Symmetry Shapes Thermodynamics of Macroscopic Quantum Systems

Vasco Cavina,[1,2,*] Ariane Soret,[1,†] Timur Aslyamov,[1] Krzysztof Ptaszyński,[1,3] and Massimiliano Esposito[1]
[1]Department of Physics and Materials Science, University of Luxembourg, L-1511 Luxembourg, Luxembourg
[2]NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, I-56126 Pisa, Italy
[3]Institute of Molecular Physics, Polish Academy of Sciences, Mariana Smoluchowskiego 17, 60-179 Poznań, Poland

(Received 22 February 2024; accepted 20 August 2024; published 23 September 2024)

We derive a systematic approach to the thermodynamics of quantum systems based on the underlying symmetry groups. We first show that the entropy of a system can be described in terms of group-theoretical quantities that are largely independent of the details of its density matrix. We then apply our technique to generic N identical interacting d-level quantum systems. Using permutation invariance, we find that, for large N, the entropy displays a universal asymptotic behavior in terms of a function $s(\boldsymbol{x})$ that is completely independent of the microscopic details of the model, but depends only on the size of the irreducible representations of the permutation group S_N. In turn, the equilibrium state of the system and macroscopic fluctuations around it are shown to satisfy a large deviation principle with a rate function $f(\boldsymbol{x}) = e(\boldsymbol{x}) - \beta^{-1}s(\boldsymbol{x})$, where $e(\boldsymbol{x})$ only depends on the ground state energy of particular subspaces determined by group representation theory, and β is the inverse temperature. We apply our theory to the transverse-field Curie-Weiss model, a minimal model of phase transition exhibiting an interplay of thermal and quantum fluctuations.

DOI: 10.1103/PhysRevLett.133.130401

Symmetries play a fundamental role in shaping physical theories, from quantum mechanics to thermodynamics [1]. In the context of nonequilibrium physics, symmetries are crucial for the understanding of stochastic energetics [2,3], dynamical phase transitions [4–6], and quantum thermodynamics [7–9]. Studying the entropic, energetic, or dynamic signatures of underlying symmetries in quantum systems is an active field of research, from fundamental questions about entropy scalings, ground state properties, thermalization and open quantum systems thermodynamics [10–19], to the optimization of quantum computing or numerical simulation procedures [20–26]. This research is gaining momentum due to rapid experimental advances [16,22], particularly in cold atoms [27,28].

In this Letter, we give a systematic description of how the underlying symmetry of a quantum system determines its entropic and energetic profile. First, we show that if the density matrix of a system is invariant under the action of a group G, its von Neumann entropy simplifies into a contribution that only depends on a structural property of the group, namely the dimension of its irreducible representations, and another that is model dependent but can be bounded from above by other group theoretical quantities. Second, we consider many-body quantum systems composed of N identical interacting d-level systems and invariant under the action of the permutation group $G = \mathrm{S}_N$. At equilibrium, we prove that the only extensive contributions to entropy are completely independent of the microscopic details of the system and are only shaped by the structure of S_N. The extensive contributions to energy are, in turn, given by the ground states of particular restrictions of the Hamiltonian, defined from group representation theory. Third, we illustrate the usefulness of our results on the transverse-field Curie-Weiss model, which—despite the apparent simplicity of its Hamiltonian—exhibits a highly nontrivial behavior resulting from the interplay of quantum and thermal fluctuations [29–32]. We highlight that the main strength of our group theory approach is that, unlike standard mean-field methods, it enables us to perform a large deviation analysis [33,34] and to obtain the rate function of the free energy, giving access to the full statistics.

von Neumann entropy using group theory—If a quantum state (i.e., density matrix) $\hat{\rho}$ is symmetric with respect to a given finite group G, we can exploit representation theory to write $\hat{\rho}$ in block diagonal form. This will be instrumental in understanding the impact that symmetry-related effects have on thermodynamics. Given a quantum system with Hilbert space $\mathcal{H}$ and a group G [35] acting on it, Maschke's theorem [36,37] allows us to decompose $\mathcal{H}$ into a direct sum of *representations* of G, that is, subspaces of $\mathcal{H}$ that remain invariant under the action of any element $\hat{g} \in G$. A representation that cannot be further decomposed (i.e., does not contain invariant subspaces) is called *irreducible*. The Hilbert space has a unique decomposition in terms of irreducible representations, hereby called V_k^{λ}:

*Contact author: vasco.cavina@sns.it
†Contact author: ariane.soret@gmail.com

$$\mathcal{H} = \bigoplus_{\lambda} \bigoplus_{k=1}^{\deg \lambda} V_k^{\lambda}, \tag{1}$$

where $\boldsymbol{\lambda}$ is a suitable index [38] labeling the irreducibles of G in the decomposition of $\mathcal{H}$. Note that, in general, the same irreducible can appear with multiplicity, denoted as $\deg \boldsymbol{\lambda}$ in Eq. (1). A symmetric state $\hat{\rho}$ is, by definition, invariant under the action of G, namely $[\hat{g}, \hat{\rho}] = 0$ for all $\hat{g} \in G$ [39], so it can be fully characterized by Schur's lemma [36,37]. The latter implies that the density matrix is block diagonal in the sectors with fixed $\boldsymbol{\lambda}$. Defining the total probability associated with a given block as $p_{\boldsymbol{\lambda}} = \mathrm{Tr}[\hat{\rho} \sum_k \hat{\Pi}_{\boldsymbol{\lambda},k}]$, with $\hat{\Pi}_{\boldsymbol{\lambda},k}$ the projector on $V_k^{\boldsymbol{\lambda}}$, we obtain

$$\hat{\rho} = \sum_{\lambda} p_{\lambda} \hat{\rho}_{\lambda}, \tag{2}$$

where $\hat{\rho}_{\boldsymbol{\lambda}} \equiv \sum_{k,k'} \hat{\Pi}_{\boldsymbol{\lambda},k} \hat{\rho} \hat{\Pi}_{\boldsymbol{\lambda},k'} / p_{\boldsymbol{\lambda}}$ are the conditional (normalized) density matrices associated to the blocks. A second implication of Schur's lemma is that every subblock of $\hat{\rho}_{\boldsymbol{\lambda}}$ with fixed k, k' is proportional to the identity matrix

$$\hat{\Pi}_{\boldsymbol{\lambda},k} \hat{\rho}_{\boldsymbol{\lambda}} \hat{\Pi}_{\boldsymbol{\lambda},k'} = \frac{1}{\dim \boldsymbol{\lambda}} [\tilde{\rho}_{\boldsymbol{\lambda}}]_{kk'} \mathbb{I}_{\boldsymbol{\lambda} kk'}, \tag{3}$$

where we gathered the proportionality coefficients in a new operator $\tilde{\rho}_{\boldsymbol{\lambda}}$ (with elements $[\tilde{\rho}_{\boldsymbol{\lambda}}]_{kk'}$) and $\mathbb{I}_{\boldsymbol{\lambda} kk'}$ denotes the identity map between $V_k^{\boldsymbol{\lambda}}$ and $V_{k'}^{\boldsymbol{\lambda}}$. Note that $\tilde{\rho}_{\boldsymbol{\lambda}}$ is defined on a reduced Hilbert space indexed by $k, k' \in [1, \deg \boldsymbol{\lambda}]$, and that the normalization $\dim \boldsymbol{\lambda} \equiv \dim V_k^{\boldsymbol{\lambda}} = \dim V_{k'}^{\boldsymbol{\lambda}}$ ensures that $\mathrm{Tr}[\tilde{\rho}_{\boldsymbol{\lambda}}] = 1$. Using the new "coarse-grained" state $\tilde{\rho}_{\boldsymbol{\lambda}}$, we may rewrite $\hat{\rho}_{\boldsymbol{\lambda}}$ as

$$\hat{\rho}_{\boldsymbol{\lambda}} = \frac{1}{\dim \boldsymbol{\lambda}} \tilde{\rho}_{\boldsymbol{\lambda}} \otimes \mathbb{I}_{\boldsymbol{\lambda}}, \tag{4}$$

where we used that $\mathbb{I}_{\boldsymbol{\lambda} kk'}$ are all copies of a unique identity matrix of dimension $\dim \boldsymbol{\lambda}$, above denoted by $\mathbb{I}_{\boldsymbol{\lambda}}$. We now use Eqs. (2) and (4) to compute the von Neumann entropy [40], given by $S(\hat{\rho}) \equiv -\mathrm{Tr}[\hat{\rho} \ln \hat{\rho}]$. Since the blocks with different $\boldsymbol{\lambda}$ are disconnected [see Eq. (2)], the final result will be a sum over the contributions of the different blocks. In addition, the inside-block degeneracy emerging from Eq. (4) yields an additional $\dim \boldsymbol{\lambda}$ factor, resulting in our first main result (see Sec. I in [41]):

$$S(\hat{\rho}) = \sum_{\lambda} p_{\lambda} [\ln \dim \boldsymbol{\lambda} + S(\tilde{\rho}_{\boldsymbol{\lambda}})] + H(p), \tag{5}$$

where $H(p) = -\sum_{\lambda} p_{\lambda} \ln p_{\lambda}$ is the Shannon entropy [42] of the classical probability distribution $p = \{p_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}}$. The first contribution in Eq. (5) depends on the dimension of a given irreducible representation, i.e., a universal property of the group G. On the contrary, the second and third terms depend on the details of $\hat{\rho}$, yet they can be bounded from above by other state-independent parameters coming from Eq. (1). Using that $\tilde{\rho}_{\boldsymbol{\lambda}}$ is defined on a reduced Hilbert space of dimension $\deg \boldsymbol{\lambda}$,

$$S(\tilde{\rho}_{\boldsymbol{\lambda}}) \leq \ln \deg \boldsymbol{\lambda}. \tag{6}$$

Similarly, the Shannon entropy $H(p)$ cannot exceed the value of the logarithm of the total number of irreducible representations appearing in the direct sum (1), that we call Λ

$$H(p) \leq \ln \Lambda. \tag{7}$$

Equations (5)–(7) allow us to give a universal description of the von Neumann entropy of any quantum system given its symmetry group G.

Universality of macroscopic entropy—For a collection of N identical d-level systems, the Hilbert space has the form

$$\mathcal{H}_N^{(d)} = \bigotimes_{i=1}^{N} \mathbb{C}_d^{(i)}, \tag{8}$$

where $\mathbb{C}_d^{(i)}$ is the d-dimensional complex vector space of the ith system. The set G of symmetries of a given system depends on the properties of the system itself. Common examples in many-body physics are the cyclic symmetry (e.g., in Ising chains with periodic boundary conditions [43,44]) and the permutation symmetry, associated, respectively, with the cyclic group Z_N and with the symmetric group S_N.

From now on, we focus on the permutation group. Each element $\pi \in \mathrm{S}_N$ acts on the tensor product basis of $\mathcal{H}_N^{(d)}$ by exchanging its local components:

$$\pi(|e_1\rangle \otimes |e_2\rangle .. \otimes |e_N\rangle) = |e_{\pi(1)}\rangle \otimes |e_{\pi(2)}\rangle .. \otimes |e_{\pi(N)}\rangle, \tag{9}$$

where $\pi(j)$ is the particle occupying the position j after the permutation took place and $|e_i\rangle$ with $e_i = 1, \ldots, d$ is a basis vector for $\mathbb{C}_d^{(i)}$. To obtain a decomposition of the form (1) for $\mathcal{H}_N^{(d)}$ and $G = \mathrm{S}_N$, let us first introduce the *occupation numbers* μ_j, which represent the number of particles in the state j for $j = 1, \ldots, d$. We then introduce the subspaces $M_{\boldsymbol{\mu}}$ with fixed occupation number vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$. By construction, the basis elements of $M_{\boldsymbol{\mu}}$ are vectors $|e_1\rangle \otimes |e_2\rangle .. \otimes |e_N\rangle$ such that the value j is taken exactly μ_j times by the e_i's, for $j = 1, \ldots, d$. From the relation (9), it is clear that the subspaces $M_{\boldsymbol{\mu}}$ are invariant under the action of S_N, so we can preliminarily decompose

$$\mathcal{H}_N^{(d)} = \bigoplus_{\boldsymbol{\mu}} M_{\boldsymbol{\mu}}, \tag{10}$$

where $\pi(M_{\boldsymbol{\mu}}) \subseteq M_{\boldsymbol{\mu}}$ for all $\pi \in \mathrm{S}_N$. To further reduce the susbspaces $M_{\boldsymbol{\mu}}$, we do the crucial observation (see Sec. II in

[41]) that they coincide with standard objects of representation theory, called *permutation modules* of index $\boldsymbol{\mu}$. Permutation modules can be completely reduced, using textbook approaches [37], as

$$M_{\boldsymbol{\mu}} = \bigoplus_{\boldsymbol{\lambda}} \bigoplus_{h=1}^{K_{\boldsymbol{\lambda},\boldsymbol{\mu}}} \mathcal{S}^{\boldsymbol{\lambda}}_{h,\boldsymbol{\mu}}, \tag{11}$$

where $\mathcal{S}^{\boldsymbol{\lambda}}_{h,\boldsymbol{\mu}}$ are the irreducible representations of S_N. While the superscript $\boldsymbol{\lambda}$ spans different irreducibles, the subscript $\boldsymbol{\mu}$ keeps track of the original $\boldsymbol{\mu}$ sector and the subscript h is a degeneracy index running between 1 and a value $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ that can be computed analytically and is known as the *Kostka number* [37]. The vector indices $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ are d-dimensional partitions of N (decreasing vectors of positive integers summing to N, $\sum_{j=1}^{d} \mu_j = \sum_{j=1}^{d} \lambda_j = N$), and, together with h in Eq. (11), constitute a complete set of quantum numbers labeling the irreducibles in the decomposition of $\mathcal{H}_N^{(d)}$.

To study the behavior of the von Neumann entropy of a permutation-invariant state in the large N limit we apply Eqs. (5)–(7) to the decomposition (10). After replacing k with the specific notation for S_N given by $\boldsymbol{\mu}$, h, the asymptotic of the entropic contributions can be obtained from known combinatorial formulas: (i) $\dim \boldsymbol{\lambda}$ is given by the *Hook length formula* [36], a combinatorial rule based on the concept of *Young tableau*. The Vershik-Kerov theorem and related results [45–47] show that $\dim \boldsymbol{\lambda}$ is exponentially large in N for $N \to \infty$. After introducing a rescaled variable $\boldsymbol{x} \equiv (\boldsymbol{\lambda}/N)$ and an intensive volume entropy associated with $\dim \boldsymbol{\lambda}$, we find (see Sec. III in [41])

$$s(\boldsymbol{x}) \equiv \lim_{N\to\infty} \frac{1}{N} \ln \dim(N\boldsymbol{x}) = -\sum_{j=1}^{d} x_j \ln x_j, \tag{12}$$

where $s(\boldsymbol{x})$ turns out to be a Shannon entropy over the rescaled variable $\boldsymbol{x}$. (ii) $\deg \boldsymbol{\lambda}$ is the multiplicity of the irreducible labeled by the partition $\boldsymbol{\lambda}$, which can be obtained from Eqs. (10) and (11). An upper bound for this quantity is derived (see Sec. III in [41]) by noting that it coincides with the logarithm of the Schur polynomial of index $\boldsymbol{\lambda}$ and all arguments equal to 1 [36]

$$\ln \deg \boldsymbol{\lambda} = \ln \sum_{\boldsymbol{\mu}} K_{\boldsymbol{\lambda},\boldsymbol{\mu}} \leq \frac{d(d-1)}{2} \ln N, \tag{13}$$

so the contribution of $\deg \boldsymbol{\lambda}$ is nonextensive, as long as the thermodynamic limit does not involve any scaling of the local Hilbert spaces dimension d. (iii) Λ is the number of distinct irreducible representations appearing in Eq. (11). If the Hilbert spaces of the single constituents have dimension d, then $\boldsymbol{\lambda}$ spans all the partitions of N composed by d elements. This quantity has a known asymptotic value for large N [48], that we can use to derive

$$\ln \Lambda \sim (d-1) \ln N. \tag{14}$$

Plugging Eqs. (13) and (14) into Eqs. (5)–(7) it is clear that the only extensive contribution to $S(\hat{\rho})$ comes from the term proportional to $\dim \boldsymbol{\lambda}$, thus

$$\lim_{N\to\infty} \frac{S(\hat{\rho})}{N} = \lim_{N\to\infty} \sum_{\boldsymbol{\lambda}} \frac{p_{\boldsymbol{\lambda}}}{N} \ln \dim \boldsymbol{\lambda} = \int p(\boldsymbol{x}) s(\boldsymbol{x}) d\boldsymbol{x}, \tag{15}$$

where $p(\boldsymbol{x}) \equiv N^d p_{\boldsymbol{\lambda}=N\boldsymbol{x}}$. We stress that $s(\boldsymbol{x})$ is a universal property of S_N, and as such is completely independent from any detail of the state $\hat{\rho}$, that are gauged only through $p(\boldsymbol{x})$ [49].

To show the versatility of our approach, we complete this section with a concise discussion of the cyclic group case $G = \mathrm{Z}_N$. Since Z_N is an Abelian group, the dimension of irreducible representations $\dim \boldsymbol{\lambda}$ is always 1. The number of distinct irreducibles Λ can be shown to be upper-bounded by the number of elements of Z_N [50], that is equal to N. We conclude that the contributions related to $\ln \dim \boldsymbol{\lambda}$ and $H(p)$ are subextensive for Z_N, meaning that the only extensive contributions to the entropy come from $S(\tilde{\rho}_{\boldsymbol{\lambda}})$ and are strongly state dependent, as opposed to what happens for the permutation group S_N.

Macroscopic equilibrium thermodynamics—A system at equilibrium is described by the free energy

$$F \equiv -\frac{1}{\beta} \ln \mathrm{Tr}[e^{-\beta \hat{H}}], \tag{16}$$

where $\hat{H}$ is the Hamiltonian and β is the inverse temperature of the system (we set $k_B = 1$). If the Hamiltonian is invariant under the action of a generic group G, i.e., $[\hat{H}, \hat{g}] = 0$ for all $\hat{g} \in G$, we can characterize $\hat{H}$ using Schur's lemma. The procedure for decomposing $\hat{H}$ is easier than the one followed for states in Eqs. (2)–(4). In the latter case, we needed to interpret $\hat{\rho}_{\boldsymbol{\lambda}}$ and $\tilde{\rho}_{\boldsymbol{\lambda}}$ as legitimate density matrices, adding proper normalization factors [e.g., introducing $p_{\boldsymbol{\lambda}}$ in Eq. (2)]. In the present case, we can simply write $\hat{H} = \sum_{\boldsymbol{\lambda}} \tilde{H}_{\boldsymbol{\lambda}} \otimes \mathbb{I}_{\boldsymbol{\lambda}}$, where the definition of $\tilde{H}_{\boldsymbol{\lambda}}$ is obtained replacing $\hat{\rho}$ with $\hat{H}$ in Eq. (3) and removing the factor $\dim \boldsymbol{\lambda}$. We can use the decomposition above in the free energy (16). Since the different $\boldsymbol{\lambda}$ subspaces are orthogonal, we have $e^{-\beta \hat{H}} = \sum_{\boldsymbol{\lambda}} e^{-\beta \tilde{H}_{\boldsymbol{\lambda}} \otimes \mathbb{I}_{\boldsymbol{\lambda}}}$, leading to

$$F = -\frac{1}{\beta} \ln \Big\{ \sum_{\boldsymbol{\lambda}} \dim \boldsymbol{\lambda} \mathrm{Tr}[e^{-\beta \tilde{H}_{\boldsymbol{\lambda}}}] \Big\}, \tag{17}$$

where $\dim \boldsymbol{\lambda}$ emerges from the trace of $\mathbb{I}_{\boldsymbol{\lambda}}$. We now consider $G = \mathrm{S}_N$. To study the contributions of $\tilde{H}_{\boldsymbol{\lambda}}$ to Eq. (17) in the large N limit we introduce the associated intensive free energy

$$e(\boldsymbol{x}) \equiv -\lim_{N\to\infty}\frac{1}{\beta N}\ln\{\mathrm{Tr}[e^{-\beta\tilde{H}(\boldsymbol{x})}]\}, \tag{18}$$

where $\tilde{H}(\boldsymbol{x}) = \tilde{H}_{\lambda/N}$ is an operator acting on the space spanned by $\boldsymbol{\mu}$, h. However, as discussed in Eqs. (6) and (13), the intensive entropy associated with such contributions vanishes when $N \to \infty$, and $e(\boldsymbol{x})$ reduces to the sole energetic contribution, that is, $\lim_{N\to\infty} N^{-1}\mathrm{Tr}[\tilde{H}(\boldsymbol{x})e^{-\beta\tilde{H}(\boldsymbol{x})}]/\mathrm{Tr}[e^{-\beta\tilde{H}(\boldsymbol{x})}]$. Moreover, assuming that at least the ground state energy, $E^0_{\boldsymbol{x}}$, of $\tilde{H}(\boldsymbol{x})$ is extensive [51], the latter will dominate $e(\boldsymbol{x})$ (since the contributions due to the excited states are exponentially suppressed):

$$e(\boldsymbol{x}) = \lim_{N\to\infty}\frac{\mathrm{Tr}[\tilde{H}(\boldsymbol{x})e^{-\beta\tilde{H}(\boldsymbol{x})}]}{N\mathrm{Tr}[e^{-\beta\tilde{H}(\boldsymbol{x})}]} = \lim_{N\to\infty}\frac{E^0_{\boldsymbol{x}}}{N}. \tag{19}$$

This reduces the calculation of $e(\boldsymbol{x})$ to a much simpler ground state problem. Using Eq. (17) with Eqs. (12) and (18), we can express the intensive free energy as

$$\lim_{N\to\infty}\frac{F}{N} = -\lim_{N\to\infty}\frac{1}{\beta N}\ln\int e^{-N\beta f(\boldsymbol{x})}d\boldsymbol{x} = f(\boldsymbol{x}^*), \tag{20}$$

where we introduced the free energy function

$$f(\boldsymbol{x}) \equiv e(\boldsymbol{x}) - \beta^{-1}s(\boldsymbol{x}) \tag{21}$$

and its minimum $\boldsymbol{x}^* = \mathrm{argmin}_{\boldsymbol{x}} f(\boldsymbol{x})$. Since the equilibrium density matrix of the system is $\hat{\rho} = e^{-\beta\hat{H}}/\mathrm{Tr}[e^{-\beta\hat{H}}]$, which, using (16), may be rewritten as $\hat{\rho} = e^{-\beta(\hat{H}-F)}$, then, using Eq. (2), we find that $p_{\boldsymbol{\lambda}} = \dim\boldsymbol{\lambda}\,\mathrm{Tr}[e^{-\beta(\tilde{H}_{\boldsymbol{\lambda}}-F)}]$. This implies

$$\lim_{N\to\infty}\frac{1}{N}\ln p_{\boldsymbol{\lambda}} = -\beta[f(\boldsymbol{x}) - f(\boldsymbol{x}^*)], \tag{22}$$

so that $f(\boldsymbol{x})$ corresponds to a genuine rate function [33]. The dominant contribution to the entropy of the system can then be calculated, using Eq. (15), as

$$\lim_{N\to\infty}\frac{S}{N} = s(\boldsymbol{x}^*), \tag{23}$$

and the energy, $E = F + \beta^{-1}S$, as

$$\lim_{N\to\infty}\frac{E}{N} = \lim_{N\to\infty}\frac{\mathrm{Tr}\left[\hat{H}e^{-\beta(\hat{H}-F)}\right]}{N} = e(\boldsymbol{x}^*). \tag{24}$$

The macroscopic thermodynamics of the system is thus fully characterized by the minimum of the free energy function (21) that can be obtained from the knowledge of the ground state of the coarse-grained Hamiltonians $\tilde{H}(\boldsymbol{x})$ [see Eq. (19)] and the universal $s(\boldsymbol{x})$ resulting from the structure of the permutation group [see Eq. (12)].

Application—To illustrate the power of our approach, we use it to describe the phase diagram of the transverse-field Curie-Weiss model, consisting of N spin-1/2 particles interacting via the Hamiltonian

$$\hat{H} = -\omega\sum_{i=1}^{N}\hat{S}_z^{(i)} - \frac{\alpha}{N}\sum_{i,j=1}^{N}\hat{S}_x^{(i)}\hat{S}_x^{(j)}, \tag{25}$$

where $\hbar = 1$, $\hat{S}^{(i)}_{z,x}$ are spin-1/2 operators, ω is the transverse magnetic field, and α is the Ising-type ferromagnetic (antiferromagnetic) interaction for $\alpha > 0$ ($\alpha < 0$). Despite its apparent simplicity, this model exhibits a highly nontrivial behavior resulting from the interplay of thermal and quantum fluctuations [29–32]. The model is permutation invariant, with $d = 2$ in Eq. (8). In this case, $\boldsymbol{\lambda}$ is a two-dimensional partition of N and can be fully parametrized in terms of a single rescaled variable $l = [(\lambda_1 - \lambda_2)/2N] = (\lambda_1/N) - \frac{1}{2} \in [0, \frac{1}{2}]$. Physically, the representations $\boldsymbol{\lambda}$ are the subspaces corresponding to a definite eigenvalue L of the total angular momentum operator $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, where $\hat{L}_{x,y,z} = \sum_{i=1}^{N}\hat{S}^{(i)}_{x,y,z}$. The variable $l = L/N$ is then the rescaled total angular momentum. In particular, in the magnetic system it corresponds to the rescaled total magnetization.

For the model considered, the entropic factor $s(l)$ is expressed, using Eq. (12), as

$$s(l) = -\left(\frac{1}{2} - l\right)\ln\left(\frac{1}{2} - l\right) - \left(\frac{1}{2} + l\right)\ln\left(\frac{1}{2} + l\right). \tag{26}$$

To compute the energetic factor $e(l)$ we use the mentioned correspondence between l and the total angular momentum. The Hamiltonian (25) can then be written in a block-diagonal form [see discussion below Eq. (16)] with

$$\tilde{H}_l = -\omega\hat{L}_z^{(l)} - \frac{\alpha}{N}\hat{L}_x^{(l)}\hat{L}_x^{(l)}, \tag{27}$$

where $\hat{L}^{(l)}_{x,z}$ are spin-(lN) operators. The matrix elements of the operator (27) in the $\hat{L}_z^{(l)}$ eigenbasis can be written in terms of the Clebsch-Gordan coefficients. After this step, for finite system sizes, Eqs. (18) and (19) can be evaluated numerically. For $N \to \infty$, we can further use a quantum-classical correspondence for large $L = lN$ to calculate the ground state energy E_l^0 as the minimum energy of a corresponding classical Hamiltonian (see Sec. IV in [41]). This yields

$$e(l) = \begin{cases} -\omega l & \text{for } l \le \frac{\omega}{2\alpha}, \\ -l^2\alpha - \frac{\omega^2}{4\alpha} & \text{for } l > \frac{\omega}{2\alpha}. \end{cases} \tag{28}$$

Interestingly, $e(l)$ is nonanalytic at $l = \omega/(2\alpha)$. This may seem surprising, as the equilibrium free energy function is

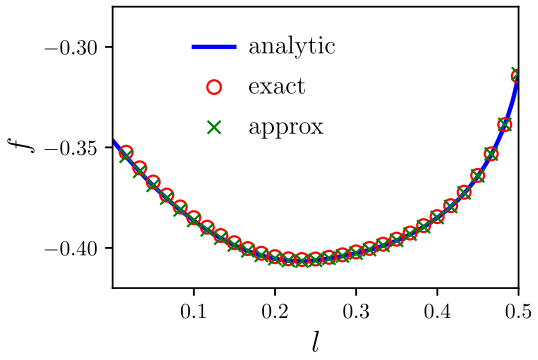

FIG. 1. Free energy function $f(l)$ calculated using analytic theory (blue solid line), exact numerical calculations (red circles), and approximate numerics (green crosses); see details in the main text. Parameters: $\alpha = 1$, $\omega = 0.5$, $\beta = 2$, $N = 1500$.

often assumed to be an analytic function of its arguments (e.g., in Landau theory of phase transitions [52]), and similar nonanalyticies of the rate function are often regarded as a feature of nonequilibrium systems [53–55].

To test our method, in Fig. 1 we present the free energy function $f(l)$ calculated for large $N = 1500$ using three different approaches: without any approximation using Eq. (17), then with asymptotic formulas (12) and (19), and finally using the analytic expression (28) for $e(l)$. All three approaches match well, which confirms the validity of our theory.

In Fig. 2(a) we present the phase diagram of the model in (β, α) coordinates, specifically, the behavior of the argument l^* minimizing the free energy function, which corresponds to equilibrium total magnetization. As one can observe, l^* is strongly enhanced in the ferromagnetic regime ($\alpha > 0$) for $\beta > \beta_c$, where $\beta_c = 2\tanh^{-1}(\omega/\alpha)/\omega$ is the critical inverse temperature (top right corner of the plot). This region corresponds to the ordered phase, where the Ising interaction induces magnetization in the x direction. Below β_c and in the antiferromagnetic regime ($\alpha < 0$), l^* is

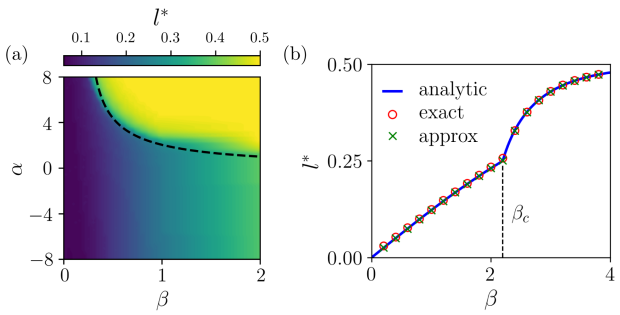

FIG. 2. (a) Density plot l^* in (β, α) coordinates, representing the phase diagram of the quantum Curie-Weiss model. Dashed line denotes the border of the ordered phase (top right corner of the figure). Plot obtained using exact numerics for $N = 100$. (b) Cross section of the density plot for $\alpha = 1$ and $N = 1500$. Methods and symbols used as in Fig. 1.

well described by the formula for the magnetic-field-induced magnetization in the z direction in the paramagnetic phase: $l^* = \tanh(\omega\beta/2)/2$. In Fig. 2(b) we plot a cross section of this graph for the constant α, calculated using the same approaches as for Fig. 1. As shown, the analytic solution exhibits a nonanalytic behavior at the phase transition point $\beta = \beta_c$, in agreement with both exact and asymptotic numerical results.

Final remarks—Our findings are particularly relevant to characterize emergent phenomena in many-body quantum systems, where they can be used to drastically simplify problems that have so far resisted analytical or numerical treatment. We focused here on the permutation group, with an emphasis on the case $d = 2$, but the scalings obtained can be straightforwardly applied to the case of all-to-all interacting d-level systems with $d \geq 2$. The case $d = 3$ was recently experimentally studied using Rydberg atoms [56]. If d is not finite, bounds of the forms (6), (7) still exist (see, e.g., the Hardy-Ramanujan formula for counting the number of unrestricted partitions [57]), and our theory can be easily extended to the case where the local Hilbert spaces have an extensive spectrum. It would also be interesting to study other groups, in particular those where $\ln \dim \boldsymbol{\lambda}$ is extensive; in such cases, we anticipate nontrivial phase transitions to occur. Finally, extending our framework to open quantum systems could be insightful for the perspective of building a quantum counterpart to classical macroscopic fluctuation theory [34,58–62].

Acknowledgments—This research was supported by the Luxembourg National Research Fund: A. S. by Project ThermoQO (C21/MS/15713841) and T. A. by ChemComplex (C21/MS/16356329).

[1] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics* (Elsevier, New York, 2013).
[2] M. Polettini, G. Bulnes-Cuetara, and M. Esposito, Conservation laws and symmetries in stochastic thermodynamics, Phys. Rev. E **94**, 052117 (2016).
[3] R. Rao and M. Esposito, Conservation laws shape dissipation, New J. Phys. **20**, 023007 (2018).
[4] M. Heyl, Dynamical quantum phase transitions: A review, Rep. Prog. Phys. **81**, 054001 (2018).
[5] M. Heyl, Dynamical quantum phase transitions in systems with broken-symmetry phases, Phys. Rev. Lett. **113**, 205701 (2014).
[6] S. A. Weidinger, M. Heyl, A. Silva, and M. Knap, Dynamical quantum phase transitions in systems with continuous symmetry breaking, Phys. Rev. B **96**, 134313 (2017).
[7] C. Aron, G. Biroli, and L. F. Cugliandolo, (Non) equilibrium dynamics: A (broken) symmetry of the Keldysh generating functional, SciPost Phys. **4**, 008 (2018).
[8] Z. Fei and H. T. Quan, Group-theoretical approach to the calculation of quantum work distribution, Phys. Rev. Res. **1**, 033175 (2019).

[9] V. Cavina, S. S. Kadijani, M. Esposito, and T. L. Schmidt, A convenient Keldysh contour for thermodynamically consistent perturbative and semiclassical expansions, SciPost Phys. **15**, 209 (2023).

[10] M. Goldstein and E. Sela, Symmetry-resolved entanglement in many-body systems, Phys. Rev. Lett. **120**, 200602 (2018).

[11] P. Calabrese, J. Dubail, and S. Murciano, Symmetry-resolved entanglement entropy in Wess-Zumino-Witten models, J. High Energy Phys. 10 (2021) 067.

[12] G. Gour, D. Jennings, F. Buscemi, R. Duan, and I. Marvian, Quantum majorization and a complete set of entropic conditions for quantum thermodynamics, Nat. Commun. **9**, 5352 (2018).

[13] O. Shpielberg, Universal entanglement entropy in the ground state of biased bipartite systems, Phys. Rev. A **106**, L030401 (2022).

[14] T. Brydges, A. Elben, P. Jurcevic, B. Vermersch, C. Maier, B. P. Lanyon, P. Zoller, R. Blatt, and C. F. Roos, Probing Rényi entanglement entropy via randomized measurements, Science **364**, 260 (2019).

[15] J. Decamp, J. Jünemann, M. Albert, M. Rizzi, A. Minguzzi, and P. Vignolo, Strongly correlated one-dimensional Bose–Fermi quantum mixtures: Symmetry and correlations, New J. Phys. **19**, 125001 (2017).

[16] M. Ueda, Quantum equilibration, thermalization and prethermalization in ultracold atoms, Nat. Rev. Phys. **2**, 669 (2020).

[17] A. Aragoneses, A. Kapulkin, and A. K. Pattanayak, Permutation entropy of indexed ensembles: Quantifying thermalization dynamics, J. Phys. Complex **4**, 02LT02 (2023).

[18] X.-J. Yu, S.-H. Shi, L. Xu, and Z.-X. Li, Emergence of competing orders and possible quantum spin liquid in SU(n) fermions, Phys. Rev. Lett. **132**, 036704 (2024).

[19] B. Yadin, B. Morris, and K. Brandner, Thermodynamics of permutation-invariant quantum many-body systems: A group-theoretical framework, Phys. Rev. Res. **5**, 033018 (2023).

[20] M. Esposito and P. Gaspard, Emergence of diffusion in finite quantum systems, Phys. Rev. B **71**, 214302 (2005).

[21] M. Esposito and P. Gaspard, Exactly solvable model of quantum diffusion, J. Stat. Phys. **121**, 463 (2005).

[22] G. Tóth, W. Wieczorek, D. Gross, R. Krischek, C. Schwemmer, and H. Weinfurter, Permutationally invariant quantum tomography, Phys. Rev. Lett. **105**, 250403 (2010).

[23] T. Moroder, P. Hyllus, G. Tóth, C. Schwemmer, A. Niggebaum, S. Gaile, O. Gühne, and H. Weinfurter, Permutationally invariant state reconstruction, New J. Phys. **14**, 105001 (2012).

[24] R. Lo Franco and G. Compagno, Indistinguishability of elementary systems as a resource for quantum information processing, Phys. Rev. Lett. **120**, 240403 (2018).

[25] N. Shammah, S. Ahmed, N. Lambert, S. De Liberato, and F. Nori, Open quantum systems with local and collective incoherent processes: Efficient numerical simulations using permutational invariance, Phys. Rev. A **98**, 063815 (2018).

[26] M. B. Mansky, S. L. Castillo, V. R. Puigvert, and C. Linnhoff-Popien, Permutation-invariant quantum circuits, arXiv:2312.14909.

[27] C. Dlaska, K. Ender, G. B. Mbeng, A. Kruckenhauser, W. Lechner, and R. van Bijnen, Quantum optimization via four-body Rydberg gates, Phys. Rev. Lett. **128**, 120503 (2022).

[28] R. Islam, R. Ma, P. Preiss, M. E. Tai, A. Lukin, M. Rispoli, and M. Greiner, Measuring entanglement entropy in a quantum many-body system, Nature (London) **528**, 77 (2015).

[29] F. C. SáBarreto and I. P. Fittipaldi, Thermodynamical properties of the transverse Ising model, Physica (Amsterdam) **129A**, 360 (1985).

[30] L. Chayes, N. Crawford, D. Ioffe, and A. Levit, The phase diagram of the quantum Curie-Weiss model, J. Stat. Phys. **133**, 131 (2008).

[31] T. Jörg, F. Krzakala, J. Kurchan, A. C. Maggs, and J. Pujos, Energy gaps in quantum first-order mean-field–like transitions: The problems that quantum annealing cannot solve, Europhys. Lett. **89**, 40004 (2010).

[32] J. Strečka and M. Jaščur, A brief account of the Ising and Ising-like models: Mean-field, effective-field and exact results, Acta Phys. Slovaca **65**, 235 (2015); arXiv:1511.03031.

[33] H. Touchette, The large deviation approach to statistical mechanics, Phys. Rep. **478**, 1 (2009).

[34] G. Falasco and M. Esposito, Macroscopic stochastic thermodynamics, arXiv:2307.12406.

[35] Formally, when we say that a group G acts on the Hilbert space of the system $\mathcal{H}$, we mean that there is a group homomorphism between G and $GL(\mathcal{H})$, the group of linear invertible operators acting on $\mathcal{H}$. To simplify the notation, we will never explicitly write this homomorphism and just treat G as a subgroup of $GL(\mathcal{H})$. With this in mind, the elements of G will be accompanied by a hat to remark that they are operators acting on $\mathcal{H}$.

[36] W. Fulton and J. Harris, *Representation Theory: A First Course* (Springer-Verlag, Berlin, 1991).

[37] B. Sagan, *The Symmetric Group* (Springer, New York, 2001).

[38] The domain of $\boldsymbol{\lambda}$ depends on the group of interest. In the case of the group S_N it takes the form of a vector, as we will see later.

[39] See Ref. [35].

[40] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2010).

[41] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.133.130401 for more details.

[42] C. E. Shannon, A mathematical theory of communication, Bell Syst. Tech. J. **27**, 379 (1948).

[43] P. Pfeuty, The one-dimensional Ising model with a transverse field, Ann. Phys. (N.Y.) **57**, 79 (1970).

[44] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, England, 2011).

[45] S. V. Kerov and A. M. Vershik, The characters of the infinite symmetric group and probability properties of the Robinson–Schensted–Knuth algorithm, SIAM J. Algebr. Discrete Methods **7**, 116 (1986).

[46] A. Bufetov, M. Zhitlukhin, and N. Kozin, *Young Diagrams and Their Limit Shape* (MCCME, Moscow, 2013).

[47] A. Nazarov, P. Nikitin, and O. Postnova, Limit shape for infinite rank limit of tensor power decomposition for lie algebras of series $\mathfrak{so}_{2n+1}$, J. Phys. A Math. Theor. **56**, 134001 (2023).

[48] C. Knessl and J. B. Keller, Partition asymptotics from recursion equations, SIAM J. Appl. Math. **50**, 323 (1990).
[49] We note that in the literature one often considers systems with collective dissipation processes (due to their computational simplicity), whose evolution preserves the probabilities p_λ [19,25]. In such systems all entropy changes in time are subextensive with the system size.
[50] The number of distinct irreducible representations of a group corresponds to the number of *conjugacy classes* of that group [36]. For Z_N and for abelian groups in general this corresponds to the number of elements of the group.
[51] This is motivated by the empirical observation that many physical systems display a well-defined macroscopic thermodynamics. If the ground state energy is subextensive, our approach still applies, with $e(\boldsymbol{x})$ going to zero. Situations where the ground-state energy is superextensive have no well-defined macroscopic thermodynamics, as commonly observed in systems with long-range interactions.
[52] L. D. Landau, *Collected Papers of L.D. Landau*, edited by D. Ter Haar (Pergamon, Oxford, 1965).
[53] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Lagrangian phase transitions in nonequilibrium thermodynamic systems, J. Stat. Mech. (2010) L11001.
[54] J. P. Garrahan and I. Lesanovsky, Thermodynamics of quantum jump trajectories, Phys. Rev. Lett. **104**, 160601 (2010).
[55] C. Aron and C. Chamon, Landau theory for non-equilibrium steady states, SciPost Phys. **8**, 074 (2020).
[56] D. S. Wild, S. Drăgoi, C. McElhanney, J. Wurtz, and S.-T. Wang, Quantum dynamics of a fully blockaded Rydberg atom ensemble, Phys. Rev. A **109**, 043111 (2024).
[57] G. E. Andrews, *The Theory of Partitions* (Cambridge University Press, Cambridge, England, 1998).
[58] B. Derrida, Non-equilibrium steady states: Fluctuations and large deviations of the density and of the current, J. Stat. Mech. (2007) P07023.
[59] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory, Rev. Mod. Phys. **87**, 593 (2015).
[60] D. Bernard, Can the macroscopic fluctuation theory be quantized?, J. Phys. A Math. Theor. **54**, 433001 (2021).
[61] T. Herpich, T. Cossetto, G. Falasco, and M. Esposito, Stochastic thermodynamics of all-to-all interacting many-body systems, New J. Phys. **22**, 063005 (2020).
[62] J. Meibohm and M. Esposito, Minimum-dissipation principle for synchronised stochastic oscillators far from equilibrium, arXiv:2401.14982.