A CUSPED HYPERBOLIC 4-MANIFOLD WITHOUT SPIN STRUCTURES

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Abstract. We build a non-compact, orientable, hyperbolic four-manifold of finite volume that does not admit any spin structure.

Introduction

It follows from a couple of works of Deligne and Sullivan [5, 20] of the 1970s that every hyperbolic manifold M is finitely covered by a stably parallelisable manifold M'. In particular, the Stiefel–Whitney classes satisfy $w_k(M') = 0$ for all $k > 0$. Unless otherwise stated, all manifolds in the paper are smooth, connected and orientable (i.e. with $w_1 = 0$), and all hyperbolic manifolds are complete and of finite volume.

The existence of hyperbolic n-manifolds that do not admit spin structures (i.e. with $w_2 \neq 0$) has been proved in 2020: there are closed for all $n \geq 4$ [13] and cusped for all $n \geq 5$ [9]. Recall instead that surfaces are stably parallelisable and 3-manifolds are parallelisable. Then several examples of hyperbolic manifolds with non-trivial Stiefel–Whitney classes have been produced with different techniques [2, 3, 4, 9, 15], but the existence of cusped 4-manifolds with $w_2 \neq 0$ appears open. We fill here the gap:

Theorem 1. *There exists a cusped orientable (arithmetic) hyperbolic* 4*-manifold* M *that does not admit any spin structure.*

Since M is arithmetic and even-dimensional, we can iteratively apply the embedding theorem of Kolpakov, Reid and Slavich [7] as in [13, Section 5], to get a sequence of totally geodesic embeddings $M = \mathbb{H}^4/\Gamma_4 \subset \mathbb{H}^5/\Gamma_5 \subset \ldots$ of n-manifolds with $\Gamma_n \subset \mathrm{PSO}(1, n; \mathbb{Q})$ commensurable with $\mathrm{PO}(1, n; \mathbb{Z})$. None of them admits a spin structure because an orientable hypersurface does not, so:

Corollary 2. *For every* $n \geq 4$*, there exists a cusped orientable (arithmetic) hyperbolic* n*-manifold that does not admit any spin structure.*

This has already been proved by Long and Reid for $n \geq 5$ [9] as follows: (1) there is a closed flat 4-manifold F^4 with $w_2(F^4) \neq 0$, so $F^{n-1} = F^4 \times S^1 \times \ldots \times S^1$ has $w_2(F^{n-1}) \neq 0$ for all $n \geq 5$; (2) as every closed flat manifold, F^{n-1} is diffeomorphic to a cusp section of a cusped hyperbolic manifold M^n [8, 10], so as before $w_1(F^{n-1}) = 0,\ w_2(F^{n-1}) \neq 0 \implies w_2(M^n) \neq 0$.

To prove Theorem 1, we instead proceed as done in the closed case by Martelli, Slavich and the first author in [13] (see also [14]), explicitly constructing a hyperbolic 4-manifold M satisfying a stronger condition: its intersection form is *odd*; equivalently, there is a closed oriented surface $S \subset M$ with odd self-intersection $S \cdot S$ (the Euler number of the normal bundle). Then $w_2(M) \neq 0$ because the result of clashing $w_2(M)$ with the $\mathbb{Z}/2\mathbb{Z}$-homology class of S is $S \cdot S \mod 2$. Note that

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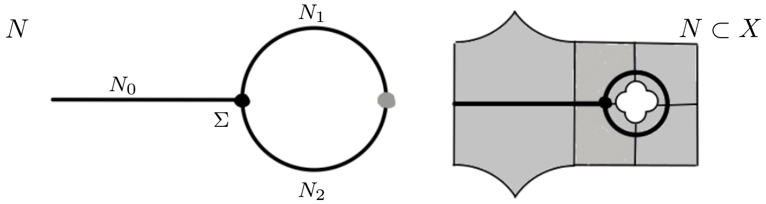

FIGURE 1. On the left, a schematic picture of the three-dimensional thickening $N = N_0 \cup N_1 \cup N_2$ of the piecewise geodesic surface $S = S_0 \cup S_1 \cup S_2$, where $S_i \subset N_i$ are totally geodesic manifolds with corners. It is not a manifold near the auxiliary surface with corners $\Sigma = N_0 \cap N_1 \cap N_2$ (represented by a black dot). On the right, the thickening X of N: a 4-manifold with corners, neighbourhood of S in M, tessellated by some copies of P^4 (represented by 10 gray pentagons)

S must necessarily be closed, otherwise $S \cdot S = 0$. Moreover, S is not homologous to any immersed totally geodesic surface in M, since such surfaces have even self-intersection (see [13]).

As in [13, 14], we build M by gluing some copies of a right-angled hyperbolic polytope P^4 in such a way that S is contained in the 2-skeleton of the tessellation. For this purpose, we need that P^4 has a compact 2-face P^2. The only unbounded, right-angled, hyperbolic 4-polytope of finite volume with a compact 2-face that we know is introduced in Section 2. It belongs to a continuous family of hyperbolic 4-polytopes discovered in 2010 by Kerckhoff–Storm [6], further studied in [12] and later used for different purposes [16, 17, 18, 19]. The polytope P^4 has 22 facets and octahedral symmetry. Its reflection group is arithmetic, and like for the well-known ideal 24-cell, is commensurable with the integral lattice $\mathrm{PO}(1, 4; \mathbb{Z})$. The manifold M belongs to this commensurability class. We thank Leone Slavich for pointing out that a conjugate of Γ_4 lies in $\mathrm{PSO}(1, 4; \mathbb{Q})$, which gives Corollary 2.

Like in [11, 13, 14], we use some right-angled polytopes $P^2 \subset P^3 \subset P^4$ (where P^n is a facet of P^{n+1}) to build some auxiliary hyperbolic manifolds with right-angled corners of increasing dimension. These objects have been fruitfully used in four- and five- dimensional hyperbolic geometry in the very last years [1, 2, 4, 16]. The surface S is piecewise geodesic and tessellated by copies of P^2, and the cells of M intersecting S form a 4-manifold with right-angled corners X (see Figure 1–right). Like in [13, 14], the construction ensures the following:

Theorem 3. *There exists a geometrically finite hyperbolic 4-manifold (of infinite volume) that covers a cusped manifold (of finite volume) and deformation retracts onto a closed surface with non-trivial normal bundle.*

Theorem 3 follows from the fact that M contains X as a convex submanifold, and the latter deformation retracts onto S. So $\pi_1(S)$ injects in $\pi_1(M)$ and induces a covering $\hat{M} \to M$ such that $\hat{M}$ is geometrically finite and diffeomorphic to the interior of X. For a proof, substitute "compact" with "complete and finite-volume" and "convex cocompact" with "geometrically finite" in the proof of [14, Proposition 6, Corollary 8].

The paper is organised as follows: the proof of Theorem 1 is summarised in Section 1, the polytope is introduced in Section 2, and the construction is performed in Section 3.

1. SUMMARY

As already explained, like in [13, 14] for the compact case, our goal is to prove the following:

Theorem 4. *There exists a cusped, oriented, arithmetic, hyperbolic* 4*-manifold* M *that contains an oriented surface* S *with self-intersection* $S \cdot S = 1$.

A *(hyperbolic) manifold with (right-angled) corners* is a complete hyperbolic manifold with boundary X, locally modelled on an orthant of $\mathbb{H}^n$. The connected submanifolds with boundary that naturally stratify ∂X are called *faces*. We call *facets* and *corners* the $(n-1)$-dimensional and $(n-2)$-dimensional faces, respectively. Each face is naturally the image under a local isometry of a manifold with corners. These local isometries are all embeddings precisely when every corner is the intersection of two facets.

An n-manifold with corners and embedded facets X is contained in a hyperbolic n-manifold M without boundary, obtained in a standard way by iteratively doubling and re-doubling X along its facets (see Section 3.5). So, to prove Theorem 4, we are reduced to building a cusped 4-manifold with corners X with embedded faces and a surface $S \subset X$ such that $S \cdot S = 1$.

The surface S cannot be contained in an orientable 3-manifold in M, otherwise $S \cdot S = 0$. Similarly to [13], it will instead be contained in a "locally Y-shaped piece" N obtained by gluing three 3-manifolds with corners N_0, N_1 and N_2 (also) along an isometric facet Σ (see Figure 1–left). The intersection $\Theta = \Sigma \cap S = \gamma_0 \cup \gamma_1 \cup \gamma_2$ is a theta-graph that trisects S in three pieces S_0, S_1 and S_2, with S_i properly embedded in N_i and $\gamma_i = \Sigma \cap S_i$ a boundary component of S_i (see Figure 3). The 4-manifold with corners X will be a thickening of N (see Figure 1–right), and will contain S with $S \cdot S = \pm 1$ by construction (see Figure 10).

All Θ, Σ, S and N will be contained in the skeleta of the tessellation of X in copies of P^4. The auxiliary surface Σ is totally geodesic, while S is pleated. Moreover, Σ and S are tessellated by P^2's and N by P^3's. Each N_i is totally geodesic in X, and $N_0 \perp N_1, N_2$. The thickenings $S \subset N \subset X$ are built via the sequence $P^2 \subset P^3 \subset P^4$.

2. The polytope

We introduce here Kerckhoff and Storm's right-angled hyperbolic 4-polytope P^4 [6]. Let us identify the hyperbolic 4-space $\mathbb{H}^4$ with the upper sheet of the hyperboloid $\langle x, x\rangle = -1$ in the Minkowski 5-space $\mathbb{R}^{1,4}$. Here $\langle x, y\rangle = -x_0y_0 + x_1y_1 + \ldots + x_4y_4$ for $x = (x_0, \ldots, x_4), y = (y_0, \ldots, y_4) \in \mathbb{R}^{1,4}$. Given a spacelike vector $v \in \mathbb{R}^{1,4}$, the inequality $\langle x, v\rangle \leq 0$ defines a half-space of $\mathbb{H}^4$. Let[1] $P^4 \subset \mathbb{H}^4$ be the intersection of the 22 half-spaces given by the vectors in Table 1. It is an unbounded, right-angled polytope of finite volume [6, Proposition 13.1].

Note that the isometry a defined by $a(x_0, x_1, \ldots, x_4) = (x_0, -x_1, \ldots, -x_4)$ is a symmetry of P^4. Moreover, the notation (taken from [16]) is such that $E'_i = a(E_i)$, $H'_i = a(H_i)$, and $C_{ij} = a(C_{kl})$ for all distinct i, j, k, l. The combinatorics of P^4 has been studied in detail in [12, Proposition 3.16]. Each vector in Table 1 corresponds to a facet of P^4, denoted with the same symbol. The 22 facets, depicted in Figure 2, are partitioned up to symmetry into three sets:[2]

1. the *extremal facets* $E_1, E_2, E_3, E_4, E'_1, E'_2, E'_3, E'_4$,
2. the *half-height facets* $H_1, H_2, H_3, H_4, H'_1, H'_2, H'_3, H'_4$,
3. the *central facets* $C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, C_{34}$.

Lemma 5. *Every combinatorial automorphism of* P^4 *is realised by an isometry of* P^4*, and every hyperbolic orbifold* O *tessellated by finitely-many copies of* P^4 *is commensurable with* $\mathbb{H}^4/\mathrm{PO}(1,4;\mathbb{Z})$.

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[1] In [6, 12], P^4 is denoted by P_t, where $t = t_4 = \bar{t} = \sqrt{3}/3$.

[2] In [6, 12], these are called: the "positive walls", the "negative walls", and the "letter walls", respectively.

$E_1 = \left(\sqrt{2}, +1, +1, +1, +\sqrt{3}\right)$	$H_1 = \left(\sqrt{2}, -1, -1, -1, +\sqrt{3}/3\right)$	$C_{12} = \left(1, +\sqrt{2}, 0, 0, 0\right)$
$E_1' = \left(\sqrt{2}, -1, -1, -1, -\sqrt{3}\right)$	$H_1' = \left(\sqrt{2}, +1, +1, +1, -\sqrt{3}/3\right)$	$C_{34} = \left(1, -\sqrt{2}, 0, 0, 0\right)$
$E_2 = \left(\sqrt{2}, +1, -1, -1, +\sqrt{3}\right)$	$H_2 = \left(\sqrt{2}, -1, +1, +1, +\sqrt{3}/3\right)$	$C_{13} = \left(1, 0, +\sqrt{2}, 0, 0\right)$
$E_2' = \left(\sqrt{2}, -1, +1, +1, -\sqrt{3}\right)$	$H_2' = \left(\sqrt{2}, +1, -1, -1, -\sqrt{3}/3\right)$	$C_{24} = \left(1, 0, -\sqrt{2}, 0, 0\right)$
$E_3 = \left(\sqrt{2}, -1, +1, -1, +\sqrt{3}\right)$	$H_3 = \left(\sqrt{2}, +1, -1, +1, +\sqrt{3}/3\right)$	$C_{14} = \left(1, 0, 0, +\sqrt{2}, 0\right)$
$E_3' = \left(\sqrt{2}, +1, -1, +1, -\sqrt{3}\right)$	$H_3' = \left(\sqrt{2}, -1, +1, -1, -\sqrt{3}/3\right)$	$C_{23} = \left(1, 0, 0, -\sqrt{2}, 0\right)$
$E_4 = \left(\sqrt{2}, -1, -1, +1, +\sqrt{3}\right)$	$H_4 = \left(\sqrt{2}, +1, +1, -1, +\sqrt{3}/3\right)$	
$E_4' = \left(\sqrt{2}, +1, +1, -1, -\sqrt{3}\right)$	$H_4' = \left(\sqrt{2}, -1, -1, +1, -\sqrt{3}/3\right)$	

TABLE 1. The spacelike vectors of $\mathbb{R}^{1,4}$ that define the polytope $P^4 \subset \mathbb{H}^4$.

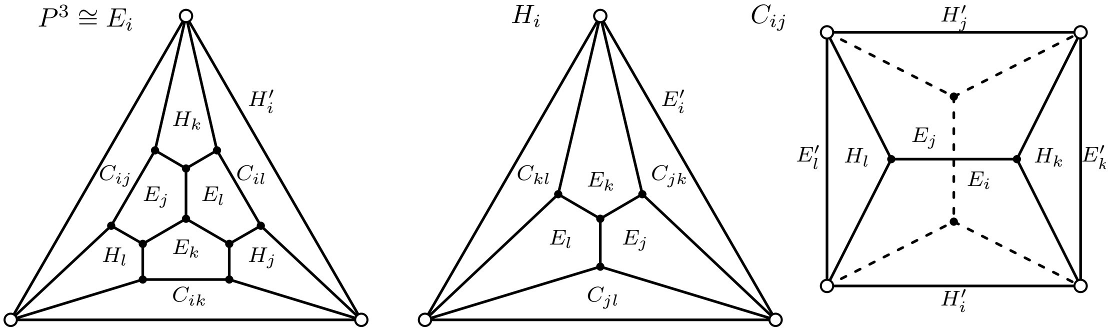

FIGURE 2. The extremal, half-height and central facets $E_i \cong P^3$, H_i and C_{ij} of P^4, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. The ideal vertices are in white. Note the compact pentagon $E_i \cap E_j \cong P^2$.

Proof. The poof of the first statement (relying on [19, Proposition 2.4] and [12, Lemma 4.15]) is the same of [16, Lemma 1.2] by [12, Section 3.2 and Proposition 3.16]. In particular (see Figure 2), every isometry between two facets of P^4 is the restriction of an isometry of P^4. Since, by hypothesis, O can be obtained by gluing the facets of some copies P^4 in pairs via isometries, O covers the orbifold $P^4/\mathrm{Isom}(P^4)$, and so it is commensurable with $P^4 = \mathbb{H}^4/\Gamma$. The reflection group $\Gamma < \mathrm{PO}(1, n)$ of P^4 is arithmetic [6, Theorem 13.2] and commensurable with $\mathrm{PO}(1, 4; \mathbb{Z})$ [12, Proposition 4.25]. □

Note from Figure 2 that the compact 2-faces of P^4 are 12 isometric pentagons $E_i \cap E_j$, $E_i' \cap E_j'$, $i \neq j$. Defining

$$P^2 = E_1 \cap E_2 \text{ and } P^3 = E_1,$$

we have a sequence of right-angled polytopes:

$$P^2 \subset P^3 \subset P^4.$$

We shall think of P^{n+1} as sitting above its *bottom facet* P^n, and call the remaining facets *vertical facets* and *top facets*, depending on whether they are adjacent to P^n or not, respectively. For example, P^3 has 5 vertical facets and 4 top facets, while P^4 has 10 vertical facets and 11 top facets.

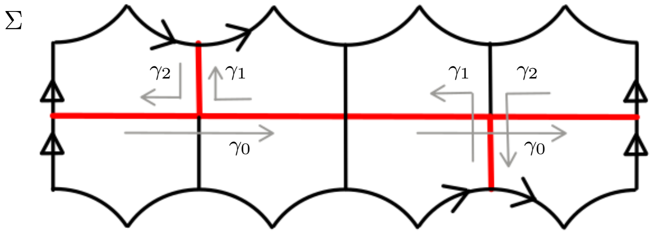

Figure 3. The surface Σ with corners obtained by gluing 8 copies of the right-angled pentagon P^2 (four edges of the big dodecagon are glued in pairs as indicated by the black arrows). It is a holed torus, and deformation retracts onto the red theta-graph $\Theta = \gamma_0 \cup \gamma_1 \cup \gamma_2$. The three red oriented curves γ_0, γ_1 and γ_2 go as indicated by the gray arrows.

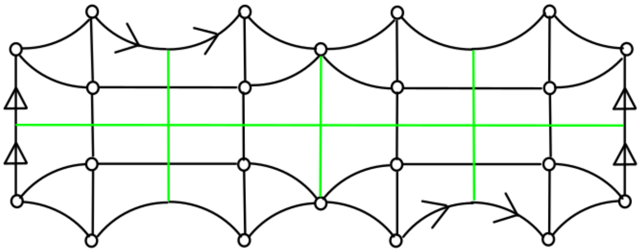

Figure 4. The top of the 3-manifold with corners Σ^{thick}. The green lines indicate its tessellation into 8 copies of P^3. As usual, the ideal vertices are in white.

3. The construction

In this section, we prove Theorem 4. We first build the auxiliary surface with corners Σ, and thicken it to a 3-manifold with corners Σ^{thick} homeomorphic to $\Sigma \times [0,1]$. Then, we build the 3-manifolds with corners N_0, N_1 and N_2 by gluing some of the top facets of Σ^{thick} in three different ways, and the 3-manifold with corners N_{12} by gluing together N_1 and N_2. After that, we glue the 3-manifolds with corners N_0 and N_{12} and thicken the resulting "locally Y-shaped piece" N to a 4-manifold with corners X. Then we study X, and finally build the 4-manifold M.

3.1. **The surface with corners Σ and its thickening Σ^{thick}.** Let Σ be the surface with corners obtained by gluing in pairs some edges of 8 copies of P^2 via the identity map, as indicated in Figure 3. Topologically, Σ is a once-holed torus. Consider the three oriented curves γ_0, γ_1 and γ_2 in the 1-skeleton of Σ as in Figure 3. The surface Σ is a thickening of the theta-graph $\Theta = \gamma_0 \cup \gamma_1 \cup \gamma_2$.

We now place a copy of P^3 "above" each P^2 in Σ, to get a 3-manifold with corners Σ^{thick} homeomorphic to $\Sigma \times [0,1]$: the vertical faces of the P^3's containing the paired edges of the P^2's in Σ are glued correspondingly via the identity map. So Σ^{thick} has three types of facets: the *bottom facet* Σ, and the *vertical* and *top facets* tessellated by the facets of P^3 of the corresponding type. The top facets are 8 ideal triangles, 4 ideal rectangles and 3 ideal hexagons, pleated with right angles along the pattern showed in Figure 4.

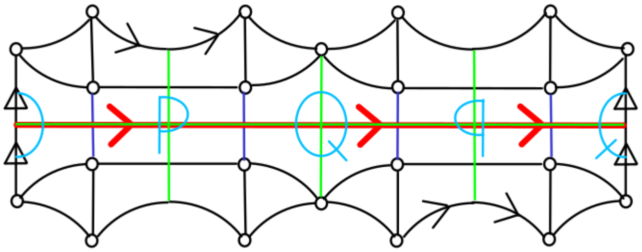

FIGURE 5. The 3-manifold with corners N_0 is built by gluing some top facets of Σ^{thick} as indicated by the blue letters P and Q. It has 5 top facets. The four vertical blue edges are glued making an angle of 2π.

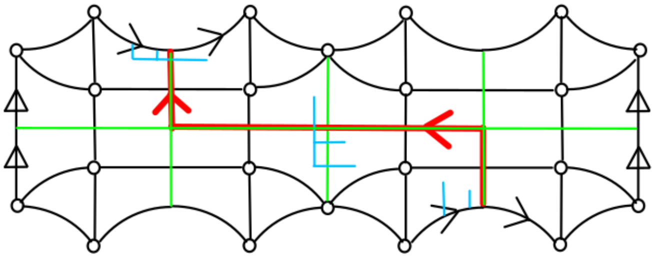

FIGURE 6. The top of the 3-manifold with corners N_1.

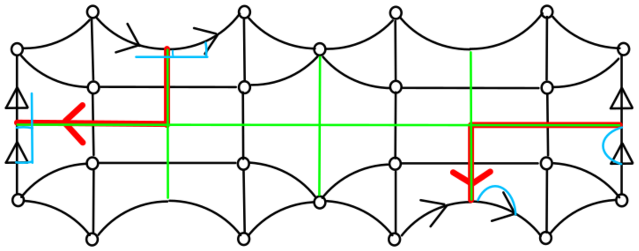

FIGURE 7. The top of the 3-manifold with corners N_2.

3.2. **The 3-manifolds with corners N_0, N_1, N_2 and N_{12}.** Let N_0, N_1 and N_2 be obtained by gluing some top facets of Σ^{thick} in pairs via the identity map, as indicated by Figures 5, 6 and 7, respectively.

Figure 5 helps to verify that N_0 is a 3-manifold with corners and embedded facets: the four glued corners are cyclically glued together in the interior of N_0, and each of the remaining corners is right-angled and belongs to two distinct facets. Moreover, the 8 copies of P^3 in N_0 that are adjacent to Σ are distinct. The check for N_1 and N_2 is even simpler, and is left to the reader.

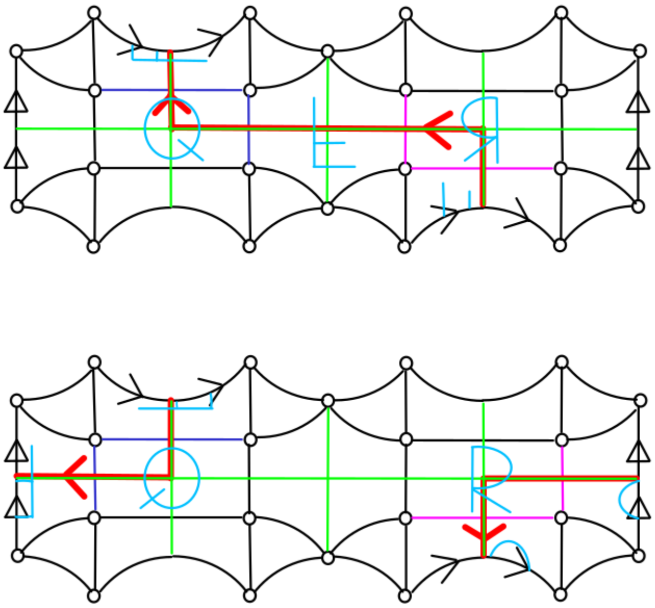

FIGURE 8. The top of the 3-manifold with corners N_{12}, obtained by pairing some top facets of N_1 (top) and N_2 (bottom) as indicated by the symbols P, Q, R and F, and the two bottom facets. It has 15 top facets. The four blue (resp. pink) edges are glued making an angle of 2π.

For $i = 0, 1, 2$, consider the surface with corners $S_i' \cong \gamma_i \times [0, 1]$ in the 2-skeleton of Σ^{thick}, tessellated by the vertical pentagons that have an edge in $\gamma_i \subset \Sigma \subset \Sigma^{\text{thick}}$. The red line in Figures 5, 6 and 7 is the top of S_i'. We call S_i the surface in N_i obtained from S_i' after the gluing. Both N_i and S_i are orientable, since the gluings reverse the orientation of both the glued polygons and the red curve.

We conclude by gluing together N_1 and N_2 as follows: we glue their two bottom facets (copies of Σ) via the identity map, and some of their top facets as in Figure 8. We call N_{12} the resulting 3-manifold with corners. Again, it is easy to check that N_{12} is an orientable 3-manifold with corners and embedded facets, that the 16 copies of P^3 in N_{12} incident to $\Sigma \subset N_{12}$ are distinct, and that $S_{12} = S_1 \cup S_2$ is an orientable surface embedded in N_{12} with $\partial S_{12} = \gamma_1 \sqcup \gamma_2$.

3.3. **The spine N and its thickening X.** Let N be obtained by gluing N_0 and N_{12} via the identity map along their two isometric copies of Σ: the bottom facet of N_0 and the properly embedded surface in N_{12} obtained by identifying the two bottom facets of N_1 and N_2. It is not a manifold (see Figure 1–left).

We now want to thicken N to a 4-manifold with corners X in which N_0 and N_{12} are totally geodesic and orthogonal. Similarly to [13, 14], this can be done in two steps.

We first thicken N_0 and N_{12} separately: we place two copies of P^4 on every copy of P^3, one “below” and the other “above”, and get two 4-manifolds with corners X_0 and X_{12}; see Figure 9–left. These are obtained by pairing some vertical facets of some copies of P^4 with the identity as gluing maps.

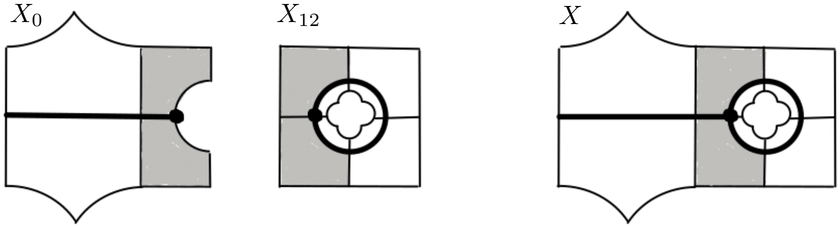

FIGURE 9. A schematic picture of X (right), obtained by gluing the thickenings X_0 and X_{12} of N_0 and N_{12} (left). The pentagons, thick segment, circle and dot represent the copies of P^4, the 3-manifolds N_0 and N_{12}, and the surface Σ, respectively.

Then, we identify in pairs the copies of P^4 in X_0 incident to Σ with the copies of P^4 in X_{12} incident to Σ from below, as in Figure 9–right. We can do this since for every pentagonal face F of P^4, there exists an isometry of P^4 that exchanges the two facets (isometric to P^3) that share F. The resulting complex X contains N as desired.

3.4. **The 4-manifold with corners X.** By construction, X is a complete and orientable hyperbolic 4-manifold with boundary. Since it is tessellated by copies of P^4, which is right angled, a priori the angles at the corners are multiples of $\pi/2$. Our aim is now to show that the angles are $\pi/2$ and the facets are embedded.

Proposition 6. *The thickening X is a hyperbolic manifold with right-angled corners.*

Proof. Every copy of a pentagonal face $F_i \cap F_j \cong P^2$ of P^4 contained in ∂X is shared by at most two copies of P^4 in X. Indeed, as we see from Figure 2, if F and F' are two facets of P^4 such that $F \cap E_i \neq \emptyset$, $F \cap E_j = \emptyset$, $F' \cap E_i = \emptyset$, $F' \cap E_j \neq \emptyset$, then $F \cap F' = \emptyset$. ☐

We now want to show that X has embedded facets. We begin by showing that X_0 and X_{12} have embedded facets. Let Y be a facet of X_0 or X_{12}. Since the latter are obtained by gluing copies of P^4 along facets with the identity, Y is a union of copies of a facet F of P^4. Consider a corner C of X contained in Y. It is not possible that both sides of C are in Y. Indeed, P^4 is right-angled, hence both sides of C are in the same copy of P^4 and, of course, there is only one facet F in P^4.

Proposition 7. *The facets of X are embedded.*

Proof. We argue similarly to the previous paragraph. Let $i\colon X_0 \to X$ and $j\colon X_{12} \to X$ be the natural inclusion embeddings. Let Y be a facet of X. If Y is entirely contained in $i(X_0)$ or $j(X_{12})$, then we easily conclude as in the previous paragraph. Otherwise, $Y \cap i(X_0)$ is union of copies of the facet F of P^4 and $Y \cap j(X_{12})$ is union of copies of the facet $f(F)$ of P^4, where f is the isometry used for the identification in the construction of X starting from X_0 and X_{12}. We conclude as before, since P^4 has only one facet F and one facet $f(F)$. ☐

The construction ensures the following.

Proposition 8. *The surface $S = S_0 \cup S_{12}$ has self-intersection ± 1 in X.*

Proof. We isotope N inside a regular neighbourhood U of N in X as follows. Say that $U = U_0 \cup U_{12}$ for two tubular neighbourhoods U_0 and U_{12} of N_0 and N_{12}. The latter are two-sided. Call U_+ and U_- the two sides of U_{12}, with $U_+ \cup U_- = U_{12}$ and $U_+ \cap U_- = N_{12}$.

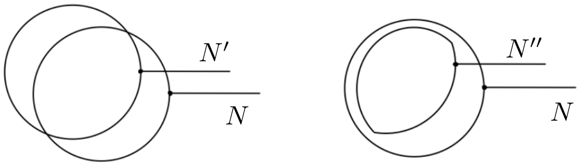

FIGURE 10. On the left, N and its isotopic copy N'. On the right, N and its isotopic copy N'', which transversely intersects N in the point $S \cap S''$, so $S \cdot S = S \cdot S'' = \pm 1$.

We first move N in one direction as in Figure 10–left, obtaining an isotopic copy $N' = N'_0 \cup N'_{12}$ of N transverse to it, with $N' \cap N \subset U_{12}$. Then, to remove the intersection with N'_{12}, we "push" $N' \cap U_+$ in the interior of U_- as in Figure 10–right, obtaining $N'' = N'_0 \cup N''_{12}$.

Then the surface $\Sigma''' = N \cap N'' = N_{12} \cap N'_0$ is a surface parallel to Σ and Σ''. Moreover, S and its isotopic copy $S'' \subset N''$ intersect transversely at one point, corresponding to the transverse intersection of the simple closed curves $\gamma_0''' = S_0 \cap \Sigma'''$ and $\gamma_2''' = S_2 \cap \Sigma'''$ in Σ'''. Therefore $S \cdot S = S \cdot S'' = \pm 1$. □

3.5. **The 4-manifold M.** Let $Y_1, \ldots, Y_m$ the facets of X. We now double X along Y_1, then double the result along the copies of Y_2, and continue iteratively, until we get a 4-manifold M without boundary tessellated by 2^m copies of X. Since the facets of X are embedded, M is hyperbolic manifold (see e.g. [11, Proposition 6]). Moreover, M is arithmetic by Lemma 5. To complete the proof of Theorem 4, it suffices to choose the orientation of M such that $S \cdot S = +1$.

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