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Decomposition theorems for the K-theory of algebraic stacks

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Introduction

In this thesis we will explore and further investigate the problem of localisation in equivariant geometry.

The first instance of this phenomenon already appeared more that fifty years ago; the localisation formula of Atiyah and Bott ([3]) states that for a smooth manifold X with an action of a compact connected Lie group G admitting a finite set of isolated fixed points $i\colon X^G \hookrightarrow X$, then

$$\int_X \alpha = \int_{X^G} \frac{i^*\alpha}{e(\nu)}$$

where $e(\nu)$ is the Euler class of the conormal bundle of X^G in X and α is any cohomology class.

Subsequently, further generalisations were developed in a great many ways: first of all in the context of index theory thanks to the work of Berline-Getzler-Vergne (see [5] or [6]) and Baum-Connes ([4], 1.6). In particular the latter showed that for a finite group G acting on a smooth manifold X the even-dimensional equivariant cohomology localizes on the fixed points sets for the elements of G:

$$H^{ev}_{\mathbb{C}}(X, G) \simeq (\bigoplus_{g\in G} H^{ev}_{\mathbb{C}}(X^g))^G.$$

Second, in equivariant stable homotopy theory it became soon clear that there are universal decompositions in rational and p-adic $G-$equivariant spectra, indexed by the prime ideals of the Burnside ring $A(G)$ of G (defined by Dieck in general, see for example [18], V). Those prime ideals are easily seen to be indexed by conjugacy classes of finite subgroups of G. In particular, for any cohomology theory we can express its rationalised equivariant version as a wedge over some suitable subgroups of G; for example Morava E-theory localises at products of cyclic groups by the work of Hopkins-Kuhn-Ravenel ([13]).

Now let us turn our attention of the algebraic case. An equivariant localisation formula was proposed by Vistoli in his paper [26] for the rational equivariant $K-$theory of schemes with an action of a finite discrete group; it has a form which is the algebraic analogue of the Baum-Connes formula above.

In their paper [25], G. Vezzosi and A. Vistoli proved a decomposition theorem for the rational equivariant K-theory of an algebraic space X by an affine algebraic group G. The pieces of this decomposition are indexed by the conjugacy classes of subgroups of G which are isomorphic to the groups $\boldsymbol{\mu}_n$ of multiplicative type, called *dual cyclic subgroups*.

Assuming a technical hypothesis (that the action is *sufficiently rational*, see the introduction of [25]), they proved the following theorem: defining $\tilde{R}(\sigma)$ to be $\mathbb{Q}(\zeta_n)$ when σ has order n and $w_G(\sigma) := N_G(\sigma)/C_G(\sigma)$

$$K(X,G) \simeq \prod_{\sigma \in \mathcal{C}(G)} (K(X^{\sigma}, C_G(\sigma))_{geom} \otimes \tilde{R}(\sigma))^{w_G(\sigma)}$$

In the formula, $K(-)_{geom}$ is a suitable localization of the rational equivariant K-theory, that Vezzosi-Vistoli conjecture to be isomorphic to the K-theory of the moduli space of the quotient.

In [24], B. Toen defined a *Riemann-Roch map* from the rational algebraic K-theory of a tame Deligne-Mumford quotient stack to the étale K-theory of its inertia. He proved that his map is an isomorphism, respects multiplication in the regular case and that it is covariant with respect to proper-push-forwards. In particular his formulation allows to make calculations with diagrams of the type

$$\begin{tikzcd}
\mathrm{K}'_*(\mathcal{X}) \arrow[r] \arrow[d, "f_*"] & K_{\acute{e}t}(\mathcal{I}\mathcal{X}) \arrow[d, "\mathcal{I}f_*"] \\
\mathrm{K}'_*(\mathcal{Y}) \arrow[r] & K_{\acute{e}t}(\mathcal{I}\mathcal{Y})
\end{tikzcd}$$

In this thesis we first extend the work of Toen to other cases; more precisely, we allow for some cases where the order of the stabilizer of a point is divisible by the characteristic of its residue field (thus the stabilizers may be infinitesimal in nonzero characteristic; in particular we cover the case of *tame* stacks, that is we allow the stabilizers to contain infinitesimal diagonalizable groups).

In the first section (1.1) we introduce (following the work [1] for the DM case) the fundamental object of our localisation formula, that is the *cyclotomic inertia*, defined as the hom-stack

$$\mathcal{I}_{\boldsymbol{\mu}}(-) := \bigsqcup_{n} \underline{Hom}(B\mu_n, -).$$

This has a map $\rho\colon \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \to \mathcal{X}$, and we will compare the rational K-theory of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ to that of $\mathcal{X}$ via the push-forward ρ_*. In order to do that, we need to assume that ρ is finite. For a Deligne-Mumford stack, for example, it is not necessarily the case that ρ is finite: the classifying stack $B\mathbb{Z}/p\mathbb{Z}$ over the p-adic integers $Spec(\mathbb{Z}_p)$, for instance, has the p-component of cyclotomic inertia equal to $B_{\mathbb{Q}_p(\zeta_p)}\mu_p$. Its image lies over $Spec(\mathbb{Q}_p) \to Spec(\mathbb{Z}_p)$, and it is clearly not finite.

We make use of a suitable localisation of the algebraic K-theory of a stack, which plays the same role as that of étale K-theory in Toen's paper: inspired by [25], we define the geometric rational K-theory of an algebraic stack $\mathcal{X}$ with finite stabilizers, and we show that it is a piece in an intrinsic decomposition of the K-theory of the stack. This is the content of section 1.3.

The content of the sections 1.1, 1.2, 1.3 is essentially due to Vistoli and Schadeck, and a detailed exposition of it can be found in the thesis of L. Schadeck ([20]). In the subsequent four sections (1.4, 1.5, 1.6, 1.7) we prove a localisation formula for tame stacks, and we prove that our formula agrees with that of Toen when the stack is Deligne-Mumford and tame. Moreover we show that we can produce a (non canonical) isomorphism from the rational K-theory of an algebraic stack $\mathcal{X}$ with finite cyclotomic inertia to the rational Chow group of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$.

In the second chapter we make some explicit calculation, and we show how to produce explicit formulas from the work that we exposed in the preceding chapter. Some of these caluclations were done in collaboration with Qiangru Kuang.

In the final chapter, we prove that - analogously to the topological case ([18]) - an action of the Burnside algebra is sufficient to ensure a decomposition of any algebraic equivariant (co)homological functor with the Mackey property.

Finally we apply this machinery to show that such a decomposition can be produced for the modular K-theory functor of Borne ([7]). In particular, in this case the subgroups at which we localise are not necessarily abelian, thus displaying a difference with the previously known formulas for algebraic K-theory.

This project was proposed to me by my phd advisor Angelo Vistoli. I warmly thank him for all the time he spent discussing all this material with me, and for all the help and advises he gave me for the duration of my doctoral studies.

Chapter 1

The decomposition theorem for algebraic K-theory of stacks

In what follows we will assume that $\mathcal{X}$ is a separated algebraic stack with finite stabilizers, and that it can be written as a quotient $\mathcal{X} = [X/G]$ where X is a noetherian algebraic space over an affine connected base scheme $A = Spec(R)$, with R excellent, and $G = GL_{n,R}$. The last condition is actually equivalent to the apparently weaker one that $\mathcal{X} = [X/G]$, where G is a flat $R-$linear algebraic group: indeed by choosing an embedding $G \subseteq \mathrm{GL}_n \overset{\mathrm{def}}{=} \mathrm{GL}_{n,R}$, and replacing X with $X \times^G \mathrm{GL}_n$, we may suppose that $G = \mathrm{GL}_n$. We will always write $\mathcal{X} = [X/G]$, where G is a product of general linear groups on R.

As in [25], we call a subgroup scheme $\sigma \subseteq G$ *dual cyclic* when it is isomorphic to $\boldsymbol{\mu}_{r,R}$ for some $r \geq 1$.

For each $r \geq 1$, the datum of a monomorphism from a dual cyclic subgroup $\boldsymbol{\mu}_{r,R} \to \mathrm{GL}_{n,R}$ is equivalent to a locally free splitting $R^n = \bigoplus_{i=1}^r S_i$; the equivalence is obtained considering the decomposition of R^n into eigenspaces for the action of $\boldsymbol{\mu}_{r,R}$.

Remark. *Consider a map* $\boldsymbol{\mu}_{r,R} \to \mathrm{GL}_{n,R}$ *and the corresponding decomposition* $R^n = \bigoplus_{i=1}^r S_i$. *The* S_i*'s are not necessarily free, and in those cases we say that the cyclic subgroup is not constant; otherwise we call it constant.*

Let $\overline{\mathcal{C}}_r(G)$ be the set of *conjugacy classes* of monomorphisms of dual cyclic subgroups of G of order r. More precisely, the functor which assigns to an $R-$scheme S the set of splittings $R^n = \bigoplus_{i=1}^r S_i$ is represented by an open subscheme U of a product of grassmanians (we shall elaborate more on this in the following sections), and it has a natural G-action (the "change of coordinates" action). The connected components of the quotient $[U/G]$ are indexed by the set of partitions $\nu = (\nu_1, \ldots, \nu_r)$ of n in r parts (where each ν_i is the dimension of the ith eigenspace of $\boldsymbol{\mu}_{r,R}$); for any σ we will call the corresponding ν its *type*, and $\overline{\mathcal{C}}_r(G)$ will indicate the set of types. The moduli space of each ν-component is isomorphic to A; indeed locally on A every dual cyclic subgroup is constant, and conjugacy classes of constant dual subgroup schemes are determined by their *type*.

If $G = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$ is a product of general linear groups then the same discussion applies, with the only difference that the types are indexed by $r-$partitions of each set $\{1, \ldots, n_i\}$.

1.1 The cyclotomic inertia of a separated stack

In this section we define an explore the notion of *cyclotomic inertia*. This notion was first defined in [1] in order to define Gromov-Witten invariants for Deligne-Mumford stacks.

Let $\mathcal{X}$ be as above, r be a positive integer. The r^{th} *cyclotomic inertia* is a stack $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ fibered in groupoids over the category of schemes over R, defined as follows.

An object of $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}(S)$, where S is a scheme, is a pair (ξ, a), where ξ is an object of $\mathcal{X}(S)$ and $a\colon \boldsymbol{\mu}_{r,S} \to \underline{\mathrm{Aut}}_S\,\xi$ is a monomorphism of group schemes. An arrow $f\colon (\xi, a) \to (\eta, b)$ from $(\xi, a) \in \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}(S)$ to $(\eta, b) \in \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}(T)$ consists of an arrow $f\colon \xi \to \eta$ in $\mathcal{X}$, such that the diagram

$$\begin{tikzcd}
\boldsymbol{\mu}_{r,S} \arrow[r, equal] \arrow[d, "a"] & \phi^*\boldsymbol{\mu}_{r,T} \arrow[d, "\phi^* b"] \\
\underline{\mathrm{Aut}}_S\,\xi \arrow[r] & \phi^*\,\underline{\mathrm{Aut}}_T\,\eta
\end{tikzcd}$$

commutes. Here $\phi\colon S \to T$ is the image of f in the category of schemes, and the bottom row is induced by f.

There is an obvious morphism of fibered categories $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \to \mathcal{X}$ that sends (ξ, a) into ξ.

For the main properties of this construction, we refer to [20]. Here we will just mention the relevant definitions and facts.

Suppose that Δ is a finite diagonalizable group scheme over R, and G is an affine group scheme of finite type over R. Consider the contravariant functor $\mathrm{H}_\Delta(G)$ from schemes over R to sets, sending a scheme S into the set of homomorphisms of group schemes $\Delta_S \to G_S$. We will think of $\mathrm{H}_\Delta(G)$ as a Zariski sheaf. We denote by $\mathrm{H}^{\mathrm{in}}_\Delta(G) \subseteq \mathrm{H}_\Delta(G)$ the subsheaf consisting of the set of monomorphisms of group schemes $\Delta_S \to G_S$.

If $\phi\colon \Delta \to \Delta'$ is a homomorphism of diagonalizable group schemes, composing with ϕ gives a natural transformation $\mathrm{H}_{\Delta'}(G) \to \mathrm{H}_\Delta(G)$. Call $\mathrm{Q}(\Delta)$ the set of quotients of Δ. For each $\Delta' \in \mathrm{Q}(\Delta)$, consider the composite $\mathrm{H}^{\mathrm{in}}_{\Delta'}(G) \subseteq \mathrm{H}_{\Delta'}(G) \to \mathrm{H}_\Delta(G)$, which is immediately seen to be a monomorphism. This induces a G-equivariant morphism of Zariski sheaves

$$\coprod_{\Delta' \in \mathrm{Q}(\Delta)} \mathrm{H}^{\mathrm{in}}_{\Delta'}(G) \longrightarrow \mathrm{H}_\Delta(G)\,.$$

Proposition 1.1.1.

(1) *The functors* $\mathrm{H}_\Delta(G)$ *and* $\mathrm{H}^{\mathrm{in}}_\Delta(G)$ *are represented by a quasi-projective scheme over* R.

(2) *If* $\Delta' \in \mathrm{Q}(\Delta)$*, then* $\mathrm{H}^{\mathrm{in}}_{\Delta'}(G)$ *is open and closed in* $\mathrm{H}_\Delta(G)$.

(3) *The morphism* $\coprod_{\Delta' \in \mathrm{Q}(\Delta)} \mathrm{H}^{\mathrm{in}}_{\Delta'}(G) \to \mathrm{H}_\Delta(G)$ *is an isomorphism.*

(4) *If G is finite and linearly reductive, then* $\mathrm{H}_\Delta(G)$ *and* $\mathrm{H}^{\mathrm{in}}_\Delta(G)$ *are finite over* $\operatorname{Spec} R$. *If G is étale and every localization R_p is equicharacteristic,* $\mathrm{H}_\Delta(G)$ *and* $\mathrm{H}^{\mathrm{in}}_\Delta(G)$ *are finite over* $\operatorname{Spec} R$.

The proof can be found in the appendix, Proposition A.2.

There is an action of G on $\mathrm{H}_{\Delta}(G)$, by conjugation, leaving $\mathrm{H}^{\mathrm{in}}_{\Delta}(G)$ invariant. Consider the closed subscheme $X_{\boldsymbol{\mu}_r} \subseteq X \times \mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G)$ defined as follows. Let (x, ϕ) be a point of $\bigl(X \times \mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G)\bigr)(S)$; that is, x is a morphism of schemes $S \to X$, and $\phi\colon \boldsymbol{\mu}_{r,S} \to G_S$ is a monomorphism of group schemes. Then ϕ induces an action of $\boldsymbol{\mu}_{r,S}$ on the S-scheme $S \times X$; we say that (x, ϕ) is in $X_{\boldsymbol{\mu}_r}(S)$ if the section $S \to S \times X$ defined by x is fixed under the action of $\boldsymbol{\mu}_{r,S}$.

The subscheme $X_{\boldsymbol{\mu}_r} \subseteq X \times \mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G)$ is G-invariant; the projection $X_{\boldsymbol{\mu}_r} \to X$ is G-equivariant, and defines a morphism of algebraic stacks $[X_{\boldsymbol{\mu}_r}/G] \to [X/G] = \mathcal{X}$.

Proposition 1.1.2. *The quotient stack $[X_{\boldsymbol{\mu}_r}/G]$ is naturally equivalent to $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$.*

Proof. We can immediately exhibit an equivalence. Namely, let (ξ, a) be a $S-$point of $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$: then ξ correspond to an equivariant morphism $g\colon P \to X$, where P is a principal $G-$bundle over S; moreover a gives a morphism $\boldsymbol{\mu}_{r,S} \to G_S$ such that g factors through $P/\boldsymbol{\mu}_{r,S} \to X$.

By definition these data give a unique equivariant morphism $P \to X \times \mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G)$, and it is immediate to see that it factors through $X_{\boldsymbol{\mu}_r}$. In particular they provide a well-defined $S-$point of $[X_{\boldsymbol{\mu}_r}/G]$, and we immediately see that this map can be extended to the morphisms $(\xi, a) \to (\eta, b)$, giving a stack map.

Conversely, given a principal $G-$bundle $P \to S$ and an equivariant map $P \to X_{\boldsymbol{\mu}_r}$ we can compose it with the inclusion $X_{\boldsymbol{\mu}_r} \subseteq X \times \mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G)$, giving two maps $\xi\colon P \to X$ and $P \to \mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G)$. From the second one, quotienting by G, we get a map $\boldsymbol{\mu}_{r,s} \to G_S$; moreover, $P \to X$ being $\boldsymbol{\mu}_{r,P}$-invariant, we have a map $P/\boldsymbol{\mu}_{r,S} \to X$ and $\boldsymbol{\mu}_{r,S} \to G_S$ factors through $\underline{\mathrm{Aut}}_S\,\xi$. This gives a morphism $[X_{\boldsymbol{\mu}_r}/G] \to \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$, and we immediately see that it is inverse to the above map. ♠

From this we can deduce the following important property.

Proposition 1.1.3. *Let $\mathcal{X}$ be an algebraic stack over R. Then the morphism $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \to \mathcal{X}$ is representable; moreover*

1. *If $\mathcal{X}$ is tame, then $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ is also a tame algebraic stack; furthermore, the morphism $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \to \mathcal{X}$ is finite.*
2. *If $\mathcal{X}$ is Deligne-Mumford and R is equicharacteristic, then $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ is also Deligne-Mumford; furthermore, the morphism $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \to \mathcal{X}$ is finite.*

Proof. Let us show that, in both cases, the morphism $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \to \mathcal{X}$ is representable and finite. Formation of $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ commutes with base change on M, so the question is fppf local on M. Locally on M the stack $\mathcal{X}$ is of the form $[X/\Gamma]$, where $X \to M$ is a finite morphism and $\Gamma \to M$ is a linearly reductive finite group scheme (in the first case, [2]) or an étale group scheme (in the second case, [16], Chapter 6); by further refining in the fppf topology, we can assume that Γ is obtained by pullback from a finite linearly reductive group scheme G over A. From the construction above we have a factorization

$$\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \subseteq [(X \times \mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G))/G] \longrightarrow [X/G] = \mathcal{X}$$

where the first homomorphism is a closed embedding, and the second is finite because of Proposition A.2(4). ♠

Remark. *The proofs in the above paragraph show, in particular, the following: if* $\mathcal{X} = [X/G]$ *with* $G = \mathrm{GL}_{n,R}$ *then* $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ *is the disjoint union*

$$\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} = \coprod_{\substack{\nu \in \overline{\mathcal{C}}_r(G) \\ |\nu| = r}} [X^\sigma / C_G(\sigma)].$$

where σ *is any dual cyclic subgroup of type* ν.

Indeed, for any S *the set* $\mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G)(S)$ *is parametrized by the dual cyclic subgroups of* $\mathrm{GL}_n(S)$. *For each such* σ, G *acts on the set* $(X^\sigma \times H_\mu G)(S)$ *with stabilizer* $C_G(\sigma)$ *(where* H_μ *is the subscheme of* $\mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G)$ *parametrizing subgroups of type* μ*). The result follows.*

Functoriality of cyclotomic inertia stacks

If $f\colon \mathcal{Y} \to \mathcal{X}$ is a representable morphism of stacks, there is an obvious induced morphism $\mathcal{I}_{\boldsymbol{\mu}_r} f\colon \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{Y} \to \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$: an object $(\eta, b) \in \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{Y}$ is sent into $(f\eta, f_\eta \circ b) \in \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$, where $f_\eta\colon \mathrm{Aut}_S\,\eta \to \mathrm{Aut}_S(f\eta)$ is the homomorphism induced by f. The point here is that f_η is a monomorphism, because f is representable. If f is not representable then f_η is not a monomorphism, so $(f\eta, f_\eta \circ b)$ is not an object of $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}(S)$.

However, while the single r^{th} inertia stacks are not functorial for nonrepresentable morphisms, their disjoint union is functorial. Let us define the *cyclotomic inertia* $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ of $\mathcal{X}$ as the disjoint union of the $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$. Since there a finite number of r such that $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \neq \emptyset$, the morphism $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \to \mathcal{X}$ is finite.

The stack $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ has an alternate description that makes its functoriality properties evident. Set $\boldsymbol{\mu}_\infty \overset{\mathrm{def}}{=} \varprojlim_n \boldsymbol{\mu}_n$, where the index set is the set of positive integers ordered by divisibility, and homomorphisms $\boldsymbol{\mu}_n \to \boldsymbol{\mu}_m$ are defined as $z \mapsto z^{n/m}$. Then $\boldsymbol{\mu}_\infty$ is a affine group scheme over S, which is not of finite type.

Denote by $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$ the stack that is defined as follows. An object (ξ, ϕ) of $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$ over a scheme S consists of an object ξ of $\mathcal{X}(S)$ and a homomorphism of group schemes $\phi\colon \boldsymbol{\mu}_{\infty,S} \to \underline{\mathrm{Aut}}_S\,\xi$. The arrows are defined in the obvious way. Forgetting ϕ gives a morphism $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X} \to \mathcal{X}$.

There is a morphism $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \to \mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$ defined as follows. If (ξ, a) is an object of $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}(S)$, we obtain an object (ξ, ϕ) of $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}(S)$, where ϕ is the composite of $a\colon \boldsymbol{\mu}_{r,S} \hookrightarrow \underline{\mathrm{Aut}}_S\,\xi$ with the projection $\boldsymbol{\mu}_{\infty,S} \to \boldsymbol{\mu}_{r,S}$. The definition for arrows is clear.

These give a morphism $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \to \mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$.

Proposition 1.1.4. *The morphism* $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \to \mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$ *defined above is an equivalence of stacks over* $\mathcal{X}$.

Proof. Choose a positive integer N that is divisible by the orders of the automorphism group schemes of all the geometric point of $\mathcal{X}$. Then every homomorphism $\boldsymbol{\mu}_{\infty,S} \to \underline{\mathrm{Aut}}_S\,\xi$ factors uniquely through $\boldsymbol{\mu}_{N,S}$. Hence we can restate the definition of $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$ by taking objects (ξ, ϕ), where ξ is an object of $\mathcal{X}(S)$ and $\phi\colon \boldsymbol{\mu}_{N,S} \to \underline{\mathrm{Aut}}_S\,\xi$ is a homomorphism (this will actually give a stack that is strictly isomorphic to $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$).

This gives a description of the stack $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$ as a quotient stack, similar to the one we have given for $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$. Write $Y \subseteq X \times \mathrm{H}_N(G)$ for the closed locus defined by the subfunctor of pair (ξ, ϕ) such that ξ is fixed under the action of $\boldsymbol{\mu}_N$ defined by ϕ; then $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X} = [Y/G]$.

For each positive integer r that divides N, consider $\boldsymbol{\mu}_r$ as a quotient of $\boldsymbol{\mu}_N$; there is an embedding $\mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G) \subseteq \mathrm{H}_{\boldsymbol{\mu}_N}(G)$. Furthermore, the intersection of Y with $X \times \mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G) \subseteq X \times \mathrm{H}_{\boldsymbol{\mu}_N}(G)$ is exactly the subscheme $X_{\boldsymbol{\mu}_r} \subseteq X \times \mathrm{H}^{\mathrm{in}}_{\boldsymbol{\mu}_r}(G)$ that intervenes in the proof of Proposition 1.1.2, so that $[X_{\boldsymbol{\mu}_r}/G] = \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$, and the morphism $[X_{\boldsymbol{\mu}_r}/G] \subseteq [Y/G]$ is isomorphic to the morphism $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \to \mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$ defined above. The the result follows from Proposition A.2(3). ♠

From now on we will identify $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ with $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$, via the equivalence above.

This description of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ makes its functorial property evident. If $f\colon \mathcal{Y} \to \mathcal{X}$ is a morphism of stacks and (η, b) is an object of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}(S) = \mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{Y}(S)$, then $(f\eta, b \circ f_\eta)$ is in fact an object of $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}(S)$. This function from the objects of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y}$ to the objects of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ extends easily to a functor $\mathcal{I}_{\boldsymbol{\mu}}f\colon \mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y} \to \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$, such that the diagram

$$\begin{tikzcd}
\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y} \arrow[r, "\mathcal{I}_{\boldsymbol{\mu}}f"] \arrow[d] & \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \arrow[d] \\
\mathcal{Y} \arrow[r, "f"] & \mathcal{X}
\end{tikzcd}$$

is strictly commutative. This gives a strict 2-functor from stacks to stacks, sending $\mathcal{X}$ into $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$.

Yet another description of $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ and $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ is as follows. Suppose that (ξ, a) is an object of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}(S)$; here ξ is an object of $\mathcal{X}(S)$, and $a\colon \boldsymbol{\mu}_{\infty,S} \to \underline{\mathrm{Aut}}_S\,\xi$ is a homomorphism of group schemes. This gives an action of $\boldsymbol{\mu}_\infty$ on ξ, which, in turn, gives a morphism of stacks $\phi\colon \mathcal{B}_S\boldsymbol{\mu}_{\infty,S} \to \mathcal{X}$. Conversely, such a morphism of stacks gives an object ξ of $\mathcal{X}(S)$, defined as the image of the trivial torsor $\boldsymbol{\mu}_{\infty,S} \to S$ through ϕ, while the action of $\boldsymbol{\mu}_{\infty,S}$ on ξ is defined as the homomorphism $\boldsymbol{\mu}_{\infty,S} = \underline{\mathrm{Aut}}_S\,\boldsymbol{\mu}_{\infty,S} \to \underline{\mathrm{Aut}}_S\,\xi$. This gives an equivalence between $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ and the stack $\underline{\mathrm{Hom}}_k(\mathcal{B}_k\boldsymbol{\mu}_\infty, \mathcal{X})$, whose objects over S are morphisms $\mathcal{B}_S\boldsymbol{\mu}_{\infty,S} \to \mathcal{X}$. An arrow from $\phi\colon \mathcal{B}_S\boldsymbol{\mu}_{\infty,S} \to \mathcal{X}$ to $\psi\colon \mathcal{B}_S\boldsymbol{\mu}_{\infty,S} \to \mathcal{X}$ consists a morphism of schemes $f\colon S \to T$, and a 2-commutative diagram

$$\begin{tikzcd}
\mathcal{B}_S\boldsymbol{\mu}_{\infty,S} \arrow[dd] \arrow[dr, "\phi"] & \\
& \mathcal{X} \\
\mathcal{B}_S\boldsymbol{\mu}_{\infty,T} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

in which the column is the homomorphism $\mathcal{B}_S\boldsymbol{\mu}_{\infty,S} \to \mathcal{B}_T\boldsymbol{\mu}_{\infty,T}$ induced by $f\colon S \to T$.

A similar description can be given for $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$. Consider a morphism $\phi\colon \mathcal{B}_S\boldsymbol{\mu}_{r,S} \to \mathcal{X}$; call $\xi \in \mathcal{X}(S)$ the image of the trivial torsor, and $a\colon \boldsymbol{\mu}_{r,S} \to \underline{\mathrm{Aut}}_S\,\xi$ the induced homomorphism. Then ϕ is representable if and only if a is an embedding of group schemes. Hence $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ can be described as follows: the objects of $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ consist of representable morphisms $\mathcal{B}_S\boldsymbol{\mu}_{r,S} \to \mathcal{X}$. The arrows can be described as in the case of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$.

For each $r > 0$ consider the automorphism group $\mathrm{Aut}\,\boldsymbol{\mu}_r$ of $\boldsymbol{\mu}_r$, which is the group of units in the ring $\mathbb{Z}/r\mathbb{Z}$. There is obvious strict right action of $\mathrm{Aut}\,\boldsymbol{\mu}_r$ on $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$, in which an automorphism u of $\boldsymbol{\mu}_r$ sends an object (ξ, a) of $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}(S)$ into $(\xi, a \circ u)$.

Consider now the profinite group $\mathrm{Aut}\,\boldsymbol{\mu}_\infty = \varprojlim_r \mathrm{Aut}\,\boldsymbol{\mu}_r$; this acts on each $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ through its quotient $\mathrm{Aut}\,\boldsymbol{\mu}_r$. This induces an action of $\mathrm{Aut}\,\boldsymbol{\mu}_\infty$ on $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} = \coprod_r \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$.

This can also be described using the identification of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ with $\mathcal{I}_{\boldsymbol{\mu}_\infty}\mathcal{X}$ by the same formula as above.

The stack-theoretic quotient $[\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}/\operatorname{Aut}\boldsymbol{\mu}_r]$ can be described as follows. Denote by $\mathcal{D}_{\boldsymbol{\mu}_r}\mathcal{X}$ the stack whose object over S are triples (ξ, σ, a), where ξ is an object of $\mathcal{X}(S)$, $\sigma \to S$ is a finite group scheme, étale-locally isomorphic to $\boldsymbol{\mu}_{r,S}$, and $a \colon \sigma \to \underline{\operatorname{Aut}}_S\,\xi$ is an embedding of group schemes over S. The arrows are defined in the obvious way.

There is a natural morphism $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \to \mathcal{D}_{\boldsymbol{\mu}_r}\mathcal{X}$ that associates with each object (ξ, a) of $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}(S)$ the object $(\xi, \boldsymbol{\mu}_{r,S}, a)$ of $\mathcal{D}_{\boldsymbol{\mu}_r}\mathcal{X}$. This is easily seen to make $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ into a $\operatorname{Aut}\boldsymbol{\mu}_r$-torsor over $\mathcal{D}_{\boldsymbol{\mu}_r}\mathcal{X}$; the corresponding action of $\operatorname{Aut}\boldsymbol{\mu}_r$ on $\mathcal{I}_{\boldsymbol{\mu}}$ is exactly the one described above. This construction will not be used in what follows.

1.2 Conventions and notation

Let us fix the notation. Let $A = Spec(R)$ be a noetherian excellent connected scheme; all the objects in the subsequent sections will be intended relative to A.

In particular let $\mathcal{X} = [X/G]$ be a separated algebraic stack, with $G = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$ a finite product of general linear groups. Moreover, we will always assume that the cyclotomic inertia map $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \to \mathcal{X}$ is finite: this is fundamental to compare the K-theory of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ and that of $\mathcal{X}$ via the push-forward map.

It is easy to see that the cyclotomic inertia is not finite over $\mathcal{X}$ whenever there is a stabilizer diagnalizable group scheme degenerating to a non-diagonalizable group over a closed subset: for example when $\mathcal{X} = B_R G$ with R a dvr such that G is diagonalizable over the generic fiber and unipotent over the special fiber. We may consider $G = \mathbb{Z}/p\mathbb{Z}$ over $Spec(\mathbb{Z}_p)$ or $G = Spec(\mathbb{F}_p[x, y]_{(x)}/(y^p - 1))$ as a group scheme over $Spec(\mathbb{F}_p[x]_{(x)})$ with group law $(y_1, y_2) \to y_1 + y_2 + x \cdot y_1 y_2$. In those cases the fiber of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \to \mathcal{X}$ over the closed point of A is empty, while it is nonempty over the generic point of A: in particular $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \to \mathcal{X}$ cannot be proper.

Let $\overline{\mathcal{C}}_r(G)$ be the set of *types* of dual cyclic subgroups of G, or equivalently the set of constant dual cyclic subgroups of G. From now on, when we pick a dual cyclic subgroup $\sigma \subset G$ of any given type, we will mean a *constant* representative of the type.

The group $Aut(\boldsymbol{\mu}_r)$ has a natural action on $\overline{\mathcal{C}}_r(G)$, and we let $\mathcal{C}_r(G)$ be the set of orbits for this action.

Finally, for a group scheme G of multiplicative type over A, let $\mathrm{R}G$ be the rational ring of characters of G. If $G = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$, let $\mathrm{R}G$ be the rational Grothendieck ring of highest-weight algebraic representations of G (which is isomorphic to $\mathbb{Z}[\hat{T}]^W$, where $\hat{T}$ denotes the characters of a maximal torus and W is the Weyl group of G).

Finally, for a stack $\mathcal{X}$, let $\mathrm{K}'_*(\mathcal{X})$ denote its algebraic K-theory of coherent sheaves.

1.3 The intrinsic decomposition of the K-theory of a quotient stack

Fix now $G = \mathrm{GL}_{n,R}$.

For any $r \geq 1$ we have a decomposition $\mathrm{R}\boldsymbol{\mu}_r = \prod_{s|r} \mathbb{Q}(\zeta_s)$. We denote by $\mathrm{R}\sigma \to \widetilde{\mathrm{R}}\sigma$ the projection onto the $\mathbb{Q}(\zeta_r)$ factor. Recall from [25] that if $\sigma \subseteq G$ is a constant dual cyclic subgroup, $\mathfrak{m}_\sigma \subseteq \mathrm{R}G$ is the maximal ideal that is the kernel of the composite $\mathrm{R}G \to \mathrm{R}\sigma \to \widetilde{\mathrm{R}}\sigma$; this only depends on the *type* of σ. If M is an $\mathrm{R}G$-module, the *σ-localization* M_σ is the localization $M_{\mathfrak{m}_\sigma}$. If $\sigma \subseteq G$ is a constant dual cyclic subgroup, then $\mathrm{K}'_*(X,G)_\sigma \neq 0$ if and only if $X^\sigma \neq \emptyset$.

We define the multiplicative system $\Sigma_r^{\mathcal{X}} = \Sigma_r \subseteq \mathrm{K}_0\,\mathcal{X}$ as follows. An element $\alpha \in \mathrm{K}'_*\,\mathcal{X}$ is in Σ_r if for all representable morphisms $\phi\colon \mathcal{B}_S\boldsymbol{\mu}_r \to \mathcal{X}$, where S is an R-scheme, the projection of $\phi^*a \in \mathrm{R}\boldsymbol{\mu}_r$ in $\widetilde{\mathrm{R}}\boldsymbol{\mu}_r = \mathbb{Q}(\zeta_r)$ is non-zero.

In particular, Σ_1 admits the following description. If $\mathcal{X}_1, \ldots, \mathcal{X}_m$ are the connected components of $\mathcal{X}$, then $\mathrm{K}'_*\,\mathcal{X} = \bigoplus_{i=1}^m \mathrm{K}'_*\,\mathcal{X}_i$. There is a rank map $\mathrm{rk}_i\colon \mathrm{K}'_*\,\mathcal{X}_i \to \mathbb{Q}$. Then α is in Σ_1 if and only if $\mathrm{rk}_i\,\alpha \neq 0$ for all i.

A morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of stacks induces a pullback homomorphism $f^*\colon \mathrm{K}_0\,\mathcal{X} \to \mathrm{K}_0\,\mathcal{Y}$ (here by K_0 we denote the naive ring of locally free sheaves on $\mathcal{X}$). If f is representable, one immediately sees that f^* carries $\Sigma_r^{\mathcal{X}}$ into $\Sigma_r^{\mathcal{Y}}$. If f is not representable this is certainly not true in general, but it is if $r = 1$.

Definition. *The $\boldsymbol{\mu}_r$-localization $\mathrm{K}'_*(\mathcal{X})_{(\boldsymbol{\mu}_r)}$ of $\mathrm{K}'_*\,\mathcal{X}$ is the $\mathrm{K}_0\,\mathcal{X}$-module $\Sigma_r^{-1}\,\mathrm{K}'_*(\mathcal{X})$.*

Set $\mathrm{K}'_*(X,G)_{(r)} \overset{\mathrm{def}}{=} \prod_{\sigma\in\mathcal{C}_r(G)} \mathrm{K}'_*(X,G)_\sigma$, so that $\mathrm{K}'_*(X,G) = \prod_{r\geq 1} \mathrm{K}'_*(X,G)_{(r)}$. The support of $\mathrm{K}'_*(X,G)_\sigma$ in $\operatorname{Spec}\mathrm{R}G$ is the maximal ideal $\mathfrak{m}_\sigma = \ker(\mathrm{R}G \to \widetilde{\mathrm{R}}\sigma)$.

Let $\sigma_1, \ldots, \sigma_m$ be (constant) representatives of dual cyclic subgroups $\sigma \in \mathcal{C}(G)$ with $|\sigma| = r$ such that $\mathrm{K}'_*(X,G)_\sigma \neq 0$. Then

$$\mathrm{K}'_*(X,G)_{(r)} = \prod_{\substack{\sigma\in\mathcal{C}(G)\\ |\sigma|=r}} \mathrm{K}'_*(X,G)_\sigma = \prod_{i=1}^m \mathrm{K}'_*(X,G)_{\sigma_i}.$$

Suppose that $\phi\colon \mathcal{B}_S\sigma \to \mathcal{X}$ is a representable morphism, where σ is a constant dual cyclic group over S. We may assume that there is an embedding $\sigma \subseteq G_S$, and a rational point $p \in X(S)$ which is fixed under the action of σ. Hence $X_S^g \neq \emptyset$, so σ is conjugate to some σ_i (that is it belongs to the same type). We say in this case that ϕ is *associated* to σ_i.

Let σ be any of the σ_i's; we define another multiplicative system $\Sigma_\sigma^{\mathcal{X}} = \Sigma_\sigma \subseteq \mathrm{K}_0\,\mathcal{X}$ as follows: $\alpha \in \mathrm{K}_0\,\mathcal{X}$ lies in Σ_σ if and only if for any S/A and any $\phi\colon \mathcal{B}_S\boldsymbol{\mu}_r \to \mathcal{X}$ associated to σ the pullback $\phi^*\alpha \in \mathrm{R}\boldsymbol{\mu}_r$ has a nonzero projection to $\widetilde{\mathrm{R}}\boldsymbol{\mu}_r$.

Proposition 1.3.1. *Let r be a fixed positive integer and $\sigma \in \mathcal{C}(G)$ be a constant dual cyclic subgroup with $|\sigma| = r$ such that $\mathrm{K}'_*(X,G)_\sigma \neq 0$.*

1. *The projection $\mathrm{K}'_*(X,G) \to \prod_{\sigma\in\mathcal{C}(G)} \mathrm{K}'_*(X,G)_\sigma$ is an isomorphism.*
2. *The $\boldsymbol{\mu}_r$-localization $\mathrm{K}'_*\,\mathcal{X} \to \mathrm{K}'_*(\mathcal{X})_{(\boldsymbol{\mu}_r)}$ isomorphic to the projection $\mathrm{K}'_*(X,G) \to \mathrm{K}'_*(X,G)_{(r)}$.*
3. *The localization $\Sigma_\sigma^{-1}\,\mathrm{K}'_*(\mathcal{X})$ is isomorphic to $\mathrm{K}'_*(X,G)_\sigma$.*

As a corollary, we get the following.

Theorem 1.3.2. *The projections* $\mathrm{K}'_*(\mathcal{X}) \to \mathrm{K}'_*(\mathcal{X})_{(\boldsymbol{\mu}_r)}$ *induce an isomorphism*

$$\mathrm{K}'_*(\mathcal{X}) \simeq \prod_{r\geq 1} \mathrm{K}'_*(\mathcal{X})_{(\boldsymbol{\mu}_r)}\,.$$

Another consequence is this.

Corollary 1.3.3. *We have* $\mathrm{K}'_*(\mathcal{X})_{(\boldsymbol{\mu}_r)} \neq 0$ *if and only if there exists a geometric point of* $\mathcal{X}$ *whose automorphism group scheme contains a dual cyclic subgroup of order* r*.*

In particular when $\mathcal{X}$ *is tame,* $\mathrm{K}'_*(\mathcal{X})_{(\boldsymbol{\mu}_r)} = 0$ *for all* $r > 1$ *if and only if* $\mathcal{X}$ *is an algebraic space.*

Proof of Proposition 1.3.1. We are going to prove the three stataments jointly. Let σ_1, …, σ_m be all the constant dual cyclic subgroups $\sigma \in \mathcal{C}(G)$ with $|\sigma| = r$ such that $\mathrm{K}'_*(X,G)_\sigma \neq 0$.

Call $S_\sigma := \mathrm{R}G \smallsetminus \mathfrak{m}_\sigma$ and $S_r \subseteq \mathrm{R}G$ the multiplicative system $\mathrm{R}G \smallsetminus (\mathfrak{m}_{\sigma_1} \cup \cdots \cup \mathfrak{m}_{\sigma_m})$; the induced map $S_r^{-1}\,\mathrm{K}'_*(X,G) \to \prod_{i=1}^m \mathrm{K}'_*(X,G)_{\sigma_i}$ is an isomorphism (this follows easily from the fact that the support of $\mathrm{K}'_*(X,G)$ in $\operatorname{Spec}\mathrm{R}G$ consists of a finite number of closed points).

We claim that the images of $S_r, S_\sigma \subseteq \mathrm{R}G$ in $\mathrm{K}_0(X,G)$ through the homomorphism $\mathrm{R}G \to \mathrm{K}_0(X,G)$ are contained in Σ_r, Σ_σ respectively. In fact, let $\alpha \in S_r$, suppose that $\mathcal{B}_S\sigma \to \mathcal{X}$ is a representable morphism, where σ is a dual cyclic group over S; it is associated to one of the σ_i. The morphisms $\mathrm{R}G \to \widetilde{\mathrm{R}}\sigma_i = \mathbb{Q}(\zeta_r)$ defined by the embedding $\sigma_i \subseteq G$ and the composite

$$\mathrm{R}G \longrightarrow \mathrm{K}_0(X,G) \longrightarrow \mathrm{K}_0\,\mathcal{B}_S\sigma = \mathrm{R}\sigma \longrightarrow \widetilde{\mathrm{R}}\sigma = \mathbb{Q}(\zeta_r)$$

defined by the morphism $\mathcal{B}_S\sigma \to \mathcal{X}$ coincide; since by hypothesis α does not map to 0 in $\mathbb{Q}(\zeta_r)$, it follows that the image of α in $\mathrm{K}_0\,\mathcal{X}$ is in Σ_r, as claimed.

If α lies in Σ_σ the reasoning is completely analogous.

Notice that if $\alpha \in \mathrm{K}_0\,\mathcal{X} = \mathrm{K}_0(X,G)$, the multiplication by α on $\mathrm{K}'_*(X,G)$ gives a homomorphism of $\mathrm{R}G$-modules, so that it preserves the multiplication above. We get a factorization

$$\begin{tikzcd}
\mathrm{K}'_*(X,G) \arrow[r] & S_r^{-1}\,\mathrm{K}'_*(X,G) \arrow[r] \arrow[d, equal] & \Sigma_r^{-1}\,\mathrm{K}'_*(X,G) \arrow[d, equal] \\
 & \prod\limits_{\substack{\sigma\in\mathcal{C}(G)\\|\sigma|=r}} \mathrm{K}'_*(X,G)_\sigma & \mathrm{K}'_*(X,G)_{(\boldsymbol{\mu}_r)}
\end{tikzcd}$$

we need to show that the resulting homomorphism

$$\prod_{\substack{\sigma\in\mathcal{C}(G)\\|\sigma|=r}} \mathrm{K}'_*(X,G)_\sigma \longrightarrow \mathrm{K}'_*(X,G)_{(\boldsymbol{\mu}_r)}$$

is an isomorphism.

Similarly we have a morphism $\mathrm{K}'_*(X,G)_\sigma \to \Sigma_\sigma^{-1}\,\mathrm{K}_0(\mathcal{X})$ and we want it to be a bijection.

These are equivalent to the following. First of all, notice that if $\alpha \in \mathrm{K}_0(X, G)$, multiplication by α gives an endomorphism of the $\mathrm{R}G$-modules $\mathrm{K}'_*(X, G)$, hence it descends to endomorphisms of $\mathrm{K}'_*(X, G)_\sigma$ for each $\sigma \in \mathcal{C}(G)$. We need to prove that the if α is in Σ_r or in Σ_σ, then α induces an automorphism of $\mathrm{K}'_*(X, G)_\sigma$. Actually, $\Sigma_r \subseteq \Sigma_\sigma$, so me may just suppose $\alpha \in \Sigma_\sigma$.

The proof of this fact is fairly long, and will be split into steps.

Step 1. Assume that G is a finite diagonalizable group over R acting trivially on an affine scheme X. Then $\mathcal{X}$ is of the form $X \times_A \mathcal{B}_R G$ (here the tensor products are intended over $K_0(A)$). Then we have $\mathrm{K}'_* \mathcal{X} = \mathrm{K}'_* X \otimes \mathrm{R}G$ and $\mathrm{K}_0(\mathcal{X}) = \mathrm{K}_0 X \otimes \mathrm{R}G$. Then $\mathcal{C}(G)$ is the set of dual cyclic subgroups of G. We have a decomposition $\mathrm{R}G = \prod_{\sigma \in \mathcal{C}(G)} \widetilde{\mathrm{R}}\sigma$. Then $\mathrm{K}'_*(X, G)_\sigma = \mathrm{K}'_* X \otimes \widetilde{\mathrm{R}}\sigma$ and $\mathrm{K}'_*(X, G)_{(r)} = \mathrm{K}'_* X \otimes \prod_{\substack{\sigma \in \mathcal{C}(G)\\ |\sigma| = r}} \widetilde{\mathrm{R}}\sigma$. In particular, (1) follows immediately.

The action of $\mathrm{K}_0(X, G)$ on $\mathrm{K}'_*(X, G)_\sigma$ is induced by the action of $\mathrm{K}_0(X, G) = \mathrm{K}_0 X \otimes \mathrm{R}\sigma$ on $\mathrm{K}'_*(X, G) = \mathrm{K}'_* X \otimes \widetilde{\mathrm{R}}\sigma$, and this in turn is induced by the action of $\mathrm{K}_0 X$ on $\mathrm{K}'_* X$. This factors through an action of $\mathrm{K}_0 X \otimes \widetilde{\mathrm{R}}\sigma$; thus it is enough to show that the image of α in $\mathrm{K}_0 X \otimes \widetilde{\mathrm{R}}\sigma$ is invertible.

Let $X_1, \ldots, X_m$ be the connected components of X. Then $\mathrm{K}_0 X = \prod_{i=1}^m \mathrm{K}_0 X_i$. There is a rank homomorphism $\mathrm{rk}\colon \mathrm{K}_0 X_i \to \mathbb{Q}$, whose kernel is nilpotent ([11], V, sect. 3); from this we obtain a homomorphism

$$\mathrm{K}'_* X \otimes \widetilde{\mathrm{R}}\sigma \longrightarrow (\widetilde{\mathrm{R}}\sigma)^m$$

with nilpotent kernel. Hence it is enough to show that the image of α in each copy of $\widetilde{\mathrm{R}}\sigma$ is non-zero. But α_i is obtained as follows: choose a point $\operatorname{Spec} S \to X_i$. This gives a morphism $\mathcal{B}_S G_S \to X \times \mathcal{B}_R G$; by composing with the morphism $\mathcal{B}_S \sigma_S \to \mathcal{B}_S G_S$ induced by the embedding $\sigma \subseteq G$; the resulting homomorphism $\mathrm{K}_0(X, G) \to \widetilde{\mathrm{R}}\sigma$ is immediately seen to coincide with the i^{th} homomorphism $\mathrm{K}'_* X \otimes \widetilde{\mathrm{R}}\sigma \to \widetilde{\mathrm{R}}\sigma$. Since this is not zero, by hypothesis, this concludes the proof.

Step 2: the case of a torus action. Here we assume that $G = \mathbb{G}_{\mathrm{m}}^n$ is a split torus. If $Y \subseteq X$ is a G-invariant subscheme of X, we denote by α_Y the restriction of α to $\mathrm{K}_0(Y, G)$. By noetherian induction, we can assume that α_Y induces an automorphism of $\mathrm{K}'_*(Y, G)_\sigma$ for all proper closed G-invariant subschemes $Y \to X$. By [22] there exists an open G-invariant subscheme U that is G-equivariantly isomorphic to a scheme of the form $V \times (G/\Gamma)$, where $\Gamma \subseteq G$ is a finite diagonalizable subgroup scheme, V is a scheme over R, and the action of G on $V \times (G/\Gamma)$ is induced by the trivial action of G on V and the action on G/Γ by translation. By shrinking again V we may assume that V is affine and connected.

Assume that $X = U$. In this case $\mathrm{K}'_*(X, G) = \mathrm{K}'_*(U, \Gamma) = \mathrm{K}'_* U \otimes \mathrm{R}\Gamma$. If $\sigma \nsubseteq \Gamma$, then $X^\sigma = \emptyset$, so that $\mathrm{K}'_*(X, G)_\sigma = 0$, and the result is obvious. If $\sigma \subseteq \Gamma$, then it is easy to convince oneself that $\mathrm{K}'_*(X, G)_\sigma = \mathrm{K}'_*(V, \Gamma)_\sigma$, and so the result follows from the preceding case.

If $X \neq U$, call Y the complement of U, with its reduced scheme structure. This is G-invariant. For each $i \geq 0$ we get a commutative diagram

$$\begin{tikzcd}
\mathrm{K}'_{i+1}(U, G) \arrow[r] \arrow[d] & \mathrm{K}'_i(Y, G) \arrow[r] \arrow[d] & \mathrm{K}'_i(X, G) \arrow[r] \arrow[d] & \mathrm{K}'_i(U, G) \arrow[r] \arrow[d] & \mathrm{K}'_{i-1}(Y, G) \arrow[d] \\
\prod_\sigma \mathrm{K}'_{i+1}(U, G)_\sigma \arrow[r] & \prod_\sigma \mathrm{K}'_i(Y, G)_\sigma \arrow[r] & \prod_\sigma \mathrm{K}'_i(X, G)_\sigma \arrow[r] & \prod_\sigma \mathrm{K}'_i(U, G)_\sigma \arrow[r] & \prod_\sigma \mathrm{K}'_{i-1}(Y, G)_\sigma
\end{tikzcd}$$

with exact rows. Since the result is true for Y and U, so that the first, second, fourth and fifth columns are isomorphisms, we see that the central arrow is also an isomorphism by the "five lemma"; so we have (1) as claimed.

As for the remaining points, we also have a diagram

$$\begin{tikzcd}
\mathrm{K}'_{i+1}(U,G) \arrow[r] \arrow[d, "\cdot\alpha_U"] & \mathrm{K}'_i(Y,G) \arrow[r] \arrow[d, "\cdot\alpha_Y"] & \mathrm{K}'_i(X,G) \arrow[r] \arrow[d, "\cdot\alpha"] & \mathrm{K}'_i(U,G) \arrow[r] \arrow[d, "\cdot\alpha_U"] & \mathrm{K}'_{i-1}(Y,G) \arrow[d, "\cdot\alpha_Y"] \\
\mathrm{K}'_{i+1}(U,G) \arrow[r] & \mathrm{K}'_i(Y,G) \arrow[r] & \mathrm{K}'_i(X,G) \arrow[r] & \mathrm{K}'_i(U,G) \arrow[r] & \mathrm{K}'_{i-1}(Y,G)
\end{tikzcd}$$

with exact rows. As before, the results follow.

Step 3: the case $G = \mathrm{GL}_n$. Consider the standard maximal torus $T \subseteq G$ with its Weyl group S_n. The subgroup $\sigma \subseteq G$ is conjugate to a subgroup of T, well defined, up to conjugation by an element of S_n thanks to the following easy

Observation 1.3.4. *$\mathcal{C}(G)$ is in natural bijective correspondence with the set of orbits for the action of S_n on $\mathcal{C}(T)$.*

We have that the restriction map induces an isomorphism $\mathrm{K}'_*(X,G) = \mathrm{K}'_*(X,T)^{\mathrm{S}_n}$ (see Appendix B); moreover the map $\mathrm{R}\mathrm{GL}_n \to \mathrm{R}T$ is a Galois cover with Galois group S_n. In particular for any $\mathrm{R}T$-module with finite support and $\sigma \in \mathcal{C}(\mathrm{GL}_n)$ there is an isomorphism $M_\sigma \simeq \prod_{\sigma' \in S_n\sigma} M_{\sigma'}$.

We conclude that

$$\prod_{\sigma\in\mathcal{C}(G)} \mathrm{K}'_*(X,G)_\sigma \simeq \Big(\prod_{\sigma\in\mathcal{C}(T)} \mathrm{K}'_*(X,T)_\sigma\Big)^{\mathrm{S}_n}.$$

and (1) follows.

If $\Gamma \subseteq \mathrm{S}_n$ is the stabilizer of $\sigma \subseteq T$ under the action of S_n, we deduce that

$$\mathrm{K}'_*(X,G)_\sigma \simeq \mathrm{K}'_*(X,T)_\sigma^\Gamma.$$

If β is the image in $\mathrm{K}_0(X,T)$ of $\alpha \in \mathrm{K}_0(X,G)$, then multiplication by β on $\mathrm{K}'_*(X,T)_\sigma$ is Γ-equivariant, and the its restriction to $\mathrm{K}'_*(X,G)_\sigma = \mathrm{K}'_*(X,T)_\sigma^\Gamma$ is multiplication by α. Since multiplication by β on $\mathrm{K}'_*(X,T)_\sigma$ is an automorphism, by the previous step, we deduced that multiplication by α is also an automorphism, as claimed.

♠

In the decomposition $\mathrm{K}'_*\mathcal{X} = \prod_r \mathrm{K}'_*(\mathcal{X})_{(\boldsymbol{\mu}_r)}$ we split off the factor $\mathrm{K}'_*(\mathcal{X})_{(\boldsymbol{\mu}_1)}$, which we call the *geometric K-theory* of $\mathcal{X}$, and denote by $\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}$, and the factor $\prod_{r\geq 2} \mathrm{K}'_*(\mathcal{X})_{(\boldsymbol{\mu}_r)}$, which we call the *algebraic part* of the K-theory of $\mathcal{X}$, and denote by $\mathrm{K}'_*(\mathcal{X})_{\mathbf{a}}$. Thus we have the *fundamental decomposition*

$$\mathrm{K}'_*\mathcal{X} = \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}} \oplus \mathrm{K}'_*(\mathcal{X})_{\mathbf{a}}.$$

As we already remarked, $\mathrm{K}'_*(\mathcal{X})_{\mathbf{a}} = 0$ if and only if $\mathcal{X}$ is an algebraic space.

Proposition 1.3.5. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable morphism of stacks.*

(1) *Assume that f has finite flat dimension. Then the pullback $f^*\colon \mathrm{K}'_* \mathcal{X} \to \mathrm{K}'_* \mathcal{Y}$ preserves the fundamental decomposition.*

(2) *Assume that f is proper and relatively tame. Then the pushforward $f_*\colon \mathrm{K}'_* \mathcal{Y} \to \mathrm{K}'_* \mathcal{X}$ preserves the fundamental decomposition.*

Proof. Write $\mathcal{X} = [X/G]$. Then we have the decomposition

$$\mathrm{K}'_* \mathcal{X} = \mathrm{K}'_*(X, G) = \prod_{\sigma \in \mathcal{C}(G)} \mathrm{K}'_*(X, G)_\sigma\,;$$

we have $\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}} = \mathrm{K}'_*(X, G)_{\{1\}}$ and $\mathrm{K}'_*(\mathcal{X})_{\mathbf{a}} = \prod_{\substack{\sigma \in \mathcal{C}(G) \\ \sigma \neq \{1\}}} \mathrm{K}'_*(X, G)_\sigma$; hence it is enough to show that f^* and f_* preserve the decomposition

$$\mathrm{K}'_*(X, G) = \prod_{\sigma \in \mathcal{C}(G)} \mathrm{K}'_*(X, G)_\sigma\,.$$

Set $Y \overset{\text{def}}{=} \mathcal{Y} \times_{\mathcal{X}} X$; then Y is an algebraic space, since f is representable; furthermore there is an action of G on Y such that $\mathcal{Y} = [Y/G]$ and the map f is G-equivariant. Then $\mathrm{K}'_* \mathcal{Y} = \mathrm{K}'_*(Y, G)$ and $\mathrm{K}'_* \mathcal{Y}$ acquires a structure of $\mathrm{R}G$-module. Furthermore, f^* and f_* are homomorphism of $\mathrm{R}G$-modules, and the result follows from this. ♠

If f is not representable, it is immediate to give examples to show that the result above can fail. However, we have the following important fact. The fundamental decomposition $\mathrm{K}'_* \mathcal{X} = \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}} \oplus \mathrm{K}'_*(\mathcal{X})_{\mathbf{a}}$ gives both a projection $\mathrm{K}'_* \mathcal{X} \to \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}$ and an embedding $\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}} \to \mathrm{K}'_* \mathcal{X}$.

Proposition 1.3.6. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a homomorphism of stacks.*

(1) *Assume that f has finite flat dimension. Then there exists a homomorphism*

$$f^*\colon \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}} \longrightarrow \mathrm{K}'_*(\mathcal{Y})_{\mathbf{g}}$$

such that the diagram

$$\begin{tikzcd}
\mathrm{K}'_* \mathcal{X} \arrow[r, two heads] \arrow[d, "f^*"] & \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}} \arrow[d, "f^*"] \\
\mathrm{K}'_* \mathcal{Y} \arrow[r, two heads] & \mathrm{K}'_*(\mathcal{Y})_{\mathbf{g}}
\end{tikzcd}$$

commutes.

(2) *Suppose that f is proper. Then there exists a homomorphism*

$$f_*\colon \mathrm{K}'_*(\mathcal{Y})_{\mathbf{g}} \longrightarrow \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}$$

such that the diagram

$$\begin{tikzcd}
\mathrm{K}'_*(\mathcal{Y})_{\mathbf{g}} \arrow[r, hook] \arrow[d, "f_*"] & \mathrm{K}'_* \mathcal{Y} \arrow[d, "f_*"] \\
\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}} \arrow[r, hook] & \mathrm{K}'_* \mathcal{X}
\end{tikzcd}$$

commutes.

The maps $\mathrm{K}'_*(-)_{\mathbf{g}} \hookrightarrow \mathrm{K}'_*(-)$ *and* $\mathrm{K}'_*(-) \twoheadrightarrow \mathrm{K}'_*(-)_{\mathbf{g}}$ *in the diagrams above are the embeddings and the projections coming from the fundamental decomposition.*

The homomorphism f^* and f_* defined above are clearly unique; they make $\mathrm{K}'_*(-)_{\mathbf{g}}$ into a contravariant functor for maps of finite flat dimension, and a covariant functor for proper maps.

Proof. The first part is a consequence of the fact that $f\colon \mathrm{K}_0\,\mathcal{X} \to \mathrm{K}_0\,\mathcal{Y}$ carries $\Sigma_1^{\mathcal{X}}$ into $\Sigma_1^{\mathcal{Y}}$.

For the second part, write $\mathcal{X} = [X/G]$ and $\mathcal{Y} = [Y/H]$, where X and Y are algebraic spaces, and G and H are affine algebraic groups acting sufficiently rationally on X and Y. The projection $X \to \mathcal{X}$ and the composite $Y \to \mathcal{Y} \xrightarrow{f} \mathcal{X}$ are respectively G and H invariant. Set $Z = X \times_{\mathcal{X}} Y$; there is a natural action of $G \times H$ on Z, and it is easy to see that $[Z/G \times H] = \mathcal{Y}$. The projection $Z \to X$ is equivariant for the projection $\mathrm{pr}_1\colon G \times H \to G$, and the induced morphism

$$\mathcal{Y} = [Z/G \times H] \longrightarrow [X/G] = \mathcal{X}$$

is isomorphic to f. The homomorphism $f_*\colon \mathrm{K}'_*(Z, G \times H) \to \mathrm{K}'_*(X, G)$ is a homomorphism of $\mathrm{R}G$-modules, where $\mathrm{K}'_*(Z, G \times H)$ is considered as an $\mathrm{R}G$-module through the homomorphism $\mathrm{pr}_1^*\colon \mathrm{R}G \to \mathrm{R}(G \times H)$.

Consider the decompositions

$$\mathrm{K}'_*(Z, G \times H) = \prod_{\rho \in \mathcal{C}(G \times H)} \mathrm{K}'_*(Z, G \times H)_\rho$$

and

$$\mathrm{K}'_*(X, G) = \prod_{\sigma \in \mathcal{C}(G \times H)} \mathrm{K}'_*(X, G)_\sigma\,.$$

If $\sigma \in \mathcal{C}(G)$, we can also consider the σ-localization

$$\mathrm{K}'_*(Z, G \times H)_\sigma = (\mathrm{R}G \smallsetminus \mathfrak{m}_\sigma)^{-1}\,\mathrm{K}'_*(Z, G \times H)$$

of $\mathrm{K}'_*(Z, G \times H)$; it is immediate to see that it coincides with the quotient

$$\prod_{\substack{\rho \in \mathcal{C}(G \times H) \\ \rho \twoheadrightarrow \sigma}} \mathrm{K}'_*(Z, G \times H)_\rho$$

of $\mathrm{K}'_*(Z, G \times H)$. If

$$\eta \in \mathrm{K}'_*(\mathcal{Y})_{\mathbf{g}} = \mathrm{K}'_*(Z, G \times H)_{\{1\}} \subseteq \mathrm{K}'_*(Z, G \times H)$$

the by definition the image of η in $\mathrm{K}'_*(Z, G \times H)_\sigma$ is zero for every $\sigma \in \mathcal{C}(G)$ with $\sigma \neq \{1\}$; this implies that the image of $f_*\eta \in \mathrm{K}'_*\,\mathcal{X}$ in

$$\mathrm{K}'_*(\mathcal{X})_{\mathbf{a}} = \prod_{\substack{\rho \in \mathcal{C}(G \times H) \\ \rho \neq \{1\}}} \mathrm{K}'_*(X, G)_\sigma$$

is zero, which implies $f_*\xi \in \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}$, as claimed. ♠

Remark. *Actually, we can make the above proposition more precise.Suppose that we have a morphism of stacks* $f\colon [Z/G\times H]\to[X/G]$ *induced by an equivariant map* $Z\to X$.

Let ρ *be a dual cyclic subgroup of* $G\times H$ *and* σ *be its projection to* G.

(1) *The pull-back* f^* *is a map of* $\mathrm{R}G$*-modules, so it induces a map*

$$f^*: \mathrm{K}'_*(X,G)_\sigma = S_\sigma^{-1}\,\mathrm{K}'_*(X,G) \longrightarrow S_\sigma^{-1}\,\mathrm{K}'_*(Z,G\times H) = \prod_{\substack{\rho\in\mathcal{C}(G\times H)\\ \rho\twoheadrightarrow\sigma}} \mathrm{K}'_*(Z,G\times H)_\rho$$

(2) *The push-forward* $f_*\colon \mathrm{K}'_*(Z,G\times H)\to\mathrm{K}'_*(X,G)$ *is a morphism of* $\mathrm{R}G$*-modules. If* $\eta\in\mathrm{K}'_*(Z,G\times H)$ *has support* ρ *(so that its projections to the* ρ'*-localizations are zero for any* $\rho'\neq\rho$*), then its image* $f_*(\eta)$ *projects to zero in* $\mathrm{K}'_*(X,G)_{\sigma'}$ *for every* $\sigma'\neq\sigma$*: indeed for any such* σ' *and* $\rho'\twoheadrightarrow\sigma'$ *it holds that* $\rho'\neq\rho$*, and* η *has null image in* $\mathrm{K}'_*(Z,G\times H)_{\rho'}$*. We conclude that* η *is zero when localized at* σ'*, since* $\mathrm{K}'_*(Z,G\times H)_{\sigma'}=\prod_{\substack{\rho'\in\mathcal{C}(G\times H)\\ \rho'\twoheadrightarrow\sigma'}}\mathrm{K}'_*(Z,G\times H)_{\rho'}$*; being* f_* *a map of* $\mathrm{R}G-$*modules,* $f_*(\eta)$ *is also zero when localized at* σ'.

In particular $f_*\eta$ *has support at* σ*, and thus lies in* $\mathrm{K}'_*(X,G)_\sigma$.

We will use the following facts, whose proof can be found in [20].

Proposition 1.3.7. *Let* $i\colon\mathcal{X}_{\mathrm{red}}\hookrightarrow\mathcal{X}$ *be the reduction of* $\mathcal{X}$*. Then*

$$i_*\colon \mathrm{K}'_*(\mathcal{X}_{\mathrm{red}})_{\mathbf{g}}\longrightarrow\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}$$

is an isomorphism.

Proposition 1.3.8. *Let* $\pi\colon\mathcal{X}'\to\mathcal{X}$ *a representable finite faithfully flat morphism of stacks. Then the sequences*

$$0\longrightarrow\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}\xrightarrow{\pi^*}\mathrm{K}'_*(\mathcal{X}')_{\mathbf{g}}\xrightarrow{\mathrm{pr}_1^*-\mathrm{pr}_2^*}\mathrm{K}'_*(\mathcal{X}'\times_{\mathcal{X}}\mathcal{X}')_{\mathbf{g}}$$

and

$$\mathrm{K}'_*(\mathcal{X}'\times_{\mathcal{X}}\mathcal{X}')_{\mathbf{g}}\xrightarrow{\mathrm{pr}_{1*}-\mathrm{pr}_{*2}}\mathrm{K}'_*(\mathcal{X}')_{\mathbf{g}}\xrightarrow{\pi_*}\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}\longrightarrow 0$$

are exact.

This is easily seen to fail for K-theory, for example, for the representable finite faithfully flat morphism $\operatorname{Spec}k\to\mathcal{B}_kG$, where G is a nontrivial finite group scheme.

Corollary 1.3.9. *Let* Γ *be a finite group,* $\pi\colon\mathcal{X}'\to\mathcal{X}$ *a Galois cover with group* Γ*. Then the pullback*

$$\pi^*\colon\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}\longrightarrow\mathrm{K}'_*(\mathcal{X}')_{\mathbf{g}}^{\Gamma}$$

and the pushforward

$$\pi_*\colon(\mathrm{K}'_*(\mathcal{X}')_{\mathbf{g}})_\Gamma\longrightarrow\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}$$

are isomorphisms.

Here of course $\mathrm{K}'_*(\mathcal{X}')_{\mathbf{g}}^{\Gamma}$ is the group of invariants, and $\pi_*(\mathrm{K}'_*(\mathcal{X}')_{\mathbf{g}})_{\Gamma}$ is the group of covariants.

This applies in particular when $\mathcal{X}$ is a quotient stack of the form $[X/\Gamma]$, where Γ is a finite group acting on a scheme X. In this case we have isomorphism $\mathrm{K}'_*([X/\Gamma])_{\mathbf{g}} \simeq (\mathrm{K}'_* X)^{\Gamma}$ and $(\mathrm{K}'_* X)_{\Gamma} \simeq \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}$.

Another way of thinking about $\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}} \subseteq \mathrm{K}'_* \mathcal{X}$ and $\mathrm{K}'_*(\mathcal{X})_{\mathbf{a}} \subseteq \mathrm{K}'_* \mathcal{X}$ is the following: $\mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}$ is formed of elements that come from schemes via pushforward, while $\mathrm{K}'_*(X)_{\mathbf{a}}$ is formed of all the elements that pull back to 0 to schemes. This is probably true in general, but we can only prove it for tame with quasi-projective moduli space.

Proposition 1.3.10. *Assume that a tame stack $\mathcal{X}$ is of finite type over S, and has a quasi-projective moduli space. Let ξ be an element of $\mathrm{K}'_* \mathcal{X}$. Then*

(a) $\xi \in \mathrm{K}'_*(\mathcal{X})_{\mathbf{a}}$ *if and only if for every morphism $f\colon Y \to \mathcal{X}$ of finite flat dimension, where Y is a scheme, we have $f^*\xi = 0$, and*

(b) $\xi \in \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}}$ *if and only if there exists a proper morphism $f\colon Y \to \mathcal{X}$, where Y is a scheme, and a class $\eta \in \mathrm{K}'_* Y$ such that $\xi = f_*\eta$.*

Let $\pi\colon \mathcal{X} \to M$ be the moduli space of $\mathcal{X}$; then we get a homomorphism

$$\pi_*\colon \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}} \longrightarrow \mathrm{K}'_*(M)_{\mathbf{g}} = \mathrm{K}'_* M\,.$$

Theorem 1.3.11. *The pushforward $\pi_*\colon \mathrm{K}'_*(\mathcal{X})_{\mathbf{g}} \to \mathrm{K}'_* M$ is an isomorphism.*

1.4 The tautological part of the K-theory of the cyclotomic inertia

Suppose that K is an extension of k, and let (ξ, a) be an object of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}(K)$. The homomorphism $a\colon \boldsymbol{\mu}_{\infty,K} \to \underline{\mathrm{Aut}}_K\,\xi$ induces an action of $\boldsymbol{\mu}_{\infty,K}$ on ξ, which commutes with itself, since $\boldsymbol{\mu}_{\infty,K}$ is abelian; thus a can be considered as a homomorphism $\boldsymbol{\mu}_{\infty,K} \to \underline{\mathrm{Aut}}_K(\xi, a)$. We can think of this as follows: a morphism $\mathcal{B}_K\boldsymbol{\mu}_{\infty,K} \to \mathcal{X}$ has a canonical lifting to a morphism $\mathcal{B}_K\boldsymbol{\mu}_{\infty,K} \to \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$, which we call its *tautological lifting.*

A morphism $\mathcal{B}_K\boldsymbol{\mu}_{\infty,K} \to \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ is called *tautological* if it is isomorphic to the tautological lifting of the composite $\mathcal{B}_K\boldsymbol{\mu}_{\infty,K} \to \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \to \mathcal{X}$. Equivalently, we can define a tautological morphism as follows. A morphism $\mathcal{B}_K\boldsymbol{\mu}_{\infty,K} \to \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ corresponds to an object (ξ, a) of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}(K)$, together with a homomorphism $b\colon \boldsymbol{\mu}_{\infty,K} \to \underline{\mathrm{Aut}}_K(\xi, a)$. This gives two action of $\boldsymbol{\mu}_{\infty,K}$ on ξ: one given by $a\colon \boldsymbol{\mu}_{\infty,K} \to \underline{\mathrm{Aut}}_K\,\xi$, and the other by the composite $\boldsymbol{\mu}_{\infty,K} \xrightarrow{b} \underline{\mathrm{Aut}}_K(\xi, a) \subseteq \underline{\mathrm{Aut}}_K\,\xi$. The morphism $\mathcal{B}_K\boldsymbol{\mu}_{\infty,K} \to \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ is tautological if and only if the two actions coincide.

As we saw, there is an equivalence between morphisms $\mathcal{B}_K\boldsymbol{\mu}_{\infty,K} \to \mathcal{X}$ and representable morphisms $\mathcal{B}_K\boldsymbol{\mu}_{r,K} \to \mathcal{X}$ for some r; hence every representable morphism $\mathcal{B}_K\boldsymbol{\mu}_{r,K} \to \mathcal{X}$ has a tautological lifting $\mathcal{B}_K\boldsymbol{\mu}_{r,K} \to \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$. Therefore we can also talk about tautological representable morphisms $\mathcal{B}_K\boldsymbol{\mu}_{r,K} \to \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$.

Let us define two multiplicative systems $\Theta_{\mathcal{X}} \subseteq \mathrm{K}_0(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$ and $\widetilde{\Theta}_{\mathcal{X}} \subseteq \mathrm{K}_0(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$ as follows. An element $\alpha \in \mathrm{K}_0(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$ is in $\Theta_{\mathcal{X}}$ if for every representable tautological morphism

$\phi\colon \mathcal{B}_K\boldsymbol{\mu}_{r,K} \to \mathcal{X}$ the image of $\phi^*\alpha \in \mathrm{R}\boldsymbol{\mu}_r$ in $\mathbb{Q}(\zeta_r)$ is non-zero. An element $\alpha \in \mathrm{K}_0(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$ is in $\widetilde{\Theta}_{\mathcal{X}}$ if for every non-tautological representable morphism $\phi\colon \mathcal{B}_K\boldsymbol{\mu}_{r,K} \to \mathcal{X}$ the image of $\phi^*\alpha \in \mathrm{R}\boldsymbol{\mu}_r$ in $\mathbb{Q}(\zeta_r)$ is non-zero. (Here K is an extension of k.)

Definition. *The* tautological part *of the K-theory of the cyclotomic inertia stack* $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ *is the localization* $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}} \overset{\mathrm{def}}{=} \Theta_{\mathcal{X}}^{-1}\,\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$. *The* non-tautological part *of the K-theory of the cyclotomic inertia stack* $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ *is the localization* $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{nt}} \overset{\mathrm{def}}{=} \widetilde{\Theta}_{\mathcal{X}}^{-1}\,\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$.

Alternatively, one can define $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{t}}$ (respectively $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{nt}}$) as the localization of $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})$ along the multiplicative systems in $\mathrm{K}_0\,\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ consisting of elements whose image in $\mathbb{Q}(\zeta_r)$ is non-zero for every non-tautological (respectively tautological) representable morphism tautological representable morphisms $\mathcal{B}_K\boldsymbol{\mu}_{r,K} \to \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$. Clearly we have $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}} = \bigoplus_{r\geq 1}\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{t}}$ and $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{nt}} = \bigoplus_{r\geq 1}\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{nt}}$. Furthermore, if $\mathcal{X}$ is an algebraic space, then $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}} = \mathrm{K}'_*\,X$ and $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{nt}} = 0$.

The following property is also evident.

Proposition 1.4.1.

(1) *The projection* $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}) \to \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{t}}$ *factors through the projection*

$$\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}) \longrightarrow \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{(\boldsymbol{\mu}_r)}\,.$$

(2) *The projection* $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}) \to \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{nt}}$ *factors through the projection* $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}) \to \bigoplus_{s\neq\, r}\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{(\boldsymbol{\mu}_s)}$.

It is easy to check that the multiplicative systems $\Theta_{\mathcal{X}} \subseteq \mathrm{K}_0\,\mathcal{X}$ and $\widetilde{\Theta}_{\mathcal{X}} \subseteq \mathrm{K}_0\,\mathcal{X}$ are invariant under the action of $\operatorname{Aut}\boldsymbol{\mu}_\infty$ on K_0; hence we get an action of $\operatorname{Aut}\boldsymbol{\mu}_\infty$ on $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}}$. By restriction, this gives an action of $\operatorname{Aut}\boldsymbol{\mu}_\infty$ on each $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{t}}$.

Let us describe the tautological and non-tautological parts of equivariant K-theory of the cyclotomic inertia stack in equivariant terms. Assume that $\mathcal{X} = [X/G]$, where G is a product of general linear groups. Let $\mathcal{U} \subseteq \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ be a connected component of the cyclotomic inertia stack $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$. Recall that $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ is the disjoint union

$$\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} = \coprod_{\substack{\sigma\in\overline{\mathcal{C}}_r(G)\\ |\sigma|=r}} [X^\sigma/C_G(\sigma)]$$

so $\mathcal{U}$ corresponds to some embedding $\sigma \in \overline{\mathcal{C}}_r(G)$. In $\mathrm{K}'_*([X^\sigma/C_G(\sigma)]) = \mathrm{K}'_*(X, C_G(\sigma))$ the multiplicative system $\Theta_{\mathcal{X}}$ restricts to the Σ_σ relative to the embedding $\sigma \hookrightarrow C_G(\sigma)$. In particular, from Proposition 1.3.1 we get

Proposition 1.4.2. *If G is a product of general linear groups then*

$$\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{t}} = \bigoplus_{\sigma\in\overline{\mathcal{C}}(G)} \mathrm{K}'_*(X^\sigma, C_G(\sigma))_\sigma.$$

From this description we obtain the following.

Proposition 1.4.3. *The homomorphism* $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}) \to \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}} \times \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{nt}}$ *is an isomorphism.*

From this splitting we get an embedding $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}} \subseteq \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{(\boldsymbol{\mu}_r)}$.

Proposition 1.4.4. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of stacks.*

(1) *If $\mathcal{X}$ and $\mathcal{Y}$ are regular, the pullback $\mathcal{I}_{\boldsymbol{\mu}} f^*\colon \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}) \to \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$ descends to an $\operatorname{Aut}\boldsymbol{\mu}_\infty$-equivariant homomorphism*

$$\mathcal{I}_{\boldsymbol{\mu}} f^*\colon \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y})_{\mathbf{t}} \longrightarrow \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}}\,.$$

(2) *If f is proper and relatively tame, the pushforward $\mathcal{I}_{\boldsymbol{\mu}} f_*\colon \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y}) \to \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$ carries $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}} \subseteq \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$ into $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y})_{\mathbf{t}} \subseteq \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y})$, and induces an $\operatorname{Aut}\boldsymbol{\mu}_\infty$-equivariant ring homomorphism $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}} \to \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y})_{\mathbf{t}}$.*

Proof. The first part follows from the fact that $\mathcal{I}_{\boldsymbol{\mu}} f^*(\Theta_{\mathcal{X}}) \subseteq \Theta_{\mathcal{Y}}$. Let us concentrate now on the second one.

Suppose that $\mathcal{Y} = [Y/H]$ and $\mathcal{X} = [X/G]$ where X, Y are algebraic spaces and G, H are general linear groups. Recall the following from the proof of Proposition 1.3.6: there is an $H-$invariant map $Y \to \mathcal{Y} \xrightarrow{f} \mathcal{X}$; if $Z = X \times_{\mathcal{X}} Y$, there is a natural action of $G \times H$ on it, and $[Z/G \times H] = \mathcal{Y}$. The projection $Z \to X$ is equivariant with respect to $\pi\colon G \times H \to G$ and induces a map

$$\mathcal{Y} = [Z/G \times H] \longrightarrow [X/G] = \mathcal{X}$$

that is isomorphic to f.

Let $\rho \hookrightarrow G \times H$ be a dual cyclic subgroup, whose projections onto G, H are σ_G, σ_H respectively. Then $C_{G\times H}(\rho) = C_G(\sigma_G) \times C_H(\sigma_H)$.

Let $\sigma := \sigma_G$; the map $Z \to X$ induces a morphism

$$[Z^\rho/C_G(\sigma_G) \times C_H(\sigma_H)] \longrightarrow [X^\sigma/C_G(\sigma)]$$

and this is exactly the restriction of $\mathcal{I}_{\boldsymbol{\mu}} f$ to the component of $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y}$ relative to ρ. The induced map $\mathcal{I}_{\boldsymbol{\mu}} f_*$ is a morphism of $\mathrm{R}C_G(\sigma)$-modules.

We can now argue as in our Observation 1.3: if $\eta \in \mathrm{K}'_*(Z^\rho, C_G(\sigma) \times C_H(\sigma_H))$ has support ρ (so that its projections to the ρ'-localizations are zero for any $\rho' \neq \rho$), then its image $\mathcal{I}_{\boldsymbol{\mu}} f_*(\eta)$ has support at σ, and thus projects to zero in $\mathrm{K}'_*([X^\sigma/C_G(\sigma)])_{\mathbf{nt}}$.

This settles the proof. ♠

One can also prove that if f is representable, then $\mathcal{I}_{\boldsymbol{\mu}} f_*\, \mathrm{K}'_*\, \mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y} \to \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$ also induces a pushforward $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y})_{\mathbf{nt}} \to \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{nt}}$, but we will not need this fact.

Let $\mathcal{X}$ be a stack, and denote by $\rho = \rho_{\mathcal{X}}\colon \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \to \mathcal{X}$ the canonical morphism.

Proposition 1.4.5. *Let $\mathcal{X}$ be an algebraic stack with finite cyclotomic inertia. For any $r > 0$, the restriction $\rho_*\colon \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{t}}^{\operatorname{Aut}\boldsymbol{\mu}_\infty} \to \mathrm{K}'_*(\mathcal{X})_{(\boldsymbol{\mu}_r)}$ of the pushforward $\rho_*\colon \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}) \to \mathrm{K}'_* X$ is an isomorphism.*

Proof. Let us begin with the following

Lemma 1.4.6. *Let X be a $G-$equivariant algebraic space and $\sigma \subset G$ a dual cyclic subgroup. Then the push-forward map $(j_\sigma)_*\colon \mathrm{K}'_*(X^\sigma, C_G(\sigma))_\sigma \to \mathrm{K}'_*(X, C_G(\sigma))_\sigma$ is an isomorphism.*

Proof. Let $Y := X - X^\sigma$; we have an exact sequence

$$\mathrm{K}'_{i+1}(Y, C_G(\sigma))_\sigma \longrightarrow \mathrm{K}'_i(X^\sigma, C_G(\sigma))_\sigma \xrightarrow{(j_\sigma)_*} \mathrm{K}'_i(X, C_G(\sigma))_\sigma \longrightarrow \mathrm{K}'_i(Y, C_G(\sigma))_\sigma$$

and $Y^\sigma = 0$, so $\mathrm{K}'_i(Y, C_G(\sigma))_\sigma = 0$ for all i by Corollary 1.3.3. ♠

Step 1. Suppose first that $\mathcal{X} = [X/T]$, where T is a split torus and X is a separated scheme.

By Lemma 1.4.6, for any $\sigma \in \mathcal{C}_r(T)$ the push-forward $(j_\sigma)_* \colon \mathrm{K}'_*(X^\sigma, T)_\sigma \to \mathrm{K}'_*(X, T)_\sigma$ is an isomorphism.

But ρ_* factors as

$$\bigoplus_{\sigma \in \mathcal{C}_r(T)} \mathrm{K}'_*(X^\sigma, T)_\sigma \xrightarrow{\oplus (j_\sigma)_*} \bigoplus_{\sigma \in \mathcal{C}_r(T)} \mathrm{K}'_*(X, T)_\sigma \longrightarrow \mathrm{K}'_*(X, T)_{(\boldsymbol{\mu}_r)}$$

and the thesis follows.

Step 2. We can finally suppose that $\mathcal{X} = [X/\mathrm{GL}_n]$, where X is a separated algebraic space; now the naive application of Lemma 1.4.6 does not give the full result, but we will reduce to the previous case.

Let T be a maximal split suborus of GL_n and let $\mathcal{Y} := [X/T]$.

Let us describe the cartesian diagram

$$\begin{tikzcd}
\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \times_{\mathcal{X}} \mathcal{Y} \arrow[r, "\rho'"] \arrow[d] & \mathcal{Y} \arrow[d] \\
\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \arrow[r, "\rho"] & \mathcal{X}
\end{tikzcd}$$

We have

$$\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} = \coprod_{\sigma \in \overline{\mathcal{C}}(\mathrm{GL}_n)} [X^\sigma / C_{\mathrm{GL}_n}(\sigma)].$$

Furthermore any $\sigma \in \overline{\mathcal{C}}(\mathrm{GL}_n)$ has a representative whose image is a subgroup of T, whence we may assume that $T \subseteq C_{\mathrm{GL}_n}(\sigma)$. Fix any such σ.

Lemma 1.4.7. *Let X be a $G-$equivariant scheme and G', G'' subgroups of G. Consider the cartesian diagram*

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "p''"] \arrow[d, "p'"] & {[X/G'']} \arrow[d] \\
{[X/G']} \arrow[r] & {[X/G]}
\end{tikzcd}$$

Then $\mathcal{S} = [(X \times G)/(G' \times G'')]$, with an action given by $(x, g)(\mathbf{g}', \mathbf{g}'') = (x\mathbf{g}', \mathbf{g}'^{-1} g \mathbf{g}'')$.

The map p' is induced by $(x, g) \to x$, while p'' is induced by $(x, g) \to xg$.

Remark. *The definition of $\mathcal{S}$ might look asymmetrical, since we could make $G' \times G''$ act by $(x, g)(\mathbf{g}', \mathbf{g}'') = (x\mathbf{g}'', \mathbf{g}''^{-1} g \mathbf{g}')$; however this is not the case. In fact the automorphism of $X \times G$ given by $(x, g) \to (xg, g^{-1})$ is equivariant with respect to the two actions.*

Proof. Let us consider the following diagram, where every square is cartesian:

$$\begin{tikzcd}
X \times G \arrow[r] \arrow[d] & \mathcal{S}' \arrow[r] \arrow[d] & X \arrow[d] \\
\mathcal{S}'' \arrow[r] \arrow[d] & \mathcal{S} \arrow[r, "p''"] \arrow[d, "p'"] & {[X/G'']} \arrow[d] \\
X \arrow[r] & {[X/G']} \arrow[r] & {[X/G]}
\end{tikzcd}$$

The horizontal and vertical maps $X \times G \to X$ are the multiplication and the projection onto the first factor, respectively.

By base-change, $X \times G \to \mathcal{S}'$ is a $G'-$principal bundle. We conclude that $\mathcal{S}' = [(X \times G)/G']$, where G' acts as $(x, g)\mathbf{g}' = (x\mathbf{g}', \mathbf{g}'^{-1}g)$.

Analogously, $\mathcal{S}'' = [(X \times G)/G'']$, where G'' acts as $(x, g)\mathbf{g}'' = (x, g\mathbf{g}'')$.

Finally, the map $\mathcal{S}'' \to \mathcal{S}$ is a G'-principal bundle, so that

$$\mathcal{S} = [\mathcal{S}''/G'] = [(X \times G)/(G' \times G'')],$$

where the action is easily seen to be the one given in the statement. ♠

Thanks to this lemma, we have a commutative diagram

$$\begin{tikzcd}
{[(X^\sigma \times \mathrm{GL}_n)/(C_{\mathrm{GL}_n}(\sigma) \times T)]} \arrow[r] \arrow[d, "\pi'"] & {[(X \times \mathrm{GL}_n)/(C_{\mathrm{GL}_n}(\sigma) \times T)]} \arrow[r] \arrow[d, "\pi"] & {[X/T]} \arrow[d, "\pi"] \\
{[X^\sigma/C_{\mathrm{GL}_n}(\sigma)]} \arrow[r] & {[X/C_{\mathrm{GL}_n}(\sigma)]} \arrow[r] & {[X/\mathrm{GL}_n]}
\end{tikzcd}$$

where the squares are cartesian.

Let us consider the conjugacy classes of dual cyclic subgroups $\tilde{\sigma} \subset (C_{\mathrm{GL}_n}(\sigma) \times T)$ who lie over σ and have fixed points in $X^\sigma \times \mathrm{GL}_n$. If $\tilde{\sigma}$ is such an element, let $G' \hookrightarrow \mathrm{GL}_n$ be the projection onto GL_n of the fixed locus of $\tilde{\sigma}$. Calling σ' the projection of $\tilde{\sigma}$ on T, we must have that the conjugation map $G' \times \mathrm{GL}_n \to \mathrm{GL}_n$, $(g, s) \to g^{-1}sg$, sends σ to σ'.

We conclude that the sought $\tilde{\sigma}$ are exactly those given by the embeddings $\sigma \to (\sigma, \sigma') \subset (C_{\mathrm{GL}_n}(\sigma) \times T)$, where σ' is a conjugate of σ lying in T. Besides, we can assume that $\sigma' = g^{-1}\sigma g$, where $g \in S_n$ is a permutation matrix, thanks to our Observation 1.3.4.

The Weyl group $W = S_n = N_{\mathrm{GL}_n}(T)/T$ acts on the set of the $\tilde{\sigma}$'s, and the stabilizer of any element is the Weyl group of $C_{\mathrm{GL}_n}(\sigma)$, which we will denote as Δ_σ.

For any such $\tilde{\sigma} = (\sigma, g^{-1}\sigma g)$ there is a commuting diagram

$$\begin{tikzcd}
{[X^\sigma/T]} \arrow[r, "i_{\tilde{\sigma}}"] \arrow[dr] & {[(X^\sigma \times \mathrm{GL}_n)/(C_{\mathrm{GL}_n}(\sigma) \times T)]} \arrow[d, "\pi"] \\
& {[X^\sigma/C_{\mathrm{GL}_n}(\sigma)].}
\end{tikzcd}$$

There is a group morphism $T \to C_{\mathrm{GL}_n}(\sigma)$, $t \to (t, g^{-1}tg)$, and $i_{\tilde{\sigma}}$ is induced by $x \to (x, g)$, which is equivariant with respect to the previous map.

The subscheme of $\tilde{\sigma}$-fixed points in $X^\sigma \times \mathrm{GL}_n$ is $X^\sigma \times C_{\mathrm{GL}_n}(\sigma)g$ and $\tilde{\sigma}$ is central in $C_{\mathrm{GL}_n}(\sigma) \times T$. So the map

$$[X^\sigma/T] = [X^\sigma \times C_{\mathrm{GL}_n}(\sigma)/C_{\mathrm{GL}_n}(\sigma) \times T] \longrightarrow [X^\sigma \times \mathrm{GL}_n/C_{\mathrm{GL}_n}(\sigma) \times T]$$

is just the morphism induced from the inertia map

$$\rho\colon \mathcal{I}_{\boldsymbol{\mu}}[X^\sigma \times \mathrm{GL}_n/C_{\mathrm{GL}_n}(\sigma) \times T] \longrightarrow [X^\sigma \times \mathrm{GL}_n/C_{\mathrm{GL}_n}(\sigma) \times T].$$

Let us call, again, σ' the dual cyclic subgroup $g^{-1}\sigma g \subset T$. By Lemma 1.4.6, the maps

$$(j_{\tilde{\sigma}})_* = (\rho' \circ i_{\tilde{\sigma}})_*\colon\ \mathrm{K}'_*(X^\sigma, T)_{\sigma'} \longrightarrow \mathrm{K}'_*(X, T)_{\sigma'}$$

and

$$(i_{\tilde{\sigma}})_*\colon\ \mathrm{K}'_*(X^\sigma, T)_{\sigma'} \longrightarrow \mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}}$$

are isomorphisms.

In particular $\rho'_*\colon\ \mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}} \to \mathrm{K}'_*(X, T)_{\sigma'}$ is an isomorphism.

Since the map π^* induces an isomorphism $\mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma)) \simeq \mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)^W$, we conclude by Observation 1.3 that for any $\tilde{\sigma}$ above σ it gives an isomorphism

$$\mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))_\sigma \simeq \mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}}^{\Delta_\sigma}$$

Let $\Gamma_\sigma = (N_{\mathrm{GL}_n}(T) \cap N_{\mathrm{GL}_n}(\sigma))/T$ be the stabilizer of $\tilde{\sigma} \subset T$ for the action of W on $\mathcal{C}(T)$. Again by Observation 1.3 we infer that π^* induces an isomorphism (see Appendix B)

$$\mathrm{K}'_*(X, \mathrm{GL}_n)_\sigma \simeq \mathrm{K}'_*(X, T)_{\sigma'}^{\Gamma_\sigma}$$

Now, by Lemma 1.3.4 there is an exact sequence

$$0 \longrightarrow \Delta_\sigma \longrightarrow \Gamma_\sigma \longrightarrow w_{\mathrm{GL}_n}(\sigma) \longrightarrow 0$$

so we have a commuting diagram (as flat pull-backs and proper push-forwards commute in a cartesian square)

$$\begin{tikzcd}
\mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))_\sigma^{w_{\mathrm{GL}_n}(\sigma)} \arrow[r, "\rho_*"] \arrow[d, "\pi^*"] & \mathrm{K}'_*(X, \mathrm{GL}_n)_\sigma \arrow[d, "\pi^*"] \\
\mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}}^{\Gamma_\sigma} \arrow[r, "\rho'_*"] & \mathrm{K}'_*(X, T)_{\sigma'}^{\Gamma_\sigma}
\end{tikzcd}$$

The two vertical arrows are isomorphisms and so is ρ'_*. We see then that ρ_* is also an isomorphism, and this concludes the proof.

♠

Corollary 1.4.8. *The pushforward ρ_* induces a group isomorphism*

$$\rho_*\colon \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}}^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty} \simeq \mathrm{K}'_*\,\mathcal{X}\,.$$

Furthermore, ρ_* is covariant for proper morphisms.

1.5 The twist map

If $\mathcal{X}$ is a noetherian algebraic stack, we denote by $\mathfrak{Coh}\,\mathcal{X}$ the category of coherent sheaves on $\mathcal{X}$. If G is a finite diagonalizable group scheme over R with character group $\widehat{G} = \operatorname{Hom}(G, \mathbb{G}_{\mathrm{m}})$, then the K-theory group of the product $\mathcal{X} \times \mathcal{B}_R G$ is the tensor product $\mathrm{K}'_*(\mathcal{X}) \otimes_{\mathrm{K}'_*(A)} \mathrm{R}G$. The point here is that the category $\mathfrak{Coh}\,(\mathcal{X} \times \mathcal{B}_k G)$ is the category of coherent sheaves on $\mathcal{X}$ with an action of G, which in turn can be identified with the direct sum of categories $\oplus_{\chi \in \widehat{G}} \mathfrak{Coh}\,\mathcal{X}$, by the usual splitting of quasi-coherent sheaves with an action of G into eigensheaves. If F is a coherent sheaf on $\mathcal{X}$ and $\chi \in \widehat{G}$, we denote by $F \otimes \chi$ the sheaf F with the action of G defined by χ, considered as a sheaf on $\mathcal{X} \otimes \mathcal{B}_R G$.

Fix a positive integer r. There is a morphism

$$\alpha_{\mathcal{X}} \colon \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \times \mathcal{B}_R\boldsymbol{\mu}_r \longrightarrow \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$$

that is defined, at the level of objects, as follows. Let (ξ, a, P) be an object of $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \times \mathcal{B}_R\boldsymbol{\mu}_r$ over an affine scheme S; here ξ is an object of $\mathcal{X}(S)$, $a \colon \boldsymbol{\mu}_{r,S} \to \underline{\mathrm{Aut}}_S\,\xi$ is a monomorphism of group schemes over S, and $P \to S$ is a $\boldsymbol{\mu}_r$-torsor. By descent along $P \to S$ we obtain another object ξ_P of $\mathcal{X}(S)$, which we can think about as the quotient $(\xi \times P)/\boldsymbol{\mu}_r$; the automorphism group scheme $\underline{\mathrm{Aut}}_S\,\xi_P \to S$ is the twisted form $(\underline{\mathrm{Aut}}_S\,\xi)_P$ obtained by descent along $P \to S$ using the action of $\boldsymbol{\mu}_{r,S}$ on $\underline{\mathrm{Aut}}_S\,\xi$ by conjugation. Since this action by conjugation leaves a invariant, we obtain a homomorphism $a_P \colon \boldsymbol{\mu}_{r,S} \to \underline{\mathrm{Aut}}_S\,\xi_P$. This gives an object $\alpha_{\mathcal{X}}(\xi, a, P) = (\xi_P, a_P, P)$. The effect of $\alpha_{\mathcal{X}}$ on arrows is clear.
We will refer to this morphism as the *twist map*.

Equivariantly, if $\mathcal{X} = [X/G]$, on any component $[X^\sigma/C_G(\sigma)] \times \mathcal{B}_R\sigma$ the map $\alpha_{\mathcal{X}}$ is given as follows: there is a projection

$$m \colon C_G(\sigma) \times \sigma \longrightarrow C_G(\sigma),$$

corresponding to the multiplication map (this is a morphism of groups), which induces a map $[X^\sigma/C_G(\sigma) \times \sigma] \to [X^\sigma/C_G(\sigma)]$. This coincides with $\alpha_{\mathcal{X}}$.

Now consider the disjoint union

$$\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}} = \coprod_r \mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \times \mathcal{B}_R\boldsymbol{\mu}_r.$$

For any $f \colon \mathcal{Y} \to \mathcal{X}$ the map $\mathcal{I}_{\boldsymbol{\mu}} f$ induces a morphism $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y} \times \mathcal{B}_R\boldsymbol{\mu}_{\infty,R} \to \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X} \times \mathcal{B}_R\boldsymbol{\mu}_{\infty,R}$ and by projection a map

$$\widetilde{\mathcal{I}_{\boldsymbol{\mu}} f} \colon \widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y}} \longrightarrow \widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}}.$$

Let us define a multiplicative system $\Omega_{\mathcal{X}} \subseteq \mathrm{K}'_0(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}})$ as follows: for any component $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \times \mathcal{B}_R\boldsymbol{\mu}_r$ consider the representable morphisms ϕ from $\mathcal{B}_R\boldsymbol{\mu}_r$ into it, that are trivial on $\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}$ and the identity on $\mathcal{B}_R\boldsymbol{\mu}_r$; $\alpha \in \mathrm{K}'_0(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \times \mathcal{B}_R\boldsymbol{\mu}_r)$ lies in $\Omega_{\mathcal{X}}$ if and only if $\phi^*\alpha$ has a nontrivial projection on $\widetilde{\mathrm{R}\boldsymbol{\mu}_r}$ for any such ϕ. $\Omega_{\mathcal{X}}$ is the multiplicative systems generated by those elements for any r.

The localization $\Omega_{\mathcal{X}}^{-1}\,\mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}})$ can be described in equivariant terms. If $\mathcal{X} = [X/G]$ (G a product of general linear groups) and $\sigma \subset G$ is a dual cyclic subgroup then $\Omega_{\mathcal{X}}^{-1}\,\mathrm{K}'_*([X^\sigma/C_G(\sigma)] \times \mathcal{B}_R\boldsymbol{\mu}_r)$ coincides with the localization - as a $\mathrm{R}(C_G(\sigma) \times \boldsymbol{\mu}_r)-$module at the dual cyclic subgroup $\tilde{\sigma} = \mathrm{Id} \times \boldsymbol{\mu}_r \hookrightarrow C_G(\sigma) \times \boldsymbol{\mu}_r$.

Definition. *Let the* tautological part $\mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}})_{\mathbf{t}}$ *of* $\mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}})$ *be the localization* $\Omega_{\mathcal{X}}^{-1}\,\mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}})$.

The projection $\mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}}) \to \mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}})_{\mathbf{t}}$ has a natural splitting, which we shall describe below.

For any $d|r$ we have that $\mathrm{K}'_*(\boldsymbol{\mu}_r)_{(\boldsymbol{\mu}_d)} \simeq \mathrm{K}'_*(A) \otimes \mathbb{Q}(\zeta_d)$, hence there exists an immersion

$$i_d : \mathbb{Q}(\zeta_d) \hookrightarrow \mathrm{R}(\boldsymbol{\mu}_r).$$

We can describe it explicitly: let us define the polynomial

$$\psi_d := \frac{t^r - 1}{\phi_d} = \prod_{\substack{d_1|r \\ d_1 \neq d}} \phi_{d_1}$$

and note that $\psi_d(\zeta_d) \neq 0$; then for any $x \in \mathbb{Q}(\zeta_d)$ we can choose a polynomial p_x such that $p_x(\zeta_d) = x \cdot \psi_d(\zeta_d)^{-1}$ and it holds that

$$i_d(x) = p_x \cdot \psi_d(t) \in \mathbb{Q}[t]/(t^r - 1) = \mathrm{R}(\boldsymbol{\mu}_r)$$

This definition is manifestly independent of the choice of p_x, since it is well-defined up to multiples of $\phi_d(t)$ and $\psi_d \cdot \phi_d(t) = 0$ in $\mathrm{R}(\boldsymbol{\mu}_r)$.

In particular $i_1 \colon \mathbb{Q} \to \mathbb{Q}[t]/(t^r - 1)$ sends $q \in \mathbb{Q}$ to $q\frac{1+t+\cdots+t^{r-1}}{r}$.

Since we also have an immersion $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \hookrightarrow \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})$ we can combine it with i_r to get a map

$$\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \hookrightarrow \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}) \otimes_{K_0(A)} \mathrm{R}(\boldsymbol{\mu}_r) = \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X} \times \mathcal{B}_R\boldsymbol{\mu}_r).$$

This is the required splitting, since $S_{\tilde{\sigma}}^{-1}\,\mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}}) = \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)$.

Proposition 1.5.1. *Let* $f \colon \mathcal{Y} \to \mathcal{X}$ *be a proper map of quotient stacks. Then* $\widetilde{\mathcal{I}_{\boldsymbol{\mu}}f_*}$ *carries* $\mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y}})_{\mathbf{t}}$ *into* $\mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}})_{\mathbf{t}}$ *and induces a ring homomorphism on the tautological parts.*

Proof. As we already did, we can suppose that $\mathcal{Y} = [Y/H]$ and $\mathcal{X} = [X/G]$ where X, Y are schemes and G, H are general linear groups. If $Z = X \times_{\mathcal{X}} Y$, there is an action of $G \times H$ on it, and $[Z/G \times H] = \mathcal{Y}$. The projection $Z \to X$ is equivariant with respect to $\pi \colon G \times H \to G$ and induces a map

$$\mathcal{Y} = [Z/G \times H] \longrightarrow [X/G] = \mathcal{X}$$

that is isomorphic to f.

Let $\rho \hookrightarrow G \times H$ be a dual cyclic subgroup, whose projections onto G, H are σ_G, σ_H respectively. Then $C_{G\times H}(\rho) = C_G(\sigma_G) \times C_H(\sigma_H)$.

Let $\sigma := \sigma_G$; the maps $Z \to X$ and $\mathcal{B}_R\rho \to \mathcal{B}_R\sigma$ induce a morphism

$$[Z^\rho/C_G(\sigma_G) \times C_H(\sigma_H)] \times \mathcal{B}_R\rho \longrightarrow [X^\sigma/C_G(\sigma)] \times \mathcal{B}_R\sigma$$

and this is exactly the restriction of $\widetilde{\mathcal{I}_{\boldsymbol{\mu}}f}$ to the component of $\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y}}$ relative to ρ. The induced map $\widetilde{\mathcal{I}_{\boldsymbol{\mu}}f_*}$ is a morphism of $\mathrm{R}(C_G(\sigma) \times \sigma)$-modules.

Suppose that $\nu \in \mathrm{K}'_*([Z^\rho/C_G(\sigma_G) \times C_H(\sigma_H)] \times \mathcal{B}_k\rho)$ has support at $\tilde{\rho} = \mathrm{id} \times \rho \subset (C_G(\sigma_G) \times C_H(\sigma_H)) \times \rho$. Then $\widetilde{\mathcal{I}_{\boldsymbol{\mu}}f_*}(\nu)$ has support at $\tilde{\sigma}(= \mathrm{id} \times \sigma)$, by the argument of Observation 1.3.

♠

Composing the covariant inclusion $\mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}})_{\mathbf{t}} \hookrightarrow \mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}})$ with $(\alpha_{\mathcal{X}})_*\colon \mathrm{K}'_*(\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}}) \to \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$ we get a map

$$\beta_{\mathcal{X}}\colon \bigoplus_r \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \longrightarrow \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}).$$

This is clearly covariant with respect to proper push-forwards.

1.6 The main result

We are now ready to state the main results of this paper, that is a Riemann-Roch formula for the $K-$theory of quotient stacks with finite cyclotomic inertia.

Let $\mathcal{X}$ be such a quotient stack. Then we have a map given by the composition

$$\mathrm{K}'_*(\mathcal{X}) \xrightarrow{\rho^*} \mathrm{K}'_*(\mathcal{I}\mathcal{X})_{\mathbf{t}}^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty} \xrightarrow{\alpha^*_{\mathcal{X}}} \Big(\bigoplus_r \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)\Big)^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty}$$

(the last arrow is the composition of $\alpha^*_{\mathcal{X}}$ and the projections $\mathrm{R}\boldsymbol{\mu}_r \to \mathbb{Q}(\zeta_r)$). The main result of [25] says that this is a controvariant isomorphism of algebras. Our goal is to replace it with a covariant isomorphism of $\mathbb{Q}-$modules, which we will call the *Lefschetz-Riemann-Roch* isomorphism. We will prove the following

Proposition 1.6.1. *$\beta_{\mathcal{X}}$ factors through $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}} \hookrightarrow \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})$ and induces an isomorphism*

$$\beta_{\mathcal{X}}\colon \bigoplus_r \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \longrightarrow \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}}.$$

Recall from Proposition 1.4.5 that there is a group isomorphism

$$\rho_*\colon \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}}^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty} \simeq \mathrm{K}'_*(\mathcal{X})$$

so we have a covariant map

$$\rho_*^{-1}\colon \mathrm{K}'_*(\mathcal{X}) \longrightarrow \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}}\mathcal{X})_{\mathbf{t}}^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty}.$$

Let $r_*\colon \Big(\bigoplus_r \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)\Big)^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty} \to \mathrm{K}'_*(\mathcal{X})$ be the isomorphism given by the composition $\rho_* \circ \beta_{\mathcal{X}}$. Then the map $\mathcal{L} := r_*^{-1}$ gives

Theorem 1.6.2. *Let $\mathcal{X}$ be a quotient stack with finite cyclotomic inertia over a base scheme $A = Spec(R)$. Then the above map gives a* Lefschetz-Riemann-Roch *isomorphism, which is covariant with respect to proper push-forwards of relatively tame morphisms:*

$$\mathcal{L}\colon \mathrm{K}'_*(\mathcal{X}) \longrightarrow \Big(\bigoplus_r \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)\Big)^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty}$$

The formula $\mathcal{L} = \rho_* \circ \beta_{\mathcal{X}}$ is, of course, valid for any quotient stack. However if $\mathcal{X}$ is regular we can express $\mathcal{L}$ in a different way, much more feasible for calculations.

Let us consider the map r^* given by the composite

$$\mathrm{K}'_*(\mathcal{X}) \xrightarrow{\lambda_{-1}(\mathcal{N})^{-1}\cdot\rho^*} \mathrm{K}'_*(\mathcal{I}\mathcal{X})_{\mathbf{t}}^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty} \xrightarrow{\alpha^*_{\mathcal{X}}} \Big(\bigoplus_r \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)\Big)^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty}$$

where $\mathcal{N}$ is the conormal sheaf for $\rho\colon \mathcal{I}\mathcal{X} \to \mathcal{X}$ and the last map is - obviously - composed with the projections $\mathrm{R}(\boldsymbol{\mu}_r) \to \mathbb{Q}(\zeta_r)$.

Our comparison theorem is as follows:

Theorem 1.6.3. *Let $\mathcal{X}$ be a regular stack with finite cyclotomic inertia over A. Suppose that $\mathcal{X}$ is either*

1. *of the form $[X/G]$, where X is a scheme and G is a finite flat group scheme over A,*
2. *Deligne-Mumford*
3. *tame.*

Then the composition

$$r^* \circ r_*\colon \Big(\bigoplus_r \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)\Big)^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty} \longrightarrow \Big(\bigoplus_r \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)\Big)^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty}$$

is equal to the endomorphism that on each component $\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)$ is a multiplication by the rational number $\frac{\phi(r)}{r}$.

In particular the Lefschetz-Riemann-Roch map, in the regular case, is equal to $\bigoplus_d r^* \cdot \frac{d}{\phi(d)}$.

Remark. *If $\mathcal{X}$ is regular, then the cyclotomic inertia is also regular. Indeed, taking $\mathcal{X} = [X/\mathrm{GL}_n]$ we have that X is regular and, by [14], Lemma 2.3, X^σ is also regular for any dual cyclic subgroup $\sigma \subset \mathrm{GL}_n$.*

We are now ready to give the proof of our main result, which amounts to the proof of Proposition 1.6.1:

Proof. We can suppose $\mathcal{X} = [X/G]$, where X is a scheme and G is an algebraic group.

Recall that $\mathrm{K}'_*(X^\sigma, C_G(\sigma))_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)$ is the localization of $\mathrm{K}'_*(X^\sigma, C_G(\sigma) \times \sigma)$ at $\tilde{\sigma} = \mathrm{id} \times \sigma$. The image of $\tilde{\sigma}$ for the multiplication map $m\colon C_G(\sigma) \times \sigma \to C_G(\sigma)$ is σ so that, by Observation 1.3, $\beta_{\mathcal{X}}$ induces for any $\sigma \in \overline{\mathcal{C}}_r(G)$ a map

$$\beta_{\mathcal{X},\sigma}\colon \mathrm{K}'_*(X^\sigma, C_G(\sigma))_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \longrightarrow \mathrm{K}'_*(X^\sigma, C_G(\sigma))_\sigma.$$

We are left to show that these are isomorphisms.

Step 1. We treat the case where $\mathcal{X}$ is the classifying space of a finite diagonalizable group scheme. For any i we have $\mathrm{K}'_i(\mathcal{X}) = \mathrm{K}'_i(A) \otimes_{\mathrm{K}'_0(A)} \mathrm{K}'_0(\mathcal{X})$, so we just need to prove the theorem for $i = 0$.

Suppose first that $\mathcal{X} = \mathcal{B}_R\boldsymbol{\mu}_n$, for any n. In this case we can just verify the claim with a very easy calculation.

For any $d|n$ the map $\beta_{\mathcal{X},\langle d\rangle}$ does the following: firstly we have an immersion

$$\mathrm{K}'_0(\boldsymbol{\mu}_n)_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_d) = \mathbb{Q} \otimes \mathbb{Q}(\zeta_d) \hookrightarrow \mathrm{R}(\boldsymbol{\mu}_n) \otimes \mathrm{R}(\boldsymbol{\mu}_d) = \mathbb{Q}[t]/(t^n - 1) \otimes \mathbb{Q}[s]/(t^d - 1)$$

which sends $q \otimes x$ to $\left(\frac{1+t+\cdots+t^{n-1}}{n}\right) \otimes (p_x \cdot \psi_d(s))$.

Finally we compose this map with $(\alpha_{\mathcal{X}})_*\colon \mathrm{R}(\boldsymbol{\mu}_n) \otimes \mathrm{R}(\boldsymbol{\mu}_d) \to \mathrm{R}(\boldsymbol{\mu}_n)$.

In this case we have a very explicit characterization of $(\alpha_{\mathcal{X}})_*$: indeed it is the push-forward induced by the multiplication morphism

$$m_d\colon \boldsymbol{\mu}_d \times \boldsymbol{\mu}_n \longrightarrow \boldsymbol{\mu}_n$$

and it just takes the invariants relative to the subgroup $\ker(m_d) = \{(x, x^{-1})\colon x \in \boldsymbol{\mu}_d\}$. In particular it is quite easy to convince ourself that $(\alpha_{\mathcal{X}})_*$ sends $t^i \otimes s^j$ to t^i if $i \equiv j \pmod d$ and to 0 otherwise.

We see then that the restriction

$$(\alpha_{\mathcal{X}})_*\colon \mathrm{R}(\boldsymbol{\mu}_d) \simeq \Big(\frac{1+t+\cdots+t^{n-1}}{n}\Big) \otimes \mathrm{R}(\boldsymbol{\mu}_d) \longrightarrow \mathrm{R}(\boldsymbol{\mu}_n)$$

sends the image of a polynomial p to

$$p \cdot \frac{1}{n}(1 + t^d + \cdots + t^{n-d}) = p \cdot \frac{t^n - 1}{n(t^d - 1)}$$

which is nothing but $1/n$ times the push-forward $\mathrm{R}(\boldsymbol{\mu}_d) \to \mathrm{R}(\boldsymbol{\mu}_n)$ induced by the immersion $\boldsymbol{\mu}_d \hookrightarrow \boldsymbol{\mu}_n$.

Now consider the map $\mathcal{B}\boldsymbol{\mu}_d \to \mathcal{B}\boldsymbol{\mu}_n$ induced by the immersion $\boldsymbol{\mu}_d \hookrightarrow \boldsymbol{\mu}_n$; it is a finite representable map, and induces isomorphisms $\mathbb{Q} \simeq (\mathrm{R}\boldsymbol{\mu}_d)_{\mathbf{g}} \to (\mathrm{R}\boldsymbol{\mu}_n)_{\mathbf{g}} \simeq \mathbb{Q}$ and $(\mathrm{R}\boldsymbol{\mu}_d)_{\boldsymbol{\mu}_d} \to (\mathrm{R}\boldsymbol{\mu}_n)_{\boldsymbol{\mu}_d}$. We thus get a commutative diagram

$$\begin{tikzcd}
\mathbb{Q}(\zeta_d) \otimes (\mathrm{R}\boldsymbol{\mu}_d)_{\mathbf{g}} \arrow[r] \arrow[d] & \mathrm{R}\boldsymbol{\mu}_d \otimes \mathrm{R}\boldsymbol{\mu}_d \arrow[r, "\beta_{\mathcal{B}\boldsymbol{\mu}_d}"] \arrow[d] & (\mathrm{R}\boldsymbol{\mu}_d)_{\boldsymbol{\mu}_d} \arrow[d] \\
\mathbb{Q}(\zeta_d) \otimes (\mathrm{R}\boldsymbol{\mu}_n)_{\mathbf{g}} \arrow[r] & \mathrm{R}\boldsymbol{\mu}_d \otimes \mathrm{R}\boldsymbol{\mu}_n \arrow[r, "\beta_{\mathcal{B}\boldsymbol{\mu}_n}"] & (\mathrm{R}\boldsymbol{\mu}_n)_{\boldsymbol{\mu}_d}
\end{tikzcd}$$

By the above computation, the top row is equal to $\frac{1}{d}\colon \mathbb{Q}(\zeta_d) \to \mathbb{Q}(\zeta_d)$, and is an isomorphism. We conclude that $\beta_{\mathcal{X},\langle d\rangle}\colon \mathbb{Q}(\zeta_d) \to \mathrm{R}(\boldsymbol{\mu}_n)_{\boldsymbol{\mu}_d}$ is also an isomorphism.

Suppose now that $\mathcal{X} = \mathcal{B}_R\boldsymbol{\Delta}$ is the classifying stack of a finite diagonalizable group scheme of order n.

Let us fix an immersion $i\colon \boldsymbol{\mu}_d \to \boldsymbol{\Delta}$. The injection $\mathbb{Q} = \mathrm{R}(\boldsymbol{\Delta})_{\mathbf{g}} \hookrightarrow \mathrm{R}(\boldsymbol{\Delta})$ sends 1 to $1/n$ times the regular representation of $\boldsymbol{\Delta}$: this follows immediately from the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{Q} = \mathrm{K}_0'(\mathcal{B}_R\mathbf{1})_{\mathbf{g}} \arrow[r, "j_*"] \arrow[d] & \mathbb{Q} = \mathrm{K}_0'(\mathcal{B}_R\boldsymbol{\Delta})_{\mathbf{g}} \arrow[d] \\
\mathbb{Q} = \mathrm{K}_0'(\mathcal{B}_R\mathbf{1}) \arrow[r, "j_*"] & \mathrm{K}_0'(\mathcal{B}_R\boldsymbol{\Delta})
\end{tikzcd}$$

where $j\colon \mathcal{B}_R\mathbf{1} \to \mathcal{B}_R\boldsymbol{\Delta}$ is the map induced by the inclusion of the identity subgroup. Indeed $j^*j_*\colon \mathrm{K}_0'(\mathcal{B}_R\mathbf{1})_{\mathbf{g}} \to \mathrm{K}_0'(\mathcal{B}_R\mathbf{1})_{\mathbf{g}}$ is the multiplication by n map and j^* is an isomorphism of fields, so the above j_* is a multiplication by n.

Then we conclude, as before, that the restriction

$$(\alpha_{\mathcal{X}})_* \colon \mathrm{R}(\boldsymbol{\mu}_d) \simeq \Big(\frac{\sum_{\delta \in \mathcal{X}(\boldsymbol{\Delta})} \delta}{n}\Big) \otimes \mathrm{R}(\boldsymbol{\mu}_d) \longrightarrow \mathrm{R}(\boldsymbol{\Delta})$$

is $1/n$ times the push-forward map i_* induced by $i\colon \boldsymbol{\mu}_d \to \boldsymbol{\Delta}$: indeed $(\alpha_{\mathcal{X}})_*(\delta \otimes s^i) = \delta$ if $\delta_{|\boldsymbol{\mu}_d} = s^i$ and is 0 otherwise.

We then see, exactly as before, that in the general case $\beta_{\mathcal{X},\boldsymbol{\mu}_d} \colon \mathbb{Q}(\zeta_d) \to \mathrm{R}(\boldsymbol{\Delta})$ is still a multiple of the immersion induced by the canonical decomposition of $R(\boldsymbol{\Delta})$ and thus an isomorphism into the $\boldsymbol{\mu}_d$-part.

Step 2. Now we can handle the case where $\mathcal{X} = [X/T]$ is the quotient of an algebraic space by a split torus. We proceed by noetherian induction on X^σ, the base case where X^σ is empty being tautological.

By Thomason's generic slice theorem for torus actions there exists a T-invariant nonempty open subscheme $U \subset X^\sigma$, a subgroup T' of T and a T-equivariant isomorphism

$$U^\sigma = U \simeq T/T' \times (U/T).$$

Since T acts on X with finite stabilizers T' is finite and diagonalizable. Let $Z = Z^\sigma := X^\sigma \backslash U$; there is a localization exact sequence

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{K}'_i(Z,T)_{\mathbf{g}} \otimes \widetilde{\mathrm{R}}(\sigma) \arrow[r] \arrow[d, "\beta_{Z,\sigma}"] & \mathrm{K}'_i(X,T)_{\mathbf{g}} \otimes \widetilde{\mathrm{R}}(\sigma) \arrow[r] \arrow[d, "\beta_{X,\sigma}"] & \mathrm{K}'_i(U,T)_{\mathbf{g}} \otimes \widetilde{\mathrm{R}}(\sigma) \arrow[r] \arrow[d, "\beta_{U,\sigma}"] & {} \\
{} \arrow[r] & \mathrm{K}'_i(Z,T)_\sigma \arrow[r] & \mathrm{K}'_i(X,T)_\sigma \arrow[r] & \mathrm{K}'_i(U,T)_\sigma \arrow[r] & {}
\end{tikzcd}$$

By the five lemma and the inductive hypothesis we just need to prove the theorem for U. But in this case $\mathrm{K}'_*(U,T) \simeq \mathrm{K}'_*(U/T)_{\mathbf{g}} \otimes \mathrm{R}(T')$ and the map $\beta_{\mathcal{U},\sigma}$ is induced by

$$\beta_{\mathcal{B}_R(T'),\sigma} \colon \mathrm{R}(T')_{\mathbf{g}} \otimes \widetilde{\mathrm{R}}(\sigma) \longrightarrow \mathrm{R}(T')_\sigma$$

We have reduced ourselves to the case of the first step, which has already been handled.

Step 3. We can now conclude the proof dealing with the case $\mathcal{X} = [X/\mathrm{GL}_n]$: it follows almost immediately from the previous one.

Let $T \subset \mathrm{GL}_n$ be a maximal split torus and $\mathcal{Y} := X/T$; as always we can assume that σ factors through T. Let $\pi\colon X^\sigma/C_{\mathrm{GL}_n}(\sigma) \to X^\sigma/T$ be the projection; this map is representable, so π^* is a map of $\mathrm{R}(\mathrm{GL}_n)$-modules and preserves the fundamental decomposition of $\mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))$.

Let us be more precise. Let Δ_σ be the Weyl group of $C_{\mathrm{GL}_n}(\sigma)$; any $\sigma' \in C(C_{\mathrm{GL}_n}(\sigma))$ can be identified with an orbit of the action of Δ_σ on $C(T)$: we can pick any representative for the elements of this orbit, and with an abuse of notation we will call it σ' again. Let $\Gamma_{\sigma'} \subseteq \Delta_\sigma$ be the stabilizer of σ'; the map π^* sends $\mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))_{\sigma'}$ isomorphically to $\mathrm{K}'_*(X^\sigma, T)_{\sigma'}^{\Gamma_{\sigma'}}$.

Obviously $\Gamma_{\mathbf{1}} = \Gamma_\sigma = \Delta_\sigma$ and there is a commutative diagram

$$\begin{tikzcd}
\mathrm{K}'_*(X^\sigma, C_G(\sigma))_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \arrow[r, "\beta_{\mathcal{X},\sigma}"] \arrow[d, "\pi^*"] & \mathrm{K}'_*(X^\sigma, C_G(\sigma))_\sigma \arrow[d, "\pi^*"] \\
\mathrm{K}'_*(X^\sigma, T)^{\Delta_\sigma}_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \arrow[r, "\beta_{\mathcal{Y},\sigma}"] & \mathrm{K}'_*(X^\sigma, T)^{\Delta_\sigma}_\sigma
\end{tikzcd}$$

To conclude the proof we just need to see that the bottom map is an isomorphism: this follows from the previous case. ♠

Now we prove Theorem 1.6.3. It is an immediate consequence of the two Lemmas we give below.

Lemma 1.6.4. *The composition*

$$\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty}_{\mathbf{t}} \xrightarrow{\rho_*} \mathrm{K}'_*(\mathcal{X}) \xrightarrow{\lambda_{-1}(\mathcal{N})^{-1}\cdot\rho^*} \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty}_{\mathbf{t}}$$

is a multiplication by $\phi(r)$.

Remark. *This result is nontrivial, even if we assume the self-intersection formula. Consider for example the case $\mathcal{X} = \mathcal{B}S_3$ and $r = 2$; we have $\mathcal{I}_{\boldsymbol{\mu}_2}\mathcal{B}S_3 = \mathcal{B}\boldsymbol{\mu}_2$ and $\rho^*\rho_*\colon \mathrm{R}\boldsymbol{\mu}_2 \to \mathrm{R}\boldsymbol{\mu}_2$ sends $f \in \mathbb{Q}[t]/(t^2-1)$ to $f + f(1)(1+t)$. Thus the result only holds, in general, after passing to the tautological part.*

Proof. Consider first the case where $\mathcal{X} = [X/T]$ is the quotient of an algebraic space by the action of a torus. Let us fix a dual cyclic subgroup $\sigma \simeq \boldsymbol{\mu}_r \subset T$ and consider the $\sigma-$local part of the k-theory $\mathrm{K}'_*(\mathcal{X})$. We have a map

$$\Big(\bigoplus_{\phi\in\mathrm{Aut}(\boldsymbol{\mu}_r)} \mathrm{K}'_*(X^{\phi(\sigma)}, T)_\sigma\Big)^{\mathrm{Aut}(\boldsymbol{\mu}_r)} \xrightarrow{\rho_*} \mathrm{K}'_*(X, T)_\sigma$$

and, obviously, $\mathrm{Aut}(\boldsymbol{\mu}_r)$ acts freely on the left factors. In particular an invariant element is of the form $(x, x, \dots)$ (where we have identified the factors via the obvious isomorphisms) and its image in $\mathrm{K}'_*([X/T])$ is $\phi(r)\cdot i_*(x)$, where $i\colon [X^\sigma/T] \hookrightarrow [X/T]$ is a closed embedding. By the self-intersection formula we have

$$\lambda_{-1}(\mathcal{N})^{-1}\cdot i^* i_*(x) = x$$

so the composition

$$\Big(\bigoplus_{\phi\in\mathrm{Aut}(\boldsymbol{\mu}_r)} \mathrm{K}'_*(X^{\phi(\sigma)}, T)_\sigma\Big)^{\mathrm{Aut}(\boldsymbol{\mu}_r)} \xrightarrow{\rho_*} \mathrm{K}'_*(X, T)_\sigma \xrightarrow{\lambda_{-1}(\mathcal{N})^{-1}\cdot\rho^*} \Big(\bigoplus_{\phi\in\mathrm{Aut}(\boldsymbol{\mu}_r)} \mathrm{K}'_*(X^{\phi(\sigma)}, T)_\sigma\Big)^{\mathrm{Aut}(\boldsymbol{\mu}_r)}$$

sends $(x, x, \dots)$ to $(\phi(r)x, \phi(r)x, \dots)$, as we wanted to see.

Now let us deal with the general case, and suppose $\mathcal{X} = [X/\mathrm{GL}_n]$. Let us fix a split maximal torus $T \subset \mathrm{GL}_n$ and a dual cyclic subgroup $\sigma \simeq \boldsymbol{\mu}_r \subset T$. We saw that there are commutative diagrams

$$\begin{tikzcd}
{[X^\sigma/T]} \arrow[r] \arrow[dr, "\pi_\sigma"'] & {[(X^\sigma \times \mathrm{GL}_n)/(C_{\mathrm{GL}_n}(\sigma) \times T)]} \arrow[r, "\rho'"] \arrow[d, "\pi'"] & {[X/T]} \arrow[d, "\pi"] \\
 & {[X^\sigma/C_{\mathrm{GL}_n}(\sigma)]} \arrow[r, "\rho"] & {[X/\mathrm{GL}_n]}
\end{tikzcd}$$

where the right square is cartesian. The action of $C_{\mathrm{GL}_n}(\sigma) \times T$ on $X^\sigma \times \mathrm{GL}_n$ is $(\mathbf{c}, \mathbf{t})(x, g) = (x\mathbf{c}, \mathbf{c}^{-1} g \mathbf{t})$, π' is induced by the projection on the first factor and ρ' is induced by $(x, g) \to x \cdot g$.

For any $g \in \mathrm{GL}_n$ permutation matrix we can form a dual cyclic subgroup $\tilde{\sigma} = (\sigma, \sigma') = (\sigma, g^{-1}\sigma g) \subset C_{\mathrm{GL}_n}(\sigma) \times T$ and these are the only ones who lie over σ. Moreover, for any such g, there is a homomorphism $T \to C_{\mathrm{GL}_n}(\sigma) \times T$, $t \to (t, g^{-1}tg)$ and a closed embedding $i_{\tilde{\sigma}} \colon [X^\sigma/T] \hookrightarrow [(X^\sigma \times \mathrm{GL}_n)/(C_{\mathrm{GL}_n}(\sigma) \times T)]$ induced by the equivariant morphism $x \to (x, g)$. The diagrams

$$\begin{tikzcd}
{[X^\sigma/T]} \arrow[r, "i_{\tilde{\sigma}}"] \arrow[dr, "\pi_{\tilde{\sigma}}"'] & {[(X^\sigma \times \mathrm{GL}_n)/(C_{\mathrm{GL}_n}(\sigma) \times T)]} \arrow[d, "\pi'"] \\
 & {[X^\sigma/C_{\mathrm{GL}_n}(\sigma)]}
\end{tikzcd}$$

are commutative.

We now prove that the composition

$$\mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}} \xrightarrow{\rho'_*} \mathrm{K}'_*(X, T)_{\sigma'} \xrightarrow{(\rho')^*} \mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}}$$

is the multiplication by $\lambda_{-1}((\pi')^*(\mathcal{N}))_{\tilde{\sigma}}$. Since $(i_{\tilde{\sigma}})_* \colon \mathrm{K}'_*(X^\sigma, T)_{\sigma'} \to \mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}}$ and $i^*_{\tilde{\sigma}} \colon \mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}} \to \mathrm{K}'_*(X^\sigma, T)_{\sigma'}$ are isomorphisms, it is sufficient to prove that for any $x \in \mathrm{K}'_*(X^\sigma, T)_{\sigma'}$ it holds that

$$i^*_{\tilde{\sigma}}((\rho')^* \rho'_*((i_{\tilde{\sigma}})_*(x))) = i^*_{\tilde{\sigma}}((i_{\tilde{\sigma}})_*(x) \cdot \lambda_{-1}((\pi')^*(\mathcal{N}))_{\tilde{\sigma}}).$$

Let $\mathcal{N}_{\sigma'}$ be the conormal bundle of $i_{\tilde{\sigma}} \colon [X^\sigma/T] \hookrightarrow [(X^\sigma \times \mathrm{GL}_n)/(C_{\mathrm{GL}_n}(\sigma) \times T)]$, and $\mathcal{N}_T$ be the conormal bundle of $\rho' \circ i_{\tilde{\sigma}} \colon [X^\sigma/T] \hookrightarrow [X/T]$. Finally, let $\mathcal{N}'$ be the conormal bundle of $\rho' \colon [(X^\sigma \times \mathrm{GL}_n)/(C_{\mathrm{GL}_n}(\sigma) \times T)] \to [X/T]$. Since the square of the above diagram is cartesian, we have $(\pi')^*\mathcal{N} = \mathcal{N}'$, so there is an exact sequence

$$0 \longrightarrow \mathcal{N}_{\sigma'} \longrightarrow \mathcal{N}_T \longrightarrow i^*_{\tilde{\sigma}}(\pi')^*(\mathcal{N}) \longrightarrow 0$$

which implies that $\lambda_{-1}(i^*_{\tilde{\sigma}}(\pi')^*(\mathcal{N})) \cdot \lambda_{-1}(\mathcal{N}_{\sigma'}) = \lambda_{-1}(\mathcal{N}_T)$.

Using the self-intersection formula twice, we then see that

$$i^*_{\tilde{\sigma}}((\rho')^* \rho'_*((i_{\tilde{\sigma}})_*(x))) = x \cdot \lambda_{-1}(\mathcal{N}_T)_{\sigma'} = x \cdot \lambda_{-1}(\mathcal{N}_{\sigma'})_{\sigma'} \cdot \lambda_{-1}(\pi^*_{\sigma'}(\mathcal{N}))_{\sigma'} = i^*_{\tilde{\sigma}}(i_{\tilde{\sigma}})_*(x) \cdot i^*_{\tilde{\sigma}}(\pi')^*(\lambda_{-1}(\mathcal{N}))_{\sigma}$$

as we wanted.

Now, we note that $W = S_n$ acts on the set of the $\tilde{\sigma}$'s, and the stabilizer of each of them is Δ_σ, the Weyl group of $C_{\mathrm{GL}_n}(\sigma)$. Moreover W acts on the set of the dual cyclic subgroups $\sigma' \subset T$ that are in the same (GL_n -) conjugacy class of σ, and the stabilizer of this action is Γ_σ, the normalizer of σ in GL_n. Finally we saw that there is an exact sequence

$$0 \longrightarrow \Delta_\sigma \longrightarrow \Gamma_\sigma \longrightarrow w_{\mathrm{GL}_n}(\sigma) \longrightarrow 0$$

Now we want to compute the composition of

$$\Big(\bigoplus_{\phi \in \mathrm{Aut}(\sigma)/w_{\mathrm{GL}_n}(\sigma)} \mathrm{K}'_*(X^{\phi(\sigma)}, C_{\mathrm{GL}_n}(\sigma))_\sigma \Big)^{\mathrm{Aut}(\boldsymbol{\mu}_r)} \xrightarrow{\rho_*} \mathrm{K}'_*(X, \mathrm{GL}_n)_\sigma$$

with

$$\mathrm{K}'_*(X, \mathrm{GL}_n)_\sigma \xrightarrow{\lambda_{-1}(\mathcal{N})^{-1} \cdot \rho^*} \Big(\bigoplus_{\phi \in \mathrm{Aut}(\sigma)/w_{\mathrm{GL}_n}(\sigma)} \mathrm{K}'_*(X^{\phi(\sigma)}, C_{\mathrm{GL}_n}(\sigma))_\sigma \Big)^{\mathrm{Aut}(\boldsymbol{\mu}_r)}.$$

By the same argument of the case $\mathcal{X} = [X/T]$, the above composition sends $(x, x, \dots)$ to $\big(\frac{\phi(r)}{|w_{\mathrm{GL}_n}(\sigma)|} \cdot \lambda_{-1}(\mathcal{N})^{-1} \cdot \rho^* \rho_*(x), \frac{\phi(r)}{|w_{\mathrm{GL}_n}(\sigma)|} \cdot \lambda_{-1}(\mathcal{N})^{-1} \cdot \rho^* \rho_*(x), \dots \big)$, where $\rho^* \rho_*$ denotes the composition

$$\mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))_\sigma^{w_{\mathrm{GL}_n}(\sigma)} \xrightarrow{\rho_*} \mathrm{K}'_*(X, \mathrm{GL}_n)_\sigma \xrightarrow{\lambda_{-1}(\mathcal{N})^{-1} \cdot \rho^*} \mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))_\sigma^{w_{\mathrm{GL}_n}(\sigma)}.$$

To compute this, let us consider the commutative diagrams

$$\begin{tikzcd}
\mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))_\sigma^{w_{\mathrm{GL}_n}(\sigma)} \arrow[r, "\rho_*"] \arrow[d, "(\pi')^*"] & \mathrm{K}'_*(X, \mathrm{GL}_n)_\sigma \arrow[d, "\pi^*"] \\
\Big(\bigoplus_{g \in W/\Delta_\sigma} \mathrm{K}'_*(X^{\phi(\sigma)} \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}}^{\Gamma_\sigma} \Big)^W \arrow[r, "\rho'_*"] & \Big(\bigoplus_{g \in W/\Gamma_\sigma} \mathrm{K}'_*(X, T)_{\sigma'} \Big)^W
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{K}'_*(X, \mathrm{GL}_n)_\sigma \arrow[r, "\lambda_{-1}(\mathcal{N})^{-1} \cdot \rho^*"] \arrow[d, "\pi^*"] & \mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))_\sigma^{w_{\mathrm{GL}_n}(\sigma)} \arrow[d, "(\pi')^*"] \\
\Big(\bigoplus_{g \in W/\Gamma_\sigma} \mathrm{K}'_*(X, T)_{\sigma'} \Big)^W \arrow[r, "\lambda_{-1}(\mathcal{N})^{-1} \cdot \rho^*"] & \Big(\bigoplus_{g \in W/\Delta_\sigma} \mathrm{K}'_*(X^{\phi(\sigma)} \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}}^{\Gamma_\sigma} \Big)^W
\end{tikzcd}$$

Fix any $g \in W/\Delta_\delta$ (recalling $\tilde{\sigma} = (\sigma, \sigma') = (\sigma, g^{-1}\sigma g)$) the map ρ'_* sends

$$\bigoplus_{\phi \in g w_{\mathrm{GL}_n}(\sigma)} \mathrm{K}'_*(X^{\phi(\sigma)} \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}}^{\Gamma_\sigma} \longrightarrow \mathrm{K}'_*(X, T)_{\sigma'}$$

The map $(\pi')^* \lambda_{-1}(\mathcal{N})^{-1} \rho^* \rho_*$ takes an element $(x, \dots, \phi(x), \dots)_{\phi \in g w_{\mathrm{GL}_n}(\sigma)}$; but when x is Γ_σ-invariant it is of the form $(x, x, \dots)$, so the above map is $|w_{\mathrm{GL}_n}(\sigma)|$ times the composition

$$\mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}}^{\Gamma_\sigma} \xrightarrow{\rho'_*} \mathrm{K}'_*(X, T)_{\sigma'}^{\Gamma_\sigma} \xrightarrow{(\rho')^*} \mathrm{K}'_*(X^\sigma \times \mathrm{GL}_n, C_{\mathrm{GL}_n}(\sigma) \times T)_{\tilde{\sigma}}^{\Gamma_\sigma}$$

We have showed that this composition is the multiplication by $\lambda_{-1}((\pi')^*(\mathcal{N}))_{\tilde{\sigma}}$, thus the map

$$\mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))_\sigma^{w_{\mathrm{GL}_n}(\sigma)} \xrightarrow{\rho_*} \mathrm{K}'_*(X, \mathrm{GL}_n)_\sigma \xrightarrow{\lambda_{-1}(\mathcal{N})^{-1}\cdot\rho^*} \mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))_\sigma^{w_{\mathrm{GL}_n}(\sigma)}$$

is a multiplication by $|w_{\mathrm{GL}_n}(\sigma)|$. Given $x \in \mathrm{K}'_*(X^\sigma, C_{\mathrm{GL}_n}(\sigma))_\sigma^{w_{\mathrm{GL}_n}(\sigma)}$, we conclude that

$$\Big(\frac{\phi(r)}{|w_{\mathrm{GL}_n}(\sigma)|} \cdot \lambda_{-1}(\mathcal{N})^{-1} \cdot \rho^*\rho_*(x), \frac{\phi(r)}{|w_{\mathrm{GL}_n}(\sigma)|} \cdot \lambda_{-1}(\mathcal{N})^{-1} \cdot \rho^*\rho_*(x), \dots\Big)$$

is equal to $(\phi(r) \cdot x, \phi(r) \cdot x, \dots)$, as we wanted to prove. ♠

Lemma 1.6.5. *Let $\mathcal{X} = [X/G]$ and $\sigma \simeq \boldsymbol{\mu}_r$ be a dual cyclic subgroup; let $\mathcal{I}_\sigma\mathcal{X} := [X^\sigma/C_G(\sigma)]$. Then the composition*

$$\mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \hookrightarrow \mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X}) \otimes \mathrm{R}\boldsymbol{\mu}_r \xrightarrow{\beta_{\mathcal{X}}} \mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_\sigma \xrightarrow{\alpha^*_{\mathcal{X}}} \mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X}) \otimes \mathrm{R}\boldsymbol{\mu}_r \longrightarrow \mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)$$

is a multiplication by $\frac{1}{r}$.

Proof. Suppose first that $\mathcal{X}$ is of the form $[X/G]$, where X is a scheme and G a finite group over *Spec*(R). Consider the stack $\mathcal{Y} := [X^\sigma/\sigma]$; then we have an obvious map $\mathcal{Y} \to \mathcal{X}$, which induces a map $\mathcal{I}_\sigma\mathcal{Y} \to \mathcal{I}_\sigma\mathcal{X}$. The latter is the finite covering $[X^\sigma/\sigma] \xrightarrow{\pi} [X^\sigma/C_G(\sigma)]$ by the group $\Gamma := C_G(\sigma)/\sigma$. By Proposition 1.3.8 the push-forward π_* is a surjection in geometric K-theory.

The stack $\mathcal{I}_\sigma\mathcal{Y} = X^\sigma \times \mathcal{B}\sigma$ is the classifying stack of the constant group σ over $S := X^\sigma$. In particular there is a surjection $\pi_*\colon \mathrm{K}'_*(S) \simeq (\mathrm{R}\sigma_S)_{\mathbf{g}} \to \mathrm{K}'_*(X^\sigma, C_G(\sigma))_{\mathbf{g}}$.

We get a commutative diagram

$$\begin{tikzcd}
\mathbb{Q}(\zeta_r) \otimes (\mathrm{R}\sigma_S)_{\mathbf{g}} \arrow[r] \arrow[d] & \mathrm{R}\boldsymbol{\mu}_r \otimes \mathrm{R}\sigma_S \arrow[r, "\beta_{\mathcal{B}\sigma}"] \arrow[d] & (\mathrm{R}\sigma_S)_\sigma \arrow[r, "\alpha^*_{\mathcal{B}\sigma}"] \arrow[d] & \mathrm{R}\boldsymbol{\mu}_r \otimes \mathrm{R}\sigma_S \arrow[r] \arrow[d] & \mathbb{Q}(\zeta_r) \otimes (\mathrm{R}\sigma_S)_{\mathbf{g}} \arrow[d] \\
\mathbb{Q}(\zeta_r) \otimes \mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_{\mathbf{g}} \arrow[r] & \mathrm{R}\boldsymbol{\mu}_r \otimes \mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X}) \arrow[r, "\beta_{\mathcal{X}}"] & \mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_\sigma \arrow[r, "\alpha^*_{\mathcal{X}}"] & \mathrm{R}\boldsymbol{\mu}_r \otimes \mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X}) \arrow[r] & \mathbb{Q}(\zeta_r) \otimes \mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_{\mathbf{g}}
\end{tikzcd}$$

To see that the right half commutes, note that the push-forward π_* of an element of the form $\mathcal{E} \otimes V$, where $\mathcal{E}$ is a sheaf on X^σ and $V \in \mathrm{R}\sigma_R$, is just the virtual sheaf $\mathcal{E} \otimes (\mathrm{Ind}_\sigma^{C_G(\sigma)} V)$ seen as a $C_G(\sigma)$-equivariant sheaf on X^σ; then we may just check the commutativity when $X = R$, where it is clear.

As we have already seen, the composition of the top rows is

$$\mathrm{K}'_*(S) \otimes \mathbb{Q}(\zeta_r) \hookrightarrow \mathrm{R}\boldsymbol{\mu}_{r,S} \simeq R\boldsymbol{\mu}_{r,S} \otimes \frac{1 + x + \dots + x^{r-1}}{r} \xrightarrow{\Delta} \mathrm{R}\boldsymbol{\mu}_{r,S}$$

where Δ sends $x^i \otimes x^j$ to $\delta_{ij}x^i$, and is equal to $\frac{1}{r}$ times the canonical immersion $\mathrm{K}'_*(S) \otimes \mathbb{Q}(\zeta_r) \hookrightarrow \mathrm{R}\boldsymbol{\mu}_{r,S}$, whence our claim holds - trivially - for the top row. But the map $(\mathrm{R}\sigma_S)_{\mathbf{g}} \to \mathrm{K}'_*(X^\sigma, C_G(\sigma))_{\mathbf{g}}$ is a surjection, so it also holds for the bottom row and the claim is established.

Suppose now that $\mathcal{X}$ is Deligne-Mumford or tame. We use the following lemma (see Appendix C), based on a technique of Toen ([24], Proposition 4.9) and a technical result by Vistoli ([27], Lemma 2.7):

Definition. *Let $\mathcal{X}$ be an algebraic stack. We call* Chow envelope *a finite representable map such that the push-forward $\mathcal{I}p_*\colon \mathcal{I}_{\boldsymbol{\mu}}\mathcal{F} \to \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ is surjective.*

Lemma 1.6.6. *Let $\mathcal{X}$ be a DM or tame quotient stack. Then there exists a Chow envelope $p\colon \mathcal{F} \to \mathcal{X}$ with $\mathcal{F}$ a disjoint union of neutral gerbes.*

From the previous discussion it follows that we have the result in the case when $\mathcal{X} = \mathcal{B}_S G$ for a finite group-scheme G over a scheme S; in particular we have it for $\mathcal{F}$. Now consider the diagram

$$\begin{tikzcd}
\mathbb{Q}(\zeta_r) \otimes \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{F})_{\mathbf{g}} \arrow[r] \arrow[d, "\mathcal{I}p_*"] & \mathbb{Q}(\zeta_r) \otimes \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{F})_{\mathbf{g}} \arrow[d, "\mathcal{I}p_*"] \\
\mathbb{Q}(\zeta_r) \otimes \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \arrow[r] & \mathbb{Q}(\zeta_r) \otimes \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}}
\end{tikzcd}$$

which is commutative since p is representable. Since $\mathcal{I}\mathcal{X}$ is regular we have a projection formula for $f = \mathcal{I}p$, that is $f_* f^*(\xi) = \xi \cdot f_*(\mathcal{O}_{\mathcal{I}\mathcal{F}})$. But f is finite and surjective, so that $f_*(\mathcal{O}_{\mathcal{I}\mathcal{F}})$ has everywhere nonzero rank in $K_0(\mathcal{I}\mathcal{X})$. By Proposition 1.3.1 it is invertible in $\mathrm{K}'_*(\mathcal{I}\mathcal{X})_{\mathbf{g}}$, so we conclude that the vertical maps $f = \mathcal{I}p_*$ are surjective.

Now, the top horizontal map is a multiplication by $\frac{1}{r}$ and $\mathcal{I}p_*$ is surjective, so the claim also holds for the bottom map.

♠

The combination of the two above lemmas gives immediately the proof of the theorem.

We conclude this section mentioning that we can compare our morphism with the Toen-Riemann-Roch map ([24]). Let $\Lambda = \mathbb{Q}(\zeta_\infty)$ be the field generated by the roots of unity; for any $\mathbb{Z}$-module M we denote M_Λ the $\Lambda-$module $M \otimes_{\mathbb{Z}} \Lambda$.

Toen, when $\mathcal{X}$ is tame and DM over a field, defined a map

$$\mathrm{K}'_*(\mathcal{X})_\Lambda \longrightarrow \left(\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}}\right)_\Lambda$$

by composing

$$r^*\colon\ \mathrm{K}'_*(\mathcal{X})_\Lambda \xrightarrow{\lambda_{-1}(\mathcal{N})^{-1}\cdot\rho^*} \mathrm{K}'_*(\mathcal{I}\mathcal{X})_\Lambda \xrightarrow{\alpha^*_{\mathcal{X}}} \bigoplus_r (\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X}))_\Lambda \otimes \mathrm{R}\boldsymbol{\mu}_r \longrightarrow \bigoplus_r (\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_r)$$

with the tautological maps $(\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_r) \to (\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}})_\Lambda$ given by $x \otimes \zeta_r^i \to x \cdot \zeta_r^i$.

This morphism is clearly the composition of our Lefschetz-Riemann-Roch morphism $\mathcal{L}$ with the map

$$t\colon ((\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_r))^{\mathrm{Aut}(\boldsymbol{\mu}_r)} \longrightarrow (\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}})_\Lambda$$

given by $x \otimes \zeta_r^i \to \frac{\phi(r)}{r} \cdot x \cdot \zeta_r^i$.

With the machinery we have developed, we can immediately give a proof of the covariance of Toen's map: since we already know that $\mathcal{L}$ is covariant, we just need to verify the following

Proposition 1.6.7. *The map t is covariant with respect to proper push-forwards of relatively tame morphisms.*

We conclude that the composition of t with our Riemann-Roch map is a well-defined covariant morphism, and it coincides with the Toen-Riemann-Roch map.

Before giving the proof, we begin with an observation. Consider two positive integers $r|n$ and the projection map $f\colon \mathcal{B}\boldsymbol{\mu}_n \to \mathcal{B}\boldsymbol{\mu}_r$.

As we have seen $\mathcal{I}f_*\colon \mathrm{K}'_*(\mathcal{I}\mathcal{B}\boldsymbol{\mu}_n) \to \mathrm{K}'_*(\mathcal{I}\mathcal{B}\boldsymbol{\mu}_r)$ induces a map $\mathcal{I}f_*\colon \mathrm{K}'_*(\mathcal{I}\mathcal{B}\boldsymbol{\mu}_n)_{\mathbf{t}} \to \mathrm{K}'_*(\mathcal{I}\mathcal{B}\boldsymbol{\mu}_r)_{\mathbf{t}}$. Let us consider the identity immersion $\sigma = \mathrm{id}\colon \boldsymbol{\mu}_n \to \boldsymbol{\mu}_n$: it induces via f the identity $\sigma' = \mathrm{id}\colon \boldsymbol{\mu}_r \to \boldsymbol{\mu}_r$, so by the above remark we have a map

$$f_*\colon \mathbb{Q}(\zeta_n) = (\mathrm{R}\boldsymbol{\mu}_n)_\sigma \longrightarrow (\mathrm{R}\boldsymbol{\mu}_r)_{\sigma'} = \mathbb{Q}(\zeta_r).$$

Lemma 1.6.8. *The map $f_*\colon \mathbb{Q}(\zeta_n) \to \mathbb{Q}(\zeta_r)$ is equal to $\frac{r}{n}\cdot\mathrm{tr}$, where $\mathrm{tr}\colon \mathbb{Q}(\zeta_n) \to \mathbb{Q}(\zeta_r)$ is the field-theoretic trace map.*

Proof. Let us make the following observation: the ring $\mathrm{R}\boldsymbol{\mu}_n$ is a $\mathrm{R}\boldsymbol{\mu}_r$-module via the pull-back $f^*\colon \mathrm{R}\boldsymbol{\mu}_r \to \mathrm{R}\boldsymbol{\mu}_n$ and - by the projection formula - f_* is a map of $\mathrm{R}\boldsymbol{\mu}_r$-modules. Moreover f^* gives by localization the immersion $i\colon \mathbb{Q}(\zeta_r) \hookrightarrow \mathbb{Q}(\zeta_n)$; thus the $\mathbb{Q}(\zeta_r)$ action on $\mathbb{Q}(\zeta_n)$ is induced localizing the action of $\mathrm{R}\boldsymbol{\mu}_r$. We conclude that the push-forward $f_*\colon \mathbb{Q}(\zeta_n) \to \mathbb{Q}(\zeta_r)$ is $\mathbb{Q}(\zeta_r)$-linear.

We now recall that the bilinear pairing $\mathrm{tr}\colon \mathbb{Q}(\zeta_n)\times\mathbb{Q}(\zeta_n) \to \mathbb{Q}(\zeta_r)$ given by $(x,y) \to \mathrm{tr}(x\cdot y)$ is nondegenerate. In particular the $\mathbb{Q}(\zeta_r)$-linear functional f_* must be of the form $x \to \mathrm{tr}(k\cdot x)$ for a suitable $k \in \mathbb{Q}(\zeta_n)$. What we are left to prove is that $k = \frac{r}{n}$.

Let us now consider the case $r = 1$. Recall that the immersion $\mathbb{Q}(\zeta_n) \hookrightarrow \mathrm{R}\boldsymbol{\mu}_n$ sends an element $p(\zeta_n)$ (where p is a polynomial modulo ϕ_n) to $\psi_n \cdot p \in \mathrm{R}\boldsymbol{\mu}_n$, where ψ_n is the unique polynomial (modulo $x^n - 1$) such that $\psi_n(\zeta_n^i) = 1$ if $(n,i) = 1$ and is 0 otherwise.

The push-forward $f_*\colon \mathrm{R}\boldsymbol{\mu}_n \to \mathbb{Q}$ sends a monomial x^i to 1 if $n|i$ and to 0 otherwise (this is obviously well-defined modulo $x^n - 1$). This is equivalent to the map $p \to \frac{1}{n}\sum_{i=0}^{n-1} p(\zeta_n^i)$. Combining the definitions, we see that $f_*\colon \mathbb{Q}(\zeta_n) \to \mathbb{Q}$ sends $p(\zeta_n)$ to

$$\frac{1}{n}\sum_{i=0}^{n-1}\psi_n(\zeta_n^i)p(\zeta_n^i) = \frac{1}{n}\sum_{(n,i)=1} p(\zeta_n^i) = \frac{1}{n}\mathrm{tr}(p(\zeta_n)).$$

Let us return to the general case. We have $f_*\colon \mathbb{Q}(\zeta_n) \to \mathbb{Q}(\zeta_r) = \mathrm{tr}(k\cdot -)$; in particular the composition $\mathbb{Q}(\zeta_n) \to \mathbb{Q}(\zeta_r) \to \mathbb{Q}$ is equal to

$$\frac{1}{r}\cdot\mathrm{tr}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_r)} \circ \mathrm{tr}_{\mathbb{Q}(\zeta_r)}^{\mathbb{Q}(\zeta_n)}(k\cdot -) = \frac{1}{r}\cdot\mathrm{tr}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_n)}(k\cdot -).$$

However it is also equal to $\frac{1}{n}\cdot\mathrm{tr}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_n)}$ and by the nondegeneracy of the trace form we conclude that $k = \frac{r}{n}$, as we wanted. ♠

We can now prove Proposition 1.6.7:

Proof. Let us recall how the push-forward is defined on the left-hand side. Suppose that we have a proper map f from $\mathcal{Y} = [Y/H]$ to $\mathcal{X} = [X/G]$ where X, Y are schemes and

G, H are general linear groups. If $Z = X \times_{\mathcal{X}} Y$, there is an action of $G \times H$ on it, and $[Z/G \times H] = \mathcal{Y}$. The projection $Z \to X$ is equivariant with respect to $\pi \colon G \times H \to G$ and gives

$$f \colon \mathcal{Y} = [Z/G \times H] \longrightarrow [X/G] = \mathcal{X}.$$

Let $\rho \simeq \boldsymbol{\mu}_n \hookrightarrow G \times H$ be a dual cyclic subgroup, whose projections onto G, H are σ_G, σ_H respectively. Then $C_{G \times H}(\rho) = C_G(\sigma_G) \times C_H(\sigma_H)$.

Let $\sigma \simeq \boldsymbol{\mu}_r = \sigma_G$; the maps $Z \to X$ and $\mathcal{B}_R\rho \to \mathcal{B}_R\sigma$ induce a morphism

$$[Z^\rho/C_G(\sigma_G) \times C_H(\sigma_H)] \times \mathcal{B}_R\rho \longrightarrow [X^\sigma/C_G(\sigma)] \times \mathcal{B}_R\sigma$$

and this is the restriction of $\widetilde{\mathcal{I}_{\boldsymbol{\mu}} f}$ to the component of $\widetilde{\mathcal{I}_{\boldsymbol{\mu}}\mathcal{Y}}$ relative to ρ. As we have seen it induces a map $(\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_n}\mathcal{Y})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_n) \to (\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_r)$.

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_n) : \mathbb{Q}(\zeta_r))$. Then acting on ρ by elements of Γ gives all the conjugacy classes of dual cyclic subgroups which lie over σ. Let Γ_1 be the stabilizer of ρ in Γ (that is the subgroup of elements γ such that $\gamma(\rho)$ and ρ are conjugated). Let $\mathcal{I}_\rho\mathcal{Y} := [Z^\rho/C_G(\sigma_G) \times C_H(\sigma_H)]$ and $\mathcal{I}_\sigma\mathcal{X} := [X^\sigma/C_G(\sigma)]$; we want to prove that the following diagram commutes:

$$\begin{tikzcd}
\Big(\bigoplus_{\gamma\in\Gamma/\Gamma_1} (\mathrm{K}'_*(\mathcal{I}_{\gamma(\rho)}\mathcal{Y})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_n)\Big)^{\mathrm{Aut}(\boldsymbol{\mu}_n)} \arrow[r] \arrow[d] & \bigoplus_{\gamma\in\Gamma/\Gamma_1} (\mathrm{K}'_*(\mathcal{I}_{\gamma(\rho)}\mathcal{Y})_{\mathbf{g}})_\Lambda \arrow[d] \\
(\mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_r) \arrow[r] & (\mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_{\mathbf{g}})_\Lambda
\end{tikzcd}$$

Any element of the top left box is of the form $\tilde{x} = (x, \dots, \gamma(x), \dots)$ for $x \in ((\mathrm{K}'_*(\mathcal{I}_\rho\mathcal{Y})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_n))^{\Gamma_1}$.

Let $x = \sum x_i \otimes \zeta_n^i$. All the elements $(\gamma(x))$ have the same image in $(\mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_r)$, since γ has trivial image in $\mathrm{Aut}(\boldsymbol{\mu}_r)$. Thus the composition

$$\Big(\bigoplus_{\gamma\in\Gamma/\Gamma_1} (\mathrm{K}'_*(\mathcal{I}_{\gamma(\rho)}\mathcal{Y})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_n)\Big)^{\mathrm{Aut}(\boldsymbol{\mu}_n)} \longrightarrow (\mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_{\mathbf{g}})_\Lambda \otimes \mathbb{Q}(\zeta_r) \longrightarrow (\mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_{\mathbf{g}})_\Lambda$$

sends $\tilde{x}$ to

$$|\Gamma/\Gamma_1| \cdot \sum_i \big(\mathcal{I}f_*(x_i) \cdot \frac{\phi(r)}{r} \cdot \frac{r}{n}\mathrm{tr}(\zeta_n^i)\big) = |\Gamma/\Gamma_1| \cdot \frac{\phi(r)}{n} \cdot \sum_i \sum_{\gamma\in\Gamma} \mathcal{I}f_*(x_i) \cdot \gamma(\zeta_n^i)$$

However, as we observed before, $\mathcal{I}f_*(x_i) = \gamma(\mathcal{I}f_*(x_i)) = \mathcal{I}_{\boldsymbol{\mu}}f_*(\gamma(x_i))$ so the above term is equal to

$$\begin{aligned}
|\Gamma/\Gamma_1| \cdot \frac{\phi(r)}{n} \cdot \sum_i \sum_{\gamma\in\Gamma} \mathcal{I}f_*(\gamma(x_i)) \cdot \gamma(\zeta_n^i) = {} & |\Gamma/\Gamma_1| \cdot \frac{\phi(r)}{n} \cdot \sum_{\gamma\in\Gamma/\Gamma_1} \sum_{\gamma_1\in\Gamma_1} \gamma\Big(\sum_i (\mathcal{I}f_*(\gamma_1(x_i \cdot \zeta_n^i)))\Big) \\
= {} & |\Gamma/\Gamma_1| \cdot \frac{\phi(r)}{n} \cdot |\Gamma_1| \cdot \sum_{\gamma\in\Gamma/\Gamma_1} \gamma\Big(\sum_i \Big(\mathcal{I}f_*(x_i) \cdot \zeta_n^i\Big)\Big)
\end{aligned}$$

recalling that $\gamma_1(x) = x$ for any $\gamma_1 \in \Gamma_1$. But $|\Gamma| = \frac{\phi(n)}{\phi(r)}$ so we can rewrite this as

$$\frac{\phi(n)}{n} \cdot \sum_{\gamma\in\Gamma/\Gamma_1} \gamma\Big(\sum_i \Big(\mathcal{I}f_*(x_i) \cdot \zeta_n^i\Big)\Big)$$

that is the image of $\tilde{x}$ under the composition

$$\Big(\bigoplus_{\gamma\in\Gamma/\Gamma_1}(\mathrm{K}'_*(\mathcal{I}_{\gamma(\rho)}\mathcal{Y})_{\mathbf{g}})_\Lambda\otimes\mathbb{Q}(\zeta_n)\Big)^{\mathrm{Aut}(\boldsymbol{\mu}_n)}\longrightarrow\bigoplus_{\gamma\in\Gamma/\Gamma_1}(\mathrm{K}'_*(\mathcal{I}_{\gamma(\rho)}\mathcal{Y})_{\mathbf{g}})_\Lambda\longrightarrow(\mathrm{K}'_*(\mathcal{I}_\sigma\mathcal{X})_{\mathbf{g}})_\Lambda.$$

The Proposition is thus proved. ♠

1.7 The rational Toen-Riemann-Roch map

Let I be the moduli space of $\mathcal{I}\mathcal{X}$. Using the results of the previous section, we can produce a covariant isomorphism

$$\mathrm{K}'_*(\mathcal{X})\longrightarrow A^*(I).$$

with image in the Chow groups of I. This map, with respect to the classical Toen-Riemann-Roch morphism, has the advantage of having $\mathbb{Q}$-coefficients and the disadvantage of being non-canonical.

We start by providing a $\mathbb{Q}$-isomorphism

$$\mathrm{K}'_*(\mathcal{X})\longrightarrow\mathrm{K}'_*(\mathcal{I}\mathcal{X})_{\mathbf{g}}.$$

In general, given a finite group G and a finitely generated $k[G]-$module M, there is a canonical isomorphism

$$M\longrightarrow(M\otimes k[G])^G$$

given by $m\to\frac{1}{|G|}\sum_{g\in G}g(m)\otimes g$.

In particular for any r we have an isomorphism

$$\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}}\longrightarrow\Big(\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}}\otimes\mathbb{Q}(\zeta_r)\Big)^{\mathrm{Aut}(\boldsymbol{\mu}_r)}$$

once we can provide an isomorphism $\mathbb{Q}(\zeta_r)\simeq k[\mathrm{Aut}(\boldsymbol{\mu}_r)]$. This is given by a choice of a *normal basis* $(x,\dots,\gamma(x),\dots)_{\gamma\in\mathrm{Aut}(\boldsymbol{\mu}_r)}$ for $\mathbb{Q}(\zeta_r)$.

These maps combine into an isomorphism $\mathrm{K}'_*(\mathcal{X})\longrightarrow\mathrm{K}'_*(\mathcal{I}\mathcal{X})_{\mathbf{g}}$ for any $\mathcal{X}$. In order to make these maps covariant for proper push-forwards, we invoke the following result of Lenstra ([17]):

Proposition 1.7.1. *There exist, for any n, an element $x_n\in\mathbb{Q}(\zeta_n)$ such that*

1. *The set $\{\gamma(x_n)\}_{\gamma\in Aut(\boldsymbol{\mu}_r)}$ is a normal basis for $\mathbb{Q}(\zeta_n)$.*
2. *For any n,m with $m|n$ we have $Tr^{\mathbb{Q}(\zeta_n)}_{\mathbb{Q}(\zeta_m)}(x_n)=x_m$.*

Having this result, we can now take the norml bases $\{\gamma\big(r\cdot x_r\big)\}_{\gamma\in\mathrm{Aut}(\boldsymbol{\mu}_r)}$ in our construction of the map

$$\phi\colon\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}}\longrightarrow\Big(\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}}\otimes\mathbb{Q}(\zeta_r)\Big)^{\mathrm{Aut}(\boldsymbol{\mu}_r)}.$$

Proposition 1.7.2. *The map ϕ is covariant with respect to proper push-forward.*

Proof. The proof is very similar to the proof of Proposition 1.6.7. Suppose that we have a proper map f from $\mathcal{Y} = [Y/H]$ to $\mathcal{X} = [X/G]$. As usual, if $Z = X \times_{\mathcal{X}} Y$, there is an action of $G \times H$ on it, and $[Z/G \times H] = \mathcal{Y}$. The projection $Z \to X$ is equivariant with respect to $\pi\colon G \times H \to G$ and gives

$$f\colon \mathcal{Y} = [Z/G \times H] \longrightarrow [X/G] = \mathcal{X}.$$

Let $\rho \simeq \boldsymbol{\mu}_n \hookrightarrow G \times H$ be a dual cyclic subgroup, whose projections onto G, H are σ_G, σ_H respectively. Then $C_{G\times H}(\rho) = C_G(\sigma_G) \times C_H(\sigma_H)$.

Let $\sigma \simeq \boldsymbol{\mu}_r = \sigma_G$ and suppose that $x \in \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_\rho}\mathcal{Y})_{\mathbf{g}}$, so that $\mathcal{I}f_*(x) \in \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_\sigma}\mathcal{X})_{\mathbf{g}}$. Let $n = |\rho|$ and $r = |\sigma|$.

Then we want to prove that the diagram

$$\begin{tikzcd}
\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_n}\mathcal{Y})_{\mathbf{g}} \arrow[r, "\phi"] \arrow[d, "\mathcal{I}f_*"] & \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_n}\mathcal{Y})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_n) \arrow[d, "\widetilde{\mathcal{I}f}_*"] \\
\mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \arrow[r, "\phi"] & \mathrm{K}'_*(\mathcal{I}_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)
\end{tikzcd}$$

is commutative. The composition of the bottom horizontal and left vertical arrows send

$$x \longrightarrow \frac{r}{\phi(r)} \sum_{\gamma \in \mathrm{Aut}(\boldsymbol{\mu}_r)} \gamma(\mathcal{I}f_*(x)) \otimes \gamma(x_r).$$

On the other hand, using Lemma 1.6.8 we immediately see that the composition of the other arrows send

$$x \longrightarrow \frac{n}{\phi(n)} \sum_{\gamma \in \mathrm{Aut}(\boldsymbol{\mu}_n)} \mathcal{I}f_*(\gamma(x)) \otimes \frac{r}{n}\mathrm{Tr}^{\mathbb{Q}(\zeta_n)}_{\mathbb{Q}(\zeta_r)}(x_n).$$

Let Γ be the Galois group of $\boldsymbol{\mu}_n$ over $\boldsymbol{\mu}_r$; then for any $\gamma \in \Gamma$ we have $\mathcal{I}f_* = \mathcal{I}f_* \circ \gamma$, so we can rewrite the last expression as

$$\frac{n}{\phi(n)} \cdot |\Gamma| \sum_{\gamma \in \mathrm{Aut}(\boldsymbol{\mu}_n)/\Gamma} \mathcal{I}f_*(\gamma(x)) \otimes \frac{r}{n}\mathrm{Tr}^{\mathbb{Q}(\zeta_n)}_{\mathbb{Q}(\zeta_r)}(x_n)$$

But $\mathrm{Aut}(\boldsymbol{\mu}_n)/\Gamma = \mathrm{Aut}(\boldsymbol{\mu}_r)$ and $\mathrm{Tr}^{\mathbb{Q}(\zeta_n)}_{\mathbb{Q}(\zeta_r)}(x_n) = x_r$, so this is equal to

$$\frac{r}{\phi(r)} \sum_{\gamma \in \mathrm{Aut}(\boldsymbol{\mu}_r)} \gamma(\mathcal{I}f_*(x)) \otimes \gamma(x_r)$$

as we wanted. ♠

Now, let I be the coarse moduli space of $\mathcal{I}\mathcal{X}$. We have an isomorphism $\pi_*\colon \mathrm{K}'_*(\mathcal{I}\mathcal{X})_{\mathbf{g}} \to \mathrm{K}'_*(I)_{\mathbf{g}} \simeq \mathrm{K}'_*(I)$, and the usual covariant Riemann-Roch map $\tau_I\colon \mathrm{K}'_*(I) \to A^*(I)$, where $A^*(-)$ is the Chow group functor. Summarizing, the composition

$$\mathrm{K}'_*(\mathcal{X}) \to \mathrm{K}'_*(\mathcal{I}\mathcal{X})_{\mathbf{g}} \xrightarrow{\pi_*} \mathrm{K}'_*(I)_{\mathbf{g}} \simeq \mathrm{K}'_*(I) \xrightarrow{\tau_I} A^*(I)$$

provides a Riemann-Roch map which is covariant and has $\mathbb{Q}-$coefficients.

Chapter 2

An explicit formula for Deligne-Mumford stacks

2.1 Introduction

Let X be a smooth projective variety over a field equipped with an action of a finite group $G \subseteq Aut(X)$ of order n. Denote the stack quotient by $\mathcal{X} = [X/G]$ and the scheme quotient Y. The stack $\mathcal{X}$ is Deligne-Mumford, so given a G-equivariant vector bundle $\mathcal{E}$ on X we can use the results of the previous chapter to compute the class of $\chi(\mathcal{E})$ in the representation ring of G.

Given an element $g \in G$, let σ be the cyclic subgroup generated by g. We denote by X^σ the fixed loci by the subgroup, which inherits an action by the centraliser $C(\sigma)$.

Then

$$\chi_G(X, \mathcal{E}) = \bigoplus_\sigma \bigoplus_i \frac{\chi(A^{\sigma,i})}{\varphi(|\sigma|)} \frac{|\sigma|}{|C(\sigma)|} \cdot \operatorname{Ind}_\sigma^G \iota(x^i)$$

where $A^{\sigma,i}$ is a $C(\sigma)$-equivariant vector bundle on X^σ that can be given explicitly.

The result is obtained by computing the euler characteristics of $\mathcal{E}$ in two ways via the Grothendieck-Riemann-Roch theorem for stacks:

$$\begin{tikzcd}
K(\mathcal{X}) \arrow[r, "\mathcal{L}_\mathcal{X}"] \arrow[d, "\pi_*"] & \bigoplus_{r,\sigma\in\overline{C}_r} K_{\mathbf{g}}(\tilde{I}_\sigma\mathcal{X}) \otimes \mathbb{Q}(\zeta_r) \arrow[d] \\
K(BG) \arrow[r, "\mathcal{L}_{BG}"] & \bigoplus_{r,\sigma\in\overline{C}_r} K_{\mathbf{g}}(\tilde{I}_\sigma BG) \otimes \mathbb{Q}(\zeta_r)
\end{tikzcd}$$

Since $\mathcal{X}$ is smooth, the upper row can be identified with the composition

$$K(\mathcal{X}) \xrightarrow{\lambda_{-1}(\mathcal{N}^\vee)^{-1}\cdot\rho^*} K(I\mathcal{X}) \xrightarrow{m^*} K(\tilde{I}\mathcal{X}) \xrightarrow{\frac{r}{\phi(r)}} K(\tilde{I}\mathcal{X}),$$

which by constuction lands in the tautological part of the K-theory of the inertia stack. On the other hand, $\mathcal{L}_{BG}^{-1}$ is the composition

$$\bigoplus_\sigma K(BC(\sigma))_{\mathbf{g}} \otimes \tilde{R}\sigma \to \bigoplus_\sigma K(BC(\sigma)) \otimes R\sigma \xrightarrow{m_*} \bigoplus_\sigma K(BC(\sigma)) \xrightarrow{\operatorname{Ind}_{C(\sigma)}^G} K(BG).$$

2.2 Computation

2.2.1 Lower composition

Inclusion of geometric part of BG The inclusion $K_{\mathbf{g}}(BH) \to K(BH)$ sends the unit to $\frac{kH}{H}$.

Inclusion $\tilde{R}\sigma \to R\sigma$ Suppose σ is dual cyclic of order r. We denote by $\iota : \mathbb{Q}[x]/\Phi_r(x) \to \mathbb{Q}[x]/x^r - 1$ the section of the projection $\mathbb{Q}[x]/x^r - 1 \to \mathbb{Q}[x]/\Phi_r(x)$.

Inertia stack of BG and representations Let σ be a dual cyclic subgroup of G of order r and let $H = C(\sigma)$ be its centraliser. The twisting operation on the rth cyclotomic inertia stack is given by the pullback induced by multiplication $m : H \times \sigma \to H$. The anti-twist is given by m_*, whose effect is taking invariants relative to the subgroup $\ker(m) = \{(x^{-1}, x) : x \in \sigma\}$. In formula, for an H-representation V and a character χ of σ

$$\begin{aligned} m_* : RH \otimes R\sigma &\to RH \\ V \otimes \chi &\mapsto \chi\text{-isotypical part of } \operatorname{Res}^H_\sigma V, \end{aligned}$$

which inherits a natural H-action.

Since H is the centraliser of σ, we have $\operatorname{Res}^H_\sigma \operatorname{Ind}^H_\sigma \chi = \frac{|H|}{|\sigma|}\chi$ for any σ-character χ and

$$\operatorname{Res}^H_\sigma k[H] = \operatorname{Res}^H_\sigma \operatorname{Ind}^H_\sigma k[\sigma] = k[\sigma]^{\oplus |H/\sigma|} = \frac{|H|}{|\sigma|} \bigoplus_{\chi \in \widehat{\sigma}} \chi = \bigoplus_{\chi \in \widehat{\sigma}} \operatorname{Res}^H_\sigma \operatorname{Ind}^H_\sigma \chi.$$

As a consequence we note that the $\chi-$isotypic part of $\operatorname{Res}^H_\sigma k[H]$ is exactly $\operatorname{Ind}^H_\sigma \chi$, whence

$$m_*(\frac{k[H]}{|H|} \otimes \chi) = \frac{1}{|H|} \operatorname{Ind}^H_\sigma \chi.$$

In particular, the composition

$$\tilde{R}\sigma \simeq K(BH)_{\mathbf{g}} \otimes \tilde{R}\sigma \hookrightarrow K(BC(\sigma)) \otimes R\sigma \xrightarrow{m_*} K(BC(\sigma))$$

is the same as

$$\tilde{R}\sigma \simeq K(B\sigma)_{\mathbf{g}} \otimes \tilde{R}\sigma \hookrightarrow K(B\sigma) \otimes R\sigma \xrightarrow{m_*} K(B\sigma) \xrightarrow{\operatorname{Ind}_\sigma^{C(\sigma)}} K(BC(\sigma)).$$

up to a correcting factor of $\frac{r}{|H|} = \frac{|\sigma|}{|C(\sigma)|}$.

Finally, by 1.6.3, the map

$$\tilde{R}\sigma \simeq K(B\sigma)_{\mathbf{g}} \otimes \tilde{R}\sigma \hookrightarrow K(B\sigma) \otimes R\sigma \xrightarrow{m_*} K(B\sigma)$$

is equal to $\frac{1}{r} \cdot \iota \colon \mathbb{Q}(\zeta_r) \to \mathbb{Q}[x]/x^r - 1$.

Now we are ready to compute the lower composition. We index the inertia stack of BG by $\overline{C} = \coprod \overline{C}_r$, where $\overline{C}_r$ is the conjugacy classes of monomorphisms $\mu_r \to G$:

$$IBG = \coprod_{\sigma \in \overline{C}_r} BC(\sigma).$$

Then the morphism $\mathcal{L}_{BG}$ is the composition

$$K(BG) \to \bigoplus_{\sigma} K(BC(\sigma)) \to \bigoplus_{\sigma} K(BC(\sigma)) \otimes R\sigma \to \bigoplus_{\sigma} K_{\mathbf{g}}(BC(\sigma)) \otimes \tilde{R}\sigma$$

whose inverse on the summand labelled by $\sigma \in \overline{C}_r$ given by

$$\begin{tikzcd}
K_{\mathbf{g}}(BC(\sigma)) \otimes \tilde{R}\sigma \arrow[r, hook] \arrow[d, equal] & K(BC(\sigma)) \otimes R\sigma \arrow[r, "m_*"] \arrow[d, equal] & K(BC(\sigma)) \arrow[r, "\operatorname{Ind}_{C(\sigma)}^{G}"] & K(BG) \\
\mathbb{Q} \otimes \mathbb{Q}[x]/(\Phi_r(x)) & K(BC(\sigma)) \otimes \mathbb{Q}[x]/x^r - 1 & & \\
1 \otimes x \arrow[r, mapsto] & \frac{kC(\sigma)}{|C(\sigma)|} \otimes \iota(x) \arrow[r, mapsto] & \frac{r}{|C(\sigma)|} \cdot \frac{1}{r} \operatorname{Ind}_{\sigma}^{C(\sigma)} \iota(x) \arrow[r, mapsto] & \frac{1}{|C(\sigma)|} \operatorname{Ind}_{\sigma}^{G} i(x)
\end{tikzcd}$$

2.2.2 Upper composition

The upper composition is considerably easier, thanks to 1.6.3 and the fact that the twisting map m^* is easier to describe. Write

$$I\mathcal{X} = \bigoplus_{r, \sigma \in \overline{C}_r} [X^{\sigma}/C(\sigma)].$$

Given a G-equivariant vector bundle $\mathcal{E}$ on X, let $(\mathcal{E}_{\sigma})_{\sigma} = \rho^*\mathcal{E}$ be its pullback along $\rho : I\mathcal{X} \to \mathcal{X}$ and let $(N_{\sigma})_{\sigma}$ be the normal bundle of ρ. On each component of the inertia stack we split the virtual bundle

$$\frac{\mathcal{E}_{\sigma}}{\lambda_{-1}(N_{\sigma}^{\vee})} := A^{\sigma} = \bigoplus_{i \in \hat{\sigma}} A^{\sigma,i}$$

into isotypical components as σ-representations, obtained by restriction from $C(\sigma)$. Here $\hat{\sigma}$ is the character group of σ, which can be identified with $\mathbb{Z}/r\mathbb{Z}$.

The upper composition thus sends

$$\mathcal{E} \longrightarrow \bigoplus_{r, \sigma \in \overline{C}_r, i \in \hat{\sigma}} \frac{r \cdot \zeta_r^i}{\phi(r)} \cdot A_{\mathbf{g}}^{\sigma,i}$$

where $A_{\mathbf{g}}^{\sigma,i}$ denotes the projection to the geometric part.

2.2.3 Total composition

We have the vertical map

$$\mathcal{I}\pi_* \colon \bigoplus_{r, \sigma \in \overline{C}_r} K_{\mathbf{g}}(I_{\sigma}\mathcal{X}) \otimes \mathbb{Q}(\zeta_r) \longrightarrow \bigoplus_{r, \sigma \in \overline{C}_r} K_{\mathbf{g}}(I_{\sigma}BG) \otimes \mathbb{Q}(\zeta_r) \simeq \bigoplus_{r, \sigma \in \overline{C}_r} \mathbb{Q}(\zeta_r)$$

which, since π is representable, is the identity on the $\mathbb{Q}(\zeta_r)$ components and sends $A_{\mathbf{g}}^{\sigma,i}$ to $\chi(A^{\sigma,i})$.

Indeed we have a commuting diagram

$$\begin{tikzcd}
K(I_\sigma \mathcal{X}) \arrow[r] \arrow[d, "\pi_*"] & K_{\mathbf{g}}(I_\sigma \mathcal{X}) \arrow[d, "I\pi_*"] \\
K(I_\sigma BG) \arrow[r] & K_{\mathbf{g}}(I_\sigma BG)
\end{tikzcd}$$

the horizontal maps being projections to the geometric part; the bottom-left composition is then easily seen to be exactly the Euler characteristic map.

Combining this with the previous calculation of the lower composition, we immediately get:

$$\chi_G(X, \mathcal{E}) = \bigoplus_{r, \sigma \in \overline{C}_r} \bigoplus_{i \in \hat{\sigma}} \frac{\chi(A^{\sigma,i})}{\phi(r)} \frac{r}{|C(\sigma)|} \cdot \operatorname{Ind}_\sigma^G \iota(\zeta_r^i).$$

2.3 Explicit formula for curves

In this section we specialise to the case where X is a smooth projective curve. Then Y is also a smooth projective curve. Let $n = |G|$. First note that we can give more concrete description of the $\chi(A^{\sigma,i})$'s. Over the identity sector corresponding to $\sigma = 1$ it is the holomorphic Euler characteristic of $\mathcal{E}$. For $\sigma \neq 1$, as the coarse moduli space of $[X^\sigma / C(\sigma)]$ is zero-dimensional, $\chi(A^{\sigma,i})$ is nothing but the dimension of the virtual representation $A^{\sigma,i}$.

Thus

$$\chi_G(X, \mathcal{E}) = \chi(X, \mathcal{E}) \frac{kG}{n} + \bigoplus_{r > 1, \sigma \in \overline{C}_r} \bigoplus_{i \in \hat{\sigma}} \frac{\dim A^{\sigma,i}}{\phi(r)} \frac{r}{|C(\sigma)|} \cdot \operatorname{Ind}_\sigma^G \iota(\zeta_r^i).$$

Let us reindex the summation. We fix a representative for each $\tilde{\sigma} \in \overline{C}_r$, which we also call $\tilde{\sigma}$. Choose also a set of representatives $\{\tilde{x}\}$ for each G-orbit with non-trivial stabilisers. The action being effective, for each $1 \neq \tilde{\sigma}$ the fixed locus X^σ is zero-dimensional, so we can regroup uniquely the $\{\tilde{\sigma}\}$'s to the sets $\{\tilde{\sigma} \subseteq G_{\tilde{x}}\}_{\tilde{x}}$. For each $x \in X$, denote by e_x (resp. e_x^t) the ramification index (resp. the *tame* ramification index). We have:

$$\begin{aligned}
\chi_G(X, \mathcal{E}) &= \chi(X, \mathcal{E}) \frac{kG}{n} + \sum_{\tilde{x}} \bigoplus_{1 \neq \tilde{\sigma} \subseteq G_{\tilde{x}}} \bigoplus_{i \in \hat{\sigma}} \frac{|X^\sigma| \cdot \dim A^{\sigma,i}}{\phi(r)} \frac{r}{|C(\sigma)|} \cdot \operatorname{Ind}_\sigma^G \iota(\zeta_r^i) \\
&= \chi(X, \mathcal{E}) \frac{kG}{n} + \sum_{x \in X} \frac{e_x}{n} \operatorname{Ind}_{G_x}^G \bigoplus_{1 \neq \tilde{\sigma} \subseteq G_x} \bigoplus_{i \in \hat{\sigma}} \frac{|X^\sigma| \cdot \dim A^{\sigma,i}}{\phi(r)} \frac{r}{|C(\sigma)|} \cdot \operatorname{Ind}_\sigma^{G_x} \iota(\zeta_r^i).
\end{aligned}$$

At this point we observe that making use of 1.6.3, we can rewrite the expression

$$\bigoplus_{1 \neq \tilde{\sigma} \subseteq G_x} \bigoplus_{i \in \hat{\sigma}} \frac{\dim A^{\sigma,i}}{\phi(r)} \frac{r}{|C(\sigma)|} \cdot \operatorname{Ind}_\sigma^{G_x} \iota(\zeta_r^i) \qquad (*)$$

Indeed, let $A^\sigma = \operatorname{Res}_{C(\sigma)}^{G_x} V^\sigma$, where V^σ is the G_x-virtual sheaf $\frac{\mathcal{E}_\sigma}{\lambda_{-1}(N_\sigma^\vee)}$.

Then we know from 1.6.3, that the following composition of maps

$$R(G_x) \xrightarrow{\qquad\text{Res}\qquad} \bigoplus_{\substack{r,\sigma \\ \tilde{\sigma}\subseteq G_x}} R(C_{G_x}(\sigma)) \xrightarrow{m^*} \bigoplus_{\substack{r,\sigma \\ \tilde{\sigma}\subseteq G_x}} R(C_{G_x}(\sigma))_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)$$

$$\bigoplus_{\substack{r,\sigma \\ \tilde{\sigma}\subseteq G_x}} R(C_{G_x}(\sigma))_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \xrightarrow{m_*} \bigoplus_{\substack{r,\sigma \\ \tilde{\sigma}\subseteq G_x}} R(C_{G_x}(\sigma)) \xrightarrow{\qquad\text{Ind}\qquad} R(G_x)$$

given explicitly by

$$V^\sigma \xrightarrow{\qquad\qquad\qquad} \sum_{\substack{r,\sigma \\ \tilde{\sigma}\subseteq G_x}} \text{Res}^{G_x}_{C_{G_x}(\sigma)}(V^\sigma) \xrightarrow{\qquad\qquad\qquad} \sum_{\substack{r,\sigma \\ i\in\hat{\sigma}}} \dim(V^{\sigma,i}) \cdot \zeta_r^i \cdot \tfrac{r}{\phi(r)}$$

$$\sum_{\substack{r,\sigma \\ i\in\hat{\sigma}}} \dim(V^{\sigma,i}) \cdot \zeta_r^i \cdot \tfrac{r}{\phi(r)} \longrightarrow \sum_{\substack{r,\sigma \\ i\in\hat{\sigma}}} \dim(V^{\sigma,i}) \cdot \tfrac{\iota(\zeta_r^i)}{|C_{G_x}(\sigma)|} \cdot \tfrac{r}{\phi(r)} \longrightarrow \sum_{\substack{r,\sigma \\ i\in\hat{\sigma}}} \tfrac{\dim(V^{\sigma,i})}{\phi(r)} \tfrac{r}{|C_{G_x}(\sigma)|} \cdot \text{Ind}^{G_x}_\sigma(\iota(\zeta_r^i))$$

is the identity. Note that we computed the map m_* exactly as we did at the end of Section 2.1.

In particular we can write

$$\bigoplus_{1\neq\tilde{\sigma}\subseteq G_x} \bigoplus_{i\in\hat{\sigma}} \frac{|X^\sigma| \cdot \dim A^{\sigma,i}}{\phi(r)} \frac{r}{|C(\sigma)|} \cdot \text{Ind}^{G_x}_\sigma \, \iota(\zeta_r^i) = A_x - \iota_{\sigma=1} \dim A_x$$

noting that the missing term corresponding to $\sigma = 1$ corresponds exactly to the geometric part of A_x.

Remark: We must have $\frac{|X^\sigma|}{|C(\sigma)|} = \frac{1}{|C_{G_x}(\sigma)|}$, since the connected components of the inertia stack corresponding to $1 \neq \tilde{\sigma} \subseteq G_x$ are equal to those of $\mathcal{I}BG_x$.

Now we invoke the following lemma:

Lemma 2.3.1. *Let H be a cyclic group of order m and χ a non-trivial character. Then*

$$\frac{1}{1-\chi} = -\frac{1}{m}\sum_{d=1}^{m-1} \chi^{-d}.$$

Proof. Same as [15] Lemma 1.2. ♠

Noting that the action of G_x on $N_x^\vee$ factors through the tame part, we use the Lemma to write

$$\dim A_x = \dim \frac{\mathcal{E}_x}{1-N_x^\vee} = \frac{1}{e_x^t} \dim \sum_{d=1}^{e_x^t-1} (-d)\mathcal{E}_x \otimes N_x^{-d} = -\frac{1}{e_x^t} \cdot \text{rk}\, \mathcal{E}_x \frac{e_x^t(e_x^t-1)}{2} = -\,\text{rk}\, \mathcal{E} \frac{e_x^t-1}{2}.$$

and

$$A_x - \iota_{\sigma=1} \dim A_x = A_x - \dim A_x \frac{kG_x}{e_x} = A_x + \operatorname{rk} \mathcal{E} \frac{e_x^t - 1}{2} \frac{kG_x}{e_x}.$$

Putting all ingredients together,

$$\begin{aligned}
\chi_G(X, \mathcal{E}) &= \chi(X, \mathcal{E}) \frac{kG}{n} + \sum_{x \in X} \frac{e_x}{n} \operatorname{Ind}_{G_x}^G (A_x + \operatorname{rk} \mathcal{E} \cdot \frac{e_x^t - 1}{2} \frac{kG_x}{e_x}) \\
&= \left(\chi(X, \mathcal{E}) + \frac{\operatorname{rk} \mathcal{E}}{2} \sum_x (e_x^t - 1) \right) \frac{kG}{n} + \sum_{x \in X} \frac{e_x}{n} \operatorname{Ind}_{G_x}^G \frac{\mathcal{E}_x}{1 - N_x^\vee}.
\end{aligned}$$

Thus we have recovered [10], Theorem 4.2. When the G-action is tame, we can recover the main theorem of [15]. Indeed one can use Hirzebruch-Riemann-Roch and Riemann-Hurwitz to express $\chi(X, \mathcal{E})$ in terms of rank and degree of $\mathcal{E}$ and genus of Y:

$$\chi(X, \mathcal{E}) = \deg \mathcal{E} + \operatorname{rk} \mathcal{E} (1 - g_X) = \deg \mathcal{E} + \operatorname{rk} \mathcal{E} \left(n(1 - g_Y) - \frac{1}{2} \sum_{x \in X} (e_x - 1) \right)$$

and expand $\frac{\mathcal{E}}{1 - N_x^\vee}$ using 2.3.1.

Chapter 3

A decomposition theorem for Deligne-Mumford stacks

In the following discussion, we will always assume that we are given a quasi-projective scheme X over a fixed base scheme $A = Spec(R)$, equipped with an action of a finite constant group G.

3.1 The general form of a decomposition theorem for quotient stacks

Here we discuss a general method for obtaining decomposition theorems with respect to some cohomological theory of a quotient DM stack $\mathcal{X} = [X/G]$.

This discussion should heuristically encompass the well-known decomposition theorems in equivariant algebraic K-theory (by Thomason, Vistoli, Vezzosi, Toen, ...), as well as the decomposition theorems in equivariant homotopy theory (such as Hopkins-Kuhn-Ravenel work about equivariant Morava K-theories and the Minami-Webb formula in equivariant cohomology).

3.1.1 Conlon's decomposition

In this section we briefly discuss our main tool, that is Conlon's results about a decomposition of the Burnside algebra and the split K-theory of a finite group. We will follow [8], making some slight simplification somewhere.

Definition. *Let G be a finite group. The* Burnside algebra $B(G)$ *is the free abelian group over the set of finite $G-$sets, quotiented by relations of the type $[S] + [T] - [S \sqcup T]$. It has a monoidal structure given by $[S] \otimes [T] := [S \times T]$.*

The Burnside algebra can be identified with the free abelian group generated by sets of the form $\{G/\Gamma\}$, where Γ ranges over conjugacy classes of subgroups of G. Let us call $\langle\Gamma\rangle$ the class of G/Γ; then the product is given by $\langle\Gamma\rangle\langle\Gamma'\rangle = \bigoplus_{g\in\mathcal{C}}\langle\Gamma \cap g^{-1}\Gamma g\rangle$, where $\mathcal{C} := \Gamma'\backslash G/\Gamma$ is the set of double cosets.

It is easy to see the following:

Proposition 3.1.1. *Let Λ be a $\mathbb{Z}-$algebra such that for any $\Gamma \hookrightarrow G$ the number $\nu^G_\Gamma = |N_G(\Gamma)/\Gamma|$ is invertible in Λ.*

Then the ring map $B(G)_\Lambda \to \Lambda$ given by sending any $\langle\Gamma\rangle$ to zero for Γ a proper subgroup has a canonical section s_G.

From now on we will always assume that $B(G)$ has coefficients in some Λ, and we will omit the subscript.

Definition. *Let ${}_GB(G)$ be the ideal generated by s_G; we will refer to it as the* tautological part *of the Burnside ring.*

Conlon proves the following ([8], Theorem 2.6)

Proposition 3.1.2. *Let ${}_HB(G)$ be the extension of ${}_HB(H)$ along the canonical induction $B(H) \hookrightarrow B(G)$. Then there is a direct sum decomposition $B(G) = \bigoplus_\Gamma {}_\Gamma B(G)$, induced by a complete set of idempotents.*

This decomposition is preserved by push-forward and is also invariant by pull-back, in the sense that $Res^G_\Gamma u^G_{\Gamma'}$ is zero unless $\Gamma' < \Gamma^g$ for some g, and in that case $Res^G_\Gamma u^G_{\Gamma'}$ is a multiple of $\sum_{\Gamma'^g} u^\Gamma_{\Gamma'^g}$, where the Γ'^g are all the $\Gamma-$conjugacy classes of $G-$conjugates of Γ'.

The result essentially reduces to the following: for any Γ, its tautological part is generated by an idempotent u_Γ, and the induced element $u^G_\Gamma := Ind^G_\Gamma(u_\Gamma)$ is *almost* idempotent, in the sense that $u^G_\Gamma \cdot u^G_\Gamma = \nu^G_\Gamma \cdot u^G_\Gamma$.

For the sake of completeness we include a revised proof:

Proof. This is easily shown by induction on the size of G (the base case being trivial). Suppose thus by induction that for any $\Gamma < G$ we have a decomposition of $B(\Gamma)$ induced by a set of orthogonal idempotents $u^\Gamma_{\Gamma'}$ (indexed by conjugacy classes of subgroups Γ'). Then we set $u^G_\Gamma := \frac{1}{\nu^G_\Gamma} Ind^G_\Gamma(u^\Gamma_\Gamma)$ for any proper subgroup Γ, while we set $u^G_G := 1 - \bigoplus\limits_{\Gamma<G} u^G_\Gamma$.

Finally we observe that the span of the $u^\Gamma_{\Gamma'}$ contains the induced image of any $B(\Gamma')$ for all proper $\Gamma' < \Gamma$.

We first prove that the u^G_Γ form a complete set of idempotents. By the projection and Mackey formulas we have

$$u^G_\Gamma \cdot u^G_\Gamma = \frac{1}{\nu^G_\Gamma} \cdot Ind^G_\Gamma\left(\frac{1}{\nu^G_\Gamma} u^\Gamma_\Gamma \cdot Res^G_\Gamma Ind^G_\Gamma u^\Gamma_\Gamma\right) = \frac{1}{\nu^G_\Gamma} \cdot Ind^G_\Gamma\left(\frac{1}{\nu^G_\Gamma} u^\Gamma_\Gamma \cdot \left(\nu^G_\Gamma \cdot u^\Gamma_\Gamma + \sum_{\Gamma'} Ind^\Gamma_{\Gamma'} u'_{\Gamma'}\right)\right)$$

for some $u'_{\Gamma'} \in B(\Gamma')$. Now, the sum $\sum_{\Gamma'} Ind^\Gamma_{\Gamma'} u'_{\Gamma'}$ is in the span of the $u^\Gamma_{\Gamma'}$ and thus has zero product with u^Γ_Γ, by the induction hypothesis. This proves that the u^G_Γ are idempotents for $\Gamma < G$.

We now want to check that the u^G_Γ's are orthogonal. To see it fix two subgroups $\Gamma, \Gamma' \leq G$. If one of them is G the orthogonality is true by definition if we know the result for proper subgroups; therefore we may suppose without loss of generality that the two subgroups are proper and that $\Gamma' \not< \Gamma$ (meaning that no $G-$conjugate of Γ' is a subgroup of Γ) and proceed as before:

$$u^G_\Gamma \cdot u^G_{\Gamma'} = \frac{1}{\nu^G_\Gamma} \cdot Ind^G_\Gamma\left(\frac{1}{\nu^G_{\Gamma'}} u^\Gamma_\Gamma \cdot Res^G_\Gamma Ind^G_{\Gamma'} u^{\Gamma'}_{\Gamma'}\right) = \frac{1}{\nu^G_\Gamma} \cdot Ind^G_\Gamma\left(\frac{1}{\nu^G_{\Gamma'}} u^\Gamma_\Gamma \cdot \left(\sum_{\Gamma''} Ind^\Gamma_{\Gamma''} u''_{\Gamma''}\right)\right)$$

for some $u''_{\Gamma''} \in B(\Gamma'')$, where the Γ'' represent proper subgroups of Γ. Again by induction hypothesis the above product is zero. Finally, the combination of the two above results shows that u_G^G is also idempotent, as we wanted.

The invariance of this decomposition by push-forward follows trivially from the definitions, while to see the invariance by pull-back we set up another induction. We want to prove that $Res_\Gamma^G u_{\Gamma'}^G$ is a rational multiple of $u_{\Gamma'}^\Gamma$; again, if $\Gamma' < G$ the above calculation with Mackey formula gives the result: indeed we may write (calling $\Gamma_g := \Gamma \cap^g \Gamma'$)

$$Res_\Gamma^G u_{\Gamma'}^G = Res_\Gamma^G Ind_{\Gamma'}^G u_{\Gamma'}^{\Gamma'} = \sum_{g \in \Gamma \backslash G / \Gamma'} Ind_{\Gamma_g}^\Gamma Res_{\Gamma_g}^{\Gamma'} u_{\Gamma'}^{\Gamma'}.$$

By the induction hypothesis, the restriction $Res_{\Gamma_g}^{\Gamma'} u_{\Gamma'}^{\Gamma'}$ is zero unless $\Gamma' = \Gamma_g$; in that case the above sum reduces to $\sum_{\Gamma'^g} u_{\Gamma'^g}^\Gamma$.

Finally, the above results imply that any restriction $Res_\Gamma^G u_G^G$ is zero. Indeed the restriction is a ring map and $1 \in B(\Gamma)$ is a linear combination of elements that come from the set $\{u_{\Gamma'}^G\}_{\Gamma' < G}$ by restriction: $1 = \sum_{\Gamma'} a_{\Gamma'} Res_\Gamma^G u_{\Gamma'}^G$; this gives readily that $Res_\Gamma^G u_G^G \cdot 1 = \sum_{\Gamma'} Res_\Gamma^G (u_G^G \cdot u_{\Gamma'}^G) = 0$, as we wanted. ♠

Suppose that we have a cohomological theory $\mathcal{H}^*(G)$ with an action of the Burnside ring, making $\mathcal{H}^*(G)$ a graded $B(G)-$module. Then the splitting of $B(G)$ which we described above gives a decomposition $\mathcal{H}^*(G) = \bigoplus_\Gamma H^*(G)_\Gamma$, where the subscript $-_\Gamma$ indicates the elements with support on ${}_\Gamma B(G)$.

Suppose for example that $\mathcal{H}^*(G) = K^*(R[G]-\mathbf{mod}^{\mathbf{split}})$ or $\mathcal{H}^*(G) = K^*(R[G]-\mathbf{mod})$. In those cases the action of the Burnside algebra is induced by the map $B(G) \to R[G]-\mathbf{mod}$ sending $\langle \Gamma \rangle$ to the representation $Ind_\Gamma^G 1$. Conlon ([8]) shows the following

Theorem 3.1.3. *Let R be a ring, $\mathcal{A} = R[G] - \mathbf{mod}$.*

(1) *Suppose that $\mathcal{H}^*(G) = K^*(\mathcal{A}^{split})$ or $\mathcal{H}^*(G) = K^*(\mathcal{A})$.*

For any $H < G$ the induction and restriction maps

$$\mathcal{H}^*(H)_H^{N_G(H)} \xrightarrow{Ind_H^G} \mathcal{H}^*(G)_H \xrightarrow{Res_H^G} \mathcal{H}^*(H)_H^{N_G(H)}$$

are isomorphisms.

(2) *Suppose that R is a complete local ring with residue field of characteristic p and $\mathcal{H}^*(G) = K^*(\mathcal{A}^{split})$. Then $\mathcal{H}^*(G)_H$ is zero unless H is a semidirect product $P \rtimes C$, where P is a $p-$group and C is a cyclic group of order coprime with p.*

(3) *Suppose that $\mathcal{H}^*(G) = K^*(\mathcal{A})$. Then $\mathcal{H}^*(G)_H$ is zero unless H is a cyclic group of order coprime with the characteristic of R.*

Note that the first point is not completely trivial, since even though the restriction of u_H^G to $K^*(H)$ is u_H the composition $Res_H^G Ind_H^G(a \cdot u_H)$ is not the identity for a general a (indeed it is equal to $u_H \cdot \sum_{\gamma \in N(H)/H} a^\gamma$).

Now, given a suitable cohomology theory $\mathcal{H}^*(G, X)$, suppose that there is an action of the Burnside algebra $B(G)$ on it. Then, by the prevoius section, there is a corresponding decomposition $\mathcal{H}^*(G, X) = \bigoplus_H \mathcal{H}^*(G, X)_H$.

We would like to have some result providing an isomorphism of the H-parts $\mathcal{H}^*(\mathcal{A}, X)_H$ with something localized on the fixed-point locus X^H. In orded to have a meaningful and functorial definition, we need to explore a variant of the inertia stack construction.

3.1.2 The "wild " inertia

Let us recall given a DM stack $\mathcal{X}$, there is a classical notion of $n-$inertia

$$\mathcal{I}_n\mathcal{X} := \underline{Hom}^{rep}(B\mathbb{Z}/n\mathbb{Z}, \mathcal{X}),$$

a stack that classifies representable morphisms from the classifying stack of a cyclic group. Similarly we have a *cyclotomic inertia*, that is defined analogously with μ_n instead of $\mathbb{Z}/n\mathbb{Z}$.

We can generalise this construction. Let us consider, for a finite discrete group G, the *universal* classifying stack $B\mathcal{G}_{[G]}$, that is the classifying stack BG seen over the stack classifying automorphisms of G: in other words the stack

$$B\mathcal{G}_{[G]} := B(G \rtimes (Aut(G)))$$

with the tautological action of $Aut(G)$ on G, which classifies the "twisted" étale group schemes, locally isomorphic to G in the étale topology (see [28], Section 3). Over $B\mathcal{G}_{[G]}$ lives the following object: .

Definition. *Let us define the stack $I^{[G]}\mathcal{X}$, so that an object of $I^{[G]}\mathcal{X}(S)$ is a pair $(\xi, G') \subseteq (\mathcal{X} \times B\mathcal{G}_{[G]})(S)$ such that G' is a subgroup of $\underline{Aut}_S(\xi)$. Moreover, let the* wild inertia *be $\mathcal{I}^{[\cdot]}\mathcal{X} := \bigsqcup_G I^{[G]}\mathcal{X}$.*

For a stack of the form $\mathcal{X} = [X/G]$, with G an affine group, it is easy to give an explicit description of $\mathcal{I}^{[\cdot]}\mathcal{X}$ (see for example [28], Lemma 6.16):

Theorem 3.1.4. *Let $\mathcal{X} = [X/G]$; then we have an isomorphism*

$$I^{[H]}\mathcal{X} \simeq \bigsqcup_{H' \in \mathcal{C}} [X^{H'}/N_G(H')]$$

where $\mathcal{C}$ is the set of conjugacy classes of subgroups $H' < G$ such that $H' \simeq H$.

3.1.3 The general decomposition theorem

Now, suppose that we have a sufficiently *well-behaved* cohomological theory $\mathcal{H}^*(G, -)$; in this case we can look for a result of the following form: for each $H < G$ conjugacy class of subgroups the induction

$$i_*\colon \mathcal{H}^*(H, X^H)_H^{N_G(H)} \longrightarrow \mathcal{H}^*(G, X)_H$$

is an isomorphism.

We shall be more precise:

Theorem 3.1.5. *Let $A = Spec(R)$ be a base scheme $\mathcal{H}^*(-,G)$ be an equivariant (co)homology theory for A-schemes with coefficients in $\mathbb{Q}$ and the following properties:*

(1) *It admits functorial pull-backs and push-forwards for equivariant maps $(Y,H) \to (X,G)$.*

(2) *It admits an action of the Burnside algebra such that any pull-back or push-forward is a map of $B(G)$-modules.*

(3) *It admits localization long exact sequences.*

(4) *Depends on $[X/G]$, in the sense that if $X = Y \times_H G$ then $\mathcal{H}^*(X,G) \simeq \mathcal{H}^*(Y,H)$.*

(5) *When we consider the map $i\colon [X/H] \to [X/G]$, then the compositions i^*i_* and i_*i^* are consistent with the expectation from representation theory; this means that they satisfy a suitable form of the Mackey formula*

$$i^*i_*(\xi) = \sum_i Ind^H_{H_i^g} \mu_g^* Res^H_{H_i}(\xi)$$

(where μ_g indicates the morphism of multiplication by $g \in G$) and of the projection formula

$$i_*i^*(\xi) = \xi \cdot i_*(1).$$

Then by (1) and (2) there is a decomposition $\mathcal{H}^*(-,G) = \bigoplus_H \mathcal{H}^*(-,G)_H$ indexed by $H < G$ conjugacy classes of subgroups.

We have that, for any H, the induction

$$i_*\colon \mathcal{H}^*(H, X^H)_H^{N_G(H)} \longrightarrow \mathcal{H}^*(G,X)_H$$

is an isomorphism.

Moreover, if the action of $B(G)$ is induced, for example, by an action of $\mathcal{H}^0(G,pt)$, then the only relevant H's are those for which the $H-$parts $\mathcal{H}^0(G,pt)_H$ are non-zero. This in general leaves out only some restricted class of subgroups.

Proof. We prove the theorem for a homology theory. The dual case is completely analogous.

We begin by observing that any X has an open dense subset U such that $[U/G]$ is a gerbe trivialized by a finite Galois cover. In particular X admits a stratification

$$U_0 \xrightarrow{j_0} U_1 \xrightarrow{j_1} \cdots \xrightarrow{j_{n-1}} U_n = X$$

by open $G-$invariant sets such that for any stratum $X_i := U_{i+1} \backslash U_i$ the quotient stack $[X_i/G]$ is a gerbe trivialized by a Galois cover with group Γ_i.

By (3) we have fon any l a commuting diagram

$$\begin{tikzcd}
\mathcal{H}_{l+1}(U_i^H, H) \arrow[r] \arrow[d] & \mathcal{H}_l(X_i^H, H) \arrow[r, "i_{i,*}"] \arrow[d] & \mathcal{H}_l(U_{i+1}^H, H) \arrow[r, "j_i^*"] \arrow[d] & \mathcal{H}_l(U_i^H, H) \arrow[r] \arrow[d] & \mathcal{H}_{l-1}(X_i^H, H) \arrow[d] \\
\mathcal{H}_{l+1}(U_i, G) \arrow[r] & \mathcal{H}_l(X_i, G) \arrow[r, "i_{i,*}"] & \mathcal{H}_l(U_{i+1}, G) \arrow[r, "j_i^*"] & \mathcal{H}_l(U_i, G) \arrow[r] & \mathcal{H}_{l-1}(X_i, G)
\end{tikzcd}$$

From the five lemma we immediately infer that if we know the thesis for U_i and for any X such that $[X/G]$ is a gerbe Galois-locally trivial then we also have it for U_{i+1}.

All we have to do, then, is to show the thesis in the case when $[X/G]$ is a gerbe trivial on a Galois cover of the moduli space M.

Let Γ be the Galois group of the trivializing cover $N \to M$ and H be the stabilizer. Let p be the projection $[N/H] \to [X/G]$. Then by (4) and (5) we immediately deduce that the pull-back p^* induces an isomorphism

$$\mathcal{H}^*(X,G) \simeq \mathcal{H}^*(N,H)^\Gamma$$

Indeed, the compositions p^*p_* and p_*p^* are easily seen to be equal to the multiplication by $|\Gamma|$ map.

We are left with proving the statement for a neutral gerbe $[X/G]$, where G acts trivially. For that we just repeat the proof of Proposition 3.1.3. This is actually quite immediate from (5): first of all, by the projection formula we have $Ind_H^G Res_H^G(\xi) = \xi \cdot Ind_H^G(1)$ and $Ind_H^G(1)$ is invertible in $B(G)_H$, making the composition an isomorphism; moreover, by the Mackey formula, we have $Res_H^G Ind_H^G(\xi) = \sum_i Ind_{H_i^g}^H Res_{H_i}^H(\xi)$; but when ξ is in the $H-$part its restriction to any proper subgroup of H is zero, so that we can rewrite $Res_H^G Ind_H^G(\xi) = \sum_{n \in N_G(H)/H} n \cdot \xi = |N_G(H)/H| \cdot \xi$ (where the action of $N_G(H)/H$ is induced by the conjugation action on H) because ξ is $N_G(H)-$invariant. We conclude that in this case the induction and restriction maps

$$\mathcal{H}_*(H)_H^{N_G(H)} \xrightarrow{Ind_H^G} \mathcal{H}_*(G)_H \xrightarrow{Res_H^G} \mathcal{H}_*(H)_H^{N_G(H)}$$

are isomorphisms, and the thesis follows. ♠

3.1.4 The decomposition of algebraic K-theory of DM stacks

Suppose for instance that $\mathcal{H}_*(X,G) = \mathrm{K}'_*([X/G])$. K-theory is well-known to satisfy all the properties of the Theorem, so we conclude that the result holds for the K-theory of quotient Deligne-Mumford stacks.

If the action of G on X is tame and the base ring contains all roots of unity we can make a comparison with the more classical decompositions (see e.g. [26]). In that case we have a decomposition of the representation ring of free G-representations $\mathrm{R}G = \bigoplus_{\boldsymbol{\mu}_r \in G} \mathbb{Q}(\zeta_r)^{N_G(\boldsymbol{\mu}_r)}$ indexed by conjugacy classes of cyclic subgroups of G; for any $H \simeq \boldsymbol{\mu}_r$ let us call $\mathrm{R}G'_H$ the $H-$piece in this classical decomposition.

Theorem 3.1.6. *The classical decomposition coincides with the one induced by the Burnside algebra.*

Proof. This is immediate to see that when G is itself cyclic, by induction on the dimension of G. More precisely, we know by Theorem 3.1.3 that Conlon's decomposition of $\mathrm{R}G$ is also indexed by conjugacy classes of cyclic subgroups, and that for any H we have an isomorphism $\mathrm{R}G_H \simeq \mathrm{R}H_H^{N_G(H)}$ induced by the induction $\mathrm{R}H \to \mathrm{R}G$; thus it sufficies to see that $\mathrm{R}G_G$ coincides with $\mathrm{R}G'_H$. But we may suppose that we know it for all the smaller groups (by induction hypothesis); if we call v^G_H the idempotent generating $\mathrm{R}G'_H$, then $\mathrm{R}G'_H$ is generated by the only rational multiple of $Ind^G_H v^H_H$ which is itself idempotent, so it is uniquely determined. In particular this generator coincides with u^G_H, by inductive hypothesis: this implies that $u^G_G = 1 - \sum_H u^G_H = 1 - \sum_H v^H_H = v^G_G$, as we wanted.

Finally, the covariance by push-forward immediately settles the general case. ♠

In particular the classical decomposition $\mathrm{K}'_*([X/G]) = \bigoplus_H \mathrm{K}'_*([X/G])_H$ coincides with the one induced by the Burnside ring, since they are both obtained localizing $\mathrm{K}'_*([X/G])$ with respect to this common $\mathrm{R}G$-decomposition.

Now, using the self-intersection formula, we can give inverse maps

$$\mathrm{K}'_*(X,G)_H \longrightarrow \mathrm{K}'_*(X^H,H)^{N(H)}_H.$$

First of all, by Proposition 3.1.3 the only relevant H's are the cyclic groups. Given one such H, let now $\mathcal{N}$ be the conormal sheaf associated to the closed immersion $i\colon X^H \hookrightarrow X$.

Suppose that i is a regular immersion. By the self-intersection formula, we know that the composition

$$\mathrm{K}'_*(X^H,H)^{N(H)}_H \xrightarrow{i_*} \mathrm{K}'_*(X,H)^{N(H)}_H \xrightarrow{i^*} \mathrm{K}'_*(X^H,H)^{N(H)}_H$$

is equal to the multiplication by the element $\lambda_{-1}(\mathcal{N})$. But we have

Lemma 3.1.7. *The element $\lambda_{-1}(\mathcal{N})$ is invertible in $\mathrm{K}'_*(X^H,H)_H$.*

Proof. Let $Y := X^H$. If Y does not have points in characteristic not coprime with $|H|$ then the result is known from the classical theory (see [24] or [26]).

Otherwise, let $Y' \hookrightarrow Y$ be the closed subscheme of points whose residue characteristic is not coprime with $|H|$. By Proposition 3.1.3 we have that $\mathrm{K}'_*(Y',H)_H = 0$; by the localization exact sequence, naming $j\colon U = (Y \backslash Y') \hookrightarrow Y$, we have that $j^*\colon \mathrm{K}'_*(Y,H)_H \to \mathrm{K}'_*(U,H)_H$ is an isomorphism. But j^* clearly preserves the multiplication by $\lambda_{-1}(\mathcal{N})$, so the thesis follows. ♠

Let us consider the composition

$$\mathrm{K}'_*(X^H,H)^{N(H)}_H \xrightarrow{i_*} \mathrm{K}'_*(X,H)^{N(H)}_H \xrightarrow{Ind^G_H} \mathrm{K}'_*(X,G)_H \xrightarrow{Res^G_H} \mathrm{K}'_*(X,H)^{N(H)}_H \xrightarrow{i^*} \mathrm{K}'_*(X^H,H)^{N(H)}_H.$$

By Mackey formula the central composition $\mathrm{K}'_*(X,H)^{N(H)}_H \xrightarrow{Ind^G_H} \mathrm{K}'_*(X,G)_H \xrightarrow{Res^G_H} \mathrm{K}'_*(X,H)^{N(H)}_H$ sends ξ to $\sum_i Ind^H_{H^g_i} Res^H_{H_i}(\xi)$; but when ξ is in the $H-$part its restriction to any proper subgroup of H is zero, so that we can rewrite $Res^G_H Ind^G_H(\xi) = \sum_{n \in N_G(H)/H} n \cdot \xi = |N(H)/H| \cdot \xi$. We conclude that the composition is equal to the multiplication by the element $|N(H)/H| \cdot \lambda_{-1}(\mathcal{N})$. In particular the modified restriction map

$$\frac{i^*}{|N(H)/H| \cdot \lambda_{-1}(\mathcal{N})}\colon \mathrm{K}'_*(X,G)_H \longrightarrow \mathrm{K}'_*(X^H,H)^{N(H)}_H$$

is an isomorphism inverse to i_*.

Summarizing, we have the following

Theorem 3.1.8. *Let X be an $R-$scheme with an action of a finite constant group G. Then we have a decomposition* $\mathrm{K}'_*(X,G) = \bigoplus_{H \text{ cyclic}} \mathrm{K}'_*(X,G)_H$ *and the push-forward induces isomorphisms*

$$i_* \colon \mathrm{K}'_*(X^H,H)_H^{N(H)} \longrightarrow \mathrm{K}'_*(X,G)_H.$$

If the immersion $X^H \hookrightarrow X$ is regular there is an inverse

$$\frac{i^*}{|N(H)/H| \cdot \lambda_{-1}(\mathcal{N})} \colon \mathrm{K}'_*(X,G)_H \longrightarrow \mathrm{K}'_*(X^H,H)_H^{N(H)}.$$

3.2 The modular K-theory of a quotient

We will now discuss a closely related example in equivariant algebraic geometry, that of modular K-theory.

In this section we briefly recall the theory of modular K-theory, along the lines of the original article [7] by Niels Borne. Here we will suppose that X is defined over a field k.

3.2.1 Sheaves of modules over a noncommutative algebra

We first recall the definitions: suppose that $\mathcal{A}$ is a subcategory of the category of finite dimensional G-representations $\mathcal{A}_{tot}$.

Let $\mathbf{Mod}(\mathcal{A})$ be the category $[\mathcal{A}^{op}, \mathbf{Ab}]$ of contravariant functors from $\mathcal{A}$ to the category of abelian groups (with natural transformations as morphisms). A (right) $\mathcal{A}$-module is by definition an object in $\mathbf{Mod}(\mathcal{A})$. $\mathcal{A}$-modules obviously form an abelian category.

The Yoneda embedding $\mathcal{A} \to \mathbf{Mod}(\mathcal{A})$ sends an object V to the contravariant representable functor $\underline{V} = \mathcal{A}(\cdot, V)$. An $\mathcal{A}$-module is said to be of finite type if it is a quotient of a finite direct sum of representable functors, and we will call $\mathbf{mod}(\mathcal{A})$ the full subcategory of $\mathbf{Mod}(\mathcal{A})$ consisting of objects of finite type.

Definition. *The projective completion $Q\mathcal{A}$ of $\mathcal{A}$ is the full subcategory of* $\mathbf{Mod}(\mathcal{A})$ *whose objects are the projective $\mathcal{A}$-modules of finite type (direct summands of finite direct sums of representable functors).*

Definition. *We say that $\mathcal{A}$ admits a finite set of additive generators if it is Morita equivalent to the full subcategory generated by a finite set of its objects.*

Now, let $\mathcal{A}$ be a projectively complete full subcategory of $\mathcal{A}_{tot}$ admitting a finite set of additive generators. Starting from $\mathcal{A}$ we can form two distinct exact categories: the first is $\mathcal{A}^{split}$, that is $\mathcal{A}$ with split exact sequences of G-modules; the second is $\mathbf{mod}(\mathcal{A})$ with its natural exact structure induced by the abelian category structure.

The Yoneda embedding induces a morphism of exact categories $\mathcal{A}^{split} \to \mathbf{mod}(\mathcal{A})$, and thus also a map in K-theory $\mathrm{K}'_*(\mathcal{A}^{split}) \to \mathrm{K}'_*(\mathbf{mod}(\mathcal{A}))$. We can formulate (see [**?**], section 2.2) the following

Theorem 3.2.1. (1) *The two groups $\mathrm{K}'_*(\mathcal{A}^{split})$ and $\mathrm{K}'_*(\mathbf{mod}(\mathcal{A}))$ are abelian free of the same rank.*

(2) *If G is of finite representation type and $\mathcal{A} = \mathcal{A}_{tot}$ then the Yoneda embedding is an isomorphism.*

In particular we see that the K-theory of $\mathbf{mod}(\mathcal{A})$ is a new invariant that we can associate to the category $\mathcal{A}_{tot} = k[G] - \mathbf{mod}$ and it is in general finer than the classical algebraic K-theory.

We can generalize this discussion to the case of a $G-$scheme Y, so that what we have done so far corresponds to $Y = Spec(k)$.

Definition. *Let X be a scheme.*

(1) *A ring (with several objects) on Y is, by definition, a category $\mathcal{A}$ enriched on $\mathbf{Qcoh(Y)}$. The pair $(Y, \mathcal{A})$ is called a ringed scheme.*

(2) *A morphism of ringed schemes $(Y', \mathcal{A}') \to (Y, \mathcal{A})$ is a couple $(f, f^\sharp)$ where $f \colon Y' \to Y$ is a scheme morphism and $f^\sharp \colon \mathcal{A} \to f_*\mathcal{A}'$ is a morphism of $\mathbf{Qcoh(Y)}$-categories.*

(3) *A quasicoherent sheaf on $(Y, \mathcal{A})$ is by definition an enriched functor from $\mathcal{A}^{op}$ to $\mathbf{Qcoh(Y)}$.*

(4) *A morphism of $\mathcal{A}$-sheaves is an enriched natural transformation.*

We denote by $\mathbf{Qcoh(Y, \mathcal{A})} = [\mathcal{A}^{\mathbf{op}}, \mathbf{Qcoh(Y)}]$ the corresponding enriched category, and by $Qcoh(Y, \mathcal{A})$ the underlying category.

This definition of ringed scheme allows for a notion of functoriality, which is essentially induced by the functoriality of $Qcoh(Y)$.

Consider first the case of a morphism $(1, i) : (Y, \mathcal{A}') \to (Y, \mathcal{A})$. In this case it is easy to see that left and right Kan extensions along $\mathcal{A}' \to \mathcal{A}$ exist and give left and right adjoints to the restriction functor $\mathbf{Qcoh(Y, \mathcal{A})} \to \mathbf{Qcoh(Y, \mathcal{A}')}$. The left adjoint will be denoted by $\otimes_{\mathcal{A}'}\mathcal{A}$.

We then extend this definition to the general case:

Definition. *Let us be given a morphism of ringed schemes $(Y', \mathcal{A}') \to (Y, \mathcal{A})$.*

(1) *We define $f_\triangle : f_*\mathbf{Qcoh(Y', \mathcal{A}')} \to \mathbf{Qcoh(Y, \mathcal{A})}$ as the $\mathbf{Qcoh(Y)}$ functor making the following diagram commute:*

$$\begin{tikzcd}
f_*[\mathcal{A}'^{op}, \mathbf{Qcoh(Y')}] \arrow[r, "f_*"] \arrow[dr, "f_\triangle"'] & {[f_*\mathcal{A}'^{op}, f_*\mathbf{Qcoh(Y')}]} \arrow[d, "f^\sharp"] \\
 & {[\mathcal{A}^{op}, \mathbf{Qcoh(Y)}]}
\end{tikzcd}$$

(2) *We define $f^\triangle \colon \mathbf{Qcoh(Y, \mathcal{A})} \to \mathbf{f_*Qcoh(Y', \mathcal{A}')}$ as the $\mathbf{Qcoh(Y)}$ -functor making the following diagram commute:*

$$\begin{tikzcd}
{[\mathcal{A}'^{op}, \mathbf{Qcoh(Y')}]} & \\
{[f^*\mathcal{A}^{op}, \mathbf{Qcoh(Y')}]} \arrow[u, "\otimes_{f^*\mathcal{A}}\mathcal{A}'"] & \\
{[f^*\mathcal{A}^{op}, f^*\mathbf{Qcoh(Y)}]} \arrow[u, "f^*"] & f^*[\mathcal{A}^{op}, \mathbf{Qcoh(Y)}] \arrow[l, "f^*"] \arrow[uul, "adj(f^\triangle)"']
\end{tikzcd}$$

These functors satisfy the natural adjunction requirement:

Proposition 3.2.2. *The couple* $(f^{\triangle}, f_{\triangle})$ *is part of a* $\mathbf{Qcoh}(\mathbf{Y})$*-adjunction between* $\mathbf{Qcoh}(\mathbf{Y}, \mathcal{A})$ *and* $f_*\mathbf{Qcoh}(\mathbf{Y}', \mathcal{A}')$.

For each object V of $\mathcal{A}$, we denote by $\langle V \rangle$ the full subcategory of $\mathcal{A}$ containing only the object V.

Proposition 3.2.3. (1) *The category* $Qcoh(Y, \mathcal{A})$ *is an abelian category.*

(2) *A sequence* $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ *of* $\mathcal{A}$*-sheaves on* Y *is exact if and only if for each object* V *of* $\mathcal{A}$ *the sequence* $\mathcal{F}'(V) \to \mathcal{F}(V) \to \mathcal{F}''(V)$ *is exact in* $Qcoh(Y, \langle V \rangle)$.

Since localization commutes with evaluation at each V, we immediately get

Corollary 3.2.4. *A sequence of* $\mathcal{A}$*-sheaves* $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ *is exact if and only if for each* $p \in Y$ *the sequence of stalks* $\mathcal{F}'_p \to \mathcal{F}_p \to \mathcal{F}''_p$ *is exact.*

3.2.2 Sheaves over the Auslander algebra

Now we are ready to return to our $G-$scheme X. We have an equivalence between the category of equivariant $G-$sheaves on X, $QCoh(X, G)$, and the category of sheaves on the quotient stack $\mathcal{X}$. We will denote by $\pi\colon \mathcal{X} \to Y$ the map from $\mathcal{X}$ to the moduli space Y (that is, the quotient scheme). In particular the category $QCoh(X, G)$ is thus naturally enriched over $QCoh(Y)$ by $\mathbf{QCoh}(\mathcal{F}, \mathcal{F}') := \pi_*\mathbf{QCoh}_{\mathcal{X}}(\mathcal{F}, \mathcal{F}')$.

Definition. *Let* (X, G) *be as above,* $s_X : X \to Spec(k)$ *be the structure morphism, and* $\mathcal{A}$ *a category equipped with a functor* $F\colon \mathcal{A} \to k[G] - \mathbf{mod}$.

(1) *The* Auslander algebra over Y *associated to* $(\mathcal{A}, F)$ *is the ring* $\mathcal{A}_X$ *over* Y *equal with the same objects of* $\mathcal{A}$ *and morphisms of the form* $\mathbf{Qcoh}(\mathbf{G}, \mathbf{X})(\mathbf{s}^*_\mathbf{X}\mathbf{F}(\mathbf{V}), \mathbf{s}^*_\mathbf{X}\mathbf{F}(\mathbf{V}'))$*, for all objects* V, V' *of* $\mathcal{A}$.

(2) *The* constant Auslander algebra over Y *is the ring* $\mathcal{A}^{\mathrm{c}}_X = s^*_Y\mathcal{A}$ *(that is, any morphism* $Hom_{\mathcal{A}^{\mathrm{c}}_X}(V, V')$ *is given by* $QCoh_Y(s^*_YF(V), s^*_YF(V'))$*).*

(3) *The* "free" Auslander algebra over Y *is the ring* $\mathcal{A}^f_X$ *over* Y *with the same objects of* $\mathcal{A}$ *and such that any morphism* $Hom_{\mathcal{A}^f_X}(V, V')$ *is given by* $QCoh_Y(\pi_*s^*_XF(V), \pi_*s^*_XF(V'))$*).*

Clearly $\mathcal{A}_X \simeq \mathcal{A}^{\mathrm{c}}_X$ when the G-action is trivial and $\mathcal{A}_X \simeq \mathcal{A}^f_X$ when the $G-$action is free.

The Auslander algebra is functorial:

Proposition 3.2.5. (1) *Any map* $f : X' \to X$ *induces a morphism of ringed schemes* $(\tilde{f}, \tilde{f}^\sharp)\colon (Y', \mathcal{A}_{X'}) \to (Y, \mathcal{A}_X)$.

(2) $adj(\tilde{f}^\sharp) : \tilde{f}^*\mathcal{A}_X \to \mathcal{A}_{X'}$ *is an isomorphism if* $X' = X \times_Y Y'$.

Proof. (1) To construct $\tilde{f}^\sharp$, we start from the isomorphism $f^*\mathbf{Qcoh}(\mathbf{X})(\mathbf{s_X}^*\mathbf{V}, \mathbf{s_X}^*\mathbf{W}) \to \mathbf{Qcoh}(\mathbf{X}')(\mathbf{s_{X'}}^*\mathbf{V}, \mathbf{s_{X'}}^*\mathbf{W})$. This is in fact a G-isomorphism and give by adjunction a G-morphism $\mathbf{Qcoh}(\mathbf{X})(\mathbf{s_X}^*\mathbf{V}, \mathbf{s_X}^*\mathbf{W}) \to \mathbf{f}_*\mathbf{Qcoh}(\mathbf{X}')(\mathbf{s_{X'}}^*\mathbf{V}, \mathbf{s_{X'}}^*\mathbf{W})$. Applying π_* and using the fact that $\pi_*f_* = \tilde{f}_*\pi'$, we get the map $\mathcal{A}_X(s_X{}^*V, s_X{}^*W) \to \tilde{f}_*\mathcal{A}_{X'}(s_{X'}{}^*V, s_{X'}{}^*W)$ that we needed.

(2) By base change, the canonical 2-arrow $\tilde{f}^*\pi_* \Longrightarrow {\pi'}_* f^*$ is an isomorphism, and the result follows.

♠

Definition. *We define* $\mathbf{Qcoh}(\mathcal{A},\mathbf{X})$ *as the category of quasi-coherent sheaves for the ringed scheme* $(Y,\mathcal{A}_X)$*. For any* $V \in \mathcal{A}$ *call* $\mathcal{A}_V$ *the corresponding representable sheaf.*

There is an obvious map from classical equivariant sheaves to $\mathcal{A}-$sheaves:

$$\begin{aligned} U_{\mathcal{A}}\colon \mathbf{Qcoh}(\mathbf{G},\mathbf{X}) \to{}& \quad \mathbf{Qcoh}(\mathcal{A},\mathbf{X}) \\ \mathcal{F} \to{}& \quad \underline{\mathcal{F}} = (V \to \quad \mathbf{Qcoh}(\mathbf{G},\mathbf{X})(\mathbf{s}_{\mathbf{X}}^*\mathbf{V},\mathcal{F})) \end{aligned}$$

We now briefly discuss the functorial properties of this definition.

First of all, given an equivariant morphism $f\colon X' \to X$, it produces an adjoint pair $(f^{\triangle}, f_{\triangle})\colon \tilde{f}_*\mathbf{Qcoh}(\mathcal{A},\mathbf{X}') \to \mathbf{Qcoh}(\mathcal{A},\mathbf{X})$, which we rename as $(f^{\mathcal{A}}, f_{\mathcal{A}})$.

Moreover, as shown by N. Borne, there are natural induction and restriction functors from and to subcategories:

Proposition 3.2.6. *Let* $\mathcal{A}'$ *be a full subcategory of* $\mathcal{A}$*.*

(1) *The restriction functor* $R : \mathbf{Qcoh}(\mathcal{A},\mathbf{X}) \to \mathbf{Qcoh}(\mathcal{A}',\mathbf{X})$ *admits as a right adjoint the functor* K *defined on objects by :*

$$\begin{aligned} K\colon \mathbf{Qcoh}(\mathcal{A}',\mathbf{X}) \to{}& \quad \mathbf{Qcoh}(\mathcal{A},\mathbf{X}) \\ \mathcal{F} \to{}& \qquad (V \to \qquad \mathbf{Qcoh}(\mathcal{A}',\mathbf{X})(\mathcal{A}_{\mathbf{X}}(\cdot,\mathbf{V})_{|\mathcal{A}'_{\mathbf{X}}},\mathcal{F})) \end{aligned}$$

Moreover $RK \simeq 1$ *(equivalently,* K *is fully faithful).*

(2) *If* $\mathcal{A}$ *is projectively complete, and every object of* $\mathcal{A}$ *is a direct summand of a finite direct sum of objects of* $\mathcal{A}'$*, this adjunction is an equivalence* $\mathbf{Qcoh}(\mathcal{A},\mathbf{X}) \simeq \mathbf{Qcoh}(\mathcal{A}',\mathbf{X})$*.*

In particular, when $\mathcal{A}$ is projectively complete we can assume that $\mathcal{A}$ is generated by a single object.

Finally, with our definitions, there is an obvious notion of change of group: indeed for any morphism $\alpha\colon H \to G$ we can form the "restricted" functor $\alpha^*\colon \mathcal{A} \to k[G] - \mathbf{mod} \xrightarrow{\alpha^*} \mathbf{k[H]} - \mathbf{mod}$. Naming $Z := X/H$, there are a morphism $\tilde{\alpha}\colon Z \to Y$ and a natural map $\alpha^{\sharp}\colon \mathcal{A}_X \to \tilde{\alpha}_*(\alpha^*\mathcal{A})_X$, giving a morphism of ringed schemes

$$(\tilde{\alpha}, \tilde{\alpha}^{\sharp}) : (Z, (\alpha^*\mathcal{A})_X) \longrightarrow (Y, \mathcal{A}_X)$$

Thus we have restriction functor $\alpha^{\mathcal{A}}\colon \mathbf{Qcoh}(\mathcal{A},\mathbf{X}) \to \tilde{\alpha}_*\mathbf{Qcoh}(\alpha^*\mathcal{A},\mathbf{X}_{|\mathbf{H}})$ and an induction functor $\alpha_{\mathcal{A}}\colon \tilde{\alpha}_*\mathbf{Qcoh}(\alpha^*\mathcal{A},\mathbf{X}_{|\mathbf{H}}) \to \mathbf{Qcoh}(\mathcal{A},\mathbf{X})$. Again, this gives an adjoint pair $(\alpha^{\mathcal{A}}, \alpha_{\mathcal{A}})$.

Let us make some comment on these induction and restriction functors. Following [7], Lemma 7.11, we see that when G is of finite representation type, $X = \mathit{Spec}(k)$ and

$\mathcal{A} = k[G]-\mathbf{mod}$ then (identifying $k[G]-\mathbf{mod}$ with a fully faithful subcetegory of $\mathcal{A}-\mathbf{mod}$ via the Yoneda embedding) the induction and restriction functors coincide with the usual induction and restriction for group representations.

More precisely, for any subgroup $H \hookrightarrow G$ we have an isomorphism $k[H]-\mathbf{mod} \simeq \alpha^*\mathcal{A}$ (this follows immediately by the prevoius proposition and Mackey formula); there is a commuting diagram

$$\begin{tikzcd}
k[H]-\mathbf{mod} \arrow[r, "Ind_H^G"] \arrow[d] & k[G]-\mathbf{mod} \arrow[d] \\
\alpha^*\mathcal{A}-\mathbf{mod} \arrow[r, "\alpha_{\mathcal{A}}"] & \mathcal{A}-\mathbf{mod}
\end{tikzcd}$$

and a similar one for restriction.

In general, from the induction-restriction (co-)adjunction it is immediate to see that $\alpha_{\mathcal{A}}$ (resp. $\alpha^{\mathcal{A}}$) sends a representable sheaf $\mathcal{A}_V$ to the functor $Hom_G(-, Ind_H^G(V))$ (resp. $Hom_H(-, Res_H^G(V))$ *if the latter is representable.*

Consider now the case where $\mathcal{A}$ is generated by a single element I; in this case $\mathcal{A}_X$ can be identified with an actual (non-commutative) algebra $\mathcal{A}_X = (s_X^* End_k(I))^G$. Let us call $A := s_X^* End_k(I)$; then for any $\mathcal{F}$ in $\mathbf{Qcoh}(\mathcal{A}, \mathbf{X})$ the induced functor $\alpha_{\mathcal{A}}\mathcal{F}$ is $\alpha_{\mathcal{A}}\mathcal{F}(I) = \tilde{\alpha}_*\mathcal{F}(I)$, seen as a A^H-module. Conversely the restricted functor is $\alpha^{\mathcal{A}}\mathcal{F}(I) = \tilde{\alpha}^* F(I) \otimes_{\tilde{\alpha}^* A^G} A^H$.

We end this section with a description of monoidal structures. Suppose that $\mathcal{A}$ is a full submonoidal category of $k[G]\mathbf{mod}$, and X is a G-scheme. Then there is a canonical structure of monoidal closed symmetric category on $\mathbf{Qcoh}(\mathcal{A}, \mathbf{X})$, whose unit object is

$$\underline{\mathcal{O}_X} \colon V \longrightarrow \pi_*^G(s_X^* V^\vee)$$

and whose tensor product is given by:

$$\mathcal{F} \otimes \mathcal{G}(V) = \int^W \mathcal{F}(W) \otimes_{\mathcal{O}_Y} \mathcal{G}(V \otimes_k W^\vee)$$

Tensor product with a representable sheaf can be easily calculated:

$$(\mathcal{F} \otimes \mathcal{A}_V)(W) \simeq \mathcal{F}(W \otimes_k V^\vee).$$

With this it is immediate to see that pull-backs and push-forwards preserve the product with any representable sheaf.

3.2.3 Modular K-theory

As for classical $G-$sheaves, there is a notion of coherence for $\mathcal{A}_X-$sheaves:

Definition. *A quasicoherent $\mathcal{A}$-sheaf $\mathcal{F}$ on X is said* coherent *if for each G-invariant open affine $U = Spec(R)$ of X, the restriction $\mathcal{F}_{|U}$ is of finite type in* $\mathbf{Qcoh}(\mathcal{A}, \mathbf{U}) \simeq [\mathcal{A}_{\mathbf{X}}(\mathbf{U})^{\mathbf{op}}, \mathbf{R}^{\mathbf{G}} - \mathbf{Mod}]$ *(i.e. if, seen in* $[\mathcal{A}_X(U)^{op}, R^G - \mathbf{Mod}]$*, it is a quotient of a finite sum of representable objects).*

We denote by $Coh(\mathcal{A}, X)$ the whole subcategory of $Qcoh(\mathcal{A}, X)$ whose objects are the coherent $\mathcal{A}$-sheaves.

Proposition 3.2.7. *Let $\mathcal{A}'$ be a full subcategory of $\mathcal{A}$. Suppose that $\mathcal{A}$ is projectively complete, and that every object of $\mathcal{A}$ is a direct summand of a finite direct sum of objects of $\mathcal{A}'$. Then restriction along $\mathcal{A}' \to \mathcal{A}$ induces an equivalence $Coh(\mathcal{A}, X) \simeq Coh(\mathcal{A}', X)$.*

Proposition 3.2.8. *Suppose $\mathcal{A}$ admits a finite set of additive generators. Then $Coh(\mathcal{A}, X)$ is an abelian category.*

With this notion at our service it is immediate to define a K-theory functor:

Definition. *Let X be a G-scheme over k and $\mathcal{A}$ a full subcategory of $k[G]-\mathbf{mod}$ admitting a finite set of additive generators. We denote by $K_i(\mathcal{A}, X)$ the Quillen i-th group of the abelian category $Coh(\mathcal{A}, X)$.*

We will refer to this functor as *modular K-theory.*

It admits well-defined notions of pull-back and push-forward:

Proposition 3.2.9. *Let $f : X' \to X$ be a morphism of G-schemes over k such that the morphism $\tilde{f} \colon Y' \to Y$ between quotient schemes is flat, and $X' = X \times_Y Y'$.*

Then the functor $f^{\mathcal{A}} \colon Coh(\mathcal{A}, X) \to Coh(\mathcal{A}, X')$ is exact, hence induces a map in K-theory.

Proposition 3.2.10. *Let $f : X' \to X$ be a morphism of G-schemes over k and $\mathcal{A}$ a full subcategory of $k[G]-\mathbf{mod}$, admitting a finite set of additive generators. Suppose that the morphism $\tilde{f} \colon Y' \to Y$ between quotient schemes is proper, then:*

1. *For each coherent $\mathcal{A}$-sheaf $\mathcal{F}$ on X', and each nonnegative integer i, the $\mathcal{A}$- sheaf $R^i f_{\mathcal{A}} \mathcal{F}$ is coherent.*
2. *There exists an integer n, such for any integer $i > n$, and any coherent $\mathcal{A}$-sheaf $\mathcal{F}$ on X', we have $R^i f_{\mathcal{A}} \mathcal{F} = 0$.*
3. *If $\tilde{f}$ is finite or Y' admits an ample line bundle then there is a well-defined push-forward $f_{\mathcal{A}} \colon K_i(\mathcal{A}, X') \to K_i(\mathcal{A}, X)$.*

It is immediate to see that these definitions are functorial.

We conclude this section citing a very important property of modular K-theory, that is it satisfies a form of localization ([7], Theorem 6.7):

Theorem 3.2.11. *Let $i \colon X' \to X$ be a morphism of G-schemes over k, and $\mathcal{A}$ a full subcategory of $k[G]-\mathbf{mod}$, admitting a finite set of additive generators.*

Suppose that the morphism $\tilde{i} \colon Y' \to Y$ between quotient schemes is a closed immersion, and that $i^\sharp \colon \mathcal{A}_X \to \tilde{i}_* \mathcal{A}_{X'}$ is an epimorphism (*).

Denote by U the pullback by $\pi : X \to Y$ of the complement of Y' in Y, and by $j : U \to X$ the canonical inclusion.

Then there is a long exact sequence:

$$\dots \longrightarrow K_{i+1}(\mathcal{A}, U) \longrightarrow K_i(\mathcal{A}, X') \longrightarrow K_i(\mathcal{A}, X) \longrightarrow K_i(\mathcal{A}, U) \longrightarrow K_{i-1}(\mathcal{A}, X') \longrightarrow \dots$$

The application of this theorem is limited by the condition of $i^\sharp\colon \mathcal{A}_X \to \tilde{i}_*\mathcal{A}_{X'}$ being an epimorphism. However in some case we can ensure that; for instance, when the substack $[X'/G] \hookrightarrow [X/G]$ is a gerbe ([7], Proposition 6.13):

Proposition 3.2.12. *Let $i\colon X' \to X$ be a morphism of G-schemes over k, such that there is a normal subgroup H of G that acts trivially on X', and G/H acts freely on X'; suppose moreover that the morphism $\tilde{i}\colon Y' \to Y$ between quotient schemes is a closed immersion.*

Then the canonical morphism $i^\sharp\colon \mathcal{A}_X \to \tilde{i}_*\mathcal{A}_{X'}$ is an epimorphism.

3.3 A localization theorem for modular K-theory

Suppose now that X is a quasi-projective variety over an algebraically closed field, equipped with an action of a finite group G.

Let $\mathcal{A}$ be a finitely generated projectively complete full subcategory of $k[G]-\mathbf{mod}$ containing the trivial representation and equipped with an action of the Burnside algebra (that is, for any V in $\mathcal{A}$ and any $\Gamma < G$ the tensor product $V \otimes Ind_\Gamma^G 1$ is also in $\mathcal{A}$).

Then we also have an action of the Burnside algebra on $Coh(\mathcal{A}, X)$, induced by the internal tensor product with representable sheaves. It is immediate to see that this action passes to K-theory, making $K^*(\mathcal{A}, X)$ a $B(G)$-module. In particular we have a decomposition $K^*(\mathcal{A}, X) = \bigoplus_{H<G} K^*(\mathcal{A}, X)_H$. Since pull-backs and push-forwards are maps of $B(G)$-modules, they preserve this decomposition.

3.3.1 Localization in modular K-theory

Now, following the ideas expressed in the previous paragraph, we would like to compare the modular $K-$theory of X with that of X^H for any $H < G$. We do so by comparing the modular K-theory of $[X/G]$ with that of its wild inertia.

For any $\mathcal{A}$ (finitely generated projectively complete full subcategory of $k[G]-\mathbf{mod}$ containing the trivial representation and equipped with an action of the Burnside algebra), we have an induction map $\alpha_{\mathcal{A}}\colon K^*(\alpha^*\mathcal{A}, X) \to K^*(\mathcal{A}, X)$ associated to the restriction $\alpha\colon k[G]-\mathbf{mod} \to \mathbf{k}[\mathbf{N_G}(\mathbf{H})]-\mathbf{mod}$, and an induction $i_{\mathcal{A}}\colon K^*(\alpha^*\mathcal{A}, X^H) \to K^*(\alpha^*\mathcal{A}, X)$. Composing them we get an induction functor

$$\hat{i}_{\mathcal{A}}\colon K^*(\alpha^*\mathcal{A}, X^H) \longrightarrow K^*(\mathcal{A}, X).$$

Our main result is the following localization theorem:

Theorem 3.3.1. *Let $(X, \mathcal{A})$ be as above. Suppose that X admits a stratification*

$$U_0 \xrightarrow{j_0} U_1 \xrightarrow{j_1} \cdots \xrightarrow{j_{n-1}} U_n = X$$

by open $G-$invariant sets such that for any stratum $X_i := U_{i+1} \backslash U_i$ the quotient stack $[X_i/G]$ is a gerbe with stabilizer normal in G.

Then for any $H < G$ the induction

$$\hat{i}_{\mathcal{A}}\colon K^*(\alpha^*\mathcal{A}, X^H)_H \longrightarrow K^*(\mathcal{A}, X)_H$$

is an isomorphism.

Proof. First of all we use that X admits a stratification

$$U_0 \xrightarrow{j_0} U_1 \xrightarrow{j_1} \cdots \xrightarrow{j_{n-1}} U_n = X$$

by open $G-$invariant sets such that for any stratum $X_i := U_{i+1} \backslash U_i$ the quotient stack $[X_i/G]$ is a gerbe with normal stabilizer. Using this fact, we can prove that the theorem holds for each U_i by induction on i. Indeed, by Propositon 3.2.12 any closed immersion $i_i \colon X_i \to U_{i+1}$ satisfies the condition $(*)$ of Theorem 3.2.11, and so we have fon any l a commuting diagram

$$\begin{tikzcd}
K_{l+1}(\alpha^*\mathcal{A}, U_i^H) \arrow[r] \arrow[d] & K_l(\alpha^*\mathcal{A}, X_i^H) \arrow[r, "i_{i,\mathcal{A}}"] \arrow[d] & K_l(\alpha^*\mathcal{A}, U_{i+1}^H) \arrow[r, "j_i^{\mathcal{A}}"] \arrow[d] & K_l(\alpha^*\mathcal{A}, U_i^H) \arrow[r] \arrow[d] & K_{l-1}(\alpha^*\mathcal{A}, X_i^H) \arrow[d] \\
K_{l+1}(\mathcal{A}, U_i) \arrow[r] & K_l(\mathcal{A}, X_i) \arrow[r, "i_{i,\mathcal{A}}"] & K_l(\mathcal{A}, U_{i+1}) \arrow[r, "j_i^{\mathcal{A}}"] & K_l(\mathcal{A}, U_i) \arrow[r] & K_{l-1}(\mathcal{A}, X_i)
\end{tikzcd}$$

From the five lemma we immediately infer that if we know the thesis for U_i and for any X such that $[X/G]$ is a gerbe then we also have it for U_{i+1}.

All we have to do, then, is to show the thesis in the case when $[X/G]$ is a gerbe. In that case there is a normal subgroup $Q < G$ such that Q acts trivially on X (so that $[X/G]$ is banded by Q) and $P := G/Q$ acts freely on X. Let us consider the pull back $\alpha^*\mathcal{A}$ with respect to $\alpha : k[G] - \mathbf{mod} \to \mathbf{k[Q] - mod}$; note that P has an action both on X and on $\mathcal{A}_{|Q}$, and so (by pull-back) it has an action on $\mathcal{A}_X$ and on $K^*(\mathcal{A}_{|Q}, X)$.

Our first step will be to show that $\alpha^{\mathcal{A}}$ induces an isomorphism

$$\alpha^{\mathcal{A}} \colon K^*(\mathcal{A}, X) \xrightarrow{\simeq} K^*(\mathcal{A}_{|Q}, X)^P.$$

To see it, we will show that both $\alpha_{\mathcal{A}}\alpha^{\mathcal{A}}$ and $\alpha^{\mathcal{A}}\alpha_{\mathcal{A}}$ are isomorphisms.

Consider the former. Let $Y := X/P$; we have a commuting square

$$\begin{tikzcd}
{[X/Q]} \arrow[r] \arrow[d] & {[X/G]} \arrow[d, "\pi"] \\
X \arrow[r, "\tilde{\alpha}"] & Y
\end{tikzcd}$$

and we have $(\mathcal{A}_{|Q})_X \simeq \tilde{\alpha}^*\mathcal{A}_X$. In particular the composition $\alpha_{\mathcal{A}}\alpha^{\mathcal{A}}$ is just induced by the composition $\tilde{\alpha}_*\tilde{\alpha}^*$ on $Qcoh(Y)$. By the classical projection formula, this functor is just the tensor product with the locally free sheaf $\mathcal{E} := \tilde{\alpha}_*\mathcal{O}_X = \pi_* s_X^*(Ind_Q^G 1)$. By Morita theory this is a local progenerator, so it induces an isomorphism of the exact category $Coh(\mathcal{A}, X)$ and thus an isomorphism in K-theory; alternatively we can observe that the tensor product with a locally free sheaf induces an action of $K^*(Y)$ on $K^*(\mathcal{A}, X)$, and the class $[\mathcal{E}]$ is a unit in $K_0(Y)$.

Now consider the latter composition. The push-forward $\alpha_{\mathcal{A}}\mathcal{F}$ of an $\mathcal{A}_{|Q}-$sheaf $\mathcal{F}$ sends s_X^*V to the sheaf $\tilde{\alpha}_*\mathcal{F}(V)$, with $(\mathcal{A}_{|Q})_X$-action induced by the map $\mathcal{A}_X \to \tilde{\alpha}_*\tilde{\alpha}^*\mathcal{A}_X$. Then composing with the pull-back we get

$$s_X^*V \longrightarrow \bigotimes_{p \in P} p^*F(V)^p$$

where $(-)^p$ means that the $\alpha^*\mathcal{A}_X$-action is twisted by p. We conclude that $\alpha^{\mathcal{A}}\alpha_{\mathcal{A}}\mathcal{F} = \sum_{p\in P} p\cdot\mathcal{F}$; thus the functor $\alpha^{\mathcal{A}}\alpha_{\mathcal{A}}$ coincides with $\sum_{p\in P} p\cdot(-)$ in $Coh(\mathcal{A}, X)$, and this relation must then hold also in K-theory. We conclude, since $\sum_{p\in P} p\cdot(-)$ is an isomorphism in $K^*(\alpha^*\mathcal{A}, X)^P$.

Now, from this fact it follows trivially that we just need to consider the case when $H < Q$ (otherwise $K^*(\mathcal{A}_{|Q}, X)_H$ is zero and X^H is empty); moreover we have $N_G(H)/N_Q(H) \simeq P$, so that also

$$\alpha^{\mathcal{A}}\colon K^*(\mathcal{A}_{|N_G(H)}, X^H) \xrightarrow{\simeq} K^*(\mathcal{A}_{|N_Q(H)}, X^H)^P$$

and we just need to prove the claim for $K^*(\mathcal{A}_{|Q}, X)$. In particular we can suppose that G acts trivially and the Auslander algebra is constant on X.

In this case, the result follows from the following

Lemma 3.3.2. *Suppose that the Auslander algebra is trivial. Then the induction $\alpha_{\mathcal{A}}$ and restriction $\alpha^{\mathcal{A}}$ induce an isomorphism*

$$K^*(\mathcal{A}, X)_H \simeq K^*(\mathcal{A}_{|H}, X)^{N_G(H)}.$$

Proof. To avoid confusion, let us call $\hat{i}_{\mathcal{A}} = \alpha_c A$ and $\hat{i}^{\mathcal{A}} = \alpha^{\mathcal{A}}$. Again, we would like to calculate the two compositions $\hat{i}_{\mathcal{A}}\hat{i}^{\mathcal{A}}$ and $\hat{i}^{\mathcal{A}}\hat{i}_{\mathcal{A}}$.

Let I be the direct sum of a set of generators for $\mathcal{A}$. Then the push-forward $\hat{i}_{\mathcal{A}}$ corresponds to the push-forward from $End_H(I)-$modules to $End_G(I)$-modules, whereas the pull-back corresponds to $\otimes_{End(I)^G} End(I)^H$, where $End(I)^G$ acts on $\mathcal{F}(I)$ on the left and on $End(I)^H$ on the right. The left $End(I)^H$-module structure on the pull-back is induced by the left action of $End(I)^H$ on itself.

Consider first $\hat{i}_{\mathcal{A}}\hat{i}^{\mathcal{A}}$, and let $T := Ind_H^G(1)$, which by hypothesis is an element of $\mathcal{A}$. By definition the sheaf $\hat{i}_{\mathcal{A}}\hat{i}^{\mathcal{A}}\mathcal{F}$ sends an element $V \in \mathcal{A}$ to the coend

$$\int^W Hom_H(V, W) \otimes \mathcal{F}(W).$$

But $Hom_H(V, W) = Hom_H(Hom(V, W), 1) = Hom_G(Hom(V, W), T)$, so $\hat{i}_{\mathcal{A}}\hat{i}^{\mathcal{A}}\mathcal{F}(V)$ is equal to

$$\int^W Hom_G(Hom(V, W), T) \otimes \mathcal{F}(W)$$

In particular, $\hat{i}_{\mathcal{A}}\hat{i}^{\mathcal{A}}\mathcal{F} = \mathcal{F} \otimes_{\mathcal{A}_X} \mathcal{A}_T$. We conclude that $\hat{i}_{\mathcal{A}}\hat{i}^{\mathcal{A}}$ is equal to the multiplication by $\mathcal{A}_T$ in $K^*(\mathcal{A}, X)_H$; this is an isomorphism, since T is invertible in the $H-$part $B(G)_H$ (it is actually a multiple of the generator).

In order to deal with $\hat{i}^{\mathcal{A}}\hat{i}_{\mathcal{A}}$, we want to prove a Mackey-type formula. Clearly it is sufficient to prove it in the exact category $Coh(\mathcal{A}_H, X)$ and, following the definitions, for the representable sheaf $\mathcal{F} = \mathcal{A}_{I|H}$. Indeed, by hypothesis, we have $Res_H^G Ind_H^G \mathcal{F} = \mathcal{F} \otimes_{End(I)^G} End(I)^H$, where $End(I)^H$ is seen as a right $End(I)^G-$module and as a left $End(I)^H$-module. But $\mathcal{F} \otimes_{End(I)^G} End(I)^H = \mathcal{F} \otimes_{End(I)^H} (End(I)^H \otimes_{End(I)^G} End(I)^H)$, where $End(I)^H \otimes_{End(I)^G} End(I)^H$ has a right $End(I)^H-$module structure given by the right action of itself on the first copy of $End(I)^H$.

Now, the induced representation $I' := Ind_H^G I_{|H} = I \otimes_G T$ is representable in $\mathcal{A}$, so the push-forward sends $\alpha^*\mathcal{A}_I$ to $\mathcal{A}_{I'}$, and composing with the pull-back ge get $\alpha^*\mathcal{A}_{Res_H^G Ind_H^G I_{|H}}$.

By the classical Mackey formula, this is equal to a sum $\bigoplus_{g\in H\backslash G/H} Ind^H_{{}^gH_i} Res^H_{H_i}(I_{|H})$. By Proposition 3.2.6 we can assume that each of the $I_i := Ind^H_{H_i^g} Res^H_{H_i}(I_{|H})$ lies in $\mathcal{A}_{|H}$; we see then that

$$Res^G_H Ind^G_H \mathcal{A}_{I|H}(I_{|H}) = \bigoplus Hom_H(I, I_i) = \bigoplus \alpha^* \mathcal{A}_{I_i}(I_{|H}).$$

By the same reasoning as above, we have that

$$Ind^H_{H_i^g} Res^H_{H_i} \alpha^* \mathcal{A}_{I|H}(I) = \alpha^* \mathcal{A}_{I_i}(I),$$

so the result holds, since this is an isomorphism of $End(I)^H-$bimodules.

We saw that we have a Mackey-type formula

$$Res^G_H Ind^G_H \mathcal{F} = \bigoplus_{g\in H\backslash G/H} Ind^H_{{}^gH_i} Res^H_{H_i} \mathcal{F}$$

in the category $Coh(\mathcal{A}_{|H}, X)$, and thus also in K-theory. However when $H_i = H \cap H^g$ is not equal to H, the restriction $Res^H_{H_i}$ is zero on the $H-$part, hence we have $Res^G_H Ind^G_H \mathcal{F} = \bigoplus_{n\in N_G(H)/H} n \cdot \mathcal{F}$ (where, again, the $N_G(H)$-action is induced by the conjugation action on $\mathcal{A}_{|H}$). However this functor is a multiple of the identity on $K^*(\mathcal{A}_{|H}, X)^{N_G(H)}$, and we conclude. ♠

Comparing the result of the Lemma applied to G with the one applied to $N_G(H)$, (and noting that obviously $N_{N_G(H)}(H) = N_G(H)$) we immediately have our thesis. ♠

Remark. *By following the proof of the Theorem, we can also prove that the induction*

$$\alpha_{\mathcal{A}} \colon K^*(\mathcal{A}_{|H}, X^H)^{N_G(H)} \longrightarrow K^*(\mathcal{A}_{N_G(H)}, X^H)$$

is an isomorphism.

Proof. Exactly as in the proof of the preceding Theorem we use the existence of a stratification of the quotient $[X/G]$ by gerbes and the five lemma to reduce to the case where there exists Q normal in G acting trivially on X and such that $P = G/Q$ acts freely on X.

But then we have a commuting diagram

$$\begin{tikzcd}
K^*(\mathcal{A}_{|H}, X)^{N_G(H)} \arrow[r, "\simeq"] \arrow[d] & (K^*(\mathcal{A}_{|H}, X)^{N_Q(H)})^P \arrow[d] \\
K^*(\mathcal{A}_{|N_G(H)}, X) \arrow[r, "\simeq"] & K^*(\mathcal{A}_{|N_Q(H)}, X)^P
\end{tikzcd}$$

so we can suppose that $H < Q$ or in other words that G acts trivially on X.

In this case the result is exactly the content of Lemma 3.3.2 . ♠

Remark. *Suppose that G is of finite representation type and $\mathcal{A} = k[G] -$ **mod**. Then $K^*(\mathcal{A} - \mathbf{mod}) \simeq \mathbf{K}^*(\mathcal{A}^{\mathbf{split}})$.*

By Theorem 3.1.3, in this case the only relevant subgroups are those of the form $P \rtimes C$, where P is a $p-$group and C is a cyclic group of order coprime with p if k is a characteristic p field.

Appendix A

Homomorphism schemes from diagonalizable groups

Let us analyze the structure of $\mathrm{H}_\Delta(G)$ when $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ is a product of general linear groups.

Proposition A.1. *Assume that G is a product of general linear groups. Then $\mathrm{H}_\Delta(G)$ is represented by an open subscheme in a disjoint union of product of Grassmannians. The action of G on each connected component of $\mathrm{H}_\Delta(G)$ is transitive, and each connected component contains a k-rational point. Furthermore, $\mathrm{H}^{\mathrm{in}}_\Delta(G)$ is a union of connected components of $\mathrm{H}_\Delta(G)$.*

Given two homomorphism f, $g\colon \Delta \to G$, thought of as k-rational points of $\mathrm{H}_\Delta(G)$, then f and g are in the same connected component of $\mathrm{H}_\Delta(G)$ if and only if then are conjugate by an element of $G(k)$.

Proof. Let $G = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. Clearly $\mathrm{H}^{\mathrm{in}}_\Delta(G) \subseteq \mathrm{H}_\Delta(G)$ is $G-$invariant, so it is a union of connected components of $\mathrm{H}_\Delta(G)$ if we know that they are $G-$orbits. We are then left to prove the remaining claims.

A homomorphism $\Delta_S \to G_S$ corresponds to eigenspaces decompositions $\mathcal{O}_S^{n_i} = \bigoplus_{\chi \in \widehat{\Delta}} V_{\chi,i}$ for every $i = 1, \ldots, r$. For every function $d\colon \widehat{\Delta} \to \mathbb{N}^r$, let $\mathrm{H}^d_\Delta(\mathrm{GL}_n) \subseteq \mathrm{H}_\Delta(\mathrm{GL}_n)$ be the subfunctor of the homomorphisms $\Delta \to G$ such that for all i's the eigenspaces $V_{\chi,i}$ have constant rank $\pi_i d(\chi)$ (where π_i denotes the ith component of d). We have a decomposition $\mathrm{H}_\Delta(G) = \coprod_d \mathrm{H}^d_\Delta(G)$. If $0 \leq m \leq n$, denote by $\mathbb{G}(m,n)$ the Grassmannian of m-dimensional subspaces of k^n. It has a tautological action of GL_n, and here is a $G-$equivariant open embedding

$$\mathrm{H}^d_\Delta(\mathrm{GL}_n) \subseteq \prod_{\chi,i} \mathbb{G}\big(\pi_i d(\chi), n_i\big)$$

which identifies the former with the open subscheme of $\prod_{\chi,i} \mathbb{G}\big(\pi_i d(\chi), n\big)$ such that for every fixed i the subspaces given by $\mathbb{G}\big(\pi_i d(\chi), n_i\big)$ are in direct sum. The latter is an integral scheme and the $G-$action on it is clearly transitive, proving the first claim.

The spaces $\mathcal{O}_R^{n_i}$ obviously have a d-decomposition for every d such that $\sum_\chi \pi_i d(\chi) = n_i$, so every connected component has a rational point, and we conclude exactly as above that the $G(R)$ action is transitive on the rational points. ♠

If $\phi\colon \Delta \to \Delta'$ is a homomorphism of diagonalizable group schemes, composing with ϕ gives a natural transformation $\mathrm{H}_{\Delta'}(G) \to \mathrm{H}_{\Delta}(G)$. Call $\mathrm{Q}(\Delta)$ the set of quotients of Δ. For each $\Delta' \in \mathrm{Q}(\Delta)$, consider the composite $\mathrm{H}^{\mathrm{in}}_{\Delta'}(G) \subseteq \mathrm{H}_{\Delta'}(G) \to \mathrm{H}_{\Delta}(G)$, which is immediately seen to be a monomorphism. This induces a G-equivariant morphism of Zariski sheaves

$$\coprod_{\Delta' \in \mathrm{Q}(\Delta)} \mathrm{H}^{\mathrm{in}}_{\Delta'}(G) \longrightarrow \mathrm{H}_{\Delta}(G)\,.$$

Proposition A.2.

(1) *The functors* $\mathrm{H}_{\Delta}(G)$ *and* $\mathrm{H}^{\mathrm{in}}_{\Delta}(G)$ *are represented by a quasi-projective scheme over* R.

(2) *If* $\Delta' \in \mathrm{Q}(\Delta)$*, then* $\mathrm{H}^{\mathrm{in}}_{\Delta'}(G)$ *is open and closed in* $\mathrm{H}_{\Delta}(G)$.

(3) *The morphism* $\coprod_{\Delta' \in \mathrm{Q}(\Delta)} \mathrm{H}^{\mathrm{in}}_{\Delta'}(G) \to \mathrm{H}_{\Delta}(G)$ *is an isomorphism.*

(4) *If* G *is finite and linearly reductive, then* $\mathrm{H}_{\Delta}(G)$ *and* $\mathrm{H}^{\mathrm{in}}_{\Delta}(G)$ *are finite over* $\operatorname{Spec} R$. *If* G *is étale and* R *equicharacteristic,* $\mathrm{H}_{\Delta}(G)$ *and* $\mathrm{H}^{\mathrm{in}}_{\Delta}(G)$ *are finite over* $\operatorname{Spec} R$.

Proof. Choose an embedding $G \subseteq \mathrm{GL}_n$ for some n; this gives an embedding of functors $\mathrm{H}_{\Delta}(G) \subseteq \mathrm{H}_{\Delta}(\mathrm{GL}_n)$.

Lemma A.3. *The inclusion* $\mathrm{H}_{\Delta}(G) \subseteq \mathrm{H}_{\Delta}(\mathrm{GL}_n)$ *is a closed embedding.*

Proof. This is standard. ♠

Clearly we have $\mathrm{H}^{\mathrm{in}}_{\Delta}(G) = \mathrm{H}_{\Delta}(G) \cap \mathrm{H}^{\mathrm{in}}_{\Delta}(\mathrm{GL}_n)$. More generally, if $\Delta' \in \mathrm{Q}(\Delta)$, the inverse image of $\mathrm{H}_{\Delta}(G) \subseteq \mathrm{H}^{\mathrm{in}}_{\Delta}(\mathrm{GL}_n)$ in $\mathrm{H}^{\mathrm{in}}_{\Delta'}(\mathrm{GL})$ equals $\mathrm{H}^{\mathrm{in}}_{\Delta'}(G)$. Hence, to prove (1), (2) and (3) we can assume that $G = \mathrm{GL}_n$.

So it is enough to prove that $\mathrm{H}_{\Delta}(\mathrm{GL}_n)$ is represented by a quasi-projective scheme over k. Let $\widehat{\Delta}$ be the group of characters $\Delta \to \mathbb{G}_{\mathrm{m}}$ of Δ. By the standard description of representations of diagonalizable groups, a representation $\Delta_S \to \mathrm{GL}_{n,S}$ corresponds to a decomposition $\mathcal{O}_S^n = \bigoplus_{\chi \in \widehat{\Delta}} V_\chi$ into eigenspaces. If $d\colon \widehat{\Delta} \to \mathbb{N}$ is a function, denote by $\mathrm{H}^{d}_{\Delta}(\mathrm{GL}_n) \subseteq \mathrm{H}_{\Delta}(\mathrm{GL}_n)$ the subfunctor of those representations of Δ such that the corresponding eigenspace V_χ has constant rank $d(\chi)$. We have a decomposition of Zariski sheaves $\mathrm{H}_{\Delta}(\mathrm{GL}_n) = \coprod_d \mathrm{H}^{d}_{\Delta}(\mathrm{GL}_n)$. If $0 \le m \le n$, denote by $\mathbb{G}(m,n)$ the Grassmannian of m-dimensional subspaces of A^n. There is an obvious embedding of functors

$$\mathrm{H}^{d}_{\Delta}(\mathrm{GL}_n) \subseteq \prod_d \mathbb{G}\bigl(d(\chi), n\bigr)$$

which is easily seen to be an open embedding. This proves (1).

Furthermore, if $d\colon \widehat{\Delta} \to \mathbb{N}$ is a function, denote by Δ'_d the quotient of Δ such that $\widehat{\Delta'_d} \subseteq \widehat{\Delta}$ is the group generated by the $\chi \in \widehat{\Delta}$ with $d(\chi) > 0$. Then it is easy to see that $\mathrm{H}^{\mathrm{in}}_{\Delta'}(\mathrm{GL}_n) \subseteq \mathrm{H}_{\Delta}(\mathrm{GL}_n)$ is the union of the components $\mathrm{H}^{d}_{\Delta}(\mathrm{GL}_n)$ with $\Delta'_d = \Delta'$. This proves (2) and (3).

To prove (4), assume that G is finite and linearly reductive or étale.

If $\Delta = \Delta' \times \Delta''$ is a decomposition into the product of two diagonalizable subgroups, and assume that $\mathrm{H}_{\Delta'}(G)$ and $\mathrm{H}_{\Delta''}(G)$ are finite over R; let us show that $\mathrm{H}_{\Delta}(G)$ is also finite. We get an obvious morphism $\mathrm{H}_{\Delta}(G) \to \mathrm{H}_{\Delta'}(G) \times \mathrm{H}_{\Delta''}(G)$; let us show that this is a closed embedding. In fact, let $S \to \mathrm{H}_{\Delta'}(G) \times \mathrm{H}_{\Delta''}(G)$ be a morphism, corresponding to an object (f', f'') of $\bigl(\mathrm{H}_{\Delta'}(G) \times \mathrm{H}_{\Delta''}(G)\bigr)(S)$; here, $f' \colon \Delta'_S \to G_S$ and $f'' \colon \Delta''_S \to G_S$ are homomorphisms of group schemes. Then (f', f'') comes from a (unique) object of $\mathrm{H}_\Delta(G)$ if and only if f' and f'' commute, that is, the morphism $(\Delta' \times \Delta'')_S \to G_S$ that sends a pair (δ', δ'') into $f'(\delta')f''(\delta'')f'(\delta')^{-1}f''(\delta'')^{-1}$ factors through the identity section $S \to G_S$. Then the result follows from the following standard fact.

Lemma A.4. *Let $X \to S$ and $Y \to S$ be morphisms of schemes, f, $g \colon X \to Y$ morphisms of S-schemes. Assume that $X \to S$ is finitely presented, finite and flat, while $Y \to S$ is separated. Then the functor from schemes to sets, sending a scheme T into the set of morphisms $T \to S$ such that the pullbacks f_T, $g_T \colon X_T \to Y_T$ coincide is represented by a closed subscheme of S.*

Now let us prove the result in general. We may assume that R is a local ring, with residue field k. If G is linearly reductive then after extending the base we may assume that G is well-split, that is, a semidirect product $G_1 \ltimes G_0$, where G_1 is constant, of order not divisible by $\operatorname{char} k$, and G_0 is diagonalizable. We can split Δ into a finite product of group schemes of type $\boldsymbol{\mu}_{p^r}$, where p is a prime; because of the previous step, we can assume that $\Delta = \boldsymbol{\mu}_{p^r}$.

If $p = \operatorname{char} k$, suppose first that G is well-split: then $\underline{\mathrm{Hom}}_A(\boldsymbol{\mu}_{p^r}, G_1) = \operatorname{Spec} A$, so $\mathrm{H}_{\boldsymbol{\mu}_{p^r}}(G) = \mathrm{H}_{\boldsymbol{\mu}_{p^r}}(G_0)$; and, because of Cartier duality, $\mathrm{H}_{\boldsymbol{\mu}_{p^r}}(G_0)$ is a finite union of copies of $\operatorname{Spec} A$. On the other hand if G is étale then $\underline{\mathrm{Hom}}_A(\boldsymbol{\mu}_{p^r}, G) = \operatorname{Spec} A$, since $\boldsymbol{\mu}_{p^r}$ is infinitesimal when A is equicharacteristic.

If $p \neq \operatorname{char} k$, then $\boldsymbol{\mu}_{p^r}$ is a constant cyclic group scheme of order p^r; hence $\mathrm{H}_{\boldsymbol{\mu}_{p^r}}(G)$ is represented by the inverse image of the identity $\operatorname{Spec} A \subseteq G$ via the map $G \to G$ defined by $x \mapsto x^{p^r}$. This ends the proof of (4). ♠

Appendix B

The Weyl group action on torus-equivariant K-theory

In this section we briefly discuss the isomorphism $\mathrm{K}'_*(X, GL_n) \simeq \mathrm{K}'_*(X, T)^W$, where T is a maximal torus and W the associated Weyl group. Let $G := GL_n$.

To see this, we first note that $\mathrm{K}'_*(X, T) \simeq \mathrm{K}'_*(X, B)$ where B is the associated Borel subgroup of G (since $[X/T] \to [X/B]$ is an affine bundle).

Now let $E \to \mathcal{X} := [X/G]$ be the associated vector bundle; we recall here how to resume the flag bundle $[X/B] \to \mathcal{X}$ as a composition of projective bundles $\mathbb{P}_i(E) \to \mathcal{X}$.

Let $\mathbb{P}_1(E) := \mathbb{P}(E) \xrightarrow{\pi_1} \mathcal{X}$ be the projective bundle associated to E. Then we define E_2 as the tatutological vector bundle on $\mathbb{P}_1(E)$ assoc--isted to $\mathbb{P}_1(E)$ (it is a codimension 1 subbundle of $\pi_1^* E$), and $\mathbb{P}_2(E) := \mathbb{P}(E_2) \xrightarrow{\pi_2} \mathbb{P}_1(E)$; inductively it is then clear how, given $\mathbb{P}_i(E) \xrightarrow{\pi_i} \mathbb{P}_{i-1}(E)$, we can define E_{i+1} as the tatutological bundle of $\mathbb{P}_i(E)$ (a codimension 1 subbundle of $\pi_i^* E_i$) and $\mathbb{P}_{i+1}(E) := \mathbb{P}(E_{i+1})$. At the end of this process we get $\mathbb{P}_n(E) = [X/B]$.

By the projective bundle theorem in equivariant K-theory (see [23]), we have isomorphisms $\mathrm{K}'_*(\mathbb{P}_{i+1}(E)) \simeq \prod_{k=0}^{n-i} \mathrm{K}'_*(\mathbb{P}_i(E))$, and by induction we get that $\mathrm{K}'_*([X/B])$ is isomorphic to a product of $n!$ factors, each isomorphic to $\mathrm{K}'_*(\mathcal{X})$; The pull-back identifies $\mathrm{K}'_*(\mathcal{X})$ as the diagonal subgroup.

These factors correspond to the pieces of the Bruhat decomposition of G/B, and the induced action of the Weyl group is just the tautological permutation of the factors. Then it is clear that $\mathrm{K}'_*(X, G) \simeq \mathrm{K}'_*(X, B)^W \simeq \mathrm{K}'_*(X, T)^W$.

Appendix C

Existence of envelopes for DM and tame stacks

In this appendix we want to prove the existence of finite envelopes by unions of neutral gerbes for reduced DM and tame algebraic stacks.

Let $\mathcal{X}$ be such a stack, over a fixed base ring R, and let M be its space of moduli. By [9], Theorem 2.7, there exists a finite surjective cover $U \to \mathcal{X}$ from a scheme. Let us consider the cartesian square

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & U \arrow[d] \\
\mathcal{X} \arrow[r] & M
\end{tikzcd}$$

where $U \to M$ is the composition $U \to \mathcal{X} \to M$. It is immediate, by [2], Corollary 3.3, to see that U is the moduli space of $\mathcal{F}$.

Let us take the normalization $\overline{\mathcal{F}}$ of $\mathcal{F}$ (which exists by [21]), and its moduli space $\overline{U}$, that is again normal (since a categorical quotient of a normal variety is normal). We want to prove that $\overline{\mathcal{F}}$ is a neutral gerbe over $\overline{U}$; this is sufficient to conclude. Indeed, the map $\overline{\mathcal{F}} \to \mathcal{X}$ admits a fppf-local section over an open dense substack $\mathcal{V}$ of $\mathcal{X}$ that is a gerbe. In particular, the map $\mathcal{I}_{\boldsymbol{\mu}}\mathcal{F} \to \mathcal{I}_{\boldsymbol{\mu}}\mathcal{X}$ is generically surjective. By noetherian induction we can ensure that the closed substack $\mathcal{X}' := \mathcal{X} \backslash \mathcal{V}$ admits a finite envelope $\mathcal{F}'$, with $\mathcal{F}'$ a finite union of trivial gerbes; then $\overline{\mathcal{F}} \sqcup \mathcal{F}'$ is the sought envelope.

Thus we just need to prove that $\overline{\mathcal{F}} \to \overline{U}$ is a gerbe.

Consider the commuting square

$$\begin{tikzcd}
\overline{\mathcal{F}} \arrow[r] \arrow[d] & \overline{U} \arrow[d] \\
\mathcal{F} \arrow[r] & U \arrow[l, bend left, dashed]
\end{tikzcd}$$

and note that by construction the arrow $\mathcal{F} \to U$ admits a section. From the universal property of normalization, the top row $\overline{\mathcal{F}} \to \overline{U}$ also admits a section. More precisely, take a smooth atlas $C \to \mathcal{F}$ and define $V := C \times_{\mathcal{F}} \overline{U}$. Consider the following diagram with cartesian quadrangles:

$$\begin{tikzcd}
\overline{C \times_{\mathcal{F}} C} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \overline{C} \arrow[r] \arrow[d] & \overline{\mathcal{F}} \arrow[d] \\
C \times_{\mathcal{F}} C \arrow[r, shift left] \arrow[r, shift right] & C \arrow[r] & \mathcal{F} \\
V \times_{\overline{U}} V \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & V \arrow[r] \arrow[u] & \overline{U} \arrow[u]
\end{tikzcd}$$

By definition the top horizontal row is a presentation for $\overline{\mathcal{F}}$ and the two bottom-left vertical arrows are dominant, hence by the universal property of normalization all the bottom vertical arrows admit a lift to the top row. In particular the moduli space map $\overline{\mathcal{F}} \to \overline{U}$ admits a section $\overline{U} \to \overline{\mathcal{F}}$.

We conclude with the following

Lemma C.1. *Let $\mathcal{F}$ be a normal DM or tame algebraic stack such that the projection $\mathcal{F} \to U$ to its moduli space admits a section. Then $\mathcal{F}$ is a trivial gerbe.*

Proof. First of all, we note that we can take a rigidification $\mathcal{F} \to \mathcal{G}$; in other words we can exhibit $\mathcal{F}$ as a gerbe over a generically trivial stack $\mathcal{G}$ (see the Appendix to [2] for a reference). Actually, to prove the Lemma we can work étale-locally on U, and thus we may suppose that $\mathcal{F}$ is connected and of the form $[X/G]$, where G is a finite étale or linearly reductive group (see [16], Theorem 6.1 or [2], Theorem 3.2, respectively). In that case we can explicitly construct the rigidification.

Passing to a further étale cover, we may take $G = P \rtimes T$, where T is discrete and P of multiplicative type ([2], Lemma 2.17). Moreover we can make some simplifying assumptions:

1. First, we can assume that G is either discrete or of multiplicative type. Indeed, let $\mathcal{F} = [X/G]$ and $\mathcal{F}_{\mathbf{dm}} = [(X/P)/(G/P)]$ (where the square bracket indicates the stacky quotient and the round bracket the schematic quotient). Consider the cartesian diagram

$$\begin{tikzcd}
\mathcal{U} \arrow[r] \arrow[d] & U \arrow[d] \arrow[dl] \\
\mathcal{F} \arrow[r] & \mathcal{F}_{\mathbf{dm}}
\end{tikzcd}$$

where the diagonal arrow is the section $U \to \mathcal{F}$. Then $\mathcal{F}_{\mathbf{dm}}$ has only étale stabilizers; moreover U is the moduli space of $\mathcal{U}$ and $\mathcal{F}_{\mathbf{dm}}$ and by construction in both cases the projection to U admits a section. In particular if we know the result for stacks that have either étale or diagonalizable stabilizers we can deduce the general case: indeed we can infer that $\mathcal{F}_{\mathbf{dm}} \to U$ is a gerbe and $\mathcal{F} \to \mathcal{F}_{\mathbf{dm}}$ is also a gerbe; for the second, it is sufficient to show that its base-change $\mathcal{U} \to U$ is a gerbe, which we infer looking at the cartesian cube

$$\begin{tikzcd}
 & \mathcal{V} \arrow[rr] \arrow[dl] \arrow[dd] & & V \arrow[dl] \arrow[dd] \\
\mathcal{U} \arrow[rr] \arrow[dd] & & U \arrow[dd] & \\
 & {[X/P]} \arrow[rr] \arrow[dl] & & (X/P) \arrow[dl] \\
\mathcal{F} \arrow[rr] & & \mathcal{F}_{\mathbf{dm}} &
\end{tikzcd}$$

Indeed $\mathcal{V}$, being a base-change of $[X/P]$, has only diagonalizable stabilizers and $\mathcal{V} \to V$ by construction admits a section, so we can deduce that it is a gerbe; this implies that $\mathcal{U} \to U$ is also a gerbe.

If $\mathcal{F}_{\mathbf{dm}} \to U$ and $\mathcal{F} \to \mathcal{F}_{\mathbf{dm}}$ are both gerbes, then $\mathcal{F} \to U$ is also a gerbe, which must then be trivial.

2. Second, we note that we may assume that X is connected. Indeed, let X be a disjoint union of connected components $X = X_1 \sqcup \cdots \sqcup X_n$; since $[X/G]$ is connected, the map $X_1 \times G \xrightarrow{m} X$ is surjective. Let $G_1 \leq G$ be the maximal subgroup sending X_1 to itself, defined by the cartesian square

$$\begin{tikzcd}
Z = X_1 \times G_1 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_1 \times G \arrow[r, "m"] & X
\end{tikzcd}$$

More precisely, it is readily checked that Z is an open and closed subgroup functor of $X_1 \times G$; moreover a flat subgroup of a group which is constant or of multiplicative type is also of the same type; if the base is connected then it must come from a subgroup of G, which we call G_1.

Then $X_1 \times G \xrightarrow{m} X$ is a G_1-principal bundle, so that $X_1 \times_{G_1} G \to X$ is an isomorphism. Taking $G-$quotients we get that

$$[X_1/G_1] \longrightarrow [X/G]$$

is an isomorphism, so me can assume without loss of generality that $X = X_1$.

Now let H be the stabilizer of a generic point of X; we may assume that H is a normal subgroup of G. Indeed, if G is of multiplicative type this follows from the fact that the stabilizer of any generic point of X must be of multiplicative type and come from a subgroup of G; moreover G is abelian, so H is certainly normal.

If G is discrete then X is integral (as it is étale over a normal stack, hence normal itself), so H is the stabilizer of the unique generic point of X (and as before comes from a subgroup of G); again, H must be normal in G.

Let now $Z \subseteq X$ be the fixed locus of H, which is closed. Then we have that Z is G-invariant, since H is normal in G; but $[X/G]$ is reduced (since it is normal) and $[Z/G]$ is a closed substack containing the generic point of $|X/G|$: this implies that $[Z/G] = [X/G]$. We conclude that $Z = X$, as we wanted.

Summarizing, H acts trivially on X, so there is a morphism

$$[X/G] \longrightarrow [X/(G/H)]$$

This is easily seen to be the rigidification map.

Still working étale-locally and mantaining the above notation we see that the rigidification $\mathcal{G}$ is again normal. Indeed, assuming again without loss of generality that H is normal, we have $\mathcal{G} = [X/(G/H)]$ when $\mathcal{F} = [X/G]$; taking a faithful representation $G \to \mathrm{GL}_n$ we can write $\mathcal{G} = [(X \times_G \mathrm{GL}_n)/(\mathrm{GL}_n/H)]$. But GL_n/H is smooth and thus the map $X \times_G \mathrm{GL}_n \to \mathcal{G}$ is smooth; the former scheme is normal by assumption and we conclude that $\mathcal{G}$ is normal as well.

Then we have that $\mathcal{G} \to U$ is generically an isomorphism and admits a section, that must be finite and representable. Since $\mathcal{G}$ is normal we conclude that $U \to \mathcal{G}$ is an isomorphism. In particular $\mathcal{G}$ is a scheme.

Finally, $\mathcal{F} \to U$ is a gerbe over a scheme with a section, so it is a trivial gerbe, as we wanted. ♠

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