Infinitary logic with infinite sequents: syntactic investigations

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The present paper deals with a purely syntactic analysis of infinitary logic with infinite sequents. In particular, we discuss sequent calculi for classi-cal and intuitionistic infinitary logic with good structural properties based on sequents possibly containing infinitely many formulas. A cut admissibility proof is proposed which employs a new strategy and a new inductive parameter. We conclude the paper by discussing related issues and possible themes for future research.

1 INTRODUCTION

Infinitary logics are described by languages including expressions of infinite length. In particular, we shall be concerned with languages augmented with countable conjunctions and disjunctions [15]. The proof theory for systems with rules with infinitely many premises has been systematically exploited in order to determine the proof-theoretic strength of arithmetical theories such as Peano arithmetic and ramified analysis (cf., e.g., [14]). However, also non-classical infinitary logics are interesting. Indeed, infinitary intuitionistic logic is the natural ground for geometric theories [13] which often constitute axiomatizations of large portions of algebra. Moreover, geometric theories form a conservative class with respect to classical over intuitionistic derivability.

From a proof-theoretic point of view, infinitary logics have been investigated with various different approaches. There are different conceptions of infinitary derivations in Gentzen-style proof theory. We list three of them:

1. Derivations are well-founded trees possibly infinitely branching with finite sequents.
2. Derivations are well-founded trees possibly infinitely branching in which every node is occupied by a possibly infinite sequent.
3. Derivations are non well-founded trees.

The first approach is the one which has proved to be the most flexible in the context of predicative proof theory and ordinal analysis (cf. [12, 14]). The structural analysis of such systems allowed to establish cut admissibility and therefore analyticity (for another, more recent approach to the issue, cf. [9] and [13]).

Derivations as non well-founded trees are the central ingredient of cyclic proof theory[1] in which derivations are allowed to contain branches of infinite length. This method has proved to be particularly promising to accommodate the proof theory of modal fixpoint logics [1].

Finally, derivations with infinite sequents have been considered in the literature in the work of Takeuti [16] and Lopez-Escobar [4]. However, the structural properties of the systems were established by means of semantic approaches, such as by showing that every sequent admits either a proof or a countermodel. This approach cannot be regarded as completely satisfactory for two distinct reasons.

[1] We observe that—as pointed out by a referee—cyclic proof theory does not deal with infinitary logics, but employs a kind of infinitary notion of derivation to reason on fixpoint logics.

First, cut admissibility is a syntactic property of a system and a semantic proof thereof is not conceptually pure as it uses tools which are external to the system itself. Second, the use of semantics to establish cut-free completeness is a move available only in the presence of a suitable structure to interpret the logic, whereas syntactic cut admissibility does not require it.

There is an evident difficulty concerning the cut admissibility theorem in the context of infinitary logic with infinite sequents. To witness this, consider the reduction in which the cut formula formula is an infinitary conjunction principal in both the premises of the cut.

$$\dfrac{\dfrac{\{\Gamma \Rightarrow \Delta, A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{k>0} A_k}\,\mathrm{R}\bigwedge \qquad \dfrac{[A_k]_{k>0}, \Pi \Rightarrow \Sigma}{\bigwedge_{k>0} A_k, \Pi \Rightarrow \Sigma}\,\mathrm{L}\bigwedge}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,\mathrm{Cut}$$

A naive attempt at replacing the cut could consist in replacing it with infinitely many cuts on the formulas A_k. However, this strategy is not available insofar as derivations have to respect the well-foundedness conditions and this would lead to non-terminating cut reductions.

In the present paper we address this issue, by providing, to the best of our knowledge, the first purely syntactic analysis of infinitary logic with infinite sequents. In particular, beside showing the admissibility of the structural rules of weakening and contraction, we show that the syntactic proof of the admissibility of the cut rule requires to consider an infinitary generalization of the cut (introduced in [16]) and to devise a new reduction strategy, together with a new inductive parameter.

We shall focus on intuitionistic and classical infinitary logic. We start by discussing classical infinitary logic in a G3-style sequent setting with infinite sequents. In classical infinitary logic every rule is height-preserving invertible and the rules of weakening and contraction are height-preserving admissible. Finally, we show that the cut rule can be proved admissible for the propositional fragment of the calculus. Next, we consider intuitionistic infinitary logic.

Infinitary intuitionistic logic has been investigated by Kalicki [3] and Nadel [8] who provided an interpretation in terms of complete Heyting algebras and showed that Kripke semantics turned out to be inadequate. The closure under infinite intersections of opens in the underlying topology forces the intuitionistically unacceptable distributivity principle:

$$\bigwedge_{k>0}(p_k \vee q) \to \bigwedge_{k>0} p_k \vee q$$

In the work [18] a new semantics for intuitionistic infinitary logic was provided. In particular, starting from the topological interpretation of intuitionistic logic—cf. [6] for a topological semantics of intuitionistic propositional logic—and building on the work of Moniri and Maleki [7] a topological and a neighborhood semantics (cf. [11] for an introduction) were introduced and studied.

We establish the admissibility of the rules of weakening and contraction with preservation of the height as well as the invertibility of some rules of the calculus. The cut rule is shown to be admissible for the entire intuitionistic calculus. This, together with an extended version of the Gödel-Gentzen negative translation, allows ourselves to obtain a proof of the cut admissibility for the classical calculus.

The results contained in the paper show that systems with infinite sequents are conservative with respect to those with finite ones (at least considering rules in which there is a single principal formula). Furthermore, soundness and completeness are preserved. This result is not trivial, because it shows that in the case of intuitionistic and classical infinitary logics working with infinite sequents does not alter the structural properties of the systems nor their deductive strength. This has to be contrasted, e.g., with the case of infinitary modal logic, where a calculus with infinite sequents has been shown not to admit cut admissibility [5].

The plan of the paper is as follows. In § 2 we provide a gentle introduction to the notion of infinite sequents and we introduce the classical calculus. The usual structural properties are established and cut is shown admissible for the propositional fragment. § 3 is devoted to the analysis of the intuitionistic calculus: a full cut admissibility theorem is proved. Next, § 4 comes full circle by inducing a syntactic cut admissibility in the full system for classical logic *modulo* an extension of the negative translation. § 5 presents an equivalence between the calculi with finite and infinite sequents. Finally, § 6 discusses some themes which may be object of future research.

Initial Sequents

$$\frac{}{p, \Gamma \Rightarrow \Delta, p}\ Ax \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\ \mathrm{L}\bot$$

Logical Rules

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ \mathrm{L}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ \mathrm{R}\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ \mathrm{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ \mathrm{R}\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ \mathrm{L}\to \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ \mathrm{R}\to$$

$$\frac{[A_k]_{k>0}, \Gamma \Rightarrow \Delta}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}\ \mathrm{L}\bigwedge \qquad \frac{\{\Gamma \Rightarrow \Delta, A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{k>0} A_k}\ \mathrm{R}\bigwedge$$

$$\frac{\{A_k, \Gamma \Rightarrow \Delta \mid k > 0\}}{\bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta}\ \mathrm{L}\bigvee \qquad \frac{\Gamma \Rightarrow \Delta, [A_k]_{k>0}}{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k}\ \mathrm{R}\bigvee$$

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ \mathrm{L}\forall \qquad \frac{\Gamma \Rightarrow \Delta, A[x/y]}{\Gamma \Rightarrow \Delta, \forall x A}\ \mathrm{R}\forall, y!$$

$$\frac{A[x/y], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ \mathrm{L}\exists, y! \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A}\ \mathrm{R}\exists$$

FIGURE 1 The calculus $\mathsf{G3C}_\omega$. The fragment $\mathsf{G3Cp}_\omega$ is obtained by removing the rules R∃, L∃, R∀ and L∀.

2 INFINITE SEQUENTS

The language of infinitary logic is built from predicate letters, connectives and quantifiers as usual and two infinitary connectives $\bigwedge$ and $\bigvee$ which denote the countable infinitary conjunction and disjunction, respectively. Formally, the language contains a countable set of n-ary predicates $p_1^{n_1}, p_2^{n_2}, \ldots$ with $n_i > 0$ and a countable set of function letters $f_1^{n_1}, f_2^{n_2}, \ldots$ with $n_i \geq 0$. The language contains a denumerable set of variables $x_1, x_2, \ldots$, connectives $\wedge, \vee, \to, \bot, \bigwedge, \bigvee$ and quantifiers $\forall, \exists$. Terms $t_1, t_2, \ldots$ are built as usual and the set of formulas FM_ω is built according to the following grammar:

$$A \ ::= p^n(t_1, \ldots, t_n) \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \forall x A \mid \exists x A \mid \bigwedge_{k>0} A_k \mid \bigvee_{k>0} A_k$$

we use small Latin letters $p, q, r, \ldots$ to denote atomic formulas and capital Latin letters $A, B, C, D, E, \ldots$ to denote compound formulas.

Definition 2.1. A multiset of formulas is here defined as a function $\Gamma : \mathsf{FM}_\omega \to \omega + 1$.

It is immediate from this stipulation that we accept multisets containing countably many formulas. With a slight abuse of notation we shall write $A \in \Phi$ instead of $\Phi(A) > 0$ to denote the fact that A occurs (possibly infinitely many times) in Φ. Multisets of formulas are denoted with square brackets, e.g. $[A, A, B, C]$. Finitary (infinitary) multiset union is denoted as $\uplus$ ($\biguplus$). We use capital Greek letters to denote multisets of formulas. As a consequence, sequents are now conceivable as syntactic objects of the shape:

$$\Gamma \Rightarrow \Delta$$

where Γ and Δ are multisets of formulas. As usual, we consider negation to be defined, i.e., $\neg A$ abridges the expression $A \to \bot$.

We present a sequent calculus $\mathsf{G3C}_\omega$ for infinitary classical logic in Figure 1. As usual, the expression $y!$ denotes the fact that the variable y does not occur in the conclusion. We denote with $\mathsf{G3Cp}_\omega$ (the infinitary propositional fragment) the calculus obtained by removing the rules governing the quantifiers. The most striking difference between the system $G3C_\omega$ and the usual presentations of infinitary logic lies in the fact that we opt for a *multiplicative* formulation of the unary rules for the infinitary connectives. In infinitary logic the multiplicative formulation of such rules imposes a change in the structures which are manipulated.

Namely, sequents are now built from infinite multisets of formulas. This requires a new approach to the structural analysis of the calculus. The principal tool in our investigations will be transfinite induction which we shall use to study the properties of derivations.

Definition 2.2. A derivation is a (possibly infinitely branching) rooted tree, where the leaves are initial sequents and are occupied by initial sequents and the other nodes are occupied by sequents constructed following the rules.

Notice that this definition rules out the possibility of proofs with branches of infinite length. To measure the length of the derivations we assign countable ordinals.

Definition 2.3. Given a derivation $\mathcal{D}$, its *height* $h(\mathcal{D})$ is defined as follows:

1. If $\mathcal{D}$ is an initial sequent, then $h(\mathcal{D}) = 0$.
2. If $\mathcal{D}$ is of the form:

$$\frac{\dots \quad \begin{array}{c} \vdots\, \mathcal{D}_n \\ \Gamma_n \Rightarrow \Delta_n \end{array} \quad \dots}{\Gamma \Rightarrow \Delta}$$

with possibly countable many premises, then $h(\mathcal{D}) = \sup_n(h(\mathcal{D}_n)) + 1$, where the latter is a countable ordinal.

A rule is *admissible* if, whenever the premises are derivable, so is the conclusion. A rule is *height-preserving* admissible if it is admissible and the height of the conclusion is less or equal to the height of each of the premise(s).

We shall also use measures to assess the complexity of formulas. Once again, these will be countable ordinals.

Definition 2.4. The *weight* of a formula A is inductively defined:

1. $w(p) = 0$ if p atomic.
2. $w(\bot) = 0$.
3. $w(A\#B) = \sup(w(A), w(B)) + 1$, where $\# \in \{\wedge, \vee, \rightarrow\}$.
4. $w(\bigwedge_{k>0} A_k) = w(\bigvee_{k>0} A_k) = (\sup_{k>0} w(A_k)) + 1$.
5. $w(QxA) = w(A) + 1$, where $Q \in \{\forall, \exists\}$.

In what follows the use of the induction hypothesis will be labeled as IH. In other cases, we shall write the name of the rule which is shown to be admissible and then a comment which motivates the application of the induction hypothesis. We also recall some standard terminology in proof theory.

Definition 2.5. The principal formula of any inference rule in the calculus $G3C_\omega$ in Figure 1 is the formula which is displayed in the conclusion of the rule. An atomic formula p is active in an initial sequent if the sequent is of the shape $\Gamma, p \Rightarrow \Delta, p$.

Lemma 2.6. The sequent $\Gamma, A \Rightarrow \Delta, A$ is derivable for every formula A and multiset Γ and Δ.

Proof. The proof is by transfinite induction on the weight of the formula A. We detail the case in which A is an infinite disjunction.

$$\dfrac{\dfrac{\{\Gamma, A_k \Rightarrow \Delta, [A_k]_{k>0} \mid k > 0\}}{\Gamma, \bigvee_{k>0} A_k \Rightarrow \Delta, [A_k]_{k>0}}\,L\bigvee}{\Gamma, \bigvee_{k>0} A_k \Rightarrow \Delta, \bigvee_{k>0} A_k}\,R\bigvee$$

Every topmost sequent is provable by induction hypothesis. □

The substitution of a variable x with a term t is defined as usual.

Lemma 2.7. For every variable x and term t for the language, the rule:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma[x/t] \Rightarrow \Delta[x/t]}\ \mathrm{Sub}[x/t]$$

is height-preserving admissible in $\mathsf{G3C}_\omega$.

Proof. The proof runs by transfinite induction on the height of the derivation of the sequent $\Gamma \Rightarrow \Delta$. □

Next, we need to establish the admissibility of the rule of weakening. In this case, we need to be able to add infinite multisets of formulas.

Lemma 2.8. The rule

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ \mathrm{Weak}$$

is height-preserving admissible in $\mathsf{G3C}_\omega$.

Proof. We argue by transfinite induction, possibly using the admissibility of the rule of substitution in order to avoid clashes of variables whenever the last rule applied is L∃ or R∀. □

We now need to establish the invertibility of the rules of the calculus with preservation of the height. This is a relevant point, as usual formulations of the unary rules for infinitary connectives are invertible due to the repetition of the principal formula in the premise of the rule [9].

Lemma 2.9. Every rule is height-preserving invertible in $\mathsf{G3C}_\omega$.

Proof. The proof is by transfinite induction on the height of the derivation. We detail the case of the rule R$\bigvee$. If $\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k$ is an initial sequent, then so is $\Gamma \Rightarrow \Delta, [A_k]_{k>0}$. If it is the conclusion of a rule, we apply the induction hypothesis and then the rule again. For example, if the last rule applied is R$\bigwedge$, we have:

$$\frac{\{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k, B_i \mid i>0\}}{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k, \bigwedge_{i>0} B_i}\ \mathrm{R}\bigwedge$$

We proceed as follows:

$$\cfrac{\cfrac{\{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k, B_i \mid i>0\}}{\{\Gamma \Rightarrow \Delta, [A_k]_{k>0}, B_i \mid i>0\}}\ \mathrm{IH}}{\Gamma \Rightarrow \Delta, [A_k]_{k>0}, \bigwedge_{i>0} B_i}\ \mathrm{R}\bigwedge$$

The other cases are dealt with analogously. □

Next key step is the admissibility of the rule of contraction. In this case we need to contract infinitely many formulas.

Lemma 2.10. The rule

$$\frac{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \mathrm{Ctr}$$

is height-preserving admissible in $\mathsf{G3C}_\omega$.

Proof. We argue by induction on the height of the derivation of the sequent $\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma$. If it is an initial sequent, then so is $\Gamma, \Pi \Rightarrow \Delta, \Sigma$. If it is the conclusion of a rule, we need to distinguish two subcases. If the principal formula is in Γ or Δ, then we apply the induction hypothesis to each of the premises of the rule and then the rule again. The general structure of the argument is as follows:

$$\dfrac{\dots \quad \Gamma', \Pi, \Pi \Rightarrow \Delta', \Sigma, \Sigma \quad \dots}{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma}\, \rho$$

We proceed as follows:

$$\dfrac{\dots \quad \dfrac{\Gamma', \Pi, \Pi \Rightarrow \Delta', \Sigma, \Sigma}{\Gamma', \Pi \Rightarrow \Delta', \Sigma}\,\mathrm{IH} \quad \dots}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\, \rho$$

If one of the formulas in Π or Σ is principal, then we need to distinguish cases according to the shape of the formulas. The strategy consists in applying the invertibility of the corresponding rule and then the induction hypothesis. Let us consider the case in which the principal formula is in Σ and is $\bigvee_{k>0} A_k$.

$$\dfrac{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma', \Sigma', \bigvee_{k>0} A_k, [A_k]_{k>0}}{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma', \Sigma', \bigvee_{k>0} A_k, \bigvee_{k>0} A_k}\, \mathrm{R}\bigvee$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma', \Sigma', \bigvee_{k>0} A_k, [A_k]_{k>0}}{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma', \Sigma', [A_k]_{k>0}, [A_k]_{k>0}}\,\mathrm{Inv}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma', [A_k]_{k>0}}\,\mathrm{IH}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma', \bigvee_{k>0} A_k}\, \mathrm{R}\bigvee$$

This is the critical case which requires to prove a stronger version of the admissibility of contraction, namely the contraction of infinite multisets of formulas. □

Finally, the last part of this section is devoted to the structural analysis of the classical calculus for infinitary classical logic. We shall prove the admissibility of a generalized form of the cut rule only for the propositional fragment $\mathsf{G3Cp}_\omega$. In essence, the crucial move is the shift from a cut between formulas to a more complex cut between multisets of formulas. This rule can be found in the literature, cf. [4, 16] and [5].[2] Recently, a variant of such rule was shown to be admissible in a system for multiplicative quantifiers through a purely proof-theoretic argument [10].

The rule is

$$\dfrac{\{\Pi_D \Rightarrow \Sigma_D, D \mid D \in \Phi\} \qquad \Phi, \Gamma \Rightarrow \Delta, \Psi \qquad \{E, \Theta_E \Rightarrow \Lambda_E \mid E \in \Psi\}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\, \mathrm{Cut}$$

where $\Pi = \biguplus_{D\in\Phi} \Pi_D$, $\Sigma = \biguplus_{D\in\Phi} \Sigma_D$, $\Theta = \biguplus_{D\in\Phi} \Theta_D$ and $\Lambda = \biguplus_{D\in\Phi} \Lambda_D$. We say that Φ and Ψ are the multisets of cut formulas. We need to introduce an ordinal measure for multisets of formulas. Essentially we define the degree of a multiset of formulas to be the supremum of the weight of the formulas in it plus one. Formally,

$$\deg(\Xi) = \sup_{D\in\Xi} w(D) + 1$$

[2] However, a syntactic proof of its admissibility is—to the best of our knowledge—not present in the literature.

for any multiset Ξ. The proof will run by double transfinite induction, with main induction on $deg(\Phi, \Psi)$ and a secondary induction hypothesis on the height of the derivation of the sequent $\Phi, \Gamma \Rightarrow \Delta, \Psi$.

Remark 2.11. It is straightforward to observe that this generalized cut rule encompasses the standard one. Indeed, the cut rule coincides with the case in which Φ is empty and Ψ contains a single formula. This generalization is crucial in order to show the admissibility of cuts in the case in which the cut formula is an infinitary formula principal in both premises of the cut. A standard cut would be:

$$\dfrac{\dfrac{\{\Gamma \Rightarrow \Delta, A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{k>0} A_k}\,\mathrm{R}\bigwedge \qquad \dfrac{[A_k]_{k>0}, \Gamma \Rightarrow \Delta}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}\,\mathrm{L}\bigwedge}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}$$

With the standard cut we cannot replace the cut with infinitely many ones, because this would transform a well-founded object in a non well-founded one. On the contrary, the generalized cut solves the problem by cutting infinitely many formulas simultaneously.

We now give an example of an application of the generalized cut.

Example 2.12. Consider the derivation of $\bigwedge_{k>0} p_k \Rightarrow \bigwedge_{k>0}(p_k \vee q)$ containing an instance of the generalized cut:

$$\dfrac{\dfrac{\{p_k \Rightarrow p_k \vee q \mid k > 0\} \qquad [p_k \vee q]_{k>0} \Rightarrow \bigwedge_{k>0}(p_k \vee q)}{[p_k]_{k>0} \Rightarrow \bigwedge_{k>0}(p_k \vee q)}\,\mathrm{Cut}}{\bigwedge_{k>0} p_k \Rightarrow \bigwedge_{k>0}(p_k \vee q)}\,\mathrm{L}\bigwedge$$

The premises of the cut rule are easily seen to be derivable. The endsequent is also derivable via root-first applications of the rules of the calculus.

By A^∞ we denote the infinite repetition of the formula A and given a multiset Γ, by Γ^∞ we denote the infinite repetition of the multiset Γ, i.e., $\Gamma^\infty = [A^\infty \mid A \in \Gamma]$. We first have to prove a preliminary theorem which involves the admissibility of atomic cuts.

Theorem 2.13. The rule

$$\dfrac{\Gamma \Rightarrow \Delta, p \qquad p, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,\mathrm{Cutat}$$

is admissible in $\mathsf{G3Cp}_\omega$.

Proof. The proof is by induction on the height of $p, \Pi \Rightarrow \Sigma$. If it is an initial sequent and p is not active, we remove it. Otherwise, we take the premise $\Gamma \Rightarrow \Delta, p$ and we obtain the desired conclusion via height-preserving admissibility of weakening. If $p, \Pi \Rightarrow \Sigma$ is the conclusion of a rule, then we notice that p is never principal by the design of the rules, so the cut can be permuted upwards and removed by the induction hypothesis (if necessary making use of the substitution lemma in order to avoid clashes of variables). □

We are now ready to prove the cut admissibility theorem. We shall establish the result for the propositional fragment of $\mathsf{G3C}_\omega$, i.e., the calculus $\mathsf{G3Cp}_\omega$, obtained from the previous one by dropping the rules for the quantifiers.[3]

Theorem 2.14. The cut rule is admissible in $\mathsf{G3Cp}_\omega$.

[3] The full cut admissibility theorem will be proved in § 4 exploiting the result obtained for the intuitionistic calculus.

Proof. The proof is by double transfinite induction as explained above. If $\Phi, \Gamma \Rightarrow \Delta, \Psi$ is an initial sequent, four subcases have to be distinguished.

1. The active formula p is neither in Φ, nor in Ψ. In this case the conclusion of the cut is an initial sequent too.
2. The active formulas are in Φ and in Δ. In this case, we consider the premise $\Pi_p \Rightarrow \Sigma_p, p$ and we apply weakening in order to obtain the desired conclusion.
3. The active formulas are in Γ and in Ψ. Symmetric to the previous case.
4. The active formulas are in Φ and in Ψ. We consider the premises $\Pi_p \Rightarrow \Sigma_p, p$ and $p, \Theta_p \Rightarrow \Lambda_p$ and we perform the following transformation:

$$\dfrac{\dfrac{\Pi_p \Rightarrow \Sigma_p, p \qquad p, \Theta_p \Rightarrow \Lambda_p}{\Pi_p, \Theta_p \Rightarrow \Lambda_p, \Sigma_p}\,\text{Cutat}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Lambda, \Sigma}\,\text{Weak}$$

If $\Phi, \Gamma \Rightarrow \Delta, \Psi$ is the conclusion of a rule and the principal formula is neither in Φ or in Ψ, then the cut is permuted upwards and replaced by possibly infinite cuts of lesser height. The general structure of the reduction is:

$$\dfrac{\{\Pi_D \Rightarrow \Sigma_D, D \mid D \in \Phi\} \qquad \dfrac{\ldots \quad \Phi, \Gamma_i \Rightarrow \Delta_i, \Psi \quad \ldots}{\Phi, \Gamma \Rightarrow \Delta, \Psi}\,\varrho \qquad \{E, \Theta_E \Rightarrow \Lambda_E \mid E \in \Psi\}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\text{Cut}$$

where ϱ is a rule (possibly with infinitely many premises). We construct the following derivation:

$$\dfrac{\ldots \qquad \dfrac{\{\Pi_D \Rightarrow \Sigma_D, D \mid D \in \Phi\} \quad \Phi, \Gamma_i \Rightarrow \Delta_i, \Psi \quad \{E, \Theta_E \Rightarrow \Lambda_E \mid E \in \Psi\}}{\Gamma_i, \Pi, \Theta \Rightarrow \Delta_i, \Sigma, \Lambda}\,\text{Cut} \qquad \ldots}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\varrho$$

The cuts are removed invoking the secondary induction hypothesis on the height of the premise $\Phi, \Gamma_i \Rightarrow \Delta_i, \Psi$ which has strictly decreased.

The final and crucial case is the one in which one of the formulas in Φ or Ψ is principal in the last rule applied. The general strategy consists in removing all the non principal formulas via cuts which are permuted upwards and then finishing with cuts on multisets of lesser degree. Of course, we need to distinguish cases according to the shape of the principal formula and to its position in the sequent.

We consider the case in which the principal formula is $\bigwedge_{k>0} A_k$ in Φ. Hence $\Phi = \Phi' \uplus [\bigwedge_{k>0} A_k]$ and we have:

$$\dfrac{\{\Pi_D \Rightarrow \Sigma_D, D \mid D \in \Phi\} \qquad \dfrac{\Phi', [A_k]_{k>0}, \Gamma \Rightarrow \Delta, \Psi}{\Phi', \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta, \Psi}\,\text{L}\bigwedge \qquad \{E, \Theta_E \Rightarrow \Lambda_E \mid E \in \Psi\}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\text{Cut}$$

In this case we first permute the cut upwards as follows:

$$\dfrac{\{\Pi_D \Rightarrow \Sigma_D, D \mid D \in \Phi'\} \qquad \Phi', [A_k]_{k>0}, \Gamma \Rightarrow \Delta, \Psi \qquad \{E, \Theta_E \Rightarrow \Lambda_E \mid E \in \Psi\}}{[A_k]_{k>0}, \Gamma, \Pi', \Theta \Rightarrow \Delta, \Sigma', \Lambda}\,\text{Cut}$$

where $\Pi' = \biguplus_{D \in \Phi'} \Pi_D$ and $\Sigma' = \biguplus_{D \in \Phi'} \Sigma_D$. This cut is removed by the secondary induction hypothesis on the height of the derivation. We then consider the premise $\Pi_{\bigwedge_{k>0} A_k} \Rightarrow \Sigma_{\bigwedge_{k>0} A_k}, \bigwedge_{k>0} A_k$ and we apply height-preserving invertibility of the rule R$\bigwedge$ to get a derivation of $\Pi_{\bigwedge_{k>0} A_k} \Rightarrow \Sigma_{\bigwedge_{k>0} A_k}, A_k$ for every $k > 0$. Hence we conclude the reduction as follows:

$$\dfrac{\dfrac{\{\Pi_{\bigwedge_{k>0} A_k} \Rightarrow \Sigma_{\bigwedge_{k>0} A_k}, A_k \mid k > 0\} \qquad [A_k]_{k>0}, \Gamma, \Pi', \Theta \Rightarrow \Delta, \Sigma', \Lambda}{(\Pi_{\bigwedge_{k>0} A_k})^{\infty}, \Gamma, \Pi', \Theta \Rightarrow \Delta, \Sigma', \Lambda, (\Sigma_{\bigwedge_{k>0} A_k})^{\infty}}\,\text{Cut}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\text{Ctr}$$

Since, by definition of degree,

$$\deg([A_k]_{k>0}) = \sup_k w(A_k) + 1 < \sup_k w(A_k) + 2 = w(\bigwedge_{k>0} A_k) + 1 = \deg([\bigwedge_{k>0} A_k]) \leq \deg(\Phi, \Psi)$$

the application of cut can be removed invoking the primary induction hypothesis on the degree of the multiset of cut formulas.

We discuss the case in which $\bigwedge_{k>0} A_k$ is principal in Ψ. In this case, the procedure is similar.

$$\dfrac{\{\Pi_D \Rightarrow \Sigma_D, D \mid D \in \Phi\} \qquad \dfrac{\{\Phi, \Gamma \Rightarrow \Delta, \Psi', A_k \mid k > 0\}}{\Phi, \Gamma \Rightarrow \Delta, \Psi', \bigwedge_{k>0} A_k}\,R\bigwedge \qquad \{E, \Theta_E \Rightarrow \Lambda_E \mid E \in \Psi\}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\text{Cut}$$

We start by permuting the cut upwards as before. First, for every $k > 0$ we construct the following derivation:

$$\dfrac{\{\Pi_D \Rightarrow \Sigma_D, D \mid D \in \Phi\} \qquad \Phi, \Gamma \Rightarrow \Delta, \Psi', A_k \qquad \{E, \Theta_E \Rightarrow \Lambda_E \mid E \in \Psi'\}}{\Gamma, \Pi, \Theta' \Rightarrow \Delta, \Sigma, \Lambda', A_k}\,\text{Cut}$$

where $\Theta' = \biguplus_{E\in\Psi'} \Theta_E$ and $\Lambda' = \biguplus_{E\in\Psi'} \Lambda_E$. For every $k > 0$ the cut is removed by secondary induction hypothesis on the height of the premise $\Phi, \Gamma \Rightarrow \Delta, \Psi', A_k$. We thus get the derivations of sequents $\{\Gamma, \Pi, \Theta' \Rightarrow \Delta, \Sigma, \Lambda', A_k \mid k > 0\}$

Next, we complete the reduction as follows:

$$\dfrac{\dfrac{\{\Gamma, \Pi, \Theta' \Rightarrow \Delta, \Sigma, \Lambda', A_k \mid k > 0\} \qquad \dfrac{\bigwedge_{k>0} A_k, \Theta_{\bigwedge_{k>0} A_k} \Rightarrow \Lambda_{\bigwedge_{k>0} A_k}}{[A_k]_{k>0}, \Theta_{\bigwedge_{k>0} A_k} \Rightarrow \Lambda_{\bigwedge_{k>0} A_k}}\,\text{Inv}}{\Gamma^\infty, \Pi^\infty, \Theta'^\infty, \Theta_{\bigwedge_{k>0} A_k} \Rightarrow \Delta^\infty, \Sigma^\infty, \Lambda'^\infty, \Lambda_{\bigwedge_{k>0} A_k}}\,\text{Cut}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\text{Ctr}$$

The cut can be removed by primary induction hypothesis. □

Remark 2.15. As discussed in the § 5, the full system G3C$_\omega$ can be shown to be equivalent to systems with finite sequents in [16]. This yields immediately a soundness and completeness result for the system and an indirect form of cut admissibility, i.e., cut-free completeness. However, we deem that a fully syntactic proof of the result is interesting both on the technical and the conceptual side.

3 INFINITARY INTUITIONISTIC LOGIC

We introduce a sequent calculus for intuitionistic logic with infinite sequents which is depicted in Figure 2. The calculus is a single-succedent sequent calculus, i.e., only one formula is allowed in the succedent. We opted for this formulation, instead of a multisuccedent one, because the presence of a single formula in the succedent streamlines the proof of cut admissibility.

Preliminary results still need to be established for the system G3I$_\omega$. The notions of height, admissibility and weight are left unchanged.

Lemma 3.1. The sequent $\Gamma, A \Rightarrow A$ is provable for every formula A in G3I$_\omega$.

Proof. By transfinite induction on the weight of A. We discuss the case of the infinitary conjunction:

$$\dfrac{\dfrac{\{[A_k]_{k>0}, \Gamma \Rightarrow A_k \mid k > 0\}}{[A_k]_{k>0}, \Gamma \Rightarrow \bigwedge_{k>0} A_k}\,R\bigwedge}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow \bigwedge_{k>0} A_k}\,L\bigwedge$$

□

Initial Sequents

$$\frac{}{p, \Gamma \Rightarrow p}\text{Ax} \qquad \frac{}{\bot, \Gamma \Rightarrow C}\text{L}\bot$$

Logical Rules

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\text{L}\wedge \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\text{R}\wedge$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\text{L}\vee \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\text{R}\vee_i$$

$$\frac{A \rightarrow B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \rightarrow B, \Gamma \Rightarrow C}\text{L}\rightarrow \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\text{R}\rightarrow$$

$$\frac{[A_k]_{k>0}, \Gamma \Rightarrow C}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow C}\text{L}\bigwedge \qquad \frac{\{\Gamma \Rightarrow A_k \mid k > 0\}}{\Gamma \Rightarrow \bigwedge_{k>0} A_k}\text{R}\bigwedge$$

$$\frac{\{A_k, \Gamma \Rightarrow C \mid k > 0\}}{\bigvee_{k>0} A_k, \Gamma \Rightarrow C}\text{L}\bigvee \qquad \frac{\Gamma \Rightarrow A_k}{\Gamma \Rightarrow \bigvee_{k>0} A_k}\text{R}\bigvee_k$$

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\text{L}\forall \qquad \frac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \forall x A}\text{R}\forall, y!$$

$$\frac{A[x/y], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\text{L}\exists, y! \qquad \frac{\Gamma \Rightarrow A[x/t]}{\Gamma \Rightarrow \exists x A}\text{R}\exists$$

FIGURE 2 The sequent calculus $\mathsf{G3I}_\omega$.

The rule of substitution is height-preserving admissible too.

Lemma 3.2. The rule

$$\frac{\Gamma \Rightarrow C}{\Gamma[x/t] \Rightarrow C[x/t]}\text{Sub}[x/t]$$

is height-preserving admissible in $\mathsf{G3I}_\omega$.

Proof. The proof is by transfinite induction on the height of the derivation of the premise $\Gamma \Rightarrow C$. □

The weakening rule is admissible too.

Lemma 3.3. The rules of weakening

$$\frac{\Gamma \Rightarrow C}{\Pi, \Gamma \Rightarrow C}\text{LW} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow C}\text{RW}$$

are height-preserving admissible in $\mathsf{G3I}_\omega$.

Proof. The proof is by transfinite induction on the height of the derivation of the premise of the rule. □

We now have to discuss the invertibility of the rules of the calculus. As it is well known, in intuitionistic logic we lose invertibility of some rules. This actually ensures the constructive reading of the connectives.

Lemma 3.4. The rule L→ is height-preserving invertible with respect to its right premise in $G3I_\omega$. Every other rule except for $R\vee_i$, $R\bigvee_k$ and R∃ is height-preserving invertible in $G3I_\omega$.

Proof. The rule L∀ is invertible by height-preserving admissibility of weakening. In the remaining cases the proof is by induction on the height of the derivation. □

Contraction is here restricted to the antecedent. Indeed, the rule is of the shape:

$$\frac{\Pi, \Pi, \Gamma \Rightarrow C}{\Pi, \Gamma \Rightarrow C}\ \mathrm{Ctr}$$

Theorem 3.5. The rule Ctr is height-preserving admissible in $G3I_\omega$.

Proof. The proof runs by induction on the height of the derivation of $\Gamma, \Pi, \Pi \Rightarrow C$. If it is an initial sequent, then so is $\Pi, \Gamma \Rightarrow C$. If it is the conclusion of a rule, we distinguish cases. If no formula in Π is principal, then we apply the induction hypothesis to each of the premises of the rule and then the rule again. If a formula in Π is principal, one needs to distinguish cases according to the shape of the principal formula.

We consider the case in which the formula is $\bigwedge_{k>0} A_k$. We have:

$$\frac{[A_k]_{k>0}, \bigwedge_{k>0} A_k, \Pi', \Pi', \Gamma \Rightarrow C}{\bigwedge_{k>0} A_k, \bigwedge_{k>0} A_k, \Pi', \Pi', \Gamma \Rightarrow C}\ \mathrm{L}\bigwedge$$

where $\Pi = \bigwedge_{k>0} A_k, \Pi'$. We construct the following derivation:

$$\cfrac{\cfrac{\cfrac{[A_k]_{k>0}, \bigwedge_{k>0} A_k, \Pi', \Pi', \Gamma \Rightarrow C}{[A_k]_{k>0}, [A_k]_{k>0}, \Pi', \Pi', \Gamma \Rightarrow C}\ \mathrm{Inv}}{[A_k]_{k>0}, \Pi', \Gamma \Rightarrow C}\ \mathrm{Ctr}}{\bigwedge_{k>0} A_k, \Pi', \Gamma \Rightarrow C}\ \mathrm{L}\bigwedge$$

The application of the rule Ctr is removed by induction on the height of the premise. □

We then have to prove the crucial result, cut admissibility. In this case, the cut is of the following shape:

$$\frac{\{\Pi_D \Rightarrow D \mid D \in \Phi\} \qquad \Phi, \Gamma \Rightarrow C}{\Pi, \Gamma \Rightarrow C}\ \mathrm{Cut}$$

where the subscript D in Π_D is used to distinguish the multisets and $\Pi = \biguplus_{D\in\Phi} \Pi_D$. As we shall see, the proof is easier then the one detailed for classical logic.

Theorem 3.6. The cut rule is admissible in $G3I_\omega$.

Proof. The proof is by double induction, with main induction on the degree of the multiset of cut formulas Φ and secondary induction hypothesis on the height of the derivation of $\Phi, \Gamma \Rightarrow C$. If $\Phi, \Gamma \Rightarrow C$ is an initial sequent, then the proof is immediate. If $\Phi, \Gamma \Rightarrow C$ is the conclusion of a rule, but no formula in Φ is principal, then the conclusion follows by permuting the cut upwards. Hence the cut is replaced by (possibly infinitely many) cuts which are removed by secondary induction hypothesis. The last case to consider is the one in which a formula in Φ is principal. In this case we need to distinguish cases according to the shape of the formula.

1. The principal formula is $A \to B$, then we have:

$$\frac{\Phi', A \to B, \Gamma \Rightarrow A \qquad \Phi', B, \Gamma \Rightarrow C}{\Phi', A \to B, \Gamma \Rightarrow C}\ \mathrm{L}\!\to$$

with $\Phi = \Phi', A \to B$. In this case, we first construct the following derivation $\mathcal{D}'$:

$$\dfrac{\{\Pi_D \Rightarrow D \mid D \in \Phi\} \qquad \Phi', A \to B, \Gamma \Rightarrow A}{\Gamma, \Pi \Rightarrow A}\,\text{Cut}$$

The cut is removed by induction on the height of the premise of the cut. Analogously, we construct a derivation $\mathcal{D}''$ of $\Pi', B, \Gamma \Rightarrow C$ where $\Pi' = \biguplus_{D \in \Phi'} \Pi_D$. Next, we perform the following reduction:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}'}{\Gamma, \Pi \Rightarrow A} \qquad \dfrac{\Pi_{A\to B} \Rightarrow A \to B}{\Pi_{A\to B}, A \Rightarrow B}\,\text{Inv}}{\Gamma, \Pi_{A\to B}, \Pi \Rightarrow B}\,\text{Cut} \qquad \dfrac{\mathcal{D}''}{\Pi', B, \Gamma \Rightarrow C}}{\dfrac{\Gamma, \Pi, \Pi \Rightarrow C}{\Gamma, \Pi \Rightarrow C}\,\text{Ctr}}\,\text{Cut}$$

The cuts are removed by induction on the degree of the multiset of cut formulas. Indeed, we have:

$$\deg([A]), \deg([B]) < \deg([A \to B]) \leq \deg([\Phi', A \to B]).$$

The cases in which the principal formula is $A \wedge B$ is dealt with analogously.

2. The cases in which the principal formula is $\bigvee_{k>0} A_k$ is as follows:

$$\dfrac{\{\Phi', A_k, \Gamma \Rightarrow C \mid k > 0\}}{\Phi', \bigvee_{k>0} A_k, \Gamma \Rightarrow C}\,\text{L}\bigvee$$

with $\Phi = \Phi', \bigvee_{k>0} A_k$. For every $k > 0$, we construct the derivation $\mathcal{D}_k$:

$$\dfrac{\{\Pi_D \Rightarrow D \mid D \in \Phi'\} \qquad \Phi', A_k, \Gamma \Rightarrow C}{A_k, \Pi', \Gamma \Rightarrow C}\,\text{Cut}$$

where $\Pi' = \biguplus_{D \in \Phi'} \Pi_D$. We then consider the premise $\Pi_{\bigvee A_k} \Rightarrow \bigvee_{k>0} A_k$. We distinguish two cases. Either the formula $\bigvee_{k>0} A_k$ is never principal in the subderivation above $\Pi_{\bigvee A_k} \Rightarrow \bigvee_{k>0} A_k$ or it is principal (in possibly infinitely many inferences). In the first case, then $\Pi_{\bigvee A_{k>0}} \Rightarrow$ is derivable and the desired conclusion follows by weakening. For any of the countable branches in which $\bigvee_{k>0} A_k$ is principal, we trace this application and we have:

$$\begin{array}{c} \dfrac{\Pi''_{\bigvee_{k>0} A_k} \Rightarrow A_k}{\Pi''_{\bigvee_{k>0} A_k} \Rightarrow \bigvee_{k>0} A_k}\,\text{R}\bigvee \\ \vdots\, \pi \\ \Pi_{\bigvee_{k>0} A_k} \Rightarrow \bigvee_{k>0} A_k \end{array}$$

Notice that by the design of the rules we can assume that in the branch in π from $\Pi_{\bigvee_{k>0} A_k} \Rightarrow \bigvee_{k>0} A_k$ to $\Pi''_{\bigvee_{k>0} A_k} \Rightarrow \bigvee_{k>0} A_k$ the formula $\bigvee_{k>0} A_k$ was never principal. We construct the following derivation:

$$\begin{array}{c} \dfrac{\Pi''_{\bigvee_{k>0} A_k} \Rightarrow A_k \qquad A_k, \Pi', \Gamma \Rightarrow C}{\Pi''_{\bigvee_{k>0} A_k}, \Pi', \Gamma \Rightarrow C}\,\text{Cut} \\ \vdots\, \pi' \\ \Gamma, \Pi \Rightarrow C \end{array}$$

where π' is obtained from π by adding whenever needed the weakened contexts Γ, Π'. The cuts can be removed by invoking the primary induction hypothesis on the degree of the multiset of cut formulas. The cases in which the principal formula is of the shape $A \vee B$ or $\exists xA$ are dealt with analogously (in the latter case one only needs to apply height-preserving substitution in order to perform the reduction, we leave the details to the reader).

3. If the principal formula is $\bigwedge_{k>0} A_k$, we have:

$$\dfrac{[A_k]_{k>0}, \Phi', \Gamma \Rightarrow C}{\bigwedge_{k>0} A_k, \Phi', \Gamma \Rightarrow C}\ \mathrm{L}\bigwedge$$

with $\Phi = \Phi', \bigwedge_{k>0} A_k$. We perform the following reduction:

$$\dfrac{\dfrac{\dfrac{\Pi_{\bigwedge_{k>0} A_k} \Rightarrow \bigwedge_{k>0} A_k}{\{\Pi_{\bigwedge_{k>0} A_k} \Rightarrow A_k \mid k > 0\}}\ \mathrm{Inv} \qquad \dfrac{\{\Pi_D \Rightarrow D \mid D \in \Phi'\} \quad [A_k]_{k>0}, \Phi', \Gamma \Rightarrow C}{[A_k]_{k>0}, \Pi', \Gamma \Rightarrow C}\ \mathrm{Cut}}{(\Pi_{\bigwedge_{k>0} A_k})^{\infty}, \Pi', \Gamma \Rightarrow C}\ \mathrm{Cut}}{\Pi, \Gamma \Rightarrow C}\ \mathrm{Ctr}$$

The topmost cut is removed by secondary induction hypothesis, whereas the lowermost one is removed by primary induction hypothesis since:

$$\deg([A_k]_{k>0}) < \deg([\bigwedge_{k>0} A_k]) \leq \deg([\Phi', \bigwedge_{k>0} A_k]).$$

4. The only remaining case is the one in which the principal formula is $\forall xA$.

$$\dfrac{A[x/t], \forall xA, \Phi', \Gamma \Rightarrow C}{\forall xA, \Phi', \Gamma \Rightarrow C}\ \mathrm{L}\forall$$

with $\Phi = \Phi', \forall xA$. We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Pi_{\forall xA} \Rightarrow \forall xA}{\Pi_{\forall xA} \Rightarrow A[x/t]}\ \mathrm{Inv} \qquad \dfrac{\{\Pi_D \Rightarrow D \mid D \in \Phi\} \quad A[x/t], \forall xA, \Phi', \Gamma \Rightarrow C}{A[x/t], \Pi, \Gamma \Rightarrow C}\ \mathrm{Cut}}{\Pi_{\forall xA}, \Pi, \Gamma \Rightarrow C}\ \mathrm{Cut}}{\Pi, \Gamma \Rightarrow C}\ \mathrm{Ctr}$$

The topmost cut is removed by secondary induction hypothesis, whereas the lowermost one is removed by primary induction hypothesis since:

$$\deg([A[x/t]]) < \deg([\forall xA]) \leq \deg([\Phi', \forall xA]).$$ □

Remark 3.7. The cut admissibility strategy for infinitary classical propositional logic extends to the case of full intuitionistic infinitary logic due to the lack of repetition of the principal formula in the right rule for the existential quantifier. Indeed, this enables to treat the case of the existential quantifier as the one of the infinitary disjunction. It is still unclear if a further modification of the cut rule may be employed so as to get a direct cut admissibility for full classical infinitary logic. We leave this theme for future research.

As a corollary to cut admissibility we get the subformula property and the disjunction property.

Corollary 3.8. If $\mathsf{G3I}_\omega \vdash\ \Rightarrow \bigvee_{k>0} A_k$, then $\mathsf{G3I}_\omega \vdash\ \Rightarrow A_k$ for some $k > 0$.

Proof. Immediate by inspection of the rules. □

As shown in [8], a distinctive feature of infinitary intuitionistic logic lies in the refutation of an infinitary distributivity principle, namely:

$$\bigwedge_{k>0}(p_k \vee q) \to \bigwedge_{k>0} p_k \vee q.$$

One may wonder whether the present version of intuitionistic infinitary logic is still sound. Indeed, the calculus $\mathsf{G3I}_\omega$ is sound with respect to the semantics of complete Heyting algebras and this can be easily shown via a routine induction on the height of the derivation (we will prove soundness and completeness in the final section of the paper, by showing the equivalence with the systems with finite sequents). Furthermore, exploiting the analyticity of the system resulting from the cut admissibility theorem, we can also show that the sequent $\Rightarrow \bigwedge_{k>0}(p_k \vee q) \to \bigwedge_{k>0} p_k \vee q$ is not derivable.

Lemma 3.9. The sequent $\Rightarrow \bigwedge_{k>0}(p_k \vee q) \to \bigwedge_{k>0} p_k \vee q$ is not derivable.

Proof. By invertibility of R$\to$ and L$\bigwedge$, it is equivalent to consider the derivability of the sequent $[p_k \vee q]_{k>0} \Rightarrow \bigwedge_{k>0} p_k \vee q$.

By the design of the sequent rules, the only way to reach an initial sequent is to apply the rule R$\vee_i$. In this case, we shall encounter two kind of sequents:

1. $[p_k \vee q]_{k>0}, \Pi \Rightarrow \bigwedge_{k>0} p_k$,
2. $[p_k \vee q]_{k>0}, \Pi \Rightarrow q$,

where Π contains $p_1, \dots, p_n$ and q for some n. It is now a trivial task to check that both the sequents are not derivable. □

Remark 3.10. As pointed out by a reviewer, the underivability result is trivially entailed by the completeness of the system $\mathsf{G3I}_\omega$ which is established in the next sections. However, we believe that the proof is interesting as it is a purely syntactic underivability result.

4 SYNTACTIC CUT ADMISSIBILITY MODULO NEGATIVE TRANSLATION

We have obtained a direct cut admissibility for intuitionistic logic, but not for classical infinitary logic. Therefore, we shall prove our result by embedding classical infinitary logic into intuitionistic infinitary logic via a natural extension of Gödel-Gentzen's negative translation. Let us recall that $\neg A$ abridges $A \to \bot$.

Definition 4.1. The infinitary Gödel-Gentzen translation $g : \mathsf{FM}_\omega \to \mathsf{FM}_\omega$ is inductively defined:

1. $(\bot)^g = \bot$,
2. $(p)^g = \neg\neg p$,
3. $(A \# B)^g = A^g \# B^g$, where $\# \in \{\wedge, \to\}$,
4. $(A \vee B)^g = \neg(\neg A^g \wedge \neg B^g)$,
5. $(\forall x A)^g = \forall x A^g$,
6. $(\exists x A)^g = \neg\forall x \neg A^g$,
7. $(\bigwedge_{k>0} A_k)^g = \bigwedge_{k>0} A_k^g$,
8. $(\bigvee_{k>0} A_k)^g = \neg\bigwedge_{k>0} \neg A_k^g$.

We prove the following lemma.

Lemma 4.2. For every formula A, we have: $\neg\neg A^g \Rightarrow A^g$ in $\mathsf{G3I}_\omega$.

Proof. The proof is by transfinite induction on the degree of the formula A. We limit ourselves to considering the case in which A is of the shape $\bigwedge_{k>0} A_k$ as an example. First, for every $k > 0$ the sequent $\neg\neg \bigwedge_{k>0} A^g_k \Rightarrow A^g_k$ is derivable:

$$\dfrac{\neg\neg \bigwedge_{k>0} A^g_k \Rightarrow \neg\neg A^g_k \qquad \overset{\vdots\ \mathrm{IH}}{\neg\neg A^g_k \Rightarrow A^g_k}}{\neg\neg \bigwedge_{k>0} A^g_k \Rightarrow A^g_k}\ \mathrm{Cut}$$

where IH denotes the application of the induction hypothesis and the leftmost premise is derivable by root-first application of the rules. Next we obtain the following derivation:

$$\dfrac{\{\neg\neg \bigwedge_{k>0} A^g_k \Rightarrow A^g_k \mid k > 0\}}{\neg\neg \bigwedge_{k>0} A^g_k \Rightarrow \bigwedge_{k>0} A^g_k}\ \mathrm{R}\bigwedge$$

□

We write Δ^g to denote the multiset of formulas $g(A)$ for every A in Δ. Furthermore, given a multiset Δ, we denote by $\neg\Delta$ which contains $\neg A$ for every formula A in Δ. Next, we establish the embedding.

Theorem 4.3. If $\mathsf{G3C}_\omega \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{G3I}_\omega \vdash \Gamma^g, \neg\Delta^g \Rightarrow$.

Proof. The proof is by transfinite induction on the height of the derivation of the sequent $\Gamma \Rightarrow \Delta$ in the calculus $\mathsf{G3C}_\omega$. If the sequent is an initial sequent the proof is immediate. We focus on the cases involving the infinitary connectives.

If the last rule is $R\bigvee$, we have:

$$\dfrac{\neg\neg \bigwedge_{k>0} \neg A^g_k \Rightarrow \bigwedge_{k>0} \neg A^g_k \qquad \dfrac{\overset{\vdots\ \mathrm{IH}}{\Gamma^g, \neg\Delta'^g, [\neg A^g_k]_{k>0} \Rightarrow}}{\Gamma^g, \neg\Delta'^g, \bigwedge_{k>0} \neg A^g_k \Rightarrow}\ \mathrm{L}\bigwedge}{\Gamma^g, \neg\Delta'^g, \neg\neg \bigwedge_{k>0} \neg A^g_k \Rightarrow}\ \mathrm{Cut}$$

where $\Delta = \Delta', \bigvee A_k$ the leftmost sequent is provable by root-first applications of the rules.

If the last rule applied is $L\bigvee$ we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\ \mathrm{IH}}{\{A^g_k, \Gamma'^g, \neg\Delta^g \Rightarrow\ \mid k > 0\}}}{\{\Gamma'^g, \neg\Delta^g \Rightarrow \neg A^g_k \mid k > 0\}}\ \mathrm{R}{\to}}{\Gamma'^g, \neg\Delta^g \Rightarrow \bigwedge_{k>0} \neg A^g_k}\ \mathrm{R}\bigwedge}{\Gamma'^g, \neg\Delta^g, \neg \bigwedge_{k>0} \neg A^g_k \Rightarrow \bigwedge_{k>0} \neg A^g_k}\ \mathrm{LW}}{\Gamma'^g, \neg\Delta^g, \neg \bigwedge_{k>0} \neg A^g_k \Rightarrow}\ \mathrm{L}{\to}$$

where $\Gamma = \Gamma', \bigvee_{k>0} A_k$. □

In order to obtain full cut admissibility for classical infinitary logic with infinite sequent a last move has to be made.

Lemma 4.4. For every sequent $\Pi, \Gamma \Rightarrow \Delta, \Lambda$, if $\Pi, \Gamma^g \Rightarrow \Delta^g, \Lambda$ is derivable in $\mathsf{G3C}_\omega$, then so is $\Pi, \Gamma \Rightarrow \Delta, \Lambda$, where Π and Λ are (possibly empty) multisets containing only atomic formulas.

Proof. This result is indeed trivial in the presence of the cut rule, but we need to establish it without resorting to it. We argue by induction on the height of the derivation. If $\Pi, \Gamma^g \Rightarrow \Delta^g, \Lambda$ is an initial sequent, so is $\Pi, \Gamma \Rightarrow \Delta, \Lambda$.

If the principal formula of the last rule applied is not of the shape $\neg\neg p$, $\neg \bigwedge_{k<0} \neg A^g_k$, $\neg\forall x \neg A^g$ or $\neg(\neg A^g \wedge \neg B^g)$, then the proof follows by applying the induction hypothesis and then the rule again. For example, if the last rule applied is R→ and the formula is of the shape $A^g \to B^g$, we have:

$$\frac{\Pi, A^g, \Gamma^g \Rightarrow \Delta'^g, B^g, \Lambda}{\Pi, \Gamma^g \Rightarrow \Delta'^g, A^g \to B^g, \Lambda}\ \mathrm{R}{\to}$$

We apply the induction hypothesis to get a derivation of $\Pi, A, \Gamma \Rightarrow \Delta', B, \Lambda$ and the desired conclusion follows from an application of the rule R→.

If the last rule applied is L→ or R→ and the principal formula is of the shape $\neg\neg p$, $\neg \bigwedge_{k>0} \neg A^g_k$, $\neg\forall x \neg A^g$ or $\neg(\neg A^g \wedge \neg B^g)$, then we need to distinguish cases. We discuss the case of L→ (the right premise of the rule is omitted to save space), the case of R→ is symmetric. If the principal formula is $\neg\neg p$, we have:

$$\frac{\Pi, \Gamma'^g \Rightarrow \Delta^g, \Lambda, \neg p}{\Pi, \neg\neg p, \Gamma'^g \Rightarrow \Delta^g, \Lambda}\ \mathrm{L}{\to}$$

We proceed as follows:

$$\cfrac{\cfrac{\Pi, , \Gamma'^g \Rightarrow \Delta^g, \Lambda, \neg p}{\Pi, p, \Gamma'^g \Rightarrow \Delta^g, \Lambda}\ \mathrm{Inv}}{\Pi, p, \Gamma' \Rightarrow \Delta, \Lambda}\ \mathrm{IH}$$

The application of the induction hypothesis is justified, because the invertibility of the rule preserves the height of the derivation. If the principal formula is $\neg \bigwedge_{k>0} \neg A^g_k$, we have:

$$\frac{\Pi, \Gamma'^g \Rightarrow \Delta^g, \Lambda, \bigwedge_{k>0} \neg A^g_k}{\Pi, \neg \bigwedge_{k>0} \neg A^g_k, \Gamma'^g \Rightarrow \Delta^g, \Lambda}\ \mathrm{L}{\to}$$

We construct the following derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Pi, , \Gamma'^g \Rightarrow \Delta^g, \Lambda, \bigwedge_{k>0} \neg A^g_k}{\{\Pi, \Gamma'^g \Rightarrow \Delta^g, \Lambda, \neg A^g_k \mid k > 0\}}\ \mathrm{Inv}}{\{\Pi, A^g_k, \Gamma'^g \Rightarrow \Delta^g, \Lambda \mid k > 0\}}\ \mathrm{Inv}}{\{\Pi, A_k, \Gamma' \Rightarrow \Delta, \Lambda \mid k > 0\}}\ \mathrm{IH}}{\Pi, \bigvee_{k>0} A_k, \Gamma' \Rightarrow \Delta, \Lambda}\ \mathrm{L}\bigvee$$

The cases of in which the principal formulas is of the shape $\neg(\neg A^g \wedge \neg B^g)$ and $\neg\forall x \neg A^g$ are analogously dealt with and we omit the details. □

Remark 4.5. Notice that the inclusion of the multisets of atomic formulas is crucial in the formulation of the previous lemma. Indeed, the statement to prove is strengthened in a way which allows us to apply the induction hypothesis. If we removed the multisets of atomic formulas, this would not be possible, because atomic formulas are not in the image of the g-translation.

We can now obtain a purely syntactic proof of the cut admissibility theorem by exploiting the negative translation of classical into intuitionistic logic.

Theorem 4.6. The cut rule is admissible in $\mathsf{G3C}_\omega$.

Proof. Suppose we have derivations of $\Phi, \Gamma \Rightarrow \Delta, \Psi$ and of the sequents $\{A, \Pi_A \Rightarrow \Sigma_A \mid A \in \Psi\}$ and $\{\Theta_A \Rightarrow \Lambda_A, A \mid A \in \Phi\}$ in G3C$_\omega$. By Theorem 4.3, we obtain derivations of $\Phi^g, \Gamma^g, \neg\Delta^g, \neg\Psi^g \Rightarrow$ and of the sequents $\{A^g, \Pi^g_A, \neg\Sigma^g_A \Rightarrow \mid A \in \Psi\}$ and $\{\Theta^g_A, \neg A^g, \neg\Lambda^g_A \Rightarrow \mid A \in \Phi\}$ in G3I$_\omega$. We first construct the following derivations in G3I$_\omega$:

$$\dfrac{\{A^g, \Pi^g_A, \neg\Sigma^g_A \Rightarrow \mid A \in \Psi\}}{\{\Pi^g_A, \neg\Sigma^g_A \Rightarrow \neg A^g \mid A \in \Psi\}}\ \text{R}\to$$

Furthermore, for every formula A in Φ, we get:

$$\dfrac{\dfrac{\Theta^g_A, \neg A^g, \neg\Lambda^g_A \Rightarrow}{\Theta^g_A, \neg\Lambda^g_A \Rightarrow \neg\neg A^g}\ \text{R}\to \qquad \neg\neg A^g \Rightarrow A^g}{\Theta^g_A, \neg\Lambda^g_A \Rightarrow A^g}\ \text{Cut}$$

where the rightmost sequent is derivable by Lemma 4.2. Hence we obtain the derivations of $\{\Theta^g_A, \neg\Lambda^g_A \Rightarrow A^g \mid A \in \Phi\}$ in G3I$_\omega$. Finally, we proceed as follows:

$$\dfrac{\{\Theta^g_A, \neg\Lambda^g_A \Rightarrow A^g \mid A \in \Phi\} \qquad \Phi^g, \Gamma^g, \neg\Delta^g, \neg\Psi^g \Rightarrow \qquad \{\Pi^g_A, \neg\Sigma^g_A \Rightarrow \neg A^g \mid A \in \Psi\}}{\Pi^g, \Gamma^g, \neg\Delta^g, \neg\Lambda^g \Rightarrow}\ \text{Cut}$$

where $\Pi = \biguplus_{A\in\Psi} \Pi_A, \Sigma = \biguplus_{A\in\Psi} \Sigma_A, \Lambda = \biguplus_{A\in\Phi} \Lambda_A$ and $\Theta = \biguplus_{A\in\Phi} \Theta_A$. Since G3I$_\omega$ is a subsystem of G3C$_\omega$, we get a derivation of $\Pi^g, \Gamma^g, \neg\Delta^g, \neg\Lambda^g \Rightarrow$ in G3C$_\omega$. The desired conclusion is obtained through the following proof transformations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi^g, \Gamma^g, \neg\Delta^g, \neg\Lambda^g \Rightarrow}{\Pi^g, \Gamma^g, \bigwedge\neg\Delta^g, \bigwedge\neg\Lambda^g \Rightarrow}\ \text{L}\bigwedge}{\Pi^g, \Gamma^g, \bigwedge\neg\Delta^g, \bigwedge\neg\Lambda^g \Rightarrow \bot, \bot}\ \text{Weak}}{\Pi^g, \Gamma^g, \bigwedge\neg\Lambda^g \Rightarrow \neg\bigwedge\neg\Delta^g, \bot}\ \text{R}\to}{\Pi^g, \Gamma^g \Rightarrow \neg\bigwedge\neg\Delta^g, \neg\bigwedge\neg\Lambda^g}\ \text{R}\to}{\Pi^g, \Gamma^g \Rightarrow (\bigvee\Delta)^g, (\bigvee\Lambda)^g}\ \text{rewriting}}{\Pi, \Gamma \Rightarrow \bigvee\Delta, \bigvee\Lambda}\ \text{Lemma 4.4}}{\Pi, \Gamma \Rightarrow \Delta, \Lambda}\ \text{Inv R}\bigvee$$

□

5 EQUIVALENCE WITH THE FINITE SEQUENTS CALCULI

In this section we show the equivalence of the calculi for infinitary logic with finite and infinite sequents. We focus on the case of the calculus for intuitionistic logic (the classical case is similar and thus we omit the details). We recall the single succedent calculus G3if$_\omega$ for intuitionistic infinitary logic with finite sequents (cf. also [9] for a multisuccedent version). It is depicted in Figure 3.

Essentially, G3if$_\omega$ is obtained from the calculus G3I$_\omega$ by replacing the rule L$\bigwedge$ with the infinitely many rules L$\bigwedge_k$.

Theorem 5.1. The calculus G3if$_\omega$ satisfies the admissibility of the structural rules of weakening, contraction and cut.

One direction of the embedding is easier. Indeed, if a sequent is derivable in the calculus with finite sequents, it is derivable also in the one based on infinite ones.

Lemma 5.2. If $\Gamma \Rightarrow C$ is derivable in G3if$_\omega$, then so is in G3I$_\omega$.

Initial Sequents

$$\frac{}{p, \Gamma \Rightarrow p}\,\mathrm{Ax} \qquad \frac{}{\bot, \Gamma \Rightarrow C}\,\mathrm{L}\bot$$

Logical Rules

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\,\mathrm{L}\wedge \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,\mathrm{R}\wedge$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\,\mathrm{L}\vee \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\,\mathrm{R}\vee_i$$

$$\frac{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C}\,\mathrm{L}{\to} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\,\mathrm{R}{\to}$$

$$\frac{A_k, \bigwedge_{k>0} A_k, \Gamma \Rightarrow C}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow C}\,\mathrm{L}\bigwedge_k \qquad \frac{\{\Gamma \Rightarrow A_k \mid k > 0\}}{\Gamma \Rightarrow \bigwedge_{k>0} A_k}\,\mathrm{R}\bigwedge$$

$$\frac{\{A_k, \Gamma \Rightarrow C \mid k > 0\}}{\bigvee_{k>0} A_k, \Gamma \Rightarrow C}\,\mathrm{L}\bigvee \qquad \frac{\Gamma \Rightarrow A_k}{\Gamma \Rightarrow \bigvee_{k>0} A_k}\,\mathrm{R}\bigvee_k$$

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\,\mathrm{L}\forall \qquad \frac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \forall x A}\,\mathrm{R}\forall, y!$$

$$\frac{A[x/y], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\,\mathrm{L}\exists, y! \qquad \frac{\Gamma \Rightarrow A[x/t]}{\Gamma \Rightarrow \exists x A}\,\mathrm{R}\exists$$

FIGURE 3 The sequent calculus $\mathsf{G3if}_\omega$.

Proof. The proof is by induction on the height of the derivations. The only new cases to detail are the ones in which the last rule applied is L$\bigwedge$. We have:

$$\cfrac{\cfrac{\cfrac{\bigwedge_{k>0} A_k, A_k, \Gamma \Rightarrow C}{[A_k]_{k>0}, A_k, \Gamma \Rightarrow C}\,\mathrm{Inv}}{[A_k]_{k>0}, \Gamma \Rightarrow C}\,\mathrm{Ctr}}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow C}\,\mathrm{L}\bigwedge$$

□

The other direction is slightly more complex. We have:

Lemma 5.3. If $\mathsf{G3I}_\omega$ proves $\Gamma \Rightarrow C$, then $\mathsf{G3if}_\omega$ proves $\bigwedge \Gamma \Rightarrow C$.

Proof. The proof is by induction on the height of the derivation. If it is an initial sequent, the proof is trivial. Otherwise, we distinguish cases according to the last rule applied. If the last rule is R→, we have:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\,\mathrm{R}{\to}$$

The induction hypothesis yields a derivation of:

$$\bigwedge \Gamma \wedge A \Rightarrow B$$

in $\mathsf{G3if}_\omega$. We apply a cut with the derivable sequent $\bigwedge \Gamma, A \Rightarrow \bigwedge \Gamma \wedge A$ and we conclude the transformation via an application of the rule R→. The other cases involving right rules are immediate (apply the induction hypothesis and then the rule again). If the last rule applied is L$\bigwedge$, the proof follows immediately by an application of the induction hypothesis.

The case in which the last rule applied is $L\bigvee$, we have:

$$\frac{\{A_k, \Gamma \Rightarrow C \mid k > 0\}}{\bigvee_{k>0} A_k, \Gamma \Rightarrow C} \, L\bigvee$$

For every $k > 0$, we get: $A_k \wedge \bigwedge \Gamma \Rightarrow C$. So we proceed as follows:

$$\frac{\bigvee_{k>0} A_k \wedge \bigwedge \Gamma \Rightarrow \bigvee_{k>0}(A_k \wedge \bigwedge \Gamma) \qquad \dfrac{\{A_k \wedge \bigwedge \Gamma \Rightarrow C \mid k > 0\}}{\bigvee_{k>0}(A_k \wedge \bigwedge \Gamma) \Rightarrow C} \, L\bigvee}{\bigvee_{k>0} A_k \wedge \bigwedge \Gamma \Rightarrow C} \, \mathrm{Cut}$$

where the leftmost sequent is derivable in $\mathsf{G3if}_\omega$ via straightforward root-first applications of the rules. □

The above theorem immediately yields a completeness result in the following form, where $\vDash$ is the validity relation in the semantics based on complete Heyting algebras.

Theorem 5.4. $\mathsf{G3I}_\omega \vdash \Gamma \Rightarrow A$ if and only if $\vDash \bigwedge \Gamma \to A$.

Proof. From left to right we argue by induction on the height of the derivation in the calculus $\mathsf{G3I}_\omega$. From right to left, if $\vDash \bigwedge \Gamma \to A$ we get $\mathsf{G3if}_\omega \vdash \Rightarrow \bigwedge \Gamma \to A$ and via the embedding $\mathsf{G3I}_\omega \vdash \Rightarrow \bigwedge \Gamma \to A$. The desired conclusion follows from invertibility of the rules $R\!\to$ and $L\bigwedge$. □

Notice that the formulation of the completeness theorem is crucial. Indeed, the sequent $[p_k]_{k>0} \Rightarrow \bigwedge_{k>0} p_k$ would not be valid if interpreted as expressing logical consequence. In fact, there are various counterexamples to $[p_k]_{k>0} \vDash \bigwedge_{k>0} p_k$ which rest on the fact that infinite intersections of open sets need not be open [8].

Remark 5.5. As observed by the reviewer, the completeness result could be obtained directly, exploiting the admissibility of the cut rule to simulate modus ponens and obtain the equivalence with axiomatic calculi. We opted to obtain completeness via the equivalence with cut-free sequent calculi with finite sequents because axiomatic calculi for infinitary classical and intuitionistic logic are not explicitly present in the literature (although they could be easily definable from the corresponding sequent calculi).

6 CONCLUDING REMARKS AND FUTURE WORKS

We have discussed and analyzed the proof theory of infinitary logic with infinite sequents. We have provided a structural analysis of the calculi for classical and intuitionistic logic. The calculi enjoy admissibility of the structural rules of weakening and contraction. Furthermore, cut is shown admissible employing a new strategy which runs by a double transfinite induction with a new parameter, the degree of a multiset of cut formulas.

The cut admissibility for full classical infinitary logic (including quantifiers) is obtained via the negative translation into full infinitary intuitionistic logic which enjoys a full and direct cut admissibility theorem. The results presented in the paper show that the extension of an infinitary calculus for classical and intuitionistic logic with infinite sequents is, in a sense, inessential as it can be interpreted with finite sequents.

However, this is not always the case. As shown by Minari in [5], working with infinite sequents in the context of infinitary modal logic marks a difference as it enables the derivability of the (infinitary variant of the) Barcan formula (cf. [17, 19] for cut-free sequent calculi for infinitary modal logic). Furthermore, in substructural logics the presence of infinitary conjunctions and disjunctions can be used to simulate contraction and exponential modalities, cf. [2].

Therefore we deem that the techniques developed in the present paper can be interesting as they might be employed to investigate other areas of infinitary logic. An interesting point to be addressed is the possibility of a full fledged syntactic approach to the proof theory of infinitary logic with infinite sequents and with rules which can act on infinite multisets of formulas. A similar approach was pursued in [16], but only with semantic methods.

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