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Filling Volumes and Simplicial Volume of Mapping Tori

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Candidata
dr.ssa Federica Bertolotti

Relatore	Supervisione interna
Prof. Roberto Frigerio	Prof. Andrea Malchiodi

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Introduction

Simplicial volume is a homotopy invariant of topological oriented compact manifolds introduced by Gromov in his proof of Mostow rigidity [Gro82]. Since then, it has garnered significant attention, with numerous results emphasizing its deep connections to the geometry of manifolds (see, e.g., [CW20, GBC91, BR21]). This invariant is notoriously difficult to compute, and exact values are known only in a limited number of cases, such as closed hyperbolic manifolds [Gro82] or closed manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$ [BK07].

If a closed manifold $E = F \times B$ is a product of two manifolds F and B, its simplicial volume is positive if and only if the simplicial volumes of both F and B are positive. The situation becomes more complex when E is a fiber bundle with base B and fiber F. Precise estimates are available for specific cases, such as surface bundles [HK01, Buc09, BC21] or manifolds with a connected structure group [Kas]. Furthermore, it is known that the vanishing of the simplicial volume of E is implied by the amenability of the fundamental group of F [Gro82], but not necessarily by the vanishing of the simplicial volume of F [KR23].

On the other hand, the amenability of the fundamental group of B does not guarantee the vanishing of the simplicial volume of E, even in the case when the base $B = S^1$ is the circle (e.g., hyperbolic 3–manifolds that fiber over S^1). Bucher and Neofytidis posed the following question in [BN20]:

Question 1. *Let M be an oriented closed manifold of dimension greater than two. When does the simplicial volume of an M–bundle over an oriented closed manifold with an amenable fundamental group vanish?*

The filling volume is a numerical invariant defined for orientation–preserving self–homotopy equivalences of oriented closed manifolds, as introduced in [BF24b]. It can be regarded, in simple terms, as a type of stable filling norm for maps. This invariant is of particular interest because it provides an alternative method to estimate the simplicial volume of manifolds that fiber over the circle by using the monodromy and a fundamental cycle of the fiber as inputs. The objective of this thesis is to define and investigate the filling volume for orientable compact manifolds.

Notation. All manifolds considered in this thesis are oriented, compact, and topological. The set of natural numbers $\mathbb{N}$ consists of all integers $n \geq 1$.

Integral and Real Filling Volume

Let $R = \mathbb{R}$ or $\mathbb{Z}$, and let M be an orientable compact n–dimensional manifold with (possibly empty) boundary ∂M. Let $C_i(M, \partial M; R)$ denote the complex of singular chains on M relative to the boundary ∂M with coefficients in R, and for every $i \in \mathbb{N}$ denote by $Z_i(M, \partial M; R) \subseteq C_i(M, \partial M; R)$ (resp. $B_i(M, \partial M; R) \subseteq C_i(M, \partial M; R)$) the subspace of cycles (resp. boundaries) in degree i. We endow $C_i(M, \partial M; R)$ with the usual ℓ^1–norm (see Section 1.1).

On the space $B_i(M, \partial M; R)$ of boundaries there is also defined the *filling norm* $\|\cdot\|_{\mathrm{fill},R}$ such that, for every $z \in B_i(M, \partial M; R)$, we have that

$$\|z\|_{\mathrm{fill},R} = \inf\{\|b\|_1 \,|\, b \in C_{i+1}(M, \partial M; R)\,,\ \partial b = z\}.$$

Recall that the relative singular homology $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the *fundamental class* $[M]_{\mathbb{Z}}$ of M relative to ∂M (in the literature usually denoted by $[M, \partial M]_{\mathbb{Z}}$). An integral fundamental cycle for M (or $\mathbb{Z}$–fundamental cycle) is any representative of $[M]_{\mathbb{Z}}$. The image of the fundamental class $[M]_{\mathbb{Z}}$ via the change of coefficient map $H_n(M, \partial M; \mathbb{Z}) \to H_n(M, \partial M; \mathbb{R})$ is denoted by $[M]_{\mathbb{R}} \in H_n(M, \partial M; \mathbb{R})$ and is called *real fundamental class* of M. A *real* fundamental cycle (or $\mathbb{R}$–fundamental cycle) of M is any representative of $[M]_{\mathbb{R}}$ in $Z_n(M, \partial M; \mathbb{R})$.

If $f\colon (M, \partial M) \to (M, \partial M)$ is a continuous map of pairs, then we denote by $f_*\colon C_\bullet(M, \partial M; R) \to C_\bullet(M, \partial M; R)$ the induced map on singular chains. The map f_* is norm non–increasing both with respect to the ℓ^1–norm, and (on the subspace of boundaries) with respect to the filling norm (see Section 2.1 for details).

Definition 2. Let $f\colon (M, \partial M) \to (M, \partial M)$ be an orientation–preserving self–homotopy equivalence of an oriented compact manifold M preserving the boundary ∂M, and let $z \in C_n(M, \partial M; R)$ be an R–fundamental cycle for M. The R–filling volume of f with coefficient in R, denoted by $\mathrm{FV}_R(f)$, is defined as

$$\mathrm{FV}_R(f) = \lim_{m \to \infty} \frac{\|f_*^m(z) - z\|_{\mathrm{fill},R}}{m}.$$

In Section 2.1 it is proved that the invariant $\mathrm{FV}_R(f)$ is well defined, i.e., the above limit exists, is finite, and does not depend on the choice of the fundamental cycle z.

We denote by $\mathrm{MCG}(M, \partial M)$ the (positive) mapping class group of M, i.e., the group of homotopy classes (relative to the boundary ∂M) of orientation–preserving self–homotopy equivalences of M that preserve the boundary ∂M (in the literature, the mapping class group often denotes the group of isotopy classes of orientation–preserving self–homeomorphisms; since there is a natural homomorphism between this last group and $\mathrm{MCG}(M, \partial M)$ as defined above, our invariants are also defined on isotopy classes of orientation–preserving self–homeomorphisms of M preserving the boundary ∂M). See Section 1.2 for details about the definition of mapping class group.

As proved in [BF24b] in the closed case, it turns out that $\mathrm{FV}_R(f)$ only depends on the homotopy class of f (see Proposition 2.4). With a small abuse of notation, for every $\varphi \in \mathrm{MCG}(M, \partial M)$ we denote by $\mathrm{FV}_R(\varphi)$ also the value $\mathrm{FV}_R(f)$, where f is any representative of φ. In this way, the invariant FV_R defines a map

$$\mathrm{FV}_R \colon \mathrm{MCG}(M, \partial M) \to [0, +\infty).$$

The filling volume becomes a real useful tool when studying the simplicial volume, of which we recall here the definition.

Definition 3 (Definition 1.3)**.** The *R–simplicial volume* of an oriented compact manifold M with (possibly empty) boundary ∂M is

$$\|M\|_R = \inf\{\|z\|_1 \mid z \text{ is an } R\text{–fundamental cycle for } (M, \partial M)\}.$$

The real simplicial volume $\|M\|_{\mathbb{R}}$ is just the classical simplicial volume as defined by Gromov in [Gro82], and it is usually denoted simply by $\|M\|$. Henceforth, when omitting the choice of the coefficients we understand that $R = \mathbb{R}$.

For every homeomorphism $f \colon (M, \partial M) \to (M, \partial M)$ of M preserving the boundary ∂M, let us denote by M_f the mapping torus of f, i.e., the manifold obtained from $M \times [0, 1]$ by identifying $(x, 0)$ with $(f(x), 1)$ for every $x \in M$ (Section 1.3).

The following result (proved for closed manifolds in [BF24b] when $R = \mathbb{R}$, and in [BF24a] when $R = \mathbb{Z}$) establishes a very neat relationship between $\mathrm{FV}_R(f)$ and the R–simplicial volume of the mapping torus M_f of f.

Theorem 4 (Theorem 2.7 and Corollary 2.8)**.** *Let M be an orientable compact manifold with (possibly empty) boundary ∂M, and let $f \colon (M, \partial M) \to (M, \partial M)$ be an orientation–preserving homeomorphism preserving the boundary ∂M. Then*

$$\mathrm{FV}_R(f) = \lim_{k \to \infty} \frac{\|M_{f^k}\|_R}{k}.$$

In particular,

$$\mathrm{FV}_{\mathbb{R}}(f) = \|M_f\|.$$

Now that we have this neat relation, the R–filling volume may appear redundant inside the literature. However, we highlight here the following: the filling volume $\mathrm{FV}_R(f)$ is computed by working on an n–dimensional manifold M, while $\|M_{f^m}\|_R$ deals with simplices in a $(n+1)$–dimensional manifold. In this sense, the filling volume allows us to reduce of 1 the dimension of the problem of the study of the simplicial volume of mapping tori. In Chapter 3 we will rely on this fact to compute the integral filling volume $\mathrm{FV}_{\mathbb{Z}}(f)$ of an orientation–preserving self–homeomorphism $f\colon S \to S$ of a closed surface: indeed, the dimension 2 allows us to explicitly construct chains that compute the integral filling volume of f.

Real Filling Volume

Thanks to Theorem 4, one may employ the knowledge of the R–simplicial volume of manifolds that fiber over the circle to get information about the R–filling volume. For example, in the case of surfaces it is possible to completely understand the vanishing or non–vanishing of $\mathrm{FV}_{\mathbb{R}}$. Indeed, it is well–known that every self–homotopy equivalence of a closed surface is homotopic to a homeomorphism, and the simplicial volume of mapping tori of surfaces is completely understood.

Theorem 5 (Theorem 3.1)**.** *Let $\varphi \in \mathrm{MCG}(S, \partial S)$, where S is an orientable compact surface with (possibly empty) boundary ∂S. Then $\mathrm{FV}_{\mathbb{R}}(\varphi) > 0$ if and only if S admits a hyperbolic metric and φ is pseudo–Anosov or virtually pseudo–Anosov.*

See Section 3.1 for the definition of (virtually) pseudo–Anosov maps.

On the contrary, it is shown in [BN20] that the simplicial volume of any orientable closed 4–dimensional manifold fibering over the circle vanishes. This already shows that, if M is an orientable closed 3–manifold and φ is an element of $\mathrm{MCG}(M)$ which may be represented by a homeomorphism, then $\mathrm{FV}_{\mathbb{R}}(\varphi) = 0$. Moreover, as shown in [Ber] and proved in Chapter 4, every orientation–preserving self–homotopy equivalence of an orientable closed 3–manifold admits a power homotopic to a homeomorphism. Building on these two results and on the fact that $\mathrm{FV}_R(\varphi^n) = |n| \, \mathrm{FV}_R(\varphi)$ for every $\varphi \in \mathrm{MCG}(M)$ and $n \in \mathbb{Z}$ (Proposition 2.5), we deduce the following result.

Theorem 6 (Theorem 4.1)**.** *For every orientable closed 3–dimensional manifold M and every $\varphi \in \mathrm{MCG}(M)$ we have $\mathrm{FV}_{\mathbb{R}}(\varphi) = 0$.*

As shown in [BN20], 2 and 4 are the only dimensions in which every mapping torus has zero simplicial volume. By translating this result in the language of the real filling volume, one gets the following.

Theorem 7 (Theorem 5.1)**.** *Let n be a positive integer, $n \neq 1, 3$. Then, there exist an orientable closed n–manifold M and a class $\varphi \in \mathrm{MCG}(M)$ such that* $\mathrm{FV}_{\mathbb{R}}(\varphi) > 0$.

Integral Filling Volume, (Stable) Integral Simplicial Volume and Delta–complexity

While the simplicial volume is closely related to the geometric structures that a manifold M can carry, integral simplicial volume $\|M\|_{\mathbb{Z}}$ and Δ–complexity $\Delta(M)$, i.e., the minimal number of simplices in a (loose) triangulation of M (see Definition 3.13), are more combinatorial in nature, and may be exploited to estimate algebraic topological invariants: for example, both Δ–complexity and integral simplicial volume provide upper bounds for Betti numbers, hence for the Euler characteristic. Since the Euler characteristic is multiplicative with respect to finite coverings, it is bounded from above also by the stable Δ–complexity and by the stable integral simplicial volume (which are obtained by stabilizing the corresponding invariants over finite coverings). These facts, together with a long–standing conjecture by Gromov on the relationship between simplicial volume and Euler characteristic for aspherical manifolds, have driven a lot of attention to stable integral simplicial volume, which has recently been proven to be equal to the classical simplicial volume for all aspherical 3–manifolds [FLMQ21] (in higher dimension, stable integral simplicial volume is not equal to simplicial volume even on the smaller class of hyperbolic manifolds [FFM12]).

A consequence of Theorem 4 is that

$$\|M_f\|_{\mathbb{Z}}^{\infty} \leq \mathrm{FV}_{\mathbb{Z}}(f) \leq \|M_f\|_{\mathbb{Z}} \leq \Delta(M_f),$$

where $\|M_f\|_{\mathbb{Z}}^{\infty}$ denotes the stable integral simplicial volume of M.

It is clear that the integral filling volume $\mathrm{FV}_{\mathbb{Z}}(f)$ of an orientation–preserving self–homeomorphism $f \colon M \to M$ and the integral simplicial volume $\|M_f\|_{\mathbb{Z}}$ of the corresponding mapping torus are very distinct invariants: for example, the integral simplicial volume is always positive, while the integral filling volume can also be zero (for example on the identity map).

The rest of the introduction is devoted to the comparison between $\mathrm{FV}_{\mathbb{Z}}(f)$ and $\|M_f\|_{\mathbb{Z}}^{\infty}$, and between integral simplicial volume and Δ–complexity.

Integral Filling Volume and Stable Integral Simplicial Volume

As we saw above, for every oriented closed n–manifold M and every orientation–preserving homeomorphism $f \colon M \to M$, it holds that $\|M_f\|_{\mathbb{Z}}^{\infty} \leq \mathrm{FV}_{\mathbb{Z}}(f)$. On the other hand, a result in [BF24b] establishes that a linear upper bound on $\mathrm{FV}_{\mathbb{Z}}(f)$ in terms of $\|M_f\|_{\mathbb{Z}}^{\infty}$ is not possible.

Theorem 8 (Theorem 5.2)**.** *For every $n \geq 2$ there exist an oriented closed n–manifold M and an orientation–preserving homeomorphism $f\colon M \to M$ such that*

$$\mathrm{FV}_{\mathbb{Z}}(f) > 0\,, \qquad \|M_f\|_{\mathbb{Z}}^{\infty} = 0.$$

The cases $n = 2$ and $n \geq 3$ are considered separately in the proof: in the latter one, we consider the product $M = S^1 \times S \times X$, where S is a hyperbolic surface and X is any manifold, and we build a suitable map $f\colon M \to M$ satisfying the theorem (see Proposition 5.7). The case $n = 2$ is, instead, a consequence of the following vanishing criterion for integral filling volume of maps of surfaces.

Theorem 9 (Theorem 3.4)**.** *Let S be an oriented compact surface, and let $f\colon S \to S$ be an orientation–preserving self–homeomorphism of S. Then $\mathrm{FV}_{\mathbb{Z}}(f) > 0$ if and only if one of the following conditions is satisfied:*

- *the surface S is a torus and f is Anosov;*
- *the surface S admits an hyperbolic metric and f is a pseudo–Anosov or a virtually pseudo–Anosov homeomorphism.*

Notice that if $f\colon T \to T$ is an Anosov map on the torus, then the corresponding mapping torus is a sol 3–manifold, thus with zero stable integral simplicial volume [Som81].

Integral Simplicial Volume and Delta–Complexity

Let us say that two numerical invariants h_1, h_2 defined on the class of oriented closed n–dimensional manifolds are *equivalent* if there exists a constant $k \geq 1$ such that $h_1(M) \leq k \cdot h_2(M)$ and $h_2(M) \leq k \cdot h_1(M)$ for every oriented closed n–manifold.

Integral simplicial volume and Δ–complexity are equivalent (in fact, they coincide) in dimension one (trivially, since the only closed 1–manifold is S^1 and $\Delta(S^1) = \left\|S^1\right\|_{\mathbb{Z}} = 1$) and in dimension two, since $\Delta(S^2) = \left\|S^2\right\|_{\mathbb{Z}} = 2$ and $\Delta(S_g) = \|S_g\|_{\mathbb{Z}} = 4g - 2$ for every oriented closed surface S_g of genus $g \geq 1$ (see [Löh18, Proposition 4.3]).

In [BF24a] the following is proved.

Theorem 10 (Theorem 3.14)**.** *In dimension $n = 3$, integral simplicial volume and Δ–complexity are not equivalent. More precisely, there exists a sequence $\{M_i\}_{i\in\mathbb{N}}$ of oriented closed 3–manifolds such that*

$$\lim_{i\to\infty} \frac{\Delta(M_i)}{\|M_i\|_{\mathbb{Z}}} = +\infty.$$

The manifolds appearing in the sequence described in Theorem 10 are all torus bundles over the circle: let $T = S^1 \times S^1$ be the 2–dimensional torus and let $f\colon T \to T$ be a Dehn twist along a nontrivial curve in T. Then the manifold M_i mentioned in Theorem 10 is the mapping torus of f^i, i.e., $M_i = T_{f^i}$.

The proof relies on Theorem 9 and on the separate study of the Δ–complexity. Indeed, if $f\colon T \to T$ is an infinite–order map on the torus, then it can be proved, using techniques developed by Lackenby and Purcell in [LP24b, LP24a], that the Delta–complexity of T_{f^i} grows linearly with i (see Theorem 3.16).

In dimension $n \geq 4$ the Δ–complexity is not always well defined, as there exist manifolds which do not admit any triangulation. One could then try to restrict the problem of the equivalence between integral simplicial volume and Δ–complexity to the category of orientable closed *triangulable* n–manifolds. However, in this case some pathologies make the problem much easier (and less elementary): for $n \geq 5$ there is an infinite set of n–manifolds that are pairwise homotopically equivalent, but not homeomorphic; in particular, the integral simplicial volume is constant on this set, while the Δ–complexity is unbounded. See Section 3.4 for further details.

Further Works

Further projects undertaken by the author during the PhD include:

- *The action of mapping class groups on de Rham quasimorphisms* — a collaboration with Giuseppe Bargagnati, Pietro Capovilla, and Francesco Milizia [BBCM24].

- *Dehn functions of subgroups of products of free groups: 3–factor case, F_{n-1} case, and Bridson–Dison group* — a collaboration with Dario Ascari, Giovanni Italiano, Claudio Llosa Isenrich, and Matteo Migliorini [ABI$^+$].

Organization of this Thesis

In Chapter 1, we introduce the classical concepts necessary for the study of the filling volume. These include (relative) singular homology and related definitions, simplicial volume and its variations, the mapping class group of a manifold, and mapping tori.

In Chapter 2, we define the R–filling volume for any normed ring R with unity. In the same chapter, we establish basic properties of this invariant and prove Theorem 4.

Chapter 3 focuses on the study of real and integral filling volumes of orientation–preserving self–homeomorphisms of compact surfaces. Leveraging the extensive literature on 3–manifolds, we apply Theorem 4 to establish Theorem 5.

In the second part of Chapter 3, we prove Theorem 9. The proof employs various techniques tailored to different types of maps: analyzing homological torsion for Anosov maps on the torus, comparing real filling volumes for pseudo–Anosov maps on hyperbolic surfaces, and constructing explicit efficient cycles for Dehn twists. We use this result, along with techniques developed by Lackenby and Purcell in [LP24b, LP24a], to compare integral simplicial volume and Delta–complexity, proving Theorem 10.

In Chapter 4, we study the real filling volume of the mapping class group of 3–manifolds. A result by Bucher and Neofytidis shows that the simplicial volume of any orientable closed 4–manifold that fibers over the circle is zero [BN20]; this directly implies that the real filling volume of orientation–preserving self–homeomorphisms of orientable closed 3–manifolds is also zero. The remainder of the chapter is dedicated to proving that every self–homotopy equivalence of a closed 3–manifold admits a power homotopic to a homeomorphism, thereby establishing Theorem 6.

Chapter 5 presents a collection of results about the R–filling volume that hold in all dimensions. In particular, we prove Theorem 7 and Theorem 8. Additionally, we examine the special cases where the n–manifold satisfies the n–uniform boundary condition or has a Gromov–hyperbolic fundamental group.

Chapter 1

Preliminaries

In this chapter, we present the classical tools required to define and study the filling volume. For the benefit of expert readers who may already be familiar with some or all of the topics discussed here, we provide a summary of the chapter's structure below.

In Section 1.1, we review the definition of (relative) homology, ℓ^1–norm, and (variations of) simplicial volume of manifolds. Along the way, we also introduce the filling norm on the space of boundaries of the singular chain complex.

In Section 1.2, we discuss how maps between spaces induce maps in homology and how these affect the ℓ^1–norm and the filling norm on the singular chain complex. In particular, we revisit the definition of the mapping class group and its action on the homology of a compact manifold.

Finally, in Section 1.3, we define mapping tori and establish some notation related to their cyclic coverings.

1.1 Relative Homology

Let R be a normed ring with unit (for example R=$\mathbb{Z}$ or $R = \mathbb{R}$) and let (X, A) be a topological pair, with $A \subset X$ possibly empty. In this thesis we will often consider $X = M$ be an oriented compact manifold and $A = \partial M$ its boundary. We denote by $C_\bullet(X; R)$ the complex of singular chains on X with coefficients in R, and we endowed it with the ℓ^1–norm: a chain $c \in C_k(X; R)$ is a formal sum $c = \sum_{i=1}^{t} a_i\sigma_i$, where $a_i \in R$ are the coefficients and $\sigma_i \colon \Delta^k \to X$ are singular simplices; if c is written in the reduced form, meaning that there are no possible simplifications in the sum, then the ℓ^1–norm of c is defined as

$$\|c\|_1 = \sum_{i \in I} |a_i|.$$

As $C_i(A;R)$ is closed inside $C_i(X;R)$, the complex of relative chains

$$C_i(X,A;R)=\frac{C_i(X;R)}{C_i(A;R)}$$

inheritances a norm, usually called ℓ^1–norm, and defined in the following way: if $\alpha\in C_i(X,A;R)$, then

$$\|\alpha\|_1=\inf\left\{\|c\|_1 \mid c\in C_i(X;R),\ \alpha=[c]\in C_i(X,A;R)\right\}.$$

We prove the following useful fact on the ℓ^1–norm on the set of relative chains.

Lemma 1.1. *Let $\alpha\in C_i(X,A;R)$ be a nontrivial relative chain of (X,A), and let c be the unique representative of α whose reduced expression contains only simplices whose image is not entirely contained in A. Then*

$$\|\alpha\|_1=\|c\|_1.$$

Proof. The representative c of α as in the statement always exists: let c' be any representative of α with reduced expression given by

$$c'=\sum_{i\in I}a_i\sigma_i$$

and set

$$I_A=\{i\in I|\,\mathrm{im}(\sigma_i)\not\subseteq A\}\subseteq I.$$

Then $c=\sum_{i\in I_A}a_i\sigma_i$ is the desired representative of c, as $c-c'\in C_i(A;R)$.

Let w be any other representative of α; then $a\coloneqq w-c\in C_i(A;R)$, implying, by definition of c, that

$$\|w\|_1=\|c+a\|_1=\|c\|_1+\|a\|_1\geq\|c\|_1;$$

this proves the statement. □

Notation. To simplify the notation, in the following we will often denote a chain $c\in C_\bullet(X,A;R)$ with the same symbol as its preferred (in the sense of Lemma 1.1) representative $c\in C_\bullet(X;R)$.

We denote by $\partial_d\colon C_d(X;R)\to C_{d-1}(X;R)$ the usual boundary operator on singular chains. With this notation, $(C_\bullet(A;R),\partial_\bullet)$ is a subcomplex of the singular complex $(C_\bullet(X;R),\partial_\bullet)$; thus, the map $\partial_d\colon C_d(X,A;R)\to C_{d-1}(X,A;R)$, given for a chain $c\in C_d(X;R)$ by $\partial_d[c]=[\partial_d c]$, is a well–defined boundary operator that makes $(C_\bullet(X,A;R),\partial_\bullet)$ into a normed chain complex. Whenever the degree d of the boundary operator is clear from the context, we will omit it.

For every $d \in \mathbb{N}$ we denote by $Z_d(X, A; R) \subseteq C_d(X, A; R)$ the space of relative cycles of degree d, i.e., the kernel of ∂_d. Notice that a representative $z \in C_d(X; R)$ of an element $[z] \in Z_d(X, A; R)$ is such that $\partial z \in C_{d-1}(A; R)$.

We denote by $B_d(X, A; R) \subseteq C_d(X, A; R)$ the subspace of degree d relative boundaries, i.e., the image of the map ∂_{d+1}. Again, a representative b of an element $[b] \in B_d(X, A; R)$ is a boundary modulo $C_d(A; R)$, meaning that there exist $z \in C_{d+1}(X; R)$ and $a \in C_d(A; R)$ such that $b = \partial z + a$.

In this context the relative homology of the pair (X, A) is defined as

$$H_d(X, A; R) = Z_d(X, A; R)/B_d(X, A; R).$$

If $\alpha \in H_d(X, A; R)$ is a class in singular homology with coefficient in R, then the ℓ^1–norm of α is defined as

$$\|\alpha\| = \inf \left\{ \|c\|_1 \mid c \in Z_d(X, A; R), \alpha = [c] \in H_d(X, A; R) \right\}.$$

Remark 1.2. Despite the name, the ℓ^1–norm is a semi–norm and generally not a norm on $H_d(X, A; \mathbb{R})$: indeed, it may happen that $\|\alpha\| = 0$ even if α is not the zero class.

(Variations of) Simplicial Volume

In this section we restrict ourself to the case in which $X = M$ is an oriented compact n–manifold and $A = \partial M$ is the (possibly empty) boundary of M. In this case, $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the *integral fundamental class* $[M]_{\mathbb{Z}}$ of M. The image of $[M]_{\mathbb{Z}}$ under the change of coefficient map $H_n(M, \partial M; \mathbb{Z}) \to H_n(M, \partial M; R)$ is called *R–fundamental class* and denoted by $[M]_R$. Any representative $z \in Z_n(M, \partial M; R)$ of the R–fundamental class ($[z] = [M]_R$) is called *R–fundamental cycle.*

Definition 1.3 (Definition 3)**.** The R–simplicial volume is then defined as the ℓ^1–norm of the R–fundamental class, i.e., $\|M\|_R = \|[M]_R\|_1$.

When the set of coefficients is not specified, then the object should be intended with real coefficients. For example, the real fundamental class and the real simplicial volume are just called *fundamental class* and *simplicial volume* and denoted by $[M]$ and $\|M\|$.

Moreover, in the non–closed case, the R–fundamental class and R–simplicial volume are often referred in the literature as *relative* R–fundamental class and *relative* R–simplicial volume and denoted by $[M, \partial M]_R$, $\|M, \partial M\|_R$. However, to the sake of readability, we will just keep the lighter notation above, without the specification of the boundary.

As the integral simplicial volume is not multiplicative under finite coverings, in the literature it is often considered its stabilized version, namely the *stable integral simplicial volume*:

$$\|M\|_{\mathbb{Z}}^{\infty} = \inf \left\{ \frac{\|N\|_{\mathbb{Z}}}{d} \;\middle|\; d \in \mathbb{N}, N \to M \text{ } d\text{–sheeted covering} \right\}.$$

Stable integral simplicial volume and real simplicial volume are stable under finite–coverings, that is $\|N\| = d\|M\|$ and $\|N\|_{\mathbb{Z}}^{\infty} = d\|M\|_{\mathbb{Z}}^{\infty}$ whenever $N \to M$ is a d–sheeted covering.

Filling Norm

Let us go back to the general case in which (X, A) is a topological pair. Using the ℓ^1–norm on the chain complex, it is also possible to define a norm on the space of boundaries $B_d(X, A; R)$: the *filling norm* $\|\cdot\|_{\mathrm{fill},R}$ of a relative boundary $\beta \in B_d(X, A; R)$ is defined as

$$\|\beta\|_{\mathrm{fill},R} = \inf\left\{\|\alpha\|_1 \mid \alpha \in C_{d+1}(X, A; R),\ \partial\alpha = \beta\right\}.$$

Lemma 1.4. *Let $\beta \in B_d(X, A; R)$ be a relative boundary of (X, A). Then*

$$\|\beta\|_{\mathrm{fill},R} \geq \frac{1}{d+2}\|\beta\|_1.$$

Proof. Let b be the representative of β whose reduced expression does not contain any simplex contained in A, so that $\|\beta\|_1 = \|b\|_1$ (Lemma 1.1). If $[z] \in C_{d+1}(X, A; R)$ is such that $\partial[z] = [b]$, then $\partial z = b + a$ for some $a \in C_d(A; R)$. Thus,

$$(d+2)\|z\|_1 \geq \|\partial z\|_1 = \|b + a\| \geq \|b\|_1 = \|\beta\|_1,$$

where the last inequality follows from the fact that the reduced expressions of b and of a share no common simplices. The statement is now a consequence of the definition of filling norm. □

As a direct consequence, we get that the filling norm is always positive on nontrivial boundaries.

Corollary 1.5. *The filling norm is a norm on $B_d(X, A; R)$.*

1.2 Maps in Homology

Let (X, A), (Y, B) be two pairs of topological spaces and let

$$f\colon (X, A) \to (Y, B)$$

be a continuous map between pairs, that is a map $f\colon X \to Y$ satisfying $f(A) \subseteq B$. The map f induces a map

$$f_*\colon C_\bullet(X; R) \to C_\bullet(Y; R),$$

on the chain $C_\bullet(X; R)$, that restricts to a map

$$f_*\big|_{C_\bullet(A;R)} = \left(f\big|_A\right)_*\colon C_\bullet(A; R) \to C_\bullet(B; R)$$

So, there is an induced map

$$f_*\colon C_\bullet(X,A;R) \to C_\bullet(Y,B;R)$$

on relative chains that is norm non–increasing. As the map f_* commutes with the boundary operator, it is a chain map that induces a norm non–increasing map in homology

$$H_d(f;R)\colon H_d(X,A;R) \to H_d(Y,B;R).$$

Moreover, $f_*|_{B_\bullet(X,A;R)}$ is norm non–increasing also with respect to the filling norm.

Two continuous maps $f,g\colon (X,A) \to (Y,B)$ are relatively homotopic if there exists a map of pairs

$$F\colon (X \times [0,1], A \times [0,1]) \to (Y;B)$$

such that $F|_{(X\times\{0\},A\times\{0\})} = f$ and $F|_{(X\times\{1\},A\times\{1\})} = g$. The map F is called *relative homotopy.* Associate to each relative homotopy, there is a chain homotopy

$$P_\bullet\colon C_\bullet(X,A;R) \to C_{\bullet+1}(Y,B;R)$$

satisfying $\partial \circ P = g_* - f_* - P \circ \partial$, so that $\partial P(z) = g_*(z) - f_*(z)$ for every cycle $z \in Z_\bullet(X,A;R)$. We refer to this map as the *prism map* associate to F. The existence of such map implies that the induced maps $H_\bullet(f;R)$ and $H_\bullet(g;R)$ coincide in homology. Moreover, the norm of P_d is not bigger than $(d+1)$, meaning that for every $\alpha \in C_d(X,A;R)$, it holds that $\|P(\alpha)\|_1 \leq (d+1)\cdot\|\alpha\|_1$.

Mapping Class Group

Let us move once again to the manifold setting. If M is an orientable compact manifold with (possibly empty) boundary ∂M, the (positive) *mapping class group*, denoted by $\mathrm{MCG}(M,\partial M)$, is the set of orientation–preserving relative self–homotopy equivalences $f\colon (M,\partial M) \to (M,\partial M)$, considered up to relative homotopy.

It follows from the discussion above that for every mapping class $\varphi \in \mathrm{MCG}(M,\partial M)$, there is a norm non–increasing isomorphism

$$H_\bullet(\varphi;R)\colon H_\bullet(M,\partial M;R) \to H_\bullet(M,\partial M;R)$$

induced by any representative $f\colon (M,\partial M) \to M,\partial M)$ of φ.

Remark 1.6. In the literature the mapping class group denotes often the isotopy classes of self–homeomorphisms of the manifold M.

1.3 Mapping Tori

Let M be a manifold with (possibly empty) boundary ∂M. For every self–homeomorphism $f\colon (M, \partial M) \to (M, \partial M)$, let us denote by M_f the mapping torus of f, i.e., the manifold

$$M_f = M \times [0,1]/\sim\ ,$$

where $(x, 0) \sim (f(x), 1)$ for every $x \in M$. The boundary of M_f is given by $\partial(M_f) = (\partial M)_{f|_{\partial M}}$. In the following chapters we will often omit the brackets or the restriction on the map when considering the boundary.

For every integer $d \in \mathbb{N}$, let us consider the mapping torus relative to the map f^d and written as

$$M_{f^d} = M \times [0, d]/\sim_d,$$

where $(x, 0) \sim (f^d(x), d)$ for every $x \in M$. We define the map

$$\pi_d\colon (M_{f^d}, \partial M_{f^d}) \to (M_f, \partial M_f),$$

which sends (x, t) to $(f^{-m}(x), t - m)$ for every integer $0 \leq m < d$ and every $t \in [m, m+1]$. This map is a degree d–covering which we refer as the *d–cyclic covering* over f.

The map,

$$\pi_\infty\colon (M \times \mathbb{R}, \partial M \times \mathbb{R}) \to (M_f, \partial M_f),$$

defined analogously as π_d, is called *infinite–cyclic covering* over f.

Chapter 2

Filling Volume

In this chapter, we introduce the main object of this thesis, namely the *Filling Volume.* This invariant, introduced in [BF24b], is defined on the set of orientation–preserving homotopy equivalences of an oriented compact manifold and it tries to capture the complexity of a map in terms of the induced action on fundamental cycles.

As in the previous chapter, R denotes a normed ring with unity.

2.1 Definition of Filling Volume

Let us consider $f\colon (M, \partial M) \to (M, \partial M)$ be an orientation–preserving self–homotopy equivalence of an orientable compact n–manifold M with (possibly empty) boundary ∂M. If $z \in Z_n(M, \partial M; R)$ is an R–fundamental cycle, then $[f_*(z)] = [z] \in H_n(M, \partial M; R)$, so that $z - f_*^k(z)$ is a boundary for every $k \in \mathbb{N}$; in particular, it makes sense to consider the filling norm $\left\| z - f^k(z) \right\|_{\text{fill},R}$.

Definition 2.1. Let M be an orientable compact n–manifold with (possibly empty) boundary ∂M, and let $f\colon (M, \partial M) \to (M, \partial M)$ be an orientation–preserving homotopy equivalence. The *R–filling volume* of M is

$$\mathrm{FV}_R(f) = \inf \left\{ \frac{\|f_*^m(z) - z\|_{\text{fill},R}}{m} \;\middle|\; [z] = [M]_R,\ m \in \mathbb{N} \right\}.$$

As for the simplicial volume, when the set of coefficients is not specified, then we refer to the *real* filling volume.

Remark 2.2. It follows directly from the definition that, for every orientation–preserving self–homotopy equivalence $f\colon (M, \partial M) \to (M, \partial M)$ of an orientable compact manifold M with boundary ∂M, it holds that

$$\mathrm{FV}_{\mathbb{R}}(f) \leq \mathrm{FV}_{\mathbb{Z}}(f).$$

We will see in the next chapters that there are examples for which the inequality is strict (Corollary 5.3).

With the following, we prove that Definition 2.1 of filling volume is equivalent to Definition 2 given in the introduction.

Proposition 2.3. *Let M, f be as in Definition 2.1 and consider an R–fundamental cycle $z \in Z_n(M, \partial M; R)$ of M. Then*

$$\mathrm{FV}_R(f) = \lim_{m\to\infty} \frac{\|f_*^m(z) - z\|_{\mathrm{fill},R}}{m}.$$

Proof. First, we prove that the numerator of the argument of the limit is subadditive: for every $m, m' \in \mathbb{N}$, we have

$$\begin{aligned}\left\|f_*^{m+m'}(z) - z\right\|_{\mathrm{fill},R} &\le \left\|f_*^{m+m'}(z) - f_*^m(z)\right\|_{\mathrm{fill},R} + \|f_*^m(z) - z\|_{\mathrm{fill},R}\\ &= \left\|f_*^m\left(f_*^{m'}(z) - z\right)\right\|_{\mathrm{fill},R} + \|f_*^m(z) - z\|_{\mathrm{fill},R}\\ &\le \left\|f_*^{m'}(z) - z\right\|_{\mathrm{fill},R} + \|f_*^m(z) - z\|_{\mathrm{fill},R}.\end{aligned}$$

By Fekete's Lemma,

$$\lim_{m\to\infty} \frac{\|f_*^m(z) - z\|_{\mathrm{fill},R}}{m} = \inf_{m\in\mathbb{N}} \frac{\|f_*^m(z) - z\|_{\mathrm{fill},R}}{m}.$$

We now prove that the limit (hence the infimum) does not depend on the fixed cycle z: let z' be another relative R–fundamental cycle for $(M, \partial M)$, so that $z - z' \in B_n(M, \partial M; R)$. Let $c \in C_{n+1}(M, \partial M; R)$ such that $\partial c = z - z'$. Thus

$$\begin{aligned}\left|\|f_*^m(z) - z\|_{\mathrm{fill},R} - \left\|f_*^m(z') - z'\right\|_{\mathrm{fill},R}\right| &\le \left\|(f_*^m(z) - z) - (f_*^m(z') - z')\right\|_{\mathrm{fill},R}\\ &\le \left\|f_*^m(z - z')\right\|_{\mathrm{fill},R} + \left\|z - z'\right\|_{\mathrm{fill},R}\\ &\le \|f_*^m(c)\|_1 + \|c\|_1\\ &\le 2\|c\|_1.\end{aligned}$$

By dividing by m and computing the limit for $m \to \infty$, one gets

$$\lim_{m\to\infty} \frac{\|f_*^m(z) - z\|_{\mathrm{fill},R}}{m} = \lim_{m\to\infty} \frac{\|f_*^m(z') - z'\|_{\mathrm{fill},R}}{m},$$

proving the statement. ☐

The following proposition shows that the invariant FV_R defined on the set of orientation–preserving homotopy equivalences yields a well–defined invariant on mapping class groups.

Proposition 2.4. *Let M be an orientable compact manifold with (possibly empty) boundary ∂M and let $f, g\colon (M, \partial M) \to (M, \partial M)$ be two orientation–preserving self–homotopy equivalences. If f and g are homotopic relative to ∂M then*

$$\mathrm{FV}_R(f) = \mathrm{FV}_R(g).$$

Proof. Let us fix $k \in \mathbb{N}$. Since f and g are homotopic, f^k and g^k are homotopic as well. Let $P_\bullet\colon C_\bullet(M, \partial M; R) \to C_{\bullet+1}(M, \partial M; R)$ be the prism operator associated to a relative homotopy between f^k and g^k (see Section 1.2). Recall that the norm of P_n is not bigger than $n+1$.

Let z be a fundamental cycle for $(M, \partial M)$. Then

$$\begin{aligned}\left\|(f^k_*(z) - z) - (g^k_*(z) - z)\right\|_{\mathrm{fill},R} &= \left\|f^k_*(z) - g^k_*(z)\right\|_{\mathrm{fill},R} \\ &= \|\partial P_n(z)\|_{\mathrm{fill},R} \leq \|P_n(z)\|_1 \leq (n+1)\|z\|_1,\end{aligned}$$

which implies

$$\left|\frac{\|f^k_*(z) - z\|_{\mathrm{fill},R}}{k} - \frac{\|g^k_*(z) - z\|_{\mathrm{fill},R}}{k}\right| \leq \frac{(n+1)\|z\|_1}{k},$$

whence the conclusion. ☐

With a slight abuse of notation, we will call *R–filling volume* also the map

$$\mathrm{FV}_R\colon\ \mathrm{MCG}(M, \partial M) \to \mathbb{R}$$

defined for $\varphi \in \mathrm{MCG}(M, \partial M)$ by $\mathrm{FV}_R(\varphi) = \mathrm{FV}_R(f)$, where f is an orientation–preserving homotopy equivalence representing the mapping class φ.

2.2 Properties of Filling Volume

In this section, we prove that the R–filling volume on $\mathrm{MCG}(M, \partial M)$ is a length function.

Proposition 2.5. *Let M be an orientable compact manifold with (possibly empty) boundary ∂M, and consider $\varphi, \psi \in \mathrm{MCG}(M, \partial M)$ and $k \in \mathbb{Z}$. It holds that*

1. $\mathrm{FV}_R(\varphi^k) = |k|\,\mathrm{FV}_R(\varphi)$;
2. $\mathrm{FV}_R(\varphi\psi\varphi^{-1}) = \mathrm{FV}_R(\psi)$;
3. *if* $\varphi\psi = \psi\varphi$, *then* $\mathrm{FV}_R(\varphi\psi) \leq \mathrm{FV}_R(\varphi) + \mathrm{FV}_R(\psi)$.

Proof. Let $f, g\colon (M, \partial M) \to (M, \partial M)$ be two orientation–preserving homotopy equivalences representing φ and ψ, and let h be a homotopy inverse of f (so that it represents the mapping class φ^{-1}). Consider an R–fundamental cycle z for $(M, \partial M)$.

Item 1. By definition of filling volume, we clearly have $\mathrm{FV}_R(\varphi^m) = m\,\mathrm{FV}_R(\varphi)$ for every $m \in \mathbb{N}$, so it is enough to prove $\mathrm{FV}_R(\varphi^{-1}) = \mathrm{FV}_R(\varphi)$. To this end, fix $k \in \mathbb{N}$. Note that h^k is the homotopy inverse of f^k, and let

$$P_\bullet\colon C_\bullet(M, \partial M; R) \to C_{\bullet+1}(M, \partial M; R)$$

be the prism operator associated with some relative homotopy between $h^k \circ f^k$ and id_M. As observed previously, P_n has norm at most $n+1$, thus

$$\begin{aligned}
\left\| h^k_*(z) - z \right\|_{\mathrm{fill},R} &= \left\| h^k_*(z) - h^k_*(f^k_*(z)) - (z - h^k_*(f^k_*(z))) \right\|_{\mathrm{fill},R} \\
&\leq \left\| h^k_*(z) - h^k_*(f^k_*(z)) \right\|_{\mathrm{fill},R} + \left\| z - h^k_*(f^k_*(z)) \right\|_{\mathrm{fill},R} \\
&\leq \left\| z - f^k_*(z) \right\|_{\mathrm{fill},R} + \|\partial P_n(z)\|_{\mathrm{fill},R} \\
&\leq \left\| f^k_*(z) - z \right\|_{\mathrm{fill},R} + \|P_n(z)\|_1 \\
&\leq \left\| f^k_*(z) - z \right\|_{\mathrm{fill},R} + (n+1)\|z\|_1.
\end{aligned}$$

By dividing by k and taking the limit for k going to infinity, one gets the inequality $\mathrm{FV}_R(h) \leq \mathrm{FV}_R(f)$. The opposite inequality follows analogously, thus $\mathrm{FV}_R(h) = \mathrm{FV}_R(f)$.

Item 2. Let $T_\bullet\colon C_\bullet(M, \partial M; R) \to C_{\bullet+1}(M, \partial M; R)$ be the prism operator associated to some relative homotopy between $h \circ f$ and id_M. Observe that for every $k \in \mathbb{N}$ the map $\left(\varphi\psi\varphi^{-1}\right)^k = \varphi\psi^k\varphi^{-1}$ is represented by the map $f \circ g^k \circ h$. By setting $z' = h_*(z)$, we have

$$\begin{aligned}
\left\| f_*(g^k_*(h_*(z))) - z \right\|_{\mathrm{fill},R} &= \left\| f_*(g^k_*(h_*(z))) - f_*(h_*(z)) + f_*(h_*(z)) - z \right\|_{\mathrm{fill},R} \\
&\leq \left\| f_*(g^k_*(z')) - f_*(z') \right\|_{\mathrm{fill},R} + \| f_*(h_*(z)) - z) \|_{\mathrm{fill},R} \\
&= \left\| f_*(g^k_*(z') - z') \right\|_{\mathrm{fill},R} + \|\partial T_n(z)\|_{\mathrm{fill},R} \\
&\leq \left\| g^k_*(z') - z' \right\|_{\mathrm{fill},R} + \|T_n\,(z)\|_1 \\
&\leq \left\| g^k_*(z') - z' \right\|_{\mathrm{fill},R} + (n+1)\|z\|_1.
\end{aligned}$$

Hence, dividing by k and taking the limit for k going to infinity, one gets $\mathrm{FV}_R(\varphi\psi\varphi^{-1}) \leq \mathrm{FV}_R(\psi)$. On the other hand,

$$\mathrm{FV}_R(\psi) = \mathrm{FV}_R(\varphi^{-1}\left(\varphi\psi\varphi^{-1}\right)\varphi) \leq \mathrm{FV}_R(\varphi\psi\varphi^{-1}),$$

whence the conclusion.

Item 3. Since $\varphi\psi = \psi\varphi$, for every $k \in \mathbb{N}$ the map $(fg)^k$ is homotopic to $f^k g^k$. If $S_\bullet \colon C_\bullet(M, \partial M; R) \to C_{\bullet+1}(M, \partial M; R)$ is the prism operator associated to some homotopy between $(fg)^k_*$ and $f^k_* g^k_*$, then we have

$$\begin{aligned}
\left\|(fg)^k_*(z) - z\right\|_{\mathrm{fill},R} &= \left\|(fg)^k_*(z) - f^k_*(g^k_*(z)) + f^k_*(g^k_*(z)) - z\right\|_{\mathrm{fill},R} \\
&\le \left\|(fg)^k_*(z) - f^k_*(g^k_*(z))\right\|_{\mathrm{fill},R} + \left\|f^k_*(g^k_*(z)) - z\right\|_{\mathrm{fill},R} \\
&= \|\partial S_n(z)\|_{\mathrm{fill},R} + \left\|f^k_*(g^k_*(z) - z) + f^k_*(z) - z\right\|_{\mathrm{fill},R} \\
&\le \|S_n(z)\|_1 + \left\|f^k_*(g^k_*(z) - z)\right\|_{\mathrm{fill},R} + \left\|f^k_*(z) - z\right\|_{\mathrm{fill},R} \\
&\le (n+1)\|z\|_1 + \left\|g^k_*(z) - z\right\|_{\mathrm{fill},R} + \left\|f^k_*(z) - z\right\|_{\mathrm{fill},R} .
\end{aligned}$$

By dividing by k both sides of the previous inequality and taking the limit as $k \to +\infty$ we get $\mathrm{FV}_R(fg) \le \mathrm{FV}_R(f) + \mathrm{FV}_R(g)$, as desired. □

Remark 2.6. The statement of the previous lemma is equivalent to say that the R–filling volume is a length function according to the definition given in [Ye].

Notice that in the literature the definition of length function $\ell \colon G \to \mathbb{R}$ on a group G may be found with the additional condition that the inequality $\ell(gh) \le \ell(g) + \ell(h)$ holds for every $g, h \in G$ (see, for example [Pol18]). We will see that this last requirement fails to be true in general for the fillilng volume (Proposition 3.12).

2.3 Filling Volume and Simplicial Volume of Mapping Tori

The following result establishes a very neat relationship between $\mathrm{FV}_R(f)$ and the R–simplicial volume of the mapping torus of f.

Theorem 2.7 (Theorem 4)**.** *Let $R = \mathbb{R}$ or $R = \mathbb{Z}$. Let M be an orientable compact manifold with (possibly empty) boundary ∂M, and let $f \colon (M, \partial M) \to (M, \partial M)$ be an orientation–preserving self–homeomorphism of M preserving the boundary ∂M. Then*

$$\mathrm{FV}_R(f) = \inf_{k \in \mathbb{N}} \frac{\left\|M_{f^k}\right\|_R}{k}. \tag{2.1}$$

Before proving this result, let us state some straightforward consequences.

Consequences

As the real simplicial volume is multiplicative under finite covering and M_{f^k} is a k–sheeted covering of M_f, when $R = \mathbb{R}$ the right–hand side of Eq. (2.1) in Theorem 2.7 is the simplicial volume of M_f.

Corollary 2.8. *Let M be an orientable compact manifold with (possibly empty) boundary ∂M, and let $f\colon (M, \partial M) \to (M, \partial M)$ be an orientation–preserving self–homeomorphism of M preserving the boundary ∂M. Then*

$$\mathrm{FV}_{\mathbb{R}}(f) = \|M_f\|_{.}$$

If, instead, we consider $R = \mathbb{Z}$, then the right–hand side of Eq. (2.1) might appear similar to the definition of the stable integral simplicial volume: indeed, instead of taking the stabilization of the integral simplicial volume of M_f over all possible finite coverings, we are just considering stabilization over a special family of finite cyclic coverings. As a direct consequence, we have the following corollary.

Corollary 2.9. *Let M be an orientable compact manifold with (possibly empty) boundary ∂M, and let $f\colon (M, \partial M) \to (M, \partial M)$ be an orientation–preserving self–homeomorphism of M preserving the boundary ∂M. Then*

$$\|M_f\|_{\mathbb{Z}}^{\infty} \leq \mathrm{FV}_{\mathbb{Z}}(f) \leq \|M_f\|_{\mathbb{Z}}.$$

The rest of this section is devoted to the proof of Theorem 2.7. One of the two inequalities, namely $\inf_{k\in\mathbb{N}} \left\|M_{f^k}\right\|_R / k \leq \mathrm{FV}_R(f)$, admits a rather simple proof. The proof of the converse inequality is more involved, and boils down to showing that mapping tori admit efficient fundamental cycles that are compatible (in a suitable sense) with the fibration (actually, with *any* fixed fibration) of the mapping torus on the circle.

Proof of the Main Theorem - Easy Inequality

Let us start with two simple lemmas.

Lemma 2.10. *Let R be a normed ring with unity. Let $f\colon (M, \partial M) \to (M, \partial M)$ be an orientation–preserving homeomorphism, and let z be an R–fundamental cycle for $(M, \partial M)$. Then*

$$\|M_f\|_R \leq (n+1)\|z\|_1 + \|f_*(z) - z\|_{\mathrm{fill},R}\,.$$

Proof. For $j = 0, 1$, let

$$i_j\colon (M, \partial M) \to (M \times [0,1], \partial M \times [0,1])$$

be the inclusion defined by $i_j(x) = (x, j)$, and let

$$\pi\colon (M \times [0,1], (\partial M) \times [0,1]) \to (M_f, \partial M_f)$$

be the quotient map which identifies $(x, 0)$ with $(f(x), 1)$ for every $x \in M$. The standard chain homotopy between $(i_0)_*$ and $(i_1)_*$ provides a chain $\overline{z} \in C_{n+1}(M \times [0,1], (\partial M) \times [0,1]; R)$ such that $\partial\overline{z} = (i_0)_*(z) - (i_1)_*(z)$ and $\|\overline{z}\|_1 \leq (n+1)\|z\|_1$. In order to project $\overline{z}$ onto an R–fundamental cycle for $(M_f, \partial M_f)$ we need to suitably replace the summand $(i_0)_*(z)$ with $(i_0 \circ f)_*(z)$. To this aim, for any given $\varepsilon > 0$ we choose a chain $b \in C_{n+1}(M, \partial M; R)$ such that $\partial b = z - f_*(z)$ and $\|b\|_1 \leq \|z - f_*(z)\|_{\text{fill},R} + \varepsilon$, and we set $\widehat{z} = \overline{z} - (i_0)_*(b) \in C_{n+1}(M \times [0,1], (\partial M) \times [0,1]; R)$. By construction, we have

$$\partial\widehat{z} = (i_0 \circ f)_*(z) - (i_1)_*(z),$$

hence $\pi_*(\widehat{z})$ is an R–fundamental cycle for $(M_f, \partial(M_f))$. We thus get

$$\|M_f\|_R \leq \|\widehat{z}\|_1 \leq \|\overline{z}\|_1 + \|(i_0)_*(b)\|_1 \leq (n+1)\|z\|_1 + \|z - f_*(z)\|_{\text{fill},R} + \varepsilon\ .$$

The conclusion follows thanks to the arbitrariness of ε. □

Lemma 2.11. *Let R be a normed ring with unity. Let $f\colon (M, \partial M) \to (M, \partial M)$ be an orientation–preserving homeomorphism preserving the boundary ∂M. Then*

$$\inf_{m \in \mathbb{N}} \frac{\|M_{f^m}\|_R}{m} \leq \mathrm{FV}_R(f).$$

Proof. Let z be any R–fundamental cycle for $(M, \partial M)$, and let $m \in \mathbb{N}$. By applying Lemma 2.10 to the homeomorphism f^m we get

$$\|M_{f^m}\|_R \leq (n+1)\|z\|_1 + \|f^m_*(z) - z\|_{\text{fill},R}\,.$$

Thus, by dividing by m and by taking the infimum, one gets the desired inequality. □

Proof of the Main Theorem - Less Easy Inequality

In the rest of this chapter we set $R = \mathbb{R}$ or $R = \mathbb{Z}$.

In order to conclude the proof of Theorem 2.7, we are left to show the inequality

$$\mathrm{FV}_R(f) \leq \inf_{k \in \mathbb{N}} \frac{\|M_f\|_R}{k},$$

which is more involved and its proof will occupy the rest of the chapter.

For every $m \in \mathbb{N}$, let

$$\pi_m\colon (M_{f^m}, \partial M_{f^m}) \to (M_f, \partial M_f)$$

and

$$\pi_\infty\colon (M \times \mathbb{R}, \partial M \times \mathbb{R}) \to (M_f, \partial M_f)$$

be the m–cyclic covering and the infinite–cyclic covering of M_f as described in Section 1.3.

Let us fix $[c]$ be an R–fundamental cycle of $(M_f, \partial M_f)$ with

$$c = \sum_{i=1}^{s} x_i \sigma_i \in C_{n+1}(M_f; R).$$

be such that $\|c\|_1 = \|[c]\|_1$ (Lemma 1.1).

We define the *length* of a singular simplex σ in M_f as follows: we consider a lift $\tilde{\sigma}$ of σ in $M \times \mathbb{R}$ and project its image on $\mathbb{R}$, thus obtaining an interval $[a_{\tilde{\sigma}}, b_{\tilde{\sigma}}]$; we then set $\mathrm{length}(\sigma) = b_{\tilde{\sigma}} - a_{\tilde{\sigma}}$ (this definition does not depend on the choice of the lift $\tilde{\sigma}$). We finally denote by N the maximal length of simplices appearing in the expression of c, i.e., we set

$$N = \max \{\mathrm{length}(\sigma_i) \mid 1 \leq i \leq s\}.$$

Consider $m > 2N + 1$ and let $h\colon [0, m] \to [0, m]$ be a continuous map such that $h([0, N]) = \{0\}$ and $h([m-N, m]) = \{m\}$. We extend h by m–periodicity to a map (still denoted by h) defined on the whole real line. By construction, for every $k \in \mathbb{Z}$ the map h shrinks the interval $[km - N, km + N]$ onto the single value km. We denote by

$$\overline{g}\colon M \times \mathbb{R} \to M \times \mathbb{R}$$

the map defined by $\overline{g}(x, t) = (x, h(t))$, so that $\overline{g}(M \times [km - N, km + N]) = M \times \{km\}$ for every $k \in \mathbb{Z}$. Being equivariant with respect to the action of the automorphisms of the covering $M \times \mathbb{R} \to M_{f^m}$, and preserving the boundary, the map $\overline{g}$ induces a continuous map

$$g\colon (M_{f^m}, \partial M_{f^m}) \to (M_{f^m}, \partial M_{f^m}).$$

Let $c_m \in C_{n+1}(M_{f^m}; R)$ be the lift of c on M_{f^m}, so that c_m represents an R–fundamental cycle for $(M_{f^m}, \partial M_{f^m})$, it satisfies $(\pi_m)_*(c_m) = m \cdot c$ and $\|c_m\|_1 = m\|c\|_1$. Observe that every simplex appearing in c_m has length not bigger than N.

Lemma 2.12. *The chain $g_*(c_m)$ is a representative of an R–fundamental cycle for $(M_{f^m}, \partial M_{f^m})$. Moreover, $\|g_*(c_m)\|_1 \leq \|c_m\|_1$.*

Proof. Since $\overline{g}$ is equivariantly homotopic to the identity of $M \times \mathbb{R}$ (relatively to the boundary), the map g is relative homotopic to the identity of $(M_{f^m}, \partial M_{f^m})$. Thus, the lemma follows. □

Recall that the *support* of a singular simplex s in a topological space X, denoted by $\mathrm{supp}(s)$, coincides with the image of the simplex in X. The *support* of a chain $c \in C_\bullet(X; R)$, denoted always as $\mathrm{supp}(c)$, is the union of the supports of the simplices appearing in the reduced expression of c.

Lemma 2.13. *If $\widetilde{s}$ is a lift of a simplex appearing in c_m under the covering $M \times \mathbb{R} \to M_{f^m}$, then $\operatorname{supp}(\overline{g}_*(\widetilde{s}))$ in $M \times \mathbb{R}$ is contained in a subset of the form $M \times [k_0 m, (k_0+1)m]$ for some $k_0 \in \mathbb{Z}$.*

Proof. Since the length of any simplex appearing in c_m is not bigger than N, if $\operatorname{supp}(\widetilde{s}) \cap (M \times \{km\}) \neq \emptyset$ for some $k \in \mathbb{Z}$, then $\operatorname{supp}(\widetilde{s}) \subseteq M \times [km - N, km + N]$, hence the image of $\overline{g}_*(\widetilde{s})$ is contained in $M \times \{km\}$.

On the other hand, if $\operatorname{supp}(\widetilde{s}) \cap (M \times \{km\}) = \emptyset$ for every $k \in \mathbb{Z}$, then there exists $k_0 \in \mathbb{Z}$ such that $\operatorname{supp}(\widetilde{s}) \subseteq M \times (k_0 m, (k_0+1)m)$. Since h maps the open interval $(k_0 m, (k_0+1)m)$ onto the closed interval $[k_0 m, (k_0+1)m]$, this implies that $\operatorname{supp}(\overline{g}_*(\widetilde{s})) \subseteq M \times [k_0 m, (k_0+1)m]$. ☐

Let us write $g_*(c_m)$ as a linear combination of simplices:

$$g_*(c_m) = \sum_{i=1}^{ms} y_i s_i \ .$$

For every $i = 1, \dots, ms$, we denote by $\widetilde{s}_i \colon \Delta^{n+1} \to M \times \mathbb{R}$ the unique lift of s_i such that the following condition holds: the image of $\widetilde{s}_i$ is disjoint from $M \times (-\infty, 0)$, but is not disjoint from $M \times [0, m)$. By Lemma 2.13, we have that $\operatorname{supp}(\widetilde{s}_i) \subseteq M \times [0, m]$ for every $i = 1, \dots, ms$.

Let us now consider the chain

$$w = \sum_{i=1}^{ms} y_i \tilde{s}_i \in C_{n+1}(M \times \mathbb{R}; R).$$

By construction, we have that

$$\|w\|_1 = \|g_*(c_m)\|_1 \leq \|c_m\|_1 = m\|c\|_1.$$

We will see that w projects to a representative of an R–fundamental cycle for $(M_{f^m}, \partial M_{f^m})$.

Let $\tau \colon M \times \mathbb{R} \to M \times \mathbb{R}$ be the generator of the automorphisms group of the covering $M \times \mathbb{R} \to M_{f^m}$, i.e., $\tau(x,t) = (f^m(x), t+m)$ for every $(x,t) \in M \times \mathbb{R}$.

Lemma 2.14. *It holds that*

$$\partial w = b^+ - b^- + a \in C_n(M \times [0,m]; R),$$

where

- $\operatorname{supp}(b^+) \subset M \times \{m\}$*;*
- $\operatorname{supp}(b^-) \subset M \times \{0\}$*;*
- $\operatorname{supp}(a) \subset (\partial M) \times [0,m]$*;*
- $[b^+] = [\tau_*(b^-)] \in C_n(M \times \{m\}, (\partial M) \times \{m\}; R)$.

Proof. Since $\mathrm{supp}(w) \subseteq M \times [0, m]$, we have that

$$\mathrm{supp}(\partial \tau_*^j(w)) \subseteq M \times [jm, (j+1)m].$$

It follows that $\mathrm{supp}(\partial \tau_*^j(w)) \cap \mathrm{supp}(\partial \tau_*^i(w))$ is empty if $|i - j| > 1$ and it is contained in $M \times \{(j+1)m\}$ if $i = j + 1$.

Since $\sum_{j \in \mathbb{Z}} \tau_*^j(w)$ is obtained by lifting $g_*(c_m)$ to $M \times \mathbb{R}$, it holds that

$$\partial \left(\sum_{j \in \mathbb{Z}} \tau_*^j(w) \right) \in C_n(\partial M \times \mathbb{R}; R). \tag{2.2}$$

Thus, $\partial w = b^+ - b^- + a \in C_n(M \times [0, m]; R)$, where b^+, b^- and a satisfy the first three items of the lemma. The last item is a direct consequence of Eq. (2.2). □

As w is obtained by considering exactly one lift (in the cyclic covering $M \times \mathbb{R}$) of every simplex in $g_*(c_m)$, w projects to $g_*(c_m)$, which is an R–fundamental cycle for $(M_{f^m}, \partial M_{f^m})$ (Lemma 2.12).

Lemma 2.15. *The chain* $[b^-] \in C_n(M \times \{0\}, \partial M \times \{0\}; R)$ *is an* R*–fundamental cycle for* $(M \times \{0\}, \partial M \times \{0\})$.

Before proving this lemma, let us recall that, given an oriented manifold M with non–empty boundary ∂M, the double $\mathcal{D}_S M$ of M with respect to $S \subseteq \partial M$ is given by gluing to M a copy of M with the opposite orientation along S (with the obvious identification), i.e., $\mathcal{D}_S M = M \cup_S M$.

In this context, it is also possible to define the double of a map and the double of an R–chain. If $f \colon (M, \partial M) \to (M, \partial M)$ is a homotopy equivalence sending S inside S, then the double $\mathcal{D}_S f \colon \mathcal{D}_S M \to \mathcal{D}_S M$ is obtained by sending each of the two copies of M to itself via the map f. If $c \in C_\bullet(M, \partial M; R)$ is an R–chain in M, then the double of c is $\mathcal{D}_S c = (\iota_1)_*(c) - (\iota_2)_*(c)$, where $\iota_1, \iota_2 \colon M \to \mathcal{D}_S M$ are the obvious inclusion into the two copies of M inside $\mathcal{D}_S M$.

Proof. First, suppose $\partial M = \emptyset$ and let $q \in M \times (0, m)$. Being an R–fundamental cycle for M_{f^m}, the chain $g_*(c_m)$ represents the positive generator of the homology $H_{n+1}(M_{f^m}, M_{f^m} \setminus \{p_m(q)\}; R) \cong R$, where $p_m \colon M \times \mathbb{R} \to M_{f^m}$ is the covering projection. Using that p_m is an orientation–preserving local homeomorphism and the fact that w includes all the lifts of simplices of $g_*(c_m)$ that intersect $(0, m)$, we have that w is a representative of the positive generator of the homology $H_{n+1}(M \times \mathbb{R}, (M \times \mathbb{R}) \setminus \{q\}; R)$. This implies that w represents an R–fundamental cycle for $(M \times [0, m], M \times \{0, m\})$. As $\mathrm{supp}(b^+) \cap (M \times \{0\}) = \emptyset$, we obtain that b^- is an R–fundamental cycle for $M \times \{0\}$.

If $\partial M \neq \emptyset$, then we can consider the double of $M \times [0, m]$ with respect to the subset of the boundary $S = (\partial M) \times [0, m]$. Thanks to the discussion above, we have that $\mathcal{D}_{(\partial M)\times\{0\}} b^- = (\iota_1)_*(b^-) - (\iota_2)_*(b^-)$ is an R–fundamental cycle for $\mathcal{D}_{(\partial M)\times\{0\}}(M \times \{0\})$. As $\mathrm{supp}((\iota_2)_*(b^-)) \cap \iota_1(M \times \{0\}) = \iota_1((\partial M) \times \{0\})$, we have that $(\iota_1)_*(b^-)$ is a representative of an R–fundamental cycle for $\iota_1(M \times \{0\}, (\partial M) \times \{0\})$, which is equivalent to the thesis.

□

If $\kappa\colon M \times [0, m] \to M$ is the projection onto the first factor, then $\beta = [\kappa_*(b^-)] \in C_n(M, \partial M; R)$ is an R–fundamental cycle for $(M, \partial M)$. Moreover, thanks to Lemma 2.14, we have that $[\partial \kappa_*(w)] = f_*^m(\beta) - \beta$. We thus get

$$\frac{\|f_*^m(\beta) - \beta\|_{\mathrm{fill},R}}{m} \leq \frac{\|\kappa_*(w)\|_1}{m} \leq \frac{\|w\|_1}{m} \leq \|c\|_1.$$

Let us summarize what we have proven so far: for any given R–fundamental cycle $[c]$ for $(M_f, \partial M_f)$, we have constructed a fundamental cycle β for $(M, \partial M)$ and a natural number $m \in \mathbb{N}$ such that

$$\frac{\|f_*^m(\beta) - \beta\|_{\mathrm{fill},R}}{m} \leq \|[c]\|_1.$$

This, together with the definition of R–filling volume, implies that

$$\mathrm{FV}_R(f) \leq \frac{\|M_{f^m}\|_R}{m}.$$

This inequality, together with Lemma 2.11, concludes the proof of Theorem 2.7.

Chapter 3

Surfaces

This chapter is devoted to the study of the real and integral simplicial volume of orientation–preserving self–homotopy equivalences of surfaces. Theorem 2.7 plays a fundamental role in this chapter: indeed, we can employ the vast literature on simplicial volume of 3–manifolds fibering over the circle to extract information regarding the real filling volume $\mathrm{FV}_{\mathbb{R}}$.

On the other way around, we will extimate the integral filling volume $\mathrm{FV}_{\mathbb{Z}}(f)$ for a self–homotopy equivalence $f\colon S \to S$ on a surface S and exploit it to get new results on the integral simplicial volume of 3–dimensional manifolds fibering over the circle.

Recall that every self–homotopy equivalence of a closed surface is homotopic to a homeomorphism, thus it is not restrictive to consider only orientation–preserving self–homeomorphisms.

3.1 Real Filling Volume of Homeomorphisms of Surfaces

As mentioned above, we can employ the vast literature on 3–manifolds to understand the behavior of the real filling volume on self–homeomorphisms of surfaces: Soma proved that the simplicial volume of a prime 3–manifold N is positive if and only if it admits a hyperbolic metric, or it admits a nontrivial JSJ–decomposition with at least one hyperbolic piece. Moreover, the simplicial volume of N is the sum of the volumes of the hyperbolic pieces of N times a constant [Som81]:

$$\|N\| = \frac{\mathrm{vol}_{\mathrm{hyp}}(N)}{v_3},$$

where v_3 denotes the volume of the (unique up to isometry) regular ideal geodesic simplex in the hyperbolic 3–space.

Let S be a surface of genus $g \geq 1$ with (possibly empty) boundary ∂S. If $f\colon (S, \partial S) \to (S, \partial S)$ is an orientation–preserving homeomorphism, then the corresponding mapping torus S_f is a prime 3–manifold (as it is aspherical, by an easy check on Serre long exact sequence of the fiber bundle $S_f \to S^1$) with (possibly empty) toroidal boundary.

The manifold S_f admits a hyperbolic structure if and only if f is pseudo–Anosov, i.e., for any positive integer $m \geq 2$ and any loop γ inside S that is not homotopic to neither a point nor a boundary component of S, the curves γ and $f^m(\gamma)$ are not homotopic. More generally, the manifold M_f admits a JSJ–decomposition with at least one hyperbolic piece if and only if f is virtually pseudo–Anosov, that is: there exist a homeomorphism $f'\colon S \to S$ homotopic to f and a subsurface $S' \subseteq S$ such that a power of f' leaves S' invariant (i.e., $(f')^k(S') = S'$) and acts as a pseudo–Anosov homeomorphism on S'. See, e.g., [Koj12, Section 2.6] for a statement which perfectly fits with our terminology.

Theorem 3.1 (Theorem 5). *Let* $\varphi \in \mathrm{MCG}(S, \partial S)$*, where* S *is an orientable compact surface with (possibly empty) boundary* ∂S*. Then* $\mathrm{FV}_{\mathbb{R}}(\varphi) > 0$ *if and only if* S *admits a hyperbolic metric and* φ *is pseudo–Anosov or virtually pseudo–Anosov.*

There are plenty of papers relating the hyperbolic volume (hence, via the Proportionality Principle, the simplicial volume) of a 3–manifold fibering over the circle S_f to other length functions on the mapping class $[f] \in \mathrm{MCG}(S)$, like the minimal topological entropy, or the translation length with respect to the Teichmüller or the Weil–Petersson distance on Teichmüller space (see, e.g., [KKT09, Koj12, KM18, LP24b, LP24a]). In a rather indirect way, these results build a bridge between the invariant $\mathrm{FV}_{\mathbb{R}}$ and other classical invariants of homeomorphisms of surfaces. For example, Theorem 2.7 and the main results of [KKT09, KM18] readily imply the following:

Corollary 3.2. *Let* S *be a closed hyperbolic surface, and let* $\varphi \in \mathrm{MCG}(S)$*. Then, there exists a constant* $C > 0$ *only depending on the genus of* S *such that*

$$C^{-1}\|\varphi\|_{WP} \leq \mathrm{FV}_{\mathbb{R}}(\varphi) \leq C\|\varphi\|_{WP}$$

and

$$\mathrm{FV}_{\mathbb{R}}(\varphi) \leq C\|\varphi\|_T,$$

where $\|\varphi\|_{WP}$ *(resp.* $\|\varphi\|_T$*) denotes the translation length of the action of* φ *on the Teichmüller space of* S *with respect to the Weil–Petersson metric (resp. to the Teichmüller metric).*

Furthermore, if φ *is pseudo–Anosov, then*

$$\mathrm{FV}_{\mathbb{R}}(\varphi) \leq \frac{3\pi|\chi(S)|}{v_3}\|\varphi\|_T$$

$$\mathrm{FV}_{\mathbb{R}}(\varphi) \leq \frac{3\sqrt{\pi|\chi(S)|}}{v_3\sqrt{2}} \|\varphi\|_{WP} \,.$$

It is well known that, for every $\varphi \in \mathrm{MCG}(S)$, where S is any closed hyperbolic surface, the translation length $\|\varphi\|_T$ is equal to the *topological entropy* $\mathrm{ent}(\varphi)$ of φ (see, e.g., [Koj12]). The topological entropy is a well–defined topological invariant of self–homeomorphisms of topological spaces, and the topological entropy of a mapping class is the infimum of the topological entropies of its representatives. In the case of surfaces, Corollary 3.2 provides an upper bound for the ratio $\mathrm{FV}_{\mathbb{R}}(\varphi)/\,\mathrm{ent}(\varphi)$ that only depends on the manifold. A natural question is whether similar results hold in higher dimensions. More precisely, we ask here the following:

Question 3.3. *Let M be an orientable closed n–manifold. Does a constant $C > 0$ (depending only on M) exist such that*

$$\mathrm{FV}_{\mathbb{R}}(\varphi) \leq C \cdot \mathrm{ent}(\varphi)$$

for every $\varphi \in \mathrm{MCG}(M)$?

When S is a surface, the quantity $2\pi|\chi(S)|\,\mathrm{ent}(\varphi) = \pi\|S\| \cdot \mathrm{ent}\,\varphi$ is usually called *normalized entropy* of φ, and Corollary 3.2 states that, for surfaces, the ratio between $\mathrm{FV}_{\mathbb{R}}(\varphi)$ and the normalized entropy of φ is bounded from above by a universal constant. Notice that such an estimate (with a constant only depending on the dimension) does not hold in higher dimension. Indeed, should a constant $C > 0$ exist such that

$$\mathrm{FV}(\varphi) \leq C \cdot \|M\| \cdot \mathrm{ent}(\varphi)$$

for every $\varphi \in \mathrm{MCG}(M)$ and every orientable closed n–dimensional manifold M, we would have $\|M_f\| = 0$ for every mapping torus such that $\|M\| = 0$. However, in the proof of [KR23, Theorem A and Theorem 2.2] examples of mapping tori M_f with $\|M_f\| > 0$ and M rationally inessential (hence, $\|M\| = 0$) are exhibited.

3.2 Integral Filling Volume of Homeomorphisms of Surfaces

Thanks to Remark 2.2 and Theorem 3.1, we already know that $\mathrm{FV}_{\mathbb{Z}}(f)$ is nonzero for every pseudo–Anosov $\varphi \in \mathrm{MCG}(S)$ of a compact surface S admitting a hyperbolic metric. This section is devoted to the proof of the following result, which completely characterizes the vanishing of the integral filling volume on homeomorphisms of closed surfaces.

Theorem 3.4 (Theorem 9)**.** *Let S be an oriented compact surface, and let $f\colon S \to S$ be an orientation–preserving self–homeomorphism of S. Then $\mathrm{FV}_{\mathbb{Z}}(f) > 0$ if and only if one of the following conditions is satisfied:*

- *the surface S is a torus and f is Anosov;*
- *the surface S admits a hyperbolic metric and f is a pseudo–Anosov or a virtually pseudo–Anosov homeomorphism.*

Notice that if $f\colon T \to T$ is an Anosov homeomorphism, then $\mathrm{FV}_{\mathbb{R}}(f) = 0$, while $\mathrm{FV}_{\mathbb{Z}}(f) > 0$. In particular, the integral filling volume succeeds in distinguishing an Anosov automorphism of the torus from a Dehn twist, while the real filling volume does not.

Even more than the real simplicial volume, the integral simplicial volume is hard to compute, and there are very few manifolds for which the exact value is known [Löh18]. For this reason, computing the integral filling volume requires more work. The more involved step in the proof of Theorem 3.4 is given by Proposition 3.5, stating that the integral filling volume of a Dehn twist of a torus vanishes. This result can then be extended to Dehn twists on higher genus surfaces (Proposition 3.8). We conclude the proof of Theorem 3.4 by showing that the integral filling volume of an Anosov homeomorphism on the torus is positive (Proposition 3.10).

Filling Volume of Dehn Twists on the Torus

We devote the whole subsection to the proof of the following result:

Proposition 3.5. *Let T be the torus and let $f\colon T \to T$ be a Dehn twist along any homotopically nontrivial simple closed curve of T. Then* $\mathrm{FV}_{\mathbb{Z}}(f) = 0$.

In order to prove this statement, we fix a fundamental cycle $c \in C_2(T;\mathbb{Z})$ for the torus, and we construct efficient chains that fill the boundaries $c - f_*^{2^{2k}}(c)$.

Let $e_i \in \mathbb{R}^{k+1}$ be the i–th vector in the canonical basis of $\mathbb{R}^{k+1}$ and $\Delta^k \subset \mathbb{R}^{k+1}$ the convex hull of $e_1, \dots, e_{k+1}$, that is

$$\Delta^k = \left\{ \sum_{i=1}^{k+1} \lambda_i e_i \;\middle|\; 0 \leq \lambda_i \leq 1,\ \sum_{i=1}^{k+1} \lambda_i = 1 \right\}.$$

For $n, m \in \mathbb{N}$ and $p_1, \dots, p_{n+1} \in \mathbb{R}^m$, we denote by $\mathrm{str}_m(p_1, \dots, p_{n+1})$ the straight n–simplex in $\mathbb{R}^m$ with vertices $p_1, \dots, p_{n+1}$, that is the singular simplex defined by the map

$$\begin{array}{rccc} \mathrm{str}_m(p_1, \dots, p_{n+1})\colon & \Delta^n & \to & \mathbb{R}^m \\ & \sum_{i=1}^{n+1} \lambda_i e_i & \mapsto & \sum_{i=1}^{n+1} \lambda_i p_i. \end{array}$$

Let $T \cong \mathbb{R}^2/\mathbb{Z}^2$ be the torus, $\pi\colon \mathbb{R}^2 \to T$ the quotient map. We denote by

$$[p_1, \dots, p_{n+1}] \in C_n(T;\mathbb{Z})$$

the n–simplex in T given by $\pi\left(\mathrm{str}_2(p_1, \dots, p_{n+1})\right)$.

Remark 3.6. As $T = \mathbb{R}^2/\mathbb{Z}^2$, we have that for every $i, j \in \mathbb{Z}$ the two symbols $[(x_1, y_1), (x_2, y_2), \ldots, (x_{n+1}, y_{n+1})]$ and $[(x_1 + i, y_1 + j), (x_2 + i, y_2 + j), \ldots, (x_{n+1} + i, y_{n+1} + j)]$ represent the same simplex in $C_n(T; \mathbb{Z})$.

With this notation, a fundamental cycle of the torus is given by

$$c = [(0,0), (1,0), (1,1)] - [(0,0), (0,1), (1,1)]$$

(see Fig. 3.1).

Let $f\colon T = \mathbb{R}^2/\mathbb{Z}^2 \to T = \mathbb{R}^2/\mathbb{Z}^2$ be a Dehn twist along a homotopically nontrivial simple closed curve. Up to conjugation by an element of the mapping class group, we can suppose f is represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $f_*\colon C_\bullet(T; \mathbb{Z}) \to C_\bullet(T; \mathbb{Z})$ be the induced map on the set of chains.

We have

$$f_*^k(c) = [(0,0), (1,0), (k+1,1)] - [(0,0), (k,1), (k+1,1)],$$

that is a fundamental cycle of T as well (Fig. 3.1).

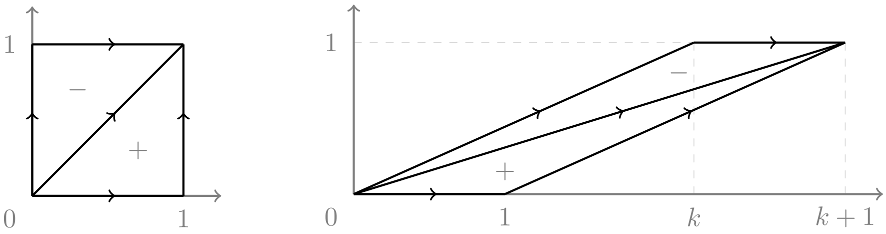


Figure 3.1: Lifts of the cycles c and $f_*^k(c)$ in $\mathbb{R}^2$.

The chain $f_*^{2^{2k}}(c) - c \in C_2(T; \mathbb{Z})$ is a boundary as it is the difference of two fundamental cycles. For $k \in \mathbb{N}$, we will construct chains $w_k \in C_3(T; \mathbb{Z})$ such that $\partial w_k = f_*^{2^{2k}}(c) - c$ and with the property that $\|w_k\|_1$ grows linearly with k. This will prove that the filling norm of $f_*^k(c) - c$ grows less than linearly.

We subdivide the construction into steps.

Step 1. Let

$$\begin{aligned} \tau_k = &+ [(0,0), (1,0), (2^{2k}+1, 0), (2^{2k}+1, 1)] \\ &- [(0,0), (2^{2k}, 0), (2^{2k}+1, 0), (2^{2k}+1, 1)] \\ &+ [(0,0), (2^{2k}, 0), (2^{2k}, 1), (2^{2k}+1, 1)] \end{aligned}$$

(see Fig. 3.2) and set

$$b_k = [(0,0), (1,0), (2^{2k}+1, 0)] - [(0,0), (2^{2k}, 0), (2^{2k}+1, 0)].$$

Then we have $\|\tau_k\|_1 = 3$ and $\partial\tau_k + b_k = f_*^{2^{2k}}(c) - c$, since

$$\begin{aligned}\partial\tau_k + b_k = &+[(1,0),(2^{2k}+1,0),(2^{2k}+1,1)] - [(0,0),(2^{2k},0),(2^{2k},1)]\\ &+[(0,0),(1,0),(2^{2k}+1,1)] - [(0,0),(2^{2k},1),(2^{2k}+1,1)]\\ &+[(2^{2k},0),(2^{2k},1),(2^{2k}+1,1)]\\ &-[(2^{2k},0),(2^{2k}+1,0),(2^{2k}+1,1)],\end{aligned}$$

where the right–hand side of the first line is equal to 0 since the triple given by $((1,0),(2^{2k}+1,0),(2^{2k}+1,1))$ is obtained from $((0,0),(2^{2k},0),(2^{2k},1))$ via the translation by $(1,0)$ (see Remark 3.6 with $(i,j)=(1,0)$), the second line is equal to $f_*^{2^{2k}}(c)$, and the sum of the third and the fourth line is equal to $-c$ (see again Remark 3.6 with $(i,j)=(2^k,0)$).

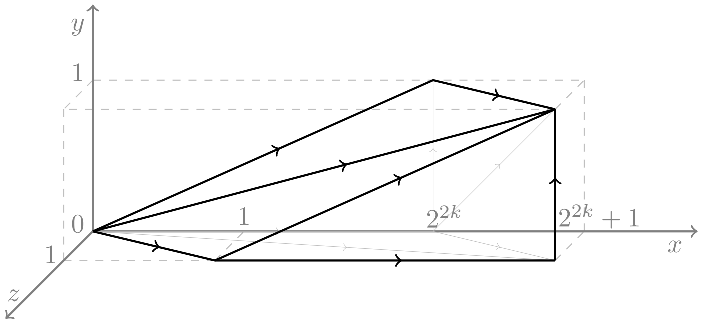


Figure 3.2: The chain τ_k is obtained by projecting the triangulation of the prism in the picture on the xy–plane.

Let $S^1 = \mathbb{R}/\mathbb{Z}$ and denote by $\gamma\colon S^1 = S^1\times\{0\} \hookrightarrow T = S^1\times S^1$ the simple closed curve in T defined by $\gamma([t]) = [(t,0)]$. The boundary b_k is supported in the image of γ, hence it represents a chain $s_k^0 \in C_2(S^1;\mathbb{Z})$ in S^1 as well, where we identify S^1 with the image of γ. Notice that with this notation we have $\gamma_*(s_k^0) = b_k$.

In order to find an efficient chain $\omega \in C_3(T;\mathbb{Z})$ whose boundary is b_k, it is enough to find an efficient chain $\omega_{S^1} \in C_3(S^1;\mathbb{Z})$ whose boundary is s_k^0 and set $\omega = \gamma_*(\omega_{S^1})$.

As we did for the torus, we denote by $\pi_{S^1}\colon \mathbb{R}\to\mathbb{R}/\mathbb{Z} = S^1$ the quotient map, and, if $x_1,\dots,x_{n+1}\in\mathbb{R}$, then we denote by $[x_1,\dots,x_{n+1}]$ the straight n–simplex in S^1 given by $\pi_{S^1}(\mathrm{str}_1(x_1,\dots,x_{n+1}))$. According to this notation, we have

$$s_k^0 = \left[0,1,2^{2k}+1\right] - \left[0,2^{2k},2^{2k}+1\right].$$

Remark 3.7. As in Remark 3.6, for every $i\in\mathbb{Z}$, the symbols $[x_1,\dots,x_{n+1}]$ and $[x_1+i,\dots,x_{n+1}+i]$ denote the same simplex in S^1.

Step 2. For every $i = 0, \dots, 2k$ let

$$s_k^i = \left[0, 2^i, 2^{2k-i} + 2^i\right] - \left[0, 2^{2k-i}, 2^{2k-i} + 2^i\right].$$

Then there exist two chains $\alpha, \beta, \in C_3(T;\mathbb{Z})$ (not depending on k and i) and maps $\varphi_k^i \colon T \to S^1$ such that $(\varphi_k^i)_*(\partial\alpha) + (\varphi_k^{i+1})_*(\partial\beta) = s_k^{i+1} - s_k^i$.

Proof. Recall that

$$c = [(0,0),(1,0),(1,1)] - [(0,0),(0,1),(1,1)]$$

and set

$$\begin{aligned} a =& + [(0,0),(1,0),(1,1/2)] - [(0,0),(0,1/2),(1,1/2)] + \\ &+ [(0,1/2),(1,1/2),(1,1)] - [(0,1/2),(0,1),(1,1)], \\ b =& + [(0,0),(1/2,0),(1/2,1)] - [(0,0),(0,1),(1/2,1)] + \\ &+ [(1/2,0),(1,0),(1,1)] - [(1/2,0),(1/2,1),(1,1)]. \end{aligned}$$

The chains a, b, c are all fundamental cycles of the torus, thus they are cobordant (Fig. 3.3). In particular, there exist $\alpha, \beta \in C_3(T;\mathbb{Z})$ such that

$$\begin{aligned} \partial\alpha =& a - c, \\ \partial\beta =& c - b. \end{aligned}$$

Let us set

$$\begin{array}{rccc} \widetilde{\varphi}_k^i \colon & \mathbb{R}^2 & \to & \mathbb{R}^1 \\ & (x,y) & \mapsto & 2^i x + 2^{2k-i} y. \end{array}$$

We define $\varphi_k^i \colon T = \mathbb{R}^2/\mathbb{Z}^2 \to S^1 = \mathbb{R}/\mathbb{Z}$ as the map induced by $\widetilde{\varphi}_k^i$ on the quotient. Then (see Fig. 3.3)

$$\left(\varphi_k^i\right)_*(c) = \left[0, 2^i, 2^{2k-i} + 2^i\right] - \left[0, 2^{2k-i}, 2^{2k-i} + 2^i\right] = s_k^i$$

and

$$\begin{aligned} \left(\varphi_k^i\right)_*(a) =& + \left[0, 2^i, 2^{2k-i-1} + 2^i\right] - \left[0, 2^{2k-i-1}, 2^{2k-i-1} + 2^i\right] + \\ &+ \left[2^{2k-i-1}, 2^{2k-i-1} + 2^i, 2^{2k-i} + 2^i\right] + \\ &- \left[2^{2k-i-1}, 2^{2k-i}, 2^{2k-i} + 2^i\right] \\ =& + \left[0, 2^i, 2^{2k-i-1} + 2^i\right] - \left[0, 2^{2k-i-1}, 2^{2k-i-1} + 2^i\right] + \\ &+ \left[2^i, 2^{i+1}, 2^{2k-i-1} + 2^{i+1}\right] - \left[2^i, 2^{2k-i-1} + 2^i, 2^{2k-i-1} + 2^{i+1}\right] \\ =& \left(\varphi_k^{i+1}\right)_*(b). \end{aligned}$$

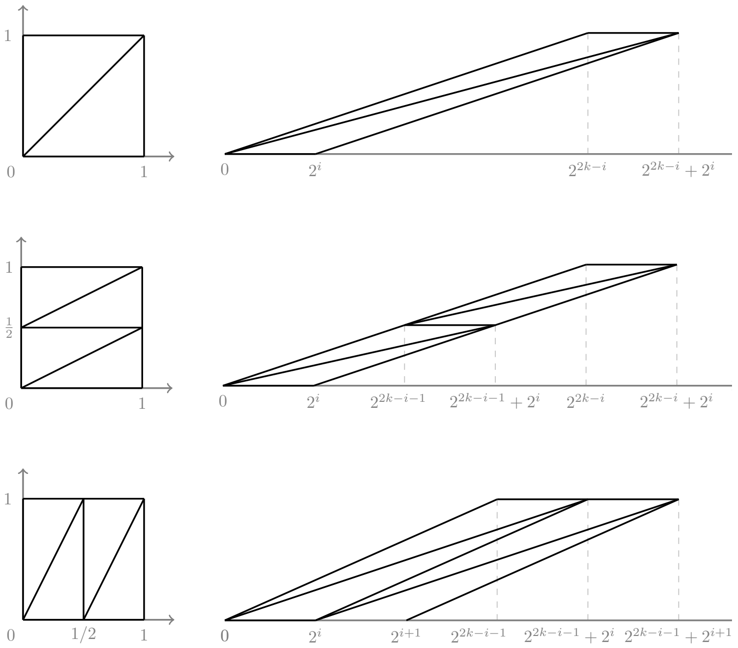


Figure 3.3: Lifts in $\mathbb{R}^2$ of the fundamental cycles c, a, b are represented on the left. The parallelograms on the right, once projected on the horizontal axis, represent lifts in $\mathbb{R}$ of the cycles $\varphi_k^i(c), \varphi_k^i(a), \varphi_k^{i+1}(b)$. Notice that the lift of $\varphi_k^{i+1}(b)$ is obtained from the lift of $\varphi_k^i(a)$ by translating the upper parallelogram to the right of the lower parallelogram.

Putting everything together, we have

$$\begin{aligned}\left(\varphi_k^i\right)_*(\partial\alpha) + \left(\varphi_k^{i+1}\right)_*(\partial\beta) &= \left(\varphi_k^i\right)_*(a-c) + \left(\varphi_k^{i+1}\right)_*(c-b)\\ &= (\varphi_k^{i+1})_*(b) - s_k^i + s_k^{i+1} - (\varphi_k^{i+1})_*(b) = s_k^{i+1} - s_k^i.\end{aligned}$$

□

Step 3. Set

$$\omega_k = \sum_{i=0}^{k-1} (\varphi_k^i)_*(\alpha) + (\varphi_k^{i+1})_*(\beta).$$

Then

$$\partial\omega_k = \sum_{i=0}^{k-1}\left((\varphi_k^i)_*(\partial\alpha) + (\varphi_k^{i+1})_*(\partial\beta)\right) = \sum_{i=0}^{k-1}(s_k^{i+1} - s_k^i) = s_k^k - s_k^0 = -s_k^0,$$

where in the last equality we used the fact that $s_k^k = 0$. Moreover,

$$\|\omega_k\|_1 \le \sum_{i=0}^{k-1}\left(\left\|(\varphi_k^i)_*(\alpha)\right\|_1 + \left\|(\varphi_k^{i+1})_*(\beta)\right\|_1\right) \le k\left(\|\alpha\|_1 + \|\beta\|_1\right).$$

We are now ready to conclude the proof of Proposition 3.5. Thanks to Step 1 and Step 3 we have

$$f_*^{2^{2k}}(c) - c = \partial\tau_k + b_k = \partial\tau_k + \gamma_*\left(s_k^0\right) = \partial\tau_k + \gamma_*\left(-\partial\omega_k\right) = \partial\left(\tau_k - \gamma_*\left(\omega_k\right)\right),$$

and

$$\left\|\tau_k - \gamma_*\left(\omega_k\right)\right\|_1 \leq \left\|\tau_k\right\|_1 + \left\|\gamma_*\left(\omega_k\right)\right\|_1 \leq 3 + k\left(\|\alpha\|_1 + \|\beta\|_1\right).$$

Thus

$$\begin{aligned}
\mathrm{FV}_{\mathbb{Z}}(f) &= \lim_{k\to\infty} \frac{\left\|f_*^k(c) - c\right\|_{\mathrm{fill},\mathbb{Z}}}{k} \\
&= \lim_{k\to\infty} \frac{\left\|f_*^{2^{2k}}(c) - c\right\|_{\mathrm{fill},\mathbb{Z}}}{2^{2k}} \\
&\leq \lim_{k\to\infty} \frac{\left\|\tau_k - \gamma_*\left(\omega_k\right)\right\|_{\mathbb{Z}}}{2^{2k}} \\
&\leq \lim_{k\to\infty} \frac{3 + k\left(\|\alpha\|_1 + \|\beta\|_1\right)}{2^{2k}} = 0.
\end{aligned}$$

□

Filling Volume of Dehn Twists on Higher Genus Surfaces

In this subsection, we exploit the vanishing of the integral filling volume for Dehn twists on the torus to obtain an analogous result for higher genus surfaces.

Let us first fix some notation. Let S be a surface. With a slight abuse of notation, if $\gamma\colon S^1 \to S$ is a closed curve, then we denote by γ also the image $\mathrm{im}(\gamma) \subset S$ of the map γ. If $S' \subset S$ is a closed subset of S, then we denote by $S\backslash\backslash S'$ the completion of $S \setminus S'$ with respect to some auxiliary Riemannian metric on S. In this way, when γ is a simple closed curve, the surface $S\backslash\backslash\gamma$ is a (not necessarily connected) compact surface with two boundary components, both canonically identified with γ.

Proposition 3.8. *Let S be an orientable compact surface of genus $g \geq 1$, $\gamma\colon S^1 \to S$ be a simple closed curve in S which is not homotopic neither to a point nor to a boundary component. If $f\colon S \to S$ is a Dehn twist along γ, then* $\mathrm{FV}_{\mathbb{Z}}(f) = 0$.

Before proving Theorem 3.8, we recall a result due to Foozwell and Rubinstein that gives a different description of S_f.

Lemma 3.9 ([FR16, Lemma 3.3])**.** *Let S, γ and f as in Proposition 3.8. Let $N' = (S\backslash\backslash\gamma) \times S^1$, that is a 3–manifold with two boundary components* *canonically homeomorphic to the torus* $T = \gamma \times S^1$. *Let* $g\colon T \to T$ *be the Dehn twist along* γ *and* N_k *be the manifold obtained from* N' *by gluing the two boundaries components along the map* g^k. *Then* S_{f^k} *is homeomorphic to the manifold* N_k.

Proof. Let us consider a regular neighborhood $A \subset S$ of γ and fix an identification $\alpha\colon S^1 \times [0,1] \xrightarrow{\sim} A$ such that $\alpha(r, 1/2) = \gamma(r)$. For simplicity, we will denote by $(\gamma(r), s)$ the point $\alpha(r,s)$ and by A^s the curve $\gamma \times \{s\}$.

For $i = 0, 1$, let $\iota_i\colon A^i \hookrightarrow S\backslash\backslash A$ be the inclusion into the boundary component $B^i = \iota_i(A^i)$ of $S\backslash\backslash A$.

Let

$$\begin{array}{rccc} f'\colon & A & \to & A \\ & (\gamma(r), s) & \mapsto & (\gamma(r+s), s), \end{array}$$

where we identify $S^1 = \mathbb{R}/\mathbb{Z}$. Up to isotopy we can assume that the Dehn twist f is given by $f\big|_{S\backslash A} = \mathrm{id}$ and $f\big|_{A} = f'$. Then, by definition,

$$S_{f^k} \cong \big((S\backslash\backslash A) \times S^1\big) \bigcup_{\substack{\iota_0 \times \mathrm{id} \\ \iota_1 \times \mathrm{id}}} A_{(f')^k}$$

where we glue $A^i \times S^1$ to $B^i \times S^1$ according to the map $\iota_i \times \mathrm{id}$ (Fig. 3.4).

Let

$$\begin{array}{rccc} g'\colon & A^1 \times S^1 & \to & A^1 \times S^1 \\ & ((\gamma(r), 1), t) & \mapsto & ((\gamma(r-t), 1), t), \end{array}$$

and let $g_k = (\iota_1 \times \mathrm{id}) \circ (g')^k\colon A^1 \times S^1 \to B^1 \times S^1$. Then, we have (Fig. 3.4)

$$N_k = \big((S\backslash\backslash A) \times S^1\big) \bigcup_{\substack{\iota_0 \times \mathrm{id} \\ g_k}} A_{\mathrm{id}},$$

where we glue $A^0 \times S^1$ to $B^0 \times S^1$ along the map $\iota_0 \times \mathrm{id}$, and $A^1 \times S^1$ to $B^1 \times S^1$ along g_k.

Now we construct the homeomorphism between S_{f^k} and N_k:

$$\begin{array}{rccc} \varphi'_k\colon & A_{\mathrm{id}} & \to & A_{(f')^k} \\ & (\gamma(r), s, t) & \mapsto & (\gamma(r + kst), s, t), \end{array}$$

where $r, t \in S^1$ and $s \in [0,1]$. We have

$$\begin{aligned} \varphi'_k(\gamma(r), s, 0) = (\gamma(r), s, 0) = \Big((f')^k(\gamma(r), s), 1\Big) = \\ = (\gamma(r + ks), s, 1) = \varphi'_k(\gamma(r), s, 1), \end{aligned}$$

thus the map is well defined. Moreover, it is a homeomorphism with inverse given by

$$\begin{array}{rccc} (\varphi'_k)^{-1}\colon & A_{(f')^k} & \to & A_{\mathrm{id}} \\ & (\gamma(r), s, t) & \mapsto & (\gamma(r-kst), s, t). \end{array}$$

We extend φ'_k on N_k with a map $\varphi_k\colon N_k \to S_{f^k}$ defined by $\varphi_k\big|_{(S\backslash\backslash A)\times S^1} = \mathrm{id}$ and $\varphi_k\big|_{A\times S^1} = \varphi'_k$.

Once again, the map φ_k is well defined; indeed, clearly $\varphi_k\big|_{A^0\times S^1} = \varphi_k\big|_{B^0\times S^1}$ and

$$g_k\big(\varphi_k\big|_A(\gamma(r),1,t)\big) = g_k(\gamma(r+kt),1,t) = \iota_1(\gamma(r),1,t).$$

So, $\varphi_k\big|_{A^1\times S^1} = \varphi_k\big|_{B^1\times S^1}$. Finally, the map is a homeomorphism: the inverse φ_k^{-1} is defined as $(\varphi'_k)^{-1}$ on $A_{(f')^k}$ and as the identity everywhere else. ☐

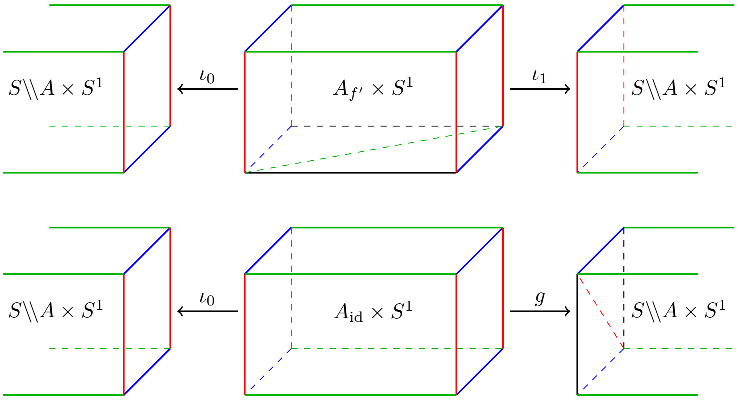


Figure 3.4: On the top it is represented how to glue the annulus $A_{f'}$ to the manifold $S\backslash\backslash A$ in order to obtain S_f. On the bottom it is represented how to glue the annulus A_{id} to the manifold $S\backslash\backslash A$ in order to obtain N_1.

We are now ready to prove Proposition 3.8.

Proof of Proposition 3.8. We prove the statement only for closed surfaces, as the proof in the compact case is completely analogous (but with more notation). By Theorem 2.7, it suffices to show that

$$\lim_{k\to\infty} \frac{\|S_{f^k}\|_{\mathbb{Z}}}{k} = 0.$$

To this aim, we employ the description of S_{f^k} given by Lemma 3.9. The compact manifold $N' = (S\backslash\backslash\gamma) \times S^1$ has two boundary components T_1, T_2, both canonically identified with the torus $T = \gamma \times S^1$; we denote by

$$i\colon T_1 \hookrightarrow T_2$$

the map given by the composition of the two canonical identifications $T_1 \to T$ and $T \to T_2$. If we endow T_1 and T_2 with the orientations induced by N', then the map i is orientation–reversing.

Let $c' \in C_3(N'; \mathbb{Z})$ be a chain representing an integral fundamental cycle for the pair $(N', \partial N')$. We have $\partial c' = b_1 + b_2$, where $b_i \in C_2(T_i; \mathbb{Z})$ is a fundamental cycle. The chains $-i_*(b_1)$ and b_2 are both fundamental cycles for T_2, thus there exists $w \in C_3(T_2; \mathbb{Z}) \subseteq C_3(N'; \mathbb{Z})$ such that $\partial w = i_*(b_1) + b_2$. Set $c = c' - w$, so that c still represents a relative fundamental cycle for $(N', \partial N')$ and $\partial c = b_1 - i_*(b_1)$.

Let $g' \colon T_1 \to T_1$ be the Dehn twist along γ, let $g_k \colon T_1 \to T_2$ be the composition $i \circ (g')^k$, and N_k be the manifold obtained from N' by gluing the two boundary components along g_k. Let us denote by $p \colon N' \to N_k$ the projection. Then, $p_*(c) \in C_3(N_k; \mathbb{Z})$ is a chain whose boundary in N_k is contained inside $p(T_1)$, and it is given by

$$\partial p_*(c) = p_*(b_1) - p_*(i_*(b_1)) = p_*(b_1) - p_*(g'_*)^{-k}(b_1) = p_*(b_1 - (g'_*)^{-k}(b_1)).$$

In particular, the filling norm of $\partial p_*(c)$ as a boundary in N_k is bounded from above by the filling norm of $p_*(b_1 - (g'_*)^{-k}(b_1))$ as a boundary in $p(T_1) \cong T$. By Lemma 3.9, the manifold N_k is homeomorphic to the manifold S_{f^k}. Thus,

$$\begin{aligned}
\mathrm{FV}_{\mathbb{Z}}(f) &= \lim_{k\to\infty} \frac{\|S_{f^k}\|_{\mathbb{Z}}}{k} \\
&= \lim_{k\to\infty} \frac{\|N_k\|_{\mathbb{Z}}}{k} \\
&\leq \lim_{k\to\infty} \frac{\|p_*(c)\|_1 + \|\partial p_*(c)\|_{\mathrm{fill},\mathbb{Z}}}{k} \\
&\leq \lim_{k\to\infty} \frac{\|c\|_1 + \left\|p_*(b_1 - (g'_*)^{-k}(b_1))\right\|_{\mathrm{fill},\mathbb{Z}}}{k} \\
&\leq \lim_{k\to\infty} \frac{\|c\|_1}{k} + \lim_{k\to\infty} \frac{\left\|b_1 - (g'_*)^{-k}(b_1)\right\|_{\mathrm{fill},\mathbb{Z}}}{k} \\
&= 0 + \mathrm{FV}_{\mathbb{Z}}(g') \\
&= 0,
\end{aligned}$$

where the last equality is due to Theorem 3.5. □

Anosov Maps on the Torus

We dedicate this subsection to the study of the integral filling volume of Anosov maps on the torus.

Proposition 3.10. *Let $f \colon T \to T$ be an orientation–preserving Anosov self–homeomorphism of the torus. Then* $\mathrm{FV}_{\mathbb{Z}}(f) > 0$.

The proof is based on a result by Sauer, which gives a lower bound on the number of simplices in an integral fundamental cycle in terms of the torsion of the homology groups ([Sau16, Theorem 3.2]).

Proof. Let $f\colon T \to T$ be an Anosov map on the 2–dimensional torus T. Since f is Anosov, the induced map in homology $f_*\colon H_1(T;\mathbb{Z}) \to H_1(T;\mathbb{Z})$ is a matrix in $\mathrm{SL}_2(\mathbb{Z})$ with eigenvalues λ and $1/\lambda$, where $|\lambda| > 1$; up to replacing f^2 with f, we can suppose $\lambda > 1$. Thanks to [Sak81, Lemma 10], we get

$$|\operatorname{tors}(H_1(T_{f^m};\mathbb{Z}))| = \operatorname{tr}(f_*^m) - 2 = \lambda^m + \lambda^{-m} - 2. \tag{3.1}$$

If $c \in Z_3(T_{f^m};\mathbb{Z})$ is any integral fundamental cycle and k is the number of 3–simplices appearing in the reduced form of c, then [Sau16, Theorem 3.2] gives

$$\|c\|_1 \geq k \geq (6\log(4))^{-1}\log|\operatorname{tors} H_1(T_{f^m};\mathbb{Z}))|,$$

hence

$$\|T_{f^m}\|_{\mathbb{Z}} \geq (6\log(4))^{-1}\log|\lambda^m + \lambda^{-m} - 2|.$$

If z is an integral fundamental cycle for T, then Lemma 2.10 gives

$$\begin{aligned}\frac{\|f_*^m(z) - z\|_{\mathrm{fill},\mathbb{Z}}}{m} &\geq \frac{\|T_{f^m}\|_{\mathbb{Z}}}{m} - \frac{3\|z\|_1}{m} \\ &\geq \frac{(6\log(4))^{-1}\log|\lambda^m + \lambda^{-m} - 2|}{m} - \frac{3\|z\|_1}{m}.\end{aligned}$$

The result follows by taking the limit as $m \to \infty$. ☐

Remark 3.11. The same proof does not work if one considers a reducible map instead of an Anosov map: indeed, if $f\colon T \to T$ is, for example, a Dehn twist, then a computation gives $\operatorname{tors}|H_1(T_{f^m};\mathbb{Z})| = m$ and so, applying the analogous reasoning, one would get $\mathrm{FV}_{\mathbb{Z}}(f) \geq 0$.

Integral Filling Volume on Surfaces – Summary

We are finally ready to prove Theorem 3.4.

Proof. If S is a sphere or a disc, then $\mathrm{MCG}(S)$ is the trivial group and so the integral filling volume is always zero.

If S is a torus, then φ is periodic, Anosov or reducible [FM11, Theorem 13.1]:

- if φ is periodic, then $\mathrm{FV}_{\mathbb{Z}}(\varphi) = 0$ (Proposition 2.5);
- if φ is Anosov, then $\mathrm{FV}_{\mathbb{Z}}(\varphi) > 0$ (Proposition 3.10);
- if φ is reducible, then it is represented by some power h^k of some Dehn twist $h\colon T \to T$ ([FM11, Section 13.1]). By Proposition 3.5 and Proposition 2.5, it follows that $\mathrm{FV}_{\mathbb{Z}}(\varphi) = k \cdot \mathrm{FV}_{\mathbb{Z}}(h) = 0$.

If S is a surface admitting a hyperbolic metric, then φ is periodic, pseudo–Anosov or reducible [FM11, Theorem 13.2]:

- If φ is periodic, then $\mathrm{FV}_{\mathbb{Z}}(\varphi) = 0$ (Proposition 2.5);
- if φ is pseudo–Anosov, then $\mathrm{FV}_{\mathbb{Z}}(\varphi) \geq \mathrm{FV}_{\mathbb{R}}(\varphi) > 0$ (Theorem 3.1 and Remark 2.2);
- If φ is reducible, then there exists some positive power φ^k of φ that can be represented by a map h with the following property: there exist pairwise disjoint annuli $A_1, \ldots, A_m \subset S$ such that $h\big|_{A_i}$ is a power of the Dehn twist on A_i (for every $1 \leq i \leq m$) and the restriction of h on $S \setminus \cup_{i=1}^m A_i$ preserves each connected component, and acts on it as a pseudo–Anosov map or as the identity map [FM11, Corollary 13.3].

  If on at least one of these connected components, say $S' \subset S$, the restriction $h\big|_{S'}$ of h is pseudo–Anosov, then, by Theorem 3.1 and Proposition 2.5, we have

  $$\mathrm{FV}_{\mathbb{Z}}(\varphi) = \frac{\mathrm{FV}_{\mathbb{Z}}(h)}{k} \geq \frac{\mathrm{FV}_{\mathbb{R}}(h)}{k} > 0.$$

  Otherwise, h restricts to the identity on each connected component of $S \setminus \cup_{i=1}^m A_i$: if we denote by g_i the Dehn twist on A_i, then

  $$h = g_1^{k_1} \circ \ldots \circ g_m^{k_m}.$$

  As $g_1, \ldots, g_m$ are Dehn twists on pairwise disjoint annuli, they pairwise commute. It follows from Proposition 2.5 and Proposition 3.5 that

  $$\mathrm{FV}_{\mathbb{Z}}(\varphi) = \frac{\mathrm{FV}_{\mathbb{Z}}(h)}{k} \leq \frac{\sum_{i=1}^m (k_i \cdot \mathrm{FV}_{\mathbb{Z}}(g_i))}{k} = 0.$$

□

3.3 Applications to Surfaces

As promised in Remark 2.6, we show that FV_R, for $R = \mathbb{Z}$ or $R = \mathbb{R}$ is not a length function in the sense of [Pol18].

Proposition 3.12. *Let $R = \mathbb{R}$ or $R = \mathbb{Z}$, and let S be a hyperbolic surface. Then, there exist elements $\varphi, \psi \in \mathrm{MCG}(S)$ such that*

$$\mathrm{FV}_R(\varphi\psi) > \mathrm{FV}_R(\varphi) + \mathrm{FV}_R(\psi).$$

Proof. As the mapping class group $\mathrm{MCG}(S)$ of a hyperbolic surface is generated by Dehn twists [Deh38], every pseudo–Anosov class $\varphi \in \mathrm{MCG}(S)$ can be written as a composition $\varphi = \psi_1 \circ \ldots \circ \psi_k$ of Dehn twists $\psi_1, \ldots, \psi_k \in \mathrm{MCG}(S)$. If, by contradiction, we suppose that $\mathrm{FV}_R(\alpha\beta) \leq \mathrm{FV}_R(\alpha) + \mathrm{FV}_R(\beta)$ for every $\alpha, \beta \in \mathrm{MCG}(S)$, then

$$\mathrm{FV}_R(\varphi) \leq \mathrm{FV}_R(\psi_1) + \ldots + \mathrm{FV}_R(\psi_k).$$

However, thanks to Theorem 3.1 and Theorem 3.4, we know that the left–hand side of this equation is positive, while the right–hand side is zero, which gives a contradiction. ☐

3.4 Integral Simplicial Volume versus Delta–Complexity

Let M be an oriented closed n–manifold. Borrowing the terminology from low–dimensional topology, we understand that a *triangulation* of M is an expression of M as a union of n–dimensional simplices with some of their facets identified in pairs via affine homeomorphisms; in particular, self–adjacences of faces of the same simplex are allowed, i.e., closed simplices need not be embedded in M. The complexity $\Delta(\mathcal{T})$ of a triangulation $\mathcal{T}$ of M is the number of n–simplices in $\mathcal{T}$.

Definition 3.13. The Δ*–complexity* $\Delta(M)$ of a manifold M is defined as the minimum of $\Delta(\mathcal{T})$, as $\mathcal{T}$ ranges over the triangulations of M. If M does not admit any triangulation, then we set $\Delta(M) = \infty$.

In this chapter, we focus on comparing Δ–complexity and integral simplicial volume. Let us say that two numerical invariants h_1, h_2 on the class of oriented closed n–manifolds are *equivalent* if there exists a constant $k \geq 1$ such that $h_1(M) \leq k \cdot h_2(M)$ and $h_2(M) \leq k \cdot h_1(M)$ for every oriented closed n–manifold M.

As any triangulation of M gives rise to an integral fundamental cycle of M, it is clear that $\|M\|_\mathbb{Z} \leq \Delta(M)$. Moreover, integral simplicial volume and Δ–complexity are equivalent (in fact, they coincide!) in dimension one (trivially, since the only closed 1–manifold is S^1 and $\Delta(S^1) = \|S^1\|_\mathbb{Z} = 1$) and in dimension two, since $\Delta(S^2) = \|S^2\|_\mathbb{Z} = 2$ and $\Delta(S_g) = \|S_g\|_\mathbb{Z} = 4g - 2$ for every oriented closed surface of genus $g \geq 1$ (see [Löh18, Proposition 4.3]). In this section, we prove the following.

Theorem 3.14 (Theorem 10)**.** *In dimension* 3*, integral simplicial volume and* Δ*–complexity are not equivalent. More precisely, there exists a sequence* $\{M_i\}_{i\in\mathbb{N}}$ *of oriented closed* 3*–manifolds such that*

$$\lim_{i\to\infty} \frac{\Delta(M_i)}{\|M_i\|_\mathbb{Z}} = +\infty.$$

Remark 3.15. It is easy to check that the second barycentric subdivision of a triangulation (in our sense) of M yields a strict triangulation of M, i.e., a description of M as the geometric realization of a simplicial complex. As a consequence, if $\Delta'(M)$ denotes the minimal number of top–dimensional simplices in a strict triangulation of M, then

$$\Delta(M) \leq \Delta'(M) \leq (n+1!)^2 \Delta(M),$$

where $n = \dim M$. In particular, the numerical invariants Δ' and Δ are equivalent according to the definition given above.

Let $T = S^1 \times S^1$ be the 2–dimensional torus and let $f\colon T \to T$ be a Dehn twist along a homotopically nontrivial simple closed curve in T. Then the manifold M_i mentioned in Theorem 3.14 is the mapping torus of f^i, i.e., $M_i = T_{f^i}$. As we proved in Proposition 3.8 that $\mathrm{FV}_{\mathbb{Z}}(f) = 0$, we already know that $\|T_{f_i}\|_{\mathbb{Z}}$ grows less than linearly in i. Thus, Theorem 3.14 is a corollary of the following theorem.

Theorem 3.16. *Let $f\colon T \to T$ be a self–homeomorphism of the torus representing an infinite–order element in* $\mathrm{MCG}(T)$*. Then, the Δ–complexity of the mapping torus of f^i has linear growth with respect to i, i.e., there exist constants $0 < k_1 \leq k_2$ such that*

$$k_1 \leq \frac{\Delta\left(T_{f^i}\right)}{i} \leq k_2$$

for every $i \geq 1$.

The proof of this theorem is based on techniques developed by Lackeny and Purcell in [LP24b, LP24a], in which the Δ–complexity of manifolds fibering over the circle has been extensively studied.

Notice that the homeomorphism $f\colon T \to T$ satisfies the hypothesis of the theorem if and only if f is a nontrivial power of a Dehn twist or an Anosov homeomorphism. In particular, when $f\colon T \to T$ is an Anosov automorphism of the torus, the conclusion of Theorem 3.16 is proved to hold in [LP24a, Theorem 1.4], where k_1 and k_2 are also explicitly described. Hence, our contribution mainly consists in adapting Lackenby's and Purcell's strategy to the Dehn twist case.

A Glimpse in Higher Dimensions

In higher dimensions, the equivalence between integral simplicial volume and Δ–complexity breaks down dramatically. For example, if an n–manifold M does not admit a triangulation (a phenomenon that begins to occur in dimension 4), then $\Delta(M) = \infty$, while $\|M\|_{\mathbb{Z}}$ remains finite. Moreover, in dimensions $n \geq 5$, this non–equivalence also holds within the class of triangulable manifolds.

Proposition 3.17. *Let $n \geq 5$. There exists a sequence $\{M_i\}_{i \in \mathbb{N}}$ of oriented closed triangulable n–manifolds such that*

$$\lim_{i \to \infty} \frac{\Delta(M_i)}{\|M_i\|_{\mathbb{Z}}} = +\infty.$$

Proof. In [KS04, Theorem 1.2], the authors construct, for every $n \geq 4$, an infinite collection $\mathcal{C}$ of oriented closed topological n–manifolds that are all simple homotopy equivalent to each other but pairwise not homeomorphic (if $n > 4$, these manifolds can be chosen to be smooth, hence triangulable). Being homotopy equivalent, the manifolds in $\mathcal{C}$ share the same simplicial volume.

On the other hand, given k n–simplices, there is only a finite number of ways to glue together their facets via affine homeomorphisms; hence, the number of manifolds (considered up to homeomorphism) whose Δ–complexity is bounded from above by k is finite. Thus, the Δ–complexities of the manifolds in the infinite set $\mathcal{C}$ cannot be bounded, implying the statement.

□

We ask here the following:

Question 3.18. *Let $n \geq 4$ and $k \in \mathbb{N}$ be fixed. Is the number of homotopy classes of orientable closed n–manifolds M such that*

$$\|M\|_{\mathbb{Z}} \leq k$$

finite?

Recall that a manifold M *d–dominates* a manifold N if there exists a degree–d map $f\colon M \to N$. As observed by Löh in [Löh18], in dimension 3 the fact that the integral simplicial volume is finite–to–one follows from the following fact: for any fixed orientable closed 3–manifold M and any fixed $d > 0$, the number of homeomorphism classes of 3–manifolds that are d–dominated by M is finite [Liu20]. Also in higher dimensions Question 3.18 is related to the (probably difficult) question whether a fixed manifold M could d–dominate an infinite number of pairwise non–homotopic manifolds.

Spine Graph on a Surface

Before going into the proof of Theorem 3.16 we introduce the spine graph of a surface. As discussed in [LP24b, LP24a] this object plays a fundamental role in the computation of the Δ–complexity of surface bundles over the circle.

Let S be a closed surface. A *spine* Γ on S is an embedded graph without vertices of degree 1 and 2, and whose complement $S \backslash\backslash \Gamma$ is a disc. If S is endowed with a fixed cellular structure, then we say that the spine Γ is cellular if it is a subcomplex of S.

An *edge contraction* on a spine Γ of S is the move inside S that collapses an edge of Γ to a point (thus collapsing two vertices of Γ to a unique vertex). An *edge expansion* is the reverse of this operation.

The *spine graph* $\mathrm{Sp}(S)$ on a closed surface S is the graph whose vertices are spines on S, considered up to isotopy, and in which two vertices are joined by an edge if and only if they differ by an edge expansion/contraction.

Note that the mapping class group $\mathrm{MCG}(S)$ acts by isometries on $\mathrm{Sp}(S)$.

Proposition 3.19 ([LP24b, Proposition 2.7]). *Let S be a closed surface of genus $g \geq 1$, and fix a spine Γ in S. Then, the map*

$$\begin{array}{ccc} \mathrm{MCG}(S) & \to & \mathrm{Sp}(S) \\ \gamma & \mapsto & \gamma \cdot \Gamma \end{array}$$

is a quasi–isometry between the mapping class group $\mathrm{MCG}(S)$ and the spine graph $\mathrm{Sp}(S)$.

The statement of [LP24b, Proposition 2.7] only ensures that $\mathrm{MCG}(S)$ and $\mathrm{Sp}(S)$ are quasi–isometric to each other. However, in the proof Lackenby and Purcell show that the action of $\mathrm{MCG}(S)$ on $\mathrm{Sp}(S)$ verifies all the hypotheses of the Milnor–Švarc Lemma and thus the latter guarantees that the map described in Proposition 3.19 is a quasi–isometry.

Lemma 3.20. *Let S be a closed surface and let $\varphi \in \mathrm{MCG}(S)$ be an element of infinite order. Then there exists a constant $k_{\mathrm{spin}} > 0$ such that, for every spine Γ of S and every $n \in \mathbb{N}$, we have*

$$\frac{d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma), \Gamma)}{n} \geq k_{\mathrm{spin}}.$$

Proof. Let us fix a spine Γ of S. The map

$$n \mapsto d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma), \Gamma),$$

is subadditive, hence, by Fekete's Lemma, it holds that

$$k_{\mathrm{spin}} := \lim_{n\to\infty} \frac{d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma), \Gamma)}{n} = \inf \left\{ \frac{d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma), \Gamma)}{n} \,,\ n \geq 1 \right\}.$$

Let us first show that k_{spin} does not depend on the chosen spine Γ. Indeed, if Γ' is any spine of S, then

$$\begin{aligned} d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma), \Gamma) \leq & d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma), \varphi^n(\Gamma')) + d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma'), \Gamma') + d_{\mathrm{Sp}(S)}(\Gamma', \Gamma) \\ = & d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma'), \Gamma') + 2 d_{\mathrm{Sp}(S)}(\Gamma', \Gamma) \end{aligned}$$

and, analogously,

$$d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma'), \Gamma') \leq d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma), \Gamma) + 2 d_{\mathrm{Sp}(S)}(\Gamma', \Gamma).$$

These inequalities readily imply that

$$\liminf_{n\to\infty} \frac{d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma),\Gamma)}{n} = \liminf_{n\to\infty} \frac{d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma'),\Gamma')}{n},$$

i.e., k_{spin} does not depend on Γ.

In order to conclude we are left to show that $k_{\mathrm{spin}} > 0$. To this aim, let us fix a finite set of generators of $\mathrm{MCG}(S)$. By [FLM01, Theorem 1.2], elements of infinite order in $\mathrm{MCG}(S)$ are undistorted, i.e., there exists a constant $c > 0$ such that

$$d_{\mathrm{MCG}(S)}(\varphi^n, \mathrm{id}) \geq cn.$$

By Proposition 3.19, there are two constants $a > 0, b \geq 0$ such that

$$d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma),\Gamma) \geq a \cdot d_{\mathrm{MCG}(S)}(\varphi^n, \mathrm{id}) - b \geq acn - b,$$

hence

$$k_{\mathrm{spin}} = \lim_{n\to\infty} \frac{d_{\mathrm{Sp}(S)}(\varphi^n(\Gamma),\Gamma)}{n} \geq \lim_{n\to\infty} \frac{acn-b}{n} = ac > 0.$$

□

Delta–Complexity of Nilmanifolds and Solmanifolds

In this subsection, we prove Theorem 3.16.

Let $f\colon T \to T$ be a homeomorphism of the torus, and let $\mathcal{T}$ be a triangulation of $M = T_f$. We denote by $\mathcal{H}'$ the dual handle decomposition induced by $\mathcal{T}$, whose i–dimensional handles, $i = 0, 1, 2, 3$, are obtained from a thin neighborhood of the $(3 - i)$–skeleton of $\mathcal{T}$ by removing a thin neighborhood of the $(3 - i - 1)$–skeleton (see, e.g., [LP24a, Section 2]).

Recall that the *weight* of a surface S embedded in M and in general position with respect to $\mathcal{T}$ is defined as the cardinality of the intersection between S and the 1–skeleton of $\mathcal{T}$. We choose a fiber $S \subseteq M$ of least weight with respect to $\mathcal{T}$. Up to isotopy, we may assume that S is in normal position with respect to the handle decomposition $\mathcal{H}'$ (see [LP24b, Section 4] for the definition of normal position).

As explained in [LP24b, Section 4], the surface S intersects the handles of $\mathcal{H}'$ in properly embedded discs, thus inheriting a cellular structure form $\mathcal{H}'$. Moreover, the handle decomposition $\mathcal{H}'$ of M induces a handle decomposition $\mathcal{H}$ of the compact manifold with boundary $M\backslash\backslash S$ obtained by cutting M along S. Since S is a fiber of a torus bundle over the circle, we may identify $M\backslash\backslash S$ with the product $T \times [0, 1]$. In this way we obtain natural identifications between S and $T \times \{0\}$, $T \times \{1\}$. If we identify both $T \times \{0\}$ and $T \times \{1\}$ with the torus T via the obvious projections, then the cellular structure of S induces two cellular structures on T, which are taken one onto the other by the homeomorphism f. We fix on T the cellular structure induced by S via the identification $S \cong T \times \{1\}$.

We choose a cellular spine Γ_0 of T. By construction, the spine $f(\Gamma_0)$ is cellular with respect to the cellular structure induced by the identification of T with $T \times \{0\}$. Therefore, by [LP24a, Lemma 2.2] (which ensures that the handle decomposition $\mathcal{H}$ is *pre–tetrahedral* according to the terminology therein), [LP24a, Theorem 2.16] (which implies that $\mathcal{H}$ admits no annular simplification, see again [LP24a] for the definition of such a notion), and [LP24a, Lemma 2.5 and Theorem 9.1], there exists a cellular spine Γ_1 in T such that

$$d_{\mathrm{Sp}(T)}(f(\Gamma_0), \Gamma_1) \leq k_{\mathrm{hand}} \Delta(\mathcal{T}),$$

where k_{hand} is a universal constant, which is independent of the chosen cellular spine Γ_0 (see [LP24a, Theorem 9.1]). By iterating this process, we obtain a sequence of cellular spines $\{\Gamma_i\}$ satisfying

$$d_{\mathrm{Sp}(T)}(f(\Gamma_{i-1}), \Gamma_i) \leq k_{\mathrm{hand}} \Delta(\mathcal{T}),$$

where at each step of the iteration we take as input Γ_{i-1} instead of Γ_0. As there are only finitely many cellular spines in $T \times \{1\}$, there are two integers $r < s$ such that $\Gamma_r = \Gamma_s$. By relabeling, we can assume $r = 0, s = m$. Thus,

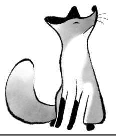
$$\begin{aligned} d_{\mathrm{Sp}(T)}(f^m(\Gamma_0), \Gamma_0) =& d_{\mathrm{Sp}(T)}(f^m(\Gamma_0), \Gamma_m) \\ \leq& \sum_{i=0}^{m-1} d_{\mathrm{Sp}(T)}(f^{m-i}(\Gamma_i), f^{m-i-1}(\Gamma_{i+1})) \\ =& \sum_{i=0}^{m-1} d_{\mathrm{Sp}(T)}(f(\Gamma_i), \Gamma_{i+1}) \\ \leq& m k_{\mathrm{hand}} \Delta(\mathcal{T}). \end{aligned}$$

Let $f \colon T \to T$ be a self–homeomorphism representing an infinite–order element in $\mathrm{MCG}(T)$, and let $\mathcal{T}_i$ be a triangulation of $M_i = T_{f^i}$ such that $\Delta(M_i) = \Delta(\mathcal{T}_i)$. By the inequality above, there exist a positive integer m_i and a cellular spine $\Gamma_0^{(i)}$ such that

$$\Delta(M_i) = \Delta(\mathcal{T}_i) \geq \frac{d_{\mathrm{Sp}(T)}\left((f^i)^{m_i}\left(\Gamma_0^{(i)}\right), \Gamma_0^{(i)}\right)}{m_i k_{\mathrm{hand}}} \geq \frac{k_{\mathrm{spin}}}{k_{\mathrm{hand}}} i, \tag{3.2}$$

where the last inequality is due to Lemma 3.20. This proves that $\Delta(M_i)$ grows at least linearly in i.

On the other hand, if $\mathcal{T}$ is a triangulation of M, then we can lift it to the i–sheeted covering $M_i \to M$, obtaining a triangulation of M_i with $i\Delta(\mathcal{T})$ tetrahedra. Therefore, $\Delta(M_i) \leq i\Delta(\mathcal{T})$, proving that $\Delta(M_i)$ grows at most linearly with respect to i.



Chapter 4

Three–Manifolds

In this chapter, we prove that the real filling volume on the mapping class group of oriented closed 3–dimensional manifolds is trivial.

Theorem 4.1 (Theorem 6)**.** *For every orientable closed* 3*–dimensional manifold* M *and every* $\varphi \in \mathrm{MCG}(M)$ *we have* $\mathrm{FV}_{\mathbb{R}}(\varphi) = 0$*.*

As we will see in the next chapter, this phenomenon occurs only in dimension 3 on the *real* filling volume (see Theorem 5.1 for a non–vanishing result on $\mathrm{FV}_{\mathbb{R}}$ in dimension $n \neq 1, 3$, and Theorem 5.2 for a statement on $\mathrm{FV}_{\mathbb{Z}}$ in dimension $n \geq 2$).

It is shown in [BN20] that the simplicial volume of any orientable closed 4–dimensional manifold fibering over the circle vanishes. Thus, Corollary 2.8 implies that for any orientable closed 3–manifold M and any $\varphi \in \mathrm{MCG}(M)$ that can be represented by a homeomorphism, it holds that $\mathrm{FV}_{\mathbb{R}}(\varphi) = 0$.

Thus, Theorem 4.1 is a corollary of Proposition 2.5 and the following theorem, stating that every orientation–preserving self–homotopy equivalence of an orientable closed 3–manifold admits a power homotopic to a homeomorphism.

Theorem 4.2. *Let* M *be an oriented closed* 3*–manifold* M*. Then, there exists a constant* A_M *(depending on the manifold* M*) such that for every homotopy equivalence* $f\colon M \to M$ *there exists an integer* k *satisfying* $1 \leq k \leq A_M$ *and such that* f^k *is homotopic to a homeomorphism.*

Let M be an oriented closed connected 3–manifold. We describe here a constant A_M satisfying the previous theorem. If $M = S^3$, then it suffices to take $A_M = 1$. If M is not homeomorphic to the sphere S^3, then let $M = M_1 \# \dots \# M_n$ be the prime decomposition of M. If there are no prime factors with finite fundamental group, then we set once again $A_M = 1$. Otherwise, up to rearranging the indices, we can assume $M_1, \dots, M_r$

have finite fundamental groups with cardinality $c_1, \dots, c_r$ respectively, and $M_{r+1}, \dots, M_n$ have infinite fundamental groups; in this case we set

$$A_M = 2 \cdot r! \cdot (c_1 \cdot c_2 \cdots c_r)! \ .$$

Given a manifold M, let us denote by $\mathrm{MCG}_{\mathrm{h}}(M)$ the subgroup of the mapping class group $\mathrm{MCG}(M)$ containing the classes representable by a homeomorphism.

Remark 4.3. According to the (still open) Borel conjecture, if M is a closed aspherical manifold, then $\mathrm{MCG}_{\mathrm{h}}(M) = \mathrm{MCG}(M)$. On the other hand, there are examples of non–aspherical 3–manifolds admitting homotopy equivalences that are not homotopic to any homeomorphism ([McC86]).

Thanks to a result by Kreck and Lück we already know that if the fundamental group of an oriented closed connected 3–manifold M is torsion–free, then $\mathrm{MCG}(M) = \mathrm{MCG}_{\mathrm{h}}(M)$ ([KL05, Theorem 0.7]). It should be noted that every oriented closed prime 3–manifold not homeomorphic to a quotient of the 3–sphere S^3 has torsion–free fundamental group.

On the other hand, if M is a quotient of S^3, then the group $\mathrm{MCG}(M)$ is finite (see [Sma74] for quotients of the 3–sphere or [KO21] for a more general statement about quotients of spheres of odd dimensions); thus, every homotopy equivalence $f\colon M \to M$ admits a power f^k homotopic to a homeomorphism, and the subgroup $\mathrm{MCG}_{\mathrm{h}}(M)$ has finite index inside $\mathrm{MCG}(M)$.

In particular, Theorem 4.2 holds when M is prime.

Remark 4.4. Up to replacing f with f^2, we can assume that the map f in Theorem 4.2 preserves the orientation (we just need to remember to add an additional coefficient 2 to the expression of A_M at the end of the proof).

Moreover, if M has $q > 1$ connected components, then there exists an integer i such that $1 \leq i \leq q!$ and f^i sends each connected component to itself. Thus, it suffices to prove the statement for oriented closed *connected* manifolds.

Plan of the Proof

In Section 4.1 we give a quick overview about decompositions of 3–manifolds, distinguishing between two kinds of prime summands: the *elliptic* ones, consisting of quotients of the 3–sphere S^3, and the *non–elliptic* ones, that are all the other prime summands. We also recall the definition of splitting homotopy equivalence and we state a result by Hendriks and Laudenbach asserting that orientation–preserving homotopy equivalences always split along decomposing systems of spheres ([HL74]).

In Section 4.2 we see how the results in [KL05] and [Sma74] imply Theorem 4.2 for prime 3–manifolds.

In Section 4.3 we prove the main theorem: first, given an orientation–preserving self–homotopy equivalence $f\colon M \to M$ of an oriented closed connected 3–manifold M, we show that there exists a sequence of powers f^{β_k} of f preserving some decomposing system of spheres Σ and sending each piece of the decomposition induced by Σ into a homeomorphic piece; out of this sequence we extract a power of f that is homotopic to a homeomorphism on each summand and, thus, it is itself homotopic to a homeomorphism.

4.1 Decomposition along Spheres

In this section, we collect well–known results on decompositions of 3–manifolds and homotopy equivalences between them.

Decompositions of Three–Manifolds

We recall here basic facts about decompositions of 3–manifolds in prime summands. More details can be found in, e.g., [Mar16].

Let M be an oriented closed connected 3–manifold. A *separating sphere* in M is an embedded 2–sphere $S \subset M$ such that $M \setminus S$ consists of two connected components. A *decomposing system of spheres* is a disjoint union of separating spheres in M.

If $\Sigma = S_1 \sqcup \ldots \sqcup S_{n-1} \subset M$ is a decomposing system of spheres, then $M \setminus\setminus \Sigma$ consists of exactly n connected components (recall from Section 3.2 that $M \setminus\setminus \Sigma$ is the completion of $M \setminus \Sigma$ with respect to some auxiliary Riemannian metric on M). Let us denote by $M'_1, \ldots, M'_n$ these connected components; in this way every M'_i is a compact manifold whose boundary consists of a disjoint union of spheres, all corresponding to a sphere in Σ; we denote by M_i the oriented closed connected manifold obtained from M'_i by gluing a 3–ball to every boundary component of M'_i. If a sphere S_j in Σ corresponds to a boundary component of M'_i, then we still denote by S_j the corresponding embedded sphere in M_i (which, now, bounds a ball). According to this construction, we have

$$M = M_1 \# \ldots \# M_n.$$

In this context, we say that M is *decomposed along* Σ, or that the *decomposition is induced* by Σ.

An oriented closed 3–manifold P not homeomorphic to S^3 is called *prime* if every decomposition of P is trivial (i.e., if $P = M_1 \# M_2$, then $M_i = S^3$ for $i = 1$ or $i = 2$). Given an oriented closed 3–manifold M, we call *prime decomposition* of M a decomposition $M = M_1 \# \ldots \# M_n$ in which each summand M_i is prime. Recall that every oriented closed connected 3–manifold M not homeomorphic to S^3 admits a prime decomposition; moreover, this decomposition is unique, meaning that whenever

$$M = M_1^1 \# \ldots \# M_n^1 \cong M_1^2 \# \ldots \# M_m^2$$

are two prime decompositions, then $m = n$ and there exists a permutation $\sigma \in \mathcal{S}_n$ such that M_i^1 is homeomorphic to $M_{\sigma(i)}^2$ for every $i \in \{1, \ldots, n\}$.

We distinguish between prime 3–manifolds with finite fundamental group and those with infinite fundamental group: the former are all obtained as quotients of S^3 and are usually referred as *elliptic manifolds*. A prime 3–manifold with infinite fundamental group is called *non–elliptic prime manifold*: these spaces have non–compact universal cover (that is homeomorphic either to $S^2 \times \mathbb{R}$ or to $\mathbb{R}^3$) and torsion–free fundamental group.

Connected Sum of Homotopy Equivalences

In this subsection, we define the decomposition along spheres of homotopy equivalences between 3–manifolds. We first define the connected sum of maps.

For $i = 1, 2$, let us consider an orientation–preserving homotopy equivalence $f_i \colon N_i \to M_i$ between two oriented closed connected manifolds N_i, M_i. Up to homotopy, we can suppose there exist two closed balls $B_i \subset M_i$, $C_i \subset N_i$ such that f_i maps homeomorphically (and preserving the orientation) the ball C_i to the ball B_i. Let h_i be a homotopy inverse of f_i; since the degree of f_i is 1, we can choose h_i in such a way that $h_i^{-1}(C_i) = B_i$ and it sends B_i homeomorphically to C_i. We set $M_i' = M_i \backslash\backslash B_i$, $N_i' = N_i \backslash\backslash C_i$, and we fix an orientation–reversing identification $\partial C_1 \cong \partial C_2$. Since $f_1\big|_{C_1}$ and $f_2\big|_{C_2}$ are orientation–preserving homeomorphisms, the identification $\partial C_1 \cong \partial C_2$ induces an orientation–reversing identification $\partial B_1 \cong \partial B_2$ in such a way that whenever two points $x_1 \in C_1$, $x_2 \in C_2$ are identified, then $f_1(x_1) \in B_1$ and $f_2(x_2) \in B_2$ are identified as well.

Let us define

$$N = N_1 \# N_2 = N_1' \bigcup_{\partial C_1 \cong \partial C_2} N_2' ,$$
$$M = M_1 \# M_2 = M_1' \bigcup_{\partial B_1 \cong \partial B_2} M_2' ,$$

which are oriented closed connected manifolds.

In this setting, the connected sum

$$f_1 \# f_2 \colon N_1 \# N_2 \to M_1 \# M_2$$

is defined by gluing $f_1\big|_{N_1'}$ and $f_2\big|_{N_2'}$ along $\partial C_1 \cong \partial C_2$. This map is a well–defined orientation–preserving homotopy equivalence: indeed, from cellular approximation it follows that $h_1 \# h_2$ is a homotopy inverse of f.

In the following, whenever we write a connected sum $f_1 \# f_2$ between orientation–preserving homotopy equivalences, we always have a (possibly implicit) choice of balls B_1, B_2, C_1, C_2 as above and an identification $\partial C_1 \cong \partial C_2$. On the other hand, the homotopy class of $f_1 \# f_2$ does not depend on these choices (this is a consequence of cellular approximation and the fact that all balls inside a connected 3–manifold are isotopic).

If more than two orientation–preserving homotopy equivalences are given, then the connected sum is defined by summing iteratively starting from the last two terms; namely, if $f_i\colon N_i \to M_i$ are orientation–preserving homotopy equivalences for $i \in \{1, \ldots, n\}$, then

$$f_1 \# \ldots \# f_n = f_1 \# (f_2 \# (\ldots \# (f_{n-1} \# f_n) \cdots)) \, .$$

As before, this is a well defined orientation–preserving homotopy equivalence whose homotopy class does not depend on the choices done.

For $i \in \{1, \ldots, n\}$, let $g_i\colon N_i \to M_i$ be another orientation–preserving homotopy equivalence and suppose that the connected sum $g_1 \# \ldots \# g_n$ is well defined. If f_i and g_i are homotopic for every $i \in \{1, \ldots, n\}$, then the two maps $g_1 \# \ldots \# g_n$ and $f_1 \# \ldots \# f_n$ are homotopic as well.

Analogously, if the map f_i is homotopic to a homeomorphism for every $i \in \{0, \ldots, n\}$, then the map $f_1 \# \ldots \# f_n$ is homotopic to a homeomorphism.

Splitting Homotopy Equivalences

We are now ready to define decompositions of homotopy equivalences and to state a result by Hendriks and Laudenbach assuring that every orientation–preserving self–homotopy equivalence of an oriented closed connected 3–manifold splits (up to homotopy) along any separating sphere.

Let M, N be two oriented closed connected 3–manifolds and let $\Sigma = S_1 \sqcup \ldots \sqcup S_{n-1} \subset M$ be a decomposing system of spheres in M inducing the decomposition $M = M_1 \# \ldots \# M_n$. An orientation–preserving homotopy equivalence $f\colon N \to M$ *splits* along Σ if

- for every $i \in \{1, \ldots, n-1\}$, the map f is transverse to S_i and $S'_i = f^{-1}(S_i)$ is an embedded 2–sphere in N;
- $\Sigma' = S'_1 \sqcup \ldots \sqcup S'_{n-1}$ is a decomposing system of spheres inducing a decomposition $N = N_1 \# \ldots \# N_n$;
- up to rearranging the indices, for every $i \in \{1, \ldots, n\}$ there exists an orientation–preserving homotopy equivalence $f_i\colon N_i \to M_i$ such that $f = f_1 \# \ldots \# f_n$, where the connected sum is taken along the decomposing system of spheres Σ' (meaning that the connected sum is obtained by gluing the maps $f_1, \ldots, f_{n-1}$ along the spheres $S'_1, \ldots, S'_{n-1}$ according to the definition given above).

In this case we say that f sends the decomposition along Σ' to the decomposition along Σ and we denote by

$$f||_{N_i}\colon N_i \to M_i$$

the map f_i. Let us highlight that $f\big|\big|_{N_i}$ is not a restriction of f and N_i is not a subset of N.

Whenever we have a splitting homotopy equivalence $f\colon N \to M$ sending a decomposition $N_1 \# \ldots \# N_n$ of N to a decomposition $M_1 \# \ldots \# M_n$ of M, we always assume that the map $f\big|\big|_{N_i}$ goes from N_i to M_i (with the same index).

In addition, if for every $i \in \{1, \ldots, n\}$ the summands N_i and M_i are homeomorphic, then we say that *f preserves the types of homeomorphism* of the decomposition induced by Σ; notice that preserving the types of homeomorphism of a decomposition does not imply being homotopic to a homeomorphism on every summand of the decomposition.

An orientation–preserving homotopy equivalence $g\colon N \to M$ *homotopically splits* along Σ if it is homotopic to a homotopy equivalence $f\colon N \to M$ that splits along Σ.

It was proved by Hendriks and Laudenbach in [HL74] that every orientation–preserving homotopy equivalence between 3–manifolds homotopically splits along an embedded nontrivial sphere.

Theorem 4.5 ([HL74, Théorème de Scindement])**.** *Let M and N be two oriented closed connected 3–manifolds and let $S \subset M$ be an embedded separating sphere not homotopic to a point. Then every orientation–preserving homotopy equivalence $f\colon N \to M$ homotopically splits along S.*

In their paper, Hendriks and Laudenbach considered also the case in which S is not separating inside M.

We extend this result to decompositions with more than two summands.

Corollary 4.6. *Let M and N be two oriented closed connected 3–manifolds and Σ a decomposing system of spheres of M inducing a decomposition $M = M_1 \# \ldots \# M_n$ with no summands homeomorphic to S^3. Then every orientation–preserving homotopy equivalence $f\colon N \to M$ homotopically splits along Σ.*

Proof. The proof is by induction on the number of spheres of Σ. If Σ is empty, then there is nothing to prove. Fix $n > 0$ and suppose the statement is true whenever the system of spheres consists of less than n spheres. Let

$$\Sigma = S_1 \sqcup \ldots \sqcup S_n$$

be a decomposing system of spheres for M as in the statement, and

$$M = M_1 \# \ldots \# M_{n+1}$$

be the decomposition induced by Σ; without loss of generality, we can assume S_n separates M_{n+1} from M_n. We set $Y = M_n \# M_{n+1}$.

The decomposing system of spheres

$$\Sigma_1 = S_1 \sqcup \ldots \sqcup S_{n-1},$$

with $n-1$ components, induces the decomposition

$$M = M_1 \# \ldots \# M_{n-1} \# Y,$$

where, by construction, none of the summands is homeomorphic to S^3. By inductive hypothesis the map f homotopically splits along Σ_1: there exist a decomposition

$$N = N_1 \# \ldots \# N_{n-1} \# X, \tag{4.1}$$

orientation–preserving homotopy equivalences $f_i\colon N_i \to M_i$ (for $i \in \{1, \ldots, n-1\}$), and $h\colon X \to Y$ such that the map f is homotopic to

$$g_1 = f_1 \# \ldots \# f_{n-1} \# h;$$

moreover, $\Sigma_1' = g_1^{-1}(\Sigma_1) = S_1' \sqcup \ldots \sqcup S_{n-1}'$ is a decomposing system of spheres inducing the decomposition (4.1).

Since $Y = M_n \# M_{n+1}$ is a decomposition along the sphere S_n, Theorem 4.5 yields a splitting of the map $h\colon X \to Y$ along S_n: we get two 3–manifolds N_n and N_{n+1} and two orientation–preserving homotopy equivalences $f_n\colon N_n \to M_n$ and $f_{n+1}\colon N_{n+1} \to M_{n+1}$ such that $X = N_n \# N_{n+1}$, the map h is homotopic to $h' = f_n \# f_{n+1}$, and $(h')^{-1}(S_n) = S_n'$ is an embedded separating sphere.

Now we need to modify the map h' and the sphere S_n' in order to guarantee the well–definition of the connected sum $f_1 \# \ldots \# f_{n+1}$ along the spheres $S_1, \ldots, S_n$. Since Σ_1' is a decomposing system of spheres, every sphere in $S_i' \subset X$ corresponding to a sphere in Σ_1' bounds a ball in X; thus we can homotope h' in such a way that the sphere $S_n' = (h')^{-1}(S_n)$ does not intersect Σ_1'.

Moreover, if $S_i \in \{S_1, \ldots, S_{n-1}\}$ is contained in M_n or in M_{n+1}, (let us say in M_n), then we can suppose f_n splits along S_i ([HL74, Proposition 1.1]) so that $f_n^{-1}(S_i)$ is a sphere inside N_n. Thus, up to homotopy, we can suppose $h\big|_{S_i'} = f_n\big|_{S_i'}$.

Recollecting all the pieces together, one gets that:

- f is homotopic to $f_1 \# \ldots \# f_{n-1} \# h$, which, in turn, is homotopic to

  $$g = f_1 \# \ldots \# f_{n-1} \# h' = f_1 \# \ldots \# f_{n+1},$$

  where each $f_i\colon N_i \to M_i$ is an orientation–preserving homotopy equivalence;

- the union

  $$\Sigma' = g^{-1}(\Sigma) = \Sigma_1' \sqcup S_n' = S_1' \sqcup \ldots \sqcup S_n'$$

  is a decomposing system of spheres;

- the system Σ' induces the decomposition

$$N = N_1 \,\#\, \ldots \,\#\, N_{n+1}.$$

Therefore, the map f homotopically splits along Σ. $\square$

4.2 Homotopy Equivalences of Prime Three–Manifolds

In this section we recall several results that imply Theorem 4.2 for prime 3–manifolds.

We start with the following theorem, due to Kreck and Lück, which already takes care of the non–elliptic case.

Theorem 4.7 ([KL05]). *If M is an oriented closed connected 3–manifold with torsion–free fundamental group, then* $\mathrm{MCG}(M) = \mathrm{MCG}_{\mathrm{h}}(M)$.

Every non–elliptic prime manifold has torsion–free fundamental group. Moreover, if $M = M_1 \,\#\, \ldots \,\#\, M_n$ is a prime decomposition of an oriented closed connected 3–manifold M, then the fundamental group of M is given by

$$\pi_1(M) = \pi_1(M_1) \,*\, \ldots \,*\, \pi_1(M_n).$$

Thus, $\pi_1(M)$ is torsion–free if and only if for every $i \in \{1, \ldots, n\}$ the group $\pi_1(M_i)$ is torsion–free or, equivalently, the prime manifold M_i is non–elliptic.

We now deal with the elliptic case. Let M be an oriented closed connected manifold and $p \in M$ a base point. Recall from Section 1.2 that $\mathrm{MCG}(M, p)$ denotes the set of orientation–preserving self–homotopy equivalences in M which preserve the point p up to homotopy relative to the point p. Since every class $\varphi \in \mathrm{MCG}(M, p)$ induces an automorphism

$$\varphi_{\#} \colon \pi_1(M, p) \to \pi_1(M, p)$$

of the fundamental group $\pi_1(M, p)$ of M, we have a well–defined homomorphism

$$\begin{array}{rccc} \Theta_{M,p} \colon & \mathrm{MCG}(M, p) & \to & \mathrm{Aut}(\pi_1(M, p)) \, . \\ & \varphi & \mapsto & \varphi_{\#} \end{array}$$

In general, this map is neither injective nor surjective ([Rut97]).

Smallen proved that whenever M is an elliptic 3–manifold, the map $\Theta_{M,p}$ is injective.

Theorem 4.8 ([Sma74]). *Let G be a finite group that acts on S^3 without fixed points and let $p \in S^3/G$. Then $\Theta_{S^3/G,\,p}$ is injective.*

Let $M = S^3/G$. Since the fundamental group $\pi_1(M,p) = G$ is a finite group, every element in $\mathrm{Aut}(G)$ has order less than $c!$, where $c = |G|$ is the cardinality of G. In particular, for every self–homotopy equivalence $f\colon (M,p) \to (M,p)$, there exists an integer k satisfying $1 \le k \le c!$ and $(f_\#)^k = \mathrm{id}_G$. Thanks to the result by Smallen, the map f^k is homotopic to the identity (that is a homeomorphism).

As every self–homotopy equivalence of a connected manifold is homotopic to a self–homotopy equivalence fixing a point, we have the following.

Proposition 4.9. *Let $M = S^3/G$ be an elliptic manifold and $f\colon M \to M$ a homotopy equivalence. Then there exists an integer k such that $1 \le k \le |G|!$ and f^k is homotopic to a homeomorphism.*

Remark 4.10. It should be noted that $|G|!$ is not the optimal bound. For example, every automorphism of G fixes the identity element $e_G \in G$ and acts as a permutation on the remaining $|G| - 1$ elements. Thus, a better (but still very loose) bound on the order of elements in $\mathrm{Aut}(G)$ is $(|G| - 1)!$. However, we believe this number is still far from being optimal: for example, it does not take into account that many homotopy equivalences are homotopic to a homeomorphism.

4.3 The Non–Prime Case

So far, we proved that for every prime 3–manifold M there exists a constant A_M (depending only on the manifold M) with the following property: for every orientation–preserving homotopy equivalence $f\colon M \to M$ there exists a positive integer $k \in \{1, \ldots, A_M\}$ such that f^k is homotopic to a homeomorphism. The constant A_M is 1 when M is non–elliptic and it may be chosen to be $c!$ when M is elliptic and the fundamental group $\pi_1(M)$ is a finite group of cardinality c.

Moreover, we saw that for every oriented closed connected 3–manifold M, every decomposing system of spheres $\Sigma \subset M$ and every orientation–preserving homotopy equivalence $f\colon M \to M$, the map f homotopically splits along Σ.

We need to put these results together. One of the main problems is that the prime decomposition is unique only up to homeomorphism: given two prime decompositions, $M = M_1^1 \# \ldots \# M_n^1$ along Σ_1 and $M = M_1^2 \# \ldots \# M_n^2$ along Σ_2, there exists a permutation $\sigma \in \mathcal{S}_n$ so that M_i^1 is homeomorphic to $M_{\sigma(i)}^2$ (for every $1 \le i \le n$); however, there is no way to identify the spheres in Σ_1 with the spheres in Σ_2 and there is no canonical homeomorphism between the summands of the first decomposition and the ones of the second decomposition. Moreover, even if $f\colon M \to M$ is an orientation–preserving homotopy equivalence sending the decomposition induced by Σ_2 to the decomposition induced by Σ_1, there is no guarantees that f preserves the types of homeomorphism of the decomposition along the system Σ_1.

With the following lemma we find powers f^{β_k} of f homotopic to some orientation–preserving homotopy equivalences g_k that preserve the types of homeomorphism of some decompositions.

Lemma 4.11. *Let M be an oriented closed connected 3–manifold, m a positive integer and $f\colon M \to M$ an orientation–preserving homotopy equivalence. Then there exist $m+1$ decomposing systems of spheres $\Sigma_0, \ldots, \Sigma_m$ inducing a prime decomposition on M, m positive integers $\beta_1, \ldots, \beta_m$ and m orientation–preserving self–homotopy equivalences $g_1, \ldots, g_m\colon M \to M$ of M such that for every $k \in \{1, \ldots, m\}$ the map g_k*

1. *splits along Σ_{k-1} and $g_k^{-1}(\Sigma_{k-1}) = \Sigma_k$,*
2. *preserves the types of homeomorphism of Σ_{k-1},*
3. *is homotopic to f^{β_k}.*

Moreover, the integers β_k satisfy $\sum_{k=1}^{m} \beta_k \leq m \cdot r!$, where r is the number of elliptic manifolds appearing in a prime decomposition of M.

Before proving this lemma, we should remark that if a homotopy equivalence g splits along Σ_1, with $g^{-1}(\Sigma_1) = \Sigma_2$, and preserves the types of homeomorphism of the decomposition induced by Σ_2, then we do not know whether the map g (or some homotopic map splitting along Σ_2) preserves the types of homeomorphism of the decomposition induced by the system $g^{-1}(\Sigma_2)$. Thus, even if the map g preserves the types of homeomorphism of Σ_1 and it is homotopic to f^{β} for some integer $\beta > 0$, we cannot obtain the desired result by taking powers of g.

The statement is still true when $m = \infty$.

Proof. When $M = S^3$ the statement is clear, so let us suppose M is not homeomorphic to S^3 and fix a prime decomposition

$$M = M_1^0 \# \ldots \# M_n^0$$

induced by a decomposing system of spheres $\Sigma'_0 \subset M$.

Step 1. For every integer $k \geq 1$ we construct by recursion a decomposing system of spheres $\Sigma'_k \subset M$, inducing a prime decomposition $M = M_1^k \# \ldots \# M_n^k$, and a homotopy equivalence $f_k\colon M \to M$, homotopic to f and sending the decomposition along Σ'_k to the decomposition along Σ'_{k-1}.

Suppose we constructed them for every integer $k \in \{1, \ldots, t-1\}$. By Corollary 4.6 there exists a homotopy equivalence f_t homotopic to f and splitting along Σ'_{t-1}; set $\Sigma'_t = f_t^{-1}(\Sigma'_{t-1})$, that is a decomposing system of spheres inducing a decomposition

$$M = M_1^t \# \ldots \# M_n^t.$$

As for every $s \in \{1, \dots, n\}$ the map $f_t||_{M_s^t} \colon M_s^t \to M_s^{t-1}$ is a homotopy equivalence and M_s^{t-1} is not homeomorphic to S^3 (by definition of prime decomposition), then M_s^t is not homeomorphic to S^3 either; hence Σ_t induces a prime decomposition.

Let us set

$$f_{s,k} = f_k||_{M_s^k} \colon M_s^k \to M_s^{k-1}.$$

As for $k \in \{1, \dots, t\}$ the map $f_k = \#_{s=1}^{n} (f_{s,k})$ is homotopic to f, the map f^t is homotopic to

$$f_1 \circ \dots \circ f_t = \mathop{\#}_{s=1}^{n} (f_{s,1} \circ \dots \circ f_{s,t}).$$

Step 2. We will find a sequence of integers $(\alpha_k)_{0 \le k \le m}$ for which $M_s^{\alpha_k}$ and $M_s^{\alpha_j}$ are homeomorphic for every $j, k \in \{0, \dots, m\}$ and every $s \in \{1, \dots, n\}$.

Set $M_s := M_s^0$; up to rearranging the indices, we can suppose M_s is elliptic for $s \in \{1, \dots, r\}$ and non–elliptic for $s \in \{r+1, \dots, n\}$.

Thanks to Theorem 4.7, for every $k > 0$ and $s \in \{r+1, \dots, n\}$ the orientation–preserving homotopy equivalence $f_{s,1} \circ \dots \circ f_{s,k}$ is homotopic to a homeomorphism and the summand M_s^k is homeomorphic to M_s.

On the other hand, because of the uniqueness of prime decompositions, for every integer $k \ge 0$ there exists a permutation $\sigma_k \in \mathcal{S}_r$ such that for every $s \in \{1, \dots, r\}$ the prime summand M_s^k is homeomorphic to $M_{\sigma_k(s)}$. Since the elements in $\mathcal{S}_r$ are exactly $r!$, the pigeonhole principle provides a permutation in $\mathcal{S}_r$ appearing at least $m+1$ times in $(\sigma_k)_{0 \le k \le m \cdot r!}$; let us denote by τ this permutation and let

$$0 \le \alpha_0 < \alpha_1 < \dots < \alpha_m \le m \cdot r!$$

be such that $\sigma_{\alpha_k} = \tau$ for every $k \in \{0, \dots, m\}$.

By construction, for every $s \in \{1, \dots, r\}$ and $j, k \in \{0, \dots, m\}$, the prime summand $M_s^{\alpha_j}$ is homeomorphic to $M_{\sigma_{\alpha_j}(s)} = M_{\tau(s)} = M_{\sigma_{\alpha_k}(s)}$, which, in turn, is homeomorphic to $M_s^{\alpha_k}$.

Conclusion. Let us set $\Sigma_k = \Sigma'_{\alpha_k}$ (for $0 \le k \le m$), $\beta_k = \alpha_k - \alpha_{k-1}$ (for $1 \le k \le m$) and

$$g_k = f_{\alpha_{k-1}+1} \circ f_{\alpha_{k-1}+2} \circ \dots \circ f_{\alpha_k} = \mathop{\#}_{s=1}^{n} \left(f_{s,\alpha_{k-1}+1} \circ f_{s,\alpha_{k-1}+2} \circ \dots \circ f_{s,\alpha_k} \right),$$

(for $1 \le k \le m$), where

$$f_{s,\alpha_{k-1}+1} \circ f_{s,\alpha_{k-1}+2} \circ \dots \circ f_{s,\alpha_k} \colon M_s^{\alpha_k} \to M_s^{\alpha_{k-1}}$$

is between homeomorphic summands.

Now, Items 1 to 3 are all direct consequences of the construction; moreover,

$$\sum_{k=1}^{m} \beta_k = \sum_{k=1}^{m} (\alpha_k - \alpha_{k-1}) = \alpha_m - \alpha_0 \leq m \cdot r!\,.$$

□

The following lemma is useful to understand how the maps g_k act on the fundamental groups of the elliptic summands of the manifold M.

Lemma 4.12. *Let* $G_1, \ldots, G_r$ *be finite groups of cardinality* $c_1, \ldots, c_r$ *respectively, and set* $m = (c_1 \cdots c_r)!$. *Suppose that for any* $s \in \{1, \ldots, r\}$ *and* $k \in \{1, \ldots, m\}$ *an automorphism* $\varphi_{s,k} \colon G_s \to G_s$ *is given. Then, there exist two integers* $0 \leq n_1 < n_2 \leq m$ *such that the map*

$$\varphi_{s,n_1+1} \circ \varphi_{s,n_1+2} \circ \ldots \circ \varphi_{s,n_2} \colon G_s \to G_s$$

is the identity id_{G_s} *for every* $s \in \{1, \ldots, r\}$.

Proof. We first prove the statement for $r = 1$: let G be a finite group of cardinality c and $(\varphi_k)_{1 \leq k \leq c!}$ a (finite) sequence of automorphisms of G. Set

$$\sigma_k = \varphi_1 \circ \ldots \circ \varphi_k$$

for $k \in \{1, \ldots, c!\}$ and $\sigma_0 = \mathrm{id}$.

As G has cardinality c, $\mathrm{Aut}(G)$ has at most $c!$ elements. By the pigeonhole principle, there exist two integers $0 \leq n_1 < n_2 \leq c!$ with the property that $\sigma_{n_1} = \sigma_{n_2}$. It follows that

$$\varphi_{n_1+1} \circ \ldots \circ \varphi_{n_2} = \sigma_{n_1}^{-1} \circ \sigma_{n_2} = \mathrm{id}_G,$$

as desired.

If $r \geq 1$, then set

$$G = G_1 \times \ldots \times G_r,$$

that is a finite group with $c = c_1 \cdots c_r$ elements, and consider for every $k \in \{1, \ldots, c!\}$ the automorphism given by

$$\varphi_k = (\varphi_{1,k}, \ldots, \varphi_{r,k}) : G \to G.$$

The thesis is now a consequence of the case $r = 1$. □

We are ready to prove Theorem 4.2. As seen in Remark 4.4, it is enough to prove it with the further assumptions that the manifold is connected and the homotopy equivalence is orientation–preserving.

Proof. Let $M = M_1 \# \dots \# M_n$ be a prime decomposition of an oriented closed connected 3–manifold M and suppose $M_1, \dots, M_r$ have finite fundamental groups of cardinality $c_1, \dots, c_r$ respectively, and $M_{r+1}, \dots, M_n$ have infinite fundamental groups.

For every $s \in \{1, \dots, r\}$ let us fix a base point $p_s \in M_s$ and set $G_s = \pi_1(M_s, p_s)$. Set $m = (c_1 \cdots c_r)!$. By applying Lemma 4.11 to the manifold M, the map f, and the integer m, we find $m+1$ decomposing systems of spheres $\Sigma_0, \dots, \Sigma_m$, m positive integers $\beta_1, \dots, \beta_m$ and m orientation–preserving self–homotopy equivalences $g_1, \dots, g_m$ of M such that for every $k \in \{1, \dots, m\}$ the map g_k

- splits along Σ_{k-1} and $g_k^{-1}(\Sigma_{k-1}) = \Sigma_k$,
- preserves the types of homeomorphism of Σ_{k-1},
- is homotopic to f^{β_k}.

Moreover, it holds that $\sum_{k=1}^m \beta_k \leq m \cdot r!$. Let

$$M = M_1^k \# \dots \# M_n^k$$

be the decomposition induced by Σ_k. As the map g_k preserves the types of homeomorphism of Σ_{k-1} for every $k \in \{0, \dots, m\}$, the summand M_s^k is homeomorphic to M_s for every $s \in \{1, \dots, n\}$.

For every $s \in \{1, \dots, r\}$ there exists a finite sequence of orientation–preserving homeomorphisms

$$\left(\theta_{s,k} \colon M_s \to M_s^k\right)_{0 \leq k \leq m}$$

such that the map

$$\tau_{s,k} := \left(\theta_{s,k-1}^{-1} \circ g_k\big|\big|_{M_s^k} \circ \theta_{s,k}\right) : M_s \to M_s$$

is an orientation–preserving self–homotopy equivalence of M_s fixing the point p_s.

We denote by

$$\varphi_{s,k} = (\tau_{s,k})_\# \colon G_s \to G_s$$

the map induced by $\tau_{s,k}$ on the fundamental group $\pi_1(M_s, p_s) = G_s$. As $\tau_{s,k}$ is a homotopy equivalence, the map $\varphi_{s,k}$ is an automorphism of G_s.

Thus, Lemma 4.12 applied to the finite groups $G_1, \dots, G_r$ and the automorphisms $\varphi_{s,k} \colon G_s \to G_s$ provides two integers $0 \leq n_1 < n_2 \leq m$ such that the map

$$\varphi_{s,n_1+1} \circ \varphi_{s,n_1+2} \circ \dots \circ \varphi_{s,n_2} \colon G_s \to G_s$$

is the identity on G_s for every $s \in \{1, \dots, r\}$; since M_s is elliptic, Theorem 4.8 implies that the homotopy equivalence

$$\tau_{s,n_1+1} \circ \dots \circ \tau_{s,n_2} = \theta_{s,n_1}^{-1} \circ \left(g_{n_1+1}\big|\big|_{M_s^{n_1+1}} \circ \dots \circ g_{n_2}\big|\big|_{M_s^{n_2}}\right) \circ \theta_{s,n_2} \colon M_s \to M_s$$

is homotopic to the identity of M_s. It follows that for every $s \in \{1, \ldots, n\}$ the map

$$g_{n_1+1}\big|\big|_{M_s^{n_1+1}} \circ \ldots \circ g_{n_2}\big|\big|_{M_s^{n_2}} : M_s^{n_2} \to M_s^{n_1}$$

is homotopic to a homeomorphism h_s (when $r < s \leq n$, the existence of h_s is ensured by Theorem 4.7).

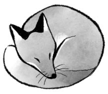
Let us set $t = \sum_{k=n_1+1}^{n_2} \beta_k$; the map $f^t = f^{\beta_{n_1+1}} \circ \ldots \circ f^{\beta_{n_2}}$ is homotopic to

$$g_{n_1+1} \circ \ldots \circ g_{n_2} = \mathop{\#}_{s=1}^{n} \left(g_{n_1+1}\big|\big|_{M_s^{n_1+1}} \circ \ldots \circ g_{n_2}\big|\big|_{M_s^{n_2}} \right),$$

which, in turn, is homotopic to the map $h = \#_{s=1}^{n} h_s$. Being a connected sum of homeomorphisms, the map h is a homeomorphism as well. Moreover, by construction it holds that

$$t = \sum_{k=n_1+1}^{n_2} \beta_k \leq \sum_{k=1}^{m} \beta_k \leq m \cdot r! = r! \cdot (c_1 \cdots c_r)! = A_M.$$

□



Chapter 5

General Results

In this Chapter, we employ what we have done so far to prove several results on manifolds of any dimensions.

5.1 Non–Vanishing of Real Filling Volume

In Chapter 4, we proved that the filling volume is trivial on orientation–preserving self–homotopy equivalences of oriented closed 3–manifolds. The following result shows that this phenomenon is an exception of dimension 3.

Theorem 5.1 (Theorem 7)**.** *Let $n \neq 1, 3$ be a positive integer. Then, there exist an orientable closed n–manifold M and a class $\varphi \in \mathrm{MCG}(M)$ such that $\mathrm{FV}_{\mathbb{R}}(\varphi) > 0$.*

Proof. Thanks to Corollary 2.8, it suffices to find examples of n–dimensional mapping tori with positive simplicial volume for every $n \neq 2, 4$. Such examples are given in [BN20, Corollary 1.4]. $\square$

The examples of mapping tori with non–vanishing simplicial volume described in [BN20, Corollary 1.4] are obtained by taking the product of 3–dimensional mapping tori with arbitrary manifolds with positive simplicial volume. On the other hand, the 3–dimensional case is well known: indeed, for every hyperbolic surface S and every pseudo–Anosov map $f \colon S \to S$, the mapping torus S_f is hyperbolic, hence it has positive simplicial volume.

Other interesting examples have been recently discovered in any odd dimension $n \geq 5$. For $n = 5$, one can consider the mapping torus V constructed in [Fuj] starting from the finite volume (non–compact) hyperbolic manifold described in [IMM22]. The closed 5–manifold V has positive simplicial volume: indeed, V is aspherical (being non–positively curved) and $\pi_1(V)$ is relatively hyperbolic, hence $\|V\| > 0$ by [BH13, Proposition 1.6]. As an alternative, in order to show that $\|V\| > 0$ one can apply [CW20, Corollary 1.6], since V

has non–positive sectional curvature and there exists a point $p \in V$ so that the sectional curvature along each tangent plane at p is negative. Finally, we refer the reader to [KR23, Corollary B] for other interesting constructions in odd dimensions $n \geq 7$.

5.2 Stable Integral Simplicial Volume and Integral Filling Volume

Thanks to Corollary 2.9 we know that $\|M_f\|_{\mathbb{Z}}^{\infty} \leq \mathrm{FV}_{\mathbb{Z}}(f)$ for every orientable compact manifold M and every orientation–preserving self–homotopy equivalence $f\colon M \to M$. The next natural question is whether the vanishing of the stable integral simplicial volume implies the vanishing of the integral filling volume. In this section, we address this problem by proving the following theorem.

Theorem 5.2 (Theorem 8)**.** *For every $n \geq 2$ there exist an oriented closed n–manifold M and an orientation–preserving homeomorphism $f\colon M \to M$ such that*

$$\mathrm{FV}_{\mathbb{Z}}(f) > 0\,, \qquad \|M_f\|_{\mathbb{Z}}^{\infty} = 0.$$

Since $\|M_f\|_{\mathbb{Z}}^{\infty} \geq \|M_f\| = \mathrm{FV}_{\mathbb{R}}(f)$, a consequence of the theorem above is the following:

Corollary 5.3. *For every $n \geq 2$ there exist an oriented closed n–manifold M and an element $\varphi \in \mathrm{MCG}(M)$ such that $\mathrm{FV}_{\mathbb{R}}(\varphi) = 0$, while $\mathrm{FV}_{\mathbb{Z}}(\varphi) \neq 0$.*

From Corollary 5.3 it is easily deduced the following:

Corollary 5.4. *For every $n \geq 2$ there exists an n–manifold M such that the restriction to $B_n(M;\mathbb{Z})$ of the filling norm on $B_n(M;\mathbb{R})$ is* not *equivalent to the integral filling norm. In other words, for every $\varepsilon > 0$ there exists an integral boundary $c \in B_n(M;\mathbb{Z})$ such that*

$$\frac{\|c\|_{\mathrm{fill},\mathbb{R}}}{\|c\|_{\mathrm{fill},\mathbb{Z}}} < \varepsilon.$$

In a different context, the fact that coefficients make a difference when computing filling norms has been recently pointed out in [LM22].

Proof of Corollary 5.4. Let M and φ be as in Corollary 5.3 and consider $f\colon M \to M$ a representative of φ. If z is an integral fundamental cycle for M and $c_m = f_*^m(z) - z$, then

$$\lim_{m\to+\infty} \frac{\|c_m\|_{\mathrm{fill},\mathbb{R}}}{\|c_m\|_{\mathrm{fill},\mathbb{Z}}} = \lim_{m\to+\infty} \frac{\|c_m\|_{\mathrm{fill},\mathbb{R}}/m}{\|c_m\|_{\mathrm{fill},\mathbb{Z}}/m} = \frac{\mathrm{FV}_{\mathbb{R}}(f)}{\mathrm{FV}_{\mathbb{Z}}(f)} = 0.$$

□

In order to prove Theorem 5.2 for $n \geq 3$ we need to exploit another variation of the simplicial volume:

Definition 5.5 ([Löh19])**.** The *weightless* R–simplicial volume is defined by

$$\|M\|_{(R)} = \min \left\{ m \in \mathbb{N} \;\middle|\; \sum_{i=1}^{m} a_i \sigma_i \text{ is an } R\text{–fundamental cycle for } M \right\}.$$

Roughly speaking, weightless simplicial volumes count only the number of simplices in a chain and ignore the (absolute value of the) coefficients. It is proved in [Löh19, Corollary 4.5] that the weightless simplicial volume of an even–dimensional hyperbolic manifold M (with coefficients in any principal ideal domain) is bounded from below by the volume of the manifold, up to a constant depending only on the dimension. We extend this result to hyperbolic manifolds of any dimension by showing the following.

Proposition 5.6. *Let M be an orientable closed n–dimensional hyperbolic manifold, and let $R = \mathbb{R}, \mathbb{Z}$ (in fact, R could be any principal ideal domain). Then*

$$\|M\|_{(R)} \geq \|M\| = \frac{\operatorname{vol}(M)}{v_n},$$

where v_n denotes the volume of the (unique up to isometry) regular ideal geodesic simplex in the hyperbolic n–space.

Proof. Let $\sum_{i=1}^{k} a_i \sigma_i$ be an R–fundamental cycle for M realizing the weightless simplicial volume of M, i.e., assume that $k = \|M\|_{(R)}$. The straightening operator on singular simplices (see, e.g., [Fri17, Chapter 8]) induces a chain map $S_* \colon C_\bullet(M; R) \to C_\bullet(M; R)$ which is homotopic to the identity; hence the cycle $z = \sum_{i=1}^{k} a_i S_*(\sigma_i)$ is still an R–fundamental cycle for M.

Let us denote by $\operatorname{supp}(S_*(\sigma_i))$ the image of $S_*(\sigma_i)$ in M. We observe that $\bigcup_{i=1}^{k} \operatorname{supp}(S_*(\sigma_i)) = M$. Indeed, otherwise z would be supported in $M \setminus \{p\}$ for some point $p \in M$, hence its class $[z] \in H_n(M; R)$ would lie in the image of the map $H_n(M \setminus \{p\}; R) \to H_n(M; R)$ induced by the inclusion. But $H_n(M \setminus \{p\}; R) = 0$, hence $[z]$ would be the zero class, against the hypothesis that z is an R–fundamental cycle for M.

Now the conclusion follows from an easy computation involving the volume of straight simplices: by definition, $S_*(\sigma_i)$ is the projection in M (via a locally isometric map) of a geodesic simplex in the hyperbolic n–space, and the volume of every geodesic simplex in the hyperbolic n–space is bounded from above by v_n [HM81, Pey02]; hence

$$\operatorname{vol}(M) = \operatorname{vol}\left(\bigcup_{i=1}^{k} \operatorname{supp}(S_*(\sigma_i)) \right) \leq \sum_{i=1}^{k} \operatorname{vol}(\operatorname{supp}(S_*(\sigma_i))) \leq k v_n \;,$$

which implies

$$\|M\|_{(R)} = k \geq \frac{\operatorname{vol}(M)}{v_n} = \|M\|. \qquad \square$$

We show the following result, which in turn proves Theorem 5.2 for $n \geq 3$:

Proposition 5.7. *Let S be an orientable closed hyperbolic surface, and $g\colon S \to S$ be a pseudo–Anosov homeomorphism. Set $M = S \times X \times S^1$, where X is any orientable closed manifold (possibly a point), and let $f\colon M \to M$ be given by $f(x, \alpha, \beta) = (g(x), \alpha, \beta)$. Then*

$$\|M_f\|_{\mathbb{Z}}^{\infty} = 0 \ , \qquad \mathrm{FV}_{\mathbb{Z}}(f) > 0.$$

Proof. For every $m \in \mathbb{N}$, let $N_m = S_{g^m}$ be the mapping torus of g^m. Since g is pseudo–Anosov, N_m is an orientable closed hyperbolic 3–manifold. Since f^m acts as the identity on the factor $X \times S^1$, it is readily seen that the mapping torus M_{f^m} splits as a product $M_{f^m} = N_m \times (X \times S^1)$. In particular, since $\left\|V \times S^1\right\|_{\mathbb{Z}}^{\infty} = 0$ for every orientable closed manifold V, we have that

$$\|M_f\|_{\mathbb{Z}}^{\infty} = \left\|N_1 \times X \times S^1\right\|_{\mathbb{Z}}^{\infty} = 0.$$

Let us now prove that $\mathrm{FV}_{\mathbb{Z}}(f) > 0$. Let us fix $m \in \mathbb{N}$ and an integral fundamental cycle z for M. Observe that [Löh19, Proposition 2.10] implies that

$$\begin{aligned}
\|M_{f^m}\|_{\mathbb{Z}} &= \left\|N_m \times (X \times S^1)\right\|_{\mathbb{Z}} \\
&\geq \left\|N_m \times (X \times S^1)\right\|_{(\mathbb{Z})} \\
&\geq \max\left\{\|N_m\|_{(\mathbb{Z})}, \left\|X \times S^1\right\|_{(\mathbb{Z})}\right\} \\
&\geq \|N_m\|_{(\mathbb{Z})}.
\end{aligned}$$

By Proposition 5.6 we have $\|N_m\|_{(\mathbb{Z})} \geq \|N_m\| = m \cdot \|N_1\|$, where the last equality is due to the fact that N_m is the total space of a degree m covering of N_1. We thus get

$$\|M_{f^m}\|_{\mathbb{Z}} \geq m \cdot \|N_1\|.$$

Therefore, by applying Lemma 2.10 to the map f^m we obtain

$$\|f_*^m(z) - z\|_{\mathrm{fill},\mathbb{Z}} \geq \|M_{f^m}\|_{\mathbb{Z}} - (n+1)\|z\|_1 \geq m \cdot \|N_1\| - (n+1)\|z\|_1,$$

hence

$$\frac{\|f_*^m(z) - z\|_{\mathrm{fill},\mathbb{Z}}}{m} \geq \|N_1\| - \frac{(n+1)\|z\|_1}{m}.$$

By taking the limit of this inequality as $m \to +\infty$ we conclude that

$$\mathrm{FV}_{\mathbb{Z}}(f) \geq \|N_1\| > 0,$$

as desired. □

We can finally prove Theorem 5.2.

Proof of Theorem 5.2. For $n = 2$ we consider $M = T$ the 2–dimensional torus and $f\colon T \to T$ an Anosov map. Then we have that $\mathrm{FV}_{\mathbb{Z}}(f) > 0$ by Proposition 3.10, while $\|T_f\|_{\mathbb{Z}}^{\infty} = \|T_f\| = 0$ by [FLMQ21, Theorem 1].

For $n > 2$, it suffices to take the manifolds constructed in Proposition 5.7. □

5.3 Gromov–Hyperbolic Fundamental Group

Being a length function (Proposition 2.5), the invariant FV_R vanishes on finite–order elements in $\mathrm{MCG}(M, \partial M)$. Using this fact we readily deduce the following.

Corollary 5.8. *Let M be an orientable closed connected aspherical manifold of dimension at least 3, let R be a normed ring with unity, and suppose that $\pi_1(M)$ is Gromov hyperbolic. Then $\mathrm{FV}_R(f) = 0$ for every orientation–preserving self–homotopy equivalence $f\colon M \to M$.*

Examples of manifolds that satisfy the hypothesis of this corollary are orientable closed connected negatively curved manifolds of dimension at least 3.

Proof. By [Gro87, Theorem 5.4A], the group $\mathrm{Out}(\pi_1(M))$ is finite and it holds that $\mathrm{MCG}(M) = \mathrm{Out}^+(\pi_1(M)) < \mathrm{Out}(\pi_1(M))$. In particular, every element in $\mathrm{MCG}(M)$ has finite order. ☐

5.4 Uniform Boundary Condition

We say that a pair of topological spaces (X, A) satisfies the *uniform boundary condition* in dimension n (or, briefly, *n–UBC*) over R (a normed ring with unity) if the norms $\|\cdot\|_1$ and $\|\cdot\|_{\mathrm{fill},R}$ on $B_n(X, A; R)$ are Lipschitz equivalent, i.e., if there exists $K > 0$ such that, for every boundary $b \in B_n(X, A; R)$, there exists a chain $c \in C_{n+1}(X, A; R)$ such that $\partial c = b$ and $\|c\|_1 \leq K \cdot \|b\|_1$.

Note that when $A = \emptyset$ the definition of n–UBC coincides with the one given in [MM85].

Proposition 5.9. *Let R be a normed ring with unity. Suppose that M is an orientable compact n–manifold with (possibly empty) boundary ∂M. If $(M, \partial M)$ satisfies n–UBC over R, then $\mathrm{FV}_R(\varphi) = 0$ for every mapping class $\varphi \in \mathrm{MCG}(M, \partial M)$.*

Proof. If $f\colon (M, \partial M) \to (M, \partial M)$ is an orientation–preserving self–homotopy equivalence, and $K > 0$ is as in the definition of n–UBC, then for every R–fundamental cycle of $(M, \partial M)$ it holds that

$$\begin{aligned}
\mathrm{FV}_R(f) &= \lim_{m\to\infty} \frac{\|f_*^m(z) - z\|_{\mathrm{fill},R}}{m} \leq \lim_{m\to\infty} \frac{K \cdot \|f_*^m(z) - z\|_1}{m} \\
&\leq \lim_{m\to\infty} \frac{K \cdot (\|f_*^m(z)\|_1 + \|z\|_1)}{m} \leq \lim_{m\to\infty} \frac{2K \cdot \|z\|_1}{m} = 0.
\end{aligned}$$

☐

It was shown in [MM85] that if $\partial M = \emptyset$ and $\pi_1(M)$ is amenable, then M satisfies n–UBC over $\mathbb{R}$ for every $n \geq 1$. We then get the following:

Corollary 5.10. *If M is an orientable closed manifold and $\pi_1(M)$ is amenable, then* $\mathrm{FV}_{\mathbb{R}}(\varphi) = 0$ *for every* $\varphi \in \mathrm{MCG}(M)$.

Corollary 5.10 may be easily deduced also from Corollary 2.8, together with the fact that the simplicial volume of any fiber bundle with an amenable fiber of positive dimension vanishes [Gro82, FM23, LS20].

Since $\mathrm{FV}_{\mathbb{Z}}$ is positive for any Anosov map on the torus (Proposition 3.10) and the torus has an amenable fundamental group, Corollary 5.10 cannot hold if one replaces $\mathrm{FV}_{\mathbb{R}}$ with $\mathrm{FV}_{\mathbb{Z}}$. Moreover, it follows from Proposition 5.9 that the 2–dimensional torus does not satisfy the 2–UBC over $\mathbb{Z}$. This gives a negative answer in the case $n = 2$ to [FL21, Question 9.4] asked by Fauser and Löh.

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