq \psi \Rightarrow h(\phi) \leq h(\psi).$$

However, for those logics more significant representation results can be obtained. Indeed, something more can be done

(i) at the level of the representation: we can obtain numerical, and thus genuinely quantitative, representations in the interval $[0,1]$;

(ii) at the level of the definitions: having fixed a logic, we can explore alternative axiomatizations for the relation *more* or *less true than* and discuss the philosophical desirability thereof.

These two points are object of the next subsections.

3.1.1 Pointwise valuations and representation

In order to show how we can improve on the strength of representation theorems, I focus on Łukasiewicz logic. We know from Theorem 2.22, that given an admissible family of pairwise Ł-valuations $\mathcal{A} \subseteq \mathcal{P}$, for every $\preceq$ in $\mathcal{A}$ there exists at least one homomorphism $h\colon \mathcal{SL} \to A$ where $A \in Var_{\vdash_{\text{Ł}}}$, representing the preorder. More specifically, A is an MV-algebra endowed with its natural order $\leq$ which preserves $\preceq$.

MV-algebras have been defined to capture the properties of the real unit interval $[0,1]$ equipped with the operations $\neg x = 1-x$ and $x \oplus y = \min\{1, x+y\}$. The real unit interval with the described operations is the standard truth-value semantics for Łukasiewicz logic. So, a pointwise **L**-valuation in $[0,1]$ is a homomorphism $h\colon \mathcal{SL} \to [0,1]_{MV}$, namely such that for all $\phi, \psi \in \mathcal{SL}$

1. $v(\bot) = 0$,
2. $v(\phi \oplus \psi) = \min\{1, v(\phi) + v(\psi)\}$.

The algebra $([0,1], \neg, \oplus, 0)$ is usually referred to as the *standard MV-algebra*. And, indeed, it plays a distinguished role: any other MV-algebra, as long as it is non-trivial, can be mapped to it, as stated in Hölder's Theorem for MV-algebras (Theorem 1.18 on page 27):

Let A be a non-trivial MV-algebra. The followings hold

(i) there exists at least one homomorphism $m\colon A \to [0,1]_{MV}$.

(ii) if A is linearly ordered then m is unique.

(iii) if A is linearly ordered and Archimedean then m is one-to-one. And if m is one-to-one then A is linearly ordered and Archimedean.

As already stated, these peculiarities of Łukasiewicz logic can be exploited in order to strengthen the representation theorem. Starting with pairwise valuations for the logic we can obtain functions representing the truth rankings which assign to sentences numbers in the real unit interval, thus acting as *measures* (see Section 3.2 for a discussion). The codomain of the representing function is no longer a generic set of truth values equipped with compatible operations, rather, the representation result is genuinely quantitative.[1]

Theorem 3.3. *For every non-trivial pairwise Ł-valuation there exists at least one pointwise Ł-valuation* $h\colon \mathcal{SL} \to [0,1]_{MV}$ *such that for all* $\phi, \psi \in \mathcal{SL}$

$$\phi \preceq \psi \Rightarrow h(\phi) \leq h(\psi).$$

This follows as corollary from Theorem 2.22 and Hölder's Theorem for MV-algebras. The former assures the existence of a homomorphism from sentences to an MV-algebra representing the pairwise valuation, the latter from the MV-algebra to the standard MV-algebra. By composing those homomorphisms and by noticing that the order is preserved, we get the desired pointwise valuation in $[0,1]$. Observe that in order to fulfil the non-triviality condition, it suffices to assume that pairwise valuations are non-trivial, by assuming for example that $\perp \prec \top$, expressing the fact that absolutely false sentences are strictly less true that absolutely true sentences.

Alternatively, we can use Theorem 1.18 in order to directly improve on the construction of Theorem 2.22 stating that for any pairwise **L**-valuation there exists at least one pointwise **L**-valuation representing it, and thus showing that there exists at least one pointwise valuation in $[0,1]$. Recall that representation theorems of this kind are obtained by considering congruences over the algebra and performing quotients. The first (and smallest) congruence we consider is the congruence induced by $\preceq$, namely $\sim$. The quotient algebra $\mathcal{SL}/\sim$ is an algebra in the variety of the logic and the canonical map from the set of sentences to it is already a pointwise valuation for the logic carrying all the informations contained in $\preceq$. In order to obtain a representation in a set of values which has additional desirable properties, for example linearity, further congruences must be considered. These will be "larger" congruences, namely extensions of $\sim$ which add more information by identifying new pairs of sentences. These congruences make the quotient set more and more precise. This process is, in general, non-unique and non-constructive, because a single congruence among a set of possible congruences must be chosen. In the special case of Łukasiewicz logic the maximal amount of information is obtained by considering congruences corresponding to maximally consistent theories (those that, intuitively, cannot be further extended). The existence of maximally consistent theories for quotienting sentences guarantees the possibility of being maximally precise in the assignment of values,

[1] We have seen that a similar strong characterisation can be obtained for classical logic. Corollary 2.27 on page 47 states that classical valuations are represented by valuations in $\{0,1\}$.

up to the point that the valuation can be done in the set $[0,1]$.[2] In the case of Łukasiewicz logic we can consider special congruences, that is *maximal ideals* (see Definition 1.20), which assure the existence of an embedding of the quotient algebra in the standard MV-algebra. Let $\preceq$ be a non-trivial pairwise **Ł**-valuation and $\mathcal{SL}/{\sim}$ be the quotient algebra generated by its symmetric part $\sim$. This is an MV-algebra, so the quotient by maximal ideals generates algebras which are linearly ordered and Archimedean, and, being such, it can be embedded in $[0,1]$ (see the proof of Theorem 1.18).

This suggests that a even stronger characterisation can be obtained by adding further axioms on pairwise valuations in order to make the quotient algebra $\mathcal{SL}/{\sim}$ being linear and Archimedean as required in points (ii) and (iii) of Hölder's Theorem for MV-algebras.

Definition 3.4. *A* linear *pairwise* **L**-*valuation is a pairwise* **L**-*valuation such that for all* $\phi, \psi \in \mathcal{SL}$

(A.L) *Either* $\phi \preceq \psi$ *or* $\psi \preceq \phi$.

An Archimedean *pairwise* **L**-*valuation is a pairwise* **L**-*valuation such that for all* $\phi, \psi \in \mathcal{SL}$

(A.A) *if* $\phi \preceq \psi$ *and* $\phi \nsim \perp$ *then* $\exists n$ $\underbrace{\phi \oplus \cdots \oplus \phi}_{n} \npreceq \psi$.

These new axioms say that every pair of sentences are comparable with respect to their truth (A.L) and that no sentence is *infinitely* less true than the others (A.A). Notice that if $\preceq$ is linear and Archimedean also $\leq_{\sim}$ defined over $\mathcal{SL}/{\sim}$ is such. As a consequence, $(\mathcal{SL}/{\sim}, \leq_{\sim})$ can be uniquely embedded in $[0,1]$:

Corollary 3.5. *For every non-trivial* linear *pairwise* **Ł**-*valuation there exists a* unique *pointwise valuation* $h\colon \mathcal{SL} \to [0,1]$ *representing it.*

Corollary 3.6. *For every non-trivial* linear and Archimedean *pairwise* **Ł**-*valuation there exists a unique pointwise valuation* $h\colon \mathcal{SL} \to [0,1]$ *such that for all* $\phi, \psi \in \mathcal{SL}$:

$$\phi \preceq \psi \Leftrightarrow h(\phi) \leq h(\psi).$$

If there are pairs of sentences which are not comparable with respect to the relation $\preceq$, namely sentences $\phi, \psi \in \mathcal{SL}$ such that $\phi \npreceq \psi$ and $\psi \npreceq \phi$, then the function in $[0,1]$ representing $\preceq$ will be in general non-unique. Intuitively, this is because values in $[0,1]$ are linearly ordered by the natural order of the real numbers. So, there will be different possible "completions" of a preorder which is silent about how to compare certain pairs. More precisely, the order induced by $\preceq$ over the quotient and

[2]The crucial role of maximally consistent theories has been pointed out in Marra (2014).

obtained by identifying sentences which are *as true as* each other, being a partial order, has more than one maximal ideal and there will be a representing function for each ideal in the set. If the truth ranking is linear, then there is a unique maximal ideal M and a unique canonical map q_M from $\mathcal{SL}/\!\sim$ to the quotient. Accordingly, the composition with the unique map into $[0, 1]$ will give a unique representation.

Moreover, numerical degrees of truth with addition and the natural ordering relation form an Archimedean ordered group. This means that every two positive real numbers are bounded by integer multiples of each other. Hölder proved that every Archimedean ordered group is isomorphic to a subgroup of the real numbers. In the MV-algebraic version, this theorem says that every linear Archimedean algebra MV-algebra can be embedded uniquely into the real unit interval. When it comes to the truth ordering over the sentences, also provided that it is linear, we cannot exclude that there are non-Archimedean elements: sentences which are *infinitely* more or less true than some other, that intuitively cannot be reached by further application of disjunctions or conjunctions (which raise and lower the truth value, respectively). However, this can be excluded by assuming Axiom (A.L), which makes the ordering $\leq_\sim$ Archimedean and guarantees the existence of an injective map into the reals, giving a strong representation result as stated in Corollary 3.6.

The desirability of these additional axioms can be questioned. That is to say, provided that we know the domain of application, we can discuss whether it makes sense to take the relation *more* or *less true than* as being linear or Archimedean in that domain (we shall see an example of this discussion for the case of vagueness in Section 3.3). This is one of the main advantages of reasoning in terms of pairwise valuations instead of pointwise valuations: some questionable properties of degrees of truth become, at any rate, dispensable. Truth from comparisons is a way for taking a step back from the formalisation in terms of numerical values and it makes clear that the mathematical structure embedded in the set of truth values somehow exceeds what we actually need for talking about many-valuedness and graded truth (see Section 3.2).

Representation theorems state the existence of morphisms between qualitative and quantitative structures, or put in other way, the existence of a functional object representing an ordering. These theorems yield functions which are unique up to certain classes of transformations depending on the axioms governing the ordering. This means that, given a function representing an ordering, every transformation of a specific kind of the function still represents the ordering. Recall that the kind of transformations available determine different types of measurement scale: ordinal, interval and absolute scale for monotone, linear and identity transformations, respectively (see Section 1.2.1). It is of interest to investigate which sort of uniqueness result we get when we prove representation theorems for truth from comparisons. We shall see that this depends on the chosen logic, and Łukasiewicz and Gödel make an interested case in point.

Let $\preceq$ be a pairwise valuation for Łukasiewicz logic and $h\colon \mathcal{SL} \to [0,1]$ a pointwise **Ł**-valuations representing it. A transformation of h is a map $k\colon [0,1] \to [0,1]$. In order for the transformation to be admissible, the resulting map should at least still preserve the ordering, that is

$$\phi \preceq \psi \ \Rightarrow k(h(\phi)) \leq k(h(\psi)).$$

If this were enough, then any weakly monotone map keeping fixed 0 and 1 would do the job. However, for our purposes, in order to be admissible, the transformation should also yield a function $k \circ h$ that is a valuation for the logic at stake.[3] The question is, thus, whether there are available transformations which are admissible in this sense. In the case of Łukasiewicz, we already proved (Corollary 3.5) that for any non-trivial linear pairwise **Ł**-valuation there exists a unique pointwise valuation $h\colon \mathcal{SL} \to [0,1]$ representing it. The uniqueness stated here is absolute, meaning that the only permissible transformation available for Łukasiewicz valuations is the identity function.[4]

In order to show to what extent this is a peculiarity of Łukasiewicz logic, I shall briefly consider the case of Gödel logic. We already know that for every pairwise **G**-valuation there exists at least one representing pointwise valuation $h\colon \mathcal{SL} \to A$, where A is a Gödel algebra. The proof is analogous to the proof of Theorem 3.3. A stronger representation result with pointwise valuations evaluating in [0,1] is obtained by building on the following

Theorem 3.7. *Let A be a non-trivial Gödel algebra, if A is linearly ordered and its universe is countable, then there exists at least one injective homomorphism to* $[0,1]$.[5]

Linearly ordered Gödel algebras can be obtained by quotienting the algebra $\mathcal{SL}/{\sim}$ modulo *prime filters* (see Definition 1.20). Accordingly, for every pairwise **G**-valuation there exists at least one pointwise **G**-valuation $h\colon \mathcal{SL} \to [0,1]_G$ such that for all $\phi, \psi \in \mathcal{SL}$

$$\phi \preceq \psi \Leftrightarrow h(\phi) \leq h(\psi).$$

It is worth noticing that the representing function whose existence is guaranteed by the theorem in the case of Gödel algebra is not absolutely unique:

Proposition 3.8. *Let $\preceq$ be a pairwise* **G***-valuation and v a pointwise* **G***-valuation representing it. Then any* strictly monotone transformation *of v which keeps 0 and*

[3]We are not interested in functions simply representing the truth ranking, they should also be logical valuations.

[4]This has been already informally noticed by Keefe (2000) in Chapter 5.

[5]The proof is an easy variant of the standard result (Rosenstein, 1982, Theorem 2.5).

1 fixed still represents $\preceq$. These are the only transformations available.[6]

Linearity of the preorder, namely the assumption that any two sentences are comparable with respect to truth, leads to unique truth-value assignments which are *absolutely* unique for Łukasiewicz logic and unique *up to strictly monotone transformations* for Gödel logic. This difference is related to the nature of operations of $[0,1]_G$, which, unlike those of $[0,1]_{MV}$, are preserved under strictly monotone transformations. In other words, Łukasiewicz logic makes an essential use of the real numbers, whereas Gödel logic is sensitive just to the truth ordering. Such a formal distinction stimulates an important clarification, crucial for the forthcoming discussion: degrees of truth are not always alike. Many of their properties depend upon the logical system that generates them.

3.1.2 Pairwise valuations for infinite-valued logics

In this section I explore possible alternative axiomatizations for pairwise valuations for infinite-valued logics, in particular for Łukasiewicz logic. I keep the structural properties fixed: since they are a formalisation of the idea of truth from comparisons, pairwise valuations are congruent preorders. I rather focus here on the assumptions of soundness and compatibility with the implication, that is

(A.1) $\vdash \phi \Rightarrow \phi \sim \top$,

(A.2) $(\phi \to \psi) \sim \top \Leftrightarrow \phi \preceq \psi$.

The right-to-left direction of Axiom (A.2), i.e.

$$\phi \preceq \psi \Rightarrow (\phi \to \psi) \sim \top,$$

might be considered counterintuitive or problematic. Indeed it might not seem totally convincing that a truth relation between sentences, for example the fact that ϕ is less true that ψ, determines the subsistence of an implication, for example the fact that the implicative sentence $\phi \to \psi$ is absolutely true. Put in another way, one might be willing to accept that a sentence is less true than another, and yet have no reasons for accepting that one implies the other, especially when the sentences are unrelated in their content.[7]

[6] This fact may be considered folklore. One possible proof can be obtained using the notion of *equivalent assignments* introduced in Codara et al. (2009), see Definition 2.1. Proposition 2.4, which is couched in algebraic language, essentially subsumes Proposition 3.8 above. In turn, the proof of Proposition 2.4 in Codara et al. (2009) is reduced to D'Antona and Marra (2006), Remark 2 and Proposition 2.4. A fully elementary proof is also possible, but I do not include details here.

[7] Weatherson (2005) points out that "It just doesn't seem true that intuitions about whether A is truer than B fit together nicely with intuitions about whether *If B then A* is determinately true." He takes this implausibility as a further reason to drop linearity (of the truth ordering).

However, being an issue of relevance, the critical aspect of Axiom (A.2) does not emerge only in the context of pairwise valuations. Non-bivalent contexts inherit problematic aspects related to the materiality of the implication, already present in classical contexts. The implausibilities typical of material implication generalise to a framework in which there are degrees of truth. If there are reasons for rejecting it in this context, then it should be already refused at the classical level, for there are no reasons why the generalisation from two truth values to degrees of truth should make the objection as to the interpretation of material implication more compelling. Also, pairwise valuations are in general not linear: if two sentences are thus unrelated to make the implication of one from the other unacceptable, than also the truth ranking can be silent about them, so that they can result incomparable with respect to their truth. This is allowed in the framework.

For the reasons just stated, I do not consider problematic the right-to-left direction of axiom (A.2). Nevertheless, in what follows I shall show that it can be dropped in favour of weaker conditions. Although, it cannot be completely disposed, since it follows from those. In the alternative formalisation I shall propose, the right-to-left direction of axiom (A.2) is still derivable, namely the truth ranking modelled through pairwise valuations coincides with the order given by the true implications. This guarantees that all the results apply.

Consider the following alternative axiomatization:

Definition 3.9. *A relation* $\preceq \subset \mathcal{SL}^2$ *is a* pairwise Ł-valuation *if and only if it is a preorder and it satisfies for all sentences* $\phi, \psi \in \mathcal{SL}$

(A.1) $\vdash_{Ł} \phi \Rightarrow \phi \sim \top$,

(A.2!) $(\phi \to \psi) \sim \top \Rightarrow \phi \preceq \psi$,

(A.3) $\phi_1 \succeq \phi_2,\ \psi_1 \preceq \psi_2 \Rightarrow \phi_1 \to \psi_1 \preceq \phi_2 \to \psi_2$.

Axiom (A.3) is a monotonicity condition stating that implication is non-increasing in the first component and non-decreasing in the second. In other words, the more true the antecedent is, the less true the whole implication is; and the more true the consequent is, the more true the whole implication is. This matches with the idea that an implication is true when the antecedent is less true than the consequent, without explicitly assuming that.

The important structural properties of the relation remain unvaried:

Proposition 3.10. *1.* $\preceq$ *is bounded,*

2. $\sim$ *is an equivalence relation,*

3. $\sim$ *is a congruence, namely*

3.a $\phi \sim \psi \Rightarrow \neg\phi \sim \neg\psi$,

3.b $\phi_1 \sim \phi_2, \psi_1 \sim \psi_2 \Rightarrow \phi_1 \star \psi_1 \sim \phi_2 \star \psi_2 \quad \star \in \{\rightarrow, \oplus, \odot\}$.

Also, since connectives are inter-definable and $\bot$ and $\top$ are the smallest and greatest element of truth ranking, monotonicity, or compositionality, conditions for defined connectives can be derived from the axioms.

Proposition 3.11. *If $\preceq$ satisfies (A.1), (A.2!) and (A.3) then*

1. $\phi \preceq \psi \Rightarrow \neg\psi \preceq \neg\phi$,
2. $\phi_1 \preceq \phi_2,\ \psi_1 \preceq \psi_2 \Rightarrow \phi_1 \oplus \psi_1 \preceq \phi_2 \oplus \psi_2$,
3. $\phi_1 \preceq \phi_2,\ \psi_1 \preceq \psi_2 \Rightarrow \phi_1 \odot \psi_1 \preceq \phi_2 \odot \psi_2$.

These conditions are counterparts of the compositionality clauses over pointwise valuations: the comparisons in truth values of complex sentences depends on the comparisons among the components. This will offer a starting point for discussing the *truth-functionality objection*, which is one of the objections raised against degree-theoretic approaches to vagueness (see Section 3.3).

The right-to-left direction of (A.2) can still be proved:

Proposition 3.12. *If $\preceq$ is a preorder and satisfies (A.1), (A.2!) and (A.3) then for all $\phi, \psi \in \mathcal{SL}$*

$$\phi \preceq \psi \Rightarrow (\phi \rightarrow \psi) \sim \top.$$

Proof. Assume $\phi \preceq \psi$. By reflexivity of $\preceq$ we have $\phi \preceq \phi$. Then by Axiom (A.3) we can conclude that $\phi \rightarrow \phi \preceq \phi \rightarrow \psi$. Since $\vdash \phi \rightarrow \phi$, by Axiom (A.1) we get $\phi \rightarrow \phi \sim \top$. By transitivity we conclude $\phi \rightarrow \psi \succeq \top$, and by boundedness that $\phi \rightarrow \psi \sim \top$. $\square$

In what follows I focus on Axiom (A.1) and possible relaxations thereof. I discuss and compare a number of reasonable routes that can be taken in this direction. In Chapter 2, I moved from a very abstract framework to one in which the underlying logic was fixed. The definitions and the results were meant to be general enough to encompass different many-valued logics. This was achieved by explicitly introducing the logic, i.e. the deducibility relation, in the axiomatization of pairwise valuation, by means of the Axiom (A.1)

$$\vdash \phi \Rightarrow \phi \sim \top.$$

Recall that this axiom expresses a soundness condition: pairwise valuations evaluates as absolutely true all (but not only!) the sentences provable in the logic. The spirit and the aim of the investigation into pairwise valuations was to provide a pairwise alternative to the standard truth-value semantics rather than justifying the underlying logic. Accordingly, the logic and the intuitively desirable condition of soundness were assumed as starting point. Nevertheless, the semantic character of pairwise valuations is not in agreement with the presence of syntactic elements. Therefore, once a specific logic has been fixed, it is natural to investigate the relaxation of Axiom (A.1).

Following the axiomatization originally proposed, we said that a pairwise valuation for Łukasiewicz logic is a pairwise valuation which satisfies

(A.1) $\vdash_{Ł} \phi \Rightarrow \phi \sim \top$,

(A.2) $(\phi \to \psi) \sim \top \Leftrightarrow \phi \preceq \top$,

or a weakening of Axiom (A.2) as described before. Since the logic is finitely axiomatizable (see Section 1.3.3), we can replace Axiom (A.1) with the following list of conditions:

(B.1) $\bot \preceq \phi$,

(B.2) $\phi \preceq \psi \to \phi$,

(B.3) $\phi \to \psi \preceq (\psi \to \chi) \to (\phi \to \chi)$,

(B.4) $\phi \to \psi \preceq \neg\psi \to \neg\phi$,

(B.5) $(\phi \to \psi) \to \psi \preceq (\psi \to \phi) \to \phi$,

(A.2) $(\phi \to \psi) \sim \top \Leftrightarrow \phi \preceq \psi$.

(B.1)–(B.5) are the transposition of a particular axiomatization of Łukasiewicz logic in the language of truth from comparisons. Conditions (B.1)–(B.5) can be justified as axioms for pairwise valuations just after a previous commitment to Łukasiewicz logic, whereas the single axiom (A.1) is justifiable as desirable requirement over the intuitive notion of *more* or *less true than*, being a condition of soundness with respect to the underlying logic, that is a condition stating that all the theorems of the logic are absolutely true according all pairwise valuations. This soundness requirement can be unpacked by building on the definition of *theorem* of a logical system. The set of theorems is inductively generated: axioms are theorems, and sentences derivable from axioms by means of inference rules are also theorems. Nothing else is a theorem. So, instead of directly assuming that every theorem of Łukasiewicz logic is absolutely true according the pairwise valuation (Axiom (A.1)), we assume that

(i) every axiom is absolutely true: for instance, (B.1) and (A.2) give that $(\bot \to \phi) \sim \top$. By reasoning in the same way for conditions (B.2)–(B.5), we have that all the axioms of Łukasiewicz logic are absolutely true.

(ii) Modus Ponens is sound: if the premisses are absoultely true also the conclusion is such, namely

$$(\phi \to \psi) \sim \top, \phi \sim \top \Rightarrow \psi \sim \top.$$

This is an immediate consequence of Axiom (A.2), transitivity and boundedness.

From this, with a simple induction, it can be proved that all the sentences provable in Łukasiewicz logic are absolutely true according to the pairwise valuation, namely

(A.1) $\vdash_{Ł} \phi \Rightarrow \phi \sim \top$.

Therefore, if we define pairwise valuations for Łukasiewicz logic as a pairwise valuations satisfying conditions (B.1)–(B.5) and (A.2), then all the main results stated before follow.

Axiomatization in terms of (B.1)–(B.5) is the most conservative alternative to (A.1). This axiomatization keeps the feature of being syntactically oriented. A different possible route, with a more semantic flavour, is to assume properties of the connectives with respect to truth ranking. Properties that would then, in the pointwise representation, translate as compositional clauses, that is truth-functionality constraints. For example, a list of properties over $\neg$ and $\preceq$ such that a representing compatible function is such that $v(\phi) = 1 - v(\phi)$. Let us assume that the aim is to obtain a infinite-valued valuation respecting Łukasiewicz clauses; or, to put it differently, to define a pairwise counterpart of Łukasiewicz valuations. Then, this can be done by looking at the equational properties characterising the pointwise algebraic semantics for the logic, MV-algebras in this case. Following Definition 1.17, we assume that the relation *no more true than* confers specific properties to the connectives:

(C.1) $\phi \oplus \psi \sim \psi \oplus \phi$,

(C.2) $\phi \oplus (\psi \oplus \chi) \sim (\phi \oplus \psi) \oplus \chi$,

(C.3) $\top \oplus \phi \sim \top$,

(C.4) $\bot \oplus \phi \sim \phi$,

(C.5) $\neg(\neg\phi \oplus \psi) \oplus \psi \sim \neg(\neg\psi \oplus \phi) \oplus \phi$,

(C.6) $\neg\neg\phi \sim \phi$,

(A.2) $(\phi \to \psi) \sim \top \Leftrightarrow \phi \preceq \psi$, or $(\neg\phi \oplus \psi) \sim \top \Leftrightarrow \phi \preceq \psi$,

and, as usual, that $\preceq$ is a congruent preorder. If $\preceq$ satisfies those conditions, then the quotient algebra $\mathcal{SL}/{\sim}$ is an MV-algebra whose natural order refines the order induced by $\preceq$.

As we have seen, Axiom (A.1) can be unpacked in different ways. What remains unchanged is the commitment with the underlying logic. In the starting abstract framework we have seen that being a congruent preorder is sufficient for a binary relation to be a pairwise valuation. But if we ask for a valuation of a specific logic, than we need further assumptions. I did not defend the plausibility of conditions (B.1)–(B.5) or (C.1)–(C.6) in terms of *more or less true* because assuming them rather then other conditions is not a matter of axiomatization of the intuitive notion of truth from comparisons, it rather depends on the chosen logic. In other words, at this stage, this investigation does not solve or remove the issue of the choice of the logic.

3.2 Degrees of truth explained away

Truth from comparison is a way to deal with many-valuedness and to avoid philosophical problems related to the status of truth values, especially non-classical truth values. When it comes to degrees of truth, the problem of the philosophical significance is more pressing and, therefore, the solution in terms of truth from comparison is more meaningful and the philosophical contribution more substantial. In this section I shall explain in details why it is so, by considering two different aspects: the nature of degrees of truth and their role as formal model of the informal notion of graded truth.

3.2.1 The nature of degrees of truth

The very idea of degrees of truth is sometimes considered problematic or unclear, or at any rate something requiring an explanation, a justification. For instance, Sainsbury (1995, p. 59) writes:

> A full defense of the degree of truth theory would require the consideration of a number of issues that I shall briefly mention. First, it is necessary to say something about what a degree of truth is. Second, some account must be given of the source and justification of the numbers that are to be assigned as degrees. Third, the full implications of the degree theory for logic must be set out and defended.

Graff (2000) points out "the absence of some substantial philosophical account of what degrees of truth are". I claim that the pairwise perspective adopted here and the resulting formal apparatus and results do provide such a philosophical account.

In Subsection 1.1.1, I distinguished three kinds of objections about the *nature* of degrees of truth: what are degrees of truth? how do we interpret the fact that a sentence is true to a certain degree? how can truth be measured? If we reason pairwise, it does not actually matter what degrees of truth are or how they should be interpreted. They are not in the picture. However, I showed that if a set of comparisons meets certain conditions then degrees of truth can be assigned to sentences in a compatible way. Through representation results, degrees of truth come back into the picture and this triggers a positive feedback on their nature and allows us to provide satisfactory answers to these questions.

In the pairwise conceptual and formal framework that I have proposed, degrees of truth are not primitive objects; rather, they arise as *measures*, that is as quantitative counterparts of a qualitative structure. Degrees of truth arise as numbers assigned in a compatible way with the truth ordering among sentences, they are formal instruments we use to *cardinalise* truth from comparisons. The fact that a sentence receives value 0.7 is not interpreted as an isolated fact. Together with the other assignments it forms a numerical relational structure which reflects the fact that some sentences are more or less true than others. In the framework of pairwise valuations, the interpretation of 0.7 is not explained in relation to the sentence that receives it. The value 0.7 encodes all the logical relations between the sentence to which is attached and the other sentences under consideration. Concerning the measurability of truth, I argue that what is going on is something analogous to measurement in other domains, where numerical measures are taken to be conceptually and formally founded on qualitative structures (see Section 1.2.1 and especially the case of utility theory). Analogously, degrees of truth are the endpoint of a process of measurement which elevates the qualitative to the quantitative.

Degrees of truth emerging from representation theorems are not always alike: they are different objects according to the strength of the characterisation and these differences should be taken into account in the philosophical discussion about their nature. That is why, in order to argue that the numerical aspect of mapping sentences into degrees of truth can be considered secondary with respect to the comparative aspect, which consists in ranking sentences by performing pairwise comparisons, I distinguish weak and strong representation results. Weak representation results, like Theorem 3.3, state the existence of numerical functions mapping sentences to the real unit interval that preserve (or agree) with the preorder over the sentences, namely for all $\phi, \psi \in \mathcal{SL}$

$$\phi \preceq \psi \Rightarrow v(\phi) \leq v(\psi).$$

Intuitively, there is no way back from the valuation to the ordering: if you know a truth-value assignment $v\colon \mathcal{SL} \to [0,1]$ then you have just a partial knowledge of the corresponding ordering, since it might be that $v(\phi) = v(\psi)$ while $\phi \prec \psi$. These results ensure that an ordering is *quantifiable*, even though it can not be identified with the numerical assignment. Numerical assignments are essentially more informative than the orderings, since they make possible finer distinctions, but still, numerical assignments are not meaningful *per se*, they rather reflect differences in comparisons. In this case, we can justify the use of numbers, as values of a measure, starting from plausible properties of the comparative notion.

Some additional conditions, like Archimedean axioms or certain continuity assumptions, are required in order to have an injective map, namely stating the existence of a unique infinite-valued valuation v such that for all $\phi, \psi \in \mathcal{SL}$

$$\phi \preceq \psi \Leftrightarrow v(\phi) \leq v(\psi).$$

Strong representation results of this kind, like Corollary 3.6, prove an equivalence between the qualitative and the quantitative side, intuitively, one can go back and forth from the ordering to the numerical assignment. In this case, linear Archimedean pairwise valuations and pointwise valuations are proven to be different formulations of the same mathematical object. Nevertheless, despite the *formal equivalence* between pairwise valuations of this kind and pointwise valuations, I defend the *philosophical priority* of the former. Comparative or pairwise judgements with respect to truth are philosophically more plausible and justifiable than real-valued valuations because they do not presuppose philosophically problematic and mysterious objects like degrees of truth. Moreover, the pairwise formulation permits to look at properties of the set of degrees as possible axioms for pairwise valuations (e.g. linerity and Archimedean assumption), and to interpret them as properties of the truth ranking (regardless to the discussion as whether they are desirable properties or not). Therefore, the relation *more* or *less true than* provides a philosophical underpinning for degrees of truth. This foundation, in turn, sheds light on the very nature of degrees of truth and provides us with the philosophical justification whose lack has been reported.

3.2.2 The role of degrees of truth

Clarifying the role of degrees of truth and how it evolves in the light of the qualitative foundation involves an investigation of the relations between formalisation and the subject matter being formalised. I take this subject matter being the informal notion of *graded truth*.

Some sentences of the everyday language are clearly true (e.g. 'Aristotle is mortal') and some sentences are clearly not true (e.g. 'Aristotle was born in Rome'). There are also sentences about which it is unclear whether they are true or not. Consider for example:

'Italy is shaped like a boot',
'Stanley Kubrick at the end of his life was bald'.

When it comes to evaluate the truth of such sentences, we can experience the *hedging response* ("Sort of", "almost") typical of borderline cases, with some hesitations and interpersonal disagreement. This has been taken as starting point or justification for generalising the standard bivalent account of truth and for considering truth as a partial notion.[8] Moreover, we can establish more-or-less relations among sentences with respect to their truth, for example we can acknowledge that 'Italy is shaped like a boot' is more true than 'France is shaped like a boot' or the sentence 'Stanley Kubrick at the end of his life was bald' is less true than 'Alfred Hitchcock at the end of his life was bald'. Accordingly, truth would not just be partial, but also susceptible of comparisons, capable of being ranked and thus graded. The just outlined cluster of everyday use, linguistic aspects and naive theoretic aspects is what I call the informal notion of graded truth.

Notice that I am considering graded truth as a legitimate and autonomous philosophical notion, distinguished both from degrees of truth, which are a possible mathematical model for it, and also distinguished from vagueness, which is its main context of application. Opponents to the meaningfulness of graded truth advocate the fact that truth being a gradable property is supervenient upon the gradability of the predicates contained in the sentences, which are in this case 'being shaped like' and 'being bald'. Fine writes about the vagueness of the truth predicate that "there can be no independent grounds for its having borderline cases" (Fine, 1975, p. 296). I ignore this aspect here, and I take graded truth as sufficiently clear and distinguishable from vagueness, regardless of its presumed philosophical dispensability.

Graded truth can be formalised in two different ways according to which preformal aspects of the informal notion is stressed: on the one hand there is the qualitative aspect, namely the fact that some sentences can be compared with respect to truth, with the true (false) sentences being more (less) true than all the others; on the other hand, there is the quantitative aspect, that is the fact that truth can vary in intensity, from none at all to some, to eventually all. These two aspects give rise to different formal models: pairwise valuations and pointwise infinite-valued valuations, respectively. The formal relations between pairwise and pointwise valuations have been extensively explored in Chapter 2 and in Section 3.1. This investigation has a philosophical feedback on the notions involved, as consequence of which also the role played by degrees of truth takes shape.

Pointwise infinite-valued valuations, as models of graded truth, introduce real numbers as indicators of the extent to which a sentence is true. Absolutely true and absolutely false sentences receive value 1 and 0, respectively. Values between 0 and 1 represent intermediate truth degrees indicating *how true* a sentence is, and, accordingly, given any two sentences which one is less (or more) true. This is meant to model the idea that sentences can be more or less true, namely the idea that truth is graded. After having considered, in the previous section, issues related to the *nature* of degrees of truth, I move to considerations about the *role* they play with respect to

[8] See Millgram (2009) for a defence of the pervasiveness of the phenomenon of partial truth.

the intuitive idea of graded truth. As a theoretical premiss to this discussion, recall the analysis of the role of formal methods put forward in Subsection 1.1.2, which was based on Cook (2002) and on the idea that there is a middle way between a descriptivist and instrumentalist attitude to formal models: the logic-as-modelling view, according to which some aspects of the model are representative of the phenomenon, whereas some others are merely artefacts (representors versus artefacts distinction).

The non-descriptive nature of degrees of truth has been clearly pointed out by Edgington (1997):

> We may likewise idealize by representing [...] the degree to which a judgement is close to clearly true, by a number, between 1 (for clear cases, clear truths) and 0 (for clear non-cases, clear falsehoods). Likewise, this is of instrumental value; likewise, we must not forget that it is an idealization. [...] The numbers serve a purpose as a theoretical tool, even if there is no perfect mapping between them and the phenomena; they give us a way of representing significant and insignificant differences, and the logical structure of combinations of these. This use of the real numbers as a theoretical tool, whether or not they are isomorphic with the phenomenon they represent, is common scientific practice. (Edgington, 1997, p. 297.)

and by Cook himself:

> In essence, the idea is to treat the problematic parts of the degree-theoretic picture, namely the assignment of particular real numbers to sentences, as mere artefacts. (Cook, 2002, p. 237.)

Being such a precise and artificial device it seems naturally questionable that real numbers provide an accurate description of what is going on when we reason adopting an idea of graded truth, since there is nothing in the intuitive idea of graded truth involving numbers, or, even less, real numbers. So, degrees of truth, as mathematical model for graded truth, are seen as an idealisation, theoretical tools with an instrumental role.

If not degrees of truth, what is representative of the graded truth? Cook claims that *verities* are, namely intermediate degrees of truth between falsity and truth. Sentences have verities, and these verities are not real numbers, but they are modelled by real numbers, in the same way in classical logic falsity and truth are modelled by 0 and 1.

> Truth comes in degrees. Thus, the fact that degree-theoretic semantics represents truth as coming in more varieties than the traditional two

> (absolute truth and absolute falsity) is representative; in other words, the assignment of verity is a representor, and there are real verities in the world. We use the real numbers to model these verities, however, as a matter of convenience, and many (but not all) of the properties holding of them are artefactual. (Cook, 2002, p. 239.)

Artefactual are for example small differences in real numbers. Edgington and, more explicitly, Cook locate the artificiality in the fact that a particular number, rather than a any other number close to it, is assigned to sentences. The idea is that small differences in real number assignments should not matter or, at any rate, should not be representative of the phenomenon of graded truth. "The idealisation must be robust enough to be independent of small numerical differences" (Edgington, 1997, p. 297). The proposed solution consists in considering small differences in degrees of truth as not always representative of the phenomenon of graded truth.

This view encounters at least three major difficulties. First, what kind of objects are verities, if they are not real numbers? Cook introduces them as a generalisation of the notion of truth values, but nothing is said about their nature or their properties. As a consequence, ontological problems related to the philosophical status of non-numerical truth values strike back. Secondly, as Keefe (1998, 2012) objects, numbers are not arbitrary in the way Edgington and Cook claim, since in virtue of the numerical framework and the definitions of the connectives they are committed to unique values for sentences. The third difficulty, related to the second, is the lack of a general strategy for determining in a natural way which parts of the model are essential and which are artefacts. Smith (2012) suggests that the modelling perspective on a formal system works only if there is a part of systems which plays the descriptive role. Accordingly, specifying which aspects of the model are artefacts and which are not is a fundamental requirement for a model to be useful. Cook himself notices that in his proposal there is the risk of considering as artefacts just the parts of the model emerging as problematic, without a general strategy for drawing a line.

Some properties of real numbers are taken as representative of actual properties of verities, for example the fact that they are more than two and the fact that they are ordered – nothing is said about the fact that they are *linearly* ordered. The ordering is representative, but some inequalities, the ones symbolising large differences between two numbers, are representative of a real difference in verity; whereas some others, symbolising small differences, are "often" artefacts. For example, differences in value among sentences generated by logical clauses of compositions are representative even though small. This happens because the logical relations between complex sentences and their components are taken to be representative.[9] Nevertheless, there is no systematic way for determining that these are all and only the meaningful small differences.

[9] See the Arbitrarily Close Verity Theorem (Cook, 2002, p. 244).

There might be a strictly logical way for determining the role of degrees of truth, and in particular for determining which parts of the model matter: looking at the invariance properties of truth-degree assignments. The uniqueness results for Gödel and Łukasiewicz logic illustrated in the previous section make a case in point. If Łukasiewicz logic is in the background, given the nature of the operations used for computing the truth values of complex sentences, there are no admissible transformations for logical valuations. This can be rephrased by saying that no difference in value, however small, is irrelevant. Whereas Gödel logic has truth-functionality clauses that make the assignments of truth valuses immune to order-preserving transformations which keep 0 and 1 fixed. In order words, in Gödel logic differences in truth value do not matter: values can be shifted monotonically. Cook (2002) adopts Edgington's rules for sentential connectives, according to which negation is computed as follows: $v(\neg\phi) = 1 - v(\phi)$. The truth-functionality clause of the negation rules out monotone transformations, because $v(\neg\phi) = 1 - v(\theta)$ implies that the only transformations $f\colon [0,1] \to [0,1]$ are such that $f(0.5) = 0.5$. In other words, the only option consists in shifting the values monotonically, keeping both the endpoints and 0.5 fixed. This means that a transformation on $[0,1]$ transforming 0.51 in 0.99 would be admissible, whereas a transformation sending 0.49 in 0.51 would be inadmissible, violating the idea that large differences are always essential and small differences are sometimes not. Therefore, to look at the available transformations on $[0,1]$ is not compatible with the idea of distinguishing small and big differences as proposed in the paper. There should be extra-logical reasons, perhaps based on our understanding of what counts as small. But it is clear that this leaves room for arbitrariness and makes the proposal unsatisfactory.

I take from Cook the taxonomy of the possible roles degrees of truth can play with respect to graded truth, but I do not share the diagnosis. I propose a different solution based on the idea of truth from comparisons. If we accept pairwise valuations as model for graded truth, then we can provide a satisfactory account of the role of degrees of truth. In a nutshell, on this view, the truth ordering among sentences is taken to be descriptive of the phenomenon of graded truth and degrees of truth to be mere artefacts with an instrumental value.

The comparative aspect is an essential part of the phenomenon of graded truth. For example, if we say that the sentence 'Italy is shaped like a boot' is somewhat true we do not have directly in mind an intensity or an extent to which it is so. We can acknowledge that this sentence has somehow an intermediate status: it is more true than the definitely false sentence 'The sun is shaped like a boot' and less true than the definitely true sentence 'A boot is shaped like a boot'. We can also collocate it on an imaginary scale and say for example that it is pretty much true by determining its relative position with respect to other sentences expressing possible attributions of the predicate 'being shaped like a boot', or even with sentences containing other predicates. I describe graded truth as the idea that there are intermediate sentences with respect to truth and falsity and some sentences are more or less true than other sentences. This is captured by the fact that in the formal model we define a binary relation ranging over the set of sentences which is transitive and has tautologies and contradictions as upper and lower bound, respectively. That is why, I consider this part of the model descriptive of the fact that we compare sentences with respect to their truth.

All other formal properties of the relation are modelling choices, more than descriptions of the phenomenon of graded truth. For example, reflexivity is such a choice since I could have equivalently taken a strict order as formal counterpart of graded truth. Other structural properties, such as for example linearity, are such that one can argue for or against them. Among non-structural properties, there are some which presuppose a logical system, like the soundness condition, and some which express relation with connectives, for example congruence or monotonicity. The plausibility of these properties can be defended given the notion they are meant to model, inasmuch as these axioms, in their general versions, express plausible interactions of pairwise valuations with the underlying logical system or with the logical structure of the language, whatever they are. Nevertheless, any descriptivist claim should be substantiated with a previous justification of these pieces of formal language as description of the corresponding informal notions and would require a totally different approach more in the spirit of experimental philosophy. The formal properties I am choosing as axioms have rather a normative character. They are a logical model for the relation *more* or *less true than* which, however plausible, is at any rate, an idealisation.[10]

We have seen that truth rankings over sentences thus formalised can be represented numerically, that is to say, real numbers can be assigned to sentences in a way that is compatible with the ordering. These numerical assignments are artefacts of the model, namely they do not represent something really occurring in the phenomenon. Here is a more radical view with respect to Cook's. On the present account, the artificiality does not lie in the assignment of a particular real number instead of a number sufficiently close to it; the artificiality rather lies in the assignment of numbers *tout court*. They are numerical truth values which arise as possible cardinalisations of the relation *no more true than* (see the previous subsection).

There is no need here to postulate new entities like verities and there is a systematic way for distinguishing representors and artefacts, such as a way for distinguishing the descriptive part and the instrumental part of the model. Moreover, thanks to the previous investigation, the role of degrees of truth takes shape. They do not carry

[10]Later I shall propose to consider the double nature of the informal notion of graded truth by distinguishing a linguistic aspect and a naive theoretical aspect. Once this distinction is in place, we can refine the dichotomy representors/artefacts and conclude that there is a sense in which the properties that I call here "modelling choices" are descriptive. They are representative of the naive theoretical aspect which is part of the informal concept of graded truth (see Subsection 3.3.1).

per se much philosophical weight or significance since they do not represent any actual aspect of the phenomenon of graded truth. However, it is convenient that the model contemplates degrees of truth. They are instrumentally useful for applications in diverse fields, ranging from linguistics to artificial intelligence. But also, and more interestingly here, they have an important logical role. Numerical degrees of truth allows for logical and mathematical manipulations, for example the value of complex sentences can be easily computed. Also, degrees of truth highly increase the expressive power of the model and allow for more sophisticated judgements. Degrees of truth make it possible to express intensities in truth comparisons, namely not just that a sentence is more true than another, but also how much more true it is.

This leads to an important point: do distances in truth values matter? Cook's answer is that some are representative of real differences in verity and some are not. In particular he argues that some small differences in truth values are merely artefacts of the model. I illustrated some philosophical and logical difficulties that this view encounters. It might seem that the present account, being focused on comparisons, ignores throughout intensities or distances in truth values. This is not the case because the distinction ordinal/cardinal does not overlap with the distinction qualitative/quantitative (which in my terminology would be pairwise/pointwise). The conflict 'metric intuition' versus 'mere-ordering intuition' is already present at the level of degrees of truth. Some degree theorists (for example Cook (2002), Smith (2008)) argue for the importance of distance considerations and assign them a crucial philosophical value. However, there are degree theorists maintaining that the ordering alone matters (for example Goguen (1969); Machina (1976), Machina (1976)). A logical system should be chosen accordingly. We have seen that there is a precise logical sense in which a degree-theoretic model may or may not ignore distances in truth values: in Łukasiewicz logic small differences do matter, whereas in Gödel logic differences in truth values do not matter, just the ordering of the sentences matters. So, truth-value assignments can be ordinal, even though they are pointwise or quantitative. On the other hand, pairwise valuations can be cardinal to the extent that they can express more than simple comparisons. For example, the presence of a connective behaving like a sum suffices to make differences among values significant. Therefore, opting for a model that takes distances into account over a model that does not is a modelling choice which might be philosophically or mathematically motivated. Both ways are compatible with the comparative perspective.

In conclusion, pairwise valuations as a model for graded truth allow to combine the philosophical plausibility of the comparative aspect with the mathematical convenience of the numerical aspect. This foundation sheds light on the very nature and role of degrees of truth. In the next section I shall show how this philosophical account of degrees of truth can be brought to bear on the problem of modelling vagueness.

3.3 Vagueness and graded truth

There is general agreement (e.g. Keefe, 2000; Smith, 2008) on the fact that a predicate P is vague if it

1. has *borderline cases*: there are objects in its domain to which the predicate clearly applies and some to which it clearly does not apply, but there are also objects to which it is unclear whether or not the predicate applies;

2. has *blurred boundaries*: there is no sharp boundary between the things to which it applies and the things to which does not apply;

3. generates *sorites paradoxes*: a series of objects ranging from one which is clearly P to one which is clearly not P, but such that for any object in the series, if it is P, then so is the next object.

For example, the predicate 'being bald' is a vague predicate and Stanley Kubrick is a borderline case for it, as he falls neither clearly in not clearly out of the extension of the predicate. Borderline cases also generate disagreement among competent speakers: some competent speakers would judge that the predicate applies and some others that it does not. Saying that the predicate has blurred boundaries amounts to saying that it lacks a well-defined extension. On a scale of 'baldness', there is no sharp division between the bald people and the rest. Also, the predicate 'being bald' is susceptible to sorites paradoxes: intuitively, a single hair cannot make a difference between being and not being bald. So if we start with a person who is clearly not bald and we remove a single one of his hairs, he would still not be bald. In a finite number of steps, we would be forced to conclude that a person with no hair is not bald.

Other predicates behaving like 'bald' are for example 'tall', 'far', 'being a heap', 'red', 'young' 'nice', and so on. Vagueness is so pervasive that it is actually hard to find predicates that are not vague according to the previous characterisation. Examples are 'being a prime number', 'being older than 30 years', and so on. We cope quite well with vague predicates in every-day use, we understand each other and we also make inferences; and yet when we try to capture their behaviour in a logical system a very small set of plausible-looking assumptions is sufficient to lead to paradoxical conclusions. This is the philosophical challenge posed by vagueness. Among the philosophical theories that have been proposed to meet this challenge (see Keefe, 2000, for an overview.), I focus on degree-theoretic approaches, based on degrees of truth and infinite-valued logics. I shall point out the main objections that have been raised against this approach, and I shall show the extent to which the formal framework introduced in Section 3.1 together with the philosophical account put forward in Section 3.2 can be brought to bear on this proposal and can rehabilitate graded truth as a plausible theory for vagueness.

3.3.1 Degree-theoretic approaches to vagueness

Rejecting bivalence is a possible strategy for dealing with vagueness in a formal framework. A crucial assumption for the sorite paradox to arise is the fact that all sentences in a formal language, including vague sentences, are either true or false. A possible way-out consists thus in generalising the standard bivalent account of truth and considering truth as a partial notion. In such an approach, sentences containing borderline predications are taken to be neither true nor false and are assigned intermediate, non-classical truth values, that is values lying between full truth and full falsehood. When it comes to the question as to how many values we should admit, the most common positions are: three or infinitely many values. Degree-theoretic approaches toward vagueness opt for infinitely many values, which are then called degrees of truth.[11]

The conceptual core of degree-theoretic proposals is the notion of graded truth, described at the beginning of Subsection 3.2.2. I argue that graded truth has a twofold nature, typical of semantic concepts like truth: on the one hand, it is a linguistic phenomenon of natural languages related to the (first-order) vagueness of the predicate 'true'.[12] On the other hand, the linguistic aspect does not exhaust the concept of graded truth: there is a naive theoretical aspect which, as explained in Subsection 1.1.2, is a cluster of ideas and intuitions on graded truth, which may be incomplete, unstable, may lack internal coherence or inter-subjective agreement. This makes graded truth a metalinguistic category for dealing with the vagueness of other predicates. Graded truth, in the second meaning, is a naive possible theory for vagueness that can be made philosophically precise. Philosophical elaboration is a first refinement of our intuitions; being the result of a theoretical elaboration, it turns an intuitive concept in a philosophical concept. Graded truth as philosophical theory collocates itself to a different level with respect to both linguistic and naive theoretical aspect. Together with them it constitute the *target system* of the formalisation. Since degrees of truth are a possible model for graded truth, they are also a possible model for analysing vagueness, a model that should account for all those aspects: the phenomenon itself, i.e. vagueness of natural languages (vagueness of 'true' included), and the (first intuitive and then philosophical) pre-formal theory of vagueness in terms of graded truth.

Let P be a vague predicate and x be a borderline case of P. In what follows I summarise how graded truth is used in order to account for the peculiarities of vague predicates.

1. P has borderline cases. Borderline cases of P are neither P nor not-P. The

[11]It is also possible to consider finitely many values but this choice has not been fully articulated in the literature on vagueness because of some difficulties it encounters, starting with the risk of arbitrariness in the choice of the cardinality of the set.

[12]More on the vagueness of 'true' in Footnote 14.

sentence 'x is P' is neither true nor false. Truth is *partial.*

2. P lacks sharp boundaries. This is accounted for in terms of *graded membership*: objects in the domain of the predicate are such that they are either P or not-P or they are P to a certain extent. In other words, graded membership consists in assuming that objects can fall under concepts to greater or lesser degrees. Accordingly, the sentence 'x is P' is taken to be either true or false or true to greater or lesser degrees. Truth is *graded.*

3. P generates a sorites paradox. We start with an object, say x_1, which is a clear case of P. The sentence 'x_1 is P' is absolutely true. We move on in the series by considering objects $x_2, x_3, \ldots$ that are less and less P, so that the sentence 'x_n is P' is less true than or at most equal to the sentence 'x_{n-1} is P'. In other words, as we go on through the sorites series the amount of truth decreases continuously. Continuity of truth captures the gradualness and the soritical aspect typical of the phenomenon of vagueness. Truth is *continuously* graded.

As we have seen, graded truth can be formalised by introducing a continuum of intermediate truth degrees between truth and falsehood. Formally, the real unit interval $[0, 1]$ is taken as the set of truth-values. These are interpreted as degrees of truth, so that sentences which are clear cases of falsehood and truth receive values 0 and 1, respectively, and borderline sentences receive an intermediate value accordingly to how true they are. Moreover, an infinite-valued semantics with truth-functionality clauses is adopted for computing the truth degree of complex sentences and making valid inferences. The semantics for connectives and the notion of validity can be defined in a variety of ways, and as we have seen this gives rise to a number of distinct infinite-valued logics. My discussion will be independent of the choice of a particular logic, unless otherwise specified.

It is of interest to have a closer look at the degree-theoretic solution to the sorites. Consider a vague predicate P and a series of objects $x_1, \ldots, x_n$ in its domain forming a sorites series, namely such that x_1 is P and for each $i < n$ if x_i is P then also x_{i+1} is such. A propositional version of the sorites is obtained by considering sentences $\phi_1, \ldots, \phi_n$ such that each ϕ_i is $P(x_i)$. Then the argument goes as follows:

(P1) ϕ_1,

(P2) $\phi_i \to \phi_{i+1}$ (for every i such that $1 \leq i \leq n-1$)

(C) ϕ_n.

From (P1) and (P2) the conclusion (C) follows by successive applications of Modus Ponens. Recall that (C) is a blatantly false statement. Hence the paradox. The possible ways out are considering the argument *invalid*, namely saying that the conclusion does not follows from the premisses, by blaming for example Modus Ponens, mathematical induction or universal instantiation, or alternatively considering the argument *unsound*, namely saying that one or more of the premisses are not true. As we know, degree-theoretical approaches allow for truth values intermediate between 0 and 1, representing truth degrees. The solution to the sorites goes as follows: the first premiss is assigned value 1, nonetheless, the argument is unsound because some of the instances of the premiss (P2) are not absolutely true. Moving along the series, there is a point in which the sentences start leaking truth so that the antecedents are slightly more true than the consequents. Regardless to the logic chosen for computing the truth value of the implication, in all infinite-valued logics it holds that an implicative sentence is absolutely true just in case the truth value of the antecedent is less than or equal to the truth value of the consequent. The most popular choice for the truth condition of the implication is Łukasiewicz implication, according to which

$$v(\phi \to \psi) = \begin{cases} 1, & \text{if } v(\phi) \leq v(\psi); \\ 1 - v(\phi) + v(\psi), & \text{otherwise.} \end{cases}$$

So, if at a certain point there is an i such that $v(\phi_{i+1}) = v(\phi_i) - \epsilon$, the truth value of the implication would be $v(\phi_i \to \phi_{i+1}) = 1 - \epsilon$. This makes some of the instances of premiss (P2) slightly less than absolutely true, and successive applications of the Modus Ponens lead to an absolutely false conclusion.

This solution to the paradox is appealing because it gives a semantic explanation of what is going on in the sorites which also accounts for the plausibility of the argument. Using Graff's terminology, we might say that a solution to the sorites paradox must provide an answer also to the Psychological question, namely the question why we are inclined to accept the sorites premisses as true in the first place, and the Epistemological question, why we are unable to say exactly which instances of a sorites sentence are not true (Graff, 2000). Proponents of degrees of truth can meet the challenge by noticing that the sorites is a valid argument whose premisses are *almost* true, or slightly less than absolute truth. They looked persuasive in the first place because we tend to mistake near truth for truth and small differences in degrees of truth as no difference at all.[13] Also, we are unable to say exactly which instances of a sorites sentence fail to be absolutely true because that would involve locating a boundary in a sorites series. Furthermore, degree theories do justice to the intuitions behind the paradox according to which soritical phenomenon are a matter of degrees. A graded truth predicate mirrors the gradualness of vague predicates. Truth fades as well as the application of vague properties does.[14] So, degree theorists are able to solve the paradox since they explain what is wrong with the argument (it

[13] Smith (forthcoming) argues that in taking very nearly true sentences as true speakers are not mistaken, they are rather rounding up or down values, or ignoring small differences for practical purposes.

[14] This adherence of levels can be made even more precise by referring to the vagueness of

is unsound) and why competent speaker find it compelling and convincing (it is valid and the premisses are very close to be absolutely true). These are two requirements that any solution to the paradox should meet. In addition, degree solutions to the sorites have the merit of providing a faithful picture of what is happening in the grain-to-grain shift from a heap to a non-heap.

Nevertheless, the just outlined degree-theoretic approach toward vagueness has been the object of long-standing criticisms. In the previous section I have addressed objections related to the nature of degrees of truth and their role. I deal now with more specific objections questioning the adequacy of degrees of truth as a model for vagueness. I distinguish five main objections of this kind: ambiguity of degrees objection, artificial precision objection, higher-order vagueness objection, linearity objection and truth-functionality objection (see e.g. Keefe, 2000, Chapter 4).

A first, well-known, objection questions the motivation of degrees of truth by arguing that degree-theorists mistake degrees of truth with degrees of applicability of the underlying predicate:

> Consider again the vague predicate 'tall': I claim that any numbers assigned in an attempt to capture the vagueness of 'tall' do no more than serve as another measure of *height*. More generally, in so far as it is possible to assign numbers which respect certain truths about e.g. comparative relations, this is no more than a measure of an attribute related to, or underlying, the vague predicate. (Keefe, 2000, p. 134.)

There would be, therefore, no need for degrees of truth, because they would be nothing but measures of P-ness – e.g. measures of height if the predicate at stake is 'tall'. I call this objection *ambiguity of degrees objection*.

The *artificial precision objection* consists in pointing out the existence of a "mismatch between the precision found in degree-theoretic semantics and the lack of precision in vague natural language." (Cook, 2002, p. 233). This is a major objection and has been pointed out by different authors:

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truth. It has been noticed since the first writings on vagueness that the predicate 'being true' as used in English is itself vague (e.g. Russell, 1923). The vagueness of the truth predicate has been acknowledged in terms of second-order vagueness, in contrast with the fist-order vagueness of ordinary predicates (see, e.g., Fine, 1975; Raffman, 2014). However, if the vagueness of truth considered as a predicate is taken seriously (as I think it should) then any theory which aims for a general account of the phenomenon of vagueness should encompass the predicate 'being true' itself among its objects. It can be argued that this perspective provides a novel element for assessing different theories which have been so far proposed for dealing with vague predicates. For example, it can be easily seen that degree theories meet the *desiderata* of avoiding a mismatch between the object-level and the meta-level: vague predicates, including the truth predicate, are graded, and the truth predicate, used as semantic predicate in the analysis of vagueness, is also graded. More work is clearly needed here.

> [Fuzzy logic] imposes artificial precision [...]. [T]hough one is not obliged to require that a predicate either definitely applies or definitely does not apply, one is obliged to require that a predicate definitely applies to such-and-such, rather than to such-and-such other, degree (e.g. that a man 5 ft 10 in tall belongs to tall to degree 0.6 rather than 0.5). (Haack, 1979)

> One serious objection to [the many-valued approach] is that it really replaces vagueness with the most incredible and refined precision. (Tye, 1989)

> [T]he degree theorist's assignments impose precision in a form that is just as unacceptable as a classical true/false assignment. [...] All predications of "is red" will receive a unique, exact value, but it seems inappropriate to associate our vague predicate "red" with any particular exact function from objects to degrees of truth. For a start, what could determine which is the correct function, settling that my coat is red to degree 0.322 rather than 0.321? (Keefe, 2000)

In a nutshell, the problem is that it would be artificial, and thus implausible, to associate vague sentences with a degree of truth, namely with a real number between 0 and 1.

A related worry is the *higher-order vagueness objection*:

> [I]f the semantics for a many-valued logic is described using a precise metalanguage, then sentences will always be assigned exact values, since sentences of the metalanguage ascribing degrees of truth will themselves be true or false simpliciter. For example the metalinguistic claim ' "this coat is red" is true to degree x' will be (completely) true for a single value of x and (completely) false for all other values of x, and that single value will be the exact and uniquely correct value to assign to 'this coat is red'. And similarly all other predications of 'is red' will receive a unique, exact value. But it seems inappropriate to associate our vague predicate 'red' with any particular exact function from objects to degrees of truth, as this requires. (Keefe, 2000, p. 113.)

The concern here is that degree-theoretical theories do no better than the classical theory in capturing vagueness, since assigning some unique value to sentences imposes sharp boundaries which would be incompatible with the fact that the metalanguage is itself vague. The problem then is how "avoiding sharp boundaries and accommodating borderline borderline cases [second-order borderline cases]." (Keefe, 2000, p. 202). As Williamson (1994, Chapter 4) notices, the problem cannot be solved by iterating the theory.

A further objection, related to the fact that a rich numerical framework is employed, is the *linearity objection*: degrees of truth, being real numbers, are linearly ordered. As a consequence, if an infinite-valued valuation is fixed, for any two sentences whatsoever, they are always comparable in respect of truth, that is to say either they are as true as one another, or one is more true than the other. The assumption that any two sentences must be comparable in respect of truth is taken to be highly problematic and counter-intuitive.

> I claim that again the connectedness axiom is incompatible with the nature of the vagueness of these comparisons: we cannot assume that there is always a fact of the matter about which of two borderline sentences is more true. (Keefe, 2000, p. 129.)

This difficulty concerns in the first place multidimensional predicates, predicates whose applicability can be determined in a number of different dimensions or respects. An example is the predicate 'intelligent'. Since different people can be intelligent in different ways, it is difficult to decide which is more true between 'Einstein is intelligent' and 'Mozart is intelligent'. Forcing comparability would misrepresent the peculiarities of these predicate. Also, sentences expressing facts about unrelated subjects, for example Stanley Kubrick's being bald and my coat's being red, can hardly be compared with respect to their degree of truth. These examples suggest that a partial order, which allows for indeterminate instances of 'the sentence ϕ is more true than the sentence ψ', would better represent the truth relations among vague sentences. However, in a degree-theoretic approach employing degrees of truth as arising in infinite-valued logics, linearity is always assumed.

Lastly, there is a further worry related to the numerical aspect of degrees of truth and especially with the fact that infinite-valued theories proposed for modelling vagueness are in general truth-functional. They differ in the functions that are taken as truth conditions for connectives, but it is generally assumed that the truth value of a complex sentence is a function of the truth values of the components. The main reason for making this assumption is to obtain a complete generalisation of the classical, two-valued case, in which truth-functionality holds. The *truth-functionality objection* consists in pointing out that values that are assigned to complex sentences through truth-functionality clauses are often counter-intuitive and they do not match the values that one would expect the sentences to have in the presence of borderline cases.

All the objections are somehow related to the fact that a unique, exact value is assigned to sentences which are imprecise and vague. A possible response consists in advocating the logic-as-modelling view described in Section 3.2, namely the idea that the numerical assignments are merely artefacts of the logical model. This position is well-described in the following passages:

> Our models are typical purely exact constructions, and we use ordinary exact logic and set theory freely in their development. This amounts to assuming we can have at least certain kinds of exact knowledge of inexact concepts. (When we say something, others may know exactly what we say, but not know exactly what we mean.) It is hard to see how we can study our subject at all rigorously without such assumptions. (Goguen, 1969, p. 327.)

> Degree-theoretic semantics is immune to Sainsbury and Tye's criticisms [artificial precision objection] if we view it not as a description but as a fruitful model of vagueness. [...] we have seen how we can have mathematical precision in the semantics without attributing it to the natural language being studied by making use of the logic as modelling picture. (Cook, 2002, p. 235 and p. 246.)

> The many-valued system is only a model, and the super-precise truth values, along with the sharp borders, are artefacts of this model. We use the mathematical precision of the continuum to model the fuzzy slide from truth to falsity as we go through a sorites series. There is nothing untoward about using a precise structure to model a vague one. (Shapiro, 2006, p. 53.)

Such a defence of degree-based treatments of vagueness against the outlined objections is based on two crucial arguments: (i) a model can be precise even though the subject matter is vague, and (ii) the unwanted precision is an artefact of the model. Nevertheless, whereas (i) might be considered a satisfactory point, the same cannot be said for (ii) – not in this model at least. Indeed, the distinction representors/artefacts in the case of degrees of truth runs into difficulties: the necessity of postulating the existence of further objects, verities, whose nature is mysterious, as representors, the fact that the feature that is is considered artefactual, i.e. numerical assignments, plays instead a crucial role in the theory and the lack of a general strategy for telling representors and artefacts apart (see Section 3.2). This makes the main argument against the objections flawed. As I argued in the previous section, a qualitative perspective on degrees of truth does a better job in accommodating the distinction representors/artefacts and in clarifying the role of degrees of truth. Therefore, it can also play a crucial role in the defence of degree-theoretic approaches to vagueness against the outlined objections.

3.3.2 Pairwise valuations and vagueness

I developed a formal and conceptual framework that allows to shift the focus from the numerical aspect, which consists in mapping sentences into mathematical structures, to the comparative aspect of graded truth, which consists in comparing sentences in respect to truth. The proposal introduced prompts a comparison with Smith (2012). In this paper, Smith suggests two ways for spelling out the idea of measuring truth on an ordinal scale, namely the idea that the only meaningful thing about assignments of degrees of truth is their relative ordering. These two ways are realism and nominalism toward measuring. According to the realist view, there are certain entities, for example lengths, such that each object has a unique length, and we refer to them by assigning real numbers in a such a way that the relations between the numbers mirror the relations between lengths. According to the nominalist view, there are no such entities as lengths: there are only objects which are susceptible of length ascriptions and the real numbers. With respect to this picture, some remarks are in order. First, I insist more on the pairwise or comparative perspective instead of the ordinal one which can also be pointwise (see discussion at the end of Subsection 3.2.2). Moreover, I do take an anti-realist attitude toward the process of measuring truth, namely I do not postulate the existence of objects like truth values or verities assigned to sentences, to which we refer by real numbers (contrast with Edgington, 1992; Cook, 2002). In my approach real numbers are assigned directly to sentences, through representation theorems that cardinalise a comparative notion of truth. The approach is not completely nominalist though, because the existence of a relational structure defined over the objects is postulated: the relation *more* or *less true than*, which is then mirrored by numerical assignments.

I take the comparative notion *more* or *less true than* as a primitive, non-defined, object. This is in agreement with Weatherson (2005), who emphasises the pre-theoretical role of the relation:

> The concept *truer than* will play a crucial role in what follows, so I should say a few more words about its nature. I mean not to give a reductive analysis of *truer*. The hopes for doing that are no better than the hopes of giving a reductive analysis of *true*. What I do aim to show is that we intuitively understand the concept well enough that it can be used in an informative philosophical theory of vagueness. Indeed, I hope the following range of considerations will convince you that I'm latching onto a concept you already possess, and these considerations will help isolate the concept, if not fully explicate it a la *Meno*. (p. 51.)

Recall the double nature of graded truth described in Subsection 3.3.1: it is a pre-theoretical concept related to the vagueness of the predicate 'true', and it provides an informal theory of vagueness, which is then made philosophically precise. Pairwise valuations are a formal counterpart for graded truth, alternative to degrees of truth. I take *all* the formal properties of pairwise valuations (the axioms stated in the definition) as representors of the informal concept of graded truth: some of these are descriptive of the linguistic phenomenon of graded truth – graded truth in its first meaning, such as for example that fact that pairwise valuations are bounded rankings; others are descriptive of graded truth as philosophical concept – graded truth in its second meaning, like the fact that pairwise valuations preserve and reflect the implication order, are sound with respect to the underlying logic, and their symmetric parts are congruences with respect to the sentential operations.[15] We know that, given these properties, pairwise valuations can be numerically represented by degrees of truth. Numerical degrees of truth are the artefactual aspect of the model, as argued in Section 3.2: they not only do not represent something really occurring in the phenomenon, but they are even absent in the informal theory. However, degrees of truth are not a byproduct of the formalisation, something we would prefer to avoid but we are forced to buy; on the contrary they make the formal model more appealing given their valuable instrumental role.

How can this framework be used in order to accommodate vagueness? We have symbols in the language for expressing logical truths and logical falsehood, $\top$ and $\bot$, respectively. I assume that contradictions are strictly less true than tautologies, namely that $\bot \prec \top$, and that all the other sentences of the languages are intermediate with respect to them, namely they are no less true than contradictions and no more true than tautologies (boundedness of $\preceq$). Absolutely true (false) sentences, including sentences of the form 'x is P' where P is a vague predicate and x is a clear case of P (not-P), are as true as logical truths (falsehoods). Sentences of the form 'x is P' where x is a borderline case for P can be taken to be *genuinely* intermediate in respect to truth, namely strictly more true than clearly false sentences and strictly less true than clearly true sentences.

So, the model that I have proposed accounts for borderline cases and for blurred boundary typical of vague predicates. I analyse now the soritical susceptibility and a pairwise treatment thereof. Recall that a sorites series is such that the first object in the series is P – premiss (P1), and if an object in the series is P so is the next object – premiss (P2), therefore the last object in the series is P – conclusion (C). The premiss (P1) is true since the fist object is supposed to be a clear case of P, whereas the conclusion is paradoxical since the last object is supposed to be a clear case of non-P. The premiss (P2) seems true or convincing since we move from an object to the successive one by slightly altering it in some respect which is relevant for the applicability of the predicate P, like for example the removal of a grain when the predicate at stake is the predicate 'being a heap'. Adjacent objects in the series are similar in respects relevant to P and the idea is that such a small step cannot make a substantial difference in the applicability of the predicate P.

We have seen the numerical solution to the paradox in the previous subsection and how simple and appealing it is. The applicability of P comes in degrees and decreases as we move along the series. The truth values of the corresponding sentences

[15] In Section 3.2 I called the latter properties 'modelling choices'.

drop accordingly. The problem now is whether we do have a likewise appealing solution in terms of pairwise valuations. Since pairwise valuations are numerically representable by pointwise valuations, I could in principle resort to the numerical solution. However, this is not a philosophically justifiable option since I argued in Section 3.2 for the artefactual nature of the numerical representation and artefacts cannot have a substantial or explicative role in the analysis of the phenomenon:

> It should be clear that a successful solution to the sorites paradox should make use of characteristics of the semantics that are representative. If the solution took advantage of too much that is artefactual, then it would be no solution at all since the artefacts do not correspond to genuine features of the phenomenon being modelled. (Cook, 2002, p. 245.)

I argue that the truth ordering among sentences suffices in providing a semantic solution to the sorites paradox which also partially accounts for the plausibility of the argument. Moreover, this solution to some extent affords the same intuitive grasp of a solution based on degrees of truth.

In order to show that this is the case, let me emphasise the centrality of *pairwise* judgements in the sorites series. It is common to different theories of vagueness to analyse the sorites by shifting the focus from the predicates P and non-P to the binary relation 'more or less P than' defined over the objects in the domain of P. As Raffman puts it:

> [P]roponents of the standard analysis conceive of borderline cases in terms of a certain kind of ordering. They suppose that for any predicate 'Φ' having borderline cases, there is some linear ordering of items (values) on a dimension decisive of the application of 'Φ', progressing from an item that is definitely Φ to an item that is definitely not-Φ. Call such an ordering a Φ-ordering. (Raffman, 2014, p. 27.)

The P-ordering, corresponding to a given predicate P, in order to generate a sorites series should be non-trivial and it should have at least one intermediate element (borderline case). Furthermore, accepting that the applicability of P is a matter of (possibly infinitely many) degrees amounts to saying that there may be indefinitely many steps between the top and the bottom element. Also, the P-ordering is taken to be linear. One can object to this by noticing that the objects in the domain of the predicate 'big' are only partially ordered, because they are big in different ways. However, once a single decisive dimension or respect is isolated, e.g. height or length, objects in the domain of the predicate can be linearly ordered.[16]

[16] I set aside issues related to *irreducibly* multidimensional predicates, that is predicates for which it is unclear which dimensions or respects are relevant, like 'nice', 'clever', etc. I do not consider

So we have a chain in which each object is no P-er than the previous one. There might be items that are marginally indistinguishable, namely that are either indistinguishable or just noticeably different, accordingly, the differences between them are negligible or insignificant, and we are entitled to ignore them. In terms of the P-ordering we would say that some objects are *as* P *as* the previous ones in the series. However in a series, there should be at least some significant differences, such that some objects are strictly less P than the previous ones, in order to reach a non-P object. So, sorites paradoxes can be analysed as being a matter of comparisons, and steps in the sorites series are well understood from a pairwise perspective.

The degree-theoretic solution to the paradox is founded on the idea of graded truth. Crucially, this is *already* a comparative solution which mirrors the comparative nature of the sorites: all that matters are differences in truth values between successive sentences. The first sentence of the series is a clear case of P, so it is as true as logical truth (absolutely true). At first it may be the case that sentences keep being absolutely true, namely as true as the previous ones. Then from a certain step onward, the sentences begin to be strictly more true than their successors, namely there are $1 \leq i \leq n-1$ such that $\phi_i \succ \phi_{i+1}$. This makes the corresponding conditional premiss not absolutely true, namely $(\phi_i \rightarrow \phi_{i+1}) \not\sim \top$, because an implicative sentence is absolutely true just in case the antecedent is no more than the consequent – this is axiom (A2): $(\phi \rightarrow \psi) \sim \top \leftrightarrow \phi \preceq \psi$. The situation is represented in the following chain:

$$\top \sim \phi_1 \sim \phi_2 \sim \phi_3 \sim \cdots \succ \phi_i \succ \phi_{i+1} \succ \phi_{i+2} \succ \cdots \sim \phi_{n-2} \sim \phi_{n-1} \sim \phi_n \sim \bot. \tag{3.1}$$

We have seen that in the pairwise reading the soritical argument is unsound because one or more premisses are not true. But why does it look convincing in the fist place? The degree-theoretic story of why the argument looks compelling resorts to the idea of near truth: some premisses are not true, but they seem to be such (or are taken to be such) because they are *almost* true (see Subsection 3.3.1). Notice that for this to be the case it is crucial that the implication is defined as Łukasiewicz implication, which codifies by definition a metric distance among values. If the sorites were analysed by adopting Gödel implication, for example, the non absolutely true premisses would not come out as almost true. What can be said for the pairwise reading? Weatherson (2005, p. 52) claims, as I do, that "the concept *truer*, and the associated concept *as true as*, are the only theoretical tools we need to provide a complete theory of vagueness". However, Weatherson admits that he does not have

them here because they are not soritical in the sense just described. I follow Raffman (2014) in saying that multidimensional predicates do not "threaten the definition of soritical borderlines in terms of linear orderings on decisive dimensions." (p. 29). However, I point out that pairwise valuations, being non linear, can also account for those predicates.

a distinguished story which appeals to those concepts in order to explain the sorites, because no notion of *almost true* can be expressed in comparative terms. Indeed, discourses of closeness to the absolute truth rest on the availability of a topology on degrees of truth. If degrees of truth are modelled by real numbers, then the mathematical structure of the reals, in particular, the topology defined on them, is necessary in order to be able to formulate distance considerations. The availability of such a topology is related to the choice of the underlying logic because, as I already said, there are pointwise real-valued semantics which are nonetheless ordinal or comparative and do not induce a metric on the set of degrees of truth. Since the present approach seeks to emancipate itself from numerical degrees of truth, and there is no commitment toward a specific logic, such a solution is not available. Nevertheless, the model based on pairwise valuations is not completely silent on the topic, because considerations of relative positions of sentences with respect to truth can be used to say something on the plausibility of the sorites, at least for discrete cases.

How does the explanation of the plausibility of the paradox in qualitative terms go? Let P be a vague predicate forming a standard sorites series, namely such that the P-ordering is bounded, linear and discrete. Such conditions permit to define a meaningful notion of *adjacency* as follows: two objects in the domain of P are adjacent in the P-ordering if they are such that $x \leq_P y$ and there is no z such that $x \leq_P z \leq_P y$ and $z \neq x$ and $z \neq y$.[17] If x_i and x_j are adjacent in the P-ordering then the corresponding sentences ϕ_i and ϕ_j saying that x_i is P and x_j is P, respectively, are adjacent in the truth ordering or they are in the same equivalence class, namely $\phi_i \sim \phi_j$.[18] This can be used to say something on the status of the implicative premisses of the sorites by resorting to Axiom (A.3), which makes the relative positions of implicative sentences depend on the relative positions of their components:

$$\text{(A.3)} \quad \phi_1 \succeq \phi_2,\ \psi_1 \preceq \psi_2 \Rightarrow \phi_1 \to \psi_1 \preceq \phi_2 \to \psi_2.$$

Recall that this is a monotonicity condition stating that implication is non-increasing in the first component and non-decreasing in the second. From axiom (A.3) and reflexivity of $\preceq$ it follows that $\phi_1 \succeq \phi_2 \Rightarrow \psi \to \phi_1 \succeq \psi \to \phi_2$, stating that the more true the consequent is, the more true the whole implication is.

[17] Notice that this does not apply to continuous sorites (Weber and Colyvan, 2010), since the definition would be trivially empty: the P-ordering would be dense, namely by definition we would have that given any two elements x and y such that $x \leq_P y$ there is z distinct from x and y, such that $x \leq_P z \leq_P y$.

[18] Adjacency in the truth ordering of two given sentences does not entail that the difference between their truth values in an arbitrary characterisation is small, since, as I already said, distance information among truth values requires a metric or a topology on the members of the domain that in a general context is not available.

Imagine that in sorites series we have three borderline objects for P such that $x_i >_P x_j >_P x_k$ and that the corresponding sentences are such that $\phi_i \succ \phi_j \succ \phi_k$. We know that both $\phi_i \to \phi_j \not\sim \top$ and $\phi_j \to \phi_k \not\sim \top$, that is to say both are not absolutely true so that the soritical step does not go through. Although we cannot express the fact that they are almost true, we can say something on their relative position with respect to other possible implicative statements. In particular, we have that $\phi_i \to \phi_j$ is no less true than $\phi_i \to \phi_k$. This suggests that the truth of an implication increases (or at least does not decrease) as we consider elements which are more and more proximal in the ordering. If x_i and x_j are also adjacent, namely as proximal as possible without coinciding (maximally proximal), then the premiss of the form $\phi_i \to \phi_j$ is more true than (or at most as true as) all other premisses with ϕ_i as antecedent except for $\phi_i \to \phi_i$, which is absolutely true. Therefore, also in pairwise terms we can express the correlation between the proximity of two elements in the P-ordering and the truth value assigned to the implicative sentence associated to them. This fact can be used to explain the plausibility of the soritical step, namely to explain why implicative premisses of the form $\phi_i \to \phi_{i+1}$ look plausible in the first place.

In the analysis of vagueness in terms of pairwise valuations we have that if x_i and x_j are adjacent in the P-ordering then the corresponding sentences ϕ_i and ϕ_j are adjacent in the truth ordering or they are in the same equivalence class, namely $\phi_i \sim \phi_j$. This prompts a comparison with the well-know criterion of closeness (Smith, 2005):

Closeness: For any objects x and y, if they are very close in P-relevant respects, then '$P(x)$' and '$P(y)$' are very close in respect of truth.

We are not justified to conclude in general for any given predicate P that if two elements x_i and x_j are adjacent with respect to the P-ordering then they are close in P-relevant respects. For example, in the case of truth predicate, it is not the case that if two sentences are adjacent in the truth ordering than they are close in respect of truth, because in a bivalent truth ordering (corresponding to a precise predicate) absolutely true and absolutely false sentences are adjacent but not close. If P is a precise predicate, such as for example 'being 1.70 m tall', then two objects x and y might be close in P-relevant respect, for example two persons who are 1.70 m and 1.71 m tall, and yet it might be that the corresponding sentences are not close in respect of truth, because it is true that x is P and false that y is P. However, also in this case, the two sentences will be adjacent in the truth ordering. So, precise predicates satisfy the adjacency condition, whereas they may not satisfy Closeness (Smith, 2005, p. 165). However, if we restrict attention to vague predicates generating a genuine sorites series, we do have that adjacent members of the series (x_i and $x_i + 1$) are close (or very close, following Smith's phrasing) in P-relevant respects. Also, if we start with a predicate with a coarse domain so that the adjacency in the P-order seems not to imply closeness in P-relevant respect, we can always refine the domain of the predicate P, and add objects to the series, for example we can imagine to subtract one millimetre of height instead of one centimetre in a sorites series for the predicate 'tall'. In other words, we need a significant numbers of steps between an object which is P and one which is non-P in order to justifiably say that adjacent objects in the P-ordering are close in P-relevant respects.

I showed that also an order-based theory can provide a satisfactory analysis of the sorites paradox. It actually suffices to focus on the comparative aspect of the standard degree-theoretical solution and to show that what counts in the solution are comparisons between the truth values of the sentences involved. Elements which are artefactual, that is which precise numbers are assigned to sentences, do not play a role in the treatment of vague predicate or in the solution to the paradox, as it is required when a distinction representors/artefacts is in play:

> Suppose a theorist wants to meet an objection by claiming that some feature of their model is merely an artefact, so can be ignored. In that situation [. . .] the theorist must meet the challenge of giving a compelling story about what we can trust in the theory and show that a substantive theory of vagueness is thereby given. (Keefe, 2012, p. 458.)

However, the qualitative reformulation of the solution to the sorites contributes in pointing out some unwelcome aspects of the solution itself. I focus here on two related difficulties: the lack of homogeneity and the mismatch with the P-ordering.

The lack of homogeneity refers to the fact that there is no uniform drop in truth value whereas the drop in P-ness is homogeneous. The soritical process is homogeneous or uniform, in spite of this, a block of sentences in the middle of the series are strictly less true than the previous ones whereas some others are as true as the previous ones (see Equation 3.1). Homogeneity is a characteristic of the sorites series: at each step we remove a grain from a heap, a single hair from a non-bald person, a millimetre from a tall person. The process is uniform, all the steps are alike, so how it comes that some of those yield to significant differences in the truth ranking, whereas some others do not?

> [T]he degree-theoretical solution is quite appealing because it explains how a definitely false conclusion can be reached by valid principles of inference from nearly absolutely true premisses, i.e., by a progressive accumulation of minute inaccuracies. But the substantial disomogeneity which we just hinted to casts some shadows on such a brilliant explanation. (Paoli, 2003, p. 367.)

A related difficulty is the mismatch between P-ordering and the truth ordering. In the above analysis of the sorites, both in its qualitative and quantitative version, we have that for any two objects x_i and x_j in the domain of P such that $i \neq j$, it holds that if x_i is (strictly!) P-er than x_j then the sentence ϕ_i (stating that x_i is P) is *at least as true as* ϕ_j, namely

$$\text{if } x_i >_P x_j \text{ then } \phi_i \succeq \phi_j.$$

A reason for allowing this mismatch is the well-known problem of tall people or basketball players: Kareem Abdul Jabbar is taller than Michael Jordan (2.18 m versus *only* 1.98), nonetheless it somehow seems counter-intuitive to conclude that the sentence 'Kareem Abdul Jabbar is tall' is more true than 'Michael Jordan is tall', since intuitively they are both absolutely true. That is why it is generally assumed that differences in P can sometimes – though not always! – make no difference in truth value.

Homogeneity can be restored by making the truth relations always weak, namely

$$\top \sim \phi_1 \succeq \phi_2 \succeq \phi_3 \succeq \cdots \succeq \phi_i \succeq \phi_{i+1} \succeq \phi_{i+2} \succeq \cdots \succeq \phi_{n-2} \succeq \phi_{n-1} \succeq \phi_n \sim \bot,$$

or always strict:

$$\top \sim \phi_1 \succ \phi_2 \succ \phi_3 \succ \cdots \succ \phi_i \succ \phi_{i+1} \succ \phi_{i+2} \succ \cdots \succ \phi_{n-2} \succ \phi_{n-1} \succ \phi_n \sim \bot.$$

I call these two options weak and strict route respectively. The weak route is not a solution to the paradox. It cannot be the case that each step is less true *or as true as* the previous one: as we have seen, there should be at least one step in which the truth value strictly decreases in order to block the paradox.

On the other hand, the strict route guarantees that the paradox does not arise: none of the implicative premisses would be absolutely true. It would also solve the problem of mismatch between the P-ordering and the truth ordering: we would have that any change in P-relevant respects makes a difference in the truth ordering of the corresponding sentences:

$$\text{if } x_i >_P x_j \text{ then } \phi_i \succ \phi_j.$$

However, the strict route has some unwelcome consequences. We would have that the sentence 'Micheal Jordan is tall' is not absolutely true, against our intuitions about the use of the predicate 'tall' according to which Michael Jordan is fully, clearly tall. More generally, we would have that all the objects in the domain of P, but for the P-est and the least P, are borderline cases for P. Again, this fails to capture our intuitions on clear cases and borderline cases. However, it can be defended as a viable route if a totally relational perspective is adopted. More discussion would be needed in this direction.

To conclude, we have seen that a theory of vagueness which appeals only to the relation *more* or *less true than* is possible. This theory is compatible with our pre-theoretical intuitions about vagueness and explains what is wrong with sorites arguments. We have also seen that it inherits some difficulties typical of the numerical solution, but nonetheless does much better than the latter in blocking the crucial objections described in the previous subsection, as I shall argue in what follows.

3.3.3 Objections addressed

I start by considering the ambiguity of degrees objection, according to which degrees of truth are unmotivated since they simply are another measure of the underlying P attribute. Smith (2003, 2008) has already shown that degree-theorists can coherently talk about degrees of truth without being committed to the identification of degrees of truth and degrees of P-ness. However, I argue that a model that uses pairwise valuations offers an even better grasp of this. In this framework, degrees of truth and degrees of P-ness are formally and conceptually distinct because they are the result of different measuring procedures. Degrees of P-ness (e.g. degrees of tallness) are numbers assigned in the attempt to measure the attribute P (e.g. height), i.e., they are numbers assigned to objects in the domain of P, $\mathcal{D}_P$, in a compatible way with the ordering *more* or *less P than* defined over the objects in the domain. Formally, we have a relational structure, $(\mathcal{D}_P, >_P)$, which is embedded into a numerical structure through a map $h\colon \mathcal{D}_P \to \mathbb{R}$, so that for all $x, y \in \mathcal{D}_P$

$$x >_P y \text{ if and only if } h(x) > h(y).$$

I defended that degrees of truth are also the result of a process of measurement, where the underlying attribute is truth, and the relational structure, $(\mathcal{SL}, \preceq)$, is given by the set of sentences and the relation *more* or *less true than*. Degrees of truth are assigned to sentences by pointwise valuations which are compatible with the ordering.

As things stand, I have an additional theoretical tool for arguing against the objection, because instead of focusing on degrees of P-ness and degrees of truth, I can focus on the underlying orderings. And the two orderings are formally and conceptually distinct. We have already seen that a mismatch between the P-ordering and truth ordering is required because of the basketball players problem. We have that the truth ordering coheres or agrees with the P-ordering and with the other possible comparisons with respect to attributes, but cannot be reduced to them. For example it might be the case that $x >_P y$ and $P(x) \sim P(y)$. If the strict route were taken to avoid this mismatch we would still have an important distinction among orderings due to the transpredicative nature of the truth ordering, which makes its domain much wider than the simple ascriptions of predicate P to the objects in $\mathcal{D}_P$. For example we may have two sentences saying that 'x is R' is more true than 'y is P' meaning that x is more R than y is P. In this respect, the truth ordering is not supervenient upon or dispensable in favour of the underlying P-, R-orderings because it is also carrying this additional information. We can, therefore, conclude that degrees of truth are *not* a measure of the attribute P because they arise as possible measures for the ordering *more or less true*, which is conceptually and formally distinct from the P-ordering.

I move now to consider the artificial precision objection which points out the implausibility of assigning numerical degrees of truth to vague sentences. Two crucial aspects of the degree-theoretic theory are blamed here: the fact that sentences receive real numbers as truth value (artificiality of the assignment) and the fact that this value is unique (precision of the assignment). I spell out the artificial precision objection by saying that it maintains that a semantics based on functions from sentences to degrees of truth coded by real numbers misrepresents the phenomenon of vagueness. As an alternative, I proposed a semantics based on pairwise comparisons among sentences. It is immediate to notice that in such a framework the artificial precision objection loses much of its force. Sentences do not receive numbers as values, they are compared in respect of truth. One of the motivations for doing that is given by the need to avoid undesired artificiality: comparative judgements are less artificial or less arbitrary than absolute judgements because they do not require the same precision (see Chapter 1). Real numbers as truth values are artificial and impose a precision which jars with the lack thereof typical of vagueness. That is why, an equivalent model in which degrees of truth are merely artefacts (see Section 3.2) is more adequate as model of vagueness.

In some cases, comparative judgements can be artificial too, for example when, given a pair of sentences, there seems to be no fact of the matter as to which sentence is more or less true than the other. This artificiality can be avoided by letting pairwise valuations be incomplete, that is by allowing for incompatible pairs of sentences with respect to truth. This, by the way, blocks the linearity objection. In the qualitative framework the linearity assumption can be questioned and dropped: it is no longer the case that any two sentences are mutually comparable with respect to truth. One of the main advantages of the pairwise perspective is exploited here, that is the fact that the relation *more* or *less true* among sentences can be taken to be structurally weaker than the natural order relation among truth values. Once we drop linearity we allow for incomparable sentences with respect to truth. We have seen that if such a preorder has to be cardinalised then the representing function is not unique: there will be a family of valuations compatible with it. Intuitively, a non-linear ordering can be 'linearised' in more than one way. Every representing function must reflect the ordering, in presence of indeterminacy as to how a pair of sentences is ordered with respect to truth then there will be different possible completions which assign different values to the sentences in the pair, and order them in different ways. Lack of uniqueness is a fair price to pay if one has to avoid unwanted precision.

A possible response to the higher-order vagueness objection goes along the same lines. The objection blames the fact that a vague predicate is associated with an exact function from objects in the domain of the predicates to degrees of truth. As a consequence there is an exact, single value which has to be assigned to a given sentence in the intended model. At the higher level, it will be absolutely(!) true that the sentence has that precise value, and absolutely false that it has any other value. On the present account this worry does not apply since a vague predicate P is not associated with an exact function from objects in $\mathcal{D}_P$ to precise truth values, it is rather associated with a preorder, the truth ordering over sentences, which is compatible with the P-ordering and interacts with the underlying orderings of the other predicates. Nevertheless, an aspect of the objection which is not solved in this framework is the presence of sharp boundaries, or sharp cut-offs, between the objects to which the predicate clearly applies and its borderline cases. Since the truth preorder is bounded, absolute truth and absolute falsity still have a privileged status.

A further charge against degree-theorists is that the way in which truth values of compound sentences are computed (truth-functionality clauses) would be counter-intuitive or incompatible with ordinary usage. The truth functionality objection has been successfully addressed by Paoli (Forthcoming); Smith (2015). I refer to the quoted paper for a defence of degree-theoretic approaches against this objection. Recall that the pairwise semantics, if taken in its general version involving Axiom (A.1) saying that all the logical truths of a given logic are evaluated true to maximum degree (as true as $\top$), is general enough to accommodate different infinite-valued systems. It is neutral with respect to the choice of the logic and, consequentially, neutral with respect to the clauses of truth-functionality for computing truth values of compound sentences. The representing functions will be logical valuations of the chosen logic. In this case, a previous commitment with the underlying logic is necessary, and the pairwise semantics intervenes in providing a more plausible semantics for that logic. In this respect, there is not much to say about the truth-functionality objection. However, I showed in Section 1.3.3 that, once one has a precise logic and truth-functionality clauses in mind (e.g. Łukasiewicz logic), an alternative axiomatization of the relation *more* or *less true than* can be provided so that it does not make explicit recourse to a given logic. This alternative axiomatization rather focuses on structural properties of connectives: whether they are increasing or decreasing functions, how they behave with respect to $\top$ and $\bot$, whether they are, for example, commutative or associative. A set of properties of this kind fixes the corresponding function to the extent that there is a unique function compatible with them. For instance, consider the following list of axioms:

(N.1) $\neg\top \sim \bot$, $\neg\bot \sim \top$

(N.2) $\phi \preceq \psi \Rightarrow \neg\psi \preceq \neg\phi$

(N.3) $\neg\neg\phi \sim \phi$

saying that the negation of an absolutely false statement should be certainly true and the negation of an absolutely true statement certainly false (N.1), that the negation is a non-increasing (N.2) and involutive (N.3) function. It can be proved (see Bennett et al., 2000, Theorem 1, p. 33) that the unique function compatible with these requirements is a function $f_\neg \colon [0,1] \to [0,1]$ such that $f_\neg(\neg x) = 1 - x$. This is an example of how the qualitative foundation provides an alternative, non-numerical, way for imposing truth-functionality bonds. The comparative reformulation of the clauses can be seen as less problematic than the numerical one, and can offer new grounds for discussion and also for testability, since it is reasonable to assume that comparative judgements are easier to be performed instead of intensity assignments (how true/how agreed something is).

3.4 Conclusions

The possibility of a measure-theoretic approach to vagueness has been considered in the literature, see in particular Keefe (2000, Chapter 5 *Vagueness by numbers*), Smith (2008, Subsection 6.1.5 *Measuring truth*). However, the idea of providing such a foundation has never been fully articulated. In particular, the linearity of the ordering and the uniqueness of the representation have been considered insurmountable issues. In Section 3.1 I showed that such a foundation is *formally* possible: it follows as a special case of a more general project of providing pairwise semantics for non-classical logics which aims at liberating discourses about many-valuedness from explicit recourse to truth values (see Chapter 2). As a consequence of this, a substantial philosophical account of what degrees of truth are and what their role is becomes possible (Section 3.2). Moreover, this foundation can be brought to bear on the analysis of degree-theoretic treatments of vagueness.

Pairwise valuations do not require the acceptance of the idea that truth is graded, since, as we have seen, also classical sentences can be evaluated pairwise. However, in presenting pairwise valuations as a possible model for vagueness there is commitment to the very idea of *graded truth*, and in particular, it should be accepted from the beginning that graded truth provides a good (informal) theory of vagueness. The main contribution is, then, to show that there is nothing essentially numerical about graded truth, nothing that forces us to formalise it by using real numbers. I share with Weatherson (2005) the idea that graded truth is *prima facie* a comparative notion and that there is an intuitive, pre-theoretical notion of *more* or *less true than* which is a ranking, but is in general not linear. Indeed, I take pairwise valuations, the formal counterpart of the relation *more* or *less true than*, as primitive objects of the formal semantics. However, contra Weatherson, I require pairwise valuations to be numerically representable by degrees of truth, so that any pairwise valuation is associated with a set of infinite-valued valuations compatible with it (a unique valuation if the ordering is linear). I want the truth ordering to be representable by infinite-valued valuation functions because representation theorems of this kind make the whole machinery of mathematical fuzzy logic available, which has great logical and instrumental value. At the same time, representation theorems isolate the portion of structure underlying the formalism which plays an explicative or descriptive role in the treatment of the phenomenon. This also counters a certain scepticism or pessimism about the possibility of a modelling perspective expressed in the literature:

> What is needed is an explicit, systematic account of how the model corresponds to or applies to natural language, stating which aspects of the model are representational, and justifying the treatment of others as mere artefacts. It is far from clear how this could be done. (Keefe, 2000, p. 55.)

This new model for vagueness retains the properties that make the standard degree-theoretic account appealing: intuitions about a graded notion of truth, analysis of the paradoxes which accounts for their original plausibility, mathematical convenience; and still mitigates some of its deficiencies or weak points: artificial precision and related objections and the lack of a philosophical account of what degrees of truth are.

Chapter 4

Degrees of truth and probabilities

In Chapter 3 I provided a philosophical underpinning for the notion of degrees of truth in terms of the relation *more* or *less true than*. This sheds new light on the philosophical discussion surrounding degrees of truth, like for example their alleged artificiality (discussed in Section 3.2) or their controversial relation with vagueness (Section 3.3). Within the philosophical discussion surrounding degrees of truth it falls also their debated relation with probabilities, ranging from their formal overlap to the resulting conceptual confusion. This is the topic of this chapter.

My proposal is to articulate the contrast between probabilities and degrees of truth at a qualitative level of analysis, namely by investigating the properties of the two orderings *more* or *less probable than* and *more* or *less true than*. The key shift in focus is from infinite-valued functions – logical valuations and probability functions, representing pointwise evaluations of sentences, to binary relations over the set of sentences, representing pairwise evaluations or comparative judgements. The overall aim of this inquiry is to shed light on the quantitative side by means of representation theorems. Moreover, I argue that a deeper understanding of the distinction between degrees of truth and probabilities can shed new light on their possible interactions. In the second part of the chapter I shall propose a new framework for connecting them, whose core consists in providing a probabilistic interpretation for the notion of graded truth.

4.1 Degrees of truth and probabilities

There is a widespread formal overlap between degrees of truth and probabilities: both can be formalised as functions from sentences to real numbers, and they do share important formal properties. In order to have a better grasp of this, I remind the structure of the propositional language I am adopted. $\mathcal{L}$ is an infinite propositional language and $\mathcal{SL}$ the set of sentences built recursively by means of a binary connective $\to$ for implication, along with the constant $\perp$ for *falsum*. As usual, negation, constant for *verum*, disjunction, conjunction and biimplication are defined as $\neg\phi := \phi \to \perp$, $\top := \neg\perp$, $\phi \vee \psi := \neg\phi \to \psi$, $\phi \wedge \psi := \neg(\neg\phi \vee \neg\psi)$ and $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$, respectively. Notice that in this chapter I use the symbol $\vee$ for the disjunction defined from the implication and the negation for Łukasiewicz logic as well, instead of the strong disjunction symbol $\oplus$. Also the dual connective is denoted by $\wedge$ instead of $\odot$. This is done in order to have the same language and syntax as classical logic, so that degrees of truth and probability are directly comparable.

Probability can be expressed in a sentential framework by taking sentences as probability-bearers: propositional variables in $\mathcal{L}$ represent events, with $\perp$ and $\top$ representing the impossible and certain event, respectively. Connectives are operations generating complex events. Classical logical consequence, denoted by $\models \subseteq \mathcal{P}(\mathcal{SL}) \times \mathcal{SL}$, determines what is true in all possible worlds (classical tautologies) and what logically follows from what. From a logical point of view, the incompatibility among two events ϕ and ψ is formalised by the fact that it is never the case that ϕ and ψ are both true. Given this, a *logical probability function* over $\mathcal{L}$ is a map $P\colon \mathcal{SL} \to [0,1]$ satisfying for all $\phi, \psi \in \mathcal{SL}$

(P1) if $\models \phi$ then $P(\phi) = 1$,

(P2) if $\models \neg(\phi \wedge \psi)$ then $P(\phi \vee \psi) = P(\phi) + P(\psi)$.

Probability is a function assigning real numbers to events and it is such that all classical tautologies receive value 1 (it is normalised) and it is additive over incompatible events, namely if two events are incompatible the probability that at least one of the two occurs is given by the sum of the probabilities of the events considered singularly.

Though degrees of truth do not behave in general like probabilities, if we restrict the attention to Łukasiewicz logic, we can notice that infinite-valued Łukasiewicz valuations are probability functions of a special kind. Let $\models_{\text{Ł}}$ be Łukasiewicz consequence relation, then for all Łukasiewicz valuations $v\colon \mathcal{SL} \to [0,1]$:

(P1*) if $\models_{\text{Ł}} \phi$ then $v(\phi) = 1$,

(P2*) if $\models_{\text{Ł}} \neg(\phi \wedge \psi)$ then $v(\phi \vee \psi) = v(\phi) + v(\psi)$.

Recall that the semantics of strong disjunction and strong conjunction, here exceptionally denoted by $\vee$ and $\wedge$, is given by

$$v(\phi \vee \psi) = \min\{1, v(\phi) + v(\psi)\},$$
$$v(\phi \wedge \psi) = \max\{0, v(\phi) + v(\psi) - 1\}.$$

When two events are incompatible the disjunction is simply the sum of their truth values, because the condition that this sum should not exceed 1 is always satisfied. Since Łukasiewicz clauses are compatible with the classical truth assignments, namely 1 and 0 behave classically, all the classical valuations are also Łukasiewicz valuations, but not the other way around. As a consequence of this and of the definitions, Łukasiewicz tautologies are a proper subset of the classical ones:

$$\forall \phi \in \mathcal{SL} \quad \models_{Ł} \phi \Rightarrow \models \phi.$$

The formal overlap between degrees of truth and logical probabilities backed up some conceptual confusion between the phenomena they are meant to model, that is imprecise and uncertain reasoning, respectively; especially when fuzzy logics first arose. This confusion goes in two directions: on the one hand, degrees of truth have been taken to express some sort of truth-functional probabilities, and, on the other hand, the theory of probability has be seen as a many-valued logic. Nowadays this kind of confusion seems no longer a risk, thanks to the better understanding of both concepts. However, philosophical and formal problems remain when it comes to how degrees of truth (many-valued logics in general) and probabilities should be combined. The qualitative foundation I proposed for degrees of truth opens a new level of analysis of the distinction degrees of truth versus probabilities and ultimately adds a novel argument supporting the distinction.

4.2 Probability from comparison

There is a long-standing tradition investigating the notion of *qualitative probability*, whose aim consists in laying down conditions under which probability measures can be proved to arise (uniquely) from qualitative or pairwise comparisons (for a survey see Fishburn, 1986). This measurement-theoretic approach to probability dates back to de Finetti (1931) and it is summarised in the following:

> There are occasions, on the other hand, when it seems preferable to start from a purely ordinal relation – i.e. a qualitative one – which either replaces the quantitative notion (should one consider it to be meaningless, or, anyway, if one simply wishes to avoid it), or is used as a first step towards its definition. For example, given two commodities (or two economic alternatives) A and B, one can ask which is preferable (or whether they are equally preferable) before defining utility (or perhaps even rejecting the very idea of measurable utility); and the same can be said for temperature, the pitch of a note, the length of intervals, etc. [...]
>
> One could proceed in a similar manner for probabilities, too. (de Finetti, 1974, vol I, p. 363.)

The relation *more* or *less probable than* is formally defined as a binary relation between sentences representing events. I consider the axiomatisation proposed by Savage (1972). He does not consider a sentential framework, rather it takes events as being elements of a Boolean algebra (the algebra of events). I introduce some notation. Let $\mathcal{A}$ be a Boolean algebra of subsets $A, B, \ldots$ of a universal set S. Each A in $\mathcal{A}$ is an *event*. The empty or impossible event is $\emptyset$ and the universal or certain event is S. Notice that, for any A in $\mathcal{A}$, $\emptyset \subseteq A \subseteq S$. Recall that $\mathcal{A}$ is complemented and closed under union, we write A^c for the complement of A and $A \cup B$ and $A \cap B$ for the union and the intersection of A and B, respectively. A *probability measure* on $\mathcal{A}$ is an infinite-valued function P on $\mathcal{A}$ such that for all A and $B \in \mathcal{A}$

1. $P(S) = 1$,
2. $P(A) \geq 0$,
3. if $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$.

Savage, following de Finetti's axiomatization, introduces the notion of *qualitative probability* as a binary relation $\preceq$ over $\mathcal{A}$ satisfying the following conditions for all $A, B, C \in \mathcal{A}$

(QP1) $\preceq$ is linear and transitive

(QP2) $\emptyset \preceq A$, $\emptyset \prec S$

(QP3) if $A \cap C = \emptyset$, $B \cap C = \emptyset$ and $A \preceq B$ then $A \cup C \preceq B \cup C$.[1]

The intuitive interpretation of the axioms is as follows. (QP1) requires the relation to be a total preorder, (QP2) says that any event is no less probable than the impossible event and that the certain event is strictly more probable than the impossible event (in other words, the relation is bounded – or non-negative – and non-trivial). (QP3) is a qualitative version of additivity.

As noted by de Finetti, in order to prove the required representation result we also need to assume that S can be partitioned into an arbitrarily large number of equiprobable subsets (*uniform partition*). The assumption is formulated as follows:

(QP$\star$) for each n, S can be partitioned into a a complete class of n incompatible events equally probable.

This can be justified by imagining a situation in which the agent is certain that there is a fair coin so that any finite sequence of heads and tails is considered equally likely by her. Savage replaces assumption (QP$\star$) with the weaker

[1] The original axiomatization proposed by de Finetti differs from this one in two respects: any possible event is taken to be strictly less probable than the certain event and the right-to-left direction of (QP3) is also assumed.

(QP$\star\star$) for each $n \geq 2$, S can be partitioned into n events such that the union of no r events is more probable than the union of any $r+1$.

It can be proved that either de Finetti's or Savage's postulate implies the existence of a unique probability function representing a qualitative probability.

Theorem 4.1. *If $\succeq$ satisfies (QP1)–(QP3) and (QP$\star\star$) then there exists a unique probability measure P such that for all $A, B \in \mathcal{A}$*

$$\textit{if } A \preceq B \textit{ then } P(A) \leq P(B).$$ [2]

In this case we say that the function P *weakly represents* (or *almost agrees* with) the ordering $\preceq$.

In general, we say that the function P *strongly represents* (or *agrees* with) the ordering $\preceq$ if for all $A, B \in \mathcal{A}$

$$A \preceq B \text{ if and only if } P(A) \leq P(B).$$

Weak representation ensures that given a certain ordering among sentences it is possible to deduce the unambiguous assignment of a numerical probability to each event compatible with it. For the purposes of this investigation, we confine ourselves to weak representation results. This is because in order to have an injective map (strong representation) it is necessary to add some additional conditions, typically an Archimedean axiom or certain continuity assumptions, which are not relevant for our discussion and might be misleading. Also, weak representations do not establish an equivalence between the pairwise and the pointwise concepts, rather an embedding, so that one cannot go back and forth from one to the other. In a sense, if pairwise comparisons enjoy just weak representation results, they are irreducibly comparative, they are not just a pairwise reformulations of absolute assignments. Moreover, from a philosophical point of view we are interested in justifying the use of numbers, as values of a measure, starting from plausible properties of the comparative notions therefore it suffices to have results which ensure that the orderings are *quantifiable* or *representable* by numerical functions.[3]

4.2.1 Logical probability from comparison

In what follows I reformulate the notion of qualitative probability in a logical setting, which is more suitable for the purposes of the comparison, by building on the known duality between Boolean algebras and classical propositional logic.[4]

[2] For a sketch of the proof see Savage (1972), pp. 35-36.

[3] Narens (1980, p. 145) argues that "weak probability representation rather than probability representation is the natural quantitative concept."

[4] The elements of the algebra are seen as propositional sentences; complement, union and intersection as negation, disjunction and conjunction respectively; empty set and universe set are $\perp$ and

Let *qualitative logical probability* be a binary relation $\preceq$ over the set of sentences $\mathcal{SL}$ which are assumed to represent events. Notice that Savage's formulation assumes that the events form a Boolean algebra; in the logical reformulation this is ensured by requiring agreement (or compatibility) with the classical logic. In particular, it suffices to assume that

(QLP0) $\models \phi \Rightarrow \phi \sim \top$,

which says that events represented by classical tautologies are maximally probable, namely certain. The axioms of qualitative probability then can be restated as

(QLP1) $\preceq$ is linear and transitive,

(QLP2) $\bot \preceq \phi$, $\bot \prec \bot$,

(QLP3) $\models \neg(\phi \wedge \chi), \models \neg(\psi \wedge \chi), \phi \preceq \psi \Rightarrow \phi \vee \chi \preceq \psi \vee \chi$,

Furthermore, a logical version of the assumption (QP$\star$) goes as follows

(QLP$\star$) for all n, there exist n events, $\phi_1, \ldots, \phi_n \in \mathcal{SL}$ such that

(i) $\models \bigvee_{i=1}^{n} \phi_i$ — collectively exhaustive,

(ii) $\models \neg(\phi_i \wedge \phi_j)$ for $i \neq j$ — mutually exclusive,

(iii) $\phi_i \sim_w \phi_j$ for $i \neq j$ — equiprobable.

The assumption does not say that the whole set of sentences should be partitioned into complete classes, rather that it contains a partition of n elements for all n. In other words, we assume that we can always isolate an arbitrarily big subset of the set of sentences behaving like a partition, such that it contains sentences which are collectively exhaustive mutually exclusive with respect to the logical consequence. Moreover, we assume that those sentences are equiprobable. This can be done because the set of sentences is infinite (recursively generated from an infinite set of propositional variables). Koopman (1940), who axiomatises comparative probability in a logical framework, also assumes (QLP$\star$) in order to obtain a numerical representation (see Definition 1 on page 283). It is a strong structural assumption, however its intuitive meaning is more compelling than other proposals (Kraft et al., 1959; Scott, 1921, compare with).

Both de Finetti's and Savage's proofs of the representation theorem do not make essential use of the structure of Boolean algebra over the set of events. This assures that the relation captured by the axioms (QLP0)–(QLP3), (QLP$\star$) is the qualitative counterpart of the notion of *logical probability* as defined before.

$\top$ and the subset relation is translated as the entailment relation $\theta \models \phi$ or as the logical implication $\models \theta \rightarrow \phi$.

4.2.2 Comparison between comparisons

As we have seen, by following the same intuitions and methods, numerical degrees of truth can be proved to arise from comparisons of sentences with respect to their truth. For the sake of comparison with the probability case, I consider the following version of the representation theorem:

Theorem 4.2. *If $\preceq \subseteq \mathcal{SL}^2$ satisfies the following*

(T0) $\models_Ł \phi \Rightarrow \phi \sim \top$,

(T1) $\preceq$ *is linear and transitive,*

(T2) $\bot \preceq \phi$, $\bot \prec \top$,

(T3) $\phi \preceq \psi \Rightarrow \phi \vee \chi \preceq \psi \vee \chi$,

(T4) $\phi \preceq \psi \Rightarrow \neg\psi \preceq \neg\phi$,

(T5) $(\phi \to \psi) \sim \top \Rightarrow \phi \preceq \psi$.

then there exists a unique Łukasiewicz valuation $v\colon \mathcal{SL} \to [0,1]$ *that weakly represents the order* $\preceq$*, namely such that for all* $\phi, \psi \in \mathcal{SL}$

$$\phi \preceq \psi \Rightarrow v(\phi) \leq v(\psi).[5]$$

The conditions are similar to those stated for qualitative probability (we will compare them directly later on), with the main difference that they are to be interpreted in terms of graded truth. In the previous versions of the representation theorem for truth (see Chapter 3), axiom (T0) was expressed in syntactical terms: if a sentence is provable in the logic then it is true to maximum degree. In this version of the theorem, in order to be on the same page with the probability case I consider normalisation with respect to the set of tautologies of logic (semantic notion) instead of the set of theorems. Formally nothing changes for we are dealing with sound and complete logics (the sets of theorems and tautologies coincide). However, the notion of truth from comparison thus formulated cannot count any longer as a foundation for the truth-value semantics because it actually presupposes it, and it would be a circular argument rather than a proper reduction. But the main purpose here is not to defend the plausibility of this foundation, because I have already done that, rather is to compare truth from comparison and probability from comparison. Hence this stretch. Moreover, we assume that all the sentences are pairwise comparable with respect to their truth and that the relation is transitive.[6] Moreover, every sentence is

[5] In order to have a strong representation result an Archimedean condition should be added (see Corollary 3.6 on page 57).

[6] As we have seen linearity is a very strong assumption and it can be dropped at the price of giving up the uniqueness of the representation. However, we are not interested here in issues of plausibility or desirability of the axioms, so we keep the conditions as their are.

no more true than the tautology and the relation is non-trivial. There are then conditions which regulate the behaviour of the relation with respect to the connectives, additivity and complementarity, which in terms of truth can be read as follows: the truth value of a disjunction is a non-decreasing function of the truth values of the components and the negation is non-increasing. In previous versions of the result, I let both (T3) and (T4) follow from the monotonicity condition for the implication. Moreover, I assume that whenever an implication is true the antecedent is less true than the consequent. This also has a probabilistic interpretation: if an implicative event is certain then the consequent should be at least as probable as the antecedent.

We know that if as underlying logic we take classical logic then the valuation representing the ordering takes just extremal values (Theorem 2.27 on page 47):

Corollary 4.3. *If* $\preceq \subseteq \mathcal{SL}^2$ *satisfies (T1)–(T5) and (T0'):* $\models \phi \Rightarrow \phi \sim \top$*, then there exists a unique classical valuation* $v\colon \mathcal{SL} \to \{0,1\}$ *representing it.*

There is a clear parallelism between qualitative logical probability and qualitative truth, as defined above, even if different strategies are used in proving representation theorems 4.1 and 4.2. The former makes use of the structural assumption of uniform partitions, while the latter takes an algebraic route. We will consider (QLP$\star$) as a technical condition to not be discussed further and in what follows we will focus on the other axioms.

One of the main differences is the form of additivity we require, compare the following axioms:

(QLP3) $\models \neg(\phi \wedge \chi), \models \neg(\psi \wedge \chi), \phi \preceq \psi \Rightarrow \phi \vee \chi \preceq \psi \vee \chi$,

(T3) $\phi \preceq \psi \Rightarrow \phi \vee \chi \preceq \psi \vee \chi$.

From (T3) we can derive that $\phi_1 \sim \phi_2, \psi_1 \sim \psi_2 \Rightarrow \phi_1 \vee \psi_1 \sim \phi_2 \vee \psi_2$, namely the symmetric part, interpreted as *as true as* is a congruence with respect to the disjunction (and the negation, and, thus, all the other connectives). This amounts to saying that equivalences with respect to truth of two disjunctive sentences are determined just by the equivalence of the components. Recall that gathering sentences in equivalence classes with respect to truth is the first step for assigning them a truth value (a degree of truth in this case). That is why the additivity condition is a compositionality constraint: the truth value of a complex sentence is fully determined by the truth values of the components. In the case of probability this condition is restricted (QLP3), it holds just for incompatible sentences. The restriction on incompatible events in the formalisation of qualitative probability corresponds to the lack of full compositionality of the probability function representing the order. The probability-values of the components only partially constrains the probability-value of the compound, whereas Łukasiewicz semantics is fully compositional.

Both qualitative logical probability and qualitative truth are normalised with respect to the set of tautologies, namely

(QLP0) $\models \phi \Rightarrow \phi \sim \top$,

(T0) $\models_{Ł} \phi \Rightarrow \phi \sim \top$.

For logical probability, condition (QLP0) corresponds to the requirement of considering classical tautologies as certain events, to be ranked as top in the probability order. The condition (T0) is best understood as a requirement of compatibility with the underlying logic, according to which the tautologies, or equivalently the theorems, are true to maximum degree. Crucially, qualitative probability is defined over classical logic whereas qualitative truth presupposes an idea of truth coming in degrees and it is, therefore, required to be sound with respect to a nonclassical logic, in our case Łukasiewicz logic. Recall that (QLP0) implies (T0) and not *vice versa* because $\{\phi \in \mathcal{SL} \mid \models_{Ł} \phi\} \subset \{\phi \in \mathcal{SL} \mid \models \phi\}$.

It is worth underlying that additivity and normalisation are related, in particular there is a tension between (QLP0) and (T3) to the extent that removing the restriction on incompatible events, and thus allowing for full compositionality for the probability functions, while retaining classical logic as underling logic leads to binary assignments, namely to trivial probability functions assigning just extremal values to sentences. Let $P(\cdot)$ be a infinite-valued function which is fully compositional, in particular the value of a disjunction is a fixed function $f_{\vee}$ of the values of the disjuncts. Assume that $P(p) = P(\neg p) = 0.5$. Then we would have

$$P(p \vee p) = f_{\vee}(P(p), P(p)) = f_{\vee}(P(p), P(\neg p)) = P(p \vee \neg p).$$

As noted in Paris (1994), if $P(\cdot)$ is meant to model degrees of belief, this looks implausible since one would expect $p \vee \neg p$ to be certain and $p \vee p$ to have the same probability as p. The implausibility turns in impossibility if $P(\cdot)$ is normalised with respect to classical logic (condition (P1)), because if that is the case then $P(p \vee \neg p) = 1$ and $P(p \vee p) = P(p) = 0.5$. It can be then proved that the classical probability functions which are fully compositional collapse in $\{0, 1\}$-probability assignments and they coincide with binary truth-assignments (or classical valuation functions) as Corollary 4.3 states. Notice that if Łukasiewicz infinite-valued logic is taken as underlying logic, instead of classical logic, then there is no incompatibility between infinite-valued functions and full compositionality.

Although the axiomatisations are similar, they are meant to capture two different concepts, that is *more* or *less probable than* and *more* or *less true than*. The first attempt to draw a line can be to recognise the presence of an objective versus subjective distinction between the orderings. Indeed, whereas qualitative truth is supposed to be an objective, agent-independent ordering, qualitative probability can be interpreted subjectively as comparative confidence, namely as an agent ranking (sentences representing) events according to how likely she thinks they are. Numbers arising from the cardinalisation of this ordering would be subjective probabilities, or degrees of belief. However, nothing in the formalisation of qualitative probability forces a subjectivist interpretation, the probability ordering can be also interpreted as comparative objective likelihood and it would thus have the same objective and agent-independent character of qualitative truth. The key distinction between the orderings is to be found elsewhere.

I argue that the qualitative perspective contributes in elucidating an important conceptual difference between probabilities and degrees of truth in relation to the background logic, irrespective of it being classical or infinite-valued. The probability ordering builds on the logic, it has the logic as constraint, in particular logical (classical) valuations. However, the function representing the probability ordering is distinct from valuation functions. In contrast with this, in the case of truth this double dimension is not present, rather we can see a sort of circularity as I mentioned: the function representing the truth ordering is constrained by logical (infinite-valued) valuations and it is itself a logical valuation. This is best clarified with reference to a scheme proposed by de Finetti in which different layers of analysis for an event are distinguished: logical, epistemic and subjective level (de Finetti, 1980). The logical level is bivalent since, on his account, events can be either true or false; the epistemic level is instead three-valued, since the agent might not know whether the event is true or false. However, the agent can be more or less uncertain and this uncertainty can be measured by using probabilities, that is why the third level, the subjective one, is infinite-valued. There is no room in this analysis for (non-trivial) objective probability, de Finetti argues that for any event the only objective probability is 1 if the event obtains and 0 if it doesn't obtain. Since, in his view, the objective probability of an event is nothing more than the truth-value of the corresponding sentence, the only objective level in his story is the logical one ("Probability does not exist." (de Finetti, 1974)). This skepticism is nowadays obsolete and an objective version of de Finetti's scheme can be imagined in which objective (rather than epistemic) uncertainty is measured by using degrees of objective probabilities.

This picture offers a key for articulating the distinction between degrees of truth and probabilities. The probability ordering and the emerging probability assignments collocate themselves to a different level with respect to the logical one: the level of uncertainty, be it subjective or objective. It is not immediately clear, instead, how the truth ordering and the emerging degrees of truth are collocated in this picture. The standard way for combining the two concepts goes as follows. Degrees of truth are introduced by directly generalising the logical level and allowing events to be absolutely true, absolutely false, but also partially true or true to a certain degree. The main purpose of this generalisation is to model reasoning containing probability ascriptions to vague expressions like e.g. the events described by the sentences 'the next person entering the door is tall' or 'the team will score soon in the game'. Betting over this fuzzy events requires a theory of probability over non-classical logics and over infinite-valued logics in particular, as has been developed by e.g. in Mundici (2006). In what follow I put forward a different approach which consists in keeping a bivalent logical level and introducing an objective one in which truth comes in degrees. The resulting graded notion of truth will be interpreted as *(graded) objective probability.*

4.3 A probabilistic interpretation for graded truth

In the first part of this chapter I compared the formal properties of qualitative truth with those of qualitative probability. The above analysis accounted for the substantial formal overlap observed between the two notions and, at the same time, added a novel argument in support of the traditional distinction between probabilities and graded truth. In what follows I propose a new framework for connecting them, whose core consists in providing a probabilistic interpretation for the notion of graded truth.

4.3.1 Motivation

Consider and compare the following

1. A coin has been tossed, but its outcome has not yet been observed.
2. A coin will be tossed tomorrow and will land Heads.
3. There will be a sea-battle tomorrow.

The first sentence describes a case of *epistemic or subjective uncertainty.* The coin actually landed Heads or Tails and the outcome, though already determined, is unknown. Whereas the second and the third describe a case of what I call *objective uncertainty*: the outcome of a future coin toss or dice thrown is not only unknown but also undetermined, not yet decided, at the present time. This uncertainty or indeterminacy looks *prima facie* as an all-or-nothing or categorical concept, however it allows for comparisons, for example it can be argued that (2) is less undetermined or less uncertain than (3) to the extent that it is more likely to happen. The picture emerging from this analysis can be seen as an *objective version* of de Finetti's schema.

I wish to develop a conceptual and formal framework in which these differences are accounted for in terms of differences in *truth value* of the sentences involved. In this framework some sentences will be determined, namely either true or false (even if there is epistemic uncertainty), and some other sentences will be objectively undetermined with respect to a certain context. Undetermined sentences will be neither true nor false, but they will rather be more or less true accordingly to how likely they are in an objective sense. As soon as the events unfold the truth values of the corresponding sentences will collapse into a determined one. Coming back to the example, whilst sentence (1) is now either true or false, sentences (2) and (3) receive an intermediate truth value, and tomorrow also (2) and (3) will be determined.

4.3.2 Formal framework

In order to formalise the just outlined idea I develop a dynamic semantics for sentences, in which the truth values depend on a certain context and as soon as the context changes the truth values change. To this aim I build on the framework developed in Flaminio et al. (2014) and I consider the context as given by a set of sentences $w \subset \mathcal{SL}$. I indifferently call it context, world or state.

Each world w can be uniquely associated with a partial valuation $e_w\colon \mathcal{SL} \to \{0, 1, u\}$ stating which sentences are true and false at that world:

$$e_w(\phi) = \begin{cases} 1, & \text{if } w \models \phi; \\ 0, & \text{if } w \models \neg\phi; \\ u, & \text{otherwise.} \end{cases} \tag{4.1}$$

The true sentences at the world w are the sentences in the deductive closure of w, whereas are false all the sentences whose negations are in the deductive closure. In a very straightforward way we call *w-determined* the sentences receiving either true or false as value and *w-undetermined* the others; namely for every $\phi \in \mathcal{SL}$

- ϕ is *w-determined* if and only if $e_w(\phi) \in \{0, 1\}$,
- ϕ is *w-undetermined* if and only if $e_w(\phi) = u$.

It is worth underlying that we are assuming classical logic in the background since $\models$ is the classical consequence relation. As a consequence, we have that if ϕ is w-determined then also $\neg\phi$ is so and if ϕ and ψ are w-determined then also $\phi \star \psi$ with $\star \in \{\to, \vee, \wedge\}$ is so. Moreover over the set of (complex) determined sentences $e_w(\cdot)$ is a classical valuation function.

In order to make this framework dynamic I need to specify how the transition among contexts works. To this aim I introduce a binary relation R over the set of contexts W. We require R to coincide with the subset relation, namely for all $w, w' \in W$

$$(w, w') \in R \Leftrightarrow w \subseteq w',$$

hence R is reflexive and transitive. Moreover, R is *monotone* to the extent that once a sentence is determined with respect to a certain context w then it stays determined if larger contexts are considered. What is more, in this framework all the sentences are determinable: if $e_w(\phi) = u$ then $\exists w'$ such that $(w, w') \in R$ and $e_{w'}(\phi) \in \{0, 1\}$.[7]

[7] The case of events which are ultimately undeterminable will not be considered here.

Notice that if $w \subseteq w'$ then $e_{w'}$ extends e_w, namely it coincides with e_w on the w-determined sentences and the number of sentences which are given a binary truth value under $e_{w'}$ is larger than the number of w-determined sentences.

In this framework we can formalise the idea presented above according to which sentences which lack a determined value are ordered according their *probabilistic degree of truth.* I start from a qualitative perspective and I consider a new hybrid relation over the set of sentences which share some properties with qualitative truth and some others with qualitative probability as stated in the following

Definition 4.4. *Given $w \in W$, let a* probabilistic truth ordering *is a binary relation $\preceq_w \subseteq \mathcal{SL}$ such that*

(PT0) $T_w \models \phi \Leftrightarrow \phi \sim_w \top$,

(PT1) $\preceq_w$ *is linear and transitive,*

(PT2) $\bot \preceq_w \phi$, $\bot \prec_w \top$,

If ϕ, ψ are w-determined

(PT3) $\phi \preceq_w \psi \Rightarrow \phi \vee \chi \preceq_w \psi \vee \chi$,

(PT4) $\phi \preceq_w \psi \Rightarrow \neg\psi \preceq_w \neg\phi$,

(PT5) $(\phi \rightarrow \psi) \sim_w \top \Rightarrow \phi \preceq \psi$,

If ϕ is w-undetermined

(PT6) $T_w \models \neg(\phi \wedge \chi), T_w \models \neg(\psi \wedge \chi)$, $\phi \preceq_w \psi \Rightarrow \phi \vee \chi \preceq_w \psi \vee \chi$,

(PT⋆) for all n, there exist n events, $\phi_1, \ldots, \phi_n \in \mathcal{SL}$ such that

(i) $T_w \models \bigvee_{i=1}^{n} \phi_i$ *— collectively exhaustive,*

(ii) $T_w \models \neg(\phi_i \wedge \phi_j)$ *for $i \neq j$ — mutually exclusive,*

(iii) $\phi_i \sim_w \phi_j$ *for $i \neq j$ — equiprobable.*

The set T_w is the set of all the sentences which are determinately true given the context w, $T_w = \{\phi \in \mathcal{SL} \mid e_w(\phi) = 1\}$, this set clearly contains w itself but also its deductive closure. T_w allows to exploit all the information we have and improves on the strength of the axioms. (PT0) assures that what is determinately true in that world (this clearly includes classical tautologies), or given that context, is maximally true; moreover, nothing else is maximally true. Once again we assume as structural properties linearity, transitivity, non-triviality and boundedness – (PT1), (PT2). When w-determined sentences are compared then the order behaves like a qualitative truth order: the connectives are monotone and there is the classical truth condition for the implication, (PT3)–(PT5). When one of the sentences is w-undetermined the relation behaves like a probability order – (PT3), (QLP$\star$).

The following representation theorem is an immediate consequence of Theorem 4.1 and Corollary 4.3.

Theorem 4.5. *If $\preceq_w \subseteq \mathcal{SL}^2$ is a probabilistic truth ordering then there exists a unique function $v_w \colon \mathcal{SL} \to [0,1]$ such that*

(1) $T_w \models \phi \Leftrightarrow v_w(\phi) = 1$

(2) If ϕ, ψ are w-determined

- $v_w(\phi) \in \{0, 1\}$,
- $v_w(\neg\phi) = f_\neg(v_w(\phi))$,
- $v_w(\phi \vee \psi) = f_\vee(v_w(\phi), v_w(\psi))$,

(3) if ϕ is w-undetermined

- $v_t(\phi) \in]0, 1[$
- $T_w \models \neg(\phi \wedge \psi) \Rightarrow v_w(\phi \vee \psi) = v_w(\phi) + v_w(\psi)$

(4) $\phi \preceq_w \psi \Rightarrow v_w(\phi) \leq v_w(\psi)$

The function weakly representing the ordering is a special infinite-valued function which takes into account the different objective levels of knowledge of an event described in Subsection 4.3.1. Given a certain context w, it gives value 1 to classical tautologies and to all the sentences which are determinately true in that context. If just determined sentences are considered, then the function is a bivalent, truth-functional valuation function and in particular coincides with the partial evaluation e_w. When undetermined sentences are involved then the function assigns intermediate truth values, which behave like probabilities. These functions form a dynamic semantics for the language. Consider, e.g. a context $w' \supseteq w$. For all w-determined sentences ϕ, $v_{w'}(\phi) = v_w(\phi)$. At the same time, $v_{w'}$ also assigns a determined value to some w-undetermined sentences.

Notice that, once a certain context is fixed, the truth ordering among determined sentences is given by the partial evaluation corresponding to the context; therefore, for each context it is uniquely determined. The case in which one of the two sentences is determined also constrains the order: the undetermined sentence will be strictly more true than any determinately false sentence and less true than any determinately true sentence. The central question is then how undetermined sentences are compared. In the following section I argue that the answer constitutes an interesting case for objective probability.

4.3.3 A case for objective probability

I argue that the notion of graded truth that emerges from the above analysis can be interpreted in terms of objective probability. I focus on two possible instantiations of this concept:

1. objective probability as chance (Beisbart and Hartmann, 2011),
2. objective probability as arising from *intersubjective agreement* (Williamson, 2010).

In what follows I show that both the interpretations are compatible with the outlined framework depending on the interpretation of the context, respectively,

1. in terms of time,
2. in terms of evidence.

Interpreting the context in terms of time consists in seeing the pair $< W, R >$ as a temporal frame, the w_i as times, or equivalently sets of the sentences that are true at a certain time, and R as a earlier/later than relation. Recall that, since R is assumed to be transitive and monotone, if something is determinately true (or false) it stays true (or false) as time passes by, whereas undetermined sentences can (and will) turned into determined ones. The interpretation of W as set of times and R as temporal relation is clearly compatible with the outlined formal framework and, moreover, I argue that it is rich in philosophical consequences.

First, under this interpretation the presented account can be also seen as a degree-theoretic treatment of future contingents, namely the problem of ascribing truth values to sentences concerning future events which are not necessary. In this framework, sentences representing classically certain (impossible) events are true (false) to maximum degree. For example, all the instances of excluded middle are true. Moreover, given a certain moment of time and given the history up to that time, past sentences are determined (either true or false) whereas future contingent events are undetermined. However, certain future contingents appear to be, at the present moment, more undetermined than others and this is modelled by assigning them different degrees of truth.

This brings to the second point. This proposal reconnects many-valuedness with issues related to determinism. Łukasiewicz, prompted by Aristotle's discussion on future contingents, proposed to drop bivalence for treating sentences such as 'I shall be in Warsaw at noon on 21 December of the next year'. Thus, many-valuedness has been introduced in the first place in order to avoid logical determinism, namely the fact that the truth value of a sentence concerning the future is already fixed at the present time. In line with this, my proposal consists in bringing in many-valuedness (three-valued and infinite-valued semantics) for modelling an idea of truth values as *unfolding* as soon as the event described in the sentence obtains or does not.

Furthermore, the interpretation in terms of time connects the notion of graded truth with the notion of objective chances, which is in turn related to future contingents and determinism. Chances are often used as synonymous with objective probabilities, as distinguished from subjective probabilities. There are different philosophical accounts of chance, the two dominant being the *frequency* account and the *propensity* account. Frequency interpretation consists in identifying chance with relative frequency, namely with the absolute frequency normalised by the total number of events. Under this interpretation, chances are relativised to a suitably chosen reference class which is assumed to be finite. Well-known difficulties related to the *reference class problem* prompted some frequentists to refine their position and consider infinite reference classes. Chance is then identified with limiting relative frequencies of events therein. On the other hand, according to the propensity interpretation, chances are thought as propensities, or dispositions, of a physical system to yield a specific outcome. This outcome can be a single-case event, or, according to long-run propensity theorists, a frequency over repeated trials.[8]

In what follows we will be neutral with respect to these accounts, this is possible since irrespectively of the interpretations there is a certain philosophical consensus on some key features of the notion of chance. To start with, there is agreement on some basic principles regulating chances, for example the idea that chance should somehow connect with actual frequencies and with possibility. It is also generally accepted that the main role of chance is to be a guide for graded belief, as codified in the Principal Principle firstly proposed by Lewis (1980) or other different versions of this principle (see Pettigrew, 2012, for a survey). I take the notion of chance as captured by the following points, which constitute a shared core across the different interpretations:

- chances are objective and agent-independent, i.e., they refer to objective states of affair, rather than to an agent's epistemic attitude with respect to states of affair.

- Chance is seen as a property of events which admits degrees. If an event is given an extremal chance (conventionally, 0 or 1) the interpretation is that its occurrence (or non-occurrence) is determined, or, in other words, that the event is certain.

- Intermediate, non-trivial chances are assigned to events which are *chancy* or uncertain, and are assumed to behave as probabilities: infinite-valued normalised functions which are additive on incompatible events.

[8]See Hájek (2012) for a more detailed discussion.

- Context-dependency: chance functions are normally relativised to a world w and a time t:

  > [...] a proposition about history up to that time; and further, it is a complete proposition about history up to that time, so that it either implies or else is incompatible with any other proposition about history up to that time. It fully specifies a segment, up to the given time, of some possible course of history. (Lewis, 1980, p. 275)

  Formally, this amounts to considering conditional probabilities with respect to an event representing the history of the world up to that time (H_{tw}).

I claim that the just outlined notion of chance is somehow related to the notion of truth. First, truth, such as chance, is objective and *prima facie* agent-independent, although for both notions we will see that an objectivity-as-intersubjectivity account is possible. Furthermore, events which get extremal chances can be naturally be considered as the truths (and falsehoods) of that specific world. In other words, trivial chances can be seen as classical truth-assignments stating what sentences are true and what false of the world. By exploiting this intuition and accepting that there are intermediate truth values, I put forward the idea of interpreting the degrees of truth emerging from Theorem 4.5 as the objective chances that the events described in the sentences will obtain.

An alternative approach consists in interpreting the context in terms of evidence: $\langle W, R\rangle$ is an informational frame, w_i are informational states, or equivalently all the sentences that are known to be true, and R is a transition relation between informational states. The base of information, or evidence base, cannot be a single agent's knowledge base, since we are not dealing with epistemic ignorance or uncertainty, rather with objective uncertainty. Thus, we will consider the case of an *ultimate evidence base*, containing complete information on the world.

Lewis (1980) put forward the idea of seeing chance as *objectified* graded belief, meaning the beliefs that an agent with total evidence about the world would adopt. The idea of ultimate evidence is clearly spelled out in Williamson (2010). In the framework of Objective Bayesian Epistemology, objectivity is reached by imposing normative constraints on degrees of belief thus forcing intersubjective agreement among agents. Accepting all the norms proposed by objective Bayesianism – Probabilism, Calibration and Equivocation – is still not enough for getting total agreement among agents' beliefs, since they still ultimately depend on the agent's language and evidence. Ideally, we can imagine that if the language is fixed and some ultimate evidence is considered then at the end of this process full objectivity is achieved. That is, we finally end up with a ultimate, unique probability function. This is the *ultimate-belief* notion of probability, in which objectivity is gained by intersubjective agreement. I refer to Williamson (2010) for the details.

Under the interpretation of the context in terms of ultimate evidence, probabilistic degrees of truth would be *ultimate* degrees of belief, in the sense just described. In this perspective, graded truth can be seen as rational graded belief any agent ought to adopt if she accepts the norms on belief and has total evidence about the world. I argue that this interpretation is of philosophical interest because of the very peculiar idea of truth involved. It reflects a 'bottom-up' and anti-metaphysical notion of truth, according to which truth is not something out there in world, it is rather built up by rational agents investigating the world itself. These agents ultimately agree on assigning their degrees of belief to a set of sentences (representing events) and we can say that this tells us *how much true* those sentences are at the world. It is, therefore, also a consensus-based notion of truth.

4.3.4 Comparison with similar frameworks

The paper Ciucci and Dubois (2013) points out that there is a conceptual conflict between the epistemic reading of the third value as *unknown* and its treatment as a semantical value in the same way as *true* and *false*. Accordingly, two different layers are isolated, pretty much along the same line as de Finetti's scheme: from an ontological point of view sentences can be either *true* or *false*, from an epistemic point of view they can be *certainly true*, *certainly false* or *unknown*. It is argued that the latter level is best expressed in a modal setting by introducing a modal operator $\Box$ for certainty ranging over atomic sentences, from which the modality $\Diamond$ expressing possibility can be defined in the usual way. The core idea is that a modal framework is conceptually more appropriate for modelling the lack of information than a three-valued approach. This motivates the reformulation of a class of three-valued semantics in terms of modal semantics based on epistemic states. This can be achieved by means of a translation of the truth-qualified statement into a modal system following a simple intuition: a true (false) propositional variable p is translated as $\Box p$ ($\Box\neg p$) representing the value certainty true (false); if p is unknown then its translation would be $\Diamond p \wedge \Diamond\neg p$. Thus the three-valued truth assignments are translated into sentences of a modal system with a bivalent semantics. The resulting semantics can be proved extensionally equivalent to the original one, in other words the translation preserves tautologies and inferences.

The argument of the paper does not apply immediately to framework introduced above. I introduce a third value as an intermediate step toward a probabilistic interpretation of graded truth, however this value is not to be interpreted epistemically, as *unknown*, rather in terms of objective uncertainty or indeterminateness. A treatment in modal terms would still be possible though, by considering a modal translation of the three-valued assignment into the modalities *determinately true*, *determinately false*, *indeterminate*. Nevertheless, in this case we would lack the main motivation which lies in the discrepancy between ontological and epistemic level, since there are not epistemic consideration involved. Indeed, the authors acknowledge that if the third value is interpreted ontologically, as e.g. *half-true*, then the confusion between truth and epistemic certainty motivating the translation simply disappears.

Also, considering the third value as a modality would undermine one of the crucial step of our argument, namely the treatment of objective uncertainty in terms of intermediate truth values beyond true and false. In our framework cases of objective uncertainty are modelled by introducing a new *semantic value*, accordingly, differences in uncertainty are accounted for in terms of difference in truth (more or less true than). This ultimately allows us to identify degrees of truth and objective probabilities. To this aim, it is essential that the intermediate state between true and false is taken as a truth value.

This also helps in clarifying the differences between my proposal and the framework of possibility theory (Dubois and Prade, 2001). Possibility theory formalises a comparative or graded notion of possibility in contrast with the treatment of necessity and possibility in modal logics as all-or-nothing syntactical notions. Again, the main difference is the subjective/objective interpretation. Possibility measures are meant to capture agent's epistemic attitude toward an event by measuring her degrees of belief or degrees of surprise (as an indicator of disbelief or subjective implausibility). An objective counterpart of possibility theory can be easily imagined and this would not be conceptually distant from our proposal, whose main element of novelty would still lie in the formalisation in terms of truth values instead of modalities.

Conclusions

The purpose of these conclusive remarks is two-fold: on the one hand, I summarise the main results and contributions of this work and, on the other, I suggest new related directions of research. I isolate the latter in paragraphs with bold titles.

The aim of the research reported in this doctoral dissertation was to lay down the formal foundations and explore the philosophical implications of *truth from comparison*, the idea that the truth of the sentences in a given formal language can be evaluated by means of binary comparisons between the sentences themselves, such as "the sentence ϕ is *less (more) true than* the sentence ψ". The standard approach consists instead in evaluating sentences by assigning them a certain value, the truth value, by means of functions, logical valuations. The motivation for this investigation lies on doubts and concerns surrounding the notion of truth values as objects on the one hand, and, on the greater philosophical plausibility of looking at comparative judgements instead, on the other.

Having motivated the interest of such an investigation, the aim was to provide a formalisation of truth from comparison which could act to all effects (i) as a method for evaluating sentences of a given formal language, or as a semantics if a logic is given, and (ii) as a foundation for the standard truth-value approach, in a precise sense borrowed from measurement theory. The idea is that starting with a purely comparative notion (qualitative perspective) we can define a functional counterpart thereof (quantitative perspective), by assigning values to objects being compared, in such a way that if an object is 'more' than the other with respect to a certain attribute then the value assigned to the former is greater than the value assigned to the latter (representation theorems). The main idea is taking qualitative structures as mathematical and conceptual foundation for quantitative structures (qualitative foundation). I proposed pairwise valuations as the formal counterpart of truth from comparison. Whereas pointwise valuations are truth-functional maps assigning possibly numerical values to sentences, or, in other words, homomorphisms from the algebra of terms to the algebra of truth values, pairwise valuations are binary relations defined over the set of sentences. I showed that the two goals, (i) and (ii), can be successfully pursued at three levels of generality:

1. language level: we consider a generic language with connectives,
2. abstract logic level: we consider a class of logics with given properties,
3. specific logic level: we consider the special case of infinite-valued logics.

I briefly go through them again in what follows.

At first, in order to lay down the minimal conditions for pairwise valuations, I considered an abstract setting with a propositional language and a set of connectives defined over it. Pairwise valuations over this language are axiomatically defined by requiring some structural properties: they are congruent preorders, namely reflexive, transitive and such that their symmetric part is a congruence with respect to the operations. These assumptions reflect the fact that truth from comparison is a ranking respecting the structure of the language. Since these comparisons are meant to be to all effects logical valuations, we are lead to consider families of pairwise valuations and admissibility conditions over them. In particular, I required the intersection pairwise valuation, the one expressing the comparisons which hold in all possible words, to be substitution-invariant. An admissible family of pairwise valuations induces a class of compatible models, that is algebras of the same type as the algebra of terms endowed with a compatible ordering. Among these models, a special role is played by irreducible algebras, the simplest ones. They are the best candidate to play the role of the set of truth values. I proved that any pairwise valuation in an admissible family induces at least one map from the sentences to an irreducible model of the family (Theorem 2.11 on page 35). These maps are the pointwise valuations representing the ordering. Pairwise valuations yield compatible sets of truth values, that is the universes of algebras similar to the algebra of terms, which cannot be further reduced, endowed with an ordering which preserves the truth comparisons. At this level we cannot properly talk of semantics (there is no logic yet) but it is clear that families of pairwise valuations provide a method for evaluating sentences which yield a compatible pointwise counterpart.

From the algebra to the logic. Theorem 2.11 establishes the minimal conditions necessary for a pairwise valuation in order to have a pointwise valuation as counterpart. If we wished to discriminate more among the possible models, then we should impose further constraints on pairwise valuations. It would be philosophically interesting to be able to choose desirable properties of the relation *more* or *less true* (on the basis of linguistic, philosophical, logical considerations), isolate accordingly a class of preordered algebras serving as models, and look at which logic is characterised by those algebras. The resulting logic could be rightly considered the logic of *more* or *less true* as axiomatised at the beginning. However, in order to do this a piece of mathematics is to be developed: a complete argebraization of pairwise valuations. This is a viable and promising way for future research.

As a second step, I moved to a less abstract framework by considering a specific propositional language with a 0-ary connective for *falsum* and a binary connective for implication. Moreover, I assumed a logic to be given abstractly, namely as a pair formed by language and deducibility relation defined over the language. I restricted attention to the class of Tarskian and strongly algebraizable logics which have as intended semantics a set of truth values partially ordered by the implication and with a greatest and smallest element, representing absolute truth and absolute falsity, respectively, which behave classically. Given a logic in this class, pointwise valuations for it are homomorphisms from the algebra of terms to algebras in the variety characterising the logic. Pairwise valuations are still congruent preorders; moreover, compatibility with the new logical setting is obtained by assuming that all theorems of the logic are true to maximum degree (Axiom (A.1)) and that the pairwise valuations preserve and reflect the ordering given by the implication (Axiom (A.2)). As a consequence of these conditions, pairwise valuations are also bounded preorders. Then I showed that an admissible family of pairwise valuations thus defined induces in a natural way a semantics for the logic at stake, by defining the notions of tautology and semantic consequence in terms of absolute truth preservation – goal (i). This semantics, that I called pairwise, can be proved to be strongly sound and complete with respect to the logic. Moreover, Theorem 2.22 on page 44 assures that pairwise valuations are representable by pointwise valuations for the logic – goal (ii).

Other consequence relations. In this work I considered a classical definition of logical consequence in terms of absolute truth preservation. In the literature, alternative definitions of logical consequence have been proposed for many-valued logic: for example validity as preservation of degrees of truth or verities (Edgington, 1992; Font, 2003) according to which there should be no drop in truth value from the premisses to the conclusion. Pairwise valuations are particularly suitable for expressing it as I pointed out in Footnote 2. It is certainly worth investigating this further and seeing how the pairwise semantics changes accordingly.

Among the wide class of logics for which I proved these results, I focused on an interesting special case: infinite-valued logics, also called fuzzy logics. Goals (i) and (ii) are clearly achievable for these logics: they follow as corollaries from the previous results. However, I also showed that for them more significant results can be obtained. First, having fixed a logic, we can propose alternative axiomatizations for truth from comparisons with a more semantic flavour, including for example specific properties of connectives. This has the positive upshot of making assumptions underlying a specific fuzzy logic transparent, even if it does not solve the problem of choosing a logic. Secondly, we can obtain numerical, and thus genuinely quantitative, representation results in the interval $[0, 1]$ (Theorems 3.3 and 3.7). From a formal point of view, considering pairwise valuations for fuzzy logic substantiates the claim that the latter are meant to model a comparative notion of truth.

Semilinear logics. The ordinal or comparative character of fuzzy logics has been emphasised in the mathematical literature on the topic. In recent works, fuzzy logics have been presented as the logics characterised by completeness with respect to a semantics based on linearly ordered algebras, thus abandoning the idea that the real-valued algebras are the only intended semantics for them (Běehounek and Cintula, 2006; Cintula and Noguera, 2011). Nevertheless, the philosophical discussion on the role and significance of fuzzy logics, almost exclusively related to vagueness, remains focused on real-valued semantics and numerical degrees of truth. As a future line of research, exploring formal and conceptual connections of the present approach with these recent developments in the study of fuzzy logics might contribute bridging the distance among this advanced piece of mathematics and the philosophical discussion which employs it as a model.

In this dissertation I argued that the just retraced formal investigation into truth from comparison has a great philosophical significance for discussions related to truth and logical valuedness. This philosophical relevance is best understood within the logic-as-modelling view, proposed by Cook, according to which formal systems do not have either a exclusively descriptive or instrumental purpose, they are models for the phenomenon they are meant to explicate, and, as such, they contain aspects which are representative of the phenomenon, with a descriptive role, and aspects which are merely artefact, with an instrumental role. I suggested that the many facets of the pre-formal notions (linguistic and naive theoretical aspects) are to be taken into account when distinguishing between representors and artefacts. With this in mind, I noted that all the pre-formal phenomena we have been considering – logical valuedness, many-valuedness, graded truth, vagueness – present this double nature which helps in clarifying the inadequacy of models based on the functional approach and paves the way for the alternative model based on pairwise comparisons. Indeed, I suggested that the charge of implausibility or interpretative difficulties for truth values follows from mistaking for representative aspects which are merely artefactual in the formalisation process. Truth from comparison and pairwise valuations provide a framework in which pointwise assignments of values are artefactual, whereas the comparative aspect is taken to be representative. That is why, the comparative perspective I propose proves to be a viable way of dealing with logical valuedness while avoiding explicit recourse to a given set of truth values. Moreover, by means of representation theorems, the comparative perspective ultimately also provides a philosophical justification of the use of exact values for evaluating the truth of sentences.

When this is applied to degrees of truth we have a clear example of how a process of formalisation triggers a positive feedback on the notion being formalised and contributes to clarifying it. I put forward pairwise valuations as an alternative formalisation for graded truth instead of infinite-valued pointwise valuations and I argued that this model allows to gain the mathematical, instrumental convenience of the numerical aspect and, at the same time, retain the plausibility of the comparative aspect, plausibility due to the fact that the comparative aspect is representative of the phenomenon of graded truth. This provides a philosophical account of what degrees of truth are and what their role in the model is, which was missing in the discussion concerning fuzzy logics: degrees of truth can be thought of as possible measures (or cardinalisations) for the relation *more* or *less true than*.

A substantial philosophical account of degrees of truth permits to shed new light on the philosophical discussion surrounding them. For example, one of the contributions of this investigation consists in taking a step toward the rehabilitation of infinite-valued logics as a model for vagueness. I showed that pairwise valuations can make a good model for vagueness. This new model for vagueness retains the properties that make the standard degree-theoretic account appealing: intuitions about a graded notion of truth, solution of the paradoxes which accounts for their original plausibility, mathematical convenience; and still mitigates some of its deficiencies or weak points: artificial precision and related objections and the lack of a philosophical account of what degrees of truth are.

Comparative logics. We have seen that the pairwise analysis of the sorites paradox inherits some difficulties typical of the numerical solution: the lack of homogeneity in the process of truth fading away and the mismatch between the P-ordering and the truth order, both related to the problem of basketball players. Degree-theoretical approaches to vagueness based on comparative logics introduced by Casari (1987), instead of fuzzy logics, do far better in avoiding this problem than those based on fuzzy logics, because they, in addition to degrees of approximate truth, consider also degrees of definite truth and falsity (Paoli, 1999, 2003). However, these accounts present problems when it comes to the interpretation of degrees (see Paoli, 2003, Section 3.3). Seeking for a combined approach having the benefit of both would be a step toward a complete and satisfactory treatment of graded predicates by appealing to the idea of graded truth.

First-order generalisation. In this investigation I restricted myself to propositional languages. However a better understanding of vagueness and vague predicates might require a first-order treatment. To this aim, a further step in the development of the theory of truth from comparison consists in generalising the goals (i) and (ii) also to predicative languages and logics.

Another issue related to degrees of truth that can be clarified thanks to the qualitative foundation concerns their relation to probabilities. I put forward the distinction between degrees of truth and degrees of belief as grounded on the differences between the orderings *more* or *less true than* and *more* or *less probable than*. This analysis provides strong reasons, both formal and conceptual, for maintaining the distinction, and, it also suggests how the distinction can be bridged. I explored a new possible way for combining degrees of truth and probabilities (and, in general, many-valuedness and probability theory) by proposing a probabilistic interpretation for graded truth meant to model undeterminateness in truth values of sentences. The philosophical feedback on the notions involved triggered by this new interpretation for graded truth is still to be fully explored, and so are possible formal connections with conditional probability and with other accounts of how degrees of truth and probabilities should be combined.

In this dissertation, I showed by means of philosophical discussions backed by formal results the fruitfulness and the philosophical interest of truth from comparison. I developed the general theory and considered the remarkable case of infinite-valued logics and degrees of truth. The general framework I propose can be also employed to deal with other logics and other philosophical problems related to logical valuedness and truth values, towards a better understanding of logico-philosophical issues related to truth.

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