Inviscid limit for Stochastic Navier-Stokes Equations under general initial conditions

Eliseo Luongo[1]
[1] Scuola Normale Superiore, Piazza dei Cavalieri,7, 56126 Pisa, Italy
Email: eliseo.luongo@sns.it

May 24, 2024

Abstract

We consider in a smooth and bounded two dimensional domain the convergence in the L^2 norm, uniformly in time, of the solution of the stochastic Navier-Stokes equations with additive noise and no-slip boundary conditions to the solution of the corresponding Euler equations. We prove, under general regularity on the initial conditions of the Euler equations, that assuming the dissipation of the energy of the solution of the Navier-Stokes equations in a Kato type boundary layer, then the inviscid limit holds.

Keywords: Inviscid limit; turbulence; additive noise; no-slip boundary conditions; boundary layer; energy dissipation.

1 Introduction

The study of the inviscid limit for the solutions of the Navier-Stokes equations is a classical topic in fluid mechanics. The Euler equations have very large classes of weak solutions, including non-dissipative ones [2], but the inviscid limit can in some cases furnish a selection principle [3]. In the case of domains without boundary several results are available in the deterministic case, see, for instance [5], [7], [9], [11], [17]. In the case of domains with boundary the difficulty of the problem changes drastically considering different boundary conditions also in the two dimensional case. Indeed, if we consider the so called Navier boundary conditions, some results are available both in the deterministic and in the stochastic case (see for example [4], [13]). The no-slip boundary conditions are more challenging. This is due to the appearance of the boundary layer. So far, only few results are available in this framework. They can be splitted in two macro-categories:

1. Conditioned results, namely proving that if the solutions of the Navier-Stokes equations have a particular behavior in the boundary layer, then the inviscid limit holds true. These are the most common kind of results available for what concern the inviscid limit with no-slip boundary conditions. See for instance [6], [12], [20], [21].
2. Unconditioned results. They are based on strong assumptions about the symmetry of the domain and of the data [14], [15], or real analytic data [18], or the vanishing of the Eulerian initial vorticity in a neighborhood of the boundary [16].

The results of this paper go in the first direction. In particular, our goal is to generalize the results of [12] to the stochastic framework and to not classical solutions of the Euler equations. We consider the stochastic Navier-Stokes equations with additive noise and no-slip boundary conditions in a smooth, bounded, two dimensional domain, proving that, under suitable assumptions on the behavior of their solutions in the boundary layer and of the additive noise, we have strong convergence to the solution of the deterministic Euler equations for vanishing viscosity.
In Section 2 we introduce the mathematical problem, giving some well known results about the well posedness of the stochastic Navier-Stokes equations with additive noise and the Euler equations, and stating our main theorems. In Sections 3 and 4 we prove our main theorems. Lastly in Section 5 we add some deterministic results related to Theorem 6 and Theorem 9.
Theorem 8 and Theorem 9 can be seen as introductory results for the analysis of the zero noise-zero viscosity limit following the idea of [1]. This kind of results are relevant for the analysis of a selection principle for the solutions of the Euler equations.

2 Main Results

Let $D \subseteq \mathbb{R}^2$ be smooth and bounded, $T > 0$ fixed and $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a filtered probability space.
Let Z be a separable Hilbert space, denote by $L^2(\mathcal{F}_{t_0}, Z)$ the space of square integrable random variables with values in Z, measurable with respect to $\mathcal{F}_{t_0}$. Moreover, denote by $C_{\mathcal{F}}([0,T]; Z)$ the space of continuous adapted processes $(X_t)_{t\in[0,T]}$ with values in Z such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_Z^2\right] < \infty$$

and by $L^2_{\mathcal{F}}(0,T;Z)$ the space of progressively measurable processes $(X_t)_{t\in[0,T]}$ with values in Z such that

$$\mathbb{E}\left[\int_0^T \|X_t\|_Z^2 \, dt\right] < \infty.$$

Denote by $L^2\left(D;\mathbb{R}^2\right)$ and $H^k\left(D;\mathbb{R}^2\right)$ the usual Lebesgue and Sobolev spaces and by $H_0^k\left(D;\mathbb{R}^2\right)$ the closure in $W^k(D)$ of smooth compact support vector valued functions. Set

$$H = \{f \in L^2(D;\mathbb{R}^2),\ \operatorname{div} f = 0,\ f \cdot n|_{\partial D} = 0\},\ V = H_0^1(D;\mathbb{R}^2) \cap H,\ D(A) = H^2 \cap V.$$

We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and the norm in H respectively. With a slight abuse of notations, sometimes, we will denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ also the inner product and the norm in $L^2\left(D;\mathbb{R}^2\right)$.
Denote by P the projection of $L^2\left(D;\mathbb{R}^2\right)$ on H and define the unbounded linear operator $A : D(A) \subseteq H \to H$ by the identity

$$\langle Av, w\rangle = \langle \Delta v, w\rangle$$

for all $v \in D(A),\ w \in H$. A will be called the Stokes operator. It is well known (see for example [19]) that A generates an analytic semigroup of negative type on H and moreover $V = D\left(\left(-A\right)^{1/2}\right)$.
Let us consider a sequence of real, independent Brownian motions $\{W_t^k\}_{k=1}^N$ adapted to $\mathcal{F}_t$ and a sequence of functions $\{\sigma_k\}_{k=1}^N \subseteq D(A)$. Let us, moreover, assume that $u_0^\nu \in L^2(\mathcal{F}_0, H)$.
Let us consider the stochastic Navier-Stokes equations below. Some physical motivations for the introduction of this model can be found in [8].

$$\begin{cases} du^\nu &= -(-\nu\Delta u^\nu + \nabla u^\nu \cdot u^\nu + \nabla p^\nu)dt + \nu^{\frac{1}{2}} \sum_{k=1}^N \sigma_k dW_t^k \ t \in [0,T] \\ u^\nu(0) &= u_0^\nu. \end{cases} \tag{1}$$

Definition 1 *Given $u_0^\nu \in L^2(\mathcal{F}_0, H)$, we say that a stochastic process u^ν is a weak solution of equation (1) if*

$$u^\nu \in C_{\mathcal{F}}([0,T]; H) \cap L^2_{\mathcal{F}}(0,T;V)$$

and for every $\phi \in D(A)$, we have

$$\langle u^\nu(t), \phi\rangle - \int_0^t b(u^\nu(s), \phi, u^\nu(s))\ ds = \langle u_0^\nu, \phi\rangle + \nu \int_0^t \langle u^\nu(s), A\phi\rangle\ ds + \nu^{\frac{1}{2}} \sum_{k=1}^N \langle \sigma_k, \phi\rangle W_t^k,$$

for every $t \in [0,T]$, $\mathbb{P} - a.s.$

Under previous assumptions on the coefficient σ_k, equation (1) is well posed. Indeed the following theorem holds, see [8].

Theorem 2 *If $u_0^\nu \in L^2(\mathcal{F}_0, H)$, $\{\sigma_k\}_{k=1}^N \subseteq D(A)$, there exists a unique weak solution of equation (1). Moreover the following relations hold:*

$$\mathbb{E}\left[\|u^\nu(t)\|^2\right] + 2\nu \int_0^t \mathbb{E}\left[\|u^\nu(s)\|_V^2\right]\ ds = \mathbb{E}\left[\|u_0^\nu\|^2\right] + t\nu \sum_{k=1}^N \|\sigma_k\|^2 \tag{2}$$

$$\mathbb{E}\left[sup_{t\in[0,T]} \|u^\nu(t)\|^2\right] \le \mathbb{E}\left[\|u_0^\nu\|^2_{L^2(D)}\right] + T\nu \sum_{k=1}^N \|\sigma_k\|^2 + K\nu^{\frac{1}{2}} \sqrt{\sum_{k=1}^N \mathbb{E}\left[\int_0^T \langle u^\nu(s), \sigma_k\rangle^2\ ds\right]}, \tag{3}$$

$$\|u^\nu(t)\|^2 + 2\nu \int_0^t \|\nabla u^\nu(s)\|^2_{L^2(D)}\ ds = \|u_0^\nu\|^2 + t\nu \sum_{k=1}^N \|\sigma_k\|^2 + 2\nu^{\frac{1}{2}} \sum_{k=1}^N \int_0^t \langle u^\nu(s), \sigma_k\rangle dW_s^k. \tag{4}$$

where K is independent from ν.

Equation (2) will be called energy equality in the following, instead equation (4) will be called Itô formula.
For our purposes we will need a different relation satisfied by u^ν that will be clarified by the following lemma.

Lemma 3 *Under the same assumptions of Theorem 2, if u^ν is a weak solution of equation (1), then for each $\phi \in C([0,T]\,;V) \cap C^1([0,T]\,;H)$*

$$\begin{aligned}\langle u^\nu(t), \phi(t)\rangle &= \langle u^\nu(0), \phi(0)\rangle + \int_0^t \langle u^\nu(s), \partial_s \phi(s)\rangle \, ds \\ &- \nu \int_0^t \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}} \phi(s)\rangle \, ds + \int_0^t b(u^\nu(s), \phi(s), u^\nu(s)) \, ds \\ &+ \nu^{\frac{1}{2}} \sum_{k=1}^N \langle \sigma_k, \phi(t)\rangle W_t^k - \nu^{\frac{1}{2}} \sum_{k=1}^N \int_0^t \langle \sigma_k, \phi(s)\rangle W_s^k \, ds\end{aligned}$$

for every $t \in [0,T]\,,\ \mathbb{P}-a.s.$

Proof. Thanks to the regularity of the weak solution u^ν, by density we have that for each $\phi \in V$

$$\begin{aligned}\langle u^\nu(t), \phi\rangle - \int_0^t b(u^\nu(s), \phi, u^\nu(s)) \, ds &= \langle u^\nu(0), \phi\rangle + \nu^{\frac{1}{2}} \sum_{k=1}^N \langle \sigma_k, \phi\rangle W_t^k \\ &- \nu \int_0^t \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}} \phi\rangle \, ds,\end{aligned}$$

for every $t \in [0,T]\,,\ \mathbb{P}-a.s.$ Therefore, due to previous relation and the regularity properties of u^ν and $\{W^k\}_{k=1}^N$, it exists a measurable set $N \subseteq \Omega$ such that

1. $\mathbb{P}(N) = 0$.
2. For each $\omega \in N^c$, $u^\nu(t)$ and W_t^k have continuous trajectories taking values in H and $\mathbb{R}$ respectively.
3. $$\begin{aligned}\langle u^\nu(t), \phi\rangle - \int_0^t b(u^\nu(s), \phi, u^\nu(s)) \, ds &= \langle u^\nu(0), \phi\rangle + \nu^{\frac{1}{2}} \sum_{k=1}^N \langle \sigma_k, \phi\rangle W_t^k \\ &- \nu \int_0^t \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}} \phi\rangle \, ds \quad \forall t \in [0,T],\ \phi \in V,\ \omega \in N^c.\end{aligned}$$
4. $$\sup_{t\in[0,T]} \|u^\nu(t)\| + \int_0^T \|u^\nu(s)\|_V^2 \, ds + \max_{k\in\{1,\dots N\}} \sup_{t\in[0,T]} |W_t^k| \le C(\omega,\nu) < +\infty \ \forall \omega \in N^c,$$
   where $C(\omega,\nu)$ is a constant possibly depending from $\omega \in \Omega,\ \nu > 0$.

The relation stated in the lemma then holds for each $\omega \in N^c$. Indeed, let $\phi(t) \in C^1([0,T]\,;H) \cap C([0,T]\,;V)$. Let, moreover, $\pi = \{0 = t_0 < t_1 < \dots < T_n = T\}$ be a partition of $[0,T]$. Thus, using the identities

$$\langle u^\nu(t_{i+1}), \phi(t_{i+1})\rangle - \langle u^\nu(t_{i+1}), \phi(t_i)\rangle = \int_{t_i}^{t_{i+1}} \langle u^\nu(t_{i+1}), \partial_s \phi(s)\rangle \, ds,$$

$$\langle \sigma_k W_{t_{i+1}}^k, \phi(t_{i+1})\rangle - \langle \sigma_k W_{t_{i+1}}^k, \phi(t_i)\rangle = \int_{t_i}^{t_{i+1}} \langle \sigma_k W_{t_{i+1}}^k, \partial_s \phi(s)\rangle \, ds,$$

we get

$$\begin{aligned}\langle u^\nu(t_{i+1}), \phi(t_{i+1})\rangle &= \langle u^\nu(t_i), \phi(t_i)\rangle - \int_{t_i}^{t_{i+1}} \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}} \phi(t_i)\rangle \, ds \\ &+ \int_{t_i}^{t_{i+1}} b(u^\nu(s), \phi(t_i), u^\nu(s)) \, ds \\ &+ \int_{t_i}^{t_{i+1}} \langle u^\nu(t_{i+1}), \partial_s \phi(s)\rangle \, ds \\ &- \nu^{\frac{1}{2}} \sum_{k=1}^N \int_{t_i}^{t_{i+1}} \langle \sigma_k W^k(t_{i+1}), \partial_s \phi(s)\rangle \, ds \\ &+ \nu^{\frac{1}{2}} \sum_{k=1}^N \left(\langle \sigma_k W_{t_{i+1}}^k, \phi(t_{i+1})\rangle - \langle \sigma_k W_{t_i}^k, \phi(t_i)\rangle \right).\end{aligned}$$

It implies

$$\begin{aligned}\langle u^\nu(T),\phi(T)\rangle &= \langle u^\nu(0),\phi(0)\rangle - \int_0^T \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}}\phi(s_\pi^-)\rangle\, ds \\ &+ \int_0^T b(u^\nu(s),\phi(s_\pi^-),u^\nu(s))\, ds \\ &+ \int_0^T \langle u^\nu(s_\pi^+),\partial_s\phi(s)\rangle\, ds - \nu^{\frac{1}{2}}\sum_{k=1}^N \int_0^T \langle \sigma_k W^k_{s_\pi^+},\partial_s\phi(s)\rangle\, ds \\ &+ \nu^{\frac{1}{2}}\sum_{k=1}^N \Big(\langle \sigma_k W^k_T,\phi(T)\rangle - \langle \sigma_k W^k_0,\phi(0)\rangle\Big).\end{aligned}$$

where $s_\pi^-(s) = t_i$ if $s \in [t_i, t_{i+1}]$ and $s_\pi^+(s) = t_{i+1}$ if $s \in [t_i, t_{i+1}]$. Taking the limit over a sequence of partitions π_l with size going to zero, we get, thanks to the regularity of u, ϕ, $\{W^k\}_{k=1}^N$ and dominated convergence theorem,

$$\begin{aligned}\langle u^\nu(T),\phi(T)\rangle &= \langle u^\nu(0),\phi(0)\rangle - \int_0^T \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}}\phi(s)\rangle\, ds \\ &+ \int_0^T b(u^\nu(s),\phi(s),u^\nu(s))\, ds \\ &+ \int_0^T \langle u^\nu(s),\partial_s\phi(s)\rangle\, ds - \nu^{\frac{1}{2}}\sum_{k=1}^N \int_0^T \langle \sigma_k W^k_s,\partial_s\phi(s)\rangle\, ds \\ &+ \nu^{\frac{1}{2}}\sum_{k=1}^N \Big(\langle \sigma_k W^k_T,\phi(T)\rangle - \langle \sigma_k W^k_0,\phi(0)\rangle\Big).\end{aligned}$$

The assumptions for applying dominated convergence theorem are satisfied for each $\omega \in N^c$. Indeed, thanks to property 2. we have

$$\begin{aligned}& b(u^\nu(s),\phi(s_{\pi_l}^-),u^\nu(s)) + \langle u^\nu(s_{\pi_l}^+),\partial_s\phi(s)\rangle \ - \nu^{\frac{1}{2}}\sum_{k=1}^N \langle \sigma_k W^k_{s^+_{\pi_l}},\partial_s\phi(s)\rangle \\ & \overset{|\pi_l|\to 0}{\to} b(u^\nu(s),\phi(s),u^\nu(s)) + \langle u^\nu(s),\partial_s\phi(s)\rangle \ - \nu^{\frac{1}{2}}\sum_{k=1}^N \langle \sigma_k W^k_s,\partial_s\phi(s)\rangle\end{aligned}$$

and by property 4.

$$\begin{aligned}&\left| b(u^\nu(s),\phi(s_{\pi_l}^-),u^\nu(s)) + \langle u^\nu(s_{\pi_l}^+),\partial_s\phi(s)\rangle \ - \nu^{\frac{1}{2}}\sum_{k=1}^N \langle \sigma_k W^k_{s^+_{\pi_l}},\partial_s\phi(s)\rangle \right| \\ &\le K(\omega,\nu,\phi)\left(\|u^\nu\|_V^2 + \sup_{t\in[0,T]}\|u^\nu(t)\| + \sum_{k=1}^N \|\sigma_k\|\right) \in L^1(0,T),\end{aligned}$$

where $K(\omega,\nu,\phi)$ is a constant depending from ω,ν,ϕ. The argument applies to a generic $t \in [0,T]$, hence we have the thesis. ■

Let us now consider the Euler equations

$$\begin{cases}\partial_t \bar{u} + \nabla\bar{u}\cdot\bar{u} + \nabla p = \bar{f} \ (x,t) \in D\times(0,T) \\ \operatorname{div}\bar{u} = 0 \\ \bar{u}\cdot n|_{\partial D} = 0 \\ \bar{u}(0) = \bar{u}_0\end{cases} \tag{5}$$

Definition 4 *Given $\bar{u}_0 \in H$, $f \in L^2(0,T;H)$ we say that $\bar{u} \in C(0,T;H)$ is a weak solution of equation (5) if for every $\phi \in C([0,T];V)\cap C^1([0,T];H)$*

$$\langle \bar{u}(t),\phi(t)\rangle = \langle \bar{u}_0,\phi(0)\rangle + \int_0^t \langle \bar{u}(s),\partial_s\phi(s)\rangle\, ds + \int_0^t b(\bar{u}(s),\phi(s),\bar{u}(s))\, ds + \int_0^t \langle \bar{f}(s),\phi(s)\rangle\, ds$$

for every $t \in [0,T]$ and the energy inequality

$$\|\bar{u}(t)\|^2 \leq \|\bar{u}_0\|^2 + 2\int_0^t \langle f(s), \bar{u}(s)\rangle \, ds$$

holds.

For what concern the well posedness of the Euler equations the following results hold true, see [10], [13].

Theorem 5 *If $\bar{u}_0 \in C^{1,\epsilon}(\bar{D}) \cap H$ and $\bar{f} \in C^{1,\epsilon}([0,T]\times\bar{D}) \cap L^2(0,T;H)$, then there exist $\bar{u}, \bar{p}$ classical solutions of equation (5). Moreover, $\bar{u}$, $\nabla\bar{u}$, p, $\nabla p \in C([0,T]\times\bar{D})$, $\bar{u}$ is unique and p is unique up to an arbitrary function of t which can be added to p.*

Theorem 6 *If $f = 0$, u is a weak solution of the Euler equations with initial condition $u_0 \in H$ and $\bar{u}$ is the unique weak solution of the Euler equations with initial condition $\bar{u}_0 \in H \cap C^{1,\epsilon}(\bar{D})$, then*

$$\|(u-\bar{u})(t)\|^2 \leq e^{2t\|\nabla\bar{u}\|_{L^\infty([0,T]\times D)}} \|u_0 - \bar{u}_0\|^2.$$

Calling

$$O_n = \left\{ u_0 \in H : \ \exists \bar{u}_0 \in H \cap C^{1,\epsilon}(\bar{D}), \ \|u_0 - \bar{u}_0\| < \frac{1}{n} e^{-3T\|\nabla\bar{u}\|_{L^\infty([0,T]\times D)}} \right\}$$

where $\bar{u}$ is the solution of the Euler equations with initial condition $\bar{u}_0$, then for each $u_0 \in \bigcap_{n\geq 1} O_n =: \tilde{O}$ there exists a unique $u \in C([0,T], H)$ weak solution of the Euler equations with initial condition u_0. Moreover the energy equality

$$\|u(t)\|^2 = \|u_0\|^2$$

holds.

For each $u_0 \in \tilde{O}$ we will say that the sequence $\{\bar{u}_0^m\}_{m\in\mathbb{N}}$ approximates u_0 in the sense of Theorem 6 if $\bar{u}_0^m \in H \cap C^{1,\epsilon}(\bar{D})$ and

$$\|u_0 - \bar{u}_0^m\| < \frac{1}{m} e^{-3T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}}$$

where $\bar{u}^m$ is the solution of the Euler equations with initial condition $\bar{u}_0^m$.
Lastly we introduce some results related to the boundary layer corrector of the solution of the Euler equations, see [12].

Proposition 7 *Under the assumptions of Theorem 5:*

- *it exists a smooth skew-symmetric matrix a such that $\bar{u} = \operatorname{div} a$ on ∂D and $a = 0$ on ∂D;*
- *let $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function such that $\xi(0) = 1$, $\xi(r) = 0$ if $r \geq 1$,*

  $$z : D \to \mathbb{R}^+, \ z(x) = \xi(\rho/\delta) \quad with \quad \rho = dist(x, \partial D)$$

  and δ a parameter which goes to 0 when ν goes to 0. Let, moreover, $v = \operatorname{div}(za)$. Then,

  $$\bar{u} - v \in C([0,T]; V) \cap C^1([0,T]; H).$$

  $\{x \in D : \rho(x) \leq \delta\}$ is the boundary strip of width δ that we denote by Γ_δ;
- *the following estimates hold true*

  $$\|v(t)\|_{L^\infty(D)} \leq K, \ \|v(t)\| \leq K\delta^{\frac{1}{2}}, \ \|\partial_t v(t)\| \leq K\delta^{\frac{1}{2}},$$

  $$\|\nabla v(t)\|_{L^\infty(D)} \leq K\delta^{-1}, \ \|\nabla v(t)\|_{L^2(D)} \leq K\delta^{-1/2}, \ \|\rho\nabla v(t)\|_{L^\infty(D)} \leq K,$$

  $$\|\rho^2\nabla v(t)\|_{L^\infty(D)} \leq K\delta, \ \|\rho\nabla v(t)\|_{L^2(D)} \leq K\delta^{\frac{1}{2}}.$$

  where the coefficient K depends from $\bar{u}$ and it is independent from t and δ.

Now we can state our main theorems. Since the stochastic term in equation (1) goes to 0 as $\nu \to 0$, we assume that the external force in the Euler equations is identically 0. Theorem 8 is a generalization of the results in [12] to this stochastic framework and also the idea of the proof is similar. In Theorem 9 we consider a wilder set of initial conditions.

Theorem 8 *If $\bar{u}_0 \in C^{1,\epsilon}(\bar{D})$ and under previous assumptions on u_0^ν and σ_k, if*

$$\lim_{\nu\to 0} \mathbb{E}\left[\|u_0^\nu - \bar{u}_0\|^2\right] = 0,$$

then the following conditions are equivalent:

1. $\lim_{\nu\to 0}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\nu-\bar{u}\|^2\right]=0.$
2. $u^\nu(t)\rightharpoonup\bar{u}(t)$ *in* $L^2(\Omega\times D)$ *for each* $t\in[0,T]$.
3. $\lim_{\nu\to 0}\nu\int_0^T\mathbb{E}\left[\|\nabla u^\nu(t)\|^2_{L^2(D)}\right]\,dt=0.$
4. $\lim_{\nu\to 0}\nu\int_0^T\mathbb{E}\left[\|\nabla u^\nu(t)\|^2_{L^2(\Gamma_{c\nu})}\right]\,dt=0.$

Theorem 9 *Assume* $u_0\in\tilde{O}$, $u_0^n\in L^2(\mathcal{F}_0,H)$, $\lim_{n\to+\infty}\mathbb{E}\left[\|u_0^n-u_0\|^2\right]=0$. *Let* u *be the solution of the Euler equations with initial condition* u_0, u^n *be the solution of the stochastic Navier-Stokes equations with viscosity* ν_n *and initial condition* u_0^n. *If*

$$\lim_{n\to+\infty}\nu_n=0,\quad \lim_{n\to+\infty}\nu_n\int_0^T\mathbb{E}\left[\|\nabla u^n(t)\|^2_{L^2(\Gamma_{c\nu_n})}\right]\,dt=0,$$

then

$$\lim_{n\to+\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^n-u\|^2\right]=0.$$

Remark 10 *Theorem 8 means that if the convergence does not take place, the energy dissipation within the boundary layer of width* $c\nu$ *must remain finite as* $\nu\to 0$. *This suggests that something violent must have happened.*

Remark 11 *Theorem 9 is new also in the deterministic framework, namely taking* $\sigma_k=0\ \forall\ k=1,\dots,n$. *In Section 5 we will prove this result in the deterministic framework for a non-zero external force.*

Remark 12 *In the following* K *will denote several constants, possibly changing their value line by line, but depending only from the solution of the Euler equations and its data,* $\{\sigma_k\}_{k=1}^N$ *and* T.

3 Proof of Theorem 8

The proof of theorem 8 follows from a preliminary weaker result, namely under the same assumptions

$$\lim_{\nu\to 0}\sup_{t\in[0,T]}\mathbb{E}\left[\|u^\nu-u\|^2\right]=0.$$

This is the analogous of the Kato's result in this stochastic framework.

Proposition 13 *Under the same assumptions of Theorem 8, if*

$$\lim_{\nu\to 0}\mathbb{E}\left[\|u_0^\nu-\bar{u}_0\|^2\right]=0,$$

then the following conditions are equivalent:

1. $\lim_{\nu\to 0}\sup_{t\in[0,T]}\mathbb{E}\left[\|u^\nu-\bar{u}\|^2\right]=0.$
2. $u^\nu(t)\rightharpoonup\bar{u}(t)$ *in* $L^2(\Omega\times D)$ *for each* $t\in[0,T]$.
3. $\lim_{\nu\to 0}\nu\int_0^T\mathbb{E}\left[\|\nabla u^\nu(t)\|^2_{L^2(D)}\right]\,dt=0.$
4. $\lim_{\nu\to 0}\nu\int_0^T\mathbb{E}\left[\|\nabla u^\nu(t)\|^2_{L^2(\Gamma_{c\nu})}\right]\,dt=0.$

Proof. 1. $\Rightarrow$ 2. and 3. $\Rightarrow$ 4. are obvious. We need only prove that 2. $\Rightarrow$ 3. and 4. $\Rightarrow$ 1.

2. $\Rightarrow$ 3. By the energy equality, for each $t=T$ we have

$$\nu\mathbb{E}\left[\int_0^T\|\nabla u^\nu(s)\|^2_{L^2(D)}\right]=\frac{1}{2}\mathbb{E}\left[\|u_0^\nu\|^2\right]-\frac{1}{2}\mathbb{E}\left[\|u^\nu(T)\|^2\right]+T\nu\sum_{k=1}^N\|\sigma_k\|^2.$$

Taking the limsup of this expression and exploiting the fact that under our assumptions it follows that

$$\mathbb{E}\left[\|u_0^\nu-\bar{u}_0\|^2\right]\to 0$$

$$\|\bar{u}(T)\|^2\le\liminf_{\nu\to 0}\mathbb{E}\left[\|u^\nu(T)\|^2\right],$$

we get the thesis.

4. $\Rightarrow$ 1. For each time t we have

$$\begin{aligned}
\mathbb{E}\left[\|(u^\nu-\bar{u})(t)\|^2\right] &= \mathbb{E}\left[\|u^\nu(t)\|^2\right] + \|\bar{u}(t)\|^2 - 2\mathbb{E}\left[\langle u^\nu(t),\bar{u}(t)\rangle\right] \\
&\overset{energy\ eq.}{\leq} \mathbb{E}\left[\|u_0^\nu\|^2\right] + t\nu\sum_k \|\sigma_k\|^2 + \|\bar{u}_0\|^2 - 2\mathbb{E}\left[\langle u^\nu(t),\bar{u}(t)\rangle\right] \\
&\overset{\mathbb{E}\left[\|u_0^\nu-\bar{u}_0\|^2\right]\to 0}{\leq} o(1) + 2\|\bar{u}_0\|^2 + t\nu\sum_k \|\sigma_k\|^2 - 2\mathbb{E}\left[\langle u^\nu(t),\bar{u}(t)\rangle\right] \\
&= o(1) + 2\|\bar{u}_0\|^2 + t\nu\sum_k \|\sigma_k\|^2 - 2\mathbb{E}\left[\langle u^\nu(t),\bar{u}(t)-v(t)\rangle\right] - 2\mathbb{E}\left[\langle u^\nu(t),v(t)\rangle\right].
\end{aligned}$$

Then

$$\mathbb{E}\left[\|(u^\nu-\bar{u})(t)\|^2\right] \leq o(1) + 2\|\bar{u}_0\|^2 + t\nu\sum_k \|\sigma_k\|^2 - 2\mathbb{E}\left[\langle u^\nu(t),\bar{u}(t)-v(t)\rangle\right] - 2\mathbb{E}\left[\langle u^\nu(t),v(t)\rangle\right]. \tag{6}$$

To analyze the second-last term we use the weak formulation satisfied by u^ν for the test function $\bar{u}-v$.

$$\begin{aligned}
&-2\langle u^\nu(t),(\bar{u}-v)(t)\rangle = -2\langle u^\nu(0),(\bar{u}-v)(0)\rangle - 2\int_0^t \langle u^\nu(s),\partial_s(\bar{u}-v)(s)\rangle\, ds + \\
&2\nu\int_0^t \langle (-A)^{\frac{1}{2}}u^\nu(s),(-A)^{\frac{1}{2}}(\bar{u}-v)(s)\rangle\, ds - 2\int_0^t b(u^\nu(s),(\bar{u}-v)(s),u^\nu(s))\, ds - \\
&2\nu^{\frac{1}{2}}\sum_k \langle \sigma_k,(\bar{u}-v)(t)\rangle W_t^k + 2\nu^{\frac{1}{2}}\sum_{k=1}^N \int_0^t \langle \sigma_k,(\bar{u}-v)(s)\rangle W_s^k\, ds.
\end{aligned}$$

Taking the expected value of the last expression we obtain

$$\begin{aligned}
&-2\mathbb{E}\left[\langle u^\nu(t),(\bar{u}-v)(t)\rangle\right] + 2\mathbb{E}\left[\|u_0^\nu\|^2\right] \overset{\mathbb{E}\left[\|u_0^\nu-\bar{u}_0\|^2\right]\to 0,\ \|v(t)\|\leq K\delta^{\frac{1}{2}}}{=} \\
&o(1) - 2\mathbb{E}\left[\int_0^t \langle u^\nu(s),\partial_s(\bar{u}-v)(s)\rangle\, ds\right] + 2\nu\mathbb{E}\left[\int_0^t \langle (-A)^{\frac{1}{2}}u^\nu(s),(-A)^{\frac{1}{2}}(\bar{u}-v)(s)\rangle\, ds\right] \\
&-2\mathbb{E}\left[\int_0^t b(u^\nu(s),(\bar{u}-v)(s),u^\nu(s))\, ds\right].
\end{aligned}$$

Moreover we have

$$\begin{aligned}
-\mathbb{E}\left[\langle u^\nu(s),\partial_s(\bar{u}-v)(s)\rangle\right] &\overset{energy\ eq.\ \|\partial_t v(t)\|\leq K\delta^{\frac{1}{2}}}{=} o(1) - \mathbb{E}\left[\langle u^\nu(s),\partial_s\bar{u}(s)\rangle\right] \\
&\overset{Euler\ eq.}{=} o(1) + \mathbb{E}\left[\langle u^\nu(s),\nabla\bar{u}(s)\cdot\bar{u}(s)\rangle\right],
\end{aligned}$$

$$\mathbb{E}\left[\langle u^\nu(t),v(t)\rangle\right] \leq \|v(t)\|\mathbb{E}\left[\|u^\nu(t)\|^2\right]^{\frac{1}{2}} \overset{energy\ eq.}{\leq} K\|v(t)\| \overset{\|v(t)\|\leq K\delta^{\frac{1}{2}}}{\leq} K\delta^{\frac{1}{2}} \to 0,$$

$$\mathbb{E}\left[\int_0^t b(u^\nu(s)-\bar{u}(s),\bar{u}(s),u^\nu(s)-\bar{u}(s))\, ds\right] \leq \|\nabla\bar{u}\|_{L^\infty([0,T]\times D)}\mathbb{E}\left[\int_0^t \|(u^\nu-\bar{u})(s)\|^2\, ds\right].$$

Inserting the relations above in equation (6) we get

$$\begin{aligned}
\mathbb{E}\left[\|(u^\nu-\bar{u})(t)\|^2\right] &\leq o(1) + 2\|\bar{u}_0\|^2 - 2\mathbb{E}\left[\|u_0^\nu\|^2\right] + t\nu\sum_{k=1}^N \|\sigma_k\|^2 + \\
&2\nu\mathbb{E}\left[\int_0^t \langle (-A)^{\frac{1}{2}}u^\nu(s),(-A)^{\frac{1}{2}}(\bar{u}-v)(s)\rangle\, ds\right] + 2\mathbb{E}\left[\int_0^t b(u^\nu(s),v(s),u^\nu(s))\, ds\right] + \\
&2\mathbb{E}\left[\int_0^t (b(u^\nu(s),\bar{u}(s),\bar{u}(s)) - b(u^\nu(s),\bar{u}(s),u^\nu(s)))\, ds\right] \\
&\overset{b(u^\nu,\bar{u},u^\nu)-b(u^\nu,\bar{u},\bar{u})=b(u^\nu-\bar{u},\bar{u},u^\nu-\bar{u})}{=} o(1) + t\nu\sum_{k=1}^N \|\sigma_k\|^2 \\
&+ 2\nu\mathbb{E}\left[\int_0^t \langle (-A)^{\frac{1}{2}}u^\nu(s),(-A)^{\frac{1}{2}}(\bar{u}-v)(s)\rangle\, ds\right] \\
&+ 2\mathbb{E}\left[\int_0^t b(u^\nu(s),v(s),u^\nu(s))\, ds\right] - 2\mathbb{E}\left[\int_0^t b(u^\nu(s)-\bar{u}(s),\bar{u}(s),u^\nu(s)-\bar{u}(s))\, ds\right].
\end{aligned}$$

If we call

$$R(s) = \nu \sum_k \|\sigma_k\|^2 + 2\nu \mathbb{E}\left[\langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}}(\bar{u} - v)(s)\rangle\right] + \mathbb{E}\left[b(u^\nu(s), v(s), u^\nu(s))\right]$$

we arrive to the integral relation

$$\mathbb{E}\left[\|(u^\nu - \bar{u}(t)\|^2\right] \leq o(1) + \int_0^t \left(K\mathbb{E}\left[\|(u^\nu - \bar{u})(s)\|^2\right] + R(s)\right) ds.$$

If we are able to prove that $\lim_{\nu\to 0} \int_0^t R(s)\ ds = 0$, then the thesis follows immediately via Gronwall's inequality. The convergence of the term related to σ_k is obvious. For what concerns the others:

$$\begin{aligned}
|\mathbb{E}\left[b(u^\nu(s), v(s), u^\nu(s))\right]| &\leq \mathbb{E}\left[\int_{\Gamma_\delta} |\nabla v(s)| \rho^2 \frac{|u^\nu(s)|^2}{\rho^2}\ dx\right] \\
&\overset{\|\rho^2 \nabla v(t)\|_{L^\infty(D)} \leq K\delta}{\leq} K\delta \mathbb{E}\left[\left\|\frac{u^\nu(s)}{\rho}\right\|^2_{L^2(\Gamma_\delta)}\right] \\
&\overset{Hardy-Littlewood\ ineq.}{\leq} K\delta \mathbb{E}\left[\|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)}\right],
\end{aligned}$$

$$\begin{aligned}
\left|\mathbb{E}\left[\nu\langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}}(\bar{u} - v)(s)\rangle\right]\right| &\leq \nu\mathbb{E}\left[\|\nabla u^\nu(s)\|_{L^2(D)} \|\nabla \bar{u}(s)\|_{L^2(D)}\right] + \nu\mathbb{E}\left[\|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)} \|\nabla v(s)\|_{L^2(\Gamma_\delta)}\right] \\
&\overset{\|\nabla v(t)\|_{L^2(D)} \leq K\delta^{-1/2}}{\leq} K\nu\mathbb{E}\left[\|\nabla u^\nu(s)\|_{L^2(D)}\right] + \nu K\delta^{-1/2}\mathbb{E}\left[\|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)}\right].
\end{aligned}$$

Taking $\delta = c\nu$, we have

$$R(t) \leq K\nu + K\nu\mathbb{E}\left[\|\nabla u^\nu(s)\|_{L^2(D)}\right] + \nu^{\frac{1}{2}} K\mathbb{E}\left[\|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)}\right] + K\nu\mathbb{E}\left[\|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)}\right].$$

Exploiting the assumption

$$\lim_{\nu\to 0} \nu \int_0^T \mathbb{E}\left[\|\nabla u^\nu(t)\|^2_{L^2(\Gamma_{c\nu})}\right]\ dt = 0,$$

previous estimates and energy equality, via Holder inequality we get $\lim_{\nu\to 0} \int_0^t R(s)\ ds = 0$ and then the thesis.

■

Corollary 14 *Under the same assumptions of Proposition 13, if*

$$\lim_{\nu\to 0} \nu \int_0^T \mathbb{E}\left[\|\nabla u^\nu(t)\|^2_{L^2(\Gamma_{c\nu})}\right]\ dt = 0,$$

then

$$\lim_{\nu\to 0} \mathbb{E}\left[\sup_{t\in[0,T]} \|u^\nu - \bar{u}\|^2\right] = 0.$$

Proof. Preliminarily, note that, starting from equation (3), we have

$$\begin{aligned}
\mathbb{E}\left[sup_{t\in[0,T]} \|u^\nu(t)\|^2\right] &\leq \mathbb{E}\left[\|u_0^\nu\|^2\right] + T\nu \sum_{k=1}^N \|\sigma_k\|^2 + K\nu^{\frac{1}{2}} \sqrt{\sum_{k=1}^N \mathbb{E}\left[\int_0^T \langle u^\nu(s), \sigma_k\rangle^2\ ds\right]} \\
&\leq \mathbb{E}\left[\|u_0^\nu\|^2\right] + T\nu \sum_{k=1}^N \|\sigma_k\|^2 + K\sqrt{\nu \sum_{k=1}^N \|\sigma_k\|^2} \mathbb{E}\left[\int_0^T \|u^\nu(s)\|^2\ ds\right]^{\frac{1}{2}} \\
&\leq \mathbb{E}\left[\|u_0^\nu\|^2\right] + T\nu^{\frac{1}{2}} \sum_{k=1}^N \|\sigma_k\|^2 + K\sqrt{\nu \sum_{k=1}^N \|\sigma_k\|^2} \left(\int_0^T \mathbb{E}\left[\|u_0^\nu\|^2\right] + s\nu \sum_{j=1}^N \|\sigma_j\|^2\ ds\right)^{\frac{1}{2}} \\
&\leq K < +\infty.
\end{aligned}$$

Now the proof is similar to the previous one. For each time t we have

$$\begin{aligned}
\|(u^\nu-\bar{u})(t)\|^2 &= \|u^\nu(t)\|^2+\|\bar{u}(t)\|^2-2\langle u^\nu(t),\bar{u}(t)\rangle \\
&\overset{Itô\ formula}{\leq} \|u_0^\nu\|^2+t\nu\sum_{k=1}^N\|\sigma_k\|^2+2\sum_{k=1}^N\nu^{\frac{1}{2}}\int_0^t\langle u^\nu(s),\sigma_k\rangle dW_s^k+\|\bar{u}_0\|^2-2\langle u^\nu(t),\bar{u}(t)\rangle \\
&= \|u_0^\nu\|^2+t\nu\sum_{k=1}^N\|\sigma_k\|^2+2\sum_{k=1}^N\nu^{\frac{1}{2}}\int_0^t\langle u^\nu(s),\sigma_k\rangle dW_s^k+\|\bar{u}_0\|^2-2\langle u^\nu(t),\bar{u}(t)-v(t)\rangle-2\langle u^\nu(t),v(t)\rangle.
\end{aligned}$$

Let us rewrite $\langle u^\nu(t),\bar{u}(t)-v(t)\rangle$ thanks to the weak formulation satisfied by u^ν

$$\begin{aligned}
-2\langle u^\nu(t),\bar{u}(t)-v(t)\rangle &= -2\langle u_0^\nu,(\bar{u}-v)(0)\rangle-2\int_0^t\langle u^\nu(s),\partial_s(\bar{u}-v)(s)\rangle\ ds \\
&\quad+2\nu\int_0^t\langle(-A)^{\frac{1}{2}}u^\nu(s),(-A)^{\frac{1}{2}}(\bar{u}-v)(s)\rangle\ ds-2\int_0^t b(u^\nu,\bar{u}-v,u^\nu)(s)\ ds \\
&\quad-2\nu^{\frac{1}{2}}\sum_{k=1}^N\langle\sigma_k,(\bar{u}-v)\rangle W_t^k+2\nu^{\frac{1}{2}}\sum_{k=1}^N\int_0^t\langle\sigma_k,(\bar{u}-v)(s)\rangle W_s^k\ ds.
\end{aligned}$$

Moreover, thanks to previous relation and the relations below:

$$-2\langle u^\nu(s),\partial_s(\bar{u}-v)(s)\rangle = 2\langle u^\nu(s),\partial_s v(s)\rangle+2\langle u^\nu(s),\nabla\bar{u}(s)\cdot\bar{u}(s)\rangle,$$

$$b(u^\nu,\bar{u},u^\nu)-b(u^\nu,\bar{u},\bar{u}) = b(u^\nu-\bar{u},\bar{u},u^\nu-\bar{u}),$$

we have at time t

$$\begin{aligned}
\|(u^\nu-\bar{u})(t)\|^2 &= \left(\|u_0^\nu\|^2+\|\bar{u}_0\|^2-2\langle u_0^\nu,(\bar{u}-v)(0)\rangle\right)+(t\nu\sum_{k=1}^N\|\sigma_k\|^2+2\nu^{\frac{1}{2}}\sum_{k=1}^N\int_0^t\langle u^\nu(s),\sigma_k\rangle dW_s^k \\
&\quad-2\nu^{\frac{1}{2}}\sum_{k=1}^N\langle\sigma_k,(\bar{u}-v)(t)\rangle W_t^k+2\nu^{\frac{1}{2}}\sum_{k=1}^N\int_0^t\langle\sigma_k,(\bar{u}-v)(s)\rangle W_s^k\ ds) \\
&\quad+\left(2\int_0^t b(u^\nu(s),v(s),u^\nu(s))\ ds-2\int_0^t b((u^\nu-\bar{u})(s),\bar{u}(s),(u^\nu-\bar{u})(s))\ ds\right)+ \\
&\quad(-2\langle u^\nu(t),v(t)\rangle+2\nu\int_0^t\langle(-A)^{\frac{1}{2}}u^\nu(s),(-A)^{\frac{1}{2}}(\bar{u}-v)(s)\rangle\ ds+2\int_0^t\langle u^\nu(s),\partial_s v(s)\rangle\ ds) \\
&= I_1(t)+I_2(t)+I_3(t)+I_4(t).
\end{aligned}$$

Thus

$$\mathbb{E}\left[sup_{t\in[0,T]}\|(u^\nu-u)(t)\|^2\right]\leq\mathbb{E}\left[sup_{t\in[0,T]}I_1(t)\right]+\mathbb{E}\left[sup_{t\in[0,T]}I_2(t)\right]+\mathbb{E}\left[sup_{t\in[0,T]}I_3(t)\right]+\mathbb{E}\left[sup_{t\in[0,T]}I_4(t)\right].$$

We will analyze all the terms in the right hand side one by one in order to get the thesis.

$$\begin{aligned}
E\left[\sup_{t\in[0,T]}I_1(t)\right] &= \mathbb{E}\left[\|u_0^\nu\|^2+\|\bar{u}_0\|^2-2\langle u_0^\nu,(\bar{u}-v)(0)\rangle\right] \\
&\leq -\mathbb{E}\left[\|u_0^\nu\|^2\right]+\|\bar{u}_0\|^2+2\mathbb{E}\left[\|u_0^\nu\|\|u_0^\nu-\bar{u}_0\|\right]+2\mathbb{E}\left[\|\bar{u}_0^\nu\|\|v(0)\|\right] \\
&\overset{\|v(t)\|\leq K\delta^{\frac{1}{2}},\ \mathbb{E}\left[\|u_0^\nu-\bar{u}_0\|^2\right]\to 0}{=} o(1)+K\delta^{\frac{1}{2}}.
\end{aligned}$$

The analysis of I_3 is similar to what we have done in Proposition 13. For this reason, in the computations below, some details have been omitted.

$$\begin{aligned}
\mathbb{E}\left[sup_{t\in[0,T]}I_3(t)\right] &\leq 2\mathbb{E}\left[sup_{t\in[0,T]}\int_0^t b(u^\nu(s),v(s),u^\nu(s))\ ds\right]+2\mathbb{E}\left[sup_{t\in[0,T]}\int_0^t b((u^\nu-\bar{u})(s),\bar{u}(s),(u^\nu-\bar{u})(s))\ ds\right] \\
&\leq 2\mathbb{E}\left[\int_0^T|b(u^\nu(s),v(s),u^\nu(s))|\ ds\right]+2\mathbb{E}\left[\int_0^T|b((u^\nu-\bar{u})(s),\bar{u}(s),(u^\nu-\bar{u})(s))|\ ds\right] \\
&\leq \mathbb{E}\left[\int_0^T\|\rho^2\nabla v(s)\|_{L^\infty(D)}\|u^\nu(s)\rho^{-1}\|_{L^2(D)}^2\right]+\mathbb{E}\left[\int_0^T\|(u^\nu-\bar{u})(s)\|^2\|\bar{u}(s)\|_{L^\infty(D)}\ ds\right] \\
&\leq 2K\delta\mathbb{E}\left[\int_0^T\|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)}^2\ ds\right]+K\mathbb{E}\left[\int_0^T\|(u^\nu-\bar{u})(s)\|^2\ ds\right].
\end{aligned}$$

The last term goes to 0 thanks to Proposition 13, the first term goes to 0 thanks to the assumptions choosing δ properly. More details will follow.
Let us analyze all the elements of I_2 exploiting previous energy equalities and some properties of Brownian motion:

$$\mathbb{E}\left[\sup_{t\in[0,T]} T\nu \sum_{k=1}^{N} \|\sigma_k\|^2\right] \leq K\nu$$

$$\mathbb{E}\left[sup_{t\in[0,T]} 2\sum_{k=1}^{N} \nu^{\frac{1}{2}} \int_0^t \langle u^\nu(s), \sigma_k\rangle dW_s^k\right] \overset{Doob's\ ineq.}{\leq} K\nu^{\frac{1}{2}}$$

$$\mathbb{E}\left[sup_{t\in[0,T]} 2\nu^{\frac{1}{2}} \sum_{k=1}^{N} \langle \sigma_k, (\bar{u}-v)(t)\rangle W_t^k\right] \leq K\nu^{\frac{1}{2}}\mathbb{E}\left[sup_{t\in[0,T]}|W_t^k|\right] \leq K\nu^{\frac{1}{2}}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} 2\nu^{\frac{1}{2}} \sum_{k=1}^{N} \int_0^t \langle \sigma_k, (\bar{u}-v)(s)\rangle W_s^k\ ds\right] \leq K\nu^{\frac{1}{2}}.$$

It remains only to analyze I_4. Some of the estimates below use ideas already presented. For this reason some details have been omitted.

$$2\nu\mathbb{E}\left[\sup_{t\in[0,T]} \int_0^t \langle (-A)^{\frac{1}{2}} u^\nu(s), (-A)^{\frac{1}{2}}(\bar{u}-v)(s)\rangle\ ds\right] \leq 2\nu\mathbb{E}\left[\int_0^T K\|\nabla u^\nu(s)\|_{L^2(D)} + K\delta^{-\frac{1}{2}}\|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)}\ ds\right]$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \langle u^\nu(t), v(t)\rangle\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]}\|u^\nu(t)\| \sup_{t\in[0,T]}\|v(t)\|\right] \leq \delta^{\frac{1}{2}}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\nu(t)\|^2\right]^{\frac{1}{2}} \leq K\delta^{\frac{1}{2}}$$

$$\mathbb{E}\left[sup_{t\in[0,T]} \int_0^t \langle u^\nu(s), \partial_s v(s)\rangle\ ds\right] \leq \|\partial_s v\|_{L^\infty(0,T;L^2(D))}\mathbb{E}\left[\int_0^T \|u^\nu(s)\|\ ds\right] \leq K\delta^{\frac{1}{2}}.$$

In conclusion, if we take $\delta = c\nu$, then

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} \|(u^\nu - u)(t)\|^2\right] &\leq o(1) + K\nu^{\frac{1}{2}} + 2K\nu\mathbb{E}\left[\int_0^T \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_{c\nu})}\ ds\right] + K\mathbb{E}\left[\int_0^T \|(u^\nu - \bar{u})(s)\|^2\ ds\right] \\ &\quad + K\nu + \nu K\mathbb{E}\left[\int_0^T \|\nabla u^\nu(s)\|_{L^2(D)}\ ds\right] + K\nu^{\frac{1}{2}}\mathbb{E}\left[\int_0^T \|\nabla u^\nu(s)\|_{L^2(\Gamma_{c\nu})}\ ds\right].\end{aligned}$$

It is clear the almost all the terms go to 0 thanks to the assumptions and Proposition 13. We need just to check that $\nu\mathbb{E}\left[\int_0^T \|\nabla u^\nu(s)\|_{L^2(D)}\ ds\right]$ and $\nu^{\frac{1}{2}}\mathbb{E}\left[\int_0^T \|\nabla u^\nu(s)\|_{L^2(\Gamma_{c\nu})}\ ds\right]$ behave properly. The required convergences are satisfied, indeed:

$$\nu\mathbb{E}\left[\int_0^T \|\nabla u^\nu(s)\|_{L^2(D)}\ ds\right] \leq \nu T\mathbb{E}\left[\int_0^T \|\nabla u^\nu(s)\|^2_{L^2(D)} ds\right]^{\frac{1}{2}} = \nu^{\frac{1}{2}} T\mathbb{E}\left[\nu\int_0^T \|\nabla u^\nu(s)\|^2_{L^2(D)} ds\right]^{\frac{1}{2}} \leq K\nu^{\frac{1}{2}}$$

$$\nu^{\frac{1}{2}}\mathbb{E}\left[\int_0^T \|\nabla u^\nu\|_{L^2(\Gamma_{c\nu})}\ ds\right] \leq K\mathbb{E}\left[\nu\int_0^T \|\nabla u^\nu\|^2_{L^2(\Gamma_{c\nu})}\ ds\right]^{\frac{1}{2}} \overset{\nu\to 0}{\to} 0.$$

The computations above complete the proof. ■
Theorem 8 follows immediately by Proposition 13 and Corollary 14.

4 Proof of Theorem 9

Similarly to Section 3, we start with a weaker result with the supremum in time outside the expected value to obtain the stronger one with the supremum in time inside the expected value. The idea behind both the proofs is simply to introduce an approximation of u_0 in the sense of Theorem 6, then

$$\|u^n - u\|^2 \leq 2\|u^n - \bar{u}^m\|^2 + 2\|\bar{u}^m - u\|^2,$$

where $\bar{u}^m$ is the solution of the Euler Equations with initial condition $\bar{u}_0^m \in H \cap C^{1,\epsilon}(\bar{D})$. Thus, the second term can be estimate via Theorem 6, the first one is analyzed exploiting techniques similar to the ones of the previous section.

Remark 15 *If $u_0 \in \tilde{O}$ and $\{\bar{u}_0^m\}_{m\in\mathbb{N}}$ approximates u_0 in the sense of Theorem 6, then*

$$\|u_0 - \bar{u}_0^m\| e^{2T\|\nabla \bar{u}^m\|_{L^\infty([0,T]\times D)}} \leq \frac{1}{m} e^{-T\|\nabla \bar{u}^m\|_{L^\infty([0,T]\times D)}}$$

$$\|u_0 - \bar{u}_0^m\| e^{2T\|\nabla \bar{u}^m\|_{L^\infty([0,T]\times D)}} T\|\nabla \bar{u}^m\|_{L^\infty([0,T]\times D)} \leq \frac{1}{m}.$$

Lemma 16 *Under the same assumptions of Theorem 9*

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}\left[\|(u^n - u)(t)\|^2\right] = 0.$$

Proof. Let $\{\bar{u}_0^m\}_{m\in\mathbb{N}}$ approximating u_0 in the sense of Theorem 6 and $\{\bar{u}_m\}_{m\in\mathbb{N}}$ the corresponding solutions of the Euler equations, then for each $t,\ n,\ m$ we have

$$\begin{aligned}\mathbb{E}\left[\|u^n(t) - u(t)\|^2\right] &\leq 2\mathbb{E}\left[\|u^n(t) - \bar{u}^m(t)\|^2\right] + 2\|\bar{u}^m(t) - u(t)\|^2 \\ &\overset{Thm\ 6}{\leq} \frac{2}{m^2} + 2\mathbb{E}\left[\|u^n(t) - \bar{u}^m(t)\|^2\right].\end{aligned}$$

We adapt the computations of the proof of Proposition 13 to analyze the second term, hence some explanations will be omitted. For each m and $\delta > 0$ fixed, let us introduce the corrector of the boundary layer v_m. v_m satisfies the estimates of Proposition 7 with respect to a constant depending from m and independent from t and δ, namely

$$\|v_m(t)\|_{L^\infty(D)} \leq K_m,\ \|v_m(t)\| \leq K_m\delta^{\frac{1}{2}},\ \|\partial_t v_m(t)\| \leq K_m\delta^{\frac{1}{2}},$$

$$\|\nabla v_m(t)\|_{L^\infty(D)} \leq K_m\delta^{-1},\ \|\nabla v_m(t)\|_{L^2(D)} \leq K_m\delta^{-1/2},\ \|\rho\nabla v_m(t)\|_{L^\infty(D)} \leq K_m,$$

$$\|\rho^2\nabla v_m(t)\|_{L^\infty(D)} \leq K_m\delta,\ \|\rho\nabla v_m(t)\|_{L^2(D)} \leq K_m\delta^{\frac{1}{2}}.$$

For each time t we have

$$\begin{aligned}\mathbb{E}\left[\|(u^n - \bar{u}^m(t)\|^2\right] &= \mathbb{E}\left[\|u^n(t)\|^2\right] + \|\bar{u}^m(t)\|^2 - 2\mathbb{E}\left[\langle u^n(t), \bar{u}^m(t)\rangle\right] \\ &\overset{energy\ eq.}{\leq} \mathbb{E}\left[\|u_0^n\|^2\right] + t\nu_n\sum_k\|\sigma_k\|^2 + \|\bar{u}_0^m\|^2 - 2\mathbb{E}\left[\langle u^n(t), (\bar{u}^m - v_m)(t)\rangle\right] + 2\mathbb{E}\left[\langle u^n(t), v_m(t)\rangle\right] \\ &= \mathbb{E}\left[\|u_0^n - u_0\|^2\right] + \|u_0\|^2 + 2\mathbb{E}\left[\langle u_0^n - u_0, u_0\rangle\right] + t\nu_n\sum_k\|\sigma_k\|^2 + \|\bar{u}_0^m - u_0\|^2 + \|u_0\|^2 \\ &+ 2\langle \bar{u}_0^m - u_0, u_0\rangle - 2\mathbb{E}\left[\langle u^n(t), \bar{u}^m(t) - v_m(t)\rangle\right] + 2\mathbb{E}\left[\langle u^n(t), v_m(t)\rangle\right] \\ &\leq \mathbb{E}\left[\|u_0^n - u_0\|^2\right] + 2\|u_0\|^2 + 2\mathbb{E}\left[\|u_0^n - u_0\|^2\right]^{1/2}\|u_0\| + \|\bar{u}_0^m - u_0\|^2 + 2\|\bar{u}_0^m - u_0\|\|u_0\| \\ &+ t\nu_n\sum_k\|\sigma_k\|^2 - 2\mathbb{E}\left[\langle u^n, \bar{u}^m - v_m\rangle\right] + 2\mathbb{E}\left[\langle u^n, v_m\rangle\right] \\ &\overset{\|v_m(t)\|\leq K_m\delta^{\frac{1}{2}}}{\leq} \mathbb{E}\left[\|u_0^n - u_0\|^2\right] + 2\|u_0\|^2 + K\mathbb{E}\left[\|u_0^n - u_0\|^2\right]^{1/2} + \|\bar{u}_0^m - u_0\|^2 + K\|\bar{u}_0^m - u_0\| \\ &+ K\nu_n - 2\mathbb{E}\left[\langle u^n(t), (\bar{u}^m - v_m)(t)\rangle\right] + K_m\delta^{\frac{1}{2}}.\end{aligned}$$

To analyze the second-last term we use the weak formulation satisfied by u^n, taking $\bar{u}^m - v_m$ as test function. We consider directly the expected value of the weak formulation. Exploiting the relation

$$\mathbb{E}\left[\langle u_0^n, \bar{u}_0^m\rangle\right] = \|u_0\|^2 + \mathbb{E}\left[\langle u_0^n - u_0, \bar{u}_0^m - u_0\rangle\right] + \mathbb{E}\left[\langle u_0^n - u_0, u_0\rangle\right] + \langle u_0, \bar{u}_0^m - u_0\rangle,$$

we get

$$
\begin{aligned}
-2\mathbb{E}\left[\langle u^n(t),(u^m-v_m)(t)\rangle\right]+2\mathbb{E}\left[\|u_0\|^2\right] &= -2\mathbb{E}\left[\langle u_0^n-u_0,\bar{u}_0^m-u_0\rangle\right]-2\mathbb{E}\left[\langle u_0^n-u_0,u_0\rangle\right]\\
&\quad -2\langle u_0,\bar{u}_0^m-u_0\rangle+2\mathbb{E}\left[\langle u_0^n,v_m(0)\rangle\right]\\
&\quad -2\mathbb{E}\left[\int_0^t\langle u^n(s),\partial_s(\bar{u}^m-v_m)(s)\rangle\,ds\right]\\
&\quad +2\nu_n\mathbb{E}\left[\int_0^t\langle(-A)^{\frac{1}{2}}u^n(s),(-A)^{\frac{1}{2}}(\bar{u}^m-v_m)(s)\rangle\,ds\right]\\
&\quad -\mathbb{E}\left[\int_0^t 2b(u^n(s),(\bar{u}^m-v_m)(s),u^n(s))\,ds\right]\\
&\overset{\|v_m(t)\|\le K_m\delta^{\frac{1}{2}}}{\le} 2\|\bar{u}_0^m-u_0\|\mathbb{E}\left[\|u_0^n-u_0\|^2\right]^{1/2}\\
&\quad +2\|u_0\|\mathbb{E}\left[\|u_0^n-u_0\|^2\right]^{1/2}+2\|u_0\|\|\bar{u}_0^m-u_0\|+2\mathbb{E}\left[\|u_0^n\|^2\right]^{1/2}K_m\delta^{\frac{1}{2}}\\
&\quad -2\mathbb{E}\left[\int_0^t\langle u^n(s),\partial_s(\bar{u}^m-v_m)(s)\rangle\,ds\right]\\
&\quad +2\nu_n\mathbb{E}\left[\int_0^t\langle(-A)^{\frac{1}{2}}u^n(s),(-A)^{\frac{1}{2}}(\bar{u}^m-v_m)(s)\rangle\,ds\right]\\
&\quad -2\mathbb{E}\left[\int_0^t b(u^n(s),(\bar{u}^m-v_m)(s),u^n(s))\,ds\right].
\end{aligned}
$$

Moreover

$$
\begin{aligned}
-\mathbb{E}\left[\langle u^n(s),\partial_s(\bar{u}^m-v_m)(s)\rangle\right] &\overset{energy\ eq.,\ \|\partial_t v_m(t)\|\le K_m\delta^{\frac{1}{2}}}{=} -\mathbb{E}\left[\langle u^n(s),\partial_s\bar{u}^m(s)\rangle\right]+K_m\delta^{\frac{1}{2}}\\
&\overset{Euler\ eq}{=} \mathbb{E}\left[\langle u^n(s),\nabla\bar{u}^m(s)\cdot\bar{u}^m(s)\rangle\right]+K_m\delta^{\frac{1}{2}}.
\end{aligned}
$$

Thanks to previous relations and noting that

$$
b(u^n(s),\bar{u}^m(s),u^n(s))-b(u^n(s),\bar{u}^m(s),\bar{u}^m(s))=b((u^n-\bar{u}^m)(s),\bar{u}^m(s),(u^n-\bar{u}^m)(s))
$$

we can continue the estimate of $\mathbb{E}\left[\|(u^n-\bar{u}^m)(t)\|^2\right]$:

$$
\begin{aligned}
\mathbb{E}\left[\|(u^n-\bar{u}^m)(t)\|^2\right] \le & K_m\delta^{\frac{1}{2}}+K\mathbb{E}\left[\|u_0^n-u_0\|^2\right]+\|\bar{u}_0^m-u_0\|^2+K\|\bar{u}_0^m-u_0\|+K\nu_n\\
&+K\mathbb{E}\left[\|u_0^n-u_0\|^2\right]^{1/2}+2\|\bar{u}_0^m-u_0\|\mathbb{E}\left[\|u_0^n-u_0\|^2\right]^{1/2}\\
&-2\mathbb{E}\left[\int_0^t b((u^n-\bar{u}^m)(s),\bar{u}^m(s),(u^n-\bar{u}^m)(s))\,ds\right]+2\mathbb{E}\left[\int_0^t b(u^n(s),v_m(s),u^n(s))\,ds\right]\\
&+2\nu_n\mathbb{E}\left[\int_0^t\langle(-A)^{\frac{1}{2}}u^n(s),(-A)^{\frac{1}{2}}(\bar{u}^m-v_m)(s)\rangle\,ds\right]\\
&\le K_m\delta^{\frac{1}{2}}+K\mathbb{E}\left[\|u_0^n-u_0\|^2\right]+\|\bar{u}_0^m-u_0\|^2+K\|\bar{u}_0^m-u_0\|+K\nu_n\\
&+K\mathbb{E}\left[\|u_0^n-u_0\|^2\right]^{1/2}+2\|\bar{u}_0^m-u_0\|\mathbb{E}\left[\|u_0^n-u_0\|^2\right]^{1/2}\\
&+2\mathbb{E}\left[\int_0^t b(u^n(s),v_m(s),u^n(s))\,ds\right]\\
&+2\nu_n\mathbb{E}\left[\int_0^t\langle(-A)^{\frac{1}{2}}u^n(s),(-A)^{\frac{1}{2}}(\bar{u}^m-v_m)(s)\rangle\,ds\right]\\
&+2\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}\mathbb{E}\left[\int_0^t\|(u^n-\bar{u}^m)(s)\|^2\,ds\right].
\end{aligned}
$$

Arguing as in the proof of Proposition 13 we have

$$\mathbb{E}\left[\int_0^t b(u^n(s), v_m(s), u^n(s))\ ds\right] \overset{\|\rho^2(t)\nabla v_m(t)\|_{L^\infty(D)}\leq K_m\delta}{\leq} K_m\delta\mathbb{E}\left[\int_0^T \|\nabla u^n(s)\|^2_{L^2(\Gamma_\delta)}\ ds\right]$$

$$\nu_n\mathbb{E}\left[\int_0^t \langle(-A)^{\frac{1}{2}}u^n(s), (-A)^{\frac{1}{2}}(\bar{u}^m - v_m)(s)\rangle\ ds\right]$$

$$\overset{\|\nabla v^m(t)\|_{L^2(D)}\leq K_m\delta^{-1/2}}{\leq} \|\nabla\bar{u}^m\|_{L^\infty(0,T;L^2(D))}\nu_n\mathbb{E}\left[\int_0^T \|\nabla u^n\|_{L^2(D)}\ ds\right] + \nu_n K_m\delta^{-1/2}\mathbb{E}\left[\int_0^T \|\nabla u^n\|_{L^2(\Gamma_\delta)}\ ds\right].$$

Thanks to the relations abobe we can continue the estimate of $\mathbb{E}\left[\|(u^n - \bar{u}^m)(t)\|^2\right]$:

$$\begin{aligned}\mathbb{E}\left[\|(u^n - \bar{u}^m)(t)\|^2\right] \leq{}& K_m\delta^{\frac{1}{2}} + K\mathbb{E}\left[\|u_0^n - u_0\|^2\right] + \|\bar{u}_0^m - u_0\|^2 + K\|\bar{u}_0^m - u_0\| + K\nu_n\\ &+ K\mathbb{E}\left[\|u_0^n - u_0\|^2\right]^{1/2} + 2\|\bar{u}_0^m - u_0\|\mathbb{E}\left[\|u_0^n - u_0\|^2\right]^{1/2} + K_m\delta\mathbb{E}\left[\int_0^T \|\nabla u^n(s)\|^2_{L^2(\Gamma_\delta)}\ ds\right]\\ &+ \|\nabla\bar{u}^m\|_{L^\infty(0,T;L^2(D))}\nu_n\mathbb{E}\left[\int_0^T \|\nabla u^n\|_{L^2(D)}\ ds\right] + \nu_n K_m\delta^{-1/2}\mathbb{E}\left[\int_0^T \|\nabla u^n(s)\|_{L^2(\Gamma_\delta)}\ ds\right]\\ &+ 2\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}\mathbb{E}\left[\int_0^t \|(u^n - \bar{u}^m)(s)\|^2\ ds\right].\end{aligned}$$

Taking $\delta = c\nu_n$, by Gronwall's inequality and Holder's inequality we have

$$\begin{aligned}sup_{t\in[0,T]}\mathbb{E}\left[\|(u^n - \bar{u}^m)(t)\|^2\right] \leq{}& \Bigg(K_m\nu_n^{1/2} + K\mathbb{E}\left[\|u_0^n - u_0\|^2\right] + \|\bar{u}_0^m - u_0\|^2 + K\|\bar{u}_0^m - u_0\| + K\nu_n\\ &+ K\mathbb{E}\left[\|u_0^n - u_0\|^2\right]^{1/2} + K_m\nu_n\mathbb{E}\left[\int_0^T \|\nabla u^n(s)\|^2_{L^2(\Gamma_\delta)}\ ds\right]\\ &+ \|\nabla\bar{u}^m\|_{L^\infty(0,T;L^2(D))}\sqrt{\nu_n}\mathbb{E}\left[\int_0^T \nu_n\|\nabla u^n(s)\|^2_{L^2(D)}\ ds\right]^{1/2}\\ &+ K_m\mathbb{E}\left[\nu_n\int_0^T \|\nabla u^n(s)\|^2_{L^2(\Gamma_\delta)}\ ds\right]^{\frac{1}{2}}\Bigg)e^{2T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}}.\end{aligned}$$

Taking the limsup with respect to n of the expression above for m fixed we have

$$\limsup_{n\to+\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\|(u^n - \bar{u}^m)(t)\|^2\right] \leq K\|\bar{u}_0^m - u_0\|e^{2T\|\nabla\bar{u}^m\|_{L^\infty([0,T\times D])}} \overset{Remark\ 15}{\leq} \frac{K}{m}e^{-T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}}. \tag{7}$$

Coming back to

$$\mathbb{E}\left[\|u^n(t) - u(t)\|^2\right] \leq \frac{2}{m^2} + 2\mathbb{E}\left[\|u^n(t) - \bar{u}^m(t)\|^2\right],$$

if we fix $\epsilon > 0$ and $\bar{m}$ such that $2\frac{1+K\bar{m}}{\bar{m}^2} < \epsilon$, then taking the limsup with respect to n of previous expression for $m = \bar{m}$ we have

$$\limsup_{n\to+\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\|u^n(t) - u(t)\|^2\right] \leq \epsilon.$$

We have the thesis from the arbitrariness of ϵ. ■

Proof of Theorem 9. Let $\{\bar{u}_0^m\}_{m\in\mathbb{N}}$ approximating u_0 in the sense of Theorem 6 and $\{\bar{u}_m\}_{m\in\mathbb{N}}$ the corresponding solutions of the Euler equations, then for each $t,\ n,\ m$ we have

$$\begin{aligned}\|u^n(t) - u(t)\|^2 &\leq 2\|u^n(t) - \bar{u}^m(t)\|^2 + 2\|\bar{u}^m(t) - u(t)\|^2\\ &\overset{Thm\ 6}{\leq} \frac{2}{m^2} + 2\|u^n(t) - \bar{u}^m(t)\|^2.\end{aligned}$$

We will adapt some computations presented in the proof of Corollary 14 and Lemma 16 to analyze the last term. For this reason we will omit some explanations. For each m and $\delta > 0$ fixed, let us introduce the corrector of the boundary layer

v_m, it satisfies previous estimates. For each time t we have

$$\begin{aligned}\|(u^n-\bar{u}^m)(t)\|^2 &= \|u^n(t)\|^2+\|\bar{u}^m(t)\|^2-2\langle u^n(t),\bar{u}^m(t)\rangle\\ &\overset{Itô\ formula}{\leq} \|u_0^n\|^2+t\nu_n\sum_{k=1}^N\|\sigma_k\|^2+2\sum_{k=1}^N\nu_n^{\frac{1}{2}}\int_0^t\langle u^n(s),\sigma_k\rangle dW_s^k\\ &+\|\bar{u}_0^m\|^2-2\langle u^n(t),(\bar{u}^m-v_m)(t)\rangle-2\langle u^n(t),v_m(t)\rangle.\end{aligned}$$

Let us rewrite $\langle u^n(t),(\bar{u}^m-v_m)(t)\rangle$ thanks to the weak formulation satisfied by u^n

$$\begin{aligned}-2\langle u^n(t),(\bar{u}^m-v_m)(t)\rangle &= -2\langle u_0^n,(\bar{u}^m-v_m)(0)\rangle-2\int_0^t\langle u^n(s),\partial_s(\bar{u}^m-v_m)(s)\rangle\ ds\\ &+2\nu_n\int_0^t\langle(-A)^{\frac{1}{2}}u^n(s),(-A)^{\frac{1}{2}}(\bar{u}^m-v_m)(s)\rangle\ ds-2\int_0^t b(u^n(s),(\bar{u}^m-v_m)(s),u^n(s))\ ds\\ &-2\sqrt{\nu_n}\sum_{k=1}^N\langle\sigma_k,(\bar{u}^m-v_m)(t)\rangle W_t^k+2\sqrt{\nu_n}\sum_{k=1}^N\int_0^t\langle\sigma_k,(\bar{u}^m-v_m)(s)\rangle W_s^k\ ds.\end{aligned}$$

Moreover

$$-2\langle u^n(s),\partial_s(\bar{u}^m-v_m)(s)\rangle = 2\langle u^n(s),\partial_s v^m(s)\rangle+2\langle u^n(s),\nabla\bar{u}^m(s)\cdot\bar{u}^m(s)\rangle.$$

Thanks to the relations above and noting that

$$b(u^n(s),\bar{u}^m(s),u^n(s))-b(u^n(s),\bar{u}^m(s),\bar{u}^m(s)) = b((u^n-\bar{u}^m)(s),\bar{u}^m(s),(u^n-\bar{u}^m)(s))$$

we have

$$\begin{aligned}\|(u^n-\bar{u}^m)(t)\|^2 &= (\|u_0^n\|^2+\|\bar{u}_0^m\|^2-2\langle u_0^n,(\bar{u}^m-v_m)(0)\rangle)+(t\nu_n\sum_{k=1}^N\|\sigma_k\|^2+2\sqrt{\nu_n}\sum_{k=1}^N\int_0^t\langle u^n(s),\sigma_k\rangle dW_s^k\\ &-2\sqrt{\nu_n}\sum_{k=1}^N\langle\sigma_k,(\bar{u}^m-v_m)(t)\rangle W_t^k+2\sqrt{\nu_n}\sum_{k=1}^N\int_0^t\langle\sigma_k,(\bar{u}^m-v_m)(s)\rangle W_s^k\ ds)\\ &+(2\int_0^t b(u^n(s),v_m(s),u^n(s))\ ds-2\int_0^t b((u^n-\bar{u}^m)(s),\bar{u}^m(s),(u^n-\bar{u}^m)(s))\ ds)+\\ &(-2\langle u^n(t),v_m(t)\rangle+2\nu_n\int_0^t\langle(-A)^{\frac{1}{2}}u^n(s),(-A)^{\frac{1}{2}}(\bar{u}^m-v_m)(s)\rangle\ ds+2\int_0^t\langle u^n(s),\partial_s v_m(s)\rangle\ ds)\\ &= I_1(t)+I_2(t)+I_3(t)+I_4(t).\end{aligned}$$

Thus

$$\mathbb{E}\left[sup_{t\in[0,T]}\|(u^n-\bar{u}^m)(t)\|^2\right]\leq\mathbb{E}\left[sup_{t\in[0,T]}I_1(t)\right]+\mathbb{E}\left[sup_{t\in[0,T]}I_2(t)\right]+\mathbb{E}\left[sup_{t\in[0,T]}I_3(t)\right]+\mathbb{E}\left[sup_{t\in[0,T]}I_4(t)\right]$$

We will analyze all the terms in the right hand side one by one in order to get the thesis.

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}I_1(t)\right] &= \mathbb{E}\left[\|u_0^n\|^2+\|\bar{u}_0^m\|^2-2\langle u_0^n,(\bar{u}^m-v_m)(0)\rangle\right]\\ &= -\mathbb{E}\left[\|u_0^n\|^2\right]+\|\bar{u}_0^m\|^2-2\mathbb{E}\left[\langle u_0^n,\bar{u}_0^m-u_0^n\rangle\right]+2\mathbb{E}\left[\langle u_0^n,v_m(0)\rangle\right]\\ &\leq -\mathbb{E}\left[\|u_0-u_0^n\|^2\right]-\|u_0\|^2-2\mathbb{E}\left[\langle u_0^n-u_0,u_0\rangle\right]+\|u_0\|^2+\|u_0-\bar{u}_0^m\|^2\\ &+2\langle u_0,\bar{u}_0^m-u_0\rangle+2\mathbb{E}\left[\|u_0^n\|\|\bar{u}_0^m-u_0^n\|\right]+2\mathbb{E}\left[\|u_0^n\|\right]\|v_m(0)\|\end{aligned}$$

$$\begin{aligned}&\overset{\|v_m(t)\|\leq K_m\delta^{\frac{1}{2}}}{\leq}\mathbb{E}\left[\|u_0^n-u_0\|^2\right]+K\mathbb{E}\left[\|u_0^n-u_0\|^2\right]^{1/2}+\|\bar{u}_0^m-u_0\|^2\\ &+K\|\bar{u}_0^m-u_0\|+K_m\delta^{\frac{1}{2}}\mathbb{E}\left[\|u_0^n\|^2\right]^{1/2}.\end{aligned}$$

The analysis of the others is similar to what we have done before:

$$\begin{aligned}
\mathbb{E}\left[sup_{t\in[0,T]} I_3(t)\right] &\leq 2\mathbb{E}\left[sup_{t\in[0,T]}\int_0^t b(u^n(s), v^m(s), u^n(s))\ ds\right] \\
&+ 2\mathbb{E}\left[sup_{t\in[0,T]}\int_0^t b((u^n-\bar{u}^m)(s), \bar{u}^m(s), (u^n-\bar{u}^m)(s))\ ds\right] \\
&\leq 2\mathbb{E}\left[\int_0^T |b(u^n(s), v^m(s), u^n(s))|\ ds\right] + 2\mathbb{E}\left[\int_0^T |b((u^n-\bar{u}^m)(s), \bar{u}^m(s), (u^n-\bar{u}^m)(s))|\ ds\right] \\
&\leq 2\mathbb{E}\left[\int_0^T \|\rho^2\nabla v_m(s)\|_{L^\infty(D)}\|u^n(s)\rho^{-1}\|^2\right] + 2\mathbb{E}\left[\int_0^T \|(u^n-\bar{u}^m)(s)\|^2\|\nabla\bar{u}^m(s)\|_{L^\infty(D)}\ ds\right] \\
&\leq 2K_m\delta\mathbb{E}\left[\int_0^T \|\nabla u^n(s)\|^2_{L^2(\Gamma_\delta)}\ ds\right] + 2\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}\mathbb{E}\left[\int_0^T \|(u^n-\bar{u}^m)(s)\|^2\ ds\right].
\end{aligned}$$

Let us analyze all the elements of I_2 exploiting previous energy equalities and properties of Brownian motion:

$$\mathbb{E}\left[\sup_{t\in[0,T]} T\nu_n\sum_{k=1}^N\|\sigma_k\|^2\right] < K\nu_n$$

$$\mathbb{E}\left[sup_{t\in[0,T]} 2\sum_{k=1}^N\sqrt{\nu_n}\int_0^t\langle u^n(s), \sigma_k\rangle dW_s^k\right] \overset{Doob's\ ineq}{\leq} K\sqrt{\nu_n}$$

$$\begin{aligned}
\mathbb{E}\left[sup_{t\in[0,T]} 2\sqrt{\nu_n}\sum_{k=1}^N\langle\sigma_k, (\bar{u}^m - v_m)(t)\rangle W_t^k\right] &\leq K\sqrt{\nu_n}\|\bar{u}^m - v_m\|_{L^\infty(0,T;L^2(D))}\mathbb{E}\left[sup_{t\in[0,T]}|W_t^k|\right] \\
&\leq K\sqrt{\nu_n}\|\bar{u}^m - v_m\|_{L^\infty(0,T;L^2(D))}
\end{aligned}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} 2\sqrt{\nu_n}\sum_{k=1}^N\int_0^t\langle\sigma_k, (u-v)(s)\rangle W_s^k\ ds\right] \leq K\sqrt{\nu_n}\|\bar{u}^m - v_m\|_{L^\infty(0,T;L^2(D))}.$$

It remains only to analyze I_4. Some of the estimates below use ideas already presented. For this reason some details have been omitted.

$$\begin{aligned}
2\nu_n\mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t\langle(-A)^{\frac{1}{2}}u^n(s), (-A)^{\frac{1}{2}}(\bar{u}^m - v_m)(s)\rangle\ ds\right] &\leq 2\nu_n\mathbb{E}\left[\int_0^T\|\nabla\bar{u}^m\|_{L^\infty(0,T;L^2(D))}\|\nabla u^n(s)\|\ ds\right] \\
&+ 2\nu_n\mathbb{E}\left[\int_0^T K_m\delta^{-\frac{1}{2}}\|\nabla u^n(s)\|_{L^2(\Gamma_\delta)}\ ds\right]
\end{aligned}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\langle u^n(t), v_m(t)\rangle\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]}\|u^n(t)\|\sup_{t\in[0,T]}\|v_m(t)\|\right] \leq K_m\delta^{\frac{1}{2}}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^n(t)\|^2\right]^{\frac{1}{2}} \leq K_m\delta^{\frac{1}{2}}$$

$$\mathbb{E}\left[sup_{t\in[0,T]}\int_0^t\langle u^n(s), \partial_s v_m(s)\rangle\ ds\right] \leq \|\partial_s v_m\|_{L^\infty(0,T;L^2(D))}\mathbb{E}\left[\int_0^T\|u^n(s)\|\ ds\right] \leq K_m\delta^{\frac{1}{2}}.$$

In conclusion, if we take $\delta = c\nu_n$, then by Holder's inequality

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\|(u^n-\bar{u}^m)(t)\|^2\right] &\leq \mathbb{E}\left[\|u_0^n - u_0\|^2\right] + K\mathbb{E}\left[\|u_0^n - u_0\|^2\right]^{1/2} + \|\bar{u}_0^m - u_0\|^2 + K\|\bar{u}_0^m - u_0\| \\
&+ K_m\sqrt{\nu_n} + K\nu_n + K\sqrt{\nu_n} + K\sqrt{\nu_n}\|\bar{u}_m - v_m\|_{L^\infty(0,T;L^2(D))} \\
&+ 2K_m\nu_n\mathbb{E}\left[\int_0^T\|\nabla u^n(s)\|^2_{L^2(\Gamma_{c\nu_n})}\ ds\right] + 2\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}\mathbb{E}\left[\int_0^T\|(u^n-\bar{u}^m)(s)\|^2\ ds\right] \\
&+ \|\nabla\bar{u}^m\|_{L^\infty(0,T;L^2(D))}\sqrt{\nu_n}\mathbb{E}\left[\nu_n\int_0^T\|\nabla u^n(s)\|^2_{L^2(D)}\ ds\right]^{1/2} \\
&+ 2K_m\mathbb{E}\left[\int_0^T\nu_n\|\nabla u^n(s)\|^2_{L^2(\Gamma_{c\nu_n})}\ ds\right]^{1/2} + K_m\nu_n.
\end{aligned}$$

Taking the limsup with respect to n of this expression for m fixed we have

$$\begin{aligned}\limsup_{n\to+\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\|(u^n-\bar{u}^m)(t)\|^2\right] &\leq \|\bar{u}_0^m-u_0\|^2+K\|\bar{u}_0^m-u_0\| \\ &+2T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}\limsup_{n\to+\infty}\left(\sup_{t\in[0,T]}\mathbb{E}\left[\|(u^n-\bar{u}_m)(t)\|^2\right]\right) \\ &\overset{eq.\ (7)}{\leq} \|\bar{u}_0^m-u_0\|^2+K\|\bar{u}_0^m-u_0\| \\ &+2T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}\frac{K}{m}e^{-T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}} \\ &\overset{Remark\ 15}{\leq} \|\bar{u}_0^m-u_0\|^2+K\|\bar{u}_0^m-u_0\|+\frac{K}{m}.\end{aligned}$$

Coming back to

$$\|u^n(t)-u(t)\|^2\leq\frac{2}{m^2}+2\|u^n(t)-\bar{u}^m(t)\|^2.$$

If we fix $\epsilon>0$ and $\bar{m}$ such that

$$2\|\bar{u}_0^{\bar{m}}-u_0\|^2+2K\|\bar{u}_0^{\bar{m}}-u_0\|+2\frac{K\bar{m}+1}{\bar{m}^2}<\epsilon,$$

then considering the expected value of the supremum in time of the previous expression for $m=\bar{m}$ we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u^n(t)-u(t)\|^2\right]\leq\frac{2}{\bar{m}^2}+2\mathbb{E}\left[\sup_{t\in[0,T]}\|u^n(t)-\bar{u}^{\bar{m}}(t)\|^2\right].$$

From the relations above, taking the limsup with respect to n of the last inequality we have

$$\limsup_{n\to+\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^n(t)-u(t)\|^2\right]<\epsilon.$$

We have the thesis from the arbitrariness of ϵ. ■

5 A Deterministic Remark

As anticipated in Remark 11, in this section we prove an inviscid limit result in the deterministic framework, analogous to Theorem 9 for a particular class of external forces. This result extends the setting considered by Kato in [12] and it is the object of Theorem 19.

Lemma 17 *If u is a weak solution of the Euler equations with initial condition $u_0\in H$ and external force $f\in L^2(0,T;H)$ and $\bar{u}$ is the unique weak solution of the Euler equations with initial condition $\bar{u}_0\in H\cap C^{1,\epsilon}(\bar{D})$ and external force $\bar{f}\in L^2(0,T;H)\cap C^{1,\epsilon}([0,T]\times\bar{D})$, then*

$$\begin{aligned}\|(u-\bar{u})(t)\|^2 \leq & e^{2t\|\nabla\bar{u}\|_{L^\infty([0,T]\times D)}}\Big(\|u_0-\bar{u}_0\|^2 \\ &+2\sqrt{T}\|f-\bar{f}\|_{L^2(0,T;H)}\left(\sqrt{2\|u_0\|^2+4T\|f\|^2_{L^2(0,T;H)}}+\sqrt{2\|\bar{u}_0\|^2+4T\|\bar{f}\|^2_{L^2(0,T;H)}}\right)\Big).\end{aligned}$$

For each $K\geq 1$, calling

$$\begin{aligned}O_n^K=\{&u_0\in H,\ f\in L^2(0,T;H):\ \exists\bar{u}_0\in H\cap C^{1,\epsilon}(\bar{D}),\ \bar{f}\in L^2(0,T;H)\cap C^{1,\epsilon}([0,T]\times\bar{D}), \\ &\|u_0-\bar{u}_0\|<\frac{1}{n}e^{-KT\|\nabla\bar{u}\|_{L^\infty([0,T]\times D)}},\ \|f-\bar{f}\|<\frac{1}{n^2}e^{-2KT\|\nabla\bar{u}\|_{L^\infty([0,T]\times D)}}\}\end{aligned}$$

where $\bar{u}$ is the solution of the Euler equations with initial condition $\bar{u}_0$ and external force $\bar{f}$, then for each $(u_0,f)\in\bigcap_{n\geq1}O_n^K=:\widetilde{O^K}$ there exists a unique $u\in C([0,T],H)$ weak solution of the Euler equations with initial condition u_0 and external force f. Moreover the energy equality

$$\|u(t)\|^2=\|u_0\|^2+2\int_0^t\langle f(s),u(s)\rangle\,ds$$

holds.

Proof.

Estimate: For what concern the solution with smooth initial condition and external force, thanks to Theorem 5, $\bar{u}$ is a classical solution and the energy equality holds, namely

$$\begin{aligned}\|\bar{u}(t)\|^2 &= \|\bar{u}_0\|^2 + 2\int_0^t \langle \bar{f}(s), \bar{u}(s)\rangle \, ds \le \|\bar{u}_0\|^2 + 2\int_0^t \|\bar{f}(s)\|\|\bar{u}(s)\| \, ds \\ &\overset{Young's\ ineq.}{\le} \|\bar{u}_0\|^2 + 2T\int_0^T \|\bar{f}(s)\|^2 \, ds + \int_0^T \frac{\|\bar{u}(s)\|^2}{2T} \, ds \\ &\le \|\bar{u}_0\|^2 + 2T\|\bar{f}\|^2_{L^2(0,T;H)} + \frac{\|\bar{u}\|^2_{L^\infty(0,T;H)}}{2}.\end{aligned}$$

Thus

$$\sup_{t\in[0,T]} \|\bar{u}(t)\|^2 \le 2\|\bar{u}_0\|^2 + 4T\|\bar{f}\|^2_{L^2(0,T;H)}. \tag{8}$$

The same estimate holds for any weak solution of the Euler equations (if it exists) with initial condition u_0 and external force f thanks to energy inequality and the same computations. Let us consider the weak formulation satisfied by u using as test function $\bar{u}$. For each time t we have

$$\begin{aligned}-\langle u(t), \bar{u}(t)\rangle &= -\langle u_0, \bar{u}_0\rangle - \int_0^t \langle u(s), \partial_s \bar{u}(s)\rangle \, ds - \int_0^t b(u(s), \bar{u}(s), u(s)) \, ds - \int_0^t \langle f(s), \bar{u}(s)\rangle \, ds \\ &= -\langle u_0, \bar{u}_0\rangle - \int_0^t b((u-\bar{u})(s), \bar{u}(s), (u-\bar{u})(s)) \, ds - \int_0^t \langle f(s), \bar{u}(s)\rangle \, ds - \int_0^t \langle \bar{f}(s), u(s)\rangle \, ds \\ &\le -\langle u_0, \bar{u}_0\rangle + \|\nabla \bar{u}\|_{L^\infty([0,T]\times D)} \int_0^t \|(u-\bar{u})(s)\|^2 \, ds - \int_0^t \langle f(s), \bar{u}(s)\rangle \, ds - \int_0^t \langle \bar{f}(s), u(s)\rangle \, ds.\end{aligned}$$

Therefore, considering the difference between u and $\bar{u}$, we have

$$\begin{aligned}\|u(t)-\bar{u}(t)\|^2 &= \|u(t)\|^2 + \|\bar{u}(t)\|^2 - 2\langle u(t), \bar{u}(t)\rangle \\ &\overset{energy\ ineq.}{\le} \|\bar{u}_0\|^2 + 2\int_0^t \langle \bar{f}(s), \bar{u}(s)\rangle \, ds + \|u_0\|^2 + 2\int_0^t \langle f(s), u(s)\rangle \, ds - 2\langle u(t), \bar{u}(t)\rangle \\ &\overset{weak\ form.}{\le} \|u_0-\bar{u}_0\|^2 + 2\|\nabla\bar{u}\|_{L^\infty([0,T]\times D)} \int_0^t \|(u-\bar{u})(s)\|^2 \, ds \\ &+ 2\int_0^t \|(u-\bar{u})(s)\|\|\left(f-\bar{f}\right)(s)\| \, ds \\ &\le \|u_0-\bar{u}_0\|^2 + 2\|\nabla\bar{u}\|_{L^\infty([0,T]\times D)} \int_0^t \|(u-\bar{u})(s)\|^2 \, ds \\ &+ 2\sqrt{T}\|f-\bar{f}\|_{L^2(0,T;H)} \left(\|u\|_{L^\infty(0,T;H)} + \|\bar{u}\|_{L^\infty(0,T;H)}\right) \\ &\overset{(8)}{\le} \|u_0-\bar{u}_0\|^2 + 2\|\nabla\bar{u}\|_{L^\infty([0,T]\times D)} \int_0^t \|(u-\bar{u})(s)\|^2 \, ds \\ &+ 2\sqrt{T}\|f-\bar{f}\|_{L^2(0,T;H)} \left(\sqrt{2\|\bar{u}_0\|^2 + 4T\|\bar{f}\|^2_{L^2(0,T;H)}} + \sqrt{2\|u_0\|^2 + 4T\|f\|^2_{L^2(0,T;H)}}\right).\end{aligned}$$

By Grownall's inequality the thesis follows, namely

$$\begin{aligned}\|(u-\bar{u})(t)\|^2 \le e^{2t\|\nabla\bar{u}\|_{L^\infty([0,T]\times D)}} \Bigg(&\|u_0-\bar{u}_0\|^2 \\ &+ 2\sqrt{T}\|f-\bar{f}\|_{L^2(0,T;H)} \left(\sqrt{2\|u_0\|^2 + 4T\|f\|^2_{L^2(0,T;H)}} + \sqrt{2\|\bar{u}_0\|^2 + 4T\|\bar{f}\|^2_{L^2(0,T;H)}}\right)\Bigg).\end{aligned}$$

Existence: Let $(u_0,\ f) \in \widetilde{O^K}$ and $\{(\bar{u}_0^n, \bar{f}^n)\}_{n\in\mathbb{N}}$ a sequence which approximates (u, f) in the sense of the lemma, namely $\bar{u}_0^n \in H \cap C^{1,\epsilon}(\bar{D}),\ \bar{f}^n \in L^2(0,T;H) \cap C^{1,\epsilon}([0,T]\times\bar{D})$ and

$$\|u_0 - \bar{u}_0^n\| < \frac{1}{n} e^{-KT\|\nabla\bar{u}^n\|_{L^\infty([0,T]\times D)}}$$

$$\|f - \bar{f}^n\|_{L^2(0,T;H)} < \frac{1}{n^2} e^{-2KT\|\nabla\bar{u}^n\|_{L^\infty([0,T]\times D)}},$$

where $\bar{u}^n$ is the solution of the Euler equations with initial condition $\bar{u}_0^n$ and external force $\bar{f}^n$. We will prove that $\{\bar{u}^n\}$ is a Cauchy sequence in $C(0,T;H)$ and the solution of the Euler equations with initial condition u_0 and external force f is unique.
Preliminarily, note that if $\|a-b\|^2 \leq \alpha$, then $\|a\|^2 \leq 4\|b\|^2 + \frac{4}{3}\alpha$.
For what concern uniqueness, if u^1 and u^2 are two solutions of the Euler equations with initial condition u_0 and external force f, then at time we have t

$$\begin{aligned}
\|\left(u^1-u^2\right)(t)\|^2 &\leq 2\|\left(u^1-\bar{u}^n\right)(t)\|^2 + 2\|\left(u^2-\bar{u}^n\right)(t)\|^2 \\
&\leq 4e^{2t\|\nabla\bar{u}^n\|_{L^\infty([0,T]\times D)}}\Bigg(\|u_0-\bar{u}_0^n\|^2 \\
&+ 2\sqrt{T}\|f-\bar{f}^n\|_{L^2(0,T;H)}\left(\sqrt{2\|u_0\|^2+4T\|f\|^2_{L^2(0,T;H)}}+\sqrt{2\|\bar{u}_0^n\|^2+4T\|\bar{f}^n\|^2_{L^2(0,T;H)}}\right)\Bigg) \\
&\leq \frac{4}{n^2}\left(1+2\sqrt{T}\left(\sqrt{2\|u_0\|^2+4T\|f\|^2_{L^2(0,T;H)}}+\sqrt{8\|u_0\|^2+16T\|f\|^2_{L^2(0,T;H)}+\frac{8+16T}{3}}\right)\right).
\end{aligned}$$

From the last inequality and the arbitrariness of n the uniqueness of the solution is evident. Lastly, let us consider $\|\bar{u}^n(t)-\bar{u}^m(t)\|^2$ for $n \geq m$. We have

$$\begin{aligned}
\|\bar{u}^n(t)-\bar{u}^m(t)\|^2 &\leq e^{2T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}}\Bigg(\|\bar{u}_0^n-\bar{u}_0^m\|^2 \\
&+ 2\sqrt{T}\|\bar{f}^n-\bar{f}^m\|_{L^2(0,T;H)}\left(\sqrt{2\|\bar{u}_0^m\|^2+4T\|\bar{f}^m\|^2_{L^2(0,T;H)}}+\sqrt{2\|\bar{u}_0^n\|^2+4T\|\bar{f}^n\|^2_{L^2(0,T;H)}}\right)\Bigg) \\
&\leq e^{2T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}}\Bigg(2\|\bar{u}_0^n-u_0\|^2+2\|\bar{u}_0^m-u_0\|^2 \\
&+ 2\sqrt{T}\|\bar{f}^n-f\|_{L^2(0,T;H)}\left(\sqrt{2\|\bar{u}_0^m\|^2+4T\|\bar{f}^m\|^2_{L^2(0,T;H)}}+\sqrt{2\|\bar{u}_0^n\|^2+4T\|\bar{f}^n\|^2_{L^2(0,T;H)}}\right) \\
&+ 2\sqrt{T}\|f-\bar{f}^m\|_{L^2(0,T;H)}\left(\sqrt{2\|\bar{u}_0^m\|^2+4T\|\bar{f}^m\|^2_{L^2(0,T;H)}}+\sqrt{2\|\bar{u}_0^n\|^2+4T\|\bar{f}^n\|^2_{L^2(0,T;H)}}\right)\Bigg) \\
&\leq K\left(T,\|u_0\|,\|f\|_{L^2(0,T;H)}\right)\left(\frac{1}{n^2}+\frac{1}{m^2}\right),
\end{aligned}$$

where $K\left(T,\|u_0\|,\|f\|_{L^2(0,T;H)}\right)$ is a constant depending from T, $\|u_0\|$ and $\|f\|_{L^2(0,T;H)}$. The last inequality implies existence.

Energy: Let $(u_0,\ f) \in \widetilde{O^K}$ and $\{(\bar{u}_0^n,\bar{f}^n)\}_{n\in\mathbb{N}}$ a sequence which approximates (u,f) in the sense of the lemma as described in the previous step. Then for each $n \in \mathbb{N}$

$$\|\bar{u}^n(t)\|^2 = \|\bar{u}_0^n\|^2 + \int_0^t \langle \bar{u}^n(s), \bar{f}^n(s)\rangle\, ds.$$

Exploiting the fact that $\bar{u}^n \overset{C([0,T];H)}{\longrightarrow} u,\ \bar{f}^n \overset{L^2(0,T;H)}{\longrightarrow} f$ we get easily the thesis.

■

Calling $\tilde{O} := \widetilde{O^2}$, for each $(u_0,f) \in \tilde{O}$ we will say that $\{(\bar{u}_0^m,\bar{f}^m)\}_{m\in\mathbb{N}}$ approximates (u_0,f) in the sense of Lemma 17 if $\bar{u}_0^m \in H \cap C^{1,\epsilon}(\bar{D}),\ \bar{f}^m \in L^2(0,T;H)\cap C^{1,\epsilon}([0,T]\times\bar{D})$ and

$$\|u_0-\bar{u}_0^m\| < \frac{1}{m}e^{-2T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}}$$

$$\|f-\bar{f}^m\|_{L^2(0,T;H)} < \frac{1}{m^2}e^{-4T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}}$$

where $\bar{u}^m$ is the solution of the Euler equations with initial condition $\bar{u}_0^m$ and external force $\bar{f}^m$.

Remark 18 *If $(u_0, f) \in \tilde{O}$ and $\{(\bar{u}_0^m, \bar{f}^m)\}_{m\in\mathbb{N}}$ approximates (u_0, f) in the sense of Lemma 17, then*

$$\begin{aligned}
\|(u-\bar{u}^m)(t)\|^2 &\le e^{2t\|\nabla \bar{u}^m\|_{L^\infty([0,T]\times D)}}\Bigg(\|u_0-\bar{u}_0^m\|^2 \\
&+ 2\sqrt{T}\|f-\bar{f}^m\|_{L^2(0,T;H)}\left(\sqrt{2\|u_0\|^2+4T\|f\|^2_{L^2(0,T;H)}}+\sqrt{2\|\bar{u}_0^m\|^2+4T\|\bar{f}^m\|^2_{L^2(0,T;H)}}\right)\Bigg) \\
&\le \frac{1}{m^2}\left(1+2\sqrt{T}\left(\sqrt{2\|u_0\|^2+4T\|f\|^2_{L^2(0,T;H)}}+\sqrt{8\|u_0\|^2+16T\|f\|^2_{L^2(0,T;H)}+\frac{8+16T}{3}}\right)\right) \\
&\le \frac{K\left(T,\|u_0\|,\|f\|_{L^2(0,T;H)}\right)}{m^2},
\end{aligned}$$

where $K\left(T,\|u_0\|,\|f\|_{L^2(0,T;H)}\right)$ is a constant depending from $T, \|u_0\|$ and $\|f\|_{L^2(0,T;H)}$.

Thanks to Lemma 17 we are able to prove a Kato type inviscid limit result also in the deterministic framework.

Theorem 19 *If $(u_0, f) \in \tilde{O}$, $u_0^n \in H$, $f^n \in L^2(0,T;H)$,*

$$\lim_{n\to+\infty}\|u_0^n-u_0\|=0, \quad \lim_{n\to+\infty}\|f^n-f\|_{L^2(0,T;H)}=0.$$

Let u be the solution of the Euler equations with initial condition u_0 and external force f, u^n be the solution of the deterministic Navier-Stokes equations with viscosity ν_n, initial condition u_0^n and external force f^n. If

$$\lim_{n\to+\infty}\nu_n=0, \quad \lim_{n\to+\infty}\nu_n\int_0^T\|\nabla u^n(t)\|^2_{L^2(\Gamma_{c\nu_n})}\,dt=0,$$

then

$$\lim_{n\to+\infty}\sup_{t\in[0,T]}\|u^n(t)-u(t)\|^2=0.$$

Proof. The proof is an adaptation of previous stochastic arguments, the only novelty is the presence of deterministic external forces. For this reason we just give details on the new elements.
Let $\{(\bar{u}_0^m, \bar{f}^m)\}_{m\in\mathbb{N}}$ approximating (u_0, f) in the sense of Lemma 17 and $\{\bar{u}_m\}_{m\in\mathbb{N}}$ the corresponding solutions of the Euler equations, then for each t, n, m we have

$$\begin{aligned}
\|u^n(t)-u(t)\|^2 &\le 2\|u^n(t)-\bar{u}^m(t)\|^2+2\|\bar{u}^m(t)-u(t)\|^2 \\
&\overset{Remark\ 18}{\le} \frac{K}{m^2}+2\|u^n(t)-\bar{u}^m(t)\|^2.
\end{aligned}$$

For each m and $\delta>0$ fixed, let us introduce the corrector of the boundary layer v_m, it satisfies the estimates described in Lemma 16. For each time t we have

$$\begin{aligned}
\|(u^n-\bar{u}^m)(t)\|^2 &= \|u^n(t)\|^2+\|\bar{u}^m(t)\|^2-2\langle u^n(t),\bar{u}^m(t)\rangle \\
&\overset{energy\ eq.,\ \|v_m\|\le K_m\delta^{\frac{1}{2}}}{\le} \|u_0^n-u_0\|^2+2\|u_0\|^2+K\|u_0^n-u_0\|+\|\bar{u}_0^m-u_0\|^2+K\|\bar{u}_0^m-u_0\| \\
&+2\int_0^t\langle f^n(s),u^n(s)\rangle\,ds+2\int_0^t\langle \bar{f}^m(s),\bar{u}^m(s)\rangle\,ds-2\langle u^n,\bar{u}^m-v_m\rangle+K_m\delta^{\frac{1}{2}}.
\end{aligned}$$

To analyze the second-last term we use the weak formulation satisfied by u^n, taking $\bar{u}^m-v_m$ as test function. Exploiting the relation

$$\langle u_0^n,\bar{u}_0^m\rangle=\|u_0\|^2+\langle u_0^n-u_0,\bar{u}_0^m-u_0\rangle+\langle u_0^n-u_0,u_0\rangle+\langle u_0,\bar{u}_0^m-u_0\rangle,$$

we get

$$
\begin{aligned}
-2\langle u^n(t),(u^m-v_m)(t)\rangle + 2\|u_0\|^2 &\overset{\|v_m\|\le K_m\delta^{\frac12}}{\le} K\|u_0^n-u_0\| + K\|\bar u_0^m-u_0\| + K_m\delta^{\frac12}\\
&- 2\int_0^t \langle u^n(s),\partial_s(\bar u^m-v_m)(s)\rangle\, ds\\
&+ 2\nu_n\int_0^t \langle (-A)^{\frac12}u^n(s),(-A)^{\frac12}(\bar u^m-v_m)(s)\rangle\, ds\\
&- \int_0^t 2b(u^n(s),(\bar u^m-v_m)(s),u^n(s))\, ds\\
&- 2\int_0^t \langle f^n(s),(\bar u^m-v_m)(s)\rangle\, ds.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
-\langle u^n(s),\partial_s(\bar u^m-v_m)(s)\rangle &\overset{energy\ eq.,\ \|\partial_t v_m(t)\|\le K_m\delta^{\frac12}}{=} -\langle u^n(s),\partial_s\bar u^m(s)\rangle + K_m\delta^{\frac12}\\
&\overset{Euler\ eq.}{=} \langle u^n(s),\nabla\bar u^m(s)\cdot\bar u^m(s)\rangle - \langle \bar f^m(s),u^n(s)\rangle + K_m\delta^{\frac12}.
\end{aligned}
$$

Thanks to the relations above and noting that

$$
b(u^n(s),\bar u^m(s),u^n(s)) - b(u^n(s),\bar u^m(s),\bar u^m(s)) = b((u^n-\bar u^m)(s),\bar u^m(s),(u^n-\bar u^m)(s))
$$

we can continue the estimate of $\|(u^n-\bar u^m)(t)\|^2$:

$$
\begin{aligned}
\|(u^n-\bar u^m)(t)\|^2 &\overset{\|v_m(t)\|\le K_m\delta^{\frac12}}{\le} K_m\delta^{\frac12} + K\|u_0^n-u_0\|^2 + \|\bar u_0^m-u_0\|^2 + K\|\bar u_0^m-u_0\| + K\|u_0^n-u_0\|\\
&+ 2\int_0^t b(u^n(s),v_m(s),u^n(s))\, ds + 2\nu_n\int_0^t \langle (-A)^{\frac12}u^n(s),(-A)^{\frac12}(\bar u^m-v_m)(s)\rangle\, ds\\
&+ 2\|\nabla\bar u^m\|_{L^\infty([0,T]\times D)}\int_0^t \|(u^n-\bar u^m)(s)\|^2\, ds\\
&+ 2\int_0^t \langle (f^n-\bar f^m)(s),(u^n-\bar u^m)(s)\rangle\, ds.
\end{aligned}
$$

Arguing as we have done in the stochastic case, we have

$$
\int_0^t b(u^n(s),v_m(s),u^n(s))\, ds \overset{\|\rho^2\nabla v_m(t)\|_{L^\infty(D)}\le K_m\delta}{\le} K_m\delta\int_0^T \|\nabla u^n(s)\|^2_{L^2(\Gamma_\delta)}\, ds
$$

$$
\begin{aligned}
\nu_n\int_0^t \langle (-A)^{\frac12}u^n(s),(-A)^{\frac12}(\bar u^m-v_m)(s)\rangle\, ds &\overset{\|\nabla v^m(t)\|_{L^2(D)}\le K_m\delta^{-1/2}}{\le} \|\nabla\bar u^m\|_{L^\infty(0,T;L^2(D))}\nu_n\int_0^T \|\nabla u^n(s)\|_{L^2(D)}\, ds\\
&+ \nu_n K_m\delta^{-1/2}\int_0^T \|\nabla u^n(s)\|_{L^2(\Gamma_\delta)}\, ds.
\end{aligned}
$$

For what concern the new term, we can estimate it as follows

$$
\begin{aligned}
\int_0^t \langle (f^n-\bar f^m)(s),(u^n-\bar u^m)(s)\rangle\, ds &\le \int_0^t \|(f^n-\bar f^m)(s)\|\|(u^n-\bar u^m)(s)\|\, ds\\
&\le \sqrt{T}\|f^n-\bar f^m\|_{L^2(0,T;H)}\left(\|u^n\|_{L^\infty(0,T;H)} + \|\bar u^m\|_{L^\infty(0,T;H)}\right)\\
&\overset{energy\ eq.}{\le} K\left(\|f^n-f\|_{L^2(0,T;H)} + \|\bar f^m-f\|_{L^2(0,T;H)}\right).
\end{aligned}
$$

Thanks to the relations above we can continue the estimate of $\|u^n(t)-\bar u^m(t)\|^2$:

$$
\begin{aligned}
\|u^n(t)-\bar u^m(t)\|^2 &\le K_m\delta^{\frac12} + K\|u_0^n-u_0\|^2 + \|\bar u_0^m-u_0\|^2 + K\|\bar u_0^m-u_0\| + K\|u_0^n-u_0\|\\
&+ K_m\delta\int_0^T \|\nabla u^n(s)\|^2_{L^2(\Gamma_\delta)}\, ds + K\left(\|f^n-f\|_{L^2(0,T;H)} + \|\bar f^m-f\|_{L^2(0,T;H)}\right)\\
&+ 2\|\nabla\bar u^m\|_{L^\infty(0,T;L^2(D))}\nu_n\int_0^T \|\nabla u^n(s)\|_{L^2(D)}\, ds + \nu_n K_m\delta^{-1/2}\int_0^T \|\nabla u^n(s)\|_{L^2(\Gamma_\delta)}\, ds\\
&+ 2\|\nabla\bar u^m\|_{L^\infty([0,T]\times D)}\int_0^t \|u^n(s)-\bar u^m(s)\|^2\, ds.
\end{aligned}
$$

Taking $\delta = c\nu_n$, by Gronwall's inequality and Holder's inequality we have

$$
\begin{aligned}
sup_{t\in[0,T]}\|u^n(t)-\bar{u}^m(t)\|^2 \leq \Bigg(& K_m\nu_n^{1/2} + K\|u_0^n - u_0\|^2 + \|\bar{u}_0^m - u_0\|^2 + K\|\bar{u}_0^m - u_0\| + K\|u_0^n - u_0\| \\
& + K_m\nu_n \int_0^T \|\nabla u^n(s)\|^2_{L^2(\Gamma_{c\nu_n})}\, ds + K\left(\|f^n - f\|_{L^2(0,T;H)} + \|\bar{f}^m - f\|_{L^2(0,T;H)}\right) \\
& + \|\nabla\bar{u}^m\|_{L^\infty(0,T;L^2(D))} K\sqrt{\nu_n}\left(\int_0^T \nu_n\|\nabla u^n\|^2_{L^2(D)}\, ds\right)^{1/2} \\
& + K_m\left(\nu_n\int_0^T \|\nabla u^n\|^2_{L^2(\Gamma_{c\nu_n})}\, ds\right)^{\frac{1}{2}}\Bigg) e^{2T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times D)}}.
\end{aligned}
$$

Taking the limsup with respect to n of this expression for m fixed we have

$$
\limsup_{n\to+\infty}\sup_{t\in[0,T]}\|u^n(t)-\bar{u}^m(t)\|^2 \leq \left(K\|\bar{u}_0^m - u_0\| + \|\bar{u}_0^m - u_0\|^2 + \|\bar{f}^m - f\|_{L^2(0,T;H)}\right) e^{2T\|\nabla\bar{u}^m\|_{L^\infty([0,T\times D])}} \tag{9}
$$

$$
\leq \frac{K}{m} + \frac{K}{m^2}. \tag{10}
$$

Coming back to

$$
\|u^n(t) - u(t)\|^2 \leq \frac{K}{m^2} + 2\|u^n(t) - \bar{u}^m(t)\|^2.
$$

If we fix $\epsilon > 0$ and $\bar{m}$ such that $\frac{5K}{\bar{m}} < \epsilon$, then taking the limsup with respect to n of previous expression for $m = \bar{m}$ we have

$$
\limsup_{n\to+\infty}\sup_{t\in[0,T]}\|u^n(t) - u(t)\|^2 \leq \epsilon.
$$

We have the thesis from the arbitrariness of ϵ. ■

Acknowledgement

I want to thank Professor Franco Flandoli for useful discussions and valuable insight into the subject.

References

[1] R. Bafico and P. Baldi. Small random perturbations of peano phenomena. *Stochastics*, 6(3-4):279–292, 1982.

[2] C. Bardos, L. Székelyhidi Jr, and E. Wiedemann. Non-uniqueness for the euler equations: the effect of the boundary. *Russian Mathematical Surveys*, 69(2):189, 2014.

[3] C. Bardos, E. S. Titi, and E. Wiedemann. The vanishing viscosity as a selection principle for the euler equations: the case of 3d shear flow. *Comptes Rendus Mathematique*, 350(15-16):757–760, 2012.

[4] F. Cipriano and I. Torrecilla. Inviscid limit for 2d stochastic navier–stokes equations. *Stochastic Processes and their Applications*, 125(6):2405–2426, 2015.

[5] P. Constantin and C. Foias. *Navier-Stokes Equations*. University of Chicago Press, 1988.

[6] P. Constantin, I. Kukavica, and V. Vicol. On the inviscid limit of the navier-stokes equations. *Proceedings of the American Mathematical Society*, 143(7):3075–3090, 2015.

[7] H. B. da Veiga. On the sharp vanishing viscosity limit of viscous incompressible fluid flows. In *New directions in mathematical fluid mechanics*, pages 113–122. Springer, 2009.

[8] F. Flandoli and E. Luongo. *Stochastic Partial Differential Equations in Fluid Mechanics*. Springer, to appear.

[9] K. K. Golovkin. Vanishing viscosity in cauchy's problem for hydromechanics equations. *Trudy Matematicheskogo Instituta imeni VA Steklova*, 92:31–49, 1966.

[10] T. Kato. On classical solutions of the two-dimensional non-stationary euler equation. *Archive for Rational Mechanics and Analysis*, 25(3):188–200, 1967.

[11] T. Kato. Nonstationary flows of viscous and ideal fluids in r3. *Journal of functional Analysis*, 9(3):296–305, 1972.

[12] T. Kato. Remarks on zero viscosity limit for nonstationary navier-stokes flows with boundary. In *Seminar on nonlinear partial differential equations*, pages 85–98. Springer, 1984.

[13] P.-L. Lions. *Mathematical Topics in Fluid Mechanics: Volume 1: Incompressible Models*, volume 1. Oxford University Press on Demand, 1996.

[14] M. C. Lopes Filho, A. L. Mazzucato, and H. N. Lopes. Vanishing viscosity limit for incompressible flow inside a rotating circle. *Physica D: Nonlinear Phenomena*, 237(10-12):1324–1333, 2008.

[15] M. C. Lopes Filho, A. L. Mazzucato, H. N. Lopes, and M. Taylor. Vanishing viscosity limits and boundary layers for circularly symmetric 2d flows. *Bulletin of the Brazilian Mathematical Society, New Series*, 39(4):471–513, 2008.

[16] Y. Maekawa. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *Communications on Pure and Applied Mathematics*, 67(7):1045–1128, 2014.

[17] F. McGrath. Nonstationary plane flow of viscous and ideal fluids. *Archive for Rational Mechanics and Analysis*, 27(5):329–348, 1968.

[18] M. Sammartino and R. E. Caflisch. Zero viscosity limit for analytic solutions of the navier-stokes equation on a half-space.p ii. construction of the navier-stokes solution. *Communications in mathematical physics*, 192(2):463–491, 1998.

[19] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*, volume 343. American Mathematical Soc., 2001.

[20] R. Temam and X. Wang. On the behavior of the solutions of the navier-stokes equations at vanishing viscosity. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 25(3-4):807–828, 1997.

[21] X. Wang. A kato type theorem on zero viscosity limit of navier-stokes flows. *Indiana University Mathematics Journal*, pages 223–241, 2001.