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Reasoning with Relevant Epistemic Logics

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Candidato
dr. Pietro VIGIANI

Relatori

Prof. Mario PIAZZA

Prof. Igor SEDLÁR

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Reality and human nature, sir, are very important things, and oh how they sometimes bring down the most perspicacious calculations!

— Fëdor Dostoevskij, *Crime and Punishment*

Abstract

The present dissertation investigates the relationships between classical and non-classical logics within a common logical platform, in which it is possible to address philosophical questions. More specifically, the focus of the dissertation lies in applications of relevant logics to problems in formal epistemology, such as the problem of logical omniscience, the distinction between explicit and implicit beliefs, and the modeling of belief revision. As the main contribution of the present work, I devise so-called *contextual modal logics*, i.e. modal logics where modal operators individuate the range of application of a given logic, to be chosen from the class of extensions of weak relevant logics. In this way, I argue that it is possible to adequately formalise the context-sensitivity of reasoning, a feature which arises when we confront reasoning about the factual and the epistemic domain. Contextual modal logics are presented both semantically and proof-theoretically, by means of Hilbert-style axiom systems. The language of the logics is modularly extended, from the basic relevant language, so as to include topic sensitive operators, implicit belief operators and dynamic operators. Characterisation results are obtained for all extensions of the logics in the contextual modal family.

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Chapter 1

Introduction: Relevant Theories and Contexts of Inference

The introduction to the present dissertation serves the purpose of introducing the material in a cohesive and well motivated manner. To this purpose, in Section [1.1](#) I address the main philosophical problem I shall be dealing with, i.e. devise logics which formalise epistemic reasoning. In Section [1.2](#) I introduce a specific interpretation of relevant theories, which is useful for modeling inference in epistemic logic. In Section [1.3](#) I sketch how to use the theory building interpretation of relevant logic to devise logics which combine classical propositional reasoning and relevant epistemic reasoning. In Section [1.4](#) I present the material contained in each chapter of the dissertation.

1.1 The Challenge of Epistemic Logic

Modal logics are extensions of propositional logics with operators which serve to qualify the *mode of truth* of formulas they are prefixed to. Modal logic is widely used in epistemology to formalize reasoning about *epistemic attitudes* of agents (cf. [58](#), [89](#)). At the heart of epistemic applications of modal logic lies the fact that modal formulas $\Box\varphi$ may be read, with a fixed agent in mind, as “the agent *knows, believes, has evidence, etc.* that φ holds”. Inference principles embodied in specific modal logics then constitute models of correct reasoning with information containing such epistemic notions.

As with scientific models in general, models of epistemic reasoning based on modal logic may be more or less faithful to the phenomena modeled. Among the possible criteria of faithfulness which can be formulated, we can distinguish a *conceptual*

and a *predictive criterion*. According to the former, epistemic logics should give a philosophically sound explanation of epistemic notions; while according to the latter, the logically valid formulas prescribed by a given epistemic logic should accord with our intuitions regarding everyday inferential patterns. Now, these two criteria have been proven hard to be jointly met.

1. *Standard Epistemic Logic*. As a first case in point, consider standard *possible worlds semantics* for normal modal logics (cf. [105]). According to possible worlds models of reasoning, assessing whether φ is, e.g., believed, requires the modeler to check whether φ holds across all possible situations which are compatible with the current one. Intuitively, a situation is compatible with another if according to the information contained in the latter, the former might be the case. This explanation complies to the conceptual criterion. The popularity of Kripke's idea in explaining the mechanics of epistemic reasoning is so unmatched that normal modal logics characterised by possible worlds semantics constitute the standard for epistemic logic (cf. [89]). However, standard epistemic logics are famously predictively inadequate, since they are beset by the problem of *logical omniscience*, as formulated in full generality by [58]. With notation adjusted:

An agent is fully logically omniscient with respect to a class \mathbf{M} of structures if, whenever he knows all of the formulas in a set Γ , and Γ logically implies the formula φ with respect to \mathbf{M} , then the agent also knows φ .
[58, p.335]

Logically omniscient agents suffer from idealisations which are at odds with our epistemic intuitions regarding the reasoning properties of everyday agents. Instances of the general problem give rise to closure properties of \Box . To give just a simple example, classical normal modal logics prescribe the validity of $\Box\varphi$ for every logical truth φ . This is clearly at odds with our intuitions, given that ordinary agents fail to believe some logical truths, such as complex mathematical propositions. More generally, real-life agents are cognitively and computationally bounded, since e.g. their memory, attention, inference capabilities, storage and vocabulary are limited: all these features fail to be captured by standard models of epistemic reasoning.

2. *Syntactic models*. At the other end of the spectrum, if one's objective is simply to devise a predictively accurate epistemic logic, the task is easy. It is sufficient to endow standard models of classical propositional logic with a fixed set, consisting of the formulas which an agent believes (cf. [58, 100]). Then, if one

thinks that an agent believing φ should thereby believe ψ , just assume that the fixed set contains ψ whenever it contains φ . However, the so-called *sentence storage model* of belief is highly ad hoc (cf. [190]): for one, no explanation is retrievable from the model as to what grounds the attribution of belief contents, which are distinct from sentences; for two, no significant constraints are imposed on the internal logic of belief. Granted this, it is hard for epistemic logic to account for the general ways in which rational capacities may fail without appeal to syntactic considerations:

It is highly questionable whether any background theory short of syntax [...] could do the job. For consider any two sentences φ and ψ logically equivalent in classical logic but differing syntactically in whatever arbitrary manner (e.g. ψ might be any of $\varphi \wedge \varphi$, $\varphi \vee (\varphi \wedge \varphi)$, $\varphi \wedge \dots \wedge \varphi$, $\varphi \wedge ((\chi \rightarrow \xi) \rightarrow \chi) \rightarrow \chi$, or the like): then, for whatever accidental reasons, the cognitive architecture of some agent might be such that she is able to believe that φ without believing that ψ . [109, p.308]

For the above reasons, the problem of logical omniscience for epistemic logic is a serious one. On the one hand, logical omniscience reveals that standard models of epistemic logic set an unattainable standard of rationality. On the other hand, it is hardly seen how to replace the ideal standard of rationality with a more modest one, which resists the syntactic identification of the content of epistemic attitudes with formulas. The epistemic logician is uncomfortably navigating between the Scylla of logical anarchism and the Charybdis of closure under normal modal logic.

Taking stock of these observations, in this dissertation I will be interested in devising a general framework of epistemic logic which meets the following desiderata (cf. [87] for a similar distinction):

1. *Lower bound.* The internal logic of epistemic notions is not trivial, but there is a *logical criterion of epistemic competence*. That is, there are syntactically distinct formulas φ and ψ such that $\Box\varphi \rightarrow \Box\psi$ is provable. Moreover, there is a logical relationship holding between φ and ψ which suffices to infer $\Box\varphi \rightarrow \Box\psi$.
2. *Upper bound.* The internal logic of epistemic notions is weak enough to comply with real-life agents' inference capabilities. That is, \Box satisfies minimal *closure properties*, which should not exceed the minimal criterion of epistemic competence. For brevity, the assessment of many closure properties of interest is postponed to Chapter [5].

Many epistemic logics reacting to the logical omniscience problem, situated within Scylla and Charybdis, struggle to jointly meet the lower and upper bound (cf. [23,

[24, 57, 58, 97, 106, 111, 126, 128, 134, 151, 167, 202, 203]). These proposals are examined in Part II of the dissertation, where the general framework I am after is applied to longstanding epistemological problems. For now, let me focus on a family of proposals, which have a radical approach to logical omniscience.

According to a family of proposals dealing with logical omniscience, the classical foundation of standard epistemic models is the main culprit of the problem. For example, classical logic is explosive, i.e. $(\varphi \wedge \neg\varphi) \rightarrow \psi$ is logically true. This is at odds with ordinary information processing, which is not explosive in light of contradictory input information, as e.g. they may come from distinct equally reliable sources. Moreover, classical logic is monotonic, i.e. $\varphi \rightarrow (\psi \rightarrow \varphi)$ is logically true. This is at odds with the revisability of ordinary information, which may be retracted once new input information is acquired. Consequently, revisionist approaches to omniscience prescribe to build epistemic logic on a non-classical, e.g. relevant, basis (cf. [26, 27, 193]). However, the move of relevant epistemic logicians seems hard to swallow. After all, while our theories about the world, as encoded e.g. in belief systems, may well contain inconsistencies, the world itself does not (cf. [51]). The issue is masterfully summarised by Goble. With notation adjusted:

I find relevant logic attractive, and think that $[\varphi \wedge \neg\varphi \rightarrow \psi]$ is indeed the real culprit behind [epistemic] explosion $[\Box(\varphi \wedge \neg\varphi) \rightarrow \Box\psi]$. Nevertheless, this is a radical departure from standard [epistemic] logics, and requires defending a non-standard approach to logic in general that goes well beyond [epistemic] considerations. [79]

The epistemic logician is then faced with a dilemma: while non-classical logics aptly model the weak inference patterns of fallible agents, classical logic aptly models the relationships between facts of the world. The dilemma requires a careful act of balance. It is perhaps surprising that the right balance between non-classical and classical considerations in epistemic logic is offered by one of the very motivations for *relevant logic*.

1.2 Building Theories with Relevant Logic

Relevant logic (cf. [54, 120, 156] for overviews) consists of a family of logics, which were mainly devised as a response to the so-called paradoxes of material and strict implication (cf. [114]), such as the above $\varphi \rightarrow (\psi \rightarrow \varphi)$ and $(\varphi \wedge \neg\varphi) \rightarrow \psi$. Among the many angles from which one can look at the foundations of relevant logic (cf. [192]), I will be primarily interested in the so-called *theory-building interpretation*.

The theory building interpretation of relevant logic was offered by [117], which in turns draws on [14, 15], led by the intuition that relevant logic can serve as a universal tool for building theories. In a nutshell, a theory is a set of sentences, representing some data, organised with some degree of coherence. Formally, data are imparted coherence by means of their being closed under some consequence relation. Relevant consequence being an extremely weak consequence relation, it can be assigned a special role in building theories: it is, so to speak, contained in every theory reasoners devise to make sense of the world. More robust theories are obtained by closing the data under stronger consequence relations, encoding our extra-logical assumptions regarding the specific domain of application of the theory. In order to motivate why this picture is useful in epistemic domains, I will preliminarily assess how relevant logic can be used to ground a theory of epistemic reasoning, which distinguishes logical implications from fallible inferences (cf. [85]).

The problem with inferences like $\varphi \rightarrow (\psi \rightarrow \varphi)$ and $(\varphi \wedge \neg\varphi) \rightarrow \psi$ has often been individuated in the fact that an appropriate connection between the premises and the conclusion seems to be lacking. This intuitive criterion of relevance is syntactically embodied by the *variable sharing property*: $\varphi \rightarrow \psi$ is provable in any relevant logic only if φ and ψ share some propositional variable (cf. [4]). However, variable sharing – and refinements thereof (cf. [30, 66]) – only provides a necessary condition on provable implications, which can hardly be taken as the unifying feature of the relevant family. Moreover, insisting on the *relevance* of relevant logic was found inaccurate already in [166], the locus classicus of relevant logic:

Irrelevance is merely a symptom of that disease and not of the essence. It is not what is really wrong with those systems [of material and strict implication], and they are not cured merely by removing irrelevance. [166, p.232]

If relevance is only a byproduct of a good implication – such as “relevant” implication – the question remains as to what are the distinctive features of an inferentially adequate implication. In order to answer this question, I want to emphasise that the very notion of implication in relevant logic is multifarious (cf. [166]), since it encompasses both *logical implication* and *fallible inference*.¹

¹A famous example pointing out the difference between logical implication and fallible inference is reported by [34], and assessed in [86]. In a nutshell, the difference is explained in the little bearings that rules like modus ponens have to actual reasoning: even accepting φ and accepting that φ entails ψ , logic on its own is not enough to *infer* ψ . Indeed, one should further accept that φ and φ entails ψ together entail ψ . But this again is not sufficient, since one should further accept that φ and $(\varphi$ entails $\psi)$ and $(\varphi$ and φ entails ψ together entail $\psi)$ together entail ψ . And so on.

1. *Entailment*. Broadly speaking, logic concerns a relation \vdash , usually called entailment or logical consequence, which determines what formulas follows from a given set of sentences (cf. Definition 2.4). Logical implication is the object language correlate of entailment, where the correlation is generally given by the deduction theorem, according to which $\varphi \vdash \psi$ iff $\vdash \varphi \rightarrow \psi$ (cf. Lemma 2.2 for a semantic version). Entailment is usually taken to be monotonic (indeed Tarskian) and to give rise to theories (cf. Definition 2.11), i.e. sets of formulas X closed under entailment, so that $Th(X) = \{\varphi \mid X \vdash \varphi\}$. That is, every theory *complies* with logical consequence. The resulting picture is the one put forward by [15, 117], according to which logic grounds theories in that entailment plays the role of *basement-level closure relation* [15, p.31] within a theory.
2. *Inference*. Compared to entailment, inference is a much weaker relation between sentences, which provides a fallible link between theories. The picture is as follows. Theories are not static entities: in the course of reasoning, they can be related to one another, e.g. they can be combined, they can exclude each other, they can be marked as special in some sense or another, etc. Theory combination is the salient theory operation underlying inference, and its object language correlate is implication \rightarrow . A theory *contains* an implication $\varphi \rightarrow \psi$ iff the theory which results from combining the original theory with a theory containing φ contains ψ (cf. Definition 2.8). As a result, implication in general expresses the total sufficiency of an input to the output of theory combination (cf. [159, 166]). Theory combination is a very general operation, e.g. it is non monotonic: it may be the case that combining a theory with another supporting $\varphi \rightarrow \psi$, does not result in a theory supporting $\varphi \wedge \chi \rightarrow \psi$, even if $(\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \chi \rightarrow \psi)$ is a logical implication.

The distinction between logical implication and fallible inference runs parallel to a distinction, established in [85], between *logic* and *theory of reasoning*.

Rational reasoning is a complex process, which is regimented by conflicting principles, such as conservatism, coherence, logical closure and clutter avoidance (cf. [85, Ch.2]). Each principle is defeasible and only holds other things being equal. For this reason, Harman's own idea is that, by virtue of its exceptionless and universal nature, logic is not especially relevant to rational reasoning. However, I will argue for a more modest position. With a minimal degree of idealisation, which is implicit in every modeling practice, I will consider the possibility that logic does provide a foundation to reasoning: it constraints a theory by providing immediately intelligible inferences, which enhance the coherence of any theory. For this reason, the kind of

logical consequence which plays the role of universal closure relation should be as weak as possible. It should be weak so as to serve as “the minimal relation that can plausibly be interpreted as a way of building theories” [117, p.8].

Because it is endowed with minimal assumptions concerning valid entailments, relevant logic is especially apt to the task of constraining rational inferences, chiefly among them epistemic inferences. In particular, some of the weak relevant modal logics in the family, such as BMC (cf. Chapter 2) and FE (cf. Chapter 4) provide criteria of epistemic competence.

So much for a motivation of how non-classical logical consequence – e.g. that of paraconsistent and non monotonic logics – can provide guidance in regimenting rational inquiry. It does because it sanctions some fundamental (weak) inferences that agents rationally perform. What I did not mention is the place of classical logic in this picture. For this, a refined analysis of our theory building practices is needed, which again draws on [15, 117]. When investigating the world, we build theories which are tailored to the targeted phenomenon of inquiry. Building a theory requires (i) collecting information about a target phenomenon and (ii) constructing a suitable closure relation for it. I emphasise the fact that each theory’s consequence relation depends not only on basement-level closure relation, but also on domain-specific entailments, which dictate further extra-logical principles:

In most (all?) cases the full/complete truth of the target phenomenon requires a closure relation that goes well beyond the basement-level consequence relation of logic. [...] And this is what the theorist’s second task involves: she needs not to build the logical part of the theory’s consequence/closure relation; that part is indeed already built and present (so to speak). What the theorist’s second task involves is the construction of the right extra-logical consequence/closure relation that serves to complete (as far as possible) the true theory. [15, p.33]

There is enough groundwork to advance a philosophical thesis. Taking stock of the above observation, theories can be used to figure out distinct domains of inquiry, and relevant logic is, so to speak, contained in every domain-specific consequence relation. Two domains on inquiry are prominent from the perspective of the present work.²

²The following is another example of theories which may require different closure relations. In building the theory of the beliefs of a risk averse chess player, it is optimal to take the player’s beliefs about the opponent’s abilities to be closed under a strong logic. However, the player’s beliefs about their own ability may be subject to lower standards of rationality.

1. *The epistemic domain.* First, one may use theories as a guide to epistemic reasoning, e.g. to tell what beliefs an agent entertains in virtue of entertaining some other beliefs. The language of such a theory includes the modal operator \Box , whose closure properties are determined by doxastic entailments, which involve the interaction between \Box and the logical vocabulary. Because of the logical omniscience problem, extra-logical doxastic entailments to be supplemented to relevant consequence should be kept to a minimum. Chapters 5 and 6 flesh out two such epistemic logics of decreasing strength, where the criterion of logical competence for \Box (i.e. the theory-specific closure relation) is given by, respectively, *relevant equivalence* and *on-topic relevant equivalence*.
2. *The factual domain.* Second, one may use theories as a guide to the facts, i.e. to tell how the world stands. To this aim, the basement-level closure relation is not strong enough, since e.g. it does not discount paraconsistent theories (containing both φ and $\neg\varphi$) from representing how the world stands. More generally, relevant propositional logics like **BM** and **F** are strongly subclassical. However, the fact that relevant consequence is the basement-level closure relation does not mean that all our theories should comply just to relevant consequence. Indeed, the established metaphysical tradition (cf. 147 for an exception) holds that factual theories, corresponding to possible worlds, are classical, i.e. they are maximally consistent theories (cf. Definition 2.11). This is not to say that relevant and classical consequence are incompatible relations.

The philosophical thesis falling out of the present picture is that relevant logic and classical logic are by no means incompatible. Quite the contrary, relevant and classical consequence are just tailored to different types of theories. However, relevant logic has a conceptual priority over classical logic, since it determines the universal theory closure relation, on top of which classical consequence arises for a specific class of theories, i.e. those which describe possible ways the world can be. Specifying under which conditions relevant theories give rise to classical theories is the primary task of *contextual modal logics*, to be developed in Part II of the dissertation.

1.3 Contextual Modal Logic, in a Nutshell

To recapitulate, if one wishes to regiment epistemic reasoning by some logic, then logical omniscience makes it necessary to resort to a weak one. Relevant consequence seems apt to the task. Even granted that relevant logic provides the desired modest standard of rationality, rational agents still collect their information within

possible worlds, which beliefs aim to correctly represent. Contextual modal logic is motivated by the consideration that classical logic is well suited to describe the relationship between facts of the world, but it is ill suited to formalise the reasoning of computationally and cognitively bounded agents. For epistemic reasoning, a more modest standard of rationality is needed. In particular, such standard should single out a criterion of epistemic closure weaker than classical consequence.³

Contextual modal logic is best motivated model theoretically, where again the starting point are models for relevant logic. The most established semantic framework for relevant logic is Routley–Meyer semantics (cf. [160, 161, 162, 166]) – see Section 2.3 for a brief historical survey. These models take inspiration from Kripke–style possible world semantics for modal logic, and generalise it by devoting a modal treatment not just to \Box but also to other connectives (cf. Chapter 2). Drawing from [70, 117, 201], these models lend themselves to a perspicuous interpretation in terms of spaces of theories, endowed with operations like theory combination and theory exclusion. Hence, it turns out that models can be interpreted in a rather syntactic fashion. The key insight of contextual modal logic is that the space of all theories, complying to relevant consequence, can be further distinguished as follows:

1. *Logical theories.* One of the technical innovations of Routley–Meyer semantics is the characterisation of the basement–level closure relation by a set of logically valid implications. That is, in the context of Routley–Meyer models, it is possible to identify those theories which contain all logically valid implications, according to a relevant notion of validity. To do so, *Logical theories* are obtained by imposing on them stronger constraints than on standard theories. The corresponding set L of logical theories, then, is then used to define validity in relevant models.
2. *Worldly theories.* The innovation of contextual modal logics – which hardly is a completely new idea (cf. Remark 3.1) – is that not only relevant consequence, but also classical consequence can be characterised by a class of theories in Routley–Meyer models. That is, the idea is to identify those theories which contain all classically valid implications. To do so, *worldly* theories are obtained by imposing further constraints on logical theories. In particular, these

³Note that the present approach to epistemic logic is not primarily aimed at modelling computationally weaker agents (for which see [9, 29, 186]), although avoidance of some computationally strong inferences embedded in classical consequence is attained. Rather, relevant consequence sets a criterion of epistemic competence which is a modest – conceptually adequate – idealisation of the practice of building epistemic theories.

constraints make relevant negation and relevant implication in *worldly theories* inferentially equivalent to Boolean negation and material implication. The corresponding set W of worldly theories is used to define validity in models for contextual modal logics.

In contextual modal logics, worldly theories are maximally consistent theories, with respect to a consequence relation which extends that of classical propositional logic. The consequence relation needs to properly extend classical propositional logic so as to provide closure principles to epistemic \Box -formulas. As remarked above, these closure principles somehow need to comply with relevant logic. The question then becomes, how is it possible for a logic C.L building on classical propositional logic to prescribe relevant epistemic closure?

The answer in contextual modal logics is given by a technically motivated modality, \Box_L , which provides a mediation between the classical and the relevant layer of C.L. To be precise, \Box_L internalises at the object language level provability in a given relevant logic L: $\Box_L\varphi$ is provable in the full system C.L iff φ is provable in the relevant logic L. With the aid of \Box_L , it is possible to restrict closure principles for epistemic modalities to relevant logic, in a classical setting. For example, in Chapter 3, a sufficient condition to believe ψ , for an agent believing φ , is that φ relevantly *implies* ψ . That is, the following rule of inference is derivable:

$$\text{from } \Box_L(\varphi \rightarrow \psi), \text{ infer } \Box\varphi \rightarrow \Box\psi.$$

Moreover, this closure principle is refined in Chapter 4 from relevant logical implication to relevant logical *equivalence*, and in Chapter 6 from relevant logical equivalence to *on topic* relevant equivalence. These epistemic closure principles are supplemented to classical propositional logic. As a result, contextual modal logic complies to a desired *principle of minimal mutilation of classical logic*, according one should retain as much of classical propositional logic when revising logic so as to accord to context-specific, e.g. epistemic, considerations. The above epistemic closure principles provide one minimal criteria of epistemic competence, which comply with the lower bound I set for the internal logic of epistemic notions. Moreover, all instances of logical omniscience not complying to these principles will turn out invalid (cf. Proposition 5.2).

From an axiomatic point of view, contextual modal logics are taken to piggyback ride on the axiomatisation of a given relevant logic L. The contextual modal logic based on L, C.L, is defined as follows, where the \Box_L -version of a formula (rule) is obtained by prefixing \Box_L to the formula (each premise and the conclusion of the rule):

1. An axiomatisation of classical propositional logic CPC (cf. Figure 3.1);
2. The \Box_L -version of the axioms and rules of L ;
3. The bridge rule: from $\Box_L\varphi$, infer φ .

One may note that the axiom system for contextual modal logic accords with the picture of relevant logic as the basement-level theory building closure relation and the principle of minimal mutilation of classical logic. For one, the epistemic domain requires modal formulas in theories to be only closed under relevant consequence. For two, extra-logical principles are supplemented to the specific class of theories which are used to describe how the world stands. The factual domain requires (i) that theories be closed under classical propositional logic and (ii) that indeed the epistemic be a subdomain of the factual domain, which is embodied by the bridge rule.

1.4 Presentation of the Material

The material of the dissertation is organised in two parts. In Part I of the dissertation I lay the technical foundations of contextual modal logics. More specifically, in Chapter 2 I recall the technical preliminaries needed to introduce various logics in the family of relevant logics, characterised by relational frame semantics. In Chapter 3 I show how to completely characterise contextual modal logics, formulated parametrically on a given logic in the relevant family, with respect to relational frame semantics. In Chapter 4 I show how to extend the results of Chapter 3 to a wider range of relevant logics. Moreover, I simplify the model theory of contextual modal logics by resorting to neighborhood frame semantics.

Part II shows the fruitfulness of contextual modal logics for a number of applications in epistemic logic. More specifically, in Chapter 5 I apply contextual modal logics to the problem of logical omniscience, and I show that epistemic attitudes are closed under few instances of the problem and comply to a criterion of epistemic competence. In Chapter 6 I extend the results of Chapter 5 by considering logics where the truth and topic both account for formulas' semantic content. I then devise philosophically and technically motivated topic-sensitive relevant logics, on which to base contextual modal logic. In Chapter 7 I show how to extend the basic framework with additional modal operators, which allow to trace a distinction between explicit and implicit belief. Finally, in Chapter 8 I extend the basic framework with dynamic

modalities, which model the action of updating agents' epistemic state with new information. I then argue that relevant updates furnish a good model of conditional reasoning, which encodes a form of belief revision.

The present dissertation is based on a number of published papers and unpublished manuscripts. Each piece of work contributed to the contents of the dissertation's chapters, with harmonised notation. In what follows, I summarise what piece of work each chapter is primarily build upon. I remark that Chapter 3 constitutes an upgraded version of [175]. Moreover, Chapter 4 builds on [173], but incorporates some new technical result.

Chapter 3.

[175] Igor Sedlár and Pietro Vigiani. “Relevant Reasoners in a Classical World”. In: *Advances in Modal Logic, Volume 14*. Ed. by David Fernández Duque, Alessandra Palmigiano, and Sophie Pichinat. London: College Publications, 2022, pp. 697–718.

Chapter 4.

[173] Igor Sedlár and Pietro Vigiani. “Epistemic Logics for Relevant Reasoners”. In: *Journal of Philosophical Logic* 53 (2024), pp. 1383–1411.

Chapter 5.

[204] Pietro Vigiani. “A Hyperintensional Logic of Non-prime Evidence”. In: *Journal of Philosophical Logic* 53.3 (2024), pp. 761–788.

Chapter 6.

[206] Pietro Vigiani. “Relevant Epistemic Logic with State-Sensitive Topics”. In: *Synthese* (2024);

[207] Pietro Vigiani and Thomas Macaulay Ferguson. *Contextual Entailment and Containment: A Ternary Approach to Information and Topic Inclusion*. Under review.

Chapter 7.

[176] Igor Sedlár and Pietro Vigiani. “Relevant Reasoning and Implicit Beliefs”. In: *Logic, Language, Information, and Computation*. Ed. by Helle Hvid Hansen, Andre Scedrov, and Ruy J.G.B. de Queiroz. Cham: Springer Nature Switzerland, 2023, pp. 336–350;

[174] Igor Sedlár and Pietro Vigiani. “Explicit and Implicit Belief in First Degree Entailment With Strict Implication”. In: *New Directions in Relevant Logic*. Ed. by

Andrew Tedder Igor Sedlár and Shawn Standaefer. Dordrecht: Springer Netherlands, 2024, pp. 425–452.

Chapter 8.

205 Pietro Vigiani. “Conditional Logic with Relevant Updates”. In: (Under review).

Part I

The Basic Framework

Chapter 2

Preliminaries on Relevant Modal Logics

In this chapter, I will introduce the family of relevant modal logics I shall work with. To this aim, in Section 2.1 I introduce the basic definitions and motivations for relevant logics. In section 2.1 I introduce the Routley–Meyer semantics for the basic relevant modal logic BMC. In Section 2.2 I show how to characterise extensions of BMC, obtained by extending BMC with various propositional and modal axioms and rules of inference. Section 2.3 concludes with some historical and conceptual remarks on relevant logic.

2.1 The Basic Relevant Modal Logic

The standard starting point of relevant logic is the thesis that: α implies β only if α is relevant to β . As it is, however, the relevance thesis leaves a number of issues underspecified. Some of them, such as the multitude of interpretations, in relevant logic, of implication and relevance, were pointed out in Chapter 1. In order to give more precise answers to these and related questions, some basic formal definitions are required.

2.1.1 Languages, Proof Systems, Derivations

In this dissertation, I will consider many logics, formulated with respect to different languages. Therefore, it will come in handy to summarise such languages in one place.

Definition 2.1 (Full Language). *Let the full language \mathbf{Lan} be defined recursively in BNF from a denumerable set of atomic propositional formulas At as follows:*

$$\begin{aligned} \varphi \in \mathcal{L} ::= & p \mid \top \mid \mathbf{t} \mid c \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \mid \neg \varphi \mid \varphi \otimes \varphi \\ & \varphi \rightarrow \varphi \mid \varphi \leftarrow \varphi \mid \varphi \supseteq \psi \mid \Box \varphi \mid \Box_L \varphi \mid [\forall] \\ & \Box_I \varphi \mid \varphi \rightarrow \exists \varphi \mid \varphi \& \varphi \mid [\cdot]. \end{aligned}$$

I will adopt the usual conventions and define content equivalence $\varphi \equiv \psi := (\varphi \supseteq \psi) \wedge (\psi \supseteq \varphi)$, extensional falsehood $\perp := \neg \top$, intensional falsehood $\mathbf{f} := \neg \mathbf{t}$, intensional disjunction $\varphi \oplus \psi := \neg \varphi \rightarrow \psi$ and the existential epistemic modality $\diamond \varphi := \neg \Box \neg \varphi$. Moreover, I will use $\forall, \exists, \implies, \iff$ as universal quantification, existential quantification, implication and equivalence, respectively, in the metalanguage.

Connectives can be grouped as follows, according to their intuitive reading:

- \top and \mathbf{t} are truth constants, extensional and intensional respectively;
- c is the classical constant;
- \wedge and \otimes are conjunctions, extensional and intensional;
- \vee and \oplus are disjunctions, extensional and intensional;
- \neg is the intensional negation;
- \rightarrow and \leftarrow are intensional implications, right and left;
- \supseteq is the intensional content inclusion implication;
- \Box, \Box_L, \Box_I and $[\forall]$ are universal modalities;
- $\rightarrow \exists$ and $\&$ are strict modalities, universal and existential.
- $[\cdot]$ is the dynamic modality.

The truth constant \mathbf{t} is the Ackermann constant, and represents in the language the conjunction of all tautologies. It is to be distinguished from the Church constant \top , which represents the weakest proposition: \top , differently from \mathbf{t} , is entailed by every formula. Similarly, $\mathbf{f} := \neg \mathbf{t}$ represents the weakest provably false proposition and, differently from \perp , it does not entail every formula. This is a first example of an ubiquitous phenomenon in relevant and substructural logic, i.e. the disentanglement of notions conflated by classical logic. A similar behaviour can be spotted when considering the properties of the intensional conjunction and disjunction \otimes, \oplus in comparison with the lattice-theoretic counterparts \wedge, \vee . In line with the linear logic tradition [78], I will denote \otimes and \oplus with *fusion* and *fission*. The readings for the connectives are discussed more at length in the context of the formalism where they are introduced.

Different sublanguages of **Lan** are obtained by focusing on a subset of the above logical connectives, specified by the language signature. I will use the convention of denoting a sublanguage of **Lan** in the signature σ_X by \mathfrak{L}_X . Table 2.1 summarises the languages appearing in the following Chapters, in order of appearance.

Signature	Logic	Chapter
$\sigma_{\text{Rel}} = \langle \mathbf{t}, \wedge, \vee, \neg, \otimes, \rightarrow, \leftarrow, \Box, \Box_L \rangle$	<i>Relevant Modal</i>	2–4
$\sigma_{\text{RRel}} = \langle \mathbf{t}, \wedge, \vee, \rightarrow, \leftarrow, \Box, \Box_L \rangle$	<i>Reduced Relevant Modal</i>	4
$\sigma_{\text{PHB}} = \langle \top, \wedge, \vee, \neg, \rightarrow, \Box, [\forall] \rangle$	<i>Plain Hyperintensional Belief</i>	5
$\sigma_{\text{CMod}} = \langle \wedge, \vee, \neg, \rightarrow, \Box \rangle$	<i>Classical Modal</i>	5
$\sigma_{\text{Rel}\top} = \langle \top, \mathbf{t}, \wedge, \vee, \neg, \otimes, \rightarrow, \leftarrow, \Box, \Box_L \rangle$	<i>Relevant Modal with \top</i>	5
$\sigma_{\text{CCont}} = \langle \wedge, \vee, \neg, \rightarrow, \supseteq, \Box \rangle$	<i>Classical Containment</i>	6
$\sigma_{\text{RCont}} = \langle \mathbf{t}, \wedge, \vee, \neg, \rightarrow, \supseteq \rangle$	<i>Relevant Containment</i>	6
$\sigma_{\text{CRel}} = \langle \mathbf{t}, \wedge, \vee, \neg, \rightarrow, \supseteq, \Box, \Box_L \rangle$	<i>Containment Relevant Modal</i>	6
$\sigma_{\text{Ent}} = \langle \wedge, \vee, \neg \rangle$	<i>Entailment</i>	7
$\sigma_{\text{SEnt}} = \langle c, \wedge, \vee, \neg, \neg\exists, \& \rangle$	<i>Strict Entailment</i>	7,8
$\sigma_{\text{IRel}} = \langle \mathbf{t}, \wedge, \vee, \neg, \rightarrow, \Box, \Box_L, \Box_I \rangle$	<i>Implicit Relevant Modal</i>	7
$\sigma_{\text{Cond}} = \langle \wedge, \vee, \neg, \rightarrow, > \rangle$	<i>Conditional</i>	8
$\sigma_{\text{DRel}} = \langle \mathbf{t}, \wedge, \vee, \neg, \rightarrow, \Box, \Box_L, [\cdot] \rangle$	<i>Dynamic Relevant Modal</i>	8

Figure 2.1: Table of languages.

In order to outline the motivations for relevant logics, let me introduce some general definitions which enable one to work with formal proofs. In this dissertation, I will work with axiomatic proof systems, but more general formulations are available.¹

Definition 2.2 (Rules, Axioms, Proof systems). *Let a rule of inference be a set $\{\langle \Gamma, \varphi \rangle \mid \Gamma \in \mathcal{P}(\mathfrak{L}), \varphi \in \mathfrak{L}\}$. Let each member of a rule $\varphi_1, \dots, \varphi_n \Rightarrow \psi$ be an*

¹Generalisations of Hilbert systems proceed by taking more structured entities than formulas. A first generalisation comes from natural deduction (cf. e.g. [153]), which employ SET – FMLA consecutions in the formulation of inference rules, but premises and conclusions can be more structured and/or higher order entities. More structured entities are found already in Gentzen’s sequent calculi [76], which use multi-sets (or lists) in place of sets and single conclusion in consecutions. Lambek calculi [108] are particularly interesting applications of sequent calculi for the study of substructural logics, where the structural rules of commutation, association, weakening and contraction manage the reorganisation of consecutions’ antecedents and succedents. Moreover, natural deduction systems for full relevant logic with distributivity usually employ bunches instead of sets [153]. Dunn–Mints calculi [49, 125] and display calculi [18] generalise sequent calculi for substructural logic by allowing for a controlled amount of weakening and contraction. On the other hand, higher order systems generalise consecutions (or sequents) to e.g. hypersequents [7] and (linear) nested sequents [33, 140]. A further generalisation, n -sided sequent calculus [8], considers more sides than just antecedents and succedents in consecutions.

instance of that rule, where $\varphi_1, \dots, \varphi_n$ are taken together as a set of premises and ψ is the conclusion. Let an axiom be an inference rule with empty set of premises. Let a proof system \mathbf{L} defined over $\mathfrak{L} \subseteq \mathbf{Lan}$ consist of a set of inference rules.

Definition 2.3 (Derivations). *For a given proof system \mathbf{L} defined over $\mathfrak{L} \subseteq \mathbf{Lan}$, let a derivation of $\varphi \in \mathfrak{L}$ from assumptions $\Gamma \subseteq \mathfrak{L}$ be a finite tree such that (i) φ is the root of the tree; (ii) every node in the tree is an \mathfrak{L} -formula; (iii) every node is obtained by means of an instance of an inference rule or is an \mathfrak{L} -formula from Γ .*

Definition 2.4 (Provability, derivability, admissibility). *For a given proof system \mathbf{L} defined over \mathfrak{L} , let a formula φ be provable in \mathbf{L} – denoted $\vdash_{\mathbf{L}} \varphi$ – iff there is a derivation of φ from no assumptions. A formula $\varphi \in \mathfrak{L}$ is provable in \mathbf{L} from assumptions $\Gamma \subseteq \mathfrak{L}$ – denoted $\Gamma \vdash_{\mathbf{L}} \varphi$ – iff there are $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $\vdash_{\mathbf{L}} (\bigwedge_{i \leq n} \gamma_i) \rightarrow \varphi$. A rule of inference $\Gamma \Rightarrow \varphi$ is derivable in \mathbf{L} iff there is a derivation in \mathbf{L} of φ from assumptions Γ . A rule of inference $\Gamma \Rightarrow \varphi$ is admissible iff φ is provable whenever all $\gamma \in \Gamma$ are provable.*

In what follows, I shall be concerned with derivations in tree-form – which for reasons of space I sometimes present as Lemmon-style lists. Moreover, it will be assumed that Γ is finite for simplicity.

Throughout the dissertation, I will only work with *Tarskian* consequence relations, i.e. it will always be the case that derivability $\vdash_{\mathbf{L}}$ is reflexive ($\varphi \vdash_{\mathbf{L}} \varphi$), transitive ($\Gamma \vdash_{\mathbf{L}} \varphi$ and $\Delta \cup \{\varphi\} \vdash_{\mathbf{L}} \psi$ implies $\Gamma \cup \Delta \vdash_{\mathbf{L}} \psi$) and monotonic ($\Gamma \vdash_{\mathbf{L}} \varphi$ implies $\Gamma \cup \Delta \vdash_{\mathbf{L}} \varphi$). As usual, a logic will be syntactically understood as the set of theorems, i.e. provable formulas. Hence, I will interchangeably use \mathbf{L} for either a proof system or for a logic generated by that proof system. Later in the dissertation, a logic will be identified semantically by the set of formulas valid in all models characterising a given proof system, which I suggestively denote \mathbf{L} . Hence, I will interchangeably use \mathbf{L} for either the class of all models of \mathbf{L} and the set of formulas valid in all models of \mathbf{L} .

Frame semantics for relevant logics deploys so-called Routley–Meyer models, which can be seen as a generalisation of the possible-worlds semantics for classical and intuitionistic modal logic (cf. [105]), where connectives are analysed via point-shifting operators between points in the domain. The distinctive feature of Routley–Meyer semantics is the use of the so-called Routley star function to analyse negation and the Routley–Meyer ternary relation to analyse implication.

2.1.2 Frame Semantics for Full Relevant Logic

Frame semantics for relevant logics is based on partially ordered sets, i.e. ordered structures (S, \leq) such that \leq is reflexive, transitive and anti-symmetric. Before defining more precisely the semantics for the full relevant modal language $\mathfrak{L}_{\text{Rel}}$, I will introduce some compact notation for semantic operations on partially ordered sets, by which I will be able to encode each frame component's monotonicity conditions with respect to \leq .

Definition 2.5 (Arrow notation). *For two given posets (S_1, \leq_1) and (S_2, \leq_2) and $k_1, \dots, k_n, k_{n+1} \in \{\downarrow, \uparrow\}$, let an n -ary function f from (S_1, \leq_1) to (S_2, \leq_2) be of type $k_1 \dots k_n \mapsto k_{n+1}$ iff:*

$$\bigwedge_{i \leq n} (s_i Z_i t_i) \implies f(s_1, \dots, s_n) Z_{n+1} f(t_1, \dots, t_n)$$

where $Z_i = \leq$ in case $k_i = \uparrow$ and $Z_i = \geq$ in case $k_i = \downarrow$. I denote as $S_1(k_1 \dots k_n, S_2(k_{n+1}))$ the set of n -ary functions from S_1 to S_2 of type $k_1 \dots k_n \mapsto k_{n+1}$.

As a special case, n -ary relations on (S, \leq) are n -ary operations from (S, \leq) to $T = (\{\mathfrak{t}, \mathfrak{f}\}, \sqsubseteq)$, where it is assumed that $\mathfrak{f} \sqsubseteq \mathfrak{t}$. Note however that S_1 and S_2 need not be identical. The following are notable examples of the above notation which will be employed:

- $S(\uparrow, T(\uparrow))$ denotes the set of all subsets of S that are closed upwards under \leq ;
- $S(\downarrow \uparrow, T(\uparrow))$ denotes the set of binary relations on S that are anti-monotonic in the first position and monotonic in the second position;
- $S(\downarrow \downarrow \uparrow, T(\uparrow))$ denotes the set of ternary relations on S that are anti-monotonic in the first and second position and monotonic in the second position;
- $S(\downarrow \downarrow, T(\uparrow))$ denotes the set of binary relations on S that are anti-monotonic in the first and second position;
- $S(\uparrow, S(\downarrow))$ denotes the set of anti-monotonic unary functions on S ;
- $S_1(\uparrow, S_2(\uparrow))$ will denote the set of monotonic unary functions $f : (S_1, \leq_1) \rightarrow (S_2, \leq_2)$.

To improve readability, I will usually omit $T(\uparrow)$; hence $S(\uparrow)$ means $S(\uparrow, T(\uparrow))$. As a final convention, if B is a binary relation on S , then $B(s)$ denotes the set $\{t \mid Bst\}$, and if $X \subseteq S$, then $B(X) := \bigcup_{s \in X} B(s)$.

With the above notation in play, it is possible to define Routley–Meyer models for relevant propositional logics on the basis of frames, i.e. relational structures consisting of a set equipped with a number of relations interpreting the connectives

of the language. L -models for relevant modal logics are obtained in this way, as follows.

Definition 2.6 (L -models). *Let an L -frame be a tuple of the following form, where $Q_{(L)} \in \{Q, Q_L\}$:*

$$(S, L, \leq, *, R, Q_{(L)})$$

- (S, \leq) is a non-empty partially ordered set of information states;
- $L \in S(\uparrow)$ is a non-empty subset of logical information states;
- $* \in S(\uparrow, S(\downarrow))$ is the Routley star, interpreting \neg ;
- $R \in S(\downarrow\downarrow\uparrow)$ is the Routley–Meyer ternary relation, interpreting $\rightarrow, \leftarrow, \otimes$;
- $Q_{(L)} \in S(\downarrow\uparrow)$ is a Kripke binary relation, interpreting \Box, \Box_L .

Moreover, let every L -frame satisfy the following frame properties:

$$\forall s \in S \exists l \in L (Rlss) \tag{L1}$$

$$\forall s, t \in S \forall l \in L (Rlst \implies s \leq t) \tag{L2}$$

For every L -frame \mathfrak{F} , let (\mathfrak{F}, V) be the L -model based on \mathfrak{F} , where $V : At \rightarrow S(\uparrow)$ is an atomic valuation function. Let **BMC** denote the class of all L -models.

Definition 2.7 (L -frame operations). *For a given L -frame \mathfrak{F} , the following frame operations on \mathfrak{F} are defined, for $X, Y \in \mathcal{P}(S)$, where $\Box_{(L)} \in \{\Box, \Box_L\}$:*

$$X \wedge^{\mathfrak{F}} Y = X \cap Y$$

$$X \vee^{\mathfrak{F}} Y = X \cup Y$$

$$\neg^{\mathfrak{F}} X = \{s \in S \mid s^* \notin X\}$$

$$X \otimes^{\mathfrak{F}} Y = \{s \in S \mid \exists t, u \in S (Rtus, t \in X \text{ and } u \in Y)\}$$

$$X \rightarrow^{\mathfrak{F}} Y = \{s \in S \mid \{s\} \otimes^{\mathfrak{F}} X \subseteq Y\}$$

$$X \leftarrow^{\mathfrak{F}} Y = \{s \in S \mid Y \otimes^{\mathfrak{F}} \{s\} \subseteq X\}$$

$$\Box_{(L)}^{\mathfrak{F}} X = \{s \in S \mid Q_{(L)}(s) \subseteq X\}$$

With an abuse of notation, in the following chapters I will adopt the same symbols, e.g. \models , for corresponding model-theoretic components across different structures and different languages. Moreover, I will omit reference to models when clear from context and I will let s, t, u , etc. range over elements of S .

Definition 2.8 (L -satisfaction). *Let the proposition expressed by an arbitrary formula $\varphi \in \mathfrak{L}_{\text{Rel}}$ in an L -model \mathfrak{M} – denoted $\llbracket \varphi \rrbracket_{\mathfrak{M}}$ – be recursively defined as follows, where $\otimes \in \{\neg, \wedge, \vee, \otimes, \rightarrow, \leftarrow, \square_{(L)}\}$ and n is the arity of \otimes :*

$$\begin{aligned} \llbracket p \rrbracket_{\mathfrak{M}} &= V(p) \\ \llbracket \mathbf{t} \rrbracket_{\mathfrak{M}} &= L \\ \llbracket \otimes(\varphi_1, \dots, \varphi_n) \rrbracket_{\mathfrak{M}} &= \otimes^{\mathfrak{F}} (\llbracket \varphi_1 \rrbracket_{\mathfrak{M}}, \dots, \llbracket \varphi_n \rrbracket_{\mathfrak{M}}) \end{aligned}$$

Let satisfaction in an L -model \mathfrak{M} – denoted $\models_{\subseteq} (\mathbf{BMC} \times S) \times \mathfrak{L}_{\text{Rel}}$ – be defined so that $\mathfrak{M}, s \models \varphi$ iff $s \in \llbracket \varphi \rrbracket_{\mathfrak{M}}$.

Definition 2.9 (Validity, Consequence). *For a given L -model \mathfrak{M} and a formula $\varphi \in \mathfrak{L}_{\text{Rel}}$, let φ be valid in \mathfrak{M} – denoted $\mathfrak{M} \models \varphi$ – iff $\mathfrak{M}, s \models \varphi$ for all $s \in L$. Let φ be valid in a relevant frame \mathfrak{F} – denoted $\mathfrak{F} \models \varphi$ – iff $\mathfrak{M} \models \varphi$ for all \mathfrak{M} based on \mathfrak{F} . For a given class of L -models \mathbf{M} (frames \mathbf{F}), let φ be valid in \mathbf{M} (\mathbf{F}) – denoted $\mathbf{M} \models \varphi$ ($\mathbf{F} \models \varphi$) – iff $\mathfrak{M} \models \varphi$ ($\mathfrak{F} \models \varphi$) for all $\mathfrak{M} \in \mathbf{M}$ ($\mathfrak{F} \in \mathbf{F}$). Let φ be a consequence of Γ in an L -model \mathfrak{M} – denoted $\Gamma \models^{\mathfrak{M}} \varphi$ – iff for all $s \in S$ such that $\mathfrak{M}, s \models \gamma$ for all $\gamma \in \Gamma$, $\mathfrak{M}, s \models \varphi$. For a given class of models \mathbf{M} , let φ be a consequence of Γ in \mathbf{M} – denoted $\Gamma \models^{\mathbf{M}} \varphi$ – iff $\Gamma \models^{\mathfrak{M}} \varphi$ for all $\mathfrak{M} \in \mathbf{M}$.*

The reader should wait until Part [III](#) for a more profound epistemological explanation of the semantics. As in Part [I](#) I will be concerned with laying down the theoretical framework, here I will only explain technical role played by each component of models for relevant modal logic. To anticipate, \square will play the role of the salient epistemic modality, and \square_L is conferred mainly a technical role, although an epistemic interpretation can be attached to it (cf. Section [5.2.3](#)). In what follows, whenever $\square_{(L)} = \square$ ($\square_{(L)} = \square_L$), $Q_{(L)}$ is assumed to take the value Q (Q_L).

L -models can be conceived of as, so to speak, “generalised” Kripke models for classical modal logic.² This is for at least four reasons.

1. *Partial order.* Models for relevant logic consist of a partially ordered set of information states, ordered by the amount of information they contain (support). In particular, $s \leq t$ means that t contains at least as much information as s . Accordingly, propositions are defined as sets of information states closed upward with respect to the information ordering, i.e. as UCLA propositions

²Recall that a Kripke model for classical modal logic is just a triple (W, Q, V) consisting of a set of possible worlds W , a binary relation $Q \subseteq W^2$ interpreting \square, \diamond , and a propositional valuation $V : At \rightarrow \mathcal{P}(S)$.

[154], in the sense specified by Lemma 2.1. This is a generalisation with respect to Kripke models for classical modal logic, which can be seen as ordered structures where \leq is the identity relation.

2. *Modalities.* Models for relevant logic consider logical connectives $\neg, \otimes, \oplus, \rightarrow, \leftarrow$ as modal operators, with their distinctive accessibility relations $*$ and R . Accessibility relations are responsible, thanks to their point shifting behaviour, for the intensional nature of $\neg, \otimes, \oplus, \rightarrow$, which contrasts the extensional Boolean nature of Boolean negation, \wedge, \vee and the material conditional. This is a generalisation with respect to Kripke models, in that a modal treatment is extended from \Box, \Diamond to a wider range of intensional connectives, underwriting the thesis that a connective with arity n is analysed via an $n + 1$ -ary accessibility relation.³
3. *Validity.* Given the prominent interest in an axiomatic study of relevant logic, the key semantic notion to parallel theoremhood is that of validity. Relevant models then generalise Kripke models, where validity is defined as truth in all possible worlds, in that validity is defined with respect to a subset of the domain, i.e. the set of logical states L . Therefore, while semantic consequence is defined classically, validity is defined non-classically.

Elements of the domain S are seen as bodies of information – which I will call indifferently *situations* or *information states*. Information states contain exactly the pieces of information corresponding to the formulas which they semantically support. Following [70, 117], an information state can then be interpreted as a fixed agent’s theory, supporting all the formulas which are contained in the theory. At the model theoretic level, theories are taken as primitive terms. However, as it often happens, there are specific syntactic models, i.e. canonical models (cf. Definition 2.13), which provide special insights into the nature of theories. To anticipate, from a syntactic point of view a theory is a set of sentences with some internal coherence. For example, by Definition 2.11 theories are closed under logical implications, while they may fail to be closed under relevant implication, as $\varphi \rightarrow \psi \in s$ and $\varphi \in s$ needs not imply that $\psi \in s$. Information states, or relevant theories, are not closed under the usual laws of classical logic, thus generalising possible worlds, and the relations used to define the operations corresponding to $\neg, \rightarrow, \otimes, \oplus$ and \Box are introduced to achieve this.

³Compared to e.g. Kripke models for intuitionistic logic, relevant models further liberate the two-place accessibility for implication into a three-place relation.

1. *Routley star.* The unary operation $*$ is the “Routley star”, mapping each state s to its maximally compatible state, i.e. the state which is maximal with respect to the information order \leq among those states that do not support the negation of any formula supported by s . In other words, s^* does not verify anything that s falsifies. By means of $*$, Routley–Meyer models make it possible that information states support inconsistent and incomplete information – since in general $s \not\leq s^*$ and $s^* \not\leq s$ respectively. Moreover, negation is not involutive, i.e. $\varphi \rightarrow \neg\neg\varphi$ and $\neg\neg\varphi \rightarrow \varphi$ are not valid – if $s \not\leq s^{**}$ and $s^{**} \not\leq s$, respectively. Note that the satisfaction clause for negation by means of $*$ clarifies in what sense \neg is understood as a point shifting operator.
2. *The ternary relation.* The Routley–Meyer relation R is usually seen as representing *combination* of information supported by situations. For example, Dunn and Restall point out that “perhaps the best reading [of $Rstu$] is to say that the combination of the pieces of information s and t (not necessarily the union) is a piece of information in u ” [54, p. 67]. Hence, $Rstu$ can be seen as representing the fact that the body of information that results from combining the context information supported by s with the input information supported by t is contained in the output information supported by u .
3. *Modal accessibility.* The Kripke accessibility relation $Q_{(L)}$ associates an information state s with the proposition consisting of all the information modally compatible with s . That is, by unraveling the definition of $\Box_{(L)}^{\mathfrak{S}}$, we obtain that $s \models \Box_{(L)}\varphi$ iff φ is satisfied at all states t such that, according to the information contained in s , it is possible that the information contained in t holds. The specific interpretation of modal compatibility and possibility depends on the reading given to \Box and \Box_L . Since the propositional basis is relevant, some of the principles valid in classical Kripke models fail to be valid in L–models, most notably the necessitation rule ($\Box_{(L)}\mathbf{N}$) $\varphi \Rightarrow \Box_{(L)}\varphi$ (as $L \neq S$) and the K–axiom ($\Box_{(L)}\mathbf{K}$) $\Box_{(L)}(\varphi \rightarrow \psi) \rightarrow (\Box_{(L)}\varphi \rightarrow \Box_{(L)}\psi)$.

Focusing on the reading of R as information combination leads to an intensional reading of \otimes , \oplus and \rightarrow :

1. *Fusion.* By unraveling the definition of $\otimes^{\mathfrak{S}}$, we obtain:

$$\mathfrak{M}, s \models \varphi \otimes \psi \iff \exists t, u \in S(Rtus, \mathfrak{M}, t \models \varphi \text{ and } \mathfrak{M}, u \models \psi).$$

That is, $s \models \varphi \otimes \psi$ iff s is obtained by combination of a φ –state with a ψ –state. Hence, fusion behaves much like a conjunction, except that the conjoined

propositions are supported at possibly distinct information states, one serving as context and the other as input state. This enables the failure of the equivalents of standard lattice properties of \wedge , such as $\varphi \otimes \psi \rightarrow \varphi$ (if there are t, u such that $Rstu$ and $s \not\leq u$). Moreover, seen as an operation on proposition, \otimes^{δ} is not a semilattice operator – in particular it is neither idempotent, nor commutative nor associative. Finally, note that in models for **R** and stronger relevant logics, $\varphi \otimes \psi$ is equivalent to $\neg(\varphi \rightarrow \neg\psi)$.

2. *Fission*. By unraveling the definition of \oplus , we obtain:

$$\mathfrak{M}, s \models \varphi \oplus \psi \iff \forall t, u \in S(Rstu \implies \mathfrak{M}, t^* \models \varphi \text{ or } \mathfrak{M}, u \models \psi).$$

That is, $s \models \varphi \oplus \psi$ iff s is obtained by combination of states such that either the first is not a $\neg\varphi$ -state or the second is a ψ -state. Hence, fission behaves much like a disjunction, except that the disjoined propositions are evaluated sequentially in distinct positions, as input and output information, linked to s through R . This enables the failure of the intensional equivalents of standard lattice properties of \vee , such as $\varphi \rightarrow \varphi \oplus \psi$ (if $\exists t, u(Rstu, s \not\leq t^*)$). A parallel treatment of disjunctive information is provided by models for **DW** and stronger relevant logics.

3. *Implications*. By unraveling the definition of \rightarrow^{δ} , we obtain:

$$\mathfrak{M}, s \models \varphi \rightarrow \psi \iff \forall t, u \in S(Rstu, \mathfrak{M}, t \models \varphi \implies \mathfrak{M}, u \models \psi).$$

That is, $s \models \varphi \rightarrow \psi$ iff combining s with any φ -state results in a ψ -state. $\mathfrak{M}, s \models \varphi \leftarrow \psi$ is read in a similar way, with s featuring as the input of information combination. Interpretation of implication in terms of a ternary relation R enables the failure of the paradoxes of material implication, such as $\varphi \rightarrow (\psi \rightarrow \varphi)$ (if $\exists t, u(Rstu, s \not\leq u)$); and the paradoxes of strict implication, such as $\varphi \wedge \neg\varphi \rightarrow \psi$ (if $s \not\leq s^*$), $\varphi \rightarrow (\psi \vee \neg\psi)$ (if $s^* \not\leq s$) and $\varphi \rightarrow (\psi \rightarrow \psi)$ (if $\exists t, u(Rstu, t \not\leq u)$).

To conclude the presentation of **L**-models, note that the combinatorial reading of R motivates the plausibility of Conditions **(L1)** and **(L2)**: according to **(L1)**, there is always a logical information state such that its combination with s does not produce new information, while according to **(L2)** combining a logical information state with s does not result in loss of information. Together, Conditions **(L1)**–**(L2)** are sufficient for the validity of Lemma **2.2**, according to which an implication is valid iff the corresponding entailment holds. That is, logical states can be seen as encapsulating

informational links from premises to conclusions of relevantly valid implications. As stressed by [120, p.52], no “deep philosophical motivation for the distinction between logical and non-logical situations” is claimed other than the desirability of Lemma 2.2 for a satisfactory model theory of implication. In Chapter 5 I will try to make the case for a deeper connection between logical states and information processing.

Before introducing the Hilbert system for the basic relevant modal logic **BMC**, the following two classic results are crucial preliminaries in proving that **BMC** is sound with respect to the class of L -models **BMC**.

Lemma 2.1 (Hereditiy). *For all $\mathfrak{M} \in \mathbf{BMC}$ and all $\varphi \in \mathfrak{L}_{\text{Rel}}$: $\varphi \in S(\uparrow)$.*

Proof. The result is established by induction on the structure of φ , where it is assumed for some arbitrary $s, t \in S$ that $\mathfrak{M}, s \models \varphi$ and $s \leq t$. The base case $\varphi \in At$ holds by definition of V .

$\varphi := \mathbf{t}$. By the assumption that $s \models \mathbf{t}$ we infer that $s \in L$, which by $L \in S(\uparrow)$ and $s \leq t$ implies $t \in L$. We conclude that $t \models \mathbf{t}$.

$\varphi := \neg\psi$. By the assumption that $s \models \neg\psi$ we infer that $s^* \not\models \psi$. By $s \leq t$ and $* \in S(\uparrow, S(\downarrow))$ we infer that $t^* \leq s^*$. Hence, by the induction hypothesis we infer that $t^* \not\models \psi$. We conclude that $t \models \neg\psi$.

$\varphi := \psi \wedge \chi$. By the assumption that $s \models \psi \wedge \chi$ we infer that $s \models \psi$ and $s \models \chi$. Hence, by the induction hypothesis we infer that $t \models \psi$ and $t \models \chi$. We conclude that $t \models \psi \wedge \chi$. The case $\varphi := \psi \vee \chi$ is established similarly.

$\varphi := \psi \otimes \chi$. By the assumption that $s \models \psi \otimes \chi$ we infer that there are u, v such that $Ruvs$, $u \models \psi$ and $v \models \chi$. By $R \in S(\downarrow\uparrow)$, $Ruvs$ and $s \leq t$ we infer that Rwt . Hence, by $u \models \psi$ and $v \models \chi$ we conclude that $t \models \psi \otimes \chi$.

$\varphi := \psi \rightarrow \chi$. Assume for some arbitrary $u, v \in S$ that $Rtuv$ and $u \models \psi$ to show that $v \models \chi$. By $R \in S(\downarrow\uparrow)$, $Rtuv$ and $s \leq t$ we infer that Rsu . By the assumption that $s \models \psi \rightarrow \chi$ and $t \models \psi$ we conclude that $v \models \chi$. The case $\varphi := \psi \leftarrow \chi$ is established similarly, using the fact that $Rutv$ and $s \leq t$ implies $Rusv$.

$\varphi := \Box_{(L)}\psi$. Assume for some arbitrary $s, t, u \in S$ that $s \models \Box_{(L)}\psi$, that $s \leq t$ and that $Q_{(L)}tu$. By $Q_{(L)} \in S(\downarrow\uparrow)$ and $s \leq t$ we infer that $Q_{(L)}su$. By $s \models \Box_{(L)}\psi$ and $Q_{(L)}su$ we infer that $u \models \psi$. We conclude that $t \models \Box_{(L)}\psi$.

□

Lemma 2.2 (Verification). *For all $\mathfrak{M} \in \mathbf{BMC}$: $\mathfrak{M} \models \varphi \rightarrow \psi \iff [\varphi]_{\mathfrak{M}} \subseteq [\psi]_{\mathfrak{M}}$.*

Proof. (\implies). Assume $\models \varphi \rightarrow \psi$. Assume also for some arbitrary $s \in S$ that $s \models \varphi$ to show that $s \models \psi$. By (L1) we infer that there is $l \in L$ such that $Rlss$. By $\models \varphi \rightarrow \psi$, $Rlss$ and $s \models \varphi$ we conclude that $s \models \psi$. (\impliedby). Assume $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$. Assume also for some arbitrary $l \in L$ and $s, t \in S$ that $Rlst$ and $s \models \varphi$ to show that $t \models \psi$. By $l \in L$, $Rlst$ and (L2) we infer that $s \leq t$. Hence, by $s \models \varphi$ and Lemma 2.1 we infer that $t \models \psi$. We conclude by $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$ that $t \models \psi$. \square

Remark 2.1 (Relevance and Paraconsistency). To see why the existence of inconsistent situations can be used to motivate relevance, consider disjunctive syllogism $(\varphi \vee \psi) \wedge \neg\varphi \rightarrow \psi$. Even if $\varphi \vee \psi$ holds one the ground of, say, φ holding, it might well be the case that $\neg\varphi$ holds too. Belnap's motto to *keep data clean* prescribes us to keep the inconsistency local, i.e. to conclude only what is already conceded, e.g. φ , rather than the stronger step of inference to ψ . Belnap's motto is especially on point when viewed with epistemic lenses. According to [17, 19], relevant logic can be used to formalise the inferential behaviour of (artificial) agents, who are often faced with contradictory and incomplete information. Such agents, upon receiving deficient information, should not pollute their data by spreading contradictions everywhere in their database, but they should keep the inconsistencies local. In light of this consideration, theories may well be inconsistent yet non-trivial, contrary to what is prescribed by the validity of $\varphi \wedge \neg\varphi \rightarrow \psi$ – which entails the validity of disjunctive syllogism.

2.1.3 Axiomatisation of BMC

With the adequate definitions from Section 2.1 in play, let me turn to the axiomatisation of the basic modal relevant logic characterised by relational semantics, BMC.

Definition 2.10 (BMC axiom system). *Let the Hilbert system for the basic relevant propositional logic BMC consist of the axioms and rules of inference of Figure 2.2.*

BMC is defined in two steps, by (i) providing an Hilbert system for the basic relevant propositional logic BM ((BMC1)–(BMC8) and (BMC10)–(BMC16)), and (ii) by supplementing it with the distinctive modal principles ((BMC9) and (BMC17)).

BM contains basic principles like modus ponens (BMC10), adjunction (BMC11), the usual introduction and elimination principles for extensional conjunction and disjunction (BMC2)–(BMC5) and the contraposition rule (BMC13). These principles seem intuitively entrenched in the inferential meanings of \rightarrow , \wedge , \vee , \neg . As for the remaining principles, some remarks are in order.

(BMC1)	$\varphi \rightarrow \varphi$
(BMC2)	$\varphi \wedge \psi \rightarrow \varphi(\psi)$
(BMC3)	$(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi \wedge \chi)$
(BMC4)	$(\psi)\varphi \rightarrow \varphi \vee \psi$
(BMC5)	$(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)$
(BMC6)	$(\varphi \wedge (\psi \vee \chi)) \rightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
(BMC7)	$\neg(\varphi \wedge \psi) \rightarrow (\neg\varphi \vee \neg\psi)$
(BMC8)	$(\neg\varphi \wedge \neg\psi) \rightarrow \neg(\varphi \vee \psi)$
(BMC9)	$\Box_{(L)}\varphi \wedge \Box_{(L)}\psi \rightarrow \Box_{(L)}(\varphi \wedge \psi)$
(BMC10)	$\varphi, \varphi \rightarrow \psi \Rightarrow \psi$
(BMC11)	$\varphi, \psi \Rightarrow \varphi \wedge \psi$
(BMC12)	$\chi \rightarrow \varphi, \psi \rightarrow \xi \Rightarrow (\varphi \rightarrow \psi) \rightarrow (\chi \rightarrow \xi)$
(BMC13)	$\varphi \rightarrow \psi \Rightarrow \neg\psi \rightarrow \neg\varphi$
(BMC14)	$\varphi \Leftarrow \mathbf{t} \rightarrow \varphi$
(BMC15)	$\varphi \rightarrow (\psi \rightarrow \chi) \Leftarrow \psi \rightarrow (\chi \leftarrow \varphi)$
(BMC16)	$\varphi \rightarrow (\psi \rightarrow \chi) \Leftarrow (\varphi \otimes \psi) \rightarrow \chi$
(BMC17)	$\varphi \rightarrow \psi \Rightarrow \Box_{(L)}\varphi \rightarrow \Box_{(L)}\psi$

Figure 2.2: Hilbert system for BMC.

1. *Negation.* (BMC7), (BMC8) and (BMC13) are the only principles characterising \neg . As a result, negation in **BM** is very weak. However, both weaker and stronger negations are viable options in non-classical logics – cf. [47, 55, 91] for a full taxonomy of negations. In particular, note that De Morgan negation results from adding to **BM** $\varphi \leftrightarrow \neg\neg\varphi$ and that sub-intuitionistic relevant logics are obtained by removing (BMC7) from **BM**.
2. *Entailment and implication.* It is easy to see that entailment in **BMC**, as the converse of deducibility, is a tarskian consequence relation. This contrasts the fact that implication as a connective is reflexive but neither transitive nor monotonic, e.g. $((\varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)$ neither $(\varphi \rightarrow (\psi \rightarrow \varphi))$ are theorems. As such, relevant logic pairs a classical theory of entailment with a non-classical theory of implication.
3. *Intensional truth constant and logical truth.* (BMC14) provides the characteristic rule of inference governing **t**. (BMC14) explains in what sense relevant logic

adopts a non-classical theory of logical truth: by (BMC14) φ is logically true iff $\mathbf{t} \rightarrow \varphi$ is logically true. Therefore, the logical truth of φ is distinguished from the classical notion – entailment from no assumptions (or \top entailing φ) – in favour of the intensional reading $\mathbf{t} \vdash \varphi$ (\mathbf{t} entailing φ).

4. *Residuation.* [124] discovered that (BMC16), the algebraic counterpart of introduction and elimination conditions for \rightarrow , can be conservatively added to **BM**. In the jargon of gaggle theory [28], (BMC15) and (BMC16) characterise $\otimes, \rightarrow, \leftarrow$ as a residuated triple. Note that relevant \rightarrow (\leftarrow) is the right (left) residual of \otimes , in that disentangling the classical and intuitionistic residuation law, according to which intuitionistic and material \rightarrow is the (left and right) residual of \wedge .
5. *Conjunctive regularity.* Since the basic principles of normal modal logics, i.e. $(\Box_{(L)}\mathbf{N})$ and $(\Box_{(L)}\mathbf{K})$, do not hold in all L -models, the basic modal logic characterised by relational semantics cannot be a normal modal logic. Indeed, the distinctive principles for $\Box_{(L)}$ in **BMC** are the conjunction distributivity axiom (BMC9) and the monotonicity rule (BMC17). These principles, more suggestively called $(\Box_{(L)}\mathbf{C})$ and $(\Box_{(L)}\mathbf{M})$ characterise $\Box_{(L)}$ as a *conjunctively regular* modality.⁴

I will now turn to the task of showing that the theorems, i.e. the provable formulas, of **BMC** coincide with the formulas valid in the class of all L -models **BMC**. As usual, soundness is the easy part.

Theorem 2.1 (BMC soundness). $\vdash_{\mathbf{BMC}} \varphi \implies \mathbf{BMC} \models \varphi$.

Proof. The result is established by induction on the length of **BMC**-derivations. Lemma 2.2 is used implicitly throughout the proof. The validity of (BMC1), (BMC2) and (BMC4) holds trivially by $\llbracket \varphi \rrbracket \subseteq \llbracket \varphi \rrbracket$, $\llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket \subseteq \llbracket \varphi \rrbracket$ and $\llbracket \varphi \rrbracket \subseteq \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket$, respectively. (BMC10), (BMC11) and (BMC14) trivially preserve validity. The remaining cases are established as follows.

⁴Note that by defining $\diamond\varphi := \neg\Box\neg\varphi$, \diamond is the dual of \Box , but $\Box\varphi \leftrightarrow \neg\diamond\neg\varphi$ is not a theorem of **BMC**. Presently, this peculiarity is not problematic, since I will be mainly concerned with the epistemic properties of \Box . However, the concerned reader may note that there are a number of viable strategies to restore the duality properties of \Box and \diamond in a relevant setting, either by working with **B** – which has double negation laws – or by defining \diamond as a primitive and imposing some coordination principles between \Box and \diamond 's accessibility relation (cf. [179]). Similar considerations apply to the reader who finds inelegant the modeling of $\otimes, \oplus, \rightarrow$ by means of a single accessibility relation. The concerned reader is referred to [73], where a general treatment of the basic relevant logic featuring is undertaken, in which \otimes, \oplus are analysed by distinct accessibility relations than \rightarrow .

(BMC3). Assume for some arbitrary $s \in S$ that $s \models (\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)$. Assume also for some arbitrary $t, u \in S$ that $Rstu$ and $t \models \varphi$ to show $u \models \psi \wedge \chi$. By $s \models (\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)$ we infer that $s \models \varphi \rightarrow \psi$ and $s \models \varphi \rightarrow \chi$, which implies by $Rstu$ and $t \models \varphi$ that $u \models \psi$ and $u \models \chi$. We conclude that $u \models \psi \wedge \chi$. The validity of (BMC5) is established similarly.

(BMC6). Assume for some arbitrary $s \in S$ that $s \models \varphi \wedge (\psi \vee \chi)$. Hence, $s \models \varphi$ and either $s \models \psi$ or $s \models \chi$. If $s \models \psi$ ($s \models \chi$), we infer that $s \models \varphi \wedge \psi$ ($s \models \varphi \wedge \chi$). In both cases, we conclude that $s \models (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$.

(BMC7). The following chain of equivalences holds: $s \models \neg(\varphi \wedge \psi)$ iff $s^* \not\models \varphi \wedge \psi$ iff $s^* \not\models \varphi$ or $s^* \not\models \psi$ iff $s \models \neg\varphi$ or $s \models \neg\psi$ iff $s \models \neg\varphi \vee \neg\psi$. The validity of (BMC8) is established similarly.

(BMC9) Assume for some arbitrary $s \in S$ that $s \models \Box_{(L)}\varphi \wedge \Box_{(L)}\psi$. Assume also for some arbitrary $t \in S$ that $Q_{(L)}st$. By $s \models \Box_{(L)}\varphi \wedge \Box_{(L)}\psi$ we infer that $s \models \Box_{(L)}\varphi$ and $s \models \Box_{(L)}\psi$. By $s \models \Box_{(L)}\varphi$ and $Q_{(L)}st$ we infer that $t \models \varphi$, and by $s \models \Box_{(L)}\psi$ and $Q_{(L)}st$ we infer that $t \models \psi$. We conclude that $t \models \varphi \wedge \psi$.

(BMC12). Assume that $\llbracket \chi \rrbracket \subseteq \llbracket \varphi \rrbracket$ and $\llbracket \psi \rrbracket \subseteq \llbracket \xi \rrbracket$. Assume also for some arbitrary $s \in S$ that $s \models \varphi \rightarrow \psi$ to show $s \models \chi \rightarrow \xi$. Hence, assume for some arbitrary $t, u \in S$ that $Rstu$ and $t \models \chi$ to show $u \models \xi$. By $t \models \chi$ and $\llbracket \chi \rrbracket \subseteq \llbracket \varphi \rrbracket$ we infer that $t \models \varphi$, which by $s \models \varphi \rightarrow \psi$ and $Rstu$ implies that $u \models \psi$. We conclude by $\llbracket \psi \rrbracket \subseteq \llbracket \xi \rrbracket$ that $u \models \xi$.

(BMC13). Assume by the induction hypothesis that $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$ and, by contradiction, that there is $s \in S$ such that $s \models \neg\psi$ and $s \not\models \neg\varphi$. By $s \not\models \neg\varphi$ we infer that $s^* \models \varphi$, which by $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$ implies that $s^* \models \psi$. We conclude that $s \not\models \neg\psi$, which is a contradiction.

(BMC15). Assume $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rightarrow \chi \rrbracket$. Assume also for some arbitrary $s \in S$ that $s \models \psi$ to show that $s \models \chi \leftarrow \varphi$. Hence, assume for some arbitrary $t, u \in S$ that $Rtsu$ and $t \models \varphi$ to show that $u \models \chi$. By $t \models \varphi$ and $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rightarrow \chi \rrbracket$ we infer that $t \models \psi \rightarrow \chi$. We conclude by $Rtsu$ and $s \models \psi$ that $u \models \chi$. The other direction is established similarly.

(BMC16). Assume $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rightarrow \chi \rrbracket$. Assume also for some arbitrary $s \in S$ that $s \models \varphi \otimes \psi$ to show that $s \models \chi$. By $s \models \varphi \otimes \psi$ we infer that there are $t, u \in S$ such that $Rtus$, $t \models \varphi$ and $u \models \psi$. By $t \models \varphi$ and $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rightarrow \chi \rrbracket$ we infer that $t \models \psi \rightarrow \chi$. We conclude by $Rtus$ and $u \models \psi$ that $s \models \chi$. The other direction is established similarly.

(BMC17) Assume $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$. Assume also for some arbitrary $s \in S$ that $s \models \Box_{(L)}\varphi$ and for some arbitrary $t \in S$ that $Q_{(L)}st$. By $s \models \Box_{(L)}\varphi$ and $Q_{(L)}st$ we infer that $t \models \varphi$. By $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$ we conclude that $t \models \psi$.

□

In order to establish the converse of soundness, completeness, I will adopt the standard method of constructing a canonical model which provides a countermodel for every non-theorem. The usual Henkin method used in classical logic applies, with slight modifications, in the substructural setting: information states in the canonical model apt for a Hilbert system \mathbf{L} are syntactic \mathbf{L} -theories, defined as follows. In the following definitions I assume that the signature of \mathfrak{L} contains at least \wedge, \vee, \neg .

Definition 2.11 (Theories). *For a given Hilbert system \mathbf{L} , let $\Gamma \subseteq \mathfrak{L}$ be an \mathbf{L} -theory iff Γ is*

- *Closed under \mathbf{L} -deducibility: $\varphi \in \Gamma$ and $\varphi \vdash_{\mathbf{L}} \psi$ implies $\psi \in \Gamma$;*
- *Closed under adjunction: $\varphi, \psi \in \Gamma$ only if $\varphi \wedge \psi \in \Gamma$.*

Let an \mathbf{L} -theory Γ be:

- *Consistent iff for all $\varphi \in \mathfrak{L}$ Γ does not contain both φ and $\neg\varphi$;*
- *Maximal iff Γ is consistent and all $\Delta \supset \Gamma$ are not consistent;*
- *Proper iff $\Gamma \neq \mathfrak{L}$;*
- *Regular iff $\mathbf{L} = \{\varphi \mid \vdash_{\mathbf{L}} \varphi\} \subseteq \Gamma$;*
- *Prime iff for all $\varphi, \psi \in \mathfrak{L}$ $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

The kind of theories Γ needed in the canonical model for the completeness of **BMC** are structured so as to respect the extensional connectives: $\varphi \wedge \psi \in \Gamma$ iff $\varphi, \psi \in \Gamma$ and $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$. That is, **BMC**-theories need to be prime.

Just like in classical logic it is possible to construct maximal consistent theories from consistent theories through the Lindenbaum lemma, in relevant logic it is possible for a given relevant logic \mathbf{L} to construct prime theories from \mathbf{L} -pairs, defined below, through the Pair Extension lemma. Given the technical importance of the Pair Extension lemma, I will detour a bit and present in full its proof, adapted from [153]. The result is stated with respect to an arbitrary distributive lattice logic \mathbf{L} , i.e. any axiom system (i) whose language contains \wedge, \vee and (ii) in which the following conditions hold:

- (E1). $\varphi \vdash_{\mathbf{L}} \varphi$;
- (E2). $\varphi \wedge \psi \vdash_{\mathbf{L}} \varphi(\psi)$;
- (E3). $\varphi \vdash_{\mathbf{L}} \psi, \varphi \vdash_{\mathbf{L}} \chi \implies \varphi \vdash_{\mathbf{L}} \psi \wedge \chi$;
- (E4). $\varphi \vdash_{\mathbf{L}} \psi, \chi \vdash_{\mathbf{L}} \psi \implies \varphi \vee \chi \vdash_{\mathbf{L}} \psi$;
- (E5). $(\psi)\varphi \vdash_{\mathbf{L}} \varphi \vee \psi$;
- (E6). $\varphi \wedge (\psi \vee \chi) \vdash_{\mathbf{L}} (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$;
- (E7). $\varphi \vdash_{\mathbf{L}} \psi, \psi \vdash_{\mathbf{L}} \chi \implies \varphi \vdash_{\mathbf{L}} \chi$.

It is easy to check that all axiom systems of the present dissertation satisfy conditions (i) and (ii).

Definition 2.12 (Independent L-pair). *Let $\langle \Gamma, \Delta \rangle \in \mathcal{P}(\mathfrak{L})^2$ be an independent L-pair iff there are no $\varphi_1, \dots, \varphi_n \in \Gamma$ and $\psi_1, \dots, \psi_m \in \Delta$ such that $\bigwedge_{i \leq n} \varphi_i \vdash_{\mathfrak{L}} \bigvee_{j \leq m} \psi_j$.*

Lemma 2.3 (Pair extension). *For all pairs $\langle \Gamma, \Delta \rangle \subseteq \mathfrak{L}^2$, if $\langle \Gamma, \Delta \rangle$ is an independent L-pair, then there is a prime L-theory s such that $\Gamma \subseteq s$ and $s \cap \Delta = \emptyset$.*

Proof. Item 1 is established as follows. Assume $\langle \Gamma, \Delta \rangle$ is an independent L-pair. Take an enumeration $\varphi_1, \dots, \varphi_n$ of the formulas of \mathfrak{L} . If $\Gamma = \emptyset$ or $\Delta = \emptyset$, the result is trivial by letting $s = \emptyset$, $s = \mathfrak{L}$ respectively. If both Γ and Δ are non-empty, we reason as follows. Let $\langle \Gamma_i, \Delta_i \rangle$ be recursively defined as follows:

$$\begin{aligned} \langle \Gamma_0, \Delta_0 \rangle &= \langle \Gamma, \Delta \rangle \\ \langle \Gamma_{i+1}, \Delta_{i+1} \rangle &= \begin{cases} \langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle & \text{if } \langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle \text{ is an independent L-pair} \\ \langle \Gamma_i, \Delta_i \cup \{\varphi_i\} \rangle & \text{otherwise.} \end{cases} \end{aligned}$$

Then, note that for every $\varphi \in \mathfrak{L}$ and every independent L-pair $\langle \Gamma, \Delta \rangle$, either $\langle \Gamma \cup \{\varphi\}, \Delta \rangle$ or $\langle \Gamma, \Delta \cup \{\varphi\} \rangle$ is an independent L-pair. For otherwise there would be some ψ_1, ψ_2 from Γ and χ_1, χ_2 from Δ such that $\psi_1 \wedge \varphi \vdash_{\mathfrak{L}} \chi_1$ and $\psi_2 \vdash_{\mathfrak{L}} \varphi \vee \chi_2$. We would then be able to derive $\psi_1 \wedge \psi_2 \vdash_{\mathfrak{L}} \chi_1 \vee \chi_2$, contradicting the fact that $\langle \Gamma, \Delta \rangle$ is an independent L-pair:

- | | |
|---|----------------|
| (1) $\psi_1 \wedge \varphi \vdash_{\mathfrak{L}} \chi_1$ | Ass. |
| (2) $\psi_2 \vdash_{\mathfrak{L}} \varphi \vee \chi_2$ | Ass. |
| (3) $\psi_1 \wedge \varphi \vdash_{\mathfrak{L}} \chi_1 \vee \chi_2$ | (E5) (E7) (1) |
| (4) $(\psi_1 \wedge \varphi) \vee \chi_2 \vdash_{\mathfrak{L}} \chi_1 \vee \chi_2$ | (E4)(E5) (3) |
| (5) $\psi_1 \wedge \psi_2 \vdash_{\mathfrak{L}} \varphi \vee \chi_2$ | (E2)(E7) (2) |
| (6) $\psi_1 \wedge \psi_2 \vdash_{\mathfrak{L}} \psi_1 \wedge (\varphi \vee \chi_2)$ | (E2)(E3) (5) |
| (7) $\psi_1 \wedge (\varphi \vee \chi_2) \vdash_{\mathfrak{L}} (\psi_1 \wedge \varphi) \vee \chi_2$ | (E2) (E4)–(E7) |
| (8) $\psi_1 \wedge \psi_2 \vdash_{\mathfrak{L}} (\psi_1 \wedge \varphi) \vee \chi_2$ | (E7) (6,7) |
| (9) $\psi_1 \wedge \psi_2 \vdash_{\mathfrak{L}} \chi_1 \vee \chi_2$ | (E7) (4,8) |

As a result, if $\langle \Gamma_i, \Delta_i \rangle$ in the above construction is an independent L-pair, then also $\langle \Gamma_{i+1}, \Delta_{i+1} \rangle$ is an independent L-pair. It follows by induction on i that every

$\langle \Gamma_i, \Delta_i \rangle$ is an independent L-pair. Hence, $\langle s, t \rangle = \langle \bigcup_i \Gamma_i, \bigcup_i \Delta_i \rangle$ is an independent L-pair such that $s \cup t = \mathfrak{L}$.

Hence, $\Delta \cap s = \emptyset$, since $\langle s, t \rangle$ is independent and $s \supseteq \Gamma$ by construction. It remains to be shown that s is a prime L-theory. If, by contradiction, s is not closed under L-implications, then there is $\varphi \in s$ such that $\varphi \vdash_L \psi$ and $\psi \notin s$. Then since $s \cup t = \mathfrak{L}$ we infer that $\psi \in t$, which together with $\varphi \vdash_L \psi$ implies that $\langle s, t \rangle$ is not independent, which is a contradiction. To show that s is closed under adjunction, assume for some arbitrary $\varphi, \psi \in \mathfrak{L}$ that $\varphi, \psi \in s$. By (E1) we infer that $\varphi \wedge \psi \vdash_L \varphi \wedge \psi$. by $\varphi, \psi \in s$ and the fact that $\langle s, t \rangle$ is independent we infer that $\varphi \wedge \psi \notin t$. We conclude by $s \cup t = \mathfrak{L}$ that $\varphi \wedge \psi \in s$. If, by contradiction, s is not prime, then there are $\varphi, \psi \in \mathfrak{L}$ such that $\varphi \vee \psi \in s$ and $\varphi, \psi \notin s$. Then since $s \cup t = \mathfrak{L}$ we infer that $\varphi, \psi \in t$, which together with $\varphi \vee \psi \vdash_L \varphi \vee \psi$ and $\varphi \vee \psi \in s$ implies that $\langle s, t \rangle$ is not independent, which is a contradiction. \square

The proof of completeness of BMC with respect to L-models is obtained by showing that the canonical BMC-model $\mathfrak{M}_{\text{BMC}}^c$ is indeed an L-model such that φ is satisfied at s exactly when φ is contained in s . Note that BMC satisfies (E1)–(E7) by (BMC1)–(BMC6) and (BMC12), since $\varphi \vdash_{\text{BMC}} \psi$ is defined as $\vdash_{\text{BMC}} \varphi \rightarrow \psi$. Hence, Lemma 2.3 can be applied to construct prime BMC-theories, which turns out crucial in the completeness proof. Again, I remark that superscripts like c are dropped whenever the context allows for it.

Definition 2.13 (Canonical BMC-model). *Let the canonical BMC-model be a tuple of the following form:*

$$\mathfrak{M}_{\text{BMC}}^c = (S^c, L^c, \leq^c, *^c, R^c, Q^c, Q_L^c, V^c)$$

- $S^c = \{s \mid s \text{ is a prime BMC-theory over } \mathfrak{L}_{\text{Rel}}\};$
- $L^c = \{s \in S^c \mid \mathbf{t} \in s\};$
- $\leq^c = \{(s, t) \in (S^c)^2 \mid s \subseteq t\};$
- $s^{*^c} = \{\varphi \in \mathfrak{L}_{\text{Rel}} \mid \neg \varphi \notin s\};$
- $R^c = \{(s, t, u) \in (S^c)^3 \mid \forall \varphi, \psi \in \mathfrak{L}_{\text{Rel}} (\varphi \rightarrow \psi \in s, \varphi \in t \implies \psi \in u)\};$
- $Q_{(L)}^c = \{(s, t) \in (S^c)^2 \mid \forall \varphi \in \mathfrak{L}_{\text{Rel}} (\Box_{(L)} \varphi \in s \implies \varphi \in t)\};$
- $V^c(p) = \{s \in S^c \mid p \in s\}.$

Lemma 2.4 (BMC-Modelhood). $\mathfrak{M}_{\text{BMC}}^c \in \mathbf{BMC}$.

Proof. The result is established by showing: (i) that (S, \leq) is a partially ordered set such that $\emptyset \neq L \subseteq S$; (ii) that each component of $\mathfrak{M}_{\text{BMC}}^c$ $L, *, R, Q, Q_L, V$ satisfies the corresponding $\uparrow \downarrow$ properties; and (iii) that $\mathfrak{M}_{\text{BMC}}^c$ satisfies the required frame conditions. Note that $L \subseteq S$ holds by definition of L , and that by definition of \subseteq, \leq is a partial order and $L, V(p) \in S(\uparrow)$.

$L \neq \emptyset$. Consider the BMC–theory $\text{BMC} = \{\varphi \mid \vdash_{\text{BMC}} \varphi\}$. Then, for some arbitrary $\varphi \in \mathfrak{L}_{\text{Rel}}$ such that $\not\vdash_{\text{BMC}} \varphi$, take the pair $(\text{BMC}, \{\varphi\})$, which is a BMC–independent pair. By Lemma 2.3 we infer that there is a prime BMC–theory $l \supseteq \text{BMC}$ such that $\varphi \notin l$. By (BMC1) we infer that $\mathbf{t} \rightarrow \mathbf{t} \in l$, which implies by (BMC14) that $\mathbf{t} \in l$, by which we conclude that $l \in L$.

$s^* \in S$. To show that s^* is closed under BMC–deducibility, assume for some arbitrary $\varphi \in s^*$ that $\varphi \vdash_{\text{BMC}} \psi$. By definition of $*$ we infer that $\neg\varphi \notin s$. By $\varphi \vdash_{\text{BMC}} \psi$ and (BMC13) we infer that $\neg\psi \vdash_{\text{BMC}} \neg\varphi$. Hence, $\neg\psi \notin s$. We conclude by definition of $*$ that $\psi \in s^*$. To show that s^* is closed under adjunction, assume for some arbitrary $\varphi, \psi \in \mathfrak{L}_{\text{Rel}}$ that $\varphi, \psi \in s^*$. By definition of $*$ we infer that $\neg\varphi, \neg\psi \notin s$, which by primeness of s implies that $\neg\varphi \vee \neg\psi \notin s$. By (BMC7) we infer that $\neg(\varphi \wedge \psi) \notin s$, by which we conclude that $\varphi \wedge \psi \in s^*$. To show that s^* is prime, assume by contradiction for some arbitrary $\varphi, \psi \notin s^*$ that $\varphi \vee \psi \in s^*$. By definition of $*$ we infer that $\neg\varphi, \neg\psi \in s$, which implies that $\neg\varphi \wedge \neg\psi \in s$. By (BMC8) we infer that $\neg(\varphi \vee \psi) \in s$. We conclude by definition of $*$ that $\varphi \vee \psi \notin s^*$, which is a contradiction.

$* \in S(\uparrow, S(\downarrow))$. It suffices to show that $s \leq t$ implies $t^* \leq s^*$. Assume for some arbitrary $s, t \in S$ that $s \leq t$. Assume also for some arbitrary $\varphi \in \mathfrak{L}_{\text{Rel}}$ that $\varphi \in t^*$. Hence, by definition of $*$ we infer that $\neg\varphi \notin t$, which by $s \leq t$ implies $\neg\varphi \notin s$. We conclude again by definition of $*$ that $\varphi \in s^*$.

$R \in S(\downarrow\downarrow\uparrow)$. Assume for some arbitrary $s, t, u, s', t', u' \in S$ that $Rstu, s' \leq s, t' \leq t$ and $u \leq u'$. Assume also for some arbitrary $\varphi, \psi \in \mathfrak{L}_{\text{Rel}}$ that $\varphi \rightarrow \psi \in s'$ and $\varphi \in t'$. By $s' \leq s$ and $t' \leq t$ we infer that $\varphi \rightarrow \psi \in s$ and $\varphi \in t$, which by definition of R and $Rstu$ implies that $\psi \in u$. We conclude by $u \leq u'$ that $\psi \in u'$.

$Q_{(L)} \in S(\uparrow\downarrow)$. Assume for some arbitrary s, t, s', t' that $Q_{(L)}st, s' \leq s$ and $t \leq t'$. Assume also for some arbitrary $\varphi \in \mathfrak{L}_{\text{Rel}}$ that $\Box_{(L)}\varphi \in s'$. By $s' \leq s$ we infer that $\Box_{(L)}\varphi \in s$, which by definition of $Q_{(L)}$ and $Q_{(L)}st$ implies that $\varphi \in t$. We conclude by $t \leq t'$ that $\varphi \in t'$.

(L1). Take some arbitrary $s \in S$. In order to show that there is $l \in L$ such that $Rlss$, consider the pair $\langle \{\mathbf{t}\}, \{\varphi \rightarrow \psi \mid \varphi \in s, \psi \notin s\} \rangle$, which is an independent BMC–pair. Otherwise, there would be $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_n \in \mathfrak{L}_{\text{Rel}}$ such that $\vdash_{\text{BMC}} \bigvee_{i \leq n} (\varphi_i \rightarrow \psi_i)$, $\varphi_i \in s$ and $\psi_i \notin s$ for all $i \leq n$. Hence, $\bigwedge_{i \leq n} \varphi_i \in s$ and $\bigvee_{i \leq n} \psi_i \notin s$ by Definition 2.11. Then we would reason as follows:⁵

$$\implies \quad \vdash_{\text{BMC}} \bigwedge_{i \leq n} \varphi_i \rightarrow \bigvee_{i \leq n} \psi_i \quad \text{by } \vdash_{\text{BMC}} \bigvee_{i \leq n} (\varphi_i \rightarrow \psi_i) \rightarrow (\bigwedge_{i \leq n} \varphi_i \rightarrow \bigvee_{i \leq n} \psi_i)$$

⁵Note that the first step relies also on (BMC10) and that $\vdash_{\text{BMC}} \bigvee_{i \leq n} (\varphi_i \rightarrow \psi_i) \rightarrow (\bigwedge_{i \leq n} \varphi_i \rightarrow \bigvee_{i \leq n} \psi_i)$ holds by (BMC2, (BMC4) and (BMC12).

$$\begin{aligned}
&\implies \bigvee_{i \leq n} \psi \in s && \text{by } \bigwedge_{i \leq n} \varphi \in s \\
&\implies \exists i \leq n (\psi_i \in s) && \text{by primeness of } s
\end{aligned}$$

which would contradict $\bigvee_{i \leq n} \psi \notin s$. Hence, by Lemma [2.3](#) we infer that there is a prime BMC–theory l such that $\mathbf{t} \in l$ and $l \cap \{\varphi \rightarrow \psi \mid \varphi \in s, \psi \notin s\} = \emptyset$. By $\mathbf{t} \in l$ we infer that $l \in L$. By $l \cap \{\varphi \rightarrow \psi \mid \varphi \in s, \psi \notin s\} = \emptyset$ and the definition of R we conclude that $Rlss$.

[\(L2\)](#). Assume for some arbitrary $s, t \in S$ and $l \in L$ that $Rlst$. Assume also that for some arbitrary $\varphi \in \mathfrak{L}_{\text{Rel}}$, $\varphi \in s$. By (BMC1) and the fact that every $l \in L$ is a regular BMC–theory, we infer that $\varphi \rightarrow \varphi \in l$. We conclude by $Rlst$, $\varphi \rightarrow \varphi \in s$ and $\varphi \in s$ that $\varphi \in t$. □

Before stating the truth lemma, it will be useful to note the following fact, stating two equivalent formulations of R^c .

Lemma 2.5 (Equivalent R). *The following facts hold about $\mathfrak{M}_{\text{BMC}}^c$:*

1. R^cstu iff $\forall \varphi, \psi \in \mathfrak{L}_{\text{Rel}} (\varphi \in s, \psi \in t \implies \varphi \otimes \psi \in u)$;
2. R^cstu iff $\forall \varphi, \psi \in \mathfrak{L}_{\text{Rel}} (\psi \in s, \varphi \leftarrow \psi \in t \implies \varphi \in u)$.

Proof. We prove the items separately.

1. (\implies). By contradiction, assume for some arbitrary $s, t, u \in S$ that $Rstu$ and that there are $\varphi, \psi \in \mathfrak{L}_{\text{Rel}}$ such that $\varphi \in s$, $\psi \in t$ and $\varphi \otimes \psi \notin u$. By (BMC1) and (BMC16) we infer that $\vdash_{\text{BMC}} \varphi \rightarrow (\psi \rightarrow (\varphi \otimes \psi))$, which implies by $\varphi \in s$ that $\psi \rightarrow (\varphi \otimes \psi) \in s$. We conclude by $\psi \in t$ and $Rstu$ that $\varphi \otimes \psi \in u$, which is a contradiction. (\impliedby). Assume for some arbitrary $s, t, u \in S$ and $\varphi, \psi \in \mathfrak{L}_{\text{BMC}}$ that $\varphi \otimes \psi \in u$ if $\varphi \in s$ and $\psi \in t$. Assume also that $\varphi \rightarrow \psi \in s$ and $\varphi \in t$. By $\varphi \rightarrow \psi \in s$ and $\varphi \in t$ we infer that $(\varphi \rightarrow \psi) \otimes \varphi \in u$. By (BMC1) and (BMC16) we infer that $\vdash_{\text{BMC}} ((\varphi \rightarrow \psi) \otimes \varphi) \rightarrow \psi$. We conclude by $(\varphi \rightarrow \psi) \otimes \varphi \in u$ that $\psi \in u$.
2. (\implies). Assume for some arbitrary $s, t, u \in S$ and $\varphi, \psi \in \mathfrak{L}_{\text{Rel}}$ that $Rstu$ and that $\psi \in s$, $\varphi \leftarrow \psi \in t$. By (BMC1) and (BMC15) we infer that $\vdash_{\text{BMC}} \psi \rightarrow ((\varphi \leftarrow \psi) \rightarrow \varphi)$, which implies by $\psi \in s$ that $(\varphi \leftarrow \psi) \rightarrow \varphi \in s$. By $Rstu$ and $\varphi \leftarrow \psi \in t$ we conclude that $\varphi \in u$. (\impliedby). Assume for some arbitrary $s, t, u \in S$ and $\varphi, \psi \in \mathfrak{L}_{\text{Rel}}$ that $\varphi \in u$ if $\psi \in s$ and $\varphi \leftarrow \psi \in t$. Assume also that $\varphi \rightarrow \psi \in s$ and $\varphi \in t$. By (BMC1) and (BMC15) we infer that $\vdash_{\text{BMC}} \varphi \rightarrow (\psi \leftarrow (\varphi \rightarrow \psi))$. By $\varphi \in t$ we infer that $\psi \leftarrow (\varphi \rightarrow \psi) \in t$. We conclude by $\varphi \rightarrow \psi \in s$ that $\psi \in u$.

□

Lemma 2.6 (BMC–Truth). $\mathfrak{M}_{\text{BMC}}^c, s \models \varphi \iff \varphi \in s$.

Proof. The result is established by induction on the structure of φ . The base case holds by definition of V . The induction step is as follows.

$\varphi := \neg\psi$. The following chain of equivalences holds: $s \models \neg\psi$ iff $s^* \not\models \psi$ iff (by the induction hypothesis) $\psi \notin s^*$ iff (by definition of $*$) $\neg\psi \in s$.

$\varphi := \psi \wedge \chi$. The following chain of equivalences holds: $s \models \psi \wedge \chi$ iff $s \models \psi$ and $s \models \chi$ iff (by the induction hypothesis) $\psi, \chi \in s$ iff (by s being a BMC–theory) $\psi \wedge \chi \in s$.

The case $\varphi := \psi \vee \chi$ is established similarly.

$\varphi := \psi \rightarrow \chi$. (\implies). By contraposition, assume for some arbitrary $s \in S$ that $\psi \rightarrow \chi \notin s$. Then, consider the pair $\langle \{\xi \mid \psi \rightarrow \xi \in s\}, \{\chi\} \rangle$, which is an independent BMC–pair. Otherwise, there would be $\xi_1, \dots, \xi_n \in \mathfrak{L}_{\text{Rel}}$ such that $\psi \rightarrow \xi_i \in s$ for all $i \leq n$ and $\vdash_{\text{BMC}} \bigwedge_{i \leq n} \xi_i \rightarrow \chi$. By $\psi \rightarrow \xi_i \in s$ for all $i \leq n$ and (BMC3) we would infer that $\psi \rightarrow \bigwedge_{i \leq n} \xi_i \in s$. By $\vdash_{\text{BMC}} \bigwedge_{i \leq n} \xi_i \rightarrow \chi$ and (BMC12) we would conclude that $\psi \rightarrow \chi \in s$, which is a contradiction. Hence, by Lemma 2.3 we infer that there is a prime BMC–theory $u \supseteq \{\xi \mid \psi \rightarrow \xi \in s\}$ such that $\chi \notin u$. Then consider the pair $\langle \{\psi\}, \{\xi \mid \exists \delta (\xi \rightarrow \delta \in s, \delta \notin u)\} \rangle$, which is an independent BMC–pair. Otherwise, there would be some $\xi_1, \dots, \xi_n \in \mathfrak{L}_{\text{Rel}}$ such that $\vdash_{\text{BMC}} \psi \rightarrow \bigvee_{i \leq n} \xi_i$ and for all $i \leq n$ there is δ_i such that $\xi_i \rightarrow \delta_i \in s$ and $\delta_i \notin u$. Then we would reason as follows:⁶

$$\begin{aligned}
\implies & \bigwedge_{i \leq n} (\xi_i \rightarrow \delta_i) \in s && \text{by } \xi_i \rightarrow \delta_i \in s \text{ for all } i \leq n \\
\implies & \bigvee_{i \leq n} \xi_i \rightarrow \bigvee_{i \leq n} \delta_i \in s && \text{by } \vdash_{\text{BMC}} \bigwedge_{i \leq n} (\xi_i \rightarrow \delta_i) \rightarrow (\bigvee_{i \leq n} \xi_i \rightarrow \bigvee_{i \leq n} \delta_i) \\
\implies & \psi \rightarrow \bigvee_{i \leq n} \delta_i \in s && \text{by } \vdash_{\text{BMC}} \psi \rightarrow \bigvee_{i \leq n} \xi_i \\
\implies & \bigvee_{i \leq n} \delta_i \in u && \text{by construction of } u \\
\implies & \delta_i \in u \text{ for some } i \leq n && \text{by } u \text{ being prime}
\end{aligned}$$

which is a contradiction. Hence, by Lemma 2.3 we infer that there is a prime BMC–theory t such that $\psi \in t$ and $t \cap \{\xi \mid \exists \delta (\xi \rightarrow \delta \in s, \delta \notin u)\} = \emptyset$. We now check that $Rstu$. Otherwise, there would be $\psi, \chi \in \mathfrak{L}_{\text{Rel}}$ such that $\psi \rightarrow \chi \in s$, $\psi \in t$ and

⁶Note that $\vdash_{\text{BMC}} \bigwedge_{i \leq n} (\xi_i \rightarrow \delta_i) \rightarrow (\bigvee_{i \leq n} \xi_i \rightarrow \bigvee_{i \leq n} \delta_i)$ holds by (BMC4), (BMC5) and (BMC12).

$\chi \notin u$. By $\psi \in t$ and $t \cap \{\xi \mid \exists \delta(\xi \rightarrow \delta \in s, \delta \notin u)\} = \emptyset$ we would infer that there is no $\delta \in \mathfrak{L}_{\text{Rel}}$ such that $\psi \rightarrow \delta \in s$ and $\delta \notin u$. By $\chi \notin u$ we would conclude that $\psi \rightarrow \chi \notin s$, which is a contradiction. To sum up, there are two prime BMC-theories, t and u , such that $Rstu$, $\psi \in t$ and $\chi \notin u$. We conclude by the induction hypothesis that $s \not\models \psi \rightarrow \chi$. (\Leftarrow). Assume that $\psi \rightarrow \chi \in s$. Assume also for some arbitrary $t, u \in S$ that $Rstu$ and $t \models \psi$. By the induction hypothesis we infer that $\psi \in t$. By $Rstu$, $\psi \rightarrow \chi \in s$ and the definition of R we infer that $\chi \in u$. We conclude by the induction hypothesis that $u \models \chi$. The cases $\varphi := \psi \otimes \chi$ and $\varphi := \psi \leftarrow \chi$ are established similarly, by using Lemma 2.5 to construct the required prime theories. $\varphi := \Box_{(L)}\psi$. (\Rightarrow). By contraposition, assume $\Box_{(L)}\psi \notin s$. Then, consider the pair $(\{\chi \mid \Box_{(L)}\chi \in s\}, \{\psi\})$, which is a BMC-independent pair. Otherwise, there would be χ_1, \dots, χ_n such that $\Box_{(L)}\chi_i \in s$ for all $i \leq n$ and $\vdash_{\text{BMC}} \bigwedge_{i \leq n} \chi_i \rightarrow \psi$. By $\Box_{(L)}\chi_i \in s$ for all $i \leq n$ we would infer that $\bigwedge_{i \leq n} \Box_{(L)}\chi_i \in s$. By $\vdash_{\text{BMC}} \bigwedge_{i \leq n} \chi_i \rightarrow \psi$ and $(\Box_{(L)}\mathbf{C})$ – $(\Box_{(L)}\mathbf{M})$ we would infer that $\vdash_{\text{BMC}} \bigwedge_{i \leq n} \Box_{(L)}\chi_i \rightarrow \Box_{(L)}\psi$, which together with $\bigwedge_{i \leq n} \Box_{(L)}\chi_i \in s$ would imply $\Box_{(L)}\psi \in s$, which is a contradiction. Hence, by Lemma 2.3 we infer that there is a prime BMC-theory $t \supseteq \{\chi \mid \Box_{(L)}\chi \in s\}$ such that $\psi \notin t$. By construction of t , t is non-empty and $Q_{(L)}st$. By the induction hypothesis and $\psi \notin t$ we infer that $t \not\models \psi$, by which we conclude that $s \not\models \Box_{(L)}\psi$. (\Leftarrow). Assume that $\Box_{(L)}\psi \in s$. Assume also for some arbitrary $t \in S$ that $Q_{(L)}st$. By $\Box_{(L)}\psi \in s$ and $Q_{(L)}st$ we infer that $\psi \in t$, which by the induction hypothesis implies that $t \models \psi$. We conclude that $s \models \Box_{(L)}\psi$. □

Theorem 2.2 (BMC Completeness). $\mathbf{BMC} \models \varphi \implies \vdash_{\text{BMC}} \varphi$.

Proof. By contraposition, assume that for some arbitrary $\varphi \in \mathfrak{L}_{\text{Rel}}$ that $\not\vdash_{\text{BMC}} \varphi$. Then, consider the pair $(\{\mathbf{t}\}, \{\varphi\})$. By $\not\vdash_{\text{BMC}} \varphi$ and (BMC14) we infer that $\mathbf{t} \not\vdash_{\text{BMC}} \varphi$, which implies that the pair is an independent BMC-pair. By Lemma 2.3 we infer that there is a prime BMC-theory l such that $\mathbf{t} \in l$ and $\varphi \notin l$. By $\mathbf{t} \in l$ and Definition 2.13 we infer that $l \in L^c$. By Lemma 2.4 M_{BMC}^c is an L-model, which implies by $\varphi \notin l$ and Lemma 2.6 that $\mathfrak{M}_{\text{BMC}}^c, l \not\models \varphi$. We infer that there is $\mathfrak{M} \in \mathbf{BMC}$, namely $\mathfrak{M}_{\text{BMC}}^c$, such that $\mathfrak{M} \not\models \varphi$. We conclude that $\mathbf{BMC} \not\models \varphi$. □

2.2 Extensions of BMC

The family of relevant modal logics I will consider is defined modularly via extensions of the basic relevant modal logic BMC.

Frame condition	Axiom/rule
(DNI) $s \leq s^{**}$	$\varphi \rightarrow \neg\neg\varphi$
(DNE) $s^{**} \leq s$	$\neg\neg\varphi \rightarrow \varphi$
(X) $s \in L \implies s^* \leq s$	$\varphi \vee \neg\varphi$
(RD) Rss^*s	$(\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi$
(CP) $Rstu \implies Rsu^*t^*$	$(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$
(WB) $Rstu \implies Rs(st)u$	$((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)$
(B) $R(st)uv \implies Rs(tu)v$	$(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$
(CB) $R(st)uv \implies Rt(su)v$	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(W) $Rstu \implies R(st)tu$	$(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$
(C) $R(st)uv \implies R(su)tv$	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$
(M) $Rstu \implies (s \leq u \vee t \leq u)$	$\varphi \rightarrow (\varphi \rightarrow \varphi)$
(ER) $\forall s \exists l (l \in L, Rsls)$	$\varphi \Rrightarrow (\varphi \rightarrow \psi) \rightarrow \psi$
(□N) $s \in L, Qst \implies t \in L$	$\varphi \Rrightarrow \Box\varphi$
(□K) $RQstu \implies \exists v (QtvQRsvu)$	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(□T) Qss	$\Box\varphi \rightarrow \varphi$
(□D) $\forall s \exists t (Qst, Qs^*t^*)$	$\Box\neg\varphi \rightarrow \neg\Box\varphi$
(□4) $Qst, Qtu \implies Qsu$	$\Box\varphi \rightarrow \Box\Box\varphi$
(□5) $Qst, Qs^*u \implies Qt^*u$	$\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$

Figure 2.3: \mathbf{L} -conditions with the corresponding axioms and rules.

2.2.1 Propositional and Modal Extensions

The axiom system \mathbf{BM} constitutes the weakest propositional logic characterised by relational Routley–Meyer semantics. An axiom system \mathbf{L}' is an *extension* of an axiom system \mathbf{L} iff all axioms of \mathbf{L} are axioms of \mathbf{L}' and all inference rules of \mathbf{L} are inference rules of \mathbf{L}' . Extensions of \mathbf{BM} can then be obtained by adding axioms and inference rules to \mathbf{BM} .

Many historically important logics in the relevant family were introduced with independent motivations, each claiming to capture salient aspects of \rightarrow – cf. also [4, 166]. For this reason, even if the general level of analysis employs allows to consider extensions of \mathbf{BM} in a piecemeal fashion, it will be instructive to present here some notable extensions of \mathbf{BM} . The propositional axioms and rules I shall consider are shown in Table 2.3, where I adopt the convention of using the same label, when no confusion arises, for axioms/rules and their corresponding frame conditions (in a sense to be specified). Note that, since I will be interested in capturing the properties of the epistemic modality \Box , I will only consider modal extensions of \mathbf{BMC} involving \Box , and adopt naming conventions accordingly.

Definition 2.14 (Notable relevant propositional systems). *Let the propositional extensions of \mathbf{BM} over $\mathfrak{L}_{\text{Prop}}$ be defined by Table 2.4, where each line connects the axiom*

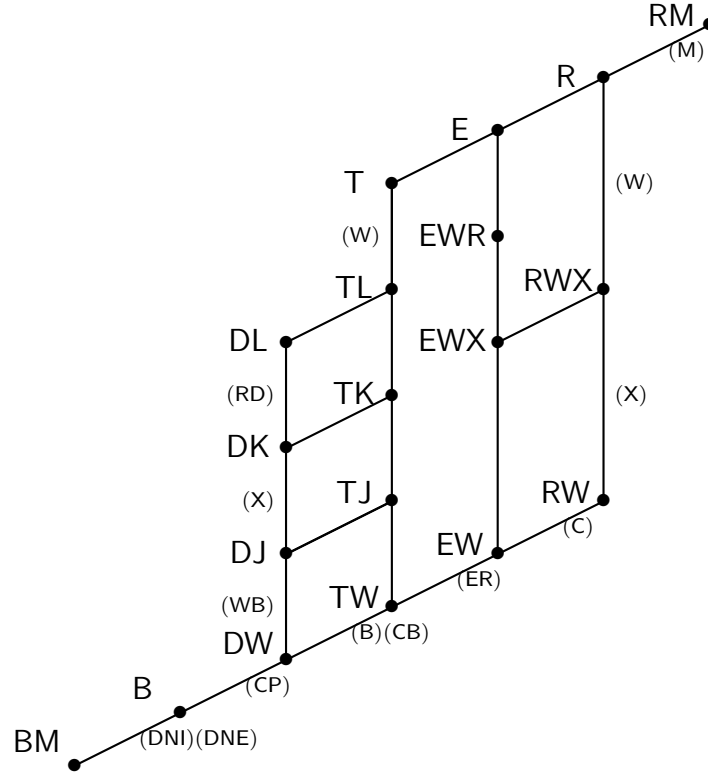


Figure 2.4: Notable propositional extensions of BM.

system below to its extension obtained by adding the principles labelling the line.

The axiom systems considered contain some redundancies – the interested reader is referred to [30, 166] for a more extensive discussion of relevant axiom systems. Hence, I will adopt the convention of denoting as L' the smallest variant L' of L such that L' has the same axioms as L and all inference rules of L are admissible in L' (note that the rules included in the definition of a system are trivially admissible in the system). Note that (BMC13) is admissible in any extension L of BMC containing (CP); (BMC12) is admissible in any L containing (B) and (CB); and (ER) is admissible in any L containing (C). Hence, in line with our convention, (BMC13) will not be considered as a basic rule of DW (and its extensions) and (BMC12) will not be considered as a basic rule of TW (and its extensions). Of course, adding the weakening axiom $\varphi \rightarrow (\psi \rightarrow \varphi)$ to RM results in classical propositional logic CPC.

As proved in Section 2.1, the axiom system BMC is the basic logic characterising relational models for relevant modal logic. Modal extensions are obtained by adding

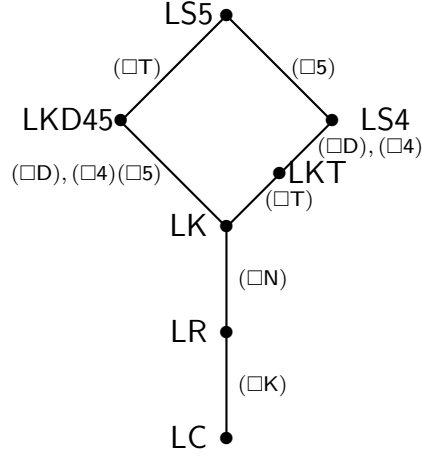


Figure 2.5: Notable modal extensions of LC.

axioms and rules of inference to BMC. The modal axioms I consider are shown in Table 2.3, where again I use the same label for axioms/rules and their corresponding frame conditions. Moreover, I use in the present chapter the convention of using L as a metavariable denoting some propositional extension of BM.

Definition 2.15 (Notable relevant modal systems). *Let the modal extensions of LC over $\mathfrak{L}_{\text{Rel}}$ be defined by Table 2.5, where L denotes a relevant propositional logic of Table 2.4.*

The extensions of BMC are characterised, in a sense specified below, by L -frames satisfying additional frame conditions. Indeed, L -models are structures with extremely weak constraints, as nothing is assumed besides the monotonicity $\uparrow\downarrow$ -conditions on $*$, R and Q . At the propositional level, a number of further conditions on L -frames can be assumed in order to characterise stronger properties of information combination and compatibility. The plausibility of these conditions is examined in Chapter 5 from an epistemological standpoint. One may require L -models to satisfy tighter conditions not just on $*$ and R , but also on the modal accessibility relation Q . In the frame conditions presented in Table 2.3, it is assumed that $R(st)uv := \exists x(Rstx, Rxuv)$, $Rs(tu)v := \exists x(Rtux, Rsvx)$ and $RQstu := \exists x(Rstx, Qxu)$ and $QRstu := \exists x(Qsx, Rxtu)$.

2.2.2 Characterisation Results

The soundness of the above propositional and modal extensions of BMC can be established by considering additional frame conditions on one side, and axioms and rules of inference on the other side. It is then sufficient to show that each additional frame conditions make the corresponding axiom/rule sound (cf. Lemma 2.7), and that each axiom/rule makes the corresponding frame condition satisfied by the corresponding canonical model (cf. Lemma 2.8).

Definition 2.16 (L -notation). *Let any selection of the frame conditions considered in Section 2.2.1 be denoted as L -conditions. Let L denote the set of formulas valid in all L -frames satisfying a given selection of L -conditions. Let an L -model (L -frame) denote any L -model (L -frame) which satisfies a given selection of L -conditions. Let L denote the extension of BMC obtained by adding the axioms and rules of inference with the same label of the frame conditions of a given selection of L -conditions.*

Then, the above propositional and modal axioms and inference rules correspond to the L -conditions with the same label in the sense specified by Theorem 2.3

Lemma 2.7 (L -soundness). *For all L -frames \mathfrak{F} , \mathfrak{F} satisfies the L -conditions only if the corresponding axioms and rules are valid in \mathfrak{F} .*

Proof. The result is established by the same argument as [71, Theorems 4.2, 8.2], where we only show the case concerning $(\Box D)$, where an incorrect frame condition is taken to correspond to $\Box\neg\varphi \rightarrow \neg\Box\varphi$.⁷ By contradiction, assume also that $\mathfrak{M} \not\models \Box\neg\varphi \rightarrow \neg\Box\varphi$ for some $\mathfrak{M} \in L$ such that $\forall s\exists t(Qst, Qs^*t^*)$ holds in \mathfrak{M} . By Lemma 2.2 we infer that there is $s \in S$ such that $\mathfrak{M}, s \models \Box\neg\varphi$ and $\mathfrak{M}, s \not\models \neg\Box\varphi$. By $\mathfrak{M}, s \models \Box\neg\varphi$ and the frame condition $(\Box D)$ we infer that there is $t \in S$ such that Qst and $\mathfrak{M}, t \models \neg\varphi$. By $\mathfrak{M}, s \not\models \neg\Box\varphi$ we infer that $\mathfrak{M}, s^* \models \Box\varphi$. Hence, by the frame condition $(\Box D)$ we infer that there is $t \in S$ such that Qs^*t^* and $\mathfrak{M}, t^* \models \varphi$. We conclude that $\mathfrak{M}, t \not\models \neg\varphi$, which is a contradiction. \square

Completeness is established by showing that the additional frame conditions hold in the canonical model for the corresponding extension of BMC. Note that the canonical L -model \mathfrak{M}_L^c , for L extension of BMC with the axioms and rules considered in Section 2.2.1, is obtained by uniformly substituting L to BMC in Definition 2.13.

Lemma 2.8 (L -canonicity). *Any selection of L -conditions holds in \mathfrak{M}_L^c .*

⁷In particular, [71]'s condition $\forall s\exists t(Qs^*t, Qst^*)$ corresponds to $\Box\varphi \rightarrow \neg\Box\neg\varphi$, and it is sufficient to prove that $\Box\neg\varphi \rightarrow \neg\Box\varphi$ is valid only if it is assumed that $t \leq t^{**}$.

Proof. It suffices to show that, for any \mathbf{L} obtained by adding an axiom or rule (X) to BMC, the canonical \mathbf{L} -model satisfies the frame condition (X). Again, I will show only the case concerning (\Box D) as an illustration of the procedure, referring the interested reader to [71, Theorems 4.2, 8.2]. Take an arbitrary \mathbf{L} -theory s , where \mathbf{L} contains the axiom (\Box D). Then, consider the pair $\langle \{\varphi \mid \Box\varphi \in s\}, \{\neg\psi \mid \Box\psi \in s^*\} \rangle$. The pair is \mathbf{L} -independent. Otherwise, there would be $\varphi_1, \dots, \varphi_n \in \mathfrak{L}_{\text{Rel}}$ and $\psi_1, \dots, \psi_m \in \mathfrak{L}_{\text{Rel}}$ such that $\vdash_{\mathbf{L}} \bigwedge_{i \leq n} \varphi_i \rightarrow \bigvee_{j \leq m} \neg\psi_j$, $\Box\varphi_1, \dots, \Box\varphi_n \in s$ and $\Box\psi_1, \dots, \Box\psi_m \in s^*$. Then we would reason as follows:

$$\begin{aligned}
&\Rightarrow \quad \vdash_{\mathbf{L}} \Box \bigwedge_{i \leq n} \varphi_i \rightarrow \Box \bigvee_{j \leq m} \neg\psi_j && \text{by (BMC17)} \\
&\Rightarrow \quad \Box \bigvee_{j \leq m} \neg\psi_j \in s && \text{by } \Box\varphi_1, \dots, \Box\varphi_n \in s \\
&\Rightarrow \quad \Box \neg \bigwedge_{j \leq m} \psi_j \in s && \text{by } \vdash_{\mathbf{L}} \bigvee_{j \leq m} \neg\psi_j \rightarrow \neg \bigwedge_{j \leq m} \psi_j \text{ and (BMC17)} \\
&\Rightarrow \quad \neg \Box \bigwedge_{j \leq m} \psi_j \in s && \text{by } (\Box\text{D}) \\
&\Rightarrow \quad \neg \bigwedge_{j \leq m} \Box\psi_j \in s && \text{by (BMC9) and (BMC13)} \\
&\Rightarrow \quad \bigvee_{j \leq m} \neg \Box\psi_j \in s && \text{by (BMC7)} \\
&\Rightarrow \quad \exists j \leq m (\neg \Box\psi_j \in s) && \text{by primeness of } s \\
&\Rightarrow \quad \exists j \leq m (\Box\psi_j \notin s^*)
\end{aligned}$$

which is a contradiction. Hence, by Lemma 2.3 there is an \mathbf{L} -theory $t \supseteq \{\varphi \mid \Box\varphi \in s\}$ and $t \cap \{\neg\psi \mid \Box\psi \in s^*\} = \emptyset$. By $t \supseteq \{\varphi \mid \Box\varphi \in s\}$ and the definition of Q we infer that Qst . By $t \cap \{\neg\psi \mid \Box\psi \in s^*\} = \emptyset$ we infer that $\psi \in t^*$ for all $\Box\psi \in s^*$. We conclude by definition of Q that Qs^*t^* . \square

With the above two lemmas, it is possible to prove the characterisation result, stating that the \mathbf{L} -conditions characterise \mathbf{L} in the following sense.

Theorem 2.3 (\mathbf{L} -Characterisation). *For any selection of \mathbf{L} -conditions: $\mathbf{L} \models \varphi \iff \vdash_{\mathbf{L}} \varphi$.*

Proof. One direction follows from Theorem 2.1 Lemma 2.7, while the other from Theorem 2.2 and Lemma 2.8. \square

2.3 History and Approaches to Relevant Logic

To conclude the chapter, let me outline some historical facts about relevant logic, which will serve as a partial justification for the material I decided to include and exclude from the dissertation. The history of relevant logics is a complicated one, and I will content myself with a brief overview of its salient events.

As is often the case in logical enterprises, we can distinguish a first proof-theoretic era, in which differently motivated deductive systems were proposed as capturing the notion of relevant consequence. According to the reconstruction of [45], axiom systems for relevant logics date back at least to [133]. Subsequently, the implicative fragment of one of the most known relevant system, R, was formulated independently in [127] and, as the weak positive implication calculus, in [37]. Finally, Ackermann's logic of entailment [1] was the starting point of Anderson and Belnap's investigations in [4].

A central event in the history of relevant logics was the development of a model-theoretic interpretation of relevant deductive systems. The first comprehensive algebraic semantics was devised in [52] for several logics – including R and Anderson and Belnap's favoured system of entailment E – and it was subsequently extended to cover more logics in the relevant family in [124]. Roughly around the same time (and place), frame semantics made its appearance. With notable predecessors in the semi-lattice semantics of [201]; the semantics of entailment of [70]; and the seminal series of articles [160, 161, 162], introducing ternary relation semantics; frame semantics culminated in [166] into what is now known as the Routley–Meyer semantics.

The standard practice in modal logic, after [105], is to employ set-theoretical semantics in the analysis of modal notions – such as necessity, belief and entailment. According to Kripke's semantics, sentences are mapped into propositions, in turn constituted by possible worlds, and connectives are interpreted as operations on propositions. Since sets of possible worlds contain more structure than points in algebraic models, possible worlds semantics for modal (and relevant) logics provide more information than algebraic semantics when it comes to the interpretation of logical properties. For this reason, frame semantics is particularly congenial to a presentation of relevant logic which is palatable to the philosophical logician.

On the other hand, the philosophical logician is often content to offer an axiomatic analysis of deductive systems. This is unfortunate for at least two reasons. For one, it is well known that axiomatic Hilbert-style calculi are far off from representing actual mathematical deductive reasoning, which famously led Gentzen to introduce natural deduction and sequent calculi (cf. [75]). For two, Hilbert systems lack of analyticity impedes a satisfactory assessment of the meta-logical properties of the calculi – such

as the subformula property, the existence of proof search procedures, interpolation theorems, etc. The drawbacks of Hilbert axiom systems are counterbalanced by their simplicity, which stems from their focus on the notion of theorem rather than that of consequence. Moreover, the formulation of the calculus syntactically mirrors in a perspicuous way the philosophical properties such calculus was devised for.

Since the present dissertation is on relevant logic from an epistemological perspective, it is sensible to delimit its scope by focusing on frame semantics and axiomatic presentations.

Chapter 3

Contextual Modal Logics based on Relational Semantics

In this chapter, I introduce the main idea of the dissertation, i.e. a family of modal logics combining classical propositional logic with some logic in the relevant modal family. To this aim, in Section 3.1 I show how to modify the Routley–Meyer semantics of relevant modal logics, so as to introduce possible worlds and classical validity. In Section 3.2 I introduce a family of Hilbert systems $C.L$, parametric on the relevant logic L at hand, and show a general soundness and completeness result.

3.1 $C.L$ –Models

I will introduce the technical machinery required to define the semantic structures of interest by introducing (i) *bounded L –models* and (ii) *possible worlds*. Before doing so, however, I will briefly recap the state of the art concerning frameworks combining classical and relevant logic.

Remark 3.1 (Related formalisms). The combination of relevant logic and classical logic has been studied in a number of papers in recent years. [123] adds a distinguished set of possible worlds in the context of Routley-Meyer semantics, where De Morgan negation behaves as Boolean negation. Their resulting logic is an extension of the classical relevant logic CR and of the classical modal logic $S4$. However, the language includes two negations. [111] puts forward an epistemic logic combining classical logic with the relevant logic of First Degree Entailment FDE . The propositional fragment of his logic is classical and the characteristic modal principle of the logic is monotonicity restricted to FDE . However, given the restriction to FDE ,

a sensible relevant conditional connective is lacking. Moreover, nesting of modal operators is not allowed in Levesque’s framework; the latter restriction is lifted in [106] (cf. Chapter 7 for a discussion). In [169, 171], combinations of classical propositional logic with relevant modal logic based on relational semantics are studied. The semantics of [169] is two-sorted and the Hilbert-style axiomatizations provided in the paper use a meta-rule of inference, which is inconvenient if the underlying relevant logic is undecidable, which is often the case. The semantics of [171] is one-sorted, thus more elegant, but the Hilbert-style axiomatizations provided in that paper have two peculiar features: proofs are defined in a non-standard way as *pairs* of finite sequences of formulas, and the completeness proof relies on \top, \perp .

3.1.1 Bounded L-Models

The first step in building **C.L**-models is the introduction of *bounds* in the context of **L**-models (recall, **L**-models satisfying the **L**-conditions considered in Table 2.3). Bounded structures, first introduced for relevant logic by [178, 179], consider two special states in **L**-models, 0 and 1, which are meant to represent the empty and the full information state, respectively. The introduction of bounds significantly complicates the models I shall be working with. Remark 3.3 at the end of the chapter explains the technical reason why it is necessary to resort to bounded models, and what are the alternatives.

Definition 3.1 (Bounded **L**-models). *Let an **L**-model \mathfrak{M} be bounded iff there are $0, 1 \in S$ such that for all $s, t \in S$ and all $p \in At$:*

$$0 \leq s \leq 1 \tag{B1}$$

$$0^* = 1 \tag{B2}$$

$$1^* = 0 \tag{B3}$$

$$Rst0 \implies (s = 0 \text{ or } t = 0) \tag{B4}$$

$$R111 \tag{B5}$$

$$R010 \tag{B6}$$

$$R1st \implies (s = 0 \text{ or } t = 1) \tag{B7}$$

$$R100 \tag{B8}$$

$$Rs1t \implies (s = 0 \text{ or } t = 1) \tag{B9}$$

$$Q_{(L)}00 \tag{B10}$$

$$Q_{(L)}1s \implies s = 1 \tag{B11}$$

$$0 \notin V(p) \tag{B12}$$

$$1 \in V(p) \tag{B13}$$

The language $\mathfrak{L}_{\text{Rel}}$ is interpreted in a uniform way by \mathbf{L} -models, bounded \mathbf{L} -models and $\mathbf{C.L}$ -models (to be introduced in Definition 3.4). The definition of satisfaction is recapitulated below.

Definition 3.2 ((Bounded) $\mathbf{C.L}$ -satisfaction). *Let the proposition expressed by an arbitrary formula $\varphi \in \mathfrak{L}_{\text{Rel}}$ in a (bounded) $\mathbf{C.L}$ -model \mathfrak{M} be recursively defined as follows, where $\otimes \in \{\neg, \wedge, \vee, \otimes, \rightarrow, \leftarrow, \square_{(L)}\}$ and n is the arity of \otimes :*

$$\begin{aligned} \llbracket p \rrbracket_{\mathfrak{M}} &= V(p) \\ \llbracket \mathbf{t} \rrbracket_{\mathfrak{M}} &= L \\ \llbracket \otimes(\varphi_1, \dots, \varphi_n) \rrbracket_{\mathfrak{M}} &= \otimes^{\mathfrak{S}}(\llbracket \varphi_1 \rrbracket_{\mathfrak{M}}, \dots, \llbracket \varphi_n \rrbracket_{\mathfrak{M}}) \end{aligned}$$

Let satisfaction in a (bounded) $\mathbf{C.L}$ -model \mathfrak{M} be defined so that $\mathfrak{M}, s \models \varphi$ iff $s \in \llbracket \varphi \rrbracket_{\mathfrak{M}}$.

Note that, in the case of bounded models, $1 \in \llbracket \mathbf{t} \rrbracket$ follows by $L \in S(\uparrow)$, $L \neq \emptyset$ and Condition (B1). Conditions (B1)–(B13) are technically necessary to ensure that 0 and 1 are indeed the empty and the full theory, satisfying no formula, all formulas respectively.

Lemma 3.1 (Empty–full). *For all bounded \mathbf{L} -models \mathfrak{M} and all $\varphi \in \mathfrak{L}_{\text{Rel}}$:*

1. $\mathfrak{M}, 0 \not\models \varphi$;
2. $\mathfrak{M}, 1 \models \varphi$.

Proof. The result is established by double induction on the structure of φ . The base case holds by Conditions (B12) and (B13), respectively. The induction step cases for $\varphi := \psi \wedge \chi$ and $\varphi := \psi \vee \chi$ are trivial.

$\varphi := \mathbf{t}$. By Definition 3.2 we infer that $1 \models \mathbf{t}$. Next, assume by contradiction that $0 \models \mathbf{t}$. Hence, $0^* \not\models \mathbf{t}$, which by Condition (B2) implies $1 \not\models \mathbf{t}$, contradicting $1 \models \mathbf{t}$.

$\varphi := \neg\psi$. The following chain of equivalences holds: $0 \not\models \neg\psi$ iff $0^* \models \psi$ iff (by Condition (B2)) $1 \models \psi$, which holds by the induction hypothesis. Next, $1 \models \neg\psi$ iff $1^* \not\models \psi$ iff (by Condition (B3)) $0 \not\models \psi$, which holds by the induction hypothesis.

$\varphi := \psi \otimes \chi$. Assume for some arbitrary $s, t \in S$ that $Rst0$. By Condition (B4), either $s = 0$ or $t = 0$. If the former (latter), by the induction hypothesis we infer that $s \not\models \psi$ ($s \not\models \chi$). We conclude that $0 \not\models \psi \otimes \chi$. Next, $1 \models \psi \otimes \chi$ since by Condition (B5) there are $s, t \in S$, namely $s = t = 1$, such that, by the induction hypothesis, $s \models \psi$ and $t \models \chi$.

$\varphi := \psi \rightarrow \chi$. $0 \not\models \psi \rightarrow \chi$ since by Condition (B6) there are $s, t \in S$, namely $s = 1$ and $t = 0$, such that, by the induction hypothesis, $s \models \psi$ and $t \not\models \chi$. Next, assume for some arbitrary $s, t \in S$ that $R1st$ and $s \models \psi$. By Condition (B7) we infer that either $s = 0$ or $t = 1$. If the former, we infer by the induction hypothesis that $s \not\models \psi$, which is a contradiction. Hence, $t = 1$, by which we conclude by the induction hypothesis that $t \models \chi$. The case $\varphi := \psi \leftarrow \chi$ is established similarly, by using Conditions (B8)–(B9).

$\varphi := \Box\psi$. $0 \not\models \Box\psi$ since by Condition (B10) there is $s \in S$, namely $s = 0$, such that, by the induction hypothesis, $s \not\models \psi$. Next, assume for some arbitrary $s \in S$ that $Q1s$. By Condition (B11) we infer that $s = 1$, by which we conclude by the induction hypothesis that $s \models \psi$. The case $\varphi := \Box_L\psi$ is established similarly.

□

3.1.2 Possible Worlds

The second step in building **C.L**-models is the definition of *possible worlds*, i.e. maximally consistent classical theories, within the context of models for relevant modal logics.

Definition 3.3 (Possible worlds). *For a given bounded **L**-model, let $w \in S$ be a possible world iff for all $s, t \in S$:*

$$w^* = w \tag{W1}$$

$$Rwww \tag{W2}$$

$$Rstw \implies (s = 0 \text{ or } t = 0 \text{ or } s \leq w) \tag{W3}$$

$$Rstw \implies (s = 0 \text{ or } t = 0 \text{ or } t \leq w) \tag{W4}$$

$$Rwst \implies (t = 1 \text{ or } s \leq w) \tag{W5}$$

$$Rwst \implies (s = 0 \text{ or } w \leq t) \tag{W6}$$

$$Rswt \implies (s = 0 \text{ or } w \leq t) \tag{W7}$$

$$Rswt \implies (t = 1 \text{ or } s \leq w) \tag{W8}$$

$$Q_L(W) = L \tag{W9}$$

(Recall that $Q_L(W) := \{s \in S \mid \exists w \in W(Q_Lws)\}$).

It is important to note that every possible world is a logical state, as proved in Proposition 3.1 below. However, the latter is provable only thanks to Condition (W8) (or, equivalently, (W3)), which is, in turn, only necessary for logics including

$\leftarrow (\otimes)$. Since in Part **II** I will consider logics in a restricted signature, I will in general assume that $W \subseteq L$, where W is the set of possible worlds, even if in some cases it turns out to be redundant, i.e. $W \subseteq L$ is provable from other conditions on models.

Proposition 3.1 (Logical worlds). *For all bounded models \mathfrak{M} and all possible worlds $w \in S$, $w \in L$.*

Proof. Take an arbitrary possible world $w \in S$. By Condition **(L1)**, we infer that there is $l \in L$ such that $Rlww$. By Condition **(W8)** we infer that $l \leq w$ or $w = 1$. Since the latter implies by Condition **(B1)** that $l \leq w$, we conclude in both cases that $w \in L$ by $L \in S(\uparrow)$. \square

Putting things together, **C.L**-models are defined as bounded **L**-models with a distinguished set of possible worlds W . Moreover, it is possible to define in **C.L**-models a generalised notion of validity, as truth preservation across a distinguished domain of states. As will be clear below, the privileged notion of validity I will employ considers truth preservation across possible worlds.

Definition 3.4 (**C.L**-models). *Let a **C.L**-model be a tuple of the following form:*

$$(S, L, W, \leq, *, R, Q, Q_L, V)$$

- $(S, L, \leq, *, R, Q, Q_L, V)$ is a bounded **L**-model;
- $W \subseteq L \subseteq S$ is a set of (not necessarily all) possible worlds.

Let $(S, L, W, \leq, *, R, Q, Q_L)$ be the **C.L**-frame a **C.L**-model $(S, L, W, \leq, *, R, Q, Q_L, V)$ is based on. Let **C.L** denote the class of all **C.L**-models, parametric on the corresponding class of **L**-models.

Definition 3.5 (Generalised validity). *For $X \in \{W, L, S\}$, let a formula φ be X -valid in a **C.L**-model \mathfrak{M} – denoted as $\mathfrak{M} \models^X \varphi$ – iff $\mathfrak{M}, s \models \varphi$ for all $s \in X$. Let φ be X -valid in a **C.L**-frame \mathfrak{F} – denoted as $\mathfrak{F} \models^X \varphi$ – iff $\mathfrak{M}, s \models \varphi$ for all **C.L**-models \mathfrak{M} based on \mathfrak{F} . For a given class of **C.L**-models \mathbf{M} (**C.L**-frames \mathbf{F}), let φ be X -valid in \mathbf{M} (\mathbf{F}) – denoted $\mathbf{M} \models^X \varphi$ ($\mathbf{F} \models^X \varphi$) – iff $\mathfrak{M} \models^X \varphi$ for all $\mathfrak{M} \in \mathbf{M}$ ($\mathfrak{F} \models^X \varphi$ for all $\mathfrak{F} \in \mathbf{F}$). Let φ be an X -consequence of Γ in a **C.L**-model \mathfrak{M} – denoted $\Gamma \models_{\mathfrak{M}}^X \varphi$ – iff for all $s \in X$ such that $\mathfrak{M}, s \models^X \gamma$ for all $\gamma \in \Gamma$, $\mathfrak{M}, s \models^X \varphi$. Let φ be an X -consequence of Γ in a class of **C.L**-models \mathbf{M} – denoted $\Gamma \models_{\mathbf{M}}^X \varphi$ – iff $\Gamma \models_{\mathfrak{M}}^X \varphi$ for all $\mathfrak{M} \in \mathbf{M}$.*

It is worth pausing a while to make a couple of remarks concerning the rich structure of **C.L**-models. Doing so will help one to understand the significance of the formal properties of **C.L**-models. As emphasised in Chapter 1, a prominent motivation for relevant logic is that it provides very weak closure conditions to build theories. Moreover, in Chapter 2 I remarked that information states in relevant models should be regarded as theories. Now, **C.L**-models provide the means to refine the picture concerning the theory-building practice, by individuating subdomains within the space S of all possible theories.

1. *The whole space S .* First, note that S in general contains theories which *comply* with a given relevant logic L – in the jargon of [192]. That is, by Lemma 2.2, whenever a state s supports some information φ and $\models^L \varphi \rightarrow \psi$, then s supports ψ as well. However, this does not mean in general that information states *contain* L . The point is more forcefully made by looking at a specific model, the canonical **C.L**-model (cf. Definition 3.9): on the one hand, $\vdash_L \varphi \rightarrow \psi$ and $\varphi \in s \in S$ implies $\psi \in s$ by the very definition of theories (cf. Definition 2.11). However, it may happen that $\varphi \rightarrow \psi \notin s \in S$, whence combining s with a φ -theory, through the ternary relation R , may not result in a ψ -theory.¹
2. *Logical theories.* A first refinement of S is already available in **L**-models by the set of logical theories $L \subseteq S$. While logical theories may differ in the information they contain, they all contain every relevant tautology. That is, theories in L both comply to L and contain L . Again, the canonical **C.L**-model provides more insights: since logical theories contain by definition every relevant tautology $\varphi \rightarrow \psi$, combining a logical theory, through R , with a φ -theory does result in a ψ -theory. However, note that, again by definition of theories, ψ was already contained in the original φ -theory: every theory building practice has to accord with the relevant closure relation.²
3. *Classical theories.* Possible worlds in $W \subseteq L \subseteq S$ constitute the real novelty of **C.L**-models. They correspond to a special class of logical theories, which in

¹That is, it is possible that a theory s complying with relevant logic is useless from the point of view of theory combination. This is desirable if we are to use, as we will do in Part II, theories complying to relevant logic as the basis for logics modeling the epistemic properties of real-life agents.

²[117] states the same point in algebraic terms: “given any set t that already is a theory, if we apply [the theory generated by relevant tautologies] l to t , the result should be the theory generated by t . But since t already is a theory, t should itself be the theory generated by t . Thus $l \otimes t$ should just be t , which is to say that l is a left identity for \otimes ”.

some sense contain classical information. To specify such precise sense, recall from Chapter 1 that relevant logic can be taken as providing the universal theory closure relation, on top of which to define closure operations corresponding to stronger theories. Stronger theories, in turn, may be used to do more than providing general reasoning principles, e.g. to describe how the world stands. Describing worlds, then, requires theories to satisfy tighter information constraints than just (logical) states (cf. Definition 3.3). By Lemma 3.3, theories describing the world contain classical propositional logic, i.e. they are maximally consistent. Moreover, by Lemma 3.4.2 they comply to classical propositional consequence, i.e. they are closed under W -consequence (cf. Definition 3.5).

In Chapter 1, I stressed that variable sharing is only a peripheral concern to relevance, and I argued that theory building is instead central. Other central concerns to explaining the concept of relevance, which, I submit, depend on the theory building argument, can be individuated. For example, explanations were found to rely on paraconsistency (cf. 19 and Remark 2.1), total sufficiency (cf. 159 and Remark 3.2). Here, I want to add that one can appeal to relevance as a phenomenon which highlights the presence of *intensional contexts*.

Remark 3.2 (Relevance, Intensionality, Sufficiency). The reader is now in a position to appreciate one reason why some inferences are rejected as problematic by relevant logicians, without appealing to variable sharing. For example, take disjunctive syllogism $\neg\varphi \wedge (\varphi \vee \psi) \rightarrow \psi$, the *bête noire* of relevant logicians. As noted in Chapter 2, according to the informational reading of relevant implication a state s supports an implication $\varphi \rightarrow \psi$ iff combining s with φ -inputs is *totally sufficient* to obtain ψ -outputs. According to this model-theoretic interpretation of relevance as total sufficiency, the properties of relevance are tied to those of information combination, encoded by the ternary relation R . Now, **C.L**-models allow one to make fine-grained distinctions in the properties of R , on the basis of the kind of information states R relates. In particular, Lemma 3.3 states that R and $*$ are ineffective at possible worlds, so that formulas such as $\varphi \otimes (\neg\varphi \oplus \psi) \rightarrow \psi$, as well as its extensional version $\varphi \wedge (\neg\varphi \vee \psi) \rightarrow \psi$, are W -valid. This fact provides insights on how to understand relevance. Relevance is best understood as a thesis about *intensional contexts*: φ is relevant to ψ iff the corresponding S -consequence (as opposed to W -consequence) $\varphi \models^S \psi$ holds. In view of this, both $\varphi \wedge (\neg\varphi \vee \psi)$ and $\varphi \otimes (\neg\varphi \oplus \psi)$ are not relevant to ψ .³

³In fact, $\varphi \otimes (\neg\varphi \oplus \psi) \rightarrow \psi$ is valid only in models of relevant logics stronger than DW. The

Having explained some of the main conceptual features of **C.L**-models, I summarise below some of their formal properties. In particular, note that Condition (W9) allows one to make fine-grained distinctions at the level of semantic entailments in **C.L**-models. As a consequence, modified versions of the semantic deduction theorem are provable (cf. Lemma 3.4).

Lemma 3.2 (C.L-heredity). *For all $\mathfrak{M} \in \mathbf{C.L}$ and all $\varphi \in \mathfrak{L}_{\text{Rel}}$: $\llbracket \varphi \rrbracket_{\mathfrak{M}} \in S(\uparrow)$.*

Proof. The result is established by virtually the same argument of Lemma 2.1. \square

Lemma 3.3 (C.L-flatness). *For all $\mathfrak{M} \in \mathbf{C.L}$, all $w \in W$ and all $\varphi, \psi \in \mathfrak{L}_{\text{Rel}}$:*

1. $\mathfrak{M}, w \models \mathbf{t}$;
2. $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$;
3. $\mathfrak{M}, w \models \varphi \otimes \psi$ iff $\mathfrak{M}, w \models \varphi \wedge \psi$;
4. $\mathfrak{M}, w \models \varphi \oplus \psi$ iff $\mathfrak{M}, w \models \varphi \vee \psi$;
5. $\mathfrak{M}, w \models \varphi \rightarrow \psi$ iff $(\mathfrak{M}, w \models \varphi \implies \mathfrak{M}, w \models \psi)$;
6. $\mathfrak{M}, w \models \varphi \leftarrow \psi$ iff $(\mathfrak{M}, w \models \psi \implies \mathfrak{M}, w \models \varphi)$.

Proof. We distinguish cases, where Item 1 holds trivially by $W \subseteq L$.

2. The following chain of equivalences holds: $w \models \neg\varphi$ iff $w^* \not\models \varphi$ iff (by Condition (W1)) $w \not\models \varphi$.

3. (\implies). Assume $w \models \varphi \otimes \psi$. Hence, there are $s, t \in S$ such that $Rstw$, $s \models \varphi$ and $t \models \psi$. By Condition (W3) we infer that either $s = 0$, $t = 0$ or $s \leq w$. By $s \models \varphi$ and $t \models \psi$ we infer that $s \neq 0$ and $t \neq 0$, as otherwise they would contradict Lemma 3.1. Hence, $s \leq w$. Similarly, by Condition (W4) we infer that either $s = 0$, $t = 0$ or $t \leq w$. By $s \neq 0$ and $t \neq 0$ we infer that $t \leq w$. By $s, t \leq w$, $s \models \varphi$, $t \models \psi$ and Lemma 3.2 we conclude that $w \models \varphi$ and $w \models \psi$. (\impliedby). Assume $w \models \varphi$ and $w \models \psi$. By Condition (W2) we infer that there are $s, t \in S$, namely $s = t = w$, such that $Rstw$, $s \models \varphi$ and $t \models \psi$. We conclude that $w \models \varphi \otimes \psi$.

4. (\implies). Assume by contraposition that $w \not\models \varphi$ and $w \not\models \psi$. By Condition (W1) and (W2) we infer that there are $s, t \in S$, namely $s = t = w$, such that $Rwst$ $s^* \not\models \varphi$ and $t \not\models \psi$. We conclude by Condition (W1) that $w \not\models \varphi \oplus \psi$. (\impliedby). Assume $w \models \varphi$ or $w \models \psi$. In order to show that $w \models \varphi \oplus \psi$, assume for some arbitrary $s, t \in S$ that $Rwst$. We distinguish cases. (1) If $w \models \varphi$, by $Rwst$ and Condition (W5) we infer

interested reader is referred to [152, Ch.2] for further discussion on the inferential grounds on which to accept disjunctive syllogism.

that either $t = 1$ or $s \leq w$. If the former, we conclude by Lemma 3.1 that $t \models \psi$. If the latter, by $w \models \varphi$ and Item 1 we infer that $w \not\models \neg\varphi$, which implies by $s \leq w$ and Lemma 3.2 that $s \not\models \neg\varphi$. We conclude that $s^* \models \varphi$. (2) If $w \models \psi$, by $Rwst$ and Condition (W6) we infer that either $s = 0$ or $w \leq t$. If the former, we infer by Lemma 3.1 that $s \not\models \neg\varphi$, by which we conclude that $s^* \models \varphi$. If the latter, by $w \models \psi$ and Lemma 3.2 we conclude that $t \models \psi$.

5. (\implies). Assume $w \models \varphi \rightarrow \psi$. Hence, by Condition (W2) we conclude that $w \models \varphi$ only if $w \models \psi$. (\impliedby). Assume $w \models \varphi$ only if $w \models \psi$. In order to show that $w \models \varphi \rightarrow \psi$, assume for some arbitrary $s, t \in S$ that $Rwst$ and $s \models \varphi$. We distinguish cases. (1) If $w \not\models \varphi$, by $Rwst$ and Condition (W5) we infer that either $t = 1$ or $s \leq w$. If the latter, by $w \not\models \varphi$ and Lemma 3.2 we conclude that $s \not\models \varphi$, which is a contradiction. Hence, $t = 1$, by which we conclude by Lemma 3.1 that $t \models \psi$. (2) If $w \models \psi$, by Condition (W6) we infer that $s = 0$ or $w \leq t$. If the former, by Lemma 3.1 we conclude that $s \not\models \varphi$, which is a contradiction. Hence, $w \leq t$. By $w \models \psi$ and Lemma 3.2 we conclude that $t \models \psi$. Item 5 is established similarly, using (W2), (W7) and (W8).

□

Lemma 3.4 (C.L-entailment). *The following hold:*

1. $\varphi \models_{\mathbf{C.L}}^S \psi$ iff $\mathbf{C.L} \models^W \Box_L(\varphi \rightarrow \psi)$;
2. $\varphi \models_{\mathbf{C.L}}^W \psi$ iff $\mathbf{C.L} \models^W \varphi \rightarrow \psi$.

Proof. Item 1 is established as follows. The following chain of equivalences holds: $W \subseteq \llbracket \Box_L(\varphi \rightarrow \psi) \rrbracket$ iff (by Definition 3.2) $Q_L(W) \subseteq \llbracket \varphi \rightarrow \psi \rrbracket$ iff (by Condition (W9)) $L \subseteq \llbracket \varphi \rightarrow \psi \rrbracket$ iff (by virtually the same argument of Lemma 2.2) $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$. Item 2 is an easy consequence of Lemma 3.3,5. □

To conclude this section, I will make some remarks which philosophically qualify the model theory of contextual modal logics. A close examination of Definition 3.5 reveals its affinity with *logical pluralism*, broadly conceived of as the philosophical thesis according to which alternative all-purpose logics are equally correct (cf. [13, 83] for more precise definitions). According to one influential version of pluralism (cf. [16]), the correctness of distinct logics is afforded by our pre-theoretic notion of valid argument: “A valid argument is one whose conclusion is true in every *case* in which all its premises are true” [16, p.476]. Beall and Restall, then, contend that pluralism arises from the fact that there exist at least two ways to make formally precise what one means by “case” (e.g. possible worlds and information states),

which result in as many notions of validity. Now, a challenge of any logically pluralistic theory is to distinguish multiple notions of validity and consequence, all the while maintaining uniform meaning for logical connectives.⁴ This challenge was recently taken up by *intra-theoretical pluralism* (cf. [90]). According to Hjortland, different logical theories are equally correct because they apply to different domains of inference. In particular different kinds of logical theories can be formulated in a unitary framework, with (i) a single language providing a uniform meaning to the logical vocabulary, and (ii) distinct consequence relations identifying the domain of application of each theory. Now, **C.L**-models naturally fall within the tenets of intra-theoretical pluralism.

1. *Distinct consequence relations.* The rich semantics of **C.L**-models allows one to retrieve the classical notion of validity, i.e. truth at all possible worlds, in a relevant setting. What is more, **C.L**-models constitute a semantic framework which is able to support various precisifications of Beall and Restall's pre-theoretic notion of validity, since in **C.L**-models it is possible to distinguish different notions of validity with respect to the domain of information states under consideration (cf. Definition 3.5). Importantly, the specific properties of the cases to which validity is relativised – be them (logical) information states or possible worlds – are reflected in the formal properties of validity (cf. Lemma 3.4).
2. *Uniform connectives' meaning.* Conditions (W1)–(W8) are technically necessary to ensure that all $w \in W$ indeed contain classical propositional information (cf. Lemma 3.3). That is, at possible worlds the intensional connectives, i.e. negation \neg , fusion \otimes , fission \oplus and relevant implications \rightarrow, \leftarrow turn out equivalent to their extensional counterparts, i.e. Boolean negation, conjunction \wedge , disjunction \vee , and the material conditional. Moreover, this feature is obtained by maintaining uniformity in the understanding of the meaning of each connective (cf. Definition 3.2). Note that, as a result of the conditions imposed on **C.L**-frames, the modal, point-shifting, nature of intensional connectives is flattened at possible worlds, so that the satisfaction of φ at a possible world w only requires the compositional evaluation of the subformulas of φ at the very same possible world w .

⁴According to [90], the pluralism of Beall and Restall does not meet this requirement, since the meaning of a connective is correctly determined by its *truth-at-cases* conditions, whence different precisifications of cases give rise to as many truth-at-cases conditions. This, in turn, results in different meaning for connectives across different notions of validity.

3.1.3 The + construction

This section is devoted to proving a crucial result concerning **C.L**-models, the Bridge Lemma 3.6. The lemma, which connects the classical and the relevant layers of the semantics, states that it is possible to pass from **L**-models to equivalent, in some sense, **C.L**-models.

The stipulation that the set of logical states L is exactly the set of Q_L -accessible states from W implies that $\Box_L\varphi$ is W -valid iff φ is **L**-valid. Hence, Condition (W9) allows to simulate **L**-models within **C.L**-models: it is easy to see that for every **C.L**-model \mathfrak{M} there is an **L**-model \mathfrak{M}_L such that $\mathfrak{M} \models^W \Box_L\varphi$ only if $\mathfrak{M}_L \models \varphi$ – it suffices to “forget” bounds and worlds in \mathfrak{M}_L . The converse operation is less straightforward, and makes use of the following semantic construction. In order to illustrate the procedure, let me preliminarily define, for any **L**-model \mathfrak{M}_L , a function $+ : \mathfrak{M}_L \mapsto \mathfrak{M}_L^+$ as follows.

Definition 3.6 (+-Construction). *For any **L**-model $\mathfrak{M}_L = (S, L, \leq, R, *, Q, Q_L, V)$, let \mathfrak{M}_L^+ be defined as follows for all $p \in At$:*

$$\mathfrak{M}_L^+ = (S^+, W, L^+, \leq^+, R^+, *^+, Q^+, Q_L^+, V^+)$$

- $S^+ = S \cup \{w, 0, 1\}$;
- $W = \{w\}$;
- $L^+ = L \cup \{w, 1\}$;
- $\leq^+ = \leq \cup \{(w, w)\} \cup \{(0, s), (s, 1) \mid s \in S^+\}$;
- $R^+ = R \cup \{(w, w, w)\} \cup \{(0, s, t), (s, 0, t), (s, t, 1) \mid s, t \in S^+\}$;
- $*^+ = * \cup \{(w, w), (0, 1), (1, 0)\}$;
- $Q^+ = Q \cup \{(w, w)\} \cup \{(0, s), (s, 1) \mid s \in S^+\}$;
- $Q_L^+ = Q_L \cup \{(w, w)\} \cup \{(0, s), (s, 1) \mid s \in S^+\} \cup \{(w, s) \mid s \in L\}$;
- $V^+(p) = V(p) \cup \{1\}$.

Note that in the above definition Q^+ww and $w \in L^+$ are assumed in the definition of \mathfrak{M}_L^+ in order to make sure that \mathfrak{M}_L^+ satisfies the frame conditions ($\Box T$), ($\Box N$) and (**ER**). Instead of using multiple definitions of \mathfrak{M}_L^+ depending on the properties satisfied by \mathfrak{M}_L , the latter properties are built into Definition 3.6 from the outset even though they are not necessary in some cases.

The + function transforms an **L**-model \mathfrak{M}_L into a **C.L**-model \mathfrak{M}_L^+ such that for every invalid formula φ in \mathfrak{M}_L , $\Box_L\varphi$ is invalid in \mathfrak{M}_L^+ . The proof requires a number of preliminary results, showing respectively that \mathfrak{M}^+ is a W -model (Claim 1), and that satisfaction (Claim 2) and all **L**-frame conditions (Claim 3) are preserved by the + function.

Lemma 3.5 (**L** to **C.L**). *For all $\mathfrak{M} \in \mathbf{L}$ there is $\mathfrak{M}' \in \mathbf{C.L}$ such that $\mathfrak{M} \not\models \varphi \implies \mathfrak{M}' \not\models^W \Box_L \varphi$.*

Proof. Take some arbitrary $\mathfrak{M} \in \mathbf{L}$. We claim that \mathfrak{M}^+ provides the desired \mathfrak{M}' .

Claim 1. $\mathfrak{M}^+ \in \mathbf{C.BMC}$.

Proof. It suffices to show that (i) $(S^+, L^+, \leq^+, R^+, *^+, Q^+, Q_L^+, V^+)$ is an **L**-model; (ii) 0 and 1 are bounds; (iii) w is a possible world. (ii) and (iii) are easily verified by inspection of the definition of \mathfrak{M}^+ . (i) is established as follows. That (S^+, \leq^+) is a poset can be verified by inspection of the definition of \mathfrak{M}^+ . Next, we show that $R^+ \in S^+(\downarrow\downarrow\uparrow)$. That $*^+, Q^+$ and Q_L^+ satisfy the required $\uparrow\downarrow$ -conditions is established similarly. Assume for some arbitrary $s, t, u \in S$ that $R^+stu, s' \leq s, t' \leq t$ and $u \leq u'$. We distinguish cases.

(1). If $\{s, t, u\} \subseteq S$, then either $s', t', u' \in S$ or $s' = 0$ or $t' = 0$ or $u' = 1$. If $s', t', u' \in S$ we conclude by $R \in S(\downarrow\downarrow\uparrow)$ that $Rs't'u'$. If $s' = 0$ ($t' = 0, u' = 1$ respectively) we conclude by inspection of R^+ that $R^+0t'u'$ ($R^+s'0u', R^+s't'1$, respectively).

(2). If $s \notin S$, then either $s = 0, s = w$ or $s = 1$. If $s = 0$, then $s' = 0$ and we conclude by inspection of R^+ that $R^+0t'u'$. If $s = w$, then either $s = t = u = w$ or $t = 0$ or $u = 1$. In the latter two cases we conclude by inspection of R^+ that $Rs'0u', Rs't'1$, respectively. If $s = t = u = w$, then either $s' = w$ (in which case we conclude by inspection of R^+ that R^+www) or $s' = 0$ (in which case we conclude that $R^+0t'u'$).

(3). The cases $t \notin S$ and $u \notin S$ are established by a similar tedious examination of cases as (2).

□

Claim 2. $\mathfrak{M}, s \models \varphi \iff \mathfrak{M}^+, s \models \varphi$.

Proof. The result is established by induction on the structure of φ . The base case follows from the definition of V^+ , and the induction steps $\varphi := \psi \wedge \chi$ and $\varphi := \psi \vee \chi$ are trivial.

$\varphi := \neg\psi$. The following chain of equivalences holds: $\mathfrak{M}, s \models \neg\psi$ iff $\mathfrak{M}, s^* \not\models \psi$ iff $\mathfrak{M}^+, s^{*+} \not\models \psi$ (by the induction hypothesis and the fact that $s \in S$ only if $s^* = s^{*+}$) iff $\mathfrak{M}^+, s \models \neg\psi$.

$\varphi := \psi \rightarrow \chi$. (\implies). Assume for some $s \in S^+$ that $\mathfrak{M}^+, s \not\models \psi \rightarrow \chi$. Hence, there are $t, u \in S^+$ such that R^+stu , $\mathfrak{M}^+, t \models \psi$ and $\mathfrak{M}^+, u \not\models \chi$. By definition of R^+ we infer that either $t = 0$ or $u = 1$ (which are ruled out by Lemma 3.1), or $t, u \in S$. If the latter, we infer that $Rstu$, by which we conclude by the induction hypothesis that $\mathfrak{M}, s \not\models \psi \rightarrow \chi$. (\impliedby). Assume for some $s \in S$ that $\mathfrak{M}, s \not\models \psi \rightarrow \chi$. Hence, there are $t, u \in S$ such that $Rstu$, $\mathfrak{M}, t \models \psi$ and $\mathfrak{M}, u \not\models \chi$. By $R \subseteq R^+$ and the induction hypothesis we infer that R^+stu , $\mathfrak{M}^+, t \models \psi$ and $\mathfrak{M}^+, u \not\models \chi$, by which we conclude that $\mathfrak{M}^+, s \not\models \psi \rightarrow \chi$. The case $\varphi := \psi \leftarrow \chi$ is established similarly.

$\varphi := \psi \otimes \chi$. (\implies). Assume for some arbitrary $s \in S$ that $\mathfrak{M}, s \models \psi \otimes \chi$. Hence, there are $t, u \in S$ such that $Rtus$, $\mathfrak{M}, t \models \psi$ and $\mathfrak{M}, u \models \chi$. By $R \subseteq R^+$ and the induction hypothesis we infer that R^+tus , $\mathfrak{M}^+, t \models \psi$ and $\mathfrak{M}^+, u \models \chi$, by which we conclude that $\mathfrak{M}^+, s \models \psi \rightarrow \chi$. (\impliedby). Assume for some $s \in S^+$ that $\mathfrak{M}^+, s \models \psi \otimes \chi$. Hence, there are $t, u \in S^+$ such that R^+tus , $\mathfrak{M}^+, t \models \psi$ and $\mathfrak{M}^+, u \models \chi$. By definition of R^+ we infer that either $t = 0$ or $u = 0$ (which are ruled out by Lemma 3.1), or $t, u \in S$. If the latter, we infer that $Rtus$, by which we conclude by the induction hypothesis that $\mathfrak{M}, s \models \psi \otimes \chi$.

$\varphi := \Box\psi$. (\implies). Assume for some $s \in S^+$ that $\mathfrak{M}^+, s \not\models \Box\psi$. Hence, there is $t \in S^+$ such that Q^+st and $\mathfrak{M}^+, t \not\models \psi$. By definition of Q^+ we have that either $t = 1$ or $t \in S$. The former contradicts Lemma 3.1. Hence, $t \in S$, by which we infer that Qst . We conclude by the induction hypothesis that $\mathfrak{M}, s \not\models \Box_L\psi$. (\impliedby). Assume for some $s \in S$ that $\mathfrak{M}, s \not\models \Box\psi$. Hence, there is some $t \in S$ such that Qst and $\mathfrak{M}, t \not\models \psi$. By $Q \subseteq Q^+$ and the induction hypothesis we infer that Q^+st and $\mathfrak{M}^+, t \not\models \psi$, by which we conclude that $\mathfrak{M}^+, s \not\models \Box\psi$. The case $\varphi := \Box_L\psi$ is established similarly.

□

Claim 3. $\mathfrak{M} \in \mathbf{L} \implies \mathfrak{M}^+ \in \mathbf{C.L.}$

Proof. The result is established by showing that the \mathbf{L} -frame conditions of Chapter 2 are preserved through the $+$ -construction. That is, it is shown that each frame condition holds in (the frame underlying) \mathfrak{M}^+ whenever it holds in (the frame underlying) \mathfrak{M} . For all conditions, we distinguish cases depending on whether a state belongs to S or to $\{w, 0, 1\}$.

(DNI). Assume that (DNI) holds in \mathfrak{M} . To show that (DNI) holds in \mathfrak{M}' , take some arbitrary $s \in S^+$. If $s \in S$, by definition of $^+*$ we infer that $s^{*+*} \in S$. By (DNI) holding in \mathfrak{M} we conclude that $s \leq s^{*+*}$. If $s = w$ ($s = 0$, $s = 1$ respectively)

we conclude by Condition (W1) (Condition (B2), Condition (B3) respectively) that $s \leq s^{*+*+} = s$. The cases for (DNE) and (RD) are established similarly.

(X). Assume that (X) holds in \mathfrak{M} . To show that (X) holds in \mathfrak{M}^+ , take some arbitrary $s \in L^+$. If $s \in L \subseteq S$, by definition of $*^+$ we infer that $s^{*+} \in S$. By (X) holding in \mathfrak{M} we conclude that $s^{*+} \leq s$. If $s = w$, by Condition (W1) we conclude that $w^{*+} = w \leq^+ w$. If $s = 1$, by Condition (B3) we conclude that $1^{*+} = 0 \leq^+ 1$.

(CP). Assume that (CP) holds in \mathfrak{M} . To show that (CP) holds in \mathfrak{M}^+ , assume for some arbitrary $s, t, u \in S^+$ that R^+stu . If $s \in S$, then by definition of R^+ we infer that either $t, u \in S$, $t = 0$ or $u = 1$. If $t, u \in S$, by definition of R^+ we infer that $Rstu$. By (CP) holding in \mathfrak{M} we infer that Rsu^*t^* . By $* \subseteq *^+$ and $R \subseteq R^+$ we conclude that $R^+su^{*+}t^{*+}$. If $t = 0$ ($u = 1$), by definition of $*^+$ and R^+ we conclude that $R^+su^{*+}1$ (R^+s0t^{*+}). If $s = 0$, then by definition of R^+ we conclude that $R^+0u^{*+}t^{*+}$. If $s = w$, then either $t = u = w$, $t = 0$ or $u = 1$. If $t = u = w$, by Condition (W2) we conclude that R^+www . If $t = 0$ ($u = 1$), again by definition of R^+ we conclude that $R^+su^{*+}1$ (R^+s0t^{*+}). If $s = 1$, then either $t = 0$ or $u = 1$ and we can reason as above.

(B). Assume that (B) holds in \mathfrak{M} . To show that (B) holds in \mathfrak{M}^+ , assume for some arbitrary $s, t, u, v, x \in S$ that R^+stx and R^+xuv . Let $T = \{s, t, u, v, x\}$. We distinguish cases.

- (1). If $T \subseteq S$, then by definition of R^+ we infer that $Rstx$ and $Rxuv$. By (B) holding in \mathfrak{M} and $R \subseteq R^+$ we conclude that R^+tuy and R^+syv .
- (2). If $1 \in T$ or $0 \in T$, then we distinguish three subcases:
 - (2.1). If $0 \in \{s, t, u\}$ or $v = 1$, then we conclude by choosing the appropriate y . For instance, if $s = 0$, then set $y = 1$, since R^+s1v and R^+tu1 hold by definition of R^+ . The other cases are similar.
 - (2.2). If $x = 0$, then by Condition (B4) we conclude that either $s = 0$ or $t = 0$ (case (2.1)). If $x = 1$, then by Condition (B7) we conclude that either $u = 0$ or $v = 1$ (case (2.1)).
 - (2.3). If $s = 1$, then by Condition (B7) we conclude that either $t = 0$ (case (2.1)) or $x = 1$ (case (2.2)). If $t = 1$, then by Condition (B9) we conclude that $s = 0$ (case (2.1)) or $x = 1$ (case (2.2)). If $u = 1$, then by Condition (B9) we conclude that $x = 0$ (case (2.2)) or $v = 1$ (case (2.1)). If $v = 0$, then by Condition (B4) we conclude that either $x = 0$ (case (2.2)) or $u = 0$ (case (2.1)).
- (3). If $T \subseteq S \cup \{w\}$, then either $T \subseteq S$ (case (1)) or $T = \{w\}$. In the latter case, set $y = w$, by which we conclude by Condition (W2). These three groups of cases exhaust all possibilities.

The cases for (CB),(W),(C) and (WB) follow the same strategy.

(M). Assume that (M) holds in \mathfrak{M} . To show that (M) holds in \mathfrak{M}^+ , assume for some arbitrary $s, t, u \in S$ that R^+stu . If $s \in S$, then by definition of R^+ we infer that either $t, u \in S, t = 0$ or $u = 1$. If $t, u \in S$, by definition of R^+ we infer that $Rstu$. By (M) holding in \mathfrak{M} we infer that $s \leq u$ or $t \leq u$. By $\leq \subseteq \leq^+$ we conclude that $s \leq^+ u$ or $t \leq^+ u$. If $s = 0$, then $s \leq^+ u$. If $s = w$, then either $t = u = w, t = 0$ or $u = 1$. If $t = u = w$, then we conclude that $s \leq^+ t$ and $s \leq^+ u$. If $t = 0$ ($u = 1$), we conclude by definition of R^+ that $t \leq^+ u$ ($s \leq^+ u$). If $s = 1$, then $t = 0$ or $u = 1$, and we can reason as above.

(ER). Assume that (ER) holds in \mathfrak{M} . To show that (ER) holds in \mathfrak{M}^+ , take some arbitrary $s \in S^+$. If $s \in S$, then by (ER) holding in \mathfrak{M} there is $l \in L$ such that $Rsls$. By $R \subseteq R^+$ we conclude that R^+sls by \cdot . If $s = w$ ($s = 0, s = 1$ respectively), by Condition (W2) (Condition (B6), Condition (B5) respectively) there is $t \in S$, namely $t = w$ ($t = 1$ respectively), such that R^+sts .

(□N). Assume that (□N) holds in \mathfrak{M} . To show that (□N) holds in \mathfrak{M}^+ , assume for some arbitrary $s \in L^+$ and Q^+st . By $s \in L^+$, we infer that either $s \in S, s = w$ or $s = 1$. If $s \in L$, then either $t \in S$ or $t = 1$. If $t \in S$, then by definition of Q^+ we infer that Qst . By (□N) we infer that $t \in L$. By $L \subseteq L^+$ we conclude that $t \in L^+$. If $s = w$, then by definition of Q^+ we infer that either $t = w$ or $t = 1$. In both cases, we conclude by definition of L^+ that $t \in L^+$. If $s = 1$, then by definition of Q^+ we infer that $t = 1$ and we can reason as above.

(□K). Assume that (□K) holds in \mathfrak{M} . To show that (□K) holds in \mathfrak{M}^+ , assume for some arbitrary $s, t, u, x \in S$ that R^+stx and Q^+xu ; we have to prove that there are y, z such that Q^+ty, Q^+sz and R^+zyu . Let $T = \{s, t, u, x\}$. We distinguish cases.

- (1). If $T \subseteq S$, then by definition of R^+ and Q^+ we infer that $Rstx$ and Qxu . By (□K) holding in \mathfrak{M} we infer that there are y, z such that Qty, Qsz and $Rzyu$. By $R \subseteq R^+$ and $Q \subseteq Q^+$ we conclude that Q^+ty, Q^+sz and R^+zyu .
- (2). If $0 \in T$ or $1 \in T$, then we distinguish three subcases:
 - (2.1). If $s = 0$ or $t = 0$ or $u = 1$, then we conclude by choosing the appropriate $y, z \in \{0, 1\}$ such that Q^+ty, Q^+sz and R^+zyu . For instance, if $s = 0$, then set $y = 1$ and $z = 0$, since Q^+t1, Q^+10 and R^+01u holds by definition of R^+ . The other cases are similar.
 - (2.2). If $x = 0$, then by Condition (B4) we conclude that $s = 0$ or $t = 0$ (case (2.1)). If $x = 1$, then by Condition (B11) we conclude that $u = 1$ (case (2.1)).

- (2.3). If $s = 1$, then by Condition (B7) we conclude that $t = 0$ (case (2.1)) or $x = 1$ (case (2.2)). If $t = 1$, then by Condition (B8) we conclude that $s = 0$ (case (2.1)) or $x = 1$ (case (2.2)). If $u = 0$, then $x = 0$ (case (2.2)).
- (3). If $T \subseteq S \cup \{w\}$, then either $T \subseteq S$ (case (1)) or $w \in T$. In the latter case, it can be shown that $T = \{w\}$ from which it follows easily that Q^+tw , Q^+sw and R^+wvu . This exhausts all possibilities and so we conclude.

(□T). Assume that (□T) holds in \mathfrak{M} . To show that (□T) holds in \mathfrak{M}^+ , assume $s \in S$. If $s \in S$, by (□T) in \mathfrak{M} we infer that Qss . By $Q \subseteq Q^+$ we conclude that Q^+ss . If $s \in \{w, 0, 1\}$, by the definition of Q^+ we conclude that Q^+ss .

(□D). Assume that (□D) holds in \mathfrak{M} . To show that (□D) holds in \mathfrak{M}^+ , assume $s \in S$. If $s \in S$, by (□D) holding in \mathfrak{M} we infer that there is $t \in S$ such that Qst^* and Qs^*t . By $* \subseteq *^+$ and $Q \subseteq Q^+$ we conclude that Q^+st^{*+} and $Qs^{*+}t$. If $s = w$ ($s = 0$, $s = 1$ respectively), then we conclude by setting $t = w$ ($t = 1$, $t = 0$ respectively).

(□4). Assume that (□4) holds in \mathfrak{M} . To show that (□4) holds in \mathfrak{M}^+ , assume Q^+st and Q^+tu . Let $T = \{s, t, u\}$. We distinguish cases.

- (1). If $T \subseteq S$, then by definition of Q^+ we infer that Qst and Qtu . By (□4) holding in \mathfrak{M} we infer that Qsu . By $Q \subseteq Q^+$ we conclude that Q^+su .
- (2). If $0 \in T$ or $1 \in T$, then we reason as follows. If $s = 0$ or $u = 1$, then by definition of Q^+ we conclude that Q^+su .
- (3). If $s = 1$, then by Condition (B11) we infer that $t = 1$, by which we conclude that $u = 1$ (case (2)). If $u = 0$, then by Qtu and the definition of Q^+ we infer that $t = 0$, which implies by Qst that $s = 0$ (case (2)).
- (4). If $T \subseteq S \cup \{w\}$, then either $T \subseteq S$ (case (1)) or $T = \{w\}$, in which case we conclude that Q^+su . This exhausts all possibilities and so we conclude.

The case for (□5) is established similarly.

□

Putting things together, for all \mathbf{L} -models \mathfrak{M} , if $\mathfrak{M} \not\models \varphi$, then there is $s \in L$ such that $\mathfrak{M}, s \not\models \varphi$. Then, (i) by Claim 1 there is a $\mathbf{C.L}$ -model \mathfrak{M}^+ such that (ii) by Claim 2 $\mathfrak{M}^+, s \not\models \varphi$. Then, (iii) by Condition (W9) we infer that $s \in Q_L^+(W)$, which implies $\mathfrak{M}^+, w \not\models \Box_L \varphi$. We conclude by Definition 3.5 that $\mathfrak{M}^+ \not\models^W \Box_L \varphi$. Finally, by Claim 3 steps (i)–(iii) can be performed for all \mathbf{L} -models, by which we conclude. □

3.2 Contextual Modal Logics

Having introduced the main class of models **C.L** I shall be working with, I will now turn to the task of providing an axiom system for the contextual modal logic **C.L**, parametric on a given relevant modal logic **L**.

3.2.1 Soundness of C.L

Definition 3.7 (\Box_L -versions). *For a given $\varphi \in \mathbf{Lan}$, let the \Box_L -version of φ be $\Box_L\varphi$. For a given rule of inference $\varphi_1, \dots, \varphi_n \Rightarrow \psi$, let the \Box_L -version of the rule of inference be $\Box_L\varphi_1, \dots, \Box_L\varphi_n \Rightarrow \Box_L\psi$. For a given Hilbert system **H**, let the \Box_L -version of **H** be obtained by taking the \Box_L -versions of all its axioms and rules of inference.*

Definition 3.8 (Contextual modal logics). *Let the Hilbert system for the contextual modal logic **C.L** in the signature $\sigma_{\mathbf{Rel}}$ consist of the following axioms and rules:*

1. Axioms and rules of classical propositional logic **CPC** (cf. Figure [3.1](#));
2. Rules **(BMC14)**–**(BMC16)** for **t**, \otimes and \leftarrow ;
3. The \Box_L -versions of **L** (cf. Figures [2.4](#) and [2.5](#));
4. The bridge rule **(BR)** $\Box_L\varphi \Rightarrow \varphi$.

Notice that many principles of **CPC** are superfluous, since they are derivable from their \Box_L -versions and **(BR)**. However, I decided to include them to give a modular presentation of the axiomatic system. A couple of further comments regarding the principles of **C.L** are in order.

1. **CPC** and **t**, \otimes , \leftarrow . The inclusion of **(BMC14)**–**(BMC16)** at the propositional level might raise an eyebrow. This is due to the fact that we work in a language with **t**, \otimes and \leftarrow . While these rules may seem at odds with classical considerations, note that **t**, \otimes and \leftarrow are definable in classical propositional logic, as shown by the admissibility of $\top \leftrightarrow \mathbf{t}$, $(\varphi \wedge \psi) \leftrightarrow (\varphi \otimes \psi)$ and $(\varphi \rightarrow \psi) \leftrightarrow (\psi \leftarrow \varphi)$. Hence, in effect the propositional layer is classical.
2. *Relevant and classical.* \Box_L provides a mediation between the classical and the relevant layer, which is obtained through the bridge rule **(BR)**. **(BR)** is meant

(CPC1) $\varphi \wedge \psi \rightarrow \varphi(\psi)$
(CPC2) $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$
(CPC3) $(\psi)\varphi \rightarrow \varphi \vee \psi$
(CPC4) $(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \psi) \rightarrow ((\varphi \vee \chi) \rightarrow \psi))$
(CPC5) $\varphi \rightarrow (\neg\varphi \rightarrow \psi)$
(CPC6) $\varphi \leftrightarrow \neg\neg\varphi$
(CPC7) $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$
(CPC8) $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$
(CPC9) $\varphi, \varphi \rightarrow \psi \Rightarrow \psi$

Figure 3.1: Hilbert system for CPC.

to capture the intuition that every relevantly provable formula is classically provable. This reading is confirmed by Lemma 3.6 below, according to which it is possible to interpret \Box_L as internalising L-provability in C.L. C.L is an extension of classical propositional logic, where the axioms and rules of the relevant logic L at hand are “confined to \Box_L -contexts”. Thanks to this feature, it is possible to use theorems of the form $\Box_L(\varphi \rightarrow \psi)$ as the suitable closure relation (relevant consequence) for modal operators (cf. Proposition 3.2).

3. *Different axiomatisations.* I remark that the chosen axiom system for C.L improves on the one of [173, 175, 176, 204, 206], since it takes a stronger version of the bridge rule (BR), which in the cited papers is restricted to implicative formulas $\Box_L(\varphi \rightarrow \psi) \Rightarrow \varphi \rightarrow \psi$. The strengthened version of (BR) is more appealing for both conceptual and technical purposes. For example, we note that the weak version of (BR) allows one to retrieve the admissibility of the strong version only for a limited number of logic, not including $(\Box_{(L)}\mathbf{N}) \varphi \Rightarrow \Box\varphi$.

Theorem 3.1 (C.L-soundness). $\vdash_{\text{C.L}} \varphi \Rightarrow \mathbf{C.L} \models^W \varphi$.

Proof. The result is established by induction on the length of C.L-proofs.

1. For every axiom φ of CPC, $\models^W \varphi$ holds as an immediate consequence of Lemma 3.3. In order to show that modus ponens, the only rule of the standard axiomatisation of CPC, preserves validity, assume $\mathbf{C.L} \models^W \varphi \rightarrow \psi$ and $\mathbf{C.L} \models^W \varphi$. By $\mathbf{C.L} \models^W \varphi \rightarrow \psi$ and Lemma 3.4.2 we infer that $\varphi \models_{\mathbf{C.L}}^W \psi$. By $\mathbf{C.L} \models^W \varphi$ we conclude that $\mathbf{C.L} \models^W \psi$.

2. For every axiom φ of \mathbf{L} , by virtually the same argument of Lemma 2.7 we infer that $\mathbf{C.L} \models^L \varphi$. Then, the following chain of equivalences holds: $\mathbf{C.L} \models^L \varphi$ iff $L \subseteq \llbracket \varphi \rrbracket$ iff (by Condition (W9)) $Q_L(W) \subseteq \llbracket \varphi \rrbracket$ iff $\mathbf{C.L} \models^W \Box_L \varphi$. Next, for every rule of inference $\varphi_1, \dots, \varphi_n \Rightarrow \psi$ of \mathbf{L} , assume that $\mathbf{C.L} \models^W \Box_L \varphi_1, \dots, \mathbf{C.L} \models^W \Box_L \varphi_n$. By the same argument as above, we infer that $\mathbf{C.L} \models^L \varphi_1, \dots, \mathbf{C.L} \models^L \varphi_n$. By virtually the same argument of Lemma 2.7 we infer that $\mathbf{C.L} \models^L \psi$, by which we conclude that Lemma 2.7 we infer that $\mathbf{C.L} \models^W \Box_L \psi$.

3. In order to show that (BR) preserves validity, assume $\mathbf{C.L} \models^W \Box_L \varphi$. Hence, $Q_L(W) \subseteq \llbracket \varphi \rrbracket$. By Condition (W9) we infer that $L \subseteq \llbracket \varphi \rrbracket$. We conclude by $W \subseteq L$ that $W \subseteq \llbracket \varphi \rrbracket$ iff $\mathbf{C.L} \models^W \varphi$.

□

The following Bridge Lemma 3.6 is of crucial technical and theoretical importance. Lemma 3.6 is of technical importance because it explains the role of the modality \Box_L , which is to internalise \mathbf{L} -provability in $\mathbf{C.L}$. Put differently, Lemma 3.6 provides a way to combine logics in a uniform classical platform, where \Box_L acts as a context-shifting operator from classical to relevant logic. Lemma 3.6 is of theoretical importance because it provides the closure conditions of the modalities \Box, \Box_L (cf. Proposition 3.2).⁵ Note that the Bridge Lemma 3.6 is established in one direction by means of a semantic argument, leveraging on the $+$ construction of Section 3.1.3.

Lemma 3.6 (Bridge). *For all $\varphi \in \mathfrak{L}_{\text{Rel}}$: $\vdash_{\mathbf{L}} \varphi \iff \vdash_{\mathbf{C.L}} \Box_L \varphi$.*

Proof. (\implies). The result is established by induction on the length of \mathbf{L} -proofs. If φ is an axiom of \mathbf{L} , then $\Box_L \varphi$ is a $\mathbf{C.L}$ -axiom by definition of $\mathbf{C.L}$. If φ is obtained by applying an \mathbf{L} -rule of inference to premises $\varphi_1, \dots, \varphi_n$, then by the induction hypothesis we infer that $\vdash_{\mathbf{C.L}} \Box_L \varphi_1, \dots, \vdash_{\mathbf{C.L}} \Box_L \varphi_n$. Hence, by the \Box_L -version of the rule, we infer that $\vdash_{\mathbf{C.L}} \Box_L \varphi$. (\impliedby). Assume by contraposition that $\not\vdash_{\mathbf{L}} \varphi$. Hence, by Theorem 2.3 there is $\mathfrak{M}_{\mathbf{L}} \in \mathbf{L}$ such that $\mathfrak{M}_{\mathbf{L}} \not\models \varphi$. By Lemma 3.5 we infer that there is $\mathfrak{M}' \in \mathbf{C.L}$ such that $\mathfrak{M}' \not\models^W \Box_L \varphi$. We conclude by Theorem 3.1 that $\not\vdash_{\mathbf{C.L}} \Box_L \varphi$. □

Finally, two features of \Box in $\mathbf{C.L}$ are remarkable. First, the modal principle distinctive of \Box in the contextual modal logic $\mathbf{C.L}$ is a restricted version of the monotonicity rule (\Box_M), such that \Box is closed under provable \Box_L -implications. By Lemma

⁵In Chapter 5 I will exploit this feature to identify closure under relevant logic as the the criterion for competence of agents' epistemic reasoning situated at possible worlds.

3.5, this means that \Box in C.L is closed under L-provable implications. Second, the *variable sharing principle* implies that cases where φ and ψ are “totally unrelated” have counterexamples in **C.L**-models, which can then be exploited to give counterexamples to $\Box\varphi \rightarrow \Box\psi$. These two facts are used in Chapter **5** to diminish the threat posed by the logical omniscience problem for epistemic logic, by invalidating many epistemically contentious principles, while still individuating a criterion for epistemic competence.

Proposition 3.2 (Relevant reasoning). *The following relevant reasoning rule (RR) is derivable in C.L:*

$$\Box_L(\varphi \rightarrow \psi) \Rightarrow \Box\varphi \rightarrow \Box\psi.$$

Proof. Assume $\vdash_{\text{C.L}} \Box_L(\varphi \rightarrow \psi)$. By \Box_L -(BMC17) we infer that $\vdash_{\text{C.L}} \Box_L(\Box\varphi \rightarrow \Box\psi)$, by which we conclude by (BR) that $\Box\varphi \rightarrow \Box\psi$. \square

3.2.2 Completeness of C.L

In order to obtain a completeness result for C.L with respect to **C.L**-models, a family of canonical models is defined, parametrised on the relevant multi-modal logic L.

Lemma 3.7 (Lindenbaum). *For all L, if (Γ, Δ) is C.L-independent and both Γ and Δ are non-empty, then there is a non-empty proper prime C.L-theory Σ such that $\Gamma \subseteq \Sigma$ and $\Delta \cap \Sigma = \emptyset$.*

Proof. The result can be established by the same argument of the Pair Extension Lemma **2.3**. Note that the above is a version of the Lindenbaum lemma, since a C.L-theory is non-empty proper prime iff it is maximal. \square

Definition 3.9 (Canonical C.L-model). *Let the canonical C.L-model be a tuple of the following form:*

$$\mathfrak{M}_{\text{C.L}}^c = (S^c, W^c, L^c, \leq^c, *^c, R^c, Q^c, Q_L^c, V^c)$$

- $S^c = \{s \mid s \text{ is a prime L-theory over } \mathfrak{L}_{\text{Rel}}\}$;
- $W^c = \{s \mid s \text{ is a non-empty, proper, prime C.L-theory over } \mathfrak{L}_{\text{Rel}}\}$;
- $L^c = \{s \in S^c \mid \mathbf{t} \in s\}$;
- $\leq^c = \{(s, t) \in (S^c)^2 \mid s \subseteq t\}$;
- $s^{*c} = \{\varphi \in \mathfrak{L}_{\text{Rel}} \mid \neg\varphi \notin s\}$;
- $R^c = \{(s, t, u) \in (S^c)^3 \mid \forall\varphi, \psi \in \mathfrak{L}_{\text{Rel}}(\varphi \rightarrow \psi \in s, \varphi \in t \implies \psi \in u)\}$;
- $Q^c = \{(s, t) \in (S^c)^2 \mid \forall\varphi \in \mathfrak{L}_{\text{Rel}}(\Box\varphi \in s \implies \varphi \in t)\}$;
- $Q_L^c = \{(s, t) \in (S^c)^2 \mid \forall\varphi \in \mathfrak{L}_{\text{Rel}}(\Box_L\varphi \in s \implies \varphi \in t)\}$;

- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Moreover, let $0 := \emptyset$ and $1 := \mathfrak{L}_{\text{Rel}}$.

Lemma 3.8 (C.L–Modelhood). $\mathfrak{M}_{\text{C.L}}^c \in \mathbf{C.L}$.

Proof. The result is established by showing that (i) $(S, L, \leq, *, R, Q, Q_L, V)$ is an \mathbf{L} –model; (ii) 0 and 1 are bounds, i.e. they satisfy Conditions (B1)–(B13); (iii) W is a set of possible worlds, i.e. it satisfies Conditions (W1)–(W9); (iv) $\mathfrak{M}_{\text{C.L}}^c$ satisfies the \mathbf{L} –conditions. (i) is established by virtually the same argument of Lemma 2.4 and (iv) follows by Lemma 2.8. (ii)–(iii) are established as follows.

(B1). Clearly, $\emptyset \subseteq s \subseteq \mathfrak{L}_{\text{Rel}}$ for all $s \in S$.

(B2). $0^* = \{\varphi \mid \neg\varphi \notin \emptyset\} = \mathfrak{L}_{\text{Rel}} = 1$.

(B3). $1^* = \{\varphi \mid \neg\varphi \in \mathfrak{L}_{\text{Rel}}\} = \emptyset = 0$.

(B4). Assume by contradiction that $Rst0$, $s \neq 0$ and $t \neq 0$ for some arbitrary $s, t \in S$. By $s \neq 0$ and $t \neq 0$ we infer that there are $\varphi \in s$ and $\psi \in t$. By $Rst0$ we conclude that $\varphi \otimes \psi \in 0$, which is a contradiction.

(B5). $R111$ holds trivially by definition of R since $\varphi \in 1$ for all $\varphi \in \mathfrak{L}_{\text{Rel}}$.

(B6). $R010$ holds trivially by definition of R , since $\varphi \notin 0$ for all $\varphi \in \mathfrak{L}_{\text{Rel}}$. Condition (B8) is established similarly.

(B7). Assume by contradiction that $R1st$, $s \neq 0$ and $t \neq 1$ for some arbitrary $s, t \in S$. By $s \neq 0$ and $t \neq 1$ we infer that there are $\varphi \in s$ and $\psi \notin t$. By $R1st$ we conclude that $\varphi \rightarrow \psi \notin 1$, which is a contradiction. (B9) is established similarly.

(B10). $Q_{(L)}00$ holds trivially by definition of $Q_{(L)}$, since $\varphi \notin 0$ for all $\varphi \in \mathfrak{L}_{\text{Rel}}$.

(B11). Assume by contradiction that $Q_{(L)}1s$ and $s \neq 1$ for some arbitrary $s \in S$. By $s \neq 1$ we infer that there is $\varphi \notin s$. By $Q_{(L)}1s$ we conclude that $\Box_{(L)}\varphi \notin 1$, which is a contradiction.

(B12). It holds by definition of 0 and V . Similarly, (B13) holds by definition of 1 and V .

$W \subseteq L$. It suffices to note that $\vdash_{\mathbf{L}} \mathbf{t}$. By $\Box_L - (\text{BMC1})$ and $\Box_L - (\text{BMC14})$ we infer that $\vdash_{\mathbf{L}} \Box_L \mathbf{t}$, which implies by (BR) that $\vdash_{\text{C.L}} \mathbf{t}$. From the latter, we conclude by definition of W that $\mathbf{t} \in w$ for all $w \in W$.

(W1). Take some arbitrary $w \in W$. Since w is maximal, we infer that $\varphi \notin w$ iff $\neg\varphi \in w$. We conclude that $w^* = \{\varphi \mid \neg\varphi \notin w\} = w$.

(W2). For all $w \in W$, $Rwww$ follows from the fact that $\vdash_{\text{CPC}} (\varphi \rightarrow \psi) \wedge \varphi \rightarrow \psi$.

(W3). Assume for some arbitrary $w \in W$ and $s, t \in S$ that $Rstw$, $s \neq 0$, $t \neq 0$ to show that $s \leq w$. To this aim, assume $\varphi \in s$. By $t \neq 0$ we infer that there is $\psi \in t$. By Lemma 2.5.1, $Rstw$ and $\varphi \in s$ we infer that $\varphi \otimes \psi \in w$. By $\vdash_{\text{CPC}} \varphi \otimes \psi \rightarrow \varphi$ we conclude that $\varphi \in w$. (W4) is established similarly.

(W5). Assume for some arbitrary $w \in W$ and $s, t \in S$ that $Rwst$ and $t \neq 1$ to show $s \leq w$. To this aim, assume by contradiction that there is $\varphi \in s$ such that $\varphi \notin w$. By w being maximal we infer that $\neg\varphi \in w$. By $\vdash_{\text{CPC}} \neg\varphi \rightarrow (\varphi \rightarrow \psi)$ we infer that $\varphi \rightarrow \psi \in w$, which by $Rwst$ and $\varphi \in s$ implies that $\psi \in t$. As ψ was arbitrary, we conclude that $t = 1$, which is a contradiction. (W8) is established similarly.

(W6). Assume for some arbitrary $w \in W$ and $s, t \in S$ that $Rwst$ and $s \neq 0$ to show $w \leq t$. To this aim, take some arbitrary $\varphi \in w$. By $s \neq 0$ we infer that there is some $\psi \in s$. By $\vdash_{\text{CPC}} \varphi \rightarrow (\psi \rightarrow \varphi)$ and $\varphi \in w$ we infer that $\psi \rightarrow \varphi \in w$. By $Rwst$ and $\psi \in s$ we conclude that $\varphi \in t$. (W7) is established similarly.

(W9). (\subseteq). By contraposition, take some arbitrary $s \notin L$. Hence, there is $\varphi \in \mathfrak{L}_{\text{Rel}}$ such that $\vdash_{\text{L}} \varphi$ and $\varphi \notin s$. By Lemma 3.6 we infer that $\vdash_{\text{C.L}} \Box_L \varphi$. By w being maximal, we infer that $\Box_L \varphi \in w$ for all $w \in W$. By definition of Q_L and $\varphi \notin s$ we infer that implies that not $Q_L ws$ for all $w \in W$. We conclude that $s \notin Q_L(W)$. (\supseteq). Take some arbitrary $s \in L$ to show that there is $w \in W$ such that $Q_L ws$. If $s = 1$, then it is sufficient to show that W is non-empty. This follows from the fact that each C.L considered is consistent by Theorem 3.1. If $s \neq 1$, then we reason as follows. Consider the pair $\langle \{\psi \mid \vdash_{\text{C.L}} \psi\}, \{\Box_L \varphi \mid \varphi \notin s\} \rangle$, which is an independent C.L-pair. Otherwise, there would be some $\Box_L \varphi_1, \dots, \Box_L \varphi_n$ such that $\vdash_{\text{C.L}} \bigvee_{i < n} \Box_L \varphi_i$ for some $n > 0$. Then we would reason as follows:

$$\begin{aligned}
\implies & \quad \vdash_{\text{C.L}} \Box_L \bigvee_{i < n} \varphi_i & \text{by } \vdash_{\text{C.L}} \bigvee_{i < n} \Box_L \varphi_i \rightarrow \Box_L \bigvee_{i < n} \varphi_i \\
\implies & \quad \vdash_{\text{L}} \bigvee_{i < n} \varphi_i & \text{by Lemma 3.6} \\
\implies & \quad \bigvee_{i < n} \varphi_i \in s & \text{by } s \in L \\
\implies & \quad \varphi_i \text{ for some } i < n \in s & \text{by } s \text{ being prime}
\end{aligned}$$

which contradicts $\varphi_i \notin s$. Note also that both sets in the pair are non-empty. Hence, by Lemma 3.7 that there is a non-empty proper prime C.L-theory w such that, by definition of the pair, $Q_L w s$.

□

Theorem 3.2 (C.L-Completeness). $\mathbf{C.L} \models \varphi \implies \vdash_{\mathbf{C.L}} \varphi$.

Proof. The result follows by Lemmas 3.8 and $\mathfrak{M}_{\mathbf{C.L}, s}^c \models \varphi \iff \varphi \in s$, which is established by virtually the same argument of Lemma 2.6. (The case of the induction step $\varphi := \Box_L \psi$ is established by virtually the same argument of the case $\varphi := \Box \psi$.)

□

3.2.3 A Qualifying Note on Bounds

To conclude the chapter, a close scrutiny of the completeness proof makes clear why it is technically necessary to resort to bounded structures, and what are the alternative options, with their corresponding advantages and drawbacks.

Remark 3.3 (Bounds and alternatives). In order to check that Lemma 3.8 holds, one has to check that the canonical model satisfies Conditions (W3)–(W8). Now, one may note that such conditions are spelled out by making reference to bounds 0 and 1. To illustrate why this is necessary take as an example Conditions (W5) and (W6): in the canonical model, it is not the case that $Rwst$ implies $w \leq t$ (since s may be \emptyset), nor that $Rwst$ implies $s \leq w$ (since t may be $\mathfrak{L}_{\text{Rel}}$).

To remedy the situation, a few options are available:

1. In presence of (DNI) and (DNE), one can define the canonical model so that empty and non-proper theories are excluded from the domain S^c . In particular, (DNI) and (DNE) are required to prove that s^{*c} is non-empty and proper whenever s is so. This option is not appealing for the logician who has philosophical grounds on which to reject (DNI) or (DNE), and I will not explore it for the sake of generality.
2. An option which does not require commitment to the double negation laws is to include a pair of split negations \neg, \sim [153], with characteristic axioms $\varphi \rightarrow \neg \sim \varphi$ and $\varphi \rightarrow \sim \neg \varphi$. This option is not explored here, mainly because it is not immediate to develop epistemic interpretations for two kinds of intensional negations – although a procedural interpretation of split negation is undertaken in [181].

3. Another option to the same effect is to include the extensional truth constants \top and \perp in the language. In particular, one may prove that s^{*c} is non-empty and proper by appealing to the characteristic axioms of \top, \perp . Working with non-empty and proper theories allows one to retain the definition of R^c while simplifying the conditions on worlds. This option is explored in Chapter 5.
4. Finally, one may solve the problem *ab origine* by changing the definition of the ternary relation in the canonical model. This option is available for relevant logics characterised by propositional neighborhood semantics, as detailed in the next Chapter 4.

Chapter 4

Contextual Modal Logics based on Neighborhood Semantics

In this chapter, I generalise the contextual modal logic C.L of Chapter 3 to a neighborhood semantics setting. To this aim, in Section 4.1 I show how to characterise the weakest relevant modal logic FE complete with respect to Routley–Meyer semantics, in the language $\mathfrak{L}_{\text{RRel}}$, by resorting to neighborhood semantics. In Section 4.2, I show how to characterise extensions of FE in the full language $\mathfrak{L}_{\text{Rel}}$. In Section 4.2, I show how to introduce possible worlds in neighborhood models for relevant modal logic, so as to obtain a general soundness and completeness theorem for neighborhood versions of contextual modal logics.

4.1 The Minimal Relevant Modal Logic

Neighborhood semantics for relevant logic was introduced by [163, 164] and studied later in [61, 80, 193].¹ The neighborhood setting is interesting first and foremost because it allows to characterise a wider number of logics than the relational setting. Moreover, neighborhood models are similar in spirit to algebraic models, which makes them easier to manipulate.

¹Neighborhood semantics analyses of various types of negation in the context of distributive lattice logics are undertaken also in [157]. However, I will focus on Routley–Meyer semantics for relevant logics, whose distinctive features are the Routley star function and the ternary Routley–Meyer relation. My presentation will rely on that of [61].

4.1.1 Neighborhood L-Models

In devising the appropriate neighborhood semantics, I will first restrict my attention to the relevant language $\mathfrak{L}_{\text{RRel}}$, without \otimes, \leftarrow , for the reason that the admissibility of the latter connectives requires a strenghtening of the basic relevant propositional logic (cf. Section [4.2.2](#)).

Definition 4.1 (Neighborhood L-models). *Let a neighborhood L-model be a structure of the following form:*

$$(S, L, Prop, *, R, Q, Q_L V)$$

- S is a non-empty set of information states;
- $Prop \subseteq \mathcal{P}(S)$ is a collection of subsets of S , the admissible propositions, closed under the operations of Definition [4.2](#);
- $L \in Prop$ is a non-empty subset of logical information states;
- $*$: $S \rightarrow S$ is the Routley star, interpreting \neg ;
- $R \subseteq S \times \mathcal{P}(S)^2$ is the neighborhood Routley-Meyer ternary relation, interpreting \rightarrow ;
- $Q, Q_L \subseteq S \times \mathcal{P}(S)$ are neighborhood binary relations, interpreting \Box, \Box_L ;
- $V : At \rightarrow Prop$ is an atomic valuation.

Let $(S, L, Prop, *, R, Q, Q_L)$ be the neighborhood L-frame a neighborhood L-model $(S, L, Prop, *, R, Q, Q_L, V)$ is based on. Let **FE** denote the class of all neighborhood L-models.

Definition 4.2 (Neighborhood frame operations). *For a given neighborhood L-frame \mathfrak{F} , let the relevant frame operations on \mathfrak{F} be defined as follows, for $X, Y \in \mathcal{P}(S)$:*

$$\begin{aligned} X \wedge^{\mathfrak{F}} Y &= X \cap Y \\ X \vee^{\mathfrak{F}} Y &= X \cup Y \\ \neg^{\mathfrak{F}} X &= \{s \in S \mid s^* \notin X\} \\ X \rightarrow^{\mathfrak{F}} Y &= \{s \in S \mid RsXY\} \\ \Box^{\mathfrak{F}} X &= \{s \in S \mid QsX\} \\ \Box_L^{\mathfrak{F}} X &= \{s \in S \mid Q_LsX\} \end{aligned}$$

Moreover, let the following conditions hold for all neighborhood L-frames \mathfrak{F} and all $X, Y \in Prop$, where $\otimes \in \{\wedge, \vee, \neg, \rightarrow, \Box, \Box_L\}$ and n is the arity of \otimes :

$$X_1, \dots, X_n \in Prop \implies \otimes^{\mathfrak{F}}(X_1, \dots, X_n) \in Prop \quad (\text{Prop1})$$

$$X \subseteq Y \iff L \subseteq X \rightarrow^{\mathfrak{F}} Y \quad (\text{Prop2})$$

Definition 4.3 (Neighborhood L-satisfaction). *Let the proposition expressed by an arbitrary formula $\varphi \in \mathcal{L}_{\text{Rel}}$ in a neighborhood L-model \mathfrak{M} – denoted $\llbracket \varphi \rrbracket_{\mathfrak{M}}$ – be recursively defined as follows, where $\otimes \in \{\neg, \wedge, \vee, \rightarrow, \Box, \Box_L\}$ and n is the arity of \otimes :*

$$\begin{aligned} \llbracket p \rrbracket_{\mathfrak{M}} &= V(p) \\ \llbracket \mathbf{t} \rrbracket_{\mathfrak{M}} &= L \\ \llbracket \otimes(\varphi_1, \dots, \varphi_n) \rrbracket_{\mathfrak{M}} &= \otimes^{\mathfrak{f}}(\llbracket \varphi_1 \rrbracket_{\mathfrak{M}}, \dots, \llbracket \varphi_n \rrbracket_{\mathfrak{M}}) \end{aligned}$$

Let satisfaction in a neighborhood L-model \mathfrak{M} be defined so that $\mathfrak{M}, s \models \varphi$ iff $s \in \llbracket \varphi \rrbracket_{\mathfrak{M}}$.

Let the notions of *validity* and *consequence* be defined as for relational L-models (cf. Definition 2.9). Compared to the relational framework, neighborhood models for multi-modal relevant modal logic differ in three important respects.

1. *Admissible propositions.* General frames' characteristic feature is the inclusion of a set *Prop* of *admissible propositions*, to which the propositional content of formulas is restricted. This restriction simplifies completeness proofs, as it allows one to have a uniform canonical model construction for a wide range of modal logics.² Moreover, restricting the set of admissible propositions to *Prop* is a generalisation with respect to the relational framework, which is obtained as the special case in which $\text{Prop} = S(\uparrow)$.
2. *Neighborhood R.* By adapting a suggestion by [70] to Routley–Meyer semantics, *Rstu* may be read as “from the perspective of s , t is informationally included in u ”, written $t \sqsubseteq_s u$. Neighborhood L-models differ from relational L-models in that they employ a *neighborhood ternary relation* $R \subseteq S \times \mathcal{P}(S)^2$ in the interpretation of implication. Lifting the reading of implicative support to neighborhoods, we obtain a similar reading of R as information inclusion between propositions: s supports $\varphi \rightarrow \psi$ iff from the perspective of s , $\llbracket \varphi \rrbracket$ is informationally included in $\llbracket \psi \rrbracket$, written $\llbracket \varphi \rrbracket \sqsubseteq_s \llbracket \psi \rrbracket$. As in the relational setting, the definition of R makes \sqsubseteq_s a very general operation, e.g. non-reflexive (as needed to invalidate $\psi \rightarrow (\varphi \rightarrow \varphi)$).

²Cf. [36, p.258] and [135, p.66] for a completeness-via-canonicity argument where the choice of a supplemented canonical model is essential to deal with classical modal logics containing (BMC17), more suggestively called (\Box M) in the literature. In addition, cf. [61] for one where the choice of an alternative definition of R in the canonical model is essential to deal with relevant modal logics containing the affixing rule.

3. *Neighborhood Q and Q_L .* $Q(s)$ and $Q_L(s)$ contain the propositions modally accessible from the information state s . The precise modal interpretation of accessibility depends, of course, on the reading of the salient modality (cf. Part [III](#)). However, one may note already a difference with respect to Kripke relational semantics. While in Kripke semantics $Q_{(L)}(s)$ consists of a single set of information states, corresponding to the intersection of the modally accessible propositions; according to neighborhood semantics, $Q_{(L)}(s)$ consists of distinct propositions. The interpretation of $\Box_{(L)}$ in terms of a neighborhood relation enables the failure, among others, of the conjunctive regularity principles $\Box_{(L)}\varphi \wedge \Box_{(L)}\psi \rightarrow \Box_{(L)}(\varphi \wedge \psi)$ (as $Q_{(L)}(s)$ is not closed under intersection) and $\varphi \rightarrow \psi \Rightarrow \Box\varphi \rightarrow \Box\psi$ (as $Q_{(L)}(s)$ is not closed under supersets).

To conclude the presentation of neighborhood L -models, note that working with general neighborhood frames implies that the heredity lemma and the semantic deduction theorem are built in the definition of models. For one, since neighborhood L -models work with *Prop* in place of $S(\uparrow)$, no $\uparrow\downarrow$ -conditions are needed on $L, *, R, Q$, and Lemma [4.1](#) holds by Condition [\(Prop1\)](#). For two, Lemma [4.2](#) also holds directly by Condition [\(Prop2\)](#), which replaces Conditions [\(L1\)](#) and [\(L2\)](#).

Lemma 4.1 (Neighborhood heredity). *For all $\mathfrak{M} \in \mathbf{FE}$ and all $\varphi \in \mathcal{L}_{\text{Rel}}$: $\llbracket \varphi \rrbracket_{\mathfrak{M}} \in \text{Prop}$.*

Proof. The result is established by induction on the structure of φ . The base case holds by definition of V . The induction step holds by the induction hypothesis and Condition [\(Prop1\)](#), for the corresponding connective. \square

Lemma 4.2 (Neighborhood verification). $\mathfrak{M} \models \varphi \rightarrow \psi \iff \llbracket \varphi \rrbracket_{\mathfrak{M}} \subseteq \llbracket \psi \rrbracket_{\mathfrak{M}}$.

Proof. The following chain of equivalences holds: $\mathfrak{M} \models \varphi \rightarrow \psi$ iff $L \subseteq \llbracket \varphi \rrbracket_{\mathfrak{M}} \rightarrow \llbracket \psi \rrbracket_{\mathfrak{M}}$ iff (by Lemma [4.1](#) and Condition [\(Prop2\)](#)) $\llbracket \varphi \rrbracket_{\mathfrak{M}} \subseteq \llbracket \psi \rrbracket_{\mathfrak{M}}$. \square

4.1.2 Axiomatisation of FE

The Hilbert system for FE is obtained by taking the axioms and rules for F, i.e. the minimal propositional relevant logic characterised by Routley–Meyer semantics, supplemented by the equivalence rules for \Box and \Box_L .

Definition 4.4 (FE axiom system). *Let the Hilbert system for the minimal relevant modal logic FE consist of the axioms and rules of inference of Figure [4.1](#).*

The axiom system F differs from BM both at the propositional and modal level.

(FE1)	$\varphi \rightarrow \varphi$
(FE2)	$\varphi \wedge \psi \rightarrow \varphi(\psi)$
(FE3)	$(\psi)\varphi \rightarrow \varphi \vee \psi$
(FE4)	$(\varphi \wedge (\psi \vee \chi)) \rightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
(FE5)	$\neg(\varphi \wedge \psi) \rightarrow (\neg\varphi \vee \neg\psi)$
(FE6)	$(\neg\varphi \wedge \neg\psi) \rightarrow \neg(\varphi \vee \psi)$
(FE7)	$\varphi, \varphi \rightarrow \psi \Rightarrow \psi$
(FE8)	$\varphi, \psi \Rightarrow \varphi \wedge \psi$
(FE9)	$\varphi \rightarrow \psi, \varphi \rightarrow \chi \Rightarrow \varphi \rightarrow \psi \wedge \chi$
(FE10)	$\varphi \rightarrow \chi, \psi \rightarrow \chi \Rightarrow \varphi \vee \psi \rightarrow \chi$
(FE11)	$\varphi \rightarrow \psi, \psi \rightarrow \chi \Rightarrow \varphi \rightarrow \chi$
(FE12)	$\varphi \leftrightarrow \chi, \psi \leftrightarrow \xi \Rightarrow (\varphi \rightarrow \psi) \rightarrow (\chi \rightarrow \xi)$
(FE13)	$\varphi \rightarrow \psi \Rightarrow \neg\psi \rightarrow \neg\varphi$
(FE14)	$\varphi \Leftarrow \Rightarrow \mathbf{t} \rightarrow \varphi$
(FE15)	$\varphi \leftrightarrow \psi \Rightarrow \Box\varphi \leftrightarrow \Box\psi$
(FE16)	$\varphi \leftrightarrow \psi \Rightarrow \Box_L\varphi \leftrightarrow \Box_L\psi$

Figure 4.1: Hilbert system for FE.

1. *Propositional level.* A feature cutting a difference between BM and F is that the affixing rule of the former is weakened in the latter to (FE12), and that transitivity, a special case of affixing, is explicitly added. Moreover, the conjunction introduction and disjunction elimination axioms are weakened to rule form, (FE9) and (FE10).
2. *Modal level.* In FE neither (BMC9), (BMC17), – more suggestively called (\Box C) and (\Box M) in the literature – nor (\Box N) and (\Box K) (cf. Figure 2.3) are valid, preserve validity respectively. That is, while BMC is conjunctively but not implicatively regular, and K is both conjunctively and implicatively regular (i.e. normal), F is neither, its distinctive modal principle for $\Box_{(L)}$ being the equivalence rules (FE15)–(FE16), more suggestively called ($\Box_{(L)}$ E).

Theorem 4.1 (FE soundness). $\vdash_{\text{FE}} \varphi \implies \mathbf{FE} \models \varphi$.

Proof. The result is established by induction on the length of FE–derivations. Lemma 4.2 is used implicitly throughout the proof. The base case holds by the same argument of Theorem 2.1. For the induction step, (FE7), (FE8), (FE13) and (FE14)

preserve validity by the same argument of Theorem 2.1 (FE9)–(FE11) preserve validity by the set-theoretical properties of \cap, \cup and \subseteq . The remaining cases are established as follows.

(FE12). Assume that $\llbracket \varphi \rrbracket = \llbracket \chi \rrbracket$ and $\llbracket \psi \rrbracket = \llbracket \xi \rrbracket$. Then, for all $s \in S$, we infer that $Rs\llbracket \varphi \rrbracket \llbracket \psi \rrbracket$ iff $Rs\llbracket \chi \rrbracket \llbracket \xi \rrbracket$. We conclude that $\llbracket \varphi \rightarrow \psi \rrbracket \subseteq \llbracket \chi \rightarrow \xi \rrbracket$.

(FE15). Assume that $\llbracket \varphi \rrbracket = \llbracket \psi \rrbracket$. Then, for all $s \in S$ we infer that $Qs\llbracket \varphi \rrbracket$ iff $Qs\llbracket \psi \rrbracket$. We conclude that $\llbracket \Box \varphi \rrbracket = \llbracket \Box \psi \rrbracket$. (FE16) is established similarly. □

As usual, completeness is established via canonicity, where the details of the definition of the canonical models are of particular technical interest.

Definition 4.5 (Canonical FE-model). *Let the canonical FE-model be a structure of the following form:*

$$\mathfrak{M}_{\text{FE}}^c = (S^c, L^c, \text{Prop}^c, *^c, R^c, Q^c, Q_L^c, V^c)$$

- $S^c = \{s \mid s \text{ is a prime FE-theory over } \mathfrak{L}_{\text{RRel}}\}$;
- $L^c = \{s \in S^c \mid \mathbf{t} \in s\}$;
- $\text{Prop}^c = \{[\varphi]_c \mid \varphi \in \mathfrak{L}_{\text{RRel}}\}$, where $[\varphi]_c = \{s \in S^c \mid \varphi \in s\}$
- $s^{*c} = \{\varphi \in \mathfrak{L}_{\text{RRel}} \mid \neg \varphi \notin s\}$;
- $R^c = \{(s, X, Y) \in S^c \times (\text{Prop}^c)^2 \mid \exists \varphi, \psi \in \mathfrak{L}_{\text{RRel}} (X = [\varphi]_c, Y = [\psi]_c, \varphi \rightarrow \psi \in s)\}$;
- $Q^c = \{(s, X) \in S^c \times \text{Prop}^c \mid \exists \varphi \in \mathfrak{L}_{\text{RRel}} (X = [\varphi]_c, \Box \varphi \in s)\}$;
- $Q_L^c = \{(s, X) \in S^c \times \text{Prop}^c \mid \exists \varphi \in \mathfrak{L}_{\text{RRel}} (X = [\varphi]_c, \Box_L \varphi \in s)\}$;
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Compared to the canonical BMC-model $\mathfrak{M}_{\text{BMC}}^c$, the canonical FE-model $\mathfrak{M}_{\text{FE}}^c$ restricts the set of admissible propositions to those expressible by the language $\mathfrak{L}_{\text{RRel}}$, i.e. to those X for which there is $\varphi \in \mathfrak{L}_{\text{RRel}}$ such that $X = [\varphi]_c$. A preliminary lemma turns out useful for completeness.

Lemma 4.3 (Canonical implication). *For all $\varphi, \psi \in \mathfrak{L}_{\text{RRel}}$: $\vdash_{\text{FE}} \varphi \rightarrow \psi \iff [\varphi]_c \subseteq [\psi]_c$.*

Proof. (\implies). Assume that $\vdash_{\text{FE}} \varphi \rightarrow \psi$ and $s \in [\varphi]$. Hence, $\varphi \in s$. By $\vdash_{\text{FE}} \varphi \rightarrow \psi$ and the fact that s is closed under FE-implications we conclude that $\psi \in s$ iff $s \in [\psi]$. (\impliedby). Assume by contraposition that $\not\vdash_{\text{FE}} \varphi \rightarrow \psi$. Hence, the pair $\langle \{\varphi\}, \{\psi\} \rangle$ is FE-independent, which implies by Lemma 2.3 that there is $s \in S$ such that $\varphi \in s$ and $\psi \notin s$. We conclude that $[\varphi] \not\subseteq [\psi]$. □

Lemma 4.4 (Admissible Canonical Propositions). *For all $[\varphi_1]_c, \dots, [\varphi_n]_c \in Prop^c$, where n is the arity of $\circledast \in \{\wedge, \vee, \neg, \rightarrow, \otimes, \leftarrow, \square, \square_L\}$:*

$$\circledast^{\mathfrak{F}^c_{FE}}([\varphi_1]_c, \dots, [\varphi_n]_c) = [\circledast(\varphi_1, \dots, \varphi_n)]_c.$$

Proof. We assume by definition of *Prop* that $X_1 = [\varphi_1], \dots, X_n = [\varphi_n], X = [\varphi]$ and $Y = [\psi]$ for some $\varphi_1, \dots, \varphi_n, \varphi, \psi \in \mathfrak{L}_{RRel}$. We distinguish cases, noting that if $\circledast = \wedge$ or $\circledast = \vee$ the result holds trivially.

$\circledast := \neg$. We reason as follows:

$$\begin{aligned} \neg[\varphi_1] &= \{s \mid s^* \notin [\varphi_1]\} && \text{by definition of } \neg^{\mathfrak{F}} \\ &= \{s \mid \varphi_1 \notin s^*\} && \text{by definition of } [\varphi] \\ &= \{s \mid \neg\varphi_1 \in s\} && \text{by definition of } * \\ &= [\neg\varphi_1] && \text{by definition of } [\varphi]. \end{aligned}$$

$\circledast := \rightarrow$. We reason as follows:

$$\begin{aligned} [\varphi_1] \rightarrow [\varphi_2] &= \{s \mid Rs[\varphi_1][\varphi_2]\} && \text{by definition of } \rightarrow^{\mathfrak{F}} \\ &= \{s \mid \exists\psi_1, \psi_2(\psi_1 \rightarrow \psi_2 \in s, && \text{by definition of } R \\ & \quad [\varphi_1] = [\psi_1], [\varphi_2] = [\psi_2])\} \\ &= \{s \mid \exists\psi_1, \psi_2(\psi_1 \rightarrow \psi_2 \in s, && \text{by lemma 4.3} \\ & \quad \vdash_{FE} \varphi_1 \leftrightarrow \psi_1, \vdash_{FE} \varphi_2 \leftrightarrow \psi_2)\} \\ &= \{s \mid \varphi_1 \rightarrow \varphi_2 \in s\} && \text{by (FE12)} \\ &= [\varphi_1 \rightarrow \varphi_2] && \text{by definition of } [\varphi]. \end{aligned}$$

$\circledast := \square_{(L)}$. We reason as follows, where $\square_{(L)} \in \{\square, \square_L\}$ and $N_{(L)} \in \{N, N_L\}$:

$$\begin{aligned} \square_{(L)}[\varphi_1] &= \{s \mid N_{(L)}s[\varphi_1]\} && \text{by definition of } \square_{(L)}^{\mathfrak{F}} \\ &= \{s \mid \exists\psi([\psi] = [\varphi_1], \square_{(L)}\psi \in s)\} && \text{by definition of } N_{(L)} \\ &= \{s \mid \exists\psi(\vdash_{FE} \psi \leftrightarrow \varphi_1, \square_{(L)}\psi \in s)\} && \text{by Lemma 4.3} \\ &= \{s \mid \square_{(L)}\varphi_1 \in s\} && \text{by (FE15) – (FE16)} \\ &= [\square_{(L)}\varphi_1] && \text{by definition of } [\varphi]. \end{aligned}$$

□

Lemma 4.5 (FE–Modelhood). $\mathfrak{M}_{FE}^c \in \mathbf{FE}$.

Proof. The result is established by showing that (i) $*^c$ is well defined, (ii) $L \in Prop$ and (iii) that \mathfrak{M}_{FE}^c satisfies the required frame conditions. (i) is established by the same argument of Lemma 2.4. (ii) holds by $L = \{s \in S^c \mid \mathbf{t} \in s\} = [\mathbf{t}] \in Prop$, while (iii) is established as follows. (Prop1) holds by Lemma 4.4. (Prop2) holds by the following chain of equivalences: $[\varphi] \subseteq [\psi]$ iff (by Lemma 4.3) $\vdash_{FE} \varphi \rightarrow \psi$ iff (by definition of L) $L \subseteq [\varphi \rightarrow \psi]$ iff (by Condition (Prop1)) $L \subseteq [\varphi] \rightarrow [\psi]$. \square

Lemma 4.6 (FE–Truth). $\llbracket \varphi \rrbracket_{\mathfrak{M}_{FE}^c} = [\varphi]_c$.

Proof. The result is established by induction on the structure of φ . The base case holds by definition of V . The induction step is as follows, where the cases involving $\mathbf{t}, \neg, \wedge, \vee$ are established by the same argument of Lemma 2.6. The cases involving $\rightarrow, \square, \square_L$ are established by the same strategy, so we show only the case $\varphi := \psi \rightarrow \chi$ as an illustration.

$$\begin{array}{lll}
s \in \llbracket \psi \rightarrow \chi \rrbracket & \iff & s \in \llbracket \psi \rrbracket \rightarrow \llbracket \chi \rrbracket & \text{by Definition 2.8} \\
& & \iff & s \in [\varphi] \rightarrow [\psi] & \text{by the induction hypothesis} \\
& & \iff & s \in [\varphi \rightarrow \psi] & \text{by Lemma 4.4}
\end{array}$$

\square

Theorem 4.2 (FE Completeness). $\mathbf{FE} \models \varphi \implies \vdash_{FE} \varphi$.

Proof. The result is established by the same argument of Theorem 2.2, using Lemmas 4.5 and 4.6. \square

4.2 Extensions and Connectives Admissibility

4.2.1 Extensions of FE

The logic FE is the minimal relevant modal logic characterised by general neighborhood semantics. As in Chapter 2, extensions of FE are obtained by the same method of Section 2.1.3.³ However, since neighborhood L–frames are different structures than (relational) L–frames, the frame conditions corresponding to the propositional and modal axioms and rules need to be reformulated. Given the algebraic spirit

³Some of the names of the logics are new. In particular, FF is the minimal system F plus the affixing rules, and BF is the system B minus the affixing rule. The remaining names are taken from [61] (for the propositional cases) and [71] (for the modal cases).

Frame condition	Axiom/rule
(\wedge) $(X \rightarrow Y) \cap (X \rightarrow Z) \subseteq X \rightarrow (Y \cap Z)$	$(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi \wedge \chi)$
(\vee) $(X \rightarrow Y) \cap (Z \rightarrow Y) \subseteq (X \cup Z) \rightarrow Y$	$(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)$
(PRE) $X \subseteq Y \implies Z \rightarrow X \subseteq Z \rightarrow Y$	$\varphi \rightarrow \psi \implies (\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi)$
(SUF) $X \subseteq Y \implies Y \rightarrow Z \subseteq X \rightarrow Z$	$\varphi \rightarrow \psi \implies (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)$
(DNI) $X \subseteq \neg\neg X$	$\varphi \rightarrow \neg\neg\varphi$
(DNE) $\neg\neg X \subseteq X$	$\neg\neg\varphi \rightarrow \varphi$
(X) $L \subseteq X \cup \neg X$	$\varphi \vee \neg\varphi$
(RD) $(X \rightarrow \neg X) \subseteq \neg X$	$(\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi$
(CP) $X \rightarrow Y \subseteq \neg Y \rightarrow \neg X$	$(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$
(WB) $(X \rightarrow Y) \cap (Y \rightarrow Z) \subseteq X \rightarrow Z$	$((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)$
(B) $(X \rightarrow Y) \subseteq (Y \rightarrow Z) \rightarrow (X \rightarrow Z)$	$(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$
(CB) $X \rightarrow Y \subseteq (Y \rightarrow Z) \rightarrow (X \rightarrow Z)$	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(W) $X \rightarrow (X \rightarrow Y) \subseteq X \rightarrow Y$	$(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$
(C) $X \rightarrow (Y \rightarrow Z) \subseteq Y \rightarrow (X \rightarrow Z)$	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$
(M) $X \subseteq X \rightarrow X$	$\varphi \rightarrow (\varphi \rightarrow \varphi)$
(ER) $L \subseteq X \implies L \subseteq (X \rightarrow Y) \rightarrow Y$	$\varphi \implies (\varphi \rightarrow \psi) \rightarrow \psi$
(\Box M) $X \subseteq Y \implies \Box X \subseteq \Box Y$	$\varphi \rightarrow \psi \implies \Box\varphi \rightarrow \Box\psi$
(\Box N) $L \subseteq X \implies L \subseteq \Box X$	$\varphi \implies \Box\varphi$
(\Box C) $\Box X \cap \Box Y \subseteq \Box(X \cap Y)$	$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$
(\Box K) $\Box(X \rightarrow Y) \subseteq \Box X \rightarrow \Box Y$	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(\Box T) $\Box X \subseteq X$	$\Box\varphi \rightarrow \varphi$
(\Box D) $\Box\neg X \subseteq \neg\Box X$	$\Box\neg\varphi \rightarrow \neg\Box\varphi$
(\Box 4) $\Box X \subseteq \Box\Box X$	$\Box\varphi \rightarrow \Box\Box\varphi$
(\Box 5) $\neg\Box X \subseteq \Box\neg\Box X$	$\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$

Figure 4.2: Neighborhood **nL**-conditions with the corresponding axioms and rules.

of neighborhood L-frames, this is especially straightforward. The propositional and modal axioms and rules I shall consider are shown in Figure 4.2, where X, Y, Z range over *Prop*.

Definition 4.6 (Notable neighborhood relevant propositional systems). *Let the neighborhood propositional extensions of F be defined by Figure 4.3, where each line connects the axiom system below to its extension obtained by adding the principles labelling the line (cf. Figure 4.2).*

Definition 4.7 (Notable neighborhood relevant modal systems). *Let the neighborhood modal extensions of LE, for a given relevant propositional logic of Figure 4.3, be defined by Figure 4.4, where each line connects the axiom system below to its extension obtained by adding the principles labelling the line (cf. Figure 4.2).*

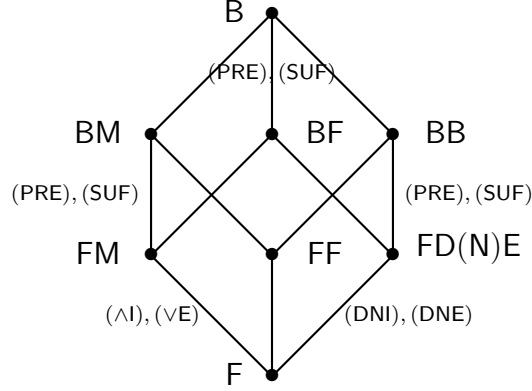


Figure 4.3: Notable propositional extensions of F.

The \mathbf{L} -notation can be extended to the neighborhood setting: let any selection of the frame conditions of Figure 4.2 on neighborhood L-frames be denoted as \mathbf{nL} -conditions. Let an \mathbf{nL} -model (\mathbf{nL} -frame) denote any neighborhood L-model (L-frame) which satisfies a given selection of \mathbf{nL} -conditions. Accordingly, let \mathbf{nL} denote the extension of FE obtained by adding the axioms and rules of inference with the same label of the given neighborhood \mathbf{nL} -conditions.

Lemma 4.7 (\mathbf{nL} -soundness). *For all \mathbf{nL} -frames \mathfrak{F} , \mathfrak{F} satisfies the \mathbf{nL} -conditions only if the corresponding axioms and rules are valid in \mathfrak{F} .*

Proof. The result is established as in [60, 163], where we note that the algebraic formulation of the frame conditions make the proof especially straightforward. \square

Lemma 4.8 (Neighborhood L-canonicity). *Any selection of \mathbf{nL} -conditions holds in $\mathfrak{M}_{\mathbf{nL}}^c$.*

Proof. The result is established again as in [60, Theorem 4.2.1]. \square

Theorem 4.3 (Neighborhood L-Characterisation). $\mathbf{nL} \models \varphi \iff \vdash_{\mathbf{nL}} \varphi$.

Proof. One direction follows from Theorem 4.1 and Lemma 4.7, while the other from Theorem 4.2 and Lemma 4.8. \square

4.2.2 Adding \otimes and \leftarrow

In [199] it is shown that, in order to extend relevant modal logics based on neighborhood semantics to include \otimes and \leftarrow , some work must be done both at the level

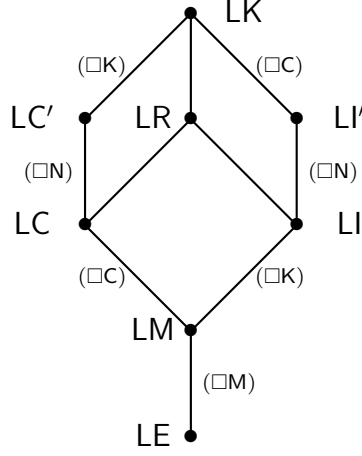


Figure 4.4: Notable modal extensions of LE.

of the proof theory and of the model theory. The procedure can be explained as follows. Extending the axiom system with sound residuation laws between \rightarrow , \otimes and \leftarrow requires one have to assume that \mathbf{nL} -models satisfy Conditions (Prop3) and (Prop4). The latter, in turn, are canonical only with respect to axiom systems containing (PRE) and (SUF) (cf. Figure 4.2), which need to be added to \mathbf{F} . Therefore, the basic relevant propositional system for logics including \otimes and \leftarrow is strengthened from \mathbf{F} to \mathbf{FF} (cf. Figure 4.3).

Definition 4.8 ($\mathbf{nL}_{\otimes\leftarrow}$ -models). *Let an $\mathbf{nL}_{\otimes\leftarrow}$ -model be an \mathbf{nL} -model \mathfrak{M} , based on a \mathbf{nL} -frame \mathfrak{F} which satisfies the following conditions, for every $X, Y \in \mathcal{P}(S)$:*

$$\bigcap_{i \leq n} (Y \rightarrow^{\mathfrak{F}} X_i) = (Y \rightarrow^{\mathfrak{F}} \bigcap_{i \leq n} X_i) \quad (\text{Prop3})$$

$$\bigcap_{i \leq n} (X_i \rightarrow^{\mathfrak{F}} Y) = (\bigcup_{i \leq n} X_i \rightarrow^{\mathfrak{F}} Y) \quad (\text{Prop4})$$

Let $\mathbf{nL}_{\otimes\leftarrow}$ be the class of $\mathbf{nL}_{\otimes\leftarrow}$ -models.

Definition 4.9 ($\mathbf{nL}_{\otimes\leftarrow}$ -satisfaction). *Let the proposition expressed by an arbitrary formula $\varphi \in \mathcal{L}_{\text{Rel}}$ in a $\mathbf{nL}_{\otimes\leftarrow}$ -model \mathfrak{M} extend that of Definition 4.3 as follows:*

$$\begin{aligned} \llbracket \varphi \otimes \psi \rrbracket^{\mathfrak{M}} &= \llbracket \varphi \rrbracket^{\mathfrak{M}} \otimes^{\mathfrak{F}} \llbracket \psi \rrbracket^{\mathfrak{M}} && \text{where } X \otimes^{\mathfrak{F}} Y = \bigcap \{Z \mid X \subseteq Y \rightarrow^{\mathfrak{F}} Z\} \\ \llbracket \varphi \leftarrow \psi \rrbracket^{\mathfrak{M}} &= \llbracket \varphi \rrbracket^{\mathfrak{M}} \leftarrow^{\mathfrak{F}} \llbracket \psi \rrbracket^{\mathfrak{M}} && \text{where } X \leftarrow^{\mathfrak{F}} Y = \bigcup \{Z \in S \mid Y \subseteq Z \rightarrow^{\mathfrak{F}} X\} \end{aligned}$$

The following lemma, which turns out useful in the soundness proof, illustrates the semantic connection between Conditions [\(Prop3\)](#) and [\(Prop4\)](#), on the one side, and the (PRE) and (SUF), on the other side.

Lemma 4.9 (Prefixing and Suffixing). *For all $\mathfrak{M} \in \mathbf{nL}_{\otimes\leftarrow}$, the following frame conditions hold for all $X, Y, Z \in Prop$:*

1. $X \subseteq Y \implies Z \rightarrow^{\delta} X \subseteq Z \rightarrow^{\delta} Y$;
2. $X \subseteq Y \implies Y \rightarrow^{\delta} Z \subseteq X \rightarrow^{\delta} Z$.

Proof. Item 1 is established as follows. Assume that $X \subseteq Y$. That is, $X = X \cap Y$, which implies that $Z \rightarrow X = Z \rightarrow (X \cap Y)$ for all $Z \in Prop$. We conclude by Condition [\(Prop3\)](#) that $Z \rightarrow X = (Z \rightarrow X) \cap (Z \rightarrow Y)$ iff $Z \rightarrow X \subseteq Z \rightarrow Y$. Item 2 is established as follows. Assume that $X \subseteq Y$. That is, $Y = X \cup Y$, which implies that $Y \rightarrow Z = X \cup Y \rightarrow Z$ for all $Z \in Prop$. We conclude by Condition [\(Prop4\)](#) that $Y \rightarrow Z = (X \rightarrow Z) \cap (Y \rightarrow Z)$ iff $Y \rightarrow Z \subseteq X \rightarrow Z$. \square

As a result of the above lemma, $\mathbf{FF} \subseteq \mathbf{nL}_{\otimes\leftarrow}$ as sets of valid formulas. At the proof-theoretic level [\[199\]](#) note that the inclusion of \otimes and \leftarrow in the relevant language makes a difference in designing a complete Hilbert system with respect to $\mathbf{nL}_{\otimes\leftarrow}$ -models. Indeed, it does not suffice to add to FE the residuation laws

$$\begin{aligned} (\otimes\text{RES}) \quad & \varphi \rightarrow (\psi \rightarrow \chi) \iff (\varphi \otimes \psi) \rightarrow \chi \\ (\leftarrow\text{RES}) \quad & \varphi \rightarrow (\psi \rightarrow \chi) \iff \psi \rightarrow (\chi \leftarrow \varphi), \end{aligned}$$

but it is also necessary to strengthen the base system from FE to FFE, which results in the system $\mathbf{FFE}_{\otimes\leftarrow}$. In particular, the newly added (PRE) and (SUF) turn out essential to show that Conditions [\(Prop3\)](#) and [\(Prop4\)](#) hold in the canonical model, while [\(Prop3\)](#) and [\(Prop4\)](#) are essential to show that $(\otimes\text{RES})$ and $(\leftarrow\text{RES})$ preserve validity in $\mathbf{nL}_{\otimes\leftarrow}$ -models. Note that (FE11) and (FE12) are redundant in FFE, hence also in $\mathbf{FFE}_{\otimes\leftarrow}$, since they are provable using (PRE) and (SUF). In the remainder of the chapter, I will assume that each $\mathbf{nL}_{\otimes\leftarrow}$ is an extension of $\mathbf{FFE}_{\otimes\leftarrow}$ and not of FE.

Theorem 4.4 ($\mathbf{nL}_{\otimes\leftarrow}$ soundness). $\vdash_{\mathbf{nL}_{\otimes\leftarrow}} \varphi \implies \mathbf{nL}_{\otimes\leftarrow} \models \varphi$.

Proof. By Theorems [4.1](#) and [4.3](#), it suffices to show that $(\otimes\text{RES})$ and $(\leftarrow\text{RES})$ preserve validity.

$(\otimes\text{RES})$. (\implies). Assume that $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rightarrow \chi \rrbracket$. We infer that $\llbracket \chi \rrbracket \in \{X \mid \llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket \rightarrow X\}$, by which we conclude that $\llbracket \varphi \rrbracket \otimes \llbracket \psi \rrbracket = \bigcap \{X \mid \llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket \rightarrow X\} \subseteq \llbracket \chi \rrbracket$. (\impliedby). Assume that $\llbracket \varphi \otimes \psi \rrbracket \subseteq \llbracket \chi \rrbracket$. By $\llbracket \varphi \otimes \psi \rrbracket = \bigcap \{X \mid \llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket \rightarrow X\}$, we

infer that $\llbracket \varphi \rrbracket \subseteq \bigcap_{i < n} (\llbracket \psi \rrbracket \rightarrow X_i)$ for $\{X_1, \dots, X_n\} = \{X \mid \llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket \rightarrow X\}$. By Condition (Prop3) we infer that $\llbracket \varphi \rrbracket \subseteq (\llbracket \psi \rrbracket \rightarrow \bigcap_{i < n} X_i)$, which implies that $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket \rightarrow \llbracket \varphi \rrbracket \otimes \llbracket \psi \rrbracket$. By Lemma 4.9.1 and $\llbracket \varphi \otimes \psi \rrbracket \subseteq \llbracket \chi \rrbracket$ we infer that $\llbracket \psi \rrbracket \rightarrow (\llbracket \varphi \otimes \psi \rrbracket) \subseteq \llbracket \psi \rrbracket \rightarrow \llbracket \chi \rrbracket$. We conclude that $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket \rightarrow \llbracket \chi \rrbracket$.

(\leftarrow RES). (\implies). Assume that $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rightarrow \chi \rrbracket$. We infer that $\llbracket \psi \rrbracket \in \{X \mid \llbracket \varphi \rrbracket \subseteq X \rightarrow \llbracket \chi \rrbracket\}$. We conclude that $\llbracket \psi \rrbracket \subseteq \bigcup \{X \mid \llbracket \varphi \rrbracket \subseteq X \rightarrow \llbracket \chi \rrbracket\} = \llbracket \chi \rrbracket \leftarrow \llbracket \varphi \rrbracket$. (\Leftarrow). Assume that $\llbracket \psi \rrbracket \subseteq \llbracket \chi \leftarrow \varphi \rrbracket$. By $\llbracket \chi \leftarrow \varphi \rrbracket = \bigcup \{X \mid \llbracket \varphi \rrbracket \subseteq X \rightarrow \llbracket \chi \rrbracket\}$, we infer that $\llbracket \varphi \rrbracket \subseteq \bigcap_{i < n} (X_i \rightarrow \llbracket \chi \rrbracket)$ for $\{X_1, \dots, X_n\} = \{X \mid \llbracket \varphi \rrbracket \subseteq X \rightarrow \llbracket \chi \rrbracket\}$. By Condition (Prop4) we infer that $\llbracket \varphi \rrbracket \subseteq (\bigcup_{i < n} X_i \rightarrow \llbracket \chi \rrbracket)$, which implies that $\llbracket \varphi \rrbracket \subseteq (\llbracket \chi \rrbracket \leftarrow \llbracket \varphi \rrbracket) \rightarrow \llbracket \chi \rrbracket$. By Lemma 4.9.2 and $\llbracket \psi \rrbracket \subseteq \llbracket \chi \leftarrow \varphi \rrbracket$ we infer that $(\llbracket \chi \rrbracket \leftarrow \llbracket \varphi \rrbracket) \rightarrow \llbracket \chi \rrbracket \subseteq \llbracket \psi \rrbracket \rightarrow \llbracket \chi \rrbracket$. We conclude that $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket \rightarrow \llbracket \chi \rrbracket$. □

Next, the extension of the characterisation Theorem 4.3 to logics in the extended language just requires to check some facts about the canonical $\mathbf{nL}_{\otimes \leftarrow}$ -model $\mathfrak{M}_{\mathbf{nL}_{\otimes \leftarrow}}^c$, obtained by uniformly substituting $\mathbf{nL}_{\otimes \leftarrow}$ for \mathbf{nL} .⁴ Namely, one needs to check that (i) $\varphi \otimes \psi$ and $\varphi \leftarrow \psi$ express admissible canonical propositions, and (ii) that (Prop3) and (Prop4) hold in $\mathfrak{M}_{\mathbf{nL}_{\otimes \leftarrow}}^c$. Completeness then follows by the same argument of Theorem 2.3.

Lemma 4.10 (\otimes, \oplus canonical propositions). *For all $[\varphi]^c, [\psi]^c \in Prop^c$:*

1. $[\varphi]^c \otimes^{\mathfrak{M}_{\mathbf{nL}_{\otimes \leftarrow}}^c} [\psi]^c = [\varphi \otimes \psi]^c$;
2. $[\varphi]^c \leftarrow^{\mathfrak{M}_{\mathbf{nL}_{\otimes \leftarrow}}^c} [\psi]^c = [\varphi \leftarrow \psi]^c$

Proof. Item 1 is established as follows. (\subseteq). Assume that $s \in [\varphi_1] \otimes [\varphi_2]$. By definition of $\otimes^{\mathfrak{M}_{\mathbf{nL}_{\otimes \leftarrow}}^c}$ we infer that $s \in [\chi]$ for all $\chi \in \mathfrak{L}_{\mathbf{Rel}}$ such that $[\varphi_1] \subseteq [\varphi_2] \rightarrow [\chi]$. By Lemma 4.3 we infer that $\chi \in s$ for all $\chi \in \mathfrak{L}_{\mathbf{Rel}}$ such that $\vdash_{\mathbf{nL}_{\otimes \leftarrow}} \varphi_1 \rightarrow (\varphi_2 \rightarrow \chi)$. By $\vdash_{\mathbf{nL}_{\otimes \leftarrow}} \varphi_1 \rightarrow (\varphi_2 \rightarrow (\varphi_1 \otimes \varphi_2))$ we conclude that $\varphi_1 \otimes \varphi_2 \in s$ iff $s \in [\varphi_1 \otimes \varphi_2]$. (\supseteq). Conversely, assume that $\varphi_1 \otimes \varphi_2 \in s$ and, for some arbitrary $\chi \in \mathfrak{L}_{\mathbf{Rel}}$, that $[\varphi_1] \subseteq$

⁴Note that, in the canonical model for \mathbf{nL} -logics and $\mathbf{nL}_{\otimes \leftarrow}$, it is possible to work with the same definition of R only because of general frames. Otherwise, the presence of \otimes, \leftarrow in the language would force us to modify the definition of R^c into, respectively, $R^c = \{(s, [\varphi]^c, X) \mid \varphi \rightarrow \psi \in s, [\psi]^c \subseteq X\}$ and $R^c = \{(s, X, [\psi]^c) \mid \varphi \rightarrow \psi \in s, X \subseteq [\varphi]^c\}$ (cf. [61]). What is worse, the new definitions of R^c would now force us to reintroduce bounds to exclude the empty and the full theory (cf. Remark 3.3). As clarified in Section 4.3 this is detrimental to the model theory of contextual modal logics.

$[\varphi_2] \rightarrow [\chi]$ to show that $s \in [\chi]$. By Lemma 4.3 we infer that $\vdash_{\mathfrak{NL}_{\otimes\leftarrow}} \varphi_1 \rightarrow (\varphi_2 \rightarrow \chi)$, which implies by $(\otimes\text{RES})$ that $\vdash_{\mathfrak{NL}_{\otimes\leftarrow}} (\varphi_1 \otimes \varphi_2) \rightarrow \chi$. We conclude by $\varphi_1 \otimes \varphi_2 \in s$ that $\chi \in s$ iff $s \in [\chi]$.

Item 2 is established as follows. (\subseteq) . Assume that $s \in [\varphi_1] \leftarrow [\varphi_2]$. By definition of $\leftarrow^{\mathfrak{S}}$ we infer that there is $\chi \in \mathfrak{L}_{\text{Rel}}$ such that $s \in [\chi]$ and $[\varphi_2] \subseteq [\chi] \rightarrow [\varphi_1]$. By Lemma 4.3 we infer that $\vdash_{\mathfrak{NL}_{\otimes\leftarrow}} \varphi_2 \rightarrow (\chi \rightarrow \varphi_1)$, which implies by $(\leftarrow\text{RES})$ that $\vdash_{\mathfrak{NL}_{\otimes\leftarrow}} \chi \rightarrow (\varphi_1 \leftarrow \varphi_2)$. We conclude by $s \in [\chi]$ that $s \in [\varphi_1 \leftarrow \varphi_2]$. (\supseteq) . Conversely, assume that $s \in [\varphi_1 \leftarrow \varphi_2]$. By $\vdash_{\mathfrak{NL}_{\otimes\leftarrow}} \varphi_2 \rightarrow ((\varphi_1 \leftarrow \varphi_2) \rightarrow \varphi_1)$ and Lemma 4.3 we infer that $[\varphi_2] \subseteq [\varphi_1 \leftarrow \varphi_2] \rightarrow [\varphi_1]$. By definition of $\leftarrow^{\mathfrak{S}}$ we conclude that $s \in [\varphi_1] \leftarrow [\varphi_2]$. \square

Lemma 4.11 (\otimes, \oplus canonical conditions). *Conditions (Prop3) and (Prop4) hold in $\mathfrak{M}_{\mathfrak{NL}_{\otimes\leftarrow}}^c$.*

Proof. (Prop3) holds by the following argument. (\implies) . Note that $\bigcap_{i \leq n} ([\psi] \rightarrow [\varphi_i]) \subseteq [\psi] \rightarrow [\varphi_i]$ for all $i \leq n$. Hence, we reason as follows:

$$\begin{aligned}
\implies & \quad \forall i \leq n (\vdash_{\mathfrak{NL}_{\otimes\leftarrow}} \bigwedge_{i \leq n} (\psi \rightarrow \varphi_i) \rightarrow (\psi \rightarrow \varphi_i)) && \text{by Lemma 4.3} \\
\implies & \quad \forall i \leq n (\vdash_{\mathfrak{NL}_{\otimes\leftarrow}} \bigwedge_{i \leq n} ((\psi \rightarrow \varphi_i) \otimes \psi) \rightarrow \varphi_i) && \text{by } (\otimes\text{RES}) \\
\implies & \quad \vdash_{\mathfrak{NL}_{\otimes\leftarrow}} \bigwedge_{i \leq n} ((\psi \rightarrow \varphi_i) \otimes \psi) \rightarrow \bigwedge_{i \leq n} \varphi_i && \text{by (FE9)} \\
\implies & \quad \vdash_{\mathfrak{NL}_{\otimes\leftarrow}} \bigwedge_{i \leq n} (\psi \rightarrow \varphi_i) \rightarrow (\psi \rightarrow \bigwedge_{i \leq n} \varphi_i) && \text{by } (\otimes\text{RES}) \\
\implies & \quad \bigcap_{i \leq n} ([\psi] \rightarrow [\varphi_i]) \subseteq ([\psi] \rightarrow \bigcap_{i \leq n} [\varphi_i]) && \text{by Lemma 4.3}
\end{aligned}$$

(\Leftarrow) . Note that by (FE2) and (FE11) $\vdash_{\mathfrak{NL}_{\otimes\leftarrow}} (\psi \rightarrow \bigwedge_{i \leq n} \varphi_i) \rightarrow (\psi \rightarrow \varphi_i)$ for all $i \leq n$. We infer by Lemma 4.3 that $[\psi] \rightarrow \bigcap_{i \leq n} [\varphi_i] \subseteq [\psi] \rightarrow [\varphi_i]$ for all $i \leq n$. We conclude that $[\psi] \rightarrow \bigcap_{i \leq n} [\varphi_i] \subseteq \bigcap_{i \leq n} ([\psi] \rightarrow [\varphi_i])$.

(Prop4) holds by the following argument. (\implies) . Note that $\bigcap_{i \leq n} ([\varphi_i] \rightarrow [\psi]) \subseteq [\varphi_i] \rightarrow [\psi]$ for all $i \leq n$. Hence, we reason as follows:

$$\begin{aligned}
\implies & \quad \forall i \leq n (\vdash_{\mathfrak{NL}_{\otimes\leftarrow}} \bigwedge_{i \leq n} (\varphi_i \rightarrow \psi) \rightarrow (\varphi_i \rightarrow \psi)) && \text{by Lemma 4.3} \\
\implies & \quad \forall i \leq n (\vdash_{\mathfrak{NL}_{\otimes\leftarrow}} \varphi_i \rightarrow (\psi \leftarrow \bigwedge_{i \leq n} (\varphi_i \rightarrow \psi))) && \text{by } (\leftarrow\text{RES})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow & \quad \vdash_{\mathbf{nL}_{\otimes\leftarrow}} \bigvee_{i \leq n} \varphi_i \rightarrow (\psi \leftarrow \bigwedge_{i \leq n} (\varphi_i \rightarrow \psi)) && \text{by (FE10)} \\
\Rightarrow & \quad \vdash_{\mathbf{nL}_{\otimes\leftarrow}} \bigwedge_{i \leq n} (\varphi_i \rightarrow \psi) \rightarrow (\bigvee_{i \leq n} \varphi_i \rightarrow \psi) && \text{by } (\leftarrow \text{RES}) \\
\Rightarrow & \quad \bigcap_{i \leq n} ([\varphi_i] \rightarrow [\psi]) \subseteq \bigcup_{i \leq n} [\varphi_i] \rightarrow [\psi] && \text{by Lemma 4.3}
\end{aligned}$$

(\Leftarrow). Note that by (FE3) $\vdash_{\mathbf{nL}_{\otimes\leftarrow}} \varphi_i \rightarrow \bigvee_{i \leq n} \varphi_i$ for all $i \leq n$. Hence, by (FE12) we infer that $\vdash_{\mathbf{nL}_{\otimes\leftarrow}} (\bigvee_{i \leq n} \varphi_i \rightarrow \psi) \rightarrow (\varphi_i \rightarrow \psi)$ for all $i \leq n$. By (FE9) we infer that $\vdash_{\mathbf{nL}_{\otimes\leftarrow}} (\bigvee_{i \leq n} \varphi_i \rightarrow \psi) \rightarrow \bigwedge_{i \leq n} (\varphi_i \rightarrow \psi)$. We conclude by Lemma 4.3 that $\bigcup_{i \leq n} [\varphi_i] \rightarrow [\psi] \subseteq \bigcap_{i \leq n} ([\varphi_i] \rightarrow [\psi])$. \square

Lemma 4.12 ($\mathbf{nL}_{\otimes\leftarrow}$ -Truth). $\llbracket \varphi \rrbracket_{\mathfrak{M}_{\mathbf{nL}_{\otimes\leftarrow}}^c} = [\varphi]_c$.

Proof. The result is established by induction on the structure of φ . By Lemma 4.6, it suffices to show the cases $\varphi := \psi \otimes \chi$ and $\varphi := \psi \leftarrow \chi$. We show only the former, where X is assumed to range over *Prop* – the latter is established similarly.

$$\begin{aligned}
\llbracket \psi \otimes \chi \rrbracket &= \bigcap \{X \mid \llbracket \psi \rrbracket \subseteq \llbracket \chi \rrbracket \rightarrow X\} && \text{by Definition 4.9} \\
&= \bigcap \{X \mid [\psi] \subseteq [\chi] \rightarrow X\} && \text{by induction hypothesis} \\
&= [\psi] \otimes [\chi] && \text{by Definition 4.9} \\
&= [\psi \otimes \chi] && \text{by Lemma 4.10}
\end{aligned}$$

\square

Theorem 4.5 ($\mathbf{nL}_{\otimes\leftarrow}$ -Characterisation). *For any selection of $\mathbf{nL}_{\otimes\leftarrow}$ -conditions: $\mathbf{nL}_{\otimes\leftarrow} \models \varphi \iff \vdash_{\mathbf{nL}_{\otimes\leftarrow}} \varphi$.*

Proof. One direction follows from Theorem 4.4 and Lemma 4.7, while the other from Lemmas 4.5, 4.8, 4.10 and 4.11. \square

4.3 Contextual Modal Logics based on Neighborhood Semantics

Having shown how to characterise weak relevant modal logics in the full relevant language via neighborhood semantics, I will now turn to the task of developing contextual modal logics based on relevant logics in the $\mathbf{nL}_{\otimes\leftarrow}$ family.

Neighborhood semantics for relevant logic is extremely useful from the perspective of contextual modal logics. To see why this is so, note that by Definition 4.5 neighborhood canonical models for extensions of $\mathbf{nL}_{\otimes\leftarrow}$ rule out the empty and the full theory as admissible propositions, as the canonical representatives of S and \emptyset in Prop^c would break variable sharing.⁵ By Remark 3.3, this is sufficient to dispense with the use of bounded structures, which greatly simplifies the model theory of contextual modal logics.

4.3.1 Possible Worlds in Neighborhood Semantics

$\mathbf{C.nL}_{\otimes\leftarrow}$ -models are defined by adapting the definition of $\mathbf{C.L}$ -models to the neighborhood setting in a straightforward way, as follows.

Definition 4.10 (Neighborhood possible worlds). *For a given $\mathbf{nL}_{\otimes\leftarrow}$ -model \mathfrak{M} based on \mathfrak{F} , let $w \in S$ be a possible world iff for all $s, t \in S$ and all $X, Y \in \mathcal{P}(S)$:*

$$w^* = w \quad (\text{W1})$$

$$w \in X \rightarrow^{\mathfrak{F}} Y \iff w \in \neg^{\mathfrak{F}} X \vee^{\mathfrak{F}} Y \quad (\text{W2})$$

$$w \in X \otimes^{\mathfrak{F}} Y \implies w \in X \wedge^{\mathfrak{F}} Y \quad (\text{W3})$$

$$w \in \neg^{\mathfrak{F}} Y \vee^{\mathfrak{F}} X \implies w \in X \leftarrow^{\mathfrak{F}} Y \quad (\text{W4})$$

$$W \subseteq \square_L^{\mathfrak{F}} X \iff L \subseteq X \quad (\text{W5})$$

Definition 4.11 (Neighborhood W -models). *Let a $\mathbf{C.nL}_{\otimes\leftarrow}$ -model be a structure of the following form:*

$$(S, L, W, \text{Prop}, *, R, Q, Q_L, V)$$

- $(S, L, \text{Prop}, *, R, Q, Q_L, V)$ is a $\mathbf{nL}_{\otimes\leftarrow}$ -model;
- $W \subseteq L$ is a set of possible worlds;

where $(S, L, W, \leq, *, R, Q, Q_L)$ is the $\mathbf{C.nL}_{\otimes\leftarrow}$ -frame a $\mathbf{C.nL}_{\otimes\leftarrow}$ -model $(S, L, W, \text{Prop}, *, R, Q, Q_L, V)$ is based on. Let $\mathbf{C.nL}_{\otimes\leftarrow}$ denote the class of all neighborhood $\mathbf{C.nL}_{\otimes\leftarrow}$ -models.

Let the notions of *proposition* expressed by φ in a $\mathbf{C.nL}_{\otimes\leftarrow}$ -model, *satisfaction*, *generalised validity* and *consequence* be defined as in Definition 4.9.

⁵This feature is in line with the exclusion, taken up in [194], of all connectives which give rise to universal formulas, like \top and \perp , from the toolbox of the relevant logician.

The properties of **C.L**-models carry over to the neighborhood setting, with either the same or simplified proofs. In particular, note that, as in Chapter 3, $W \subseteq L$ is redundant, but I will assume it since its provability, stated in the following Proposition 4.1, relies on conditions (W3)–(W4), which are only needed in presence of \otimes, \leftarrow in the language.

Proposition 4.1 (Logical worlds). *For all models $\mathfrak{M} \in \mathbf{nL}_{\otimes \leftarrow}$ and all possible worlds $w \in S$, $w \in L$.*

Proof. First, note that, for all $X \in \mathcal{P}(S)$, $X \subseteq L \otimes^{\mathfrak{s}} X$ by definition of $\otimes^{\mathfrak{s}}$ and Condition (Prop2). Then, take an arbitrary possible world $w \in S$. By $w \in \{w\}$ we infer that $w \in L \otimes^{\mathfrak{s}} \{w\}$, which by Condition (W3) implies that $w \in L \wedge^{\mathfrak{s}} \{w\}$ iff $w \in L \cap \{w\}$. We conclude that $w \in L$. \square

Lemma 4.13 (**C.nL** $_{\otimes \leftarrow}$ -heredity). *For all $\mathfrak{M} \in \mathbf{C.nL}_{\otimes \leftarrow}$ and all $\varphi \in \mathcal{L}_{\text{Rel}}$: $\llbracket \varphi \rrbracket_{\mathfrak{M}} \in \text{Prop}$.*

Proof. The result is established by the same argument of Lemma 4.1, where the cases concerning \otimes, \leftarrow hold thanks to Definition 4.9. \square

Lemma 4.14 (Neighborhood **C.nL** $_{\otimes \leftarrow}$ -flatness). *For all $\mathfrak{M} \in \mathbf{C.nL}_{\otimes \leftarrow}$, all $w \in W$ and all $\varphi, \psi \in \mathcal{L}_{\text{Rel}}$:*

1. $\mathfrak{M}, w \models \mathbf{t}$;
2. $\mathfrak{M}, w \models \neg \varphi$ iff $\mathfrak{M}, w \not\models \varphi$;
3. $\mathfrak{M}, w \models \varphi \otimes \psi$ iff $\mathfrak{M}, w \models \varphi \wedge \psi$;
4. $\mathfrak{M}, w \models \varphi \oplus \psi$ iff $\mathfrak{M}, w \models \varphi \vee \psi$;
5. $\mathfrak{M}, w \models \varphi \rightarrow \psi$ iff $(\mathfrak{M}, w \models \varphi \implies \mathfrak{M}, w \models \psi)$;
6. $\mathfrak{M}, w \models \varphi \leftarrow \psi$ iff $(\mathfrak{M}, w \models \psi \implies \mathfrak{M}, w \models \varphi)$.

Proof. Item 1 follows from $W \subseteq L$. Items 2 and 5 follow, respectively, from Conditions (W1) and (W2). Item 4 follows from Items 2 and 5. The remaining cases are established as follows.

3. (\implies). It follows from Condition (W3). (\impliedby). Assume by contradiction that $w \in \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$ and $w \notin \llbracket \varphi \rrbracket \otimes \llbracket \psi \rrbracket$. Hence, there is $X \in \text{Prop}^+$ such that $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket \rightarrow X$ and $w \notin X$. By $w \in \llbracket \varphi \rrbracket$ we infer that $w \in \llbracket \psi \rrbracket \rightarrow Z$. By Item 4 we infer that either $w \notin \llbracket \psi \rrbracket$ or $w \in Z$, which contradicts $w \in \llbracket \psi \rrbracket$ and $w \notin Z$.

6. (\implies). Assume by contradiction that $w \in \llbracket \varphi \rrbracket \leftarrow \llbracket \psi \rrbracket$ and that $w \notin \neg \llbracket \psi \rrbracket$, $w \notin \llbracket \varphi \rrbracket$. Hence, there is $Z \in \text{Prop}^+$ such that $w \in Z$ and $\llbracket \psi \rrbracket \subseteq Z \rightarrow \llbracket \varphi \rrbracket$. By $w \notin \neg \llbracket \psi \rrbracket$ and Item 1 we infer that $w \in Y$, which implies that $w \in Z \rightarrow \llbracket \varphi \rrbracket$. By Item 4 we infer that either $w \notin Z$ or $w \in \llbracket \varphi \rrbracket$, which contradict $w \in Z$ and $w \notin \llbracket \varphi \rrbracket$. (\impliedby). It follows from Condition (W4).

□

Lemma 4.15 (Semantic entailments). *For all $\varphi, \psi \in \mathfrak{L}_{\text{Rel}}$:*

1. $\varphi \models_{\mathbf{C.nL}_{\otimes\leftarrow}}^S \psi$ iff $\mathbf{C.nL}_{\otimes\leftarrow} \models^W \Box_L(\varphi \rightarrow \psi)$;
2. $\varphi \models_{\mathbf{C.nL}_{\otimes\leftarrow}}^W \psi$ iff $\mathbf{C.nL}_{\otimes\leftarrow} \models^W \varphi \rightarrow \psi$.

Proof. Item 1 is established by the following chain of equivalences: $W \subseteq \llbracket \Box_L(\varphi \rightarrow \psi) \rrbracket$ iff (by Definition 3.2) $W \subseteq \Box_L \llbracket \varphi \rightarrow \psi \rrbracket$ iff (by Condition (W5)) $L \subseteq \llbracket \varphi \rightarrow \psi \rrbracket$ iff (by Condition (Prop2)) and Lemma 4.13) $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$. Item 2 is an easy consequence of Lemma 4.14.5. □

4.3.2 The Neighborhood + Construction

To conclude the presentation of $\mathbf{C.nL}_{\otimes\leftarrow}$ -models, note that it is easy to see that for every $\mathbf{C.nL}_{\otimes\leftarrow}$ -model \mathfrak{M} there is an $\mathbf{nL}_{\otimes\leftarrow}$ -model \mathfrak{M}' such that $\mathfrak{M} \models \varphi$ only if $\mathfrak{M}' \models \Box_L \varphi$. As in the relational case, the converse operation is more delicate. However, unlike the relational case, the + construction of the suitable $\mathbf{C.nL}_{\otimes\leftarrow}$ -model from a given $\mathbf{nL}_{\otimes\leftarrow}$ -model is made easier by the absence of bounds.

Definition 4.12 (+-Construction). *For any neighborhood $\mathbf{nL}_{\otimes\leftarrow}$ -model $\mathfrak{M}_{\mathbf{nL}_{\otimes\leftarrow}} = (S, L, Prop, R^*, Q, Q_L, V)$, let the structure $\mathfrak{M}_{\mathbf{nL}_{\otimes\leftarrow}}^+$ be defined as follows for all $p \in At$ and $X, Y \in Prop^+$:*

$$\mathfrak{M}_{\mathbf{nL}}^+ = (S^+, W, L^+, Prop^+, R^+, *^+, Q^+, Q_L^+, V^+)$$

- $S^+ = S \cup \{w\}$;
- $Prop^+ = \{Z \mid Z \cap S \in Prop\}$;
- $W = \{w\}$;
- $L^+ = L \cup \{w\}$;
- $R^+ = \{(s, X, Y) \mid Rs(X \cap S)(Y \cap S)\} \cup \{(w, X, Y) \mid w \in \neg X \vee Y\}$;
- $*^+ = * \cup \{(w, w)\}$;
- $Q^+ = \{(s, X) \mid Qs(X \cap S)\} \cup \{(w, X) \mid X \in Prop^+\}$;
- $Q_L^+ = \{(s, X) \mid Q_Ls(X \cap S)\} \cup \{(w, X) \mid L \subseteq X\}$;
- $V^+(p) = V(p)$ for all $p \in At$.

Lemma 4.16 (+ construction). *For all $\mathfrak{M} \in \mathbf{nL}_{\otimes\leftarrow}$, $\mathfrak{M}^+ \in \mathbf{C.nL}_{\otimes\leftarrow}$ is such that $\mathfrak{M} \not\models \varphi \implies \mathfrak{M}^+ \not\models \Box_L \varphi$.*

Proof. The result is established by the same proof strategy of Lemma 3.6, i.e. by proving three Claims 4-6.

Claim 4. $\mathfrak{M}^+ \in \mathbf{C.nL}_{\otimes \leftarrow \otimes \leftarrow}$.

Proof. Clearly, $L^+ \in Prop^+$ by definition of $Prop^+$ and $L \in Prop$. Then, it suffices to show that Conditions (Prop1)–(Prop4) and (W1)–(W5) hold in \mathfrak{M}^+ . (W1) holds by the same argument of Claim 1. The remaining cases are established as follows, where we assume throughout the proof that X_1, X_2, X, Y are arbitrary members of $Prop^+$.

(Prop1). It is established by case distinction, where the cases $\otimes := \wedge$, $\otimes := \vee$ are trivial. If $\otimes := \neg$, we reason as follows. By definition of $*^+$, we infer that $\neg^{\delta^+} X_1 = \{s \in S \mid s^{*^+} \notin X_1 \cap S\} \cup \{w \mid w \notin X\} = \neg^{\delta}(X_1 \cap S) \cup \{w \mid w \notin X\}$. By (Prop1) holding in \mathfrak{M} we infer that $\neg^{\delta}(X_1 \cap S) \in Prop$. By definition of $Prop^+$ we conclude that $\neg^{\delta^+} X_1 \in Prop^+$. If $\otimes := \rightarrow$, we reason as follows. If $X_1 \in Prop$ or $X_2 \notin Prop$, then by definition of R^+ we infer that $X_1 \rightarrow^{\delta^+} X_2 = \{s \mid Rs(X_1 \cap S)(X_2 \cap S)\} \cup \{w\} = (X_1 \cap S) \rightarrow^{\delta}(X_2 \cap S) \cup W$. By (Prop1) holding in \mathfrak{M} we infer that $(X_1 \cap S) \rightarrow^{\delta}(X_2 \cap S) \in Prop$, by which we conclude by definition of $Prop^+$ that $X_1 \rightarrow^{\delta^+} X_2 \in Prop^+$. Otherwise, if $X_1 \notin Prop$ and $X_2 \in Prop$, then by definition of R^+ we infer that $X_1 \rightarrow^{\delta^+} X_2 = (X_1 \cap S) \rightarrow^{\delta}(X_2 \cap S)$. By definition of $Prop^+$ we conclude that $X_1 \rightarrow^{\delta^+} X_2 \in Prop \subseteq Prop^+$. If $\otimes := \otimes$, we reason as follows. By Definition 4.2, $X_1 \otimes^{\delta^+} X_2 = \bigcap \{Z \in Prop^+ \mid X_1 \subseteq X_2 \rightarrow^{\delta^+} Z\}$. Hence, by $Z_1, \dots, Z_n \in Prop^+$ and the case $\otimes := \wedge$ we conclude that $X_1 \otimes^{\delta^+} X_2 \in Prop^+$. The case $\otimes := \leftarrow$ is established similarly. If $\otimes := \square$, we reason as follows. By definition of Q^+ we infer that $\square^{\delta^+} X_1 = \{s \mid Qs(X \cap S)\} \cup \{w\} = \square^{\delta}(X \cap S) \cup W$. By definition of $Prop^+$ we conclude that $\square^{\delta^+} X_1 \in Prop^+$. If $\otimes := \square_L$, we reason as follows. By definition of Q_L^+ we infer that $\square_L^{\delta^+} X_1 \in \{\square^{\delta}(X \cap S), \square^{\delta}(X \cap S) \cup W\}$. In both cases, $\square_L^{\delta^+} X_1 \in Prop^+$ by definition of $Prop^+$.

(Prop2). (\implies). Assume that $X \subseteq Y$. Hence, by (Prop2) holding in \mathfrak{M} we infer that $L \subseteq (X \cap S) \rightarrow^{\delta}(Y \cap S)$, which implies by definition of R^+ that $L \subseteq X \rightarrow^{\delta^+} Y$. Now, by $X \subseteq Y$ we infer that $w \in X \rightarrow^{\delta^+} Y$. By the latter and $L \subseteq X \rightarrow^{\delta^+} Y$ we conclude that $L^+ \subseteq X \rightarrow^{\delta^+} Y$. (\impliedby). Assume that $L^+ \subseteq X \rightarrow^{\delta^+} Y$. Hence, $L \subseteq X \rightarrow^{\delta^+} Y$ and $w \in X \rightarrow^{\delta^+} Y$, which by definition of R^+ and (Prop2) holding in \mathfrak{M} implies that $(X \cap S) \subseteq (Y \cap S)$ and $w \notin X$ or $w \in Y$. Now, to conclude $X \subseteq Y$, we distinguish cases. If $w \notin X$, then we conclude that $X = X \cap S \subseteq Y \cap S \subseteq Y$. If $w \in X$, then $w \in Y$ by which we conclude that $X = (X \cap S) \cup \{w\} \subseteq (Y \cap S) \cup \{w\} = Y$.

(Prop3). Assume that $s \in \bigcap_{i \leq n} (Y \rightarrow^{\delta^+} X_i)$. If $Y \in Prop$ or each $X_1, \dots, X_n \notin Prop$, then by the argument for (Prop1) we infer that $\bigcap_{i \leq n} (Y \rightarrow^{\delta^+} X_i) = \bigcap_{i \leq n} ((Y \cap S) \rightarrow^{\delta}(X_i \cap S)) \cup W$. By (Prop3) holding in \mathfrak{M} we conclude that $\bigcap_{i \leq n} ((Y \cap S) \rightarrow^{\delta}(X_i \cap S)) \cup W \in Prop^+$.

$S) \rightarrow^{\delta} (X_i \cap S)) \cup W = ((Y \cap S) \rightarrow^{\delta} \bigcap_{i \leq n} (X_i \cap S)) \cup W = Y \rightarrow^{\delta^+} \bigcap_{i \leq n} X_i$.
 Otherwise, if $Y \notin Prop$ and either $X_1, \dots, X_n \in Prop$, then $\bigcap_{i \leq n} (Y \rightarrow^{\delta^+} X_i) = \bigcap_{i \leq n} ((Y \cap S) \rightarrow^{\delta} (X_i \cap S))$, by which we conclude by **(Prop3)** holding in \mathfrak{M} that $\bigcap_{i \leq n} (Y \rightarrow^{\delta^+} X_i) = Y \rightarrow^{\delta^+} \bigcap_{i \leq n} X_i$. **(Prop4)** is established similarly.

(W2). It holds by $\{(w, X, Y) \mid w \in \neg X \vee Y\} \subseteq R^+$.

(W3). Assume by contraposition that $w \notin X \wedge^{\delta^+} Y$. To show that $w \notin X \otimes^{\delta^+} Y$, it suffices to show that there is Z such that $X \subseteq Y \rightarrow^{\delta^+} Z$ and $w \notin Z$. To this aim, take $Z := (X \otimes^{\delta^+} Y) \cap S$. Clearly, $w \notin Z \subseteq S$. To show that $X \subseteq Y \rightarrow^{\delta^+} Z$, note that $X \subseteq Y \rightarrow^{\delta^+} Z$ iff $(X \cap S) \subseteq (Y \cap S) \rightarrow^{\delta} Z$ and $(w \in X \implies w \in \neg^{\delta^+} Y \vee^{\delta^+} Z)$. By $w \notin X \wedge^{\delta^+} Y$, the latter is equivalent to $(X \cap S) \subseteq (Y \cap S) \rightarrow^{\delta} Z$. Again by $w \notin X \wedge^{\delta^+} Y$, we infer that $X \subseteq Y \rightarrow^{\delta^+} Z$ iff $(X \cap S) \subseteq (Y \cap S) \rightarrow^{\delta} ((X \cap S) \otimes^{\delta} (Y \cap S))$, which holds.

(W4). Assume that $w \in \neg^{\delta^+} Y \vee^{\delta^+} X$. To show that $w \in X \leftarrow^{\delta^+} Y$, it suffices to show that there is Z such that $w \in Z$ and $Y \subseteq Z \rightarrow^{\delta^+} X$. To this aim, take $Z := (X \leftarrow^{\delta^+} Y) \cup W$. Clearly, $w \in Z \supseteq W$. To show that $Y \subseteq Z \rightarrow^{\delta^+} X$, note that $Y \subseteq Z \rightarrow^{\delta^+} X$ iff $Y \cap S \subseteq (Z \cap S) \rightarrow^{\delta} (X \cap S)$ and $(w \in Y \implies w \in \neg^{\delta^+} Z \vee^{\delta^+} X)$. By $w \in \neg^{\delta^+} X \vee^{\delta^+} Y$, the latter is equivalent to $Y \cap S \subseteq (Z \cap S) \rightarrow^{\delta} (X \cap S)$. Again by $w \in \neg^{\delta^+} X \vee^{\delta^+} Y$ we infer that $Z \cap S = (X \cap S) \leftarrow^{\delta} (Y \cap S)$, by which we conclude that $Y \subseteq Z \rightarrow^{\delta^+} X$ iff $(Y \cap S) \subseteq ((X \cap S) \leftarrow^{\delta} (Y \cap S)) \rightarrow^{\delta} (X \cap S)$, which holds.

(W5). It holds by $\{(w, X) \mid L \subseteq X\} \subseteq Q_L^+$.

□

Claim 5. $\mathfrak{M}, s \models \varphi \iff \mathfrak{M}^+, s \models \varphi$.

Proof. The result is established by induction on the structure of φ . The base case and the induction step cases $\varphi := \psi \wedge \chi$, $\varphi := \psi \vee \chi$ are trivial.

$\varphi := \neg\psi$. The following chain of equivalences holds: $\mathfrak{M}, s \models \neg\psi$ iff $\mathfrak{M}, s^* \not\models \psi$ iff (by the induction hypothesis and the fact that if $s \in S$, then $s^* = s^{*+}$) $\mathfrak{M}^+, s^{*+} \not\models \psi$ iff $\mathfrak{M}^+, s \models \neg\psi$.

$\varphi := \psi \rightarrow \chi$. The following chain of equivalences holds: $\mathfrak{M}, s \models \psi \rightarrow \chi$ iff $Rs[\psi]_{\mathfrak{M}}[\chi]_{\mathfrak{M}}$ iff (by induction hypothesis) $Rs([\psi]_{\mathfrak{M}^+} \cap S)([\chi]_{\mathfrak{M}^+} \cap S)$ iff (by $s \in S$ and definition of R^+) $R^+s[\psi]_{\mathfrak{M}^+}[\chi]_{\mathfrak{M}^+}$ iff $\mathfrak{M}^+, s \models \psi \rightarrow \chi$.

$\varphi := \psi \otimes \chi$. The following chain of equivalences holds: $\mathfrak{M}, s \models \psi \otimes \chi$ iff $s \in \bigcap \{X \mid \llbracket \psi \rrbracket_{\mathfrak{M}} \subseteq \llbracket \psi \rrbracket_{\mathfrak{M}} \rightarrow^{\tilde{\delta}} X\}$ iff (by induction hypothesis) $s \in \bigcap \{X \mid (\llbracket \psi \rrbracket_{\mathfrak{M}^+} \cap S) \subseteq (\llbracket \psi \rrbracket_{\mathfrak{M}^+} \cap S) \rightarrow^{\tilde{\delta}} X\}$ iff (by $w \notin \llbracket \psi \rrbracket_{\mathfrak{M}^+} \cap S, w \notin \llbracket \chi \rrbracket_{\mathfrak{M}^+} \cap S$ and definition of R^+) $s \in \bigcap \{X \mid \llbracket \psi \rrbracket_{\mathfrak{M}^+} \subseteq \llbracket \psi \rrbracket_{\mathfrak{M}^+} \rightarrow^{\tilde{\delta}^+} X\}$ iff $\mathfrak{M}^+, s \models \psi \otimes \chi$.

$\varphi := \psi \leftarrow \chi$. The following chain of equivalences holds: $\mathfrak{M}, s \models \psi \leftarrow \chi$ iff $s \in \bigcup \{X \mid \llbracket \chi \rrbracket_{\mathfrak{M}} \subseteq X \rightarrow^{\tilde{\delta}} \llbracket \psi \rrbracket_{\mathfrak{M}}\}$ iff (by induction hypothesis) $s \in \bigcup \{X \mid (\llbracket \chi \rrbracket_{\mathfrak{M}^+} \cap S) \subseteq X \rightarrow^{\tilde{\delta}} (\llbracket \psi \rrbracket_{\mathfrak{M}^+} \cap S)\}$ iff (by $w \notin \llbracket \psi \rrbracket_{\mathfrak{M}^+} \cap S, \llbracket \chi \rrbracket_{\mathfrak{M}^+} \cap S$ and definition of R^+) $s \in \bigcup \{X \mid \llbracket \chi \rrbracket_{\mathfrak{M}^+} \subseteq X \rightarrow^{\tilde{\delta}^+} \llbracket \psi \rrbracket_{\mathfrak{M}^+}\}$ iff $\mathfrak{M}^+, s \models \psi \leftarrow \chi$.

$\varphi := \Box_{(L)}\psi$. The following chain of equivalences holds: $\mathfrak{M}, s \models \Box_{(L)}\psi$ iff $Q_{(L)}s \llbracket \psi \rrbracket_{\mathfrak{M}}$ iff (by the induction hypothesis) $Q_{(L)}s(\llbracket \psi \rrbracket_{\mathfrak{M}^+} \cap S)$ iff (by definition of $Q_{(L)}^+$) $Q_{(L)}^+s \llbracket \psi \rrbracket_{\mathfrak{M}^+}$ iff $\mathfrak{M}^+, s \models \Box_{(L)}\psi$.

□

Claim 6. $\mathfrak{M} \in \mathbf{nL}_{\otimes \leftarrow} \implies \mathfrak{M}^+ \in \mathbf{C.nL}_{\otimes \leftarrow}$.

Proof. The result is established by showing that each of the $\mathbf{nL}_{\otimes \leftarrow}$ -frame conditions of Figure 4.2 holds in (the frame underlying) \mathfrak{M}^+ whenever it holds in (the frame underlying) \mathfrak{M} . Note that, by the presence of \otimes and \leftarrow , (PRE) and (SUF) hold by default. Throughout the proof, we assume that X, Y, Z are arbitrary members of $Prop^+$.

(\wedge I) Assume that (\wedge I) holds in \mathfrak{M} . To show that (\wedge I) holds in \mathfrak{M}^+ , take some arbitrary $s \in (X \rightarrow^{\tilde{\delta}^+} Y) \cap (X \rightarrow^{\tilde{\delta}^+} Z)$. If $s \in S$, then by definition of R and R^+ we infer that $s \in (X \cap S) \rightarrow^{\tilde{\delta}} (Y \cap S) \cap (X \cap S) \rightarrow^{\tilde{\delta}} (Z \cap S)$. By (\wedge I) holding in \mathfrak{M} we infer that $s \in (X \cap S) \rightarrow^{\tilde{\delta}} ((Y \cap S) \cap (Z \cap S)) = (X \cap S) \rightarrow^{\tilde{\delta}} ((Y \cap Z) \cap S)$. We conclude by definition of R and R^+ that $s \in X \rightarrow^{\tilde{\delta}^+} (Y \cap Z)$. If $s = w$, then by Condition (W2) we infer that $s \in \neg^{\tilde{\delta}^+} X \cup Y$ and $s \in \neg^{\tilde{\delta}^+} X \cup Z$, which implies that $s \in \neg^{\tilde{\delta}^+} X \cup (Y \cap Z)$. We conclude by Condition (W2) that $s \in X \rightarrow^{\tilde{\delta}^+} (Y \cap Z)$. The case (\vee E) is established similarly.

(DNI) Assume that (DNI) holds in \mathfrak{M} . To show that (DNI) holds in \mathfrak{M}^+ , take some arbitrary $s \in X$. If $s \in S$, then $s \in X \cap S$ we infer that $s \in \neg^{\tilde{\delta}} \neg^{\tilde{\delta}} (X \cap S)$. We conclude by definition of $*$ and $*^+$ that $s \in \neg^{\tilde{\delta}^+} \neg^{\tilde{\delta}^+} X$. If $s = w$, then by Condition (W1) we infer that $s^{*+*+} = s^{*+} = s$. We conclude that $s \in \neg^{\tilde{\delta}^+} \neg^{\tilde{\delta}^+} X$. (DNE), (RD) and (X) are established similarly.

(CP) Assume that (CP) holds in \mathfrak{M} . To show that (X) holds in \mathfrak{M}^+ , take some arbitrary $s \in X \rightarrow^{\tilde{\delta}^+} Y$. If $s \in S$, then by definition of R and R^+ we infer that

$s \in (X \cap S) \rightarrow^{\delta} (Y \cap S)$. By (CP) holding in \mathfrak{M} we infer that $s \in \neg^{\delta}(Y \cap S) \rightarrow^{\delta} \neg^{\delta}(X \cap S)$. By definition of $*$ we infer that $s \in (\neg^{\delta}Y \cap S) \rightarrow^{\delta} (\neg^{\delta}X \cap S)$. We conclude by definition of $R, R^+, *$ and $*^+$ that $s \in \neg^{\delta^+}Y \rightarrow^{\delta^+} \neg^{\delta^+}X$. If $s = w$, then by Condition (W2) we infer that $s \in \neg^{\delta^+}X \cup Y$. By definition of $*^+$ and Condition (W1) we infer that $\neg^{\delta^+}X \cup Y = \neg^{\delta^+} \neg^{\delta^+}Y \cup \neg^{\delta^+}X$. By Condition (W2) we conclude that $s \in \neg^{\delta^+}Y \rightarrow^{\delta^+} \neg^{\delta^+}X$.

(B) Assume that (B) holds in \mathfrak{M} . To show that (B) holds in \mathfrak{M}^+ , take some arbitrary $s \in X \rightarrow^{\delta^+} Y$. If $s \in S$, then by definition of R and R^+ we infer that $s \in (X \cap S) \rightarrow^{\delta} (Y \cap S)$. By (B) holding in \mathfrak{M} we infer that $s \in ((Y \cap S) \rightarrow^{\delta} (Z \cap S)) \rightarrow^{\delta} ((X \cap S) \rightarrow^{\delta} (Z \cap S))$. By definition of R and R^+ we conclude that $s \in (Y \rightarrow^{\delta^+} Z) \rightarrow^{\delta^+} (X \rightarrow^{\delta^+} Z)$. If $s = w$, then assume by contradiction that $s \in X \rightarrow^{\delta^+} Y$ and $s \notin (Y \rightarrow^{\delta^+} Z) \rightarrow^{\delta^+} (X \rightarrow^{\delta^+} Z)$. By Conditions (W1) and (W2), we infer that $s \in (Y \rightarrow^{\delta^+} Z)$ and $s \notin (X \rightarrow^{\delta^+} Z)$. The latter implies that $s \in X$ and $s \notin Z$. By $s \in X \rightarrow^{\delta^+} Y$ and $s \in X$ we infer that $s \in Y$. By $s \in (Y \rightarrow^{\delta^+} Z)$ and $s \in Y$ we infer that $s \in Z$, contradicting $s \notin Z$. The cases for (CB),(W),(C) and (WB) follow the same strategy.

(M) Assume that (M) holds in \mathfrak{M} . To show that (M) holds in \mathfrak{M}^+ , take some arbitrary $s \in X$. If $s \in S$, then by (M) holding in \mathfrak{M} we infer that $s \in (X \cap S) \rightarrow^{\delta} (X \cap S)$. By definition of R and R^+ we conclude that $s \in X \rightarrow^{\delta^+} X$. If $s = w$, then $s \in X \cup \neg^{\delta^+}X$. we conclude by Condition (W2) that $s \in X \rightarrow^{\delta^+} X$.

(ER) Assume that (ER) holds in \mathfrak{M} . To show that (ER) holds in \mathfrak{M}^+ , assume that $L^+ \subseteq X$ and take some arbitrary $s \in L^+$. If $s \in L$, then by $L \subseteq (X \cap S)$ and (ER) holding in \mathfrak{M} we infer that $s \in ((X \cap S) \rightarrow^{\delta} (Y \cap S)) \rightarrow^{\delta} (Y \cap S)$ for all $Y \in Prop^+$. We conclude by definition of R and R^+ that $s \in (X \rightarrow^{\delta^+} Y) \rightarrow^{\delta^+} Y$. if $s = w$, then by $L^+ \subseteq X$ we infer that $s \in X$. If, by contradiction, $s \notin (X \rightarrow^{\delta^+} Y) \rightarrow^{\delta^+} Y$, by Conditions (W1) and (W2) we infer that $s \in X \rightarrow^{\delta^+} Y$ and $s \notin Y$. By $s \in X$ and $s \in X \rightarrow^{\delta^+} Y$ we infer that $s \in Y$, contradicting $s \notin Y$.

(□M) Assume that (□M) holds in \mathfrak{M} . To show that (□M) holds in \mathfrak{M}^+ , assume that $X \subseteq Y$ and take some arbitrary $s \in \Box^{\delta^+}X$. If $s \in S$, then by definition of Q and Q^+ we infer that $s \in \Box^{\delta}(X \cap S)$. By $X \subseteq Y$ we infer that $X \cap s \subseteq Y \cap S$. By (□M) holding in \mathfrak{M} and $s \in \Box^{\delta}(X \cap S)$ we infer that $s \in \Box^{\delta}(Y \cap S)$. By definition of Q and Q^+ we conclude that $s \in \Box^{\delta^+}Y$. If $s = w$, then we conclude that $s \in \Box^{\delta^+}Y$ directly from the definition of Q^+ .

(□N) Assume that (□N) holds in \mathfrak{M} . To show that (□N) holds in \mathfrak{M}^+ , assume that $L^+ \subseteq X$ and take some arbitrary $s \in L^+$. If $s \in L$, then by $L^+ \subseteq X$ we infer that

$L = (L^+ \cap S) \subseteq (X \cap S)$. By $(\Box N)$ holding in \mathfrak{M} we infer that $s \in \Box^{\mathfrak{S}}(X \cap S)$. By definition of Q and Q^+ we conclude that $s \in \Box^{\mathfrak{S}^+}X$. If $s = w$, then we conclude that $s \in \Box^{\mathfrak{S}^+}X$ directly from the definition of Q^+ .

$(\Box C)$ Assume that $(\Box C)$ holds in \mathfrak{M} . To show that $(\Box C)$ holds in \mathfrak{M}^+ , assume that $s \in \Box^{\mathfrak{S}^+}X \cap \Box^{\mathfrak{S}^+}Y$. If $s \in S$, then by definition of Q and Q^+ we infer that $s \in \Box^{\mathfrak{S}}(X \cap S) \cap \Box^{\mathfrak{S}}(Y \cap S)$. By $(\Box C)$ holding in \mathfrak{M} we infer that $s \in \Box^{\mathfrak{S}}((X \cap S) \cap (Y \cap S)) = \Box^{\mathfrak{S}}((X \cap Y) \cap S)$. By definition of Q and Q^+ we conclude that $s \in \Box^{\mathfrak{S}^+}(X \cap Y)$. If $s = w$, then we conclude that $s \in \Box^{\mathfrak{S}^+}(X \cap Y)$ directly from the definition of Q^+ and Lemma 4.13. The remaining cases are established similarly. \square

Putting things together, assume for some arbitrary $\mathfrak{M} \in \mathbf{nL}_{\otimes \leftarrow}$ that $\mathfrak{M} \not\models^L \varphi$. Then, there is $l \in L$ such that $\mathfrak{M}, l \not\models \varphi$. Then, (i) by Claim 4 there is a $\mathbf{C.nFFE}_{\otimes \leftarrow}$ -model, \mathfrak{M}^+ , such that (ii) by Claim 5 $\mathfrak{M}^+, l \not\models \varphi$. Then, (iii) by Condition W5 there is $w \in W^+$ such that not $Q_L^+ w \Vdash \varphi$, which implies $\mathfrak{M}^+, w \not\models \Box_L \varphi$. We conclude that $\mathfrak{M}^+ \not\models^W \Box_L \varphi$. Finally, by Claim 6 steps (i)–(iii) can be performed for all $\mathbf{nL}_{\otimes \leftarrow}$ -models, which concludes the proof. \square

4.3.3 Soundness and Completeness of C.nL

The axiomatisation of contextual neighborhood modal logics follows the strategy of Chapter 3 (cf. Definition 3.8). That is, for each relevant multi-modal logic $\mathbf{nL}_{\otimes \leftarrow}$ extending $\mathbf{FFE}_{\otimes \leftarrow}$, we define $\mathbf{C.nL}_{\otimes \leftarrow}$ such that it extends CPC with the \Box_L -version of nL and the bridge rule (BR) linking the two layers nL and CPC. Again, I remark that, compared to [173, 175, 176, 204, 206], a strengthened version of the bridge rule is taken here.

Definition 4.13 (Contextual neighborhood modal logics). *Let the Hilbert system for the contextual neighborhood modal logic $\mathbf{C.nL}_{\otimes \leftarrow}$ in the signature σ_{Rel} consist of the following axioms and rules:*

1. Axioms and rules of CPC (cf. Figure 3.1);
2. Rules (FE14), $(\otimes \text{RES})$ and $(\leftarrow \text{RES})$;
3. The \Box_L -versions of $\mathbf{nL}_{\otimes \leftarrow}$;
4. The bridge rule (BR) $\Box_L \varphi \Rightarrow \varphi$.

Soundness and completeness of $\mathbf{C.nL}_{\otimes\leftarrow}$ with respect to $\mathbf{C.nL}_{\otimes\leftarrow}$ is established by standard methods. Hence, I will only provide details when needed.

Theorem 4.6 ($\mathbf{C.nL}_{\otimes\leftarrow}$ -soundness). $\vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \varphi \implies \mathbf{C.nL}_{\otimes\leftarrow} \models^W \varphi$.

Proof. The result is established by induction on the length of $\mathbf{C.nL}_{\otimes\leftarrow}$ -proofs. Item 1 follows from Lemma 4.14; item 2 follows from Lemmas 4.14 and 4.2; item 3 follows from Theorem 4.5 and Condition (W5); and item 4 follows from $W \subseteq L$. \square

Lemma 4.17 (Bridge). *For all $\varphi \in \mathfrak{L}_{\text{Rel}}$: $\vdash_{\mathbf{nL}_{\otimes\leftarrow}} \varphi \iff \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \Box_L \varphi$.*

Proof. As for Lemma 3.6, one direction is established by an easy induction on the length of $\mathbf{nL}_{\otimes\leftarrow}$ -proofs, while the other follows from Theorems 4.6, 4.4 and Lemma 4.16. \square

The following proposition expresses the distinctive closure property of the epistemic modality \Box in $\mathbf{C.nL}_{\otimes\leftarrow}$, i.e. closure under provable relevant equivalence, and is to be compared with its relational analogue, Proposition 3.2.

Proposition 4.2 (Relevant reasoning). *The following rule (nRR) is derivable in every $\mathbf{C.nL}$ such that $\mathbf{nL}_{\otimes\leftarrow}$ does not include ($\Box\text{N}$):*

$$\Box_L(\varphi \leftrightarrow \psi) \Rightarrow \Box\varphi \leftrightarrow \Box\psi.$$

Proof. The result follows by the same argument of Proposition 3.2, using $\Box_L(\text{FE15})$ in place of $\Box_L(\text{BMC17})$. \square

To conclude the chapter, completeness is established by constructing the $\mathbf{C.nL}_{\otimes\leftarrow}$ -canonical model $\mathfrak{M}_{\mathbf{C.nL}_{\otimes\leftarrow}}^c$. As expected, the latter is obtained by individuating within the canonical $\mathbf{nL}_{\otimes\leftarrow}$ -model a distinguished subset of $\mathbf{C.nL}_{\otimes\leftarrow}$ -theories among $\mathbf{nL}_{\otimes\leftarrow}$ -theories.

Definition 4.14 (Canonical $\mathbf{C.nL}_{\otimes\leftarrow}$ -model). *Let the canonical $\mathbf{C.nL}_{\otimes\leftarrow}$ -model be a tuple of the following form:*

$$\mathfrak{M}_{\mathbf{C.L}}^c = (S^c, W^c, L^c, \text{Prop}^c, *^c, R^c, Q^c, Q_L^c, V^c)$$

- $S^c = \{s \mid s \text{ is a prime } \mathbf{nL}_{\otimes\leftarrow}\text{-theory over } \mathfrak{L}_{\text{Rel}}\}$;
- $W^c = \{s \mid s \text{ is a non-empty, proper, prime } \mathbf{C.nL}_{\otimes\leftarrow}\text{-theory over } \mathfrak{L}_{\text{Rel}}\}$;
- $L^c = \{s \in S^c \mid \mathbf{t} \in s\}$;
- $\text{Prop}^c = \{[\varphi]_c \mid \varphi \in \mathfrak{L}_{\text{Rel}}\}$, where $[\varphi]_c = \{s \in S^c \mid \varphi \in s\}$
- $s^{*c} = \{\varphi \in \mathfrak{L}_{\text{Rel}} \mid \neg\varphi \notin s\}$;

- $R^c = \{(s, X, Y) \in S^c \times (Prop^c)^2 \mid \exists \varphi, \psi \in \mathfrak{L}_{\text{Rel}}(X = [\varphi]_c, Y = [\psi]_c, \varphi \rightarrow \psi \in s)\}$;
- $Q^c = \{(s, X) \in S^c \times Prop^c \mid \exists \varphi \in \mathfrak{L}_{\text{Rel}}(X = [\varphi]_c, \Box \varphi \in s)\}$;
- $Q_L^c = \{(s, X) \in S^c \times Prop^c \mid \exists \varphi \in \mathfrak{L}_{\text{Rel}}(X = [\varphi]_c, \Box_L \varphi \in s)\}$;
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Note that Lemma 4.3 transfers with the same argument to the new setting. However, showing that $\mathfrak{M}_{\mathbf{C.nL}_{\otimes\leftarrow}^c} \in \mathbf{C.nL}_{\otimes\leftarrow}$ is now a lot easier than in the relational case, since it requires one to check sensibly fewer conditions. Of course, this advantage originates in the absence of bounds.

Lemma 4.18 ($\mathbf{C.nL}_{\otimes\leftarrow}$ -Modelhood). $\mathfrak{M}_{\mathbf{C.nL}_{\otimes\leftarrow}^c} \in \mathbf{C.nL}_{\otimes\leftarrow}$.

Proof. By Lemmas 4.5, 4.10 and 4.11 it suffices to show that W is a set of possible worlds, i.e. that $W \subseteq L$ and Conditions (W1)–(W5) are satisfied. $W \subseteq L$ and (W1) hold by the same argument of Lemma 3.8. The remaining cases are as follows, where we assume by definition of $Prop$ that $X = [\varphi]$ and $Y = [\psi]$ for some $\varphi, \psi \in \mathfrak{L}_{\text{Rel}}$.

(W2). We reason as follows, where $w \in W$ is arbitrary and $\vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} (\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$ holds by classical reasoning:

$$\begin{aligned}
w \in \neg[\varphi] \vee [\psi] &\iff \neg\varphi \vee \psi \in w && \text{by Lemma 4.4} \\
&\iff \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \neg\varphi \vee \psi && \text{by definition of } W \text{ and Lemma 3.7} \\
&\iff \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \varphi \rightarrow \psi && \text{by } \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} (\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \\
&\iff \varphi \rightarrow \psi \in w && \text{by definition of } W \text{ and Lemma 3.7} \\
&\iff w \in [\varphi \rightarrow \psi] && \text{by definition of } [\cdot] \\
&\iff w \in [\varphi] \rightarrow [\psi] && \text{by Lemma 4.4}
\end{aligned}$$

(W3). We reason as follows, where $w \in W$ is arbitrary and $\vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} (\varphi \otimes \psi) \leftrightarrow (\varphi \wedge \psi)$ holds by (\otimes RES) and $\vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} (\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\varphi \wedge \psi) \rightarrow \chi)$:

$$\begin{aligned}
w \in [\varphi] \otimes [\psi] &\implies w \in [\varphi \otimes \psi] && \text{by Lemma 4.10} \\
&\implies \varphi \otimes \psi \in w && \text{by definition of } [\cdot] \\
&\implies \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \varphi \otimes \psi && \text{by definition of } W \text{ and Lemma 3.7} \\
&\implies \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \varphi \wedge \psi && \text{by } \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} (\varphi \otimes \psi) \rightarrow (\varphi \wedge \psi) \\
&\implies \varphi \wedge \psi \in w && \text{by definition of } W \\
&\implies w \in [\varphi \wedge \psi] && \text{by definition of } [\cdot] \\
&\implies w \in [\varphi] \wedge [\psi] && \text{by Lemma 4.4}
\end{aligned}$$

(W4). We reason as follows, where $w \in W$ is arbitrary and $\vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} (\varphi \rightarrow \psi) \leftrightarrow (\varphi \leftarrow \psi)$ holds by (\leftarrow RES) and $\vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi)$:

$$\begin{aligned}
w \in [\varphi] \leftarrow [\psi] &\implies w \in [\varphi \leftarrow \psi] && \text{by Lemma 4.10} \\
&\implies \varphi \leftarrow \psi \in w && \text{by definition of } [\cdot] \\
&\implies \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \varphi \leftarrow \psi && \text{by definition of } W \text{ and Lemma 3.7} \\
&\implies \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \psi \rightarrow \varphi && \text{by } \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} (\varphi \leftarrow \psi) \leftrightarrow (\psi \rightarrow \varphi) \\
&\implies \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \neg\psi \vee \varphi && \text{by } \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} (\psi \rightarrow \varphi) \leftrightarrow (\neg\psi \vee \varphi) \\
&\implies \neg\psi \vee \varphi \in w && \text{by definition of } W \\
&\implies w \in [\neg\psi \vee \varphi] && \text{by definition of } [\cdot] \\
&\implies w \in [\neg\varphi] \vee [\psi] && \text{by Lemma 4.4}
\end{aligned}$$

(W5). We reason as follows:

$$\begin{aligned}
W \subseteq \Box_L[\varphi] &\iff W \subseteq [\Box_L\varphi] && \text{by Lemma 4.4} \\
&\iff \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \Box_L\varphi && \text{by definition of } W \text{ and Lemma 3.7} \\
&\iff \vdash_{\mathbf{nL}_{\otimes\leftarrow}} \varphi && \text{by Lemma 4.17} \\
&\iff L \subseteq [\varphi] && \text{by definition of } L, \text{ (FE14) and Lemma 2.3}
\end{aligned}$$

□

Theorem 4.7 ($\mathbf{C.nL}_{\otimes\leftarrow}$ -Completeness). $\mathbf{C.nL}_{\otimes\leftarrow} \models \varphi \implies \vdash_{\mathbf{C.nL}_{\otimes\leftarrow}} \varphi$.

Proof. The result follows by Lemmas 4.18 and $\mathfrak{M}_{\mathbf{C.nL}_{\otimes\leftarrow}}^c, s \models \varphi \iff \varphi \in s$, which is established by virtually the same argument of Lemma 4.12. □

4.4 Concluding Remarks

In conclusion of Part II, let us take stocks of what has been accomplished so far and point out what are the future challenges. In the first part of the dissertation, contextual modal logics have been established both semantically and axiomatically for the full relevant language. Contextual modal logics were shown to provide the technical resources to model validity in classical and relevant logic, with a technically motivated modality \Box_L bridging the two. Finally, soundness and completeness were established for contextual modal logics building on a wide range of logics in the relevant family. However, the project of contextual modal logics is only at its first stages, and it needs foundational work both at a technical and philosophical level.

For example, I will point out two interesting lines of research to be pursued. From a technical point of view, the most urgent need is to provide a structural proof theoretical analysis of the framework, using Gentzen–style sequent calculus formalisms and generalisations thereof. Indeed, proofs in contextual modal logics are represented in a Hilbert-style formulation, i.e. as sequences of formulas. This is inefficient for the establishment of meta–theoretic properties of the system, such as effective decidability, the subformula property, interpolation, etc. and analytic calculi are to be devised. From a philosophical perspective, it was pointed out in Section [3.1.2](#) that contextual modal logics offer a novel perspective on the debate over logical pluralism, since they provide a unitary framework in which to specify different logics using a single language. However, pluralistic concerns in contextual modal logics are only evident at the level of the semantics, where different specifications of states yield different notions of logical consequence. On the other hand, axiomatic calculi are formulated so as to privilege the interplay between classical and relevant provability, thus hiding the possibility to express a plurality of logics. In this respect, Gentzen–style calculi are desirable, since they rest on the idea that a fixed set of operational rules determines the meaning of a connective, while a varying set of structural rules determines the properties of a given consequence relation.

Part II

Applications

Chapter 5

Relevant Reasoners in Possible Worlds

In this chapter, I apply the framework of contextual modal logics developed in Part [I](#) to epistemic logic. To this aim, in Section [5.1](#) I introduce the problem of logical omniscience and the hyperintensional challenge it poses to epistemic logic. In Section [5.2](#) I introduce some approaches dealing with the problem of logical omniscience and their limitations. In Section [5.3](#) I lay down a version of contextual modal logic, which combines relational and neighborhood semantics, and which serves the purpose of diminishing the logical omniscience threat.

5.1 The Problem of Logical Omniscience

In Part [I](#) of this dissertation, I outlined how to combine different logical theories in a common framework. In Part [II](#), I will show how to apply the framework so as to provide a model of epistemic reasoning which avoids some limitations of existing formalisms. As a disclaimer, I remark that the chapters of Part [II](#) employ different techniques from Part [I](#), sometimes combining relational and neighborhood semantics for contextual modal logics. This will serve as an illustration of the versatility of the framework.

Recall from Chapter [I](#) that epistemic logic has to deal with the problem of logical omniscience. For a given class of models \mathbf{M} , the problem can be logically stated as follows:

$$\Gamma \models_{\mathbf{M}} \varphi \implies \Box\Gamma \models_{\mathbf{M}} \Box\varphi,$$

where $\Box\Gamma := \{\Box\gamma \mid \gamma \in \Gamma\}$. In its logical form, such problem pertains then to the

closure properties of the modal operator \Box , i.e. what logical commitments are made on the information believed or known by rational agents. In what follows, I will discuss some of the most interesting instances of the logical omniscience problem from an epistemological viewpoint, by dividing the problematic closure properties of \Box in three groups.

5.1.1 Epistemic Closure and Logical Connectives

The first group of epistemic closure properties I shall consider contains axioms regimenting the distribution of \Box over the standard propositional connectives. Intuitively, what is at odds with the validity of the following principles is that propositional connectives impart different inferential properties on the formulas in their scope, depending on whether such connectives occur in propositional or modal contexts. In particular, for now I will assume that $\neg, \otimes, \oplus, \rightarrow$ are interpreted relevantly (classically) when they occur (do not occur) in the scope of epistemic modalities (cf. Section 5.3 for a justification).

Closure principle	Axiom
$(\Box\neg)$ Closure under negation	$\neg\Box\varphi \leftrightarrow \Box\neg\varphi$
$(\Box\wedge)$ Closure under conjunction	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
$(\Box\vee)$ Closure under disjunction	$(\Box\varphi \vee \Box\psi) \leftrightarrow \Box(\varphi \vee \psi)$
$(\Box\rightarrow)$ Closure under implication	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
$(\Box\otimes)$ Closure under fusion	$\Box\varphi \otimes \Box\psi \rightarrow \Box(\varphi \otimes \psi)$
$(\Box\oplus)$ Closure under fission	$(\Box\varphi \oplus \Box\psi) \leftrightarrow \Box(\varphi \oplus \psi)$

1. \Box and negations. To see why one might reject the validity of $(\Box\neg)$, note that in modal contexts, e.g. belief systems, negation can be interpreted as expressing a weak form of falsification [181, 209].¹ That is, negated formulas in belief systems can be taken as expressing that a situation excludes some information (cf. [25]), which is distinct from the lack of positive information. Under this light, the negative support of φ in epistemic contexts is adequately modeled by an epistemic base excluding φ , which is formally captured by a non-classical

¹Besides its link with the semantic notion of falsification, negation can also be linked with the linguistic notion of denial. The relationship between negation and rejection, the propositional attitude expressing denial, has been spelled out in modal terms in [95], where it is argued that the weak rejection of φ amounts to publicly refraining from committing to φ , and [142], where it is argued that weakly rejecting φ amounts to refusing to believe φ .

type of negation (cf. Chapter 2). Positive support and weak negative support are, then, compatible notions: believing that φ is ruled out by the agent’s epistemic information, i.e. believing $\neg\varphi$, is compatible with believing that φ is positively supported.² This provides evidence for the invalidity of one direction of $(\Box\neg)$, since e.g. two distinct reliable sources can independently support φ and exclude φ . For the other direction, note that the space of possible sources may be incomplete, so that there may be no ground on which to believe φ and none on which to believe $\neg\varphi$.

2. \Box and conjunctions. To see why one might reject the validity of $(\Box\wedge)$ and $(\Box\otimes)$, consider an agent bearing two distinct instances of belief attitudes towards φ and towards ψ . According to the *fragmentation theory of belief storage* (cf. also [57]), “information utilizable by cognitive processes is stored in distinct, independently accessible data structures” [21, p.80]. This means that agents may fail to *aggregate* information stored in distinct locations of their belief bases into a single belief attitude towards $\varphi \wedge \psi$. Moreover, agents may also fail to believe $\varphi \otimes \psi$, since the latter requires agents not to aggregate, but *combine* their positive information about φ and ψ – cf. Section 5.3.2 and [180] for a discussion of the distinction between aggregation and combination. Finally, note that $(\Box\rightarrow)$ can be rejected on similar grounds: an agent believing $\varphi \rightarrow \psi$ and believing φ , may fail to believe ψ because they either fail to aggregate or combine their beliefs.
3. \Box and disjunctions. Finally, there are several independent reasons why one should reject $(\Box\vee)$ and $(\Box\oplus)$. In one direction, it is often claimed that beliefs should comply with a *clutter avoidance principle* [85, p.12], according to which agents with limited storage capacity should not clutter their belief bases with any sort of superfluous information. Another explanation comes from topic-sensitive logics (cf. [23]), according to which agents’ beliefs throughout inferences should remain on-topic, without introducing new subject matter. In both cases, an agent believing φ (ψ) should not thereby believe $\varphi \vee \psi$ nor

²[181] argues for an asymmetry between verification and falsification in virtue of the fact that positive support is a direct notion, while negative support involves ruling out some information, which is an indirect procedure. [209], in turn, responds to [181] by arguing that, while non-classical negations can be seen as expressing a weak form of falsification, the symmetry between verification can be restored by pairing positive support with strong negative support (linked by a strong, contradictory-forming, negation operator) and by pairing negative support with a weak form of verification (linking ruling out with ruling in).

$\varphi \oplus \psi$. For the other direction, consider the following example, adapted from [9], which works equally well for $(\Box\vee)$ and $(\Box\oplus)$:

Consider a scenario in which an agent fails to access their mail box. The failed access provides them evidence that either the email address or the password are wrong, and that if the email address is not wrong then the password is wrong. However, they do not have thereby evidence that the email address is wrong, nor that the password is wrong.

5.1.2 Epistemic Closure and Classical Omniscience

The second group of epistemic closure properties I shall consider contains standard omniscience principles considered in the classical epistemic logic literature (cf. [58, 92]).

Closure principle	Axiom/Rule
$(\Box E)$ Closure under logical equivalence	$\varphi \leftrightarrow \psi \Rightarrow \Box\varphi \leftrightarrow \Box\psi$
$(\Box M)$ Closure under logical implication	$\varphi \rightarrow \psi \Rightarrow \Box\varphi \rightarrow \Box\psi$
$(\Box N)$ Necessitation rule	$\varphi \Rightarrow \Box\varphi$
$(\Box T)$ Factivity	$\Box\varphi \rightarrow \varphi$
$(\Box 4)$ Positive introspection	$\Box\varphi \rightarrow \Box\Box\varphi$
$(\Box 5)$ Negative introspection	$\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$

1. \Box and logical implications. Closure under $(\Box M)$ is the prototypical instance of the logical omniscience problem. Even in cases where ψ is a logical consequence of φ , an agent believing φ may fail to believe ψ because they are resource-bounded, i.e. they may not be in a cognitive position – e.g. they may lack the time, computational power, memory– to perform all the deductive steps necessary to derive the desired conclusion from premises they have epistemic access to. Failure of $(\Box E)$, the hallmark of hyperintensionality, is analysed more closely in the next section. For now, consider the following counterexample, adapted from [137]:

Consider a scenario in which an agent has evidence for some empirical statement φ , but has no information regarding some classical tautology ψ . Since conjunctions of tautologies with empirical statements are true in the same possible worlds as the empirical statements alone, φ and $\varphi \wedge \psi$ are logically equivalent. However, the agent has evidence supporting φ but not supporting $\varphi \wedge \psi$. Note that, were the agent's information maximally

consistent, they would have evidence for φ and for $\varphi \wedge \psi$ in exactly the same circumstances.

2. \Box *and valid formulas*. To see why one might reject the validity of $(\Box N)$, a priori domains are often invoked (cf. [87]). For example, consider an arbitrary complex mathematical proposition, which is usually taken to be logically valid. This does not mean, however, that agents should be attributed any epistemic property towards such proposition, since e.g. they may not be trained enough to fully grasp its meaning or they may not have any reason to believe it.
3. \Box *and specific interpretations*. The validity of the final three principles, $(\Box T)$, $(\Box 4)$ and $(\Box 5)$ is traditionally tied to the specific interpretation of \Box .³ On the one hand, it is uncontroversial to accept that knowledge is factive, thus satisfying $(\Box T)$. On the other hand, introspection principles are more slippery: an internalist interpretation of the foundation of doxastic warrant leads many researchers to conclude that the logic of beliefs should be fully introspective, thus satisfying $(\Box 4)$ and $(\Box 5)$ (cf. [189]). However, the myth of the self-illumination of certain mental activities was questioned ever since [89], since cognitively bounded agents can hardly be assumed to have higher order access to their epistemic states. At any rate, in what follows I will be interested in devising (non-trivial) logics which are general enough to allow for the restriction of these principles' validity, given the uncommittal interpretation of \Box .

5.1.3 Epistemic Closure and Relevant Omniscience

Recall from Part I that contextual modal logics are devised so as to restrict the closure principles to relevant consequence (cf. (RR) Proposition 3.2) or relevant equivalence (cf. (nRR) Proposition 4.2). The third group of epistemic closure properties I shall consider contains specific instances of (RR) and (nRR). Seen as omniscience principles, their implausibility poses a problem to contextual modal logics building on relevant epistemic logics. In general, weak logics lacking the following principles have been variously advocated (cf. e.g. [93, 136, 153, 157]), thus motivating the study of epistemic logics of agents reasoning according to these weak logics.

³For example, according to standard models of belief and knowledge, their logics are in the vicinity of, respectively, CCKD45 and CPCS4 (cf. [12]), while in most models of knowledge in the computer science and game theory literature, it is assumed that knowledge is a CPCS5-modality (cf. [122]).

Closure principle		Axiom
(\Box W)	Modal weakening	$\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$
(\Box CO)	Modal contraction	$\Box(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow \Box(\varphi \rightarrow \psi)$
(\Box MP)	Modal modus ponens	$\Box(\varphi \wedge (\varphi \rightarrow \psi)) \rightarrow \Box\psi$
(\Box S)	Modal explosion	$\Box(\varphi \wedge \neg\varphi) \rightarrow \Box\psi$
(\Box DS)	Modal disjunctive syllogism	$\Box((\varphi \vee \psi) \wedge \neg\varphi) \rightarrow \Box\psi$
(\Box AB)	Modal absorption	$\Box\varphi \leftrightarrow \Box(\varphi \vee (\varphi \wedge \psi))$
(\Box DT)	Modal distribution	$\Box(\varphi \wedge (\psi \vee \chi)) \leftrightarrow \Box((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
(\Box DM1)	Modal De Morgan 1	$\Box(\neg(\varphi \wedge \psi)) \leftrightarrow \Box(\neg\varphi \vee \neg\psi)$
(\Box DM2)	Modal De Morgan 2	$\Box(\neg(\varphi \vee \psi)) \leftrightarrow \Box(\neg\varphi \wedge \neg\psi)$
(\Box PF)	Modal Parry fallacy	$\Box((\varphi \wedge \neg\varphi) \vee \psi) \rightarrow \Box\psi$
(\Box DF)	Modal Deutsch fallacy	$\Box((\varphi \wedge \neg\varphi) \wedge \psi) \rightarrow \Box(\psi \wedge \neg\psi)$

1. \Box and information combination. To see why one might reject the validity of (\Box W), (\Box CO) and (\Box MP), note that these principles highlight the complex behaviour of information combination on which epistemic support is based. For a counterexample to (\Box W) (a modal version of weakening), it suffices to think about an agent whose non-conclusive evidence towards a proposition is defeated upon considering further information. For a counterexample to (\Box CO) (modal contraction), it suffices to adapt an argument from channel theory (cf. [155]): think about an agent whose information that 6 is even leads them to believe that 10 is even by repeated use of the information conveyed by the rule “if n is even, $n + 2$ is even”. Such agent does not believe that 10 is even with a single application of the rule. Counterexamples to (\Box MP) (modal modus ponens) involve a similar reference to the failure of idempotency of information combination.
2. \Box and consistency/completeness. An argument for the rejection of (\Box S), (\Box PF), (\Box DF) and (\Box DS) is similar to that for (\Box –). For a counterexample to (\Box S), think about an agent aggregating contradictory pieces of evidence, thereby coming to believe $\varphi \wedge \neg\varphi$, who does not become thereby altogether irrational. Drawing from [19], agents realising some inconsistency in their belief base, should keep it local and not let it spread over the whole belief base. For the same reason, (\Box PF) and (\Box DF) are also highly counterintuitive, since they prescribe that contradictions can be suppressed in disjunctive beliefs and that they are infectious in conjunctive beliefs. Finally, a counterexample to (\Box DS) comes from the admissibility of inconsistent situations (cf. Remark 2.1).

3. \Box *and topic*. To see why one might reject the validity of $(\Box AB)$, it is again useful to resort to topic-theoretic considerations. In particular, note that belief attribution is sensitive to how the logical material is arranged in believed propositions. For a concrete example contra $(\Box AB)$, note that even if φ and $\varphi \vee (\varphi \wedge \psi)$ convey the same information in any distributive lattice logic, the topic of ψ may “disturb” the independent belief in φ , as confirmed by the following counterexample, adapted from [59]:

Consider a scenario in which an agent believes that John is driving. Now, in many logics, including relevant logic, John is driving is logically equivalent to John is driving or John is driving drunk. However, the newly introduced topic is such that in many cases it blocks the agent to even entertain the disjunct that John is driving drunk, hence blocking the attribution of the belief that John is driving or John is driving drunk.

More abstractly, consider $(\Box DT)$: agents believing $\varphi \wedge (\psi \vee \chi)$ may fail to believe $(\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ because the rearrangement of the topic of ψ (χ), when ψ (χ) is conjoined with φ , may disturb the attribution of belief – cf. [32] for a rejection of distribution grounded in meaning containment. As examined in Chapter [6], the position of negation can play a similar disturbing role, thus motivating the rejection of $(\Box DM1)$ and $(\Box DM2)$.

5.2 The Hyperintensional Challenge

The most successful strategies to tackle logical omniscience at its roots have appealed to the *hyperintensionality* of epistemic modalities. Roughly speaking, epistemic modalities are hyperintensional because in the context of epistemic modalities one cannot substitute logically equivalent formulas. In order to precisely spell out what the hyperintensional challenge amounts to for epistemic logics, let me clarify what I mean by hyperintensionality.

5.2.1 Hyperintensionality: A Working Definition

The *locus classicus* of hyperintensionality is the following passage from [39]:

It is well known that it seems possible to have a situation in which there are two propositions p and q which are logically equivalent and yet are such that a person may believe the one and not the other. If we regard a proposition as a set of possible worlds then two logically equivalent propositions will be

identical, and so if “x believes that” is a genuine sentential functor, the situation described in the opening sentence could not arise. I call this the paradox of hyperintensional contexts. Hyperintensional contexts are simply contexts which do not respect logical equivalence. [39, p.25]

Unfortunately, the above passage leaves it unsettled how to logically formulate the claim that hyperintensional contexts do not respect logical equivalence.⁴ Of the many resulting definitions of hyperintensionality, the following is to be preferred on the grounds of logic and language neutrality. According to this definition, due to [129], hyperintensionality is reduced to the abstract algebraic concept of self-extensionality.

Definition 5.1 (Hyperintensionality). *Let a connective $\otimes \in \sigma$ be self-extensional in a logic L in the signature σ iff the following rule is admissible:*

$$\varphi_i \dashv\vdash_L \psi_i \Rightarrow \otimes(\varphi_1, \dots, \varphi_n) \dashv\vdash_L \otimes(\psi_1, \dots, \psi_n).$$

Let L be hyperintensional iff there is $\otimes \in \sigma$ such that \otimes is not self-extensional.

Note that, according to the definition of hyperintensionality and provability, \Box is hyperintensional in L iff $(\Box E)$ is not admissible in L . Now, the admissibility of $(\Box E)$ is entrenched in the semantic architecture of classical and non-classical modal logics. This is due to the frame semantics for both formalisms, which builds on the idea that modalities are restricted quantifiers over states of the domain. Granted this, $\varphi \leftrightarrow \psi$ are logically equivalent iff they express the same proposition, i.e. iff they are satisfied at precisely the same states. Given that modalities express the properties of propositions in those formalisms, $(\Box E)$ is unavoidable.

Following [39], the paradox of hyperintensionality relies on the assumptions that (i) *modalities are genuine sentential functors* and (ii) *propositions are sets of possible*

⁴For one, note that it is often implicitly assumed that the background logic is an extension of CPC, where hyperintensional contexts are represented by formulas in the scope of the additional operator. For example, [39] makes this implicit assumption when stating that “we may have a person so logically blind that he may believe p without believing $\neg p$ ”. However, this needs not be the case, as the issue of hyperintensionality arises for various non-classical logics [109, 191]. For two, note that the definition of hyperintensionality given by [191], which generalises that of [210], relies on the presence of \Box in the language. On a related note, observe that this language-specificity induces also a specific theory of sentential content, as logically equivalent propositions in [191, 210] are those expressing necessary equivalent formulas, i.e. formulas for which $\vdash_L \Box(\varphi \leftrightarrow \psi)$. However, the necessary equivalence expresses two distinct semantic theses in classical and relevant modal logics.

worlds. Accordingly, a way out of the paradox is offered by lifting either (i) or (ii). The two types of strategies can be divided into, roughly, two families of frameworks (a similar list appears in [170]).

1. *Structuralist approaches*. According to structuralist approaches, modalities express semantic properties of sentences, as well as of some other component of sentences. This can be further articulated, as in [170], by contending that modalities express properties of propositions *and* sentences, or by contending that propositions are variously structured entities, such as tuples of intensions [116] or thick propositions [212]. Salient examples of approaches to omniscience pursuing the structuralist strategy employ *justification logic* [6, 195], *awareness logic* [57, 167], and *topic-sensitive logic* [23, 65]. Awareness and justification logic are discussed in Chapter 7.
2. *State-based approaches*. According to state-based approaches, modalities express properties of intensions, i.e. functions from states in the model to sets of propositions, thus adhering to the standard possible-worlds semantics picture of sentential content. However, propositions comprise not only (well-behaved) of possible worlds but also of *impossible worlds* or information states⁵, i.e. states at which the laws of logic may fail. Epistemic logics based on impossible worlds models were devised by [151], generalised in [208], and popularised recently in [24]. A notable example in epistemic logic is the framework of [111], of which contextual modal logics can be viewed as a generalisation.

The next two sections are devoted to a discussion of two representatives from the structuralist and the state-based family of hyperintensional approaches to logical omniscience, *topic-sensitive semantics* and *impossible worlds semantics*.⁶ The two representatives are chosen in order to illustrate, by comparison, the extent to which contextual modal logics can be useful in formal epistemology.

5.2.2 Topic Sensitive Semantics

Theories of topic-sensitive intensional modals (TSIMs) move from the dissatisfaction with standard modal logics' modeling of propositional attitudes (cf. [23]).

⁵Note that, according to the definition of [191], relevant logic counts as a semantic approach to hyperintensionality, where the distinction between information states and logical information states is enough to make distinctions between necessary equivalent propositions.

⁶As clarified in Chapter 6, topic-sensitive logics belong to the structuralist family because the truth of a conditional $\varphi \supset \psi$ in a pointed model depends on the variables occurring in ψ being a subset of those occurring in φ .

In possible–worlds semantics, epistemic attitudes are directed toward propositions. TSIM theories argue that this view is too coarse–grained, as it overlooks the fact that epistemic attitudes bear an intentionality towards the states of affairs, issues, situations, or circumstances they are *about*. Therefore, aboutness should be a component of the semantic content of epistemic sentences, which, together with the intension, make up for the *thick content* of a proposition. According to the theory of thick content, two formulas express the same proposition just in case they are intensionally equivalent and they are about the same topic.

From a semantic point of view, models of TSIMs work by supplementing *topic models* to classical Kripke models. Topic models are simple algebraic structure which are capable to formalise the fact that what a sentence is about can be included in what another one is about.

Definition 5.2 (Topic models). *Let a topic model be a tuple of the following form:*

$$(\mathcal{T}, \oplus, \tau)$$

- (\mathcal{T}, \oplus) is a join semilattice, i.e. $\oplus : \mathcal{T}^2 \rightarrow \mathcal{T}$ is a topic fusion operator such that:

$$a \oplus b = b \oplus a \tag{T1}$$

$$(a \oplus b) \oplus c = a \oplus (b \oplus c) \tag{T2}$$

$$a \oplus a = a \tag{T3}$$

- $\tau : At \rightarrow \mathcal{T}$ is a topic valuation such that:

$$\tau(\varphi) = \tau(p_1) \oplus \dots \oplus \tau(p_n) \text{ where } At(\varphi) = \{p_1, \dots, p_n\} \tag{T4}$$

Each topic model is a join semilattice of concepts or topics, (\mathcal{T}, \oplus) , where the join operator \oplus represents topic fusion, which is commutative, associative and idempotent. It is well-known that every join semilattice (\mathcal{T}, \oplus) gives rise to a partially ordered set (\mathcal{T}, \preceq) by setting $a \preceq b$ iff $a \oplus b = b$. In this way, topic models express the mereological relations between topics.

Among the logics in the TSIM family, the framework of [134] is interesting from the present perspective because it focuses on the closure properties of a unary belief modality. Özgün and Berto’s *logic of plain hyperintensional belief* capitalises on the intuition that closure within a belief system should necessarily require topic inclusion so as to model agents who are able to work out all the consequences of what they believe, provided such consequences stay on topic with the totality of concepts already grasped by the agent. Recall that a well pre-ordering Q is a relation which is reflexive, transitive and such that $\forall X \neq \emptyset \exists x \in Min(X)$ where $Min(X) = \{x \in X \mid \forall y \in X(Qyx)\}$.

Definition 5.3 (PHB-models). *Let a plain hyperintensional belief model (PHB-models for short) be a structure of the following form:*

$$(S, Q, V, \mathcal{T}, \mathbf{b}, \oplus, \tau)$$

- (S, Q, V) is a plausibility model, i.e.:
 - S is a non-empty set of possible worlds;
 - $Q \subseteq S^2$ is a well pre-ordered plausibility relation, interpreting \square ;
 - $V : At \rightarrow \mathcal{P}(S)$ is an atomic valuation;
- $(\mathcal{T}, \mathbf{b}, \oplus, \tau)$ is a topic model with $\mathbf{b} \in \mathcal{T}$ such that $\tau : At \cup \{\top\} \rightarrow \mathcal{T}$, with $\tau(\top) = \mathbf{b}$.

Let **PHB** denote the class of **PHB**-models.

Intuitively, Qst expresses that t is deemed more plausible than s and $Min(X)$ represents the states deemed most plausible by the agent. Note also that topic models in **PHB**-models contain a designated element, \mathbf{b} , which serves the purpose of representing the topics in the agent's belief base, i.e. the topic the agent has grasped.

Definition 5.4 (PHB-satisfaction). *Let satisfaction of a formula $\varphi \in \mathcal{L}_{\text{PHB}}$ in a **PHB**-model \mathfrak{M} be recursively defined as follows:*

$$\begin{array}{lll}
 \mathfrak{M}, s \models p & \iff & s \in V(p) \\
 \mathfrak{M}, s \models \top & \iff & s \in S \\
 \mathfrak{M}, s \models \varphi \wedge \psi & \iff & \mathfrak{M}, s \models \varphi \text{ and } \mathfrak{M}, s \models \psi \\
 \mathfrak{M}, s \models \varphi \vee \psi & \iff & \mathfrak{M}, s \models \varphi \text{ or } \mathfrak{M}, s \models \psi \\
 \mathfrak{M}, s \models \neg\varphi & \iff & \mathfrak{M}, s \not\models \varphi \\
 \mathfrak{M}, s \models \varphi \rightarrow \psi & \iff & \mathfrak{M}, s \not\models \varphi \text{ or } \mathfrak{M}, s \models \psi \\
 \mathfrak{M}, s \models \square\varphi & \iff & \forall t \in Min(S)(\mathfrak{M}, t \models \varphi) \text{ and } \tau(\varphi) \preceq \mathbf{b} \\
 \mathfrak{M}, s \models [\forall]\varphi & \iff & \forall t \in S(\mathfrak{M}, t \models \varphi)
 \end{array}$$

Definition 5.5 (Classical validity). *Let a formula $\varphi \in \mathcal{L}_{\text{PHB}}$ be valid in a **PHB**-model \mathfrak{M} ($\mathfrak{M} \models_{\text{PHB}} \varphi$) iff $\mathfrak{M}, s \models \varphi$ for all $s \in S$. Let $\varphi \in \mathcal{L}_{\text{PHB}}$ be a consequence of $\Gamma \subseteq \mathcal{L}_{\text{PHB}}$ in all **PHB**-models ($\Gamma \models_{\text{PHB}} \varphi$) iff for all **PHB**-model models \mathfrak{M} , and all $s \in S$ such that $\mathfrak{M}, s \models \gamma$ for all $\gamma \in \Gamma$, $\mathfrak{M}, s \models \varphi$.*

In **PHB**-models, \top serves as a proxy in the language to indicate the totality of topics grasped by the agent, as clarified by $\tau(\top) = \mathbf{b}$. In turn, \mathbf{b} provides the desired

topic-theoretic dependency in the analysis of \Box . As in standard doxastic models, an agent believes φ only if φ is true at all maximally plausible worlds, i.e. all worlds which they consider most plausible according to the order induced by Q .⁷ However, such a veridical component is not sufficient for an agent to believe φ , as it is required that the topic of φ be among (as a mereological part) the topic grasped by the agent, i.e. $\tau(\varphi) \oplus \mathfrak{b} = \mathfrak{b}$ iff $\tau(\varphi) \preceq \mathfrak{b}$. Belief is closed under *on-topic* logical consequence, i.e. an agent believes all consequences of what they believe, as long as such consequences are about topics the agent has already grasped.

To conclude the presentation of **PHB**-models, note that $[\forall]$ is a universal modality, which is technically needed to axiomatise the logic. Instead of presenting the axiom system for the logic of plain hyperintensional belief (cf. [134]), I will turn to assess its formal and conceptual properties.

1. *Topic transparency.* In TSIM semantics the topic-theoretic material is locked entirely in the specification of the propositional variables. This means that logical connectives are transparent, i.e. they bear no contribution in determining the topic of complex sentences, as witnessed by $\tau(\Box\neg(\varphi \rightarrow \psi)) = \tau(\varphi \wedge \psi) = \tau(\varphi) \oplus \tau(\psi)$. However, a recent body of work has begun to question the topic transparency of connectives, such as intensional conditionals. For example, [64] offers the illustration of an intuitionistic conditional, which on a standard BHK-style reading is *about* the availability of a recursive method of converting proofs of the antecedent φ into proofs of the consequent ψ and is consequently *about more* than the sum the topics of φ and ψ .⁸
2. *Sufficiency thesis.* On a related note, topics of complex formulas are uniquely specified once the atomic topics are specified. However, recent work on topics has shown the limits of this assumption. In many cases, the topic of a subjunctive conditional with antecedent φ is naturally understood as incorporating those *situations selected by* φ . The topics of intensional conditionals are *state-sensitive*, i.e., can include as subject-matter the *states* that are fixed by the antecedent (or consequent).

⁷Note that the truth of a doxastic sentence in **PHB**-models is global, i.e. state-independent. As noted by [134], this limitation could be lifted by understanding \Box as an S5-modality; however the fact that this modification does not alter the logic indicates that the state at which agents form their belief does not play a significant role in determining the properties of belief. In Chapter 6 this assumption is challenged.

⁸Other examples of topic transparency failure include negation – cf. [212], where, in the context of an examination of Hempel’s paradox, it is argued that the conditional “all ravens are black” differs topic-theoretically from the contraposed “all non-black things are non-ravens”.

3. *Syntactic filter.* TSIM theories contend that the inclusion requirement on epistemic closure is between topics, which, whatever metaphysical theory one decides to adopt, are not going to be syntactic objects [23, pp.79–80]. However, this distinction does not mark any substantial difference between topics and sets of propositional variables. Topic models are simple algebraic structures, so simple that each topic can be isomorphically regarded as a set of propositional variables, and topic fusion just as set union. If so, then doubts arise that the semantics of topic models is just syntax in disguise, and that the topic–inclusion preservation is just an attempt to reduce a complex semantic matter, that of topic interrelations, to syntactical form, i.e. variable inclusion (cf. [166, pp.96–99]). At any rate, the strategy pursued by topic theorists does not seem to be committed to topic models, as defined in Definition 5.2. In Chapter 6 I will show how to define fully semantic topic models.
4. *Formal dubia.* By understanding topic inclusion as a necessary requirement for the closure of belief, **PHB**–models reduce the problem of logical omniscience, since in general it does not hold that

$$\Gamma \models_{\mathbf{PHB}} \varphi \implies \Box \Gamma \models_{\mathbf{PHB}} \Box \varphi,$$

where $\Box \Gamma := \{\Box \gamma \mid \gamma \in \Gamma\}$. However, once topic–inclusion is preserved, epistemic closure reduces just to validity in the underlying normal modal logic. That is, since $s \models \Box \bar{\varphi}$ iff $\tau(\varphi) \preceq \mathbf{b}$ ⁹ – where $\bar{\varphi} := \bigwedge_{p \in \text{At}(\varphi)} p \vee \neg p$ represent a tautological sentence *about* φ – we have that

$$\Gamma \models_{\mathbf{PHB}} \varphi \implies \Box(\Gamma \cup \{\bar{\varphi}\}) \models_{\mathbf{PHB}} \Box \varphi.$$

Moreover, the logics on which TSIM theories are built are classical. As a consequence, special cases of the above closure principle, like $\Box((\varphi \wedge \neg \varphi) \vee \psi) \rightarrow \Box \psi$ ($(\Box \text{PF})$) and $\Box((\varphi \wedge \neg \varphi) \wedge \psi) \rightarrow \Box(\psi \wedge \neg \psi)$ ($(\Box \text{DF})$) of Section 5.1, are valid in all **PHB**–models. In Chapter 6 I will show how to relieve topic–sensitive modal logics from these formal dubia.

5.2.3 Impossible Worlds Semantics

Model theories for several families of intensional logics based on frame semantics face a common challenge: representing states at which theorems fail. In many such cases,

⁹Note that $s \models \Box \bar{\varphi}$ iff $s \models \bigwedge_{p \in \text{At}(\varphi)} \Box(p)$ iff $\forall t \in \text{Min}(S) \forall p \in \text{At}(\varphi) (\mathfrak{M}, t \models p)$ and $\forall p \in \text{At}(\varphi) (\tau(\bar{p}) \preceq \mathbf{b})$ iff $\forall p \in \text{At}(\varphi) (\tau(p) \preceq \mathbf{b})$ iff $\tau(\varphi) \preceq \mathbf{b}$.

states of the models are divided into two sorts, *standard* and *non-standard*, where non-standard states are set apart in virtue of the fact that they fail to support the same truth-conditions for connectives as standard states (see e.g. [104]).

The same strategy was adopted by the *impossible worlds semantics* of [151], in order to solve the problem of logical omniscience. The key insight of impossible world semantics is that, while agents' reasoning takes place at possible worlds, which subscribe to standard principles of logic, agents may consider as epistemic alternatives "options which only look possible but which contain hidden contradictions" [88, p.476]. From a semantic point of view, impossible worlds models supplement a set of non-standard states S' to classical Kripke models. The following definition is adapted from [208].

Definition 5.6 (IW-models). *Let an impossible worlds model (IW-model for short) be a structure of the following form:*

$$(S, S', Q, V)$$

- S is a non-empty set of possible worlds;
- S' is a set of impossible worlds;
- $Q \subseteq (S \cup S')^2$ is a Kripke binary relation, interpreting \Box ;
- $V : \mathfrak{L}_{\text{CMod}} \times (S \cup S') \rightarrow \{\mathbf{f}, \mathbf{t}\}$ is an atomic valuation such that for all $s \in S \cup S'$ $V(\top, s) = \mathbf{t}$ and for all $s \in S$:

$$\begin{aligned} V(\varphi \wedge \psi, s) = \mathbf{t} & \iff V(\varphi, s) = \mathbf{t} \text{ and } V(\psi, s) = \mathbf{t} \\ V(\varphi \vee \psi, s) = \mathbf{t} & \iff V(\varphi, s) = \mathbf{t} \text{ or } V(\psi, s) = \mathbf{t} \\ V(\neg\varphi, s) = \mathbf{t} & \iff V(\varphi, s) = \mathbf{f} \\ V(\varphi \rightarrow \psi, s) = \mathbf{t} & \iff V(\varphi, s) = \mathbf{f} \text{ or } V(\psi, s) = \mathbf{t} \\ V(\Box\varphi, s) = \mathbf{t} & \iff \forall t \in S \cup S' (Qst \implies V(\varphi, t) = \mathbf{t}) \end{aligned}$$

Let \mathbf{IW} denote the class of all IW-models.

Definition 5.7 (IW-validity). *Let a formula $\varphi \in \mathfrak{L}_{\text{CMod}}$ be valid in an IW-model \mathfrak{M} ($\mathfrak{M} \models_{\mathbf{IW}} \varphi$) iff $V(\varphi, s) = \mathbf{t}$ for all $s \in S$. Let $\varphi \in \mathfrak{L}_{\text{CMod}}$ be valid in \mathbf{IW} ($\models_{\mathbf{IW}} \varphi$) iff φ is valid in all IW-models.*

The main feature of impossible worlds models is that validity is defined classically, as truth in all possible worlds, while the valuation clause for \Box quantifies over possible and impossible worlds. This is the key insight by which impossible world semantics solves the problem of logical omniscience, since there is no formula φ such that $\models_{\mathbf{IW}} \Box\varphi$. However, it does so at some price. To see this, let's address some of its conceptual and formal properties.

1. *Recursive truth.* In impossible worlds models, the truth value of a formula at impossible worlds is not specified recursively. On the contrary, formulas at impossible worlds are treated as propositional variables. While this allows for a great deal of generality (cf. [208]), it seems to contravene to basic principles of sentential content, according to which the meaning of complex formulas should be determined compositionally on the basis of subformulas occurring therein. As clarified in the next section, contextual modal logics elegantly remedy this deficit.
2. *Connectives' meaning.* On a related note, the truth-conditions of logical connectives are gerrymandered across different types of states, since at possible worlds they are specified truth-functionally, while at impossible worlds they are assigned an arbitrary truth value. Again, this feature is odd from the point of view of the logicity of connectives, whose meaning should be specified uniformly across the states of the model theory.
3. *Formal dubia.* Impossible worlds models capture the intuition that “knowledge and belief do not display even the weakest regularities describable in terms of inference patterns” [208, p.525]. For this reason, there is no general epistemic logic of these notions, which accordingly do not satisfy any closure property. However, this explanation is unsatisfactory for at least two reasons. For one, impossible worlds models provide no explanation whatsoever of the reason why each closure principle should fail. For two, a hyperintensional modal logic capturing some epistemic notion should be closed under *some* principle, which illuminates on the understanding of the given epistemic modality.

5.3 Applied Contextual Modal Logics

Like many hyperintensional frameworks, most frameworks dealing with logical omniscience assume classical propositional logic as a background logic, and explain the hyperintensional features of epistemic modalities by making fine-grained distinctions which are conflated by classical epistemic logic at the modal level. The family of contextual modal logics devised in Chapters [3] and [4] seems to fit nicely into this picture. In particular, **C.L**-models refine the classical picture by defining possible worlds within models for relevant modal logic. This move, in turn, allows to define validity and consequence in relation to specific subsets of the domain (cf. Definition [3.5]).

In order to better appreciate the properties of contextual modal logics, in what

follows I will recast them as epistemic logics and evaluate them in the context of the logical omniscience problem. Moreover, in order for the framework to be more palatable to the philosophical logician, I will adopt a hybrid relational–neighborhood semantics for relevant modal logic, where neighborhood semantics is used only in the analysis of the epistemic modality \Box .

Remark 5.1 (Extensional truth constants). The presence of \top and \perp in the context of relevant logic may raise an eyebrow, since it breaks variable–sharing. Indeed, the absence of universal formulas was deemed in [194] a necessary requirement for a connective to count as relevant. However, one may note that (i) failures of variable–sharing are only limited to formulas involving \top and (ii) there are strategies to circumvent the problem (cf. [198]). On top of that, I already remarked in Chapter 1 that (iii) variable–sharing can be regarded as a symptom, rather than an explication, of relevance. At any rate, Remark 3.3 explains how to avoid the use of \top by fully resorting to neighborhood semantics (cf. Chapter 4 for one strategy): the following section can be regarded by the suspicious reader as an illustration of an alternative method by which one can avoid bounded models.

5.3.1 Extensional–Intensional Models

First, let me recapitulate the details of the preferred semantics. In what follows, I will focus on models for the logic EI, the logic of extensional and intensional truth.

Definition 5.8 (EI–models). *Let an extensional–intensional model (EI–model for short) be a structure of the following form:*

$$(S, L, W, Prop, \leq, *, R, Q, Q_L, V)$$

- (S, \leq) is a non–empty partially ordered set of information states;
- $Prop \subseteq \mathcal{P}(S)$ is a set of admissible propositions, such that for $\otimes \in \{\wedge, \vee, \neg, \rightarrow, \Box, \Box_L\}$ and $\otimes^{\mathfrak{S}}$ defined as in Definition 2.7:
- $L \in S(\uparrow)$ is a non–empty subset of logical information states;
- $* \in S(\uparrow, S(\downarrow))$ is the Routley star, interpreting \neg ;
- $R \in S(\downarrow\downarrow\uparrow)$ is the Routley–Meyer ternary relation, interpreting $\rightarrow, \otimes, \leftarrow$;
- $Q \in S(\uparrow, \mathcal{P}(\mathcal{P}(S))(\uparrow))$ is a neighborhood binary relation, interpreting \Box ;
- $Q_L \in S(\downarrow\uparrow)$ is a Kripke binary relation, interpreting \Box_L ;
- $L \in S(\uparrow)$ is a non–empty subset of logical information states;
- $W \subseteq L$ is a set of possible worlds such that for all $w \in W$:

$$w^* = w \tag{EI1}$$

$$Rwww \quad (\text{EI2})$$

$$Rwst \implies s \leq w \leq t \quad (\text{EI3})$$

$$Rstw \implies s \leq w, t \leq w \quad (\text{EI4})$$

$$Rswt \implies s \leq w \leq t \quad (\text{EI5})$$

$$Q_L(W) = L \quad (\text{EI6})$$

- $V : At \rightarrow S(\uparrow)$ is the atomic valuation function.

Moreover, let every extensional intensional model satisfy the following usual frame properties:

$$\forall s \in S \exists l \in L(Rlss) \quad (\text{EI7})$$

$$\forall s, t \in S \forall l \in L(Rlst \implies s \leq t) \quad (\text{EI8})$$

Let **EI** denote the class of all extensional–intensional models. Let $(S, L, \leq, *, R, Q, Q_L, V)$ be the \mathbf{BME}_\top –model an EI–model is based on. Let \mathbf{BME}_\top denote the class of \mathbf{BME}_\top –models.

Conditions (EI1)–(EI6) on W account for different classical features of possible worlds. Condition (EI1) implies that worlds are consistent. Condition (EI2) flattens information combination to idempotency, so that combining a world with itself results in no new information. Similarly, according to Condition (EI3)–(EI5), worlds are maximal with respect to information combination in all postions. That is: (i) by (EI3) when inputs are processed in the *context* of possible worlds, it is only possible to apply to them information they already contain and to obtain no less information; (ii) by (EI4), taking worlds as input information or as context information does not make any difference in terms of the output information produced; and (iii) by (EI5), any possible world resulting from information combination is such that it contains the information contained in the context and input. Next, for a motivation of Condition (EI6), it is useful to assign an epistemic interpretation to Q_L : $Q_L(s)t$ contains the *logical* information which is available at s for an inference. Under this interpretation, it is reasonable to assume that the information that can be used in an inference at possible worlds comes from the relevantly acceptable deductions, i.e. from the model of logically valid implications in relevant logic.¹⁰ Finally, recall from Chapter 2 that Conditions (EI7)–(EI8), together with (EI6), are technically needed to yield the modified version of the semantic deduction theorem.

¹⁰The information that can be used in a logical inference at non-worldly states may have different properties than logical information. Therefore, in the semantic interpretation of formulas with nested occurrences of \Box_L , what counts as logical information varies, i.e. at different information

Definition 5.9 (*EI*-satisfaction). *Let \mathfrak{M} be a *EI*-model or a BME_\top -model. Let satisfaction of a formula $\varphi \in \mathcal{L}_{\text{Rel}_\top}$ in \mathfrak{M} pointed at $s \in S$ be recursively defined as follows:*

$$\begin{array}{lll}
\mathfrak{M}, s \models p & \iff & s \in V(p) \\
\mathfrak{M}, s \models \top & \iff & s \in S \\
\mathfrak{M}, s \models \mathbf{t} & \iff & s \in L \\
\mathfrak{M}, s \models \varphi \wedge \psi & \iff & \mathfrak{M}, s \models \varphi \text{ and } \mathfrak{M}, s \models \psi \\
\mathfrak{M}, s \models \varphi \vee \psi & \iff & \mathfrak{M}, s \models \varphi \text{ or } \mathfrak{M}, s \models \psi \\
\mathfrak{M}, s \models \neg\varphi & \iff & \mathfrak{M}, s^* \not\models \varphi \\
\mathfrak{M}, s \models \varphi \rightarrow \psi & \iff & \forall t, u \in S(Rstu, \mathfrak{M}, t \models \varphi \implies \mathfrak{M}, u \models \psi) \\
\mathfrak{M}, s \models \varphi \otimes \psi & \iff & \exists t, u \in S(Rtus, \mathfrak{M}, t \models \varphi \text{ and } \mathfrak{M}, u \models \psi) \\
\mathfrak{M}, s \models \varphi \leftarrow \psi & \iff & \forall t, u \in S(Rtsu, \mathfrak{M}, t \models \psi \implies \mathfrak{M}, u \models \varphi) \\
\mathfrak{M}, s \models \Box\varphi & \iff & Qs[\varphi]_{\mathfrak{M}} \\
\mathfrak{M}, s \models \Box_L\varphi & \iff & \forall t \in S(Q_Lst \implies \mathfrak{M}, t \models \varphi)
\end{array}$$

In *EI*-models, a meaningful distinction is between *extensional truth*, i.e. truth at possible worlds, and *intensional truth*, i.e. truth at information states. Intensional truth represents how worlds looks like from the perspective of an agent, while extensional truth represents ground facts holding at each possible world.¹¹ As a consequence of (E11)–(E17), the following result sanctions the fact that extensional truth turns out to be classical satisfaction. The result is established similarly to Lemma 3.3, by a simplified argument, retrievable from that proof. Recall that $\varphi \oplus \psi := \neg\varphi \rightarrow \psi$.

Lemma 5.1 (*EI*-flatness). *For all $\mathfrak{M} \in \mathbf{EI}$, all $w \in W$ and all $\varphi, \psi \in \mathcal{L}_{\text{Rel}_\top}$:*

1. $\mathfrak{M}, w \models \mathbf{t}$ iff $\mathfrak{M}, w \models \top$;
2. $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$;

states agents may have different logical resources to carry out deductions. Still, \Box_L is modeled by relational instead of neighborhood accessibility, which results in the closure under $(\Box_L C)$ and $(\Box_L M)$. Under the intended relevant provability interpretation of \Box_L (cf. Section 5.3.2), this is reasonable.

¹¹Note that the issue of whether agents can access ground facts or whether they can tell whether they are situated within a world is set aside. The distinction between worlds and information states is just a modeler's resource to distinguish whether or not the agent's information corresponds to ground truths, or ontological thought.

3. $\mathfrak{M}, w \models \varphi \otimes \psi$ iff $\mathfrak{M}, w \models \varphi \wedge \psi$;
4. $\mathfrak{M}, w \models \varphi \oplus \psi$ iff $\mathfrak{M}, w \models \varphi \vee \psi$;
5. $\mathfrak{M}, w \models \varphi \rightarrow \psi$ iff $(\mathfrak{M}, w \models \varphi \implies \mathfrak{M}, w \models \psi)$;
6. $\mathfrak{M}, w \models \varphi \leftarrow \psi$ iff $(\mathfrak{M}, w \models \psi \implies \mathfrak{M}, w \models \varphi)$.

Definition 5.10 (*EI*-validity). *Let validity and consequence in BME_\top -models be defined as L -validity and S -consequence, respectively (cf. Definition 3.5). Let validity and consequence in EI -models be defined as W -validity and S -consequence, respectively.*

Contextual modal logics, when recast as epistemic logics, are interesting hyper-intensional frameworks which are included the state-based family and lift some of the limitations of the impossible-worlds approach.

1. *State-based family.* In both impossible-worlds approaches and contextual modal logics, the domain set is divided into standard states, i.e. possible worlds, and non-standard states, which obey looser constraints. Moreover, in both formalism validity is defined with respect to standard states, while the epistemic accessibility relation Q quantifies over (sets of) possibly non-worldly states. As a consequence, contextual modal logics are hyperintensional frameworks. This fact is sanctioned by Proposition 5.1, together with soundness and completeness. $(\Box E)$ can fail because two formulas may be classically equivalent, i.e. true at the same possible worlds, but not relevantly equivalent, i.e. true at the same information states in an EI -model.
2. *Continuity.* In impossible-worlds approaches, possible worlds are distinguished from impossible worlds by stipulating a difference in types between the two domains. On the other hand, contextual modal logics avail themselves of a strategy which maintains a continuity between the two domains of states, which fall under the same type. That is, possible worlds and logical states are just more structured information states, obeying tighter constraints concerning their relationship with frame components. This continuity in the identification of possible worlds as special information states enables, in turn, to uniquely specify the meaning of each connective by (recursive) relevant satisfaction at information states. It only so happens that relevant satisfaction turns out equivalent to classical satisfaction at possible worlds.
3. *Non-classical regimentation.* In impossible worlds approaches, logical omniscience is avoided in all forms, since non-standard states are logically anarchic, thus invalidating every epistemic closure principle. On the other hand,

in EI -models epistemic attitudes are closed under relevant equivalence, which models preservation under sameness of information (cf. Proposition 3.2). Crucially, contextual modal logics achieve this in a classical propositional setting, where only formulas in the scope of \Box are regimented by relevant logic. As a consequence, epistemic attitudes are closed under a weak form of relevant omniscience (cf. Proposition 5.2), which identifies the minimal criterion of epistemic competence.

Proposition 5.1 (Hyperintensionality). *EI is hyperintensional.*

Proof. Consider the model \mathfrak{M} where $S = \{w, s, t\}$, $W = \{w\}$, $Q(w) = \{\{w, s\}\}$, R_{wst} , R_{sst} , R_{tss} , $s^* = t$, $V(p) = \{s\}$ and $V(q) = \{t\}$ (the other components are irrelevant and can be specified so that \mathfrak{M} is indeed an EI -model). Clearly, $\mathfrak{M} \models (p \rightarrow q) \leftrightarrow (\neg p \vee q)$ as $\llbracket p \rightarrow q \rrbracket_{\mathfrak{M}} \cap W = \llbracket \neg p \vee q \rrbracket_{\mathfrak{M}} \cap W = W$. However, $\mathfrak{M} \not\models \Box(p \rightarrow q) \leftrightarrow \Box(\neg p \vee q)$, as $\mathfrak{M}, w \models \Box(p \rightarrow q)$ (since $Qw \llbracket p \rightarrow q \rrbracket_{\mathfrak{M}} = \{w, s\}$) but $\mathfrak{M}, w \not\models \Box q$ (as not $Qw \llbracket \neg p \vee q \rrbracket_{\mathfrak{M}} = S$). Therefore, \mathfrak{M} is a countermodel to $(\Box E)$. \square

5.3.2 Relevant Reasoners...

Although it is of independent technical interest to study how classical and non-classical logics interact in a uniform framework, it is especially interesting from an epistemological point of view to consider the specific non-classical case of relevant logic. This is because the semantics of relevant models accommodates a great deal of intuitions about information processing performed in everyday reasoning.

In order to highlight how Routley–Meyer semantics manages to offer a general model of the complexity of inferential patterns which occur in epistemic contexts, I will review the interpretation of EI -models components.

1. *Epistemic contexts.* $Q(s)$ is the epistemic context of an agent relative to s , which contains the relevant propositions an agent has epistemic access to. In neighborhood semantics, epistemic contexts consist of distinct sets of information states. On the other hand, according to Kripke semantics epistemic contexts consist of a single set of information states. Then, Kripke semantics embodies the assumption that agents are able to pool together their information in one single piece, corresponding to the conjunction of the epistemically accessible propositions. However, the ample support against this assumption (cf. Section 5.1.1) suggests that a more adequate assumption, embodied in the failure of closure under intersection $(\Box C)$ (alternatively called $(\Box \wedge)$ is not to

require agents to *aggregate* information. Aggregation constitutes an epistemic action which agents “perform in order to bring together within the scope of a single instance of a propositional attitude those propositions that were previously within the scope of distinct instances of propositional attitudes” [180, p.227].

2. *Information combination.* Another salient epistemic action, besides aggregation, which requires effort on agents’ part, is combination. Within epistemic contexts, information can be combined through R . Recall that whenever we have $Rstu$, s can be regarded as containing the contextual information to which, by means of information combination, the information contained in t is applied. Put differently, the theory s can be combined through R with t to yield u iff s contains the informational link (expressed by a relevant implication) between the input information in t and the output information in u .¹² The properties of information combination, then, correspond to the properties of theory application. Weak commutativity ($Rstu \implies Rtsu$), idempotence ($Rsss$) and associativity ($R(st)uv \iff Rs(tu)v$) have all been advocated as characteristic features of contextual information combination, thus giving rise to strong models of relevant logics (cf. [53]). However, subtler ways of combining information arise once we make distinctions in the types of information to be combined (cf. [149, 155, 182]). For example, we may distinguish between information in implicational form, programs, and non-implicational information, data. Then, each of weak commutativity, idempotence and associativity can fail depending on whether we combine data, programs or apply programs to data.¹³
3. *Information exclusion.* Recall that the Routley–star function $*$ maps a state s to the state s^* which is maximally compatible with s . This interpretation is made precise by noting that $*$ can be defined in terms of a *compatibility relation* $C \in S(\Downarrow)$ such that $s^* = t \iff Cst, \forall u(Csu \implies u \leq t)$. The existence of

¹²This observation makes the picture of information combination reminiscent of how channel theory and situation semantics model the connection between channels, expressing implicative regularities, and situations (cf. [56, 96]).

¹³[182] argues that relevant deductive reasoning should be weakly commutative but not associative and not strongly-commutative ($R(st)uv \implies R(su)tv$). Next, according to a channel-theoretic interpretation of relevant logic [155], channels’ warrant transmission is not idempotent. Finally, it may be argued that combination has a distinctive application order, so that only applying an input signal to a program is felicitous. Since programs constitute different data types than signals and targets, information combination should not be associative nor (weakly or strongly) commutative.

s^* in the latter amounts to assuming that C is serial ($\forall s \exists t (Cst)$) and convergent ($\forall s \exists t (Cst, \forall u (Csu \implies u \leq t)$).¹⁴ Then, s^* does not support anything that s explicitly reject, capturing the idea that for a theory s to support a negated formula $\neg\varphi$ amounts to the maximally compatible theory s^* excluding the piece of information encoded in φ . The properties of theory exclusion, in the form of valid principles, are encoded in the properties of compatibility [25]: for example, assuming symmetry ($\forall s, t (Cst \implies Cts)$) yields the validity of $\varphi \rightarrow \neg\neg\varphi$; assuming $\forall s \exists t (Cst, \forall u (Ctu \implies u \leq s))$ yields the validity of $\neg\neg\varphi \rightarrow \psi$; and assuming Css yields the validity of $\varphi \wedge \neg\varphi \rightarrow \psi$.

4. *Logical information and deducibility.* Condition (EI6) implies that $\Box_L \varphi$ is true at all possible worlds iff φ is true at all the logical states. This observation lends a more epistemically perspicuous reading of \Box_L . According to the modified semantic deduction theorem, the fact that ψ is a relevant consequence of φ means that, according to any possible world the agent is situated in, the logical information at their disposal permits ψ to be deducible in a logical context fed with input φ . Pressing further on this point, we can then analyse $w \models \Box_L(\varphi \rightarrow \psi)$ as ψ being deducible in w from φ based solely on logical information, or as ψ being a priori deducible from φ .

The general moral to draw is that relevant logic constitutes a flexible framework, which can be fine-tuned to the modeler's purposes in order to obtain logics underscoring increasingly stronger closure principles. In turn, the characterisation of stronger logics shows that stronger logics correspond to stronger assumptions on agents' epistemic abilities. Each of the frame conditions discussed requires an epistemic action on the agents' part, which requires *effort*. Epistemic effort is prone to error, one of the sources of the cognitive deficiencies targeted by the logical omniscience problem. Therefore, in what follows I will stick to the logic characterised by the class of *all EI*-models.

5.3.3 ... In Classical Worlds

Contextual modal logics provide a flexible framework to combine substructural logics with stronger logics at the propositional level. Although the generality of the

¹⁴A more general semantics for relevant logic without these properties can be provided by using directly C as negation's accessibility relation [153, 204]. From the point of view of contextual modal logics, this is interesting because it allows to capture subintuitionistic logics, lacking $\neg(\varphi \wedge \psi) \rightarrow (\neg\varphi \vee \neg\psi)$, thus allowing for the failure of $\Box(\neg(\varphi \wedge \psi)) \leftrightarrow \Box(\neg\varphi \vee \neg\psi)$.

framework does not commit one to the choice of any specific propositional logic, it is especially interesting from an epistemological point of view to consider the case of classical propositional logic, as done in Part [I](#). In the epistemic logic community, there is widespread support to this claim. For example, consider the following passages:

Intuitionistic logic can be understood as the logic of scientific research (rather positivistically conceived) while on the other hand the classical logic is the logic of ontological thought. [\[82\]](#), p.596]

Nonstandard worlds serve only as epistemic alternatives; although an agent may be muddled and may consider a nonstandard world possible, we (the logicians who get to examine the situation from the outside) know that the “real world” must obey the laws of standard logic. [\[58\]](#), p.204]

[The semantic values *t*, *b*, *n*, *f* of FDE] are in some broad sense “epistemic” and not “ontic” – they are fundamentally subjective due to the limitations of humans (and as it turns out also their machines), not the world. In this we are to be contrasted with “dialethists” (see e.g., [\[143\]](#)) who defend the idea of their being “really true” contradictions. [\[48\]](#), pp.428–429]

In relevance-theoretic terms, an input is relevant to an individual when its processing in a context of available assumptions yields a positive cognitive effect. A positive cognitive effect is a worthwhile difference to the individual’s representation of the world – a true conclusion, for example. [\[211\]](#), p.607]

When we make an inference about what is happening in the world, we do so from the limited perspective of some situation. [...] When we make a hypothesis in an inference in our natural deduction system, we hypothesise that a proposition is true in a situation and that this situation obtains in a particular world. This world is held fixed for the entire inference. [\[120\]](#), p.41–42].

The above passages serve to illustrate the salient feature of contextual modal logics in general, and *EI*-models in particular, which characterise these logics as logics of relevant reasoners situated within classical worlds:

1. *Ontological/Epistemic truth*. As highlighted by [\[82\]](#), different domains of reasoning may be regimented by different logics. In Grzegorzczuk’s logic of scientific research, a specific kind of epistemic reasoning, experimental investigation, complies with the laws of intuitionistic logic, while classical logic regulates the

relationships between ground facts. In *EI*-models, the division between *ontological thought* and *epistemic thought* runs parallel to the division between *extensional truth* and *intensional truth*. The former formalises truth from the perspective of an ideal agent, which has access to all and every ground facts of the world. The latter formalises epistemic truth, or truth from the perspective of a specific agent, which is subject to all sorts of cognitive and computational deficiencies.

2. *Domains of application.* Tacking stock of the distinction between two kinds of truths, ontological and epistemic, [58] illustrates the domain of application of the two. While *standard states* should be subject to extensional truth, hence regimented by classical logic; *non-standard states* should be subject to epistemic truth, since they serve only as epistemic alternatives. As pointed out in Chapter 1 (cf. also Section 5.3.1), epistemic truth should correspond to truth according to some logic. In *EI*-models, this is done by using relevant logic to define epistemic truth, as logical connectives' meaning is fixed by relevant satisfaction. In doing so, logical connectives impart minimal commitments on the epistemic actions agents are endowed with.
3. *Negation and contradictions.* [48] provides further confirmation to the claim that relevant logic should be used only as a logic of epistemic reasoning and not an all-purpose logic. As Dunn points out, our theories about the world, as encoded e.g. in belief systems, may well contain inconsistencies, making the claim of epistemological paraconsistency just common sense. However, the world itself, which those theories aim at representing, arguably does not contain inconsistencies, making ontological paraconsistency a more fanciful position to defend. In *EI*-models, the joint thesis of epistemological paraconsistency and ontological consistency is confirmed by Lemma 5.1, according to which relevant negation, which is in general paraconsistent, turns out equivalent to Boolean negation at possible worlds.
4. *Implication and relevance.* As pointed out in Section 5.3.2, an essential feature of epistemic contexts is information combination, which requires information to be stored in implicative data-types to be applied to inputs in order to produce relevant outputs.¹⁵ Now, according to [211], a relevant output is a conclusion

¹⁵Pressing further on this point, [94] argues that a cognitively salient interpretation of situation semantics take information to consists of fallible rules, with which we conceptualize (a part of) the world.

which aims at correctly representing the world. Input–output reasoning can only be performed by including an implication connective, which represents at the object language level the informational links (be them law–like correspondences, entailments, fallible inferences, etc.) that agents can avail themselves when processing information. In *EI*–models, while these links might not in general be truth–conducive, Lemma 5.1 ensures that they preserve truth at possible worlds.¹⁶

5. *Epistemically situated inference*. According to the *theory of situated inference* [120], inference is relativised with respect to the world where it takes place. Indeed, agents collect information and process it within possible worlds. However, in models for relevant epistemic logic, one lacks the possibility to talk about facts of the world, and information states have been alternatively interpreted as data available to an agent [27], bodies of evidence [171], or even states of explicit knowledge/belief [183]. In *EI*–models, we have the resources to distinguish between information states, where inferences are performed, from possible worlds, within which agents form their beliefs. Hence, the theory of situated inference is refined so as to make the epistemic traits of inference explicit at the object language level: a relevant inference $\varphi \rightarrow \psi$ is situated at a world w in that an agent situated in w bears the salient epistemic attitude (e.g. hypothesise) towards $\llbracket \varphi \rightarrow \psi \rrbracket$, i.e. $w \models \Box(\varphi \rightarrow \psi)$.

5.3.4 Axiomatisation of EI

To conclude the chapter, I will present a sound and complete axiomatisation of the logic of extensional and intensional truth EI.

Definition 5.11 (EI axiom system). *Let the Hilbert system for the logic of extensional and intensional truth EI consist of the the following axioms and rules:*

1. *Axioms and rules of CPC;*
2. *The \Box_L –versions of BME_\top (cf. Figure 5.1);*
3. *The bridge rule: (BR) $\Box_L \varphi \Rightarrow \varphi$.*

¹⁶Another notable cognitive effect attained by combining relevant inputs with an informational context is the (static) revision of the agent’s doxastic information [211]. In *EI*–models, this cognitive effect is embodied in the failure of $\varphi \rightarrow (\psi \rightarrow \varphi)$.

(BME _⊤ 1)	<i>Axioms and rules of BM (cf. Definition 7.24)</i>
(BME _⊤ 2)	$\varphi \rightarrow \top$
(BME _⊤ 3)	$\perp \rightarrow \varphi$
(BME _⊤ 4)	$\top \rightarrow \neg\perp$
(BME _⊤ 5)	$(\Box_L\varphi \wedge \Box_L\psi) \rightarrow \Box_L(\varphi \wedge \psi)$
(BME _⊤ 6)	$\varphi \rightarrow \psi \Rightarrow \Box_L\varphi \rightarrow \Box_L\psi$
(BME _⊤ 7)	$\varphi \leftrightarrow \psi \Rightarrow \Box\varphi \rightarrow \Box\psi$

Figure 5.1: Hilbert system for BME_⊤.

EI is the logic of agents reasoning according to the relevant logic BME_⊤ and situated in classical possible worlds.¹⁷ In EI, the \Box_L -version of each axiom and rule in BME_⊤ expresses an admissible principle of epistemic reasoning. Then, \Box_L -((BME_⊤2)) (\Box_L -((BME_⊤3))) can be taken as expressing some trivial information derivable by agents no matter what their information is (absurd information implying whatsoever information). Restricted classicality then holds for \top, \perp insofar as a restricted form of double negation laws holds by \Box_L -((BME_⊤4)). Next, given that \Box_L expresses relevant provability, \Box_L -((BME_⊤5)) and \Box_L -((BME_⊤6)) sanction some basic properties of \Box_L as a provability operator. \Box_L -((BME_⊤7)) is the distinctive rule of epistemic thought in ((BME_⊤)), expressing that agents cannot epistemically distinguish relevantly equivalent propositions. Finally, (BR) provides a link between ontological and epistemic thought, stating that information about the world needs to conform to worldly facts.

As a consequence of Proposition 4.2, agents' epistemic reasoning in EI-models is closed under modal versions of absorption $\Box\varphi \leftrightarrow \Box(\varphi \vee (\varphi \wedge \psi))$, distribution $\Box(\varphi \wedge (\psi \vee \chi)) \leftrightarrow \Box((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$ and De Morgan laws $\Box(\neg(\varphi \wedge \psi)) \leftrightarrow \Box(\neg\varphi \vee \neg\psi)$, $\Box(\neg(\varphi \vee \psi)) \leftrightarrow \Box(\neg\varphi \wedge \neg\psi)$. It is easy to check that the remaining omniscience principles are invalidated.

Proposition 5.2 (Logical Omniscience). *All epistemic closure principles of Section 5.1 except $(\Box AB)$, $(\Box DT)$, $(\Box DM1)$ and $(\Box DM2)$ are not valid, do not preserve*

¹⁷The results stated here extend to stronger logics at the propositional and modal level by Theorems 2.3 and 4.3, respectively. Note however that, given the hybrid relational and neighborhood semantics, the required correspondence for the axiom $(\Box K)$ is given by the frame condition $Rstu, Ns(X \rightarrow^{\mathfrak{S}} Y), NtX \Rightarrow NuY$.

validity respectively.

The methods to establish soundness and completeness carry through without complications from Chapters 3 and 4, since EI is a logic in the family of contextual modal logics. However, it is useful to spell out the formal details of completeness, since the presence of \top in the language allows for a simplification of the proof of the Bridge Lemma 5.2 and the Model Lemma 5.3. The remaining results are presented without proof.

Theorem 5.1 (EI–soundness). $\vdash_{\text{EI}} \varphi \implies \mathbf{EI} \models \varphi$.

Lemma 5.2 (Bridge). For all $\varphi \in \mathcal{L}_{\text{Rel}\top}$: $\vdash_{\mathbf{BME}\top} \varphi \iff \vdash_{\text{EI}} \Box_L \varphi$.

Proof. First, we show the following version of the claim (3.5):

$$\forall \mathfrak{M} \in \mathbf{BME}\top \exists \mathfrak{M}' \in \mathbf{EI} (\mathfrak{M} \not\models \varphi \implies \mathfrak{M}' \not\models \Box_L \varphi) \quad (\text{LCL})$$

To establish (LCL), it suffices to take, for any $\mathfrak{M} \in \mathbf{BME}\top$, the following model \mathfrak{M}' – which results from a more straightforward semantic construction than in claim (3.5):

- $S' = S \cup W$;
- $W = \{w\}$;
- $L' = L \cup W$;
- $\text{Prop}' = \{X \mid X \cap S \in \text{Prop}\}$;
- $\leq' = \leq \cup \{(w, w)\}$;
- $*' = * \cup \{(w, w)\}$;
- $R' = R \cup \{(w, w, w)\}$;
- $Q'_L = \{(w, s) \mid s \in L\}$;
- $Q' = Q$;
- $V' = V$.

That $\mathfrak{M}' \in \mathbf{EI}$ holds by an easy inspection of the definition. Next, by $\mathfrak{M} \not\models \varphi$, we infer that there is $s \in L$ such that $\mathfrak{M}, s \not\models \varphi$. Hence, it suffices to show that (*) $\forall s \in S (\mathfrak{M}, s \models \varphi \iff \mathfrak{M}', s \models \varphi)$. By the latter and $\mathfrak{M}, s \not\models \varphi$ we infer that $\mathfrak{M}', s \not\models \varphi$, and by (EI6) we infer that $Q'_L w s$. Hence, $\mathfrak{M}', w \not\models \Box_L \varphi$, by which we conclude that $\mathfrak{M}' \not\models \Box_L \varphi$. (*) is established by an easy induction on the structure of φ . The base case holds by $V' = V$, while the induction step follows by noting that, for $\otimes \in \{\leq, *, R, Q, Q_L\}$, the restriction of \otimes' to S is the same as \otimes . Having established (LCL), the result is established by the same argument of Lemma 3.6. Note that the argument relies on the completeness of $\mathbf{BME}\top$ with respect to $\mathbf{BME}\top$ –models, which is established by standard methods, combining techniques employed in Chapters 3 and 4. \square

Definition 5.12 (Canonical EI-model). *Let the canonical EI-model be a tuple of the following form:*

$$\mathfrak{M}_{\text{EI}}^c = (S^c, W^c, L^c, \leq^c, *^c, R^c, Q^c, Q_L^c, V^c)$$

- $S^c = \{s \mid s \text{ is a non-empty, proper, prime BME}_{\top}\text{-theory over } \mathfrak{L}_{\text{Rel}_{\top}}\}$;
- $W^c = \{s \mid s \text{ is a non-empty, proper, prime EI-theory over } \mathfrak{L}_{\text{Rel}_{\top}}\}$;
- $L^c = \{s \mid s \text{ is a non-empty proper prime regular BME}_{\top}\text{-theory over } \mathfrak{L}_{\text{Rel}_{\top}}\}$;
- $\text{Prop}^c = \{[\varphi]_c \mid \varphi \in \mathfrak{L}_{\text{Rel}_{\top}}\}$, where $[\varphi]_c = \{s \in S^c \mid \varphi \in s\}$
- $\leq^c = \{(s, t) \in (S^c)^2 \mid s \subseteq t\}$;
- $s^{*c} = \{\varphi \in \mathfrak{L}_{\text{Rel}_{\top}} \mid \neg\varphi \notin s\}$;
- $R^c = \{(s, t, u) \in (S^c)^3 \mid \forall \varphi, \psi \in \mathfrak{L}_{\text{Rel}_{\top}} (\varphi \rightarrow \psi \in s, \varphi \in t \implies \psi \in u)\}$;
- $Q^c = \{(s, X) \in S^c \times \text{Prop}^c \mid \exists \varphi \in \mathfrak{L}_{\text{Rel}_{\top}} (X = [\varphi]_c, \Box\varphi \in s)\}$;
- $Q_L^c = \{(s, t) \in (S^c)^2 \mid \forall \varphi \in \mathfrak{L}_{\text{Rel}_{\top}} (\Box_L\varphi \in s \implies \varphi \in t)\}$;
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Lemma 5.3 (EI-Modelhood). $\mathfrak{M}_{\text{EI}}^c \in \mathbf{EI}$.

Proof. The result is established by showing that (i) $(S, L, \leq, *, R, Q, Q_L, V)$ is a BME_{\top} -model and that (ii) W is a set of possible worlds and $W \subseteq L$. (i) is established by standard methods (cf. Chapters 3 and 4), hence we show only the cases which require the use of the \top -axioms. (ii) is established as in Lemma 3.8, where we note that the argument is simplified by the absence of bounds.

$(s^* \neq \emptyset)$. By definition of S , $s \neq \mathfrak{L}_{\text{Rel}_{\top}}$. That is, there is $\varphi \in \mathfrak{L}_{\text{Rel}_{\top}}$ such that $\varphi \notin s$, which implies by $(\text{BME}_{\top}3)$ that $\perp \notin s$. By $\perp := \neg\top$ and the definition of $*$ we conclude that $\top \in s^*$.

$(s^* \neq \mathfrak{L}_{\text{Rel}_{\top}})$. By definition of S , $s \neq \emptyset$. That is, there is $\varphi \in \mathfrak{L}_{\text{Rel}_{\top}}$ such that $\varphi \in s$, which implies by $(\text{BME}_{\top}2)$ that $\top \in s$. By $(\text{BME}_{\top}4)$ we infer that $\neg\perp \in s$. By definition of $*$ we conclude that $\perp \notin s^*$.

□

Theorem 5.2 (EI-Completeness). $\mathbf{EI} \models \varphi \implies \vdash_{\text{EI}} \varphi$.

Proof. The result follows from Lemmas 5.3 and $\mathfrak{M}_{\text{EI}}^c, s \models \varphi \iff \varphi \in s$, which is established by the same argument of Lemma 2.6. □

Chapter 6

Topic-Sensitive Relevant Reasoners in Possible Worlds

In this chapter, I upgrade the epistemic logic of Chapter 5, by focusing on issues of topic containment. To this aim, in Section 6.1 I present two salient representatives of containment logics. In Section 6.2 I develop a version of relevant containment logic which allows one to overcome the limitations of existing approaches in the containment logic literature. In Section 6.3 I use the preferred relevant containment logic to develop a logic of topic-sensitive relevant reasoners in possible worlds.

6.1 Two Containment Logics

Although the choice to regiment epistemic competence by means of relevant logic provides reason to think that “the logical omniscience problem can be alleviated by appropriately choosing the nonstandard logic” [58], relevant logic seems to be a good but not ideal candidate logic. To appreciate this, note that relevant logic satisfies the epistemic version of the absorption law $\varphi \leftrightarrow (\varphi \wedge (\varphi \vee \psi))$, the distribution law $(\varphi \wedge (\psi \vee \chi)) \leftrightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$ and the De Morgan equivalences $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$, $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$. Moreover, epistemic contextual modal logics based on relational semantics will validate also $\Box\varphi \rightarrow \Box(\varphi \vee \psi)$. In this chapter, I will devise epistemic logics where commitments to these principles are not taken for granted but can be fine-tuned.

As pointed out in Chapter 5, part of the mentioned relevant principles are problematic due to agents cognitive deficiencies in grasping the topic of some piece of vocabulary. The fact that inference is *ampliative* (cf. [212]), and that agents obey the so-called *clutter avoidance principle* (cf. [85]) in organising their epistemic database

of information, motivates the imposition of a *topic requirement* across epistemic inference. In this section, I will present two topic-theoretic formalisms from the family of *containment logics*.

6.1.1 Parry’s Analytic Implication

Parry’s logic of analytic implication [137, 138] was one of the first attempts to model “real implication”, or analytic implication, i.e. the relation which validates formal inference within a system [138, p.2]. According to Parry, a necessary requirement on analytic implication is that the concepts involved in the consequents of implication within a given system should not exceed those involved in the antecedent within that system. This tenet finds a syntactic counterpart in a variable-inclusion principle, according to which a necessary condition for $\varphi \rightarrow \psi$ to be a logical truth is that the propositional variables occurring in ψ occur in φ as well.⁴ The variable inclusion requirement rules out the addition principle $\varphi \rightarrow \varphi \vee \psi$, thus finding a different way out of the Lewis paradox than relevant logics.

Parry’s formulations of analytic implication systems were only syntactic, providing an axiomatic calculus in the language $\mathfrak{L}_{\text{PCont}}$, where \rightarrow is a primitive connective. Later, [69] provided a semantic characterisation of Parry’s system by proving it deductively equivalent to a sound and complete axiomatic system in a language where $\varphi \rightarrow \psi$ is provably equivalent to $\Box(\varphi \rightarrow \psi) \wedge (\varphi \supseteq \psi)$. The same result is obtained here by defining $\varphi \rightarrow \psi := \Box(\varphi \rightarrow \psi) \wedge (\varphi \supseteq \psi)$ in $\mathfrak{L}_{\text{CCont}}$.

Fine’s semantics of containment models supplements S4-Kripke models with structures representing the *concepts* conceivable at each world (cf. [69]). Recent works on the conceptual, intentional content of propositions [23, 212] speaks of *subject matters* or *topics* instead of concepts, and understand subject matter-conditions and truth-conditions constitute two independent dimensions of sentential contents. In Chapter 5, I already introduced the essence of Finean concept models under the name of topic models.

Definition 6.1 (Containment models). *Let a classical containment model be a tuple of the following form:*

$$(S, Q, V, \mathcal{T}, \oplus, \tau)$$

- $(\mathcal{T}_s, \oplus_s, \tau_s)$ is a topic model for all $s \in S$;
- (S, Q, V) is an S4-Kripke model, i.e.

⁴Logics in the containment family can be individuated by the grain of the syntactic criterion they superimpose on a given carrier logic, as detailed in [62, 63, 66].

- S is a set of possible worlds;
- $Q \subseteq S^2$ is a reflexive transitive conceivability relation, interpreting \Box ;
- $V : At \rightarrow \mathcal{P}(S)$ is an atomic valuation.

Moreover, let every classical containment model satisfy the following frame condition:

$$Qst, \tau_s(\varphi) \preceq_s \tau_s(\psi) \implies \tau_t(\varphi) \preceq_t \tau_t(\psi) \quad (\text{CC1})$$

Let **PAI** denote the class of all classical containment models.

The modal accessibility relation is interpreted by Fine as conceivability: Qst if the topics used in specifying the subject matter of a world s , what s is about, are sufficient to specify the subject matter of t . The subject matter of a world can be equated with the totality of concepts available in the description of such world. Following [69], then, Qst if the world t is conceivable by resorting only to the concepts used to specify the world s . With this interpretation in mind, Fine assumes by Condition (CC1) that topic inclusion is preserved across accessible worlds, i.e. the mereological relations holding at s remain unchanged in any world t which is conceivable with respect to s .

Definition 6.2 (Classical Containment satisfaction). *Let \mathfrak{M} be a classical containment model. Let satisfaction of a formula $\varphi \in \mathcal{L}_{\text{CCont}}$ in \mathfrak{M} pointed at $s \in S$ be recursively defined as follows:*

$$\begin{array}{lll} \mathfrak{M}, s \models p & \iff & s \in V(p) \\ \mathfrak{M}, s \models \neg\varphi & \iff & \mathfrak{M}, s \not\models \varphi \\ \mathfrak{M}, s \models \varphi \wedge \psi & \iff & \mathfrak{M}, s \models \varphi \text{ and } \mathfrak{M}, s \models \psi \\ \mathfrak{M}, s \models \varphi \vee \psi & \iff & \mathfrak{M}, s \models \varphi \text{ or } \mathfrak{M}, s \models \psi \\ \mathfrak{M}, s \models \varphi \rightarrow \psi & \iff & \mathfrak{M}, s \not\models \varphi \text{ or } \mathfrak{M}, s \models \psi \\ \mathfrak{M}, s \models \Box\varphi & \iff & \forall t \in S(Qst \implies \mathfrak{M}, t \models \varphi) \\ \mathfrak{M}, s \models \varphi \supseteq \psi & \iff & \forall t \in S(Qst \implies \tau_t(\psi) \preceq_t \tau_t(\varphi)) \end{array}$$

Let validity be defined classically, as in Definition 5.5.

In classical containment models, $\varphi \supseteq \psi$ is interpreted as topic containment, which by definition of topic models boils down to the condition that the atomic variables occurring in ψ be included in those occurring in φ . In PAI then, Definition 6.2 implies that the truth conditions for analytic implication reduce to those for an S4 strict conditional with a variable inclusion syntactic filter:

$$s \models \varphi \rightarrow \psi \text{ iff } \begin{cases} s \models \Box(\varphi \rightarrow \psi) \\ At(\psi) \subseteq At(\varphi). \end{cases}$$

Although not a literal confirmation of Gödel’s conjecture that “perhaps φ analytically implies ψ can be interpreted as ψ is derivable from φ and the logical axioms and ψ does not include any other concepts than φ ” (cf. [138]), this decomposition was largely received as an affirmation of Gödel’s intuition, according to which in analytic conditionals the consequent follows from the antecedent in virtue of a type of conceptual analysis.²

I will now consider two issues which cast some doubts on the adequacy of Fine’s result in capturing Gödel’s conjecture and on PAI as a logic of inference within a formal system. These issues are formally summarised by Proposition [6.1], which contains three dubious containment formulas.

1. *Formal dubia.* In classical containment models, the rejection of $\varphi \rightarrow \varphi \vee \psi$ makes justice to Parry’s intuition that analytic implication should be carried out within a fixed domain of discourse. The validity of $\varphi \rightarrow \varphi \vee \psi$ fails exactly because of topic-inclusion considerations, since while in all possible worlds φ classically implies $\varphi \vee \psi$, ψ may be about topics which are not included in the topic of φ . However, it is not to be taken for granted that entailment should be regimented by an intensional conditional based on classical logic, since this makes analytic implication useless for paraconsistent purposes. Indeed, once topic-inclusion is satisfied, the validity of an analytic implication reduces to the validity of the underlying S4-strict implication, making $(\varphi \vee (\psi \wedge \neg\psi)) \rightarrow \varphi$ and $((\varphi \wedge \neg\varphi) \wedge \psi) \rightarrow (\psi \wedge \neg\psi)$ logically true.³ However, such principles were argued to subscribe to contentious principles of epistemic reasoning in Chapter [5].
2. *Syntactic sieve.* Modeling sentences’ subject matter via topic models, classical containment models inherit the criticism raised about TSIM semantics in Chapter [5]. To recapitulate, standard TSIM treatments of topic fail to realise that intensional operators might exert a transformative role on the topic of sentences, the latter topic depending on the states of evaluation. For example, the topic of intensional conditionals can exceed the sum of the topics of the antecedent and the consequent (cf. Chapter [5]).⁴ To further elaborate on

²To see the failure of Gödel’s conjecture, note that $((p \rightarrow p) \wedge (q \rightarrow q)) \rightarrow (p \rightarrow (q \rightarrow q))$ is an S4 theorem and has variable inclusion but is not valid in Parry’s system.

³[196] points out other related problematic formulas, such as $(\varphi \wedge \neg\varphi) \vee \psi \rightarrow \psi$ and $\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \dots \rightarrow (\varphi \rightarrow \varphi))$.

⁴The topic of a subjunctive conditional with antecedent φ , for example, is in many cases naturally understood as incorporating those *situations selected by* φ in the manner described by [102]. In

the point, it might be argued, with [31], that the topic-theoretic apparatus in classical containment models provides no insight on the notion of concept inclusion, and is essentially just syntax in disguise. Indeed, Fine’s axiomatisation includes the axiom $\varphi \supseteq \psi$ whenever $At(\psi) \subseteq At(\varphi)$, which makes variable inclusion a sufficient condition for content inclusion.

3. *Topic inclusion heredity.* In classical containment models, topic inclusion is *hereditary* by (CC1). This assumption runs counter to an epistemic reading of the accessibility relation Q . Although the latter is not Fine’s reading of S4 Kripke models, it is closer to the present aims, and it is often adopted – e.g. in the semantics of intuitionistic logic. In such a setting, (CC1) amounts to a *coarsening* of topic during the process of rational inquiry; once two topics are found to coincide, they must necessarily coincide for the duration of the investigation. However, when one conducts an investigation and learns about a topic, such learning often serves instead to *refine* the topical landscape. We have tended, for example, to identify the topics of two cryptic species until scientific investigation provides genetic evidence that the two are distinct.⁵

Proposition 6.1. *The following are valid in classical containment models (recall that $\varphi \equiv \psi := (\varphi \supseteq \psi) \wedge (\psi \supseteq \varphi)$):*

1. $(\varphi \rightarrow \psi) \equiv (\varphi \vee \psi)$;
2. $(\varphi \rightarrow \psi) \supseteq (\psi \rightarrow \varphi)$;
3. $(\varphi \rightarrow \psi) \supseteq (\neg\psi \rightarrow \neg\varphi)$.

As an upshot of the above considerations, relieving a model theory of the commitments outlined in Proposition 6.1 can be expected to make the containment logic intuition available to a broad range of conditionals in which topic transparency naturally fails.

6.1.2 Relevant Containment Logic

Containment logics qualify as relevant logics with respect to analytic implication under the syntactic criterion, since variable inclusion implies variable sharing. However,

cases in which states fixed by φ and ψ differ, then, the topics of $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$ may fail to coincide. The operation of taking the converse of a conditional is thus not always topic-preserving.

⁵Insofar as Fine’s project in [69] was largely oriented towards historical ends – that of providing a model theory for Parry’s logic – the assumption of heredity was necessary. Indeed, the syntactic counterpart of heredity of topic inclusion ($\varphi \supseteq \psi \rightarrow \Box(\varphi \supseteq \psi)$) is not independent of Parry’s axioms.

this has not spared containment logics from harsh criticism by their relevant cousins [166, p.96]. The main source of criticism, raised by [196], stems from the hidden classical features of PAI, highlighted by Fine’s double-barreled analysis of analytic implication. Nonetheless, Sylvan himself has been one of the few authors to put some effort into laying the groundwork for *relevant containment logic* in [196], which motivates the need for a relevant implication which does not introduce superfluous information.

Sylvan’s crucial observation is that worries concerning classical theories “are not repaired simply by throwing on a variable-inclusion filter” [166, p.101]. A more promising approach is then to use relevant logic as an alternative modal logic, on which a satisfactory theory of containment can piggyback ride. To this aim, [196] offers an account of a relevant logic in which Parry’s topic-inclusion intuitions were incorporated into the framework. Just as Fine grafted a topic-theoretic apparatus onto Kripke semantics for the modal logic **S4**, Sylvan grafts such a mechanism onto the semantics for relevant logic.

Definition 6.3 (Relevant neighborhood models). *Let a relevant neighborhood frame be a tuple of the following form:*

$$(S, Prop, L, R, *)$$

- S is a set of information states;
- $Prop \subseteq \mathcal{P}(S)$ is a set of admissible propositions;
- $L \in Prop$ is a set of logical states;
- $* : S \rightarrow S$ is the Routley star function, interpreting \neg ;
- $R \subseteq S \times \mathcal{P}(S)^2$ is a neighborhood ternary relation, interpreting \rightarrow .

Moreover, let the following frame conditions hold for all relevant neighborhood frames \mathfrak{F} and all $X, Y \in Prop$:

$$X \wedge^{\mathfrak{F}} Y, X \vee^{\mathfrak{F}} Y, \neg^{\mathfrak{F}} X, X \rightarrow^{\mathfrak{F}} Y \in Prop \quad (\text{TC1})$$

$$X \subseteq Y \iff L \subseteq X \rightarrow^{\mathfrak{F}} Y \quad (\text{TC2})$$

where $\wedge^{\mathfrak{F}}, \vee^{\mathfrak{F}}, \neg^{\mathfrak{F}}, \rightarrow^{\mathfrak{F}}$ are as in Definition 4.2. Let (\mathfrak{F}, V) , with $V : At \rightarrow Prop$, be a relevant neighborhood model based on a relevant neighborhood frame \mathfrak{F} .

As in Chapter 5, neighborhood Routley–Meyer models for relevant propositional logic are best understood as formalising epistemic truth, or truth from the perspective of the agent’s information state. This interpretation is especially interesting when reading relevant implication as a form of *contextual entailment*. Minimal

desiderata on the formalisation of entailment as a binary relation include that: (i) entailment relates propositions (i.e. sets of information states); and (ii) entailment between propositions interacts appropriately with implication between the corresponding formulas. Relevant neighborhood models can be used to model entailment, as done in [80], by writing $X \sqsubseteq_s Y$ in place of R_sXY , with the caveat that a *contextual entailment* $X \sqsubseteq_s Y$ may be read as “ X entails Y according to the informational context fixed by s ”.

Different informational contexts pose different constraints on contextual entailment. At non-logical states s , \sqsubseteq_s has no specific properties. Just like at non-logical states even the simplest laws of logic such as $\varphi \rightarrow \varphi$ may fail; in the same fashion, at non-logical states \sqsubseteq_s is not a partial ordering (e.g. it may fail to be reflexive, as needed to invalidate $\psi \rightarrow (\varphi \rightarrow \varphi)$). However, the set of logical states L subscribes to tighter logical connections. For one, logical states $l \in L$ support all logically valid relevant implications. For two, \sqsubseteq_l *reduces to* the partial order \subseteq in the following sense: the information inclusion common to all logical states are exactly those recorded by set-theoretic inclusion. Of course, logical states typically make more than the logical truths true, so two logical states will often differ in the conditionals they make true, but the ones in common are those corresponding to \subseteq on elements of *Prop*. The connection between logical entailment and set inclusion is recorded by the semantic deduction theorem (Lemma 6.3).

Definition 6.4 (Relevant containment model). *Let a relevant containment model be a tuple of the following form:*

$$(S, Prop, L, *, R, V, \mathcal{T}, \oplus, \tau)$$

- $(S, Prop, L, *, R, V)$ is an relevant neighborhood model with the proviso that $V : At \cup \{\varphi \supseteq \psi \mid \varphi, \psi \in \mathfrak{L}_{\text{RCont}}\} \rightarrow Prop$;
- $(\mathcal{T}_s, \oplus_s, \tau_s)$ is a topic model for all $s \in L$.

Definition 6.5 (Relevant Containment satisfaction). *Let \mathfrak{M} be a relevant containment model. Let satisfaction of a formula $\varphi \in \mathfrak{L}_{\text{RCont}}$ in \mathfrak{M} pointed at $s \in S$ be recursively defined as follows:*

$$\begin{array}{lll} \mathfrak{M}, s \models p & \iff & s \in V(p) \\ \mathfrak{M}, s \models \mathbf{t} & \iff & s \in L \\ \mathfrak{M}, s \models \neg\varphi & \iff & \mathfrak{M}, s^* \not\models \varphi \\ \mathfrak{M}, s \models \varphi \wedge \psi & \iff & \mathfrak{M}, s \models \varphi \text{ and } \mathfrak{M}, s \models \psi \\ \mathfrak{M}, s \models \varphi \vee \psi & \iff & \mathfrak{M}, s \models \varphi \text{ or } \mathfrak{M}, s \models \psi \end{array}$$

$$\begin{aligned}
\mathfrak{M}, s \models \varphi \rightarrow \psi & \iff \forall t, u \in S(Rstu, \mathfrak{M}, t \models \varphi \implies \mathfrak{M}, u \models \psi) \\
\mathfrak{M}, s \models \varphi \supseteq \psi & \iff \begin{cases} \tau_s(\psi) \preceq_s \tau_s(\varphi) & \text{if } s \in L \\ s \in V(\varphi \supseteq \psi) & \text{otherwise} \end{cases}
\end{aligned}$$

Let validity in relevant containment models be defined as L -validity, i.e. truth at all logical states.

Sylvan's relevant containment models work by leveraging on the relevant distinctions between logical and non-logical states. In Routley–Meyer models, logical theories are distinguished from non-logical theories in that the former contain all logical theorems while the latter exhibit a high degree of freedom in their closure properties. Likewise, in relevant containment models, logical theories are associated with well behaved topic structures, while non-logical theories are associated with no structure at all. While topic inclusions at logical states are evaluated by considering the mereological relationship in topic models, at non-logical states they are treated as propositional atoms, as they are assigned an arbitrary propositional value by V . As a consequence, at logical states subject matter parthood is well behaved, e.g. it enjoys the properties dictated by the join-semilattice structure; while at non-logical states topic inclusions are assessed arbitrarily, obeying no condition whatsoever.

Certainly, Sylvan's modified approach improves on classical containment models in several ways. Most importantly, Sylvan's framework liberates the resulting logic from the twin commitments of a classical picture of the conditional and the heredity of topic inclusion. However, I wish to consider two sources of criticism.

1. *Disharmony*. As noted in Chapter 5, relevant logics provide an harmonious semantic analysis of logical connectives' meaning, by means of uniform truth conditions for connectives across different types of states (logical or non-logical). Although built on a relevant logic framework, the discontinuity between logical and non-logical states is *reintroduced* in a disharmony between the truth-conditions for *topic containment*. Drawing one's attention to the definition of $\varphi \supseteq \psi$, one can take note of the case distinction: At *logical* states l , the truth-conditions for $\varphi \supseteq \psi$ are assessed naturally by a recursive definition in terms of the relations \preceq_l ; at *non-logical* states, the truth-conditions are determined piecemeal by overloading the valuation function V . This discontinuity appears to be overly syntactic and artificial, since it reproduces at a topical level what was found inadequate in truth specification for impossible worlds.
2. *Situational containment*. Sylvan's relevant containment logic shares with PAI the property that the inclusion of the variables of ψ in those of φ is a sufficient

condition for $\varphi \supseteq \psi$ to be logically true. This feature reveals the syntactic nature of topic inclusion in existing containment logics. On a related note, the binary analysis of topic inclusion fails to notice that topics are always evaluated *in situ*, i.e. in a particular discursive context, which is individuated by the information available in a given situation. In what follows, I wish to consider the thesis that the particular background discursive topic associated with a possible state influences the inclusions that will be upheld in that context. While the discursive topic of a logical state induces a discursive context where the standard lattice-theoretic treatment of topics are respected, the focal topic of a non-logical state can be more focused, interrupting the standard topic-theoretic expectations.

6.2 Upgrading Relevant Containment

The above critical remarks of classical and relevant containment models suggest a harmony between entailment and containment. Just like the truth-conditions for a relevant conditional $\varphi \rightarrow \psi$ at a state s consider *three* relata, namely, the proposition expressed by φ , the proposition expressed by ψ , and the *informational context fixed by s* ; it is natural to approach the analysis of a statement of topic inclusion $\varphi \supseteq \psi$ as requiring a relation between the topic of φ , the topic of ψ , and the *focal topic of the discursive context fixed by s* . This narrative thus suggests to consider a *ternary topic inclusion relation*, aligning the relevant analysis of topic with the relevant analysis of entailment and respecting the uniformity in the meaning of containment across states – cf. [197] for a different approach to topics in relevant logic.

6.2.1 Ternary Containment Models

In order to deliver such a ternary analysis of topic inclusion, I will devise a model theory such that (i) topic models are generalised, (ii) the semantic content of a formula is split into an informational and a topical component, and (iii) the informational content of a containment formula $\varphi \supseteq \psi$ only depends on the inclusion relationships between the topical contents of φ and ψ .

Definition 6.6 (Generalised topic models). *Let a generalised topic model be a tuple of the following form:*

$$(\mathcal{T}, \preceq, \neg, \wedge, \vee, \rightarrow, \supseteq, \tau)$$

- \mathcal{T} is a set of topics;

- $\preceq \subseteq \mathcal{T}^2$ is a topic inclusion relation;
- $\neg, \wedge, \vee, \rightarrow, \supseteq: \mathcal{T}^2 \rightarrow \mathcal{T}$ are topic functions;
- $\tau: At \cup \{\mathbf{t}\} \rightarrow \mathcal{T}$ is a topic valuation;

Let \mathbf{T} be the class of generalised topic models.

Next, *ternary containment premodels* allow one to specify the different properties of \preceq on the basis of the focal topic of discourse fixed by the information available in a given situation. In particular, the discursive contexts fixed by logical states obey tighter conditions than those fixed by non-logical states.

Definition 6.7 (Ternary containment premodel). *Let a ternary containment premodel be a tuple of the following form:*

$$(S, Prop, L, *, R, V, \mathbf{t})$$

- $(S, Prop, L, *, R, V)$ is a relevant neighborhood model;
- $\mathbf{t}: S \rightarrow \mathbf{T}$ is a function assigning the generalised topic model $(\mathcal{T}_s, \preceq_s, \neg_s, \wedge_s, \vee_s, \rightarrow_s, \supseteq_s, \tau_s)$ to all $s \in S$.

Moreover, let the following conditions hold for all generalised topic models, all $l \in L$ and $a, b \in \mathcal{T}_l$:

$$a \vee_l b = b \vee_l a \quad (\text{TC3})$$

$$(a \vee_l b) \vee_l c = a \vee_l (b \vee_l c) \quad (\text{TC4})$$

$$a \vee_l a = a \quad (\text{TC5})$$

$$a \vee_l b = a \wedge_l b \quad (\text{TC6})$$

$$a \preceq_l b \iff a \vee_l b = b \quad (\text{TC7})$$

$$\neg_l a = a \quad (\text{TC8})$$

The reader may note the differences between the topic structure described in Definitions [5.2](#) and [6.9](#).

1. *Containment and partial order.* $a \preceq_s b$ may be read as “the topic a is contained in topic b according to the discursive context fixed by s ”. A situation can be thought of as fixing a discursive context by transforming a topic into the focal discursive topic, according to the special-case concerns of the given situation.⁶

⁶When we fix a focal discursive topic, we often motivate such transformations through the use of the operation “*qua*.” Discussion about *e.g.* cats when purchasing a pet cat transforms the topic **Cats** into the topic **Cats qua Pets**; discussion about cats with the chief sales officer of a pet store transforms **Cats** into the topic **Cats qua Commodities**.

In generalised topic models, these considerations are formalised by making no assumptions on contextual containment. In particular, the join semilattice structure of topic models is abandoned and \preceq_s need not be in general a partial order. To see why this is plausible, consider reflexivity: $a \preceq_s a$ since the focal discursive topic a , i.e. the topic a transformed by the special-case concerns of s , may differ from a before the transformation. In this picture, logical contexts are understood as discursive contexts which are transparent, i.e. assessment of topical inclusion is not affected by contextual modifiers. That is, logical states are understood as possessing no special-case concerns, whence for all logical states $l \in L$, \preceq_l turns out to be a partial order. This connection is recorded by Lemma 6.2.

2. *Topic functions.* While Definition 5.2 — essentially carrying over the formalism of 69 — reduces the assignment of topic to all complex formulas to a single function \oplus , Definition 6.6 equips the topic structure with an assignment function for each connective. Although this might seem like complicating matters beyond necessity, expanding the range of topic functions is not a purposeless abandonment of considerations of economy. Rather, such an approach is a natural extension of recent work on models of topic. 64 and 65, for example, gave arguments in favor of a model of topic in which every intensional connective is associated with a distinct function on topics. The picture encapsulated in Definition 6.6 is thus a natural endpoint to this trend.
3. *Finean topic models.* In ternary containment premodels, it is possible to partially retrieve the properties of Finean topic models (cf. Definition 5.2). That is, by Conditions (TC3)-(TC5), (\mathcal{T}_l, \vee_l) is a join semilattice for all $l \in L$; by Condition (TC6), $(\mathcal{T}_l, \wedge_l)$ is a join semilattice as well; and by Condition (TC7), $(\mathcal{T}_l, \preceq_l)$ is the poset induced by either (\mathcal{T}_l, \vee_l) or $(\mathcal{T}_l, \wedge_l)$. In this way, the semilattice structure (\mathcal{T}, \oplus) of Definition 5.2 is simulated by either $(\mathcal{T}_l, \wedge_l)$ or (\mathcal{T}_l, \vee_l) . Moreover, Condition (TC8) makes the topic operator for negation inert at logical states.⁷ Note, however, that even in normal discursive contexts $a \rightarrow_l b$ and $a \supseteq_l b$ are only functionally dependent on a and b . This allows the interruption of many topic inclusions in F.C, such as the principles of Proposition 6.1.

⁷Condition (TC8) boils down to *negation transparency*, which was criticised e.g. in 5. A more general approach, driven by consideration of the transformational role exerted by negation on topics, would require one to work with ternary containment models where this assumption is dropped.

Definition 6.8 (Topical content). *Let \mathfrak{M} be a ternary containment premodel. Let the topical content of φ at \mathfrak{M} pointed at s , denoted as $(\varphi)_{\mathfrak{M}}^s$, be defined by recursively extending τ_s to $\mathfrak{L}_{\text{RCont}}$ as follows, for $\otimes \in \{\wedge, \vee, \rightarrow, \supseteq\}$:*

$$\begin{aligned} (\mathit{p})_{\mathfrak{M}}^s &= \tau_s(\mathit{p}) \\ (\neg\varphi)_{\mathfrak{M}}^s &= \neg_s(\varphi)_{\mathfrak{M}}^s \\ (\varphi \otimes \psi)_{\mathfrak{M}}^s &= (\varphi)_{\mathfrak{M}}^s \otimes_s (\psi)_{\mathfrak{M}}^s \end{aligned}$$

In ternary containment premodels, the *topical content* of φ relative to the focal discursive context fixed by a state s , $(\varphi)^s$, is recursively determined. As a consequence, the informational content of containment formulas $\varphi \supseteq \psi$ is not arbitrarily assigned by the valuation V , as in topic models. Note, however, the topic valuation τ in Definition 6.6 does assign a primitive topic not just to atoms but also to \mathbf{t} .

Next, *ternary containment models*, can be defined by ternary containment premodels and topical content.

Definition 6.9 (Ternary containment models). *Let a ternary containment model be a ternary containment premodel \mathfrak{M} such that for all $\varphi, \psi \in \mathfrak{L}_{\text{RCont}}$:*

$$\{s \mid (\psi)_{\mathfrak{M}}^s \preceq_s (\varphi)_{\mathfrak{M}}^s\} \in \text{Prop} \quad (\text{TC9})$$

Let **F.C** denote the class of all relevant containment models.

Note that one may define $\llbracket \varphi \rrbracket_{\mathfrak{M}} \supseteq^{\mathfrak{M}} \llbracket \psi \rrbracket_{\mathfrak{M}} := \{s \mid (\psi)_{\mathfrak{M}}^s \preceq_s (\varphi)_{\mathfrak{M}}^s\}$, by which *Prop* is closed under $\supseteq^{\mathfrak{M}}$, an operation which matches the interpretation of the connective \supseteq . Then, Condition (TC9) serves the same purpose of Condition (TC1) in Definition 6.3. A difference worth noting between the relevant neighborhood frame operations and $\supseteq^{\mathfrak{M}}$ is that the latter must be defined as an operation on ternary content models. Condition (TC9) is necessary to yield the Heredity Lemma 6.1. Finally, the topical contents of formulas φ, ψ relative to a discursive context s contribute to the determination of the informational content of containment formulas $\varphi \supseteq \psi$. That is, we assume that $\llbracket \varphi \supseteq \psi \rrbracket$ at s depends only on $(\varphi)^s$ and $(\psi)^s$, which are recursively and independently defined.

Definition 6.10 (Informational content). *Let \mathfrak{M} be a ternary containment model. Let satisfaction of a formula $\varphi \in \mathfrak{L}_{\text{RCont}}$ in \mathfrak{M} pointed at $s \in S$ be defined by updating Definition 6.5 as follows:*

$$\mathfrak{M}, s \models \varphi \supseteq \psi \quad \iff \quad (\psi)_{\mathfrak{M}}^s \preceq_s (\varphi)_{\mathfrak{M}}^s$$

Let the informational content of $\varphi \in \mathfrak{L}_{\text{RCont}}$ in \mathfrak{M} be defined as $\llbracket \varphi \rrbracket_{\mathfrak{M}} = \{s \in S \mid \mathfrak{M}, s \models \varphi\}$

Let validity in ternary containment models be defined as L-validity, i.e. truth at all logical states, and consequence as S-consequence, i.e. truth preservation across S . Ternary containment models display the interesting feature that

$$\mathfrak{M}, s \models \varphi \rightarrow \psi \text{ iff } \begin{cases} \llbracket \varphi \rrbracket_{\mathfrak{M}} \sqsubseteq_s \llbracket \psi \rrbracket_{\mathfrak{M}} \\ (\psi)_{\mathfrak{M}}^s \preceq_s (\varphi)_{\mathfrak{M}}^s, \end{cases}$$

which, in turn, can be interpreted as the conjunction of two theses: φ informationally entails and topically contains ψ , relative to the context fixed by s . To repeat, this analysis is able to respect uniformity in the meaning of entailment and containment across states: both operations are semantically analysed via accessibility relations \sqsubseteq_s and \preceq_s , which in case s is a logical state turn out to be partial orders.

Lemma 6.1 (Informational heredity). *For all $\mathfrak{M} \in \mathbf{F.C}$ and all $\varphi \in \mathcal{L}_{\text{RCont}}$: $\llbracket \varphi \rrbracket_{\mathfrak{M}} \in \text{Prop}$.*

Proof. By induction on the structure of φ . The base case holds by definition of V . The induction step for the cases involving $\wedge, \vee, \neg, \rightarrow$ holds by the induction hypothesis and Condition (TC1), while the case $\varphi := \psi \supseteq \chi$ holds by Condition (TC9). \square

Lemma 6.2 (Containment). $\mathfrak{M} \models \varphi \supseteq \psi \iff \forall l \in L ((\psi \vee \varphi)_{\mathfrak{M}}^l = (\varphi)_{\mathfrak{M}}^l)$.

Proof. The following chain of equivalences holds for all $l \in L$: $\mathfrak{M} \models \varphi \supseteq \psi$ iff (by Definition 6.10) $(\psi)^l \preceq_l (\varphi)^l$ iff (by Condition (TC6)) $(\psi)^l \vee_l (\varphi)^l = (\varphi)^l$ iff (by Definition 6.8) $(\psi \vee \varphi)^l = (\varphi)^l$. We conclude that $\mathfrak{M} \models \varphi \supseteq \psi$ iff for all $l \in L$ $(\psi \vee \varphi)^l = (\varphi)^l$. \square

Lemma 6.3 (Entailment). $\mathfrak{M} \models \varphi \rightarrow \psi \iff \llbracket \varphi \rrbracket_{\mathfrak{M}} \subseteq \llbracket \psi \rrbracket_{\mathfrak{M}}$.

Proof. The following chain of equivalences holds: $\mathfrak{M} \models \varphi \rightarrow \psi$ iff (by Definition 6.9) $L \subseteq \llbracket \varphi \rrbracket \rightarrow^{\delta} \llbracket \psi \rrbracket$ iff (by Lemma 6.1 and Condition (TC2)) $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$. \square

6.2.2 Axiomatisation of F.C

A sound and complete axiom system for the logic of contextual entailment and containment F.C is obtained by adopting the strategy of [196], which provides characteristic principles for \supseteq on top of an axiom system for relevant logic.

Definition 6.11 (F.C axiom system). *Let the Hilbert system for the logic of contextual entailment and containment F.C consist of the the following axioms and rules, for $\otimes \in \{\wedge, \vee, \rightarrow, \supseteq\}$:*

- (F.C1) *Axioms and rules of F (cf. Definition 4.4, principles (FE1)–(FE14))*
- (F.C2) $\varphi \supseteq \varphi$
- (F.C3) $\varphi \equiv \neg\varphi$
- (F.C4) $(\varphi \wedge \psi) \supseteq \varphi$
- (F.C5) $(\varphi \wedge \psi) \supseteq \psi$
- (F.C6) $(\varphi \vee \psi) \equiv (\varphi \wedge \psi)$
- (F.C7) $(\mathbf{t} \wedge (\varphi \supseteq \psi) \wedge (\psi \supseteq \chi)) \rightarrow (\varphi \supseteq \chi)$
- (F.C8) $(\mathbf{t} \wedge (\varphi \supseteq \psi) \wedge (\varphi \supseteq \chi)) \rightarrow (\varphi \supseteq (\psi \wedge \chi))$
- (F.C9) $(\mathbf{t} \wedge (\varphi \equiv \psi)) \rightarrow (\neg\varphi \equiv \neg\psi)$
- (F.C10) $(\mathbf{t} \wedge (\varphi \equiv \chi) \wedge (\psi \equiv \xi)) \rightarrow ((\varphi \otimes \psi) \equiv (\chi \otimes \xi))$

The containment axioms might warrant some special attention. Many of these are highly natural, appearing in *e.g.* Fine’s alternative axiomatization of PAI in [69]. In the case of (F.C7)–(F.C9), however, the presence of the Ackermann constant \mathbf{t} in the antecedent might raise an eyebrow. From a model-theoretic perspective, the presence of \mathbf{t} ensures that the truth of axioms (F.C7)–(F.C9) range over *logical states*. From a syntactic point of view, the presence of \mathbf{t} is a pointer to the fact that containment formulas obey stronger requirements (e.g. transitivity of topic inclusion). Put differently, containment axioms are enthymematic, as the lattice properties of containment implicitly depend on the availability of logical information (encoded in \mathbf{t} , which serves to represent the conjunction of theorems) The utility of this device, imported from [196], will be reinforced when reviewing the proof of soundness.

Theorem 6.1 (F.C soundness). $\vdash_{\text{F.C}} \varphi$ *only if* $\models \varphi$.

Proof. By Theorem 4.1, we need only show the validity of each of the containment axioms (F.C2)–(F.C9). In the proof, we use Lemmas 6.2 and 6.3 without mention.

(F.C2). Take some arbitrary $l \in L$. By (TC5) we infer that $(\llbracket \varphi \rrbracket^l \vee_l \llbracket \varphi \rrbracket^l) = \llbracket \varphi \rrbracket^l$, by which we infer that $(\llbracket \varphi \rrbracket^l) \preceq_l (\llbracket \varphi \rrbracket^l)$. We conclude that $\mathfrak{M}, l \models \varphi \supseteq \varphi$.

(F.C3). Take some arbitrary $l \in L$. By Definitions 6.8 and 6.9 we infer that $(\llbracket \neg\varphi \rrbracket^l) = \neg_l(\llbracket \varphi \rrbracket^l) = \llbracket \varphi \rrbracket^l$. By the latter and (TC8) we infer that $(\llbracket \varphi \rrbracket^l) \preceq_l (\llbracket \neg\varphi \rrbracket^l)$ and $(\llbracket \neg\varphi \rrbracket^l) \preceq_l (\llbracket \varphi \rrbracket^l)$. We conclude that $\mathfrak{M}, l \models \varphi \equiv \neg\varphi$.

(F.C4). The following chain of equalities holds:

$$\begin{aligned}
\langle \varphi \rangle^l \vee_l \langle \varphi \wedge \psi \rangle^l &= \langle \varphi \rangle^l \vee_l (\langle \varphi \rangle^l \wedge_l \langle \psi \rangle^l) && \text{by Definition } \boxed{6.8} \\
&= \langle \varphi \rangle^l \vee_l (\langle \varphi \rangle^l \vee_l \langle \psi \rangle^l) && \text{by Condition } \boxed{\text{TC6}} \\
&= (\langle \varphi \rangle^l \vee_l \langle \varphi \rangle^l) \vee_l \langle \psi \rangle^l && \text{by Condition } \boxed{\text{TC4}} \\
&= \langle \varphi \rangle^l \vee_l \langle \psi \rangle^l && \text{by Condition } \boxed{\text{TC5}} \\
&= \langle \varphi \rangle^l \wedge_l \langle \psi \rangle^l && \text{by Condition } \boxed{\text{TC6}} \\
&= \langle \varphi \wedge \psi \rangle^l && \text{by Definition } \boxed{6.8}
\end{aligned}$$

That is, $\langle \varphi \rangle^l \preceq_l \langle \varphi \wedge \psi \rangle^l$. We conclude that $l \models (\varphi \wedge \psi) \supseteq \varphi$. The case for (F.C5) is established similarly.

(F.C6). It is a straightforward consequence of $\boxed{\text{TC6}}$.

(F.C7). Take some arbitrary $s \in S$ such that $s \models \mathbf{t} \wedge (\varphi \supseteq \psi) \wedge (\psi \supseteq \chi)$. Hence, $s \models \mathbf{t}$, $s \models \varphi \supseteq \psi$ and $s \models \psi \supseteq \chi$. By $s \models \mathbf{t}$ we infer that $s \in L$, which implies by $s \models \varphi \supseteq \psi$ and $s \models \psi \supseteq \chi$ that $\langle \psi \rangle^s \preceq_s \langle \varphi \rangle^s$ and $\langle \chi \rangle^s \preceq_s \langle \psi \rangle^s$. By $\boxed{\text{TC7}}$ we infer that $\langle \psi \rangle^s \vee_s \langle \varphi \rangle^s = \langle \varphi \rangle^s$ and $\langle \chi \rangle^s \vee_s \langle \psi \rangle^s = \langle \psi \rangle^s$, which implies by $\boxed{\text{TC4}}$ that $\langle \chi \rangle^s \vee_s \langle \varphi \rangle^s = \langle \varphi \rangle^s$. We infer by $\boxed{\text{TC7}}$ that $\langle \chi \rangle^s \preceq_s \langle \varphi \rangle^s$, by which we conclude that $s \models \varphi \supseteq \chi$.

(F.C8). Take some arbitrary $s \in S$ such that $s \models \mathbf{t} \wedge (\varphi \supseteq \psi) \wedge (\varphi \supseteq \chi)$. Hence, $s \models \mathbf{t}$, $s \models \varphi \supseteq \psi$ and $s \models \varphi \supseteq \chi$. By $s \models \mathbf{t}$ we infer that $s \in L$, which implies by $s \models \varphi \supseteq \psi$ and $s \models \varphi \supseteq \chi$ that $\langle \psi \rangle^s \preceq_s \langle \varphi \rangle^s$ and $\langle \chi \rangle^s \preceq_s \langle \varphi \rangle^s$. By $\boxed{\text{TC7}}$ we infer that $\langle \psi \rangle^s \vee_s \langle \varphi \rangle^s = \langle \varphi \rangle^s$ and $\langle \chi \rangle^s \vee_s \langle \varphi \rangle^s = \langle \varphi \rangle^s$, which implies by $\boxed{\text{TC3}}$ – $\boxed{\text{TC4}}$ that $(\langle \psi \rangle^s \vee_s \langle \chi \rangle^s) \vee_s \langle \varphi \rangle^s = \langle \varphi \rangle^s$. We infer by Definition $\boxed{6.8}$, $\boxed{\text{TC6}}$ and $\boxed{\text{TC7}}$ that $\langle \psi \wedge \chi \rangle^s \preceq_s \langle \varphi \rangle^s$, by which we conclude that $s \models \varphi \supseteq (\psi \wedge \chi)$.

(F.C9). Take some arbitrary $s \in S$ such that $s \models \mathbf{t} \wedge (\varphi \equiv \psi)$. Hence, $s \models \mathbf{t}$ and $s \models \varphi \equiv \psi$. By $s \models \mathbf{t}$ we infer that $s \in L$, which implies by $s \models \varphi_i \equiv \psi_i$ that $\langle \varphi \rangle^s \preceq_s \langle \psi \rangle^s$ and $\langle \psi \rangle^s \preceq_s \langle \varphi \rangle^s$. We infer by the anti-symmetry of \preceq_s that $\langle \varphi_i \rangle^s = \langle \psi_i \rangle^s$, which implies by Definition $\boxed{6.8}$ that $\neg_s \langle \varphi_i \rangle^s = \neg_s \langle \psi_i \rangle^s$. We conclude that $s \models \neg \varphi \equiv \neg \psi$. The case (F.C10) is established similarly.

□

The strategy to establish completeness can be sketched as follows. The canonical ternary containment model is built out of: (i) the canonical relevant neighborhood model \mathcal{J}^c (cf. Definition $\boxed{4.5}$); and (ii) a function associating members of the domain of \mathcal{J}^c with canonical generalised topic models \mathfrak{Z}_s^c . Each \mathfrak{Z}_s^c is in turn defined so as to specify different sorts of topic structures depending on the kind of information state s considered. For all logical states $l \in L$, \mathfrak{Z}_l^c is built out of equivalence classes over

containment equivalence formulas, while for non logical states $s \notin L$, the specified equivalence relation reduces to the identity relation, so that elements of \mathfrak{T}_s^c are just formulas of the language and topic operations are just the logical connectives on those formulas (cf. Definition [6.13](#)).

Definition 6.12 (Canonical relevant neighborhood model). *Let the canonical relevant neighborhood model be a tuple of the following form:*

$$\mathfrak{J}^c = (S^c, Prop^c, L^c, *^c, R^c, V^c)$$

- $S^c = \{s \mid s \text{ is a prime F.C theory over } \mathfrak{L}_{\text{RCont}}\}$;
- $Prop^c = \{\llbracket \varphi \rrbracket_c \mid \varphi \in \mathfrak{L}_{\text{RCont}}\}$ where $\llbracket \varphi \rrbracket_c = \{s \mid \varphi \in s\}$;
- $L^c = \{s \in S^c \mid \mathbf{t} \in s\}$;
- $s^{*^c} = \{\varphi \in \mathfrak{L}_{\text{RCont}} \mid \neg\varphi \notin s\}$;
- $R^c = \{(s, X, Y) \in S^c \times (Prop^c)^2 \mid \exists \varphi, \psi \in \mathfrak{L}_{\text{RCont}} (X = \llbracket \varphi \rrbracket_c, Y = \llbracket \psi \rrbracket_c, \varphi \rightarrow \psi \in s)\}$;
- $V(p) = \{s \in S^c \mid p \in s\}$.

Definition 6.13 (Raw topic structure). *For a given F.C-theory s , let the raw topic structure associated with s be a tuple of the following form:*

$$(\mathcal{T}_s^c, \preceq_s^c, \neg_s^c, \vee_s^c, \wedge_s^c, \rightarrow_s^c, \supseteq_s^c, \tau_s^c)$$

- $\mathcal{T}_s^c = \mathfrak{L}_{\text{RCont}}$;
- $\psi \preceq_s^c \varphi \iff \varphi \supseteq \psi \in s$;
- $\otimes_s^c(\varphi_1, \dots, \varphi_n) = \otimes(\varphi_1, \dots, \varphi_n)$ for $\otimes \in \{\wedge, \vee, \supseteq, \rightarrow\}$;
- $\tau_s^c(\varphi) = \varphi$.

Definition 6.14 (Canonical equivalence). *For a given F.C-theory s , let \sim_s the canonical equivalence relation on the language $\mathfrak{L}_{\text{RCont}}$ be defined as the reflexive closure of the following relation \sim'_s :*

$$\varphi \sim'_s \psi \iff \mathbf{t} \wedge (\varphi \supseteq \psi) \wedge (\psi \supseteq \varphi) \in s$$

Note that \sim_s is indeed a congruence relation for all $s \in S$. If $s \in L$, this holds by (F.C2), (F.C5) and (F.C7)–(F.C10). If $s \notin L$, this holds trivially since we assume that $\varphi \sim_s \psi \iff \varphi = \psi$. As a consequence, the following is well defined.

Definition 6.15 (Canonical generalised topic model). *For a given s , let the canonical generalised topic model associated with s be a tuple of the following form:*

$$\mathfrak{T}_s^c = (\mathcal{T}_s^c, \preceq_s^c, \neg_s^c, \vee_s^c, \wedge_s^c, \rightarrow_s^c, \supseteq_s^c, \tau_s^c)$$

- $\mathcal{T}_s^c = \mathfrak{L}_{\text{RCont}} / \sim_s$;
- $\preceq_s = \{([\varphi]_c^s, [\psi]_c^s) \in \mathcal{T}_s^c \mid \exists \varphi' \in [\varphi]_c^s, \psi' \in [\psi]_c^s (\psi' \supseteq \varphi' \in s)\}$;
- $\otimes_s([\varphi_1]_c^s, \dots, [\varphi_n]_c^s) = [\otimes(\varphi_1, \dots, \varphi_n)]_c^s$ for $\otimes \in \{\neg, \wedge, \vee, \supseteq, \rightarrow\}$;
- $\tau_s(p) = [p]_c^s = \{\psi \mid p \sim_s \psi\}$.

Definition 6.16 (Canonical ternary containment model). *Let the canonical ternary containment model be a tuple of the following form:*

$$\mathfrak{M}^c = (\mathcal{T}^c, \mathfrak{t}^c)$$

- \mathcal{T}^c is the canonical relevant neighborhood model;
- $\mathfrak{t}^c : s \mapsto \mathfrak{T}_s^c$ maps each $s \in S^c$ with its associated canonical generalised topic model.

Note that, according to Definition [6.16](#), each non-logical state in the canonical model \mathfrak{M}^c is associated with a raw topic structure, as per Definition [6.13](#).

Lemma 6.4 (Regular theories). *For all $s \in L^c$ in \mathfrak{M}^c :*

1. $\langle \mathcal{T}_s^c, \preceq_s \rangle$ is a join semilattice;
2. \neg_s^c is the identity;
3. $\wedge_s^c = \vee_s^c = \oplus_s$.

Proof. Item 1 is established by showing that $\langle \mathcal{T}_s^c, \preceq_s \rangle$ is a poset for which there exists a definable join \oplus_s . By (F.C2) and (F.C7), we infer that if $s \in L$, then \preceq_s is reflexive and transitive. Anti-symmetry follows from the equivalence relation \sim_s driving the quotient. Next, for a finite set $\{[\varphi_0], \dots, [\varphi_i]\}$, we offer the topic $[\varphi_0 \wedge \dots \wedge \varphi_i]$ as its least upper bound, i.e. $[\varphi_0 \wedge \dots \wedge \varphi_i] = \oplus_s\{[\varphi_0], \dots, [\varphi_i]\}$. First, we show that $[\varphi_0 \wedge \dots \wedge \varphi_i]$ is an upper bound of each $[\varphi_j]$, for all $j \leq i$. $[\varphi_j] \preceq_s [\varphi_0 \wedge \dots \wedge \varphi_i]$ follows from iterated application of (F.C4)-(F.C5). Second, we show that $[\varphi_0 \wedge \dots \wedge \varphi_i]$ is the least such upper bound, i.e. $[\varphi_0 \wedge \dots \wedge \varphi_i] \preceq_s [\psi]$ for all ψ such that $[\varphi_j] \preceq_s [\psi]$ for all $j \leq i$. By $[\varphi_j] \preceq [\psi]$ for all $j \leq i$, we can infer that for all $j \leq i$ there are $\psi' \in [\psi], \varphi'_j \in [\varphi_j]$ such that $\psi' \supseteq \varphi'_j \in s$. By (F.C8) we infer that if $s \in L$, then $\psi' \supseteq \varphi'_0 \wedge \dots \wedge \varphi'_i \in s$, by which we conclude that $[\varphi_0 \wedge \dots \wedge \varphi_i] \preceq_s [\psi]$. Item 2 and 3 follows from (F.C3) and (F.C6), respectively, together with the definition of \sim_s . \square

Lemma 6.5 (Relevant containment model). $\mathfrak{M}^c \in \mathbf{F.C}$.

Proof. It suffices to show that (i) $(S, L, *, R, V)$ is a relevant neighborhood model, (ii) each $(\mathcal{T}_s, \preceq_s, \neg_s, \vee_s, \wedge_s, \rightarrow_s, \supseteq_s, \tau_s)$ is a generalised topic model, and (iii) that \mathfrak{M}^c is a ternary containment premodel satisfying Condition [\(TC9\)](#). (i) is established

as in Lemma 4.18. In order to establish (ii) and (iii), it suffices to show that \mathfrak{M}^c satisfies Conditions (TC3)–(TC9). Conditions (TC3)–(TC8) follow from Lemma 6.4. To establish (TC9), it suffices to show that $\{s \mid [\psi]_s \preceq_s [\varphi]_s\} = \{s \mid \varphi \supseteq \psi \in s\}$, since $\{s \mid \varphi \supseteq \psi \in s\} = \llbracket \varphi \supseteq \psi \rrbracket_c \in Prop$. If $s \notin L$ then by Definition 6.13 $[\psi]_s \preceq_s [\varphi]_s \iff \varphi \supseteq \psi \in s$. If $s \in L$ we reason as follows. (\implies). Assume $[\psi]_s \preceq_s [\varphi]_s$. By Definition 6.15 we infer that there are $\psi' \sim_s \psi$ and $\varphi' \sim_s \varphi$ such that $\varphi' \supseteq \psi' \in s$. Hence, by Definition 6.14, $\psi' \sim_s \psi$ and $\varphi' \sim_s \varphi$ we infer that $\mathbf{t} \wedge (\varphi \equiv \varphi') \in s$ and $\mathbf{t} \wedge (\psi \equiv \psi') \in s$. By $s \in L$ we infer that $\mathbf{t} \in s$. Hence, by (C6) and $\varphi' \supseteq \psi' \in s$ we conclude that $\varphi \supseteq \psi \in s$. (\impliedby). Assume $\varphi \supseteq \psi \in s$. By $s \in L$ and (C1), $\mathbf{t} \wedge (\varphi \equiv \varphi) \in s$ and $\mathbf{t} \wedge (\psi \equiv \psi) \in s$. Hence, by Definition 6.14 there are $\varphi' \sim_s \varphi$ and $\psi' \sim_s \psi$, namely φ and ψ , such that $\varphi' \supseteq \psi'$. We conclude by Definition 6.15 that $[\psi]_s \preceq_s [\varphi]_s$. \square

Lemma 6.6 (Truth). $\mathfrak{M}^c, s \models \varphi \iff s \in \llbracket \varphi \rrbracket_c$.

Proof. The result is established by induction on the structure of φ . The base case, as well as the induction step involving the connectives $\neg, \wedge, \vee, \rightarrow$ are established as in [199], and the case $\varphi := \psi \supseteq \chi$ holds by Condition (TC9). \square

Theorem 6.2 (Completeness). $\mathbf{F.C} \models \varphi$ only if $\vdash_{\mathbf{F.C}} \varphi$.

Proof. The result follows from Lemmas 6.5 and 6.6. \square

6.2.3 Topic Containment and Variable Inclusion

The analysis of content inclusion offered by ternary containment logic is irreducible to syntactic conditions. The general lesson to draw is that variable inclusion is neither necessary nor sufficient for content inclusion, *pace* Fine’s double-barreled analysis. Nonetheless, stronger logics where variable inclusion is a sufficient condition for topic inclusion are ready at hand. In ternary containment models, Conditions (TC3)–(TC8) ensure that the general topic functions reduce to simpler ones. Then, ternary containment models satisfying stronger conditions on the topic structure are obtained by flattening the additional structure distinguishing generalised topic models from topic models.

Definition 6.17 (Sylvan topic model). *Let a Sylvan topic model be a generalised topic model where for all $l \in L$ and $a, b \in \mathcal{T}_l$:*

$$a \rightarrow_l b = a \supseteq_l b = a \vee_l b \tag{ST1}$$

Let **ST** be the class of Sylvan topic models.

Definition 6.18 (Sylvan containment model). *Let a Sylvan containment model be a tuple of the following form:*

$$(S, Prop, L, *, R, V, \mathbf{t})$$

- $(S, Prop, L, *, R, V)$ is relevant neighborhood model;
- $\mathbf{t} : S \rightarrow \mathbf{ST}$ is a function assigning a Sylvan topic model $(\mathcal{T}_s, \preceq_s, \neg_s, \wedge_s, \vee_s, \rightarrow -s, \supseteq_s, \tau_s)$ to all $s \in S$.

Let **F.c** be the class of all Sylvan containment models.

Informational content, validity and entailment in Sylvan containment models are defined as in ternary containment models and relevant neighborhood models.

By the above definition of Sylvan topic models, all binary topic functions of generalised topic models but \vee_l are made redundant at logical states, i.e. equivalent to \vee_l , and the unary topic function \neg_l is ineffective. That is, in Sylvan topic models the rich structure of general topic functions is reduced at logical states to a simple join semilattice. However, contrary to Sylvan's own relevant containment models of [196], in Sylvan containment models we have that the join semilattice structure arises, for logical states, as a special case from the general topic structures associated with non-logical states. Hence, Sylvan's discontinuous treatment of the interpretation of containment and non-containment formulas is emended here.

Finally, Sylvan containment models are indeed models characterising Sylvan's own relevant containment logic. First, note that in Sylvan containment models Fine's modeling Condition (T4) is retrieved, i.e. topic content is uniquely determined by the content locked in propositional atoms.

Lemma 6.7 (Atomic determination). *For all $\mathfrak{M} \in \mathbf{F.c}$, $l \in L$ and $\varphi \in \mathfrak{L}_{\mathbf{RCont}}$ with $At(\varphi) = \{p_1, \dots, p_n\}$: $(\varphi)_{\mathfrak{M}}^l = \tau_l(p_1) \vee_l \dots \vee_l \tau_l(p_n)$.*

Proof. The result is established by a routine induction on the structure of φ , where the base case holds trivially by Definition 6.8 and the induction step follows by Conditions (TC6), (TC8) and (ST1). \square

The above lemma is used to prove the soundness of Sylvan's relevant containment logic **F.c** with respect to the class of Sylvan containment models.⁸

⁸To be precise, [196] puts forward two logics of relevant containment. According to the first, which is discarded by Sylvan in favour of the one proposed here, non-logical situations are associated with order-structures, weaker than semilattices. Such a strategy is attractive because, while

Definition 6.19. *Let the Hilbert system for Sylvan's relevant containment logic F.c consist of the following axioms and rules:*

- (F.c1) *Axioms and rules of F;*
(F.c2) $\varphi \supseteq \psi$ if $At(\psi) \subseteq At(\varphi)$
(F.c3) $(\mathbf{t} \wedge (\varphi \supseteq \psi) \wedge (\psi \supseteq \chi)) \rightarrow (\varphi \supseteq \chi)$
(F.c4) $(\mathbf{t} \wedge (\chi \supseteq \varphi) \wedge (\chi \supseteq \psi)) \rightarrow (\chi \supseteq \varphi \wedge \psi)$

Theorem 6.3 (F.c-soundness). $\vdash_{\mathbf{F.c}} \varphi \implies \mathbf{F.c} \models \varphi$.

Proof. By Theorem 6.1, it suffices to show that (F.c2) is valid. To this aim, take some arbitrary $l \in L$ and $\varphi, \psi \in \mathfrak{L}_{\mathbf{RCont}}$ such that $At(\psi) \subseteq At(\varphi)$. By Lemma 6.7 we infer that $\tau_l(p_1) \vee_l \dots \vee_l \tau_l(p_n) = (\varphi)^l$ and $\tau_l(p_1) \vee_l \dots \vee_l \tau_l(p_m) = (\psi)^l$ for $At(\varphi) = \{p_1, \dots, p_n\}$, $At(\psi) = \{p_1, \dots, p_m\}$ and $n \geq m$. Then, the following chain of equations holds by Definitions 6.7 and 6.8:

$$\begin{aligned}
(\psi)^l \vee_l (\varphi) &= \tau_l(p_1) \vee_l \dots \vee_l \tau_l(p_m) \vee_l \tau_l(p_1) \vee_l \dots \vee_l \tau_l(p_n) \\
&= \tau_l(p_1) \vee_l \dots \vee_l \tau_l(p_m) \vee_l \tau_l(p_1) \vee_l \dots \vee_l \tau_l(p_m) \\
&\quad \vee_l \tau_l(p_{m+1}) \vee_l \dots \vee_l \tau_l(p_n) \\
&= \tau_l(p_1) \vee_l \dots \vee_l \tau_l(p_m) \vee_l \tau_l(p_{m+1}) \vee_l \dots \vee_l \tau_l(p_n) \\
&= (\varphi)^l
\end{aligned}$$

That is, $(\psi) \preceq_l (\varphi)$. We conclude that $l \models \varphi \supseteq \psi$. □

The proof of completeness relies on the same canonical model construction as for F.C. In particular, the canonical Sylvan containment model $\mathfrak{M}_{\mathbf{F.c}}^c$, is obtained by uniformly substituting F.c to F.C in Definition 6.16.

Lemma 6.8 (Sylvan containment model). $\mathfrak{M}_{\mathbf{F.c}}^c \in \mathbf{F.c}$.

Proof. It is easy to observe that F.C's containment axioms (F.C2)–(F.C6) are special cases of (F.c2). Hence, $\mathfrak{M}_{\mathbf{F.c}}^c \in \mathbf{F.C}$ is established by the same argument of Lemma 6.5. It remains to be shown that Condition (ST1) holds in $\mathfrak{M}_{\mathbf{F.c}}^c$. To this aim,

retaining topic compositionality, it allows for a uniform semantic approach, according to which simpler topic structures correspond to stronger logical constraints on information states. However, order-structures are not general enough, as it is desirable that non-logical states underwrite no specific containment thesis.

take some arbitrary $l \in L$. To show that $a \rightarrow_l b = a \vee_l b$, it suffices to show that $[\varphi]_l \rightarrow_l [\psi]_l = [\varphi]_l \vee_l [\psi]_l$ for arbitrary $\varphi, \psi \in \mathfrak{L}_{\text{RCont}}$. By unraveling the definition of $[\varphi]_l$ in $\mathfrak{M}_{\text{F.c}}^e$, this is equivalent to showing that $[\varphi \rightarrow \psi]_l = [\varphi \vee \psi]_l$. Then, note that $At(\varphi \rightarrow \psi) = At(\varphi \vee \psi)$, which implies by (F.c2) that $\vdash_{\text{F.c}} \varphi \rightarrow \psi \equiv \varphi \vee \psi$. By $l \in L$, we infer that $(\varphi \rightarrow \psi) \equiv (\varphi \vee \psi) \in l$, by which we infer by Definition of \preceq_l and Lemma 6.4 that $[\varphi \rightarrow \psi]_l = [\varphi \vee \psi]_l$. We conclude that $[\varphi]_l \rightarrow_l [\psi]_l = [\varphi]_l \vee_l [\psi]_l$. $a \supseteq_l b = a \vee_l b$ is established similarly. \square

Theorem 6.4 (SRC-completeness). $\mathbf{F.c} \models \varphi \implies \vdash_{\text{F.c}} \varphi$.

Proof. The result follows from Lemmas 6.8 and 6.6. \square

6.3 Relevant Containment Reasoners in Possible Worlds

In Chapter 5, I showed how to devise a philosophically grounded hyperintensional framework for logical omniscience, using the state-based strategy. Now, impossible worlds and topic-sensitive frameworks have been taken as competing approaches to hyperintensional phenomena in [23, p.144], as the former wins in flexibility while the latter wins in naturalness. In this section I try to reconcile the two approaches by using a logic in the family of contextual modal logics, where epistemic contexts are regimented by *relevant logic* and *containment considerations*. The contextual modal framework proposed here accomplishes both flexibility and naturalness. Flexibility is attained by employing a substructural, rather than classical, logic of entailment and containment. Naturalness is attained by providing a fully semantic account of subject matter and information, via contextual containment and entailment.

6.3.1 Possible Worlds and Relevant Containment

One of the leading intuitions of TSIM semantics is to recast Parry's observation that analytic implication formalises inference within a formal system as an observation about the closure properties of epistemic attitudes: closure within a belief system should necessarily require topic inclusion. In the present section, this requirement is accomplished by defining the epistemic modality

$$\Box_C \varphi := \Box(\varphi \wedge (\mathbf{t} \supseteq \varphi)),$$

expressing the fact that an agent bears the salient epistemic attitude towards the thick proposition expressed by φ . The latter, in turn, consists of the intersection

between the proposition expressed by φ (the information conveyed by φ) and the proposition expressing that the topic of φ is among those grasped by the agent at situational context fixed by φ . Therefore, adapting the suggestion of [134], the Ackermann constant \mathbf{t} serves as a proxy in the language for the totality of grasped topics.

Definition 6.20 (Containment W -models). *Let a containment W -frame be a structure of the following form:*

$$(S, Prop, L, W, *, R, Q, Q_L, \mathbf{t})$$

- $(S, Prop, L, *, R, V, \mathbf{t})$ is a ternary containment model;
- $Q \subseteq S \times \mathcal{P}(S)$ is a neighborhood binary relation, interpreting \Box ;
- $Q_L \subseteq S^2$ is a Kripke binary relation, interpreting \Box_L ;
- $W \subseteq L$ is a set of possible worlds such that, for all $w \in W$, all $p, q \in At$ and all $X, Y \in \mathcal{P}(S)$:

$$w^* = w \quad (\text{CW1})$$

$$w \in X \rightarrow^{\tilde{\mathfrak{F}}} Y \iff w \in \neg^{\tilde{\mathfrak{F}}} X \vee^{\tilde{\mathfrak{F}}} Y \quad (\text{CW2})$$

$$Q_L(W) = L \quad (\text{CW3})$$

$$\tau_w(p) = \tau_w(q) \quad (\text{CW4})$$

Moreover, let the following frame conditions hold for all containment W -frames $\tilde{\mathfrak{F}}$ and all $X \in Prop$:

$$\Box^{\tilde{\mathfrak{F}}} X, \Box_L^{\tilde{\mathfrak{F}}} X \in Prop \quad (\text{CW8})$$

where $\Box^{\tilde{\mathfrak{F}}}, \Box_L^{\tilde{\mathfrak{F}}}$ are as in Definition 4.2. Let $(\tilde{\mathfrak{F}}, V)$, with $V : At \rightarrow Prop$, be a containment W -model based on a containment W -frame $\tilde{\mathfrak{F}}$. Let **EI.C** denote the class of all relevant containment models. Let $(S, Prop, L, *, R, Q, Q_L, V, \mathbf{t})$ be the containment L -model a containment W -model is based on. Let **FE.C** denote the class of containment L -models.

Containment W -models interestingly combine some of the semantic features of PAI, PHB and C.R.

1. *Grasping topics.* Like PHB-models, containment W -models contain a designated element representing the totality of topics grasped by the agent at a given epistemic alternative. To see this, note that, for any information state s , one can represent \mathbf{b} of PHB-models as $(\mathbf{t})^s$, allowing the totality of grasped

topics to vary depending on the situational context. Then, a contextually fixed agent grasps a topic a at a state s iff $a \preceq_s (\mathbf{t})^s$. Contrary to PHB-models, however, topic inclusions are represented explicitly in the object language by the connective \supseteq . This provides the resources to define a topic-sensitive epistemic modality as $\Box_C \varphi := \Box(\varphi \wedge (\mathbf{t} \supseteq \varphi))$. Then, $\Box_C \varphi$ expresses that an agent bears the salient epistemic attitude towards the *thick proposition* expressed by φ at a given situation, which requires that the topic expressed by φ is included in those topics grasped by the agent at a given situational context. The latter topics, in turn, are represented by (\mathbf{t}) , which expresses the “topic of the world”, as conceptualised by the agent at a given situation.

2. *Possible worlds and topics.* Possible worlds are associated with trivial topic structures. To see why this is plausible, consider again maximally consistent theories, i.e. theories of cognitively unbounded agents. From a topic-theoretic perspective, any possible world gives information about everything fixed by the language, the topic of the world. We can represent the fact that cognitively unbounded agents are topically omniscient by mapping the topic of each sentence to such topic of the world. In the model theory, this is accomplished in two steps. First, (CW4) and $W \subseteq L$ imply that (i) all complex formulas have the same topic at a possible world and that (ii) \preceq_w is a partial order for all $w \in W$ which gives rise to the very special one-element join semilattice (\mathcal{T}_w, \vee_w) , with \preceq_w is identity. Finally, the latter implies that every topic inclusion holds at possible worlds.

Definition 6.21 (Containment W -satisfaction). *Let \mathfrak{M} be a containment W -model. Let satisfaction of a formula $\varphi \in \mathcal{L}_{\text{CRel}}$ in \mathfrak{M} pointed at $s \in S$ be recursively defined as follows:*

$\mathfrak{M}, s \models p$	\iff	$s \in V(p)$
$\mathfrak{M}, s \models \mathbf{t}$	\iff	$s \in L$
$\mathfrak{M}, s \models \neg \varphi$	\iff	$\mathfrak{M}, s^* \not\models \varphi$
$\mathfrak{M}, s \models \varphi \wedge \psi$	\iff	$\mathfrak{M}, s \models \varphi$ and $\mathfrak{M}, s \models \psi$
$\mathfrak{M}, s \models \varphi \vee \psi$	\iff	$\mathfrak{M}, s \models \varphi$ or $\mathfrak{M}, s \models \psi$
$\mathfrak{M}, s \models \varphi \rightarrow \psi$	\iff	$Rs[\varphi]_{\mathfrak{M}}[\psi]_{\mathfrak{M}}$
$\mathfrak{M}, s \models \varphi \supseteq \psi$	\iff	$(\psi)_{\mathfrak{M}}^s \preceq_s (\varphi)_{\mathfrak{M}}^s$
$\mathfrak{M}, s \models \Box \varphi$	\iff	$Qs[\varphi]_{\mathfrak{M}}$
$\mathfrak{M}, s \models \Box_L \varphi$	\iff	$Q_L s[\varphi]_{\mathfrak{M}}$

Definition 6.22 (Containment W -validity). *Let validity and consequence in containment W -models be defined as W -validity and S -consequence, respectively (cf. Definition 3.5).*

Note that, with respect to the models of Chapter 5, I resort here to fully neighborhood models. Moreover, in order to characterise the weakest possible relevant containment logics, I restrict the language to $\mathfrak{L}_{\text{RRel}}$, excluding \otimes, \leftarrow . Containment W -models inherit all the good properties of ternary containment models, i.e. Lemmas 6.1–6.3, as well as the properties of contextual modal logics, recorded by Lemmas 6.9, 6.10 and 6.11 below.

Lemma 6.9 (Containment W -heredity). *For all $\mathfrak{M} \in \mathbf{Ei.C}$ and all $\varphi \in \mathfrak{L}_{\text{CRel}}$: $\llbracket \varphi \rrbracket_{\mathfrak{M}} \in \text{Prop}$.*

Lemma 6.10 (Containment W -entailment). $\varphi \models_{\mathbf{Ei.C}} \psi \iff \mathbf{Ei.C} \models \Box_L(\varphi \rightarrow \psi)$.

Lemma 6.11 (Containment W -flatness). *For all $\mathfrak{M} \in \mathbf{Ei.C}$, all $w \in W$ and all $\varphi, \psi \in \mathfrak{L}_{\text{CRel}}$:*

1. $\mathfrak{M}, w \models \mathbf{t}$;
2. $\mathfrak{M}, w \models \neg\varphi \iff \mathfrak{M}, w \not\models \varphi$;
3. $\mathfrak{M}, w \models \varphi \rightarrow \psi \iff (\mathfrak{M}, w \models \varphi \implies \mathfrak{M}, w \models \psi)$;
4. $\mathfrak{M}, w \models \varphi \supseteq \psi$.

Proof. The only new case is Item 4, which is established as follows. First, note that by (CW4) it is easily seen, by routine induction on the structure of φ, ψ , that $\tau_w(\varphi) = \tau_w(\psi)$ for all $\varphi, \psi \in \mathfrak{L}_{\text{CRel}}$. Then, by $W \subseteq L$ we infer that \preceq_w is a partial order, hence reflexive, by which we infer that $\tau_w(\psi) \preceq_w \tau_w(\varphi)$. We conclude that $w \models \varphi \supseteq \psi$. \square

Having introduced the semantics for relevant containment W -models, let's assess its formal and conceptual properties. These properties allow for a fully *semantic* and *state-sensitive* characterisation of topic and information inclusion.

1. *Contextual entailment.* Semantic regions in the logical space determine the properties of information inclusion. These properties are reflected in the properties of \sqsubseteq_s , which obey increasingly stronger constraints: if $s \in S$, respectively $s \in L$ and $s \in W$, \sqsubseteq_s is a general relation, set inclusion and set inclusion restricted to W , respectively. That is, Lemmas 6.10 and 6.11 imply that:

$$\mathfrak{M} \models^X \varphi \rightarrow \psi \text{ iff } \begin{cases} \llbracket \varphi \rrbracket_{\mathfrak{M}}^W \subseteq \llbracket \psi \rrbracket_{\mathfrak{M}}^W & \text{if } X = W \\ \llbracket \varphi \rrbracket_{\mathfrak{M}} \subseteq \llbracket \psi \rrbracket_{\mathfrak{M}} & \text{if } X = L \\ \llbracket \varphi \rrbracket_{\mathfrak{M}} \sqsubseteq_s \llbracket \psi \rrbracket_{\mathfrak{M}} & \text{if } X = S. \end{cases}$$

where $\llbracket \varphi \rrbracket_{\mathfrak{M}}^W = W \cap \llbracket \varphi \rrbracket_{\mathfrak{M}}$.

2. *Contextual containment.* Likewise, semantic regions in the logical space determine the properties of topic inclusion. These properties are reflected in the properties of \preceq_s , which obey to increasingly stronger constraints: if $s \in S$, respectively $s \in L$ and $s \in W$, \preceq_s is a general relation, a partial order and the identity relationship, respectively. That is, Lemmas [6.2](#) and [6.11](#) imply that:

$$\mathfrak{M} \models^X \varphi \supseteq \psi \text{ iff } \begin{cases} \text{always} & \text{if } X = W \\ \forall s \in X (\llbracket \psi \rrbracket_{\mathfrak{M}}^s \vee_s \llbracket \varphi \rrbracket_{\mathfrak{M}}^s = \llbracket \varphi \rrbracket_{\mathfrak{M}}^s) & \text{if } X = L \\ \forall s \in X (\tau_s(\psi) \preceq_s \tau_s(\varphi)) & \text{if } X = S. \end{cases}$$

3. *Hyperintensionality and Omniscience.* The language interpreted by containment W -models allows one to express two epistemic modalities, \Box and \Box_C . Focusing on the latter, by the definition $\Box_C \varphi := \Box(\varphi \wedge (\mathbf{t} \supseteq \varphi))$ it formalises a topic-sensitive intentional modality (TSIM), i.e. an attitude which is directed towards a thick content, comprising an informational as well as a topical component. That is, in order for an agent to bear that TSIM towards φ at s , the thick proposition expressed by φ , i.e. $\llbracket \varphi \wedge (\mathbf{t} \supseteq \varphi) \rrbracket$, must be in the epistemic context of the agent relative to s . Note that, in order to check the satisfaction of $\Box_C \varphi$ at s , the topic of \mathbf{t} is evaluated at all and only those φ -states t accessible from s : $(\mathbf{t})^t$ then determines the topic of the world, as conceptualised in the situational context t . The definition of \Box_C as the salient TSIM is sufficient to (i) invalidate all of the problematic closure principles of Chapter [5](#) (cf. Proposition [6.2](#)), while (ii) assuming that epistemic attitudes are closed under on-topic relevant equivalence (cf. Proposition [6.3](#)).

6.3.2 Axiomatisation of EI.C

The axiom system EI.C, improving on the system EI of Chapter [5](#), is formulated by the usual strategy of contextual modal logics. The only modification is the addition of an axiom sanctioning the fact that classical theories represent the theories of topically omniscient agents, whence all containment formulas holds.

Definition 6.23 (EI.C axiom system). *Let the Hilbert system for the logic of relevant containment reasoners EI.C consist of the the following axioms and rules:*

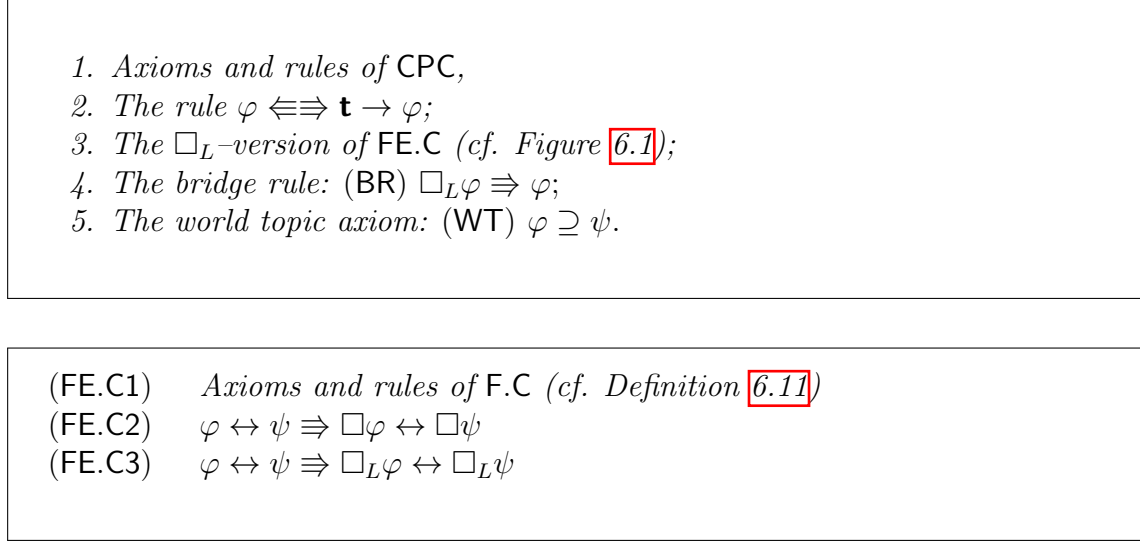


Figure 6.1: Hilbert system for FE.C.

It is easy to check that all the contentious epistemic closure principles assessed in Chapter 5, including (\Box DT) $\Box(\varphi \wedge (\psi \vee \chi)) \leftrightarrow \Box((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$, (\Box DM) $\Box(\neg(\varphi \wedge \psi)) \leftrightarrow \Box(\neg\varphi \vee \neg\psi)$ and (\Box AB), turn out invalid in containment W -models.

Proposition 6.2 (El.C omniscience). *All epistemic closure principles of Section 5.1, where \Box_C is substituted to \Box are not El.C-theorems, do not preserve El.C-provability, respectively.*

While \Box is closed under (nRR) (cf. Proposition 3.2), \Box_C poses a stronger sufficient requirement on epistemic closure. That is, agents believing φ in El.C only believe ψ provided that the thick proposition expressed by φ and ψ , i.e. the proposition expressing the information that φ and ψ and the fact that the topics of φ and ψ are within reach, are relevantly equivalent. Therefore, the following closure under on-topic relevant equivalence holds by the same argument of Proposition 4.2

Proposition 6.3 (El.C epistemic competence). *The following rule is derivable in El.C, where $\bar{\varphi} := \varphi \wedge (\mathbf{t} \supseteq \varphi)$:*

$$\Box_L(\bar{\varphi} \leftrightarrow \bar{\psi}) \Rightarrow \Box_C \varphi \leftrightarrow \Box_C \psi.$$

Soundness, completeness and the bridge lemma are established by the standard methods of Part II, where again the proof of the bridge lemma is simplified by the absence of bounds.

Theorem 6.5 (El.C–soundness). $\vdash_{\text{El.C}} \varphi \implies \mathbf{El.C} \models \varphi$.

Proof. By induction on the length El.C–proofs, where the only new cases involve showing that (WT) is valid, which follows from Lemma 6.11.4. \square

Lemma 6.12 (Bridge). *For all $\varphi \in \mathcal{L}_{\text{CRel}}$: $\vdash_{\text{FE.C}} \varphi \iff \vdash_{\text{El.C}} \Box_L \varphi$.*

Proof. The result is established once again by the argument of Lemma 3.6, with the claim (3.5) is established by a more straightforward semantic construction.⁹

$$\forall \mathfrak{M} \in \mathbf{FE.C} \exists \mathfrak{M}' \in \mathbf{El.C} (\mathfrak{M} \not\models \varphi \implies \mathfrak{M}' \not\models \Box_L \varphi) \quad (\text{LCL3})$$

To establish (LCL3), it suffices to take, for any $\mathfrak{M} \in \mathbf{FE.C}$, the following model \mathfrak{M}' , where $X, Y \in \text{Prop}'$:

- $S' = S \cup W$;
- $\text{Prop}' = \text{Prop} \cup \{X \cup W \mid X \in \text{Prop}\}$;
- $W = \{w\}$;
- $L' = L \cup W$;
- $\leq' = \leq \cup \{(w, w)\}$;
- $*' = * \cup \{(w, w)\}$;
- $R' = R \cup \{(s, X, Y) \mid Rs(X \cap S)(Y \cap S)\} \cup \{(w, X, Y) \mid w \in \neg X \cup Y\}$;
- $Q' = \{(s, X) \mid Qs(X \cap S)\}$;
- $Q'_L = \{(w, s) \mid s \in L\}$;
- $V' = V$;
- $(\mathcal{T}'_s, \leq'_s, \neg'_s, \wedge'_s, \vee'_s, \rightarrow'_s, \supseteq'_s, \tau'_s) = (\mathcal{T}_s, \leq_s, \neg_s, \wedge_s, \vee_s, \rightarrow_s, \supseteq_s, \tau_s)$ if $s \in S$;
- $\mathcal{T}'_w = \{a\}$;
- $\neg'_w a = a$;
- $a \otimes'_w a = a$ for $\otimes \in \{\leq, \wedge, \vee, \rightarrow, \supseteq\}$
- $\tau'_w(p) = a$ for all $p \in \text{At}$.

That $\mathfrak{M}' \in \text{El.C}$ holds by an easy inspection of the definition. Next, by $\mathfrak{M} \not\models \varphi$, we infer that there is $s \in L$ such that $\mathfrak{M}, s \not\models \varphi$. Hence, as in Lemma 5.2 it suffices to show that $\forall s \in S(\mathfrak{M}, s \models \varphi \iff \mathfrak{M}', s \models \varphi)$. The latter is established by an easy induction on the structure of φ . The base case holds by $V' = V$, while the induction step follows by noting that (i) for $\otimes \in \{*, R, Q, Q_L\}$, the restriction of \otimes' to S is the same as \otimes and that (ii) $a \leq'_s b \iff a \leq_s b$ for all $s \in S$. \square

⁹The argument for Lemma 3.6 relies here on the completeness of FE.C with respect to FE.C–models, which is established by standard methods, combining techniques employed in Chapters 3–4 and Section 6.2.

The canonical EI.C-model is the result of pasting together the canonical models for C.FE and F.C (together with the standard relational clause for Q_L^c), as follows.

Definition 6.24 (Canonical EI.C-model). *Let the canonical EI.C-model be a tuple of the following form:*

$$\mathfrak{M}_{\text{EI.C}}^c = (S^c, W^c, L^c, \leq^c, *^c, R^c, Q^c, Q_L^c, V^c, \mathbf{t}^c)$$

- $S^c = \{s \mid s \text{ is a prime FE.C-theory over } \mathfrak{L}_{\text{CRel}}\};$
- $\text{Prop}^c = \{[\varphi]^c \mid \varphi \in \mathfrak{L}_{\text{CRel}}\}$ where $[\varphi]^c = \{s \in S^c \mid \varphi \in s\};$
- $W^c = \{s \mid s \text{ is a non-empty, proper, prime EI.C-theory over } \mathfrak{L}_{\text{CRel}}\};$
- $L^c = \{s \in S^c \mid \mathbf{t} \in s\};$
- $\leq^c = \{(s, t) \in (S^c)^2 \mid s \subseteq t\};$
- $s^{*c} = \{\varphi \in \mathfrak{L}_{\text{CRel}} \mid \neg\varphi \notin s\};$
- $R^c = \{(s, X, Y) \in S^c \times (\text{Prop}^c)^2 \mid \exists \varphi, \psi \in \mathfrak{L}_{\text{CRel}}(X = [\varphi]_c, Y = [\psi]_c, \varphi \rightarrow \psi \in s)\};$
- $Q^c = \{(s, X) \in S^c \times \text{Prop}^c \mid \exists \varphi \in \mathfrak{L}_{\text{CRel}}(X = [\varphi]_c, \Box\varphi \in s)\};$
- $Q^c = \{(s, X) \in S^c \times \text{Prop}^c \mid \exists \varphi \in \mathfrak{L}_{\text{CRel}}(X = [\varphi]_c, \Box_L\varphi \in s)\};$
- $V^c(p) = \{s \in S^c \mid p \in s\};$
- $\mathbf{t}^c : s \mapsto (\mathcal{T}_s^c, \preceq_s^c, \neg_s^c, \wedge_s^c, \vee_s^c, \rightarrow_s^c, \supseteq_s^c, \tau_s^c)$ defined as follows:
 - $\mathcal{T}_s^c = \mathfrak{L}_{\text{CRel}} / \sim_s;$
 - $\preceq_s^c = \{([\varphi]_c^s, [\psi]_c^s) \in \mathcal{T}_s^c \mid \exists \varphi' \in [\varphi]_c^s, \psi' \in [\psi]_c^s (\psi' \supseteq \varphi' \in s)\};$
 - $\otimes_s^c([\varphi_1]_c^s, \dots, [\varphi_n]_c^s) = [\otimes(\varphi_1, \dots, \varphi_n)]_c^s$ for $\otimes \in \{\neg, \wedge, \vee, \supseteq, \rightarrow\};$
 - $\tau_s^c(p) = [p]_c^s = \{\psi \mid p \sim_s \psi\}.$

Lemma 6.13 (EI.C-Modelhood). $\mathfrak{M}_{\text{EI.C}}^c \in \mathbf{EI.C}$.

Proof. The result is established by showing that (i) $(S, L, *, R, Q, Q_L, V)$ is a FE-model; that that (ii) $(S, \text{Prop}, L, *, R, V, \mathbf{t})$ is a ternary containment model; and that (ii) W is a set of possible worlds. (i) is established by the same argument of Lemma 4.18; (ii) is established as in lemma 6.5. Finally, (iii) is established as in Lemma 3.8, except for the cases (CW4), which is established as follows. Assume $s \in W$. By $W \subseteq L$ we infer that $\mathbf{t} \in w$. By (WT) we infer that $\vdash_{\text{EI.C}} \varphi \equiv \psi$ for all $\varphi, \psi \in \mathfrak{L}_{\text{CRel}}$, which implies that $\varphi \equiv \psi \in w$. By the latter and $\mathbf{t} \in w$ we infer that $\varphi \sim_s \psi$ for all $\varphi, \psi \in \mathfrak{L}_{\text{CRel}}$, by which we conclude that $[\varphi]_s = [\psi]_s$. \square

Theorem 6.6 (EI.C-Completeness). $\mathbf{EI.C} \models \varphi \implies \vdash_{\text{EI.C}} \varphi$.

Proof. The result follows from Lemmas 6.13 and $\mathfrak{M}_{\text{EI.C}}^c, s \models \varphi \iff \varphi \in s$, which is established by induction on the structure of φ , via the same argument of Lemmas 4.6 (for $p, \mathbf{t}, \wedge, \vee, \neg, \Box$) 2.6 (for \Box_L) and 6.6 (for \supseteq). \square

Chapter 7

Explicit and Implicit Belief

In this chapter, I show how to model the distinction between explicit and implicit belief within contextual modal logics. To this aim, in Section 7.1 I enrich the relational structures and the axiomatics of Chapter 3 so as to model implicit belief. In the remainder of the chapter, I show how to model implicit belief and classical reasoning within the Australian plan for the relevant logic FDE. In Section 7.2 I give a natural deduction presentation of FDE, and recall its two main model theoretic analyses. In Section 7.3 I show how to augment FDE with strict implication \rightarrow and a classicality constant c , in terms of which (i) explicit and implicit belief turn out to be definable and (ii) it is possible to represent classical reasoning. Finally, I show that Levesque’s influential logic of explicit and implicit belief is embeddable in the resulting logic $\text{FDE}_c^{\rightarrow}$.

7.1 Contextual Modal Logics for Explicit and Implicit Belief

The material presented in this chapter is clearly indebted to Hector Levesque’s work (cf. [111, 112]). [111] famously addresses standard possible–worlds models of epistemic logic as unrealistically modeling the ordinary concept of explicit belief (cf. also Chapter 5). Such critical remarks motivate Levesque to provide a model of explicit belief based on *situation* semantics, which results in explicit belief being closed under consequence in the logic of First Degree Entailment, FDE, the implication-free fragment of Anderson and Belnap’s relevant logic of entailment E of [4]. Moreover, Levesque argues that the possible worlds account of belief realistically characterises a distinct yet related concept, that of implicit belief. Roughly speaking, implicit

beliefs correspond to what the world would be like if the agent’s explicit beliefs were to be true, without the agent necessarily realising it. Accordingly, it is plausible that implicit belief, as opposed to explicit belief, is closed under classical logical consequence.

I will take up Levesque’s idea and apply it to two distinct yet related formalisms, i.e. an extension of contextual modal logic with implicit belief, and an extension of First Degree Entailment with strict implication and a classicality constant.

7.1.1 Explicit and Implicit Beliefs in Epistemic Logic

It is commonly held in the philosophical and psychological literature that the difference between explicit and implicit attitudes is that agents are aware of the former, which are conscious, but not of the latter. Implicit attitudes, in turn, are characterised in various ways, e.g. in terms of dispositions, measures from tests like the so-called “implicit-association test”, map-like mental representations or degrees of accessibility of explicit attitudes (see the discussion in [20]). In the logical literature, the distinction between explicit and implicit belief has been spelled out in several ways.

1. *Awareness logic.* Very often it happens that the literature on implicit belief is intertwined with that on *awareness logic*, a framework for dealing with logical omniscience which identifies awareness as the missing ingredient linking implicit belief and explicit belief.¹ The generality of the concept of awareness employed in [57] is worth noting: first, awareness is modeled by an arbitrary syntactic function; second, no precise meaning is attached to being aware of φ , which can indicate that the agents take φ into account, hold a mental representation of the concept expressed by φ [126], that φ is included in the agents’ vocabulary [84], or simply that the agents have immediate, computationally-free, access to φ [57].

¹(For the sake of simplicity, in this footnote we re-use our symbols for modal operators without assuming our definitions of the associated formulas.) For example, in [122] it is argued that “by definition, one has to be aware of explicit belief”, while Fagin and Halpern [57] read Levesque’s logic of explicit and implicit belief as capturing the lack of logical omniscience that arises through lack of awareness. In [57], a modal operator for awareness in Levesque’s logic is defined as $A\varphi := \bigwedge\{\Box(p \vee \neg p) \mid p \in Pr(\varphi)\}$. This is not the only possible definition of awareness, e.g. [126] define $A\varphi := \Box_I\varphi \vee \Box_I\neg\Box_I\varphi$. Fagin and Halpern propose their own logic of general awareness, with a primitive awareness operator A such that $\Box\varphi := \Box_I\varphi \wedge A\varphi$. Again, this is not the only possible definition of explicit knowledge in terms of awareness, e.g. [22] define $\Box\varphi := \Box_I(\varphi \wedge A\varphi)$.

2. *Implicit belief as primitive.* At any rate, the distinction between explicit and implicit belief needs not make reference at all to the concept of awareness. In most logical frameworks, the working intuition is to understand explicit belief as a *conceptual primitive*, as belief actually held by agents or belief “at hand”. Then, implicit belief is what agents would believe explicitly in some *ideal state*.² Levesque’s model of explicit and implicit belief can be regarded in this fashion, as a semantic framework in which implicit belief is defined as explicit belief in some ideal state.

Levesque’s idea is made precise by the semantics of relevant logic. Employing (possibly incomplete and inconsistent) situations as the main semantic object in [111] allows to model the information relevant only to what agents actually, explicitly, believe. In what follows, I will present two formalisms which explore the idea that possible worlds constitute the limit, ideal, case of complete and consistent situations, thus modeling the information relevant to what agents implicitly believe.

7.1.2 Explicit Implicit C.L-models

Recall from Remark 3.1 that contextual modal logics can be seen as generalising [111] to the full relevant modal language $\mathfrak{L}_{\text{FRel}}$. However, a full generalisation of Levesque’s logic should encompass also implicit belief \Box_I . To this aim, I compactly write $Q_{(IL)} \in \{Q, Q_L, Q_I\}$ and correspondingly $\Box_{(IL)} \in \{\Box, \Box_L, \Box_I\}$. Note that the model theory of contextual modal logics for explicit and implicit belief eiC.L builds on relational semantics (cf. Chapter 3). Therefore, the models used in this section contain bounds. Neighborhood versions, without bounds, are easily obtainable.

Definition 7.1 (Explicit implicit C.L-models). *Let an explicit implicit C.L-model be a tuple of the following form:*

$$(S, L, W, \leq, *, R, Q, Q_L, Q_I, V)$$

²Formally, this usually results in double-barreled analyses of explicit belief (in terms of implicit belief plus e.g. awareness, justification [6], acknowledgement [202], etc.), where implicit belief is assumed as a *logical primitive*. However, the opposite route is just as viable: for example, in syntactic approaches like that of [3], a set of explicitly believed formulas is assumed, and its closure under logical consequence models implicit beliefs. In [203] explicit knowledge is modeled by means of neighborhood semantics and implicit knowledge is defined as explicit knowledge with respect to a neighborhood model extending the original one, which behaves like a relational Kripke model. Kripke models represent, then, ideal states agents arrive at after having performed all relevant deductive steps.

- $(S, L, W, \leq, *, R, Q, Q_L, V)$ is a **C.L**-model (cf. Definition 3.4) building on the (bounded) **L**-model $(S, L, \leq, *, R, Q, Q_L, V)$;
- $(S, L, W, \leq, *, R, Q_I, Q_L, V)$ is a **C.L**-model building on the (bounded) **L**-model $(S, L, \leq, *, R, Q_I, Q_L, V)$, where $Q_I \subseteq S^2$ is a Kripke binary relation, interpreting \Box_I . It is assumed that Q_I is only downward monotonic, i.e. $Q_I^{-1}(s) \in S(\downarrow)$.
- $W \subseteq L \subseteq S$ is set of logical states which are possible worlds.

Moreover, let the following conditions hold for every $w \in W$ and all $s \in S$:

$$Q_I w s \implies s \in W \quad (\text{I1})$$

$$Q_I w s \implies Q w s \quad (\text{I2})$$

Let $(S, L, W, \leq, *, R, Q, Q_L, Q_I)$ be the explicit implicit **C.L**-frame an explicit implicit **C.L**-model $(S, L, W, \leq, *, R, Q, Q_L, Q_I, V)$ is based on. Let **eiC.L** denote the class of all explicit implicit **C.L**-models, parametric on the corresponding class of **L**-models.

Definition 7.2 (eiC.L-satisfaction). Let the proposition expressed by an arbitrary formula $\varphi \in \mathfrak{L}_{\text{Rel}}$ in an **eiC.L**-model \mathfrak{M} be recursively defined as follows, where $\otimes \in \{\neg, \wedge, \vee, \otimes, \rightarrow, \leftarrow, \Box_{(IL)}\}$ and n is the arity of \otimes :

$$\llbracket p \rrbracket_{\mathfrak{M}} = V(p)$$

$$\llbracket \mathbf{t} \rrbracket_{\mathfrak{M}} = L$$

$$\llbracket \otimes(\varphi, \dots, \varphi_n) \rrbracket_{\mathfrak{M}} = \otimes^{\mathfrak{S}}(\llbracket \varphi \rrbracket_{\mathfrak{M}}, \dots, \llbracket \varphi_n \rrbracket_{\mathfrak{M}})$$

Let satisfaction in a **eiC.L**-model \mathfrak{M} be defined so that $\mathfrak{M}, s \models \varphi$ iff $s \in \llbracket \varphi \rrbracket_{\mathfrak{M}}$.

Let generalised validity and consequence be defined as in Definition 3.5. It is clear that Lemmas 3.2, 3.3 and 3.4 continue to hold in the generalised framework.

Contextual modal logics offer an appropriate way of representing the beliefs actively held by moderately idealised agents (cf. Chapter 5). Now, explicit implicit **C.L**-models provide the additional resources to model implicit beliefs. In particular, the model theory is devised so that implicit belief satisfies some minimal desiderata. Conditions (I1)–(I2) yield the intended interpretation of $Q_I(w)$, which represents implicit beliefs of the agent in a possible world w . In particular, by Condition (I1), $Q_I(w)$ contains only possible worlds and so implicit beliefs are closed under classical consequence (cf. Proposition 7.1), while by Condition (I2) the “implicit” epistemic state of the agent at w is a subset of the “explicit” epistemic state, and so every explicit belief is an implicit belief (cf. Proposition 7.2)³

³Note that, contrary to Q and Q_L , we do not assume that Q_I is upward monotone in its second argument. This condition has to do with the canonical model construction, since in the canonical model Q_I^c will not be upward monotone.

Lemma 7.1 (Classical-implicit). For all $\mathfrak{M} \in \mathbf{eiC.L}$ and $\varphi, \psi \in \mathfrak{L}_{\text{Rel}}$: $\varphi \models_{\mathfrak{M}}^W \psi \implies \Box_I \varphi \models_{\mathfrak{M}}^W \Box_I \psi$.

Proof. The result is an immediate consequence of Condition (I1). \square

Lemma 7.2 (Implicit-explicit). For all $\mathfrak{M} \in \mathbf{eiC.L}$ and $\varphi \in \mathfrak{L}_{\text{Rel}}$: $\Box \varphi \models_{\mathfrak{M}}^W \Box_I \varphi$.

Proof. The result is an immediate consequence of Condition (I2). \square

Together, the above propositions provide the following formal characterisation of implicit belief as the “classical closure” of explicit belief. That is, implicitly believed information is exactly the information classically following from the explicitly believed information.

Proposition 7.1 (Classical closure). For all $\varphi, \dots, \varphi_n, \psi \in \mathfrak{L}_{\text{Rel}}$ without occurrences of modal operators, the following are equivalent:

1. $\varphi \wedge \dots \wedge \varphi_n \rightarrow \psi$ is a classical tautology;
2. $\Box \varphi \wedge \dots \wedge \Box \varphi_n \rightarrow \Box_I \psi$ is valid in all W -models.

Proof. (1) \implies (2): If $\bigwedge_{i \leq n} \varphi_i \rightarrow \psi$ is a classical tautology, then $\bigwedge_{i \leq n} \varphi_i \models_{\mathfrak{M}}^W \psi$ for all W -models \mathfrak{M} by Lemma 3.3. Then, $\bigwedge_{i \leq n} \Box_I \varphi_i \models_{\mathfrak{M}}^W \Box_I \psi$ by Lemma 7.1, which entails $\bigwedge_{i \leq n} \Box \varphi_i \models_{\mathfrak{M}}^W \Box_I \psi$ by Lemma 7.2. Consequently, $\bigwedge_{i \leq n} \Box \varphi_i \rightarrow \Box_I \psi$ is valid in all W -models \mathfrak{M} by Lemma 3.4.2. (2) \implies (1): If $\bigwedge_{i \leq n} \varphi_i \rightarrow \psi$ is a propositional formula that is not a classical tautology, then there is a classical valuation v such that $v(\varphi_i) = 1$ for all φ_i and $v(\psi) = 0$. We may turn this valuation into a $\mathbf{eiC.L}$ -model \mathfrak{M} with the set of states $S = \{0, v, 1\}$ and V such that $v \in V(p)$ iff $v(p) = 1$ for all $p \in \text{At}$. Moreover, we assume that $W = \{v\}$, $Q_I v v$, and the rest is added so that this structure is indeed a $\mathbf{eiC.L}$ -model⁴. It is obvious that $\Box \varphi \wedge \dots \wedge \Box \varphi_n \rightarrow \Box_I \psi$ is not valid in \mathfrak{M} . \square

Two remarks are in order regarding Proposition 7.1. First, from a semantic point of view, implicit belief is stronger than the classical closure of explicit belief, as Conditions (I1–I2) ensure only that $Q_I(w) \subseteq Q(w) \cap W$ for all $w \in W$ and not the stronger condition $Q_I(w) = Q(w) \cap W$. However, the above proposition tells us that this does not matter in general. The present weaker semantics is more amenable to the canonical model technique. Second, Proposition 7.1 can be understood within a

⁴We can define \mathfrak{M} similarly as in the $+$ -construction used in the proof of Proposition 3.5, with the proviso that we do not add a new possible world w since v itself is seen as the only possible world in the model.

broader *computational interpretation* of **eiC.L**-models. According to this interpretation, defended e.g. by [41], classical logic constitutes the upper bound of deductive reasoning.⁵

The computational thesis is translated in **eiC.L**-models by the assumption $W \subseteq L \subseteq S$. The latter assumption lends itself to an epistemic reading of \Box , \Box_L and \Box_I as epistemic context-shifters. In particular, by Condition (W9) Q_L shifts from the class of ideal deductive contexts W , whose states support all classical tautologies, to L , whose states support all relevant tautologies; and by Conditions (I1)–(I2) Q_I shifts from W to (a subclass of) W itself. It is in this sense that implicit belief can be seen as modelling the explicit belief of an agent operating in the ideal epistemic state where they can avail themselves of all classical tautologies, i.e. what an ideal, unbounded agent would explicitly believe. As a consequence, (some form of) logical omniscience fails for explicit belief but is restored for implicit belief (cf. Proposition 7.2). This is a welcome result, since implicit belief captures the (classical) consequences of explicit belief.

Proposition 7.2 (Logical omniscience). *For all $\mathfrak{M} \in \mathbf{eiC.L}$ and $\varphi, \psi \in \mathcal{L}_{\text{Rel}}$:*

1. $\varphi \models_{\mathfrak{M}}^W \psi \not\Rightarrow \Box\varphi \models_{\mathfrak{M}}^W \Box\psi$;
2. $\varphi \models_{\mathfrak{M}}^W \psi \Rightarrow \Box_I\varphi \models_{\mathfrak{M}}^W \Box_I\psi$.

Proof. Item 1 is established as in Proposition 5.2. Item 2 follows from the fact that by (I1) we have that $Q_I(W) \subseteq W$, by which we conclude that $\llbracket\varphi\rrbracket \cap W \subseteq \llbracket\psi\rrbracket \cap W$ does imply $\llbracket\varphi\rrbracket \cap Q_I(W) \subseteq \llbracket\psi\rrbracket \cap Q_I(W)$. \square

7.1.3 Axiomatisation of eiC.L

The semantics of the foregoing section was set up with an eye to two crucial principles concerning the properties of implicit belief, namely, that implicit belief extends explicit belief (cf. Proposition 7.2) and that it is closed under classical consequence (cf. Proposition 7.1). From an axiomatic point of view, the desiderata are the following principles – the latter will follow from the soundness of CPC (standard) and the completeness of **eiC.L** (to be shown):

$$\vdash_{\mathbf{eiC.L}} \Box\varphi \rightarrow \Box_I\varphi \qquad \vdash_{\text{CPC}} \varphi \wedge \dots \wedge \varphi_n \rightarrow \psi \implies \vdash_{\mathbf{eiC.L}} \Box_I\varphi \wedge \dots \wedge \Box_I\varphi_n \rightarrow \Box_I\psi.$$

⁵Note that this interpretation is not alternative, but compatible with the interpretation of Chapter 5, according to which possible worlds contain agent-independent facts, while situations contain the information agents possess regarding such facts. The two views are compatible with the preferred metaphysics of possible worlds [23], where a version of linguistic ersatzism is defended for impossible worlds.

Consequently, implicit belief is the classical closure of explicit belief in the following specific sense:

$$\vdash_{\text{CPC}} \varphi \wedge \cdots \wedge \varphi_n \rightarrow \psi \implies \vdash_{\text{eiC.L}} \Box \varphi \wedge \cdots \wedge \Box \varphi_n \rightarrow \Box_I \psi.$$

In order to obtain these properties, the axiom system **eiC.L** builds on a logic **L** which is an axiomatic extension of the multi-modal relevant logic **BMC**, comprising three modalities \Box , \Box_L and \Box_I , all conjunctively regular. That is, in the present context **BMC** contains extends **BM** with $(\Box_{(IL)}\mathbf{C})$ and $(\Box_{(IL)}\mathbf{M})$. Then, **L** is obtained by extending **BMC** with the axioms and rules of inference of Figure 2.3, where $\Box_{(LI)}$ and $Q_{(LI)}$ are uniformly substituted to $\Box_{(L)}$ and $Q_{(L)}$. In effect, then, conjunctively regularity principles are assumed not only for \Box and \Box_L , but also for \Box_I , since **L** contains $(\Box_I \varphi \wedge \Box_I \psi) \rightarrow \Box_I(\varphi \wedge \psi)$ and $\varphi \rightarrow \psi \implies \Box_I \varphi \rightarrow \Box_I \psi$. **eiC.L** is then defined as follows, parametrically on **L**.

Definition 7.3 (**eiC – L** axiom system). *Let the Hilbert system for **eiC.L** consist of the the following axioms and rules:*

1. Axioms and rules of **CPC**;
2. The rule $\varphi \iff \mathbf{t} \rightarrow \varphi$;
3. The \Box_L -versions of **L**;
4. The bridge rule: **(BR)** $\Box_L \varphi \implies \varphi$;
5. The explicit implicit axiom: $(\Box \Box_I) \Box \varphi \rightarrow \Box_I \varphi$;
6. The **K**-axiom for \Box_I : $(\Box_I \mathbf{K}) \Box_I(\varphi \rightarrow \psi) \rightarrow (\Box_I \varphi \rightarrow \Box_I \psi)$;
7. The necessitation rule for \Box_I : $(\Box_I \mathbf{N}) \varphi \implies \Box_I \varphi$.

Note that \Box_I is defined so as to be not only conjunctively regular, but also a normal modality in **eiC.L**. It is immediate to check that $(\Box_I \mathbf{K})$ and $(\Box_I \mathbf{N})$ yield, together with classical reasoning, the derivability of

$$(\mathbf{IR}) \quad \varphi \wedge \cdots \wedge \varphi_n \rightarrow \psi \implies \Box_I \varphi \wedge \cdots \wedge \Box_I \varphi_n \rightarrow \Box_I \psi,$$

by which \Box_I is closed under classical consequence. When compared with **(RR)** (cf. Proposition 3.2), **(IR)** expresses the closure property of reasoning with implicit belief.

The main meta-logical properties of contextual modal logics hold in the extended setting by the same arguments of Chapter 3.

Theorem 7.1 (**eiC.L**-soundness). $\vdash_{\text{eiC.L}} \varphi \implies \mathbf{eiC.L} \models \varphi$.

Proof. The result is established by induction on the length of eiC.L–proofs. The cases corresponding to items 1–4 of Definition 7.3 are established as in Theorem 3.1. The remaining cases are as follows.

($\Box\Box_I$). Assume $w \models \Box\varphi$ and Q_Iws for some arbitrary $w \in W$ and $s \in S$. By (I2) we infer that Qws , which implies by $w \models \Box\varphi$ that $s \models \varphi$. We conclude that $w \models \Box_I\varphi$.

($\Box_I\mathbf{K}$). Assume $w \models \Box_I(\varphi \rightarrow \psi)$ and $w \models \Box_I\varphi$ for some arbitrary $w \in W$. Hence, for all s such that Q_Iws , $s \models \varphi \rightarrow \psi$ and $s \models \varphi$. By (I1) we infer that $s \in W$, which implies by $s \models \varphi \rightarrow \psi$ and Lemma 3.4.2 that $s \models \psi$. We conclude that $w \models \Box_I\psi$.

($\Box_I\mathbf{N}$). Assume $\mathfrak{M} \models^W \varphi$ for all $\mathfrak{M} \in \mathbf{eiC.L}$. Moreover, assume Q_Iws for some arbitrary $w \in W$ and $s \in S$. By (I1) we infer that $s \in W$, hence $s \models \varphi$ for all $\mathfrak{M} \in \mathbf{eiC.L}$, by which we conclude that $\mathfrak{M} \models \Box_I\varphi$.

□

Lemma 7.3 (Bridge). *For all $\varphi \in \mathfrak{L}_{\mathbf{IRel}}$: $\vdash_{\mathbf{L}} \varphi \iff \vdash_{\mathbf{eiC.L}} \Box_L \varphi$.*

The proof of Lemma 7.3 is omitted, since it transposes without substantial modifications.⁶ For Theorems 7.1 and 7.2, I show only the details of the new cases, concerning \Box_I .

Definition 7.4 (Canonical eiC.L–model). *Let the canonical eiC.L–model be a tuple of the following form:*

$$\mathfrak{M}_{\mathbf{eiC.L}}^c = (S^c, W^c, L^c, \leq^c, *^c, R^c, Q^c, Q_L^c, Q_I^c, V^c)$$

- $S^c = \{s \mid s \text{ is a prime } \mathbf{L}\text{-theory over } \mathfrak{L}_{\mathbf{IRel}}\}$;
- $W^c = \{s \mid s \text{ is a non-empty, proper, prime eiC.L-theory over } \mathfrak{L}_{\mathbf{IRel}}\}$;
- $L^c = \{s \in S^c \mid \mathbf{t} \in s\}$;
- $\leq^c = \{(s, t) \in (S^c)^2 \mid s \subseteq t\}$;
- $s^{*^c} = \{\varphi \in \mathfrak{L}_{\mathbf{IRel}} \mid \neg\varphi \notin s\}$;
- $R^c = \{(s, t, u) \in (S^c)^3 \mid \forall \varphi, \psi \in \mathfrak{L}_{\mathbf{IRel}} (\varphi \rightarrow \psi \in s, \varphi \in t \implies \psi \in u)\}$;
- $Q^c = \{(s, t) \in (S^c)^2 \mid \forall \varphi \in \mathfrak{L}_{\mathbf{IRel}} (\Box\varphi \in s \implies \varphi \in t)\}$;
- $Q_L^c = \{(s, t) \in (S^c)^2 \mid \forall \varphi \in \mathfrak{L}_{\mathbf{IRel}} (\Box_L\varphi \in s \implies \varphi \in t)\}$;

⁶To be precise, the proof of Lemma 7.3 requires the usual + construction argument of Lemma 3.5, with the new component Q_I^+ defined as $Q_I^+ = Q_I \cup \{(w, w)\} \cup \{(0, s) \mid s \in S^+\}$. The absence of the tuples $(s, 1)$, for all $s \in S^+$, is due to the fact that it is not required that Q_I , hence Q_I^+ , be closed upwards under \leq .

- $Q_I^c st$ iff $\begin{cases} \Box_I \varphi \in s \implies \varphi \in t & \text{if } s \notin W^c \\ (\Box_I \varphi \in s \implies \varphi \in t) \ \& \ t \in W^c & \text{if } s \in W^c; \end{cases}$
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Moreover, let $0 := \emptyset$ and $1 := \mathfrak{L}_{\text{IRel}}$.

Lemma 7.4 (eiC.L–modelhood). $\mathfrak{M}_{\text{eiC.L}}^c \in \text{eiC.L}$.

Proof. By Lemma 3.8, it suffices to show that Conditions (I1) and (I2) are satisfied by $\mathfrak{M}_{\text{eiC.L}}^c$. (I1) holds since, assuming $Q_I st$ and $s \in W$, by definition of Q_I we have that $t \in W$. (I2) holds since, assuming $Q_I st$, $\Box \varphi \in s$ and $s \in W$, by $(\Box \Box_I)$ we have $\Box_I \varphi \in s$, hence by definition of Q_I we conclude that $\varphi \in t$. \square

Lemma 7.5 (eiC.L–truth). $\mathfrak{M}_{\text{eiC.L}}^c, s \models \varphi \iff \varphi \in s$.

Proof. The result is established by induction on the structure of φ . By Lemma 2.6, it suffices to show the induction step case $\varphi := \Box_I \psi$. (\implies). Assume $\Box_I \psi \notin s$ and consider the pair $(\{\chi \mid \Box_I \chi \in s\}, \{\psi\})$. In case $s \in W^c$, we show that the pair is Cl.L-independent. Otherwise, there would be $\chi_1, \dots, \chi_n \in \mathfrak{L}_{\text{IRel}}$ such that $\Box_I \chi_1, \dots, \Box_I \chi_n \in s$ and $\vdash_{\text{eiC.L}} (\bigwedge_{i \leq n} \chi_i) \rightarrow \psi$. By (IR) we would infer that $\vdash_{\text{eiC.L}} (\bigwedge_{i \leq n} \Box_I \chi_i) \rightarrow \Box_I \psi$. By $\Box_I \chi_1, \dots, \Box_I \chi_n \in s$ we would infer that $\Box_I \psi \in s$, which would contradict $\Box_I \psi \notin s$. Hence, by Lemma 2.3 there is $t \in W^c$ such that $Q_I st$ and $\psi \notin t$. If $s \notin W^c$, then the argument is similar – we just need to show that $(\{\chi \mid \Box_I \chi \in s\}, \{\psi\})$ is L-independent, which follows from L containing $(\Box_I \varphi \wedge \Box_I \psi) \rightarrow \Box_I (\varphi \wedge \psi)$ and $\varphi \rightarrow \psi \implies \Box_I \varphi \rightarrow \Box_I \psi$. In both cases, we conclude by the induction hypothesis that $s \not\models \Box_I \psi$. (\impliedby) Assume $\Box_I \varphi \in s$ and $Q_I^c st$. Hence, $\varphi \in t$, by which we conclude that $t \models \varphi$ by the induction hypothesis. \square

Theorem 7.2 (C.L–completeness). $\text{eiC.L} \models \varphi \implies \vdash_{\text{eiC.L}} \varphi$.

Proof. The result follows by Lemmas 7.4 and 7.5. \square

7.2 A primer on First Degree Entailment

In this section, I will introduce the basics of First Degree Entailment, the fragment of the relevant logic E without nested implications. I will reverse the usual order of presentation, by first outlining the proof system for FDE, and then two semantic frameworks which characterise it.

7.2.1 First Degree Entailment

The most common presentations of FDE use FMLA–FMLA sequents, i.e. inference rules and derivations in natural deduction proof systems are defined by substituting sequents of the form $\varphi \vdash \psi$ to formulas in Definitions 2.2 and 2.3. A sequent $\varphi \vdash \psi$ is provable in FDE – denoted $\varphi \vdash_{\text{FDE}} \psi$ – iff there is a derivation of $\varphi \vdash \psi$ in FDE from no assumptions. Rule derivability is defined by substituting sequents to formulas in Definition 2.4. The definition of theories in natural deduction systems is as in Definition 2.11, where $\varphi \vdash_{\text{FDE}} \psi$ means that the sequent $\varphi \vdash \psi$ is FDE–provable (as opposed to $\varphi \rightarrow \psi$ being L–provable).

Definition 7.5 (First Degree Entailment). *Let a natural deduction system for FDE consist of the following sequents and rules of inference:*

- (FDE1) $\varphi \vdash \psi, \varphi \vdash \chi \Rightarrow \varphi \vdash \psi \wedge \chi$
- (FDE2) $\varphi \wedge \psi \vdash \varphi(\psi)$
- (FDE3) $(\psi)\varphi \vdash \varphi \vee \psi$
- (FDE4) $\varphi \vdash \psi, \chi \vdash \psi \Rightarrow \varphi \vee \chi \vdash \psi$
- (FDE5) $\varphi \dashv\vdash \neg\neg\varphi$
- (FDE6) $\varphi \vdash \psi \Rightarrow \neg\psi \vdash \neg\varphi$
- (FDE7) $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
- (FDE8) $\varphi \vdash \psi, \psi \vdash \chi \Rightarrow \varphi \vdash \chi$

Alternative axiomatisations trade De Morgan principles – $\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$ and $\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$ – with (FDE6) (cf. 51), or contain only sequents and no rules (cf. 184). As shown by Anderson and Belnap (cf. 4), deducibility in FDE exhibits interesting syntactic properties, completely determining when φ entails ψ . To show this, they introduce the notion of *primitive entailment*, which formally captures tautologies.

Definition 7.6 (Primitive entailment). *Let a literal be a propositional atom or a negated propositional atom. Let a consecution $\varphi \vdash \psi$ be a primitive entailment iff (i) φ is a conjunction of literals; (ii) ψ is a disjunction of literals; and (iii) $\varphi_i = \psi_j$ for some conjoined φ_i and disjoined ψ_j .*

Theorem 7.3 (Normal form 4). *Every formula $\varphi \in \mathfrak{L}_{\text{Ent}}$ is equivalent to a formula in disjunctive normal form φ_{DSJ} , i.e. a disjunction of conjunctions of literals, and*

to a formula in conjunctive normal form φ_{CNJ} , i.e. a conjunction of disjunctions of literals.

Theorem 7.4 (Variable sharing property [4]). $\varphi \vdash_{\text{FDE}} \psi$ iff $\varphi_i \vdash \psi_j$ is a primitive entailment for all disjuncts φ_i of φ_{DSJ} and conjuncts ψ_j of ψ_{CNJ} .

Remark 7.1 (Relevance and Variable Sharing). The concept of primitive entailment can be used to substantiate the concept of relevance. To see why, note that Theorem 7.4 explains on what syntactic ground disjunctive syllogism $\neg\varphi \wedge (\varphi \vee \psi) \vdash \psi$ is to be discarded. Given that $p \wedge (q \vee \neg q)$ can be turned into disjunctive normal form $(p \wedge \neg p) \vee (p \wedge q)$, $p \wedge (q \vee \neg q) \not\vdash_{\text{FDE}} q$ is explained by the fact that $p \wedge \neg p \vdash q$ is not a primitive entailment.

7.2.2 American–Plan Semantics

I will now introduce two model theories for FDE, the so-called *American plan* and *Australian plan* for negation within FDE.

The American–plan semantics of FDE starts from the observation that information can be contradictory and incomplete, and formalises it as follows: a formula can be assigned value true **t**, false **f**, both true and false **b**, and neither true nor false **n**. [48, 50] clarify that these semantic values are to be understood epistemically rather than ontologically, rooted in the fact that humans are cognitively and computationally limited agents.

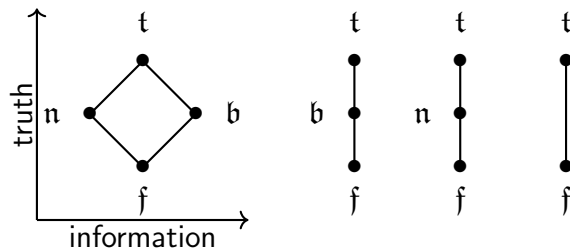


Figure 7.1: \mathbb{DML}_4 and its sublattices.

These intuitions are taken up in Belnap and Dunn’s semantics for FDE by taking **t**, **f**, **b** and **n** as primitive semantic values, thus generalising the two–valued semantics for classical propositional logic. The easiest way to conceive of the 4 values is by ordering them in the lattice \mathbb{DML}_4 shown in Figure 7.1 (left), which can be read along the axes of the truth order and the information order, so that **t** is more true

than **f** and **b** is more informative than **n**.⁷ The 4-valued semantics provides perhaps the most intuitive analysis of FDE, based on the following logical matrix.

Definition 7.7 (Logical matrix). *For a given language $\mathcal{L} \subseteq \text{Lan}$ in the signature σ , let a logical matrix over σ be a tuple of the following form:*

$$(\mathcal{V}, \mathcal{D}, \{f^{\otimes}\}_{\otimes \in \sigma})$$

- \mathcal{V} is a set of truth values;
- \mathcal{D} is a set of designated truth values;
- For all $\otimes \in \sigma$ with arity n , $f^{\otimes} : \mathcal{V}^n \rightarrow \mathcal{V}$ is a truth function.

Definition 7.8 (Valuation (Matrix)). *For a given language $\mathcal{L} \subseteq \text{Lan}$ in the signature σ , let a valuation based on a logical matrix \mathfrak{M} over σ be a homomorphism $V : \mathcal{L} \rightarrow \mathcal{V}$, i.e. such that for all $\otimes \in \sigma$ with arity n ,*

$$V(\otimes(\varphi, \dots, \varphi_n)) = f^{\otimes}(V(\varphi), \dots, V(\varphi_n)).$$

Let $V(\Gamma) = \{a \in \mathcal{V} \mid \exists \gamma \in \Gamma (V(\gamma) = a)\}$.

Definition 7.9 (FDE models). *Let an FDE-model be a tuple of the following form:*

$$(\{\mathbf{t}, \mathbf{f}, \mathbf{b}, \mathbf{n}\}, \{\mathbf{t}, \mathbf{b}\}, f^{\wedge}, f^{\vee}, f^{\neg}, V)$$

- $(\{\mathbf{t}, \mathbf{f}, \mathbf{b}, \mathbf{n}\}, \{\mathbf{t}, \mathbf{b}\}, f^{\wedge}, f^{\vee}, f^{\neg})$ is a logical matrix, where $f^{\wedge}, f^{\vee}, f^{\neg}$ are defined as follows:

\neg		\wedge	t	b	n	f	\vee	t	b	n	f
t	f	t	t	b	n	f	t	t	t	t	t
b	b	b	b	b	f	f	b	t	b	t	b
n	n	n	n	f	n	f	n	t	t	n	n
f	t	f	f	f	f	f	f	t	b	n	f

- $V : \mathcal{L}_{\text{FDE}} \rightarrow \{\mathbf{t}, \mathbf{f}, \mathbf{b}, \mathbf{n}\}$ is a valuation.

Let \mathbf{FDE}_4 be the class of all FDE matrices.

⁷FDE is closely related to a number of systems, such as Priest's logic of paradox LP [145], Kleene's strong paracomplete logic K3 [99], Angell's containment logic AC [5] and Priest's logic of *catuskoti* FDE _{φ} [146]. Specifically, LP and K3 result from adding to FDE $\varphi \vdash \psi \vee \neg \psi$ and $\varphi \wedge \neg \varphi \vdash \psi$, respectively. Of course, adding both results in CPC. Logical matrices for LP, K3 and CPC can be obtained by considering the sublattices of \mathbb{DML}_4 shown in Figure 7.1.

Definition 7.10 (FDE matrix consequence). *Let φ be a consequence of Γ in an FDE-model \mathfrak{M} – denoted $\Gamma \models^{\mathfrak{M}} \varphi$ – iff $V(\Gamma) \subseteq \{\mathbf{t}, \mathbf{b}\}$ only if $V(\varphi) \in \{\mathbf{t}, \mathbf{b}\}$. Let φ be a consequence of Γ in a class of FDE-models \mathbf{M} – denoted $\Gamma \models^{\mathbf{M}} \varphi$ – iff $\Gamma \models^{\mathfrak{M}} \varphi$ for all $\mathfrak{M} \in \mathbf{M}$.*

Note that \mathbb{DML}_4 gives rise to a De Morgan algebra, with f^\wedge, f^\vee defined as, respectively, meet and join, and f^\neg defined as an involutive and order-inverting function. \mathbb{DML}_4 has exactly three proper sublattices, which are shown in Figure 7.1 (from left to right) and which are used in the algebraic semantics of LP, K3 and CPC, respectively – cf. [141]. Matrices for LP, K3 and CPC are obtained by considering submatrices of $\mathfrak{M}_{\text{FDE}}$ obtained by removing \mathbf{n}, \mathbf{b} , both, respectively. This entails that, seen as sets of theorems, $\text{FDE} \subseteq \text{LP}, \text{K3}$ and $\text{LP}, \text{K3} \subseteq \text{CPC}$.

While \mathbb{DML}_4 offers a perspicuous representation of data classification by agents faced with incomplete and inconsistent information, truth functions are not always easy to read, especially when the language is enriched with further connectives. For this reason, it is useful to note that 4-valued semantics is effectively equivalent to the more palatable 2-valued semantics – in the strong sense that there is a mechanic procedure turning one into the other.⁸ The latter 2-valued semantics results from generalising satisfaction to *bilateral satisfaction*: valuation is generalised from a single truth assignment *function* to two independently specified truth and falsity *relations*, which are generalised to truth and falsity valuations $\models^{\mathbf{t}}$ and $\models^{\mathbf{f}}$.

Definition 7.11 (Bilateral FDE-models). *Let a bilateral FDE-model for FDE be a tuple of the following form:*

$$(V^+, V^-)$$

such that $V^+, V^- : At \times \{\mathbf{t}, \mathbf{f}\}$ are truth and falsity relations, respectively. Let \mathbf{FDE}_2 be the class of all bilateral FDE models.

Definition 7.12 (Bilateral FDE satisfaction). *Let positive and negative satisfaction in a bilateral FDE model \mathfrak{M} – denoted $\models^{\mathbf{t}}, \models^{\mathbf{f}} \subseteq \mathbf{FDE}_2 \times \mathfrak{L}_{\text{Ent}}$ – be recursively defined as follows:*

$$\begin{array}{lll} \mathfrak{M} \models^{\mathbf{t}} p & \iff & \mathbf{t} \in V^+(p) \\ \mathfrak{M} \models^{\mathbf{f}} p & \iff & \mathbf{f} \in V^-(p) \end{array}$$

⁸The procedure, formulated in [131] for a wide class of FDE-based logics, for the present case amounts in one direction to setting, for each bilateral FDE-model \mathfrak{M} , $\mathfrak{M} \models^{\mathbf{t}} \varphi$ iff $V(\varphi) \in \{\mathbf{t}, \mathbf{b}\}$ and $\mathfrak{M} \models^{\mathbf{f}} \varphi$ iff $V(\varphi) \in \{\mathbf{f}, \mathbf{b}\}$, and conversely to setting $V(\varphi) = \mathbf{t}$ iff $\mathfrak{M} \models^{\mathbf{t}} \varphi$ and $\mathfrak{M} \not\models^{\mathbf{f}} \varphi$, $V(\varphi) = \mathbf{b}$ iff $\mathfrak{M} \models^{\mathbf{t}} \varphi$ and $\mathfrak{M} \models^{\mathbf{f}} \varphi$, $V(\varphi) = \mathbf{n}$ iff $\mathfrak{M} \not\models^{\mathbf{t}} \varphi$ and $\mathfrak{M} \not\models^{\mathbf{f}} \varphi$, and $v(\varphi) = \mathbf{f}$ iff $\mathfrak{M} \not\models^{\mathbf{t}} \varphi$ and $\mathfrak{M} \models^{\mathbf{f}} \varphi$.

$$\begin{array}{lll}
\mathfrak{M} \models^t \varphi \wedge \psi & \iff & \mathfrak{M} \models^t \varphi \text{ and } \mathfrak{M} \models^t \psi \\
\mathfrak{M} \models^f \varphi \wedge \psi & \iff & \mathfrak{M} \models^f \varphi \text{ or } \mathfrak{M} \models^f \psi \\
\mathfrak{M} \models^t \varphi \vee \psi & \iff & \mathfrak{M} \models^t \varphi \text{ or } \mathfrak{M} \models^t \psi \\
\mathfrak{M} \models^f \varphi \vee \psi & \iff & \mathfrak{M} \models^f \varphi \text{ and } \mathfrak{M} \models^f \psi \\
\mathfrak{M} \models^t \neg \varphi & \iff & \mathfrak{M} \models^f \varphi \\
\mathfrak{M} \models^f \neg \varphi & \iff & \mathfrak{M} \models^t \varphi
\end{array}$$

Definition 7.13 (FDE bilateral consequence). *Let φ be a consequence of Γ in a bilateral FDE model \mathfrak{M} – denoted $\Gamma \models_2^{\mathfrak{M}} \varphi$ – iff $\mathfrak{M} \models^t \gamma$ for all $\gamma \in \Gamma$ only if $\mathfrak{M} \models^t \varphi$. Let φ be a consequence of Γ in a class of bilateral FDE models \mathbf{M} – denoted $\Gamma \models_2^{\mathbf{M}} \varphi$ – iff $\Gamma \models_2^{\mathfrak{M}} \varphi$ for all $\mathfrak{M} \in \mathbf{M}$.*

Notable semantic properties of FDE are that every formula has a bilateral FDE model, i.e. for all φ there is a bilateral FDE model \mathfrak{M} such that $\mathfrak{M} \models^t \varphi$, and that no formula is bilaterally valid, i.e. for all φ there is a bilateral FDE model \mathfrak{M} such that $\mathfrak{M} \not\models^t \varphi$.

Theorem 7.5 (FDE characterisation [51]). $\Gamma \models^{\text{FDE}_4} \varphi$ iff $\Gamma \models^{\text{FDE}_2} \varphi$ iff $\bigwedge \Gamma \vdash_{\text{FDE}} \varphi$.

7.2.3 Australian-Plan Semantics

Instead of generalising the number of truth values or the number of satisfaction relations, the *Australian-plan* semantics enriches semantic models with a domain set, and formulas are not mapped to truth values but to *propositions*, i.e. subsets of the domain.

Definition 7.14 (Routley models). *Let a Routley model be a tuple of the following form:*

$$(S, *, V)$$

- $*$: $S \rightarrow S$ is the Routley star function, interpreting \neg , with $s^{**} = s$;
- V : $At \rightarrow \mathcal{P}(S)$ is an atomic valuation function.

Let **FDE** be the class of all Routley models.

Definition 7.15 (FDE satisfaction). *Let satisfaction in a Routley model \mathfrak{M} be recursively defined as follows:*

$$\mathfrak{M}, s \models p \iff s \in V(p)$$

$$\begin{array}{lll}
\mathfrak{M}, s \models \varphi \wedge \psi & \iff & \mathfrak{M}, s \models \varphi \text{ and } \mathfrak{M}, s \models \psi \\
\mathfrak{M}, s \models \varphi \vee \psi & \iff & \mathfrak{M}, s \models \varphi \text{ or } \mathfrak{M}, s \models \psi \\
\mathfrak{M}, s \models \neg \varphi & \iff & \mathfrak{M}, s^* \not\models \varphi
\end{array}$$

Definition 7.16 (FDE consequence). *Let φ be valid in a Routley model \mathfrak{M} – denoted $\mathfrak{M} \models \varphi$ – iff $\mathfrak{M}, s \models \varphi$ for all $s \in S$. Let φ be a consequence of Γ in a Routley model \mathfrak{M} – denoted $\Gamma \models^{\mathfrak{M}} \varphi$ – iff $\mathfrak{M} \models \gamma$ for all $\gamma \in \Gamma$ only if $\mathfrak{M} \models \varphi$. Let φ be a consequence of Γ in a class of FDE models \mathbf{M} – denoted $\Gamma \models^{\mathbf{M}} \varphi$ – iff $\Gamma \models^{\mathfrak{M}} \varphi$ for all $\mathfrak{M} \in \mathbf{M}$. Let a consecution $\Gamma \vdash \varphi$ be valid in a class of Routley models \mathbf{M} iff $\Gamma \models^{\mathbf{M}} \varphi$.*

Australian-plan semantics is the frame semantics used in the dissertation. Hence, the reader is referred to the discussions of Chapter 2 for an illustration of the components of Routley models, soundness and completeness (cf. Theorems 2.1 and 2.2).

Theorem 7.6 (FDE-characterisation). $\Gamma \models^{\text{FDE}} \varphi$ iff $\bigwedge \Gamma \vdash_{\text{FDE}} \varphi$.

Having outlined the main features of the American-plan and the Australian-plan semantics for FDE, I will briefly pause to assess the differences of the resulting accounts of negation.

1. *American and Australian equivalence.* As noted in [132, Remark 8], American and Australian-style semantics are equivalent for FDE. However, things change (i) when further constraints are added to the model theory or (ii) when the language is extended with additional vocabulary. In particular, it is noteworthy that as a result of either (i) or (ii), logics may fail to be closed under the rule of contraposition (FDE8).⁹
2. *\neg as contradictory-forming.* A typical reason why logics based on American-style semantics fail to be closed under contraposition is that positive and negative semantic clauses can be formulated independently one from the other. The American plan for negation has been motivated by an account of \neg as a contradictory-forming operator, i.e. $\neg\varphi$ is true iff φ is false and $\neg\varphi$ is false iff φ

⁹For an example of (i), consider Priest’s logic of paradox LP. In [42, 130], this fact is taken as one reason to favour the American over the Australian-style semantic, since the validity of contraposition is built into the latter but not the former. For an example of (ii), consider Priest’s logic of intentionality, which extends FDE with a strict implication operator (cf. [147]), and which is also not closed under contraposition. A similar problem affects Lakemeyer’s generalisation of Levesque’s logic to arbitrary formulas $B\varphi$ for explicit belief and $L\varphi$ for implicit belief (cf. [106]), in which (contrary to Levesque’s original logic of [111]) $B\varphi$ entails $L\varphi$ but $\neg L\varphi$ does not entail $\neg B\varphi$.

is true. On this account, truth and falsity are taken as primitives and negation is taken to be a sentential operator that flip-flops truth and falsity (cf. [42]).

3. \neg as *incompatibility*. On the other hand, Australian-style negation is understood as a modality, whose distinctive accessibility relation is compatibility and whose functional role is to express exclusion (cf. [47]). On this account, compatibility is taken as a primitive and fundamental notion, and negation is grounded in compatibility, so that s supports $\neg\varphi$ iff any state compatible with s does not support φ ¹⁰. Grounding negation in compatibility has the effect that the inferential properties of negation are dictated by its semantic clause and the properties of compatibility. For example, a common assumption on compatibility, defended also in [25], is symmetry, from which it follows that negation satisfies double negation introduction. Another example of a typical feature of negation in Australian-style semantics is that it satisfies contraposition by design.¹¹

In what follows, my aim will not be to take a position in the debate between the American and Australian plan for negation. Rather, I will outline a way to model explicit and implicit beliefs within FDE, so as to obtain an Australian-style version of the logic of [111], which additionally lifts some of its limitations. Moreover, in the style of contextual modal logics, the extension of FDE will be suitable to perform classical reasoning.

7.3 Explicit and Implicit Belief in FDE

FDE can be extended with various connectives (cf. [132]). In the remainder of the chapter, I will be interested in the logic $\text{FDE}_c^{\neg\exists}$, i.e. an extension of FDE by the intensional connectives $\neg\exists$ and $\&$, expressing a strict version of implication and conjunction, as well as with a classicality constant c .¹² Such extension is interesting,

¹⁰It is well-known that $*$ can be defined in terms of a symmetric, serial and convergent compatibility relation (see e.g. [25]). The Routley star $*$ then expresses maximal compatibility with respect to a partial order \leq , which can be safely added to Routley models. Hence, Routley models fall under the Australian plan.

¹¹Note that grounding negation in compatibility is problematic for a number of paraconsistent negations for which contraposition fails, like Da Costa's negation and Nelson's N4 negation. Note also that contraposition is invalid for the Australian-style semantics of LP based on an extension of pointed models of [165] for FDE, as shown in [130].

¹²Extensions of FDE with strict implication may be at odds with the relevant roots of FDE, since they break variable sharing, but they are not unheard of – a prominent example is Priest's extension

because it allows to articulate the distinction between explicit and implicit belief in the same vein as [111], without the limitation of the latter that no nesting of doxastic operators is allowed. A similar generalisation of [111] is obtained in [106] in an American-style semantic framework. Contrasted with the latter, FDE_c^{\neg} is (i) based on Australian-plan semantics, (ii) closed under contraposition (like the original [111]), and (iii) a flexible framework where to model different properties of explicit and implicit belief (unlike [111]).¹³

7.3.1 FDE with strict implication and classicality

In what follows, strict implication \neg , in the style of [113], and the classicality constant c , in the style of [38], are introduced to introduce \Box and \Box_I as defined modality. In order to do so, I will use the abbreviations $\top := p_0 \neg p_0$, $\perp := \neg\top$, $\varphi \supset \psi := \neg\varphi \vee \psi$, $\varphi \equiv \psi := (\varphi \supset \psi) \wedge (\psi \supset \varphi)$. With these, it is possible to define:

$$\begin{aligned} \Box\varphi &:= \top \neg \varphi & \Box_I\varphi &:= c \neg \varphi \\ \Diamond\varphi &:= \varphi \& \top & \Diamond_I\varphi &:= \varphi \& c. \end{aligned}$$

Definition 7.17 (FDE_c^{\neg} -models). *Let an FDE_c^{\neg} -model be a tuple of the following form:*

$$(S, W, *, Q, V)$$

- $(S, *, V)$ is a Routley model;
- $W \subseteq S$ is a set of possible worlds, with $w^* = w$ for all $w \in W$;
- $Q \subseteq S^2$ is a Kripke accessibility relation, interpreting \neg .

Let $(S, W, *, Q)$ be the frame an FDE_c^{\neg} -model $(S, W, *, Q, V)$ is based on. Let \mathbf{FDE}_c^{\neg} be the class of all FDE_c^{\neg} -models.

Definition 7.18 (FDE_c^{\neg} satisfaction). *Let satisfaction in an FDE_c^{\neg} -model \mathfrak{M} – denoted $\models_{\subseteq} \mathbf{FDE} \times \mathfrak{L}_{\text{SEnt}}$ – be recursively defined by extending Definition [7.15] as follows:*

$$\begin{aligned} \mathfrak{M}, s \models c & \iff s \in W \\ \mathfrak{M}, s \models \varphi \neg \psi & \iff \forall t \in S(Qst, \mathfrak{M}, t \models \varphi \implies \mathfrak{M}, t \models \psi) \\ \mathfrak{M}, s \models \varphi \& \psi & \iff \mathfrak{M}, s \models \psi, \exists t \in S(Qts, \mathfrak{M}, t \models \varphi) \end{aligned}$$

of FDE, which uses a version of strict implication as a basis of a logic of intentionality [147].

¹³In Levesque's semantics explicit beliefs are modelled by a fixed set of states, which implicitly commits to certain introspection properties: for instance, if φ is explicitly believed according to a state s in Levesque's model, then φ is explicitly believed in all states of the model and if φ is not explicitly believed in s , then it is not explicitly believed according to all states.

Let consequence in (classes of) \mathbf{FDE}_c^{\neg} -models and frames be defined as in Definition 7.16. The reason for adding $\&$ as the adjoint operator with respect to \neg is technical, since it simplifies the completeness proof. \neg residuates over $\&$ from the right side:

$$\varphi \models_M \psi \neg \chi \iff \varphi \& \psi \models_M \chi$$

Note that $\varphi \vdash \psi \in \mathbf{FDE}_c^{\neg}$ only if $\varphi \neg \psi \in \mathbf{FDE}_c^{\neg}$. The converse is not true. For example, $p \neg (\top \& \top)$ is valid in all frames, but $p \vdash \top \& \top$ is not valid in frames where R is not backwards serial.

In \mathbf{FDE}_c^{\neg} -models, the reading of Q and c are epistemically driven.

1. *Explicit belief.* Qst iff all information available to some fixed agent according to state s is true in state t . Therefore, $\varphi \neg \psi$ expresses a notion close to *conditional belief*: $\varphi \neg \psi$ is true in s iff each state where the agent's information (according to s) is true, and where φ is true as well, is a state where ψ is true. This lends a particular epistemic reading to $\Box\varphi$. By unraveling the definition of \Box , we obtain:

$$\mathfrak{M}, s \models \Box\varphi \iff \forall t \in S(Qst \implies \mathfrak{M}, t \models \varphi)$$

That is, $s \models \Box$ iff φ is true in every state where the information that the agent possesses according to s is true. In line with Levesque's [111] use of the semantics for FDE to model explicit belief, $\Box\varphi$ then means that φ is explicitly believed by the agent.

2. *Implicit belief.* The classicality constant c is true in (denotes the set of) complete and consistent states W ; such states can be called "classical". This lends an epistemic reading of $\Box_I\varphi$, in line with the computational interpretation of contextual modal logics (cf. Section 7.1). By unraveling the definition of \Box_I , we obtain:

$$\mathfrak{M}, s \models \Box_I\varphi \iff \forall t \in W(Qst \implies \mathfrak{M}, t \models \varphi)$$

that is, $s \models \Box_I\varphi$ iff φ is true in every *classical* state where the information that the agent possesses according to s is true. This means that \Box_I expresses the "classical closure" of explicit beliefs held by the agent – to recall, what would be explicitly believed by the agent if the epistemically accessible states were of some ideal sort (classical possible worlds). Then, c represents classical worlds in the language.

Remark 7.2 (\mathbf{FDE}_c^{\neg} and contextual modal logics). Although the primary use of c is to model implicit belief as the classical closure of explicit belief, c can be used

to retrieve classical propositional logic in an FDE setting. To this aim, it suffices to note $c \vdash \varphi$ is valid for all classical propositional tautologies. Doing so allows one to note that, in the context of FDE_c^{\neg} -models, classical logic is “hidden” inside relevant logic (cf. [168] for further details). Contextual modal logics attain a similar combination of classical and non-classical reasoning in a specular way: by defining validity classically, the \Box_L modality allows one to appreciate that relevant logic is “hidden” inside classical logic.

7.3.2 Axiomatisation of FDE_c^{\neg} logics

One of the advantages of the present framework, compared with that of [106], is that it allows to modularly adjust the epistemic properties of explicit and implicit belief, depending on the application at hand. In fact, I will show a characterisation result for a number of axiomatic extensions of FDE_c^{\neg} , the basic axiom system extending FDE with \neg and c .

Definition 7.19 (FDE_c^{\neg} axiom system). *Let the natural deduction system for FDE_c^{\neg} consist of the the following sequents and rules:*

- ($\text{FDE}_c^{\neg}1$) *Sequents and rules of FDE (cf. Definition [7.5])*
- ($\text{FDE}_c^{\neg}2$) $c \vdash \varphi \vee \neg\varphi$
- ($\text{FDE}_c^{\neg}3$) $c \wedge \varphi \wedge \neg\varphi \vdash \perp$
- ($\text{FDE}_c^{\neg}4$) $\top \vdash c \vee \neg c$
- ($\text{FDE}_c^{\neg}5$) $\varphi \& (\psi \wedge \chi) \dashv\vdash (\varphi \& \psi) \wedge \chi$
- ($\text{FDE}_c^{\neg}6$) $\varphi \vdash \psi \rightarrow \chi \iff \varphi \& \psi \vdash \chi$
- ($\text{FDE}_c^{\neg}7$) $\varphi \vdash \psi \implies \psi \rightarrow \chi \vdash \varphi \rightarrow \chi$

A number of sequents and rules are provable in FDE_c^{\neg} (cf. Proposition [7.3]), which will turn out useful in the completeness proof. Note that most of them receive an epistemically perspicuous interpretation, once read as properties of explicit and implicit belief (cf. Proposition [7.4]). Finally, Proposition [7.5] makes precise in what sense implicit belief is defined as the closure of explicit belief under classical logic.

Proposition 7.3 (Provable sequents and derivable rules). *The following sequents and rules of inference are provable, respectively derivable, in FDE_c^{\neg} :*

1. $\varphi \vdash \psi \rightarrow (\varphi \& \psi)$;
2. $(\varphi \rightarrow \psi) \& \varphi \vdash \psi$;
3. $\varphi \& \psi \vdash \psi$
4. $\varphi \vdash \top$;
5. $\perp \vdash \varphi$;
6. $(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi) \vdash \varphi \rightarrow (\psi \wedge \chi)$;
7. $(\varphi \vee \psi) \& \chi \vdash (\varphi \& \chi) \vee (\psi \& \chi)$;
8. $\varphi \vdash \psi, \chi \vdash \xi \Rightarrow \varphi \& \chi \vdash \psi \& \xi$;
9. $\varphi \wedge \psi \vdash \chi \Rightarrow (\theta \rightarrow \varphi) \vdash (\theta \wedge \psi) \rightarrow \chi$.

Proof. Items 1 and 2 are easy consequences of (FDE_c³6). Item 3 is established using (FDE_c³5) and Item 8. Item 4 is an easy consequence of Item 3, and Item 5 follows from Item 4 and (FDE6). Item 6 and 7 are established by using (FDE1)–(FDE2), respectively (FDE3)–(FDE4), and (FDE_c³6). Items 8 and 9 are established as follows (note that the proof of the former is independent of Item 3):

$$\begin{array}{c}
\frac{\varphi \vdash \psi \quad \frac{\psi \& \xi \vdash \psi \& \xi}{\psi \vdash \xi \rightarrow (\psi \& \xi)} \text{(FDE}_c^3\text{6)}}{\varphi \vdash \xi \rightarrow (\psi \& \xi)} \text{(FDE8)} \quad \frac{\chi \vdash \xi}{\xi \rightarrow (\psi \& \xi) \vdash \chi \rightarrow (\psi \& \xi)} \text{(FDE}_c^3\text{7)}}{\varphi \vdash \chi \rightarrow (\psi \& \xi)} \text{(FDE8)} \\
\frac{\varphi \vdash \chi \rightarrow (\psi \& \xi)}{\varphi \& \chi \vdash \psi \& \xi} \text{(FDE}_c^3\text{6)} \\
\frac{\frac{\theta \rightarrow \varphi \vdash \theta \rightarrow \varphi}{(\theta \rightarrow \varphi) \& \theta \vdash \varphi} \text{(FDE}_c^3\text{6)}}{\frac{\dots}{((\theta \rightarrow \varphi) \& \theta) \wedge \psi \vdash \varphi \wedge \psi}} \\
\frac{\varphi \wedge \psi \vdash \chi}{((\theta \rightarrow \varphi) \& \theta) \wedge \psi \vdash \chi} \text{(FDE8)} \\
\frac{\frac{\dots}{(\theta \rightarrow \varphi) \& (\theta \wedge \psi) \vdash \chi} \text{(FDE}_c^3\text{5)}}{\theta \rightarrow \varphi \vdash \theta \wedge \psi \rightarrow \chi} \text{(FDE}_c^3\text{6)}
\end{array}$$

□

Proposition 7.4 (Epistemic sequents). *The following sequents (rules of inference) are provable (derivable) in FDE_c³:*

1. $\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$;
2. $\varphi \vdash \Box\Diamond\varphi$;
3. $\Diamond\Box\varphi \vdash \varphi$;
4. $\Box_I\varphi \wedge \Box_I\psi \vdash \Box_I(\varphi \wedge \psi)$;
5. $\Box\varphi \vdash \Box_I\varphi$;
6. $\varphi \vdash \psi \rightarrow \chi \Rightarrow \Diamond\varphi \wedge \psi \vdash \chi$;
7. $\varphi \vdash \psi \Rightarrow \Box\varphi \vdash \Box\psi$;
8. $\varphi \wedge \psi \vdash \chi \Rightarrow \Box\varphi \vdash \psi \rightarrow \chi$;
9. $\varphi \vdash \psi \Rightarrow \Box_I\varphi \vdash \Box_I\psi$.

Proof. Items 1 and 4 are easy consequences of Proposition 7.3.6, while Item 2 and 3 are instances of Proposition 7.3.1 and 7.3.2, respectively. Item 5 is an easy consequence of (FDE_c³7) and Proposition 7.3.4. Items 7–9 are easy consequences of Proposition 7.3.9. Finally, item 6 is established as follows:

$$\begin{array}{c}
\frac{\varphi \vdash \psi \multimap \chi \quad \frac{\top \wedge \psi \vdash \psi}{\psi \multimap \chi \vdash \top \wedge \psi \multimap \chi} \text{(FDE}_c^{\multimap 3} \text{7)}}{\varphi \vdash \top \wedge \psi \multimap \chi} \text{(FDE}_c^{\multimap 3} \text{8)}} \\
\frac{\varphi \vdash \top \wedge \psi \multimap \chi}{\varphi \& (\top \wedge \psi) \vdash \chi} \text{(FDE}_c^{\multimap 3} \text{6)}} \\
\frac{\varphi \& (\top \wedge \psi) \vdash \chi}{\Diamond \varphi \wedge \psi \vdash \chi} \text{(FDE}_c^{\multimap 3} \text{5)}}
\end{array}$$

□

For the sake of the following two propositions, we assume familiarity with frame semantics for CPC and the classical modal logic \mathbf{K} , as well completeness of CPC and \mathbf{K} .

Proposition 7.5 (Classical closure). *For all $\varphi_1, \dots, \varphi_n, \psi \in \mathcal{L}_{\text{Ent}}$, the following are equivalent:*

1. $\Box \varphi_1 \wedge \dots \wedge \Box \varphi_n \vdash_{\text{FDE}_c^{\multimap 3}} \Box_I \psi$;
2. $\varphi_1 \wedge \dots \wedge \varphi_n \vdash_{\text{CPC}} \psi$.

Proof. (1) \implies (2). By contraposition, assume (2) does not hold with counter-model $(\{s\}, V)$. Then, take the $\text{FDE}_c^{\multimap 3}$ -model $\mathfrak{M} = (\{s\}, W, *, Q, V)$ such that $W = \{s\}$ and $R = \{(s, s)\}$. Clearly, we have that $\mathfrak{M}, s \models \Box \varphi_1 \wedge \dots \wedge \Box \varphi_n$, since by assumption $\mathfrak{M}, s \models \varphi_1 \wedge \dots \wedge \varphi_n$. However, $\mathfrak{M}, s \not\models \Box_I \psi$, since by assumption $\mathfrak{M}, s \not\models \psi$ (note that this is possible only because $\varphi_1, \dots, \varphi_n, \psi$ are propositional formulas). Hence, we conclude by Theorem 7.8 that (1) does not hold. (2) \implies (1). Assume (2). Hence, $\Box_I \varphi_1 \wedge \dots \wedge \Box_I \varphi_n \vdash_{\text{FDE}_c^{\multimap 3}} \Box_I \psi$ (otherwise from the counter-model $(S, W, *, Q, V)$ to the latter we can extract the counter-model (W, V) to the former). By Proposition 7.4 (item 5), $\Box \varphi_1 \wedge \dots \wedge \Box \varphi_n \vdash_{\text{FDE}_c^{\multimap 3}} \Box_I \varphi_1 \wedge \dots \wedge \Box_I \varphi_n$. Hence, we conclude that (1) holds. □

While classical behavior can be forced for propositional variables, this does not apply to formulas containing strict implication, as the following counter-example to a modal version of Proposition 7.5 shows.

Proposition 7.6 (Modal closure failure). *The following hold:*

1. $\top \vdash_{\mathbf{K}} (\Box \neg(p \wedge \neg p))$;
2. $\Box \top \not\vdash_{\text{FDE}_c^{\multimap 3}} \Box_I \Box \neg(p \wedge \neg p)$.

Proof. It is immediate to verify that $(\Box \neg(p \wedge \neg p))$ is valid in every Kripke model, as $p \wedge \neg p$ is true at no state in the model. Hence, by completeness of \mathbf{K} with respect to \mathbf{K} -Kripke models we infer (1). Then, consider the $\text{FDE}_c^{\multimap 3}$ -model $\mathfrak{M} = (S, W, *, Q, V)$ such that $S = \{s, t, u\}$, $W = \{t\}$, $Q = \{(s, t), (t, u)\}$, $s^* = u$ and $t^* = t$, $V(p) = \{s, t\}$.

Clearly we have that $\mathfrak{M}, s \models \Box \top$. However, $\mathfrak{M}, s \not\models \Box_I \Box \neg(p \wedge \neg p)$, since $Qst, t \in W$ and $\mathfrak{M}, t \not\models \Box \neg(p \wedge \neg p)$ (as $Qtu, u^* = s^{**} = s \in V(p)$ and $u \notin V(p)$). Hence, by Theorem [7.8](#) (proven below) we conclude that (2) holds. \square

On the one hand, the above proposition can be regarded as a drawback, in that it shows that implicit belief in \mathbf{FDE}_c^3 cannot be defined as the closure of explicit belief under the classical modal logic \mathbf{K} . On the other hand however, the above result is in line with the understanding of agents' explicit beliefs as being regimented by non-classical logics, thus not being closed under \mathbf{K} . Therefore, it is to be expected that implicit belief is not the classical closure of explicit belief when epistemic formulas are considered. What is more, when considering explicit beliefs one may note that failure of classical closure is a desirable property: \mathbf{FDE}_c^3 correctly prescribes that agents can hold explicit beliefs about some classical tautology without explicitly believing all of them, that is e.g. $\Box \top \not\models_{\mathbf{FDE}_c^3} \Box \neg(\varphi \wedge \neg\varphi)$. Then, if $\Box \top \vdash_{\mathbf{FDE}_c^3} \Box_I \Box \neg(p \wedge \neg p)$ was the case then agents would implicitly believe that they explicitly believe all classical tautologies of the form $\neg(p \wedge \neg p)$, thereby committing agents to hold an infinite number of false implicit beliefs. In fact, a similar argument can be put forward for any classical tautology, not just $\neg(p \wedge \neg p)$.

Having outlined the properties of explicit and implicit belief in \mathbf{FDE}_c^3 , let me turn to soundness and completeness.

Theorem 7.7 (\mathbf{FDE}_c^3 -soundness). $\varphi \vdash_{\mathbf{FDE}_c^3} \psi \implies \varphi \vdash \psi \in \mathbf{FDE}_c^3$.

Proof. The result is established by induction on the length of \mathbf{FDE}_c^3 -proofs. By Theorem [7.6](#), it suffices to show the induction step cases concerning $(\mathbf{FDE}_c^3 2)$ – $(\mathbf{FDE}_c^3 7)$.

$(\mathbf{FDE}_c^3 2)$. The following chain of equivalences holds: $s \models \varphi \& (\psi \wedge \chi)$ iff $s \models \psi, s \models \chi$ and there is t such that Rts and $t \models \varphi$ iff $s \models \varphi \& \psi$ and $s \models \chi$ iff $s \models (\varphi \& \psi) \wedge \chi$.

$(\mathbf{FDE}_c^3 3)$. (\implies) . Assume that $\varphi \models_{\mathbf{FDE}_c^3} \psi \multimap \chi$ and that for some arbitrary $s \in S$ $s \models \varphi \& \psi$. Hence, there is t such that $Qts, t \models \varphi$ and $s \models \psi$. By $t \models \varphi$ and $\varphi \models_{\mathbf{FDE}_c^3} \psi \multimap \chi$ we infer that $t \models \psi \multimap \chi$, which implies by Qts and $s \models \psi$ that $s \models \chi$. We conclude that $\varphi \& \psi \models_M \chi$. (\impliedby) . Assume that $\varphi \& \psi \models_{\mathbf{FDE}_c^3} \chi$ and that for some arbitrary $s, t \in S$ $s \models \varphi, Qst$ and $t \models \psi$. The latter implies that $t \models \varphi \& \psi$. We conclude by $\varphi \& \psi \models_{\mathbf{FDE}_c^3} \chi$ that $t \models \chi$.

$(\mathbf{FDE}_c^3 4)$. Assume that $\varphi \models_{\mathbf{FDE}_c^3} \psi$ and that for some arbitrary $s \in S$ $s \models \psi \multimap \chi, Qst$ and $t \models \varphi$. By $\varphi \models_{\mathbf{FDE}_c^3} \psi$ we infer that $t \models \psi$ and by $s \models \psi \multimap \chi$ we infer that $t \models \chi$. We conclude that $s \models \varphi \multimap \chi$.

(FDE_c³5). It follows from the fact that if $s \in C$, $s^* = s$, which implies that $s \models \neg\varphi$ iff $s \not\models \varphi$. (FDE_c³6) is established similarly.

(FDE_c³7). Assume by contradiction that there is $s \in S$ such that $s \not\models c$ and $s \not\models \neg c$. Then $s \notin C$ and $s^* \in C$. By $s = s^*$ we conclude that $s \in C$ and $s \notin C$, a contradiction. □

Definition 7.20 (Canonical FDE_c³-model). *Let the canonical FDE_c³-model be a tuple of the following form:*

$$\mathfrak{M}_{\text{FDE}_c^3}^c = (S^c, W^c, *^c, Q^c, V^c)$$

- $S^c = \{s \mid s \text{ is a non-empty proper prime FDE}_c^3\text{-theory over } \mathfrak{L}_{\text{Ent}}\}$;
- $W^c = \{s \in S^c \mid c \in s\}$;
- $s^{*c} = \{\varphi \in \mathfrak{L}_{\text{Ent}} \mid \neg\varphi \notin s\}$;
- $Q^c = \{(s, t) \in (S^c)^2 \mid \forall \varphi \in \mathfrak{L}_{\text{Ent}}(\varphi \rightarrow \psi \in s, \varphi \in t \implies \psi \in t)\}$;
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Lemma 7.6 (Q-equivalence). *The following are equivalent:*

1. $Q^c st$;
2. $\top \rightarrow \varphi \in s \implies \varphi \in t$;
3. $\varphi \in s \implies \varphi \& \top \in t$.

Proof. (1) \implies (2). Assume $Q^c st$ and $\top \rightarrow \varphi \in s$. Since t is non-empty, by Proposition [7.3.4](#) we infer that $\top \in t$. Hence, by the assumption Rst we infer that $\varphi \in t$ which establishes (2). (2) \implies (3). Assume (2) and $\varphi \in s$. By Proposition [7.3.1](#) we infer that $\top \rightarrow (\varphi \& \top) \in s$, which implies by (2) that $\varphi \& \top \in t$, which establishes (3). (3) \implies (1). Assume (3), $\varphi \rightarrow \psi \in s$ and $\varphi \in t$. the latter implies $((\varphi \rightarrow \psi) \& \top) \wedge \varphi \in t$, which by (FDE_c³5) implies that $(\varphi \rightarrow \psi) \& (\top \wedge \varphi) \in t$. Hence, by Proposition [7.3.8](#) we infer that $(\varphi \rightarrow \psi) \& \varphi \in t$, which implies by Proposition [7.3.2](#) that $\psi \in t$, which establishes (1). □

Lemma 7.7 (FDE_c³-modelhood). $\mathfrak{M}_{\text{FDE}_c^3}^c \in \mathbf{FDE}_c^3$.

Proof. That s^* is non-empty and proper whenever is so is established as in Lemma [6.13](#), and $s^{**} = s$ holds by (FDE5). Finally, that $s \in W$ implies $s^* = s$ is established as follows. (\implies). Assume for some arbitrary $\varphi \in s^*$. Hence, $\neg\varphi \notin s$. By $s \in W$, we infer that $c \in s$. Hence, by (FDE_c³2) we infer $\varphi \vee \neg\varphi \in s$. We conclude by $\neg\varphi \notin s$ that $\varphi \in s$. (\impliedby). Assume by contradiction that $\varphi \in s$ and $\varphi \notin s^*$. Hence, $\neg\varphi \in s$, which implies by $c \in s$, $\varphi \in s$ and (FDE_c³3) that $\perp \in s$. By Proposition [7.3.5](#) we conclude that $\varphi \in s$ for all φ , which contradicts s being proper. □

Lemma 7.8 (FDE_c³-truth). $\mathfrak{M}_{\text{FDE}_c^3}^c, s \models \varphi \iff \varphi \in s$.

Proof. The result is established by induction on the structure of φ . By Theorem [7.6](#), it suffices to show the base case $\varphi := c$ (which is trivial) and the induction step cases $\varphi := \psi \multimap \chi$ and $\varphi := \psi \& \chi$.

$\varphi := \psi \multimap \chi$. (\implies). Assume $\psi \multimap \chi \notin s$ and consider the pair $(\{\xi \mid \top \multimap \xi \in s\} \cup \{\psi\}, \{\chi\})$. The pair is FDE_c³-independent, since otherwise there would be $\xi_1, \dots, \xi_n \in \mathfrak{L}_{SEnt}$ such that $(\bigwedge_{i \leq n} \xi_i) \wedge \psi \vdash_{\text{FDE}_c^3} \chi$ and $\top \multimap \xi_i \in s$ for all $i \leq n$. we would reason as follows:

$$\begin{aligned}
\left(\bigwedge_{i \leq n} \xi_i\right) \wedge \psi \vdash_{\text{FDE}_c^3} \chi &\implies \top \multimap \left(\bigwedge_{i \leq n} \xi_i\right) \vdash_{\text{FDE}_c^3} (\top \wedge \psi) \multimap \chi \quad \text{by Prop. [7.3.9](#)} \\
&\implies \bigwedge_{i \leq n} (\top \multimap \xi_i) \vdash_{\text{FDE}_c^3} (\top \wedge \psi) \multimap \chi \quad \text{by Prop. [7.3.6](#)} \\
&\implies \bigwedge_{i \leq n} (\top \multimap \xi_i) \vdash_{\text{FDE}_c^3} \psi \multimap \chi \quad \text{by Prop. [7.3.4](#)} \\
&\implies \psi \multimap \chi \in s \quad \text{by } \bigwedge (\top \multimap \xi_i) \in s
\end{aligned}$$

which is a contradiction. Hence, by Lemma [2.3](#) and [7.6](#) there is $t \in S^c$ such that Q^cst , $\psi \in t$ and $\chi \notin t$, by which we conclude that $s \not\models \psi \multimap \chi$. (\Leftarrow). Assume $\psi \multimap \chi \in s$, Qst and $\psi \in t$. Hence, by definition of Q , $\chi \in t$.

$\varphi := \psi \& \chi$. (\implies). Assume $s \models \varphi \& \psi$. Hence, there is $t \in S$ such that Qst , $\psi \in s$ and $\chi \in t$. By Proposition [7.3.1](#) we infer that $\chi \multimap (\psi \& \chi) \in s$. We conclude by the case $\varphi := \psi \multimap \chi$ that $\psi \& \chi \in t$. (\Leftarrow). Assume that $\psi \& \chi \in t$. By Lemma [7.6](#), it is sufficient to show that $\chi \in t$ and there is s such that $\psi \in s$ and, for all $\xi \in s$, $\xi \& \top \in t$. The first part is easy: $\psi \& \chi \in t$ implies $\chi \in t$ by Proposition [7.3.3](#). To prove the second part, take the pair $(\{\psi\}, \{\xi \mid \xi \& \top \notin t\})$, which is FDE_c³-independent. Otherwise, there would be $\xi_1, \dots, \xi_n \in \mathfrak{L}_{SEnt}$ such that $\psi \vdash_{\text{FDE}_c^3} (\bigvee_{i \leq n} \xi_i)$, and $\xi_i \& \top \notin t$ for all $i \leq n$. We would as follows:

$$\begin{aligned}
\psi \vdash_{\text{FDE}_c^3} \left(\bigvee_{i \leq n} \xi_i\right) &\implies \psi \& \top \vdash_{\text{FDE}_c^3} \left(\bigvee_{i \leq n} \xi_i\right) \& \top \quad \text{by Proposition [7.3.8](#)} \\
&\implies \psi \& \top \vdash_{\text{FDE}_c^3} \bigvee_{i \leq n} (\xi_i \& \top) \quad \text{by Proposition [7.3.7](#)} \\
&\implies \psi \& \top \notin t \quad \text{by } \bigvee_{i \leq n} (\xi_i \& \top) \notin t
\end{aligned}$$

$$\implies \psi \& \chi \notin t \quad \text{by Props. } \boxed{7.3.4} \text{ and } \boxed{7.3.8}$$

which is a contradiction. Hence, by Lemma [2.3](#) and [7.6](#) there is $s \in S^c$ such that $Q^c st$, $\psi \in s$ and $\chi \in t$, by which we conclude that $t \models \psi \& \chi$.

□

Theorem 7.8 (FDE_c^{\neg} -completeness). $\varphi \models_{\text{FDE}_c^{\neg}} \psi \implies \varphi \vdash_{\text{FDE}_c^{\neg}} \psi$.

Proof. The result follows by Lemmas [7.7](#) and [7.8](#). □

The characterisation result can be extended from FDE_c^{\neg} to stronger logics via the method of frame correspondences. The additional frame properties correspond to sequents in the precise sense specified by Lemma [7.9](#) and the correspondences are summarised in Table [7.2](#). To formulate the correspondence result, I will denote with $\mathbf{FDE}_c^{\neg}.\mathbf{X}$ the class of FDE_c^{\neg} -models satisfying frame condition (X) and with $\text{FDE}_c^{\neg}.\mathbf{X}$ the axiomatic extension of FDE_c^{\neg} obtained by adding the sequent (X). Finally, recall that $\varphi \supset \psi := \neg\varphi \vee \psi$

The sequents we consider in Table [7.2](#) underlie various epistemic properties one may wish to impose on explicit (conditional) belief. Most notably, $(\Box 1)$ - $(\Box 3)$ express factivity, positive and negative introspection of explicit belief at information states; $(\Box 4)$ - $(\Box 6)$ express factivity, positive and negative introspection of explicit belief at classical information states; and $(\Box_1 1)$ - $(\Box_1 2)$ express positive and negative introspection of implicit belief. Next, $(\Box\Box_1 1)$ - $(\Box\Box_1 2)$ capture the idea that the very fact that something is explicitly believed is implicitly believed. Note that a similar list, featuring some interaction axioms between explicit and implicit belief, appears in [106](#)¹⁴. (Lprec) implies that s cannot support contradictory information about belief (either implicit or explicit), while (Rprec) implies that information about belief supported by s is complete (and similarly for (Lpost) - (Rpost)). Put differently, states satisfying pre-classicality (post-classicality) behave like possible worlds when it comes to formulas with \neg ($\&$).

Lemma 7.9 (FDE_c^{\neg} -correspondences). *For all $\mathfrak{F} \in \mathbf{FDE}_c^{\neg}$, \mathfrak{F} satisfies condition (X) of Table [7.2](#), iff (X) is valid in $\mathbf{FDE}_c^{\neg}.\mathbf{X}$.*

Proof. We proceed by distinguishing cases.

¹⁴Note that, while in [106](#) $(\Box\Box_1 1) - (\Box_1 2)$ are valid in the basic model, here models can be fine-tuned, via frame correspondences, depending on specific applications.

Label	Frame property		Sequents
(□1)	Reflexivity	Rss	$\Box\varphi \vdash \varphi$
(□2)	Transitivity	$Rst, Rtu \implies Rsu$	$\Box\varphi \vdash \Box\Box\varphi$
(□3)	*-euclideaness	$Rs^*t, Rsu \implies Ru^*t$	$\neg\Box\varphi \vdash \Box\neg\Box\varphi$
(□4)	C-reflexivity	$s \in C \implies Rss$	$c \vdash \Box\varphi \supset \varphi$
(□5)	C-transitivity	$s \in C, Rst, Rtu \implies Rsu$	$c \vdash \Box\varphi \supset \Box\Box\varphi$
(□6)	C-euclideaness	$s \in C, Rst, Rsu \implies Ru^*t$	$c \vdash \neg\Box\varphi \supset \Box\neg\Box\varphi$
(□□ _i 1)	ei-transitivity	$t \in C, Rst, Rtu \implies Rsu$	$\Box\varphi \vdash \Box_I\Box\varphi$
(□□ _i 2)	ei-euclideaness	$u \in C, Rs^*t, Rsu \implies Ru^*t$	$\neg\Box\varphi \vdash \Box_I\neg\Box\varphi$
(□ _i 1)	i-transitivity	$t, u \in C, Rst, Rtu \implies Rsu$	$\Box_I\varphi \vdash \Box_I\Box_I\varphi$
(□ _i 2)	i-euclideaness	$t, u \in C, Rs^*t, Rsu \implies Ru^*t$	$\neg\Box_I\varphi \vdash \Box_I\neg\Box_I\varphi$
(Lprec)	Left pre-classicality	$Rs^*t \implies Rst$	$(\varphi \multimap \psi) \wedge \neg(\varphi \multimap \psi) \vdash \perp$
(Rprec)	Right pre-classicality	$Rst \implies Rs^*t$	$\top \vdash (\varphi \multimap \psi) \vee \neg(\varphi \multimap \psi)$
(Lpostc)	Left post-classicality	$Rst^* \implies Rst$	$(\varphi \& \psi) \wedge \neg(\varphi \& \psi) \vdash \perp$
(Rpostc)	Right post-classicality	$Rst \implies Rst^*$	$\top \vdash (\varphi \& \psi) \vee \neg(\varphi \& \psi)$

Figure 7.2: Some frame correspondences for FDE_c^3 -frames.

(□1). (\implies). Assume $s \models \Box\varphi$. Hence, for all t such that Qst , $t \models \varphi$, from which we conclude by reflexivity that $s \models \varphi$. (\Leftarrow). Assume F is not reflexive, i.e. for some s not Qss . It suffices to take a model based on F such that $u \in V(p)$ iff Qsu to obtain a counter-model to $\Box p \vdash p$. (□4) is established similarly.

(□2). (\implies). Assume $s \models \Box\varphi$, Qst and Qtu . Hence, for all v such that Qsv , $v \models \varphi$ and by transitivity Qsu , by which we conclude $s \models \Box\Box\varphi$. (\Leftarrow). Assume F is not transitive, i.e. for some s, t, u Qst, Qtu but not Qsu . It suffices to take a model based on F such that $v \in V(p)$ iff Qsv to obtain a counter-model to $\Box p \vdash \Box\Box p$. (□5), (□□_i1) and (□_i1) are established similarly.

(□3). (\implies). Assume $s \models \neg\Box\varphi$ and Qsu . Hence, there is t such that Qs^*t and $t \not\models \varphi$ and by *-euclideaness Qu^*t , by which $u \models \neg\Box\varphi$, by which we conclude that $s \models \Box\neg\Box\varphi$. (\Leftarrow). Assume F is not *-euclidean, i.e. for some s, t, u Qs^*t, Qsu but not Qu^*t . It suffices to take any model based on F such that $V(p) = S \setminus \{t\}$ to obtain a counter-model to $\neg\Box p \vdash \Box\neg\Box p$. (□6), (□□_i2) and (□_i2) are established similarly.

(Lprec). (\implies). Assume by contradiction that $s \models \varphi \multimap \psi$ and $s \models \neg(\varphi \multimap \psi)$. By the former, for all t such that Qst and $t \models \varphi$, $t \models \psi$. By the latter, there is u such that Qs^*u , $u \models \varphi$ and $u \not\models \psi$. Hence, by left pre-classicality Qsu , by which we conclude that $u \models \psi$, which is a contradiction. (\Leftarrow). Assume F is not left pre-classical, i.e. there are s, t such that Qs^*t but not Qst . It suffices to take any model based on F such that $V(p) = S$ and $V(q) = S \setminus \{t\}$ to obtain a counter-model to $(p \multimap q) \wedge \neg(p \multimap q) \vdash \perp$. (Lpostc) is established similarly.

(Rprec). (\implies). We show that for all s , $s \models \varphi \multimap \psi$ or $s \models \neg(\varphi \multimap \psi)$. By contradiction, assume $s \not\models \varphi \multimap \psi$ and $s \not\models \neg(\varphi \multimap \psi)$. Hence, there is t such that Qst , $t \models \varphi$ and $t \not\models \psi$ and, for all u such that Qs^*u and $u \models \varphi$, $u \models \psi$. Hence, by right pre-classicality $t \models \varphi$, which is a contradiction. (\impliedby). Assume F is not right pre-classical, i.e. there are s, t such that Qst but not Qs^*t . It suffices to take any model based on F such that $V(p) = S$ and $V(q) = S \setminus \{t\}$ to obtain a counter-model to $\top \vdash (p \multimap q) \vee \neg(p \multimap q)$. (Rpostc) is established similarly. \square

Lemma 7.10 (FDE_c³-extension canonicity). *For any condition X of Table 7.2, the canonical FDE_c³.X-model satisfies (X).*

Proof. We rely on Lemma 7.6.

(□1). Assume $\Box\varphi \in s$, hence by the provability of the corresponding sequent that $\varphi \in s$, by which we conclude that Qss .

(□2). Assume Qst, Qtu . If $\Box\varphi \in s$, by the provability of the corresponding sequent $\Box\Box\varphi \in s$, hence by Qst $\Box\varphi \in t$ and by Qtu $\varphi \in u$, by which we conclude that Qsu .

(□3). Assume Qs^*t, Qsu . If $\Box\varphi \in u^*$, $\neg\Box\varphi \notin u$, hence by Qsu $\Box\neg\Box\varphi \notin s$, by which by the provability of the (□3)-sequent $\neg\Box\varphi \notin s$. Hence, $\Box\varphi \in s^*$ and by Qs^*t $\varphi \in t$, by which we conclude that Qu^*t .

(□4) – (□6). Note that if $s \in C$, $\varphi \in s$ and $c \vdash_{\text{FDE}_c^3} \varphi \supset \psi$, then we have $\psi \in s$, and so these cases can be argued for similarly as (□1)-(□3).

(Lprec). If not Qst , then there are φ, ψ such that $\varphi \multimap \psi \in s$, $\varphi \in t$ and $\psi \notin t$. By the corresponding provable sequent we have that $\neg(\varphi \multimap \psi) \notin s$, and so $\varphi \multimap \psi \in s^*$. Hence, we conclude that not Qs^*t . (Lpostc) is established similarly.

(Rprec). If not Qs^*t , then there are φ, ψ such that $\varphi \multimap \psi \in s^*$, $\varphi \in t$ and $\psi \notin t$. By $\varphi \multimap \psi \in s^*$ we have that $\neg(\varphi \multimap \psi) \notin s$, and by the corresponding provable sequent we have that $\varphi \multimap \psi \in s$. Hence, we conclude that not Qst . (Rpostc) is established similarly. \square

Theorem 7.9 (FDE_c³ soundness and completeness).

$$\varphi \vdash_{\text{FDE}_c^3.X} \psi \text{ iff } \varphi \models^{\text{FDE}_c^3.X} \psi.$$

Proof. One direction follows from Theorem 7.7 and Lemma 7.9, while the other follows from Theorem 7.8 and Lemma 7.10. \square

7.3.3 Embedding Levesque's Logic in FDE_c^3

In this section I show that Levesque's logic of explicit and implicit belief embeds into FDE_c^3 . The embedding of BL into FDE_c^3 shows that the latter is a generalisation of Levesque's original logic, thus confirming that FDE_c^3 offers an alternative approach to contextual modal logics. Moreover, in FDE_c^3 one is able to express the distinction between explicit and implicit beliefs not only for formulas without nested beliefs, but for formulas where \Box, \Box_I occur at any depth.

The embedding procedure comprises a series of steps. First, I introduce Levesque's original axiomatic system and define a (generalised) version of its semantic generating the same logic BL (in the language $\mathcal{L}_{(BL)}$, consisting of all formulas of \mathcal{L}_{BL} that do not contain nested instances of B and L). Second, I define a translation t from $\mathcal{L}_{(BL)}$ to \mathcal{L}_{SEnt} . Third, I show how to construct from the canonical generalised Levesque model \mathfrak{G} a FDE_c^3 -model \mathfrak{G}^* such that a formula $\varphi \in \mathcal{L}_{(BL)}$ is valid in \mathfrak{G} if and only if $c \vdash t(\varphi)$ is valid in \mathfrak{G}^* . This, together with the completeness results for BL and FDE_c^3 , provides an embedding result of BL into FDE_c^3 .

Definition 7.21 (BL-models). *A Levesque model (BL-model) is a structure of the following form:*

$$(S, W, B, T, F)$$

- $V^+, V^- : At \rightarrow \mathcal{P}(S)$ are truth and falsity valuations;
- $W \subseteq S$ is a set of possible worlds such that $w \in W$ implies $w \in V^+(p) \iff w \notin V^-(p)$;
- $B \subseteq S$ is a set of epistemically accessible situations.

Let **BL** be the class of all BL-models.

A BL-model is an american-style possible-worlds framework, consisting of a set of situations, together with: (i) a set B corresponding to the agent's belief state and containing situations supporting believed information; and (ii) a set of possible worlds W containing complete and consistent situations, for which truth and falsity valuations are mutually exclusive and exhaustive. Note that, in contrast with [111], W is not assumed to contain *all* complete and consistent situations.

Definition 7.22 (BL-satisfaction). *Let positive and negative satisfaction in a BL-model \mathfrak{G} be recursively defined as follows:*

$$\begin{array}{lll} \mathfrak{G}, s \models^t p & \iff & s \in V^+(p) \\ \mathfrak{G}, s \models^f p & \iff & s \in V^-(p) \end{array}$$

$\mathfrak{G}, s \models^t \neg\varphi$	\iff	$\mathfrak{G}, s \models^f \varphi$
$\mathfrak{G}, s \models^f \neg\varphi$	\iff	$\mathfrak{G}, s \models^t \varphi$
$\mathfrak{G}, s \models^t \varphi \wedge \psi$	\iff	$\mathfrak{G} \models^t \varphi$ and $\mathfrak{G}, s \models^t \psi$
$\mathfrak{G}, s \models^f \varphi \wedge \psi$	\iff	$\mathfrak{G}, s \models^f \varphi$ or $\mathfrak{G}, s \models^f \psi$
$\mathfrak{G}, s \models^t \varphi \vee \psi$	\iff	$\mathfrak{G}, s \models^t \varphi$ or $\mathfrak{G}, s \models^t \psi$
$\mathfrak{G}, s \models^f \varphi \vee \psi$	\iff	$\mathfrak{G}, s \models^f \varphi$ and $\mathfrak{G}, s \models^f \psi$
$\mathfrak{G}, s \models^t B\varphi$	\iff	$\forall t \in B(\mathfrak{G}, t \models^t \varphi)$
$\mathfrak{G}, s \models^f B\varphi$	\iff	$\mathfrak{G}, s \not\models^t B\varphi$
$\mathfrak{G}, s \models^t L\varphi$	\iff	$\forall t \in B \cap W(\mathfrak{G}, t \models^t \varphi)$
$\mathfrak{G}, s \models^f L\varphi$	\iff	$\mathfrak{G}, s \not\models^t L\varphi$

Definition 7.23 (BL–validity). *Let a formula φ be valid in a BL–model \mathfrak{G} – denoted $\mathfrak{G} \models \varphi$ – iff $\mathfrak{G}, w \models^t \varphi$ for all $w \in W$. Let $\varphi \in \mathbf{BL}$ denote that a formula is valid in all BL–model $\mathfrak{M} \in \mathbf{BL}$.*

The axiom system for BL is formulated by extending classical propositional logic, so the material conditional \supset plays the role of the turnstile \vdash . Moreover, formulas in the scope of B are closed under consequence in FDE, as recorded by Lemma [7.3.3](#). Finally, it is easily seen that contraposition is an admissible rule of inference of BL.

Definition 7.24 (BL–axiom system). *Let the Hilbert system for BL consist of the following axioms and rules:*

- (BL1) *Axioms and rules of CPC*
- (BL2) $B(\varphi \wedge \psi) \equiv B(\psi \wedge \varphi)$
- (BL3) $B(\varphi \vee \psi) \equiv B(\psi \vee \varphi)$
- (BL4) $B(\varphi \wedge (\psi \wedge \chi)) \equiv B((\varphi \wedge \psi) \wedge \chi)$
- (BL5) $B(\varphi \vee (\psi \vee \chi)) \equiv B((\varphi \vee \psi) \vee \chi)$
- (BL6) $B(\varphi \wedge (\psi \vee \chi)) \equiv B((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
- (BL7) $B\neg(\varphi \vee \psi) \equiv B(\neg\varphi \wedge \neg\psi)$
- (BL8) $B\neg(\varphi \wedge \psi) \equiv B(\neg\varphi \vee \neg\psi)$
- (BL9) $B\neg\neg\varphi \equiv B\varphi$
- (BL10) $B\varphi \wedge B\psi \equiv B(\varphi \wedge \psi)$
- (BL11) $B\varphi \vee B\psi \supset B(\varphi \vee \psi)$
- (BL12) $B\varphi \supset L\varphi$

- (BL13) $L\varphi \wedge L(\varphi \supset \psi) \supset L\psi$
 (BL14) $\varphi \Rightarrow L\varphi$
 (BL15) $(B\varphi \vee B\psi) \supset B\chi \Rightarrow B(\varphi \vee \psi) \supset B\chi$

Lemma 7.11 (FDE-BL). $\varphi \vdash_{\text{FDE}} \psi \Rightarrow \vdash_{\text{BL}} B\varphi \supset B\psi$

Proof. To prove this lemma, it is useful to switch to an equivalent formula-formula system for FDE that does not have the contraposition rule (see e.g. Dunn's \mathbf{R}_{fde} [51]). The lemma is then established by a straightforward induction on the length of proofs. \square

The proof of soundness and completeness of BL with respect to BL-models is an adaptation of the succinct proof of [112].

Theorem 7.10 (BL-soundness). $\vdash_{\text{BL}} \varphi \Rightarrow \varphi \in \text{BL}$.

Proof. The result is established by induction on the length of BL-proofs, where the only interesting cases are those involving L , (BL12)-(BL15).

(BL12). Assume for all $w \in W$ that $w \models^t B\varphi$ iff for all $t \in B$ $t \models \varphi$. Hence for all $t \in B \cap W$ we have that $t \models \varphi$, by which we conclude that $w \models L\varphi$.

(BL13). Assume for all $w \in W$ that $w \models^t L\varphi$ and $w \models L(\varphi \supset \psi)$ iff for all $t \in B \cap W$ $t \models \varphi$ and $t \models \varphi \supset \psi$. Hence for all $t \in B \cap W$ we have $t \models \psi$, by which we conclude that $w \models L\psi$.

(BL14). Assume $w \models^t \varphi$ for all $w \in W$. Hence for all $w \in W \cap B$ we have that $w \models^t \varphi$, by which we conclude that $\models^t L\varphi$.

(BL15). We preliminarily show the following claim: for all BL-models $\mathfrak{G}_1 = (S, W, B_1, V^+, V^-)$ and $\mathfrak{G}_2 = (S, W, B_2, V^+, V^-)$ and $\mathfrak{G} = (S, W, B_1 \cup B_2, V^+, V^-)$ and all $w \in W$, if $\mathfrak{G}_1, w \models^t B\chi$ and $\mathfrak{G}_2, w \models^t B\chi$, then $\mathfrak{G}, w \models B\chi$. The proof of the claim is immediate: χ is a non-modal formula and $\mathfrak{G}_1, s \models^t \chi$ iff $\mathfrak{G}_2, s \models^t \chi$ iff $\mathfrak{G}, s \models^t \chi$ for non-modal χ .

Then, assume that $B\varphi \vee B\psi \supset B\chi$ is valid in all models. Take a model $\mathfrak{G} = (S, W, B, V^+, V^-)$, and assume that $\mathfrak{G}, w \models^t B(\varphi \vee \psi)$ for some $w \in W$, i.e.

$$B \subseteq \underbrace{\{s \in S \mid \mathfrak{G}, s \models^t \varphi\}}_{X_1} \cup \underbrace{\{s \in S \mid \mathfrak{G}, s \models^t \psi\}}_{X_2}.$$

Define $\mathfrak{G}_1 = (S, W, B_1, V^+, V^-)$ and $\mathfrak{G}_2 = (S, W, B_2, V^+, V^-)$ where $B_1 = B \cap X_1$ and $B_2 = B \cap X_2$. Both \mathfrak{G}_1 and \mathfrak{G}_2 are BL-models. By our assumption, $\mathfrak{G}_1 \models^t B\varphi \vee B\psi \supset B\chi$ and $\mathfrak{G}_2 \models^t B\varphi \vee B\psi \supset B\chi$. Importantly, $\mathfrak{G}_1, w \models^t B\varphi$ and $\mathfrak{G}_2, w \models^t B\psi$ by the respective definitions of B_1 and B_2 . Hence, $\mathfrak{G}_i, w \models^t B\varphi \vee B\psi$ and so $\mathfrak{G}_i, w \models^t B\chi$ for $i \in \{1, 2\}$ by the assumption. Moreover, clearly $B = B_1 \cup B_2$ (since $B \subseteq X_1 \cup X_2$). We conclude by our initial claim that $(G, w) \models^t B\chi$.

□

Let the propositional fragment of $\mathfrak{L}_{(\text{BL})}$ be denoted as $\mathfrak{L}_{\text{Prop}}$.

Definition 7.25 (Canonical Γ -model). *Let Γ be a maximally consistent BL-theory. The canonical BL-model is a structure of the following form:*

$$\mathfrak{G}^\Gamma = (S, W, B, V^+, V^-)$$

- $S^\Gamma = \{s \mid s \text{ is a prime FDE-theories over } \mathfrak{L}_{\text{Prop}}\}$;
- $W^\Gamma = \{s \mid s \text{ is a maximally consistent FDE-theories over } \mathfrak{L}_{\text{Prop}}\}$;
- $B^\Gamma = \{s \in S^\Gamma \mid \exists \chi \in \mathfrak{L}_{\text{Prop}}(\chi \in s, \neg \chi \in s), \forall \psi \in \mathfrak{L}_{\text{Prop}}(B\psi \in \Gamma \implies \psi \in s)\} \cup \{w \in W^\Gamma \mid \forall \psi \in \mathfrak{L}_{\text{Prop}}(L\psi \in \Gamma \implies \psi \in w)\}$;
- $V_\Gamma^+(p) = \{s \in S^\Gamma \mid p \in s\}$;
- $V_\Gamma^-(p) = \{s \in S^\Gamma \mid \neg p \in s\}$.

Since Γ is a maximal consistent BL-theory, $\Gamma \cap \mathfrak{L}_{\text{Prop}}$ is a complete and consistent prime FDE-theory over $\mathfrak{L}_{\text{Prop}}$, hence $u = \Gamma \cap \mathfrak{L}_{\text{Prop}} \in W^\Gamma$. Moreover, clearly $\mathfrak{G}^\Gamma \in \mathbf{BL}$, since every complete and consistent FDE-theory u is such that $\neg \varphi \in u$ iff $\varphi \notin u$.

Lemma 7.12 (BL truth). *The following hold for all \mathfrak{G}^Γ :*

1. For all $\varphi \in \mathfrak{L}_{\text{Prop}}$: $\varphi \in s$ iff $\mathfrak{G}^\Gamma, s \models^t \varphi$ and $\neg \varphi \in s$ iff $\mathfrak{G}^\Gamma, s \models^f \varphi$;
2. For all $\varphi \in \mathfrak{L}_{(\text{BL})}$: $\varphi \in \Gamma \iff (\mathfrak{G}^\Gamma, u) \models^t \varphi$.

Proof. Item 1 is established by double induction on the complexity of φ , as in Lemma 7.8. To prove the Item 2, we reason by cases. First, if φ is a propositional formula, then we need to prove $\varphi \in u$ iff $u \models^t \varphi$. This is an instance of the Item 1.

$\varphi := B\chi$. (\implies). If $B\chi \in \Gamma$, then $\chi \in t$ for all $t \in B$ by (BL12). By Item 1 we infer that $t \models^t \chi$ for all $t \in B$. We conclude that $u \models^t B\chi$. (\impliedby). If $B\chi \notin \Gamma$, then we reason as follows. We claim that there is $s \in B$ such that $\chi \notin s$. $u \not\models^t B\chi$ follows from this claim by Item 1. The claim is established as follows. It can be shown that the pair $(\{\psi \mid B\psi \in \Gamma\} \cup \{p \wedge \neg p\}, \{\chi\})$ is FDE-independent if p is any variable not occurring in χ . If it were not, then there would be ψ_1, \dots, ψ_n such that $B\psi_1, \dots, B\psi_n \in \Gamma$

and $\psi_1 \wedge \dots \wedge \psi_n \wedge p \wedge \neg p \vdash_{\text{FDE}} \chi$. But then $\psi_1 \wedge \dots \wedge \psi_n \vdash_{\text{FDE}} \chi$ ¹⁵. It follows that $\vdash_{\text{BL}} B\psi_1 \wedge \dots \wedge B\psi_n \supset B\chi$, which contradicts the assumption that $B\chi \notin U$. Given that the pair is FDE-independent, it follows that there is an inconsistent $s \in S$ such that $\{\psi \mid B\psi \in \Gamma\} \subseteq s$, but $\chi \notin s$.

$\varphi := L\chi$. (\implies). If $L\chi \in \Gamma$, then $\chi \in t$ for all $t \in B \cap W$ by the definition of B . Hence, $u \models^t L\chi$ by the induction hypothesis. (\impliedby). If $L\chi \notin \Gamma$, then we reason as follows. We claim that there is a $w \in B \cap W$ such that $\chi \notin w$. $u \not\models^t L\chi$ follows from this claim by Item 1. The claim is established as follows. If $\chi \in w$ for all $w \in B \cap W$, then χ is entailed in classical logic by the set $\{\psi \mid L\psi \in \Gamma\}$ (since W contains all classical theories over $\mathfrak{L}_{\text{Prop}}$). By compactness of classical logic, $\psi_1 \wedge \dots \wedge \psi_n \supset \chi$ is a theorem of classical logic for some $\{\psi_1, \dots, \psi_n\} \subseteq \{\psi \mid L\psi \in \Gamma\}$. Hence, $\vdash_{\text{BL}} L\psi_1 \wedge \dots \wedge L\psi_n \supset L\chi$ by (BL13)-(BL14). But this contradicts the assumption that $L\chi \notin \Gamma$, which is a BL-theory.

The final case is the case when φ is obtained using the propositional connectives \neg, \wedge, \vee from some $\{p_1, \dots, p_n\} \cup \{B\psi_1, \dots, B\psi_m\} \cup \{L\chi_1, \dots, L\chi_k\}$. But then the claim is established easily by induction on the structure of φ . The base cases where φ is a propositional variable or a modal formula of the form $B\psi$ or $L\chi$ were established in the previous claims. The rest is straightforward. (If $\varphi := \neg\psi$, then the argument makes use of the fact that u is a complete and consistent situation in G^Γ and so $u \not\models^t \chi$ iff $u \models^f \chi$ for all $\chi \in \mathfrak{L}_{(\text{BL})}$.) \square

Theorem 7.11 (BL soundness and completeness). $\varphi \in \mathbf{BL} \implies \vdash_{\text{BL}} \varphi$.

Proof. The result follows from the fact that $\mathfrak{G}^\Gamma \in \mathbf{BL}$ and Lemma 7.12. \square

It is possible to leverage on the canonical Γ -model used in the completeness proof of BL in the embedding proof. Indeed, each canonical Γ -model \mathfrak{G} can be transformed into an FDE_c^3 -model \mathfrak{G}^* such that $\varphi \in \mathfrak{L}_{(\text{BL})}$ is valid in \mathfrak{G} iff $t(\varphi) \in \mathfrak{L}_{\text{SEnt}}$ is valid in \mathfrak{G}^* , for a natural translation function t that takes $B\varphi$ into $\Box t(\varphi)$ and $L\varphi$ into $\Box_I t(\varphi)$. In effect, \mathfrak{G}^* is the canonical BL-model \mathfrak{G} , where the negative valuation V^- is replaced by the Routley star operation $*$.

¹⁵In general, if p does not occur in χ , then $\psi \wedge (p \wedge \neg p) \vdash_{\text{FDE}} \chi$ entails $\psi \vdash_{\text{FDE}} \chi$. If the latter fails for some ψ, χ , then completeness of FDE entails that there is an FDE-model \mathfrak{M} such that $\mathfrak{M}, s \models^t \psi$ and $\mathfrak{M}, s \not\models^t \chi$. Now change s to s' by setting $s' \models^t p$ and $s' \models^f p$. An easy proof by induction on complexity of formulas shows that $s' \models^t \psi$ and $s' \not\models^t \chi$.

Definition 7.26 (c-translation). *Let the c-translation function $t : \mathfrak{L}_{(\text{BL})} \rightarrow \mathfrak{L}_{(\text{SEnt})}$ as follows, where $\mathfrak{L}_{(\text{SEnt})}$ is the subset of $\mathfrak{L}_{\text{SEnt}}$ without $\&$ where c, \top only occur as antecedents of \neg -formulas:*

$$\begin{aligned} t(p) &= p \\ t(\neg\varphi) &= \neg t(\varphi) \\ t(\varphi \wedge \psi) &= t(\varphi) \wedge t(\psi) \\ t(\varphi \vee \psi) &= t(\varphi) \vee t(\psi) \\ t(B\varphi) &= \top \neg t(\varphi) \\ t(L\varphi) &= c \neg t(\varphi) \end{aligned}$$

Definition 7.27 (Embedding model). *For every canonical BL-model $\mathfrak{G} = (S, B, W, V^+, V^-)$, let $\mathfrak{G}^* = (S, W, *, Q, V)$ be defined as follows:*

- $s^* = \{\varphi \mid \neg\varphi \notin s\}$;
- Qst iff $t \in B$;
- $s \in V(p)$ iff $p \in s$.

Lemma 7.13 (Embedding equivalence). *For all canonical BL-models \mathfrak{G} :*

$$\mathfrak{G} \models \varphi \text{ iff } c \models^{\mathfrak{G}^*} t(\varphi).$$

Proof. The result follows immediately from the following claim $\mathfrak{G}^*, s \models t(\varphi) \iff \mathfrak{G}, s \models^t \varphi$ and $\mathfrak{G}^*, s^* \not\models t(\varphi) \iff \mathfrak{G}, s \models^f \varphi$. The claim is established by double induction on the complexity of φ . We show only the base case and the induction step for cases $\varphi := \neg\psi$ and $\varphi := B\psi$ as an illustration.

$\varphi \in At$. The following chain of equivalences holds: $s \in V^+(p)$ iff (by Definition 7.26) $s \in V^+(t(p))$ iff (by Definition 7.25) $t(p) \in s$ iff (by Definition 7.27) $s \in V(t(p))$; and $s \in V^-(p)$ iff (by Definition 7.25) $\neg p \in s$ iff (by Definition 7.26) $\neg t(p) \in s$ iff (by definition of $*$ in G^*) $t(p) \notin s^*$ iff (by Definition 7.27) $s^* \notin V(t(p))$.

$\varphi := \neg\psi$. The following chain of equivalences holds: $\mathfrak{G}, s \models^t \neg\psi$ iff $\mathfrak{G}, s \models^f \psi$ iff (by induction hypothesis) $\mathfrak{G}^*, s^* \not\models t(\psi)$ iff $\mathfrak{G}^*, s \models \neg t(\psi)$ iff (by Definition 7.26) $\mathfrak{G}^*, s \models t(\neg\psi)$; and $\mathfrak{G}, s \models^f \neg\psi$ iff $\mathfrak{G}, s \models^t \psi$ iff (by induction hypothesis) $\mathfrak{G}^*, s \models t(\psi)$ iff $\mathfrak{G}^*, s^* \not\models \neg t(\psi)$ iff (by Definition 7.26) $\mathfrak{G}^*, s^* \not\models t(\neg\psi)$.

$\varphi := B\psi$. The following chain of equivalences holds: $\mathfrak{G}, s \models^t B\psi$ iff $\mathfrak{G}, t \models^t \psi$ for all $t \in B$ iff (by Definition 7.27) $\mathfrak{G}, t \models^t \psi$ for all t such that Qst iff (by induction hypothesis) $\mathfrak{G}^*, t \models t(\psi)$ for all t such that Qst iff $\mathfrak{G}^*, s \models \top \neg t(\psi)$ iff (by Definition

7.26 $\mathfrak{G}^*, s \models t(B\psi)$; and $\mathfrak{G}, s \models^f B\psi$ iff $\mathfrak{G}, s \not\models^t B\psi$ iff there is $t \in B$ such that $\mathfrak{G}, t \not\models^t \psi$ iff (by induction hypothesis) there is $t \in B$ such that $\mathfrak{G}^*, t \not\models t(\psi)$ iff (by Definition **7.27**) there is t such that Qs^*t and $\mathfrak{G}^*, t \not\models t(\psi)$ iff $\mathfrak{G}^*, s^* \not\models \top \neg t(\psi)$ iff (by Definition **7.26**) $\mathfrak{G}^*, s^* \not\models t(B\psi)$.

□

Theorem 7.12 (Embedding). $\vdash_{\text{BL}} \varphi \iff c \vdash_{\text{FDE}_c^3} t(\varphi)$.

Proof. (\implies). It suffices by Theorems **7.10** and **7.8** to show the claim that the translation of each BL-axiom (rule of inference) is c -valid (preserves c -validity) in FDE_c^3 -models. The claim is established by induction on the length of BL-proofs. The cases for (BL2)–(BL11) follow from the fact that $\varphi \models^{\text{FDE}} \psi$ only if $c \models^{\text{FDE}_c^3} \Box\varphi \supset \Box\psi$. For (BL12)–(BL15), we reason as in Theorem **7.10**. Conversely, assume $\not\vdash_{\text{BL}} \varphi$. By Theorem **7.11** we have that $\mathfrak{G} \not\models \varphi$, where \mathfrak{G} is the canonical BL-model such that $\varphi \notin \Gamma$. By Lemma **7.13**, \mathfrak{G}^* is a FDE_c^3 -model such that $c \not\models^{\mathfrak{G}^*} t(\varphi)$. We conclude by Theorem **7.7** that $c \not\vdash_{\text{FDE}_c^3} t(\varphi)$. □

Chapter 8

Contextual Conditional Logic with Relevant Updates

In this chapter, I extend the expressivity of contextual modal logics by introducing dynamic modalities, used to model belief revision. To this aim, in Section 8.1 I briefly sketch two static approaches to belief revision, the AGM theory and classical conditional logic. In Section 8.2 I argue that belief revision is best tackled within a dynamic and relevant setting, and I recast dynamic contextual modal logic as a conditional logic.

8.1 Two approaches to Conditional Reasoning

The *locus classicus* linking conditional reasoning and belief revision is a famous footnote from [150]:

If two people are arguing “If p will q ?” and are both in doubt as to p , they are adding p hypothetically to their stock of knowledge and arguing on that basis about q ; so that in a sense “If p , q ” and “If p , $\neg q$ ” are contradictories. We can say they are fixing their degrees of belief in q given p . [150, p.247]

The significance of the above passage for a theory of conditionals has been variously interpreted. For the present purposes, it is interesting to note that the so-called *Ramsey test* seems to recommend that conditional reasoning involve a form of belief revision, so that $\varphi > \psi$ is acceptable whenever φ provides reasons for ψ . However, the rational support antecedents grant to consequents of so-called *Ramsey conditionals* has been spelled out differently across logical communities, e.g. by using

probabilistic models (cf. [2, 46]), possible worlds models (cf. [103, 115, 188]) and syntactic belief revision models (cf. [3, 68]). In what follows, I review two influent approaches to revision theory, i.e. AGM belief revision and counterfactual analyses of conditionals.

8.1.1 AGM Belief Revision

Originating in the seminal [3], AGM belief revision has been a powerful and elegant theory to study the changes of agents' beliefs. According to the AGM model, beliefs are represented as unstructured databases, called *belief states*, consisting of sets of sentences in the propositional language \mathcal{L}_{Ent} , where as usual $\varphi \rightarrow \psi := \neg\varphi \vee \psi$ and $\perp := p \wedge \neg p$ for some arbitrary $p \in \text{At}$. These sets are assumed to be closed under logical consequence relative to a given logic L , so in effect any belief state s is an L -theory (cf. Definition 2.11). In the paradigmatic belief revision theory, the logical consequence under consideration is *supraclassical*, i.e. L is a Tarskian consequence relation – recall: reflexive, transitive and monotonic – at least strong as CPC (cf. [118]). Despite the many recent developments and modifications (cf. [67]), the original AGM theory focused on three types of change in belief states: *expansion*, *contraction* and *revision*.

Definition 8.1 (AGM model). *Let an AGM model be a structure of the following form:*

$$(S, +, *, -)$$

- $S \subseteq \mathcal{P}(\mathcal{L}_{\text{Ent}})$ is a set of belief states;
- $+$: $S \times \mathcal{L}_{\text{Ent}} \rightarrow S$ is the expansion function, defined by $s + \varphi := \text{Th}(s \cup \{\varphi\})$;
- $*$: $S \times \mathcal{L}_{\text{Ent}} \rightarrow S$ is the revision function, satisfying the axioms of Figure 8.1;
- $-$: $S \times \mathcal{L}_{\text{Ent}} \rightarrow S$ is the contraction function, defined by $s - \varphi := (s * \varphi) \cap s$.

In AGM models, expansion is easily characterised as the minimal change which results from adding a formula to a belief set, and closing it under logical consequence. Contraction and revision, however, are more subtle operations. For a start, note that in the above AGM models contraction is defined in terms of revision by $s - \varphi := (s * \neg\varphi) \cap s$, referred to as the *Harper identity*. However, the opposite route is just as viable, i.e. revision could be definable by a primitive contraction operator via the *Levi identity* $s * \varphi := (s - \neg\varphi) + \varphi$. Next, both revision and contraction can be characterised in two ways, either (i) by direct construction or (ii) by means of a list of syntactic postulates. Both methods are guided by the intuition that contraction (revision) changes a belief state s according to a principle of informational economy:

(*1) $s * \varphi = Th(s \cup \{\varphi\})$	(Closure)
(*2) $\varphi \in s * \varphi$	(Success)
(*3) $s * \varphi \subseteq s + \varphi$	(Inclusion)
(*4) $s \not\vdash_L \neg\varphi \implies s + \varphi \subseteq s * \varphi$	(Vacuity)
(*5) $\varphi \not\vdash_L \perp \implies s * \varphi \not\vdash_L \perp$	(Consistency)
(*6) $\vdash_L \varphi \leftrightarrow \psi \implies s * \varphi = s * \psi$	(Extensionality)
(*7) $s * \varphi \vee \psi = \begin{cases} s * \varphi \text{ or} \\ s * \psi \text{ or} \\ s * \varphi \cap s * \psi \end{cases}$	(Disjunctive factoring)

Figure 8.1: AGM postulates for revision $*$.

it removes (adds) an input sentence φ whenever φ is contingent in s (φ preserves the consistency of s). In what follows, it will be more useful to focus on the axiomatic approach.¹

Focusing on the characteristic postulates of revision, the axioms (*1)–(*7) listed in Figure 8.1 usually form the basis of the AGM theory. Axioms (*1)–(*7) encapsulate a number of assumptions on revision. (*1) sanctions that belief states are theories; (*2) states that inputs are always accepted by revision; (*3)–(*4) guarantee that expansion give lower and upper bounds of revision, when the input is consistent with the belief state; (*5) requires belief revision to preserve consistency, even when the starting state is inconsistent; (*6) states that revision is indifferent

¹Following the other route, $s - \varphi$ and $s * \varphi$ can be roughly constructed as follows (cf. [3]). Start from a belief state s . In a nutshell, the contraction $s - \varphi$ should be the largest subtheory of s not containing φ . To this aim, consider the set $s \perp \varphi$ of all *remainders of s by φ* , i.e. all maximal subsets of s not implying φ . If $s \perp \varphi = \emptyset$, it means that φ is a tautology, and therefore contraction is not successful, yielding $s - \varphi = s$. Otherwise, select a subset $\gamma(s \perp \varphi)$ consisting of the most entrenched remainders of s by φ , according to some transitive entrenchment relation. The *transitively relational partial meet contraction* of s by φ is then obtained by setting $s - \varphi = \bigcap \gamma(s \perp \varphi)$ (cf. [68]). The corresponding revision operator $*$ is obtained from a transitive relational partial meet contraction via the Levi identity. Finally, [3] prove that $*$ is a transitive relational partial meet revision if and only if $*$ satisfies (*1)–(*7).

to equivalent inputs. The last axiom (*7) intuitively states that revisions with $\varphi \vee \psi$ is just revision with the more entrenched between the two disjuncts or (if they are indifferent) the common outcomes of revising by each disjunct.

Despite the elegance and simplicity of the theory, AGM belief revision was found ill suited to deal with conditionals. Indeed, once belief sets are allowed to contain conditional sentences $\varphi > \psi$, a tension arises between the Ramsey test, formulated in AGM models as

$$\varphi > \psi \in s \iff \psi \in s * \varphi \quad (\text{Ramsey test})$$

and the AGM postulates. In particular, in presence of the other AGM postulates, (*4) is equivalent to

$$s \not\vdash_L \neg\varphi \implies s \subseteq s * \varphi \quad (\text{Preservation})$$

which expresses the intuitive conservativity principle stating that, during revision, you should continue to believe as many of the old beliefs as possible (cf. [85]). Now, the main result of [74] was showing that (*Ramsey Test*) and (*Preservation*), together with (*2) and (*5) lead to contradiction. This result, which famously goes under the name of Gärdenfors' triviality theorem, has stimulated a variety of responses, ranging from (i) the denial of genuine truth-conditions for conditional statements, (ii) weakened versions of (*Ramsey Test*) and (iii) modifications of AGM models and postulates of revision (cf. [110] for an overview). Besides Gärdenfors' triviality theorem, for the present purposes I wish to consider two general sources of criticism of AGM belief revision.

1. *Revision properties.* A critical feature of the AGM theory is that it does not subscribe to intuitive principles of revision. For example, the very fact that belief states are taken to be (supra)classical theories – a fact which is preserved through revision by (*1) – should raise an eyebrow. By this assumption, agents' beliefs are closed under *classical* logical consequence, i.e. they suffer from the most severe form of logical omniscience. Non-classical AGM belief revision tries to alleviate the problem by considering belief states as closed under e.g. paraconsistent logic, contra (*5) (cf. [72, 107, 119, 144]). Besides (*1), other axioms have received their share of criticism. For example, [40] correctly points out that the success postulate (*2) seems to place an inordinate value on novelty, and its behaviour towards what is learned seems capricious. For another example, (*3) and (*4) jointly imply the controversial claim that revision reduces to expansion, if the input is consistent with the belief state (cf. [98]).

2. *Semantic, modal, dynamic analysis.* According to a second source of criticism, the AGM theory is not able to give a semantic analysis of the modal and dynamic features of belief revision. In particular, AGM models do not provide a semantic analysis of Ramsey conditionals, since they do not provide truth conditions, but only acceptability conditions, for a conditional. Moreover, revision in AGM only changes agents' commitments towards ontic facts, modeled by propositional formulas. However, genuine belief revision triggers a change in the stock of the very beliefs previously held. That is, the AGM framework only models the revision of beliefs towards propositional information and not revision of agents' (higher order) beliefs, since belief is not included in the formal language. For this reason, AGM belief revision was taken to capture only a *static* style of revision, whence $s * \psi$ would contain the agent's revised beliefs about the old state of the world s , as it was before the revision. This remark is fleshed out in [11, 185], where AGM belief revision is translated, under some epistemic assumptions, into classical modal logic and Gärdenfors' triviality theorem is avoided by defining $\varphi > \psi := [\varphi]\Box\psi$ in the context of public announcement logic (cf. Section 8.2.1).

8.1.2 Variably strict Conditionals

In order to devise genuine truth conditions for $\varphi > \psi$, I will argue that revision should be formalised along the lines of a *counterfactual conditional*. The philosophical literature on conditional logics analyses conditionals as variably strict operators within the context of possible world semantics for modal logic (cf. [115, 188]). That is, $\varphi > \psi$ expresses the relative necessity of ψ with respect to the standards set by φ . In particular, φ sets the standards under which to evaluate the necessity of ψ by selecting a range of accessible worlds, which can be interpreted as the most plausible, or most similar, φ -worlds with respect to the actual. The selection operation can be formally modeled in various ways, which give rise to a variety of formal conditional logics. Among these, one of the most general and flexible is Chellas–Seegerberg logic CK (cf. [35, 177]). From a semantic point of view, models for CK supplement to possible-worlds models for classical logic a Kripke *ternary* accessibility relation, which abstracts away from the specific properties of similarity or closeness.

Definition 8.2 (Chellas–Seegerberg models). *Let a Chellas–Seegerberg model (CS-model for short) be a structure of the following form:*

$$(S, C, V)$$

- S is a set of possible worlds;

- $C \subseteq S \times \mathcal{P}(S) \times S$ is a ternary accessibility relation, interpreting $>$;
- $V : At \rightarrow \mathcal{P}(S)$ is an atomic valuation.

Let **CS** denote the class of **CS**-models.

Definition 8.3 (CS-satisfaction). Let satisfaction of a formula $\varphi \in \mathcal{L}_{\text{Cond}}$ in a **CS**-model \mathfrak{M} be recursively defined as follows:

$$\begin{array}{lll}
\mathfrak{M}, s \models p & \iff & s \in V(p) \\
\mathfrak{M}, s \models \varphi \wedge \psi & \iff & \mathfrak{M}, s \models \varphi \text{ and } \mathfrak{M}, s \models \psi \\
\mathfrak{M}, s \models \varphi \vee \psi & \iff & \mathfrak{M}, s \models \varphi \text{ or } \mathfrak{M}, s \models \psi \\
\mathfrak{M}, s \models \neg\varphi & \iff & \mathfrak{M}, s \not\models \varphi \\
\mathfrak{M}, s \models \varphi \rightarrow \psi & \iff & \mathfrak{M}, s \not\models \varphi \text{ or } \mathfrak{M}, s \models \psi \\
\mathfrak{M}, s \models \varphi > \psi & \iff & \forall t \in S(Cs[\varphi]_{\mathfrak{M}}t \implies \mathfrak{M}, t \models \psi)
\end{array}$$

Let validity and consequence in **CS**-models be defined classically, as usual. The semantic clause for $>$ can be taken to express a modal version of the Ramsey Test: a conditional $\varphi > \psi$ is true at a given state iff ψ is believed at all possible worlds compatible with φ which cannot be excluded at s . Intuitively, then, $Cs[\varphi]t$ expresses that the world t is doxastically accessible with respect to world s , according to the supposition that φ holds. According to the presentation of [200], **CS**-models can be simply read as classical Kripke models, endowed with denumerably many accessibility relations Q_φ for $\varphi \in \mathcal{L}_{\text{Cond}}$. As such, each accessibility relation Q_φ can be paired with a corresponding *indexed normal modality* \Box_φ such that $\Box_\varphi\psi$ is a notational variant for $\varphi > \psi$. A sound and complete axiomatisation of the logic of **CS**-models is then simply a normal multi-modal classical logic CPCK.

Definition 8.4 (CK-axiomatisation). Let the axiom system CK consist of:

- (CK1) An axiomatisation of classical propositional logic CPC
(CK2) $\varphi \leftrightarrow \psi \implies (\varphi > \chi) \leftrightarrow (\psi > \chi)$
(CK3) $(\varphi_1 \wedge \dots \wedge \varphi_n) \rightarrow \psi \implies ((\chi > \varphi_1) \wedge \dots \wedge (\chi > \varphi_n)) \rightarrow (\chi > \psi)$ for $n \geq 0$

If $>$ is seen as a revision operator, the logic of belief revision CK complies with the general tenets of AGM belief revision, while lifting some of its limitations.

1. *Comparison with AGM postulates.* The rule (CK2) can be taken as an equivalent version of (*6), stating that revision is not sensitive to syntactic differences. Next, the normality rule (CK3) prescribes that agents' beliefs after revision are closed under classical consequence, in line with (*1). Moreover, since every $t \in Q_\varphi(s)$ is a possible world, the doxastic state of an agent after revising with φ in \mathfrak{M} at s , which can be translated from AGM as $\{\psi \mid \forall t \in Q_\varphi(s)(\mathfrak{M}, t \models \psi)\}$, is consistent, thus complying to (*5). Finally, the success postulate (*2) is rightfully avoided, since $\varphi > \varphi$ is not a logical truth.
2. *Classical basis.* (CK3) also implies that revised beliefs states in CK suffer from logical omniscience principles such as:

$$\begin{aligned} (> \mathbf{K}) \quad & (\varphi > (\psi \rightarrow \chi)) \rightarrow ((\varphi > \psi) \rightarrow (\varphi > \chi)) \\ (> \mathbf{N}) \quad & \varphi \Rightarrow \psi > \varphi \end{aligned}$$

Besides exposure to logical omniscience, the fact that CK is based on CPC results in a failure to exclude from the stock of acceptable conditionals those which do not exhibit a *relevant connection* between antecedents and consequents (cf. [121]).

Motivated by the second of the above remarks, [121] adapted CK to a relevant setting. In a nutshell, the proposal is to impose a double constraint on the standards of necessity under which the truth of a conditional's consequent is to be evaluated. That is, Mares and Fuhrmann define a counterfactual conditional on the basis of (i) the relevant conditional \rightarrow and (ii) the relative modality $>$ ²

$$\varphi \gg \psi := \varphi > (\varphi \rightarrow \psi).$$

Recall from Chapter 2 that the Routley–Meyer ternary accessibility relation R is used to interpret \rightarrow in terms of information combination. Then, a state s intuitively satisfies $\varphi \gg \psi$ iff for all t doxastically accessible on the supposition that φ , combining t with a φ -state produces ψ -outputs. Recall that basing modal logic on relevant logic instead of classical logic results in weakening normal modal logic to conjunctively regular modal logic. Hence, it is no surprise that the relevant conditional logic ConR is just a multi-modal relevant logic RC, with conjunctively regular indexed modalities $\{\Box_{\varphi_i}\}_{i \in |\mathcal{L}_{\text{Cond}}|}$ such that $\Box_\varphi \psi$ is a notational variant for $\varphi > \psi$.

²Note that Mares and Fuhrmann's definition of the conditional is a bit cumbersome and difficult to parse. As such, the corresponding truth clause for $\varphi \gg \psi$ ceases to provide an adequate translation of the Ramsey Test.

Definition 8.5 (ConR-axiomatisation). *Let the axiom system ConR consist of:*

- (ConR1) *Axioms and rules of the relevant propositional logic R (cf. Figure 2.4);*
 (ConR2) $\varphi \leftrightarrow \psi \Rightarrow (\varphi > \chi) \leftrightarrow (\psi > \chi)$
 (ConR3) $(\varphi_1 \wedge \dots \wedge \varphi_n) \rightarrow \psi \Rightarrow ((\chi > \varphi_1) \wedge \dots \wedge (\chi > \varphi_n)) \rightarrow (\chi > \psi)$ for $n \geq 1$

Thanks to the affixing rule of R, a variant of (ConR3) – with \gg in place of $>$ – is derivable. Therefore, revised belief states in ConR are closed only under the principles:

- $$\begin{aligned} (\gg \text{ C}) \quad & (\varphi \gg \psi) \wedge (\varphi \gg \chi) \rightarrow (\varphi \gg (\psi \wedge \chi)) \\ (\gg \text{ M}) \quad & \varphi \rightarrow \psi \Rightarrow (\chi \gg \varphi) \rightarrow (\chi \gg \psi) \end{aligned}$$

Some natural observations fall out of the definition of the logic ConR.

1. *Relevant basis.* Being based on relevant logic, ConR alleviates the logical commitments of belief states, thus serving the purpose of modeling a more realistic kind of belief revision than CK. Moreover, ConR inherits from the paraconsistency of R the fact that formulas believed after revision can be contradictories, contra (*5). Finally, conditionals $\varphi \gg \psi$ in ConR express also that φ is relevant to ψ . Possible specifications of the required relevant connection which do not appeal to relevant logic, are found in [46, 158].
2. *All-purpose logics.* However, ConR shares with other relevant formalisms the drawback that they perform poorly as *all-purpose* logics (cf. Section 5.3.2). To repeat the remarks of Chapter 1, despite relevant logics being attractive models of information handling in doxastic states, they constitute a radical departure from standard epistemic logics, which requires defending a non-classical approach to logic in general that goes beyond epistemic considerations.

8.2 Dynamic Conditional Logic for Belief Revision

To sum up, a satisfactory approach to belief revision should comply with four general tenets:

- (i). Belief revision is a conditional operation, encoded by Ramsey conditionals, which should be given appropriate truth conditions;
- (ii). Revision should be a dynamic operation, which changes beliefs about propositional formulas (ontic information) as well as beliefs about beliefs (doxastic information);
- (iii). Revision subscribes to weaker properties than the AGM postulates, which in particular take into account the relevance of the input to the output;
- (iv). Relevance considerations are limited to the belief revision process which underlies conditional reasoning.

As highlighted in Section [8.1](#), the Ramsey conditionals falling out of standard AGM belief revision theory and variably strict conditional logics fail to attain desiderata (i)–(iv). In particular, one reason why both frameworks are doomed to fail the task is that they both model the beliefs of agents concerning the facts of the world as it was *before* the revision. That is, they do not consider the *dynamic* change of the world which revision induces. This issue was tackled by Dynamic Epistemic Logic accounts of belief revision (cf. [44](#), [97](#), [128](#)), according to which revision involves changing beliefs about a changing situation.

In order to account for the changes caused by dynamic belief revision, in this section I will consider extensions of modal logics by the dynamic modality $[\varphi]\psi$. Within this context, I will define Ramsey conditionals as

$$\varphi > \psi := [\varphi]\Box\psi,$$

expressing that a contextually fixed agent believes ψ after updating their prior beliefs with φ . In what follows, I first introduce standard relational models for Relevant Public Announcement Logic RPAL. While these models turn out to be too strong for a logic of conditionals, I show how neighborhood models suffice to develop contextual modal logic based on weak RPAL. The framework developed in this section combines the best of AGM belief revision, relevant conditional logic and public announcement logic. In particular, it does so by (i) modeling propositional information as classical theories (like AGM); (ii) modeling the closure properties of Ramsey conditionals by relevant logic (like ConR); and (iii) modeling revision of belief states as a dynamic operation (like PAL).

8.2.1 Relevant Public Announcement Logic

Dynamic epistemic logic is a flourishing research field (cf. [43](#), [97](#), [128](#) for overviews), which aims at formally modeling different styles of information flow, with various

degrees of privacy and reliability. In this section, I will be concerned exclusively with the simplest communication scenario, the public announcement of information φ to a fixed agent, as studied by Public Announcement Logic PAL.

Public announcements are modeled in PAL through a model-transforming operator in two possible ways. According to one strategy, the announcement of φ transforms a classical Kripke model by eliminating all $\neg\varphi$ -worlds and all relations targeting $\neg\varphi$ -worlds (cf. [10, 139]). Alternatively, the *arrow modification* strategy contrasts the “world elimination” strategy by holding that the public announcement of φ has only the effect of eliminating all relations targeting $\neg\varphi$ worlds but leaves intact the world structure (cf. [77, 101]).

In recent years, the arrow modification strategy has been adapted to a relevant setting through the technique of *situated updates* (cf. [148, 172]). In the context of relational semantics, situated updates model the effect of an announcement as the *information update* of all the *situations* included in the agent’s epistemic state. Information updates, in turn, are modeled by the fusion operation \otimes (cf. Definition 2.7).

To get the model theory off the ground, it suffices to interpret the dynamic language $\mathfrak{L}_{\text{DRel}}$ in relational L-models (cf. Definition 2.6).

Definition 8.6 (Dynamic relational satisfaction). *Let the proposition expressed by an arbitrary formula $\varphi \in \mathfrak{L}_{\text{DRel}}$ in an L-model \mathfrak{M} – denoted $\llbracket \varphi \rrbracket_{\mathfrak{M}}$ – be recursively defined by extending Definition 2.8 as follows:*

$$\mathfrak{M}, s \models [\varphi]\psi \iff \mathfrak{M}^\varphi, s \models \psi$$

where $\mathfrak{M}^\varphi = (S, L, \leq, *, R, Q^\varphi, Q_L, V)$ is obtained from $\mathfrak{M} = (S, L, \leq, *, R, Q, Q_L, V)$ by setting:

$$Q^\varphi(s) := Q(s) \otimes^{\mathfrak{f}} \llbracket \varphi \rrbracket_{\mathfrak{M}} = \{v \in S \mid \exists t, u \in S(Qst, Rtu, \mathfrak{M}, u \models \varphi)\}.$$

Let validity and consequence be defined as usual, by L-validity and S-consequence (cf. Definition 3.5). Note that satisfaction is well defined, since the operation $(\cdot)^\varphi$ indeed maps an L-model into an L-model.

Compared to the static language, checking the satisfaction of a dynamic formula $[\varphi]\psi$ in a model \mathfrak{M} requires to check the satisfaction of ψ in the model \mathfrak{M}^φ . Given the definition of \mathfrak{M}^φ , this matters only for \Box -formulas. By unraveling the definition of $\llbracket [\varphi]\Box\psi \rrbracket$, we obtain:

$$\mathfrak{M}, s \models [\varphi]\Box\psi \iff Q(s) \otimes^{\mathfrak{f}} \llbracket \varphi \rrbracket_{\mathfrak{M}} \subseteq \llbracket \psi \rrbracket_{\mathfrak{M}^\varphi}.$$

Now, according to the informational interpretation of $\otimes^{\mathfrak{F}}$ (cf. Chapter 2), $X \otimes^{\mathfrak{F}} Y$ contains the information obtainable by the combination, through R , of X and Y . As a result, the following dynamic reading of $[\varphi]\Box\psi$ is afforded: ψ is believed at s after the announcement of φ iff it is believed that combining a previously believed information with a φ -information results in a ψ -information. That this amounts to a dynamic way of changing one's belief is confirmed by the fact the latter ψ information is checked to hold in the model \mathfrak{M} once \mathfrak{M} incorporates the changes induced by φ . Such changes, in turn, involve considering all the possible recombinations of information, operated via R , that φ -inputs may trigger. This informal explanation is made precise by the following Proposition.

Proposition 8.1 (Conditional Reduction). **BMC** $\models [\varphi]\Box\psi \leftrightarrow \Box(\varphi \rightarrow [\varphi]\psi)$.

Proof. (\implies). By Lemma 2.2, it suffices to show that for all $\mathfrak{M} \in \mathbf{BMC}$, $\llbracket [\varphi]\Box\psi \rrbracket_{\mathfrak{M}} = \llbracket \Box(\varphi \rightarrow [\varphi]\psi) \rrbracket_{\mathfrak{M}}$. The following chain of equivalences establishes the result:

$$\begin{aligned}
\mathfrak{M}, s \not\models [\varphi]\Box\psi &\iff \mathfrak{M}^{\varphi}, s \not\models \Box\psi \\
&\iff \exists v \in Q^{\varphi}(s)(\mathfrak{M}^{\varphi}, v \not\models \psi) \\
&\iff \exists t, u, v(Qst, Rtuv, \mathfrak{M}, u \models \varphi, \mathfrak{M}^{\varphi}, v \not\models \psi) \\
&\iff \exists t, u, v(Qst, Rtuv, \mathfrak{M}, u \models \varphi, \mathfrak{M}, v \not\models [\varphi]\psi) \\
&\iff \mathfrak{M}, s \not\models \Box(\varphi \rightarrow [\varphi]\psi)
\end{aligned}$$

□

Given the definition of the Ramsey conditional $\varphi > \psi := [\varphi]\Box\psi$, relevant public announcement logic can be meaningfully recast as a conditional logic, in which conditional reasoning involves a form of belief revision. In the resulting conditional logic, belief revision is performed along the prescriptions of relevant logic, since the properties of relevant implication regiment the properties of combination triggered by updates. This seems *prima facie* to comply with the desiderata (i)–(iii) I set out at the beginning of Section 8.2, thus constituting an ideal candidate logic on the basis of which to apply the contextual modal logic strategy.

Despite the intuitive appeal, however, this proposal immediately faces a problem with requirement (iii). One of the original motivations of the counterfactual approach to conditional logics was to devise an operator which was *non-monotonic*, so as to formalise defeasible reasoning, i.e. the reasoning of agents who might retract from previously held beliefs upon learning new information. A typical example of defeasible reasoning is the following: If upon learning that Tweety is a bird an agent believes that Tweety flies, one should not be warranted to learn that Tweety

is a bird and Tweety is a penguin while still rationally concluding to the belief that Tweety flies. Now, relevant public announcement logic is in the uncomfortable position that it gives the wrong verdict with respect to these paradigmatic non-monotonic inferences, at least in cases involving propositional formulas (like the example above).³

Proposition 8.2 (Monotonicity). *For all $\psi \in \mathfrak{L}$ not containing \square : $\varphi > \psi \models_{\mathbf{BMC}} (\varphi \wedge \chi) > \psi$.*

Proof. Assume that $\mathfrak{M}, s \models \varphi > \psi$ and $Q^{\varphi \wedge \psi} sv$ for some arbitrary $\mathfrak{M} \in \mathbf{BMC}$ and $s, v \in S$. We need to show that $\mathfrak{M}^{\varphi \wedge \chi}, v \models \psi$. By $Q^{\varphi \wedge \psi} sv$, we infer that there are $t, u \in S$ such that Qst, Rtw and $\mathfrak{M}, u \models \varphi \wedge \psi$. The latter implies $\mathfrak{M}, u \models \varphi$. Since there are $t, u \in S$ such that Qst, Rtw and $\mathfrak{M}, u \models \varphi$, by the assumption that $\mathfrak{M}, s \models \varphi > \psi$ we infer that $\mathfrak{M}^\varphi, v \models \psi$. We conclude by observing that, since ψ does not contain instances of \square , $\mathfrak{M}^\varphi, v \models \psi$ iff $\mathfrak{M}^{\varphi \wedge \chi}, v \models \psi$. \square

8.2.2 Neighborhood W-Models with Updates

As it turns out, the intuitive appeal of relevant public announcement logic as a logic of conditional statements can be salvaged. In this section, I show how to achieve desiderata (i)–(iv) for belief revision by devising contextual modal logic, obtained by combining classical propositional logic and relevant public announcement logic. From a technical point of view, the work of this section is an extension of Chapter 4 with dynamic modalities. In particular, I shall use basic neighborhood **C.FE**-models (cf. Section 4.3), recapitulated below, to interpret the dynamic language $\mathfrak{L}_{\mathbf{DRel}}$.

Definition 8.7 (**C.FE**-models). *Let a **C.FE**-model be a structure of the following form:*

$$(S, L, W, Prop, *, R, Q, Q_L, V)$$

- S is a non-empty set of information states;

³This counterintuitive conclusion for relevant public announcement logic can be, to some extent, resisted. Following [180], one may contend that the right notion of conjunction to be used in epistemic contexts is the intensional \otimes and not the extensional one \wedge . As a consequence, the right formalisation of the Tweety example is that $\varphi > \psi$ should not entail $(\varphi \otimes \chi) > \psi$. With respect to this formalisation, the logic gives the right verdict: if upon learning that Tweety is a bird an agent believes that Tweety flies, one may retract such belief only upon learning that Tweety is a bird and combining it with the information that Tweety is a penguin.

- $Prop \subseteq \mathcal{P}(S)$ is a set of admissible propositions, such that for $\otimes \in \{\wedge, \vee, \neg, \rightarrow, \square, \square_L\}$ and $\otimes^{\tilde{\delta}}$ defined as in Definition 4.2:

$$X_1, \dots, X_n \in Prop \implies \otimes^{\tilde{\delta}}(X_1, \dots, X_n) \in Prop$$

- $L \in Prop$ is the set of logical information states such that for all $X, Y \in Prop$:

$$X \subseteq Y \iff L \subseteq X \rightarrow^{\tilde{\delta}} Y$$

- $*$: $S \rightarrow S$ is the Routley star, interpreting \neg ;
- $R \subseteq S \times \mathcal{P}(S)^2$ is the neighborhood Routley–Meyer relation, interpreting \rightarrow ;
- $Q, Q_L \subseteq S \times \mathcal{P}(S)$ are neighborhood binary relations, interpreting \square, \square_L ;
- $V : At \rightarrow Prop$ is an atomic valuation.
- $W \subseteq L$ is a set of possible worlds, such that for all $w \in W$ and $X, Y \in \mathcal{P}(S)$:

$$\begin{aligned} w^* &= w \\ w \in X \rightarrow^{\tilde{\delta}} Y &\iff w \in \neg^{\tilde{\delta}} X \vee^{\tilde{\delta}} Y \\ W \subseteq \square_L^{\tilde{\delta}} X &\iff L \subseteq X \end{aligned}$$

For any **C.FE**–model $(S, L, W, Prop, *, R, Q, Q_L, V)$, let $(S, L, Prop, *, R, Q, Q_L, V)$ be the corresponding **FE**–model. Let **FE** and **C.FE** denote the class of all **FE**–models and **C.FE**–models.

Definition 8.8 (Dynamic **C.FE**–satisfaction). Let the satisfaction of a formula $\varphi \in \mathcal{L}_{DRel}$ in a **C.FE**–model \mathfrak{M} pointed at $s \in S$ be recursively defined as follows:

$$\begin{array}{lll} \mathfrak{M}, s \models p & \iff & s \in V(p) \\ \mathfrak{M}, s \models \mathbf{t} & \iff & s \in L \\ \mathfrak{M}, s \models \varphi \wedge \psi & \iff & \mathfrak{M}, s \models \varphi \text{ and } \mathfrak{M}, s \models \psi \\ \mathfrak{M}, s \models \varphi \vee \psi & \iff & \mathfrak{M}, s \models \varphi \text{ or } \mathfrak{M}, s \models \psi \\ \mathfrak{M}, s \models \neg \varphi & \iff & \mathfrak{M}, s^* \not\models \varphi \\ \mathfrak{M}, s \models \varphi \rightarrow \psi & \iff & Rs[[\varphi]]_{\mathfrak{M}}[[\psi]]_{\mathfrak{M}} \\ \mathfrak{M}, s \models \square \varphi & \iff & Qs[[\varphi]]_{\mathfrak{M}} \\ \mathfrak{M}, s \models \square_L \varphi & \iff & Q_L s[[\varphi]]_{\mathfrak{M}} \\ \mathfrak{M}, s \models [\varphi]\psi & \iff & \mathfrak{M}^\varphi, s \models \psi \end{array}$$

where $\mathfrak{M}^\varphi = (S, L, W, Prop, *, R, Q^\varphi, Q_L, V)$ is obtained from $(S, L, W, Prop, *, R, Q, Q_L, V)$ by setting:

$$Q^\varphi := \{(s, X) \mid Qs([[\varphi]])_{\mathfrak{M}} \rightarrow^{\tilde{\delta}} X\}.$$

Definition 8.9 (**C.FE**–validity). *Let validity and consequence in **C.FE**–models be defined as W –validity and S –consequence, respectively.*

Again, note that $\mathfrak{M}^\varphi \in \mathbf{C.FE}$ whenever $\mathfrak{M} \in \mathbf{C.FE}$. The dynamic modality $[\cdot]$ is interpreted in **C.FE**–models by means of a model–transforming operation, which adapts to a neighborhood setting the situated updates of [148]. By unraveling Definition 8.8, updating the doxastic state of the agent with φ amounts to adding the newly announced formula φ to the agent’s doxastic state, and considering the possible information combination that such addition triggers. That is, the φ –updated doxastic state contains all propositions outputted by the combination of the states in the old doxastic state with (the propositions expressed by) input φ . The following lemma records our informal explanation.

Lemma 8.1 (Updated Belief). *For all $\mathfrak{M} \in \mathbf{C.FE}$: $\mathfrak{M}, s \models [\varphi]\Box\psi \iff \mathfrak{M}, s \models \Box(\varphi \rightarrow [\varphi]\psi)$.*

Proof. We reason as follows:

$$\begin{aligned}
\mathfrak{M}, s \models [\varphi]\Box\psi &\iff \mathfrak{M}^\varphi, s \models \Box\psi && \text{by Definition 8.8} \\
&\iff Q^\varphi s \llbracket \psi \rrbracket_{\mathfrak{M}^\varphi} && \text{by Definitions 4.2 and 8.8} \\
&\iff Qs(\llbracket \varphi \rrbracket_{\mathfrak{M}} \rightarrow^{\tilde{s}} \llbracket \psi \rrbracket_{\mathfrak{M}^\varphi}) && \text{by definition of } Q^\varphi \\
&\iff Qs(\llbracket \varphi \rrbracket_{\mathfrak{M}} \rightarrow^{\tilde{s}} \llbracket [\varphi]\psi \rrbracket_{\mathfrak{M}}) && \text{by Definition 8.8} \\
&\iff \mathfrak{M}, s \models \Box(\varphi \rightarrow [\varphi]\psi) && \text{by Definitions 4.2 and 8.8}
\end{aligned}$$

□

From a model theoretic perspective, **C.FE**–models interpreting the dynamic language inherit all the good properties of the models for the static logic of relevant reasoners in classical worlds from Chapter 4. I recap these properties, without proof, in the following Lemmas.

Lemma 8.2 (Dynamic **C.FE**–heredity). *For all $\mathfrak{M} \in \mathbf{C.FE}$ and all $\varphi \in \mathfrak{L}_{\text{DRel}}$: $\llbracket \varphi \rrbracket_{\mathfrak{M}} \in \text{Prop}$.*

Lemma 8.3 (Dynamic **C.FE**–entailments). *For all $\varphi, \psi \in \mathfrak{L}_{\text{DRel}}$:*

$$\varphi \models_{\mathbf{C.FE}}^S \psi \text{ iff } \mathbf{C.FE} \models^W \Box_L(\varphi \rightarrow \psi) \text{ iff } \mathbf{C.FE} \models^L \varphi \rightarrow \psi$$

Lemma 8.4 (Dynamic **C.FE**–flatness). *For all $\mathfrak{M} \in \mathbf{C.FE}$, all $w \in W$ and all $\varphi, \psi \in \mathfrak{L}_{\text{DRel}}$:*

1. $\mathfrak{M}, w \models \mathbf{t}$;
2. $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$;
3. $\mathfrak{M}, w \models \varphi \rightarrow \psi$ iff $(\mathfrak{M}, w \models \varphi \implies \mathfrak{M}, w \models \psi)$.

Another interesting semantic closure property, which differentiates $\varphi > \psi$ from $\Box(\varphi \rightarrow \psi)$, is recorded by the following proposition. One can interpret Proposition 8.3 as confirming that update is a truly dynamic operation, in the style of dynamic epistemic logic. That is, a necessary component of conditional reasoning is that beliefs keep track of how updates change the world, the latter including the very beliefs of agents. Updates do not simply record dispositions to change beliefs about a static world. On the contrary, beliefs are among the objects of belief change, and the meaning of formulas after updates ($\llbracket \psi \rrbracket_{\mathfrak{M}\varphi}$) need no longer be the same thing as before updates ($\llbracket \psi \rrbracket_{\mathfrak{M}}$).

Proposition 8.3 ($>$ and updates). *The following holds:*

1. $\forall \mathfrak{M} \in \mathbf{C.FE}(\llbracket [\varphi]\psi \rrbracket_{\mathfrak{M}} = \llbracket [\varphi]\chi \rrbracket_{\mathfrak{M}} \implies \llbracket \varphi > \psi \rrbracket_{\mathfrak{M}} = \llbracket \varphi > \chi \rrbracket_{\mathfrak{M}})$;
2. *not* $\forall \mathfrak{M} \in \mathbf{C.FE}(\llbracket [\varphi]\psi \rrbracket_{\mathfrak{M}} = \llbracket [\varphi]\chi \rrbracket_{\mathfrak{M}} \implies \llbracket \Box(\varphi \rightarrow \psi) \rrbracket_{\mathfrak{M}} = \llbracket \Box(\varphi \rightarrow \chi) \rrbracket_{\mathfrak{M}})$.

Proof. Item 1 is an easy consequence of Lemma 8.1. To show Item 2, it suffices to consider the following countermodel. Consider the W -model \mathfrak{M} such that $S = \{s, t, u\}$, $W = \{s\}$, $R = \{(t, \{s\}, \{s\})\}$, $Q(s) = \{\{t\}\}$, $Q(t) = \{\{u\}\}$, $V(p) = \{s\}$, $V(q) = \{t\}$ and $V(r) = \{u\}$ (the remaining components can be specified so that \mathfrak{M} is indeed a W -model). Clearly, $\llbracket [p]\Box q \rrbracket = \llbracket [p]\Box r \rrbracket$, since $\llbracket [p]\Box q \rrbracket = \llbracket \Box(p \rightarrow q) \rrbracket = \emptyset = \llbracket \Box(p \rightarrow r) \rrbracket = \llbracket [p]\Box r \rrbracket$. However, $\mathfrak{M}, s \models \Box(p \rightarrow \Box q)$ (as $Q(s) = \{\{t\}\}$, $V(p) = \{s\}$, $\llbracket \Box q \rrbracket = \{s\}$ and $Rt\{s\}\{s\}$), but $\mathfrak{M}, s \not\models \Box(p \rightarrow \Box r)$ (as $Q(s) = \{\{t\}\}$, $V(p) = \{s\}$, $\llbracket \Box r \rrbracket = \{t\}$ and not $Rt\{s\}\{t\}$). \square

It will be of interest to list some monotonicity principles, whose invalidity in $\mathbf{C.FE}$ -models is easily checked. The implausibility of most of the following principles of conditional reasoning has been widely discussed: (*Monotonicity*), (*Transitivity*) and (*Contraposition*) have been found inadequate since 81; (*Cumulative Transitivity*) and (*Cautious Monotonicity*) are rejected e.g. in 158; a counterexample to (*Rational Monotonicity*) is provided e.g. by 187; and finally (*Reflexivity*), (*Belief Monotonicity*) are discussed in 148. The failure of these principles is a desideratum for any conditional logic which aims at capturing the fallible and bounded nature of belief revision.

$$(\varphi > \psi) \rightarrow ((\varphi \wedge \chi) > \psi) \qquad (\textit{Monotonicity})$$

$(\varphi > \psi) \rightarrow ((\psi > \chi) \rightarrow (\varphi > \chi))$	(Transitivity)
$(\varphi > \psi) \rightarrow (\neg\psi > \neg\varphi)$	(Contraposition)
$(\varphi > \psi) \rightarrow ((\varphi \wedge \psi > \chi) \rightarrow (\varphi > \chi))$	(Cumulative Transitivity)
$(\varphi > \psi) \rightarrow ((\varphi > \chi) \rightarrow (\varphi \wedge \psi > \chi))$	(Cautious Monotonicity)
$(\varphi > \psi) \rightarrow (\neg(\varphi > \neg\chi) \rightarrow (\varphi \wedge \chi > \psi))$	(Rational Monotonicity)
$\varphi > \varphi$	(Reflexivity)
$\Box\varphi \wedge \Box\psi \rightarrow (\psi > \varphi)$	(Belief Monotonicity)

In light of the critical remarks of the previous sections, it is valuable that **C.FE**-models offer an alternative model of conditional reasoning, which does not suffer from the deficiencies of AGM belief revision, **CK** and **ConR**. To see this, let me address how the *dynamic logic of relevant reasoners in classical worlds* scores with respect to desiderata (i)–(iv) I set out for Ramsey conditionals.

(i),(ii). **C.FE**-models offer a *semantic* and *dynamic* formalisation of the Ramsey Test, in spirit with its original epistemic motivations:

$$\mathfrak{M}, s \models \varphi > \psi \iff Q^\varphi s \llbracket \psi \rrbracket_{\mathfrak{M}}.$$

Therefore, **C.FE**-models can be used as models of conditional reasoning, with the Ramsey Test acting as a constitutive feature of the meaning of conditionals: s supports $\varphi > \psi$ in **C.FE**-models iff the agent operating in s believes ψ after revising their prior beliefs with φ , where belief revision is performed by combining their prior doxastic information with φ -inputs. That belief revision is a dynamic operator is confirmed by Proposition [8.3](#).

(iii). The definition of $\varphi > \psi$ in **C.FE**-models accounts for the paraconsistent, ampliative and defeasible nature of belief revision. The recombination triggered by φ -updates is regimented in **C.FE**-models by relevant logic, and information combination in relevant models has a complex structure, thanks to the minimal assumptions imposed on R and $*$. Hence, relevant logic allows one to formally capture the intuition that belief revision involves complex forms of information combination, which underlie everyday inferential patterns occurring in doxastic states. As a result, the logic of **C.FE**-models, to be developed in the next section, not only invalidates $(> \mathbf{K}), (> \mathbf{N}), (> \mathbf{C})$ and $(> \mathbf{M})$, but it is able to reject as invalid almost all principles discussed in the literature on monotonic conditional operators.

- (iv). **C.FE**-models provide the resources to model the conditional reasoning of agents revising their belief according to relevant logic, all the while being situated in possible worlds. A compelling motivation for this feature is that the partiality and contradictoriness of the information collected to form beliefs about possible worlds is due to limitations of humans, not the world itself. As a result, the logic of **C.FE**-models combines the best of **CK** and **ConR**, since it extends classical propositional logic **CPC**, while prescribing that information combination in doxastic states is performed according to the relevant modal logic **F.E**.

8.2.3 Axiomatisation of $\mathbf{C.FE}_\square$

The axiom system $\mathbf{C.FE}_\square$ is obtained by applying the usual strategy for building contextual modal logic. To this aim, the base system I consider is a dynamic extension of the minimal relevant logic **FE**, which I denote \mathbf{FE}_\square . As usual in the dynamic logic literature, \mathbf{FE}_\square is obtained by supplementing a set of reduction axioms to **FE**, which intuitively allow to reduce formulas containing dynamic operators to equivalent formulas in the “static” language, i.e. the reduced relevant modal language $\mathcal{L}_{\text{RRel}}$.

Definition 8.10 ($\mathbf{C.FE}_\square$ axiom system). *Let the Hilbert system for the dynamic logic of relevant reasoners in classical worlds $\mathbf{C.FE}_\square$ consist of the following axioms and rules:*

1. Axioms and rules of **CPC**;
2. The \square_L -version of \mathbf{FE}_\square (cf. Figure 8.2);
3. The bridge rule: (BR) $\square_L\varphi \Rightarrow \varphi$.

$\mathbf{F.E}_\square$ consists of a dynamic extension of one of the weakest relevant multi-modal logic characterised by Routley-Meyer semantics. The set of reduction axioms used is not the usual one adopted in Public Announcement Logic **PAL** (cf. [128]), but corresponds to the relevant version of the axiom system of [77]. We note that in our set of reduction principles, the standard $[\cdot]$ -compositionality axiom and $[\cdot]$ -necessitation rule of **PAL** (cf. [128, p.278]), which characterise $[\cdot]$ as a normal modality, are replaced with conjunctive regularity principles.

It will be useful in view of completeness to note that the update connective $[\cdot]$ is in some sense redundant. That is, as usual in the dynamic logic literature (cf.

(FE _□ 1)	<i>Axioms and rules of FE (cf. Definition 4.4)</i>
(FE _□ 2)	$[\varphi]p \leftrightarrow p$
(FE _□ 3)	$[\varphi]\neg\psi \leftrightarrow \neg[\varphi]\psi$
(FE _□ 4)	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
(FE _□ 5)	$[\varphi](\psi \vee \chi) \leftrightarrow ([\varphi]\psi \vee [\varphi]\chi)$
(FE _□ 6)	$[\varphi](\psi \rightarrow \chi) \leftrightarrow ([\varphi]\psi \rightarrow [\varphi]\chi)$
(FE _□ 7)	$[\varphi]\Box_L\psi \leftrightarrow \Box_L[\varphi]\psi$
(FE _□ 8)	$[\varphi]\Box\psi \leftrightarrow \Box(\varphi \rightarrow [\varphi]\psi)$
(FE _□ 9)	$\varphi \leftrightarrow \chi, \psi \leftrightarrow \xi \Rightarrow [\varphi]\psi \rightarrow [\chi]\xi$

Figure 8.2: Hilbert system for FE_□

[128, p.278]), it can be proved that every formula is provably equivalent in F.E_□ and C.F.E_□ to a formula in the static language $\mathfrak{L}_{\text{Rel}}$. To this aim, complexity is defined as an inductive parameter distinct from the structure of a formula.

Definition 8.11 (Complexity). *Let the $\mathfrak{L}_{\text{DRel}}$ -complexity $c : \mathfrak{L}_{\text{DRel}} \rightarrow \mathbb{N}$ of a formula φ be defined recursively as follows, for $\otimes \in \{\neg, \wedge, \vee, \rightarrow, \Box, \Box_L, \Box, [\cdot]\}$:*

- $c(p) = 1$;
- $c(\otimes(\varphi_1, \dots, \varphi_n)) = 1 + \sum_{i \leq n} c(\varphi_i)$;

Lemma 8.5 (Syntactic reduction). *For $L \in \{F.E_{\square}, C.F.E_{\square}\}$ and all $\varphi \in \mathfrak{L}_{\text{DRel}}$, there is $\varphi' \in \mathfrak{L}_{\text{Rel}}$ such that $\vdash_L \varphi \leftrightarrow \varphi'$.*

Proof. In what follows we assume $L = F.E_{\square}$ without loss of generality. The result is established by induction on the complexity of φ . The base case is established by setting $\varphi' := \varphi$. The case $\varphi := \otimes(\psi_1, \dots, \psi_n)$, where $\otimes \in \{\neg, \wedge, \vee, \rightarrow, \Box, \Box_L\}$ is established as follows. By induction hypothesis, $\vdash_L \psi_1 \leftrightarrow \psi'_1, \dots, \vdash_L \psi_n \leftrightarrow \psi'_n$ for some $\psi'_1, \dots, \psi'_n \in \mathfrak{L}_{\text{Rel}}$. We conclude by setting $\varphi' := \otimes(\psi'_1, \dots, \psi'_n)$. Finally, the case $\varphi := [\psi]\chi$ is established as follows. By the induction hypothesis we infer that there are $\psi', \chi' \in \mathfrak{L}_{\text{Rel}}$ such that $\vdash_L \psi \leftrightarrow \psi'$ and $\vdash_L \chi \leftrightarrow \chi'$. By \Box_L -(FE_□9) we infer that $\vdash_L [\psi]\chi \leftrightarrow [\psi']\chi'$. Then, to find a suitable $\varphi' \in \mathfrak{L}_{\text{Rel}}$ such that $\vdash_L [\psi]\chi \leftrightarrow \varphi'$ it suffices to show that

$$\forall \psi', \chi' \in \mathfrak{L}_{\text{Rel}} \exists \varphi' \in \mathfrak{L}_{\text{Rel}} : \vdash_L [\psi']\chi' \leftrightarrow \varphi' \quad (*)$$

which is established by induction on the complexity of χ' . The base case, when $\chi' \in At$ holds by setting $\varphi' := \chi'$. The induction step cases where the main connective of χ' is either $\neg, \wedge, \vee, \rightarrow, \Box, \Box_L$ are similar, hence we show only the case $\chi' := \Box\xi$ as an illustration. By $\Box_L - (\text{FE}\Box 8)$ we infer that $\vdash_L [\psi']\Box\xi \leftrightarrow \Box(\psi' \rightarrow [\psi']\xi)$. By induction hypothesis we infer that there is $\gamma \in \mathfrak{L}_{\text{RRel}}$ such that $\vdash_L [\psi']\xi \leftrightarrow \gamma$, which implies by $\vdash_L [\psi']\Box\xi \leftrightarrow \Box(\psi' \rightarrow [\psi']\xi)$, together with $\Box_L - (\text{FE15})$ and $\Box_L - (\text{FE10})$, that $\vdash_L [\psi']\Box\xi \leftrightarrow \Box(\psi' \rightarrow \gamma)$. We conclude by setting $\varphi' := \Box(\psi' \rightarrow \gamma)$. Finally, the case $\chi' := [\xi]\gamma$ is ruled out by $\chi' \in \mathfrak{L}_{\text{RRel}}$. \square

The following propositions show the closure properties of $>$. For one, the Ramsey conditional is hyperintensional in both its antecedent and consequent position. This marks a contrast with **CK** and **ConR**, where e.g. $\psi \leftrightarrow \chi \Rightarrow (\varphi > \psi) \rightarrow (\varphi > \chi)$ is admissible. By Proposition [8.4](#), however, a restricted version of the equivalence rule holds for $>$ in **C.F.E \Box** .

Proposition 8.4 ($>$ -Relevant equivalence). *The following rules are derivable in **C.F.E \Box** :*

1. $\Box_L(\varphi \leftrightarrow \psi) \Rightarrow (\varphi > \chi) \leftrightarrow (\psi > \chi)$;
2. $\Box_L(\varphi \leftrightarrow \psi) \Rightarrow (\chi > \varphi) \leftrightarrow (\chi > \psi)$.

Proof. The following derivations establish the result.

Antecedent Restricted Equivalence

- (1) $\Box_L(\varphi \leftrightarrow \psi)$
- (2) $\Box_L(\Box\chi \leftrightarrow \Box\chi)$ $\Box_L - (\text{FE15}), (1)$
- (3) $\Box_L([\varphi]\Box\chi \leftrightarrow [\psi]\Box\chi)$ $\Box_L - (\text{FE}\Box 9), (2)$
- (4) $[\varphi]\Box\chi \leftrightarrow [\psi]\Box\chi$ **(BR)**, (3)
- (5) $(\varphi > \chi) \leftrightarrow (\psi > \chi)$ *definition of $>$* , (4)

Consequent Restricted Equivalence

- (1) $\Box_L(\varphi \leftrightarrow \psi)$
- (2) $\Box_L([\chi]\varphi \leftrightarrow [\chi]\psi)$ $\Box_L - (\text{FE}\Box 9), (1)$
- (3) $\Box_L((\chi \rightarrow [\chi]\varphi) \leftrightarrow (\chi \rightarrow [\chi]\psi))$ $\Box_L - (\text{FE12}), (2)$
- (4) $\Box_L(\Box(\chi \rightarrow [\chi]\varphi) \leftrightarrow \Box(\chi \rightarrow [\chi]\psi))$ $\Box_L - (\text{FE15}), (3)$
- (5) $\Box_L([\chi]\Box\varphi \leftrightarrow [\chi]\Box\psi)$ $\Box_L - (\text{FE}\Box 8), \Box_L - (\text{FE12}), (4)$
- (6) $[\chi]\Box\varphi \leftrightarrow [\chi]\Box\psi$ **(BR)**, (5)
- (7) $(\chi > \varphi) \leftrightarrow (\chi > \psi)$ *definition of $>$* , (6)

□

Having laid down the properties of the conditional in $\mathbf{C.F.E}_\square$, let me turn to the assessment of the meta-logical properties of the system.

Theorem 8.1 (Soundness). *For all $\varphi \in \mathfrak{L}_{\text{DRel}}$:*

1. $\vdash_{\mathbf{F.E}_\square} \varphi \implies \mathbf{FE} \models \varphi$;
2. $\vdash_{\mathbf{C.F.E}_\square} \varphi \implies \mathbf{C.FE} \models \varphi$.

Proof. Item 1 is established by induction on the length of $\mathbf{F.E}_\square$ -derivations. All cases are covered by the argument of Theorem 4.1, with the exception of the reduction principles $(\mathbf{FE}_\square 2)$ – $(\mathbf{FE}_\square 9)$. $(\mathbf{FE}_\square 2)$ – $(\mathbf{FE}_\square 7)$ are trivially valid, since for all $\mathfrak{M} \in \mathbf{L}^*$, R, Q_L are the same in \mathfrak{M}^φ and in \mathfrak{M} .

$(\mathbf{FE}_\square 8)$. We reason as follows:

$$\begin{aligned}
\mathfrak{M}, s \models [\varphi] \square \psi &\iff \mathfrak{M}^\varphi, s \models \square \psi && \text{by Definition 8.6} \\
&\iff Q^\varphi s \models \llbracket \psi \rrbracket_{\mathfrak{M}^\varphi} && \text{by Definitions 4.2 and 8.6} \\
&\iff Qs(\llbracket \varphi \rrbracket_{\mathfrak{M}} \rightarrow^{\mathfrak{F}} \llbracket \psi \rrbracket_{\mathfrak{M}^\varphi}) && \text{by definition of } Q^\varphi \\
&\iff Qs(\llbracket \varphi \rrbracket_{\mathfrak{M}} \rightarrow^{\mathfrak{F}} \llbracket [\varphi] \psi \rrbracket_{\mathfrak{M}}) && \text{by Definition 8.6} \\
&\iff \mathfrak{M}, s \models \square(\varphi \rightarrow [\varphi] \psi) && \text{by Definitions 4.2 and 8.6.}
\end{aligned}$$

$(\mathbf{FE}_\square 9)$. Assume by Lemma 8.3 that $\llbracket \varphi \rrbracket_{\mathfrak{M}} = \llbracket \chi \rrbracket_{\mathfrak{M}}$ and $\llbracket \psi \rrbracket_{\mathfrak{M}} = \llbracket \xi \rrbracket_{\mathfrak{M}}$ for all $\mathfrak{M} \in \mathbf{FE}$. The following chain of equivalences establishes the result: $\mathfrak{M}, s \models [\varphi] \psi$ iff $\mathfrak{M}^\varphi, s \models \psi$ iff (by $\llbracket \psi \rrbracket_{\mathfrak{M}^\varphi} = \llbracket \xi \rrbracket_{\mathfrak{M}^\varphi}$) $\mathfrak{M}^\varphi, s \models \xi$ iff (by $Q^\varphi = Q^\chi$, which holds by $\llbracket \varphi \rrbracket_{\mathfrak{M}} = \llbracket \chi \rrbracket_{\mathfrak{M}}$) $\mathfrak{M}^\chi, s \models \xi$ iff $\mathfrak{M}, s \models [\chi] \xi$.

Item 2 is established from Item 1 by the same argument of Theorem 4.1. □

Next, note that in the proof of the bridge lemma one needs to prove that model updates do not interfere with the $+$ -construction.

Lemma 8.6 (Bridge \mathbf{FE}_\square – $\mathbf{C.FE}_\square$). *For all $\varphi \in \mathfrak{L}_{\text{DRel}}$: $\vdash_{\mathbf{F.E}_\square} \varphi \iff \vdash_{\mathbf{C.F.E}_\square} \square_L \varphi$.*

Proof. The result is established by the same argument of Lemma 4.17, where we show the missing details required to establish the right-to-left direction. As usual, assume by contraposition that $\not\vdash_{\mathbf{F.E}_\square} \varphi$. Hence, by Theorem 8.2.1 (which does not rely on the present lemma) there is $\mathfrak{M} \in \mathbf{FE}$ such that $\mathfrak{M} \not\models \varphi$. To conclude by Theorem 8.1.2 that $\not\vdash_{\mathbf{C.F.E}_\square} \square_L \varphi$, it suffices to show that

$$\forall \mathfrak{M} \in \mathbf{FE} \exists \mathfrak{M}' \in \mathbf{C.FE} (\mathfrak{M} \not\models \varphi \implies \mathfrak{M}' \not\models \square_L \varphi).$$

The latter is established by the same argument of Lemma 4.16, where the only thing which is missing is to show that Claim 5 extends to the extended language. The proof goes again by induction, and we need to show that for all $\mathfrak{M} \in \mathbf{FE}$, \mathfrak{M}^+ (defined as in Definition 4.12) is such that for all $s \in S$

$$\mathfrak{M}, s \models [\varphi]\psi \iff \mathfrak{M}^+, s \models [\varphi]\psi.$$

To this aim, note that $(Q^+)^{\varphi}(s) = (Q^{\varphi})^+(s)$, which is established as follows, where (1) is the condition that $x \in W \cap (\llbracket \psi \rrbracket_{\mathfrak{M}^+} \cup X)$:

$$\begin{aligned} (Q^+)^{\varphi} s X &\iff Q^+ s (\llbracket \varphi \rrbracket_{\mathfrak{M}^+} \rightarrow^{\mathfrak{F}^+} X) && \text{by definition of } Q^{\varphi} \\ &\iff Q^+ s \{x \mid R^+ x \llbracket \varphi \rrbracket_{\mathfrak{M}^+} X\} && \text{by Definition 4.2} \\ &\iff Q^+ s \{x \mid Rx (\llbracket \varphi \rrbracket_{\mathfrak{M}^+} \cap S) (X \cap S) \text{ or (1)}\} && \text{by definition of } R^+ \\ &\iff Q^+ s \{x \mid Rx \llbracket \varphi \rrbracket_{\mathfrak{M}} (X \cap S) \text{ or (1)}\} && \text{by induction hypothesis} \\ &\iff Qs (\{Rx \llbracket \varphi \rrbracket_{\mathfrak{M}} (X \cap S) \text{ or (1)}\} \cap S) && \text{by definition of } Q^+ \\ &\iff Qs \{Rx \llbracket \varphi \rrbracket_{\mathfrak{M}} (X \cap S)\} && \text{by } S \cap W = \emptyset \\ &\iff Qs (\llbracket \varphi \rrbracket_{\mathfrak{M}} \rightarrow^{\mathfrak{F}} (X \cap S)) && \text{by Definition 4.2} \\ &\iff Q^{\varphi} s (X \cap S) && \text{by definition of } Q^{\varphi} \\ &\iff (Q^{\varphi})^+ s X && \text{by definition of } Q^+. \end{aligned}$$

Then, we reason as follows:

$$\begin{aligned} \mathfrak{M}, s \models [\varphi]\psi &\iff \mathfrak{M}^{\varphi}, s \models \psi && \text{by Definition 8.6} \\ &\iff (\mathfrak{M}^{\varphi})^+, s \models \psi && \text{by induction hypothesis} \\ &\iff (\mathfrak{M}^+)^{\varphi}, s \models \psi && \text{by } (Q^+)^{\varphi}(s) = (Q^{\varphi})^+(s) \\ &\iff \mathfrak{M}^+, s \models [\varphi]\psi && \text{by Definition 8.6.} \end{aligned}$$

□

Turning to completeness, one may eventually notice the usefulness of Lemma 8.5. By the latter, the task of proving completeness for \mathbf{FE}_{\square} and $\mathbf{C.FE}_{\square}$ can be carried out in much the same way as the proof of completeness for \mathbf{FE} and $\mathbf{C.FE}$. To this aim, I adapt the definitions of the canonical \mathbf{FE}_{\square} -model and the canonical $\mathbf{C.FE}_{\square}$ -model to Definitions 4.5 4.14). Compared with Section 4.3.3, one may note that here I work with a language excluding \otimes, \leftarrow , for the reason that the inclusion of either of the two gives rise to prefixing, respectively suffixing, systems (cf. Section 4.2.2). Affixing systems, in turn, were criticised in Section 8.2.1 because they give rise to monotonic conditionals.

Definition 8.12 (Canonical $\mathbf{C.FE}_{\square}$ -model). *Let the canonical $\mathbf{C.FE}$ -model be a tuple of the following form:*

$$\mathfrak{M}_{\mathbf{C.FE}_{\square}}^c = (S^c, W^c, L^c, \leq^c, *^c, R^c, Q^c, Q_L^c, V^c)$$

- $S^c = \{s \mid s \text{ is a prime } \mathbf{FE}_{\square}\text{-theory over } \mathfrak{L}_{\mathbf{DRel}}\};$
- $W^c = \{s \mid s \text{ is a non-empty, proper, prime } \mathbf{C.FE}_{\square}\text{-theory over } \mathfrak{L}_{\mathbf{DRel}}\};$
- $L^c = \{s \in S^c \mid \mathbf{t} \in s\};$
- $s^{*c} = \{\varphi \in \mathfrak{L}_{\mathbf{DRel}} \mid \neg\varphi \notin s\};$
- $R^c = \{(s, X, Y) \in S^c \times (\mathit{Prop}^c)^2 \mid \exists \varphi, \psi \in \mathfrak{L}_{\mathbf{DRel}} (X = [\varphi]_c, Y = [\psi]_c, \varphi \rightarrow \psi \in s)\};$
- $Q^c = \{(s, X) \in S^c \times \mathit{Prop}^c \mid \exists \varphi \in \mathfrak{L}_{\mathbf{DRel}} (X = [\varphi]_c, \Box\varphi \in s)\};$
- $Q_L^c = \{(s, X) \in S^c \times \mathit{Prop}^c \mid \exists \varphi \in \mathfrak{L}_{\mathbf{DRel}} (X = [\varphi]_c, \Box_L\varphi \in s)\};$
- $V^c(p) = \{s \in S^c \mid p \in s\}.$

Let $\mathfrak{M}_{\mathbf{FE}_{\square}} = (S^c, L^c, \leq^c, *^c, R^c, Q^c, Q_L^c, V^c)$ be the canonical \mathbf{FE}_{\square} -model.

Theorem 8.2 (Dynamic Completeness). *For all $\varphi \in \mathfrak{L}_{\mathbf{DRel}}$:*

1. $\mathbf{FE} \models \varphi \implies \vdash_{\mathbf{F.E}_{\square}} \varphi$;
2. $\mathbf{C.FE} \models \varphi \implies \vdash_{\mathbf{C.F.E}_{\square}} \varphi$.

Proof. Item 1 is established as follows. Assume by contraposition that $\not\models_{\mathbf{FE}_{\square}} \varphi$. By Lemma 8.5 we infer that $\not\models_{\mathbf{FE}_{\square}} \varphi'$ for some $\varphi' \in \mathfrak{L}_{\mathbf{RRel}}$. By the same argument of Lemma 4.5, $\mathfrak{M}_{\mathbf{FE}_{\square}}^c \in \mathbf{FE}$. By Lemma 2.3 there is $s \in L$ such that $\varphi' \notin s$, which implies by Lemma 4.6 that $\mathfrak{M}_{\mathbf{FE}_{\square}}^c, s \not\models \varphi'$. By Lemma 8.5 and Theorem 8.1.1 we conclude that $\mathfrak{M}_{\mathbf{FE}_{\square}}^c, s \not\models \varphi$. Item 2 is established similarly, using Lemmas 4.18, 3.7 and Theorem 8.1.2. \square

8.3 Concluding Remarks

In conclusion of Part III, let us take stocks of what has been accomplished so far and point out what are the future challenges. In the second part of the dissertation, contextual modal logics were applied to tackle core issues in formal epistemology, such as the representation of hyperintensional attitudes in general and topic-theoretic attitudes in particular; the problem of logical omniscience; the distinction between explicit and implicit belief; and the dynamic modeling of belief revision. Although this shows the fruitfulness of contextual modal logics, several unexplored domains of applications could benefit from taking a contextual perspective. For example, I

will point out two interesting lines of research to be pursued. For one, revisionist strategies to standard deontic logic rarely appeal to non-classical logics. One explanation of why this is so is that, while normative conflicts are part of life and should be accommodated in a deontic logic, the relationship between worldly facts is aptly modelled by classical logic. A contextual version of deontic logic would employ the additional expressive resources of contextual modal logic to give better formalisations of famous deontic puzzles and correct predictions regarding their validity. A second line of application of contextual modal logics is to devise contextual versions of dynamic epistemic logics, which goes well beyond the study of public announcements. More complex phenomena of informational dynamics to be studied include the study of action models and revision policies for contextual modal logics. The former allows to represent agents' uncertainties regarding which information is being conveyed by the source, while the latter allow to encode different degrees of reliability of sources of information.

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