Mario Piazza
Matteo Tesi

Inside Classical Logic: Truth, Contradictions, Fractionality

Abstract. Fractional semantics provides a multi-valued interpretation of a variety of logics, governed by purely proof-theoretic principles. This approach employs a method of systematic decomposition of formulas through a well-disciplined sequent calculus, assigning a fractional value that measures the "quantity of identity" (intuitively, "quantity of truth") within a sequent. A key consequence of this framework is the breakdown of the traditional symmetry between truth and contradiction. In this paper, we explore the ramifications of this novel perspective on classical logic. Specifically, we (i) introduce an alternative *paraconsistent* consequence relation, and (ii) show how the gradual character of contradictions induces a corresponding characterization of tautologies, thereby obtaining a full-fledged informational refinement of classical logic.

Keywords: Classical logic, Proof theory, Paraconsistency, Contradictions.

1. Introduction

Beginners in logic read in any elementary logic textbook that the propositional territory of classical logic is split into three regions like Caesar's Gaul: *tautologies*, *contingencies*, and *contradictions*. Tautologies invariably hold true for *any* truth-value assignment $\{0, 1\}$ to their atoms; contingencies become true for *some* assignments; contradictions are false for *any* assignment. The idea is that the notion of tautology is the semantic analogue of theoremhood in mathematics, where there is no theorem more theorem than another, while contradictions represent an undifferentiated absurdity that trivializes the underlying reasoning in its integrity.

Classical logic is indeed *explosive* due to its validation of the principle of *ex falso sequitur quodlibet*, expressed by the tautology:

$$A \wedge \neg A \to B$$

This implies that the consequence relation is explosive too, as it holds that:

$$A, \neg A \models B$$

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for any formula B. The principle of *ex falso* has been justified since the Medieval epoch (see [15]). Indeed:

$$\frac{\vdash A \qquad \vdash \neg A}{\vdash B}$$

follows from the combination of *modus ponens* and disjunctive syllogism (DS). In particular, we have:

$$\cfrac{\cfrac{\cfrac{\vdash A}{\vdash A, B}\,\text{RW}}{\vdash A \vee B}\,\vee \qquad \vdash \neg A}{\vdash B}\,\text{DS}$$

where RW and $\vee$ denote the rules of weakening of the succedent and of introduction of the disjunction, respectively.

It would be desirable to develop a method that introduces a gradation in the explosiveness of classical logic while preserving some classical features (we will clarify this issue). A promising approach to achieving this goal involves two steps: (a) introducing a distinction between contradictions within a classical landscape, and (b) defining a consequence relation that accounts for this distinction. Various approaches have been proposed to establish syntactic criteria for differentiating between formulas in classical logic, extending beyond contradictions and tautologies.[1] For instance, in [7] an isomorphism between formulas is defined, providing a purely syntactic criterion to establish a kind of *propositional identity* as in [6]. However, existing approaches do not address the paraconsistency issues central to this paper and, moreover, the distinction defined by the isomorphism lacks a quantitative aspect.

In the present work, the distinction is made effective through what we have termed *fractional semantics* [10], which is a novel, constructive, and purely proof-theoretic interpretation of classical logic assigning rational values to formulas in the interval $[0, 1]$. Such semantics allows for fine-grained distinctions among contradictions while ensuring that tautologies consistently hold the maximum value of 1. This aspect is valuable as it empowers fractional semantics to function as an interpretation for classical logic, safeguarding the essential concept of theoremhood in mathematics. Furthermore, this approach provides an interpretation for reasoning within the bounds of classical logic while accommodating nuanced analyses of contradictory statements.

[1] The problem has a long-standing history, first addressed by Carnap in [3]. For a more recent discussion, see [4] and [5], among others.

In the fractional approach, the value of a formula is determined by its decomposition into initial sequents within a sequent calculus, according to three fundamental requirements:

- Invertibility of every rule, that is the provability of its conclusion implies the provability of (each one of) its premise(s).
- Termination of the proof search, i.e. every attempt at finding a derivation gives either a proof or a refutation.
- Stability, i.e., every derivation of a sequent has the same multiset of initial sequents.

The value coincides with the ratio between the number of axiomatic initial sequents and the total number of initial sequents. The resulting decorated system is $\overline{\overline{\mathbf{GS4}}}$. Fractional semantics was first designed for classical logic [10] and then exported to a wide range of logics, including extensions of classical logic such as modal logics [11,12], and substructural restrictions like the multiplicative-additive fragment of linear logic MALL [13].

In this paper, we show how a fractional setting allows for a deeper understanding of the interplay between classical contradictions and their consequences. In particular, we aim to substantiate the strong intuition that, given two contradictions with different fractional values, the one with *lower* value should possess *greater* explosive power than the one with the higher value. Simply put, the key takeaway message is this: *the more false the contradiction, the greater its explosiveness.*

The new consequence relation $\mid\!\sim$ is first examined from a semantic standpoint, and its properties are thoroughly investigated. Subsequently, we present a proof-theoretic framework to elucidate its nature. Technically, the construction of a well-behaved sequent calculus, denoted as **GS4F**, hinges on a mixed system. Here, the rules for $\mid\!\sim$ are formulated through those governing the fractional system $\overline{\overline{\mathbf{GS4}}}$ and the calculus for classical propositional logic **GS4**. A distinctive aspect of the rules for the calculus **GS4F** is the inclusion of ternary rules of inference, where two premises are employed to disentangle the antecedent from the succedent for the evaluation of their fractional values.

Our proposal aims to fulfill two key desiderata. On the one hand, it must accommodate a graded version of paraconsistency. On the other hand, it should preserve (a form of) classicality. While one reviewer has observed that **GS4F** may not straightforwardly qualify as classical in the strictest sense,[2] we assert that classicality in this context specifically refers to the *conservativity* of classically valid formulas over the newly defined consequence relation (see Theorem 4.10). Moreover, it is important to emphasize that **GS4F** retains a core set of classical rules, underscoring its connection with classical principles despite its novel approach to handling inconsistency. The rules governing the behaviour of the connectives inside the fractional consequence relation are not classical, but they are defined through derivability in classical calculi.

The structural properties of **GS4F** are thoroughly investigated and we conclude that the new consequence relation is reflexive, right monotonic and transitive. Hence we obtain a calculus for a subclassical logic which is defined through a constraint naturally imposed from the fractional interpretation of classical logic. Our calculus therefore showcases the maximal amount of classicality one may retain while introducing a graded form of paraconsistency and modifying the rules of the calculus. Therefore this *controlled* account of paraconsistency (which varies according to the fractional truth degree of the assumptions) is defined *inside* classical logic. Paraconsistent phenomena, as induced in classical logic, have been explored through various avenues, such as the regimentation of classical logic using control sets [9] or its characterization via refutation systems [14,16]. Avron's proposal [1,2] aligns conceptually with our present approach. In particular, we share the perspective that classical logic serves as the primary logic, being the one (mostly) employed as metalogic for other systems. However, our approach diverges in that it relies on purely proof-theoretic considerations to establish the fractional interpretation of classical logic, while Avron's methodology introduces matrices for describing the new system.

The introduction of more nuanced perspective on contradictions paves the way for a method to analyze and obtain an informational refinement for tautologies. To achieve this, we define an equivalence relation between contradictions based on their shared values. This, in turn, induces an equivalence relation among classical tautologies, where two tautologies are considered equivalent if their negations have the same value.

These findings regarding contradictions and tautologies align with commonly held intuitions about truth and falsity. It is widely acknowledged that the explosive nature of contradictions is an undesirable aspect of classical reasoning. Our proposal resonates with this enduring concern, providing a

[2] Indeed, as observed by him/her, even the theorems of classical logic can be proved (whenever formulated in the $\{\wedge, \neg\}$ fragment) within an intuitionistic calculus, despite the two systems being radically different.

solution rooted in the distinctive framework of fractional semantics. This approach serves as a foundation for a flexible treatment of falsity and truth, employing combinatorial, constructive, and proof-theoretic methods without relying on external semantic structures.

This paper is structured as follows. Section 2 introduces key notions an terminology that will be employed in the sequel and it recalls the main features of fractional semantics. In Section 3, we formulate a consequence relation designed to remain sensitive to contradictions. Section 4 establishes a syntactic framework for the aforementioned consequence relation. In particular, a sequent calculus which combines different proof systems is introduced and its structural analysis occupies the central part of the section. In Section 5, the focus is on the quotient of the set of formulas in propositional classical logic: we prove that the set of fractional values for tautologies is dense. Section 6 presents final considerations on our perspective.

2. Preliminaries

We begin by establishing some basic notions that will be used throughout this paper. In particular, we work with a language containing propositional atoms and complex formulas are built using the connectives $\neg$, $\wedge$ and $\vee$. We use $p, q, r, ...$ as metavariables for propositional atomic formulas and $A, B, C, ...$ as metavariables for compound formulas. The set of formulas is denoted as $\mathscr{F}$. A multiset is a collection of formulas in which multiplicity counts and order is irrelevant. We use capital Greek letters as metavariables for multisets. A sequent is a syntactic object of the shape:

$$\Gamma \vdash \Delta$$

where Γ, Δ are multisets of formulas.

We start by recalling the rules of the system **GS4** which are found in Figure 1. The unary rules are multiplicative, whereas the binary rules for are additive and initial sequents are allowed to contain only atomic formulas. The height of a derivation is defined as usual. The properties of **GS4** are the following:

1. Generalized initial sequents $\Gamma, A \vdash A, \Delta$ are provable.
2. The rules of weakening:

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \text{LW} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ \text{RW}$$

   are admissible.

AXIOMS

$$\frac{}{p, \Gamma \vdash \Delta, p}\ ax$$

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$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\ \mathrm{R}\vee \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ L\wedge \qquad \frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ \mathrm{L}\vee$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ L\neg \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \mathrm{R}\neg$$

Figure 1. The **GS4** sequent calculus

3. Every rule is height-preserving invertible, in the sense that the derivability of the conclusion entails the derivability of each of its premises with no greater derivation height. The rules of contraction:

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \mathrm{LC} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \mathrm{RC}$$

   are height-preserving admissible.

4. The cut rule:

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \mathsf{Cut}$$

   is admissible.

Since we are interested in the analysis of any formula of the language, we add to the calculus initial sequents of the shape:

$$\Gamma \vdash \Delta$$

where Γ and Δ only contain atomic propositional formulas and no propositional atom p occurs both in Γ and in Δ. In other words, the resulting system $\overline{\mathbf{GS4}}$ thus obtained is trivial, in the sense that it derives any sequent. The final step which leads to assign proper measures to formulas consists in decorating sequents with fractional values. To this end, in [10] a sequent calculus $\overline{\overline{\mathbf{GS4}}}$ (see Figure 2) was introduced which determines the

AXIOMS

$$\frac{}{\Gamma \mid\!\frac{0}{1}\, \Delta}\ \overline{ax} \qquad \frac{}{\Gamma, p \mid\!\frac{1}{1}\, \Delta, p}\ ax$$

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$$\frac{\Gamma \mid\!\frac{m}{n}\, \Delta, A, B}{\Gamma \mid\!\frac{m}{n}\, \Delta, A \vee B}\ \mathrm{R}\vee \qquad \frac{\Gamma \mid\!\frac{m_1}{n_1}\, \Delta, A \qquad \Gamma \mid\!\frac{m_2}{n_2}\, \Delta, B}{\Gamma \mid\!\frac{m_1+m_2}{n_1+n_2}\, \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, B, \Gamma \mid\!\frac{m}{n}\, \Delta}{A \wedge B, \Gamma \mid\!\frac{m}{n}\, \Delta}\ L\wedge \qquad \frac{A, \Gamma \mid\!\frac{m_1}{n_1}\, \Delta \qquad B, \Gamma \mid\!\frac{m_2}{n_2}\, \Delta}{A \vee B, \Gamma \mid\!\frac{m_1+m_2}{n_1+n_2}\, \Delta}\ \mathrm{L}\vee$$

$$\frac{\Gamma \mid\!\frac{m}{n}\, \Delta, A}{\neg A, \Gamma \mid\!\frac{m}{n}\, \Delta}\ L\neg \qquad \frac{A, \Gamma \mid\!\frac{m}{n}\, \Delta}{\Gamma \mid\!\frac{m}{n}\, \Delta, \neg A}\ \mathrm{R}\neg$$

Figure 2. The $\overline{\overline{\mathbf{GS4}}}$ sequent calculus

ratio between tautological and total topsequents for any derivation of a sequent. Each initial sequent is decorated with either $\frac{1}{1}$ or $\frac{0}{1}$, unary rules do not alter the fractional value, whereas binary rules sum the numerators and the denominators. The essential components of $\overline{\overline{\mathbf{GS4}}}$ are the invertibility of every rule, termination of the proof search, and stability, as previously defined in the introduction.

We now fix some terminology which will be essential to our subsequent development.

Definition 2.1. Given a derivation π, $\mathsf{top}^0(\pi)$ represents the multiset of its complementary, i.e. non valid, initial sequents, while $\mathsf{top}^1(\pi)$ denotes the multiset of its tautological, i.e., valid, initial sequents. The multiset $\mathsf{top}(\pi)$ is formed by the multiset union of $\mathsf{top}^0(\pi)$ and $\mathsf{top}^1(\pi)$.

Since the decomposition of a sequent in initial sequents is unique by stability in the calculus $\overline{\overline{\mathbf{GS4}}}$, we will drop the reference to derivations and simply write $\mathsf{top}(A)$.

Definition 2.2. *(Fractional evaluation function)* Let $\mathbb{Q}^* = [0,1] \cap \mathbb{Q}$, i.e., $\mathbb{Q}^*$ is the set of the rational numbers in the closed interval $[0,1]$. The evaluation function $[\![\cdot]\!] : \mathscr{F} \to \mathbb{Q}^*$ is defined as follows: for any logical formula A,

$$[\![A]\!] = \frac{\#\mathsf{top}^1(A)}{\#\mathsf{top}(A)}$$

where $\#X$ denotes the cardinality of X for any multiset X. The fractional evaluation naturally extends to a multiset Γ of formulas as $[\![\bigvee \Gamma]\!]$.

Given a multiset of formulas Γ we will use the notation $\llbracket\Gamma\rrbracket$ to denote $\llbracket\bigvee\Gamma\rrbracket$. Note that fractional semantics is, in a sense, conservative over standard Boolean interpretations. In particular, every classically valid formula is, by completeness, derivable in **GS4** and so its decomposition will lead to tautological initial sequents.

LEMMA 2.1. *If A is derivable in* **GS4**, *then* $\llbracket A\rrbracket = 1$

PROOF. If A is derivable in **GS4**, then every topmost sequent is tautological. Hence

$$\frac{\#\mathsf{top}^1(A)}{\#\mathsf{top}(A)} = \frac{\#\mathsf{top}(A)}{\#\mathsf{top}(A)} = 1$$

which gives the desired conclusion. ■

In the next section, we shall introduce a new consequence relation that relies on the fractional values assigned by the decomposition procedure induced by the calculus $\overline{\overline{\mathbf{GS4}}}$.

3. Decomposing *ex falso*

Our aim is to define a consequence relation which is sensible to contradictions. In simple terms, we contend that not all contradictions should be treated uniformly. Depending on the shape of the formula A, the contradictory pair A and $\neg A$ may not necessarily trivialise a system. Fractional semantics enables one to introduce a fine grained distinction between contradictions, assigning them values in the set $[0, 1] \cap \mathbb{Q}$.

The underlying intuition behind the new consequence relation is that a conclusion is permissible only when the fractional value of the premises is less than or equal to that of the conclusion. This intuition is entirely consistent with the venerable principle that reasoning should be truth-preserving (when moving form valid premises to a conclusion). In our setting, what is preserved is the quantity of identity or, perhaps more intuitively, the "quantity of truth" conceived as a fractional value.

We recall that the usual argument to establish the validity of the principle of *ex falso quodlibet*:

$$A, \neg A \vdash B$$

In a standard sequent calculus for classical logic, the derivability of such a sequent is given by a simple root-first application of the rules RW and L$\neg$ (if explicitly present or built-in the initial sequents with weakened contexts).

$$\dfrac{\dfrac{A \vdash A}{A, \neg A \vdash}\ \text{L}\neg}{A, \neg A \vdash B}\ \text{R}W$$

We will show below that *ex falso* cannot be proved (in general) in the newly defined system. As opposed to the usual strategy used to block the derivability of the *ex falso* principle, we do not give up the admissibility of the rule of right weakening, or weakening of the conclusion. On the contrary, we work on the logical rules of the calculus, whose correct application will depend on the quantity of identity contained in the antecedent.

Definition 3.1. Given multisets Γ and Δ we say that Δ is a *fractional consequence* of Γ, in symbols $\Gamma \mid\!\sim \Delta$ if the following requirements are fulfilled:

1. $\Gamma \vdash \Delta$ is derivable in **GS4**.
2. $[\![\Gamma]\!] \leq [\![\Delta]\!]$

Unsurprisingly, the above definition is guided by the standard concept of logical consequence, extended to the framework of fractional semantics. Indeed, the statement *A is a classical consequence of Γ if, whenever the premises are valid, so is the conclusion* expresses the preservation of truth from premises to conclusion. Reasoning analogously in the context of fractional semantics the intended meaning of $\Gamma \mid\!\sim A$ is *A is a fractional consequence of Γ whenever the fractional value of the premisses is less or equal to that of the conclusion.*

Let us first give a look to the newly defined consequence relation. First, we observe that it is conservative over classical propositional logic.

Lemma 3.1. *If $\Gamma = \emptyset$, then $\Gamma \mid\!\sim \Delta$ iff $\Gamma \vdash \Delta$.*

Proof. We limit ourselves to discussing the right-to-left direction. If $\Gamma = \emptyset$, then $\vdash \Delta$ holds and so Δ is derivable in **GS4** and so $[\![\Delta]\!] = 1$, so $[\![\Gamma]\!] \leq [\![\Delta]\!]$, therefore $\Gamma \mid\!\sim \Delta$. ■

However, when Γ is not empty, the two consequence relations strongly differ. We now propose an analysis of the properties of the consequence relation.

Lemma 3.2. *$\mid\!\sim$ is reflexive.*

Proof. Immediate, since $A \vdash A$ is derivable for every A and also $[\![A]\!] = [\![A]\!]$. ■

Concerning monotonicity, we need to distinguish two different ways of adding information. We have two different rules:

$$\frac{\Gamma \mid\!\sim \Delta}{\Gamma, A \mid\!\sim \Delta}\text{LW} \qquad \frac{\Gamma \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta, A}\text{RW}$$

We say that a logic is *left* monotonic if the rule LW is valid in it. The definition of right monotonicity is symmetric. If a logic is left and right monotonic, we say that it is monotonic. We first start to observe that in general fractional values increase with the addition of further information.

LEMMA 3.3. *For any multiset of formulas Δ and any multiset Σ of atomic formulas, if $\Delta \subseteq \Sigma$, then $[\![\Delta]\!] \leq [\![\Delta,\Sigma]\!]$.*

PROOF. The proof is immediate by noting that the total multiset of topsequents of a derivation of $\vdash \Delta$ does not increase adding the multiset of atomic formulas Σ, whereas the multiset of tautological topsequents may increase. ∎

LEMMA 3.4. *For any multiset of formulas Δ and formula A, $[\![\Delta]\!] \leq [\![\Delta, A]\!]$.*

PROOF. We consider the sequent $\vdash \Delta, A$. Since every rule is height-preserving invertible, we can apply the rules in any order without affecting the decomposition process. We decompose the formula A, obtaining derivations $\pi_1,\ldots,\pi_n$ of the sequents $\Gamma_1 \vdash \Delta, \Theta_1,\ldots,\Gamma_n \vdash \Delta, \Theta_n$ where Γ_i, Θ_i are atomic for every $1 \leq i \leq n$. Since for every i we get $\mathsf{top}^1(\Delta) \subseteq \mathsf{top}^1(\Gamma_i \vdash \Delta, \Theta_i)$, we obtain:

$$[\![\Delta, A]\!] = \frac{\Sigma_{1\leq i\leq n}\#\mathsf{top}^1(\Gamma_i\vdash\Delta,\Theta_i)}{n\cdot\#\mathsf{top}(\Delta)} = \frac{n\cdot\#\mathsf{top}^1(\Delta)+q}{n\cdot\#\mathsf{top}(\Delta)} = \frac{\#\mathsf{top}^1(\Delta)}{\#\mathsf{top}(\Delta)} + \frac{q}{n\cdot\#\mathsf{top}(\Delta)}$$

Therefore, we conclude $[\![\Delta]\!] \leq [\![\Delta, A]\!]$. ∎

An immediate consequence of this result is the following proposition.

PROPOSITION 3.5. *If $[\![\Delta, A]\!] = q$, then $[\![\Delta]\!] = q'$ for some $q' \leq q$ in $[0, 1] \cap \mathbb{Q}$.*

PROOF. Suppose $[\![\Delta]\!] = q'$ and, towards a contradiction, $q < q'$. By Lemma 3.4, we get $q' = [\![\Delta]\!] \leq [\![\Delta, A]\!] = q < q'$ and thus $q' < q'$, which is a contradiction. ∎

LEMMA 3.6. *The relation $\mid\!\sim$ is right monotonic, but not left monotonic.*

PROOF. We prove that it is right monotonic. If $\Gamma \mid\!\sim \Delta$, then $\Gamma \vdash \Delta$ is derivable and $[\![\Gamma]\!] \leq [\![\Delta]\!]$. Since $[\![\Delta]\!] \leq [\![\Delta, A]\!]$ for every A, by the transitivity of $\leq$ we get the desired conclusion. To show that it is not left monotonic, consider the case of $p \mid\!\sim p \wedge p$ which holds and $p, q \vee \neg q \mid\!\sim p \wedge p$ which is not true. ∎

Differently from the bounded supraclassical consequence relation, the fractional consequence relation is not fully monotonic, but is transitive.

THEOREM 3.7. *The rule:*

$$\frac{\Gamma \mathrel{|\!\sim} \Delta, A \qquad \Gamma, A \mathrel{|\!\sim} \Delta}{\Gamma \mathrel{|\!\sim} \Delta}\ \mathsf{Trs}$$

holds.

PROOF. Clearly, if $\Gamma \mathrel{|\!\sim} \Delta, A$ and $\Gamma, A \mathrel{|\!\sim} \Delta$, we get by definition and by the cut rule which is admissible in **GS4** (see Section 2). $\Gamma \vdash \Delta$. Also, we have:

$$[\![\Gamma]\!] \leq [\![\Delta, A]\!] \quad \textit{and} \quad [\![\Gamma, A]\!] \leq [\![\Delta]\!]$$

Since $[\![\Gamma]\!] \leq [\![\Gamma, A]\!]$ by Lemma 3.4, we get $[\![\Gamma]\!] \leq [\![\Delta]\!]$ which yields the desired conclusion. ■

REMARK 1. The relation $\mathrel{|\!\sim}$ is not structural in the sense that it is not closed under arbitrary substitution of atomic formulas. Indeed, if $\Gamma \mathrel{|\!\sim} A$, then it might not be the case that $\Gamma[B/p] \mathrel{|\!\sim} A[B/p]$[3]. For example:

- $\Gamma = [p, q]$
- $A \equiv q$

Consider the substitution $[r \vee \neg r/p]$. It is clear that $[\![(r \vee \neg r) \vee q]\!] = 1$ and $[\![q]\!] = 0$, therefore the consequence relation fails to be structural.

We now take a look into the admissible rules of the system. Specifically, we show that the newly defined consequence relation behaves well with respect to disjunctions and negation on the right hand side, but does not with respect to conjunctions.

PROPOSITION 3.8. *The rule:*

$$\frac{\Gamma \mathrel{|\!\sim} A, B, \Delta}{\Gamma \mathrel{|\!\sim} A \vee B, \Delta}\ {\mathrel{|\!\sim}}\vee$$

is admissible.

PROOF. Indeed, if $\Gamma \mathrel{|\!\sim} A, B, \Delta$, then $\Gamma \vdash A, B, \Delta$ is derivable and so is $\Gamma \vdash A \vee B, \Delta$ by the rule $R\vee$. Furthermore, if $[\![\Gamma]\!] \leq [\![A, B, \Delta]\!]$, then $[\![\Gamma]\!] \leq [\![A, B, \Delta]\!] = [\![A \vee B, \Delta]\!]$, which yields the desired conclusion. ■

[3]Where, as usual, $\Gamma[B/p]$ denotes the multiset obtained by substituting every occurrence of the atomic formula p with the formula B.

Remark 2. We would like to highlight that while it was conceivable to define the consequence relation $\Gamma \vdash \Delta$ and $[\![\bigwedge \Gamma]\!] \leq [\![\Delta]\!]$ (where $\bigwedge \Gamma$ denotes the conjunction of the formulas in Γ), we chose an alternative approach for two specific reasons. First, the resulting consequence relation would fail to be transitive, which is arguably a desirable feature for a logical system. Second, our selected consequence relation establishes a connection between the truth content of the antecedent and the succedent. In doing so, we find it preferable to analyze the quantity of identity of the antecedent taken as a sequent.

We would like to conclude the analysis of the properties of the consequence relation by observing that it validates a form of contraction on the left, but not on the right of the turnstile.

Lemma 3.9. *If* $A, A, \Gamma \mid\!\sim \Delta$*, then* $A, \Gamma \mid\!\sim \Delta$*.*

Proof. If $A, A, \Gamma \mid\!\sim \Delta$, then $A, A, \Gamma \vdash \Delta$ is derivable in **GS4** and by contraction admissibility, we get that $A, \Gamma \vdash \Delta$ is derivable. Next, $[\![A, A, \Gamma]\!] \leq [\![\Delta]\!]$. By Proposition 3.5, we have $[\![A, \Gamma]\!] \leq [\![A, A, \Gamma]\!] \leq [\![\Delta]\!]$ which gives the desired conclusion. ■

Notice that the $\mid\!\sim$ consequence relation fails to validate contraction in the succedent as counterexamples can be found.

A comment on fractional semantics and other possible consequence relations derived from it is also pertinent. In [10] a family of bounded supraclassical system, i.e. systems stronger than classical propositional logic, was defined and proved to be reflexive, monotonic (both right and left), structural but not transitive. In contrast, the relation $\mid\!\sim$ is characterized by the following metalogical properties:

- reflexivity;
- right monotonicity;
- transitivity;
- no left monotonicity and no structurality.

The following table summarizes the difference between the three consequence relation (classical, fractional and bounded supraclassical)[4]:

[4] A reviewer questioned why we introduced the distinction between left and right monotonicity in this context. We did so to emphasize that the fractional consequence relation is right monotonic, even though it is paraconsistent.

	Classical	Bounded	Fractional
Ref	✓	✓	✓
LMon	✓	✓	×
RMon	✓	✓	✓
Trs	✓	×	✓
Str	✓	✓	×

Remark 3. We would also like to draw a comparison between the present approach and the literature on relevance logic. Notably, the rejection of monotonicity stands out as a pivotal aspect in relevant reasoning [15]. In our setting, for example, the addition of a tautology to the antecedent invalidates the the consequence unless the fractional value of the succedent is 1. When incorporating tautologies as assumptions, the only permissible outcome is a tautological conclusion, since the value of truth must not diminish in transitioning from assumptions to conclusions.

4. Combining Fractional Systems

So far, our investigations have been conducted primarily from a semantic standpoint. However, it is now desirable to establish a syntactic framework for the newly introduced consequence relation.

4.1. A Calculus for Fractional Consequence

Specifically, we aim to develop an analytic calculus that aligns with the semantic characterization provided earlier. To achieve this, we will define a proof system that incorporates explicit rules for governing the consequence relation denoted as $\mid\!\sim$. In brief, the system will integrate the fractional system $\overline{\overline{\mathbf{GS4}}}$ with specific rules designed for the consequence relation. As a result, we will be working with *three* types of sequents:

- $\mid\!\sim$-sequents, denoting derivability w.r.t. fractional consequence;
- $\vdash^{m}_{n}$-sequents, denoting derivability in the decorated system $\overline{\overline{\mathbf{GS4}}}$;
- $\vdash$-sequents, interpreting derivability in the system **GS4**.

The rules of the calculus are given in Figure 3. To provide a better understanding of the intuitive meaning of the rules, consider – as an example – the rule $\mid\!\sim\wedge$:

AXIOMS

$$\frac{}{p \mid\!\sim p, \Delta}\, ax$$

$$\frac{}{\Gamma \mathrel{|\tfrac{0}{1}} \Delta}\, \overline{ax} \qquad \frac{}{\Gamma, p \mathrel{|\tfrac{1}{1}} \Delta, p}\, ax$$

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The rules of **GS4**, plus the rules:

$$\frac{\Gamma \mathrel{|\tfrac{m}{n}} \Delta, A, B}{\Gamma \mathrel{|\tfrac{m}{n}} \Delta, A \vee B}\, \mathrm{R}\vee \qquad \frac{\Gamma \mathrel{|\tfrac{m_1}{n_1}} \Delta, A \qquad \Gamma \mathrel{|\tfrac{m_2}{n_2}} \Delta, B}{\Gamma \mathrel{|\tfrac{m_1+m_2}{n_1+n_2}} \Delta, A \wedge B}\, R\wedge$$

$$\frac{A, B, \Gamma \mathrel{|\tfrac{m}{n}} \Delta}{A \wedge B, \Gamma \mathrel{|\tfrac{m}{n}} \Delta}\, L\wedge \qquad \frac{A, \Gamma \mathrel{|\tfrac{m_1}{n_1}} \Delta \qquad B, \Gamma \mathrel{|\tfrac{m_2}{n_2}} \Delta}{A \vee B, \Gamma \mathrel{|\tfrac{m_1+m_2}{n_1+n_2}} \Delta}\, \mathrm{L}\vee$$

$$\frac{\Gamma \mathrel{|\tfrac{m}{n}} \Delta, A}{\neg A, \Gamma \mathrel{|\tfrac{m}{n}} \Delta}\, L\neg \qquad \frac{A, \Gamma \mathrel{|\tfrac{m}{n}} \Delta}{\Gamma \mathrel{|\tfrac{m}{n}} \Delta, \neg A}\, \mathrm{R}\neg$$

FRACTIONAL CONSEQUENCE

$$\frac{\Gamma, A \vee B \vdash \Delta \qquad \mathrel{|\tfrac{m_1}{n_1}} \Gamma, A \vee B \qquad \mathrel{|\tfrac{m_2}{n_2}} \Delta}{\Gamma, A \vee B \mid\!\sim \Delta}\, \vee\!\mid\!\sim,\ \tfrac{m_1}{n_1} \leq \tfrac{m_2}{n_2} \qquad \frac{\Gamma \mid\!\sim A, B, \Delta}{\Gamma \mid\!\sim A \vee B, \Delta}\, \mid\!\sim\vee$$

$$\frac{\Gamma, A \wedge B \vdash \Delta \qquad \mathrel{|\tfrac{m_1}{n_1}} \Gamma, A \wedge B \qquad \mathrel{|\tfrac{m_2}{n_2}} \Delta}{\Gamma, A \wedge B \mid\!\sim \Delta}\, \wedge\!\mid\!\sim,\ \tfrac{m_1}{n_1} \leq \tfrac{m_2}{n_2} \qquad \frac{\Gamma \vdash \Delta, A \wedge B \qquad \mathrel{|\tfrac{m_1}{n_1}} \Gamma \qquad \mathrel{|\tfrac{m_2}{n_2}} A \wedge B, \Delta}{\Gamma \mid\!\sim A \wedge B, \Delta}\, \mid\!\sim\wedge,\ \tfrac{m_1}{n_1} \leq \tfrac{m_2}{n_2}$$

$$\frac{\Gamma, \neg A \vdash \Delta \qquad \mathrel{|\tfrac{m_1}{n_1}} \Gamma, \neg A \qquad \mathrel{|\tfrac{m_2}{n_2}} \Delta}{\Gamma, \neg A \mid\!\sim \Delta}\, \neg\!\mid\!\sim,\ \tfrac{m_1}{n_1} \leq \tfrac{m_2}{n_2} \qquad \frac{\Gamma, A \mid\!\sim \Delta}{\Gamma \mid\!\sim \neg A, \Delta}\, \mid\!\sim\neg$$

STRUCTURAL RULE

$$\frac{\Gamma \mid\!\sim \Delta \qquad \mathrel{|\tfrac{m_1}{n_1}} \Gamma, B \qquad \mathrel{|\tfrac{m_2}{n_2}} \Delta}{\Gamma, B \mid\!\sim \Delta}\, \mathsf{wf},\ \tfrac{m_1}{n_1} \leq \tfrac{m_2}{n_2}$$

Figure 3. The **GS4F** sequent calculus

$$\frac{\Gamma \vdash \Delta, A \wedge B \qquad \mathrel{|\tfrac{m_1}{n_1}} \Gamma \qquad \mathrel{|\tfrac{m_2}{n_2}} A \wedge B, \Delta}{\Gamma \mid\!\sim A \wedge B, \Delta}\, \mid\!\sim\wedge,\ \frac{m_1}{n_1} \leq \frac{m_2}{n_2}$$

The three-premise rule introduces a non-classical (indeed, sub-classical) turnstile $\mid\!\sim$ and *stricto sensu* does not qualify as classical. The premises express, however, a connection with classical logic in the form of classical and fractional derivability. For this reason, the calculus naturally includes rules (and syntax) to encode derivability in the calculi **GS4** and $\overline{\overline{\mathbf{GS4}}}$.

We now prove two preliminary lemmata in order to obtain the cut-elimination result. First, we show a kind of invertibility result, namely we prove syntactically, basing on the rules of the calculus, that $\mid\!\sim\ \subset\ \vdash$. In other words, we show that given a derivation of $\Gamma \mid\!\sim \Delta$ in **GS4F** we can obtain a derivation of the sequent $\Gamma \vdash \Delta$ in **GS4**.

LEMMA 4.1. *For every multiset* Γ *and* Δ*, we have:*

$$\frac{\Gamma \mid\!\sim \Delta}{\Gamma \vdash \Delta}\ \mathsf{inv}$$

and the height is preserved.

PROOF. We argue by induction on the height of the derivation. If $\Gamma \mid\!\sim \Delta$ is an initial sequent, then it is of the shape $p \mid\!\sim p, \Delta$. Clearly, $p \vdash p, \Delta$ is derivable in **GS4**.

If $\Gamma \mid\!\sim \Delta$ is the conclusion of a unary rule, the proof follows from an application of the induction hypothesis and, possibly, admissibility of the rule of weakening in **GS4**. For instance, if the last rule applied is $\mid\!\sim\vee$, we have:

$$\frac{\Gamma \mid\!\sim A, B, \Delta}{\Gamma \mid\!\sim A \vee B, \Delta}\ {\mid\!\sim\vee}$$

and we construct the following derivation:

$$\cfrac{\cfrac{\Gamma \mid\!\sim A, B, \Delta}{\Gamma \vdash A, B, \Delta}\ \text{IH}}{\Gamma \vdash A \vee B, \Delta}\ R\vee$$

If the last rule applied is a trinary rule, we simply take the derivation of one the premises which yields the desired conclusion. ■

We now need to prove other properties of our system in order to get cut-elimination.

THEOREM 4.2. *If $\Gamma \mid\!\sim \Delta$ is derivable then there are rational numbers $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ such that $\models_{\frac{m_1}{n_1}} \Gamma$ and $\models_{\frac{m_2}{n_2}} \Delta$ and $\frac{m_1}{n_1} \leq \frac{m_2}{n_2}$.*

PROOF. The proof is by induction on the height of the derivations. If $\Gamma \mid\!\sim \Delta$ is an initial sequent, then it is of the shape $p \mid\!\sim p, \Delta$. In this case the conclusion is trivial since $[\![p]\!] = 0 \leq [\![p, \Delta]\!]$. If $\Gamma \mid\!\sim \Delta$ is the conclusion of a rule, we need to distinguish cases. If the last rule applied has three premises, by inspection of the rules, the conclusion is given by the premises of the rule. If the last rule applied is unary, we use the induction hypothesis. For example,

$$\frac{\Gamma \mid\!\sim A, B, \Delta}{\Gamma \mid\!\sim A \vee B, \Delta}\ {\mid\!\sim\vee}$$

We apply the induction hypothesis to the premise which gives $\models_{\frac{m_1}{n_1}} \Gamma$ and $\models_{\frac{m_2}{n_2}} A, B, \Delta$ for some $\frac{n_1}{m_1} \leq \frac{n_2}{m_2}$. Hence, by the rule $R\vee$ we get $\models_{\frac{m_2}{n_2}} A \vee B, \Delta$. This gives $\frac{m_1}{n_1} \leq \frac{m_2}{n_2}$, i.e., the desired conclusion. ■

We now prove the final preliminary result to syntactic cut-elimination. Indeed, we show that whenever a sequent is classically derivable and its components satisfy the fractional restrictions, we can infer the derivability of the fractional consequence relation.

THEOREM 4.3. *The rule:*

$$\frac{\Gamma \vdash \Delta \qquad \mid\!\frac{m_1}{n_1} \Gamma \qquad \mid\!\frac{m_2}{n_2} \Delta}{\Gamma \mid\!\sim \Delta} \ \mathsf{adm},\ \frac{m_1}{n_1} \leq \frac{m_2}{n_2}$$

is admissible.

PROOF. The proof is by induction on the height of the derivation of $\Gamma \vdash \Delta$ in **GS4**. If $\Gamma \vdash \Delta$ is an initial sequent, then it is of the shape $\Gamma', p \vdash \Delta', p$ where Γ' and Δ' are multisets of atomic formulas. Therefore the conclusion follows from multiple applications of the rule wf. Supposing $\Gamma' = q_1, \ldots, q_n$, we have:

$$\frac{p \mid\!\sim p, \Delta' \qquad \mid\!\frac{0}{1} p, q_1 \qquad \mid\!\frac{0}{1} p, \Delta'}{q_1, p \mid\!\sim p, \Delta'} \ \mathsf{wf}$$

$$\vdots$$

$$\frac{q_1, \ldots, q_{n-1}, p \mid\!\sim p, \Delta' \qquad \mid\!\frac{0}{1} p, \Gamma' \qquad \mid\!\frac{0}{1} p, \Delta'}{\Gamma', p \mid\!\sim p, \Delta'} \ \mathsf{wf}$$

If $\Gamma \vdash \Delta$ is the conclusion of $\wedge$, the proof follows from an application of the rule $\mid\!\sim\wedge$. If it is the conclusion of the rule $\vee$, then we have:

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \ \mathrm{R}\vee$$

We observe that $[\![\Delta, A, B]\!] = [\![\Delta, A \vee B]\!] = \frac{m_2}{n_2}$, therefore we proceed as follows:

$$\cfrac{\cfrac{\Gamma \vdash \Delta, A, B \qquad \mid\!\frac{m_1}{n_1} \Gamma \qquad \mid\!\frac{m_2}{n_2} \Delta, A, B}{\Gamma \mid\!\sim \Delta, A, B} \ IH,\ \frac{m_1}{n_1} \leq \frac{m_2}{n_2}}{\Gamma \mid\!\sim \Delta, A \vee B} \ \mid\!\sim\vee$$

■

Contraction is not - in general - admissible in **GS4F**, but we show that left contraction is admissible and a restricted version of right contraction is admissible too and useful to show syntactic admissibility of the Trs rule in the system **GS4F**.

LEMMA 4.4. *The rules:*

$$\frac{\Gamma, A, A \mathrel{|\!\sim} \Delta}{\Gamma, A \mathrel{|\!\sim} \Delta}\, LC \qquad \frac{\Gamma \mathrel{|\!\sim} \Delta, p, p}{\Gamma \mathrel{|\!\sim} \Delta, p}\, RC_{at},\ p \text{ atomic}$$

are height-preserving admissible in **GS4F**.

PROOF. By Lemma 4.1, we get the derivability of $A, A, \Gamma \vdash \Delta$ in **GS4**. By admissibility of contraction in **GS4**, we get that $A, \Gamma \vdash \Delta$ is derivable in **GS4**. By Theorem 4.2, we get that there are $\frac{m_1}{n_1} \leq \frac{m_2}{n_2}$ and $[\![A, A, \Gamma]\!] = \frac{m_1}{n_1} \leq \frac{m_2}{n_2} = [\![\Delta]\!]$. Since $[\![A, \Gamma]\!] \leq [\![A, A, \Gamma]\!]$, we can apply Theorem 4.3 to obtain the desired conclusion.

To prove the admissibility of RC_{at}, we can reason analogously. We only add that, although in general it is not true that $[\![\Delta, A, A]\!] \leq [\![\Delta, A]\!]$, if A is atomic we have:

$$[\![\Delta, p, p]\!] = [\![\Delta, p]\!]$$

(because the presence of a double occurrence of an atomic formula does not alter the number of tautological topsequents in **GS4**) which allows us to get the desired conclusion. ■

LEMMA 4.5. *The following rule:*

$$\frac{\Gamma, \Pi \mathrel{\big|\!\frac{m_1}{n_1}} \Delta, \Sigma}{\Gamma \mathrel{\big|\!\frac{m_2}{n_2}} \Delta}\, \mathsf{low},\ \frac{m_2}{n_2} \leq \frac{m_1}{n_1}$$

is admissible in $\overline{\overline{\mathbf{GS4}}}$.

PROOF. The proof is analogous to the one of Proposition 3.5 and thus we omit the details. ■

4.2. Transitivity Elimination in the Mixed System

We are now in the position to show cut (or transitivity) elimination for the combined system. When we apply the rule *Cut* we are referring to the rule admissible in **GS4**. By $\overline{@}$, where @ is a connective, we denote the application of the invertibility of the rules for the connectives in the calculus **GS4**.

THEOREM 4.6. *The rule* Trs *can be eliminated in* **GS4F**.

PROOF. The proof runs by induction on the sum of the height of the derivations of the premises of the cut. We distinguish three cases.

The left premise of the cut is an initial sequent. We assume that the cut formula is principal (otherwise the reduction is trivial) and we have:

$$\frac{p \mathrel{|\!\sim} \Delta, p \qquad p, p \mathrel{|\!\sim} \Delta}{p \mathrel{|\!\sim} \Delta}\, \mathsf{Trs}$$

We proceed as follows:

$$\dfrac{p, p \mid\!\sim \Delta}{p \mid\!\sim \Delta}\,LC$$

The right premise of the cut is an initial sequent. We have:

$$\dfrac{\mid\!\sim \Delta, p, p \qquad p \mid\!\sim \Delta, p}{\mid\!\sim \Delta, p}\,\mathsf{Trs}$$

The proof is symmetric to the one of the previous case exploiting the admissibility of the rule RC_{at}.

The cut formula is principal in both premises of the cut. We distinguish cases according to the shape of the rule. We discuss the three cases separately.

- The case in which the cut formula is of the shape $\neg A$:

$$\dfrac{\dfrac{A, \Gamma \mid\!\sim \Delta}{\Gamma \mid\!\sim \neg A, \Delta}\,{\mid\!\sim}\neg \qquad \dfrac{\Gamma, \neg A \vdash \Delta \qquad \Big|\frac{m_1}{n_1}\,\Gamma, \neg A \qquad \Big|\frac{m_2}{n_2}\,\Delta}{\neg A, \Gamma \mid\!\sim \Delta}\,\neg{\mid\!\sim},\ \tfrac{m_1}{n_1} \leq \tfrac{m_2}{n_2}}{\Gamma \mid\!\sim \Delta}\,\mathsf{Trs}$$

  We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma, \neg A \vdash \Delta}{\Gamma \vdash \Delta, A}\,\mathrm{L}\overline{\neg} \qquad \dfrac{A, \Gamma \mid\!\sim \Delta}{\Gamma, A \vdash \Delta}\,\mathsf{inv}}{\Gamma \vdash \Delta}\,Cut \qquad \dfrac{\Big|\frac{m_1}{n_1}\,\Gamma, \neg A}{\Big|\frac{m_3}{n_3}\,\Gamma}\,\mathsf{low},\ \tfrac{m_3}{n_3} \leq \tfrac{m_1}{n_1} \qquad \Big|\frac{m_2}{n_2}\,\Delta}{\Gamma \mid\!\sim \Delta}\,\mathsf{adm}$$

- The cut formula is of the shape $A \vee B$.

$$\dfrac{\dfrac{\Gamma \mid\!\sim \Delta, A, B}{\Gamma \mid\!\sim \Delta, A \vee B}\,{\mid\!\sim}\vee \qquad \dfrac{\Gamma, A \vee B \vdash \Delta \qquad \Big|\frac{m_1}{n_1}\,\Gamma, A \vee B \qquad \Big|\frac{m_2}{n_2}\,\Delta}{A \vee B, \Gamma \mid\!\sim \Delta}\,\vee{\mid\!\sim},\ \tfrac{m_1}{n^1} \leq \tfrac{m_2}{n_2}}{\Gamma \mid\!\sim \Delta}\,\mathsf{Trs}$$

  We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \mid\!\sim \Delta, A \vee B}{\Gamma \vdash \Delta, A \vee B}\,\mathsf{inv}}{\Gamma \vdash \Delta, A, B}\,\mathrm{R}\overline{\vee} \qquad \dfrac{\Gamma, A \vee B \vdash \Delta}{\Gamma, A \vdash \Delta, B}\,\mathrm{L}\overline{\vee}, \mathrm{RW}}{\Gamma \vdash \Delta, B}\,\mathrm{Cut} \qquad \dfrac{\Gamma, A \vee B \vdash \Delta}{\Gamma, B \vdash \Delta}\,\mathsf{inv}}{\Gamma \vdash \Delta}\,\mathrm{Cut} \qquad \dfrac{\Big|\frac{m_1}{n_1}\,\Gamma, A \vee B}{\Big|\frac{m_3}{n_3}\,\Gamma}\,\mathsf{low},\ \tfrac{m_3}{n_3} \leq \tfrac{m_1}{n_1} \qquad \Big|\frac{m_2}{n_2}\,\Delta}{\Gamma \mid\!\sim \Delta}\,\mathsf{adm}$$

- The cut formula is of the shape $A \wedge B$. In this case we have:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A \wedge B \qquad \Big|\frac{m_1}{n_1}\,\Gamma \qquad \Big|\frac{m_2}{n_2}\,\Delta, A \wedge B}{\Gamma \mid\!\sim \Delta, A \wedge B}\ {\mid\!\sim}\wedge,\ \tfrac{m_1}{n_1} \leq \tfrac{m_2}{n_2} \qquad \dfrac{\Gamma, A \wedge B \vdash \Delta \qquad \Big|\frac{m_3}{n_3}\,\Gamma, A \wedge B \qquad \Big|\frac{m_4}{n_4}\,\Delta}{\Gamma, A \wedge B \mid\!\sim \Delta}\ \wedge{\mid\!\sim},\ \tfrac{m_3}{n_3} \leq \tfrac{m_4}{n_4}}{\Gamma \mid\!\sim \Delta}\ \mathsf{Trs}$$

We first apply the Theorem 4.2 to the premise $A \wedge B, \Gamma \mid\!\sim \Delta$. Since $\frac{m_3}{n_3} = [\![A \wedge B, \Gamma]\!] \leq \frac{m_4}{n_4} = [\![\Delta]\!]$ and $[\![\Gamma]\!] \leq [\![\Gamma, A \wedge B]\!]$, we proceed as follows:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A \wedge B \qquad A \wedge B, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ \mathrm{Cut} \qquad \Big|\frac{m_1}{n_1}\,\Gamma \qquad \Big|\frac{m_4}{n_4}\,\Delta}{\Gamma \mid\!\sim \Delta}\ \mathsf{adm},\ \tfrac{m_1}{n_1} \leq \tfrac{m_4}{n_4}$$

The cut formula is not principal in the last rule applied. If the last rule applied is a unary rule, then we apply the induction hypothesis and then the rule again. For instance:

$$\dfrac{\dfrac{\Gamma \mid\!\sim \Delta, A, B, C}{\Gamma \mid\!\sim \Delta, A, B \vee C}\ {\mid\!\sim}\vee \qquad A, \Gamma \mid\!\sim \Delta, B \vee C}{\Gamma \mid\!\sim \Delta, B \vee C}\ \mathsf{Trs}$$

We construct the following derivation:

$$\dfrac{\dfrac{\Gamma \mid\!\sim \Delta, A, B, C \qquad \dfrac{A, \Gamma \mid\!\sim \Delta, B \vee C}{A, \Gamma \mid\!\sim \Delta, B, C}\ {\mid\!\sim}\triangledown}{\Gamma \mid\!\sim \Delta, B, C}\ \mathsf{Trs}}{\Gamma \mid\!\sim \Delta, B \vee C}\ {\mid\!\sim}\vee$$

The application of Trs is removed invoking the induction hypothesis on the sum of the height of the derivation.

If the last rule applied is a ternary inference rule we need to exploit the invertibility lemmata. We show a concrete case, the other ones are similar and left to the reader.

$$\dfrac{\dfrac{\Gamma, \neg B \vdash \Delta, A \qquad \Big|\frac{m_1}{n_1}\,\Gamma, \neg B \qquad \Big|\frac{m_2}{n_2}\,\Delta, A}{\Gamma, \neg B \mid\!\sim \Delta, A}\ \neg{\mid\!\sim}\ \ \tfrac{m_1}{n_1} \leq \tfrac{m_2}{n_2} \qquad A, \Gamma, \neg B \mid\!\sim \Delta}{\Gamma, \neg B \mid\!\sim \Delta}\ \mathsf{Trs}$$

First, we observe that by Theorem 4.2 applied to the premise $A, \Gamma, \neg B \mid\!\sim \Delta$ there are $\frac{m_3}{n_3}, \frac{m_4}{n_4}$ such that $\Big|\frac{n_3}{m_3}\,\Gamma, \neg B, A$ and $\Big|\frac{n_4}{m_4}\,\Delta$ and $\frac{m_3}{n_3} \leq \frac{m_4}{n_4}$. Next, we proceed as follows:

$$\dfrac{\dfrac{\Gamma, \neg B \vdash \Delta, A \qquad \dfrac{A, \Gamma, \neg B \mid\!\sim \Delta}{A, \neg B, \Gamma \vdash \Delta}\ \mathsf{inv}}{\Gamma, \neg B \vdash \Delta}\ \mathit{Cut} \qquad \Big|\frac{m_1}{n_1}\,\Gamma, \neg B \qquad \Big|\frac{m_4}{n_4}\,\Delta}{\Gamma, \neg B \mid\!\sim \Delta}\ \neg{\mid\!\sim},\ \tfrac{m_1}{n_1} \leq \tfrac{m_4}{n_4}$$

■

We observe that the cut-elimination result could be streamlined by combining the lemmata and the theorems above.

THEOREM 4.7. Trs *is admissible in* **GS4F**.

PROOF. We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma \mathrel{|\!\sim} \Delta, A}{\Gamma \vdash \Delta, A}\,\text{inv} \qquad \dfrac{A, \Gamma \mathrel{|\!\sim} \Delta}{A, \Gamma \vdash \Delta}\,\text{inv}}{\Gamma \vdash \Delta}\,\text{Cut} \qquad \dfrac{\Gamma \mathrel{|\!\sim} \Delta, A}{\left|\frac{m_1}{n_1}\right. \Gamma}\,\text{Theorem 4.2} \qquad \dfrac{A, \Gamma \mathrel{|\!\sim} \Delta}{\left|\frac{m_2}{n_2}\right. \Delta}\,\text{Theorem 4.2}}{\Gamma \mathrel{|\!\sim} \Delta}\,\text{Theorem 4.3}$$

where $\frac{m_1}{n_1} \leq \frac{m_2}{n_2}$ (which can be checked via a small calculation). ■

However, we deem that it is interesting to show the precise dynamic of the cut-elimination algorithm for a combined system.

Finally, we would like to sketch some examples concerning the paraconsistency of the $\mathrel{|\!\sim}$-consequence relation. Indeed, a consequence relation is *paraconsistent* whenever there is a pair of formulas A and B such that:

$$A, \neg A \mathrel{|\!\not\sim} B$$

PROPOSITION 4.8. *For every atomic formula p and any formula B, the sequent $p \wedge \neg p \mathrel{|\!\sim} B$ is derivable in* **GS4F**.

PROOF. It is enough to observe that the sequent $p \wedge \neg p \mathrel{|\!\sim} B$ is provable.

$$\dfrac{p \wedge \neg p \vdash B \qquad \left|\frac{0}{2}\right. p \wedge \neg p \qquad \left|\frac{m_1}{n_1}\right. B}{p \wedge \neg p \mathrel{|\!\sim} B}\,\wedge\!\mathrel{|\!\sim}$$

because $\frac{0}{2} = [\![p \wedge \neg p]\!] \leq \frac{m_1}{n_1} = [\![B]\!]$ for any formula B. ■

This shows that contradictions which are entirely false, i.e., which do not contain any true component, validate the ex falso sequitur quodlibet principle. Indeed, $p \wedge \neg p$ trivialises the entire system. However, let us consider the case of the formulas $(p \wedge \neg p) \wedge (p \vee \neg p)$ and $q \vee p$. The sequent $(p \wedge \neg p) \wedge (p \vee \neg p) \vdash q \vee p$ is indeed provable in **GS4**. However, $(p \wedge \neg p) \wedge (p \vee \neg p) \mathrel{|\!\sim} q \vee p$ is not provable in **GS4F**.

THEOREM 4.9. $\mathrel{|\!\sim}$ *is a paraconsistent consequence relation.*

PROOF. Consider the formulas $(p \wedge \neg p) \wedge (p \vee \neg p)$ and $q \vee p$. Let us construct a putative derivation for the sequent $(p \wedge \neg p) \wedge (p \vee \neg p) \mathrel{|\!\sim} q \vee p$.

$$
\dfrac{
\dfrac{\dfrac{p, \neg p, p \vee \neg p \vdash q, p}{p \wedge \neg p, p \vee \neg p \vdash q, p}\,{\scriptstyle L\wedge}}{(p \wedge \neg p) \wedge (p \vee \neg p) \vdash q, p}\,{\scriptstyle L\wedge}
\qquad
\dfrac{
\dfrac{\vdash^{0}_{1} p \quad \dfrac{p \vdash^{0}_{1}}{\vdash^{0}_{1} \neg p}\,{\scriptstyle R\neg}}{\vdash^{0}_{2} p \wedge \neg p}\,{\scriptstyle R\wedge}
\quad
\dfrac{\dfrac{p \vdash^{1}_{1} p}{\vdash^{1}_{1} p, \neg p}\,{\scriptstyle R\neg}}{\vdash^{1}_{1} p \vee \neg p}\,{\scriptstyle R\vee}
}{\vdash^{1}_{3} (p \wedge \neg p) \wedge (p \vee \neg p)}
\qquad
\vdash^{0}_{1} q, p
}{
\dfrac{(p \wedge \neg p) \wedge (p \vee \neg p) \mid\!\sim q, p}{(p \wedge \neg p) \wedge (p \vee \neg p) \mid\!\sim q \vee p}\,{\scriptstyle \mid\!\sim\vee}
}\,{\scriptstyle \wedge\mid\!\sim}
$$

The topmost sequent on the left is clearly derivable in **GS4**. The application of the rule $\wedge\mid\!\sim$ is not correct, because $[\![(p \wedge \neg p) \wedge (p \vee \neg p)]\!] = \frac{1}{3}$ is strictly greater than $[\![q, p]\!] = 0$, hence $(p \wedge \neg p) \wedge (p \vee \neg p) \mid\!\sim q \vee p$. ■

The fundamental idea behind the new defined consequence relation is to exploit the nuanced distinctions in contradictions offered by fractional semantics. When scrutinizing classical logic formulas from a fractional perspective, contradictions form a *dense* set. This prompts a not trivial question: how should we logically manage this variability? Bounded supraclassical logic, as introduces in [10], presents a solution by proposing systems that surpass classical logic in deductive power. These resultant calculi exhibit characteristics such as reflexivity, monotonicity, and structural properties, albeit lacking transitivity. However, being an *extension* of classical logic, they do not filter out the validities of classical logic and, consequently, prove every instance of the law of *ex falso*.

In contrast, the fractional consequence relation is not parametrized by a specific rational value and constitutes a proper *subsystem* of classical logic. Indeed, a formula A is deducible from a multiset Γ of formulas only if the fractional value associated with Γ is less than that of A. As an immediate upshot, we observe the breakdown of the *ex falso*: not every contradiction trivializes reasoning. This fact arises from the existence of contradictions whose fractional value is not 0. In fact, for every rational number $q \in [0, 1) \cap \mathbb{Q}$, there exists a contradiction A such that $[\![A]\!] = q$. Consequently, there are infinitely many cases in which not everything follows from a contradiction. In the context of relation $\mid\!\sim$, deduction is conceptualized as a process that results in conclusions with either equal or greater identity content than the premises.

Therefore, it is correct to assert that the fractional consequence relation is properly contained within that of classical logic. To be more precise, there are sequents that are derivable in classical logic but no longer derivable when we transition to the $\mid\!\sim$-turnstile. However, a strong connection persists between this calculus and classical logic. The calculus is not only built on classical logic rules, but there is also a sense in which the system is *conservative* over classical logic. Indeed, as noted Lemma 3.1, when the antecedent Γ of the fractional consequence relation is empty, derivability reverts to the classical one.

Theorem 4.10. *For every formula A, if A is a classical tautology, then $\mid\!\sim A$ is derivable in* **GS4F**.

Proof. Since A is a tautology, by completeness theorem, we obtain $\vdash A$. Furthermore, $\mid\!\frac{1}{1} A$. Hence:

$$\dfrac{\vdash A \qquad \mid\!\frac{0}{1} \qquad \mid\!\frac{1}{1} A}{\mid\!\sim A}\ \text{Theorem 4.3}$$

∎

5. From Degrees of Falsity to Degrees of Truth

In this section we will apply the methodology of fractional semantics to offer a new perspective on tautologies and truth in classical logic. As we have seen, fractional semantics can provide an approach to tame and control explosivity of classical logic while maintaining some desirable structural properties. However, we will now see that truth can - in a sense - be given a degree.

For every tautology A, we have: $[\![A]\!] = 1$. However, classically, the negation of any tautology is always a contradiction, i.e. if for every valuation v, $v(A) = 1$, then for every valuation v, $v(\neg A) = 0$.

5.1. The (Fractional) Algebra of Tautologies

Let us introduce a relation between formulas based on their fractional evaluation.

Definition 5.1. A formula A is *oppositely equivalent* to a formula B, in symbols $A \sim B$, whenever $[\![\neg A]\!] = [\![\neg B]\!]$.

Let us observe the following basic fact.

Lemma 5.1. *If A and B are contradictions, then $A \sim B$.*

Proof. If A and B are contradictions, then $\neg A$ and $\neg B$ are tautologies and so each of their initial sequents in a derivation in **GS4** will be tautological, so $[\![\neg A]\!] = [\![\neg B]\!]$ and thus $A \sim B$. ∎

On the contrary, if A and B are not contradictions, the relation is not trivial. In particular, if we deal with tautologies, it can easily be shown that this is the case.

REMARK 4. There are tautologies A and B such that $A \not\sim B$. Indeed, let us consider the tautologies $p \vee \neg p$ and $p \vee \neg p \vee (p \wedge \neg p)$. We have:

$$[\![\neg(p \vee \neg p)]\!] = 0 \neq [\![\neg((p \vee \neg p) \vee (p \wedge \neg p))]\!] = 0.\overline{3}$$

As is evident from the fact that the relation is defined by an equality, it is an equivalence relation. Indeed, $\sim$ is clearly reflexive, symmetric and transitive.

In fact, we can now study the quotient of the set of formulas of propositional classical logic. It can be then shown that just as in the case of contradiction, the set of fractional values of tautologies can be shown to be *dense*.

LEMMA 5.2. *For every tautology A and B such that $[\![\neg A]\!] < [\![\neg B]\!]$ there is a tautology C such that:*

$$[\![\neg A]\!] < [\![\neg C]\!] < [\![\neg B]\!]$$

PROOF. Since A and B are tautologies, $\neg A$ and $\neg B$ are contradictions. Therefore, since the set of contradictions is dense with respect to fractional evaluations [10], we get that there is a contradiction D such that $[\![\neg A]\!] < [\![D]\!] < [\![\neg B]\!]$. We take $\neg D$ as C and we get the desired conclusion. ■

Let us now consider the structure which emerges by taking the quotient of the tautologies, i.e.:

$$\mathcal{A} = \langle \mathsf{TAUT}_{/\sim}, \wedge, \vee \rangle$$

we denote its elements, i.e. the equivalence classes, as $[A]$.

PROPOSITION 5.3. *$\wedge$ and $\vee$ are operations on $\mathcal{A}$.*

PROOF. Indeed, $[A] \wedge [B] = [A \wedge B]$ is an operator as the conjunction of tautologies is a tautology too. The same holds for disjunctions. ■

LEMMA 5.4. *The operations $\wedge$ and $\vee$ are commutative and associative.*

PROOF. The proof is immediate by definition. ■

The structure $\mathcal{A}$ fails to be a lattice, because it does not satisfy the absorption law.

REMARK 5. $\mathcal{A}$ is not a lattice. Indeed, consider the case $[A] \wedge ([A] \vee [B]) = [A] \wedge [A \vee B] = [A \wedge (A \vee B)]$. Were the absorption law valid, we would have

$[A \wedge (A \vee B)] = [A]$, which is equivalent to $[\![\neg A]\!] = [\![\neg(A \wedge (A \vee B))]\!] = [\![\neg A \vee (\neg A \wedge \neg B)]\!]$. In general, $[\![A]\!] \leq [\![\neg A \vee (\neg A \wedge \neg B)]\!]$, but the contrary does not hold, as easy counterexamples can be found.

THEOREM 5.5. *The algebra:*

$$\mathcal{A} = \langle \mathsf{TAUT}_{/\sim}, \mathsf{min}, \mathsf{max} \rangle$$

is a linearly ordered lattice.

PROOF. The set $\mathsf{TAUT}_{/\sim}$ is linearly ordered by the following binary relation between equivalence classes $[A]$, $[B]$:

$$[A] \leq_{\sim} [B] \text{ if and only if } [\![\neg B]\!] \leq [\![\neg A]\!]$$

Therefore the infimum $inf\{[A], [B]\}$ and the supremum $sup\{[A], [B]\}$ with respect to the relation $\leq_{\sim}$ are always defined for any pair of equivalence classes and they coincide with the minimum and maximum fractional value between $[\![\neg A]\!]$ and $[\![\neg B]\!]$, respectively. ∎

We leave as an open problem the task of finding other appropriate operations on the quotient $\mathcal{A}$ in order to obtain a lattice.

REMARK 6. A reviewer suggested a more quantitative approach to the issue. In particular, they proposed directly assigning a value $\mathtt{v}(\top)$ to any tautology $\top$ as $1 - [\![\neg\top]\!]$. It is immediate to observe that the equivalence relation $A \sim B$ could then be defined as $\mathtt{v}(A) = \mathtt{v}(B)$. Indeed:

$$A \sim B \Longleftrightarrow \mathtt{v}(A) = \mathtt{v}(B) \Longleftrightarrow 1 - [\![\neg A]\!] = 1 - [\![\neg B]\!] \Longleftrightarrow [\![\neg A]\!] = [\![\neg B]\!]$$

This suggestion is certainly interesting and does not impact any of the results in the section. However, we prefer to maintain our abstract approach, as our objective is to introduce a differentiation and classification of tautologies via equivalence classes, rather than proposing a method for assigning values to them. In our view, introducing an additional value assignment to measure tautologies could conceptually conflict with one of the central principles of fractional semantics, namely its conservativity over the set of classically valid formulas.

We believe that the definition of these equivalence classes of tautologies aligns with the broader objectives of our approach. Specifically, the analysis of tautologies is conducted *inside* classical logic utilizing a refined categorization of their associated contradictions through the lenses of fractional semantics.

5.2. Computing Degrees of Truth

We have described the algebraic structure which emerges from the equivalence relation on the classes of tautologies. The simplest way to evaluate the degree of truth of the negation of a tautology consists in decomposing and evaluating the fractional value according to the rules of the classical calculus. In particular, let us consider the following toy example. Consider the pair of formulas $p \vee \neg p$ and $\neg(p \vee \neg p)$:

$$\dfrac{\dfrac{p \vdash^{1}_{1} p}{\vdash^{1}_{1} p, \neg p}\,\mathrm{R}\neg}{\vdash^{1}_{1} p \vee \neg p}\,\mathrm{R}\vee \qquad \dfrac{\dfrac{p \vdash^{0}_{1} \qquad \dfrac{\vdash^{0}_{1} p}{\neg p \vdash^{0}_{1} \neg p}\,\mathrm{L}\neg}{p \vee \neg p \vdash^{0}_{2}}\,\mathrm{L}\vee}{\vdash^{0}_{2} \neg(p \vee \neg p)}\,\mathrm{R}\neg$$

Consider the tautology $(p \wedge \neg p) \vee (p \vee \neg p)$ (an instance of the ex falso):

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash^{1}_{1} p, p \qquad \dfrac{p, p \vdash^{1}_{1} p}{p \vdash^{1}_{1} \neg p, p}\,\mathrm{R}\neg}{p \vdash^{2}_{2} p \wedge \neg p, p}\,\mathrm{R}\wedge}{\vdash^{2}_{2} p \wedge \neg p, p, \neg p}\,\mathrm{R}\neg}{\vdash^{2}_{2} p \wedge \neg p, p \vee \neg p}\,\mathrm{R}\vee}{\vdash^{2}_{2} (p \wedge \neg p) \vee (p \vee \neg p)}\,\mathrm{R}\vee$$

Its associated contradiction $\neg((p \wedge \neg p) \vee (p \vee \neg p))$ which has the following decomposition:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash^{1}_{1} p}{p, \neg p \vdash^{1}_{1}}\,\mathrm{L}\neg}{p \wedge \neg p \vdash^{1}_{1}}\,\mathrm{L}\wedge \qquad \dfrac{p \vdash^{0}_{1} \qquad \dfrac{\vdash^{0}_{1} p}{\neg p \vdash^{0}_{2}}\,\mathrm{L}\neg}{p \vee \neg p \vdash^{1}_{3}}\,\mathrm{L}\vee}{(p \wedge \neg p) \vee (p \vee \neg p) \vdash^{1}_{3}}\,\mathrm{L}\vee}{\vdash^{1}_{3} \neg((p \wedge \neg p) \vee (p \vee \neg p))}\,\mathrm{R}\neg$$

does not get the fractional value 0, because one of the three initial sequents determined by the root-first application of the rules is a tautological one and thus the value is $0.\overline{3}$.

The underlying idea is that $p \vee \neg p$ and $(p \wedge \neg p) \vee (p \vee \neg p)$ are both tautologies. However, one may observe that one of the disjunct which contributes to the tautological character of $(p \wedge \neg p) \vee (p \vee \neg p)$ is a contradiction. This is not reflected from a classical semantic point of view, but emerges whenever we consider the unique decomposition of the formula in a suitable sequent calculus. Therefore $p \vee \neg p$ can be regarded as a tautology bearing a higher degree of truth than $(p \wedge \neg p) \vee (p \vee \neg p)$, because its associated contradiction has no tautological content at all.

Below is a table presenting examples of tautologies along with the values of their corresponding contradictions: Clearly, $p \vee \neg p$ and $p \rightarrow (q \rightarrow p)$

Tautology	**Contradiction**
$p \vee \neg p$	$[\![p \wedge \neg p]\!] = 0$
$p \rightarrow (q \rightarrow p)$	$[\![p \wedge (q \wedge \neg p)]\!] = 0$
$(p \vee \neg p) \vee (p \wedge \neg p)$	$[\![(\neg p \wedge p) \vee (\neg p \vee p)]\!] = 0,\overline{3}$
$p \wedge \neg p \rightarrow p \wedge \neg p$	$[\![(p \wedge \neg p) \wedge (p \vee \neg p)]\!] = 0,\overline{3}$
$p \wedge \neg p \rightarrow (r \wedge \neg r) \vee (q \wedge \neg q)$	$[\![(p \wedge \neg p) \wedge (r \vee \neg r) \wedge (q \vee \neg q)]\!] = 0,\overline{5}$

are in the same equivalence class, whereas $p \wedge \neg p \rightarrow (r \wedge \neg r) \vee (q \wedge \neg q)$ is not. This reflects the idea that the latter tautology has more contradictory content.

The common feature between the equivalence relation here defined and the fractional consequence relation lies in the relevance of the fractional value of the contradictions:

- With respect to $\mid\!\sim$, a contradiction with value 0 validates full explositivity, i.e. *ex falso sequitur quodlibet*. Hence, the higher the fractional value of the contradiction, the weaker the explosiveness of the fractional consequence relation.
- With respect to the equivalence relation, we observe that a tautology has a stronger degree of truth (and is higher in the ordering) if the associated contradiction has a lower one.

Therefore explosivity and the degree of truth are inversely proportional to the fractional value of contradictions. The content of this observation is condensed in the following proposition.

THEOREM 5.6. *Given contradictions $A \wedge \neg A$ and $B \wedge \neg B$ such that $[\![A \wedge \neg A]\!] \leq [\![B \wedge \neg B]\!]$:*

1. If $B \wedge \neg B \mid\!\sim C$, then $A \wedge \neg A \mid\!\sim C$

2. $[\neg(B \wedge \neg B)] \leq_{\sim} [\neg(A \wedge \neg A)]$.

Proof. We discuss the two items separately. Concerning 1., if $B \wedge \neg B \mid\!\sim C$, then $[\![B \wedge \neg B]\!] \leq [\![C]\!]$ and thus $[\![A \wedge \neg A]\!] \leq [\![C]\!]$, which yields $A \wedge \neg A \mid\!\sim C$. With respect to 2., this immediately follows from the ordering defined on the quotient set of tautologies. ■

6. Conclusion: on (Non-)classical Logic(s)

In this concluding section, we aim to extract insights from the findings presented herein.

The fractional consequence relation exhibits strongly non-classical features. The resulting system is, indeed, both non-monotonic and paraconsistent, albeit to a graded extent. The emergence of these distinctive properties arises from the interplay between classical logic and fractional semantics. The fractional consequence relation, while being a constrained version of its classical counterpart, is shaped through a filtering process dictated by syntactic and computational considerations inherent to the proof theory of classical logic. According to this broad picture, paraconsistency phenomena can be also described within a well-discipline semantic and syntactic framework which maintains a strong connection to classicality.

Our analysis of the mixed system reveals significant aspects. First, the system is conservative over tautologies, that is every classically derivable formula is derivable in the fractional system too, as non-classicality comes into play when dealing with reasoning under assumptions. Second, although inducing non-monotonic and paraconsistent aspects, all the rules involve classical systems, retaining the desirable proof theoretic properties of invertibility, analyticity and symmetry characteristic of classical logic. The formulation of three-premise rules combines classical and fractional derivability, as the rules of the system **GS4F** governing the new consequence relation are defined based on classical rules. Moreover, the underlying calculus preserves classical derivability as an *integral* part of the system. This preservation occurs in two distinct ways:

- The calculus **GS4F**, as a mixed system, incorporates the rules of classical propositional logic. Consequently, by the subformula property and the design of its rules, it is immediate to observe that any sequent $\Gamma \vdash \Delta$ derivable in **GS4F** is also classically valid.

- Regarding the fractional consequence relation $\mid\!\sim$, a strong conservativity result applies to classical theorems. Specifically, if A is a tautology, it can be shown that $\mid\!\sim A$ is derivable in **GS4F**.[5]

In conclusion, our proposal illustrates how classicality can be retained by introducing controlled paraconsistency to make fine-grained distinctions between contradictions.[6] In other words, this approach enables the extraction of non-classical elements from classical logic by leveraging these distinctions, thereby achieving a balance between classical reasoning and paraconsistent flexibility.

The breaking of the symmetry between classical tautologies and contradictions has significant implications for our understanding of tautologies. Each classical tautology corresponds to a contradiction obtained by negating it, enabling the identification of distinct classes of tautologies. These degrees of truth are determined through an analysis of classical derivability, anchoring them to classicality. So we claim that the two approaches presented in this paper illustrate how fractional semantics contributes to an intensional refinement of classical logic, inducing paraconsistent and non-monotonic behaviors into the system.

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[5] Furthermore, this provability extends to the full language of classical propositional logic without requiring any preprocessing or translation of formulas, unlike in the cases of classical and intuitionistic logic [8].

[6] As also pointed out in the introduction, other approaches [7] have introduced criteria of distinction between propositions maintaining full classicality with respect to the rules of the calculus.

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M. Piazza
Scuola Normale Superiore Classe di Lettere e Filosofia
Pisa
Italy
mario.piazza@sns.it

M. Tesi
Institute of Logic and Computation
TU Wien
Vienna
Austria
matteo.tesi@tuwien.ac.at