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Nonmonotonic and normative reasoning: a unified proof-theoretic framework

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Abstract

Rational agents are mostly located in dynamic environments, thereby handling incomplete information concerning the world and the rules that should govern their behavior. When elaborating such incomplete information, agents reach defeasible conclusions, that may be withdrawn when further information is available. Nonmonotonic logics are designed with the aim of modeling this mode of reasoning. This dissertation investigates different kinds of nonmonotonic logic through the *lens* of structural proof-theory. Specifically, the thesis proposes a uniform proof-theoretic platform for monotonic and nonmonotonic extensions of classical propositional logic, based on combinations of sequents and antisequents (i.e., sequents for underivability) framed in suitable Gentzen-style calculi. The first case study is abductive reasoning, defined as the search for the missing premise in a deductively invalid argument. For any given abductive problem, we provide a syntactical procedure to generate an expected solution which does not coincide with the deductively minimal one, and which is a natural candidate for being the result of an inference to the best explanation. Next, we introduce the notion of hybrid hypersequent, where sequents and antisequents are composed in parallel to provide contrary updating on the derivation of the conclusion. We show that hybrid hypersequents are flexible enough to provide (weakly) analytic calculi for a number of logics for nonmonotonic and normative reasoning: default logics, a weak version of preferential logic **R** corresponding to base-generated belief revision, constrained I/O logics. Crucially, this proof-theoretic approach does not rely on *ad hoc* extensions of the underlying language to formalize extra-logical rules. Lastly, we present a modified notion of controlled sequent, where the turnstile is annotated with sets of formulas to prescribe what should or should not be entailed by the formulas in the antecedent. We introduce controlled sequent calculi for deontic reasoning grounded in default logic, showing that the introduction of suitable extra-logical rules permits to navigate paradoxical, dilemmatic or dynamic deontic scenarios in accordance with their intuitive assessment.

CHAPTER 1

Introduction

Traditionally, proofs and refutations are employed to represent the inferential dynamics of deductive reasoning: proofs are formal objects conveying deductively valid arguments, whereas refutations are formal objects conveying deductively invalid arguments. In classical logic (as well as in a large family of non-classical logics), deductively valid arguments are stable under the addition of new premises: if a statement can be legitimately inferred from a set of premises, it can be inferred *a fortiori* from a larger set of premises. The addition of premises whose topic is unrelated with the topic of the conclusion, or which are plainly false, does not alter the validity of the argument.

Rational agents are mostly located in dynamic environments, thereby handling incomplete information concerning the world and the rules that should govern their behavior. When elaborating such incomplete information, agents reach defeasible conclusions, that may be withdrawn when further information is available. Nonmonotonic logics are designed precisely with the aim of modeling this mode of reasoning. In such logics, valid arguments may not be stable under the addition of new premises: the information delivered by the additional premises may disable inferential pathways which could be previously followed. This dissertation investigates different kinds of nonmonotonic logic through the *lens* of structural proof-theory.

To represent the inferential dynamics of non-monotonic reasoning, we define inference systems where proofs and refutations interact. In particular, we design Gentzen-style calculi where sequents and antisequents – i.e., sequents for refutability – are suitably combined. In these systems, extra-logical information is introduced by axioms (Chapter 2) and distinct, extra-logical inference rules (Chapters 5 - 8). Next, each sequent occurring in a derivation is paired with a set of antisequents composed in parallel with it, either explicitly (as in Chapters 5 - 7) or implicitly (as in Chapter 8). Such antisequents spell out negative constraints on the provability of the associated sequents. For each derivable sequent, the addition of new premises, either in the form of new extra-logical axioms (as in Chapters 5 - 7) or in the form of additional formulas on the left-hand side of the sequent symbol (as in Chapter 8), can inhibit the derivation of the associated antisequents. Invalidating such refutations corresponds to the violation of the negative constraints on the provability of the sequent: as a result, the derivation of the latter turns out to be unsound, unfit for being a proper proof.

In this thesis, we argue that antisequents with non-empty antecedent and single succedent can express abductive problems – that is, scenarios where a goal formula cannot be derived from a set of given premises (Chapter 4). Each provable sequent in our Gentzen-style calculi is thus associated with a set of open abductive problems: the violation of negative constraints on its provability corresponds to the overgeneration of solutions to the associated abductive problems. When consistency is preserved, such inference systems model an ideal agent that is able to solve specific abductive problems, provided that she remains unable to solve certain others.

Norms are often unstable and context-dependent: this places their behavior squarely within the realm of non-monotonic logic. The Gentzen-style calculi presented in this work accomodate defeasible reasoning about facts, obligations and permissions in a uniform framework (Chapters 7 - 8). In particular, we show that our systems are flexible enough to model rational agents navigating paradoxical, dilemmatic or dynamic deontic scenarios: the dissertation ends with the discussion of a wide range of notable examples.

Overview of the thesis

Chapter 2. The first part of this chapter sets the stage by introducing a standard sequent calculus for classical propositional logic, Kleene's G4. Bottom-up reading of G4 logical rules yields a terminating and confluent decomposition procedure for any formula. On this basis, we present a procedure to attain analytic sequent calculi for axiomatic extensions of classical propositional logic [**121**, **123**, **129**]. In the second part of the chapter, we present a rewriting procedure for minimizing the number and the size of the atomic sequents obtained by G4-based decomposition. Next, we show that the algorithm computes all the irreducible conjunctive normal forms of the underlying formula. As an upshot, we obtain a proof-theoretic version of Parikh's finest splitting theorem: we prove that the finest splitting of the formula coincides with the finest splitting of any of its irreducible versions.

Chapter 3. In this chapter, we investigate certain intricacies and peculiarities of the proof theory of deduction-refutation systems (D-R systems, henceforth), namely systems integrating theorems and antitheorems of a given logic. The logics considered here are classical, first-degree entailment-based (**FDE**-based, henceforth) and intuitionistic logics, formulated within a sequent calculus framework. Our primary focus is on establishing the general conditions under which anticut rules (the contrapositive versions of the familiar cut rule) can be eliminated from D-R sequent calculi, while distinguishing between two main variants of these systems. This proof-theoretic investigation leads to the introduction of a new, Gentzen-style refutation calculus for intuitionistic logic, which is interesting in its own right.

Chapter 4. Abductive reasoning involves finding the missing premise of an "unsaturated" deductive inference, thereby selecting a possible *explanans* for a conclusion based on a set of previously accepted premises. In this chapter, we explore abductive reasoning from a structural proof-theory perspective. We present a hybrid sequent calculus for classical propositional logic that uses sequents and antisequents to define a procedure for identifying the set of analytic hypotheses that a rational agent would be expected to select as *explanans* when presented with an abductive problem. Specifically, we show that this set may not include the deductively minimal hypothesis due to the presence of redundant information. We also establish that the set of all analytic hypotheses exhausts all possible solutions to the given problem. Finally, we propose a deductive criterion for differentiating between the best *explanans* candidates and other hypotheses.

Chapter 5. This chapter is devoted to the investigation of default reasoning from a structural proof-theoretic perspective. We introduce *hybrid* hypersequent calculi for propositional default logics, where extra-logical rules directly capture default rules, while parallel composition of sequents and antisequents formalizes contrary updating on the conclusions of extra-logical rules. We establish the admissibility of structural rules and the invertibility of logical rules, showing that cut-free proofs exhibit a weakened form of analyticity. Next, we prove that specific hybrid hypersequent calculi are sound and weakly complete with respect to credulous consequence based on Łukaszewicz extensions. Lastly, we propose a hypersequent-based decision method for skeptical consequence which relies on the *abductive* search of counterexamples, thereby circumventing the need for early computation of all extensions.

Chapter 6. In this chapter, we investigate AGM belief revision from a structural proof-theoretic perspective. First, we present a syntactic characterization of maximally consistent subsets of sets of clauses: on this basis, we offer a fine-grained, constructive approach to base-generated belief revision. Next, we introduce hybrid hypersequent calculi for base-generated revision, where sequents and antisequents are composed in parallel to formalize contrary updating on the provability of extra-logical axioms. Finally, we establish the admissibility of structural rules, showing that these calculi are sound and (weakly) complete with respect to a weak version of the preferential logic R.

Chapter 7. Constrained Input/Output (I/O) logics address scenarios involving conflicting conditional obligations. By allowing the withdrawal of norms to preserve consistency, these logics exhibit a close relationship with default logics. In this chapter, we provide a formal account of this relationship to develop a uniform Gentzen-style proof theory for the *entire* family of constrained I/O logics. Specifically, we introduce hypersequent calculi that integrate extra-logical rules to directly capture conditional obligations. The parallel composition of sequents and antisequents formalizes the dynamic updating of conclusions under consistency constraints. Crucially, such approach avoids any *ad hoc* extension of the underlying language. Moreover, we establish the admissibility of structural rules and the invertibility of logical rules, showing that cut-free proofs maintain a weakened form of analyticity. Finally, we leverage straightforward translations between hypersequent calculi for constrained I/O logics and those for default logics, to provide a modular treatment of disjunctive default logics and disjunctive normative inference.

Chapter 8 In many real-life settings, agents must navigate dynamic environments while reasoning under incomplete information and acting on a *corpus* of unstable, context-dependent, and often conflicting norms. In this chapter, we introduce a general, non-modal, proof-theoretic framework for deontic reasoning grounded in default logic. The basic feature of our approach is the notion of *controlled sequent* – a sequent annotated with sets of formulas that prescribe what should or should not be entailed by the formulas in the antecedent. When combined with distinct extra-logical rules representing defaults and norms, these control sets serve to record the conditions and constraints governing their applicability, thereby enabling local soundness checks for derived sequents. We prove that controlled sequent calculi enjoy admissibility of contraction and non-analytic cuts, and we establish their strong completeness with respect to credulous consequence based on default theories and normative systems. Finally, we show that controlled sequent calculi are a flexible and expressive foundation for resolving deontic conflicts and capturing dynamic deontic notions through suitabile extra-logical rules.

Sources

Some of the chapters of this dissertation are modified versions of published articles. Chapter 3 is based on [128], whereas Chapter 4 stems from [124]. In Chapter 5, we draw material from [127]; Chapter 6 is a modified version of [147].

CHAPTER 2

Proofs: analyticity, extra-logicality and relevance

This chapter introduces preliminary notions and results which will be employed throughout the thesis. Section 2.1 sets the stage by presenting a standard sequent calculus for classical propositional logic, Kleene's $\mathsf{G4}$. Bottom-up reading of $\mathsf{G4}$ logical rules yields a terminating and confluent decomposition procedure for any formula. On this basis, we define a procedure to attain analytic sequent calculi for axiomatic extensions of classical propositional logic. This approach to *supraclassical* logics ([**93**]) has been developed in full detail in [**121**, **123**, **129**]: here, we limit ourselves to recall the basics. In Section 2.2, the $\mathsf{G4}$-based decomposition procedure is refined to generate sets of atomic (anti)sequents where redundant information *modulo* classical equivalence is eliminated. As an upshot, we obtain a proof-theoretic approach to Parikh's finest splitting theorem [**118**, **81**].

2.1. Analyticity and extra-logicality: supraclassical logics

In this thesis, we shall employ a standard propositional language, consisting of a denumerable set of atoms $p, q, r, \ldots$, the unary connective $\neg$ and the binary connectives $\wedge, \vee, \rightarrow$ for negation, conjunction, disjunction and material implication, respectively. Moreover, we shall use Latin letters $A, B, C, \ldots$ to refer to arbitrary formulas.

In this section, we use capital Greek letters $\Gamma, \Delta, \Pi, \Sigma, \ldots$ to denote finite *multisets* of formulas, and $\Theta, \Lambda, \ldots$ to denote finite multisets of *atomic* formulas. For any context Γ we shall be adopting the following conventions: if $\Gamma = \{A_1, A_2, \ldots, A_n\}$, then

$$\Gamma^{\perp} = \{\neg A_1, \neg A_2, \ldots, \neg A_n\} \qquad \bigwedge \Gamma = A_1 \wedge A_2 \wedge \cdots \wedge A_n \qquad \bigvee \Gamma = A_1 \vee A_2 \vee \cdots \vee A_n.$$

For $\Gamma = \emptyset$, we set $\Gamma^{\perp} = \Gamma$, $\bigwedge \Gamma = \top$, and $\bigvee \Gamma = \bot$, where $\top$ and $\bot$ stand for an arbitrarily chosen tautology and contradiction, respectively. The *logical complexity* $\mathrm{C}(A)$ of a formula A is 1 if A is atomic, $\mathrm{C}(B) + 1$ if A is of the form $\neg B$ and $\mathrm{C}(B) + \mathrm{C}(C) + 1$ if A is of the form $B \otimes C$, with $\otimes \in \{\wedge, \vee, \rightarrow\}$. The measure C can be easily extended to any multiset $\Gamma = A_1, \ldots, A_n$ by writing $\mathrm{C}(\Gamma) = \mathrm{C}(A_1) + \ldots + \mathrm{C}(A_n)$.

We shall be dealing with Gentzen-style sequents $\Gamma \vdash \Delta$ as well as *antisequents* $\Gamma \dashv \Delta$, where $\Gamma \dashv \Delta$ is valid if, and only if, $\Gamma \vdash \Delta$ is invalid [**56**, **58**, **138**]. In the case of classical logic, an antisequent is valid if and only if there exists some Boolean valuation verifying all the formulas in Γ and falsifying all those in Δ.

AXIOMS

$$\frac{}{\Gamma, p \vdash p, \Delta}\,ax \qquad\qquad \frac{}{\Theta \dashv \Lambda}\,\overline{ax}$$

LOGICAL RULES

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}\,L\neg \qquad\qquad \frac{\Gamma \dashv \Delta, A}{\Gamma, \neg A \dashv \Delta}\,L'\neg$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\,R\neg \qquad\qquad \frac{\Gamma, A \dashv \Delta}{\Gamma \dashv \Delta, \neg A}\,R'\neg$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,L\wedge \qquad\qquad \frac{\Gamma, A, B \dashv \Delta}{\Gamma, A \wedge B \dashv \Delta}\,L'\wedge$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,R\wedge \qquad\qquad \frac{\Gamma \dashv \Delta, A_i}{\Gamma \dashv \Delta, A_1 \wedge A_2}\,R'_i\wedge$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\,L\vee \qquad\qquad \frac{A_i, \Gamma \dashv \Delta}{A_1 \vee A_2, \Gamma \dashv \Delta}\,L'_i\vee$$

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\,R\vee \qquad\qquad \frac{\Gamma \dashv \Delta, A, B}{\Gamma \dashv \Delta, A \vee B}\,R'\vee$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,L\to \qquad\qquad \frac{\Gamma \dashv \Delta, A}{\Gamma, A \to B \dashv \Delta}\,L'_1\to \qquad\qquad \frac{B, \Gamma \dashv \Delta}{\Gamma, A \to B \dashv \Delta}\,L'_2\to$$

$$\frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\,R\to \qquad\qquad \frac{\Gamma, A \dashv \Delta, B}{\Gamma \dashv \Delta, A \to B}\,R'\to$$

FIGURE 1. $\mathsf{G4}$ and $\overline{\overline{\mathsf{G4}}}$ sequent calculi

The system $\overline{\overline{\mathsf{G4}}}$ for classical propositional logic is imported from [**123**, **138**], with logical contexts handled as multisets of formulas. In particular, $\overline{\overline{\mathsf{G4}}}$ is obtained by adding to the original Kleene's $\mathsf{G4}$[1] [**79**, pp. 289-290, p. 306] the complementary axiom $\frac{}{\Theta \dashv \Lambda}\,\overline{ax}$, where $\Theta \cap \Lambda = \varnothing$, as well as distinct rules for antisequents. Whenever generalizing over the union of sequents and antisequents, we write $\Gamma \overset{*}{\vdash} \Delta$: the measure C can be extended to any (anti)sequent $\Gamma \overset{*}{\vdash} \Delta$ by writing $\mathrm{C}(\Gamma \overset{*}{\vdash} \Delta) = \mathrm{C}(\Gamma) + \mathrm{C}(\Delta)$.

A $\overline{\overline{\mathsf{G4}}}$ derivation π may end either in a sequent $\Gamma \vdash \Delta$ or in an antisequent $\Gamma \dashv \Delta$: in the first case, we say that π is a *proof* for $\Gamma \vdash \Delta$; in the second, π qualifies as a *refutation* for

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[1] Kleene's $\mathsf{G4}$ is the same as the $\mathsf{G3}$ calculus for classical propositional logic in [**167**].

$\Gamma \vdash \Delta$. As usual, the *height* of a derivation π – denoted by $h(\pi)$ – is defined as the number of nodes in its longest branch.

We can decompose any (anti)sequent $\Gamma \overset{*}{\vdash} \Delta$ into a set of atomic (anti)sequents, using bottom-up the rules $L\neg, R\neg, L\land, R\land, L\lor, R\lor, L \to, R \to$ in Figure 1, with $\overset{*}{\vdash}$ in place of $\vdash$, until each leaf of the resulting tree ends with an atomic (anti)sequent.

We recall two crucial features of the $\overline{\overline{\mathsf{G4}}}$ proof system:

THEOREM 2.1. *$\overline{\overline{\mathsf{G4}}}$ proves (refutes) $\Gamma \vdash \Delta$ if and only if the formula $\bigwedge \Gamma \to \bigvee \Delta$ is classically valid (*in*valid).*

THEOREM 2.2. *Maximal $\overline{\overline{\mathsf{G4}}}$-decomposition yields a unique set of atomic (anti)sequents.*

PROOF. For a proof see [14, 123]. $\square$

Theorem 2.2 allows us to directly refer to a unique set of atomic (anti)sequents associated with a certain (anti)sequent $\Gamma \overset{*}{\vdash} \Delta$, being such a decomposition independent of the specific derivation delivering it. In particular, we write $\mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$ to indicate the set of atomic (anti)sequents associated with $\Gamma \overset{*}{\vdash} \Delta$, and $\mathsf{top_c}(\Gamma \overset{*}{\vdash} \Delta)$ to indicate the set of the atomic (anti)sequents $\Theta \overset{*}{\vdash} \Lambda \in \mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$ such that $\Theta \cap \Lambda = \varnothing$ – where the subscript c stands for 'complementary' – and $\mathsf{top_i}(\Gamma \overset{*}{\vdash} \Delta)$ to indicate the set of atomic sequents $\Theta \overset{*}{\vdash} \Lambda \in \mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$ such that $\Theta \cap \Lambda \neq \varnothing$ – where the subscript i stands for 'identity'. Moreover, we use $cnf(A)$ to refer to the conjunction of the formula translations of the one-sided formulations of the (anti)sequents in $\mathsf{top_c}(\overset{*}{\vdash} A)$ (if any).

EXAMPLE 2.1. *This is a decomposition-tree of $\overset{*}{\vdash} ((p \to q) \land (p \lor s)) \to r$:*

$$\dfrac{\overset{*}{\vdash} ((p \to q) \land (p \lor s)) \to r}{\dfrac{(p \to q) \land (p \lor s) \overset{*}{\vdash} r}{\dfrac{p \to q, p \lor s \overset{*}{\vdash} r}{L\lor\ \dfrac{p \lor s \overset{*}{\vdash} p, r}{p \overset{*}{\vdash} p, r \qquad s \overset{*}{\vdash} p, r} \qquad \dfrac{p \lor s, q \overset{*}{\vdash} r}{p, q \overset{*}{\vdash} r \qquad s, q \overset{*}{\vdash} r}\ L\lor}\ L\to}\ L\land}\ R\to$$

Hence, we have that $\mathsf{top}(\overset{*}{\vdash} ((p \to q) \land (p \lor s)) \to r) = \{p \overset{*}{\vdash} p, r\ ;\ s \overset{*}{\vdash} p, r\ ;\ p, q \overset{*}{\vdash} r\ ;\ s, q \overset{*}{\vdash} r\}$ *with* $\mathsf{top_c}(\overset{*}{\vdash} ((p \to q) \land (p \lor s)) \to r) = \{s \overset{*}{\vdash} p, r\ ;\ p, q \overset{*}{\vdash} r\ ;\ s, q \overset{*}{\vdash} r\}$. *Moreover,* $cnf(((p \to q) \land (p \lor s)) \to r) = (\neg s \lor p \lor r) \land (\neg p \lor \neg q \lor r) \land (\neg s \lor \neg q \lor r)$.

For any set $\mathcal{C}$ of atomic (anti)sequents, we say that $\mathcal{C}$ is closed under Cut whenever $\Phi, \Theta \overset{*}{\vdash} \Lambda, \Psi$ belongs to $\mathcal{C}$ if $\Theta \overset{*}{\vdash} \Lambda, p$ and $p, \Phi \overset{*}{\vdash} \Psi$ belong to $\mathcal{C}$, for some atom p. Furthermore, we say that $\mathcal{C}$ is closed under Contraction whenever $p, \Theta \overset{*}{\vdash} \Lambda$ and $\Phi \overset{*}{\vdash} \Psi, q$ belong to $\mathcal{C}$ if $p, p, \Theta \overset{*}{\vdash} \Lambda$ and $\Phi \overset{*}{\vdash} \Psi, q, q$ belong to $\mathcal{C}$, respectively, for some atoms p, q[2]. We write

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[2] We use $\overset{*}{\vdash}$ to highlight the fact that decomposition and closure under Cut are defined independently of any specific deduction-refutation system – i.e., independently of whether the corresponding sequent is

$\mathsf{top}^{\star}(\Gamma \vdash^{*} \Delta)$ to refer to the set of atomic (anti)sequents which is obtained from $\mathsf{top}(\Gamma \vdash^{*} \Delta)$ by maximal application of the following steps (cf. [129, p. 9]):

- (i) start with $\mathcal{C}_0 = \mathsf{top}(\Gamma \vdash^{*} \Delta)$;
- (ii) take the closure under Contraction of $\mathcal{C}_n$;
- (iii) if $\Theta \vdash^{*} \Lambda, p$ and $p, \Phi \vdash^{*} \Psi$ belong to $\mathcal{C}_n$, and $\Phi, \Theta \vdash^{*} \Lambda, \Psi$ does not belong to $\mathcal{C}_n$, then take $\mathcal{C}_{n+1} = \mathcal{C}_n \cup \{\Phi, \Theta \vdash^{*} \Lambda, \Psi\}$.

Finally, we use $\mathsf{top}^{\star}_{\mathsf{c}}(\Gamma \vdash^{*} \Delta)$ to denote the set of complementary (anti)sequents in $\mathsf{top}^{\star}(\Gamma \vdash^{*} \Delta)$.

EXAMPLE 2.2. *Let A be $(p \to q) \wedge ((q \wedge p) \to r) \wedge (r \wedge q) \to s$. We have that*

- (i) $\mathcal{C}_0 = \mathsf{top}(\vdash^{*} A) = \{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\}$*, which is closed under Contraction;*
- (ii) *we take $\mathcal{C}_1$ to be the set $\{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\} \cup \{p, p \vdash^{*} r\}$: its closure under Contraction is $\{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\, ;\, p, p \vdash^{*} r\, ;\, p \vdash^{*} r\}$;*
- (iii) *we take $\mathcal{C}_2$ to be the set $\{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\, ;\, p, p \vdash^{*} r\, ;\, p \vdash^{*} r\} \cup \{q, q, p \vdash^{*} s\}$: its closure under Contraction is $\{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\, ;\, p, p \vdash^{*} r\, ;\, p \vdash^{*} r\, ;\, q, q, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\}$;*
- (iv) *we take $\mathcal{C}_3$ to be the set $\{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\, ;\, p, p \vdash^{*} r\, ;\, p \vdash^{*} r\, ;\, q, q, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\} \cup \{r, p \vdash^{*} s\}$, which is closed under Contraction;*
- (v) *we take $\mathcal{C}_4$ to be the set $\{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\, ;\, p, p \vdash^{*} r\, ;\, p \vdash^{*} r\, ;\, q, q, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\, ;\, r, p \vdash^{*} s\} \cup \{q, p, p \vdash^{*} s\}$: its closure under Contraction is $\{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\, ;\, p, p \vdash^{*} r\, ;\, p \vdash^{*} r\, ;\, q, q, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\, ;\, r, p \vdash^{*} s\, ;\, q, p, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\}$;*
- (vi) *we take $\mathcal{C}_5$ to be the set $\{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\, ;\, p, p \vdash^{*} r\, ;\, p \vdash^{*} r\, ;\, q, q, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\, ;\, r, p \vdash^{*} s\, ;\, q, p, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\} \cup \{p, q, p \vdash^{*} s\}$: its closure under Contraction is $\{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\, ;\, p, p \vdash^{*} r\, ;\, p \vdash^{*} r\, ;\, q, q, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\, ;\, r, p \vdash^{*} s\, ;\, q, p, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\, ;\, p, q, p \vdash^{*} s\, ;\, p, q \vdash^{*} s\}$;*
- (vii) *we take $\mathcal{C}_6$ to be the set $\{p \vdash^{*} q\, ;\, q, p \vdash^{*} r\, ;\, r, q \vdash^{*} s\, ;\, p, p \vdash^{*} r\, ;\, p \vdash^{*} r\, ;\, q, q, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\, ;\, r, p \vdash^{*} s\, ;\, q, p, p \vdash^{*} s\, ;\, q, p \vdash^{*} s\, ;\, p, q, p \vdash^{*} s\, ;\, p, q \vdash^{*} s\} \cup \{p, p, p \vdash^{*} s\}$.*

As a result, we conclude that $\mathsf{top}^{\star}(\vdash^{*} A)$ is the closure under Contraction of $\mathcal{C}_6$, and that $\mathsf{top}^{\star}_{\mathsf{c}}(\vdash^{*} A) = \mathsf{top}^{\star}(\vdash^{*} A)$: notice that $\mathsf{top}^{\star}(\vdash^{*} A)$ is closed under Cut.

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provable or refutable. In particular, when employing $\vdash^{*}$ in the definition of closure under Cut, we do not suggest that closure under Cut preserves refutability in a specific deduction-refutation system: obviously, this is not the case. Rather, we want to give a presentation of closure under Cut which abstracts from the adoption of a specific deduction-refutation system.

2.1.1. Supraclassical logics. A *(propositional) supraclassical logic* is the extension of (propositional) classical logic with a finite set $\mathcal{W}$ of *extra-logical axioms* – i.e., a finite set of classically invalid formulas [93]. In this chapter, we focus on *consistent* supraclassical logics – i.e., extensions with finite, consistent sets of extra-logical axioms.

If W is the conjunction of formulas in $\mathcal{W}$ and $\Theta \overset{*}{\vdash} \Lambda$ stands for any clause in $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} W)$, then the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $\mathcal{W}$ is obtained from $\overline{\overline{\overline{\mathsf{G4}}}}$ by replacing any instance $\overline{\Gamma, \Theta \dashv \Lambda, \Delta}$ of the rule $\overline{ax}$ with an instance $\overline{\Gamma, \Theta \vdash \Lambda, \Delta}$ of the rule ax. Due to Post completeness, the set of formulas provable in a given supraclassical logic is not closed under uniform substitution: as a result, each atomic formula occurring in a $\overline{\overline{\overline{\mathsf{G4s}}}}$-derivation must be interpreted as a propositional constant [120, 121].

Analogously to a $\mathsf{G4}$-derivation, a $\overline{\overline{\overline{\mathsf{G4s}}}}$-derivation π can conclude with either a sequent $\Gamma \vdash \Delta$ or an antisequent $\Gamma \dashv \Delta$. In the first case, π is considered a *proof* for $\Gamma \vdash \Delta$, while in the second, π serves as a *refutation* of $\Gamma \vdash \Delta$. Let us say that a rule of the form

$$\frac{\Gamma_1 \overset{*}{\vdash} \Delta_1 \qquad \cdots \qquad \Gamma_n \overset{*}{\vdash} \Delta_n}{\Gamma \overset{*}{\vdash} \Delta}$$

is *admissible* in $\overline{\overline{\overline{\mathsf{G4s}}}}$ if and only if the (anti)sequent $\Gamma \overset{*}{\vdash} \Delta$ is derivable whenever the (anti)sequents $\{\Gamma_i \overset{*}{\vdash} \Delta_i\}_{1 \leq i \leq n}$ are derivable.

Let us recall a few results on the structural properties of $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculi [129].

PROPOSITION 2.1. The following statements hold:

(i) the rules of Left and Right Weakening are height-preserving admissible in $\overline{\overline{\overline{\mathsf{G4s}}}}$;

(ii) logical rules of $\overline{\overline{\overline{\mathsf{G4s}}}}$ are height-preserving invertible;

(iii) the rules of Left and Right Contraction are height-preserving admissible in $\overline{\overline{\overline{\mathsf{G4s}}}}$.

PROOF. For each statement, the proof is by induction on the height of a $\overline{\overline{\overline{\mathsf{G4s}}}}$-proof π of the premise: as usual, the height of π is taken to be the number of nodes in a branch of maximal length. Notice that the proof for (i) relies on the fact that initial sequents are closed under Weakening, and that the proof for (iii) exploits point (ii) and the fact that $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} W)$ is closed under Contraction. □

THEOREM 2.3. *The rule of Cut*

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Sigma}{\Pi, \Gamma \vdash \Delta, \Sigma}$$

is admissible in $\overline{\overline{\overline{\mathsf{G4s}}}}$.

PROOF. We consider the topmost Cut application, and reason by primary induction on the logical complexity of the Cut formula and secondary induction on the sum of the height of the premises to obtain that both premises are initial sequents.

Subsequently, we prove that the set of all initial sequents is closed under Cut (cf. [121]): by Proposition 2.1, point (ii) it is enough to prove that the set of all initial atomic sequents is closed under Cut. Since the set of initial atomic sequents comprises identity sequents and weakened versions of the atomic (anti)sequents in $\mathsf{top}^{\star}_{\mathsf{c}}(\mid\!\overset{*}{-} W)$, it suffices to notice that $\mathsf{top}^{\star}_{\mathsf{c}}(\mid\!\overset{*}{-} W)$ is closed under Cut to get the conclusion. ☐

Proposition 2.2. If there exists a $\overline{\overline{\overline{\mathsf{G4s}}}}$-proof π of $\Gamma \vdash \Delta$, then for any sequent $\Gamma' \vdash \Delta'$ occurring in π we have that formulas in $\Gamma' \vdash \Delta'$ are subformulas of formulas in Γ, Δ.

Proof. By induction on the height of π. ☐

The following result is a straightforward consequence of Proposition 2.1 and Theorem 2.3:

Theorem 2.4. $\overline{\overline{\overline{\mathsf{G4s}}}}$ *proves (refutes) the sequent* $\Gamma \vdash \Delta$ *if and only if the following conditions hold:*

- (i) *the formula* $\bigwedge \Gamma \to \bigvee \Delta$ *is (is not) a classical consequence of* W*;*
- (ii) *the formula* $\bigvee \Delta$ *is (is not) a classical consequence of* $W \cup \Gamma$*.*

Proof. The proof of both statements is straightforward, and relies on Proposition 2.1 and Theorem 2.3. ☐

Additionally, we prove a *refutation*-theoretic result about $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculi involving Strengthening rules – i.e., the contrapositive versions of the rules of Weakening [24]:

Proposition 2.3. The rules of Left and Right Strengthening

$$\frac{A, \Gamma \dashv \Delta}{\Gamma \dashv \Delta} \qquad \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta}$$

are admissible in $\overline{\overline{\overline{\mathsf{G4s}}}}$.

Proof. We prove that Left and Right Strengthening rules can be absorbed in $\overline{\overline{\overline{\mathsf{G4s}}}}$ proceeding by simultaneous induction on $\mathrm{C}(A, \Gamma \dashv \Delta) = \mathcal{C}(\Gamma \dashv \Delta, A)$. If $\mathrm{C}(A, \Gamma \dashv \Delta) = 1$, then A is atomic and $\Gamma = \Delta = \varnothing$: since $\overline{\overline{\overline{\mathsf{G4s}}}}$ is consistent, $\overline{\overline{\overline{\mathsf{G4s}}}}$ refutes the empty sequent. If $\mathrm{C}(A, \Gamma \dashv \Delta) > 1$ and A is of the form, say, $B \to C$, then by inductive hypothesis, we have that $\overline{\overline{\overline{\mathsf{G4s}}}}$ refutes $\Gamma \vdash \Delta$ when it refutes $C, \Gamma \vdash \Delta$ and when it refutes $\Gamma \vdash \Delta, B$. If $\overline{\overline{\overline{\mathsf{G4s}}}}$ refutes $B \to C, \Gamma \dashv \Delta$, then $\overline{\overline{\overline{\mathsf{G4s}}}}$ refutes either $C, \Gamma \vdash \Delta$ or $\Gamma \vdash \Delta, B$ – and we are done. ☐

Let us conclude this section with a general observation about any Gentzen-style calculus $\mathcal{S}$ that extends $\mathsf{G4}$ while ensuring the admissibility of Weakening and Cut, and the invertibility of logical rules. Following [14], we say that a rule of the form

$$\frac{\Gamma_1 \vdash \Delta_1 \qquad \cdots \qquad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

admissible in $\mathcal{S}$ is *pure* if and only if, for any Π, Σ, the rule

$$\frac{\Pi, \Gamma_1 \vdash \Delta_1, \Sigma \qquad \cdots \qquad \Pi, \Gamma_n \vdash \Delta_n, \Sigma}{\Pi, \Gamma \vdash \Delta, \Sigma}$$

is admissible in $\mathcal{S}$. Moreover, we say that $\mathcal{S}$ is *deductively complete* if and only if a rule of the form

$$\frac{\Gamma_1 \vdash \Delta_1 \qquad \cdots \qquad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

is admissible in $\mathcal{S}$ exactly when $\mathcal{S}$ proves $\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n, \Gamma \vdash \Delta$. We can establish the following, strengthened version of a result from [14]:

THEOREM 2.5. *$\mathcal{S}$ is deductively complete if and only if all the rules of $\mathcal{S}$ are pure.*

PROOF. We first notice that if $\mathcal{S}$ proves $\Gamma \vdash \Delta$ and $\Gamma = A_1, \ldots, A_n$, then admissibility of Cut suffices to ensure that a rule of the form

$$\frac{\vdash A_1 \qquad \cdots \qquad \vdash A_n}{\vdash \Delta}$$

is admissible in $\mathcal{S}$. ($\Leftarrow$) If all the rules of $\mathcal{S}$ are pure, then all rules admissible in $\mathcal{S}$ are pure: as a result, if a rule of the form displayed above is admissible, then a rule of the form

$$\frac{A_1, \ldots, A_n \vdash A_1 \qquad \cdots \qquad A_1, \ldots, A_n \vdash A_n}{A_1, \ldots, A_n \vdash \Delta}$$

is admissible. By admissibility of Weakening and invertibility of logical rules, the sequents $\{A_1, \ldots, A_n \vdash A_i\}_{1 \leq i \leq n}$ are provable – and this suffices to conclude that $\mathcal{S}$ is strongly complete. ($\Leftarrow$) See [14, p. 267]: full generality is obtained exploiting the invertibility of logical rules. ($\Rightarrow$) Suppose by contradiction that $\mathcal{S}$ has a rule

$$\frac{\Pi_1 \vdash \Sigma_1 \qquad \cdots \qquad \Pi_n \vdash \Sigma_n}{\Pi \vdash \Sigma}$$

and that there exist Π', Σ' such that $\mathcal{S}$ proves the sequents $\{\Pi', \Pi_i \vdash \Sigma_i, \Sigma'\}_{1 \leq i \leq n}$ and does not prove the sequent $\Pi', \Pi \vdash \Sigma, \Sigma'$. If $\mathcal{S}$ is deductively complete, then $\mathcal{S}$ proves the sequent

$$\bigwedge \Pi_1 \to \bigvee \Sigma_1, \ldots, \bigwedge \Pi_n \to \bigvee \Sigma_n \vdash \bigwedge \Pi \to \bigvee \Sigma \tag{2.1.1}$$

and does not prove the sequent

$$(\bigwedge \Pi' \wedge \bigwedge \Pi_1) \to (\bigvee \Sigma_1 \vee \bigvee \Sigma'), \ldots, (\bigwedge \Pi' \wedge \bigwedge \Pi_n) \to (\bigvee \Sigma_n \vee \bigvee \Sigma') \vdash (\bigwedge \Pi' \wedge \bigwedge \Pi) \to (\bigvee \Sigma \vee \bigvee \Sigma') \tag{2.1.2}$$

By the invertibility of logical rules, if $\mathcal{S}$ proves (2.1.1) then $\mathcal{S}$ proves any sequent of the form

$$\bigvee \Sigma_{h(1)}, \ldots, \bigvee \Sigma_{h(j)}, \Pi \vdash \Sigma, \bigwedge \Pi_{h(j+1)}, \ldots, \bigwedge \Pi_{h(j+j')} \tag{2.1.3}$$

with $n = j + j'$ and h being any permutation of $1, \ldots, n$. On the other hand, $\mathcal{S}$ proves the sequent

$$\bigvee \Sigma', \Pi', \Pi \vdash \Sigma, \Sigma', \bigwedge \Pi' \tag{2.1.4}$$

as well as any sequent of the form

$$\bigvee\Sigma', \bigvee\Sigma_{h(1)}, \dots, \bigvee\Sigma_{h(k)}, \Pi', \Pi \vdash \Sigma, \Sigma', \bigwedge\Pi_{h(k+1)}, \dots, \bigwedge\Pi_{h(k+k')} \bigwedge\Pi' \tag{2.1.5}$$

with $n > k + k'$ and h being any permutation of $1, \dots, k + k'$. On the other hand, the invertibility of logical rules ensures that if $\mathcal{S}$ does not prove (2.1.2) and proves the sequents (2.1.4) and (2.1.5), then $\mathcal{S}$ does not prove a sequent

$$\bigvee\Sigma_{h(1)}, \dots, \bigvee\Sigma_{h(j)}, \Pi', \Pi \vdash \Sigma, \Sigma', \bigwedge\Pi_{h(j+1)}, \dots, \bigwedge\Pi_{h(j+j')} \tag{2.1.6}$$

for some choice of j, j' and h – by admissibility of Weakening, a contradiction. ☐

2.2. Eliminating redundancy: a syntactic approach

In this section, we use capital Greek letters $\Gamma, \Delta, \Pi, \Sigma, \dots$ to denote finite *sets* of formulas, and $\Theta, \Lambda, \dots$ to denote finite sets of *atomic* formulas. We set the *syntactic equivalence relation* on contexts to be the smallest congruence relation on contexts induced by the following equation:

$$A_1, \dots, A_i, \dots, A_m = A_1, \dots, A_i, A_i, \dots, A_m$$

As a result, any occurrence of the comma in an (anti)sequent $\Gamma \vdash^{*} \Delta$ is interpreted over set-theoretic union.

When dealing with contexts as sets, $\overline{\overline{\mathsf{G4}}}$ derivations are sequences of instances of $\overline{\overline{\mathsf{G4}}}$ rules which are applied *modulo* syntactic equivalence. As a result, the height of a $\overline{\overline{\mathsf{G4}}}$ derivation is measured as the number of nodes in its longest branch *modulo* syntactic equivalence.

Example 2.3. *These are* $\overline{\overline{\mathsf{G4}}}$*-proofs with contexts as sets, where applications of the syntactic equivalence relation between contexts are made explicit:*

$$R\wedge\,\dfrac{R\vee\,\dfrac{=\,\dfrac{R\vee\,\dfrac{R\neg\,\dfrac{ax\,\dfrac{}{q, p \vdash p}}{q \vdash p, \neg p}}{q \vdash p \vee \neg p}}{q \vdash p \vee \neg p, p \vee \neg p}}{q \vdash (p \vee \neg p) \vee (p \vee \neg p)} \qquad ax\,\dfrac{}{q \vdash q}}{q \vdash ((p \vee \neg p) \vee (p \vee \neg p)) \wedge q} \qquad\qquad \dfrac{\dfrac{}{p, q \vdash p, q}\,ax \qquad \dfrac{\dfrac{}{p, q \vdash q}\,ax}{p, q \vdash q, q}\,=}{p, q \vdash p \wedge q, q}\,R\wedge$$

The adoption of the contexts-as-sets approach does not alter the structural properties of $\overline{\overline{\mathsf{G4}}}$. The only difference consists in the formulation of the Cut and Strengthening rules. From sequents $\Gamma \vdash \Delta, A$ and $A, \Pi \vdash \Sigma$, the application of Cut yields a sequent $\Pi^A, \Gamma^A \vdash \Delta^A, \Sigma^A$, where Π^A (Γ^A, Δ^A, Σ^A) is identical to Π (Γ, Δ and Σ, respectively) except for the fact that any occurrence of A is erased[3]. Analogously, from an antisequent $A, \Gamma \vdash \Delta$, the application of Strengthening yields an antisequent $A, \Gamma^A \vdash \Delta$.

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[3] In standard terminology, this rule is usually referred to as Mix [164] or Multicut [167].

Theorem 2.2 allows us to directly refer to a unique set of atomic (anti)sequents associated with a certain (anti)sequent $\Gamma \overset{*}{\vdash} \Delta$ *modulo* syntactic equivalence. We write $\mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$ ($\mathsf{top}_{\mathsf{c}}(\Gamma \overset{*}{\vdash} \Delta)$) to indicate the set of (complementary) atomic (anti)sequents obtained by maximal decomposition of $\Gamma \overset{*}{\vdash} \Delta$ *via* bottom-up applications of the rules $L\neg$, $R\neg$, $L\wedge$, $R\wedge$, $L\vee$, $R\vee$, $L \to$, $R \to$ after erasing multiple occurrences of the same formula. For any set of atomic (anti)sequents $\mathcal{C}$, closure under Cut of $\mathcal{C}$ is defined as usual, with the condition that any Cut application generates a conclusion where multiple occurrences of the same formula are erased. As a result, $\mathsf{top}^{\star}(\Gamma \overset{*}{\vdash} \Delta)$ and $\mathsf{top}^{\star}_{\mathsf{c}}(\Gamma \overset{*}{\vdash} \Delta)$ are generated without any need to ensure closure under Contraction. In what follows, we use the term *clause* to denote any atomic (anti)sequent.

In this section, we refine the $\mathsf{G4}$-based decomposition procedure to generate sets of atomic (anti)sequents where redundant information *modulo* classical equivalence is eliminated. Let us begin by introducing some relevant terminology.

DEFINITION 2.1. Let $\mathcal{C}$ be a set of complementary clauses. $\mathcal{D}$ is a *reduct under Weakening* of $\mathcal{C}$ iff $\mathcal{D}$ is obtained by maximal application of the following procedure:

(i) start with $\mathcal{C}_0 = \mathcal{C}$;

(ii) if $\Theta \overset{*}{\vdash} \Lambda$ and $\Theta', \Theta \overset{*}{\vdash} \Lambda, \Lambda'$ belong to $\mathcal{C}_n$, take $\mathcal{C}_{n+1} = \mathcal{C}_n \setminus \{\Theta', \Theta \overset{*}{\vdash} \Lambda, \Lambda'\}$.

EXAMPLE 2.4. *Let* $\mathcal{C}$ *be* $\{p \vdash q\,;\, r, p \vdash q, s\,;\, r \vdash s\,; p, r \vdash\,;\, r \vdash\ \}$. *We can perform the following reduction sequences to obtain the reduct under Weakening of* $\mathcal{C}$*:*

(i) $\mathcal{C}_1 = \{p \vdash q\,;\, r \vdash s\,;\, p, r \vdash\ ;\, r \vdash\ \}$, $\mathcal{C}_2 = \{p \vdash q\,;\, r \vdash s\,;\, r \vdash\ \}$ *and* $\mathcal{C}_3 = \{p \vdash q\,;\, r \vdash\ \}$*;*

(ii) $\mathcal{C}'_1 = \mathcal{C}_1$, $\mathcal{C}'_2 = \{p \vdash q\,;\, p, r \vdash\ ;\, r \vdash\ \}$, $\mathcal{C}'_3 = \mathcal{C}_3$*;*

(iii) $\mathcal{C}''_1 = \{p \vdash q\,;\, r, p \vdash q, s\,;\, r \vdash s\,;\, r \vdash\ \}$, $\mathcal{C}''_2 = \{p \vdash q\,;\, r, p \vdash q, s\,;\, r \vdash\ \}$, $\mathcal{C}''_3 = \mathcal{C}_3$*;*

(iv) $\mathcal{C}'''_1 = \{p \vdash q\,;\, r, p \vdash q, s\,;\, p, r \vdash\ ;\, r \vdash\ \}$, $\mathcal{C}'''_2 = \mathcal{C}''_2$, $\mathcal{C}'''_3 = \mathcal{C}_3$.

A set of clauses $\mathcal{D}$ is the *closure under Cut* of a set $\mathcal{C}$, in symbols $\mathcal{C}^{\star}$, whenever $\Phi, \Theta \vdash \Lambda, \Psi$ belongs to $\mathcal{C}$ if $\Theta \vdash \Lambda, p$ and $p, \Phi \vdash \Psi$ belong to $\mathcal{C}$, for some atom p. We write $\mathcal{C}^{\star}_{\mathsf{c}}$ to denote the set of complementary clauses in $\mathcal{C}^{\star}$.

DEFINITION 2.2. Let $\mathcal{C}$ be a set of complementary clauses. $\mathcal{D}$ is a *reduct under Weakening and Cut* of $\mathcal{C}$ if and only if $\mathcal{D}$ is obtained by maximal application of the following procedure:

(i) start with $\mathcal{C}_0$ being the reduct under Weakening of $\mathcal{C}$;

(ii) if $\Theta', \Theta \overset{*}{\vdash} \Lambda, \Lambda'$ belongs to $\mathcal{C}_n$ and $\Theta \overset{*}{\vdash} \Lambda$ belongs to $(\mathcal{C}_n \setminus \{\Theta', \Theta \overset{*}{\vdash} \Lambda, \Lambda'\})^{\star}$, take $\mathcal{C}_{n+1} = \mathcal{C}_n \setminus \{\Theta', \Theta \overset{*}{\vdash} \Lambda, \Lambda'\}$.

Example 2.5. *Let* $\mathcal{C} = \{p \vdash q, r\,;\, r, s \vdash t\,;\, s, p \vdash q, t, u\,;\, u \vdash v\,;\, s, p \vdash q, t, v\}$*: notice that the reduct under Weakening of* $\mathcal{C}$ *is* $\mathcal{C}$ *itself. We perform the following reduction sequences to obtain the only reduct under Weakening and Cut of* $\mathcal{C}$*:*

(i) $\mathcal{C}_1 = \{p \vdash q, r\,;\, r, s \vdash t\,;\, u \vdash v\,;\, s, p \vdash q, t, v\}$, $\mathcal{C}_2 = \{p \vdash q, r\,;\, r, s \vdash t\,;\, u \vdash v\}$*;*

(ii) $\mathcal{C}'_1 = \{p \vdash q, r\,;\, r, s \vdash t\,;\, s, p \vdash q, t, u\,;\, u \vdash v\}$, $\mathcal{C}'_2 = \mathcal{C}_2$.

Example 2.6. *Let* $\mathcal{C} = \{p \vdash q\,;\, q \vdash r\,;\, p \vdash r\,;\, r \vdash q\}$*: notice that the reduct under Weakening of* $\mathcal{C}$ *is* $\mathcal{C}$ *itself. We perform the following reduction sequences to obtain the reducts under Weakening and Cut of* $\mathcal{C}$*:*

(i) $\mathcal{C}_1 = \{p \vdash q\,;\, q \vdash r\,;\, r \vdash q\}$*;*

(ii) $\mathcal{C}'_1 = \{q \vdash r\,;\, p \vdash r\,;\, r \vdash q\}$.

Definition 2.3. Let $\mathcal{C}$ be a set of complementary clauses. We say that $\mathcal{D}$ is a *strengthening under Cumulative Cut* of $\mathcal{C}$ iff $\mathcal{D}$ is obtained from $\mathcal{C}$ by maximal application of the following rewriting procedure:

(i) start with $\mathcal{C}_0 = \mathcal{C}$;

(ii) for any $\mathcal{C}_n$, apply exactly one of the following steps:

  (a) if $\Theta', \Theta \overset{*}{\vdash} \Lambda, \Lambda', p$ belongs to $\mathcal{C}_n$ and $p, \Theta \overset{*}{\vdash} \Lambda$ belongs to $\mathcal{C}^\star$, take $\mathcal{C}_{n+1} = (\mathcal{C}_n \setminus \{\Theta', \Theta \overset{*}{\vdash} \Lambda, \Lambda', p\}) \cup \{\Theta', \Theta \vdash \Lambda, \Lambda'\}$;

  (b) if $p, \Theta', \Theta \overset{*}{\vdash} \Lambda, \Lambda'$ belongs to $\mathcal{C}_n$ and $\Theta \overset{*}{\vdash} \Lambda, p$ belongs to $\mathcal{C}^\star$, take $\mathcal{C}_{n+1} = (\mathcal{C}_n \setminus \{p, \Theta', \Theta \overset{*}{\vdash} \Lambda, \Lambda'\}) \cup \{\Theta', \Theta \overset{*}{\vdash} \Lambda, \Lambda'\}$;

Example 2.7. *Let* $\mathcal{C} = \{p \vdash \ ;\, r \vdash s, q\,;\, q \vdash p\}$*: notice that the only reduct under Weakening and Cut of* $\mathcal{C}$ *is* $\mathcal{C}$ *itself, and that* $\mathcal{C}^\star = \mathcal{C} \cup \{r \vdash s, p\,;\, q \vdash \ \}$*. We perform the following reduction sequences to obtain the strengthening under Cumulative Cut of* $\mathcal{C}$*:*

(i) $\mathcal{C}_1 = \{p \vdash \ ;\, r \vdash s, q\,;\, q \vdash \ \}$, $\mathcal{C}_2 = \{p \vdash \ ;\, r \vdash s\,;\, q \vdash \ \}$*;*

(ii) $\mathcal{C}'_1 = \mathcal{C}_2$.

2.2.1. Minimization of normal forms, proof-theoretically. In this subsection, for any classically invalid formula A we present a proof-theoretic procedure to minimize $cnf(A)$ – i.e, the G4-canonical conjunction normal form of A whose clauses are all and only the formula translations of the one-sided formulations of the (anti)sequents in $\mathsf{top_c}(\overset{*}{\vdash} A)$.

Proposition 2.4. Let $\mathcal{C}$ be a set of complementary clauses. Then, there exists a unique subset $\mathcal{D}$ which is a reduct under Weakening of $\mathcal{C}$.

Proof. Suppose by contradiction that there exists two distinct subsets $\mathcal{D}_1, \mathcal{D}_2$ of $\mathcal{C}$ which are reducts under Weakening of $\mathcal{C}$. This implies that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are proper subsets of $\mathcal{C}$, and that there must exist a clause $\Theta \overset{*}{\vdash} \Lambda$ which belongs to (say) $\mathcal{D}_1$ and does not belong to $\mathcal{D}_2$. If $\Theta \mid\!\!\overset{*}{-} \Lambda$ belongs to $\mathcal{D}_1$, then it belongs to $\mathcal{C}$; on the other hand, if $\Theta \mid\!\!\overset{*}{-} \Lambda$ does not belong to $\mathcal{D}_2$, then there must be a clause $\Theta' \mid\!\!\overset{*}{-} \Lambda'$ in $\mathcal{C}$ and $\mathcal{D}_2$ such that $\Theta' \subset \Theta$ or $\Lambda' \subset \Lambda$. We infer that $\Theta' \mid\!\!\overset{*}{-} \Lambda'$ does not belong to $\mathcal{D}_1$: this entails that there exists a clause $\Theta'' \mid\!\!\overset{*}{-} \Lambda''$ in $\mathcal{C}$ and $\mathcal{D}_1$ such that $\Theta'' \subset \Theta'$ or $\Lambda'' \subset \Lambda'$ – a contradiction. □

PROPOSITION 2.5. Let $\mathcal{C}$ be a set of complementary clauses. There may be more than one subset of $\mathcal{C}$ which is a reduct under Weakening and Cut of $\mathcal{C}$.

PROOF. It suffices to consider Example 2.6. □

PROPOSITION 2.6. Let $\mathcal{C}$ be a set of complementary clauses. There may be more than one strengthening under Cumulative Cut of $\mathcal{C}$.

PROOF. It suffices to consider the following example. Let $\mathcal{C} = \{p \vdash q, r\,;\, q \vdash r\,;\, r \vdash q\}$: notice that the only reduct under Weakening and Cut of $\mathcal{C}$ is $\mathcal{C}$ itself. The reduction sequences yielding the strengthening under Cumulative Cut of $\mathcal{C}$ are the following:

(*i*) $\mathcal{C}_1 = \{p \vdash q\,;\, q \vdash r\,;\, r \vdash q\}$;

(*ii*) $\mathcal{C}'_1 = \{p \vdash r\,;\, q \vdash r\,;\, r \vdash q\}$.

□

Let $\mathcal{C}$ be a set of complementary clauses and $\mathcal{S}$ be the set of formula translations of the clauses in $\mathcal{C}$: we say the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $\mathcal{C}$ is just the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for the supraclassical logic obtained by adding the formulas in $\mathcal{S}$ as extra-logical axioms.

THEOREM 2.6. *Let $\mathcal{C}$ be a set of complementary clauses. If $\mathcal{D} = \{\Theta_1 \mid\!\!\overset{*}{-} \Lambda_1, \ldots, \Theta_n \mid\!\!\overset{*}{-} \Lambda_n\}$ is a reduct under Weakening and Cut of $\mathcal{C}$, then the following conditions hold:*

(*i*) *the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $\mathcal{D}$ proves the clauses in $\mathcal{C}$, and vice versa;*

(*ii*) *the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $\mathcal{D} \setminus \{\Theta_i \mid\!\!\overset{*}{-} \Lambda_i\}$ refutes $\Theta_i \vdash \Lambda_i$, for any $1 \leq i \leq n$.*

PROOF. We establish the two statements separately.

(*i*) It suffices to notice that for any clause $\Theta \mid\!\!\overset{*}{-} \Lambda$ in $\mathcal{C} \setminus \mathcal{D}$ there is (at least) one clause $\Theta' \mid\!\!\overset{*}{-} \Lambda'$ in $\mathcal{D}^{\star}$ such that $\Theta' \subseteq \Theta$ and $\Lambda' \subseteq \Lambda$.

(*ii*) Suppose by contradiction that the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $\mathcal{D} \setminus \{\Theta_i \mid\!\!\overset{*}{-} \Lambda_i\}$ proves $\Theta_i \vdash \Lambda_i$ for some $1 \leq i \leq n$. This implies that $\Theta_i \vdash \Lambda_i$ is of the form $\Theta''_i, \Theta'_i \vdash \Lambda'_i, \Lambda''_i$, with (say) $\Theta'_i \mid\!\!\overset{*}{-} \Lambda'_i$ belonging to $(\mathcal{D} \setminus \{\Theta_i \mid\!\!\overset{*}{-} \Lambda_i\})^{\star}$. As a result, any reduction sequence ending with $\mathcal{D}$ can be extended so as to end with $\mathcal{D} \setminus \{\Theta_i \mid\!\!\overset{*}{-} \Lambda_i\}$: this entails that $\mathcal{D}$ is not a reduct under Weakening and Cut of $\mathcal{C}$ – a contradiction.

□

Lemma 2.1. *Let* $\mathcal{C} = \{\Theta_1 \overset{*}{\vdash} \Lambda_1, \dots, \Theta_n \overset{*}{\vdash} \Lambda_n\}$ *be a set of complementary clauses. The* $\overline{\overline{\mathsf{G4s}}}$ *calculus for* $\mathcal{C}$ *proves* $\Theta \vdash \Lambda$ *iff* $\overline{\overline{\mathsf{G4}}}$ *proves the sequent* $(\bigwedge \Theta_1 \to \bigvee \Lambda_1), \dots, (\bigwedge \Theta_n \to \bigvee \Lambda_n) \vdash \bigwedge \Theta \to \bigvee \Lambda$.

Proof. ($\Rightarrow$) If the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{C}$ proves $\Theta \vdash \Lambda$, then there exists (at least) one $\Phi \overset{*}{\vdash} \Psi$ in $\mathcal{C}^{\star}$ such that $\Phi \subseteq \Theta$ and $\Psi \subseteq \Lambda$. Let π be any derivation of $\Phi \vdash \Psi$ in $\overline{\overline{\mathsf{G4s}}} + \mathit{Cut}$: we proceed by induction on $h(\pi)$ to get the result. The base case is obvious, and thus we focus on the inductive step. If $h(\pi) > 1$, then π has the form

$$\dfrac{\begin{array}{c}\vdots\\ \Phi' \vdash \Psi', p\end{array} \qquad \begin{array}{c}\vdots\\ p, \Phi'' \vdash \Psi''\end{array}}{\Phi \vdash \Psi}\,\mathit{cut}$$

By inductive hypothesis, $\overline{\overline{\mathsf{G4}}}$ proves $(\bigwedge \Theta_1 \to \bigvee \Lambda_1), \dots, (\bigwedge \Theta_n \to \bigvee \Lambda_n) \vdash \bigwedge \Phi' \to (\bigvee \Psi' \vee p)$ and $(\bigwedge \Theta_1 \to \bigvee \Lambda_1), \dots, (\bigwedge \Theta_n \to \bigvee \Lambda_n) \vdash (p \wedge \bigwedge \Phi'') \to \bigvee \Psi''$. By invertibility of logical rules, this implies that $\overline{\overline{\mathsf{G4}}}$ proves $(\bigwedge \Theta_1 \to \bigvee \Lambda_1), \dots, (\bigwedge \Theta_n \to \bigvee \Lambda_n), \Phi' \vdash \Psi', p$ and $(\bigwedge \Theta_1 \to \bigvee \Lambda_1), \dots, (\bigwedge \Theta_n \to \bigvee \Lambda_n), p, \Phi'' \vdash \Psi''$: since $\overline{\overline{\mathsf{G4}}}$ is closed under Cut, $\overline{\overline{\mathsf{G4}}}$ proves $(\bigwedge \Theta_1 \to \bigvee \Lambda_1), \dots, (\bigwedge \Theta_n \to \bigvee \Lambda_n), \Phi \vdash \Psi$ – as desired.
($\Leftarrow$) If $\overline{\overline{\mathsf{G4}}}$ proves the sequent $(\bigwedge \Theta_1 \to \bigvee \Lambda_1), \dots, (\bigwedge \Theta_n \to \bigvee \Lambda_n) \vdash \bigwedge \Theta \to \bigvee \Lambda$, then it is also provable in the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{C}$. On the other hand, $\overline{\overline{\mathsf{G4s}}}$ proves the sequent $\vdash \bigwedge \Theta_i \to \bigvee \Lambda_i$ for any $1 \leq i \leq n$: it suffices to notice that $\overline{\overline{\mathsf{G4s}}}$ is closed under Cut to reach the conclusion. □

We say that $\mathcal{D} = \{\Theta_1 \overset{*}{\vdash} \Lambda_1, \dots, \Theta_n \overset{*}{\vdash} \Lambda_n\}$ is an *irreducible version* of a set of complementary clauses $\mathcal{C}$ such that

(*i*) the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{C}$ proves the clauses in $\mathcal{D}$, and vice versa;

(*ii*) the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{D} \setminus \{\Theta_i \overset{*}{\vdash} \Lambda_i\}$ refutes $\Theta_i \vdash \Lambda_i$, for any $1 \leq i \leq n$;

(*iii*) the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{D}$ refutes $\Theta'_i \vdash \Lambda'_i$, with $\Theta'_i \subset \Theta_i$ or $\Lambda'_i \subset \Theta_i$.

The following result ensures that, for any set of complementary clauses $\mathcal{C}$, a strengthening under Cumulative Cut of one of its reducts under Weakening and Cut is an irreducible version of $\mathcal{C}$.

Theorem 2.7. *Let* $\mathcal{C}$ *be a set of complementary clauses and* $\mathcal{D}$ *be a reduct under Weakening and Cut of* $\mathcal{C}$. *If* $\mathcal{E} = \{\Theta_1 \overset{*}{\vdash} \Lambda_1, \dots, \Theta_n \overset{*}{\vdash} \Lambda_n\}$ *is a strengthening under Cumulative Cut of* $\mathcal{D}$ *and* $\Theta'_i \subset \Theta_i$ *or* $\Lambda'_i \subset \Lambda_i$, *then the following conditions hold:*

(*i*) *the* $\overline{\overline{\mathsf{G4s}}}$ *calculus for* $\mathcal{E}$ *proves the clauses in* $\mathcal{C}$, *and vice versa;*

(*ii*) *the* $\overline{\overline{\mathsf{G4s}}}$ *calculus for* $\mathcal{E} \setminus \{\Theta_i \overset{*}{\vdash} \Lambda_i\}$ *refutes* $\Theta_i \vdash \Lambda_i$, *for any* $1 \leq i \leq n$;

(*iii*) *the* $\overline{\overline{\mathsf{G4s}}}$ *calculus for* $\mathcal{E}$ *refutes* $\Theta'_i \vdash \Lambda'_i$, *for any* $1 \leq i \leq n$.

Proof. We establish the two statements separately.

(i) It suffices to notice that any clause in $\mathcal{E}$ belongs to $\mathcal{C}^\star$ – the proof being by induction on the length of the reduction sequence generating $\mathcal{E}$.

(ii) We reason by induction on the length of the reduction sequence generating $\mathcal{E}$ – in symbols, $lh(\sigma_{\mathcal{E}})$. If $lh(\sigma_{\mathcal{E}}) = 0$, then $\mathcal{E} = \mathcal{D}$: by hypothesis, if $\mathcal{D} = \{\Phi_1 \overset{*}{\vDash} \Psi_1, \ldots, \Phi_n \overset{*}{\vDash} \Psi_n\}$, then the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{D} \setminus \{\Phi_i \overset{*}{\vDash} \Psi_i\}$ refutes $\Phi_i \vdash \Psi_i$, for any $1 \le i \le n$. If $lh(\sigma_{\mathcal{E}}) > 0$, then there exists a set of clauses $\mathcal{D}_k = \{\Phi'_1 \overset{*}{\vDash} \Psi'_1, \ldots, \Phi'_n \overset{*}{\vDash} \Psi'_n\}$ such that $\Phi'_i \overset{*}{\vDash} \Psi'_i$ is (say) of the form $\Phi''_i, \Phi'''_i \overset{*}{\vDash} \Psi'''_i, \Psi''_i, p$ with $p, \Phi''_i \overset{*}{\vDash} \Psi''_i$ belonging to $\mathcal{D}^\star$, for a unique $1 \le i \le n$. By inductive hypothesis, the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{D}_k \setminus \{\Phi'_i \overset{*}{\vDash} \Psi'_i\}$ refutes $\Phi'_i \vdash \Psi'_i$: since the $\overline{\overline{\mathsf{G4s}}}$ calculus admits Strengthening, it refutes also the sequent $\Phi''_i, \Phi'''_i \vdash \Psi'''_i, \Psi''_i$ – as desired.

(iii) Suppose by contradiction that the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{E}$ proves (say) $\Theta'_n \vdash \Lambda'_n$, with $\Theta'_n = \Theta_n$ and $\Lambda'_n = \Lambda_n \setminus \{p\}$: this implies the existence of (at least) one clause $\Phi \overset{*}{\vDash} \Psi$ in $\mathcal{E}^\star$ such that $\Phi \subseteq \Theta'_n$ and $\Psi \subseteq \Lambda'_n$. Lemma 2.1 ensures that $\overline{\overline{\mathsf{G4}}}$ proves $\bigwedge \Theta_1 \to \bigvee \Lambda_1, \ldots, \bigwedge \Theta'_n \to (\bigvee \Lambda'_n \vee p) \vdash \bigwedge \Phi \to \bigvee \Psi$: by invertibility of logical rules we infer that $\overline{\overline{\mathsf{G4}}}$ proves the sequent

$$\bigwedge \Theta_1 \to \bigvee \Lambda_1, \ldots, \bigwedge \Theta_{n-1} \to \bigvee \Lambda_{n-1}, p \vdash \bigwedge \Phi \to \bigvee \Psi \tag{2.2.1}$$

On the other hand, point (ii) and Lemma 2.1 guarantee that $\overline{\overline{\mathsf{G4}}}$ refutes the sequent

$\bigwedge \Theta_1 \to \bigvee \Lambda_1, \ldots, \bigwedge \Theta_{n-1} \to \bigvee \Lambda_{n-1} \vdash \bigwedge \Theta_n \to \bigvee \Lambda_n$: by invertibility of logical rules and admissibility of Strengthening, this entails that $\overline{\overline{\mathsf{G4}}}$ refutes the sequent

$$\bigwedge \Theta_1 \to \bigvee \Lambda_1, \ldots, \bigwedge \Theta_{n-1} \to \bigvee \Lambda_{n-1} \vdash \bigwedge \Phi \to \bigvee \Psi \tag{2.2.2}$$

If $\overline{\overline{\mathsf{G4}}}$ proves (2.2.1) and refutes (2.2.2), then Lemma 2.1 establishes that

(a) there exists (at least) one clause $\Phi' \overset{*}{\vDash} \Psi'$ belonging to $((\mathcal{E} \setminus \{\Theta_n \overset{*}{\vDash} \Lambda_n\}) \cup \{\overset{*}{\vDash} p\})^\star$, with $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$;

(b) $\Phi' \overset{*}{\vDash} \Psi'$ does not belong to $(\mathcal{E} \setminus \{\Theta_n \overset{*}{\vDash} \Lambda_n\})^\star$.

As a result, any derivation π of $\Phi' \vdash \Psi'$ in the calculus $\overline{\overline{\mathsf{G4s}}} + Cut$ for $((\mathcal{E} \setminus \{\Theta_n \overset{*}{\vDash} \Lambda_n\}) \cup \{\overset{*}{\vDash} p\})^\star$ features (at least) one Cut application involving $\overset{*}{\vDash} p$ as left premise. Suppose without loss of generality that π has the form

$$\dfrac{ax\,\dfrac{}{\vdash p} \qquad \begin{array}{c}\vdots\,\pi_1\\ p, \Xi \vdash \Omega\end{array}}{\Xi \vdash \Omega}\,cut$$

$$\begin{array}{c}\dfrac{ax\,\dfrac{}{\vdash p} \qquad\qquad \begin{array}{c}\vdots\,\pi_2'\\ p, \Xi' \vdash \Omega'\end{array}}{\Xi' \vdash \Omega'}\,cut\\ \vdots\,\pi_2''\\ \Phi' \vdash \Psi'\end{array}$$

with no Cut application in π_1 or π_2' involving $\vdash p$. We can turn π into a derivation of the form

$$\begin{array}{c}\vdots\,\pi_1\\ p, \Xi \vdash \Omega\end{array}$$

$$\begin{array}{c}\dfrac{ax\,\dfrac{}{\vdash p} \qquad \begin{array}{c}\vdots\,\pi_2'\\ p, \Xi' \vdash \Omega'\end{array}}{\Xi' \vdash \Omega'}\,cut\\ \vdots\,\pi_2''\\ \Phi' \vdash \Psi'\end{array}$$

Iterated application of this proof-transformation yields a derivation π' of $\Phi' \vdash \Psi'$ with exactly one Cut application involving $\vdash p$. Cut applications can be freely permuted: this implies that we can turn π' into a derivation π'' where the only Cut application involving $\vdash p$ is permuted last. As a result, π'' has the form

$$\dfrac{ax\,\dfrac{}{\vdash p} \qquad \begin{array}{c}\vdots\\ p, \Phi' \vdash \Psi'\end{array}}{\Phi' \vdash \Psi'}$$

Hence, we conclude that $p, \Phi' \mathrel{|\!\overset{*}{-}} \Psi'$ belongs to $(\mathcal{E} \setminus \{\Theta_n \vdash \Lambda_n\})^\star$, and thus to $\mathcal{D}^\star$: by Definition 2.3, it turns out that $\Theta_n \mathrel{|\!\overset{*}{-}} \Lambda_n$ cannot belong to any strengthening under Cumulative Cut of $\mathcal{D}$ – a contradiction.

□

2.2.2. On irreducible normal forms. We can strengthen Theorem 2.7, showing that any irreducible version of a set of complementary clauses $\mathcal{C}$ can be obtained as a strengthening under Cumulative Cut of a reduct under Weakening and Cut of $\mathcal{C}^\star$.

THEOREM 2.8. *Let $\mathcal{C}$ be a set of complementary clauses and $\mathcal{D}$ be an irreducible version of $\mathcal{C}$. Then, $\mathcal{D}$ is a strengthening under Cumulative Cut of a reduct under Weakening and Cut of $\mathcal{C}^\star$.*

Proof. Let $\mathcal{E} = \{\Theta_1 \overset{*}{\vdash} \Lambda_1, \dots, \Theta_n \overset{*}{\vdash} \Lambda_n\}$ be any set of complementary clauses such that the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $\mathcal{E}$ proves the clauses in $\mathcal{C}$ and vice versa: without loss of generality, we may assume that $\mathcal{E}$ is reduced under Weakening and Cut.
Any irreducible version $\mathcal{D}$ of $\mathcal{C}$ is a strengthening under Cumulative Cut of $\mathcal{E}$. Now, for each clause $\Theta_i \overset{*}{\vdash} \Lambda_i$ in $\mathcal{E}$ there exists (at least) one clause $\Theta'_i \overset{*}{\vdash} \Lambda'_i$ in $\mathcal{C}^\star$ such that $\Theta'_i \subseteq \Theta_i$ and $\Lambda'_i \subseteq \Lambda_i$: for any $\mathcal{E}' = \{\Theta'_1 \overset{*}{\vdash} \Lambda'_1, \dots, \Theta'_n \overset{*}{\vdash} \Lambda'_n\}$ we infer that the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $\mathcal{E}'$ proves the clauses in $\mathcal{C}$, and that the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $\mathcal{E}$ proves the clauses in $\mathcal{E}'$. Once more, we take $\mathcal{E}'$ to be reduced under Weakening and Cut without loss of generality.
Any strengthening under Cumulative Cut of $\mathcal{E}$ is a strengthening under Cumulative Cut of $\mathcal{E}'$: it suffices to notice that $\mathcal{E}'$ is a reduct under Weakening and Cut of $\mathcal{C}^\star$ to reach the conclusion. □

Proposition 2.7. Let $\mathcal{C}$ be a set of complementary clauses, and $\mathcal{C}_1, \dots, \mathcal{C}_n$ be the irreducible versions of $\mathcal{C}$. Then, the following conditions hold:

(i) $\mathcal{C}_i \subseteq \mathcal{C}^\star_j$, for any $1 \leq i, j \leq n$;

(ii) $\mathcal{C}^\star_i = \mathcal{C}^\star_j$, for any $1 \leq i, j \leq n$.

Proof. We prove each statement separately.

(i) Suppose by contradiction that (say) $\Theta \overset{*}{\vdash} \Lambda$ in $\mathcal{C}_i$ does not belong to $\mathcal{C}^\star_j$. By Theorem 2.7, point (i) we infer that there is (at least) one clause $\Theta' \overset{*}{\vdash} \Lambda'$ in $\mathcal{C}^\star_j$ such that either $\Theta' \subset \Theta$ or $\Lambda' \subset \Lambda$. On the other hand, we have that the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $(\mathcal{C}_i \setminus \{\Theta \overset{*}{\vdash} \Lambda\}) \cup \{\Theta' \overset{*}{\vdash} \Lambda'\}$ proves the clauses in $\mathcal{C}_j$, and thus in $\mathcal{C}_i$, and vice versa: however, this contradicts Theorem 2.7, point (iii).

(ii) By point (i) we have that $\mathcal{C}_i \subseteq \mathcal{C}^\star_j$. As a result, we have that $\mathcal{C}^\star_i \subseteq \mathcal{C}^{\star\star}_j$: we reach the conclusion by noticing that $\mathcal{C}^{\star\star}_j = \mathcal{C}^\star_j$.

□

Example 2.8. *Let* $\mathcal{C}$ *be defined as in Example 2.6.* $\mathcal{C}^\star_1 = (\mathcal{C}'_1)^\star = \{p \vdash q\,;\, q \vdash r\,;\, r \vdash q\,;\, p \vdash r\,;\, q \vdash q\,;\, r \vdash r\}$.

The set of complementary clauses in the closure under Cut of one of $\mathcal{C}$'s irreducible versions coincides with the (unique) set of *relevant* consequences of $\mathcal{C}$ according to [151].

Proposition 2.8. Let $\mathcal{C}$ be the set of relevant consequences of a set of complementary clauses $\mathcal{D}$. If $\mathcal{E} = \{\Theta_1 \overset{*}{\vdash} \Lambda_1, \dots, \Theta_n \overset{*}{\vdash} \Lambda_n\}$ is the reduct under Weakening of $\mathcal{C}$ and $\Theta'_i \subset \Theta_i$ or $\Lambda'_i \subset \Lambda_i$, then the following conditions hold:

(i) the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $\mathcal{E}$ proves the clauses in $\mathcal{C}$, and vice versa;

(ii) the $\overline{\overline{\overline{\mathsf{G4s}}}}$ calculus for $\mathcal{E}$ refutes $\Theta'_i \vdash \Lambda'_i$, for any $1 \leq i \leq n$.

Proof. We prove each statement separately.

- (i) Straightforward from Theorem 2.6.
- (ii) Suppose by contradiction that the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{E}$ proves (say) $\Theta'_n \vdash \Lambda'_n$, with $\Theta'_n = \Theta_n$ and $\Lambda'_n = \Lambda_n \setminus \{p\}$: this implies the existence of (at least) one clause $\Phi \overset{*}{\vdash} \Psi$ in $\mathcal{E}^\star$ such that $\Phi \subseteq \Theta'_n$ and $\Psi \subseteq \Lambda'_n$. If $\Phi \overset{*}{\vdash} \Psi$ belongs to $\mathcal{E}^\star$, then it belongs to $\mathcal{C}^\star$, and thus to $\mathcal{C}$. On the other hand, $\Theta'_n \overset{*}{\vdash} \Lambda'_n$ belongs to $\mathcal{C}$: since $\mathcal{E}$ is a reduct under Weakening of $\mathcal{C}$, we get a contradiction.

□

Proposition 2.8 ensures that the reduct under Weakening of the set of relevant consequences of $\mathcal{C}$ is the set of *prime implicates* of $\mathcal{C}$ – namely, the set of the shortest clauses logically implied by $\mathcal{C}$ [105].

2.2.3. Parikh's theorem, proof-theoretically. For any clause $\Theta \overset{*}{\vdash} \Lambda$, we use $lp(\Theta \overset{*}{\vdash} \Lambda)$ to denote Θ, and $rp(\Theta \overset{*}{\vdash} \Lambda)$ to denote Λ. If $\mathcal{C} = \{\Theta_1 \overset{*}{\vdash} \Lambda_1, \ldots, \Theta_n \overset{*}{\vdash} \Lambda_n\}$, we take

$$lp(\mathcal{C}) = \bigcup_{i=1}^{n} \big(lp(\Theta_i \overset{*}{\vdash} \Lambda_i)\big)$$

$$rp(\mathcal{C}) = \bigcup_{i=1}^{n} \big(rp(\Theta_i \overset{*}{\vdash} \Lambda_i)\big)$$

We say that the *letter set* of $\mathcal{C}$, in symbols $\mathcal{L}(\mathcal{C})$, is $(lp(\mathcal{C}))^\perp \cup rp(\mathcal{C})$. The following result shows that the letter set of any irreducible version of $\mathcal{C}$ is the (unique) least letter set of $\mathcal{C}$:

Proposition 2.9. Let $\mathcal{C}$ be a set of complementary clauses, $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be the irreducible versions of $\mathcal{C}$ and $\mathcal{D}$ be a set of complementary clauses such that the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{D}$ proves the clauses in $\mathcal{C}$ and vice versa. Then, the following conditions hold:

- (i) $\mathcal{L}(\mathcal{C}_i) \subseteq \mathcal{L}(\mathcal{D})$, for any $1 \leq i \leq m$;
- (ii) $\mathcal{L}(\mathcal{C}_i) = \mathcal{L}(\mathcal{C}_j)$, for any $1 \leq i, j \leq m$.

Proof. We prove each statement separately.

- (i) Straightforward from Theorems 2.7 and 2.8.
- (ii) By Proposition 2.7 we have that $\mathcal{C}_i \subseteq \mathcal{C}_j^\star$ and $\mathcal{C}_j \subseteq \mathcal{C}_i^\star$: as a result, we infer that $lp(\mathcal{C}_i) \subseteq lp(\mathcal{C}_j)$ and $lp(\mathcal{C}_j) \subseteq lp(\mathcal{C}_i)$ – as well as that $rp(\mathcal{C}_i) \subseteq rp(\mathcal{C}_j)$ and $rp(\mathcal{C}_j) \subseteq rp(\mathcal{C}_i)$.

□

Definition 2.4. Let $\mathcal{C}$ be a set of complementary clauses and $\mathcal{P} = \{\mathcal{L}_i\}_{i=1}^n$ be a partition of $\mathcal{L}(\mathcal{C})$. $\mathcal{P}$ is a *splitting* of $\mathcal{C}$ iff there are sets of clauses $\mathcal{C}_1, \ldots, \mathcal{C}_n$ such that (*i*) $\mathcal{L}(\mathcal{C}_i) \subseteq \mathcal{L}_i$, for any $1 \leq i \leq n$;

(*ii*) the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{C}_1, \ldots, \mathcal{C}_n$ proves the clauses in $\mathcal{C}$, and vice versa.

We say that a splitting is *immediate* iff $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n = \mathcal{C}$. Moreover, we posit that a splitting $\mathcal{P} = \{\mathcal{L}_i\}_{i=1}^{n}$ of $\mathcal{C}$ is *at least as fine* than a splitting $\mathcal{P}' = \{\mathcal{L}'_i\}_{i=1}^{n}$ of $\mathcal{C}$ iff any $\mathcal{L}'_i$ is the union of some $\mathcal{L}_i$.

We present a constructive version of Parikh's theorem on the uniqueness of the finest splitting of a set of complementary clauses:

THEOREM 2.9. *Let $\mathcal{C}$ be a set of complementary clauses, and $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be the irreducible versions of $\mathcal{C}$. Then, the following statements hold:*

(*i*) *$\mathcal{P}$ is an immediate splitting of $\mathcal{C}_i$ iff $\mathcal{P}$ is an immediate splitting of $\mathcal{C}_j$, for any $1 \leq i, j \leq m$;*

(*ii*) *the finest immediate splitting of $\mathcal{C}_i$ is unique, for any $1 \leq i \leq m$;*

(*iii*) *the finest immediate splitting of $\mathcal{C}_i$ is the finest splitting of $\mathcal{C}_i$, for any $1 \leq i \leq m$.*

PROOF. We prove each statement separately.

(*i*) We prove the left-to-right direction: the proof for the other direction is analogous. If $\mathcal{P}$ is an immediate splitting of $\mathcal{C}_i$, then there exist disjoint sets of clauses $\mathcal{C}_{i1}, \ldots, \mathcal{C}_{in}$ such that $\mathcal{L}(\mathcal{C}_{ih}) \cap \mathcal{L}(\mathcal{C}_{ik}) = \varnothing$, for any $1 \leq h \neq k \leq n$, and $\mathcal{C}_{i1} \cup \cdots \cup \mathcal{C}_{in} = \mathcal{C}_i$. Proposition 2.7 ensures that for any clause $\Theta \vdash^{*} \Lambda$ in $\mathcal{C}_j$ there exists exactly one $\mathcal{C}_{ih}$ such that $\mathcal{L}(\Theta \vdash^{*} \Lambda) \subseteq \mathcal{L}(\mathcal{C}_{ih})$. On the other hand, Proposition 2.9 guarantees that $\mathcal{L}(\mathcal{C}_i) = \mathcal{L}(\mathcal{C}_j)$: this implies that for any $\mathcal{C}_{ih}$ there exists (at least) one $\Theta \vdash^{*} \Lambda$ in $\mathcal{C}_j$ such that $\mathcal{L}(\Theta \vdash^{*} \Lambda) \subseteq \mathcal{L}(\mathcal{C}_{ih})$. As a result, we infer that $\mathcal{P}$ is an immediate splitting of $\mathcal{C}_j$.

(*ii*) It suffices to notice that the finest immediate splitting of $\mathcal{C}_i$ is the family of the smallest subsets $\mathcal{C}_{i1}, \ldots, \mathcal{C}_{in}$ of $\mathcal{C}_i$ such that $\mathcal{C}_{i1} \cup \cdots \cup \mathcal{C}_{in} = \mathcal{C}_i$ and $\mathcal{L}(\mathcal{C}_{ih}) \cap \mathcal{L}(\mathcal{C}_{ik}) = \varnothing$, for any $1 \leq h \neq k \leq n$.

(*iii*) Suppose by contradiction that there exists a splitting $\mathcal{P}$ of $\mathcal{C}_i$ finer than its finest immediate splitting. This entails the existence of a set of clauses $\mathcal{D}$ such that the following conditions hold:

(*a*) the $\overline{\overline{\mathsf{G4s}}}$ calculus for $\mathcal{D}$ proves the clauses in $\mathcal{C}$, and vice versa;

(*b*) the finest immediate splitting of $\mathcal{D}$ is finer than the finest immediate splitting of $\mathcal{C}_i$.

We reason by cases over $\mathcal{D}$ to get the contradiction. If $\mathcal{D}$ is irreducible, it suffices to notice that the finest immediate splitting of $\mathcal{C}_i$ is the finest immediate splitting of $\mathcal{D}$ (by point (*i*) above). On the other hand, if $\mathcal{D}$ is not irreducible, we consider an irreducible version $\mathcal{D}'$ of $\mathcal{D}$: the finest immediate splitting of $\mathcal{D}'$ is at least as fine as the finest immediate splitting of $\mathcal{D}$. However, $\mathcal{D}'$ is also an irreducible version of $\mathcal{C}$: we infer that the finest immediate splitting of $\mathcal{D}'$ is the finest immediate splitting of $\mathcal{C}_i$ (again, by point (i) above).

□

Corollary 2.10. *Let $\mathcal{C}$ be a set of complementary clauses, and $\mathcal{D}$ be any of its irreducible versions. Then, the finest immediate splitting of $\mathcal{D}$ is the finest splitting of $\mathcal{C}$.*

PROOF. Straightforward from Theorems 2.7 and 2.8. □

For any set $\mathcal{C}$ of complementary clauses, any of its irreducible versions plays the role of the *most modular version* of $\mathcal{C}$ [**94**, pp. 277-279].

CHAPTER 3

Refutations: unmixed versus hybrid

While the idea of a refutation calculus traces back to Łukasiewicz's study of Aristotelian syllogistic in the early 1950s [89], deduction-refutation systems (D-R systems, henceforth) have garnered increasing attention only in recent decades as a burgeoning area within structural proof theory [56, 58][1]. D-R systems are syntactic frameworks designed to derive both valid and refutable formulas. These formulas are filtered by their respective deducibility and refutability relations, denoted by the turnstile symbols ($\vdash$) and ($\dashv$) for sequents and antisequents.

For reasons that will become soon evident, we find it useful to distinguish between *hybrid* and *unmixed* D-R systems. Hybrid systems include (binary) rules that combine both $\vdash$ and $\dashv$, so that an antisequent follows from a sequent and an antisequent as premises, while unmixed systems lack such combined rules, instead featuring rules that involve either sequents or antisequents exclusively.

In this chapter, we aim to provide a comprehensive analysis of the so-called anticut rules in D-R sequent calculi for classical, **FDE**-based and intuitionistic logics. This is a topic that has recently been highlighted in the context of D-R sequent calculi for **FDE**-based logics [107]. In traditional sequent calculi, the cut rule is a generalisation of the venerable rule of *modus ponens*, corresponding to the ubiquitous mathematical tactics of using intermediate lemmas or general theorems within a proof. It composes two sequents while eliminating the cut formula, namely the formula that appears among in the succedent of the first sequent and the antecedent of the second. In hybrid sequent calculi, the anticut rules have the following form:

$$acut_1 \, \frac{\Gamma \vdash \Delta, A \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{A, \Pi \dashv \Sigma} \qquad\qquad \frac{A, \Gamma \vdash \Delta \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{\Pi \dashv \Sigma, A} \, acut_2$$

If $\dashv$ is interpreted as the metatheoretic negation of $\vdash$, then anticut rules can be understood as contrapositive versions of the cut rule for $\vdash$[2]:

$$acut_1 \, \frac{\Gamma \vdash \Delta, A \qquad \Pi, \Gamma \nvdash \Delta, \Sigma}{A, \Pi \nvdash \Sigma} \qquad\qquad \frac{A, \Gamma \vdash \Delta \qquad \Pi, \Gamma \nvdash \Delta, \Sigma}{\Pi \nvdash \Sigma, A} \, acut_2$$

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[1] D-R systems may be considered part of the broader class of bilateral systems, as argued elsewhere [123, 125, 126]. However, we do not intend to pursue this point further here.

[2] It is worth stressing that our metatheory is grounded in classical logic, ensuring that the metatheoretic negation of $\vdash$ obeys the law of contraposition.

The anticut rules operate simultaneously on a sequent and an antisequent, producing a refutational conclusion in which the anticut formula is *introduced*. In the special case where Γ and Δ are empty, the anticut rules effectively function as controlled weakening rules within the framework of refutability:

$$acut_1\,\frac{\vdash A \qquad \Pi \dashv \Sigma}{A, \Pi \dashv \Sigma} \qquad\qquad \frac{A \vdash \qquad \Pi \dashv \Sigma}{\Pi \dashv \Sigma, A}\,acut_2$$

Analogously to the cut rule, anticut rules are impractical for refutation search. Furthermore, it is not immediately clear whether these rules are necessary to guarantee Ł-completeness, that is completeness with respect to both deducibility and refutability[3]. In a recent paper [107], the author examines a family of **FDE**-based propositional logics and raises the question of finding anticut-free hybrid sequent calculi capable of refuting all the antitheorems of these logics[4]. In what follows, we address this unresolved problem by establishing three claims:

(i) anticut rules *cannot* be eliminated from hybrid sequent calculi without sacrificing Ł-completeness;

(ii) when anticut rules are combined with an appropriate set of axioms, the resulting system is refutation-complete;

(iii) in unmixed sequent calculi, anticut rules *can* be constructively eliminated without compromising Ł-completeness.

Unlike [107], this chapter addresses intuitionistic propositional logic. We introduce the first hybrid sequent calculus for intuitionistic propositional logic, combining anticut rules with a suitable set of axioms and an (infinite) set of specific refutational rules. We prove that such rules yield contrapositive, Gentzen-style versions of (restricted) *Visser's rules* – namely, the rules from which any rule which is admissible but not derivable in a Hilbert-style calculus for intuitionistic logic can be derived [146]. Moreover, we show that the anticut-elimination strategy designed for **FDE**-based logics can be adapted to a (terminating) unmixed sequent calculus for intuitionistic logic.

The chapter is organized as follows. In Section 3.1, we set the stage by introducing hybrid calculi for classical propositional logic. In Section 3.2, we show that the anticut rules cannot be eliminated from hybrid sequent calculi for classical propositional logic, and that

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[3] In [57], a *hybrid* D-R system is said to be 'Ł-complete' exactly when it is complete with respect to deducibility and refutability. In this chapter, we employ 'Ł-completeness' to refer to completeness with respect to both deducibility and refutability, irrespective of whether *hybrid* or *unmixed* D-R systems are considered.

[4] Note that this issue is specific to sequent-based calculi and does not arise in [57], which presents a hybrid (sequent-style) natural deduction calculus for classical logic. As expected, the system in [57] achieves Ł-completeness without the need for anticut rules.

the system resulting from the complementarity axiom and the anticut rules is refutation-complete for classical propositional logic. Moreover, we introduce a strategy for eliminating anticut rules from an unmixed sequent calculus for classical propositional logic. In Section 3.3, we build on the basic concepts of hybrid and unmixed sequent calculi for **FDE**, **K3**, and **LP** to establish that results analogous to those presented in Section 3.2 can similarly be achieved for these logics. Lastly, in Section 3.4, we present a structural analysis of our hybrid calculus for intuitionistic propositional logic, together with a detailed exposition of the anticut-elimination argument for the unmixed sequent calculus for the same logic.

3.1. Hybrid systems for classical logic

In this chapter, we employ a variant of Kleene's $\mathsf{G4}$ calculus, where initial sequents feature only atomic formulas. Till the end of this chapter, we shall use the same label $\mathsf{G4}$ to denote its variant with atomic initial sequents. For our purposes, it suffices to recall the following facts about $\mathsf{G4}$:

THEOREM 3.1. $\mathsf{G4}$ *enjoys the following properties.*

(*i*) *Each logical rule of* $\mathsf{G4}$ *is height-bounded invertible with preservation of initial sequents.*

(*ii*) *The Weakening, Contraction and Cut rules are admissible in* $\mathsf{G4}$.

PROOF. For proofs see [14, 167, 123]. □

The hybrid calculus $\mathsf{G4}_{H1}$ is obtained from $\mathsf{G4}$ by incorporating the complementarity axiom ax_{cl} ⊣ along with the contrapositive versions of the logical and cut rules (see Figure 1). The method of adding contrapositive versions of deductive rules to design deduction-refutation calculi is illustrated in [57].

Conversely, the hybrid calculus $\mathsf{G4}_{H2}$ is constructed from $\mathsf{G4}$ by including the complementarity axiom ax_{cl} ⊣, refutational versions of unary logical rules, hybrid versions of binary logical rules, and contrapositive versions of the cut rule (see Figure 2). To our knowledge, the $\mathsf{G4}_{H2}$ calculus is not explicitly labeled in the literature. Since the logical rules of $\mathsf{G4}$, when read bottom-up, correspond to tableaux rules for provability [156], any refutational branch of an anticut-free derivation in $\mathsf{G4}_{H2}$ can be viewed as an open branch of the corresponding tableaux.

Let us remark that the method for constructing refutational calculi that underlies the design of $\overline{\overline{\mathsf{G4}}}$ offers an alternative approach to the one presented by [57]. Beginning with an invertible and terminating proof calculus, we obtain a refutational calculus by first converting unary rules into their refutational counterparts. Then, for each binary rule, we introduce a unary rule featuring one of the original premises as refutational premise.

$$ax_{cl}\dashv \frac{}{\Theta \dashv \Lambda}$$

$$aL\wedge \frac{A \wedge B, \Gamma \dashv \Delta}{A, B, \Gamma \dashv \Delta} \qquad aR\wedge_1 \frac{\Gamma \vdash \Delta, A \qquad \Gamma \dashv \Delta, A \wedge B}{\Gamma \dashv \Delta, B} \qquad aR\wedge_2 \frac{\Gamma \vdash \Delta, B \qquad \Gamma \dashv \Delta, A \wedge B}{\Gamma \dashv \Delta, A}$$

$$aL\vee_1 \frac{A, \Gamma \vdash \Delta \qquad A \vee B, \Gamma \dashv \Delta}{B, \Gamma \dashv \Delta} \qquad aL\vee_2 \frac{B, \Gamma \vdash \Delta \qquad A \vee B, \Gamma \dashv \Delta}{A, \Gamma \dashv \Delta} \qquad aR\vee \frac{\Gamma \dashv \Delta, A \vee B}{\Gamma \dashv \Delta, A, B}$$

$$aL \rightarrow_1 \frac{\Gamma \vdash \Delta, A \qquad A \rightarrow B, \Gamma \dashv \Delta}{B, \Gamma \dashv \Delta} \qquad aL \rightarrow_2 \frac{B, \Gamma \vdash \Delta \qquad A \rightarrow B, \Gamma \dashv \Delta}{\Gamma \dashv \Delta, A} \qquad aR \rightarrow \frac{\Gamma \dashv \Delta, A \rightarrow B}{A, \Gamma \dashv \Delta, B}$$

$$aL\neg \frac{\neg A, \Gamma \dashv \Delta}{\Gamma \dashv \Delta, A} \qquad aR\neg \frac{\Gamma \dashv \Delta, \neg A}{A, \Gamma \dashv \Delta}$$

$$acut_1 \frac{\Gamma \vdash \Delta, A \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{A, \Pi \dashv \Sigma} \qquad acut_2 \frac{A, \Gamma \vdash \Delta \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{\Pi \dashv \Sigma, A}$$

Figure 1. $\mathsf{G4}_{H1}$ sequent calculus

$$a'\wedge \dashv \frac{A, B, \Gamma \dashv \Delta}{A \wedge B, \Gamma \dashv \Delta} \qquad \dashv a'\wedge \frac{\Gamma \vdash \Delta, A \qquad \Gamma \dashv \Delta, B}{\Gamma \dashv \Delta, A \wedge B} \qquad \dashv a'\wedge \frac{\Gamma \vdash \Delta, B \qquad \Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A \wedge B}$$

$$a'\vee \dashv \frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \dashv \Delta}{A \vee B, \Gamma \dashv \Delta} \qquad a'\vee \dashv \frac{B, \Gamma \vdash \Delta \qquad A, \Gamma \dashv \Delta}{A \vee B, \Gamma \dashv \Delta} \qquad \dashv a'\vee \frac{\Gamma \dashv \Delta, A, B}{\Gamma \dashv \Delta, A \vee B}$$

$$a'\neg \dashv \frac{\Gamma \dashv \Delta, A}{\neg A, \Gamma \dashv \Delta} \qquad \dashv a'\neg \frac{A, \Gamma \dashv \Delta}{\Gamma \dashv \Delta, \neg A}$$

$$a' \rightarrow \dashv \frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \dashv \Delta}{A \rightarrow B, \Gamma \dashv \Delta} \qquad a' \rightarrow \dashv \frac{B, \Gamma \vdash \Delta \qquad \Gamma \dashv \Delta, A}{A \rightarrow B, \Gamma \dashv \Delta} \qquad \dashv a' \rightarrow \frac{A, \Gamma \dashv \Delta, B}{\Gamma \dashv \Delta, A \rightarrow B}$$

$$acut_1 \frac{\Gamma \vdash \Delta, A \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{A, \Pi \dashv \Sigma} \qquad acut_2 \frac{A, \Gamma \vdash \Delta \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{\Pi \dashv \Sigma, A}$$

Figure 2. $\mathsf{G4}_{H2}$ sequent calculus

3.2. Anticut and classical logic

In this section, we illustrate the deductive power of anticut by proving a cluster of results: to this aim, we employ the hybrid $\mathsf{G4}_{H1\cap 2}$ calculus, which gathers all the rules shared by $\mathsf{G4}_{H1}$ and $\mathsf{G4}_{H2}$.

We begin by showing that the contrapositive versions of standard weakening rules can be seen as anticut applications in disguise.

Lemma 3.1. *The rules of Strengthening*

$$str \frac{A, \Gamma \dashv \Delta}{\Gamma \dashv \Delta} \qquad \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta} str$$

are admissible in the $\mathsf{G4}_{H1\cap 2}$ *calculus.*

Proof. If $\Gamma \cup \Delta = \varnothing$, the conclusion is in both cases immediate. On the other hand, if $\Gamma = \Gamma' \cup [B]$ or $\Delta = \Delta' \cup [C]$, we proceed as follows:

$$\mathit{acut}_1\,\dfrac{\overset{\vdots}{A, B \vdash B} \qquad A, B, \Gamma' \dashv \Delta}{B, \Gamma' \dashv \Delta} \qquad\qquad \dfrac{\overset{\vdots}{A, C \vdash C} \qquad A, \Gamma \dashv \Delta', C}{\Gamma \dashv \Delta', C}\,\mathit{acut}_2$$

$$\mathit{acut}_1\,\dfrac{\overset{\vdots}{B \vdash B, A} \qquad B, \Gamma' \dashv \Delta, A}{B, \Gamma' \dashv \Delta} \qquad\qquad \dfrac{\overset{\vdots}{C \vdash C, A} \qquad \Gamma \dashv \Delta', C, A}{\Gamma \dashv \Delta', C}\,\mathit{acut}_2$$

□

Moreover, the contrapositive versions of standard contraction rules are admissible in $\mathsf{G4}_{H1\cap 2}$.

Lemma 3.2. *The rules of Duplication*

$$\mathit{dup}\,\dfrac{A, \Gamma \dashv \Delta}{A, A, \Gamma \dashv \Delta} \qquad\qquad \dfrac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A, A}\,\mathit{dup}$$

are admissible in $\mathsf{G4}_{H1\cap 2}$.

Proof. We focus on Left Duplication, the other case being analogous. Reasoning by induction on the height of the $\mathsf{G4}_{H1\cap 2}$-derivation π of $A, \Gamma \dashv \Delta$, we prove the existence of a $\mathsf{G4}_{H1\cap 2}$-derivation ρ of $A, A, \Gamma, \Gamma \dashv \Delta, \Delta$ (cf. the *copy* rule in Lemma 3.3). Hence, we apply Lemma 3.1 to ρ to get a $\mathsf{G4}_{H1\cap 2}$-derivation of $A, A, \Gamma \dashv \Delta$.

If $h(\pi) = 1$, the conclusion is trivial. If $h(\pi) > 1$ and the last rule applied is acut_1, we reason by cases over A. If A is principal in the acut_1-application, then ρ has the following form:

$$\dfrac{\dfrac{\overset{\vdots}{\Pi \vdash \Sigma, A} \qquad \overset{\vdots}{\Gamma, \Pi \dashv \Sigma, \Delta}}{A, \Gamma \dashv \Delta}\,\mathit{acut}_1}{A, A, \Gamma, \Gamma \dashv \Delta, \Delta}\,\mathit{copy}$$

We consider the following derivation:

$$\dfrac{\overset{\vdots}{\Pi \vdash \Sigma, A} \qquad \dfrac{\overset{\vdots}{\Pi \vdash \Sigma, A} \qquad \dfrac{\overset{\vdots}{\Gamma, \Pi \dashv \Sigma, \Delta}}{\Gamma, \Gamma, \Pi, \Pi \dashv \Sigma, \Sigma, \Delta, \Delta}\,\mathit{copy}}{A, \Gamma, \Gamma, \Pi \dashv \Sigma, \Delta, \Delta}\,\mathit{acut}_1}{A, A, \Gamma, \Gamma \dashv \Delta, \Delta}\,\mathit{acut}_1$$

The *copy* application can be removed by inductive hypothesis. On the other hand, if A is not principal in the acut_1-application, then ρ has the following form:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Pi \vdash \Sigma, B\end{array} \qquad \begin{array}{c}\vdots\\ A, \Gamma', \Pi \dashv \Sigma, \Delta\end{array}}{A, B, \Gamma' \dashv \Delta}\,\mathit{acut}_1}{A, A, B, B, \Gamma', \Gamma' \dashv \Delta, \Delta}\,\mathit{copy}$$

with $\Gamma = \Gamma' \cup [A]$. We take the following derivation:

$$\dfrac{\begin{array}{c}\vdots\\ \Pi \vdash \Sigma, B\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \Pi \vdash \Sigma, B\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ A, \Gamma', \Pi \dashv \Sigma, \Delta\end{array}}{A, A, \Gamma', \Gamma', \Pi, \Pi \dashv \Sigma, \Sigma, \Delta, \Delta}\,\mathit{copy}}{A, A, B, \Gamma', \Gamma', \Pi \dashv \Sigma, \Delta, \Delta}\,\mathit{acut}_1}{A, A, B, B, \Gamma', \Gamma' \dashv \Delta, \Delta}\,\mathit{acut}_1$$

Finally, if the last rule applied is acut_2, ρ has the following form:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ B, \Pi \dashv \Sigma\end{array} \qquad \begin{array}{c}\vdots\\ A, \Gamma, \Pi \dashv \Sigma, \Delta'\end{array}}{A, \Gamma \dashv \Delta', B}\,\mathit{acut}_2}{A, A, \Gamma, \Gamma \dashv \Delta', \Delta', B, B}\,\mathit{copy}$$

with $\Delta = \Delta' \cup [B]$. We consider the following derivation to get the conclusion:

$$\dfrac{\begin{array}{c}\vdots\\ B, \Pi \dashv \Sigma\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ B, \Pi \dashv \Sigma\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ A, \Gamma, \Pi \dashv \Sigma, \Delta'\end{array}}{A, A, \Gamma, \Gamma, \Pi, \Pi \dashv \Sigma, \Sigma, \Delta', \Delta'}\,\mathit{copy}}{A, A, \Gamma, \Gamma, \Pi \dashv \Sigma, \Delta', \Delta', B}\,\mathit{acut}_2}{A, A, \Gamma, \Gamma \dashv \Delta', \Delta', B, B}\,\mathit{acut}_2$$

□

Next, we show that any application of the contrapositive of a $\mathsf{G4}$ logical rule can be turned into a number of *acut* and *dup* applications:

THEOREM 3.2. *Each logical rule of* $\mathsf{G4}_{H1}$ *is admissible in the* $\mathsf{G4}_{H1\cap 2}$ *calculus.*

PROOF. We consider the refutational rules for $\wedge$ and $\rightarrow$, leaving the other cases to the reader.

$a\wedge$ Consider the following derivations, with $i = 1, 2$[5]:

---

[5] We use dashed lines to denote admissible rules, and doubled (dashed) lines to refer to multiple applications of (admissible) rules.

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A \wedge B \vdash B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A \wedge B \vdash A$}
\AxiomC{$A \wedge B, \Gamma \dashv \Delta$}
\dashedLine
\RightLabel{dup}
\UnaryInfC{$A \wedge B, A \wedge B, \Gamma \dashv \Delta$}
\RightLabel{$acut_1$}
\BinaryInfC{$A \wedge B, A, \Gamma \dashv \Delta$}
\RightLabel{$acut_1$}
\BinaryInfC{$A, B, \Gamma \dashv \Delta$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$\Gamma \vdash \Delta, A_i$}
\dashedLine
\LeftLabel{wk}
\UnaryInfC{$A_{3-i}, \Gamma \vdash \Delta, A_i$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A_{3-i}, \Gamma \vdash \Delta, A_{3-i}$}
\LeftLabel{$\vdash \wedge$}
\BinaryInfC{$A_{3-i}, \Gamma \vdash \Delta, A_1 \wedge A_2$}
\AxiomC{$\Gamma \dashv \Delta, A_1 \wedge A_2$}
\doubleLine
\RightLabel{dup}
\UnaryInfC{$\Gamma, \Gamma \dashv \Delta, \Delta, A_1 \wedge A_2$}
\RightLabel{$acut_2$}
\BinaryInfC{$\Gamma \dashv \Delta, A_{3-i}$}
\end{prooftree}$$

$a \rightarrow$ Consider the following derivations:

$$\begin{prooftree}
\AxiomC{$\Gamma \vdash \Delta, A$}
\dashedLine
\LeftLabel{wk}
\UnaryInfC{$\Gamma \vdash \Delta, A, B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B, \Gamma \vdash \Delta, B$}
\LeftLabel{$\rightarrow \vdash$}
\BinaryInfC{$A \rightarrow B, \Gamma \vdash \Delta, B$}
\AxiomC{$A \rightarrow B, \Gamma \dashv \Delta$}
\doubleLine
\RightLabel{dup}
\UnaryInfC{$A \rightarrow B, \Gamma, \Gamma \dashv \Delta, \Delta$}
\RightLabel{$acut_1$}
\BinaryInfC{$B, \Gamma \dashv \Delta$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A, \Gamma \vdash \Delta, A$}
\AxiomC{$B, \Gamma \vdash \Delta$}
\dashedLine
\RightLabel{wk}
\UnaryInfC{$B, A, \Gamma \vdash \Delta$}
\RightLabel{$\rightarrow \vdash$}
\BinaryInfC{$A \rightarrow B, A, \Gamma \vdash \Delta$}
\AxiomC{$A \rightarrow B, \Gamma \dashv \Delta$}
\doubleLine
\RightLabel{dup}
\UnaryInfC{$A \rightarrow B, \Gamma, \Gamma \dashv \Delta, \Delta$}
\RightLabel{$acut_2$}
\BinaryInfC{$\Gamma \dashv \Delta, A$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B \vdash A \rightarrow B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\vdash A \rightarrow B, A$}
\AxiomC{$\Gamma \dashv \Delta, A \rightarrow B$}
\dashedLine
\RightLabel{dup}
\UnaryInfC{$\Gamma \dashv \Delta, A \rightarrow B, A \rightarrow B$}
\RightLabel{$acut_1$}
\BinaryInfC{$A, \Gamma \dashv \Delta, A \rightarrow B$}
\RightLabel{$acut_2$}
\BinaryInfC{$A, \Gamma \dashv \Delta, B$}
\end{prooftree}$$

□

On the other hand, any application of a $\mathsf{G4}_{H2}$ refutational logical rule can be turned into a number of *acut* and *dup* applications:

THEOREM 3.3. *Each logical rule of* $\mathsf{G4}_{H2}$ *is admissible in the* $\mathsf{G4}_{H1\cap 2}$ *calculus.*

PROOF. We consider the refutational rules for $\vee$ and $\rightarrow$, leaving the other cases to the reader.

$a'\vee$ Take the following derivations:

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A_{3-i} \vdash A_1 \vee A_2$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A_i, \Gamma \vdash \Delta$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A_{3-i}, \Gamma \dashv \Delta$}
\doubleLine
\RightLabel{dup}
\UnaryInfC{$A_{3-i}, \Gamma, \Gamma \dashv \Delta, \Delta$}
\RightLabel{$acut_2$}
\BinaryInfC{$A_{3-i}, \Gamma \dashv \Delta, A_i$}
\RightLabel{$acut_1$}
\BinaryInfC{$A_1 \vee A_2, \Gamma \dashv \Delta, A_i$}
\dashedLine
\RightLabel{str}
\UnaryInfC{$A_1 \vee A_2, \Gamma \dashv \Delta$}
\end{prooftree}$$

$$\dfrac{\begin{matrix}\vdots\\ A\vee B\vdash A,B\end{matrix}\qquad\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A,B\end{matrix}}{\Gamma\dashv\Delta,A\vee B}\,\mathit{acut}_2$$

$a'\to$ Consider the following derivations:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ B\vdash A\to B\end{matrix}\qquad\dfrac{\begin{matrix}\vdots\\ \Gamma\vdash\Delta,A\end{matrix}\qquad\dfrac{\begin{matrix}\vdots\\ B,\Gamma\dashv\Delta\end{matrix}}{B,\Gamma,\Gamma\dashv\Delta,\Delta}\,\mathit{dup}}{A,B,\Gamma\dashv\Delta}\,\mathit{acut}_1}{A\to B,A,\Gamma\dashv\Delta}\,\mathit{acut}_1}{A\to B,\Gamma\dashv\Delta}\,\mathit{str}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \vdash A\to B,A\end{matrix}\qquad\dfrac{\begin{matrix}\vdots\\ B,\Gamma\vdash\Delta\end{matrix}\qquad\dfrac{\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A\end{matrix}}{\Gamma,\Gamma\dashv\Delta,\Delta,A}\,\mathit{dup}}{\Gamma\dashv\Delta,A,B}\,\mathit{acut}_2}{A\to B,\Gamma\dashv\Delta,B}\,\mathit{acut}_1}{A\to B,\Gamma\dashv\Delta}\,\mathit{str}$$

$$\dfrac{\begin{matrix}\vdots\\ A\to B,A\vdash B\end{matrix}\qquad\begin{matrix}\vdots\\ A,\Gamma\dashv\Delta,B\end{matrix}}{\Gamma\dashv\Delta,A\to B}\,\mathit{acut}_2$$

□

We conclude this section by showing that any application of a refutational logical rule of $\overline{\overline{\mathsf{G4}}}$ can be replaced by applications of *acut*:

THEOREM 3.4. *Each refutational logical rule of* $\overline{\overline{\mathsf{G4}}}$ *is admissible in the* $\mathsf{G4}_{H1\cap 2}$ *calculus.*

PROOF. We consider only the $\dashv\wedge$, $\vee\dashv$ and $\to\dashv$ rules.

$\dashv\wedge$ Consider the following derivations:

$$\mathit{acut}_2\,\dfrac{\begin{matrix}\vdots\\ A\wedge B\vdash A\end{matrix}\qquad\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A\end{matrix}}{\Gamma\dashv\Delta,A\wedge B}\qquad\mathit{acut}_2\,\dfrac{\begin{matrix}\vdots\\ A\wedge B\vdash B\end{matrix}\qquad\begin{matrix}\vdots\\ \Gamma\dashv\Delta,B\end{matrix}}{\Gamma\dashv\Delta,A\wedge B}$$

$\vee\dashv$ Take the following derivations:

$$\mathit{acut}_1\,\dfrac{\begin{matrix}\vdots\\ A\vdash A\vee B\end{matrix}\qquad\begin{matrix}\vdots\\ A,\Gamma\dashv\Delta\end{matrix}}{A\vee B,\Gamma\dashv\Delta}\qquad\dfrac{\begin{matrix}\vdots\\ B\vdash A\vee B\end{matrix}\qquad\begin{matrix}\vdots\\ B,\Gamma\dashv\Delta\end{matrix}}{A\vee B,\Gamma\dashv\Delta}\,\mathit{acut}_1$$

$\to\dashv$ Consider the following derivations:

$$\mathit{acut}_1\,\dfrac{\begin{matrix}\vdots\\ \vdash B\to C,B\end{matrix}\qquad\begin{matrix}\vdots\\ \Gamma\dashv\Delta,B\end{matrix}}{B\to C,\Gamma\dashv\Delta}\qquad\mathit{acut}_1\,\dfrac{\begin{matrix}\vdots\\ C\vdash B\to C\end{matrix}\qquad\begin{matrix}\vdots\\ C,\Gamma\dashv\Delta\end{matrix}}{B\to C,\Gamma\dashv\Delta}$$

□

Remark 1. In the proofs of Theorems 3.3 and 3.4 we establish the admissibility of the refutational logical rules of $\overline{\overline{\mathsf{G4}}}$ in $\mathsf{G4}_{H1\cap 2}$ leveraging the following provable sequents:

$$A, B \vdash A \wedge B \qquad A \wedge B \vdash A \qquad A \wedge B \vdash B \tag{3.2.1}$$

$$A \vdash A \vee B \qquad B \vdash A \vee B \qquad A \vee B \vdash A, B \tag{3.2.2}$$

$$\vdash A \to B, A \qquad B \vdash A \to B \qquad A \to B, A \vdash B \tag{3.2.3}$$

$$\vdash A, \neg A \qquad \neg A, A \vdash \tag{3.2.4}$$

This approach to logical rules is analogous to that of Gentzen's **LDK** calculus: there, the sequents (3.2.1), (3.2.2) and (3.2.4) played the role of axioms (under the label of 'logical groundsequents') and standard structural rules were sufficient to recover each logical rule for deducibility [174, 108].

3.2.1. Anticut elimination in classical logic. In this subsection, we focus on the topic of anticut elimination from D-R sequent calculi for classical logic. In order to illustrate a constructive approach to *acut* elimination in $\overline{\overline{\mathsf{G4}}}$, we first prove two preliminary results.

Theorem 3.5. *For any sequent* $\Gamma \vdash \Delta$*, either* $\overline{\overline{\mathsf{G4}}}$ *proves* $\Gamma \vdash \Delta$ *or refutes* $\Gamma \vdash \Delta$ *– but not both.*

Proof. Straightforward by Theorem 2.2. ☐

Lemma 3.3. *The copy rule*

$$\dfrac{\Gamma \dashv \Delta \qquad \Gamma \dashv \Delta}{\Gamma, \Gamma \dashv \Delta, \Delta}\ \mathit{copy}$$

is admissible in $\overline{\overline{\mathsf{G4}}}$*.*

Proof. Let us assume (without loss of generality) that for any *copy* application the two premisses have the same derivation π: we reason by induction on $2h(\pi)$ to get the conclusion. If $h(\pi) = 1$, the conclusion is immediate; otherwise, we reason by cases over the last rule applied in π.

$\vee$ If the last rule applied in π is $\vee \dashv$, we have e.g.

$$\mathit{copy}\ \dfrac{\vee\dashv\ \dfrac{\begin{matrix}\vdots\\ A, \Gamma \dashv \Delta\end{matrix}}{A \vee B, \Gamma \dashv \Delta} \qquad \vee\dashv\ \dfrac{\begin{matrix}\vdots\\ A, \Gamma \dashv \Delta\end{matrix}}{A \vee B, \Gamma \dashv \Delta}}{A \vee B, A \vee B, \Gamma, \Gamma \dashv \Delta, \Delta} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ A, \Gamma \dashv \Delta\end{matrix} \qquad \begin{matrix}\vdots\\ A, \Gamma \dashv \Delta\end{matrix}}{A, A, \Gamma, \Gamma \dashv \Delta, \Delta}\ \mathit{copy}}{A \vee B, A, \Gamma, \Gamma \dashv \Delta, \Delta}\ \vee\dashv}{A \vee B, A \vee B, \Gamma, \Gamma \dashv \Delta, \Delta}\ \vee\dashv$$

On the other hand, if the last rule applied in π is $\dashv \vee$, we have

$$\textit{copy}\,\dfrac{\dashv\vee\,\dfrac{\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A,B\end{matrix}}{\Gamma\dashv\Delta,A\vee B}\qquad\dashv\vee\,\dfrac{\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A,B\end{matrix}}{\Gamma\dashv\Delta,A\vee B}}{\Gamma,\Gamma\dashv\Delta,\Delta,A\vee B,A\vee B}\quad\rightsquigarrow\quad\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A,B\end{matrix}\qquad\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A,B\end{matrix}}{\Gamma,\Gamma\dashv\Delta,\Delta,A,B,A,B}\,\textit{copy}}{\Gamma,\Gamma\dashv\Delta,\Delta,A,B,A\vee B}\,\dashv\vee}{\Gamma,\Gamma\dashv\Delta,\Delta,A\vee B,A\vee B}\,\dashv\vee$$

$\neg$ If the last rule applied in π is $\neg\dashv$, we have

$$\textit{copy}\,\dfrac{\neg\dashv\,\dfrac{\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A\end{matrix}}{\neg A,\Gamma\dashv\Delta}\qquad\neg\dashv\,\dfrac{\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A\end{matrix}}{\neg A,\Gamma\dashv\Delta}}{\neg A,\neg A,\Gamma,\Gamma\dashv\Delta,\Delta}\quad\rightsquigarrow\quad\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A\end{matrix}\qquad\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A\end{matrix}}{\Gamma,\Gamma\dashv\Delta,\Delta,A,A}\,\textit{copy}}{\neg A,\Gamma,\Gamma\dashv\Delta,\Delta,A}\,\neg\dashv}{\neg A,\neg A,\Gamma,\Gamma\dashv\Delta,\Delta}\,\neg\dashv$$

On the other hand, if the last rule applied in π is $\dashv\neg$, we have

$$\textit{copy}\,\dfrac{\dashv\neg\,\dfrac{\begin{matrix}\vdots\\ A,\Gamma\dashv\Delta\end{matrix}}{\Gamma\dashv\Delta,\neg A}\qquad\dashv\neg\,\dfrac{\begin{matrix}\vdots\\ A,\Gamma\dashv\Delta\end{matrix}}{\Gamma\dashv\Delta,\neg A}}{\Gamma,\Gamma\dashv\Delta,\Delta,\neg A,\neg A}\quad\rightsquigarrow\quad\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A\end{matrix}\qquad\begin{matrix}\vdots\\ \Gamma\dashv\Delta,A\end{matrix}}{\Gamma,\Gamma\dashv\Delta,\Delta,A,A}\,\textit{copy}}{\neg A,\Gamma,\Gamma\dashv\Delta,\Delta,A}\,\dashv\neg}{\neg A,\neg A,\Gamma,\Gamma\dashv\Delta,\Delta}\,\dashv\neg$$

□

We are ready to prove the main result of this section:

THEOREM 3.6. *There exists an algorithm which turns any* $\overline{\overline{\overline{\mathsf{G4}}}}+acut_i$*-derivation of* $\Pi\dashv\Sigma$ *into a* $\overline{\overline{\overline{\mathsf{G4}}}}$*-derivation of* $\Pi\dashv\Sigma$*, with* $i=1,2$*.*

PROOF. We focus on the topmost $acut_i$ application, proceeding by primary induction on the logical complexity of the $acut_i$ formula and by secondary induction on the height of the derivation of the right premise.
First, there are cases of reduction of the size of the $acut_i$ formula where the inductive hypothesis applies to smaller $acut_{3-i}$ formulas:

$$acut_1\,\dfrac{\begin{matrix}\vdots\\ \Gamma\vdash\Delta,\neg B\end{matrix}\qquad\begin{matrix}\vdots\\ \Pi,\Gamma\dashv\Delta,\Sigma\end{matrix}}{\neg B,\Pi\dashv\Sigma}\quad\rightsquigarrow\quad\dfrac{\dfrac{inv\,\dfrac{\begin{matrix}\vdots\\ \Gamma\vdash\Delta,\neg B\end{matrix}}{B,\Gamma\vdash\Delta}\qquad\begin{matrix}\vdots\\ \Pi,\Gamma\dashv\Delta,\Sigma\end{matrix}}{\Pi\dashv\Sigma,B}\,acut_2}{\neg B,\Pi\dashv\Sigma}\,\neg\dashv$$

$$acut_2\,\dfrac{\begin{matrix}\vdots\\ \neg B,\Gamma\vdash\Delta\end{matrix}\qquad\begin{matrix}\vdots\\ \Pi,\Gamma\dashv\Delta,\Sigma\end{matrix}}{\Pi\dashv\Sigma,\neg B}\quad\rightsquigarrow\quad\dfrac{\dfrac{inv\,\dfrac{\begin{matrix}\vdots\\ \neg B,\Gamma\vdash\Delta\end{matrix}}{\Gamma\vdash\Delta,B}\qquad\begin{matrix}\vdots\\ \Pi,\Gamma\dashv\Delta,\Sigma\end{matrix}}{B,\Pi\dashv\Sigma}\,acut_1}{\Pi\dashv\Sigma,\neg B}\,\dashv\neg$$

On the other hand, there are cases of reduction where a single $acut_i$ application is replaced by multiple $acut_i$ and $acut_{3-i}$ applications on smaller formulas. Take e.g. the following derivations:

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta, B \wedge C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\RightLabel{$acut_1$}
\BinaryInfC{$B \wedge C, \Pi \dashv \Sigma$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B \vee C, \Gamma \vdash \Delta$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\RightLabel{$acut_2$}
\BinaryInfC{$\Pi \dashv \Sigma, B \vee C$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B \to C, \Gamma \vdash \Delta$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\RightLabel{$acut_2$}
\BinaryInfC{$\Pi \dashv \Sigma, B \to C$}
\end{prooftree}
$$

We transform these derivations as follows:

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta, B \wedge C$}
\dashedLine
\LeftLabel{inv}
\UnaryInfC{$\Gamma \vdash \Delta, C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta, B \wedge C$}
\dashedLine
\LeftLabel{inv}
\UnaryInfC{$\Gamma \vdash \Delta, B$}
\doubleLine
\dashedLine
\LeftLabel{wk}
\UnaryInfC{$\Pi, \Gamma \vdash \Delta, \Sigma, B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\dashedLine
\RightLabel{$copy$}
\BinaryInfC{$\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma$}
\RightLabel{$acut_1$}
\BinaryInfC{$B, \Pi, \Gamma \dashv \Delta, \Sigma$}
\RightLabel{$acut_1$}
\BinaryInfC{$B, C, \Pi \dashv \Sigma$}
\RightLabel{$\wedge \dashv$}
\UnaryInfC{$B \wedge C, \Pi \dashv \Sigma$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B \vee C, \Gamma \vdash \Delta$}
\dashedLine
\LeftLabel{inv}
\UnaryInfC{$C, \Gamma \vdash \Delta$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B \vee C, \Gamma \vdash \Delta$}
\dashedLine
\LeftLabel{inv}
\UnaryInfC{$B, \Gamma \vdash \Delta$}
\doubleLine
\dashedLine
\LeftLabel{wk}
\UnaryInfC{$B, \Pi, \Gamma \vdash \Delta, \Sigma$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\dashedLine
\RightLabel{$copy$}
\BinaryInfC{$\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma$}
\RightLabel{$acut_2$}
\BinaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma, B$}
\RightLabel{$acut_2$}
\BinaryInfC{$\Pi \dashv \Sigma, B, C$}
\RightLabel{$\dashv \vee$}
\UnaryInfC{$\Pi \dashv \Sigma, B \vee C$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B \to C, \Gamma \vdash \Delta$}
\dashedLine
\LeftLabel{inv}
\UnaryInfC{$\Gamma \vdash \Delta, B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B \to C, \Gamma \vdash \Delta$}
\dashedLine
\LeftLabel{inv}
\UnaryInfC{$C, \Gamma \vdash \Delta$}
\doubleLine
\dashedLine
\LeftLabel{wk}
\UnaryInfC{$C, \Pi, \Gamma \vdash \Delta, \Sigma$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\dashedLine
\RightLabel{$copy$}
\BinaryInfC{$\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma$}
\RightLabel{$acut_1$}
\BinaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma, C$}
\RightLabel{$acut_1$}
\BinaryInfC{$B, \Pi \dashv \Sigma, C$}
\RightLabel{$\dashv \to$}
\UnaryInfC{$\Pi \dashv \Sigma, B \to C$}
\end{prooftree}
$$

A different strategy must be employed to deal with the following case:

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta, B \vee C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\RightLabel{$acut_1$}
\BinaryInfC{$B \vee C, \Pi \dashv \Sigma$}
\end{prooftree}
$$

Take a $\overline{\overline{\mathsf{G4}}}$-proof π of $\Pi, \Gamma \vdash \Delta, \Sigma, B, C$: if we unthread (say) B from π, we obtain either a $\overline{\overline{\mathsf{G4}}}$-proof π' of $\Pi, \Gamma \vdash \Delta, \Sigma, C$ or (at least) one acut_i-free derivation π'' of $\Pi, \Gamma \dashv \Delta, \Sigma, C$[6]. We reason by cases to reach the conclusion:

$$\dfrac{\dfrac{\overset{\vdots}{C, \Gamma \vdash \Delta, C} \qquad \dfrac{\overset{\vdots\,\pi'}{\Pi, \Gamma \vdash \Delta, \Sigma, C} \qquad \dfrac{\overset{\vdots}{\Pi, \Gamma \dashv \Delta, \Sigma} \quad \overset{\vdots}{\Pi, \Gamma \dashv \Delta, \Sigma}}{\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma}\,\mathit{copy}}{C, \Pi, \Gamma \dashv \Delta, \Sigma}\,\mathit{acut}_1}{C, \Pi \dashv \Sigma}\,\mathit{acut}_1}{B \vee C, \Pi \dashv \Sigma}\,\vee\dashv$$

$$\dfrac{\dfrac{\mathit{inv}\,\dfrac{\overset{\vdots}{\Gamma \vdash \Delta, B \vee C}}{\Gamma \vdash \Delta, B, C} \qquad \overset{\vdots\,\pi''}{\Pi, \Gamma \dashv \Delta, \Sigma, C}}{B, \Pi \dashv \Sigma}\,\mathit{acut}_1}{B \vee C, \Pi \dashv \Sigma}\,\vee\dashv$$

The same argument applies to the following cases:

$$\mathit{acut}_2\,\dfrac{\overset{\vdots}{B \wedge C, \Gamma \vdash \Delta} \qquad \overset{\vdots}{\Pi, \Gamma \dashv \Delta, \Sigma}}{\Pi \dashv \Sigma, B \wedge C} \qquad\qquad \dfrac{\overset{\vdots}{\Gamma \vdash \Delta, B \to C} \qquad \overset{\vdots}{\Pi, \Gamma \dashv \Delta, \Sigma}}{B \to C, \Pi \dashv \Sigma}\,\mathit{acut}_1$$

We can thus reduce ourselves to the case where the topmost acut_i formula is atomic. Notice that if π is the derivation of the right premiss $\Pi, \Gamma \dashv \Delta, \Sigma$ of the topmost acut_i application, then π is acut_i-free. Now, we proceed by induction on $h(\pi)$ to show that the topmost acut_i application can be eliminated.
If $h(\pi) = 1$, the antisequent $\Pi, \Gamma \vdash \Delta, \Sigma$ has the form $\Theta \dashv \Lambda$.

- (i) If $i = 1$, then the left premise has the form $\Theta' \vdash \Lambda', A$, with $\Theta' \subseteq \Theta$ and $\Lambda' \subseteq \Lambda$. This implies that $A \in \Theta'$, and thus that $A \notin \Lambda$: as a result, we infer that $\overline{\overline{\mathsf{G4}}}$ derives $A, \Theta'' \dashv \Lambda''$, with $\Theta'' = \Theta \setminus \Theta'$ and $\Lambda'' = \Lambda \setminus \Lambda'$.
- (ii) If $i = 2$, then the left premise has the form $A, \Theta' \vdash \Lambda'$, with $\Theta' \subseteq \Theta$ and $\Lambda' \subseteq \Lambda$. This implies that $A \in \Lambda'$, and thus that $A \notin \Theta$: as a result, we infer that $\overline{\overline{\mathsf{G4}}}$ derives $\Theta'' \dashv \Lambda'', A$, with $\Theta'' = \Theta \setminus \Theta'$ and $\Lambda'' = \Lambda \setminus \Lambda'$.

On the other hand, if $h(\pi) > 1$ and some formula in $\Gamma \cup \Delta$ is principal in the last rule applied in π, we exploit the secondary inductive hypothesis as follows:

---

[6] Intuitively, a formula D occurring in the conclusion of a derivation ρ is *unthread* from ρ whenever D and its ancestors are deleted from ρ, up to the initial (anti)sequents of ρ. For a formal treatment of the notion of unthreading, see [137].

$$
{\scriptstyle acut_1}\,\dfrac{\overset{\vdots}{\Gamma \vdash \Delta, A} \qquad {\scriptstyle \dashv}\,\dfrac{\overset{\vdots}{\Pi, \Gamma' \dashv \Delta', \Sigma}}{\Pi, \Gamma \dashv \Delta, \Sigma}}{A, \Pi \dashv \Sigma}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\overset{\vdots}{\Gamma \vdash \Delta, A}}{\Gamma' \vdash \Delta', A}\,{\scriptstyle inv} \qquad \overset{\vdots}{\Pi, \Gamma' \dashv \Delta', \Sigma}}{A, \Pi \dashv \Sigma}\,{\scriptstyle acut_1}
$$

$$
{\scriptstyle acut_1}\,\dfrac{\overset{\vdots}{A, \Gamma \vdash \Delta} \qquad {\scriptstyle \dashv}\,\dfrac{\overset{\vdots}{\Pi, \Gamma' \dashv \Delta', \Sigma}}{\Pi, \Gamma \dashv \Delta, \Sigma}}{\Pi \dashv \Sigma, A}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\overset{\vdots}{A, \Gamma \vdash \Delta}}{A, \Gamma' \vdash \Delta'}\,{\scriptstyle inv} \qquad \overset{\vdots}{\Pi, \Gamma' \dashv \Delta', \Sigma}}{\Pi \dashv \Sigma, A}\,{\scriptstyle acut_1}
$$

Finally, if $h(\pi) > 1$ and some formula in $\Pi \cup \Sigma$ is principal in the last rule applied in π, we leverage the secondary inductive hypothesis as follows:

$$
{\scriptstyle acut_1}\,\dfrac{\overset{\vdots}{\Gamma \vdash \Delta, A} \qquad {\scriptstyle \dashv}\,\dfrac{\overset{\vdots}{\Pi', \Gamma \dashv \Delta, \Sigma'}}{\Pi, \Gamma \dashv \Delta, \Sigma}}{A, \Pi \dashv \Sigma}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\overset{\vdots}{\Gamma \vdash \Delta, A} \qquad \overset{\vdots}{\Pi', \Gamma \dashv \Delta, \Sigma'}}{A, \Pi' \dashv \Sigma'}\,{\scriptstyle acut_1}}{A, \Pi \dashv \Sigma}\,{\scriptstyle \dashv}
$$

$$
{\scriptstyle acut_2}\,\dfrac{\overset{\vdots}{A, \Gamma \vdash \Delta} \qquad {\scriptstyle \dashv}\,\dfrac{\overset{\vdots}{\Pi', \Gamma \dashv \Delta, \Sigma'}}{\Pi, \Gamma \dashv \Delta, \Sigma}}{\Pi \dashv \Sigma, A}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\overset{\vdots}{A, \Gamma \vdash \Delta} \qquad \overset{\vdots}{\Pi', \Gamma \dashv \Delta, \Sigma'}}{\Pi' \dashv \Sigma', A}\,{\scriptstyle acut_2}}{\Pi \dashv \Sigma, A}\,{\scriptstyle \dashv}
$$

□

The strategy for *acut*-elimination does not work for either $\mathsf{G4}_{H1}$ or $\mathsf{G4}_{H2}$: it is instructive to highlight the features hindering *acut* elimination in both cases.

For $\mathsf{G4}_{H1}$, it is clear that unthreading formulae from $\mathsf{G4}$-proofs is insufficient to produce $\mathsf{G4}_{H1}$ refutations. Take e.g. the following $\mathsf{G4}$-proof π:

$$
\dfrac{{\scriptstyle \vdash \neg}\,\dfrac{{\scriptstyle ax_{cl} \vdash}\,\dfrac{}{p \vdash p, q}}{\vdash p, \neg p, q} \qquad \dfrac{\dfrac{}{p, q \vdash q}\,{\scriptstyle ax_{cl} \vdash}}{q \vdash \neg p, q}\,{\scriptstyle \vdash \neg}}{p \to q \vdash \neg p, q}\,{\scriptstyle \to \vdash}
$$

If we unthread q from the conclusion of π, we get the following derivation π':

$$
\dfrac{{\scriptstyle \vdash \neg}\,\dfrac{{\scriptstyle ax_{cl} \vdash}\,\dfrac{}{p \vdash p}}{\vdash p, \neg p} \qquad \dfrac{\dfrac{}{p, q \dashv}\,{\scriptstyle ax_{cl} \dashv}}{q \dashv \neg p}\,{\scriptstyle a'R\neg}}{p \to q \dashv \neg p}\,{\scriptstyle a'L \to_1}
$$

Since $a'R\neg$ and $a'L \to_1$ are admissible but not derivable in $\mathsf{G4}_{H1}$, the derivation π' does not count as a $\mathsf{G4}_{H1}$-refutation. Instead, a $\mathsf{G4}_{H1}$-refutation of the same end-antisequent must include (at least) one *acut*-application – as witnessed by the following derivation:

$$\dfrac{{\scriptstyle \vdash\to}\;\dfrac{{\scriptstyle ax_{cl}\vdash}\;\dfrac{}{q,p\vdash q}}{q\vdash p\to q} \qquad \dfrac{{\scriptstyle \neg\vdash}\;\dfrac{{\scriptstyle ax_{cl}\vdash}\;\dfrac{}{p\vdash p}}{p,\neg p\vdash} \qquad \dfrac{}{q,p\dashv}\;{\scriptstyle ax_{cl}\dashv}}{q\dashv\neg p}\;{\scriptstyle acut_2}}{p\to q\dashv\neg p}\;{\scriptstyle acut_1}$$

Additionally, the secondary inductive hypothesis cannot be applied – as witnessed e.g. by the following configurations, where the *acut*$_i$ formula is atomic:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta,A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \Pi,\Gamma\vdash\Delta,\Sigma,C\end{array} \qquad \begin{array}{c}\vdots\\ \Pi,\Gamma\dashv\Delta,\Sigma,B\wedge C\end{array}}{\Pi,\Gamma\dashv\Delta,\Sigma,B}}{A,\Pi\dashv\Sigma,B}$$

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta,B,A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \Pi,\Gamma\vdash\Delta,\Sigma,C\end{array} \qquad \begin{array}{c}\vdots\\ \Pi,\Gamma\dashv\Delta,\Sigma,B\wedge C\end{array}}{\Pi,\Gamma\dashv\Delta,\Sigma,B}}{A,\Pi\dashv\Sigma}$$

On the other hand, the fact that *acut* rules cannot be eliminated from $\mathsf{G4}_{H1}$ without compromising completeness is not unexpected. As a matter of fact, the refutational logical rules of $\mathsf{G4}_{H1}$ do not allow for the introduction of logically complex formulas without assuming the introduction of even more complex formulas: since initial antisequents involve only atomic formulas, it is essential to have rules (actually, the *acut* rules) which introduce logically complex formulas following a distinct pattern.

As for $\mathsf{G4}_{H2}$, although unthreading remains well-defined, the elimination process cannot be fully executed. For instance, consider the following scenario:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta,B\vee C\end{array} \qquad \begin{array}{c}\vdots\\ \Pi,\Gamma\dashv\Delta,\Sigma\end{array}}{B\vee C,\Pi\dashv\Sigma}$$

If $\mathsf{G4}_{H2}$ proves $\Gamma\vdash\Delta,C$, we have

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta,C\end{array} \qquad \begin{array}{c}\vdots\\ \Pi,\Gamma\dashv\Delta,\Sigma\end{array}}{C,\Pi\dashv\Sigma}$$

If $\mathsf{G4}_{H2}$ refutes $B,\Pi\dashv\Sigma$, one cannot infer $B\vee C,\Pi\dashv\Sigma$ from $C,\Pi\dashv\Sigma$. Again, this is unsurprising. The binary refutational rules of $\mathsf{G4}_{H2}$ are sufficiently strong to ensure soundness, but they are too weak to achieve completeness, because they do not account for scenarios where both premises are invalid.

3.3. Anticut and FDE-based logics

In this section, we shall use capital Greek letters $\Theta,\Lambda\ldots$ to denote multisets of *literals* – i.e., atoms and negated atoms. Moreover, we take the *logical complexity* $\mathrm{C}(A)$ of a formula

$$ax_{fde} \vdash \frac{}{\Gamma, p \vdash p, \Delta} \qquad ax_{fde} \vdash \frac{}{\Gamma, \neg p \vdash \neg p, \Delta}$$

$$\wedge \vdash \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \vdash \wedge \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee \vdash \frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vdash \vee \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\neg \wedge \vdash \frac{\neg A, \Gamma \vdash \Delta \qquad \neg B, \Gamma \vdash \Delta}{\neg(A \wedge B), \Gamma \vdash \Delta} \qquad \vdash \neg \wedge \frac{\Gamma \vdash \Delta, \neg A, \neg B}{\Gamma \vdash \Delta, \neg(A \wedge B)}$$

$$\neg \vee \vdash \frac{\neg A, \neg B, \Gamma \vdash \Delta}{\neg(A \vee B), \Gamma \vdash \Delta} \qquad \vdash \neg \vee \frac{\Gamma \vdash \Delta, \neg A \qquad \Gamma \vdash \Delta, \neg B}{\Gamma \vdash \Delta, \neg(A \vee B)}$$

$$\neg\neg \vdash \frac{A, \Gamma \vdash \Delta}{\neg\neg A, \Gamma \vdash \Delta} \qquad \vdash \neg\neg \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, \neg\neg A}$$

FIGURE 3. Gfde sequent calculus

$$ax_{k3} \vdash \frac{}{\Gamma, p \vdash p, \Delta} \qquad ax_{k3} \vdash \frac{}{\Gamma, \neg p \vdash \neg p, \Delta} \qquad ax_{k3} \vdash \frac{}{\Gamma, p, \neg p \vdash \Delta}$$

$$ax_{k3} \dashv \frac{}{\Theta \dashv \Lambda} \qquad ax_{k3} \dashv \frac{}{\Theta \dashv p, \neg p, \Lambda}$$

FIGURE 4. Axioms of Gk3

$$ax_{lp} \vdash \frac{}{\Gamma, p \vdash p, \Delta} \qquad ax_{lp} \vdash \frac{}{\Gamma, \neg p \vdash \neg p, \Delta} \qquad ax_{lp} \vdash \frac{}{\Gamma \vdash p, \neg p, \Delta}$$

$$ax_{lp} \dashv \frac{}{\Theta \dashv \Lambda} \qquad ax_{lp} \dashv \frac{}{\Theta, p, \neg p \dashv \Lambda}$$

FIGURE 5. Axioms of Glp

A to be 0 if A is a literal, $\mathrm{C}(B) + 1$ if A is of the form $\neg\neg B$, $\mathrm{C}(B) + \mathrm{C}(C) + 1$ if A is of the form $B \otimes C$ and $\mathrm{C}(\neg B) + \mathrm{C}(\neg C) + 1$ if A is of the form $\neg(B \otimes C)$, with $\otimes \in \{\wedge, \vee\}$.

In this section, we shall deal with the following logics: Belnap's first-degree entailment logic [5, 6], Kleene's strong three-valued logic [78] and Priest's logic of paradox [135] – in short, **FDE**, **K3** and **LP**. The Gfde, Gk3 and Glp systems are sequent calculi for **FDE**, **K3** and **LP**[7]. The rules of Gfde are displayed in Figure 3. On the other hand, Gk3 and Glp differ from Gfde only for the axioms (see Figures 4 and 5). In the rules $ax_{k3} \dashv$ ($ax_{lp} \dashv$), $\Theta \cap \Lambda = \varnothing$ and no atom p is such that $[p, \neg p] \subseteq \Theta$ ($[p, \neg p] \subseteq \Lambda$, respectively).

---

[7] These systems are *multi-succedent* sequent calculi, modeled after the multi-succedent sequent calculus for **FDE** introduced in [140]. In contrast, [107] employs single-succedent calculi, based on the single-succedent sequent calculus for **FDE** first presented in [47].

The $\mathsf{Gfde}_{H1}, \mathsf{Gfde}_{H2}, \mathsf{Gfde}_{H1\cap 2}$ and $\overline{\overline{\mathsf{Gfde}}}$ calculi – as well as their counterparts for **K3** and **LP** – are analogous to the $\mathsf{G4}_{H1}$, $\mathsf{G4}_{H2}$, $\mathsf{G4}_{H1\cap 2}$ and $\overline{\overline{\mathsf{G4}}}$ calculi, respectively.

Let G be any system among $\mathsf{Gfde}, \mathsf{Gk3}, \mathsf{Glp}$: we recall some basic facts about $\overline{\overline{\mathsf{G}}}$.

THEOREM 3.7. *$\overline{\overline{\mathsf{G}}}$ enjoys the following properties.*

(*i*) *Each logical rule of G is height-bounded invertible with preservation of initial (anti)sequents.*

(*ii*) *The Weakening, Contraction and Cut rules are admissible in $\overline{\overline{\mathsf{G}}}$.*

(*iii*) *Maximal $\overline{\overline{\mathsf{G}}}$-decomposition yields a unique set of atomic (anti)sequents.*

PROOF. The proofs are analogous to those for $\mathsf{G4}$, except for the fact that the logical complexity of a formula A, $\mathrm{C}(A)$, is defined in a different way (see above). ☐

As a preliminary result, we establish that applications of the contrapositives of standard weakening and contraction rules are admissible in the $\mathsf{G}_{H1\cap 2}$ calculi.

LEMMA 3.4. *The Strengthening rules are admissible in the $\mathsf{G}_{H1\cap 2}$ calculus.*

PROOF. We argue as in the proof of Lemma 3.1: it suffices to prove that G proves $\Gamma, A \vdash A, \Delta$ for any formula A – proceeding by induction on the logical complexity of A. ☐

LEMMA 3.5. *The Duplication rules are admissible in the $\mathsf{G}_{H1\cap 2}$ calculus.*

PROOF. We argue as in the proof of Lemma 3.2. ☐

Now, we show that any application of the contrapositive of a logical rule in G can be interpreted as a series of *acut* and *dup* applications.

THEOREM 3.8. *Each logical rule of G_{H1} is admissible in the $\mathsf{G}_{H1\cap 2}$ calculus.*

PROOF. It suffices to consider the refutational rules which are not featured by $\mathsf{G4}_{H1}$: we focus on $a\neg\wedge$, leaving the others to the reader.

$a\neg\wedge$ Consider the following derivations:

$$\dfrac{\neg\wedge\vdash\;\dfrac{\mathit{wk}\;\dfrac{\neg A, \Gamma \vdash \Delta}{\neg A, \Gamma \vdash \Delta, \neg B} \qquad \begin{array}{c}\vdots\\ \neg B, \Gamma \vdash \Delta, \neg B\end{array}}{\neg(A \wedge B), \Gamma \vdash \Delta, \neg B} \qquad \dfrac{\neg(A \wedge B), \Gamma \dashv \Delta}{\neg(A \wedge B), \Gamma, \Gamma \dashv \Delta, \Delta}\;\mathit{dup}}{\neg B, \Gamma \dashv \Delta}\;\mathit{acut}_1$$

$$\dfrac{\neg\wedge\vdash\;\dfrac{\mathit{wk}\;\dfrac{\neg B, \Gamma \vdash \Delta}{\neg B, \Gamma \vdash \Delta, \neg A} \qquad \begin{array}{c}\vdots\\ \neg A, \Gamma \vdash \Delta, \neg A\end{array}}{\neg(A \wedge B), \Gamma \vdash \Delta, \neg A} \qquad \dfrac{\neg(A \wedge B), \Gamma \dashv \Delta}{\neg(A \wedge B), \Gamma, \Gamma \dashv \Delta, \Delta}\;\mathit{dup}}{\neg A, \Gamma \dashv \Delta}\;\mathit{acut}_1$$

$$\dfrac{\begin{array}{c}\vdots\\ \neg B \vdash \neg(A\wedge B)\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \neg A\vdash\neg(A\wedge B)\end{array}\qquad \dfrac{\Gamma\dashv\Delta,\neg(A\wedge B)}{\Gamma\dashv\Delta,\neg(A\wedge B),\neg(A\wedge B)}\,dup}{\Gamma\dashv\Delta,\neg A,\neg(A\wedge B)}\,acut_2}{\Gamma\dashv\Delta,\neg A,\neg B}\,acut_2$$

□

We can also reduce applications of logical rules in G_{H2} to applications of *acut* and *dup* rules.

THEOREM 3.9. *Each logical rule of* G_{H2} *is admissible in the* $\mathsf{G}_{H1\cap 2}$ *calculus.*

PROOF. It suffices to consider the refutational rules which do not belong to $\mathsf{G4}_{H2}$: we focus on $a'R\neg\vee$, leaving the other cases to the reader.

$a'R\neg\vee$ Take the following derivations, with $i = 1, 2$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \neg(A_1\vee A_2)\vdash\neg A_{3-i}\end{array}\qquad \dfrac{\Gamma\vdash\Delta,\neg A_i\qquad \dfrac{\Gamma\dashv\Delta,\neg A_{3-i}}{\Gamma,\Gamma\dashv\Delta,\Delta,\neg A_{3-i}}\,dup}{\neg A_i,\Gamma\dashv\Delta,\neg A_{3-i}}\,acut_1}{\neg A_i,\Gamma\dashv\Delta,\neg(A_1\vee A_2)}\,acut_2}{\Gamma\dashv\Delta,\neg(A_1\vee A_2)}\,str$$

□

Finally, we prove the following.

THEOREM 3.10. *Each refutational rule in* $\overline{\overline{\mathsf{G}}}$ *is admissible in the* $\mathsf{G}_{H1\cap 2}$ *calculus.*

PROOF. We consider only the $\neg\wedge\dashv$ and $\dashv\neg\vee$ rules.

$\neg\wedge\dashv$ Consider the following derivations:

$$acut_1\,\dfrac{\begin{array}{c}\vdots\\ \neg A\vdash\neg(A\wedge B)\end{array}\qquad\begin{array}{c}\vdots\\ \neg A,\Gamma\dashv\Delta\end{array}}{\neg(A\wedge B),\Gamma\dashv\Delta}\qquad\dfrac{\begin{array}{c}\vdots\\ \neg B\vdash\neg(A\wedge B)\end{array}\qquad\begin{array}{c}\vdots\\ \neg B,\Gamma\dashv\Delta\end{array}}{\neg(A\wedge B),\Gamma\dashv\Delta}\,acut_1$$

$\dashv\neg\vee$ Take the following derivations:

$$acut_2\,\dfrac{\begin{array}{c}\vdots\\ \neg(A\vee B)\vdash\neg A\end{array}\qquad\begin{array}{c}\vdots\\ \Gamma\dashv\Delta,\neg A\end{array}}{\Gamma\dashv\Delta,\neg(A\vee B)}\qquad\dfrac{\begin{array}{c}\vdots\\ \neg(A\vee B)\vdash\neg B\end{array}\qquad\begin{array}{c}\vdots\\ \Gamma\dashv\Delta,\neg B\end{array}}{\Gamma\dashv\Delta,\neg(A\vee B)}\,acut_2$$

□

3.3.1. Anticut elimination in FDE-based logics. We extend now the anticut-elimination strategy presented in Section 3.2 to logics based on **FDE**. First, we state a basic fact about all these systems:

THEOREM 3.11. *For any sequent* $\Gamma \vdash \Delta$*, either* $\overline{\overline{\mathsf{G}}}$ *proves* $\Gamma \vdash \Delta$ *or refutes* $\Gamma \vdash \Delta$ *– but not both.*

PROOF. The argument follows the same reasoning as in the proof of Theorem 3.5. ☐

We prove the desired result exploiting the following.

LEMMA 3.6. *The Copy rule is admissible in* $\overline{\overline{\mathsf{G}}}$.

PROOF. We argue as in the proof of Lemma 3.3. ☐

THEOREM 3.12. *There exists an algorithm which turns any* $\overline{\overline{\mathsf{G}}}+acut_i$*-derivation of* $\Pi \dashv \Sigma$ *into* $\overline{\overline{\mathsf{G}}}$*-derivation of* $\Pi \dashv \Sigma$*, with* $i = 1, 2$.

PROOF. We argue as in the proof of Theorem 3.6: here, we consider only one case arising when the $acut_i$ formula is a literal – leaving the others to the reader.
Let A be the $acut_i$ formula and π the derivation of the right premise. If $\mathcal{C}(A) = 0$, $h(\pi) = 1$, $i = 1$ and G is Gfde, consider the derivation

$$\dfrac{ax_{fde} \vdash \dfrac{}{\Theta \vdash \Lambda, A} \qquad \dfrac{}{\Phi, \Theta \dashv \Lambda, \Psi}\, ax_{fde} \dashv}{A, \Phi \dashv \Psi}\, acut_1$$

We have that $\Theta \cap \Lambda = \varnothing$ and thus that $A \in \Theta$: since $\Theta \cap \Psi = \varnothing$, we infer that $A \notin \Psi$. Hence, we can derive $A, \Phi \dashv \Psi$ by a single application of $ax_{fde} \dashv$. Notice that if G is $\mathsf{Gk3}$ (Glp) and $A \notin \Theta$, then $[p, \neg p] \subseteq \Theta$ ($[p, \neg p] \subseteq \Lambda$, respectively) – contrary to the hypothesis that $\Phi, \Theta \vdash \Lambda, \Psi$ is refutable. ☐

For the same reasons we detailed in the case of $\mathsf{G4}_{H1}$ and $\mathsf{G4}_{H2}$, the elimination strategy just presented does not apply to either G_{H1} or G_{H2} calculi.

3.3.2. A glimpse on anticut and Ł-adequacy. Let H denote any system among Kleene's $\mathsf{G4}$, Gfde, $\mathsf{Gk3}$, and Glp. Theorems 3.5 and 3.11 state that, for any pair of multisets Γ and Δ, there exists either an $\overline{\overline{\mathsf{H}}}$-derivation of $\Gamma \vdash \Delta$ or an $\overline{\overline{\mathsf{H}}}$-derivation of $\Gamma \dashv \Delta$ (but not both). Furthermore, $\overline{\overline{\mathsf{H}}}$ derives $\Gamma \vdash \Delta$ precisely when $\bigvee \Delta$ is a logical consequence of $\bigwedge \Gamma$. Consequently, Theorems 3.5 and 3.11 guarantee that $\overline{\overline{\mathsf{H}}}$ derives $\Gamma \dashv \Delta$ exactly when $\bigvee \Delta$ is not a logical consequence of $\bigwedge \Gamma$. This establishes that the (strong) Ł-adequacy of unmixed $\overline{\overline{\mathsf{H}}}$ systems can be proven without relying on anticut rules. Moreover, it is important to observe that the anticut-elimination strategy employed in the proofs of Theorems 3.6 and 3.12 presupposes Ł-adequacy.

$$ax_{int}\vdash \frac{}{\Gamma, p \vdash p, \Delta} \qquad ax_{\bot}\vdash \frac{}{\bot, \Gamma \vdash \Delta}$$

$$\wedge\vdash \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \vdash\wedge \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee\vdash \frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vdash\vee \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$at \to\vdash \frac{A, B, \Gamma \vdash \Delta}{A, A \to B, \Gamma \vdash \Delta} \qquad \wedge\to\vdash \frac{C \to (D \to B), \Gamma \vdash \Delta}{(C \wedge D) \to B, \Gamma \vdash \Delta}$$

$$\vee\to\vdash \frac{C \to B, D \to B, \Gamma \vdash \Delta}{(C \vee D) \to B, \Gamma \vdash \Delta} \qquad \to\to\vdash \frac{D \to B, C, \Gamma \vdash D \qquad B, \Gamma \vdash \Delta}{(C \to D) \to B, \Gamma \vdash \Delta}$$

$$\vdash\to_i \frac{\Gamma, A \vdash B}{\Gamma \vdash \Delta, A \to B}$$

FIGURE 6. $\mathsf{G4ip}$ sequent calculus. In rule $at \to\vdash$, A is an atom or $\bot$.

With respect to hybrid systems, Theorems 3.2, 3.3, 3.6, as well as 3.8, 3.9, and 3.12, establish the equivalence of these systems with the corresponding unmixed systems for the same logic. This equivalence indirectly confirms the (strong) Ł-adequacy of each hybrid system. A direct argument for establishing the Ł-adequacy of hybrid calculi for both classical and **FDE**-based logics can be uniformly constructed by leveraging *acut* rules.

Specifically, concerning refutational completeness, one can provide an argument which differs from the one detailed in [107, pp. 610–612]. Whenever H is one of classical, **FDE**, **K3**, or **LP** logic, if $\Gamma \not\models A$ in H, we can proceed by induction on $\mathcal{C}(\Gamma \cup [A])$ to show that $\mathsf{H}_{H1\cap H2}$ derives $\Gamma \dashv A$ (details omitted).

3.4. Anticut and intuitionistic logic

In this section, we examine anticut rules in D-R systems for intuitionistic propositional logic[8]. Till the end of the chapter, we shall consider $\bot$ as a 0-ary connective instead of an atomic formula, and use capital Greek letters $\Theta, \Lambda, \ldots$ to denote multisets of *atomic* formulas and $\bot$.

We begin by considering the unmixed sequent calculus $\overline{\overline{\mathsf{G4ip}}}$, obtained from the $\mathsf{G4ip}$ calculus for intuitionistic propositional logic (see Figure 6) by adding the refutational rules in Figure 7. In the rules $ax_{int} \dashv$ and $\to \dashv \to$ of Figure 7, the following conditions hold:

---

[8] We emphasize that this work adheres to Łukasiewicz's approach to refutation within an intuitionistic framework. This approach diverges from that of [88], where the author introduces informal conditions on the concept of refutation that are analogous to the **BHK** interpretation of constructive proof.

$$ax_{int} \dashv \dfrac{}{\Gamma^{\to}, \Theta \dashv \Lambda}$$

$$\wedge \dashv \dfrac{A, B, \Gamma \dashv \Delta}{A \wedge B, \Gamma \dashv \Delta} \qquad \dashv \wedge \dfrac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A \wedge B} \qquad \dashv \wedge \dfrac{\Gamma \dashv \Delta, B}{\Gamma \dashv \Delta, A \wedge B}$$

$$\vee \dashv \dfrac{A, \Gamma \dashv \Delta}{A \vee B, \Gamma \dashv \Delta} \qquad \vee \dashv \dfrac{B, \Gamma \dashv \Delta}{A \vee B, \Gamma \dashv \Delta} \qquad \dashv \vee \dfrac{\Gamma \dashv \Delta, A, B}{\Gamma \dashv \Delta, A \vee B}$$

$$at \to \dashv \dfrac{p, B, \Gamma \dashv \Delta}{p, p \to B, \Gamma \dashv \Delta} \qquad \wedge \to \dashv \dfrac{C \to (D \to B), \Gamma \dashv \Delta}{(C \wedge D) \to B, \Gamma \dashv \Delta}$$

$$\vee \to \dashv \dfrac{C \to B, D \to B, \Gamma \dashv \Delta}{(C \vee D) \to B, \Gamma \dashv \Delta} \qquad \to\to \dashv \dfrac{B, \Gamma \dashv \Delta}{(C \to D) \to B, \Gamma \dashv \Delta}$$

$$\to \dashv \to \dfrac{D_1 \to B_1, C_1, \Gamma_1, \Theta \dashv D_1 \quad \ldots \quad D_m \to B_m, C_m, \Gamma_m, \Theta \dashv D_m \qquad \Gamma, \Theta, E_1 \dashv F_1 \quad \ldots \quad \Gamma, \Theta, E_n \dashv F_n}{\Gamma, \Theta \dashv E_1 \to F_1, \ldots, E_n \to F_n, \Lambda}$$

FIGURE 7. $\overline{\overline{\mathsf{G4ip}}}$ sequent calculus

(*i*) $\Gamma^{\to} = A_1 \to B_1, \ldots, A_l \to B_l$, with $A_1, \ldots, A_l$ being atoms or $\perp$;

(*ii*) $\perp \notin \Theta$;

(*iii*) $\Theta \cap [A_1, \ldots, A_l] = \Theta \cap \Lambda = \varnothing$;

(*iv*) $\Gamma = (C_1 \to D_1) \to B_1, \ldots, (C_m \to D_m) \to B_m, \Gamma^{\to}$;

(*v*) $\Gamma_i = \Gamma \setminus [(C_i \to D_i) \to B_i]$, for any $1 \leq i \leq m$.

First, let us revisit some fundamental results about $\overline{\overline{\mathsf{G4ip}}}$.

THEOREM 3.13. $\overline{\overline{\mathsf{G4ip}}}$ *enjoys the following properties.*

(*i*) *Each logical rule of* $\mathsf{G4ip}$ *is height-preserving invertible, except for* $\to\to \vdash$ *(which fails to be invertible with respect to the left premise) and* $\vdash \to$ *.*

(*ii*) *The weakening, contraction and cut rules are admissible in* $\mathsf{G4ip}$.

(*iii*) *Maximal* $\overline{\overline{\mathsf{G4ip}}}$*-decomposition yields (at least) one set of irreducible (anti)sequents – namely, (anti)sequents of the form* $A_1 \to B_1, \ldots, A_l \to B_l, q_1, \ldots, q_m \mathrel{|\!\!\!\!\overset{*}{-}} r_1, \ldots, r_n$ *with* $A_1, \ldots, A_l$ *being atoms or* $\perp$ *and* $[A_1, \ldots, A_l] \cap [q_1, \ldots, q_m] = \varnothing$.

PROOF. For proofs see [40, 131]. ☐

Let us remark that the method for constructing refutational calculi that underlies the design of $\overline{\overline{\mathsf{G4ip}}}$ is analogous to the one employed in the design of $\overline{\overline{\mathsf{G4}}}$, *modulo* the lack of full invertibility of the logical rules (cf. the rule $\to \dashv \to$).

To the best of our knowledge, no hybrid sequent calculus for intuitionistic logic has been formulated so far[9]. One may be tempted to design Gentzen-style hybrid systems for intuitionistic logic modeled after the the hybrid sequent calculi for classical and **FDE**-based logics. Let us consider the following candidates:

- (i) $\mathsf{G4ip}_{H1}$, obtained from $\mathsf{G4ip}$ by adding $ax_{int} \dashv$, the *acut* rules and the contrapositive versions of the logical rules of $\mathsf{G4ip}$;
- (ii) $\mathsf{G4ip}_{H2}$, defined as the extension of $\mathsf{G4ip}$ with $ax_{int} \dashv$, the *acut* rules and the hybrid rules obtained from the logical rules of G by turning one of the premises and the conclusion into an antisequent.

The following results show that these candidates are not qualified for the job.

Proposition 3.1. The $\mathsf{G4ip}_{H1}$ is not Ł-complete.

Proof. We prove a stronger statement – namely, that if $\mathsf{G4ip}_{H1}$ derives $\Gamma \dashv \Delta$, then $\mathsf{G4}_{H1}$ derives $\Gamma \dashv \Delta$: since $\mathsf{G4}_{H1}$ does not derive (say) $\dashv p \vee (p \to \bot)$, this implies that $\mathsf{G4ip}_{H1}$ is not complete with respect to refutability.
We proceed by induction on the height of the $\mathsf{G4ip}_{H1}$-derivation π of $\Gamma \dashv \Delta$. Suppose by contradiction that $\Gamma \dashv \Delta$ is an initial antisequent $A_1 \to B_1, \ldots, A_l \to B_l, q_1, \ldots, q_m \dashv r_1, \ldots, r_n$ of $\mathsf{G4ip}_{H1}$, and that $\mathsf{G4}_{H1}$ derives $A_1 \to B_1, \ldots, A_l \to B_l, q_1, \ldots, q_m \vdash r_1, \ldots, r_n$. By Theorem 2.2, point (i) we infer that $\mathsf{G4}_{H1}$ proves $q_1, \ldots, q_m \vdash r_1, \ldots, r_n, A_1, \ldots, A_l$: since $[q_1, \ldots, q_m] \cap [r_1, \ldots, r_n] = [q_1, \ldots, q_m] \cap [A_1, \ldots, A_l] = \varnothing$, we get the contradiction. If $h(\pi) > 1$, we reason by cases over the last rule applied, exploiting the inductive hypothesis as well as Lemma 3.1, Theorem 3.2 and the following derivations to reach the conclusion.

$$\dfrac{\begin{array}{c}\vdots\\ p, p \to B \vdash B\end{array} \qquad \dfrac{p, p \to B, \Gamma \dashv \Delta}{p, p, p \to B, \Gamma \dashv \Delta}\,\mathit{dup}}{p, B, \Gamma \dashv \Delta}\,\mathit{acut}_1$$

$$\dfrac{\begin{array}{c}\vdots\\ (C \wedge D) \to B \vdash C \to (D \to B)\end{array} \qquad (C \wedge D) \to B, \Gamma \dashv \Delta}{C \to (D \to B), \Gamma \dashv \Delta}\,\mathit{acut}_1$$

$$\dfrac{\begin{array}{c}\vdots\\ (C \vee D) \to B \vdash D \to B\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ (C \vee D) \to B \vdash C \to B\end{array} \qquad \dfrac{(C \vee D) \to B, \Gamma \dashv \Delta}{(C \vee D) \to B, (C \vee D) \to B, \Gamma \dashv \Delta}\,\mathit{dup}}{C \to B, (C \vee D) \to B, \Gamma \dashv \Delta}\,\mathit{acut}_1}{C \to B, D \to B, \Gamma \dashv \Delta}\,\mathit{acut}_1$$

---

[9] This is not true of Hilbert-style D-R systems for intuitionistic logic, as witnessed by [152] and [153].

$$adp_m\ \dfrac{A_1\to B_1,\dots,A_m\to B_m,\Gamma^{\to},\Theta\dashv A_1 \qquad \dots \qquad A_1\to B_1,\dots,A_m\to B_m,\Gamma^{\to},\Theta\dashv A_m}{A_1\to B_1,\dots,A_m\to B_m,\Gamma^{\to},\Theta\dashv A_1,\dots,A_m,\Lambda}$$

FIGURE 8. Hybrid refutation rules for intuitionistic logic

$$\dfrac{\begin{matrix}\vdots\\ C\to D,(C\to D)\to B\vdash B\end{matrix} \qquad {\scriptstyle acut_1}\,\dfrac{{\scriptstyle \vdash\to}\,\dfrac{{\scriptstyle cut}\,\dfrac{\begin{matrix}\vdots\\ (C\to D)\to B\vdash D\to B\end{matrix} \quad C,D\to B,\Gamma\vdash D}{C,(C\to D)\to B,\Gamma\vdash D}}{(C\to D)\to B,\Gamma\vdash C\to D} \qquad {\scriptstyle dup}\,\dfrac{(C\to D)\to B,\Gamma\dashv\Delta}{(C\to D)\to B,(C\to D)\to B,\Gamma,\Gamma\dashv\Delta}}{C\to D,(C\to D)\to B,\Gamma\dashv\Delta}}{B,\Gamma\dashv\Delta}\,{\scriptstyle acut_1}$$

$$\dfrac{\begin{matrix}\vdots\\ (C\to D)\to B\vdash D\to B\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ (C\to D)\to B,C\to D\vdash B\end{matrix} \qquad \dfrac{B,\Gamma\vdash\Delta \qquad \dfrac{(C\to D)\to B,\Gamma\dashv\Delta}{(C\to D)\to B,(C\to D)\to B,\Gamma,\Gamma\dashv\Delta}\,{\scriptstyle dup}}{(C\to D)\to B,(C\to D)\to B,\Gamma\dashv B}\,{\scriptstyle acut_1}}{(C\to D)\to B,\Gamma\dashv C\to D}\,{\scriptstyle acut_2}}{(C\to D)\to B,\Gamma,C\dashv D}\,{\scriptstyle aR\to}}{D\to B,\Gamma,C\dashv D}\,{\scriptstyle acut_1}$$

□

PROPOSITION 3.2. The $\mathsf{G4ip}_{H2}$ is not Ł-sound.

PROOF. Theorem 3.13, point (ii) ensures that the following rule is not sound:

$$\dfrac{C,D\to B,\Gamma\dashv D \qquad B,\Gamma\vdash\Delta}{(C\to D)\to B,\Gamma\dashv\Delta}\ a'L\to\to$$

□

To obtain Ł-adequate, hybrid sequent calculi for intuitionistic logic we introduce new refutational rules – namely, the adp_m rules (see Figure 8). These are Gentzen-style reformulations of the refutational principles presented in [152] and [153]. In the rule schema adp, the following conditions hold:

(i) $\Gamma^{\to} = E_1\to F_1,\dots,E_n\to F_n$, with $E_1,\dots,E_n$ being atoms or $\bot$;

(ii) $\bot\notin\Theta$;

(iii) $\Theta\cap[E_1,\dots,E_n]=\Theta\cap\Lambda=\varnothing$;

(iv) $m\geq 2$;

(v) A_i has the form $C_i\to D_i$, for any $1\leq i\leq m$.

On this basis, we define the minimal $\mathsf{G4ip}_H$ calculus as the extension of $\mathsf{G4ip}$ with $ax_{int}\dashv$, the *acut* rules and the adp_m rules. Since this calculus makes its first appearance in this chapter, we devote the first part of this section to the investigation of its structural properties.

Henceforth, we take the logical complexity $\mathcal{C}(A)$ of a formula A to be defined as follows [40]: 0 if A is $\bot$, 1 if A is atomic, $\mathcal{C}(B)+\mathcal{C}(C)+1$ if A is $B\to C$, $\mathcal{C}(B)+\mathcal{C}(C)+2$ if A is

$B \wedge C$ and $\mathcal{C}(B) + \mathcal{C}(C) + 3$ if A is $B \vee C$. The other proof-theoretic notions are defined in the same way as in the previous sections of this chapter.

LEMMA 3.7. *The strengthening and duplication rules are admissible in the minimal* $\mathsf{G4ip}_H$ *calculus.*

PROOF. We argue as in the proofs of Lemmas 3.1 and 3.2. ☐

THEOREM 3.14. *Each refutational rule of* $\overline{\overline{\mathsf{G4ip}}}$ *is admissible in the minimal* $\mathsf{G4ip}_H$ *calculus.*

PROOF. We focus on the most interesting case, leaving the others to the reader.

$\to \dashv \to$ Take the following derivations π_i, π'_j and π''_i, for any $1 \leq i \leq m$ and $1 \leq j \leq m$:

$$\dfrac{\begin{array}{c}\vdots\\ C_i \to D_i, C_i \vdash D_i\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ D_i \to B_i, C_i \vdash (C_i \to D_i) \to B_i\end{array} \qquad \dfrac{D_i \to B_i, C_i, \Gamma_i, \Theta \dashv D_i}{D_i \to B_i, C_i, C_i, \Gamma_i, \Theta \dashv D_i}\,\mathit{dup}}{(C_i \to D_i) \to B_i, C_i, \Gamma_i, \Theta \dashv D_i}\,\mathit{acut}}{(C_i \to D_i) \to B_i, \Gamma_i, \Theta \dashv C_i \to D_i}\,\mathit{acut}$$

$$\dfrac{\begin{array}{c}\vdots\\ \vdash (E_n \to F_n) \to (E_n \to F_n)\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \vdash (E_1 \to F_1) \to (E_1 \to F_1)\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ E_j \to F_j, E_j \vdash F_j\end{array} \qquad \Gamma, \Theta, E_j \dashv F_j}{\Gamma, \Theta \dashv E_j \to F_j}\,\mathit{acut}}{(E_1 \to F_1) \to (E_1 \to F_1), \Gamma, \Theta \dashv E_j \to F_j}\,\mathit{acut}\\ \vdots\end{array}}{(E_1 \to F_1) \to (E_1 \to F_1), \ldots, (E_n \to F_n) \to (E_n \to F_n), \Gamma, \Theta \dashv E_j \to F_j}\,\mathit{acut}$$

$$\dfrac{\begin{array}{c}\vdots\\ \vdash (E_n \to F_n) \to (E_n \to F_n)\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \vdash (E_1 \to F_1) \to (E_1 \to F_1)\end{array} \qquad \begin{array}{c}\vdots{\scriptstyle \pi_i}\\ (C_i \to D_i) \to B_i, \Gamma_i, \Theta \dashv C_i \to D_i\end{array}}{(E_1 \to F_1) \to (E_1 \to F_1), (C_i \to D_i) \to B_i, \Gamma_i, \Theta \dashv C_i \to D_i}\,\mathit{acut}\\ \vdots\end{array}}{(E_1 \to F_1) \to (E_1 \to F_1), \ldots, (E_n \to F_n) \to (E_n \to F_n), (C_i \to D_i) \to B_i, \Gamma_i, \Theta \dashv C_i \to D_i}\,\mathit{acut}$$

Let $\Pi = (E_1 \to F_1) \to (E_1 \to F_1), \ldots, (E_n \to F_n) \to (E_n \to F_n)$. We obtain the conclusion by considering the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots{\scriptstyle \{\pi''_i\}_{1 \leq i \leq m}}\\ \{\Pi, \Gamma, \Theta \dashv C_i \to D_i\}_{1 \leq i \leq m}\end{array} \qquad \begin{array}{c}\vdots{\scriptstyle \{\pi'_j\}_{1 \leq j \leq n}}\\ \{\Pi, \Gamma, \Theta \dashv E_j \to F_j\}_{1 \leq j \leq n}\end{array}}{\Pi, \Gamma, \Theta \dashv C_1 \to D_1, \ldots, C_m \to D_m, E_1 \to F_1, \ldots, E_n \to F_n, \Lambda}\,\mathit{adp}}{\Pi, \Gamma, \Theta \dashv E_1 \to F_1, \ldots, E_n \to F_n, \Lambda}\,\mathit{str}}{\Gamma, \Theta \dashv E_1 \to F_1, \ldots, E_n \to F_n, \Lambda}\,\mathit{str}$$

☐

REMARK 2. In the proof of Theorem 3.14 we establish the admissibility of the refutational logical rules of $\overline{\overline{\mathsf{G4ip}}}$ in the minimal $\mathsf{G4ip}_H$ calculus leveraging the provable sequents (3.2.1) and (3.2.2) of Section 3.2 as well as the following ones:

$$p, p \to B \vdash p \wedge B \tag{3.4.1}$$

$$(C \wedge D) \to B \vdash C \to (D \to B) \tag{3.4.2}$$

$$(C \vee D) \to B \vdash (C \to B) \wedge (C \to D) \tag{3.4.3}$$

$$B \vdash (C \to D) \to B \tag{3.4.4}$$

It should be noticed that the provability of sequents (3.4.1) – (3.4.3) can be reduced to the provability of the following instances of the sequent $A, A \to B \vdash B$:

$$p, p \to B \vdash B \qquad (C \wedge D) \to B, C \wedge D \vdash B \qquad (C \vee D) \to B, C \vee D \vdash B$$

This approach to logical rules is analogous to that of Gentzen's **LIG** calculus: there, the sequents (3.2.1), (3.2.2) and $A, A \to B \vdash B$ played the role of axioms (under the label of 'logical groundsequents'), and standard structural rules together with the single-succedent version of $\vdash \to$ were sufficient to recover each logical rule for deducibility [**174**, **10**].

LEMMA 3.8. *Let* $A_1, \dots, A_m$ *be formulas such that* A_i *is not an implication, for some* $1 \leq i \leq m$, $\Gamma^{\to} = C_1 \to D_1, \dots, C_n \to D_n$ *with* $C_1, \dots, C_n$ *being atoms or* $\bot$ *and* $\Theta \cap [C_1, \dots, C_n] = \Theta \cap \Lambda = \varnothing$. *The rules*

$$\dfrac{A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_1 \qquad \dots \qquad A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_m}{A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_1, \dots, A_m, \Lambda}\, adp'_m$$

are admissible in the $\mathsf{G4ip}_H$ *calculus.*

PROOF. For any A_i which is not implicational we consider the following derivations π_j and π'_i, with $1 \leq j \leq m$:

$$\dfrac{\vdash\to\;\dfrac{cut\;\dfrac{\to\to\vdash\;\dfrac{\begin{matrix}\vdots\\ p, p \to A_i \vdash p\end{matrix}\qquad\begin{matrix}\vdots\\ A_i \vdash A_i\end{matrix}}{(p \to p) \to A_i \vdash A_i}\qquad\begin{matrix}\vdots\\ A_i \to B_i, A_i \vdash B_i\end{matrix}}{A_i \to B_i, (p \to p) \to A_i \vdash B_i}}{A_i \to B_i \vdash ((p \to p) \to A_i) \to B_i}\qquad A_1 \to B_1, \dots, A_i \to B_i, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_j}{A_1 \to B_1, \dots, ((p \to p) \to A_i) \to B_i, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_j}\, acut_1$$

$$\dfrac{\to\to\vdash\;\dfrac{\begin{matrix}\vdots\\ p, p \to A_i \vdash p\end{matrix}\qquad\begin{matrix}\vdots\\ A_i \vdash A_i\end{matrix}}{(p \to p) \to A_i \vdash A_i}\qquad\begin{matrix}\vdots_{\pi_{j=i}}\\ A_1 \to B_1, \dots, ((p \to p) \to A_i) \to B_i, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_i\end{matrix}}{A_1 \to B_1, \dots, ((p \to p) \to A_i) \to B_i, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv (p \to p) \to A_i}\, acut_2$$

A single application of adp_m yields $A_1 \to B_1, \dots, ((p \to p) \to A_i) \to B_i, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_1, \dots, (p \to p) \to A_i, \dots, A_m, \Lambda$. If $\Pi = A_1 \to B_1, \dots, A_{i-1} \to B_{i-1}, A_{i+1} \to B_{i+1}, \dots, A_m \to B_m, \Gamma^{\to}, \Theta$ and $\Sigma = A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_m, \Lambda$, we take the following derivation ρ to reach the conclusion:

$$\dfrac{\begin{array}{c}\vdots\\ A_i \vdash (p \to p) \to A_i\end{array} \qquad acut_1\;\dfrac{\vdash \to_i\;\dfrac{\to\to\vdash\;\dfrac{\begin{array}{c}\vdots\\ p \to p, A_i \to B_i, A_i \vdash A_i\end{array} \qquad \begin{array}{c}\vdots\\ B_i, A_i \vdash B_i\end{array}}{((p \to p) \to A_i) \to B_i, A_i \vdash B_i}}{((p \to p) \to A_i) \to B_i \vdash A_i \to B_i} \qquad \begin{array}{c}\vdots\\ ((p \to p) \to A_i) \to B_i, \Pi \dashv (p \to p) \to A_i, \Sigma\end{array}}{A_i \to B_i, \Pi \dashv (p \to p) \to A_i, \Sigma}}{A_i \to B_i, \Pi \dashv A_i, \Sigma}\,acut_2$$

$\square$

THEOREM 3.15. *Let* $\Gamma^{\to} = D_1 \to E_1, \ldots, D_n \to E_n$, $D_1, \ldots, D_n$ *being atoms or* $\bot$, $\Theta \cap [D_1, \ldots, D_n] = \varnothing$ *and* $m \geq 1$. *Any rule of the form*

$$\dfrac{\{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_i\}_{1 \leq i \leq m} \qquad \{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_h\}_{1 \leq h \leq k}}{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_1, \ldots, C_k}\,vis_m$$

is admissible in the minimal $\mathsf{G4ip}_H$ *calculus, with* $A_i \neq C_h$ *for any* $1 \leq i \leq m$ *and* $1 \leq h \leq k$.

PROOF. Suppose that each A_i is implicational and each C_h atomic: for any $1 \leq h \leq k$, if the minimal $\mathsf{G4ip}_H$ calculus refutes $A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_h$, then $C_h \notin \Theta$. Hence, it suffices to consider the following derivation:

$$\dfrac{\dfrac{\{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_i\}_{1 \leq i \leq m}}{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_1, \ldots, A_m, C_1, \ldots, C_k}\,adp_m}{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_1, \ldots, C_k}\,str$$

If there exists an A_i which is not implicational, we replace $A_i \to B_i$ with $((p \to p) \to A_i) \to A_i$ and A_i with $(p \to p) \to A_i$ as in Lemma 3.8: hence, we obtain a derivation π_i of $A_1 \to B_1, \ldots, ((p \to p) \to A_i) \to B_i, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv (p \to p) \to A_i$ as well as, for any h, a derivation π'_h of $A_1 \to B_1, \ldots, ((p \to p) \to A_i) \to B_i, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_h$. Moreover, if there exists a C_h which is not atomic, we take the following derivations ρ_h and ρ'_i:

$$\dfrac{\to\to\vdash\;\dfrac{p, p \to C_h \vdash p \qquad C_h \vdash C_h}{(p \to p) \to C_h \vdash C_h} \qquad \dfrac{\vdash \to\;\dfrac{\to\to\vdash\;\dfrac{p, p \to C_h \vdash p \qquad C_h \vdash C_h}{(p \to p) \to C_h \vdash C_h}}{\vdash ((p \to p) \to C_h) \to C_h} \qquad \begin{array}{c}\vdots\,\pi'_h\\ A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_h\end{array}}{((p \to p) \to C_h) \to C_h, A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_h}\,acut_1}{((p \to p) \to C_h) \to C_h, A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv (p \to p) \to C_h}\,acut_2$$

$$\dfrac{\vdash \to\;\dfrac{\to\to\vdash\;\dfrac{p, p \to C_h \vdash p \qquad C_h \vdash C_h}{(p \to p) \to C_h \vdash C_h}}{\vdash ((p \to p) \to C_h) \to C_h} \qquad \begin{array}{c}\vdots\,\pi_i\\ A_1 \to B_1, \ldots, ((p \to p) \to A_i) \to B_i, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv (p \to p) \to A_i\end{array}}{((p \to p) \to C_h) \to C_h, A_1 \to B_1, \ldots, ((p \to p) \to A_i) \to B_i, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv (p \to p) \to A_i}\,acut_1$$

Let $\Pi = ((p \to p) \to C_h) \to C_h, A_1 \to B_1, \ldots, ((p \to p) \to A_i) \to B_i, \ldots, A_m \to B_m$, $\Pi' = \Pi \setminus [((p \to p) \to C_h) \to C_h]$, $\Pi'' = \Pi' \setminus [((p \to p) \to A_i) \to B_i]$ and $\Sigma = C_1, \ldots, C_{h-1}, C_{h+1}, \ldots, C_k$. We plug ρ_h and ρ'_i into the following derivations ρ and ρ':

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \{\Pi,\Gamma^{\to},\Theta\dashv A_j\}_{1\leq j\neq i\leq m}\end{matrix}\qquad\begin{matrix}\vdots_{\rho_h}\\ \Pi,\Gamma^{\to},\Theta\dashv(p\to p)\to C_h\end{matrix}\qquad\begin{matrix}\vdots_{\rho'_i}\\ \Pi,\Gamma^{\to},\Theta\dashv(p\to p)\to A_i\end{matrix}}{\Pi,\Gamma^{\to},\Theta\dashv A_1,\ldots,A_{i-1},A_{i+1},\ldots,A_m,C_1,\ldots,C_{h-1},C_{h+1},\ldots,C_k,(p\to p)\to A_i,(p\to p)\to C_h}\,adp_{m+1}}{\Pi',\Gamma^{\to},\Theta\dashv C_1,\ldots,C_{h-1},C_{h+1},\ldots,C_k,(p\to p)\to C_h}\,str$$

$$\dfrac{C_h\vdash(p\to p)\to C_h\qquad\dfrac{((p\to p)\to A_i)\to B_i\vdash A_i\to B_i\qquad\begin{matrix}\vdots_{\rho}\\ ((p\to p)\to A_i)\to B_i,\Pi'',\Gamma^{\to},\Theta\dashv\Sigma,(p\to p)\to C_h\end{matrix}}{A_i\to B_i,\Pi'',\Gamma^{\to},\Theta\dashv\Sigma,(p\to p)\to C_h}\,acut_1}{A_i\to B_i,\Pi'',\Gamma^{\to},\Theta\dashv\Sigma,C_h}\,acut_2$$

Notice that $A_i\to B_i,\Pi'',\Gamma^{\to},\Theta\dashv\Sigma,C_h$ is identical to $A_1\to B_1,\ldots,A_m\to B_m,\Gamma^{\to},\Theta\dashv C_1,\ldots,C_k$, as desired. The same argument applies whenever there are multiple non-implicational A_i's and non-atomic C_h's. ☐

REMARK 3. The vis_m rules are Gentzen-style formulations of the contrapositive versions of (restricted) Visser's rules [73, 74][10]. If we drop the requirement that $A_i\neq C_h$, then vis_m and adp'_m rules are equivalent in presence of *acut* rules (cf. [77, p. 256] and [59, p. 339]).

We say that B is an *Harrop formula* whenever B is either an atom, or $\bot$, or a conjunction of Harrop formulas, or an implication whose consequent is Harrop [65]. It is well-known that the disjunction property of intuitionistic logic is preserved under Harrop formulas: if Γ contains only Harrop formulas, then $A\vee B$ is a logical consequence of Γ only if either A or B is a logical consequence of Γ. The following is a refutation-theoretic formulation of this result[11]:

THEOREM 3.16. *Let Γ be a multiset of Harrop formulas. The rule*

$$\dfrac{\Gamma\dashv\Delta_1\qquad\Gamma\dashv\Delta_2}{\Gamma\dashv\Delta_1,\Delta_2}\,ahdp$$

is admissible in the minimal $\mathsf{G4ip}_H$ *calculus.*

PROOF. If $\Gamma=\varnothing$, $\Delta_1=A_{11},\ldots,A_{1m}$ and $\Delta_2=A_{21},\ldots,A_{2m}$ where $m\geq 1$, we take the following derivation π_i, with $i=1,2$:

---

[10] It suffices to notice that the minimal $\mathsf{G4ip}_H$ calculus derives $A_1\to B_1,\ldots,A_m\to B_m,\Gamma^{\to},\Theta\dashv D_j$ for any $1\leq j\leq n$.

[11] Let us recall that the disjunction property of intuitionistic logic extends to the broader class of *projective formulas*, which includes non-Harrop formulas such as $p\to(q\vee r)$ [52]. Consequently, we assert that the generalized version of *ahdp*, where Γ is a multiset of projective formulas, is admissible in the minimal $\mathsf{G4ip}_H$ calculus. However, we leave this claim without a proof, as a complete inductive characterization of the class of projective formulas (if any) remains an open problem [17].

$$\dfrac{\vdash A_{i1} \to A_{i1} \qquad \dashv A_{i1}, \dots, A_{im}}{A_{i1} \to A_{i1} \dashv A_{i1}, \dots, A_{im}}\, acut_1$$

$$\vdots$$

$$\dfrac{\vdash A_{(3-i)m} \to A_{(3-i)m} \qquad A_{i1} \to A_{i1}, \dots, A_{(3-i)(m-1)} \to A_{(3-i)(m-1)} \dashv A_{i1}, \dots, A_{im}}{A_{i1} \to A_{i1}, \dots, A_{(3-i)(m-1)} \to A_{(3-i)(m-1)}, A_{(3-i)m} \to A_{(3-i)m} \dashv A_{i1}, \dots, A_{im}}\, acut_1$$

Hence, for any $1 \leq j \leq m$ we consider the following derivation π'_j:

$$\dfrac{\overset{\vdots\,\pi_i}{A_{i1} \to A_{i1}, \dots, A_{(3-i)m} \to A_{(3-i)m} \dashv A_{i1}, \dots, A_{im}}}{A_{i1} \to A_{i1}, \dots, A_{(3-i)m} \to A_{(3-i)m} \dashv A_{ij}}\, str$$

Let $\Pi = A_{i1} \to A_{i1}, \dots, A_{(3-i)m} \to A_{(3-i)m}$: we plug the π'_j's into the following derivation ρ to reach the conclusion.

$$\dfrac{\dfrac{\overset{\vdots\,\pi_{11}}{\Pi \dashv A_{11}} \qquad \cdots \qquad \overset{\vdots\,\pi_{2m}}{\Pi \dashv A_{2m}}}{\Pi \dashv \Delta_1, \Delta_2}\, adp'_{2m}}{\dashv \Delta_1, \Delta_2}\, str$$

If $\Gamma \neq \varnothing$, then $\Gamma = A_1 \to B_1, \dots, A_m \to B_m, \Gamma'$, with $m \geq 0$, $A_1, \dots, A_m$ being atoms or $\bot$ and Γ' lacking implications with atoms or $\bot$ as antecedent: to reach the conclusion, we reason by cases over $deg(\Gamma')$, defined as the maximal logical complexity of a formula in Γ', and proceed by induction on the logical complexity of Γ.
If $deg(\Gamma') = 1$, then $\Gamma' = q_1, \dots, q_n$. If $A_i \neq q_j$ for any $1 \leq i \leq m$ and $1 \leq j \leq n$, then for any $A \in \Delta_1, \Delta_2$ we apply $acut_2$ to $\vdash A \to A$ and $\Gamma \dashv \Delta_1$ as well as $\vdash A \to A$ and $\Gamma \dashv \Delta_2$: hence, we exploit adp'_m to get the conclusion. On the other hand, if $A_i = q_j$ for some $1 \leq i \leq m$ and $1 \leq j \leq n$, we apply $at \to \dashv_{inv}$ and exploit the inductive hypothesis.
If $deg(\Gamma') > 1$, we reason by cases over Γ'. If $\Gamma' = B \wedge C, \Gamma''$, we leverage $acut_1$ to replace $B \wedge C$ with (say) B, and apply the inductive hypothesis. If $\Gamma' = B \to C, \Gamma''$ and $B = D \wedge E$, we exploit $acut_1$ to replace $(D \wedge E) \to C$ with $D \to (E \to C)$; if $B = D \vee E$ or $D \to E$, we leverage $acut_1$ to replace $B \to C$ with $D \to C$. In all cases, we apply the inductive hypothesis to reach the conclusion.

□

REMARK 4. Theorems 3.15 and 3.16 establish that, in presence of *acut* rules, the admissibility of adp'_m rules entails the admissibility of contrapositive versions of the disjunction property and Visser rules vis_m, for any m. Since the only intermediate logic where the disjunction property and all the Visser rules hold is intuitionistic logic [72], we infer that for any intermediate logic **L** distinct from the latter there must exist (at least) one $n \geq 2$ such that some adp'_n rules are not admissible in any hybrid calculus for **L**. A(n algebraic) proof of the fact that there exists (at least) one $n \geq 2$ such that *all* adp'_n rules are not admissible in any hybrid calculus for $\mathbf{L}$ is detailed in [153, pp. 79-80].

3.4.1. Anticut elimination in intuitionistic logic. In this subsection, we illustrate a constructive approach to *acut* admissibility in $\overline{\overline{\overline{\mathsf{G4ip}}}}$. First, we recall a crucial result:

Theorem 3.17. *For any sequent $\Gamma \vdash \Delta$, either $\overline{\overline{\overline{\mathsf{G4ip}}}}$ proves $\Gamma \vdash \Delta$ or refutes $\Gamma \vdash \Delta$ – but not both.*

Proof. For a proof see [131, pp. 227-228]. ☐

In what follows, we establish a number of intermediate lemmas, which we will exploit in the proof of the main result – namely, Theorem 3.18.

Lemma 3.9. *The rules*

$$str\;\frac{A, \Gamma \dashv \Delta}{\Gamma \dashv \Delta} \qquad\qquad \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta}\; str$$

$$\wedge \dashv_{inv}\;\frac{B \wedge C, \Gamma \dashv \Delta}{B, C, \Gamma \dashv \Delta} \qquad\qquad \frac{\Gamma \dashv \Delta, B \vee C}{\Gamma \dashv \Delta, B, C}\; \dashv \vee_{inv}$$

$$at \to \dashv_{inv}\;\frac{p, p \to B, \Gamma \dashv \Delta}{p, B, \Gamma \dashv \Delta} \qquad\qquad \frac{(B \wedge C) \to D, \Gamma \dashv \Delta}{B \to (C \to D), \Gamma \dashv \Delta}\; \wedge \to \dashv_{inv}$$

$$\vee \to \dashv_{inv}\;\frac{(B \vee C) \to D, \Gamma \dashv \Delta}{B \to D, C \to D, \Gamma \dashv \Delta} \qquad\qquad \frac{\Gamma \dashv A \to B}{\Gamma, A \dashv B}\; \dashv \to_{inv}$$

are admissible in $\overline{\overline{\overline{\mathsf{G4ip}}}}$.

Proof. For each rule distinct from $\dashv \to_{inv}$, we establish the conclusion proceeding by induction on the height of the derivation π of the premise: in the cases of *inv* rules, we exploit the fact that the last rule applied in π cannot be $\to \dashv \to$. As far as the rule $\dashv \to_{inv}$ is concerned, let π be a $\overline{\overline{\overline{\mathsf{G4ip}}}}$-derivation of $\Gamma \dashv A \to B$: we define the *right rank* of $A \to B$ in π as the number of consecutive antisequents in π where $A \to B$ occurs on the right-hand side, from the conclusion upwards. We reach the conclusion proceeding by induction on the right rank of $A \to B$ in π (we omit the details). ☐

Lemma 3.10. *Let $\vec{A} = A_1 \to \cdots \to A_m$, with $m \geq 1$. If the rule*

$$\frac{B, \Gamma \dashv \Delta}{C, \Gamma \dashv \Delta}\; r$$

is admissible in $\overline{\overline{\overline{\mathsf{G4ip}}}}$, then the rule

$$\frac{\vec{A} \to B, \Gamma \dashv \Delta}{\vec{A} \to C, \Gamma \dashv \Delta}\; nr$$

is admissible in $\overline{\overline{\overline{\mathsf{G4ip}}}}$.

Proof. We argue by primary induction on m, secondary induction on the height of the derivation π of the premise $\vec{A} \to B, \Gamma \dashv \Delta$ and ternary induction on the logical complexity of A_m. If $m = 1$, $h(\pi) = 1$ and A_m is $\bot$ or an atom, the conclusion is immediate. If $m = 1$ and $h(\pi) > 1$, we reason by cases over the last rule applied in π. If $A_1 \to B$ is not principal in it, we apply the secondary inductive hypothesis. If A_1 is principal in it, we distinguish multiple cases according to A_1's principal connective (if any). If A_1 is an atom p belonging to Γ, we apply r and $at \to \dashv$ to get the conclusion; if $A_1 = D \wedge E$, we exploit the secondary inductive hypothesis. Whenever $A_1 = D \vee E$, or $A_1 = D \to E$ with $\to \dashv \to$ being the last rule applied, we exploit the secondary as well as the ternary inductive hypothesis. If $m > 1$, we leverage the primary inductive hypothesis and replicate the argument we made for $m = 1$ to reach the conclusion. $\square$

Lemma 3.11. *Let A be a non-implicational formula. The rule*

$$\frac{A \to B, A \to C, \Gamma \dashv \Delta}{A \to (B \wedge C), \Gamma \dashv \Delta} \to \wedge \dashv$$

is admissible in the $\overline{\overline{\mathsf{G4ip}}}$ calculus.

Proof. We reason by primary induction on the logical complexity of A and secondary induction on the height of the derivation π of the premise. If A is $\bot$ or an atom and $h(\pi) = 1$, the conclusion is immediate. If $h(\pi) > 1$ and neither $p \to B$ nor $p \to C$ is principal in the last rule applied, we exploit the secondary inductive hypothesis. If (say) $p \to B$ is principal and $p \to C$ not, we consider the following derivation:

$$\begin{array}{cl}
p, B, p \to C, \Gamma \dashv \Delta & \\
\hdashline
p, B, C, \Gamma \dashv \Delta & at \to \dashv_{inv} \\
\hline
p, B \wedge C, \Gamma \dashv \Delta & \wedge \dashv \\
\hline
p, p \to (B \wedge C), \Gamma \dashv \Delta & at \to \dashv \\
\hdashline
p \to (B \wedge C), \Gamma \dashv \Delta & str
\end{array}$$

If $A = D \wedge E$, we consider the following derivation:

$$\begin{array}{cl}
(D \wedge E) \to B, (D \wedge E) \to C, \Gamma \dashv \Delta & \\
\hdashline
D \to (E \to B), (D \wedge E) \to C, \Gamma \dashv \Delta & \wedge \to \dashv_{inv} \\
\hdashline
D \to (E \to B), D \to (E \to C), \Gamma \dashv \Delta & \wedge \to \dashv_{inv} \\
\hline
D \to ((E \to B) \wedge (E \to C)), \Gamma \dashv \Delta & \to \wedge \dashv
\end{array}$$

By primary inductive hypothesis and $\wedge \dashv_{inv}$, the rule leading from $(E \to B) \wedge (E \to C), \Gamma \dashv \Delta$ to $E \to (B \wedge C), \Gamma \dashv \Delta$ is admissible in $\overline{\overline{\mathsf{G4ip}}}$: hence, Lemma 3.10 ensures the admissibility of the rule leading from $D \to ((E \to B) \wedge (E \to C)), \Gamma \dashv \Delta$ to $D \to (E \to (B \wedge C)), \Gamma \dashv \Delta$: an application of $\wedge \to \dashv$ suffices to the conclusion.
If $A = D \vee E$, we take the following derivation:

$$
\begin{array}{cl}
(D \lor E) \to B, (D \lor E) \to C, \Gamma \dashv \Delta & \\
\hdashline
D \to B, E \to B, (D \lor E) \to C, \Gamma \dashv \Delta & \lor \to \dashv_{inv} \\
\hdashline
D \to B, E \to B, D \to C, E \to C, \Gamma \dashv \Delta & \lor \to \dashv_{inv} \\
\hline
D \to (B \land C), E \to B, E \to C, \Gamma \dashv \Delta & \to \land \dashv \\
\hline
D \to (B \land C), E \to (B \land C), \Gamma \dashv \Delta & \to \land \dashv \\
\hline
(D \lor E) \to (B \land C), \Gamma \dashv \Delta & \lor \to \dashv
\end{array}
$$

□

Lemma 3.12. *Let* $\Gamma^{\to} = A_1 \to B_1, \ldots, A_k \to B_k$, *with* $A_1, \ldots, A_k$ *atoms or* $\bot$, *and* $\Theta \cap [A_1, \ldots, A_k] = \varnothing$. *The restricted copy rule*

$$\dfrac{\Pi, \Gamma^{\to}, \Theta \dashv \Lambda, \Delta \qquad \Pi, \Gamma^{\to}, \Theta \dashv \Lambda, \Delta}{\Pi, \Gamma, \Gamma \dashv \Lambda, \Lambda, \Delta}\ \mathit{rcopy}$$

is admissible in $\overline{\overline{\mathsf{G4ip}}}$.

Proof. We assume (without loss of generality) that for any *rcopy* application the two premises have the same derivation π: we reason by induction on $2h(\pi)$ to get the conclusion. If $h(\pi) = 1$, the conclusion is immediate; otherwise, we reason by cases over the last rule applied in π. The only interesting case arises when the last rule applied in π is

$$\dfrac{\begin{array}{c}\vdots{\scriptstyle \{\pi_i\}_{1\le i\le m}} \\ \{D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Theta \dashv E_i\}_{1\le i\le m}\end{array} \qquad \begin{array}{c}\vdots{\scriptstyle \{\pi'_j\}_{1\le j\le n}} \\ \{\Gamma, \Gamma^{\to}, \Theta, F_j \dashv G_j\}_{1\le j\le n}\end{array}}{\Gamma, \Gamma^{\to}, \Theta \dashv F_1 \to G_1, \ldots, F_n \to G_n, \Lambda}\ \to\dashv\to$$

where $\Gamma_i = \Gamma \setminus [(D_i \to E_i) \to C_i]$. We apply the inductive hypothesis to remove the *rcopy* applications from the following derivations ρ_i and ρ'_j, for any $1 \le i \le m$ and $1 \le j \le m$:

$$\dfrac{\begin{array}{c}\vdots{\scriptstyle \pi_i} \\ D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Theta \dashv E_i\end{array} \qquad \begin{array}{c}\vdots{\scriptstyle \pi_i} \\ D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Theta \dashv E_i\end{array}}{D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv E_i}\ \mathit{rcopy}$$

$$\dfrac{\begin{array}{c}\vdots{\scriptstyle \pi'_j} \\ \Gamma, \Gamma^{\to}, \Theta, F_j \dashv G_j\end{array} \qquad \begin{array}{c}\vdots{\scriptstyle \pi'_j} \\ \Gamma, \Gamma^{\to}, \Theta, F_j \dashv G_j\end{array}}{\Gamma, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta, F_j \dashv G_j}\ \mathit{rcopy}$$

Hence, we plug ρ_i and ρ'_i into the following derivation:

$$\dfrac{\begin{array}{c}\vdots{\scriptstyle \{\rho_i\}_{1\le i\le m}} \\ \{D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv E_i\}_{1\le i\le m}\end{array} \qquad \begin{array}{c}\vdots{\scriptstyle \{\pi'_j\}_{1\le j\le n}} \\ \{\Gamma, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta, F_j \dashv G_j\}_{1\le j\le n}\end{array}}{\Gamma, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv F_1 \to G_1, \ldots, F_n \to G_n, \Lambda, \Lambda}\ \to\dashv\to$$

□

Lemma 3.13. *The Copy rule*

$$\frac{\Gamma \dashv \Delta \qquad \Gamma \dashv \Delta}{\Gamma, \Gamma \dashv \Delta, \Delta}\, \mathit{copy}$$

is admissible in $\overline{\overline{\mathsf{G4ip}}}$.

Proof. We argue as in the proof of Lemma 3.3: we focus on the case where the last rule applied in the derivation π of each premise is $\to \dashv \to$. We have that π has the following form:

$$\frac{\begin{matrix} \vdots\{\pi_i\}_{1\leq i\leq m} \\ \{D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Theta \dashv E_i\}_{1\leq i\leq m} \end{matrix} \qquad \begin{matrix} \vdots\{\pi'_j\}_{1\leq j\leq n} \\ \{\Gamma, \Gamma^{\to}, \Theta, F_j \dashv G_j\}_{1\leq j\leq n} \end{matrix}}{\Gamma, \Gamma^{\to}, \Theta \dashv F_1 \to G_1, \ldots, F_n \to G_n, \Lambda} \to \dashv \to$$

where Γ_i, $\Gamma^{\to}$ and Θ are defined as in the proof of Lemma 3.12. First, we apply *rcopy* to the π_i and π'_j's to obtain the derivations ρ_i and ρ'_j of $D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv E_i$ and $\Gamma, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta, F_j \dashv G_j$, respectively. For each $1 \leq i \leq m$ and $1 \leq j \leq n$ we consider a copy ρ_{m+i} and ρ'_{n+j} of ρ_i and ρ'_j such that $D_i = D_{m+i}$, $E_i = E_{m+i}$, $C_i = C_{m+i}$, $\Gamma_i = \Gamma_{m+1}$, $F_j = F_{n+j}$ and $G_j = G_{n+j}$. Hence, we plug the $\rho_1, \ldots, \rho_{2m}$ and $\rho'_1, \ldots, \rho'_{2n}$'s into the following derivation:

$$\frac{\begin{matrix} \vdots\{\rho_{i'}\}_{1\leq i'\leq 2m} \\ \{D_{i'}, E_{i'} \to C_{i'}, \Gamma_{i'}, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv E_i\}_{1\leq i'\leq 2m} \end{matrix} \qquad \begin{matrix} \vdots\{\rho'_{j'}\}_{1\leq j'\leq 2n} \\ \{\Gamma, \Gamma^{\to}, \Theta, F_{j'} \dashv G_{j'}\}_{1\leq j'\leq 2n} \end{matrix}}{\Gamma, \Gamma, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv F_1 \to G_1, \ldots, F_n \to G_n, F_1 \to G_1, \ldots, F_n \to G_n, \Lambda, \Lambda} \to \dashv \to$$

□

Lemma 3.14. *The rules*

$$\mathit{dup}\, \frac{A, \Gamma \dashv \Delta}{A, A, \Gamma \dashv \Delta} \qquad\qquad \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A, A}\, \mathit{dup}$$

are admissible in the $\overline{\overline{\mathsf{G4ip}}}$ *calculus.*

Proof. It suffices to consider the following derivations:

$$\mathit{str}\, \frac{\mathit{copy}\, \dfrac{A, \Gamma \dashv \Delta}{A, A, \Gamma, \Gamma \dashv \Delta, \Delta}}{A, A, \Gamma \dashv \Delta} \qquad\qquad \frac{\dfrac{\Gamma \dashv \Delta, A}{\Gamma, \Gamma \dashv \Delta, \Delta, A, A}\, \mathit{copy}}{\Gamma \dashv \Delta, A, A}\, \mathit{str}$$

□

Lemma 3.15. *The rule*

$$\frac{B, \Gamma \dashv \Delta}{A \to B, \Gamma \dashv \Delta} \to \dashv$$

is admissible in the $\overline{\overline{\mathsf{G4ip}}}$ *calculus.*

Proof. We assume (without loss of generality) that A is non-implicational. We reason by primary induction on the height of the derivation π of the premise, secondary induction on the logical complexity of A and ternary induction on B. If $h(\pi) = 1$ and A is $\bot$ or an atom p, the conclusion is straightforward. If $A = C \wedge D$, we take the following derivation:

$$\dfrac{\dfrac{\dfrac{B, \Gamma \dashv \Delta}{D \to B, \Gamma \dashv \Delta}\,\to\dashv}{C \to (D \to B), \Gamma \dashv \Delta}\,\to\dashv}{(C \wedge D) \to B, \Gamma \dashv \Delta}\,\wedge\to\dashv$$

On the other hand, if $A = C \vee D$, we consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{B, \Gamma \dashv \Delta}{B, B, \Gamma \dashv \Delta}\,\mathit{dup}}{C \to B, B, \Gamma \dashv \Delta}\,\to\dashv}{C \to B, D \to B, \Gamma \dashv \Delta}\,\to\dashv}{(C \vee D) \to B, \Gamma \dashv \Delta}\,\vee\to\dashv$$

If $h(\pi) > 1$ and A is an atom p such that $p \notin \Gamma$, we reason by cases over the last rule applied. If B is not principal in the last rule applied, it suffices to apply the first inductive hypothesis. If B is principal in the last rule applied, and the last rule applied is neither $\wedge \dashv$ nor $\to \dashv \to$, we apply the primary inductive hypothesis and Lemma 3.10 to reach the conclusion. Here, we focus on the remaining cases.
If $B = C \wedge D$, we consider the following derivation:

$$\dfrac{\dfrac{\dfrac{C, D, \Gamma \dashv \Delta}{A \to C, D, \Gamma \dashv \Delta}\,\to\dashv}{A \to C, A \to D, \Gamma \dashv \Delta}\,\to\dashv}{A \to (C \wedge D), \Gamma \dashv \Delta}\,\to\wedge\dashv$$

On the other hand, if $B = (C \to D) \to E$ and the last rule applied is $\to \dashv \to$, we apply the first inductive hypothesis. Finally, if $h(\pi) > 1$ and A is neither atomic nor $\bot$, we proceed in the same way as in the case where $h(\pi) = 1$ and A is neither atomic nor $\bot$. $\square$

Now, we are ready to prove the existence of an *acut* elimination strategy for $\overline{\overline{\mathsf{G4ip}}}$.

Theorem 3.18. *There exists an algorithm which turns any* $\overline{\overline{\mathsf{G4ip}}} + acut_i$*-derivation of* $\Pi \dashv \Sigma$ *into a* $\overline{\overline{\mathsf{G4ip}}}$*-derivation of* $\Pi \dashv \Sigma$*, with* $i = 1, 2$.

Proof. We focus on the topmost $acut_i$ application, proceeding by primary induction on the logical complexity of the $acut_i$ formula and by secondary induction on the height of the derivation π of the right premise. If the $acut_i$ formula is an atom, a conjunction or a disjunction, the argument runs as in the proof of Theorem 3.6. The only different case arises whenever A is an atom and the last rule applied in π is $\to \dashv \to$: we leverage the secondary

inductive hypothesis as well as the fact that $\Gamma \vdash \Delta, A$ can be derived with a single $ax_{int} \vdash$ application to reach the conclusion.
Here, we discuss in detail only the case where the $acut_i$ formula has the form $A \to B$: we assume without loss of generality that A is either an atom or an implication.

(i) Consider the following derivation:

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta, A \to B$}
\AxiomC{$\vdots_{\pi}$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\RightLabel{$acut_1$}
\BinaryInfC{$A \to B, \Pi \dashv \Sigma$}
\end{prooftree}$$

We reason by cases over the last rule applied in π to reach the conclusion. If $\Pi, \Gamma \dashv \Delta, \Sigma$ is an initial antisequent, then $\overline{\overline{\mathsf{G4ip}}}$ proves $\Gamma, A \vdash B$, and thus $\Pi, \Gamma, A \vdash B, \Delta, \Sigma$. On the other hand, Theorem 3.17 ensures that $\overline{\overline{\mathsf{G4ip}}}$ derives either $\Pi, \Gamma \vdash B, \Delta, \Sigma$ or $\Pi, \Gamma \dashv B, \Delta, \Sigma$. In the first case, we consider the following derivation:

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \vdash B, \Delta, \Sigma$}
\AxiomC{}
\RightLabel{$ax_{int} \dashv$}
\UnaryInfC{$\Pi, \Gamma \dashv \Delta, \Sigma$}
\dashedLine
\RightLabel{$copy$}
\UnaryInfC{$\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma$}
\RightLabel{$acut_1$}
\BinaryInfC{$B, \Gamma, \Pi \dashv \Delta, \Sigma$}
\doubleLine
\RightLabel{str}
\UnaryInfC{$B, \Pi \dashv \Sigma$}
\dashedLine
\RightLabel{$\to \dashv$}
\UnaryInfC{$A \to B, \Pi \dashv \Sigma$}
\end{prooftree}$$

In the second case we consider the following derivation:

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, A \vdash B, \Delta$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv B, \Delta, \Sigma$}
\RightLabel{$acut_2$}
\BinaryInfC{$\Pi \dashv \Sigma, A$}
\end{prooftree}$$

If A is atomic, it suffices to notice that $A \notin \Pi$: this implies that $A \to B, \Pi \dashv \Sigma$ can be derived with a single application of $ax_{int} \dashv$. If $A = C \to D$, we consider the following derivation:

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, C \to D \vdash B$}
\dashedLine
\LeftLabel{inv}
\UnaryInfC{$\Gamma, D \vdash B$}
\LeftLabel{$\vdash \to$}
\UnaryInfC{$\Gamma \vdash D \to B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, C \to D \vdash B, \Delta$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, \Gamma \dashv B, \Delta, \Sigma$}
\doubleLine
\RightLabel{dup}
\UnaryInfC{$\Pi, \Gamma, \Gamma \dashv B, \Delta, \Sigma$}
\RightLabel{$acut_2$}
\BinaryInfC{$\Pi, \Gamma \dashv \Sigma, C \to D$}
\RightLabel{$acut_1$}
\BinaryInfC{$\Pi, D \to B \dashv \Sigma, C \to D$}
\doubleLine
\RightLabel{str}
\UnaryInfC{$\Pi, D \to B \dashv C \to D$}
\dashedLine
\RightLabel{$\dashv \to_{inv}$}
\UnaryInfC{$\Pi, D \to B, C \dashv D$}
\RightLabel{$\to \dashv \to$}
\UnaryInfC{$(C \to D) \to B, \Pi \dashv \Sigma$}
\end{prooftree}$$

If $h(\pi) > 1$ and the last rule applied in π is not $\to \dashv \to$, it suffices to exploit the secondary inductive hypothesis to reach the conclusion. If the last rule applied in π is $\to \dashv \to$ we argue as in the case where $h(\pi) = 1$.

(ii) Consider the following derivation:

$$\dfrac{\begin{matrix}\vdots \\ A \to B, \Gamma \vdash \Delta\end{matrix} \qquad \begin{matrix}\vdots_{\pi} \\ \Pi, \Gamma \dashv \Delta, \Sigma\end{matrix}}{\Pi \dashv \Sigma, A \to B}\ acut_2$$

If $\Pi, \Gamma \dashv \Delta, \Sigma$ is an initial antisequent, we reason by cases over A. If A is atomic, we infer that the last rule applied in the derivation of $A \to B, \Gamma \vdash \Delta$ is $at \to \vdash$: if $\Gamma = \Gamma', A$, then $\overline{\overline{\mathsf{G4ip}}}$ proves $B, A, \Gamma' \vdash \Delta$. Take the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ B, A, \Gamma' \vdash \Delta\end{matrix} \qquad \dfrac{\dfrac{}{\Pi, A, \Gamma' \dashv \Delta, \Sigma}\ ax_{int} \dashv}{\Pi, A, A, \Gamma' \dashv \Delta, \Sigma}\ dup}{A, \Pi \dashv \Sigma, B}\ acut_2}{A, \Pi \dashv B}\ str}{\Pi \dashv \Sigma, A \to B}\ \to \dashv \to$$

If $A = C \to D$, the last rule applied in the derivation of $A \to B, \Gamma \vdash \Delta$ is $\to\to \vdash$. As a result, we have that $\overline{\overline{\mathsf{G4ip}}}$ proves $C, D \to B, \Gamma \vdash D$ and $B, \Gamma \vdash \Delta$: the provability of $C, D \to B, \Gamma \vdash D$ entails the provability of $B, \Pi, \Gamma \vdash C \to D, \Delta, \Sigma$. Theorem 3.17 ensures that $\overline{\overline{\mathsf{G4ip}}}$ derives either $\Pi, \Gamma \vdash C \to D, \Delta, \Sigma$ or $\Pi, \Gamma \dashv C \to D, \Delta, \Sigma$. In the first case, we leverage the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ B, \Gamma \vdash \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\vdots \\ \Pi, \Gamma \vdash \Delta, \Sigma, C \to D\end{matrix} \qquad \dfrac{\dfrac{}{\Pi, \Gamma \dashv \Delta, \Sigma}\ ax_{int} \dashv}{\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma}\ copy}{C \to D, \Pi, \Gamma \dashv \Delta, \Sigma}\ acut_1}{C \to D, \Pi \dashv \Sigma, B}\ acut_2}{C \to D, \Pi \dashv B}\ str}{\Pi \dashv \Sigma, (C \to D) \to B}\ \to \dashv \to$$

If $h(\pi) > 1$ and the last rule applied in π is not $\to \dashv \to$, it suffices to exploit the secondary inductive hypothesis to get the conclusion. If the last rule applied in π is $\to \dashv \to$ we argue as in the case where $h(\pi) = 1$.

□

REMARK 5. Consider the following rule:

$$\dfrac{\Gamma \dashv \Delta}{\bot \to B, \Gamma \dashv \Delta}\ \bot \to \dashv$$

In [110], the authors state that $\bot \to \dashv$ is redundant in $\overline{\overline{\mathsf{G4ip}}}$, due to the fact that the context $\Gamma^{\to}$ in the rules $ax_{int} \dashv$ and $\to \dashv \to$ is allowed to contain implications with $\bot$ as antecedent. Theorem 3.18 provides a constructive argument for the same claim, since any $\bot \to \dashv$ application can be seen as an *acut* application:

$$\dfrac{\begin{matrix}\vdots \\ \vdash \bot \to B\end{matrix} \qquad \Gamma \dashv \Delta}{\bot \to B, \Gamma \dashv \Delta}\ acut_1$$

Notice that if we disallow implications with antecedent $\bot$ from $\Gamma^{\to}$ in the rules $ax_{int} \dashv$ and $\to \dashv \to$ of $\overline{\overline{\mathsf{G4ip}}}$, there can be no *acut*-free $\overline{\overline{\mathsf{G4ip}}}$-derivation of (say) $(p \to \bot) \to (q \vee r) \dashv (p \to \bot) \to q, (p \to \bot) \to r$, and thus Theorem 3.18 ceases to hold.

We conclude this section with the following result, establishing that $\overline{\overline{\mathsf{G4ip}}}$ and the minimal $\mathsf{G4ip}_H$ calculus derive the same (anti)sequents:

THEOREM 3.19. *The rules adp_m are admissible in* $\overline{\overline{\mathsf{G4ip}}}$.

PROOF. Let $\Gamma = (C_1 \to D_1) \to B_1, \ldots, (C_m \to D_m) \to B_m$ and $\Gamma_i = \Gamma \setminus [(C_i \to D_i) \to B_i]$, for any $1 \leq i \leq m$. Take the following derivations π_i and π_i':

$$\dfrac{(C_i \to D_i) \to B_i, \Gamma_i, \Gamma^{\to}, \Theta \dashv C_i \to D_i}{(C_i \to D_i) \to B_i, \Gamma_i, \Gamma^{\to}, \Theta, C_i \dashv D_i} \dashv \to_{inv}$$

$$\dfrac{(C_i \to D_i) \to B_i \vdash D_i \to B_i \qquad \begin{array}{c} \vdots \pi_i \\ (C_i \to D_i) \to B_i, \Gamma_i, \Gamma^{\to}, \Theta, C_i \dashv D_i \end{array}}{D_i \to B_i, \Gamma_i, \Gamma^{\to}, \Theta, C_i \dashv D_i} \; acut$$

We plug the derivations π_i and π_i' into the following derivation:

$$\dfrac{\begin{array}{c} \vdots \{\pi_i\}_{1 \leq i \leq m} \\ \{D_i \to B_i, \Gamma_i, \Gamma^{\to}, \Theta, C_i \dashv D_i\}_{1 \leq i \leq m} \end{array} \qquad \begin{array}{c} \vdots \{\pi_i'\}_{1 \leq i \leq m} \\ \{(C_i \to D_i) \to B_i, \Gamma_i, \Gamma^{\to}, \Theta, C_i \dashv D_i\}_{1 \leq i \leq m} \end{array}}{(C_1 \to D_1) \to B_1, \ldots, (C_m \to D_m) \to B_m, \Gamma^{\to}, \Theta \dashv C_1 \to D_1, \ldots, C_m \to D_m} \to \dashv \to$$

$\square$

CHAPTER 4

Abduction as deductive saturation

Abductive processes are ubiquitous in scientific theorizing and everyday life. They involve inherent cognitive risk for a rational agent who must select possible *explanantes* for an *explanandum* based on incomplete or uncertain information. Although these processes are not deductive in nature, the ultimate goal of a rational agent in abductive reasoning can be described as the search for the missing premise of an "unsaturated" deductive inference. Charles Sanders Peirce presents this situation as an abductive scenario:

> *I once landed at a seaport in a Turkish province; and, as I was walking up to the house which I was to visit, I met a man upon horseback, surrounded by four horsemen holding a canopy over his head. As the governor of the province was the only personage I could think of who would be so greatly honored, I inferred that this was he. This was a hypothesis.* [119]

The treatment of abduction as an enthymematic deductive argument in reverse is guided by certain directives. Given a non empty set of premises Γ and a formula G such that $\Gamma \nvdash G$, we need to find a formula H satisfying three logical conditions:

$$\mathsf{A1}\colon\ \Gamma, H \vdash G \qquad \mathsf{A2}\colon\ H \nvdash G \qquad \mathsf{A3}\colon\ \Gamma, H \nvdash \bot$$

A1 – A3 are part of the tradition of twentieth-century philosophy of science as they can be traced back to Hempel's essential requirements for H to be considered an *explanans* of G given Γ ([66], pp. 277-78). Of course, the Hempelian account is no longer the prevailing approach to explanation among most contemporary philosophers of science. Over time, the Hempelian model has faced criticism and has been challenged by alternative accounts of explanation. Many philosophers now advocate for a more nuanced understanding of explanation that incorporates additional factors beyond simple deductive subsumption. Some of these alternative accounts include causal models, pragmatic approaches, and various forms of contextual explanations (for a survey see [176]). However, the Hempelian approach, with its focus on logical coherence and systematic analysis, aligns with a structured framework for understanding the problem of abduction from an abstract deductive perspective. A1 states that the formula G needs to be 'deductively reachable' from the set of premises $\Gamma \cup \{H\}$, that is H must bridge the deductive gap between Γ and G. A2 and A3 require that the formula H provides useful and non-trivial information. Specifically, A2 ensures that Γ is not a superfluous context by demanding that H alone does not imply G, while $\mathsf{A3}$ requires that adding H to Γ should not make $\Gamma \cup \{H\}$ inconsistent.

To the extent that there exist infinitely many abductive formulas obeying $\mathsf{A1}$ – $\mathsf{A3}$ for any invalid sequent $\Gamma \nvdash G$, a natural question immediately arises: what strategy should be employed by a rational agent to select just one of these formulas? The following two-step strategy seems to be a reasonable one:

(1) restrict the search space to the (finite) set of abductive hypotheses that convey information already contained in Γ and G;

(2) investigate the search space enlarged with abductive hypotheses that satisfy conditions $\mathsf{A1}$ – $\mathsf{A3}$ and provide information not in Γ or G.

Several efforts have been made to address Step (1), which aims to define an *effective* procedure for generating and justifying hypotheses that satisfy $\mathsf{A1}$ – $\mathsf{A3}$. One traditional approach relies on the use of tableaux. Essentially, it consists in writing the refutation tree associated with the set $\Gamma, \neg G$, examining the open branches, and then identifying any cluster of formulas Δ which allow for the systematic closure of each one of the open branches in the tableau under consideration [**103**, **32**, **11**]. The formula H resulting from the *maximal* cluster of such formulas satisfies *deductive minimality* (DM, henceforth):

$$\mathsf{DM}\text{: for any } H', \text{ if } \Gamma, H' \vdash G, \text{ then } H' \vdash H$$

H is regarded as the optimal hypothesis under the name of *least compromising hypothesis*.

In this chapter, we prove *inter alia* that the condition of DM is not necessary for the optimality of H. In effect, DM fails to capture something fundamental to abductive reasoning: its purpose of finding the simplest and most relevant *explanans* from among many. To illustrate this failure from the perspective of a rational agent, let's consider two simple examples.

Example 4.1. *Consider the invalid sequent $p \vee q \nvdash q$. The resulting least compromising hypothesis is $\neg p \vee q$. However, it seems reasonable to assume that a rational agent would consider $\neg p \vee q$ too weak to properly saturate $p \vee q \nvdash q$. In fact, $\neg p$ seems to provide a better explanation for $p \vee q \nvdash q$, as it appeals to an instance of the disjunctive syllogism $p \vee q, \neg p \vdash q$.*

Example 4.2. *Consider the invalid sequent $p \to q \nvdash r \to q$. Inserting among the premises the least compromising hypothesis $r \to (p \vee q)$ is a detour for a rational agent seeking an optimal explanation for $p \to q \nvdash r \to q$. Instead, $r \to p$ fits the bill by referring to an instance of the hypothetical syllogism $r \to p, p \to q \vdash r \to q$* (cf. [**119**], p. 472).

To overcome these difficulties, we design a sequent-based procedure that always *approximates* an abductive hypothesis providing a better explanation in our refined sense. Although our machinery hinges on the well-known duality between tableaux *à la* Smullyan and Kleene's sequent system [156, 123], we believe that explicitly handling sequents instead of tableaux results in a simpler formal approach, since sequents allow for a *local* control of information flow.

Furthermore, our approach can be usefully applied to Step (2), which concerns the search space expanded with abductive hypotheses that satisfy conditions A1 – A3 while providing additional information. We show how a generalized version of our procedure can track any abductive hypothesis with new information. Specifically, we establish that any formula in the expanded search space that satisfies conditions A1 – A3 must also imply one of these hypotheses that satisfy the same conditions. This result enables us to shift our attention to the (infinite) subspace of abductive hypotheses that respect conditions A1 to A3 and imply hypotheses that offer a better explanation. We hypothesize that this subspace includes the set of candidates for selection as the best explanans.

The chapter is organized as follows. In Section 4.1, we describe a sequent-based procedure for generating the least compromising hypothesis, and we provide sufficient conditions for enforcing its satisfaction of conditions A2 and A3. Section 4.2, presents another sequent-based procedure for generating optimal approximations of the hypotheses, which are analytically obtained from the abductive problem and are expected to be selected as optimal by a rational agent. We also spell out sufficient conditions for ensuring that this procedure satisfies conditions A2 and A3. In Section 4.3, we generalize the procedure in Section 4.2 to obtain any possible strengthening of the least compromising hypothesis. This generalization lays the groundwork for a logical treatment of abduction in the presence of new information. At the end of the chapter, we include a legend of the terminology we employ, in order to improve readability.

4.1. Producing the least compromising hypothesis

In what follows, by *abductive problem* we mean any expression of the form $\Gamma, \textcircled{?} \vdash G$, with $\Gamma \neq \varnothing$ and such that $\overline{\overline{\mathsf{G4}}}$ refutes $\Gamma \vdash G$. Accordingly, by *abductive algorithm* we refer to any effective procedure that, given in input an abductive problem $\Gamma, \textcircled{?} \vdash G$, returns an *abductive hypothesis* H such that $\Gamma, H \vdash G$ is provable in $\overline{\overline{\mathsf{G4}}}$. In this chapter, we adopt contexts as sets: $\Gamma, \Delta, \Pi, \Sigma, \ldots$ stand for finite sets of formulas, and $\Theta, \Lambda, \ldots$ for finite sets of atomic formulas.

In [32], the tableaux method is employed to design an elegant and effective abductive algorithm for producing what they call the *least compromising hypothesis.* We begin this section by proposing a sequent-based reading of the very same procedure. The switching from tableaux to sequents is here technically justified by the fact that sequent calculi facilitate the study of the structural properties of the algorithm. Due to the well-known duality between semantic tableaux *à la* Smullyan and Kleene's system $\mathsf{G4}$ [156], any result obtained for one system can be nonetheless imported in the other.

PROCEDURE 1 (Least Compromising Hypothesis). For any abductive problem $\Gamma, \textcircled{?} \vdash G$, the *least compromising hypothesis* $\mathsf{LCH}\big(\Gamma, \textcircled{?} \vdash G\big)$ is the formula resulting from the following steps:

(1) Decompose the antisequent $\Gamma \dashv G$ till the set of clauses $\mathsf{top_c}(\Gamma \dashv G) = \big\{\Theta_1 \dashv \Lambda_1, \dots, \Theta_n \dashv \Lambda_n\big\}$ is fully accomplished.

(2) For each clause $\Theta_i \dashv \Lambda_i \in \mathsf{top_c}(\Gamma \dashv G)$ consider the formula $C_i \equiv \bigwedge \Theta_i \to \bigvee \Lambda_i$.

(3) Finally set $\mathsf{LCH}\big(\Gamma, \textcircled{?} \vdash G\big) = C_1 \wedge \dots \wedge C_n$.

EXAMPLE 4.3. *We apply Procedure 1 to compute the formula* $\mathsf{LCH}(p \to q, p \vee q, \textcircled{?} \vdash r)$*:*

(1) By looking at the $\overline{\overline{\mathsf{G4}}}$*-proof reported in Example 2.1, we immediately get*

$$\mathsf{top_c}\big(p \to q, p \vee q, \textcircled{?} \vdash r\big) = \big\{q \dashv p, r \ ; \ p, q \dashv r \ ; \ q \dashv r\big\}$$

(2) Then we turn each clause into their corresponding formula:

$$\begin{aligned} q \dashv p, r &\quad\Rightarrow\quad q \to (p \vee r) \\ p, q \dashv r &\quad\Rightarrow\quad (p \wedge q) \to r \\ q \dashv r &\quad\Rightarrow\quad q \to r \end{aligned}$$

(3) We finally lead up to the compound formula:

$$\mathsf{LCH}(p \to q, p \vee q, \textcircled{?} \vdash r) = (q \to (p \vee r)) \wedge ((p \wedge q) \to r) \wedge (q \to r).$$

It is possible for the decomposition of the anti-sequent $\Gamma \dashv G$ to produce a set of complementary top-clauses $\Theta_1 \dashv \Lambda_1, \dots, \Theta_n \dashv \Lambda_n$ such that there exists one $\Theta_i \dashv \Lambda_i$ which is a classical consequence of $\Theta_{j_1} \dashv \Lambda_{j_1}, \dots, \Theta_{j_k} \dashv \Lambda_{j_k}$, with $1 \le i \ne j_1 \ne \dots \ne j_k \le n$. For example, consider the LCH-hypothesis of Example 4.3, and note that $q \to (p \vee r)$ and $(p \wedge q) \to r$ are classical consequences of $q \to r$, whereas $q \to r$ is a classical consequence of $q \to (p \vee r)$ and $(p \wedge q) \to r$. In general, it is reasonable to consider such a $\Theta_i \dashv \Lambda_i$ as redundant. Dropping $\bigwedge \Theta_i \to \bigvee \Lambda_i$ from the set of conjuncts of the LCH-hypothesis yields a logically equivalent formula, which is an optimized version of the former.

In [32], the authors demonstrate that one can generate an optimized version of the LCH-hypothesis by replacing Smullyan-style tableaux with KE-tableaux [31, 33], which are a dual presentation of a sequent calculus that does not enjoy admissibility of Cut [45]. In our sequent-based approach via $\mathsf{G4}$, eliminating redundant clauses can be achieved by utilizing the following rewriting rules:

$$\{\Theta \dashv \Lambda\} \cup \{\Theta', \Theta \dashv \Lambda, \Lambda'\} \to_w \{\Theta \dashv \Lambda\} \tag{4.1.1}$$

$$\bigcup_{i=1}^{n}\{\Theta_i \dashv \Lambda_i\} \cup \{\Theta'', \Theta'_1, \ldots, \Theta_j, \ldots, \Theta'_n \dashv \Lambda'_1, \ldots, \Lambda_k, \ldots, \Lambda'_n, \Lambda''\} \rightarrow_c \bigcup_{i=1}^{n}\{\Theta_i \dashv \Lambda_i\} \quad (4.1.2)$$

with $\Theta' \cup \Lambda' \neq \varnothing$, $n \geq 2$ and $1 \leq j \neq k \leq n$, provided that for any $1 \leq j' \neq j, k' \neq k, j' \neq k' \leq n$ and some Φ, Ψ we have that $\Theta'_{j'} = (\Phi \cup \Theta_{j'}) \setminus \{p\}$ and $\Lambda'_{k'} = (\Lambda_{k'} \cup \Psi) \setminus \{p\}$ for any $p \in (\Lambda_{k'} \cup \Psi) \cap (\Phi \cup \Theta_{j'})$.

The rationale for adopting these rewriting rules is that of avoiding cases in which $\overline{\overline{\mathsf{G4}}}$ derives at least one clause in $\mathcal{C}$ from other clauses in $\mathcal{C}$ either by applying (an invalidity-preserving version of) Weakening – as with the derivation of $\Theta', \Theta \dashv \Lambda, \Lambda'$ from $\Theta \dashv \Lambda$ -, or by applying in some order (invalidity-preserving versions of) Weakening and Cut - as with the derivation of $\Theta'', \Theta'_1, \ldots, \Theta_j, \ldots, \Theta'_n \dashv \Lambda'_1, \ldots, \Lambda_k, \ldots, \Lambda'_n, \Lambda''$ from $\bigcup_{i=1}^{n}\{\Theta_i \dashv \Lambda_i\}$.

For any set $\mathcal{C}$ of clauses, maximal application of the rewriting rules 4.1.1 – 4.1.2 to $\mathcal{C}$ yields a (not necessarily unique) subset $\mathcal{D}$ of clauses where all redundant clauses from $\mathcal{C}$ have been dropped *modulo* logical equivalence: we refer to $\mathcal{D}$ as a *reduct under Weakening and Cut* of $\mathcal{C}$ after [121].

We can thus refine step (1) of Procedure 1 by taking a reduct under Weakening and Cut of the set of top-clauses which results from the decomposition of the abductive problem. If we consider once more Example 4.3, this refinement forces us to consider two possible optimizations of the LCH-hypothesis $(q \rightarrow (p \vee r)) \wedge ((p \wedge q) \rightarrow r) \wedge (q \rightarrow r)$: in one case, we first apply rule (3), thus dropping $q \dashv p, r$ and $p, q \dashv r$ from $\mathsf{top_c}(p \rightarrow q, p \vee q, \textcircled{?} \dashv r)$ and getting $q \rightarrow r$ as optimized LCH-hypothesis; in the other case, we first apply rule (4), thus dropping $q \dashv r$ from $\mathsf{top_c}(p \rightarrow q, p \vee q, \textcircled{?} \vdash r)$ and getting $(q \rightarrow (p \vee r)) \wedge ((p \wedge q) \rightarrow r)$ as a distinct (but logically equivalent) optimized version of the LCH-hypothesis.
We can now turn to the proof of the first basic result about the LCH-hypothesis:

THEOREM 4.1. *For any (abductive) problem* $\Gamma, \textcircled{?} \vdash G$, $\overline{\overline{\mathsf{G4}}}$ *proves both* $\mathsf{LCH}(\Gamma \dashv G) \vdash \bigwedge \Gamma \rightarrow G$ *and* $\bigwedge \Gamma \rightarrow G \vdash \mathsf{LCH}(\Gamma \dashv G)$.

PROOF. Let $\mathsf{top_i}(\Gamma \dashv G) = \big\{\Theta_1 \vdash \Lambda_1, \ldots, \Theta_m \vdash \Lambda_m\big\}$ and $\mathsf{top_c}(\Gamma \dashv G) = \big\{\Theta_{m+1} \dashv \Lambda_{m+1}, \ldots, \Theta_{m+n} \dashv \Lambda_{m+n}\big\}$. It is a routine matter to verify that $\overline{\overline{\mathsf{G4}}}$ proves each of the following sequents:

$$\big(\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1\big) \wedge \cdots \wedge \big(\bigwedge \Theta_{m+n} \rightarrow \bigvee \Lambda_{m+n}\big) \vdash \bigwedge \Gamma \rightarrow G \quad (4.1.3)$$

$$\bigwedge \Gamma \rightarrow G \vdash \big(\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1\big) \wedge \cdots \wedge \big(\bigwedge \Theta_{m+n} \rightarrow \bigvee \Lambda_{m+n}\big) \quad (4.1.4)$$

$$\vdash \big(\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1\big) \wedge \cdots \wedge \big(\bigwedge \Theta_{m} \rightarrow \bigvee \Lambda_{m}\big) \quad (4.1.5)$$

The provability of sequents (5) and (6) is obvious, whereas the provability of (7) straightforwardly follows from the fact that each clause $\Theta_i \vdash \Lambda_i$, with $1 \leqslant i \leqslant m$, is tautological. By

$\wedge$-invertibility of $\mathsf{G4}$, provability of sequents (5) – (7) implies that the following sequents are provable:

$$(\bigwedge \Theta_1 \to \bigvee \Lambda_1), \dots, (\bigwedge \Theta_{m+n} \to \bigvee \Lambda_{m+n}) \vdash \bigwedge \Gamma \to G \tag{4.1.6}$$

$$\bigwedge \Gamma \to G \vdash \bigwedge \Theta_j \to \bigvee \Lambda_j \tag{4.1.7}$$

$$\vdash \bigwedge \Theta_1 \to \bigvee \Lambda_1, \dots, \bigwedge \Theta_m \to \bigvee \Lambda_m \tag{4.1.8}$$

with $1 \leq j \leq m+n$. By closure of $\mathsf{G4}$ under Cut, provability of (8) and (10) implies that the following sequent is provable:

$$(\bigwedge \Theta_{m+1} \to \bigvee \Lambda_{m+1}), \dots, (\bigwedge \Theta_{m+n} \to \bigvee \Lambda_{m+n}) \vdash \bigwedge \Gamma \to G \tag{4.1.9}$$

Provability of (9) and (11), together with the fact that

$$\mathsf{LCH}(\Gamma \dashv G) = (\bigwedge \Theta_{m+1} \to \bigvee \Lambda_{m+1}) \wedge \cdots \wedge (\bigwedge \Theta_{m+n} \to \bigvee \Lambda_{m+n})$$

yields the conclusion. ☐

We can now show that the LCH-hypothesis enjoys condition $\mathsf{A1}$ (cf. Lemma 3.1 and Theorem 3.1 in [32]):

Corollary 4.2. *For any problem* $\Gamma, \textcircled{?} \vdash G$*, the sequent* $\Gamma, \mathsf{LCH}(\Gamma \dashv G) \vdash G$ *is always provable in* $\overline{\overline{\mathsf{G4}}}$*.*

Proof. The claim is an immediate consequence of Theorem 4.1 and full invertibility of $\mathsf{G4}$. ☐

The previous result can be strengthened by showing that the LCH-abductive hypothesis turns out to be deductively minimal (*modulo* logical equivalence) with respect to the whole set of formulas obeying the condition $\mathsf{A1}$. It should now be clear why in the literature the resulting abductive hypothesis is classified as the "least compromising" one:

Theorem 4.3. *For any problem* $\Gamma, \textcircled{?} \vdash G$*, if* $\overline{\overline{\mathsf{G4}}}$ *proves* $\Gamma, A \overset{i}{\vdash} G$*, then it also proves* $A \overset{i}{\vdash} \mathsf{LCH}(\Gamma \dashv G)$*.*

Proof. If $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, A \vdash G$, then it proves $A \vdash \bigwedge \Gamma \to G$ as well: by Theorem 4.1 and closure of $\mathsf{G4}$ under Cut we get the desired conclusion. ☐

Minimality guarantees that if $\mathsf{LCH}(\Gamma \dashv G)$ does not satisfy $\mathsf{A2}$ and $\mathsf{A3}$, then no abductive hypothesis A can satisfy $\mathsf{A2}$ and $\mathsf{A3}$ either. Since we are interested in abductive hypotheses that comply with the complete set of *desiderata* $\mathsf{A1}$, $\mathsf{A2}$, and $\mathsf{A3}$, a natural question arises as to whether $\mathsf{LCH}(\Gamma \dashv G)$ always satisfies them simultaneously. Unfortunately, the answer is negative. For example, consider the problem $\neg p \vee \neg q, \textcircled{?} \vdash p \wedge q$. According to Procedure 1, we have that $\mathsf{LCH}(\neg p \vee \neg q, ? \vdash p \wedge q) = p \wedge q \wedge (p \vee q)$. The sequents $p \wedge q \wedge (p \vee q) \vdash p \wedge q$ and $\neg p \vee \neg q, p \wedge q \wedge (p \vee q) \vdash$ are both provable in $\overline{\overline{\mathsf{G4}}}$.

Upon closer examination, we can observe that the formula $\mathsf{LCH}(\Gamma \dashv G)$ satisfies conditions $\mathsf{A2}$ and $\mathsf{A3}$ in a limited number of cases characterized by the following result:

THEOREM 4.4. *For any problem* $\Gamma, \textcircled{?} \vdash G$*,* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\mathsf{LCH}(\Gamma \dashv G) \vdash G$ *and* $\Gamma, \mathsf{LCH}(\Gamma \dashv G) \vdash$ *just in case* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\neg G \vdash \bigwedge \Gamma$ *and* $\Gamma \vdash \neg G$*, respectively.*

PROOF. By Theorem 4.1 and $\mathsf{G4}$ being closed under Cut, $\mathsf{LCH}(\Gamma \dashv G)$ does not satisfy condition $\mathsf{A2}$ if and only if $\overline{\overline{\mathsf{G4}}}$ proves $\bigwedge \Gamma \to G \vdash G$, and $\mathsf{LCH}(\Gamma \dashv G)$ does not satisfy $\mathsf{A3}$ if and only if $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, \bigwedge \Gamma \to G \vdash$. We consider the two cases separately.

(*i*) If $\overline{\overline{\mathsf{G4}}}$ proves $\bigwedge \Gamma \to G \vdash G$, then $\overline{\overline{\mathsf{G4}}}$ proves $\vdash G, \bigwedge \Gamma$ by $\to$-invertibility of $\mathsf{G4}$, and then $\neg G \vdash \bigwedge \Gamma$ by one application of $\neg_{\mathfrak{L}}$. On the other hand, if $\overline{\overline{\mathsf{G4}}}$ proves $\neg G \vdash \bigwedge \Gamma$ then $\overline{\overline{\mathsf{G4}}}$ proves $\vdash G, \bigwedge \Gamma$ by $\neg$-invertibility of $\mathsf{G4}$, and then derives $\bigwedge \Gamma \to G \vdash G$ from $G \vdash G$ by one application of $\to_{\mathfrak{L}}$.

(*ii*) If $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, \bigwedge \Gamma \to G \vdash$, then $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, G \vdash$ by $\to$-invertibility of $\mathsf{G4}$, and then $\Gamma \vdash \neg G$ by one application of $\neg_{\mathfrak{R}}$. On the other hand, if $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma \vdash \neg G$ then $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, G \vdash$ by $\neg$-invertibility of $\mathsf{G4}$, and then derives $\Gamma, \bigwedge \Gamma \to G \vdash$ from $\Gamma \vdash \bigwedge \Gamma$ by one application of $\to_{\mathfrak{L}}$.

By contraposition, we obtain the desired conclusion.

□

From now on, we will call *explanans* any abductive hypothesis respecting conditions $\mathsf{A2}$ and $\mathsf{A3}$. Bearing in mind that B is deductively independent of A when $\overline{\overline{\mathsf{G4}}}$ refutes both the sequents $A \vdash B$ and $A, B \vdash$, we collect the following facts about any LCH-*explanans*:

PROPOSITION 4.1. For any problem $\Gamma, \textcircled{?} \vdash G$, if $\mathsf{LCH}(\Gamma \dashv G)$ is an explanans, then:

(*i*) $\bigwedge \Gamma$, G and $\mathsf{LCH}(\Gamma \dashv G)$ are all truth-functionally contingent;

(*ii*) $\mathsf{LCH}(\Gamma \dashv G)$ and $\bigwedge \Gamma$ turn out to be deductively independent of each other;

(*iii*) G is deductively independent of $\mathsf{LCH}(\Gamma \dashv G)$, but not viceversa.

PROOF. We prove each statement separately.

(*i*) If $\Gamma, \textcircled{?} \vdash G$ is an abductive problem, then $\bigwedge \Gamma$ is not contradictory and G not tautological. On the other hand, if $\mathsf{LCH}(\Gamma \dashv G)$ respects conditions $\mathsf{A2}$ and $\mathsf{A3}$ then $\mathsf{LCH}(\Gamma \dashv G)$ is not contradictory, and by Theorem 4.4 neither $\bigwedge \Gamma$ nor $\neg G$ can be tautological.

(*ii*) $\mathsf{LCH}(\Gamma \dashv G)$ is deductively independent of $\bigwedge \Gamma$: by contradiction, if $\overline{\overline{\mathsf{G4}}}$ proved $\bigwedge \Gamma \vdash \mathsf{LCH}(\Gamma \dashv G)$, then it would prove $\bigwedge \Gamma \vdash G$ by Corollary 4.2 and closure of $\mathsf{G4}$ under Cut; on the other hand, if $\overline{\overline{\mathsf{G4}}}$ proved $\bigwedge \Gamma \vdash \neg \mathsf{LCH}(\Gamma \dashv G)$, $\neg$-invertibility of $\overline{\overline{\mathsf{G4}}}$ would guarantee that $\bigwedge \Gamma, \mathsf{LCH}(\Gamma \dashv G)$ is provable – contrary to condition $\mathsf{A3}$.

The fact that $\bigwedge\Gamma$ is deductive independent of $\mathsf{LCH}(\Gamma \dashv G)$ can be proved by an analogous argument.

(*iii*) G is deductively independent of $\mathsf{LCH}(\Gamma \dashv G)$: if $\overline{\overline{\overline{\mathsf{G4}}}}$ proved $\mathsf{LCH}(\Gamma \dashv G), G \vdash$, then it would prove $\Gamma, \mathsf{LCH}(\Gamma \dashv G) \vdash$ by Corollary 4.2 and closure of $\mathsf{G4}$ under Cut – contrary to condition $\mathsf{A3}$; if $\overline{\overline{\overline{\mathsf{G4}}}}$ proved $\mathsf{LCH}(\Gamma \dashv G) \vdash G$, then condition $\mathsf{A2}$ would be violated. On the other hand, it suffices to notice that $\overline{\overline{\overline{\mathsf{G4}}}}$ proves $G \vdash \bigwedge\Gamma \to G$ to conclude, by Theorem 4.1 and closure of $\mathsf{G4}$ under Cut, that $\mathsf{LCH}(\Gamma \dashv G)$ is not deductively independent of G. ☐

As a result, (*i*) and (*ii*) of Proposition 4.1 jointly state that a rational agent uses an LCH-*explanans* only if she uses LCH to lower the number of (contingent) facts independent from a (contingent) theoretical background: according to the terminology of [3], a rational agent uses LCH as an *explanans* only if she uses it to reduce the number of *novelties* w.r.t. the theoretical background. On the other hand, (*iii*) of Proposition 4.1 shows that the minimal *explanans* LCH enjoys maximal *evidential support*, meaning that if the *explanandum* is true, then the LCH-*explanans* cannot fail to be true (cf. [39], p. 45).

We conclude this section by noticing that the LCH-abductive hypothesis is context-sensitive, that is to say the addition of premises in the theoretical background may alter the deductive strength of the LCH-abductive hypothesis:

PROPOSITION 4.2. For any distinct problems $\Gamma, \textcircled{?} \vdash G$ and $\Gamma', \Gamma, \textcircled{?} \vdash G$,

(*i*) $\overline{\overline{\overline{\mathsf{G4}}}}$ proves $\mathsf{LCH}(\Gamma \dashv G) \vdash \mathsf{LCH}(\Gamma', \Gamma \dashv G)$;

(*ii*) $\overline{\overline{\overline{\mathsf{G4}}}}$ refutes $\mathsf{LCH}(\Gamma', \Gamma \dashv G) \vdash \mathsf{LCH}(\Gamma \dashv G)$ if and only if $\overline{\overline{\overline{\mathsf{G4}}}}$ refutes $\Gamma, \neg A \vdash G$ for at least one formula $A \in \Gamma'$.

PROOF. For (1) it suffices to consider that $\overline{\overline{\overline{\mathsf{G4}}}}$ proves $\bigwedge\Gamma \to G \vdash \big(\bigwedge\Gamma' \wedge \bigwedge\Gamma\big) \to G$ and exploit Theorem 4.1. As to (2), we consider the two directions separately.

(*i*) If $\overline{\overline{\overline{\mathsf{G4}}}}$ proves $\mathsf{LCH}(\Gamma', \Gamma \dashv G) \vdash \mathsf{LCH}(\Gamma \dashv G)$, then it proves $(\bigwedge\Gamma' \wedge \bigwedge\Gamma) \to G \vdash \bigwedge\Gamma \to G$ by Theorem 4.1 and closure of $\mathsf{G4}$ under Cut. As a result, $\overline{\overline{\overline{\mathsf{G4}}}}$ proves $\bigwedge\Gamma \vdash G, A$ for any $A \in \Gamma'$ by full invertibility of $\mathsf{G4}$: by one application of $\neg_{\mathfrak{L}}$ we get the result.

(*ii*) If $\overline{\overline{\overline{\mathsf{G4}}}}$ proves $\bigwedge\Gamma, \neg A \vdash G$ for any $A \in \Gamma'$, then it proves $\bigwedge\Gamma \vdash G, A$ by $\neg$-invertibility of $\mathsf{G4}$ and thus $(\bigwedge\Gamma' \wedge \bigwedge\Gamma) \to G \vdash \bigwedge\Gamma \to G$ by applications of $\wedge_{\mathfrak{R}}, \to_{\mathfrak{L}}$ and $\to_{\mathfrak{R}}$: as a result, $\overline{\overline{\overline{\mathsf{G4}}}}$ proves $\mathsf{LCH}(\Gamma', \Gamma \dashv G) \vdash \mathsf{LCH}(\Gamma \dashv G)$ by Theorem 4.1 and closure of $\mathsf{G4}$ under Cut.

By contraposition, we get the conclusion. ☐

	ABDUCTIVE PROBLEM	MINIMAL HYPOTHESIS	EXPECTED HYPOTHESIS
1	$p \to q, \textcircled{?} \vdash q$	$q \vee p$	p
2	$p \to q, \textcircled{?} \vdash p \to r$	$(q \wedge p) \to r$	$q \to r$
3	$p \to q, t \to r, p, \textcircled{?} \vdash r$	$(p \wedge q) \to (t \vee r)$	$q \to t$
4	$(p \vee q) \to (r \wedge s) \vdash s$	$s \vee p \vee q$	$p \vee q$
5	$\neg q \to \neg p, \textcircled{?} \vdash q$	$p \vee q$	p
6	$p, \textcircled{?} \vdash p \wedge q$	$p \to q$	q
7	$p, t, \textcircled{?} \vdash (p \wedge q) \wedge (t \wedge r)$	$((p \wedge t) \to q) \wedge ((p \wedge t) \to r)$	$q \wedge r$
8	$p \vee q, p \to r, \textcircled{?} \vdash r$	$q \to (p \vee r)$	$\neg q$
9	$p \vee q, \neg p \vdash r$	$q \to (p \vee r)$	$q \to r$
10	$p \to (q \wedge t), \textcircled{?} \vdash q$	$p \vee q$	p
11	$(p \wedge q) \to (r \vee s), p, \textcircled{?} \vdash s$	$((r \wedge p) \to s) \wedge (p \to (s \vee q))$	$q \wedge \neg r$
12	$p \vee q, \textcircled{?} \vdash p \vee t$	$q \to (p \vee t)$	$q \to t$
13	$p \to q, r \to s, \textcircled{?} \vdash q \vee s$	$q \vee s \vee p \vee r$	$p \vee r$
14	$p \to q, p \vee r, \textcircled{?} \vdash q \vee s$	$r \to (p \vee q \vee \vee s)$	$r \to s$
15	$p \to q, r \to s, \textcircled{?} \vdash \neg p \vee \neg r$	$\neg p \vee \neg r \vee \neg q \vee \neg s$	$\neg q \vee \neg s$
16	$p \to q, \neg q \vee \neg s, \textcircled{?} \vdash \neg p \vee \neg r$	$(p \wedge q \wedge r) \to s$	$r \to s$
17	$(p \to r) \wedge (q \to r), \textcircled{?} \vdash r$	$r \vee p \vee q$	$p \vee q$
18	$(p \vee q) \to (p \vee t), \textcircled{?} \vdash t$	$(p \vee t \vee q) \wedge (t \vee \neg p)$	$q \wedge \neg p$
19	$p \to q, \textcircled{?} \vdash p \to (q \wedge r)$	$(p \wedge q) \to r$	$p \to r$
20	$p \to q, \textcircled{?} \vdash q \wedge r$	$(q \vee p) \wedge (q \to r) \vee (r \vee p)$	$p \wedge r$

FIGURE 1. Examples of minimal and expected solutions

4.2. Deductive minimality and expected explanation

It is easy to find problems in which a rational agent's preferred abductive hypothesis does not match the minimum deductive hypothesis. Some of these problems are illustrated in Figure 1, in addition to those presented in the Introduction . In all these cases, the expected hypothesis satisfies conditions A1 – A3 and is obtained by dropping some atomic pieces of information from the least compromising hypothesis.

For the sake of optimality, it is plausible to assume that deleted atoms correspond to *redundant* information – information in the abductive problem that the rational agent treats as irrelevant against deductive saturation. Specifically, the rational agent seems to implicitly treat as irrelevant some atomic pieces of information, whether in the theoretical background or in the goal formula, that perform partial deductive saturation even before making an abductive inference. The results presented in this section explore this intuition.

Let us begin with some terminology. For any (anti)sequent $\Gamma \overset{*}{\vdash} \Delta$, if $\mathsf{top}(\Gamma \overset{*}{\vdash} \Delta) = \{\Theta_1 \overset{*}{\vdash} \Lambda_1, \ldots, \Theta_n \overset{*}{\vdash} \Lambda_n\}$ then we have that

(*a*) $\mathsf{AT}(\Gamma \overset{*}{\vdash} \Delta) = \bigcup_{i=1}^{n} (\Theta_i \cup \Lambda_i)$;

(*b*) $\mathsf{ID}(\Gamma \overset{*}{\vdash} \Delta) = \bigcup_{i=1}^{n} (\Theta_i \cap \Lambda_i)$;

(*c*) $\mathsf{CUT}(\Gamma \overset{*}{\vdash} \Delta) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (\Lambda_i \cap \Theta_j)$;

(*d*) $\mathsf{AT}(A) = \mathsf{AT}(\overset{*}{\vdash} A) = \mathsf{AT}(A \overset{*}{\vdash})$.

Example 4.4. *Take the* $\overline{\overline{\mathsf{G4}}}$ *derivation of* $p \to q, p \vee q \dashv r$ *in Example 2.1. We have that*

(*i*) $\mathsf{AT}(p \to q, p \vee q \dashv r) = \{p, q, r\}$*;*

(*ii*) $\mathsf{ID}(p \to q, p \vee q \dashv r) = \{p\}$*;*

(*iii*) $\mathsf{CUT}(p \to q, p \vee q \dashv r) = \{p\}$*.*

For any problem $\Gamma, \textcircled{?} \vdash G$, we say that an atom $p \in \mathsf{AT}(\Gamma \dashv G)$ is *abductively redundant* if $p \in \mathsf{ID}(\Gamma \dashv G)$. In other words, an atom is abductively redundant when 'trivializes' a clause in the decomposition of the abductive problem. Intuitively, atomic sentences of this kind correspond to pieces of information which are trivially contained in the theoretical background, or trivially contained in the goal formula, or shared between theoretical background and goal formula.

If we revise $\mathsf{LCH}(\Gamma \dashv G)$ by erasing atoms in $S \subseteq \mathsf{ID}(\Gamma \dashv G)$, we can partially eliminate redundant information. According to the following proposition, a rational agent who eliminates all abductive redundant information also drops *all* the information contained in intermediate steps possibly used to "saturate" the abductive problem *via* the deductively minimal *explanans*:

Proposition 4.3. For any problem $\Gamma, \textcircled{?} \vdash G$ such that $\mathsf{LCH}(\Gamma \dashv G)$ is an *explanans*, if $p \in \mathsf{ID}(\Gamma \dashv G)$ then $p \in \mathsf{CUT}(\bigwedge \Gamma \to G \dashv)$.

Proof. Notice that $\mathsf{top}(\Gamma \dashv G) = \mathsf{top}(\dashv \bigwedge \Gamma \to G)$ due to $\to$-invertibility of $\mathsf{G4}$. Moreover, if $\mathsf{LCH}(\Gamma \dashv G)$ is an *explanans*, then $\overline{\overline{\mathsf{G4}}}$ refutes $\bigwedge \Gamma \to G \vdash$ by using Proposition 4.1 and Theorem 4.1. Based on this, we can prove a stronger statement than the one above. Namely, for any formula A, $p \in \mathsf{ID}(\dashv A)$ only if $p \in \mathsf{CUT}(A \dashv)$.

To prove this, we must perform an intermediate step. For any $\Theta \overset{*}{\vdash} \Lambda \in \mathsf{top}(\dashv A)$, suppose $\Theta = p_1, \ldots, p_m$ and $\Lambda = p_{m+1}, \ldots, p_{m+n}$ with $m, n \geq 0$ and $m + n > 0$. For any clause $\Phi \overset{*}{\vdash} \Psi \in \mathsf{top}(A \dashv)$, there is precisely one $p_h \in (\Phi \cup \Psi)$ such that $p_h \in \Psi$ if $1 \leq h \leq m$ and $p_h \in \Phi$ if $m + 1 \leq h \leq m + n$. Furthermore, for any two distinct $\Phi \overset{*}{\vdash} \Psi$ and $\Phi' \overset{*}{\vdash} \Psi' \in \mathsf{top}(A \dashv)$, there are at least two atoms p_h and $p_{h'}$ such that $p_h \in (\Phi \cup \Psi)$, $p_{h'} \in (\Phi' \cup \Psi')$, $p_{h'} \in \Psi'$ if $1 \leq h' \leq m$ and $p_{h'} \in \Phi'$ if $m + 1 \leq h' \leq m + n$, and $p_h \neq p_{h'}$.

We reason by (course-of-value) induction over the number $k \geq 0$ of connectives in A. If $k = 0$, the result is trivial. If $k = j + 1$ with $j \geq 0$, then it suffices to consider two cases.

(*i*) A is of form $\neg B$: since $\mathsf{top}(\neg B \dashv) = \mathsf{top}(\dashv B)$ and $\mathsf{top}(\dashv \neg B) = \mathsf{top}(B \dashv)$ by $\neg$-invertibility of $\mathsf{G4}$, it suffices to apply the inductive hypothesis for $k < j$.

(*ii*) A is of form $B \wedge C$: since $\mathsf{top}(B \wedge C \dashv) = \mathsf{top}(B, C \dashv)$ and $\mathsf{top}(\dashv B \wedge C) = \mathsf{top}(\dashv B) \cup \mathsf{top}(\dashv C)$ by $\wedge$-invertibility of $\mathsf{G4}$, it suffices to apply twice the inductive hypothesis – with j being $j_1 + j_2$, j_1 being the number of connectives in B and j_2 the number of connectives in C.

It can be now proved that $p \in \mathsf{ID}(\dashv A)$ only if $p \in \mathsf{CUT}(A \dashv)$ (we omit the details). Notice that $\mathsf{CUT}(\mathsf{LCH}(\Gamma \dashv G) \dashv) \subseteq \mathsf{CUT}(\bigwedge \Gamma \to G \dashv)$ by Theorem 4.1. ☐

For any problem $\Gamma, \textcircled{?} \vdash G$, the elimination of redundant information generates formulas according to the following procedure:

PROCEDURE 2 (Approximation to an expected hypothesis). For any problem $\Gamma, \textcircled{?} \vdash G$ and any subset S of $\mathsf{ID}(\Gamma \dashv G)$, the S-approximation to an expected hypothesis $\mathsf{EH}_S\big(\Gamma, \textcircled{?} \vdash G\big)$ is the formula obtained according to the following steps:

(1) Decompose the antisequent $\Gamma \dashv G$ till the set of clauses $\mathsf{top_c}(\Gamma \dashv G) = \big\{\Theta_1 \dashv \Lambda_1, \dots, \Theta_n \dashv \Lambda_n\big\}$ is fully accomplished.

(2) For each clause $\Theta_i \dashv \Lambda_i \in \mathsf{top_c}(\Gamma \dashv G)$ take the largest clause $\Theta'_i \dashv \Lambda'_i$ such that $\Theta'_i \subseteq \Theta_i, \Lambda'_i \subseteq \Lambda_i$ and $\Theta'_i \cap S = \Lambda'_i \cap S = \varnothing$.

(3) For each clause $\Theta'_i \dashv \Lambda'_i$ thus obtained consider the formula $C_i \equiv \bigwedge \Theta'_i \to \bigvee \Lambda'_i$.

(4) Finally set $\mathsf{EH}_S\big(\Gamma, \textcircled{?} \vdash G\big) = C_1 \wedge \cdots \wedge C_n$ (avoiding repetition of conjuncts).

Notice that, for any problem $\Gamma, \textcircled{?} \vdash G$, if $|\mathsf{ID}(\Gamma \dashv G)| = k$, then there are (at most) 2^k EH_S-hypotheses.

Remark that an $\mathsf{EH}_S(\Gamma \dashv G)$-hypothesis is just the $\mathsf{LCH}(\Gamma \dashv G)$-hypothesis whenever either $S = \varnothing$ or $\mathsf{AT}(\mathsf{LCH}(\Gamma \dashv G)) \cap S = \varnothing$.

EXAMPLE 4.5. *We apply Procedure 2 to compute the formula* $\mathsf{EH}_S\big((p \wedge q) \vee (r \wedge s), \textcircled{?} \vdash p \wedge r\big)$*, for any* $S \subseteq \mathsf{ID}\big((p \wedge q) \vee (r \wedge s), \textcircled{?} \vdash p \wedge r\big)$*:*

(1) By performing decomposition we get

$$\mathsf{top_c}\big((p \wedge q) \vee (r \wedge s), \textcircled{?} \vdash p \wedge r\big) = \big\{r, s \dashv p \ ; \ p, q \dashv r\big\}$$

and

$$\mathsf{ID}\big((p \wedge q) \vee (r \wedge s), \textcircled{?} \vdash p \wedge r\big) = \{p, r\}$$

(2) For any $S \subseteq \{p, r\}$ *delete all occurrences of atoms in* S *from clauses* $r, s \dashv p$ *and* $p, q \dashv r$*, and take the formula translations of the resulting clauses.*

$$\begin{tikzcd}
 & \neg s \wedge \neg q & \\
(\neg r \vee \neg s) \wedge (q \to r) \arrow[ur] & & (s \to p) \wedge (\neg p \vee \neg q) \arrow[ul] \\
 & \mathsf{LCH} = ((r \wedge s) \to p) \wedge ((p \wedge q) \to r) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

FIGURE 2. Poset of EH_S-abductive hypotheses for $(p \wedge q) \vee (r \wedge s) \dashv p \wedge r$

(3) Finally, we obtain the following set of EH_S*-abductive hypotheses:*

$$((r \wedge s) \to p) \wedge ((p \wedge q) \to r)$$

$$(\neg r \vee \neg s) \wedge (q \to r)$$

$$(s \to p) \wedge (\neg p \vee \neg q)$$

$$\neg s \wedge \neg q$$

We can refine Procedure 2 by taking a reduct under Weakening and Cut of the set of clauses resulting from step (3).

Let us define a partial order $\leq$ over the set of EH_S-hypotheses such that, for any $S, T \subseteq \mathsf{ID}(\Gamma \dashv G)$, $\mathsf{EH}_S(\Gamma \dashv G) \leq \mathsf{EH}_T(\Gamma \dashv G)$ if and only if $S \subseteq T$ (see Figure 2). It is easy to prove that $\leq$ is monotonic w.r.t. deductive strength:

THEOREM 4.5. *For any problem* $\Gamma, \textcircled{?} \vdash G$ *and any* $S, T \subseteq \mathsf{ID}(\Gamma \dashv G)$, *if* $\mathsf{EH}_S(\Gamma \dashv G) \leq \mathsf{EH}_T(\Gamma \dashv G)$ *then* $\overline{\overline{\overline{\mathsf{G4}}}}$ *proves* $\mathsf{EH}_T(\Gamma \dashv G) \vdash \mathsf{EH}_S(\Gamma \dashv G)$.

PROOF. By construction, $\mathsf{EH}_T(\Gamma \dashv G)$ is of the following form

$$(\bigwedge \Theta_1 \to \bigvee \Lambda_1) \wedge \cdots \wedge (\bigwedge \Theta_n \to \bigvee \Lambda_n) \tag{4.2.1}$$

On the other hand, $\mathsf{EH}_S(\Gamma \dashv G)$ has by construction the following form

$$((\bigwedge \Theta_1 \wedge \bigwedge \Theta_1') \to (\bigvee \Lambda_1 \vee \bigvee \Lambda_1') \wedge \cdots \wedge (\bigwedge \Theta_n \wedge \bigwedge \Theta_n') \to (\bigvee \Lambda_n \vee \bigvee \Lambda_n')) \tag{4.2.2}$$

with $\Theta_i', \Lambda_i' \subseteq \big(T \setminus S\big)$ for any $1 \leq i \leq n$. By full invertibility of $\mathsf{G4}$ we have that $\overline{\overline{\overline{\mathsf{G4}}}}$ proves

$$(\bigwedge \Theta_i \to \bigvee \Lambda_i) \vdash (\bigwedge \Theta_i \wedge \bigwedge \Theta_i') \to (\bigvee \Lambda_i \vee \bigvee \Lambda_i') \tag{4.2.3}$$

for any $1 \leq i \leq n$. Provability of (14), together with the fact that $\mathsf{EH}_T(\Gamma \dashv G)$ and $\mathsf{EH}_S(\Gamma \dashv G)$ have the form displayed by (12) and (13), respectively, implies that $\overline{\overline{\mathsf{G4}}}$ proves $\mathsf{EH}_T(\Gamma \dashv G) \vdash \mathsf{EH}_S(\Gamma \dashv G)$ by $n(n-1)$ applications of Left Weakening, $n(n-1)$ applications of $\wedge_{\mathfrak{L}}$ and $n-1$ applications of $\wedge_{\mathfrak{R}}$. ☐

Corollary 4.6. *For any problem* $\Gamma, ⓡ \vdash G$ *and any* $S \subseteq \mathsf{ID}(\Gamma \dashv G)$*,* $\overline{\overline{\mathsf{G4}}}$ *proves* $\mathsf{EH}_S(\Gamma \dashv G) \vdash \mathsf{LCH}(\Gamma \dashv G)$.

We can now show that any formula obtained according to Procedure 2 satisfies condition A1, and is thus an abductive hypothesis:

Corollary 4.7. *For any problem* $\Gamma, ⓡ \vdash G$ *and any* $S \subseteq \mathsf{ID}(\Gamma \dashv G)$*,* $\mathsf{EH}_S(\Gamma \dashv G)$ *satisfies condition* A1*.*

PROOF. Since $\mathsf{LCH}(\Gamma \dashv G)$ is always such that $\Gamma, \mathsf{LCH}(\Gamma \dashv G) \vdash G$ by Corollary 4.2, it suffices to exploit Corollary 4.6 and closure under Cut of G4 to get the result. ☐

If the elimination of redundant information from $\mathsf{top_c}(\mathsf{LCH}(\Gamma \dashv G)) = \{\Theta_1 \dashv \Lambda_1, \dots, \Theta_n \dashv \Lambda_n\}$ is non-vacuous, then an EH_S-abductive hypothesis may not be logically equivalent to the deductively minimal hypothesis:

PROPOSITION 4.4. For any problem $\Gamma, ⓡ \vdash G$ and any $S \subseteq \mathsf{ID}(\Gamma \dashv G)$, if $\mathsf{LCH}(\Gamma \dashv G)$ is an *explanans* then $\overline{\overline{\mathsf{G4}}}$ refutes $\mathsf{LCH}(\Gamma \dashv G) \vdash \mathsf{EH}_S(\Gamma \dashv G)$ if and only if one of the following holds:

- (i) $\mathsf{EH}_S(\Gamma \dashv G)$ is contradictory;
- (ii) if $\Theta_i \dashv \Lambda_i$, $\Theta_j \dashv \Lambda_j$ and $\Theta_k \dashv \Lambda_k \in \mathsf{top_c}(\dashv \mathsf{LCH}(\Gamma \dashv G))$, with $1 \leq i, j, k \leq n$, then
  - (a) for any i such that either $\Theta'_i = \Theta_i \setminus S$ or $\Lambda'_i = \Lambda_i \setminus S$ is non empty, and any $j \neq i$, either there is one non empty $\Theta'_j \subseteq \Theta_j$ such that $\Theta'_j \cap \Theta'_i = \varnothing$, or there is one non empty $\Lambda'_j \subseteq \Lambda_j$ such that $\Lambda'_j \cap \Lambda'_i = \varnothing$;
  - (b) for any j there is (at least) an atom p such that either $p \in \Lambda'_j$ and $p \notin \Theta'_k$, or $p \in \Theta'_j$ and $p \notin \Lambda'_k$ – for any $k \neq j$.

PROOF. Notice that $\mathsf{LCH}(\Gamma \dashv G)$ being an *explanans* implies that $\overline{\overline{\mathsf{G4}}}$ refutes $\mathsf{LCH}(\Gamma \dashv G) \vdash$, by Proposition 4.1. We separately prove the two directions of the biconditional.

- (i) If $\overline{\overline{\mathsf{G4}}}$ refutes $\mathsf{LCH}(\Gamma \dashv G) \vdash \mathsf{EH}_S(\Gamma \dashv G)$, then there must be (at least) one $\Theta_i \dashv \Lambda_i \in \mathsf{top_c}(\dashv \mathsf{LCH}(\Gamma \dashv G))$ such that $(\Theta_i \cup \Lambda_i) \cap S \neq \varnothing$. Suppose by contradiction that $\mathsf{EH}_S(\Gamma \dashv G)$ is not contradictory and one of the following two holds:

(a) there is (at least) one distinct $\Theta_j \dashv \Lambda_j \in \mathsf{top_c}(\dashv \mathsf{LCH}(\Gamma \dashv G))$ such that, if $\Theta'_i \neq \varnothing$, then for any non empty $\Theta'_j \subseteq \Theta_j$ we have that $\Theta'_j \cap \Theta'_i \neq \varnothing$, and, if $\Lambda'_i \neq \varnothing$, then for any non empty $\Lambda'_j \subseteq \Lambda_j$ we have that $\Lambda'_j \cap \Lambda'_i$;

(b) there is (at least) one $\Theta_j \dashv \Lambda_j \in \mathsf{top_c}(\dashv \mathsf{LCH}(\Gamma \dashv G))$ such that, for any atom p, if $p \in \Lambda'_j$, then $p \in \Theta'_k$ for (at least) one $k \neq j$ – and, if $p \in \Theta'_j$, then $p \in \Lambda'_k$ for (at least) one $k \neq j$.

Any $\Phi \mathrel{|\!\overset{*}{-}} \Psi \in \mathsf{top}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ results from the selection of one (not necessarily distinct) atom for any $\Theta \mathrel{|\!\overset{*}{-}} \Lambda \in \mathsf{top}(\dashv \mathsf{LCH}(\Gamma \dashv G))$ with permutation of side (cf. the proof of Proposition 4.3). As a consequence, if (a) is the case then for any $\Phi \mathrel{|\!\overset{*}{-}} \Psi \in \mathsf{top}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ we have that either $\Phi \cap \Lambda'_i \neq \varnothing$ or $\Psi \cap \Theta'_i \neq \varnothing$ and thus that $\mathsf{LCH}(\Gamma \dashv G) \vdash (\bigwedge \Theta'_i \to \bigvee \Lambda'_i)$ is provable – a contradiction. On the other hand, if (b) is the case then for any $\Phi \mathrel{|\!\overset{*}{-}} \Psi \in \mathsf{top}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ such that $\Phi \cap \Lambda'_i = \varnothing$ and $\Psi \cap \Theta'_i = \varnothing$ we have that $\overline{\overline{\mathsf{G4}}}$ proves $\Phi \vdash \Psi$ – again, a contradiction.

(ii) If $\mathsf{EH}_S(\Gamma \dashv G)$ is contradictory, then $\overline{\overline{\mathsf{G4}}}$ proves $\mathsf{LCH}(\Gamma \dashv G) \vdash \mathsf{EH}_S(\Gamma \dashv G)$ only if the sequent $\mathsf{LCH}(\Gamma \dashv G) \vdash$ is provable – a contradiction. On the other hand, suppose that there is (at least) one $\Theta_i \dashv \Lambda_i \in \mathsf{top_c}(\dashv \mathsf{LCH}(\Gamma \dashv G))$ such that $(\Theta_i \cup \Lambda_i) \cap S \neq \varnothing$ and, for any distinct $\Theta_j \dashv \Lambda_j \in \mathsf{top_c}(\dashv \mathsf{LCH}(\Gamma \dashv G))$, there is either a non empty $\Theta'_j \subseteq \Theta_j$ such that $\Theta'_j \cap \Theta'_i = \varnothing$ or a non empty $\Lambda'_j \subseteq \Lambda_j$ such that $\Lambda'_j \cap \Lambda'_i = \varnothing$, with $\Theta'_i = (\Theta_i \setminus S)$ and $\Lambda'_i = (\Lambda_i \setminus S)$. This means that there is (at least) one $\Phi \mathrel{|\!\overset{*}{-}} \Psi \in \mathsf{top}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ such that $\Phi \cap \Lambda'_i = \varnothing$ and $\Psi \cap \Theta'_i = \varnothing$. If for any j there is (at least) an atom p such that either $p \in \Lambda'_j$ and $p \notin \Theta'_k$ for any $k \neq j$, or $p \in \Theta'_j$ and $p \notin \Lambda'_k$ for any $k \neq j$, then one can always pick a $\Phi \mathrel{|\!\overset{*}{-}} \Psi \in \mathsf{top}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ such that $\Phi \cap \Psi = \varnothing$: as a result, $\overline{\overline{\mathsf{G4}}}$ refutes $\mathsf{LCH}(\Gamma \dashv G) \vdash (\bigwedge \Theta'_i \to \bigvee \Lambda'_i)$ – as desired.

$\square$

Given the set $\mathcal{F}$ of all formulas, for any set S of atomic sentences we use $\mathcal{F} \ominus S$ to denote the largest set of formulas in which no atom from S occurs – more formally, $\mathcal{F} \ominus S = \{A \in \mathcal{F} \mid \mathsf{AT}(A) \cap S = \varnothing\}$. We can show that any non-contradictory EH_S-hypothesis is deductively minimal w.r.t. abductive hypotheses in $\mathcal{F} \ominus S$:

Proposition 4.5. For any problem $\Gamma, \textcircled{?} \vdash G$ and any $S \subseteq \mathsf{ID}(\Gamma \dashv G)$, if A is any abductive hypothesis such that $A \in \mathcal{F} \ominus S$ and $\mathsf{EH}_S(\Gamma \dashv G)$ is not contradictory, then $\overline{\overline{\mathsf{G4}}}$ proves $A \vdash \mathsf{EH}_S(\Gamma \dashv G)$.

Proof. Since $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, A \vdash G$, if $\mathsf{top_c}(\Gamma \dashv G) = \{\Theta_1 \dashv \Lambda_1, \dots, \Theta_m \dashv \Lambda_m\}$ then each sequent $\Theta_i, A \vdash \Lambda_i$, with $1 \leq i \leq m$, is provable. On the other hand, if $\mathsf{top}(A \dashv) = \{\Theta'_1 \mathrel{|\!\overset{*}{-}} \Lambda'_1, \dots, \Theta'_n \mathrel{|\!\overset{*}{-}} \Lambda'_n\}$, then each sequent $\Theta'_j, \Theta_i \vdash \Lambda_i, \Lambda'_j$, with $1 \leq j \leq n$, is provable.

If $\mathsf{EH}_S(\Gamma \dashv G) = \big(\bigwedge \Phi_1 \to \bigvee \Psi_1\big) \wedge \cdots \wedge \big(\bigwedge \Phi_m \to \bigvee \Psi_m\big)$, with $\Phi_i = \big(\Theta_i \setminus S\big)$ and $\Psi_i = \big(\Lambda_i \setminus S\big)$, we can prove that $\overline{\overline{\mathsf{G4}}}$ proves $\Phi_i, \Theta'_j \vdash \Lambda'_j, \Psi_i$: we just reason by cases over $\Theta'_j \models^{*} \Lambda'_j$.

(i) If $\overline{\overline{\mathsf{G4}}}$ proves $\Theta'_j \vdash \Lambda'_j$, then it proves $\Phi_i, \Theta'_j \vdash \Lambda'_j, \Psi_i$ by $\mathsf{G4}$ being closed under Weakening.

(ii) If $\overline{\overline{\mathsf{G4}}}$ refutes $\Theta'_j \vdash \Lambda'_j$, then $\Theta'_j \cap \Lambda'_j = \varnothing$: since $\Theta'_j, \Theta_i \vdash \Lambda_i, \Lambda'_j$ is provable, we have that either $\Theta'_j \cap \Lambda_i \neq \varnothing$ or $\Theta_i \cap \Lambda'_j \neq \varnothing$. The fact that $A \in \mathcal{F} \ominus S$ implies that $\Theta'_j \cap S = \Lambda'_j \cap S = \varnothing$: since $\mathsf{EH}_S(\Gamma \dashv G)$ being non-contradictory is sufficient to guarantee that either $\Phi_i \neq \varnothing$ or $\Psi_i \neq \varnothing$, then we have that either $(\Theta'_j \cap \Lambda_i) \subseteq \Psi_i$ or $\big(\Theta_i \cap \Lambda'_j\big) \subseteq \Phi_i$. This implies that either $\Psi_i \cap \Theta'_j \neq \varnothing$ or $\Phi_i \cap \Lambda'_j \neq \varnothing$ – which is enough to conclude that $\overline{\overline{\mathsf{G4}}}$ proves $\Phi_i, \Theta'_j \vdash \Lambda'_j, \Psi_i$.

If $\overline{\overline{\mathsf{G4}}}$ proves $\Phi_i, \Theta'_j \vdash \Lambda'_j, \Psi_i$, with $1 \leq i \leq m$ and $1 \leq j \leq n$, then it proves each sequent $\Phi_i, \bigwedge \Theta'_j \vdash \bigvee \Lambda'_j, \Psi_i$, and, by m applications of $\to_{\mathcal{R}}$ and $m-1$ applications of $\wedge_{\mathcal{R}}$, each sequent $\Theta'_j \vdash \Lambda'_j, \mathsf{EH}_S(\Gamma \dashv G)$. As an immediate consequence, we have that $\overline{\overline{\mathsf{G4}}}$ proves $A \vdash \mathsf{EH}_S(\Gamma \dashv G)$. $\square$

The following example illustrates that even when we restrict ourselves to abductive problems where the LCH-hypothesis serves as an *explanans*, there is no guarantee that the EH_S-hypothesis is also an *explanans*, for some set of atoms $S \neq \varnothing$.

EXAMPLE 4.6. *Consider the problem* $(p \wedge \neg q) \to r, q \to \neg r, \textcircled{?} \vdash r$*: if* $S = \{q, r\}$*, Procedure 2 yields* $\top \to \bot$ *as an optimized version of* $\mathsf{EH}_S((p \wedge \neg q) \to r, q \to \neg r, \textcircled{?} \vdash r)$ *- and the sequents* $\top \to \bot \vdash r$ *and* $(p \wedge \neg q) \to r, q \to \neg r, (\top \to \bot) \vdash$ *are clearly provable in* $\overline{\overline{\mathsf{G4}}}$.

Once more, closer examination shows that the EH_S-hypothesis satisfies conditions $\mathsf{A2}$ and $\mathsf{A3}$ in a restricted number of cases, which is characterized by the following result:

THEOREM 4.8. *For any problem* $\Gamma, \textcircled{?} \vdash G$ *and any* $S \subseteq \mathsf{ID}(\Gamma \dashv G)$*,* $\mathsf{EH}_S(\Gamma \dashv G)$ *is an* explanans *just in case*

(i) *there is at least one* $\Theta \dashv \Lambda \in \mathsf{top_c}(\mathsf{EH}_S(\Gamma \dashv G) \dashv)$ *such that, for any* $\Theta' \subseteq \Theta$*,* $\Lambda' \subseteq \Lambda$*,* $\Theta' \dashv \Lambda' \notin \mathsf{top_c}(G \dashv)$*;*

(ii) *there is at least one* $\Theta \dashv \Lambda \in \mathsf{top_c}(\mathsf{EH}_S(\Gamma \dashv G) \dashv)$ *such that, for any* $\Theta' \subseteq \Theta$*,* $\Lambda' \subseteq \Lambda$*,* $\Theta' \dashv \Lambda' \notin \mathsf{top_c}(\dashv \bigwedge \Gamma)$*.*

PROOF. Note that if $\mathsf{EH}_S(\Gamma \dashv G)$ serves as an *explanans*, it cannot be contradictory. This means that $\mathsf{top_c}(\mathsf{EH}_S(\Gamma \dashv G) \dashv) \neq \varnothing$. Furthermore, observe that if $\overline{\overline{\mathsf{G4}}}$ proves either $G \vdash$ or $\vdash \bigwedge \Gamma$, then by Corollary 4.7 and closure of $\mathsf{G4}$ under Cut, it also proves either $\Gamma, \mathsf{EH}_S(\Gamma \dashv G)$ or $\mathsf{EH}_S(\Gamma \dashv G) \vdash G$. Thus, if $\mathsf{EH}_S(\Gamma \dashv G)$ is an explanans, then $\mathsf{top_c}(G \dashv)$ and $\mathsf{top_c}(\dashv \bigwedge \Gamma)$

are both non-empty.
We can focus on case (*i*), since case (*ii*) is analogous.

(*i*) Let us assume by contradiction that $\mathsf{EH}_S(\Gamma \dashv G)$ is an *explanans* and, for any $\Theta \dashv \Lambda \in \mathsf{top_c}(\mathsf{EH}_S(\Gamma \dashv G) \dashv)$, there exist $\Theta' \subseteq \Theta$ and $\Lambda' \subseteq \Lambda$ such that $\Theta' \dashv \Lambda' \in \mathsf{top_c}(G \dashv)$. Since $\overline{\overline{\mathsf{G4}}}$ refutes $\Gamma \vdash G$ and thus $\mathsf{top_c}(G \dashv) \neq \ \dashv$, we have $\big(\Theta' \cup \Lambda'\big) \neq \varnothing$. It is easy to show that for any $\Theta' \dashv \Lambda' \in \mathsf{top_c}(G \dashv)$ and any $\Theta'' \dashv \Lambda'' \in \mathsf{top_c}(\dashv G)$, either $\Theta' \cap \Lambda'' \neq \varnothing$ or $\Lambda' \cap \Theta'' \neq \varnothing$ (cf. the proof of Proposition 4.3). As a result, $\overline{\overline{\mathsf{G4}}}$ proves $\mathsf{EH}_S(\Gamma \dashv G) \vdash G$, and thus $\mathsf{EH}_S(\Gamma \dashv G)$ does not satisfy condition $\mathsf{A2}$ – a contradiction.

Now, let us assume by contradiction that there exists at least one $\Theta \dashv \Lambda \in \mathsf{top_c}(\mathsf{EH}_S(\Gamma \dashv G) \dashv)$ such that, for any $\Theta' \subseteq \Theta$, $\Lambda' \subseteq \Lambda$, $\Theta' \dashv \Lambda' \notin \mathsf{top_c}(G \dashv)$, and $\mathsf{EH}_S(\Gamma \dashv G)$ does not satisfy condition $\mathsf{A2}$. Since $\mathsf{top_c}(\mathsf{EH}_S(\Gamma \dashv G) \dashv) \neq \varnothing$, we must conclude that $\overline{\overline{\mathsf{G4}}}$ proves $\vdash G$ – another contradiction.

□

Corollary 4.9. *For any problem* $\Gamma, \textcircled{?} \vdash G$ *and any non-empty* $S \subseteq \mathsf{ID}(\Gamma \dashv G)$*, if* $\mathsf{LCH}(\Gamma \dashv G)$ *is an* explanans*, then* $\mathsf{EH}_S(\Gamma \dashv G)$ *is an* explanans *just in case*

(*i*) *there is least one* $\Theta \dashv \Lambda \in \mathsf{top_c}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ *such that, for any* $\Theta' \subseteq \Theta$*,* $\Lambda' \subseteq \Lambda$*,* $\Theta' \dashv \Lambda' \notin \mathsf{top_c}(G \dashv)$ *and* $(\Theta \cup \Lambda) \nsubseteq S$*;*

(*ii*) *there is least one* $\Theta \dashv \Lambda \in \mathsf{top_c}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ *such that, for any* $\Theta' \subseteq \Theta$*,* $\Lambda' \subseteq \Lambda$*,* $\Theta' \dashv \Lambda' \notin \mathsf{top_c}(\dashv \bigwedge \Gamma)$ *and* $(\Theta \cup \Lambda) \nsubseteq S$*.*

Theorem 4.8 and Corollary 4.9 provide some important insights into the nature of $\mathsf{EH}_S(\Gamma \dashv G)$ as an *explanans*. Specifically, they state that $\mathsf{EH}_S(\Gamma \dashv G)$ is an *explanans* if the $\mathsf{LCH}(\Gamma \dashv G)$-hypothesis respects conditions $\mathsf{A2}$ and $\mathsf{A3}$, regardless of whether or not there are abductively redundant atoms present. This is important because it shows that the number of (contingent) novelties against the (contingent) theoretical background can be reduced without necessarily depending on abductively redundant atoms. Furthermore, any EH_S-*explanans* can be used to reduce the number of novelties in a way that approximates the abductively optimal one. Additionally, we can establish that $\mathsf{EH}_S(\Gamma \dashv G)$ and $\bigwedge \Gamma$ are deductively independent of each other, and G is deductively independent of $\mathsf{EH}_S(\Gamma \dashv G)$ (cf. Proposition 4.1).

The cases where an EH_S-hypothesis fails to be maximally supported by evidence can be characterized as follows:

Proposition 4.6. For any problem $\Gamma, \textcircled{?} \vdash G$ and any $S \subseteq \mathsf{ID}(\mathsf{top}(\Gamma \dashv G))$, if $\mathsf{LCH}(\Gamma \dashv G)$ is an *explanans* then $\overline{\overline{\mathsf{G4}}}$ refutes $G \vdash \mathsf{EH}_S(\Gamma \dashv G)$ if and only if $\mathsf{AT}(\Theta \dashv \Lambda) \cap S \neq \varnothing$ for some $\Theta \dashv \Lambda \in \mathsf{top_c}(\dashv G)$.

Proof. Analogous to the proof of Proposition 4.4. □

We are now ready to give a formal rendition of the intuitive notion of 'expected hypothesis' we started this section with:

Procedure 3 (Expected hypothesis). For any problem $\Gamma, ⓘ \vdash G$ such that $\mathsf{LCH}(\Gamma \dashv G)$ is an *explanans*, and for any subset S of $\mathsf{ID}(\Gamma \dashv G)$, the set of expected hypotheses $\mathsf{EH}\big(\Gamma, ⓘ \vdash G\big)$ is obtained according to the following steps:

1. Decompose the antisequent $\Gamma \dashv G$ till (a reduct under Weakening and Cut of) the set of clauses $\mathsf{top_c}(\Gamma \dashv G)$ is fully accomplished.
2. For each S apply steps (2) – (4) of Procedure 2 so as to get the set $\mathcal{E}$ of all (optimized) EH_S-hypotheses.
3. Take the greatest $\mathcal{E}' \subseteq \mathcal{E}$ such that $\mathcal{E}'$ does not include any formula A for which each clause of $\mathsf{top_c}(A \dashv)$ is both a weakened version of a clause in $\mathsf{top_c}(G \dashv)$ and a weakened version of a clause in $\mathsf{top_c}(\dashv \bigwedge \Gamma)$ (cf. Theorem 4.8).
4. Finally, take the least $\mathcal{E}'' \subseteq \mathcal{E}'$ which contains the maximal elements of $\mathcal{E}'$ w.r.t. $\leq$.

We give some examples of how Procedure 3 works.

Example 4.7. *For any problem in Figure 1 it is easy to verify that the set of* EH-*hypotheses produced according to Procedure 3 contains only the hypothesis reported in the rightmost column. Take e.g. the abductive problem* $p \to q, r \to s, ⓘ \vdash q \vee s$*:*

1. *the only reduct under Weakening and Cut of* $\mathsf{top_c}(\Gamma \dashv G)$ *is* $\{ \dashv q, s, r, p \}$*, and thus* $\mathsf{LCH}(\Gamma \dashv G) = q \vee s \vee r \vee p$*;*
2. $\mathsf{ID}(\Gamma \dashv G) = \{q, s\}$*, and thus the greatest* EH_S*-hypothesis w.r.t* $\leq$ *is* $r \vee p$*;*
3. *since the greatest* EH_S*-hypothesis w.r.t* $\leq$ *is an* explanans*, we have that the only* EH*-hypothesis is* $r \vee p$*.*

Example 4.8. *Take the abductive problem* $(p \wedge \neg q) \to r, q \to \neg r, ⓘ \vdash r$ *of Example 4.6:*

1. *the only reduct under Weakening and Cut of* $\mathsf{top_c}(\Gamma \dashv G)$ *is* $\{ \dashv r, p \ ; \ q \dashv r\}$*, and thus* $\mathsf{LCH}(\Gamma \dashv G) = (r \vee p) \wedge (r \vee \neg q)$*;*
2. $\mathsf{ID}(\Gamma \dashv G) = \{r, q\}$*, and thus the greatest* $\mathsf{EH}_S(\Gamma \dashv G)$ *w.r.t* $\leq$ *is* $\top \to \bot$*, which is not an* explanans*;*
3. *the greatest* EH_S*-hypothesis w.r.t.* $\leq$ *which is an* explanans *is* $p \wedge \neg q$*: the only* EH*-hypothesis is* $p \wedge \neg q$*, as expected.*

Example 4.9. *Take the abductive problem* $(p \wedge \neg q) \to r, q \to \neg r, ⓘ \vdash \neg r$*:*

(1) the only reduct under Weakening and Cut of $\mathsf{top_c}(\Gamma \dashv G)$ *is* $\{r \dashv q\}$*, and thus* $\mathsf{LCH}(\Gamma \dashv G) = r \to q$*;*

(2) $\mathsf{ID}(\Gamma \dashv G) = \{r, q\}$*, and thus the greatest* EH_S*-hypothesis w.r.t.* $\leq$ *is* $\top \to \bot$*, which is not an* explanans*;*

(3) the greatest EH_S*-hypothesis w.r.t.* $\leq$ *which is an* explanans *is* q*: the only* EH*-hypothesis is* q*, as expected.*

Procedure 3 is an effective tool for tracking intuitively expected hypotheses in familiar examples of abductive problem: we propose to take it as a normative standard for the rational agent – even in cases where we lack equally strong intuitions.

4.3. Beyond analyticity

As we have seen, given a problem $\Gamma, \textcircled{?} \vdash G$, analytic decomposition can be used as a tool for generating formulas, possibly stronger than $\mathsf{LCH}(\Gamma \dashv G)$, which satisfy conditions $\mathsf{A1}$ and, possibly, $\mathsf{A2}$ – $\mathsf{A3}$: since any A among these formulas is such that $\mathsf{AT}(A) \subseteq \mathsf{AT}(\Gamma \dashv G)$ we say that they are *analytic abductive hypotheses* (possibly, *analytic explanantes*). In order to track formulas obtained through decomposition in full generality, we modify Procedure 1 as follows:

PROCEDURE 4 (Strengthened Least Compromising Hypothesis). For any problem $\Gamma, \textcircled{?} \vdash G$, the $\vec{S}$-strengthened least compromising hypothesis $\mathsf{SLCH}_{\vec{S}}\big(\Gamma, \textcircled{?} \vdash G\big)$ is the formula obtained as follows:

(1) Decompose the antisequent $\Gamma \dashv G$ till the non-empty set of complementary clauses $\mathsf{top_c}(\Gamma \dashv G) = \big\{\Theta_1 \dashv \Lambda_1, \ldots, \Theta_n \dashv \Lambda_n\big\}$ is fully accomplished.

(2) Define a sequence $\vec{S} = \langle S_1, \ldots, S_n\rangle$ of subsets of $\mathsf{AT}(\Gamma \dashv G)$, and, for each clause $\Theta_i \dashv \Lambda_i \in \mathsf{top_c}(\Gamma \dashv G)$, take the largest clause $\Theta'_i \dashv \Lambda'_i$ such that $\Theta'_i \subseteq \Lambda_i$, $\Theta'_i \subseteq \Lambda_i$ and $\Theta'_i \cap S_i = \Lambda'_i \cap S_i = \varnothing$.

(3) For each clause $\Theta'_i \dashv \Lambda'_i$ thus obtained consider the formula $C_i \equiv \bigwedge \Theta'_i \to \bigvee \Lambda'_i$.

(4) Finally set $\mathsf{SLCH}_S\big(\Gamma, \textcircled{?} \vdash G\big) = C_1 \wedge \cdots \wedge C_n$ (avoiding repetition of conjuncts).

Notice that, if $\vec{S} = \langle S_1, \ldots, S_n\rangle$, $S_i = S$ for any $1 \leq i \leq n$ and $S \subseteq \mathsf{ID}(\Gamma \dashv G)$, then $\mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G) = \mathsf{EH}_S(\Gamma \dashv G)$; if it is the case that $S = \varnothing$, then $\mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G) = \mathsf{LCH}(\Gamma \dashv G)$.

EXAMPLE 4.10. *We apply Procedure 4 to compute the formula* $\mathsf{SLCH}_{\vec{S}}\big(p \to q, \textcircled{?} \vdash p \to r\big)$ *for any* $\vec{S}$*:*

(1) By performing decomposition we get

$$\mathsf{top_c}(p \to q, \textcircled{?} \vdash p \to r) = \big\{q, p \dashv r\big\}$$

and

$$\mathsf{AT}(p \to q, \textcircled{?} \vdash p \to r) = \{p, q, r\}$$

(2) It is trivial to fix an enumeration of the elements of $\mathsf{top_c}(p \to q, \textcircled{?} \vdash p \to r)$.

(3) For any $\vec{S}$ *we obtain a single clause, which we turn into its corresponding formula to obtain the corresponding* $\mathsf{SLCH}_{\vec{S}}$*-hypothesis:*

$$\begin{aligned} q, p \dashv r \quad &\Rightarrow \quad (q \wedge p) \to r \\ p \dashv r \quad &\Rightarrow \quad p \to r \\ q \dashv r \quad &\Rightarrow \quad q \to r \\ q, p \dashv \quad &\Rightarrow \quad \neg q \vee \neg p \\ q \dashv \quad &\Rightarrow \quad \neg q \\ p \dashv \quad &\Rightarrow \quad \neg p \\ \dashv r \quad &\Rightarrow \quad r \\ \dashv \quad &\Rightarrow \quad \top \to \bot \end{aligned}$$

Remark that, for any problem $\Gamma, \textcircled{?} \vdash G$*, if* $\mathsf{top_c}(\Gamma \dashv G) = \{\Theta_1 \dashv \Lambda_1, \ldots, \Theta_n \dashv \Lambda_n\}$ *and* $\sum_{i=1}^{n} |\mathsf{AT}(\Theta_i \dashv \Lambda_i)| = k$*, then there are (at most)* 2^k $\mathsf{SLCH}_{\vec{S}}$*-hypotheses.*

We can refine Procedure 4 with the aim of optimization, similar to how we refined Procedure 2 earlier. In particular, if the set of clauses $\mathcal{C}$ generated by step (2) in Procedure 4 includes the empty antisequent, then the only reduct of $\mathcal{C}$ under Weakening and Cut is the singleton of the empty antisequent. As a result, the refined Procedure 4 sets an upper bound on the number of all $\mathsf{SLCH}_{\vec{S}}$-hypotheses to $2^k - 2(2^{n-1} - 1)$.

It is immediate to verify that any $\mathsf{SLCH}_{\vec{S}}$-hypothesis satisfies condition $\mathsf{A1}$, as shown in Theorem 4.5 and Corollaries 4.6 and 4.7. Moreover, an $\mathsf{SLCH}_{\vec{S}}$-hypothesis satisfies conditions $\mathsf{A2} - \mathsf{A3}$ if $\mathsf{top_c}(\mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G) \dashv)$ contains at least one clause that is not a (possibly) weakened version of a clause in $\mathsf{top_c}(G \dashv)$, and at least one clause which is not a (possibly) weakened version of a clause in $\mathsf{top_c}(\dashv \bigwedge \Gamma)$, as shown in Theorem 4.8.

Let us introduce a bit more of terminology: for any formula A, if $\mathsf{top}(\models^{*} A) = \{\Theta_1 \models^{*} \Theta_m, \ldots, \Theta_n \models^{*} \Lambda_m\}$ then we use *cnf*(A) to refer to $\bigwedge_{i=1}^{m}(\bigwedge \Theta_i \to \bigvee \Lambda_i)$. The following result shows that any analytic *explanans* A logically implies some $\mathsf{SLCH}_{\vec{S}}$-*explanans*:

THEOREM 4.10. *For any problem* $\Gamma, \textcircled{?} \vdash G$*, if a formula is an analytic* explanans A*, then* A *is logically equivalent to* $B \wedge C$*, where*

(i) $B = \mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_n}}(\Gamma \dashv G)$*, with* $\mathsf{SLCH}_{\vec{S_i}}(\Gamma \dashv G)$ *being an* explanans *for any* $1 \leq i \leq n$*;*

(ii) $\overline{\overline{\overline{\mathsf{G4}}}}$ *refutes* $\Gamma, C \vdash G$*.*

Proof. First, notice that if A is an (analytic) *explanans*, then it is a contingent formula. This is because if A were a tautology, then the refutability of $\Gamma \vdash G$ would imply the refutability of $\Gamma, A \vdash G$ (against condition $\mathsf{A1}$). Similarly, if A were a contradiction, then $\overline{\overline{\mathsf{G4}}}$ would prove both $A \vdash G$ and $\Gamma, A \vdash G$ (against conditions $\mathsf{A2}$ and $\mathsf{A3}$, respectively).
If A is a contingent formula, then $\mathsf{top}_{\mathsf{c}}(\dashv A) \neq \varnothing$ and $\mathsf{top}_{\mathsf{c}}(A \dashv) \neq \varnothing$. If $\mathsf{top}_{\mathsf{i}}(A \dashv) \neq \varnothing$, then we can always consider a formula A' that is logically equivalent to A and such that $\mathsf{top}(A' \dashv) = \mathsf{top}_{\mathsf{c}}(A \dashv)$. Let us assume that $\mathsf{top}(A' \dashv) = \big\{\Theta_1 \dashv \Lambda_1, \ldots, \Theta_m \dashv \Lambda_m\big\}$ and that $\mathsf{top}_{\mathsf{c}}(\Gamma \dashv G) = \big\{\Phi_1 \dashv \Psi_1, \ldots, \Phi_n \dashv \Psi_n\big\}$: if $\Gamma, A \vdash G$, and thus $\Gamma, A' \vdash G$, is provable, then for any $\Theta_i \dashv \Lambda_i \in \mathsf{top}(A' \dashv)$ and any $\Phi_j \dashv \Psi_j \in \mathsf{top}_{\mathsf{c}}(\Gamma \dashv G)$ there is either one non-empty $\Theta_{i_j} \subseteq \Theta_i$ such that $\Theta_{i_j} = \Theta_i \cap \Psi_j$ or one non-empty $\Lambda_{i_j} \subseteq \Lambda_i$ such that $\Lambda_{i_j} = \Lambda_i \cap \Phi_j$.
Bearing these facts in mind, we can proceed to prove the two statements separately.

(*i*) Consider any $\Theta' \overset{*}{\vdash} \Lambda' \in \mathsf{top}(\dashv A')$ such that, for a given j such that $1 \leq j \leq n$, if $p, q \in \Theta'$, then $p \in \Lambda_{i_j}$ and $q \in \Lambda_{i'_j}$ and, if $r, s \in \Lambda'$, then $r \in \Theta_{i_j}$ and $s \in \Theta_{i'_j}$ – with $1 \leq i \neq i' \leq m$: it is easy to see that there is (at least) one $\vec{S}$ such that $\Theta' \dashv \Lambda' \in \mathsf{top}_{\mathsf{c}}(\dashv \mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G))$. Since the set of all $\vec{S}$-strengthenings of a given clause $\Phi_j \dashv \Psi_j \in \mathsf{top}_{\mathsf{c}}(\Gamma \dashv G)$ cannot be totally ordered with respect to deductive strength (cf. Theorem 4.5), it may be the case that (a reduct under Weakening and Cut of) the set of the $\vec{S}$-strengthenings of $\Phi_j \dashv \Psi_j \in \mathsf{top}_{\mathsf{c}}(\Gamma \dashv G)$ included in $\mathsf{top}(\dashv A')$ does not narrow down to a singleton. This holds for any $1 \leq j \leq n$, and therefore, there exist $\vec{S_1}, \ldots, \vec{S_N}$ such that $\mathsf{top}_{\mathsf{c}}\big(\dashv \mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_N}}(\Gamma \dashv G)\big) \subseteq \mathsf{top}(\dashv A')$.
At this point, we must consider two possibilities: either (a) for any $\Theta_i \dashv \Lambda_i \in \mathsf{top}(A' \dashv)$ we have that $\Theta_i = (\Theta_{i_1} \cup \cdots \cup \Theta_{i_n})$ and $\Lambda_i = (\Lambda_{i_1} \cup \cdots \cup \Lambda_{i_n})$, or (b) there is at least one $\Theta_i \dashv \Lambda_i \in \mathsf{top}(A' \dashv)$ such that either $\Theta_i \supset (\Theta_{i_1} \cup \cdots \cup \Theta_{i_n})$ or $\Lambda_i \supset (\Lambda_{i_1} \cup \cdots \cup \Lambda_{i_n})$. In the first case we have $\mathsf{top}_{\mathsf{c}}\big(\dashv \mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_N}}(\Gamma \dashv G)\big) = \mathsf{top}(\dashv A')$: since $cnf(A') \equiv A'$, we have that A is logically equivalent to $\mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_N}}(\Gamma \dashv G) \wedge \top$. In the second case, there must exist a formula C such that $\mathsf{top}(\overset{*}{\vdash} C) = \mathsf{top}(\dashv A') \setminus \mathsf{top}_{\mathsf{c}}(\dashv \mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_N}}(\Gamma \dashv G))$. Therefore, A must be logically equivalent to $\mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_N}}(\Gamma \dashv G) \wedge C$. We can reach the conclusion by noting that $\mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G), \ldots, \mathsf{SLCH}_{\vec{S_N}}(\Gamma \dashv G)$ necessarily satisfy conditions $\mathsf{A2}$ – $\mathsf{A3}$.

(*ii*) Since $\overline{\overline{\mathsf{G4}}}$ refutes $\Gamma, \top \vdash G$, we focus on case (b) and assume for contradiction that $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, C \vdash G$. The provability of $\Gamma, C \vdash G$ implies that for any $\Pi \overset{*}{\vdash} \Sigma \in \mathsf{top}(C \overset{*}{\vdash})$ and any $\Phi_j \vdash \Psi_j \in \mathsf{top}_{\mathsf{c}}(\Gamma \dashv G)$, either $\overline{\overline{\mathsf{G4}}}$ proves $\Pi \vdash \Sigma$, or there exists a non-empty $\Pi_j \subseteq \Pi$ such that $\Pi_j = \Pi \cap \Psi_j$, or a non-empty $\Sigma_j \subseteq \Sigma$ such that $\Sigma_j = \Sigma \cap \Phi_j$. If $\Pi \vdash \Sigma$ were always provable, then C would be contradictory – a contradiction. As a consequence, there must be (at least) one $\Pi \dashv \Sigma \in \mathsf{top}(C \dashv)$ such that for any

$1 \leq j \leq n$, there is either one non-empty $\Pi_j \subseteq \Pi$ such that $\Pi_j = \Pi \cap \Psi_j$ or one non-empty $\Sigma_j \subseteq \Sigma$ such that $\Sigma_j = \Sigma \cap \Phi_j$. This means that there is (at least) one $\vec{S}$ such that $\mathsf{top}_{\mathsf{c}}(\dashv \mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G)) \subseteq \mathsf{top}(\mathrel{|\!\!\overset{*}{-}} C)$. By construction, for any clause $\Pi' \mathrel{|\!\!\overset{*}{-}} \Sigma' \in \mathsf{top}(\mathrel{|\!\!\overset{*}{-}} C)$, we have that there is (at least) one atom p such that if $p \in \Pi'$, then $p \notin \Phi_j$, and if $p \in \Sigma'$, then $p \notin \Psi_j$, for any $1 \leq j \leq n$. As a result, for any $\vec{S}$, $\mathsf{top}_{\mathsf{c}}(\dashv \mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G)) \nsubseteq \mathsf{top}(\mathrel{|\!\!\overset{*}{-}} C)$, which leads to a contradiction.

□

Corollary 4.11. *For any problem* $\Gamma, \textcircled{?} \vdash G$, *if a formula* A *is an analytic* explanans *and* $\mathsf{CUT}(\dashv A) = \varnothing$ *then* $cnf(A) = B \wedge C$, *where*

(*i*) $B = \mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_n}}(\Gamma \dashv G)$, *with* $\mathsf{SLCH}_{\vec{S_i}}(\Gamma \dashv G)$ *being an* explanans *for any* $1 \leq i \leq n$;

(*ii*) $\mathsf{AT}(C) \subseteq \mathsf{AT}(\Gamma \dashv G)$ *and* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\Gamma, C \vdash G$.

Example 4.11. *Take the problem* $p \to q, \textcircled{?} \vdash p \to r$ *of Example 4.10:* $(\neg q \vee \neg p) \vee (r \wedge \neg r)$ *is an analytic* explanans, *and it is logically equivalent to* $((q \wedge p) \to r) \wedge (\neg q \vee \neg p \vee \neg r)$, *with* $(q \wedge p) \to r$ *being an* $\mathsf{SLCH}_{\vec{S}}$-explanans *and the sequent* $p \to q, \neg q \vee \neg p \vee \neg r \vdash p \to r$ *being refutable.*

Theorem 4.10 establishes that, for any problem $\Gamma, \textcircled{?} \vdash G$, each analytic *explanans* A can be decomposed into a conjunction of $\mathsf{SLCH}_{\vec{S}}$-*explanantes* for $\Gamma, \textcircled{?} \vdash G$ and a 'derived' problem $\Gamma, C, \textcircled{?} \vdash G$. The following proposition shows that any $\mathsf{SLCH}_{\vec{S}}$-*explanans* for $\Gamma, \textcircled{?} \vdash G$ is an $\mathsf{SLCH}_{\vec{S}}$-*explanans* for the derived problem $\Gamma, C, \textcircled{?} \vdash G$:

Proposition 4.7. For any problem $\Gamma, \textcircled{?} \vdash G$ and each analytic explanans A, if $A \equiv B \wedge C$, $B = \bigwedge_{i=1}^{n} \mathsf{SLCH}_{\vec{S_i}}(\Gamma \dashv G)$ and $\overline{\overline{\mathsf{G4}}}$ refutes $\Gamma, C \vdash G$, then any $\mathsf{SLCH}_{\vec{S}}$-abductive hypothesis for $\Gamma, \textcircled{?} \vdash G$ is an $\mathsf{SLCH}_{\vec{S}}$-abductive hypothesis for $\Gamma, C, \textcircled{?} \vdash G$.

Proof. It is routine to show that $\Theta \dashv \Lambda \in \mathsf{top}_{\mathsf{c}}(\Gamma, C \dashv G)$ if and only if there exist $\Theta' \dashv \Lambda' \in \mathsf{top}_{\mathsf{c}}(\Gamma \dashv G)$ and $\Theta'' \dashv \Lambda'' \in \mathsf{top}_{\mathsf{c}}(C \dashv)$ such that $\Theta' \cup \Theta'' = \Theta$ and $\Lambda' \cup \Lambda'' = \Lambda$. As a result, if there is an atom $p \in (\Theta \cup \Lambda)$ such that $p \in (\Theta' \cup \Lambda') \cap (\Theta'' \cup \Lambda'')$, then $p \in \Theta'$ if and only if $p \in \Theta''$, and $p \in \Lambda'$ if and only if $p \in \Lambda''$: this means that if p is erased from $\Theta' \dashv \Lambda'$ then it is also erased from $\Theta \dashv \Lambda$. By construction, any $\mathsf{SLCH}_{\vec{S}}$-abductive hypothesis is obtained by non-uniformly deleting atoms from clauses in $\mathsf{top}_{\mathsf{c}}(\Gamma \dashv G)$: this suffices to get the conclusion. □

For any problem $\Gamma, \textcircled{?} \vdash G$ we say that a set of *explanantes* $A_1, \dots, A_n$ is a set of *alternative* abductive solutions just if $A_1, \dots, A_n$ are pairwise mutual exclusive and jointly exhaustive – i.e., such that $\overline{\overline{\mathsf{G4}}}$ proves $A_i, A_j \vdash$ and $\vdash A_1, \dots, A_n$ respectively, for any $1 \leq i \neq j \leq n$ (cf. [39], pp. 45-46): the following proposition shows that the set of all analytic *explanantes* is not a set of alternative abductive solutions.

PROPOSITION 4.8. For any problem $\Gamma, \textcircled{?} \vdash G$, if $\mathsf{LCH}(\Gamma \dashv G), A_1, \ldots, A_n$ are distinct formulas respecting condition $\mathsf{A1} - \mathsf{A3}$, then

(i) $\mathsf{LCH}(\Gamma \dashv G)$ and A_i are not mutually exclusive, for any $1 \leq i \leq n$;

(ii) $\mathsf{LCH}(\Gamma \dashv G), A_1, \ldots, A_n$ are not jointly exhaustive.

PROOF. We treat each case separately.

(i) Suppose to the contrary that $\mathsf{LCH}(\Gamma \dashv G), A_i$ respect conditions $\mathsf{A1} - \mathsf{A3}$, for any $1 \leq i \leq n$, while being mutually exclusive. This means that $\overline{\overline{\mathsf{G4}}}$ proves $\mathsf{LCH}(\Gamma \dashv G), A_i \vdash$: since Theorem 4.3 guarantees that $\overline{\overline{\mathsf{G4}}}$ proves $A_i \vdash \mathsf{LCH}(\Gamma \dashv G)$, we have that $\overline{\overline{\mathsf{G4}}}$ proves $A_i \vdash$ by closure under Cut – i.e. a contradiction.

(ii) Suppose to the contrary that $\mathsf{LCH}(\Gamma \dashv G), A_1, \ldots, A_n$ respect condition $\mathsf{A1}$ and that $\mathsf{LCH}(\Gamma \dashv G), A_1, \ldots, A_n$ are jointly exhaustive. We have that $\overline{\overline{\mathsf{G4}}}$ proves both $\Gamma, \mathsf{LCH}(\Gamma \dashv G) \vdash G$ and $\vdash \mathsf{LCH}(\Gamma \dashv G), A_1, \ldots, A_n$: by closure of $\mathsf{G4}$ under Cut it follows that $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma \vdash G, A_1, \ldots, A_n$. We can iterate n times this reasoning step, by taking at each step exactly one among $A_1, \ldots, A_n$ as cut formula: finally, we reach the conclusion that $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma \vdash G$ – a contradiction.

□

At this point it is natural to ask whether our framework can be used to investigate the (infinite) set of *non-analytic abductive hypotheses* (*explanantes*) – i.e., formulas A obeying condition $\mathsf{A1}$ (conditions $\mathsf{A1} - \mathsf{A3}$, respectively) such that $\mathsf{AT}(A) \not\subseteq \mathsf{AT}(\Gamma \dashv G)$.

EXAMPLE 4.12. *Take the problem $p \to q, \textcircled{?} \vdash p \to r$ of Example 4.10: $(\neg p \wedge s) \vee (\neg q \wedge t)$ is a non-analytic abductive hypothesis, whereas $\neg q \wedge (s \vee t)$ is a non-analytic* explanans.

First, let us notice that non-analytic *explanantes* enjoy the same kind of 'abductive normal form' as analytic ones:

THEOREM 4.12. *For any problem $\Gamma, \textcircled{?} \vdash G$, if a formula is a non-analytic* explanans *A, then A is logically equivalent to $B \wedge C$, where*

(i) *$B = \mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_n}}(\Gamma \dashv G)$, with $\mathsf{SLCH}_{\vec{S_i}}(\Gamma \dashv G)$ being an* explanans *for any $1 \leq i \leq n$;*

(ii) *$\overline{\overline{\mathsf{G4}}}$ refutes $\Gamma, C \vdash G$.*

PROOF. Analogous to the proof of Theorem 4.10. □

Corollary 4.13. *For any problem $\Gamma, \textcircled{?} \vdash G$, if a formula A is a non-analytic* explanans *and $\mathsf{CUT}(\dashv A) = \varnothing$ then $cnf(A) = B \wedge C$, where*

(i) *$B = \mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_n}}(\Gamma \dashv G)$, with $\mathsf{SLCH}_{\vec{S_i}}(\Gamma \dashv G)$ being an* explanans *for any $1 \leq i \leq n$;*

(*ii*) $\mathsf{AT}(C) \not\subseteq \mathsf{AT}(\Gamma \dashv G)$ *and* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\Gamma, C \vdash G$.

EXAMPLE 4.13. *Take the problem* $p \to q, \textcircled{?} \vdash p \to r$ *of Example 4.10:* $((q \wedge p) \vee s) \to (r \wedge t)$ *is a non-analytic* explanans*, and it is logically equivalent to* $((q \wedge p) \to r) \wedge ((q \wedge p) \to t) \wedge (s \to r) \wedge (s \to t)$*, with* $(q \wedge p) \to r$ *being an* $\mathsf{SLCH}_{\vec{S}}$-explanans. *Then, the sequent* $p \to q, ((q \wedge p) \to t) \wedge (s \to r) \wedge (s \to t) \vdash p \to r$ *being refutable and* $\mathsf{AT}\big(((q \wedge p) \to t) \wedge (s \to r) \wedge (s \to t)\big) \not\subseteq \mathsf{AT}\big(p \to q, \textcircled{?} \vdash p \to r\big)$.

Theorem 4.12 states that for any problem of the form $\Gamma, \textcircled{?} \vdash G$ and any non-analytic *explanans* A there exists a "derived" problem of the form $\Gamma, C, \textcircled{?} \vdash G$, which possibly makes all new information in A explicit in the theoretical background. Corollary 4.13 further refines this result for a specific class of non-analytic *explanantes*. It is easy to show that any $\mathsf{SLCH}_{\vec{S}}$-*explanans* for $\Gamma, \textcircled{?} \vdash G$ is also an $\mathsf{SLCH}_{\vec{S}}$-*explanans* for $\Gamma, C, \textcircled{?} \vdash G$ (as per Proposition 4.7). Therefore, we can conclude that the deductive saturation of a problem $\Gamma, \textcircled{?} \vdash G$ through a non-analytic *explanans* A can always be understood as the deductive saturation of a (possibly) distinct problem of the form $\Gamma, C, \textcircled{?} \vdash G$ through an analytic *explanans*. This implies that the set of analytic abductive solutions enjoys a certain "completeness": in the end, deductive saturation can always be performed via analytic *explanantes* including $\mathsf{SLCH}_{\vec{S}}$-*explanantes*.

Let us end this section by proposing the following conjecture: for any problem of the form $\Gamma, \textcircled{?} \vdash G$ and any *explanans* A, A is *candidate for the best explanans* only if A is logically equivalent to $B \wedge C$, where C is such that $\overline{\overline{\mathsf{G4}}}$ refutes $\Gamma, C \vdash G$ and B is a conjunction of EH-hypotheses for the problem $\Gamma, C, \textcircled{?} \vdash G$ (as described in Procedure 3).

4.4. Legend of the symbols

$\mathsf{AT}(\Gamma \vdash^{*} \Delta)$ = set of all atoms occurring in the clauses of $\mathsf{top}(\Gamma \vdash^{*} \Delta)$

$\mathsf{ID}(\Gamma \vdash^{*} \Delta)$ = set of all identity atoms occurring in the clauses of $\mathsf{top}(\vdash^{*} A)$

$\mathsf{CUT}(\Gamma \vdash^{*} \Delta)$ = set of all cut atoms occurring in the clauses of $\mathsf{top}(\vdash^{*} A)$

$\mathsf{AT}(A)$ = set of all atomic subformulas of A

$cnf(A)$ = conjunction of formula translations of clauses in $\mathsf{top}(\vdash^{*} A)$

$\Gamma, \textcircled{?} \vdash G$ = abductive problem with G as *explanandum* and Γ as theoretical background

$\mathsf{LCH}\big(\Gamma, \textcircled{?} \vdash G\big)$ = the least compromising hypothesis for $\Gamma, \textcircled{?} \vdash G$

$\mathsf{EH}_S\big(\Gamma, \textcircled{?} \vdash G\big)$ = the S-approximation to an expected hypothesis for $\Gamma, \textcircled{?} \vdash G$

$\mathsf{EH}\big(\Gamma, \textcircled{?} \vdash G\big)$ = an expected hypothesis for $\Gamma, \textcircled{?} \vdash G$

$\mathsf{SLCH}_{\vec{S}}\big(\Gamma, \textcircled{?} \vdash G\big)$ = the $\vec{S}$-strengthened least compromising hypothesis for $\Gamma, \textcircled{?} \vdash G$

CHAPTER 5

Hybrid hypersequents for propositional default logics

Default reasoning is a form of non-monotonic reasoning that allows us to draw plausible conclusions under incomplete information and in the absence of explicit evidence to the contrary: when disproved by new evidence, these conclusions can be withdrawn [**142**, **101**, **7**]. From a logical perspective, default reasoning can be formalized by extending classical logic with a collection of *extra-logical axioms* – which encompass the propositional contents of beliefs held by an ideal reasoner – along with a set of *default rules* encapsulating the informational pathways she follows to arrive at defeasible conclusions.

Starting with Reiter's work [**142**], default logic's formalism was initially developed using Hilbert-style calculi [**90**, **50**, **22**, **37**]. Later, researchers explored alternative proof-theoretic approaches, such as semantic tableaux [**4**, **143**] and sequent calculi [**19**, **42**]. However, each of these proof-theoretic methods presents certain drawbacks in various respects. To summarize:

(*i*) the proof-search space in Hilbert-style calculi is infinite: this feature makes these calculi less suitable for representing complex reasoning tasks;

(*ii*) in tableaux-based calculi, applying rules requires considering the *entire* derivation tree: this prevents local control over the information circulating in the derivation. Furthermore, tableaux are used to compute extensions as a whole, rather than representing individual derivations within extensions;

(*iii*) sequent calculi, while versatile, rely on *ad hoc* extensions of the underlying language: this feature makes them less suitable for a *modular* and *uniform* proof-theoretic treatment encompassing axiomatic extensions of classical logic [**121**].

The aim of this chapter is to introduce a novel proof-theoretic approach to default propositional logics, centered on a non-standard notion of *hypersequent*. Traditionally, hypersequents are sequences of sequents separated by a bar, originally conceived to provide analytic calculi for modal and intermediate logics lacking cut-free sequent calculi [**13**, **15**]. We modify the notion of hypersequent in order to embed within derivation trees the consistency checks involved in the application of default rules: specifically, we redefine hypersequents as *hybrid* constructs, each comprising a sequent and a *set* of *antisequents* [**58**]. Departing from the conventional disjunctive interpretation of the separating bar, we embrace a conjunctive reading [67, 126]. In this framework, antisequents within a hybrid hypersequent furnish contrary updates concerning the provability of the associated sequent.

Informally, a default is an inference rule defined by three components: a premise (the prerequisite) and a finite set of formulas (the justifications), which together lead to a conclusion. Specifically, if the prerequisite is proven, the conclusion becomes provable provided that the negations of the justifications are not provable. Our formalization of default rules through hybrid hypersequents involves specifying distinct *extra-logical rules*. Here is the general idea. Initially, we convert each default's prerequisite into conjunctive normal form while translating its conclusion into clausal form. Subsequently, for every default rule and each classically invalid clause within its prerequisite, we generate two hybrid hypersequents: the provability part of the first hypersequent includes the clause itself, whereas the provability part of the second hypersequent features a possibly weakened version of that clause. These hybrid hypersequents serve as premises for the extra-logical rule corresponding to the default. The conclusion of such a rule comprises a hybrid hypersequent, whose refutational part consists of all antisequents in the premises alongside the set of negated justifications.

On this basis, we design hybrid hypersequent HG4 calculi for default logics. We claim that these calculi overcome some drawbacks of previous formalisms:

(*i*) the proof-search space in HG4 calculi is finite;
(*ii*) HG4 calculi enable local control over the information flow and provide representations of individual derivations within extensions;
(*iii*) HG4 calculi do not rely on *ad hoc* extensions of the underlying language.

The chapter is organized as follows. Section 5.1 contains the notions and results concerning default logics that we will subsequently employ. We also introduce the formal framework for managing axiomatic extensions of classical logic, namely hybrid sequent calculi with crucial proof-theoretic properties. In Section **??**, our focus shifts to hybrid hypersequent calculi. Here, we establish admissibility of structural rules, invertibility of logical rules and a weakened version of the subformula property for cut-free proofs. We present hybrid hypersequent calculi that are sound and (weakly) complete with respect to credulous consequence based over *Łukaszewicz extensions* [90], showing that they fail to be strongly complete due to their non-monotonic behaviour in relation to the addition of extra-logical axioms. Moreover, we highlight that admissible rules fail to be encoded in provable hypersequents because of the context-sensitivity of extra-logical rules. Section **??** presents a hypersequent-based decision procedure for skeptical consequence: this method relies on the exhaustive search of counterexamples, circumventing the need for early computation of all extensions. Section **??** outlines avenues for future research, with a brief discussion on hybrid hypersequent calculi for credulous consequence based on *Reiter extensions* [142] as well as the relationship between our decision procedure for skeptical consequence and abductive reasoning.

5.1. Preliminaries on default logic

A *default* is a domain-specific inference rule of the form

$$\frac{B : C_1, \dots, C_n}{D} \tag{5.1.1}$$

where B is the *prerequisite*, the non-contradictory formulas $C_1, \dots, C_n$ are the *justifications* and the non-tautological formula D is the *conclusion* of the default. Its interpretation is that if B is proved, then D is provable, so long as the formulas $\neg C_1, \dots, \neg C_n$ are not provable. We say that a default rule is *triggered* whenever its prerequisite is proved.

If $n = 1$, then a default rule of the form (5.1.1) is *normal* if and only if $\overline{\overline{\mathsf{G4}}}$ proves both the sequent $C_1 \vdash D$ and $D \vdash C_1$, *supernormal* if it is normal and $B = \top$, *semi-normal* if and only if $\overline{\overline{\mathsf{G4}}}$ proves $C_1 \vdash D$. For convenience, when discussing normal default rules, we only focus on those for which $C_1 = D$. For semi-normal default rules, we only consider those where $C_1 = D \wedge E$ for some formula E.

A default theory is a pair $\langle \mathcal{W}, \mathcal{D} \rangle$, where $\mathcal{W}$ is a finite, consistent set of extra-logical axioms and $\mathcal{D}$ is a finite, non-empty set of default rules: we will use $req(\mathcal{D}'), just(\mathcal{D}')$ and $concl(\mathcal{D}')$ to refer to the set of prerequisites, justifications and conclusions, respectively, of the defaults in any $\mathcal{D}' \subseteq \mathcal{D}$. A default theory $\langle \mathcal{W}, \mathcal{D} \rangle$ is said to be normal (supernormal, semi-normal) if and only if all the defaults in $\mathcal{D}$ are normal (supernomal and semi-normal, respectively).

Example 5.1. *Let $\mathcal{W}$, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ be defined as follows:*

$$\mathcal{W} = \{p, q\}$$

$$\mathcal{D}_1 = \left\{ \frac{p : r}{r} \; , \; \frac{q : s}{s} \right\}$$

$$\mathcal{D}_2 = \left\{ \frac{\top : r}{r} \; , \; \frac{\top : \neg r}{\neg r} \right\}$$

$$\mathcal{D}_3 = \left\{ \frac{p : r \wedge s}{s} \; , \; \frac{q : \neg t \wedge \neg s}{\neg s} \; , \; \frac{p : u \wedge r}{r} \right\}$$

$\langle \mathcal{W}, \mathcal{D}_1 \rangle$ is a normal default theory, $\langle \mathcal{W}, \mathcal{D}_2 \rangle$ is a supernormal default theory, $\langle \mathcal{W}, \mathcal{D}_3 \rangle$ is a semi-normal default theory.

By *(Reiter) extension*, we mean any set of formulas derivable from $\mathcal{W}$ by classical logic and the maximal application of default rules in $\mathcal{D}$, whose consistency condition holds both before and after their being triggered [142]. Specifically, we say that $\mathcal{E}$ is an extension of $\langle \mathcal{W}, \mathcal{D} \rangle$ if and only if $\mathcal{E}$ is quasi-inductively defined as follows:

$$\mathcal{E}^0 = \mathcal{W}$$

$$\mathcal{E}^{k+1} = Cn(\mathcal{E}^k) \cup \{concl(\mathcal{D}') \mid \mathcal{D}' \subseteq \mathcal{D}, req(\mathcal{D}') \subseteq \mathcal{E}^k, (just(\mathcal{D}'))^{\perp} \cap \mathcal{E} = \varnothing\}$$

$$\mathcal{E} = \bigcup_{i=0}^{\omega} \mathcal{E}^i$$

with $Cn(\Gamma)$ denoting the set of classical consequences of Γ.

Example 5.2. *Let $\mathcal{W}$ and $\mathcal{D}_3$ be as in Example 5.1. The default theory $\langle \mathcal{W}, \mathcal{D}_3 \rangle$ has two extensions – i.e., $\mathcal{E}_1 = Cn(\{p, q, s, r\})$ and $\mathcal{E}_2 = Cn(\{p, q, \neg s, r\})$.*

A *modified extension* (called also *Łukaszewicz extension*) is a set of formulas derivable from $\mathcal{W}$ by classical logic and the maximal application of default rules in $\mathcal{D}$ whose consistency condition holds (both before and after their being triggered) relatively to a simultaneously defined support set ([7, pp. 75-76], [90]). Specifically, we say that $\langle \mathcal{E}, \mathcal{F} \rangle$ is a modified extension of $\langle \mathcal{W}, \mathcal{D} \rangle$ if and only if $\mathcal{E}$ and $\mathcal{F}$ are quasi-inductively defined as follows:

$$\mathcal{E}^0 = \mathcal{W} \qquad \mathcal{F}^0 = \varnothing$$

$$\mathcal{E}^{k+1} = Cn(\mathcal{E}^k) \cup \{concl(\mathcal{D}') \mid \mathcal{D}' \subseteq \mathcal{D}, req(\mathcal{D}') \subseteq \mathcal{E}^k, A \in (\mathcal{F} \cup just(\mathcal{D}')) \Rightarrow \neg A \notin Cn(\mathcal{E} \cup concl(\mathcal{D}'))\}$$

$$\mathcal{F}^{k+1} = \mathcal{F}^k \cup \{just(\mathcal{D}') \mid \mathcal{D}' \subseteq \mathcal{D}, req(\mathcal{D}') \subseteq \mathcal{E}^k, A \in (\mathcal{F} \cup just(\mathcal{D}')) \Rightarrow \neg A \notin Cn(\mathcal{E} \cup concl(\mathcal{D}'))\}$$

$$\mathcal{E} = \bigcup_{i=0}^{\omega} \mathcal{E}^i \qquad \mathcal{F} = \bigcup_{i=0}^{\omega} \mathcal{F}^i$$

Example 5.3. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as follows:*

$$\mathcal{W} = \varnothing$$

$$\mathcal{D} = \left\{ \frac{\top : p}{\neg p} \right\}$$

$\langle \mathcal{W}, \mathcal{D} \rangle$ has no extensions: for any extension $\mathcal{E}$ of $\langle \mathcal{W}, \mathcal{D} \rangle$, if $\neg p$ belongs to $\mathcal{E}$ then $\neg p$ does not belong to $\mathcal{E}$ and vice versa (since the default rule must be withheld when $\neg p$ is proved, and applied as long as $\neg p$ is not proved).

$\langle \mathcal{W}, \mathcal{D} \rangle$ has a unique modified extension $\langle \mathcal{E}, \mathcal{F} \rangle = \langle Cn(\varnothing), \varnothing \rangle$: $\neg p$ does not belong to $\mathcal{E}$ (since the justification of the default rule is not consistent with its conclusion).

If $\langle \mathcal{W}, \mathcal{D} \rangle$ is a default theory, then A is a *(modified) credulous consequence* of $\mathcal{W}$ if and only if A belongs to at least one (modified) extension of $\langle \mathcal{W}, \mathcal{D} \rangle$ and *(modified) skeptical consequence* of $\mathcal{W}$ whenever A belongs to all (modified) extensions of $\langle \mathcal{W}, \mathcal{D} \rangle$. Intuitively, (modified) credulous and skeptical consequence describe two different attitudes on the part of an ideal reasoner making defeasible inferences: the credulous agent commits to as many individually consistent beliefs as possible, whereas the skeptical one refutes any potentially conflicting conclusion [166]. It can be easily shown that (modified) credulous and skeptical consequence relations are supraclassical while being non-monotonic: they support conclusions that may not be preserved upon addition of arbitrary premises [92].

Example 5.4. *Let $\mathcal{W}, \mathcal{D}_3$ be as in Example 5.1, and $\mathcal{W}'$ be defined as follows:*

$$\mathcal{W}' = \{p, q, \neg r, t\}$$

The set of modified credulous consequences of $\mathcal{W}$ *is* $Cn(\{p, q, s, r\}) \cup Cn(\{p, q, \neg s, r\})$*, whereas the set of modified skeptical consequences of* $\mathcal{W}$ *is* $Cn(\{p, q, r\})$*. On the other hand,* $\langle \mathcal{W}', \mathcal{D} \rangle$ *has a unique modified extension* $\langle \mathcal{E}, \mathcal{F} \rangle = \langle Cn(\{p, q\}), \{r \wedge s\} \rangle$*: the set of modified credulous (as well as modified skeptical) consequences of* $\mathcal{W}'$ *is* $Cn(\{p, q\})$*.*

The following proposition establishes that, for any default theory $\langle \mathcal{W}, \mathcal{D} \rangle$, the set of modified credulous consequences of $\mathcal{W}$ is not empty, and that the set of credulous consequences of $\mathcal{W}$ is not empty when $\langle \mathcal{W}, \mathcal{D} \rangle$ is a normal default theory:

PROPOSITION 5.1. If $\langle \mathcal{W}, \mathcal{D} \rangle$ is a default theory, then the following holds:

- (i) there exists (at least) one modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W}, \mathcal{D} \rangle$;
- (ii) if $\langle \mathcal{W}, \mathcal{D} \rangle$ is a normal default theory, then $\langle \mathcal{E}, \mathcal{F} \rangle$ is a modified extension of $\langle \mathcal{W}, \mathcal{D} \rangle$ if and only if $\mathcal{E}$ is an extension of $\langle \mathcal{W}, \mathcal{D} \rangle$.

PROOF. For proofs see [7, pp. 76-77]. For a direct proof of the existence of extensions for normal default theories, see [7, pp. 49-50]. ☐

We write $C_{nm}(\Gamma)$ to denote the set of consequences of Γ according to a (supraclassical and) non-monotonic consequence relation. The following proposition establishes that the (modified) credulous and skeptical consequence relations satisfy the Consistency Preservation property – i.e., the fact that $C_{nm}(\Gamma)$ remains classically consistent when Γ is classically consistent:

PROPOSITION 5.2. If $\langle \mathcal{W}, \mathcal{D} \rangle$ is a default theory, then a (modified) extension $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W}, \mathcal{D} \rangle$ is inconsistent if and only if $\mathcal{W}$ is inconsistent.

PROOF. For extensions, a proof of the result is furnished in [7], p. 43. For modified extensions, the result trivially follows from the definition of modified extension. ☐

A non-monotonic consequence relation satisfies Cautious Monotony if $C_{nm}(\Gamma) \subseteq C_{nm}(\Delta)$ when $\Gamma \subseteq \Delta \subseteq C_{nm}(\Gamma)$, and Cumulative Cut when $C_{nm}(\Delta) \subseteq C_{nm}(\Gamma)$ if $\Gamma \subseteq \Delta \subseteq C_{nm}(\Gamma)$. A non-monotonic consequence relation is said to be *cumulative* if it satisfies both Cautious Monotony and Cumulative Cut – i.e., if the addition of consequences to the set of premises does not change the set of consequences, thus allowing the use of intermediate lemmas. It can be proved that the (modified) skeptical consequence relation always satisfies Cumulative Cut:

PROPOSITION 5.3. Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be a default theory. If $\langle \mathcal{E}, \mathcal{F} \rangle$ is a (modified) extension of $\langle \mathcal{W}, \mathcal{D} \rangle$, and $\mathcal{E}' \subseteq \mathcal{E}$, then $\langle \mathcal{E}, \mathcal{F} \rangle$ is a (modified) extension of $\langle \mathcal{W} \cup \mathcal{E}', \mathcal{D} \rangle$.

Proof. For extensions, the result can be proved as in [7, p. 44]: the proof for modified extensions is completely analogous. ☐

However, the (modified) skeptical consequence relation fails to be cumulative:

Proposition 5.4. Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be a default theory. If $\langle \mathcal{E}, \mathcal{F} \rangle$ is a (modified) extension of $\langle \mathcal{W}, \mathcal{D} \rangle$, $\mathcal{E}' \subseteq \mathcal{E}$ and $\langle \mathcal{E}'', \mathcal{F}'' \rangle$ is a (modified) extension of $\langle \mathcal{W} \cup \mathcal{E}', \mathcal{D} \rangle$, then $\langle \mathcal{E}'', \mathcal{F}'' \rangle$ may not be a (modified) extension of $\langle \mathcal{W}, \mathcal{D} \rangle$.

Proof. It suffices to consider the following classical example in [91]. Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as follows:

$$\mathcal{W} = \varnothing$$

$$\mathcal{D} = \left\{ \frac{\top : p}{p} \ , \ \frac{p \vee q : \neg p}{\neg p} \right\}$$

$\langle \mathcal{W}, \mathcal{D} \rangle$ has a unique (modified) extension $\langle \mathcal{E}_1, \mathcal{F}_1 \rangle = \langle Cn(\{p\}), \{p\} \rangle$: as a consequence, $\{p \vee q\} \subseteq \mathcal{E}_1$. On the other hand, $\langle \mathcal{W} \cup \{p \vee q\}, \mathcal{D} \rangle$ has two (modified) extensions – namely $\langle \mathcal{E}_1, \mathcal{F}_1 \rangle = \langle Cn(\{p\}), \{p\} \rangle$ and $\langle \mathcal{E}_2, \mathcal{F}_2 \rangle = \langle Cn(\{\neg p, q\}), \{\neg p\} \rangle$: we have that $\{p \vee q\} \subseteq \mathcal{E}_2$ and that $\langle \mathcal{E}_2, \mathcal{F}_2 \rangle$ is not a (modified) extension of $\langle \mathcal{W}, \mathcal{D} \rangle$. ☐

Any default theory $\langle \mathcal{W}, \mathcal{D} \rangle$ enjoys the computational property of semi-monotonicity with respect to modified extensions – namely, the fact that no modified credulous consequence of $\mathcal{W}$ can be ruled out by the addition of new default rules:

Proposition 5.5. Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be a default theory, and $\mathcal{D} \subseteq \mathcal{D}'$. For any modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W}, \mathcal{D} \rangle$ there exists (at least) one modified extension $\langle \mathcal{E}', \mathcal{F}' \rangle$ of $\langle \mathcal{W}, \mathcal{D}' \rangle$ such that $\mathcal{E} \subseteq \mathcal{E}'$ and $\mathcal{F} \subseteq \mathcal{F}'$.

Proof. For a proof see [90, pp. 12-13]. ☐

Two (modified) extensions $\langle \mathcal{E}_1, \mathcal{F}_1 \rangle$ and $\langle \mathcal{E}_2, \mathcal{F}_2 \rangle$ of $\langle \mathcal{W}, \mathcal{D} \rangle$ are orthogonal if $\mathcal{E}_1 \cup \mathcal{E}_2$ is inconsistent. It's worth noting that orthogonality of (modified) extensions may not hold:

Example 5.5. *Let* $\langle \mathcal{W}, \mathcal{D} \rangle$ *be defined as follows:*

$$\mathcal{W} = \varnothing$$

$$\mathcal{D} = \left\{ \frac{\top : p \wedge q}{q}, \ \frac{\top : \neg q \wedge \neg p}{\neg p} \right\}$$

We have that $\langle \mathcal{W}, \mathcal{D} \rangle$ *has two modified extensions* $\langle \mathcal{E}_1, \mathcal{F}_1 \rangle = \langle Cn(\{q\}), \{p \wedge q\} \rangle$ *and* $\langle \mathcal{E}_2, \mathcal{F}_2 \rangle = \langle Cn(\{\neg p\}), \{\neg q \wedge \neg p\} \rangle$.

AXIOMS

$$\frac{}{\Gamma, \Theta \vdash \Lambda, \Delta \mid \mathcal{R}}\ ax$$

LOGICAL RULES

$$\frac{\Gamma \vdash \Delta, A \mid \mathcal{R}}{\Gamma, \neg A \vdash \Delta \mid \mathcal{R}}\ L\neg \qquad\qquad \frac{\Gamma, A \vdash \Delta \mid \mathcal{R}}{\Gamma \vdash \Delta, \neg A \mid \mathcal{R}}\ R\neg$$

$$\frac{\Gamma, A, B \vdash \Delta \mid \mathcal{R}}{\Gamma, A \wedge B \vdash \Delta \mid \mathcal{R}}\ L\wedge \qquad\qquad \frac{\Gamma \vdash \Delta, A \mid \mathcal{R}_1 \qquad \Gamma \vdash \Delta, B \mid \mathcal{R}_2}{\Gamma \vdash \Delta, A \wedge B \mid \mathcal{R}_1 \mid \mathcal{R}_2}\ R\wedge$$

$$\frac{\Gamma, A \vdash \Delta \mid \mathcal{R}_1 \qquad \Gamma, B \vdash \Delta \mid \mathcal{R}_2}{\Gamma, A \vee B \vdash \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2}\ L\vee \qquad\qquad \frac{\Gamma \vdash \Delta, A, B \mid \mathcal{R}}{\Gamma \vdash \Delta, A \vee B \mid \mathcal{R}}\ R\vee$$

$$\frac{\Gamma \vdash \Delta, A \mid \mathcal{R}_1 \qquad \Gamma, B \vdash \Delta \mid \mathcal{R}_2}{\Gamma, A \to B \vdash \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2}\ L\to \qquad\qquad \frac{\Gamma, A \vdash \Delta, B \mid \mathcal{R}}{\Gamma \vdash \Delta, A \to B \mid \mathcal{R}}\ R\to$$

EXTRA-LOGICAL RULES

$$\frac{\{\Gamma, \Phi, \Theta_i \vdash \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} \qquad \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\ \delta$$

Figure 1. $\mathsf{HG4}$ hypersequent calculi for $\langle \mathcal{W}, \mathcal{D} \rangle$

5.2. Hypersequent calculi for modified credulous consequence

5.2.1. The proof-theoretic platform. To formally describe an ideal reasoner's process of augmenting her initial base through the application of default rules, we propose an alternative version of the hypersequent framework.

Hypersequents are denoted by capital Latin letters $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \ldots$. In proof-theoretic literature, a hypersequent is a sequence of sequents which are separated by the structural connective '|': e.g., $\mathcal{H} = \Gamma_1 \vdash \Delta_1 \mid \cdots \mid \Gamma_n \vdash \Delta_n$. Usually, hypersequents are taken to be multisets instead of sequences: e.g., $\mathcal{H} = \Gamma_{\pi(1)} \vdash \Delta_{\pi(1)} \mid \cdots \mid \Gamma_{\pi(n)} \vdash \Delta_{\pi(n)}$ for any permutation π on $1, \ldots, n$. The intended interpretation of a hypersequent $\Gamma_1 \vdash \Delta_1 \mid \cdots \mid \Gamma_n \vdash \Delta_n$ is that $\Gamma_i \vdash \Delta_i$ is provable, for some $1 \leq i \leq n$. In other words, the bar symbol '|' receives an interpretation in terms of disjunction: if the formula translation of a sequent $\Gamma \vdash \Delta$ is $\bigwedge \Gamma \to \bigvee \Delta$, then the formula translation of the hypersequent $\Gamma_1 \vdash \Delta_1 \mid \cdots \mid \Gamma_n \vdash \Delta_n$ is $\bigvee\limits_{i=1}^{n} (\bigwedge \Gamma_i \to \bigvee \Delta_i)$.

In this chapter, we employ a non-standard notion of hypersequent, whereby a hypersequent is a *set* made up of a sequent $\Gamma \vdash \Delta$ and a (possibly, empty) set of antisequents $\Gamma_1 \dashv \Delta_1, \cdots, \Gamma_n \dashv \Delta_n$: e.g., $\mathcal{H} = \Gamma \vdash \Delta \mid \Gamma_1 \dashv \Delta_1 \mid \cdots \mid \Gamma_n \dashv \Delta_n$ is a hypersequent, with $\Gamma_i \neq \Gamma_j$ or $\Delta_i \neq \Delta_j$ for any $1 \leq j \neq i \leq n$, and with $\mathcal{H} = \Gamma \vdash \Delta \mid \Gamma_{\pi(1)} \dashv \Delta_{\pi(1)} \mid \cdots \mid \Gamma_{\pi(n)} \dashv \Delta_{\pi(n)}$ for any permutation π defined on $1, \ldots, n$. The intended interpretation of a hypersequent $\Gamma \vdash \Delta \mid \Gamma_1 \dashv \Delta_1 \mid \cdots \mid \Gamma_n \dashv \Delta_n$ is that $\Gamma \vdash \Delta$ is provable and $\Gamma_k \vdash \Delta_k$ is refutable, for any $1 \leq k \leq n$: in this case, the bar symbol '|' is *metalogically* interpreted as conjunction. With both sequents and antisequents present, our hypersequents do not have formula translations: they serve as syntactical objects expressing *contrary updating* on the provability of the included sequents. For any hypersequent $\mathcal{H}$, we denote the sets of antisequents in $\mathcal{H}$ with capital Latin letters $\mathcal{R}, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{S}, \mathcal{S}_1, \mathcal{S}_2, \ldots$. Specifically, in any hypersequent $\mathcal{H}$ of the form $\Gamma \vdash \Delta \mid \mathcal{R}$, the letter $\mathcal{R}$ designates the *refutational part* of $\mathcal{H}$.

The following definition introduces our Genzten-style calculi featuring hybrid hypersequents.

DEFINITION 5.1. Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be any default theory. Any hypersequent calculus $\mathsf{HG4}$ for $\langle \mathcal{W}, \mathcal{D} \rangle$ can be defined by adopting the rules in Figure 1, provided that the conditions $(i) - (vi)$ below are fulfilled.

- (i) For each instance of ax, either $\Theta \cap \Lambda \neq \varnothing$ or $\Theta \models^{*} \Lambda$ is a complementary clause in $\mathsf{top}^{\star}_{\mathsf{c}}(\models^{*} W)$, with W being the conjunction of formulas in $\mathcal{W}$.
- (ii) For each instance of ax, the refutational part of the conclusion must include a set $\mathcal{R}'$ of antisequents.
- (iii) For any default rule of the form $\dfrac{B : C_1, \ldots, C_k}{D}$ in $\mathcal{D}$, if $\mathsf{top}_{\mathsf{c}}(\models^{*} B) = \{\Theta_1 \models^{*} \Lambda_1, \ldots, \Theta_m \models^{*} \Lambda_m\}$ and $\Phi \models^{*} \Psi$ belongs to $\mathsf{top}^{\star}_{\mathsf{c}}(\models^{*} D)$, then there exists an extra-logical rule of the following form:

$$\frac{\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} \qquad \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k} \, \delta$$

- (iv) For any prerequisite-free default rule of the form $\dfrac{\top : C_1, \ldots, C_k}{D}$ in $\mathcal{D}$, if $\Phi \models^{*} \Psi$ belongs to $\mathsf{top}^{\star}_{\mathsf{c}}(\models^{*} D)$, then there exists an extra-logical rule of the following form:

$$\frac{\Gamma, p \vdash p, \Delta \mid \mathcal{R}_1}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k} \, \delta$$

- (v) For any extra-logical rule

$$\frac{\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} \qquad \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k} \, \delta$$

if the atomic (anti)sequent $\Phi' \mathrel{|\!\overset{*}{-}} \Psi'$ occurs in $\mathsf{top}^{\star}_{\mathsf{c}}(\mathrel{|\!\overset{*}{-}} (\bigwedge \Phi \to \bigvee \Psi) \wedge W)$ without belonging to $\mathsf{top}^{\star}_{\mathsf{c}}(\mathrel{|\!\overset{*}{-}} W)$, then there exists an extra-logical rule of the following form:

$$\dfrac{\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} \qquad \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}{\Gamma, \Phi' \vdash \Psi', \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\;\delta$$

(vi) For any pair of extra-logical rules

$$\dfrac{\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} \qquad \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\;\delta$$

$$\dfrac{\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{m+1 \leq i \leq n} \qquad \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{m+1 \leq i \leq n}}{\Gamma, \Phi' \vdash \Psi', \Delta \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_{m+1} \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_{k'}}\;\delta$$

if the atomic (anti)sequent $\Xi \mathrel{|\!\overset{*}{-}} \Omega$ occurs in $\mathsf{top}^{\star}_{\mathsf{c}}(\mathrel{|\!\overset{*}{-}} (\bigwedge \Phi \to \bigvee \Psi) \wedge (\bigwedge \Phi' \to \bigvee \Psi'))$ without belonging to $\mathsf{top}^{\star}_{\mathsf{c}}(\mathrel{|\!\overset{*}{-}} (\bigwedge \Phi \to \bigvee \Psi))$, $\mathsf{top}^{\star}_{\mathsf{c}}(\mathrel{|\!\overset{*}{-}} (\bigwedge \Phi' \to \bigvee \Psi'))$ or $\mathsf{top}^{\star}_{\mathsf{c}}(\mathrel{|\!\overset{*}{-}} W)$, then there exists an extra-logical rule of the following form:

$$\dfrac{\{\Gamma, \Theta_j \vdash \Lambda_j, \Delta \mid \mathcal{R}_j\}_{1 \leq j \leq n} \qquad \{\Theta_j \vdash \Lambda_j \mid \mathcal{R}'_j\}_{1 \leq j \leq n}}{\Gamma, \Xi \vdash \Omega, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_{k'}}\;\delta$$

The conditions $(i) - (ii)$ in Definition 5.1 ensure that HG4 calculi incorporate the extra-logical axioms from $\mathcal{W}$, whereas the conditions $(iii) - (iv)$ guarantee that HG4 calculi feature extra-logical rules corresponding to the default rules occurring in $\mathcal{D}$. On the other hand, the closure conditions (v) and (vi) are needed in order to ensure that HG4 calculi enjoy Cut elimination (cf. Theorem 5.1).

Notice that HG4 calculi feature non-unary rules, even if the separating bar within hypersequents is interpreted conjunctively: this is due to the fact that hybrid hypersequents are defined as *single* sequents alongside (possibly, empty) sets of antisequents (cf. [126]).

Example 5.6. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as follows:*

$$\mathcal{W} = \{p \to q, q \to r\}$$

$$\mathcal{D} = \left\{ \frac{p \to q : \neg r}{s \vee t} \;,\; \frac{q \to r : r}{\neg s \vee t} \right\}$$

Any HG4 *calculus for $\langle \mathcal{W}, \mathcal{D} \rangle$ features*

$$\Gamma, p \vdash q, \Delta \mid \mathcal{R} \qquad \Gamma, q \vdash r, \Delta \mid \mathcal{R} \qquad \Gamma, p \vdash r, \Delta \mid \mathcal{R}$$

as initial hypersequents, for any $\mathcal{R}$ including some $\mathcal{R}'$, and as extra-logical rules:

$$\dfrac{\Gamma, p \vdash q, \Delta \mid \mathcal{R}_1 \qquad p \vdash q \mid \mathcal{R}'_1}{\Gamma \vdash s, t, \Delta \mid \mathcal{R}_1 \mid \mathcal{R}'_1 \mid \dashv \neg\neg r}\;\delta \qquad \dfrac{\Gamma, q \vdash r, \Delta \mid \mathcal{S}_1 \qquad q \vdash r \mid \mathcal{S}'_1}{\Gamma, s \vdash t, \Delta \mid \mathcal{S}_1 \mid \mathcal{S}'_1 \mid \dashv \neg r}\;\delta'$$

$$\dfrac{\Gamma, p \vdash q, \Delta \mid \mathcal{R}_1 \qquad \Gamma, q \vdash r, \Delta \mid \mathcal{S}_1 \qquad p \vdash q \mid \mathcal{R}'_1 \qquad q \vdash r \mid \mathcal{S}'_1}{\Gamma \vdash t, t, \Delta \mid \mathcal{R}_1 \mid \mathcal{S}_1 \mid \mathcal{R}'_1 \mid \mathcal{S}'_1 \mid \dashv \neg\neg r \mid \dashv \neg r}\;\delta''$$

$$\dfrac{\Gamma, p \vdash q, \Delta \mid \mathcal{R}_1 \qquad \Gamma, q \vdash r, \Delta \mid \mathcal{S}_1 \qquad p \vdash q \mid \mathcal{R}'_1 \qquad q \vdash r \mid \mathcal{S}'_1}{\Gamma \vdash t, \Delta \mid \mathcal{R}_1 \mid \mathcal{S}_1 \mid \mathcal{R}'_1 \mid \mathcal{S}'_1 \mid \dashv \neg\neg r \mid \dashv \neg r} \, \delta'''$$

For any HG4 calculus and each extra-logical rule δ in it, we define the label $\delta_{\mathcal{D}'}$ as follows:

- (i) if δ is generated in accordance with point (iii) in Definition 5.1, then $\mathcal{D}' = \left\{ \dfrac{B : C_1, \ldots, C_k}{D} \right\}$;
- (ii) if δ is generated in accordance with point (iv) in Definition 5.1 above, then $\mathcal{D}' = \left\{ \dfrac{\top : C_1, \ldots, C_k}{D} \right\}$;
- (iii) if δ is generated from an extra-logical rule δ' with label $\delta_{\mathcal{D}_1}$ in accordance with point (v) in Definition 5.1, then $\mathcal{D}' = \mathcal{D}_1$;
- (iv) if δ is generated from an extra-logical rule δ' with label $\delta_{\mathcal{D}_1}$ and an extra-logical rule δ'' with label $\delta_{\mathcal{D}_2}$ in accordance with point (vi) in Definition 5.1, then $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2$.

Intuitively, for any application of an extra-logical rule δ in an HG4-derivation π, the label $\delta_{\mathcal{D}'}$ keeps record of the default rules encoded in δ.

EXAMPLE 5.7. *Let* $\langle \mathcal{W}, \mathcal{D} \rangle$ *be as in Example 5.6.* $\mathcal{D}'$ *is* $\left\{ \dfrac{p \to q : \neg r}{s \vee t} \right\}$ *in* $\delta_{\mathcal{D}'}$, $\left\{ \dfrac{q \to r : r}{\neg s \vee t} \right\}$ *in* $\delta'_{\mathcal{D}'}$ *and* $\mathcal{D}$ *in* $\delta''_{\mathcal{D}'}, \delta'''_{\mathcal{D}'}$.

We conclude the general presentation of HG4 calculi with a brief comment on the structure of extra-logical rules.

REMARK 6. We may formulate extra-logical rules in HG4 calculi in an alternative way, with principal formulas repeated in the left-hand premises:

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ \{\Gamma, \Phi, \Theta_i \vdash \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} & \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m} \end{array}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k} \, \delta_{\mathcal{D}'}$$

Adopting this formulation in the definition of HG4 calculi would offer certain advantages. Firstly, repetitions in the premises of extra-logical rules would be eliminated whenever $\Gamma, \Delta = \varnothing$ and $\mathcal{R}_i = \mathcal{R}'_i$ for some $1 \leq i \leq m$. Moreover, HG4-derivations would rely exclusively on atomic information already present in the initial hypersequents. Importantly, all results regarding HG4-derivations established in Sections 5.2 and 5.3 would remain unaffected. However, this approach would result in a less economical presentation of extra-logical rules compared to the one currently adopted in Definition 5.1.

Nonetheless, this formulation remains intriguing, as it characterizes default rules as constrained instances of Strengthening. To the best of our knowledge, Strengthening rules are exclusively discussed in the context of refutational systems (cf. Proposition 2.3). This would mark the first instance where constrained versions of these rules are employed within deductive systems.

To address *modified credulous consequence* (in short, m-credulous consequence) based on $\langle \mathcal{W}, \mathcal{D} \rangle$, we first define the hypersequent calculus $\mathsf{HG4c}$ as the $\mathsf{HG4}$ calculus for which the set $\mathcal{R}'$ in the refutational part of initial hypersequents is *empty*. It is easy to find cases where there exists (at least) one $\mathsf{HG4c}$-derivation of a hypersequent $\Gamma \vdash \Delta \mid \mathcal{R}$ such that $\bigwedge \Gamma \to \bigvee \Delta$ is not an m-credulous consequence of $\mathcal{W}$.

EXAMPLE 5.8. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as follows:*

$$\mathcal{W} = \{p, q\}$$

$$\mathcal{D} = \left\{ \frac{p : r}{r}\ , \frac{q : \neg r}{\neg r} \right\}$$

$\langle \mathcal{W}, \mathcal{D} \rangle$ has two modified extensions – namely, $\langle \mathcal{E}_1, \mathcal{F}_1 \rangle = \langle Cn(\{p, q, r\}), \{r\} \rangle$ and $\langle \mathcal{E}_2, \mathcal{F}_2 \rangle = \langle Cn(\{p, q, \neg r\}), \{\neg r\} \rangle$.
On the other hand, this is a derivation in the $\mathsf{HG4c}$ *calculus for $\langle \mathcal{W}, \mathcal{D} \rangle$:*

$$\dfrac{\delta_{\mathcal{D}'}\,\dfrac{ax\,\dfrac{}{\vdash p} \qquad ax\,\dfrac{}{\vdash p}}{\vdash r \mid \dashv \neg r} \qquad \dfrac{\dfrac{\dfrac{}{\vdash q}\,ax \qquad \dfrac{}{\vdash q}\,ax}{r \vdash \mid \dashv \neg\neg r}\,\delta_{\mathcal{D}''}}{\vdash \neg r \mid \dashv \neg\neg r}\,R\neg}{\vdash r \wedge \neg r \mid \dashv \neg r \mid \dashv \neg\neg r}\,R\wedge$$

The formula $r \wedge \neg r$ belongs neither to $\mathcal{E}_1$ nor to $\mathcal{E}_2$.

As a result, we need to offer a criterion to single out $\mathsf{HG4c}$-derivations which deliver m-credulous consequences of $\mathcal{W}$ from those which do not. To this aim, we make some additions to our terminological and conceptual apparatus.

For each $\mathsf{HG4}$-derivation π we say that a default rule $\dfrac{B : C_1, \dots, C_k}{D}$ *belongs to $def(\pi)$* if and only if there exists (at least) one extra-logical rule labelled $\delta_{\mathcal{D}'}$ which is applied in π and such that $\dfrac{B : C_1, \dots, C_k}{D}$ belongs to $\mathcal{D}'$. We shall use D_π to refer to the conjunction of the formulas in *concl*$(def(\pi))$.

DEFINITION 5.2. An $\mathsf{HG4}$-derivation π of $\Gamma \vdash \Delta \mid \Pi_1 \dashv \Sigma_1 \mid \cdots \mid \Pi_n \dashv \Sigma_n$ is a *proof* if $\overline{\overline{\mathsf{G4}}}$ refutes $W, D_\pi \vdash \bigwedge \Pi_i \to \bigvee \Sigma_i$ for each $1 \leq i \leq n$, and a *paraproof* otherwise.

To make the reader familiar with the notions of $\mathsf{HG4c}$-proof and $\mathsf{HG4c}$-paraproof, we present some examples.

EXAMPLE 5.9. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be as in Example 5.8. The following* $\mathsf{HG4c}$*-derivations are proofs:*

$$\delta_{\mathcal{D}'}\,\dfrac{ax\,\dfrac{}{s \vdash p} \qquad \dfrac{}{\vdash p}\,ax}{s \vdash r \mid \dashv \neg r} \qquad\qquad \dfrac{\dfrac{ax\,\dfrac{}{s \vdash q} \qquad \dfrac{}{\vdash q}\,ax}{s, r \vdash \mid \dashv \neg\neg r}\,\delta_{\mathcal{D}''}}{s \vdash \neg r \mid \dashv \neg\neg r}\,R\neg$$

Notice that, in both HG4c*-proofs, the atom s is an auxiliary formula in the application of the extra-logical rule.*

Example 5.10. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as follows:*

$$\mathcal{W} = \{p, q\}$$

$$\mathcal{D} = \left\{ \frac{p : r}{r} \ , \ \frac{q : \neg r}{\neg r} \ , \ \frac{r : s}{s} \ , \ \frac{\neg r : \neg s}{\neg s} \right\}$$

The following HG4c*-derivations are proofs:*

$$\delta_{\mathcal{D}'''}\,\dfrac{\delta_{\mathcal{D}'}\,\dfrac{ax\,\dfrac{}{t \vdash p} \qquad ax\,\dfrac{}{\vdash p}}{t \vdash r \mid \dashv \neg r} \qquad \dfrac{\dfrac{}{\vdash p}\,ax \qquad \dfrac{}{\vdash p}\,ax}{\vdash r \mid \dashv \neg r}\,\delta_{\mathcal{D}'}}{t \vdash s \mid \dashv \neg r \mid \dashv \neg s} \qquad \dfrac{R\neg\,\dfrac{\delta_{\mathcal{D}''}\,\dfrac{ax\,\dfrac{}{t \vdash q} \qquad ax\,\dfrac{}{\vdash q}}{t, r \vdash \ \mid \dashv \neg\neg r}}{t \vdash \neg r \mid \dashv \neg\neg r} \qquad \dfrac{\dfrac{\dfrac{}{\vdash q}\,ax \qquad \dfrac{}{\vdash q}\,ax}{r \vdash \ \mid \dashv \neg\neg r}\,\delta_{\mathcal{D}''}}{\vdash \neg r \mid \dashv \neg\neg r}\,R\neg}{t \vdash \neg s \mid \dashv \neg\neg r \mid \dashv \neg\neg s}\,\delta_{\mathcal{D}''''}$$

Example 5.11. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as follows (cf. the proof of Proposition 5.5):*

$$\mathcal{W} = \varnothing$$

$$\mathcal{D} = \left\{ \frac{\top : p}{p} \ , \ \frac{p \vee q : \neg p}{\neg p} \right\}$$

The following HG4c*-derivation is a paraproof:*

$$\dfrac{\delta_{\mathcal{D}'}\,\dfrac{ax\,\dfrac{}{p \vdash p, q}}{\vdash p, q \mid \dashv \neg p} \qquad \dfrac{\dfrac{}{p \vdash p, q}\,ax}{\vdash p, q \mid \dashv \neg p}\,\delta_{\mathcal{D}'}}{p \vdash q \mid \dashv \neg p \mid \dashv \neg\neg p}\,\delta_{\mathcal{D}''}$$

The applications of extra-logical rules in HG4c-derivations correspond to applications of default rules (possibly followed by applications of modus ponens) in Hilbert-style derivations. As a result, any HG4c-derivation π can be viewed as an attempt to construct a modified extension of the underlying default theory $\langle \mathcal{W}, \mathcal{D} \rangle$. However, this attempt may fail in certain cases (cf. Example 5.11). To address such scenarios, the notions of HG4c-proof and HG4c-paraproof are introduced, providing a mechanism to determine whether the sequent at the conclusion of the derivation represents an m-credulous consequence of $\mathcal{W}$ or not.

5.2.2. Proof-theoretic results. In this subsection, we show that provability in the HG4c calculus is sound and (weakly) complete with respect to m-credulous consequence based on $\langle \mathcal{W}, \mathcal{D} \rangle$. To achieve this, we present a structural analysis of HG4 calculi for $\langle \mathcal{W}, \mathcal{D} \rangle$.

Let us begin by stating that a rule of the form

$$\frac{\Gamma_1 \vdash \Delta_1 \mid \mathcal{R}_1 \qquad \cdots \qquad \Gamma_n \vdash \Delta_n \mid \mathcal{R}_n}{\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}$$

is *admissible* in HG4 if the hypersequent $\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n$ is provable whenever the hypersequents $\{\Gamma_i \vdash \Delta_i \mid \mathcal{R}_i\}_{1 \leq i \leq n}$ are provable. Notice that, unlike G4s calculi, HG4 calculi may feature (non-binary) derivable rules which are not admissible:

EXAMPLE 5.12. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be as in Example 5.3. The following rule is not admissible in* HG4c*:*

$$\frac{\Gamma, p \vdash p, \Delta \mid \mathcal{R}}{\Gamma, p \vdash \Delta \mid \mathcal{R} \mid \dashv \neg p}\ \delta_{\mathcal{D}'}$$

PROPOSITION 5.6. The rules of Left and Right Weakening

$$\frac{\Gamma \vdash \Delta \mid \mathcal{R}}{A, \Gamma \vdash \Delta \mid \mathcal{R}} \qquad \frac{\Gamma \vdash \Delta \mid \mathcal{R}}{\Gamma \vdash \Delta, A \mid \mathcal{R}}$$

are height-preserving admissible in HG4.

PROOF. We focus on Left Weakening, reasoning by induction on the height of an HG4-proof π of $\Gamma \vdash \Delta \mid \mathcal{R}$: as usual, the height of π is taken to be the number of nodes in a branch of maximal length. If $h(\pi) \leq n+1$ and the last rule in π is an extra-logical one, then π has the form

$$\frac{\begin{matrix} \vdots_{\pi_i} & & \vdots \\ \{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} & & \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m} \end{matrix}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\ \delta_{\mathcal{D}'}$$

By induction hypothesis, the hypersequents $\{A, \Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}$ have HG4-derivations whose height does not surpass n. In particular, we consider HG4-derivations $\pi'_1, \dots, \pi'_m$ with $h(\pi'_i) \leq n$ and such that the extra-logical rules applied in π'_i are the same as those applied in π_i for any $1 \leq i \leq m$: as a result, we have that $D_{\pi'_i} = D_{\pi_i}$ for any $1 \leq i \leq m$, and thus that $\pi'_1, \dots, \pi'_m$ are HG4-proofs. Hence, by a single application of $\delta_{\mathcal{D}'}$ we get an HG4-derivation π' of $A, \Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k$, with $h(\pi') \leq n+1$: since $D_\pi = D_{\pi'}$, we conclude that π' is an HG4-proof. ☐

In standard hypersequent calculi, External Weakening rules permit the addition of a component sequent to a given hypersequent [15]. In our framework, however, hypersequents consist of only one sequent and a (potentially empty) set of antisequents. Consequently, any application of External Weakening to a hybrid hypersequent must be understood as adding a component *antisequent* to its refutational part. If an HG4-derivation π ends with an External Weakening application, and its immediate subderivation π_1 is an HG4-proof, then we consider the External Weakening application *safe* precisely when π is also an HG4-proof.

PROPOSITION 5.7. The rule of safe External Weakening

$$\frac{\Gamma \vdash \Delta \mid \mathcal{R}}{\Gamma \vdash \Delta \mid \mathcal{R} \mid \Pi \dashv \Sigma}$$

is height-preserving admissible in HG4.

Proof. We reason by induction on the height of an HG4-proof π of $\Gamma \vdash \Delta \mid \mathcal{R}$. If $h(\pi) = 1$, we exploit the fact that the refutational part of initial hypersequents of HG4 is arbitrary up to the inclusion of a specific set $\mathcal{R}'$ of antisequents – which is different for each HG4 calculus. Otherwise, we reason by cases over the last rule applied in π as in the proof of Proposition 5.6. $\square$

We say that a rule of the form

$$\frac{\Gamma_1 \vdash \Delta_1 \mid \mathcal{R}_1 \qquad \cdots \qquad \Gamma_n \vdash \Delta_n \mid \mathcal{R}_n}{\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}$$

is *invertible* if and only if a rule of the form

$$\frac{\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}{\Gamma_i \vdash \Delta_i \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}$$

is admissible in HG4, for any $1 \leq i \leq n$.

Proposition 5.8. Logical rules of HG4 are height-preserving invertible.

Proof. By induction on the height of the HG4-proofs of hypersequents $B \circ C, \Gamma \vdash \Delta \mid \mathcal{R}$ and $\Gamma \vdash \Delta, B \circ C \mid \mathcal{R}$, with $\circ \in \{\wedge, \vee, \to\}$, as well as hypersequents $\neg B, \Gamma \vdash \Delta \mid \mathcal{R}$ and $\Gamma \vdash \Delta, \neg B \mid \mathcal{R}$. Let us illustrate in detail only the case of an HG4-proof π of $B \to C, \Gamma \vdash \Delta \mid \mathcal{R}$, with $h(\pi) \leq n + 1$ and where the last rule is an extra-logical one:

$$\frac{\begin{array}{cc} \vdots_{\pi_i} & \vdots \\ \{B \to C, \Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} & \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m} \end{array}}{B \to C, \Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg D_1 \mid \cdots \mid \dashv \neg D_k}\ \delta_{\mathcal{D}'}$$

By inductive hypothesis, the hypersequents $\{C, \Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}$ and $\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta, B \mid \mathcal{R}_i\}_{1 \leq i \leq m}$ have HG4-derivations whose height does not surpass n. In particular, we consider HG4-derivations $\pi'_1, \ldots, \pi'_m$ and $\pi''_1, \ldots, \pi''_m$ of the hypersequents $\{C, \Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}$ and $\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta, B \mid \mathcal{R}_i\}_{1 \leq i \leq m}$, respectively, with $h(\pi'_i), h(\pi''_i) \leq n$ and such that $\overline{\overline{\mathsf{G4}}}$ proves $D_{\pi_i} \vdash D_{\pi'_i}$ and $D_{\pi_i} \vdash D_{\pi''_i}$ for any $1 \leq i \leq m$: as a result, we have that $\pi'_1, \ldots, \pi'_m$ and $\pi''_1, \ldots, \pi''_m$ are HG4-proofs. Hence, by single applications of $\delta_{\mathcal{D}'}$ we get HG4-derivations π', π'' of $C, \Gamma \vdash \Delta \mid \mathcal{R}$ and $\Gamma \vdash \Delta, B \mid \mathcal{R}$, respectively, with $h(\pi'), h(\pi'') \leq n + 1$ and $\mathcal{R} = \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg D_1 \mid \cdots \mid \dashv \neg D_k$. Since $\overline{\overline{\mathsf{G4}}}$ proves $D_\pi \vdash D_{\pi'}$ and $D_\pi \vdash D_{\pi''}$, we conclude that both π' and π'' are HG4-proofs. $\square$

Every HG4-proof π can be divided into an extra-logical part, where only *ax* and δ rules are applied, and a logical one with all the applications of logical rules:

Proposition 5.9. Any HG4-proof π can be turned into an HG4-proof π' where logical rules are permuted last.

Proof. Let π be an HG4-proof of the following form:

$$\dfrac{\overset{\vdots\,\pi_i}{\{\Gamma,\Theta_i \vdash \Lambda_i,\Delta \mid \mathcal{R}_i\}_{1\leq i\leq m}} \qquad \overset{\vdots\,\pi'_i}{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1\leq i\leq m}}}{\Gamma,\Phi \vdash \Psi,\Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg D_1 \mid \cdots \mid \dashv \neg D_k}\,\delta_{\mathcal{D}'}$$

with no applications of extra-logical rules in the sub-proofs π_i and π'_i, for any $1 \leq i \leq m$. Moreover, let A be a non-atomic formula in (say) Γ: if A is principal in the last rule in π_i, for some $1 \leq i \leq m$, we reason by cases over A's principal connective. For example, if A is $B \to C$, then π has the following form:

$$\dfrac{L\!\to\dfrac{\overset{\vdots\,\pi_{i_1}}{\Gamma',\Theta_i \vdash \Lambda_i,\Delta,B \mid \mathcal{S}_{i_1}} \qquad \overset{\vdots\,\pi_{i_2}}{C,\Gamma',\Theta_i \vdash \Lambda_i,\Delta \mid \mathcal{S}_{i_2}}}{B \to C,\Gamma',\Theta_i \vdash \Lambda_i,\Delta \mid \mathcal{R}_i} \qquad \overset{\vdots\,\pi_j}{\{\Gamma,\Theta_j \vdash \Lambda_j,\Delta \mid \mathcal{R}_j\}_{1\leq j\neq i\leq m}} \qquad \overset{\vdots\,\pi'_i}{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1\leq i\leq m}}}{\Gamma,\Phi \vdash \Psi,\Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg D_1 \mid \cdots \mid \dashv \neg D_k}\,\delta_{\mathcal{D}'}$$

with $\Gamma' = \Gamma \cup [B \to C]$ and $\mathcal{R}_i = \mathcal{S}_{i_1} \mid \mathcal{S}_{i_2}$. By Proposition 5.8, there exist HG4-proofs π_{j_1}, π_{j_2} of hypersequents $\Gamma',\Theta_j \vdash \Lambda_j,\Delta,B \mid \mathcal{R}_j$ and $C,\Gamma',\Theta_j \vdash \Lambda_j,\Delta \mid \mathcal{R}_j$, respectively, such that $\overline{\overline{\mathsf{G4}}}$ proves $D_{\pi_j} \vdash D_{\pi_{j_1}}$ and $D_{\pi_j} \vdash D_{\pi_{j_2}}$ for any $1 \leq j \neq i \leq m$. Now, consider the following HG4c-derivations π'_{i_1} and π'_{i_2}:

$$\dfrac{\overset{\vdots\,\pi_{i_1}}{\Gamma',\Theta_i \vdash \Lambda_i,\Delta,B \mid \mathcal{S}_{i_1}} \qquad \overset{\vdots\,\pi_{j_1}}{\{\Gamma',\Theta_j \vdash \Lambda_j,\Delta,B \mid \mathcal{R}_j\}_{1\leq j\neq i\leq m}} \qquad \overset{\vdots\,\pi'_i}{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1\leq i\leq m}}}{\Gamma',\Phi \vdash \Psi,\Delta,B \mid \mathcal{S}_{i_1} \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_{i-1} \mid \mathcal{R}_{i+1} \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg D_1 \mid \cdots \mid \dashv \neg D_k}\,\delta_{\mathcal{D}'}$$

$$\dfrac{\overset{\vdots\,\pi_{i_2}}{C,\Gamma',\Theta_i \vdash \Lambda_i,\Delta \mid \mathcal{S}_{i_2}} \qquad \overset{\vdots\,\pi_{j_2}}{\{C,\Gamma',\Theta_j \vdash \Lambda_j,\Delta \mid \mathcal{R}_j\}_{1\leq j\neq i\leq m}} \qquad \overset{\vdots\,\pi'_i}{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1\leq i\leq m}}}{C,\Gamma',\Phi \vdash \Psi,\Delta \mid \mathcal{S}_{i_1} \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_{i-1} \mid \mathcal{R}_{i+1} \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg D_1 \mid \cdots \mid \dashv \neg D_k}\,\delta_{\mathcal{D}'}$$

Since $\overline{\overline{\mathsf{G4}}}$ proves $D_{\pi_j} \vdash D_{\pi_{j_1}}$ and $D_{\pi_j} \vdash D_{\pi_{j_2}}$ for any $1 \leq j \neq i \leq m$, we have that π'_{i_1} and π'_{i_2} are HG4c-proofs. Now, let $\mathcal{S}_1$ be $\mathcal{S}_{i_1} \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_{i-1} \mid \mathcal{R}_{i+1} \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg D_1 \mid \cdots \mid \dashv \neg D_k$, and $\mathcal{S}_2$ be $\mathcal{S}_{i_1} \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_{i-1} \mid \mathcal{R}_{i+1} \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg D_1 \mid \cdots \mid \dashv \neg D_k$. We can plug π'_{i_1} and π'_{i_2} into a single HG4-derivation π' as follows:

$$\dfrac{\overset{\vdots\,\pi'_{i_1}}{\Gamma' \vdash \Delta,B \mid \mathcal{S}_1} \qquad \overset{\vdots\,\pi'_{i_2}}{C,\Gamma' \vdash \Delta \mid \mathcal{S}_2}}{\Gamma \vdash \Delta \mid \mathcal{S}_1 \mid \mathcal{S}_2}\,L\!\to$$

Given that $\overline{\overline{\mathsf{G4}}}$ proves $D_\pi \vdash D_{\pi'}$, we conclude that π' is a HG4-proof. On the other hand, since refutational parts are *sets* of antisequents, the conclusion $\Gamma \vdash \Delta \mid \mathcal{S}_1 \mid \mathcal{S}_2$ can be expressed as the hypersequent $\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg D_1 \mid \cdots \mid \dashv \neg D_k$. $\square$

Proposition 5.10. The rules of Left and Right Contraction

$$\frac{A, A, \Gamma \vdash \Delta \mid \mathcal{R}}{A, \Gamma \vdash \Delta \mid \mathcal{R}} \qquad \frac{\Gamma \vdash \Delta, A, A \mid \mathcal{R}}{\Gamma \vdash \Delta, A \mid \mathcal{R}}$$

are height-preserving admissible in $\mathsf{HG4}$.

Proof. We focus on Left Contraction, reasoning by induction on the height of an $\mathsf{HG4}$-proof π of $A, A, \Gamma \vdash \Delta \mid \mathcal{R}$. If $h(\pi) = 1$, point (i) in Definition 5.1 ensures that the set of initial hypersequents of the $\mathsf{HG4}$ calculus is closed under (Left) Contraction. If $h(\pi) \geq n+1$, we reason by cases over the last rule applied in π.

If the last rule applied is an extra-logical rule $\delta_{\mathcal{D}'}$, we must distinguish three subcases, according to whether both occurrences of A are principal in $\delta_{\mathcal{D}'}$, one occurrence of A is principal $\delta_{\mathcal{D}'}$ and the other is not, no occurrence of A is principal in $\delta_{\mathcal{D}'}$. To deal with all these subcases, we take A to be an atomic formula.

In the first case, π has the following form:

$$\dfrac{\begin{matrix}\vdots & \qquad & \vdots \\ \{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} & \qquad & \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}\end{matrix}}{A, A, \Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\;\delta_{\mathcal{D}'}$$

By conditions $(iii) - (vi)$ in Definition 5.1, there exists an extra-logical rule $\delta'_{\mathcal{D}'}$ of the form

$$\dfrac{\begin{matrix}\vdots & \qquad & \vdots \\ \{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} & \qquad & \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}\end{matrix}}{A, \Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\;\delta'_{\mathcal{D}'}$$

Hence, we replace the application of $\delta_{\mathcal{D}'}$ with an application of $\delta'_{\mathcal{D}'}$ to get an $\mathsf{HG4}$-derivation π' of $A, \Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k$: since $D_{\pi'} = D_\pi$, we infer that π' is an $\mathsf{HG4}$-proof.

If one occurrence of A is principal in $\delta_{\mathcal{D}'}$ and the other is not, then π has the following form:

$$\dfrac{\begin{matrix}\vdots & \qquad & \vdots\,\pi_i \\ \{A, \Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m} & \qquad & \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}\end{matrix}}{A, A, \Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\;\delta_{\mathcal{D}'}$$

We exploit Propositions 5.6 and 5.7 to get an $\mathsf{HG4}$-derivation π'_i of the hypersequent $\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}'_i \mid \mathcal{R}_i$ from the $\mathsf{HG4}$-proof π_i of $\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i$, with $h(\pi'_i) \leq h(\pi_i)$ for any $1 \leq i \leq m$. In particular, we leverage Propositions 5.6 and 5.7 to infer that $\overline{\overline{\overline{\mathsf{G4}}}}$ proves $D_{\pi_i} \vdash D_{\pi'_i}$ for any $1 \leq i \leq m$: as a consequence, we obtain that $\pi'_1, \ldots, \pi'_m$ are $\mathsf{HG4}$-proofs. Hence, we plug $\pi'_1, \ldots, \pi'_m$ and $\pi_1, \ldots, \pi_m$ into the following $\mathsf{HG4}$-derivation π':

$$\dfrac{\begin{array}{cc}\vdots_{\pi'_i} & \vdots_{\pi_i}\\ \{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}'_i \mid \mathcal{R}_i\}_{1\leq i\leq m} & \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1\leq i\leq m}\end{array}}{A, \Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\,\delta_{\mathcal{D}'}$$

Since $h(\pi'_i) \leq h(\pi_i)$ for any $1 \leq i \leq m$, we have that $h(\pi') \leq h(\pi)$; on the other hand, π' is an HG4-proof, due to the fact $\overline{\overline{\mathsf{G4}}}$ proves $D_\pi \vdash D_{\pi'}$.

If no occurrence of A is principal in $\delta_{\mathcal{D}'}$, then π has the following form:

$$\dfrac{\begin{array}{cc}\vdots & \vdots_{\pi_i}\\ \{A, A, \Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1\leq i\leq m} & \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1\leq i\leq m}\end{array}}{A, A, \Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\,\delta_{\mathcal{D}'}$$

As in the previous subcase, we exploit Propositions 5.6 and 5.7 to get an HG4-proof π'_i of the hypersequent $A, \Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}'_i \mid \mathcal{R}_i$ with $h(\pi'_i) \leq h(\pi_i)$ for any $1 \leq i \leq m$. Hence, we plug $\pi'_1, \ldots, \pi'_m$ and $\pi_1, \ldots, \pi_m$ into the following HG4-derivation π':

$$\dfrac{\begin{array}{cc}\vdots_{\pi'_i} & \vdots_{\pi_i}\\ \{A, \Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}'_i \mid \mathcal{R}_i\}_{1\leq i\leq m} & \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1\leq i\leq m}\end{array}}{A, \Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\,\delta_{\mathcal{D}'}$$

Since $h(\pi'_i) \leq h(\pi_i)$ and $\overline{\overline{\mathsf{G4}}}$ proves $D_\pi \vdash D_{\pi'}$, we conclude that $h(\pi') \leq h(\pi)$ and π' is an HG4-proof.

On the other hand, if the last rule in π is, say, $L \to$ and A is principal, then π has the form

$$\dfrac{\begin{array}{cc}\vdots_{\pi_1} & \vdots_{\pi_2}\\ B \to C, \Gamma \vdash \Delta, B \mid \mathcal{R}_1 & C, B \to C, \Gamma \vdash \Delta \mid \mathcal{R}_2\end{array}}{B \to C, B \to C, \Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2}\,L \to$$

with $A = B \to C$. By Proposition 5.8, if $h(\pi_1), h(\pi_2) \leq n$, then there exists HG4-derivations π'_1, π'_2 of the hypersequents $\Gamma \vdash \Delta, B, B \mid \mathcal{R}_1$ and $C, C, \Gamma \vdash \Delta \mid \mathcal{R}_2$, respectively, with $h(\pi'_i) \leq h(\pi_i)$ and $D_{\pi_i} \vdash D_{\pi'_i}$ provable in $\overline{\overline{\mathsf{G4}}}$, for $i = 1, 2$. We apply the inductive hypothesis twice to get HG4-derivations π''_1 and π''_2 of the hypersequents $\Gamma \vdash \Delta, B \mid \mathcal{R}_1 \mid \mathcal{R}_2$ and $C, \Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2$, respectively, with $h(\pi''_i) \leq h(\pi'_i)$ and $D_{\pi_i} \vdash D_{\pi'_i}$ provable in $\overline{\overline{\mathsf{G4}}}$, for $i = 1, 2$. Hence, we apply $L \to$ to get an HG4-proof π' of $B \to C, \Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2$ with $h(\pi') \leq n + 1$. ☐

Remark 7. We may formulate extra-logical rules in HG4 calculi in an alternative way, removing the left-hand premises:

$$\dfrac{\begin{array}{c}\vdots\\ \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}_i\}_{1\leq i\leq m}\end{array}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\,\delta_{\mathcal{D}'}$$

Lemmas 5.6 and 5.7 ensure that extra-logical rules of this form are interderivable with the extra-logical rules adopted in Definition 5.1. Moreover, HG4 calculi with extra-logical rules of the form just displayed enjoy the same structural properties as the HG4 calculi presented in Definition 5.1. Removing left-hand premises yields obvious computational advantages: we shall stick to this formulation in Chapter 7. However, let us remark that the formulation of extra-logical rules adopted in Definition 5.1 permits the upward permutation of logical rules – something unattainable with the formulation employed in this chapter. First, consider the following example with a unary logical rule:

$$R\vee\,\dfrac{\delta_{\mathcal{D}'}\,\dfrac{\begin{array}{c}\vdots\\ \Gamma, p \vdash q, \Delta, B, C \mid \mathcal{R}_1\end{array} \qquad \begin{array}{c}\vdots\\ p \vdash q \mid \mathcal{R}_2\end{array}}{\Gamma, r \vdash s, \Delta, B, C \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \mathcal{R}_3}}{\Gamma, r \vdash s, \Delta, B \vee C \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \mathcal{R}_3} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, p \vdash q, \Delta, B, C \mid \mathcal{R}_1\end{array}}{\Gamma, p \vdash q, \Delta, B \vee C \mid \mathcal{R}_1}\,R\vee \qquad \begin{array}{c}\vdots\\ p \vdash q \mid \mathcal{R}_2\end{array}}{\Gamma, r \vdash s, \Delta, B \vee C \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \mathcal{R}_3}\,\delta_{\mathcal{D}'}$$

Next, consider the following example with a binary logical rule:

$$\dfrac{\delta_{\mathcal{D}''}\,\dfrac{\begin{array}{c}\vdots\\ \Gamma, p \vdash q, \Delta, B \mid \mathcal{R}_1\end{array} \qquad \begin{array}{c}\vdots\\ p \vdash q \mid \mathcal{R}_2\end{array}}{\Gamma, r \vdash s, \Delta, B \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \mathcal{R}_3} \qquad \begin{array}{c}\vdots\\ \Gamma, r \vdash s, \Delta, C \mid \mathcal{R}_4\end{array}}{\Gamma, r \vdash s, \Delta, B \wedge C \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \mathcal{R}_3 \mid \mathcal{R}_4}\,L\wedge$$

We perform the permutation as follows:

$$\dfrac{\dfrac{L\wedge\,\dfrac{wk\,\dfrac{\begin{array}{c}\vdots\\ \Gamma, p \vdash q, \Delta, B \mid \mathcal{R}_1\end{array}}{\Gamma, r, p \vdash q, s, \Delta, B \mid \mathcal{R}_1} \qquad wk\,\dfrac{\begin{array}{c}\vdots\\ \Gamma, r \vdash s, \Delta, C \mid \mathcal{R}_4\end{array}}{\Gamma, r, p \vdash q, s, \Delta, C \mid \mathcal{R}_4}}{\Gamma, r, p \vdash q, s, \Delta, B \wedge C \mid \mathcal{R}_1 \mid \mathcal{R}_4} \qquad \begin{array}{c}\vdots\\ p \vdash q \mid \mathcal{R}_2\end{array}}{\Gamma, r, r \vdash s, s, \Delta, B \wedge C \mid \mathcal{R}_1 \mid \mathcal{R}_4 \mid \mathcal{R}_2 \mid \mathcal{R}_3}\,\delta_{\mathcal{D}''}}{\Gamma, r \vdash s, \Delta, B \wedge C \mid \mathcal{R}_1 \mid \mathcal{R}_4 \mid \mathcal{R}_2 \mid \mathcal{R}_3}\,ctr$$

If an HG4-derivation π ends with a Cut application, and its immediate subderivations π_1 and π_2 are HG4-proofs, then we say that the Cut application is *safe* exactly when π is an HG4-proof.

EXAMPLE 5.13. *Let $\langle \mathcal{W}, \mathcal{D}\rangle$ be defined as in Example 5.10. The Cut application in the following* HG4c-*derivation is safe:*

$$\dfrac{\delta_{\mathcal{D}'''}\,\dfrac{\delta_{\mathcal{D}'}\,\dfrac{ax\,\dfrac{}{t \vdash p} \qquad ax\,\dfrac{}{\vdash p}}{t \vdash r \mid \dashv \neg r} \qquad \delta_{\mathcal{D}'}\,\dfrac{ax\,\dfrac{}{\vdash p} \qquad ax\,\dfrac{}{\vdash p}}{\vdash r \mid \dashv \neg r}}{t \vdash s \mid \dashv \neg r \mid \dashv \neg s} \qquad \dfrac{\dfrac{}{s \vdash p}\,ax \qquad \dfrac{}{\vdash p}\,ax}{s \vdash r \mid \dashv \neg r}\,\delta_{\mathcal{D}'}}{t \vdash r \mid \dashv \neg r \mid \dashv \neg s}\,cut$$

EXAMPLE 5.14. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as in Example 5.10. The Cut application in the following* HG4c*-derivation is not safe:*

$$\dfrac{\delta_{\mathcal{D}'}\dfrac{ax\,\dfrac{}{\vdash p, s} \qquad ax\,\dfrac{}{\vdash p}}{\vdash r, s \mid \dashv \neg r} \qquad \dfrac{\dfrac{}{s \vdash q, t}\,ax \qquad \dfrac{}{\vdash q}\,ax}{s \vdash r, t \mid \dashv \neg\neg r}\,\delta_{\mathcal{D}''}}{\vdash r, r, t \mid \dashv \neg r \mid \dashv \neg\neg r}\,cut$$

THEOREM 5.1. *The rule of safe Cut*

$$\dfrac{\Gamma \vdash \Delta, A \mid \mathcal{R}' \qquad A, \Pi \vdash \Sigma \mid \mathcal{R}''}{\Pi, \Gamma \vdash \Delta, \Sigma \mid \mathcal{R}' \mid \mathcal{R}''}$$

is admissible in HG4.

PROOF. We consider the topmost safe Cut application, reasoning by primary induction on the logical complexity of the Cut formula A, and by secondary induction on the sum of the heights of the HG4-proofs of the premises. We argue as in the proof of Theorem 2.3, except for the new configurations generated by the application of extra-logical rules.

Consider the cases arising when the last rule in the HG4-proof of $\Gamma \vdash \Delta, A \mid \mathcal{R}'$ is an extra-logical one. If the Cut formula A is not principal in $\delta_{\mathcal{D}'}$, the derivation has the following form

$$\dfrac{\delta_{\mathcal{D}'}\dfrac{\overset{\vdots}{\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta, A \mid \mathcal{R}_i\}_{1 \leq i \leq m}} \qquad \overset{\vdots}{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}}{\Gamma, \Phi \vdash \Psi, \Delta, A \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k} \qquad \overset{\vdots}{A, \Pi \vdash \Sigma \mid \mathcal{R}''}}{\Pi, \Gamma, \Phi \vdash \Psi, \Delta, \Sigma \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k \mid \mathcal{R}''}\,cut$$

with $\mathcal{R}' = \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k$. If A is not atomic, we rely on Proposition 5.8 to get the conclusion. On the other hand, if A is atomic, it suffices to take the following HG4-derivation:

$$\dfrac{cut\,\dfrac{\overset{\vdots}{\Gamma, \Theta_1 \vdash \Lambda_1, \Delta, A \mid \mathcal{R}_1} \qquad \overset{\vdots}{A, \Pi \vdash \Sigma \mid \mathcal{R}''}}{\Pi, \Gamma, \Theta_1 \vdash \Lambda_1, \Delta, \Sigma \mid \mathcal{R}_1 \mid \mathcal{R}''} \quad \cdots \quad cut\,\dfrac{\overset{\vdots}{\Gamma, \Theta_m \vdash \Lambda_m, \Delta, A \mid \mathcal{R}_m} \qquad \overset{\vdots}{A, \Pi \vdash \Sigma \mid \mathcal{R}''}}{\Pi, \Gamma, \Theta_m \vdash \Lambda_m, \Delta, \Sigma \mid \mathcal{R}_m \mid \mathcal{R}''} \qquad \overset{\vdots}{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}}{\Pi, \Gamma \vdash \Delta, \Sigma \mid \mathcal{R}_1 \mid \mathcal{R}'' \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'' \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\,\delta_{\mathcal{D}'}$$

By Definition 5.2 and secondary inductive hypothesis, this is a cut-free HG4-proof.

On the other hand, if A is principal in $\delta_{\mathcal{D}'}$, the derivation has the following form

$$\dfrac{\delta_{\mathcal{D}'}\dfrac{\overset{\vdots}{\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}} \qquad \overset{\vdots}{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}}{\Gamma, \Phi \vdash \Psi, \Delta, A \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k} \qquad \overset{\vdots}{A, \Pi \vdash \Sigma \mid \mathcal{R}''}}{\Pi, \Gamma, \Phi \vdash \Psi, \Delta, \Sigma \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k \mid \mathcal{R}''}\,cut$$

We reason by cases over the last rule applied in the HG4-proof of $A, \Pi \vdash \Sigma \mid \mathcal{R}''$ to reach the conclusion.

(i) If $A, \Pi \vdash \Sigma \mid \mathcal{R}''$ is an initial hypersequent, $A, \Pi \vdash \Sigma$ is an identity sequent and A is principal, then $\Pi, \Gamma, \Phi \vdash \Psi, \Delta, \Sigma$ is a (possibly) weakened version of $\Gamma \vdash \Delta, A$, and we can exploit Proposition 5.6. On the other hand, if A is not principal, then $\Pi, \Gamma \vdash \Delta, \Sigma$ is an identity sequent. By points (i) and (ii) in Definition 5.1, we have the following $\mathsf{HG4}$-derivation:

$$\frac{}{\Pi, \Gamma, \Phi \vdash \Psi, \Delta, \Sigma \mid \mathcal{R}_1 \mid \mathcal{R}'' \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'' \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\, ax$$

Definition 5.2 ensures that this is an $\mathsf{HG4}$-proof: notice that the refutational part $\mathcal{R}_1 \mid \mathcal{R}'' \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'' \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k$ is just $\mathcal{R}' \mid \mathcal{R}''$.

(ii) If $A, \Pi \vdash \Sigma \mid \mathcal{R}''$ is an initial hypersequent, $A, \Pi \vdash \Sigma$ is not an identity sequent and A is principal, then $A, \Pi \vdash \Sigma$ is of the form $\Pi', A, \Phi' \vdash \Psi', \Sigma'$ for (at least) one clause $A, \Phi' \vdash \Psi'$ in $\mathsf{top}^{\star}_{\mathsf{c}}(\vdash W)$.

If $\overline{\overline{\mathsf{G4}}}$ proves $\Phi', \Phi \vdash \Psi, \Psi'$, we exploit points (i) and (ii) in Definition 5.1 as well as Definition 5.2 to obtain that the following $\mathsf{HG4}$-derivation is a proof:

$$\frac{}{\Pi', \Phi', \Gamma, \Phi \vdash \Psi, \Delta, \Psi', \Sigma' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\, ax$$

If $\overline{\overline{\mathsf{G4}}}$ refutes $\Phi', \Phi \vdash \Psi, \Psi'$ and $\Phi', \Phi \models^{*} \Psi, \Psi'$ does not belong to $\mathsf{top}^{\star}_{\mathsf{c}}(\models^{*} W)$, we rely on point (v) in Definition 5.1 to infer that $\mathsf{HG4}$ features an extra-logical rule $\delta_{\mathcal{D}''}$ of the following form:

$$\frac{\{\Gamma', \Theta_i \vdash \Lambda_i, \Delta' \mid \mathcal{R}_i\}_{1 \leq i \leq m} \qquad \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}{\Gamma', \Phi', \Phi \vdash \Psi, \Psi', \Delta' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\, \delta_{\mathcal{D}''}$$

Hence, we replace the application of $\delta_{\mathcal{D}'}$ in the derivation with an application of $\delta_{\mathcal{D}''}$ to obtain the following $\mathsf{HG4}$-proof (we use a compact notation for Weakening applications):

$$\frac{wk\,\dfrac{\dfrac{\vdots}{\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}}}{\{\Pi', \Gamma, \Theta_i \vdash \Lambda_i, \Delta, \Sigma' \mid \mathcal{R}_i\}_{1 \leq i \leq m}} \qquad \dfrac{\vdots}{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}}{\Pi', \Gamma, \Phi', \Phi \vdash \Psi, \Psi', \Delta, \Sigma' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\, \delta_{\mathcal{D}''}$$

On the other hand, if A is not principal, then $A, \Pi \vdash \Sigma$ is of the form $A, \Pi', \Phi \vdash \Psi, \Sigma'$ for (at least) one clause $\Phi' \vdash \Psi'$ in $\mathsf{top}^{\star}_{\mathsf{c}}(\vdash W)$. By points (i) and (ii) in Definition 5.1 as well as Definition 5.2 we have that the following $\mathsf{HG4}$-derivation is a proof:

$$\frac{}{\Pi', \Gamma, \Phi, \Phi' \vdash \Psi', \Psi, \Delta, \Sigma' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\, ax$$

(iii) If the last rule applied in the $\mathsf{HG4}$-proof of $A, \Pi \vdash \Sigma \mid \mathcal{R}''$ is an extra-logical one and A is principal in it, then the derivation has the following form:

$$\frac{\delta_{\mathcal{D}'}\,\dfrac{\dfrac{\vdots}{\{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}} \qquad \dfrac{\vdots}{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}}{\Gamma, \Phi \vdash \Psi, \Delta, A \mid \mathcal{R}'} \qquad \dfrac{\dfrac{\vdots}{\{\Pi, \Theta_i \vdash \Lambda_i, \Sigma \mid \mathcal{R}_i\}_{m+1 \leq i \leq n}} \qquad \dfrac{\vdots}{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{m+1 \leq i \leq n}}}{\Pi, A, \Phi' \vdash \Psi', \Sigma \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_{m+1} \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_{k'}}\,\delta_{\mathcal{D}'''}}{\Pi, \Phi', \Gamma, \Phi \vdash \Psi, \Delta, \Psi', \Sigma \mid \mathcal{R}' \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_{m+1} \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_{k'}}\, cut$$

with $\mathcal{R}'' = \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_{m+1} \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_{k'}$.

If $\overline{\overline{\mathsf{G4}}}$ proves $\Phi', \Phi \vdash \Psi, \Psi'$, we exploit points (i) and (ii) in the definition of the $\mathsf{HG4}$ calculus as well as Definition 5.2 we obtain that the following $\mathsf{HG4}$-derivation is a proof:

$$\dfrac{}{\Pi, \Gamma, \Phi', \Phi \vdash \Psi, \Psi', \Delta, \Sigma \mid \mathcal{R}' \mid \mathcal{R}''}\ ax$$

If $\overline{\overline{\mathsf{G4}}}$ refutes $\Phi', \Phi \vdash \Psi, \Psi'$ and $\Phi', \Phi \overset{*}{\models} \Psi, \Psi'$ belongs neither to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\models} \bigwedge \Phi \to \bigvee \Psi \vee A)$, nor to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\models} A \wedge \bigwedge \Phi' \to \bigvee \Psi')$ nor to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\models} W)$, we rely on point (vi) in Definition 5.1 to infer that $\mathsf{HG4}$ features an extra-logical rule $\delta_{\mathcal{D}' \cup \mathcal{D}''}$ of the following form:

$$\dfrac{\{\Gamma'', \Theta_j \vdash \Lambda_j, \Delta'' \mid \mathcal{R}_j\}_{1 \le j \le n} \qquad \{\Theta_j \vdash \Lambda_j \mid \mathcal{R}_j\}_{1 \le j \le n}}{\Gamma'', \Phi', \Phi \vdash \Psi, \Psi', \Delta'' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_{k'}}\ \delta_{\mathcal{D}' \cup \mathcal{D}'''}$$

Hence, we replace the $\delta_{\mathcal{D}'}$ and $\delta_{\mathcal{D}''}$ applications with an application of $\delta_{\mathcal{D}' \cup \mathcal{D}'''}$ to obtain the following $\mathsf{HG4}$-proof (again, we use a compact notation for Weakening applications):

$$\dfrac{wk\ \begin{array}{c} \vdots \\ \{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \le i \le m} \\ \hdashline\hdashline \{\Pi, \Gamma, \Theta_i \vdash \Lambda_i, \Delta, \Sigma \mid \mathcal{R}_i\}_{1 \le i \le m} \end{array} \qquad wk\ \begin{array}{c} \vdots \\ \{\Pi, \Theta_i \vdash \Lambda_i, \Sigma \mid \mathcal{R}_i\}_{m+1 \le i \le n} \\ \hdashline\hdashline \{\Pi, \Gamma, \Theta_i \vdash \Lambda_i, \Delta, \Sigma \mid \mathcal{R}_i\}_{m+1 \le i \le n} \end{array} \qquad \begin{array}{c} \vdots \\ \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \le i \le n} \end{array}}{\Pi, \Gamma, \Phi', \Phi \vdash \Psi, \Psi', \Delta, \Sigma \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_k}\ \delta_{\mathcal{D}' \cup \mathcal{D}'''}$$

On the other hand, if A is not principal in the last rule applied in the $\mathsf{HG4}$-proof of $A, \Pi \vdash \Sigma \mid \mathcal{R}''$, then the derivation has the following form:

$$\dfrac{\delta_{\mathcal{D}'}\ \dfrac{\begin{array}{c}\vdots \\ \{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \le i \le m}\end{array} \qquad \begin{array}{c}\vdots \\ \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \le i \le m}\end{array}}{\Gamma, \Phi \vdash \Psi, \Delta, A \mid \mathcal{R}'} \qquad \dfrac{\begin{array}{c}\vdots \\ \{A, \Pi, \Theta_i \vdash \Lambda_i, \Sigma \mid \mathcal{R}_i\}_{m+1 \le i \le n}\end{array} \qquad \begin{array}{c}\vdots \\ \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{m+1 \le i \le n}\end{array}}{A, \Pi, \Phi' \vdash \Psi', \Sigma \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_{m+1} \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_{k'}}\ \delta_{\mathcal{D}'''}}{\Pi, \Phi', \Gamma, \Phi \vdash \Psi, \Delta, \Psi', \Sigma \mid \mathcal{R}' \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_{m+1} \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_{k'}}\ cut$$

To get the conclusion, we take an $\mathsf{HG4}$-derivation π_j of the following form, for any $m + 1 \le j \le n$:

$$\dfrac{\delta_{\mathcal{D}'}\ \dfrac{\begin{array}{c}\vdots \\ \{\Gamma, \Theta_i \vdash \Lambda_i, \Delta \mid \mathcal{R}_i\}_{1 \le i \le m}\end{array} \qquad \begin{array}{c}\vdots \\ \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \le i \le m}\end{array}}{\Gamma, \Phi \vdash \Psi, \Delta, A \mid \mathcal{R}'} \qquad \begin{array}{c}\vdots \\ A, \Pi, \Theta_j \vdash \Lambda_j, \Sigma \mid \mathcal{R}_j\end{array}}{\Pi, \Theta_j, \Gamma, \Phi \vdash \Psi, \Delta, \Lambda_j, \Sigma \mid \mathcal{R}' \mid \mathcal{R}_j}\ cut$$

By Definition 5.2, π_j is an $\mathsf{HG4}$-proof. Hence, we exploit the secondary inductive hypothesis to remove the Cut application in π_j. Finally, we plug $\pi_{m+1}, \ldots, \pi_n$ into the following $\mathsf{HG4}$-proof:

$$\dfrac{\begin{array}{c}\vdots \\ \{\Pi, \Gamma, \Phi, \Theta_j \vdash \Lambda_j, \Psi, \Delta, \Sigma \mid \mathcal{R}' \mid \mathcal{R}_j\}_{m+1 \le j \le n}\end{array} \qquad \begin{array}{c}\vdots \\ \{\Theta_j \vdash \Lambda_j \mid \mathcal{R}'_j\}_{m+1 \le j \le n}\end{array}}{\Pi, \Gamma, \Phi, \Phi' \vdash \Psi', \Psi, \Delta, \Sigma \mid \mathcal{R}' \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}' \mid \mathcal{R}_n \mid \mathcal{R}'_{m+1} \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_k}\ \delta_{\mathcal{D}'''}$$

Notice that the refutational part $\mathcal{R}' \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}' \mid \mathcal{R}_n \mid \mathcal{R}'_{m+1} \mid \cdots \mid \mathcal{R}'_n \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_k$ is identical to $\mathcal{R}' \mid \mathcal{R}''$.

(iv) If the last rule applied in the HG4-proof of $A, \Pi \vdash \Sigma \mid \mathcal{R}''$ is a logical one, say $L \rightarrow$, then the derivation has the form

$$\dfrac{\begin{array}{c}\vdots \\ A, \Pi' \vdash \Sigma, B \mid \mathcal{S}\end{array} \qquad \begin{array}{c}\vdots \\ A, C, \Pi' \vdash \Sigma \mid \mathcal{S}'\end{array}}{A, B \rightarrow C, \Pi' \vdash \Sigma \mid \mathcal{S} \mid \mathcal{S}'}\ L \rightarrow$$

with $\Pi = \Pi' \cup [B \rightarrow C]$ and $\mathcal{R}'' = \mathcal{S} \mid \mathcal{S}'$. By hypothesis, the HG4-proofs π and π' of $A, C, \Pi' \vdash \Sigma \mid \mathcal{S}'$ and $A, \Pi' \vdash \Sigma, B \mid \mathcal{S}$, respectively, are cut-free. If π and π' do not feature extra-logical rules, then for any initial hypersequent in π and π' where A occurs we apply cases $(i) - (ii)$ above to get the result. On the other hand, consider the lowermost applications of extra-logical rules in (say) π. The formula A occurs in the conclusion of any such application: hence, we apply cases $(i) - (iii)$ above to reach the conclusion.

□

Similarly to the G4s calculi for supraclassical logics, HG4 calculi enjoy admissibility of Weakening, admissibility of Contraction and invertibility of logical rules (cf. Proposition 2.1). On the other hand, it is interesting to compare admissibility of Cut in G4s calculi (cf. Theorem 2.3) and admissibility of safe Cut in HG4 calculi.

Due to the completeness result stated in Theorem 2.4, admissibility of Cut in G4s calculi ensures that the set of classical consequences derived from extra-logical axioms by means of G4s rules is closed under *modus ponens*. As far as the HG4c calculus is concerned, Theorem 3.5 below represents a completeness result analogous to Theorem 2.4. For its part, Theorem 5.1 guarantees that the set of m-credulous consequences *within* the same modified extensions is closed under *modus ponens* (cf. Example 5.13). The failure of unrestricted Cut reflects the fact that the closure under *modus ponens* of the set of m-credulous consequences *across* different modified extensions does not hold (cf. Example 5.14).

PROPOSITION 5.11. If there exists an HG4-proof π of $\Gamma \vdash \Delta \mid \mathcal{R}$, then for any hypersequent $\Gamma' \vdash \Delta' \mid \mathcal{R}'$ in π the formulas in Γ', Δ' are either subformulas of formulas in Γ, Δ or atoms.

PROOF. By induction on the height of π. □

In our framework, $\overline{\overline{\text{G4s}}}$ sequent calculi for supraclassical logics satisfy the full subformula property (cf. Proposition 2.2), whereas HG4 hypersequent calculi for a specific class of supraclassical logics – namely, default logics – lose their analytic character due to the inclusion of extra-logical rules. It should be noticed that this feature sets our approach apart from the alternative Gentzen-style presentation of supraclassical logics in [109], where sequent calculi for supraclassical logics always enjoy the weakened version of the subformula property of Proposition 5.11.

Unlike Hilbert-style calculi for default logics, the proof-search space in the HG4 calculi is finite. By utilizing Propositions 5.8 and 5.9, we can establish a natural proof-search strategy for the HG4c calculus.

PROCEDURE 5. Let A be an arbitrary formula, $\mathcal{C}_1 = \mathsf{top}(\overset{*}{\vdash} A)$ and $\mathfrak{D}_1^h = \varnothing$ for any $h \geq 1$ in a given enumeration of the elements of $\mathcal{C}_1$. $\mathcal{C}_{s_1}, \dots, \mathcal{C}_{s_r}$ are the sets of (anti)sequents obtained from the maximal application of the following steps, starting with the set $\mathcal{C}_1$:

- (i) Select a complementary (anti)sequent $\Theta_h \overset{*}{\vdash} \Lambda_h$ in $\mathcal{C}_{1\mathbf{n}} = \{\Theta_1 \overset{*}{\vdash} \Lambda_1, \dots, \Theta_m \overset{*}{\vdash} \Lambda_m\}$ which is not a weakened version of an element of $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} W)$ and which is the conclusion of the application of an extra-logical rule among $\delta_{\mathcal{D}_{h1}}, \dots, \delta_{\mathcal{D}_{hn}} \notin \mathfrak{D}^h_{1\mathbf{n}}$ (if any). Then, select one $1 \leq j \leq n$.
- (ii) If the right-hand side premises of $\delta_{\mathcal{D}_{hj}}$ contain the sequents $\Theta_{m+1} \vdash \Lambda_{m+1}, \dots, \Theta_{m+k_j} \vdash \Lambda_{m+k_j}$ and $i = 2^h \cdot 3^j$, then
$$\mathcal{C}_{1\mathbf{n}i} = \mathcal{C}_{1\mathbf{n}} \setminus \{\Theta_h \overset{*}{\vdash} \Lambda_h\} \cup \{\Theta_{m+1} \overset{*}{\vdash} \Lambda_{m+1}, \dots, \Theta_{m+k_j} \overset{*}{\vdash} \Lambda_{m+k_j}\}$$
$$\mathfrak{D}^{m+l}_{1\mathbf{n}i} = \mathfrak{D}^h_{1\mathbf{n}} \cup \{\delta_{\mathcal{D}_{hj}}\}$$
for any $1 \leq l \leq k_j$.

Procedure 5 intuitively operates on $\mathsf{top}(\overset{*}{\vdash} A)$, generating a set of (anti)sequents where either: (1) every complementary (anti)sequent is a weakened version of elements in $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} W)$, or (2) at least one complementary (anti)sequent is not. The procedure terminates after finitely many steps, as the number of extra-logical rules in HG4c is finite, and each application of step (ii) reduces the allowable bottom-up applications of extra-logical rules in subsequent iterations.

EXAMPLE 5.15. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be as in Example 5.8. The* HG4c *calculus for $\langle \mathcal{W}, \mathcal{D} \rangle$ features the following extra-logical rules:*

$$\delta_{\mathcal{D}'}\ \dfrac{\Gamma \vdash p, \Delta \mid \mathcal{R} \qquad \vdash p \mid \mathcal{R}'}{\Gamma \vdash r, \Delta \mid \mathcal{R} \mid \mathcal{R}' \mid \dashv \neg r} \qquad \dfrac{\Gamma \vdash q, \Delta \mid \mathcal{R} \qquad \vdash q \mid \mathcal{R}'}{\Gamma, r \vdash \Delta \mid \mathcal{R} \mid \mathcal{R}' \mid \dashv \neg\neg r}\ \delta_{\mathcal{D}''}$$

$$\dfrac{\Gamma \vdash p, \Delta \mid \mathcal{R}_1 \qquad \Gamma \vdash q, \Delta \mid \mathcal{R}_2 \qquad \vdash p \mid \mathcal{R}'_1 \qquad \vdash q \mid \mathcal{R}'_2}{\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \mathcal{R}'_1 \mid \mathcal{R}'_2 \mid \dashv \neg r \mid \dashv \neg\neg r}\ \delta_{\mathcal{D}'''}$$

Let A be $(q \to r) \wedge (r \to s)$. We have that $\mathcal{C}_1 = \{q \overset{*}{\vdash} r\ ;\ r \overset{*}{\vdash} s\}$: if $\Theta_1 \overset{*}{\vdash} \Lambda_1$ is $q \overset{*}{\vdash} r$ and $\Theta_2 \overset{*}{\vdash} \Lambda_2$ is $r \overset{*}{\vdash} s$, then $\mathfrak{D}_1^1 = \mathfrak{D}_1^2 = \varnothing$. Since $\mathcal{W} = \{p, q\}$, neither $\Theta_1 \overset{*}{\vdash} \Lambda_1$ nor $\Theta_2 \overset{*}{\vdash} \Lambda_2$ is a weakened version of an element of $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} W)$. We illustrate the construction by stages of a set $\mathcal{C}_{s_t}$ resulting from maximal application of Procedure 5.

(1) By step (i), we select (say) $\Theta_1 \mathrel{|\!\overset{*}{-}} \Lambda_1$*: if* $\delta_{\mathcal{D}'} = \delta_{\mathcal{D}_{11}}$ *and* $\delta_{\mathcal{D}'''} = \delta_{\mathcal{D}_{12}}$*, we select (say)* $\delta_{\mathcal{D}_{12}}$*.*

(2) By step (ii), we have $\mathcal{C}_{1(2^1 \cdot 3^2)} = \{r \mathrel{|\!\overset{*}{-}} s \; ; \mathrel{|\!\overset{*}{-}} p \; ; \mathrel{|\!\overset{*}{-}} q\}$*; moreover, if* $\Theta_3 \mathrel{|\!\overset{*}{-}} \Lambda_3$ *is* $\mathrel{|\!\overset{*}{-}} p$ *and* $\Theta_4 \mathrel{|\!\overset{*}{-}} \Lambda_4$ *is* $\mathrel{|\!\overset{*}{-}} q$*, we have* $\mathfrak{D}^3_{1(2^1 \cdot 3^2)} = \mathfrak{D}^4_{1(2^1 \cdot 3^2)} = \{\delta_{\mathcal{D}_{12}}\}$*.*

(3) By step (i), we select $\Theta_2 \mathrel{|\!\overset{*}{-}} \Lambda_2$*: if* $\delta_{\mathcal{D}''} = \delta_{\mathcal{D}_{21}}$ *and* $\delta_{\mathcal{D}'''} = \delta_{\mathcal{D}_{22}}$*, we select (say)* $\delta_{\mathcal{D}_{22}}$*.*

(4) By step (ii), we have $\mathcal{C}_{1(2^1 \cdot 3^2)(2^2 \cdot 3^2)} = \{\mathrel{|\!\overset{*}{-}} p \; ; \mathrel{|\!\overset{*}{-}} q\}$*; moreover, if* $\Theta_5 \mathrel{|\!\overset{*}{-}} \Lambda_5$ *is* $\mathrel{|\!\overset{*}{-}} p$ *and* $\Theta_6 \mathrel{|\!\overset{*}{-}} \Lambda_6$ *is* $\mathrel{|\!\overset{*}{-}} q$*, we have* $\mathfrak{D}^5_{1(2^1 \cdot 3^2)(2^2 \cdot 3^2)} = \mathfrak{D}^6_{1(2^1 \cdot 3^2)(2^2 \cdot 3^2)} = \{\delta_{\mathcal{D}_{22}}\}$*.*

Note that if any complementary (anti)sequent in $\mathcal{C}_{s_t}$ is a weakened version of an element in $\mathsf{top}^{\star}_{\mathsf{c}}(\mathrel{|\!\overset{*}{-}} W)$ for some $1 \leq t \leq r$, there exists a HG4c-derivation π_t from the corresponding sequents (i.e., hypersequents with an empty refutational part) to $\vdash A \mid \mathcal{R}_t$, where $\mathcal{R}_t$ consists precisely of the antisequents introduced through extra-logical rule applications.

By Proposition 5.9, any HG4c-proof π can be transformed into a proof π' of the same conclusion where all logical rules, if any, are permuted last. Proposition 5.8 further ensures that if π''s extra-logical part includes non-atomic formulas $B_1, \ldots, B_m$, a proof π'' exists with these formulas replaced by atomic ones. If π'' involves repeated applications of the same extra-logical rule along a branch, Propositions 5.6 and 5.7 allow transforming it into π''' with a single such application. Consequently, if any π_t is a HG4c-paraproof, $\vdash A \mid \mathcal{R}$ is unprovable in HG4c for any $\mathcal{R}$; otherwise, it is provable for $\mathcal{R} = \mathcal{R}_t$.

We are now in position to prove that the HG4c calculus is sound and (weakly) complete with respect to m-credulous consequence based on $\langle \mathcal{W}, \mathcal{D} \rangle$:

THEOREM 5.2. *Let* $\langle \mathcal{E}, \mathcal{F} \rangle$ *be a modified extension of* $\langle \mathcal{W}, \mathcal{D} \rangle$*. Then, the formula* $A \in \mathcal{E}$ *if and only if there exists an* HG4c*-proof of* $\vdash A \mid \mathcal{R}$*, for some* $\mathcal{R}$*.*

PROOF. ($\Leftarrow$) Let π be an HG4c-proof of $\vdash A \mid \mathcal{R}$. By Proposition 5.9 there exist sub-proofs $\pi_1, \ldots, \pi_n$ of hypersequents $\Gamma_1 \vdash \Delta_1 \mid \mathcal{R}_1, \ldots, \Gamma_n \vdash \Delta_n \mid \mathcal{R}_n$ with only *ax* and extra-logical rules: for any application of an extra-logical rule, Proposition 5.6 ensures that the HG4c-proofs of the left premises do not surpass the maximal height of the HG4-proofs of the right premises.

Let $\pi'_1, \ldots, \pi'_m$ be the sub-proofs of $\pi_1, \ldots, \pi_n$ of height h, with $1 \leq h \leq max(h(\pi_1), \ldots, h(\pi_n))$, and (for any $1 \leq j \leq m$) let $\Pi_j \vdash \Sigma_j \mid \mathcal{R}'_j$ be the conclusion of π'_j. We prove by complete induction on h the existence of (at least) one modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W}, def(\pi) \rangle$ such that $\bigwedge \Pi_j \to \bigvee \Sigma_j \in \mathcal{E}$.

The base case is obvious, so we focus on the inductive step. If $h > 1$, the last rule in any π'_j is an extra-logical rule $\delta_{\mathcal{D}'}$ with premises $\Pi'_{1j} \vdash \Sigma'_{1j} \mid \mathcal{R}'_{1j}, \ldots, \Pi'_{kj} \vdash \Sigma'_{kj} \mid \mathcal{R}'_{kj}$. By inductive hypothesis, there exists (at least) one modified extension $\langle \mathcal{E}', \mathcal{F}' \rangle$ of $\langle \mathcal{W}, def(\pi) \rangle$ such that, for any $1 \leq j \leq m$, $\bigwedge \Pi'_{1j} \to \bigvee \Pi'_{1j}, \ldots, \bigwedge \Pi'_{kj} \to \bigvee \Sigma'_{kj} \in \mathcal{E}'$. Suppose that the application of $\delta_{\mathcal{D}'}$ leads to $\dashv \neg C_{1j}, \ldots, \dashv \neg C_{k'j}$ in $\mathcal{R}'_j$: by Definition 5.2, $\overline{\overline{\overline{\mathsf{G4}}}}$ refutes

$W, D_\pi \vdash \neg C_{h'j}$ for any $1 \le h' \le k'$ – and this implies that $C_{h'j} \in \mathcal{F}'$. By construction of the HG4c calculus, either $\bigwedge \Pi_j \to \bigvee \Sigma_j$ occurs in *concl*$(\mathcal{D}')$ or it can be derived from the formulas in *concl*$(\mathcal{D}')$ through suitable applications of *modus ponens*: in both cases, we can conclude that $\bigwedge \Pi_j \to \bigvee \Sigma_j \in \mathcal{E}'$.

If we take h as $max(h(\pi_1), \ldots, h(\pi_n))$, then there exists (at least) one modified extension $\langle \mathcal{E}', \mathcal{F}' \rangle$ of $\langle \mathcal{W}, def(\pi) \rangle$ such that *any* formula $\bigwedge \Gamma_i \to \bigvee \Delta_i$ belongs to $\mathcal{E}'$: from Proposition 5.5 follows the existence of (at least) one modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W}, \mathcal{D} \rangle$ such that $\mathcal{E}' \subseteq \mathcal{E}$, $\mathcal{F}' \subseteq \mathcal{F}$ and $\bigwedge \Gamma_i \to \bigvee \Delta_i \in \mathcal{E}$.

$(\Rightarrow)$ If $\langle \mathcal{E}, \mathcal{F} \rangle$ is a modified extension of $\langle \mathcal{W}, \mathcal{D} \rangle$ and $A \in \mathcal{E}$, then there exists a k such that $A \in \mathcal{E}^k$. We show by (transfinite) induction on k that there exists (at least) one HG4c-proof π of $\vdash A \mid \mathcal{R}$, for some $\mathcal{R}$, such that $D_\pi \in \mathcal{E}$.

If $k = j+1$ and A can be obtained from formulas in $\mathcal{E}^j$ *via* suitable applications of *modus ponens*, then we leverage Theorem 5.1 to reach the conclusion. The most meaningful case arises when $k = j+1$, A is not a classical consequence of formulas in $\mathcal{E}_j$ and there exists a default rule of the form $\dfrac{B : C_1, \ldots, C_k}{A}$ such that $B \in \mathcal{E}^j$. If $A \in \mathcal{E}$ and $\mathcal{D}'$ represents the maximal set of default rules triggered by formulas in $\mathcal{E}^j$, then $C_h \in (\mathcal{F} \cup just(\mathcal{D}'))$, and so $\neg C_h \notin Cn(\mathcal{E} \cup concl(\mathcal{D}'))$, for any $1 \le h \le k$. By inductive hypothesis, there exists (at least) one HG4c-proof π of $\vdash B \mid \mathcal{R}$, for some $\mathcal{R}$, such that $D_\pi \in \mathcal{E}$. If $\mathsf{top}_{\mathsf{c}}(\mathrel{|\!\overset{*}{-}} B) = \{\Theta_i \mathrel{|\!\overset{*}{-}} \Lambda_i\}_{1 \le i \le m}$, then by Proposition 5.8 and induction on the height of π we get that for each hypersequent in $\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}_i\}_{1 \le i \le m}$ there exists an HG4c-proof π'_i such that $D_{\pi'_i} \in \mathcal{E}$. For any $\Phi \mathrel{|\!\overset{*}{-}} \Psi$ in $\mathsf{top}^{\star}_{\mathsf{c}}(\mathrel{|\!\overset{*}{-}} A)$ consider the following HG4c-derivation π':

$$\dfrac{\begin{array}{c}\vdots_{\pi'_i} \\ \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}_i\}_{1 \le i \le m}\end{array} \qquad \begin{array}{c}\vdots_{\pi'_i} \\ \{\Theta_i \vdash \Lambda_i \mid \mathcal{R}_i\}_{1 \le i \le m}\end{array}}{\Phi \vdash \Psi \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k}\ \delta_{\mathcal{D}'}$$

If $\neg C_h \notin Cn(\mathcal{E} \cup concl(\mathcal{D}'))$, then $\neg C_h \notin Cn(\mathcal{W} \cup \{D_{\pi'_i}\}_{1 \le i \le m} \cup \{A\})$: this implies that $\overline{\overline{\mathsf{G4}}}$ refutes $W, D_{\pi'} \vdash \neg C_h$, for any $1 \le h \le k$. As a result, we get an HG4c-proof π'' of $\vdash A \mid \mathcal{R}'$, with $\mathcal{R}' = \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k$, such that $D_{\pi''} \in \mathcal{E}$: notice that the refutational part $\mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k$ is identical to $\mathcal{R}'$. ☐

Corollary 5.3. *The formula A is an m-credulous consequence of $\mathcal{W}$ if and only if there exists (at least) one $\mathcal{R}$ such that $\vdash A \mid \mathcal{R}$ is provable in* HG4c.

PROOF. Straightforward from Theorem 5.2. ☐

Corollary 5.4. *If $\langle \mathcal{W}, \mathcal{D} \rangle$ is a normal default theory, then A is a credulous consequence of $\mathcal{W}$ if and only if there exists (at least) one $\mathcal{R}$ such that $\vdash A \mid \mathcal{R}$ is provable in* HG4c.

PROOF. Straightforward from Corollary 5.3 and point (*ii*) of Proposition 5.1. ☐

Corollary 5.3 establishes that $\vdash A \mid \mathcal{R}$ is provable in HG4c if and only if A is an m-credulous consequence of $\mathcal{W}$. However, one cannot prove the stronger claim that $\Gamma \vdash A \mid \mathcal{R}$ is provable in HG4c if and only if A is an m-credulous consequence of $\mathcal{W}$ and a set of additional premises Γ. In short, HG4c hypersequent calculi fail to be strongly adequate with respect to m-credulous consequence – a major difference with respect to any G4s calculus (cf. Theorem 2.4, point (*ii*)). Examples 5.16 and 5.17 provide counterexamples to strong soundness and strong completeness, respectively.

Example 5.16. *Let* $\langle \mathcal{W}, \mathcal{D} \rangle$ *be as in Example 5.8. Consider the following* HG4c*-proof:*

$$\dfrac{ax\,\dfrac{}{\neg r \vdash p} \qquad \dfrac{}{\vdash p}\,ax}{\neg r \vdash r \mid \dashv \neg r}\,\delta_{\mathcal{D}'}$$

Let $\mathcal{W}'$ *be* $\mathcal{W} \cup \{\neg r\}$*:* $\langle \mathcal{W}', \mathcal{D} \rangle$ *has a unique modified extension* $\langle \mathcal{E}, \mathcal{F} \rangle = \langle Cn(\{p, q, \neg r\}), \{\neg r\} \rangle$*. As a result,* r *is not an* m*-credulous consequence of* $\mathcal{W}'$*.*

Example 5.17. *Let* $\langle \mathcal{W}, \mathcal{D} \rangle$ *be defined as follows:*

$$\mathcal{W} = \varnothing$$

$$\mathcal{D} = \left\{ \frac{\top : p}{p}, \frac{p \vee q : \neg p}{\neg p} \right\}$$

The hypersequent $p \vee q \vdash q \mid \mathcal{R}$ *is not provable, for any* $\mathcal{R}$*.*

We have proved that the hypersequent calculus HG4c is sound and (weakly) complete with respect to the class of all m-credulous consequences of $\mathcal{W}$. Let us observe that HG4 calculi can be tailored for *specific* classes of m-credulous consequences of $\mathcal{W}$.

This is accomplished by modifying the initial hypersequents to ensure that their refutational part includes a (suitably chosen) *non-empty* part $\mathcal{R}'$ (in accordance with condition (*ii*) in Definition 5.1). The HG4 calculus with initial hypersequents incorporating $\mathcal{R}' = \Pi_1 \dashv \Sigma_1 \mid \cdots \mid \Pi_n \dashv \Sigma_n$ remains sound and (weakly) complete with respect to $\mathcal{R}'$-*modified credulous consequence* based on $\langle \mathcal{W}, \mathcal{D} \rangle$. Here an $\mathcal{R}'$-*modified extension* $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W}, \mathcal{D} \rangle$ is defined as follows:

$$\mathcal{E}^0 = \mathcal{W} \qquad \mathcal{F}^0 = \{\neg(\bigwedge \Pi_i \to \bigvee \Sigma_i)\}_{1 \leq i \leq n}$$

$$\mathcal{E}^{k+1} = Cn(\mathcal{E}^k) \cup \{concl(\mathcal{D}') \mid \mathcal{D}' \subseteq \mathcal{D}, req(\mathcal{D}') \subseteq \mathcal{E}^k, A \in (\mathcal{F} \cup just(\mathcal{D}')) \Rightarrow \neg A \notin Cn(\mathcal{E} \cup concl(\mathcal{D}'))\}$$

$$\mathcal{F}^{k+1} = \mathcal{F}^k \cup \{just(\mathcal{D}') \mid \mathcal{D}' \subseteq \mathcal{D}, req(\mathcal{D}') \subseteq \mathcal{E}^k, A \in (\mathcal{F} \cup just(\mathcal{D}')) \Rightarrow \neg A \notin Cn(\mathcal{E} \cup concl(\mathcal{D}'))\}$$

$$\mathcal{E} = \bigcup_{i=0}^{\omega} \mathcal{E}^i \qquad \mathcal{F} = \bigcup_{i=0}^{\omega} \mathcal{F}^i$$

Let us pinpoint another difference between $\mathsf{G4s}$ calculi and $\mathsf{HG4}$ calculi. In Chapter 2, we established that $\mathsf{G4s}$ calculi are deductively complete (cf. Theorem 2.5). We say that an admissible rule of the form

$$\frac{\Gamma_1 \vdash \Delta_1 \mid \mathcal{R}_1 \qquad \cdots \qquad \Gamma_n \vdash \Delta_n \mid \mathcal{R}_n}{\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n} \tag{5.2.1}$$

is *pure* if and only if, for any Π, Σ, the following rule is admissible

$$\frac{\Pi, \Gamma_1 \vdash \Delta_1, \Sigma \mid \mathcal{R}_1 \qquad \cdots \qquad \Pi, \Gamma_n \vdash \Delta_n, \Sigma \mid \mathcal{R}_n}{\Pi, \Gamma \vdash \Delta, \Sigma \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}$$

Since the right-premises of extra-logical rules are context-sensitive, Theorem 2.5 guarantees that $\mathsf{HG4}$ calculi are not deductively complete. In particular, there may be an admissible rule of the form (5.2.1) such that the hypersequent $\bigwedge \Gamma_1 \to \bigvee \Delta_1, \dots, \bigwedge \Gamma_n \to \bigvee \Delta_n, \Gamma \vdash \Delta \mid \mathcal{R}$ is not provable, for any $\mathcal{R}$.

Example 5.18. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as follows:*

$$\mathcal{W} = \{p \vee r\}$$

$$\mathcal{D} = \left\{ \frac{p : q}{r} \right\}$$

By Proposition 5.10, the following rule is admissible in $\mathsf{HG4c}$*:*

$$\frac{\vdash p, r \mid \mathcal{R}_1 \qquad \vdash p \mid \mathcal{R}_2}{\vdash r \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \dashv \neg q} \, \delta_{\mathcal{D}'}$$

On the other hand, there exists no $\mathsf{HG4c}$*-proof of $p \vee r, p \vdash r \mid \mathcal{R}$, for any $\mathcal{R}$.*

5.2.3. A comparison between hybrid hypersequents and sequents for credulous consequence. We conclude this section by comparing our $\mathsf{HG4c}$ calculi with the sequent calculi for credulous consequence introduced in [19]. Since the latter focuses on Reiter's default logic, we restrict our analysis to $\mathsf{HG4c}$ calculi for normal default theories (cf. Section **??** for a discussion on hybrid hypersequent calculi for Reiter default logics).

In our framework, extra-logical axioms are treated as provable formulas, placed on equal footing with tautologies, following the methods outlined in [121] and [129]. Default rules, on the other hand, are formalized as distinct extra-logical inference rules. Consequently, a $\mathsf{HG4c}$-derivation π of $\Gamma \vdash \Delta \mid \Pi_1 \dashv \Sigma_1 \mid \cdots \mid \Pi_n \dashv \Sigma_n$ can be interpreted as an attempt to construct an extension $\mathcal{E}$ of $\langle \mathcal{W}, \mathcal{D} \rangle$ such that $\bigwedge \Gamma \to \bigvee \Delta \in \mathcal{E}$ and $(\bigwedge \Pi_i \to \bigvee \Sigma_i) \notin \mathcal{E}$, provided that $\Pi_i \dashv \Sigma_i$ was introduced in π by an extra-logical rule. By Theorem 5.2, this attempt succeeds precisely when π constitutes a $\mathsf{HG4c}$-proof. Notably, the calculus lacks explicit rules for reasoning with antisequents; instead, to verify whether π is a proof, we rely on derivations of underivability claims in the $\overline{\overline{\mathsf{G4}}}$ calculus (or, equivalently, in the $\overline{\overline{\mathsf{G4s}}}$ calculus incorporating $\mathcal{W}$, or even the $\overline{\overline{\mathsf{G4s}}}$ calculus incorporating $\mathcal{W}$ and D_π).

In contrast, [19] treats both extra-logical axioms and default rules as formulas on the left-hand side of sequents. Default rules are internalized within the object language, breaking them down into a monotonic component (the 'residue', represented by the operator $/$) and a set of constraints on extensions, managed via a special modality $\mathbf{L}$. Thus, sequents in [19] take the form $\Sigma \mid \Gamma \mathrel{\big|} \Delta$, where Σ is a set of formulas $\mathbf{L}B_1, \ldots, \mathbf{L}B_m, \neg\mathbf{L}C_1, \ldots, \neg\mathbf{L}C_n$. A derivation of $\Sigma \mid \Gamma \mathrel{\big|} \Delta$ is interpreted as a successful attempt to prove that $\bigvee \Delta$ belongs to at least one extension $\mathcal{E}$ of Γ that satisfies the constraints in Σ – i.e., $B_1, \ldots, B_m \in \mathcal{E}$ and $C_1, \ldots, C_n \notin \mathcal{E}$. Antisequents also appear in derivations within this sequent calculus to track cases where either $\bigvee \Delta$ is not a classical consequence of the *reduct* of Γ (cf. [19, p. 234] and [101, p. 54]), or a negative constraint of the form $\neg\mathbf{L}D$ is enforced. As a result, reasoning about underivability is formalized within the system itself. Furthermore, when read bottom-up, any rule detailed in [19, p. 238 and p. 242] decomposes the default theory Γ itself. The calculus can thus be seen as a sequent-based decision procedure for credulous consequence (cf. Procedure 5 and the rules in [19, p. 238]).

In our approach, determining whether a $\mathsf{HG4c}$-derivation π is a proof involves using derivations of antisequents in the $\overline{\overline{\mathsf{G4s}}}$ calculus. These derivations are not performed within the $\mathsf{HG4c}$-calculus itself. This distinction is important: the set of refutable atomic sequents in a hypothetical $\overline{\overline{\mathsf{HG4c}}}$ calculus is circularly defined, as it depends on the complement of the set of provable atomic sequents. The set of provable atomic sequents in $\mathsf{HG4c}$ is quasi-inductively definable, unlike in any $\overline{\overline{\mathsf{G4s}}}$ calculus. This contrasts with the approach in [19], where underivability is internalized within the system. However, their method relies on *ad hoc* linguistic constructs – specifically, the connective $/$ and the constant $\mathbf{L}$. In contrast, our design deliberately avoids such constructs and offers several distinct advantages. First, the absence of explicit rules for antisequents leads to a more streamlined and compact formalism. Second, the concept of a $\mathsf{HG4c}$-proof becomes more versatile, enabling the system to accommodate a broader range of default logics compared to [19]. This includes Łukaszewicz's logic (Theorem 5.2), as well as Reiter's and cumulative logics (see Section **??**). Finally, we establish a Cut elimination theorem for $\mathsf{HG4c}$ calculi – a result unattainable in [19], where the turnstile $\big|$ represents the credulous consequence relation.

5.3. A decision procedure for modified skeptical consequence

In this section, we leverage the expressive power of $\mathsf{HG4}$ calculi to offer a syntactic treatment of the *modified skeptical consequences* (in short, m-skeptical consequences) of $\mathcal{W}$. Specifically, $\mathsf{HG4}$ calculi serve as the platform for defining a syntactical decision procedure for m-skeptical consequence that does *not* presuppose the computation of *all* modified extensions of $\langle \mathcal{W}, \mathcal{D} \rangle$. To maintain focus and relevance, we confine our analysis to default theories featuring multiple modified extensions.

Let us begin by introducing a specific notion of $\mathsf{HG4}$-paraproof:

DEFINITION 5.3. Let π be an HG4-proof of $\vdash A \mid \mathcal{R}$, where $\mathcal{R}$ consists only of antisequents introduced by extra-logical rules (if any). An HG4-paraproof π' is *orthogonal* to π if the following conditions hold:

- (i) π' is a paraproof with π and π'' as immediate subproofs;
- (ii) π'' consists of
  - (a) an extra-logical part, with lowermost hypersequents $\Gamma_1 \vdash \Delta_1 \mid \mathcal{R}_1, \ldots, \Gamma_n \vdash \Delta_n \mid \mathcal{R}_n$ derived from extra-logical rules without auxiliary formulas, where each $\mathcal{R}_i$ contains only antisequents introduced by extra-logical rules;
  - (b) a logical part, whose lowermost hypersequent is $\bigwedge_{i=1}^{n}(\bigwedge \Gamma_i \to \bigvee \Delta_i) \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n$;
- (iii) the last rule applied in π' is $R\wedge$;
- (iv) any derivation ρ satisfying conditions $(i) - (iii)$ is such $|def(\pi)| \leq |def(\rho)|$.

EXAMPLE 5.19. *Let* $\langle \mathcal{W}, \mathcal{D} \rangle$ *be defined as follows:*

$$\mathcal{W} = \varnothing$$

$$\mathcal{D} = \left\{ \frac{\top : \neg q \wedge p}{p}\ ,\ \frac{\top : \neg r \wedge q}{q}\ ,\ \frac{\top : \neg p \wedge r}{r} \right\}$$

Notice that $\langle \mathcal{W}, \mathcal{D} \rangle$ *has multiple modified extensions – namely,* $\langle \mathcal{E}_1, \mathcal{F}_1 \rangle = \langle Cn(\{p\}), \{\neg q \wedge p\} \rangle$, $\langle \mathcal{E}_2, \mathcal{F}_2 \rangle = \langle Cn(\{q\}), \{\neg r \wedge q\} \rangle$ *and* $\langle \mathcal{E}_3, \mathcal{F}_3 = \langle Cn(\{r\}), \{\neg p \wedge r\} \rangle$ *– and no extensions. Let* π *be the following* HG4c*-proof:*

$$\dfrac{\dfrac{}{p \vdash p}\,ax}{\vdash p \mid \dashv \neg(\neg q \wedge p)}\,\delta_{\mathcal{D}'}$$

The following HG4c*-paraproofs are orthogonal to* π*:*

$$\dfrac{\delta_{\mathcal{D}'}\,\dfrac{ax\,\dfrac{}{p \vdash p}}{\vdash p \mid \dashv \neg(\neg q \wedge p)} \qquad \dfrac{\dfrac{}{p \vdash p}\,ax}{\vdash q \mid \dashv \neg(\neg r \wedge q)}\,\delta_{\mathcal{D}''}}{\vdash p \wedge q \mid \dashv \neg(\neg q \wedge p) \mid \dashv \neg(\neg r \wedge q)}\,R\wedge$$

$$\dfrac{\delta_{\mathcal{D}'}\,\dfrac{ax\,\dfrac{}{p \vdash p}}{\vdash p \mid \dashv \neg(\neg q \wedge p)} \qquad \dfrac{\dfrac{}{p \vdash p}\,ax}{\vdash r \mid \dashv \neg(\neg p \wedge r)}\,\delta_{\mathcal{D}'''}}{\vdash p \wedge r \mid \dashv \neg(\neg q \wedge p) \mid \dashv \neg(\neg p \wedge r)}\,R\wedge$$

Let π'_1, π'_2 denote two HG4c-paraproofs orthogonal to an HG4c-proof π, and π''_1, π''_2 be the immediate subderivations of π'_1 and π'_2, respectively, distinct from π. We say that π'_1 and π'_2 are *equivalent* if and only if π''_i can be obtained from π''_{3-i} by some permutation of rules, with $i = 1, 2$. Orthogonal paraproofs will be considered *modulo* this equivalence relation.

Suppose that π_i is an HG4-paraproof orthogonal to an HG4-proof π, with π'_i being the immediate subderivation of π_i distinct from π: in what follows, we use W_i to denote $D_{\pi'_i}$, and $\mathcal{R}_{\pi'_i}$ to refer to the refutational part of the conclusion of π'_i.

Procedure 6. Let A be an m-credulous consequence of $\mathcal{W}$, $\mathsf{HG4}^1$ stand for the $\mathsf{HG4c}$ calculus and $\mathbf{n}$ vary over finite sequences of natural numbers. $\mathsf{HG4}^{s_1}, \ldots, \mathsf{HG4}^{s_k}$ are the $\mathsf{HG4}$ calculi resulting from maximal application of the following steps, starting with the $\mathsf{HG4}^1$ calculus:

(i) Select an $\mathsf{HG4}^{\mathbf{1n}}$-proof π of $\vdash A \mid \mathcal{R}$ with minimal $\mathcal{R}$ (if any) and take the $\mathsf{HG4}^{\mathbf{1n}}$-paraproofs $\pi_1, \ldots, \pi_m$ orthogonal to π (if any).

(ii) For any $1 \leq i \leq m$, the $\mathsf{HG4}^{\mathbf{1n}i}$ calculus is the smallest extension of the $\mathsf{HG4}^{\mathbf{1n}}$ calculus obtained as follows:

  (a) if the refutational part of the initial hypersequents of $\mathsf{HG4}^{\mathbf{1n}}$ must include $\mathcal{R}$, then the refutational part of the initial hypersequents of $\mathsf{HG4}^{\mathbf{1n}i}$ must include $\mathcal{R} \mid \mathcal{R}_{\pi'_i}$;

  (b) if $\Theta \mathrel{|\!\!\stackrel{*}{\sim}} \Lambda$ belongs to $\mathsf{top}^\star_\mathsf{c}(\mathrel{|\!\!\stackrel{*}{\sim}} W \wedge W_i)$, with $\mathsf{top}^\star_\mathsf{c}(\mathrel{|\!\!\stackrel{*}{\sim}} W)$ being the extra-logical base of $\mathsf{HG4}^{\mathbf{1n}}$, then $\Theta \vdash \Lambda \mid \mathcal{S}$ is an initial hypersequent of $\mathsf{HG4}^{\mathbf{1n}i}$, for any $\mathcal{S}$ including $\mathcal{R} \mid \mathcal{R}_{\pi'_i}$.

Example 5.20. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be as in Example 5.19. Consider the following* $\mathsf{HG4c}$*-proof* π*:*

$$\dfrac{\dfrac{\dfrac{}{p \vdash p, q \vee r}\,ax}{\vdash p, q \vee r \mid \dashv \neg(\neg q \wedge p)}\,\delta_{\mathcal{D}'}}{\vdash p \vee q \vee r \mid \dashv \neg(\neg q \wedge p)}\,R\vee$$

The $\mathsf{HG4c}$*-paraproofs orthogonal to* π *are the following:*

$$\dfrac{R\vee\,\dfrac{\delta_{\mathcal{D}'}\,\dfrac{ax\,\dfrac{}{p \vdash p, q \vee r}}{\vdash p, q \vee r \mid \dashv \neg(\neg q \wedge p)}}{\vdash p \vee q \vee r \mid \dashv \neg(\neg q \wedge p)} \qquad \dfrac{\dfrac{}{p \vdash p}\,ax}{\vdash q \mid \dashv \neg(\neg r \wedge q)}\,\delta_{\mathcal{D}''}}{\vdash (p \vee q \vee r) \wedge q \mid \dashv \neg(\neg q \wedge p) \mid \dashv \neg(\neg r \wedge q)}\,R\wedge$$

$$\dfrac{R\vee\,\dfrac{\delta_{\mathcal{D}'}\,\dfrac{ax\,\dfrac{}{p \vdash p, q \vee r}}{\vdash p, q \vee r \mid \dashv \neg(\neg q \wedge p)}}{\vdash p \vee q \vee r \mid \dashv \neg(\neg q \wedge p)} \qquad \dfrac{\dfrac{}{p \vdash p}\,ax}{\vdash r \mid \dashv \neg(\neg p \wedge r)}\,\delta_{\mathcal{D}''}}{\vdash (p \vee q \vee r) \wedge r \mid\dashv \neg(\neg q \wedge p) \mid \dashv \neg(\neg p \wedge r)}\,R\wedge$$

By step (ii) *of Procedure 6 we obtain two extensions of* $\mathsf{HG4c}$*:*

(1) *the* $\mathsf{HG4}^{11}$ *calculus with* $\Gamma \vdash \Delta, q \mid \dashv \neg(\neg r \wedge q)$ *as initial hypersequent;*

(2) *the* $\mathsf{HG4}^{12}$ *calculus with* $\Gamma \vdash \Delta, r \mid \dashv \neg(\neg p \wedge r)$ *as initial hypersequent.*

Consider the following $\mathsf{HG4}^{11}$*-proof* π_1 *and* $\mathsf{HG4}^{12}$*-proof* π_2*:*

$$R\vee\,\dfrac{ax\,\dfrac{}{\vdash p, r, q \mid \dashv \neg(\neg r \wedge q)}}{\vdash p \vee q \vee r \mid \dashv \neg(\neg r \wedge q)} \qquad \dfrac{\dfrac{}{\vdash p \vee q, r \mid \dashv \neg(\neg p \wedge r)}\,ax}{\vdash p \vee q \vee r \mid \dashv \neg(\neg p \wedge r)}\,R\vee$$

There exists no $\mathsf{HG4}^{11}$*-paraproof orthogonal to* π_1 *and no* $\mathsf{HG4}^{12}$*-paraproof orthogonal to* π_2*.*

Intuitively, Procedure 6 operates by recursively extracting information from orthogonal paraproofs to identify modified extensions (if any) where the formula A cannot be proved.

If π is an HG4c-proof of $\vdash A \mid \mathcal{R}$ with minimal $\mathcal{R}$, then A is an m-skeptical consequence of $\mathcal{W}$ exactly when it is provable in HG4c from any set $\mathcal{D}' \subseteq \mathcal{D}$ which is inconsistent with $def(\pi)$ relative to the set $just(\mathcal{D}') \cup just(def(\pi))$. For any such $\mathcal{D}'$ there exists (at least) one minimal $\mathcal{D}'' \subseteq \mathcal{D}'$ which is inconsistent with $def(\pi)$ relative to the set $just(\mathcal{D}'') \cup just(def(\pi))$: if π' is an HG4c-paraproof orthogonal to π, and π'' its immediate subderivation distinct from π, there must exist (at least) one minimal $\mathcal{D}''$ for which $\mathcal{D}'' = def(\pi'')$. On the other hand, A is provable from any $\mathcal{D}''' \subseteq \mathcal{D}''$ if and only if A is an $\mathcal{R}'$-skeptical consequence of $\mathcal{W} \cup concl(def(\pi''))$, with $\mathcal{R}'$ being the refutational part of the conclusion of π''.

Correspondingly, the first application of step (i) in Procedure 6 selects all the HG4c-paraproofs orthogonal to π (if any). Subsequently, the first application of step (ii) (if any) tests skeptical provability against any $\mathsf{HG4}^{\mathbf{1n}i}$ calculus resulting from HG4c by incorporating $concl(def(\pi''))$ into the extra-logical base and $\mathcal{R}'$ into the refutational part of any derivable hypersequent. Maximal iteration of steps $(i) - (ii)$ ensures that the search for counterexamples to m-skeptical provability (if any) is exhaustive.

It is noteworthy that Procedure 6 is guaranteed to terminate after a finite number of steps. As a matter of fact, for any $\mathsf{HG4}^{\mathbf{1n}}$ calculus, a finite number of extra-logical rules is disabled in any $\mathsf{HG4}^{\mathbf{1n}i}$ calculus that extends it. This is due to the fact that the minimal refutational part of an hypersequent derivable in some $\mathsf{HG4}^{\mathbf{1n}i}$ strictly includes the minimal refutational part of any hypersequent derivable in $\mathsf{HG4}^{\mathbf{1n}}$.

We establish that Procedure 6 conducts an exhaustive search for counterexamples to the m-skeptical provability of A: specifically, our proof capitalizes on the structural properties of the HG4 calculi (cf. [165] for a different argument).

THEOREM 5.5. *A is an m-skeptical consequence of $\mathcal{W}$ if and only if for any* $\mathsf{HG4}^{s_h}$ *calculus (with $1 \leq h \leq k$), there exists an* $\mathsf{HG4}^{s_h}$*-proof of $\vdash A \mid \mathcal{R}$, for some $\mathcal{R}$.*

PROOF. Henceforth, let $\langle \mathcal{W}_{\mathbf{n}}, \mathcal{D} \rangle$ and $\langle \mathcal{W}_{\mathbf{n}i}, \mathcal{D} \rangle$ be the underlying default theories of $\mathsf{HG4}^{\mathbf{1n}}$ and $\mathsf{HG4}^{\mathbf{1n}i}$ – and $\mathcal{R}_{\mathbf{n}}$ and $\mathcal{R}_{\mathbf{n}i}$ be the minimal refutational parts of any hypersequent derivable in $\mathsf{HG4}^{\mathbf{1n}}$ and $\mathsf{HG4}^{\mathbf{1n}i}$, respectively. By construction, there exists an $\mathsf{HG4}^{\mathbf{1n}}$-proof π of $\vdash A \mid \mathcal{R}$, for some $\mathcal{R}$, along with a specific $\mathsf{HG4}^{\mathbf{1n}}$-paraproof π_i orthogonal to π, such that (1) π_i' is the immediate subderivation of π_i which is distinct from π, (2) $W_{\mathbf{n}i} = W_{\mathbf{n}} \wedge D_{\pi_i'}$ and (3) the conclusion of π_i' is an hypersequent of the form $\vdash B \mid \mathcal{R}_{\mathbf{n}i}$.

($\Rightarrow$) Suppose by contradiction that A is an m-skeptical consequence of $\mathcal{W}$, but there exists at least one sequence s_h such that every $\mathsf{HG4}^{s_h}$-derivation of $\vdash A \mid \mathcal{R}$, for any $\mathcal{R}$, is a paraproof. We reason by cases on the length lh of s_h to derive a contradiction.

If $lh(s_h) = 1$, then $\mathsf{HG4}^{s_h}$ reduces to the $\mathsf{HG4c}$ calculus. Since any $\mathsf{HG4}^{s_h}$-derivation of $\vdash A \mid \mathcal{R}$, for some $\mathcal{R}$, is a paraproof, Theorem 5.2 implies that A cannot be an m-credulous consequence of $\mathcal{W}$ – a contradiction.

If $lh(s_h) \geq n+1$, then $\mathsf{HG4}^{s_h}$ corresponds to some $\mathsf{HG4}^{\mathbf{1n}i}$ calculus. We have that $\vdash A \mid \mathcal{R}$ is not provable in $\mathsf{HG4}^{\mathbf{1n}i}$, for any $\mathcal{R}$: Proposition 5.1(i) and a generalized version of Theorem 5.2 ensure that A fails to be an $\mathcal{R}_{\mathbf{n}i}$-skeptical consequence of $\mathcal{W}_{\mathbf{n}i}$. Hence, there exists (at least) one $\mathcal{D}' \subseteq \mathcal{D}$ such that $def(\pi'_i) \subseteq \mathcal{D}'$ and, for any $\mathcal{D}'' \supseteq \mathcal{D}'$, there exists no $\mathsf{HG4}^{\mathbf{1n}}$-proof ρ of $\vdash A \mid \mathcal{R}'$ for which $def(\rho) = \mathcal{D}''$, for any $\mathcal{R}'$ including $\mathcal{R}_{\mathbf{n}}$. As a result, Proposition 5.5 guarantees that A fails to be an $\mathcal{R}_{\mathbf{n}}$-skeptical consequence of $\mathcal{W}_{\mathbf{n}}$. Starting from $\mathcal{W}_{\mathbf{n}}$, it suffices to iterate n times the same argument to obtain that A is not an m-skeptical consequence of $\mathcal{W}$ – a contradiction.

($\Leftarrow$) Suppose by contradiction that for any $1 \leq h \leq k$ there exists an $\mathsf{HG4}^{s_h}$-proof π_{s_h} of $\vdash A \mid \mathcal{R}$, for some $\mathcal{R}$, but A is not an m-skeptical consequence of $\mathcal{W}$. We distinguish two cases, according to whether any $\mathsf{HG4}^{s_h}$ calculus is just $\mathsf{HG4c}$ or each $\mathsf{HG4}^{s_h}$ is an $\mathsf{HG4}^{\mathbf{1n}i}$ calculus.

In the first case, there exists no $\mathsf{HG4c}$-paraproof orthogonal to π_{s_h}. If A is not an m-skeptical consequence of $\mathcal{W}$, there must exist modified extensions $\langle \mathcal{E}_1, \mathcal{F}_1 \rangle, \ldots, \langle \mathcal{E}_m, \mathcal{F}_m \rangle$ of $\langle \mathcal{W}, \mathcal{D} \rangle$ such that $A \notin \mathcal{E}_i$, for any $1 \leq i \leq m$. If $\mathcal{D}_i \subseteq \mathcal{D}$ is a set of defaults generating $\mathcal{E}_i$, there must exist (at least) one $\mathsf{HG4c}$-proof π' of (say) $\vdash B \mid \mathcal{R}'$, with $def(\pi') = \mathcal{D}_i$. If we plug π_{s_h} and π' into a single $\mathsf{HG4c}$-derivation *via* a single application of $R\wedge$, we obtain an $\mathsf{HG4c}$-paraproof orthogonal to π_{s_h} – a contradiction.

In the second case, we proceed as in the previous one to infer that A is an $\mathcal{R}_{\mathbf{n}i}$-skeptical consequence of each $\mathcal{W}_{\mathbf{n}i}$. Hence, for any $\mathcal{D}' \subseteq \mathcal{D}$ with $def(\pi'_i) \subseteq \mathcal{D}'$ there exists (at least) one $\mathcal{D}'' \supseteq \mathcal{D}'$ such that there exists an $\mathsf{HG4}^{\mathbf{1n}}$-proof ρ of $\vdash A \mid \mathcal{R}'$ for which $def(\rho) = \mathcal{D}''$, for some $\mathcal{R}'$ including $\mathcal{R}_{\mathbf{n}}$.

Starting from each $\mathcal{W}_{\mathbf{n}}$, we iterate $n-1$ times the same argument. Let σ be the $\mathsf{HG4c}$-proof of $\vdash A \mid \mathcal{S}$, for some $\mathcal{S}$, from which the sequence s_h is built, $\sigma_1, \ldots, \sigma_m$ be the $\mathsf{HG4c}$-paraproofs orthogonal to π and σ'_j be the immediate subderivation of σ_j distinct from σ, for any $1 \leq j \leq m$. If $\mathcal{R}_{\sigma'_j}$ is the refutational part of the conclusion of σ'_j, then A is an $\mathcal{R}_{\sigma'_j}$-skeptical consequence of $\mathcal{W}_j$. As a result, for any $\mathcal{D}' \subseteq \mathcal{D}$ with $def(\sigma'_j) \subseteq \mathcal{D}'$ there exists (at least) one $\mathcal{D}'' \supseteq \mathcal{D}'$ such that there exists an $\mathsf{HG4c}$-proof σ'' of $\vdash A \mid \mathcal{S}'$ for which $def(\sigma'') = \mathcal{D}''$, for some $\mathcal{S}'$ including $\mathcal{R}_{\sigma'_j}$. Hence, there exists an $\mathsf{HG4c}$-proof σ'' of $\vdash A \mid \mathcal{S}'$ from any set of defaults $\mathcal{D}'''$ which is inconsistent with $def(\pi)$ relative to $just(def(\pi)) \cup just(\mathcal{D}''')$. This suffices to conclude that A is an m-skeptical consequence of $\mathcal{W}$ – a contradiction. $\square$

Corollary 5.6. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be a normal default theory. A is a skeptical consequence of $\mathcal{W}$ if and only if for any $\mathsf{HG4}^{s_h}$ calculus, with $1 \leq h \leq n$, there exists an $\mathsf{HG4}^{s_h}$-proof of $\vdash A \mid \mathcal{R}$, for some $\mathcal{R}$.*

We conclude this section by proving that the existence of orthogonal paraproofs is a necessary condition for the orthogonality of modified extensions (cf. Example 5.5):

Corollary 5.7. *Let A be an m-credulous consequence of $\mathcal{W}$. If $\neg A$ is an m-credulous consequence of $\mathcal{W}$, then for (at least) one $\mathsf{HG4c}$-proof π of $\vdash A$ there exists (at least) one $\mathsf{HG4c}$-paraproof π' orthogonal to π.*

PROOF. By Theorem 5.5, if π is any $\mathsf{HG4c}$-proof of $\vdash A$ and there exists no $\mathsf{HG4c}$-paraproof π' orthogonal to π, then A is an m-skeptical consequence of $\mathcal{W}$. Furthermore, Proposition 5.2 entails that, for any modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$, the set $\mathcal{E}$ is consistent. Consequently, if A is an m-skeptical consequence of $\mathcal{W}$, then $\neg A$ is not an m-credulous consequence of $\mathcal{W}$; thus, by contraposition, we get the conclusion. $\square$

CHAPTER 6

Hybrid hypersequents for belief revision

Belief change is ubiquitous in ordinary life as well as in scientific theorizing. Historically, a prominent role in the formal analysis of belief change has been played by the framework proposed by Alchourrón, Gärdenfors and Makinson in 1985 [2].

In the AGM model, beliefs are represented by (classically invalid) formulas in a standard formal language for classical propositional logic, and attention is restricted on sets of beliefs closed under classical consequence – typically, under the assumption that a rational agent is committed to believe all the consequences of the beliefs she actually holds [85]. For any belief set $\mathcal{K}$ and any formula A, three types of belief change involving $\mathcal{K}$ and A are taken into consideration:

(i) the *expansion* of $\mathcal{K}$ by A, which results into the smallest logically closed set including $\mathcal{K}$ and A;

(ii) the *contraction* of $\mathcal{K}$ by A, which yields a subset of $\mathcal{K}$ not containing A;

(iii) the *revision* of $\mathcal{K}$ by A, which amounts to the addition of A to $\mathcal{K}$ along with the removal of sentences from $\mathcal{K}$ in order to ensure consistency of the resulting belief set.

In analogy with the previous chapter on propositional default logics, the present one introduces a proof-theoretic approach to AGM belief revision centered on hybrid hypersequents.

The chapter is organized as follows. Section 6.1 contains the proof-theoretic results concerning maximally consistent subsets of sets of clauses that we will subsequently employ. In Section 6.2, we offer a constructive presentation of base-generated belief revision in terms of maximally consistent subsets of base-generated belief expansion. In Section 6.3, our focus shifts to hybrid hypersequent calculi. Here, we establish admissibility of structural rules, invertibility of logical rules and the full subformula property for cut-free proofs. We present hybrid hypersequent calculi that are sound and (weakly) complete with respect to (refined) base-generated belief revision, showing that they fail to be strongly complete due to their non-monotonic behaviour in relation to the addition of extra-logical axioms.

6.1. A proof-theoretic approach to maximally consistent subsets

In this section, we undertake a syntactic approach to maximally consistent subsets (in short, *mcs*'s) of inconsistent sets of formulas. To improve intelligibility, we introduce *ad hoc* (hybrid) sequent calculi for supraclassical logics featuring explicit structural rules. Till the end of the chapter, we adopt contexts as sets: $\Gamma, \Delta, \Pi, \Sigma, \ldots$ stand for finite sets of formulas, and $\Theta, \Lambda, \ldots$ for finite sets of atomic formulas.

Let $\mathcal{S}$ be a finite set of extra-logical axioms, S the conjunction of the latter and $\Theta \vDash^{*} \Lambda$ be any clause in $\mathsf{top_c}(\vDash^{*} S)$: the $\overline{\overline{\mathsf{G4str}}}$ calculus for $\mathcal{S}$ is obtained from $\overline{\overline{\mathsf{G4}}}$ by adopting the Weakening and Cut rules as primitive, and replacing any instance $\dfrac{}{\Theta \vDash^{\circ} \Lambda}$ of the rule $\overline{ax}$ with an instance $\dfrac{}{\Theta \vdash \Lambda}$ of the rule ax.

Proposition 6.1. $\overline{\overline{\mathsf{G4s}}}$ proves $\Gamma \vdash \Delta$ iff $\overline{\overline{\mathsf{G4str}}}$ proves $\Gamma \vdash \Delta$.

Proposition 6.2. Logical rules are invertible in $\overline{\overline{\mathsf{G4str}}}$.

Proof. By Proposition 6.1 and Proposition 2.1. Notice that preservation of height fails to hold: consider the case of a $\overline{\overline{\mathsf{G4str}}}$-proof π of the form

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta, A\end{array} \qquad \begin{array}{c}\vdots\\ A, B \to C, \Pi \vdash \Sigma\end{array}}{B \to C, \Pi, \Gamma \vdash \Delta, \Sigma}\,cut$$

with $A = C$ and $h(\pi) \leq n+1$. If we reason by routine induction, there are $\overline{\overline{\mathsf{G4str}}}$-proofs π_1, π_2 of the sequents $A, \Pi \vdash \Sigma, B$ and $A, C, \Pi \vdash \Sigma$, with $h(\pi_1), h(\pi_2) \leq n$: if we apply Cut to the sequents $\Gamma \vdash \Delta, A$ and $A, C, \Pi \vdash \Sigma$ we get a $\overline{\overline{\mathsf{G4str}}}$-proof π' of $\Pi, \Gamma \vdash \Delta, \Sigma$ with $h(\pi') \leq n+1$. If $\overline{\overline{\mathsf{G4}}}$ refutes $\Pi, \Gamma \vdash \Delta, \Sigma$, then we get a $\overline{\overline{\mathsf{G4str}}}$-proof π'' of $C, \Pi, \Gamma \vdash \Delta, \Sigma$ of the form

$$\dfrac{\begin{array}{c}\vdots_{\pi'}\\ \Pi, \Gamma \vdash \Delta, \Sigma\end{array}}{C, \Pi, \Gamma \vdash \Delta, \Sigma}\,wk$$

with $h(\pi'') \leq n+2$. ☐

Lemma 6.1. *If there is a* $\overline{\overline{\mathsf{G4str}}}$*-derivation* π *of* $\Gamma \vdash \Delta$ *from* $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n$*, then* $\overline{\overline{\mathsf{G4str}}}$ *proves* $\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n \vdash \bigwedge \Gamma \to \bigvee \Delta$.

Proof. We proceed by induction of $h(\pi)$ to show that there is a $\overline{\overline{\mathsf{G4str}}}$-derivation of $\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n, \Gamma \vdash \Delta$ from the sequents $\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n, \Gamma_1 \vdash \Delta_1, \ldots, \bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n, \Gamma_n \vdash \Delta_n$ (the details are omitted). Hence, we infer that there exists a $\overline{\overline{\mathsf{G4str}}}$-derivation of $\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n \vdash \bigwedge \Gamma \to \bigvee \Delta$ from $\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n, \Gamma_1 \vdash \Delta_1, \ldots, \bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n, \Gamma_n \vdash \Delta_n$. We exploit Proposition 6.2 to establish that the latter sequents are provable – and this concludes the proof. ☐

We state that a rule of the form

$$\dfrac{\Gamma_1 \vdash \Delta_1 \qquad \cdots \qquad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}\,r$$

can be *eliminated* from $\overline{\overline{\mathsf{G4str}}}$ if and only if r belongs to $\overline{\overline{\mathsf{G4str}}}$ and the sequent $\Gamma \vdash \Delta$ is provable without applying r whenever the sequents $\{\Gamma_i \vdash \Delta_i\}_{1 \leq i \leq n}$ are provable without applying r. Furthermore, we say that a Cut application is *inessential* exactly when the Cut formula is not atomic.

PROPOSITION 6.3. The rule of inessential Cut can be eliminated from $\overline{\overline{\mathsf{G4str}}}$.

PROOF. We consider the topmost inessential Cut application, reasoning by induction on the logical complexity of A and exploiting Proposition 6.2. ☐

PROPOSITION 6.4. If a clause $\Theta \vdash \Lambda$ is provable in $\overline{\overline{\mathsf{G4str}}}$, then there is (at least) one $\overline{\overline{\mathsf{G4str}}}$-proof π of $\Theta \vdash \Lambda$ which contains a sequent $\Phi \vdash \Psi$ such that

(i) every rule applied above $\Phi \vdash \Psi$ is ax or Cut;

(ii) every rule applied below $\Phi \vdash \Psi$ is Weakening.

PROOF. Let π' be a $\overline{\overline{\mathsf{G4str}}}$-proof of $\Theta \vdash \Lambda$. If a non-atomic formula A is introduced by the application of a rule in some branch of π', then A occurs down the same branch as an inessential Cut formula – and this contradicts Proposition 6.3: as a result, no logical rule can be applied in π. On the other hand, let us consider the topmost Weakening application in π' which is immediately followed by a Cut application. We reason by cases over the Cut formula p to get the conclusion – e.g., as follows:

$$
\begin{array}{c}
\begin{array}{r@{\;}c}
 & \begin{array}{c@{\qquad}c}
 \begin{array}{r@{\;}c} & \vdots \\ wk & \begin{array}{c} \Theta' \vdash \Lambda' \\ \hline \Theta' \vdash \Lambda', p \end{array} \end{array} & \begin{array}{c} \vdots \\ p, \Phi' \vdash \Psi' \end{array}
 \end{array} \\
 \hline
 cut & \Phi', \Theta', \vdash \Lambda', \Psi'
\end{array}
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c@{\;}l}
\begin{array}{c} \vdots \\ \Theta' \vdash \Lambda' \\ \hline\hline \Phi', \Theta' \vdash \Lambda', \Psi' \end{array} & wk
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{r@{\;}c}
 & \begin{array}{c@{\qquad}c}
 \begin{array}{r@{\;}c} & \vdots \\ wk & \begin{array}{c} \Theta' \vdash \Lambda', p \\ \hline \Theta' \vdash \Lambda', q, p \end{array} \end{array} & \begin{array}{c} \vdots \\ p, \Phi' \vdash \Psi' \end{array}
 \end{array} \\
 \hline
 cut & \Phi', \Theta', \vdash \Lambda', q, \Psi'
\end{array}
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c@{\;}l}
\begin{array}{c@{\qquad}c} \vdots & \vdots \\ \Theta' \vdash \Lambda', p & p, \Phi' \vdash \Psi' \end{array} & \\
\hline
\Phi', \Theta' \vdash \Lambda', \Psi' & cut \\
\hline
\Phi', \Theta' \vdash \Lambda', \Psi', q & wk
\end{array}
$$

☐

LEMMA 6.2. *Let* $\mathcal{C}$ *be a set of clauses. Then* $\mathcal{C}$ *is inconsistent iff the empty clause belongs to* $\mathcal{C}^{\star}$.

PROOF. For any set $\mathcal{C}$ of clauses, let us say that the $\overline{\overline{\mathsf{G4str}}}$-calculus for $\mathcal{C}$ is the $\overline{\overline{\mathsf{G4str}}}$-calculus for the formula translations of the clauses in $\mathcal{C}$. ($\Rightarrow$) If $\mathcal{C}$ is inconsistent, Proposition 2.4 ensures that the $\overline{\overline{\mathsf{G4str}}}$ calculus for $\mathcal{C}$ proves the sequents $\vdash A$ and $\vdash \neg A$, for some formula A: by Proposition 6.2 we get the result. ☐

THEOREM 6.1. *Let $\mathcal{C}$ be a set of clauses. Then $\mathcal{D} \subseteq \mathcal{C}$ is a mcs of $\mathcal{C}$ iff for any non-empty clause $p_1, \ldots, p_m \mid\!\overset{*}{\sim} q_1, \ldots, q_n$ in $\mathcal{C} \setminus \mathcal{D}$ the clauses $\{\mid\!\overset{*}{\sim} p_i\}_{1 \leq i \leq m}$ and $\{q_j \mid\!\overset{*}{\sim}\}_{1 \leq j \leq n}$ belong to $\mathcal{D}^{\star}$*[1].

PROOF. ($\Leftarrow$) Straightforward from Lemma 6.2. ($\Rightarrow$) If $\mathcal{D}$ is a mcs of $\mathcal{C}$, then for any non-empty clause $p_1, \ldots, p_m \mid\!\overset{*}{\sim} q_1, \ldots, q_n$ in $\mathcal{C} \setminus \mathcal{D}$ we have that $\mathcal{E} = \mathcal{D} \cup \{p_1, \ldots, p_m \mid\!\overset{*}{\sim} q_1, \ldots, q_n\}$ is an inconsistent set of clauses. If D is the conjunction of the formula translations of the clauses in $\mathcal{D}$, Lemma 6.1 ensures that the $\overline{\overline{\mathsf{G4str}}}$-calculus for $\mathcal{D}$ proves the sequent $D, (p_1 \wedge \cdots \wedge p_m) \to (q_1 \vee \cdots \vee q_n) \vdash \top \to \bot$. Since the $\overline{\overline{\mathsf{G4str}}}$-calculus for $\mathcal{D}$ proves $\vdash D$, it proves also $(p_1 \wedge \cdots \wedge p_m) \to (q_1 \vee \cdots \vee q_n) \vdash \top \to \bot$: we leverage Proposition 6.2 to infer that the $\overline{\overline{\mathsf{G4str}}}$-calculus for $\mathcal{D}$ proves the sequents $\{\vdash p_i\}_{1 \leq i \leq m}$ and $\{q_j \vdash\}_{1 \leq j \leq n}$. Since $\mathcal{D}$ is consistent, Lemma 6.2 ensures that the $\overline{\overline{\mathsf{G4str}}}$-calculus for $\mathcal{D}$ does not prove the empty clause: by Proposition 6.4, we conclude that $\{\mid\!\overset{*}{\sim} p_i\}_{1 \leq i \leq m}$ and $\{q_j \mid\!\overset{*}{\sim}\}_{1 \leq j \leq n}$ belong to $\mathcal{D}^{\star}$. ☐

PROPOSITION 6.5. Let $\mathcal{C}$ be a set of clauses. If $\mathcal{D}$ is a mcs of $\mathcal{C}$, then we have that

- (*i*) $(\mathcal{C} \setminus \mathcal{D})$ is a set of complementary clauses;
- (*ii*) $(\mathcal{C} \setminus \mathcal{D}) = (\mathcal{C} \setminus \mathcal{D})^{*}$;
- (*iii*) if $\mathcal{C}$ is a set of non-empty clauses, then $\mathcal{C} \setminus \mathcal{D}$ is consistent.

PROOF. (*i*) If there is an identity clause $\Theta, p \mid\!\overset{*}{\sim} p, \Lambda$ in $\mathcal{C} \setminus \mathcal{D}$, then Theorem 6.1 implies that clauses $\mid\!\overset{*}{\sim} p$ and $p \mid\!\overset{*}{\sim}$ belong to $\mathcal{D}^{\star}$: as a result, the empty sequent belongs to $\mathcal{D}^{\star}$ – by Lemma 6.2, a contradiction. (*ii*) If clauses of the form $\Theta \mid\!\overset{*}{\sim} \Lambda, p$ and $p, \Phi \mid\!\overset{*}{\sim} \Psi$ belong to $\mathcal{C} \setminus \mathcal{D}$, then Theorem 6.1 entails that clauses $\mid\!\overset{*}{\sim} p$ and $p \mid\!\overset{*}{\sim}$ belong to $\mathcal{D}^{\star}$ – as before, a contradiction. (*iii*) Suppose by contradiction that $\mathcal{C} \setminus \mathcal{D}$ is inconsistent, and thus that there exists (at least) one non-empty mcs $\mathcal{E}$ of $\mathcal{C} \setminus \mathcal{D}$ such that $\mathcal{E} \subset (\mathcal{C} \setminus \mathcal{D})$. By Theorem 6.1, for any non-empty clause $p_1, \ldots, p_m \mid\!\overset{*}{\sim} q_1, \ldots, q_n$ in $(\mathcal{C} \setminus \mathcal{D}) \setminus \mathcal{E}$ the clauses $\{\mid\!\overset{*}{\sim} p_i\}_{1 \leq i \leq m}$ and $\{q_j \mid\!\overset{*}{\sim}\}_{1 \leq j \leq n}$ belong to $\mathcal{E}^{\star}$. By Lemma 6.2 and Proposition 6.4, this means that there exist clauses $\{\Theta_i \mid\!\overset{*}{\sim} p_i, \Lambda_i\}_{1 \leq i \leq m}$ and $\{\Phi_j, q_j \mid\!\overset{*}{\sim} \Psi_j\}_{1 \leq j \leq n}$ in $\mathcal{E}$: Theorem 6.1 entails that the clauses $\{p_i \mid\!\overset{*}{\sim}\}_{1 \leq i \leq n}$ and $\{\mid\!\overset{*}{\sim} q_j\}_{1 \leq j \leq n}$ belong to $\mathcal{D}^{\star}$. On the other hand, by Theorem 6.1 we have that also the clauses $\{\mid\!\overset{*}{\sim} p_i\}_{1 \leq i \leq m}$ and $\{q_j \mid\!\overset{*}{\sim}\}_{1 \leq j \leq n}$ belong to $\mathcal{D}^{\star}$ – a contradiction. ☐

PROPOSITION 6.6. Let $\mathcal{C}$ be a set of clauses. If $\mathcal{C} = \mathcal{C}^{\star}$ and $\mathcal{D}$ is a mcs of $\mathcal{C}$, then $\mathcal{D} = \mathcal{D}^{\star}$.

PROOF. Suppose by contradiction that there is (at least) one mcs $\mathcal{D}$ of $\mathcal{C}$ such that $\mathcal{D}^{\star} \not\subseteq \mathcal{D}$. This implies that there is (at least) one non-empty clause $\Theta \mid\!\overset{*}{\sim} \Lambda$ in $\mathcal{D}^{\star}$ which belongs to $\mathcal{C} \setminus \mathcal{D}$: by Theorem 6.1, if $\Theta = \{p_1, \ldots, p_m\}$ and $\Lambda = \{q_1, \ldots, q_n\}$, then the clauses $\{\mid\!\overset{*}{\sim} p_i\}_{1 \leq i \leq m}$ and $\{q_j \mid\!\overset{*}{\sim}\}_{1 \leq j \leq n}$ belong to $\mathcal{D}^{\star}$ – by Lemma 6.2, a contradiction. ☐

PROPOSITION 6.7. Let $\mathcal{C}$ be a set of clauses. Then we have that:

---

[1] Let us recall that we use $\mid\!\overset{*}{\sim}$ whenever abstracting from a specific deduction-refutation system.

(i) if $\mathcal{D}$ is a mcs of $\mathcal{C}$, $\mathcal{D}^\star$ may *not* be a mcs of $\mathcal{C}^\star$;

(ii) if $\mathcal{D}^\star$ is a mcs of $\mathcal{C}^\star$, $\mathcal{D}$ may *not* be a mcs of $\mathcal{C}$.

Proof. For each statement we offer an example. (i) Let $\mathcal{C}$ be $\{\ p, q \vdash r, s\ ;\ r \vdash\ ;\ s \vdash\ ;\ \vdash p\ ;\ \vdash q\ \}$. By Theorem 6.1, the set $\mathcal{D} = \{\ p, q \vdash r, s\ ;\ r \vdash\ ;\ s \vdash\ ;\ \vdash p\ \}$ is a mcs of $\mathcal{C}$. On the other hand, $\mathcal{D}^\star = \{\ p, q \vdash r, s\ ;\ r \vdash\ ;\ s \vdash\ ;\ \vdash p\ ;\ p, q \vdash\ ;\ p, q \vdash r\ ;\ p, q \vdash s\ ;\ q \vdash r, s\ ;\ q \vdash s\ ;\ q \vdash r\ ;\ q \vdash\ \}$. By Theorem 6.1, $\mathcal{D}^\star$ is *not* a mcs of $\mathcal{C}^\star$, since e.g. $\vdash r$ belongs to $\mathcal{C}^\star \setminus \mathcal{D}^\star$ and $r \vdash$ does not belong to $\mathcal{D}^\star$. (ii) Let $\mathcal{C}$ be $\{\ p \vdash q\ ;\ q \vdash r\ ;\ \vdash p\ ;\ \vdash q\ ;\ q \vdash\ ;\ r \vdash\ \}$, and thus $\mathcal{C}^\star = \{\ p \vdash q\ ;\ q \vdash r\ ;\ \vdash p\ ;\ \vdash q\ ;\ q \vdash\ ;\ r \vdash\ ;\ p \vdash r\ ;\ \vdash r\ ;\ \vdash p\ ;\ \vdash\ \}$. The set $\mathcal{E} = \{\ p \vdash q\ ;\ q \vdash r\ ;\ p \vdash\ ;\ \vdash r\ ;\ q \vdash\ ;\ p \vdash r\ \}$ is a mcs of $\mathcal{C}^\star$: as witnessed by Proposition 6.6, $\mathcal{E} = \mathcal{E}^\star$ – whereas $\mathcal{E}$ is not even a subset of $\mathcal{C}$. ☐

Proposition 6.8. If $\mathcal{C}$ is a set of clauses and $\mathcal{D}_1, \dots, \mathcal{D}_n$ are the mcs's of $\mathcal{C}$, then the following conditions hold:

(i) $\mathcal{D}_i \cup \mathcal{D}_j$ is inconsistent, for any $1 \leq i \neq j \leq n$;

(ii) if $\mathcal{C} = \mathcal{C}^\star$, then $(\mathcal{D}_1 \cap \ldots \cap \mathcal{D}_i) = (\mathcal{D}_1 \cap \ldots \cap \mathcal{D}_i)^\star$, for any $1 \leq i \leq n$.

Proof. (i) $\mathcal{D}_i$ and $\mathcal{D}_j$ are distinct mcs's of $\mathcal{C}$: as a result, there exists (at least) one non-empty clause $\Theta \vdash \Lambda$ which belongs to (say) $\mathcal{D}_i$ and not to $\mathcal{D}_j$. The clause $\Theta \vdash \Lambda$ thus belongs to $\mathcal{C} \setminus \mathcal{D}_j$: if $\Theta = p_1, \dots, p_m$ and $\Lambda = q_1, \dots, q_{m'}$, then Theorem 6.1 guarantees that the clauses $\{\overset{*}{\vdash} p_h\}_{1 \leq h \leq m}$ and $\{q_k \overset{*}{\vdash}\}_{1 \leq k \leq m'}$ belong to $\mathcal{D}_j^\star$. (ii) If $\Theta \overset{*}{\vdash} \Lambda, p$ and $p, \Phi \overset{*}{\vdash} \Psi$ belong to $\mathcal{D}_1 \cap \ldots \cap \mathcal{D}_i$, then $\Theta \overset{*}{\vdash} \Lambda, p$ and $p, \Phi \overset{*}{\vdash} \Psi$ belong to $\mathcal{D}_j$, for each $1 \leq j \leq i$. By Proposition 6.6 we have that $\mathcal{D}_j = \mathcal{D}_j^\star$: this implies that $\Phi, \Theta \overset{*}{\vdash} \Lambda, \Psi$ belongs to each $\mathcal{D}_j$ – and we are done. ☐

6.2. Base-generated belief revision

We say that $\mathcal{B}$ is a *belief base* if $\mathcal{B}$ is a finite, non-empty set of classically invalid formulae, and that $\mathcal{K}$ is a *belief set* if $\mathcal{K}$ is a set of formulae comprising (at least) one extra-logical axiom and which is closed under classical consequence. Furthermore, we state that a belief set $\mathcal{K}$ is *generated by a (belief) base* $\mathcal{B}$ exactly when $\mathcal{K} = Cn(\mathcal{B})$.

Definition 6.1. Let $\mathcal{B}$ be a belief base. The *base expansion of* $\mathcal{B}$ *by* A, in symbols $\mathcal{B} + A$, is $\mathcal{B} \cup \{A\}$.

For any base $\mathcal{B}$ we can axiomatically define the operation of contraction:

Definition 6.2. Let $\mathcal{B}$ be a belief base. The set of formulae $\mathcal{B} \div A$ is the *base contraction of* $\mathcal{B}$ *by* A iff the following postulates are satisfied:

BC1 *Success*: if A is not tautological, then $A \notin (\mathcal{B} \div A)$.

BC2 *Inclusion*: $(\mathcal{B} \div A) \subseteq \mathcal{B}$.

BC3 *Relevance*: if $B \in \mathcal{B}$ and $B \notin (\mathcal{B} \div A)$, then there is a set $\mathcal{B}'$ such that $(\mathcal{B} \div A) \subseteq \mathcal{B}' \subseteq \mathcal{B}$, $A \notin Cn(\mathcal{B}')$ and $A \in Cn(\mathcal{B}' \cup \{B\})$.

BC4 *Uniformity*: if for all $\mathcal{B}' \subseteq \mathcal{B}$ we have that $A \in Cn(\mathcal{B}')$ iff $B \in Cn(\mathcal{B}')$, then $\mathcal{B} \div A = \mathcal{B} \div B$.

Axioms BC1 and BC2 ensure that (at least) A is removed from $\mathcal{B}$ after contraction by A. By Axiom BC3, if a formula B is removed from $\mathcal{B}$ after contraction by A, then B plays some role for the fact that $\mathcal{B}$ logically implies A: in other words, the exclusion of formulas from $\mathcal{B}$ after contraction by A is blocked unless there is some good reason for the exclusion[2]. Finally, Axiom BC4 guarantees that the result of contraction of $\mathcal{B}$ by A depends only on which subsets of $\mathcal{B}$ logically imply A.

On the other hand, one can pursue a constructive approach to base contraction of $\mathcal{B}$ by A *via* the following notions.

Definition 6.3. Let $\mathcal{B}$ be a belief base, and A a formula.

(i) The *remainder set of* $\mathcal{B}$ *with respect to* A, in symbols $\mathcal{B}\perp A$, contains any $\mathcal{B}' \subseteq \mathcal{B}$ for which $A \notin Cn(\mathcal{B}')$ and there is no $\mathcal{B}'' \subseteq \mathcal{B}$ such that $\mathcal{B}' \subset \mathcal{B}''$ and $A \notin Cn(\mathcal{B}'')$.

(ii) A *selection function* for $\mathcal{B}$ is a function γ such that $\gamma(\mathcal{B}\perp A) \neq \varnothing$ and $\gamma(\mathcal{B}\perp A) \subseteq (\mathcal{B}\perp A)$, if $(\mathcal{B}\perp A) \neq \varnothing$ – and $\gamma(\mathcal{B}\perp A) = \mathcal{B}$, otherwise.

Theorem 6.2. *Let $\mathcal{B}$ be a belief base. Then $\mathcal{B} \div A = \bigcap \gamma(\mathcal{B}\perp A)$ for some selection function for $\mathcal{B}$.*

Proof. For a proof see [62, pp. 657-658]. ☐

For any base $\mathcal{B}$ we can axiomatically define the operation of revision:

Definition 6.4. Let $\mathcal{B}$ be a belief base. The set of formulae $\mathcal{B} * A$ is the *base revision of* $\mathcal{B}$ *by* A[3] iff the following postulates are satisfied:

BR1 *Success*: if A is not tautological, then $A \in (\mathcal{B} * A)$.

BR2 *Inclusion*: $(\mathcal{B} * A) \subseteq (\mathcal{B} + A)$.

BR3 *Relevance*: if $B \in \mathcal{B}$ and $B \notin (\mathcal{B} * A)$, then there is a consistent set $\mathcal{B}'$ such that $(\mathcal{B} * A) \subseteq \mathcal{B}' \subseteq (\mathcal{B} + A)$ and $\mathcal{B}' + B$ is inconsistent.

---

[2] In AGM contraction, the analogous of BC3 is the much-debated axiom of Recovery (cf. [61, pp. 219-221] for a brief comparison).

[3] We use 'base revision of $\mathcal{B}$ by A' to denote what in the literature is known as '*internal* (partial meet) base revision of $\mathcal{B}$ by A' (cf. [62, p. 649]).

BR4 *Uniformity*: if for all $\mathcal{B}' \subseteq \mathcal{B}$ we have that $\neg A \in Cn(\mathcal{B}')$ iff $\neg B \in Cn(\mathcal{B}')$, then $\mathcal{B} * A = \mathcal{B} * B$.

One can undertake a constructive approach to base revision by exploiting the following result:

THEOREM 6.3. *If* $\mathcal{B}$ *is a belief base, then* $\mathcal{B} * A = (\mathcal{B} \div \neg A) + A$.

PROOF. For a proof see [62, pp. 661-662]. ☐

PROPOSITION 6.9. Let $\mathcal{B}$ be a belief base and A a consistent formula. Then $\mathcal{B}*A = \bigcap_{i=1}^{n} \mathcal{B}_i$, for some mcs's $\mathcal{B}_1, \ldots, \mathcal{B}_n$ of $\mathcal{B} + A$ containing A.

PROOF. Theorem 6.3 and Theorem 6.2 ensure the existence of (at least) one selection function γ for $\mathcal{B}$ such that $\mathcal{B} * A = \big(\bigcap \gamma(\mathcal{B}\perp\neg A)\big) + A$. If $\gamma(\mathcal{B}\perp\neg A) = \{\mathcal{B}'_1, \ldots, \mathcal{B}'_n\}$, then the following equalities hold:

$$\mathcal{B} * A = \Big(\bigcap_{i=1}^{n} \mathcal{B}'_i\Big) \cup \{A\} = \bigcap_{i=1}^{n} (\mathcal{B}'_i \cup \{A\})$$

Suppose by contradiction that $\mathcal{B}$ contains a formula $B \notin \mathcal{B}'_i$ such that $\mathcal{B}'_i \cup \{A\} \cup \{B\}$ is consistent, for some i. This implies that $\neg A \notin Cn(\mathcal{B}'_i \cup \{B\})$, whereas Definition 6.3 guarantees that $\neg A \in Cn(\mathcal{B}')$, for any $\mathcal{B}' \subseteq \mathcal{B}$ such that $\mathcal{B}'_i \subset \mathcal{B}'$: as a result, we get that $\mathcal{B}'_i \cup \{A\}$ is a mcs of $\mathcal{B} \cup \{A\}$, for any i – as desired. ☐

In this chapter, we shall focus on belief sets generated by belief bases. Base expansion, base contraction and base revision give rise to analogous operations on the generated belief set:

DEFINITION 6.5. Let $\mathcal{B}$ a belief base, and $\mathcal{K} = Cn(\mathcal{B})$. The set of formulae

(*i*) $\mathcal{K} + A$ is the *base-generated expansion of* $\mathcal{K}$ *by* A iff $\mathcal{K} + A = Cn(\mathcal{B} + A)$;

(*ii*) $\mathcal{K} \div A$ is the *base-generated contraction of* $\mathcal{K}$ *by* A iff $\mathcal{K} \div A = Cn(\mathcal{B} \div A)$;

(*iii*) $\mathcal{K} * A$ is the *base-generated revision of* $\mathcal{K}$ *by* A iff $\mathcal{K} * A = Cn(\mathcal{B} * A)$.

Base-generated contraction of a belief set can be axiomatically characterized as follows:

THEOREM 6.4. *Let* $\mathcal{B}$ *a belief base, and* $\mathcal{K} = Cn(\mathcal{B})$. *The set* $\mathcal{K} \div A$ *is the base-generated contraction of* $\mathcal{K}$ *by* A *iff it satisfies the following postulates:*

BGC1 Closure*:* $\mathcal{K} \div A = Cn(\mathcal{K} \div A)$.

BGC2 Success*: if* A *is not tautological, then* $A \notin Cn(\mathcal{K} \div A)$.

BGC3 Inclusion*:* $\mathcal{K} \div A \subseteq \mathcal{K}$.

BGC4 Vacuity*: if* $A \notin \mathcal{K}$*, then* $\mathcal{K} \subseteq \mathcal{K} \div A$.

BGC5 Extensionality*: if A and B are classically equivalent, then* $\mathcal{K} \div A = \mathcal{K} \div B$.

BGC6 Finitude*: there is a finite set* $\mathcal{B}$ *such that* $\mathcal{K} \div A = Cn(\mathcal{B}')$ *for some* $\mathcal{B}' \subseteq \mathcal{B}$.

BGC7 Symmetry*: if for all A we have that* $B \in \mathcal{K} \div A$ *exactly when* $C \in \mathcal{K} \div A$*, then* $\mathcal{K} \div B = \mathcal{K} \div C$.

BGC8 Conservativity*: if* $\mathcal{K} \div B$ *is not a subset of* $\mathcal{K} \div A$*, then there is some C such that* $\mathcal{K} \div A \subseteq \mathcal{K} \div C$, $C \notin \mathcal{K} \div C$ *and* $A \in (\mathcal{K} \div B) \cup (\mathcal{K} \div C)$.

Proof. For a proof see [63, pp. 610-614]. ☐

Axioms BGC1 – BGC2 ensure that the result of base-generated contraction of $\mathcal{K}$ by A is a belief set which does not contain A, unless A is tautological: Axioms BGC3 – BGC4 guarantee that such a belief set is identical to $\mathcal{K}$ whenever A does not belong to $\mathcal{K}$. Axiom BGC5 states that $\mathcal{K} \div A$ is not altered by the free substitution of A with a logically equivalent formula, whereas Axiom BGC6 establishes that base-generated contraction of $\mathcal{K}$ by A yields only a finite number of base-generated belief sets – even if A varies over an infinite set of formulas. On the other hand, Axiom BGC7 requires that if there is no contraction by which a sentence B is retracted without another sentence C being retracted, and vice versa, then contraction by B is identical to contraction by C. Lastly, Axiom BGC8 ensures that every element of $\mathcal{K}$ is retained in $\mathcal{K}$ after contraction by A, unless there is some good reason to exclude it[4].

On this basis, we can give an axiomatic characterization of base-generated revision of a belief set:

Theorem 6.5. *Let $\mathcal{K}$ be a belief set. The set $\mathcal{K} * A$ is the base-generated revision of $\mathcal{K}$ by A iff it satisfies the following postulates:*

BGR1 Closure*:* $\mathcal{K} * A = Cn(\mathcal{K} * A)$.

BGR2 Success*:* $A \in \mathcal{K} * A$.

BGR3 Inclusion*:* $\mathcal{K} * A \subseteq \mathcal{K} + A$.

BGR4 Vacuity*: if* $\neg A \notin \mathcal{K}$ *then* $\mathcal{K} + A \subseteq \mathcal{K} * A$.

BGR5 Consistency Preservation*: if A is consistent, then* $\mathcal{K} * A$ *is consistent.*

BGR6 Extensionality*: if A and B are classically equivalent, then* $\mathcal{K} * A = \mathcal{K} * B$.

Proof. By Theorem 6.3, $\mathcal{B} * A = (\mathcal{B} \div \neg A) + A$: this implies that $Cn(\mathcal{B} * A) = Cn((\mathcal{B} \div \neg A) \cup \{A\})$. By reflexivity, monotony and idempotence of classical consequence we have that $Cn((\mathcal{B} \div \neg A) \cup \{A\}) = Cn(Cn(\mathcal{B} \div \neg A) \cup Cn(\{A\})) = Cn((\mathcal{K} \div \neg A) \cup Cn(\{A\})) = Cn((\mathcal{K} \div \neg A) \cup \{A\})$. By the classical result in [2, p. 513], $\mathcal{K} * A$ satisfies Axioms BGR1 – BGR6 if and only if $\mathcal{K} \div \neg A$ satisfies Axioms BGC1 – BGC5: Theorem 6.4 suffices to the conclusion. ☐

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[4] Axiom BGC8 plays for Theorem 6.4 the same role as Axiom BC3 in Definition 6.2 (cf. [63, pp. 605-606] for a detailed discussion).

We conclude this section with a classical result concerning the relationship between base-generated belief revision and a weak version of the preferential logic R:

THEOREM 6.6. *Let* $\mathcal{K}$ *be a belief set. Then* $\mathcal{K} * A$ *satisfies Axioms* $\mathsf{BGR1}$ – $\mathsf{BGR6}$ *and* $B \in \mathcal{K} * A$ *iff* $A \mathrel{|\!\sim} B$ *for a nonmonotonic consequence relation satisfying the following postulates:*

- $\mathsf{R1}$ Right Weakening*:* $A \mathrel{|\!\sim} B$ *iff* $A \mathrel{|\!\sim} C$ *and* $B \in Cn(\{C\})$*.*
- $\mathsf{R2}$ Reflexivity*:* $A \mathrel{|\!\sim} A$*.*
- $\mathsf{R3}$ Weak Conditionalization*: If* $A \mathrel{|\!\sim} B$*, then* $\top \mathrel{|\!\sim} A \rightarrow B$*.*
- $\mathsf{R4}$ Weak Rational Monotony*: If* $\top \mathrel{|\!\sim} A \rightarrow B$ *and* $\not\top \mathrel{|\!\sim} \neg A$*, then* $A \mathrel{|\!\sim} B$*.*
- $\mathsf{R5}$ Consistency Preservation*: If* A *is consistent, then there is (at least) one formula* B *such that* $\not A \mathrel{|\!\sim} B$*.*
- $\mathsf{R6}$ Left Logical Equivalence*: If* A *and* B *are classically equivalent, then* $A \mathrel{|\!\sim} B$ *iff* $A \mathrel{|\!\sim} C$*.*

PROOF. For a proof see [95]. $\square$

6.3. Hypersequent calculi for base-generated belief revision

Let $\mathcal{B}$ be a belief base, and B be the conjunction of the formulas in $\mathcal{B}$. We exploit Theorem 2.2 to define a unique, logically equivalent set $\mathcal{B}'$ comprising the formula translations of the clauses in $\mathsf{top_c}(\overset{*}{\vdash} B)$: in what follows, we shall always consider $\mathcal{B}$ to be identical to such $\mathcal{B}'$. This assumption allows for a fine-grained control over the information stored in agent's beliefs, while making the notion of belief base less intensional: we deem that the price of this move is worth paying, at least if we refrain from adhering to a foundationalist view of the agent's doxastic apparatus [64].

In our framework, the result of the base expansion of $\mathcal{B}$ by a consistent formula A is represented by the set of clauses $\mathsf{top}(\overset{*}{\vdash} B \wedge A)$. Theorem 6.1 allows us to pinpoint the mcs's $\mathcal{C}_1, \dots, \mathcal{C}_m$ of $\mathsf{top}(\vdash B \wedge A)$ which include $\mathsf{top_c}(\vdash A)$: for any $1 \leq i \leq m$, $\mathcal{C}_i$ is a subset of $\mathsf{top}(\vdash B \wedge A)$ including $\mathsf{top_c}(\vdash A)$ and such that $\{\vdash p_i\}_{1 \leq i \leq m}$ and $\{q_j \vdash\}_{1 \leq j \leq n}$ belong to $\mathcal{C}_i^\star$ for any clause $p_1, \dots, p_m \vdash q_1, \dots, q_n$ in $\mathsf{top}(\vdash B \wedge A) \setminus \mathcal{C}_i$. On the other hand, given a selection function γ for $\mathcal{B}$, Proposition 6.9 establishes that the result of the base revision of $\mathcal{B}$ by A is $\mathcal{C}_1 \cap \dots \cap \mathcal{C}_i$, for some $1 \leq i \leq m$: this yields a syntactic characterization of the result of the base revision of $\mathcal{B}$ by A.

EXAMPLE 6.1. *Let* $\mathcal{B}$ *be the set* $\{p, q \rightarrow r, \neg q, \neg r\}$ *and* A *be* $p \rightarrow q$*. By Theorem 6.1, the mcs's of* $\mathsf{top}(\vdash B \wedge A)$ *which include* $\mathsf{top_c}(\vdash A)$ *are the following:*

(i) $\mathcal{C}_1 = \{p \vdash q \ ; \ \vdash p \ ; \ q \vdash r\}$*;*

(ii) $\mathcal{C}_2 = \{p \vdash q \;;\; \vdash p \;;\; r \vdash\}$;

(iii) $\mathcal{C}_3 = \{p \vdash q \;;\; q \vdash \;;\; r \vdash \;;\; q \vdash r\}$.

If $\gamma(\mathcal{B} \perp A) = \{\mathcal{C}_1, \mathcal{C}_2\}$*, then the result of the base revision of* $\mathcal{B}$ *by* A *based on* γ *is* $\mathcal{C}_1 \cap \mathcal{C}_2 = \{p \vdash q \;;\; \vdash p\}$.

Now, consider $\mathcal{K} * A = Cn(\mathcal{B} * A)$: by Propositions 2.4 and 6.1, $B \in \mathcal{K} * A$ exactly when the $\overline{\overline{\mathsf{G4str}}}$ calculus for $\mathcal{C}_1 \cap \cdots \cap \mathcal{C}_i$ proves $\vdash B$. By Lemma 6.2 and Proposition 6.4, a clause $\Theta \vdash \Lambda$ can be derived from $\mathcal{C}_1 \cap \cdots \cap \mathcal{C}_i$ if and only if there is a $\overline{\overline{\mathsf{G4str}}}$-proof of $\Theta \vdash \Lambda$ where a (possibly, empty) sequence of Cut applications is followed by a (possibly, empty) sequence of Weakening applications: as a result, $\Theta \vdash \Lambda$ is a (possibly, weakened) clause $\Phi \vdash \Psi$ in $(\mathcal{C}_1 \cap \cdots \cap \mathcal{C}_i)^\star$. On the other hand, $(\mathcal{C}_1 \cap \cdots \cap \mathcal{C}_i)^\star$ is identical to the set $\mathcal{C}_1^\star \cap \cdots \cap \mathcal{C}_i^\star$ (cf. the proof of Proposition 6.8): this implies that the set of clauses which are classically derivable from $\mathcal{C}_1 \cap \cdots \cap \mathcal{C}_i$ is just the closure under Weakening of the set $\mathcal{C}_1^\star \cap \cdots \cap \mathcal{C}_i^\star$. In the next subsection we leverage this fact to obtain hypersequent calculi which are adequate with respect to the base-generated revision of $\mathcal{K}$ by A.

EXAMPLE 6.2. *Let* $\mathcal{B}$*,* A *and* γ *be as in Example 6.1. We have that:*

(i) $\mathcal{C}_1^\star = \{p \vdash q \;;\; \vdash p \;;\; q \vdash r \;;\; \vdash q \;;\; \vdash r \;;\; p \vdash r\}$;

(ii) $\mathcal{C}_2^\star = \{p \vdash q \;;\; \vdash p \;;\; r \vdash \;;\; \vdash q\}$

The closure under Cut of the result of the base revision of $\mathcal{B}$ *by* A *based on* γ *is* $(\mathcal{C}_1 \cap \mathcal{C}_2)^\star = \mathcal{C}_1^\star \cap \mathcal{C}_2^\star = \{p \vdash q \;;\; \vdash p \;;\; \vdash q\}$. *It's worth noting that the mcs's of* $\mathsf{top}^\star(\vdash B \wedge A)$ *including* $\mathsf{top}_\mathsf{c}(\vdash A)$ *are* $\mathcal{C}_1^\star$*,* $\mathcal{C}_2^\star$*,* $\mathcal{C}_3^\star$ *and the set* $\{p \vdash q \;;\; q \vdash \;;\; \vdash r \;;\; q \vdash r \;;\; p \vdash r \;;\; p \vdash\}$.

Let $\mathcal{B}$ be a belief base, with B being the conjunction of its formulas, and A be a consistent formula. Moreover, let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be the mcs's of $\mathsf{top}(\vdash B \wedge A)$ which include $\mathsf{top}(\vdash A)$. Given a selection function γ for $\mathcal{B}$, the hypersequent calculus $\mathsf{HG4}$ for $\mathcal{K} * A$ based on γ has the rules displayed in Figure 1, with the *proviso* that each instance of *ax* fulfills the following side conditions:

(i) either $\Theta \cap \Lambda \neq \varnothing$ or $\Theta \mathrel{|\!\overset{*}{-}} \Lambda$ is a clause in $\mathsf{top}_\mathsf{c}^\star(\mathrel{|\!\overset{*}{-}} B \wedge A)$;

(ii) the refutational part $\mathcal{R}$ is $\Theta_1 \dashv \Lambda_1 \mid \cdots \mid \Theta_k \dashv \Lambda_k$, with $\{\Theta_h \mathrel{|\!\overset{*}{-}} \Lambda_h\}_{1 \leq h \leq k} = \mathsf{top}_\mathsf{c}^\star(\mathrel{|\!\overset{*}{-}} B \wedge A) \setminus \{\mathcal{C}_1^\star \cap \cdots \cap \mathcal{C}_i^\star\}$ for some $1 \leq i \leq n$.

EXAMPLE 6.3. *Let* $\mathcal{B}$*,* A *and* γ *be as in Example 6.1. The* $\mathsf{HG4}$ *calculus for* $\mathcal{K} * A$ *based on* γ *features*

$$\Gamma, p \vdash q, \Delta \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv \qquad \Gamma, q \vdash r, \Delta \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv$$

$$\Gamma, p \vdash r, \Delta \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv \qquad \Gamma \vdash \Delta \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv$$

$$\Gamma \vdash p, \Delta \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv \qquad \Gamma \vdash q, \Delta \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv$$

$$\Gamma, p \vdash \Delta \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv \qquad \Gamma, q \vdash \Delta \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv$$

$$\Gamma \vdash r, \Delta \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv \qquad \Gamma, r \vdash \Delta \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv$$

as initial hypersequents.

It is easy to find cases where there is (at least) one derivation of a hypersequent $\Gamma \vdash \Delta \mid \mathcal{R}$ in the $\mathsf{HG4}$ calculus for $\mathcal{K} * A$ based on a selection function γ such that $\bigwedge \Gamma \to \bigvee \Delta$ does not belong to $\mathcal{K} * A$. To address this problem, we distinguish $\mathsf{HG4}$-provability from $\mathsf{HG4}$-derivability along the following lines:

Definition 6.6. An $\mathsf{HG4}$-derivation π is a *proof* if $\overline{\overline{\mathsf{G4}}}$ refutes $\bigwedge \Theta \to \bigvee \Lambda \vdash \bigwedge \Theta_i \to \bigvee \Lambda_i$ for any instance of ax in π with conclusion $\Gamma, \Theta \vdash \Lambda, \Delta \mid \Theta_1 \dashv \Lambda_1 \mid \cdots \mid \Theta_n \dashv \Lambda_n$ and each $1 \leq i \leq n$, and a *paraproof* otherwise.

Example 6.4. *Let $\mathcal{B}$, A and γ be as in Example 6.1. The following* $\mathsf{HG4}$*-derivation is a proof:*

$$\dfrac{\neg_{\mathcal{L}}\,\dfrac{ax\,\dfrac{}{\vdash p, r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv}}{\neg p \vdash r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv} \qquad \dfrac{\dfrac{}{\vdash q, r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv}\,ax}{\neg q \vdash r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv}\,\neg_{\mathcal{L}}}{\neg p \vee \neg q \vdash r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv}\,\vee_{\mathcal{L}}$$

On the other hand, the following $\mathsf{HG4}$*-derivation is a paraproof:*

$$\dfrac{\neg_{\mathcal{L}}\,\dfrac{ax\,\dfrac{}{\vdash p, r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv}}{\neg p \vdash r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv} \qquad \dfrac{\dfrac{}{\vdash r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv}\,ax}{\neg r \vdash r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv}\,\neg_{\mathcal{L}}}{\neg p \vee \neg r \vdash r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv}\,\vee_{\mathcal{L}}$$

In the remaining part of this section we exploit the structural properties of $\mathsf{HG4}$ calculi to show that $\mathsf{HG4}$-provability is sound and (weakly) complete with respect to base-generated belief revision. Let us begin by stating that a rule of the form

$$\dfrac{\Gamma_1 \vdash \Delta_1 \mid \mathcal{R}_1 \qquad \cdots \qquad \Gamma_n \vdash \Delta_n \mid \mathcal{R}_n}{\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}\,r$$

is *admissible* in $\mathsf{HG4}$ if the hypersequent $\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n$ is provable whenever the hypersequents $\{\Gamma_i \vdash \Delta_i \mid \mathcal{R}_i\}_{1 \leq i \leq n}$ are provable, and *absorbed* if the hypersequent $\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n$ is derivable whenever the hypersequents $\{\Gamma_i \vdash \Delta_i \mid \mathcal{R}_i\}_{1 \leq i \leq n}$ are derivable.

Proposition 6.10. The rules of Left and Right Weakening

$$wk\,\dfrac{\Gamma \vdash \Delta \mid \mathcal{R}}{A, \Gamma \vdash \Delta \mid \mathcal{R}} \qquad \dfrac{\Gamma \vdash \Delta \mid \mathcal{R}}{\Gamma \vdash \Delta, A \mid \mathcal{R}}\,wk$$

are height-preserving admissible in $\mathsf{HG4}$.

Proof. We reason by routine induction on the height of a $\mathsf{HG4}$-proof π of $\Gamma \vdash \Delta \mid \mathcal{R}$: as usual, the height of π is taken to be the number of nodes in a branch of maximal length. ☐

Theorem 6.7. *The rule of Cut*

AXIOMS

$$\frac{}{\Gamma, \Theta \vdash \Lambda, \Delta \mid \mathcal{R}}\, ax$$

LOGICAL RULES

$$\frac{\Gamma \vdash \Delta, A \mid \mathcal{R}}{\Gamma, \neg A \vdash \Delta \mid \mathcal{R}}\, \neg_L \qquad \frac{\Gamma, A \vdash \Delta \mid \mathcal{R}}{\Gamma \vdash \Delta, \neg A \mid \mathcal{R}}\, \neg_R$$

$$\frac{\Gamma, A, B \vdash \Delta \mid \mathcal{R}}{\Gamma, A \wedge B \vdash \Delta \mid \mathcal{R}}\, \wedge_L \qquad \frac{\Gamma \vdash \Delta, A \mid \mathcal{R}_1 \qquad \Gamma \vdash \Delta, B \mid \mathcal{R}_2}{\Gamma \vdash \Delta, A \wedge B \mid \mathcal{R}_1 \mid \mathcal{R}_2}\, \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \mid \mathcal{R}_1 \qquad \Gamma, B \vdash \Delta \mid \mathcal{R}_2}{\Gamma, A \vee B \vdash \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2}\, \vee_L \qquad \frac{\Gamma \vdash \Delta, A, B \mid \mathcal{R}}{\Gamma \vdash \Delta, A \vee B \mid \mathcal{R}}\, \vee_R$$

$$\frac{\Gamma \vdash \Delta, A \mid \mathcal{R}_1 \qquad \Gamma, B \vdash \Delta \mid \mathcal{R}_2}{\Gamma, A \to B \vdash \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2}\, \to_L \qquad \frac{\Gamma, A \vdash \Delta, B \mid \mathcal{R}}{\Gamma \vdash \Delta, A \to B \mid \mathcal{R}}\, \to_R$$

FIGURE 1. HG4 hypersequent calculi for $\mathcal{K} * A$

$$\frac{\Gamma \vdash \Delta, A \mid \mathcal{R}' \qquad A, \Pi \vdash \Sigma \mid \mathcal{R}''}{\Pi, \Gamma \vdash \Delta, \Sigma \mid \mathcal{R}' \mid \mathcal{R}''}\, cut$$

is admissible in HG4*.*

PROOF. We argue as for the proof of Theorem 2.3. ☐

PROPOSITION 6.11. If there exists a HG4-proof π of $\Gamma \vdash \Delta \mid \mathcal{R}$, then for any hypersequent $\Gamma' \vdash \Delta' \mid \mathcal{R}'$ in π the formulas in Γ', Δ' are subformulas of formulas in Γ, Δ.

PROOF. By induction on the height of π. ☐

We say that a rule of the form

$$\frac{\Gamma_1 \vdash \Delta_1 \mid \mathcal{R}_1 \qquad \cdots \qquad \Gamma_n \vdash \Delta_n \mid \mathcal{R}_n}{\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}$$

is *invertible* if and only if a rule of the form

$$\frac{\Gamma \vdash \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}{\Gamma_i \vdash \Delta_i \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}$$

is admissible in HG4, for any $1 \leq i \leq n$.

PROPOSITION 6.12. Logical rules of HG4 are height-preserving invertible.

Proof. By routine induction on the height of the $\mathsf{HG4}$-proofs of hypersequents $B \circ C, \Gamma \vdash \Delta \mid \mathcal{R}$ and $\Gamma \vdash \Delta, B \circ C \mid \mathcal{R}$, with $\circ \in \{\wedge, \vee, \to\}$, as well as hypersequents $\neg B, \Gamma \vdash \Delta \mid \mathcal{R}$ and $\Gamma \vdash \Delta, \neg B \mid \mathcal{R}$. ☐

Theorem 6.8. *A hypersequent* $\Gamma \vdash \Delta \mid \mathcal{R}$ *is provable in the* $\mathsf{HG4}$ *calculus for* $\mathcal{K} * A$ *based on a selection function* γ *iff* $\bigwedge \Gamma \to \bigvee \Delta$ *belongs to* $\mathcal{K} * A$ *based on* γ.

Proof. ($\Rightarrow$) We reason by induction on the height of a $\mathsf{HG4}$-proof π of $\Gamma \vdash \Delta \mid \mathcal{R}$. If $h(\pi) = 1$, then π is of the form

$$\frac{}{\Gamma, \Theta \vdash \Lambda, \Delta \mid \Theta_1 \dashv \Lambda_1 \mid \cdots \mid \Theta_k \dashv \Lambda_k}\ ax$$

Definition 6.6 ensures that $\overline{\overline{\mathsf{G4}}}$ refutes $\bigwedge \Theta \to \bigvee \Lambda \vdash \bigwedge \Theta_i \to \bigvee \Lambda_i$, for any $1 \leq i \leq n$: as a result, $\Theta \vdash \Lambda$ does not belong to $\mathsf{top}^{\star}_{\mathsf{c}}(\vdash B \wedge A) \setminus (\mathcal{C}^{\star}_1 \cap \cdots \cap \mathcal{C}^{\star}_i)$ – on pain of contradiction. This implies that $\bigwedge \Theta \to \bigvee \Lambda$ and thus $(\bigwedge \Gamma \wedge \bigwedge \Theta) \to (\bigvee \Lambda \vee \bigvee \Delta)$ belong to $\mathcal{K} * A$. The inductive step is obvious – and we are done.

($\Leftarrow$) We reason by induction on the length of a Hilbert-style deduction δ of $\bigwedge \Gamma \to \bigvee \Delta$ from a set of axioms for classical propositional logic along with the set of extra-logical axioms in (refined) $\mathcal{B} * A$.

[base] If $lh(\delta) = 1$ and $\bigwedge \Theta \to \bigvee \Lambda$ is an extra-logical axiom, then suppose by contradiction that $\overline{\overline{\mathsf{G4}}}$ proves $\bigwedge \Theta \to \bigvee \Lambda \vdash \bigwedge \Theta_i \to \bigvee \Lambda_i$ for some $1 \leq i \leq n$. By Proposition 6.4, this implies that the clause $\Theta_i \vdash \Lambda_i$ is a weakened version of $\Theta \vdash \Lambda$. On the other hand, $\Theta \vdash \Lambda$ belongs to some mcs $\mathcal{C}$ of $\mathsf{top}(\vdash B \wedge A)$ which does not contain $\Theta_i \vdash \Lambda_i$: if $\Theta_i = p_1, \ldots, p_m$ and $\Lambda_i = q_1, \ldots, q_n$, Theorem 6.1 ensures that the clauses $\{\vdash p_h\}_{1 \leq h \leq m}$ and $\{q_k \vdash\}_{1 \leq k \leq n}$ belong to $\mathcal{C}^{\star}$. Since $\Theta \subseteq \Theta_i$ and $\Lambda \subseteq \Lambda_i$, by Lemma 6.2 we would get a contradiction: we can thus infer that $\overline{\overline{\mathsf{G4}}}$ refutes $\bigwedge \Theta \to \bigvee \Lambda \vdash \bigwedge \Theta_i \to \bigvee \Lambda_i$ for any $1 \leq i \leq n$, as desired.

[step] If $lh(\delta) \geq n+1$, then the last rule applied is modus ponens: we exploit Proposition 6.12 and Theorem 6.7 to reach the conclusion. ☐

Corollary 6.9. *A hypersequent* $\Gamma \vdash \Delta \mid \mathcal{R}$ *is provable in the* $\mathsf{HG4}$ *calculus for* $\mathcal{K} * A$ *based on a selection function* γ *iff* $A \mid (\bigwedge \Gamma \to \bigvee \Delta)$ *for a nonmonotonic consequence relation* $\mid$ *satisfying Axioms* $\mathsf{R1}$ – $\mathsf{R6}$.

Proof. Straightforward from Theorems 6.8 and 6.6. ☐

Theorem 6.8 establishes that $\vdash A \mid \mathcal{R}$ is provable in $\mathsf{HG4}$ if A belongs to the base-generated revision of $\mathcal{K}$ by A, for some selection function γ for $\mathcal{B}$. However, one cannot prove the stronger claim that $\Gamma \vdash A \mid \mathcal{R}$ is provable in $\mathsf{HG4}$ if A belongs to the base-generated revision of $\mathcal{K} \cup \Gamma$ by A, for the same selection function γ. In short, $\mathsf{HG4}$ calculi fail to be strongly complete with respect to base-generated belief revision.

Example 6.5. *Let* $\mathcal{B}$*,* A *and* γ *be as in Example 6.1. As shown in Example 6.4, the hypersequent* $\neg p \vee \neg q \vdash r \mid p \dashv \mid q \dashv \mid \dashv r \mid r \dashv \mid q \dashv r \mid p \dashv r \mid \dashv$ *is provable: by Theorem 6.8, this means that* $(\neg p \vee \neg q) \rightarrow r$ *belongs to the revision of* $\mathcal{K}$ *by* A *based on* γ*. On the other hand, Theorem 6.1 entails that the mcs's of* $\mathsf{top}(\vdash B \wedge (\neg p \vee \neg q) \wedge A)$ *which include* $\mathsf{top_c}(\vdash A)$ *are the following:*

(*i*) $\mathcal{D}_1 = \{p \vdash q \ ; \ p, q \vdash \ ; \ q \vdash r \ ; \ q \vdash \ ; \ r \vdash\}$

(*ii*) $\mathcal{D}_2 = \{p \vdash q \ ; \ \vdash p \ ; \ q \vdash r\}$

(*iii*) $\mathcal{D}_3 = \{p \vdash q \ ; \ \vdash p \ ; \ r \vdash\}$.

If either $\gamma((\mathcal{B} \cup \{\neg p \vee \neg q\}) \bot A) = \mathcal{D}_i \cap \mathcal{D}_j$ *for any* $1 \leq i \neq j \leq 3$*, or* $\gamma((\mathcal{B} \cup \{\neg p \vee \neg q\}) \bot A) = \mathcal{D}_i$ *for* $i = 1, 3$*, or* $\gamma((\mathcal{B} \cup \{\neg p \vee \neg q\}) \bot A) = \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{D}_3$*, then* r *does not belong the revision of* $\mathcal{K} \cup \{\neg p \vee \neg q\}$ *by* A*.*

CHAPTER 7

Hybrid hypersequents for constrained I/O logics

The Input/Output (I/O) logic framework was developed 25 years ago by David Makinson and Leendert van der Torre as a formal approach to understanding how normative systems work – particularly, how input conditions (I) give rise to outputs (O) expressing what is obliged, prohibited or permitted under those conditions [97, 99]. *Constrained* I/O logics are a subclass of I/O logics designed to ensure that outputs remain consistent relative to a specified set of formulas [98]. This refinement facilitates a more precise analysis of obligations within the domain of normative reasoning. By systematically addressing potential conflicts between inputs and outputs, these logics provide a robust framework for managing complex scenarios, such as contrary-to-duty obligations (i.e., conditional obligations that arise when the associated condition is prohibited), situations involving multiple levels of violations of the same obligation and cases where obligations are logically or physically conflicting [117, pp. 518-520]. However, while a Gentzen-style proof theory exists for unconstrained I/O logics [29], the lack of an analogous framework encompassing all constrained I/O logics significantly hinders both their theoretical development and practical implementation.

On the one hand, it is well known that constrained I/O logics are closely related to default logics – a class of non-monotonic logics designed to capture plausible inferences in the absence of explicit contrary evidence. When such evidence emerges, these inferences can be withdrawn to maintain consistency [142, 101, 7]. In this chapter, we build on the close connection between I/O and default logics to develop a uniform Gentzen-style proof theory for the entire family of constrained I/O logics. In contrast to existing approaches [83, 163, 169, 29], we refrain from interpreting input/output pairs as formulas with an *ad hoc* logical connective. Instead, we remain faithful to the original meta-level formulation of conditional obligations introduced in [97].

To achieve this, we leverage the notion of *hybrid hypersequent*. We show that conditional obligations in the setting of I/O logics can be represented using distinct extra-logical rules, without any need for extensions of the underlying language. On the other hand, the parallel composition of sequents and antisequents formalizes the interaction between conditional obligations and constraints. For any derivable hypersequent, we determine whether the derivation qualifies as a proof by verifying whether the constraints imposed by antisequents are enforced. This approach provides a Gentzen-style reformulation of the notion of *safe* derivation originally introduced in [98].

The resulting calculi exhibit nice structural properties, including the admissibility of structural rules and the invertibility of logical rules. Moreover, we detail simple proof-theoretic translations between certain calculi for constrained I/O logics and corresponding calculi for default logics.

To summarize, the main technical contributions of the chapter are as follows:

- We introduce a uniform proof-theoretic framework for the analysis of all constrained I/O logics. In particular, we provide a modular treatment of the simple-minded and basic reusable output logics (with throughput) – namely, $\mathsf{out}_3^{(+)}$ and $\mathsf{out}_4^{(+)}$.
- We establish Cut elimination and a weakened subformula property across all systems considered.
- We construct a proof-theoretic bridge between normative and default reasoning.
- We develop a representation of basic reusable output with throughput which enables a more refined treatment of disjunctive reasoning in both default and normative contexts.

The chapter is organized as follows. Section 7.1 introduces preliminary notions on I/O logics and disjunctive default logic. Section 7.2 develops the proof-theoretic platform in terms of hypersequents, and Section 7.3 examines its structural properties. Section 7.4 establishes completeness results for our calculi, while Section 7.5 covers the translations of hypersequent calculi for constrained I/O logics to hypersequent calculi for default logics as well as our proof-theoretic approach to disjunctive default and normative reasoning.

7.1. Preliminaries on I/O logics and disjunctive defaults

7.1.1. Constrained I/O logics. We use Cn to denote the operation of classical consequence. Let $\mathcal{G}$ be any finite set of ordered pairs of formulas (A, B), and $\mathcal{A}$ be any finite set of formulas: we write $\mathcal{G}(\mathcal{A})$ to refer to the set of formulas B such that $(A, B) \in \mathcal{G}$ for some $A \in \mathcal{A}$. For any $(A, B) \in \mathcal{G}$, we say that A is the body and B the head.

An *I/O logic without constraints* is an operation mapping pairs $(\mathcal{G}, \mathcal{A})$ into a set out of formulas. The basic I/O logics without constraints are the following ("mce" abbreviates "maximal consistent extension"[1]):

(1) $\mathsf{out}_1(\mathcal{G}, \mathcal{A}) = Cn\big(\mathcal{G}(Cn(\mathcal{A}))\big)$

(2) $\mathsf{out}_2(\mathcal{G}, \mathcal{A}) = \bigcap \big\{Cn\big(\mathcal{G}(\mathcal{B})\big) \mid \mathcal{B} \text{ is an mce of } \mathcal{A}\big\}$

(3) $\mathsf{out}_3(\mathcal{G}, \mathcal{A}) = \bigcap \big\{Cn\big(\mathcal{G}(\mathcal{B})\big) \mid \mathcal{A} \subseteq \mathcal{B} \supseteq Cn(\mathcal{B}) \supseteq \mathcal{G}(\mathcal{B})\big\}$

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[1] The original definitions of basic I/O logics are in terms of *complete sets* [97]. We say that a *maximal consistent extension* of $\mathcal{A}$ is a maximal consistent set which includes $\mathcal{A}$ as a subset. In the limit case where $\mathcal{A}$ is inconsistent, we take any mce of $\mathcal{A}$ to be the set of all formulas. Hence, any mce of $\mathcal{A}$ is a complete set and vice versa: the definitions given here are equivalent to the ones in [97].

(4) $\mathsf{out}_4(\mathcal{G}, \mathcal{A}) = \bigcap \big\{Cn\big(\mathcal{G}(\mathcal{B})\big) \mid \mathcal{B}$ is an mce of $\mathcal{A}$ and $\mathcal{G}(\mathcal{B}) \subseteq \mathcal{B}\big\}$

(5) $\mathsf{out}_\mathsf{i}^{+}(\mathcal{G}, \mathcal{A}) = \mathsf{out}_\mathsf{i}(\mathcal{G} \cup \mathcal{I}, \mathcal{A})$ with $\mathcal{I} = \{(A, A) \mid A$ is a formula$\}$, for any $1 \leq \mathsf{i} \leq 4$

Let $\mathcal{C}$ be a set of formulas. The expression $\mathsf{maxfamily}_\mathsf{i}^{(+)}(\mathcal{G}, \mathcal{A}, \mathcal{C})$ denotes the family of maximal $\mathcal{G}' \subseteq \mathcal{G}$ such that $\mathsf{out}_\mathsf{i}^{(+)}(\mathcal{G}', \mathcal{A})$ is consistent with a finite set $\mathcal{C}$ of formulas. A *constrained I/O logic* maps triples $\langle \mathcal{G}, \mathcal{A}, \mathcal{C}\rangle$ to the family of sets $\mathsf{outfamily}_\mathsf{i}^{(+)}(\mathcal{G}, \mathcal{A}, \mathcal{C})$, which contains every set of the form $\mathsf{out}_\mathsf{i}^{(+)}(\mathcal{G}', \mathcal{A})$ for some $\mathcal{G}' \in \mathsf{maxfamily}_\mathsf{i}^{(+)}(\mathcal{G}, \mathcal{A}, \mathcal{C})$.

EXAMPLE 7.1. *Let* $\langle \mathcal{G}, \mathcal{A}, \mathcal{C}\rangle$ *be defined as follows:* $\mathcal{G} = \{(p, q), (q, p), (p \vee q, r), (r, s), (p \vee q, \neg s)\}$, $\mathcal{A} = \{p \vee q\}$ *and* $\mathcal{C} = \mathcal{A}$.

- $\mathsf{out}_1(\mathcal{G}, \mathcal{A}) = Cn(\{r, \neg s\})$ *and* $\mathsf{out}_1^{+}(\mathcal{G}, \mathcal{A}) = Cn(\{r, \neg s, p \vee q\})$.
- $\mathsf{out}_2(\mathcal{G}, \mathcal{A}) = Cn(\{r, \neg s, q \vee p\})$ *and* $\mathsf{out}_2^{+}(\mathcal{G}, \mathcal{A}) = Cn(\{p, q, r, s, \neg s\})$.
- $\mathsf{out}_3(\mathcal{G}, \mathcal{A}) = Cn(\{r, \neg s, s\})$ *and* $\mathsf{out}_3^{+}(\mathcal{G}, \mathcal{A}) = Cn(\{r, \neg s, s, p \vee q\})$.
- $\mathsf{out}_4(\mathcal{G}, \mathcal{A}) = \mathsf{out}_4^{+}(\mathcal{G}, \mathcal{A}) = Cn(\{p, q, r, s, \neg s\})$.
- $\mathsf{outfamily}_2^{+}(\mathcal{G}, \mathcal{A})$ *contains* $\{(p, q), (q, p), (p \vee q, r), (r, s)\}$, $\{(p, q), (q, p), (r, s), (p \vee q, \neg s)\}$ *and* $\{(p, q), (q, p), (p \vee q, r), (p \vee q, \neg s)\}$.

In the remainder of this subsection, we gather useful results on (constrained) I/O logics.

LEMMA 7.1. $\mathsf{out}_3(\mathcal{G}, \mathcal{A}) = \mathsf{out}_3^{\flat}(\mathcal{G}, \mathcal{A})$, *where* $\mathsf{out}_3^{\flat}(\mathcal{G}, \mathcal{A}) = \bigcup_{i=0}^{\omega} \mathcal{A}_i$ *and*

$$\mathcal{A}_0 = Cn(\mathcal{G}(Cn(\mathcal{A})))$$
$$\mathcal{A}_{n+1} = Cn(\mathcal{A}_n \cup \mathcal{G}(Cn(\mathcal{A}_n \cup \mathcal{A})))$$

PROOF. See [160]. ☐

If (A, B) is a generator from $\mathcal{G}$, we say that the *materialization* of (A, B) – in symbols, $m(\{(A, B)\})$ – is the formula $A \rightarrow B$. We write $m(\mathcal{G})$ to denote the set $\{m(\{(A, B)\}) \mid (A, B) \in \mathcal{G}\}$.

LEMMA 7.2. $\mathsf{out}_4(\mathcal{G}, \mathcal{A}) = \bigcap\{Cn(\mathcal{G}(\mathcal{B})) \mid \mathcal{B}$ is an mce of $\mathcal{A} \cup m(\mathcal{G})\} = \mathsf{out}_2(\mathcal{G}, \mathcal{A} \cup m(\mathcal{G}))$.

PROOF. See [97, p. 395]. ☐

LEMMA 7.3. $\mathsf{out}_4^{+}(\mathcal{G}, \mathcal{A}) = Cn(\mathcal{A} \cup m(\mathcal{G}))$.

PROOF. See [97, p. 400]. ☐

LEMMA 7.4. $x \in \bigcup \mathsf{outfamily}_\mathsf{i}^{(+)}(\mathcal{G}, \mathcal{A}, \mathcal{C})$ *if and only if there exists an* $\mathcal{H} \subseteq \mathcal{G}$ *such that* $x \in \mathsf{out}_\mathsf{i}^{(+)}(\mathcal{H}, \mathcal{A})$ *and* $\mathsf{out}_\mathsf{i}^{(+)}(\mathcal{H}, \mathcal{A})$ *is consistent with* $\mathcal{C}$.

PROOF. For a proof see [98, p. 161]. ☐

7.1.2. Disjunctive default theories. Modified credulous consequence does not allow reasoning by cases: if C is an m-credulous consequence of A and an m-credulous consequence of B, it may be the case that it is not an m-credulous consequence of $A \vee B$. One can modify modified credulous consequence in different ways so as to permit reasoning by cases [51, 106, 144]. In this chapter, we focus on three approaches to disjunctive default reasoning, respectively presented in [16], [80] and [106], which are conceptually related to disjunctive normative reasoning in out_4^+.

Let $\mathcal{D}$ be a set of normal defaults. The *Besnard-Quiniou-Quinton disjunctive translation* $\mathcal{D}^{bqq}$ of $\mathcal{D}$ is the smallest set of default rules which satisfies the following condition:

(b) For any default rule in $\mathcal{D}$ of the form

$$\frac{B : C}{C}$$

there exists a default rule in $\mathcal{D}^{bqq}$ of the form:

$$\frac{\top : B \to C}{B \to C}$$

We say that A is a *Besnard-Quiniou-Quinton disjunctive consequence* (in short, a *bqq*-disjunctive consequence) of $\langle \mathcal{W}, \mathcal{D} \rangle$ if and only if A is an m-credulous consequence of $\langle \mathcal{W}, \mathcal{D}^{bqq} \rangle$.

Example 7.2. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as follows:*

$$\mathcal{W} = \{p \vee q\}$$
$$\mathcal{D} = \left\{ \frac{p : r}{r}, \frac{q : r}{r} \right\}$$

The unique modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W}, \mathcal{D} \rangle$ is $\langle \{p \vee q\}, \varnothing \rangle$. On the other hand, the unique modified extension $\langle \mathcal{E}', \mathcal{F}' \rangle$ of $\langle \mathcal{W}, \mathcal{D}^{bqq} \rangle$ is $\langle Cn(\{p \vee q, r\}), \{p \to r, q \to r\} \rangle$.

The *Konolige disjunctive translation* $\mathcal{D}^k$ of $\mathcal{D}$ is the smallest set of default rules which satisfies the following condition:

(k) For any default rule in $\mathcal{D}$ of the form

$$\frac{B : C}{C}$$

there exists a default rule in $\mathcal{D}^b$ of the form:

$$\frac{\top : C}{B \to C}$$

We say that A is a *Konolige disjunctive consequence* (in short, a k-disjunctive consequence) of $\langle \mathcal{W}, \mathcal{D} \rangle$ if and only if A is an m-credulous consequence of $\langle \mathcal{W}, \mathcal{D}^k \rangle$.

Example 7.3. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as follows:*

$$\mathcal{W} = \{\neg q\}$$

$$\mathcal{D} = \left\{ \frac{p : q}{q} \right\}$$

The unique modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ *of* $\langle \mathcal{W}, \mathcal{D}^b \rangle$ *is* $\langle Cn(\{\neg q, \neg p\}, \{p \to q\} \rangle$. *Hence, we have that b-disjunctive consequence permits undesired instances of contraposition. On the other hand, the unique modified extension* $\langle \mathcal{E}', \mathcal{F}' \rangle$ *of* $\langle \mathcal{W}, \mathcal{D}^k \rangle$ *is* $\langle \{\neg q\}, \varnothing \rangle$. *This implies that* $\neg p$ *is not a k-disjunctive consequence of* $\langle \mathcal{W}, \mathcal{D} \rangle$.

The *Moinard disjunctive translation* $\mathcal{D}^m$ of $\mathcal{D}$ is the smallest set of default rules which satisfies the following condition:

(m) For any default rule in $\mathcal{D}$ of the form

$$\frac{B : C}{C}$$

there exists a default rule in $\mathcal{D}^b$ of the form:

$$\frac{\top : B \wedge C}{B \to C}$$

We say that A is a *Moinard disjunctive consequence* (in short, an m-disjunctive consequence) of $\langle \mathcal{W}, \mathcal{D} \rangle$ if and only if A is an m-credulous consequence of $\langle \mathcal{W}, \mathcal{D}^m \rangle$.

EXAMPLE 7.4. *Let* $\langle \mathcal{W}, \mathcal{D} \rangle$ *be defined as follows:*

$$\mathcal{W} = \{p \vee q\}$$

$$\mathcal{D} = \left\{ \frac{r : \neg p}{\neg p}, \frac{r : \neg q}{\neg q} \right\}$$

The unique modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ *of* $\langle \mathcal{W}, \mathcal{D}^k \rangle$ *is* $\langle Cn(\{p \vee q, r \to \neg p, r \to \neg q\}, \{\neg p, \neg q\} \rangle$: *this implies that* $\neg r$ *is a k-disjunctive consequence of* $\mathcal{W}$. *Hence, we have that k-disjunctive consequence still permits undesired instances of contraposition. On the other hand, the modified extensions* $\langle \mathcal{E}_1, \mathcal{F}_1 \rangle$ *and* $\langle \mathcal{E}_2, \mathcal{F}_2 \rangle$ *of* $\langle \mathcal{W}, \mathcal{D}^m \rangle$ *are* $\langle Cn(\{p \vee q, r \to \neg p\}), \{r \wedge \neg p\} \rangle$ *and* $\langle Cn(\{p \vee q, r \to \neg q\}), \{r \wedge \neg q\} \rangle$, *respectively. As a result, we have that* $\neg r$ *is not an m-disjunctive consequence of* $\mathcal{W}$.

The *strong* version of $\mathcal{D}^{bqq}$, in symbols $\mathcal{D}^{bqq}_s$, is obtained from $\mathcal{D}^{bqq}$ by adding a default rule

$$\frac{\top : B \to C}{B \to C \vee A}$$

for any formula A. The strong versions of $\mathcal{D}^k$ and $\mathcal{D}^m$, in symbols $\mathcal{D}^k_s$ and $\mathcal{D}^m_s$, are obtained from $\mathcal{D}^k$ and $\mathcal{D}^m$, respectively, in an analogous way.

We say that the *disjunctive closure* of a set $\mathcal{D}$ of default rules, in symbols $disj(\mathcal{D})$, is the smallest set of default rules which satisfies the following condition:

(d) For any $m \geq 1$ default rules in $\mathcal{D}$ of the form:

$$\frac{B_1 : C_1}{D_1}, \ldots, \frac{B_m : C_m}{D_m}$$

there exist a default rule in $\mathcal{D}^b$ of the form:

$$\frac{B_1 \vee \cdots \vee B_m \vee A : C_1, \ldots, C_m}{D_1 \vee \cdots \vee D_m \vee A}$$

for any formula A.

We conclude this section with the following result, which illustrates how the disjunctive translations just presented relate to the closure of m-credulous consequence under reasoning by cases. Let $\mathcal{D}$ be a set of normal default rules, and $\mathcal{D}', \mathcal{D}''$ be sets of default rules obtained from $\mathcal{D}$ by replacing the justification of each default rule δ with $req(\delta) \to concl(\delta), req(\delta) \wedge concl(\delta)$, respectively.

Theorem 7.1. *A formula A belongs to a modified extension of $\langle \mathcal{W}, \mathcal{D}_s^k \rangle$ if and only if A belongs to a modified extension of $\langle \mathcal{W}, \mathcal{D}^d \rangle$. The same holds for $\langle \mathcal{W}, \mathcal{D}_s^{bqq} \rangle$ and $\langle \mathcal{W}, (\mathcal{D}')^d \rangle$, and for $\langle \mathcal{W}, \mathcal{D}_s^m \rangle$ and $\langle \mathcal{W}, (\mathcal{D}'')^d \rangle$.*

Proof. We prove the statement for $\langle \mathcal{W}, \mathcal{D}_s^k \rangle$ and $\langle \mathcal{W}, \mathcal{D}^d \rangle$ (the proofs of the other cases are completely analogous). ($\Rightarrow$) Let $\langle \mathcal{E}_1, \mathcal{F}_1 \rangle$ be a modified extension of $\langle \mathcal{W}, \mathcal{D}_s^k \rangle$ such that $A \in \mathcal{E}_1$. Hence, there exists a Hilbert-style derivation π of A from (logical and extra-logical) axioms featuring (possibly, zero) applications of *modus ponens* and defaults from $\mathcal{D}^k$ such that, for each default rule δ applied in π (if any), $just(\{\delta\}) \subseteq \mathcal{F}_1$. There exists a derivation ρ of A from the axioms, where each application of a default δ from $\mathcal{D}^k$ in π is replaced as follows (we employ dashed lines to denote applications of admissible rules):

(I) if δ has the form $\dfrac{\top : C}{B \to C \vee D}$, we consider the following derivation:

$$\begin{array}{c} \vdots \\ \top \\ \hdashline B \vee \neg B \vee D : C \\ \hline C \vee \neg B \vee D \\ \hdashline B \to C \vee D \end{array} \ \delta'$$

The rule δ' belongs to $\mathcal{D}^d$. If $def^d(\rho)$ is the set of default rules applied in ρ, we infer the existence of a modified extension $\langle \mathcal{E}_2', \mathcal{F}_2' \rangle$ of $\langle \mathcal{W}, def^d(\rho) \rangle$ such that $A \in \mathcal{E}_2'$ and $just(def^d(\rho)) = \mathcal{F}_2'$. By Proposition 5.5, there must exist a modified extension $\langle \mathcal{E}_2, \mathcal{F}_2 \rangle$ of $\langle \mathcal{W}, \mathcal{D}^d \rangle$ such that $\mathcal{E}_2' \subseteq \mathcal{E}_2$ and $\mathcal{F}_2' \subseteq \mathcal{F}_2$.

($\Leftarrow$) We focus on the topmost applications of defaults from $\mathcal{D}^d$ in a Hilbert-style derivation π of A, performing the following replacements:

(II) if δ has the form $\dfrac{B : C}{C}$, we consider the following derivation:

$$\dfrac{\delta' \dfrac{\begin{array}{c} \vdots \\ \top : C \end{array}}{B \to C} \qquad \begin{array}{c} \vdots \\ B \end{array}}{C}$$

Notice that the rule δ' belongs to $\mathcal{D}_s^k$.

(III) if δ has the form $\dfrac{B_1 \vee \cdots \vee B_m \vee D : C_1, \dots, C_m}{C_1 \vee \cdots \vee C_m \vee D}$, we take the following derivation ρ:

$$\dfrac{\overset{\vdots\,\rho'}{B_1 \vee \cdots \vee B_m \vee D} \qquad \dfrac{\delta_1\,\dfrac{\overset{\vdots}{\top : C_1}}{B_1 \to C_1 \vee \cdots \vee C_m \vee D} \quad \cdots \quad \delta_m\,\dfrac{\overset{\vdots}{\top : C_m}}{B_m \to C_1 \vee \cdots \vee C_m \vee D} \quad \overset{\vdots}{D \to C_1 \vee \cdots \vee C_m \vee D}}{B_1 \vee \cdots \vee B_m \vee D \to C_1 \vee \cdots \vee C_m \vee D}}{C_1 \vee \cdots \vee C_m \vee D}$$

Clearly, the rules $\delta_1, \dots, \delta_m$ belong to $\mathcal{D}_s^k$. By hypothesis, we have that $\overline{\overline{\mathsf{G4}}}$ derives $W \vdash B_1 \vee \cdots \vee B_m \vee D$ and $W, C_1 \vee \cdots \vee C_m \vee D \dashv \neg C_i$ for any $1 \leq i \leq m$. Since $\overline{\overline{\mathsf{G4}}}$ derives $C_1 \vee \cdots \vee C_m \vee D \vdash B_i \to C_1 \vee \cdots \vee C_m \vee D$, it suffices an *acut* application to infer that $\overline{\overline{\mathsf{G4}}}$ derives $W, B_i \to C_1 \vee \cdots \vee C_m \vee D \dashv C_j$, for any $1 \leq j \leq m$. If π' is the subderivation of π which has δ has last rule applied, it suffices to notice that $concl(def(\pi')) = concl(def(\rho))$ to conclude.

□

7.2. Conditional obligations as extra-logical rules

Our proof-theoretic approach to constrained I/O logics employs hybrid hypersequents. To distinguish the roles of derivable formulas, we use labelled turnstiles:

- $\overset{i}{\vdash}$ for sequents tracking input formulas,
- $\overset{o}{\vdash}$ for sequents tracking output formulas, and
- $\overset{o}{\dashv}$ for antisequents tracking formulas that cannot belong to the output.

When generalizing over $\overset{i}{\vdash}$ and $\overset{o}{\vdash}$, we write $\overset{\times}{\vdash}$. In each non-unary rule, $\overset{\times}{\vdash}$ is assumed to be the same labelled turnstile in every sequent.

Let $\langle \mathcal{G}, \mathcal{A}, \mathcal{C} \rangle$ be any triple of conditional obligations, input formulas and consistency constraints. In this subsection, we define $\mathsf{HG4}^{\mathsf{i}}_{\mathsf{c}}$ hypersequent calculi for $\bigcup \mathsf{outfamily}_{\mathsf{i}}(\mathcal{G}, \mathcal{A}, \mathcal{C})$ adopting the rules in Figure 1.

(i) For any instance of ax^i and ax^o, $\mathcal{R}$ is an arbitrary set of antisequents.

(ii) Each instance of ax^i is either an identity sequent or such that $\Theta \overset{*}{\vdash} \Lambda$ belongs to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} A)$, with A being the conjunction of the formulas in $\mathcal{A}$.

(iii) Any identity sequent has an o-labelled version ax^o.

Condition (i) is required to ensure the admissibility of External Weakening (cf. Lemma 7.6). On the other hand, condition (ii) guarantees that each $\mathsf{HG4}$ calculus incorporates the extra-logical axioms from $\mathcal{A}$ as input formulas. Finally, condition (iii) together with the o-labelled versions of logical rules ensures that the output formulas include all classical tautologies.

AXIOMS

$$\frac{}{\Gamma, \Theta \overset{i}{\vdash} \Lambda, \Delta \mid \mathcal{R}}\; ax^i \qquad \frac{}{\Gamma, \Theta \overset{o}{\vdash} \Lambda, \Delta \mid \mathcal{R}}\; ax^o$$

LOGICAL RULES

$$\frac{\Gamma \overset{\times}{\vdash} \Delta, A \mid \mathcal{R}}{\Gamma, \neg A \overset{\times}{\vdash} \Delta \mid \mathcal{R}}\; L\neg \qquad \frac{\Gamma, A \overset{\times}{\vdash} \Delta \mid \mathcal{R}}{\Gamma \overset{\times}{\vdash} \Delta, \neg A \mid \mathcal{R}}\; R\neg$$

$$\frac{\Gamma, A, B \overset{\times}{\vdash} \Delta \mid \mathcal{R}}{\Gamma, A \wedge B \overset{\times}{\vdash} \Delta \mid \mathcal{R}}\; L\wedge \qquad \frac{\Gamma \overset{\times}{\vdash} \Delta, A \mid \mathcal{R}_1 \qquad \Gamma \overset{\times}{\vdash} \Delta, B \mid \mathcal{R}_2}{\Gamma \overset{\times}{\vdash} \Delta, A \wedge B \mid \mathcal{R}_1 \mid \mathcal{R}_2}\; R\wedge$$

$$\frac{\Gamma, A \overset{\times}{\vdash} \Delta \mid \mathcal{R}_1 \qquad \Gamma, B \overset{\times}{\vdash} \Delta \mid \mathcal{R}_2}{\Gamma, A \vee B \overset{\times}{\vdash} \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2}\; L\vee \qquad \frac{\Gamma \overset{\times}{\vdash} \Delta, A, B \mid \mathcal{R}}{\Gamma \overset{\times}{\vdash} \Delta, A \vee B \mid \mathcal{R}}\; R\vee$$

$$\frac{\Gamma \overset{\times}{\vdash} \Delta, A \mid \mathcal{R}_1 \qquad \Gamma, B \overset{\times}{\vdash} \Delta \mid \mathcal{R}_2}{\Gamma, A \to B \overset{\times}{\vdash} \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2}\; L\to \qquad \frac{\Gamma, A \overset{\times}{\vdash} \Delta, B \mid \mathcal{R}}{\Gamma \overset{\times}{\vdash} \Delta, A \to B \mid \mathcal{R}}\; R\to$$

EXTRA-LOGICAL RULES

$$\frac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{\times}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \Pi_1 \overset{o}{\dashv} \Sigma_1 \mid \cdots \mid \Pi_k \overset{o}{\dashv} \Sigma_k}\; \delta$$

FIGURE 1. $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ hypersequent calculi for $\langle \mathcal{G}, \mathcal{A}, \mathcal{C} \rangle$

DEFINITION 7.1. The hypersequent calculus $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{1}}$ for $\bigcup \mathsf{outfamily}_1(\mathcal{G}, \mathcal{A}, \mathcal{C})$ is defined by the rules in Figure 1, provided that conditions $(i)-(iii)$ above, as well as conditions $(iv)-(v)$ below, are fulfilled.

(iv) For any conditional obligation (B, D) in $\mathcal{G}$, if $\mathsf{top}_{\mathsf{c}}(\overset{*}{\vdash} B) = \{\Theta_1 \overset{*}{\vdash} \Lambda_1, \ldots, \Theta_m \overset{*}{\vdash} \Lambda_m\}$[2] and $\Phi \overset{*}{\vdash} \Psi$ occurs in $\mathsf{top}_{\mathsf{c}}^{\star}(\overset{*}{\vdash} D)$, then there exists a primitive extra-logical rule of the form:

$$\frac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}\; \delta$$

with C being the conjunction of formulas in $\mathcal{C}$.

(v) For any pair of primitive extra-logical rules

$$\delta\, \frac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C} \qquad \frac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{m+1 \leq i \leq n}}{\Gamma, \Phi' \overset{o}{\vdash} \Psi', \Delta \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \overset{o}{\dashv} \neg C}\; \delta$$

---

[2] If $B = \top$, we take $\mathsf{top}_{\mathsf{c}}(\overset{*}{\vdash} B) = \{p \overset{*}{\vdash} p\}$ for some atom p.

if the atomic (anti)sequent $\Xi \overset{*}{\vdash} \Omega$ occurs in $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} (\bigwedge \Phi \to \bigvee \Psi) \wedge (\bigwedge \Phi' \to \bigvee \Psi'))$ without belonging to $\mathsf{top}^{\star}_{\mathsf{c}}(\vdash^{*} (\bigwedge \Phi \to \bigvee \Psi))$ or $\mathsf{top}^{\star}_{\mathsf{c}}(\vdash^{*} (\bigwedge \Phi' \to \bigvee \Psi'))$, then there exists a primitive extra-logical rule of the form:

$$\frac{\{\Theta_j \overset{i}{\vdash} \Lambda_j \mid \mathcal{R}_j\}_{1 \leq j \leq n}}{\Gamma, \Xi \overset{o}{\vdash} \Omega, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n \mid \overset{o}{\dashv} \neg C}\ \delta$$

Condition (iv) guarantees that the $\mathsf{HG4}^1_{\mathsf{c}}$ calculus features extra-logical rules corresponding to the generators occurring in $\mathcal{G}$. For its part, the closure condition (v) is needed in order to ensure that the $\mathsf{HG4}^1_{\mathsf{c}}$ calculus enjoys Cut elimination (cf. Theorem 7.2).

EXAMPLE 7.5. *Let* $\mathcal{G} = \{(p, r \vee s), (q, \neg r \vee s), (s, t), (u, v), (v, u)\}$, $\mathcal{A} = \{p, q, u \vee v, \neg t\}$ *and* $\mathcal{C}$ *an arbitrary set of (non-contradictory) formulas. The* $\mathsf{HG4}^1_{\mathsf{c}}$ *calculus for* $\bigcup \mathsf{outfamily}_1(\mathcal{G}, \mathcal{A}, \mathcal{C})$ *features the initial hypersequents*

$$\frac{}{\Gamma \overset{i}{\vdash} p, \Delta \mid \mathcal{R}} \qquad \frac{}{\Gamma \overset{i}{\vdash} q, \Delta \mid \mathcal{R}} \qquad \frac{}{\Gamma \overset{i}{\vdash} u, v, \Delta \mid \mathcal{R}} \qquad \frac{}{\Gamma, t \overset{i}{\vdash} \Delta \mid \mathcal{R}}$$

as well as the primitive extra-logical rules

$$\frac{\overset{i}{\vdash} p \mid \mathcal{R}}{\Gamma \overset{o}{\vdash} r, s, \Delta \mid \mathcal{R} \mid \overset{o}{\dashv} \neg C} \qquad \frac{\overset{i}{\vdash} q \mid \mathcal{R}}{\Gamma, r \overset{o}{\vdash} s, \Delta \mid \mathcal{R} \mid \overset{o}{\dashv} \neg C} \qquad \frac{\overset{i}{\vdash} s \mid \mathcal{R}}{\Gamma \overset{o}{\vdash} t, \Delta \mid \mathcal{R} \mid \overset{o}{\dashv} \neg C}$$

$$\frac{\overset{i}{\vdash} u \mid \mathcal{R}}{\Gamma \overset{o}{\vdash} v, \Delta \mid \mathcal{R} \mid \overset{o}{\dashv} \neg C} \qquad \frac{\overset{i}{\vdash} v \mid \mathcal{R}}{\Gamma \overset{o}{\vdash} u, \Delta \mid \mathcal{R} \mid \overset{o}{\dashv} \neg C}$$

$$\frac{\overset{i}{\vdash} p \mid \mathcal{R}_1 \qquad \overset{i}{\vdash} q \mid \mathcal{R}_2}{\Gamma \overset{o}{\vdash} s, s, \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \overset{o}{\dashv} \neg C} \qquad \frac{\overset{i}{\vdash} p \mid \mathcal{R}_1 \qquad \overset{i}{\vdash} q \mid \mathcal{R}_2}{\Gamma \overset{o}{\vdash} s, \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \overset{o}{\dashv} \neg C}$$

DEFINITION 7.2. The hypersequent calculus $\mathsf{HG4}^2_{\mathsf{c}}$ for $\bigcup \mathsf{outfamily}_2(\mathcal{G}, \mathcal{A}, \mathcal{C})$ is defined by the rules in Figure 1, provided that conditions (i) – (v) above, as well as conditions (vi) below, are satisfied.

(vi) For any $m \geq 2$ distinct primitive extra-logical rules of the form:

$$\frac{\Theta_{1_1} \overset{i}{\vdash} \Lambda_{1_1} \mid \mathcal{R}_{1_1} \qquad \cdots \qquad \Theta_{n_1} \overset{i}{\vdash} \Lambda_{n_1} \mid \mathcal{R}_{n_1}}{\Gamma, \Phi_1 \overset{o}{\vdash} \Psi_1, \Delta \mid \mathcal{R}_{1_1} \mid \cdots \mid \mathcal{R}_{n_1} \mid \mathcal{R}_1}$$

$$\cdots$$

$$\frac{\Theta_{1_m} \overset{i}{\vdash} \Lambda_{1_m} \mid \mathcal{R}_{1_m} \qquad \cdots \qquad \Theta_{n_m} \overset{i}{\vdash} \Lambda_{n_m} \mid \mathcal{R}_{n_m}}{\Gamma, \Phi_m \overset{o}{\vdash} \Psi_m, \Delta \mid \mathcal{R}_{1_m} \mid \cdots \mid \mathcal{R}_{n_m} \mid \mathcal{R}_m}$$

if $\mathbf{J}$ is the set of all sets of the form $\{k^1_1, \ldots, k^m_m\}$, where each k^i_j is selected from $1_j, \ldots, n_j$ and $1 \leq j \leq m$, the atomic (anti)sequent $\Phi \overset{*}{\vdash} \Psi$ is in the closure under Contraction of $\Phi_m, \ldots, \Phi_1 \overset{*}{\vdash} \Psi_1, \ldots, \Psi_m$, then there exists a *non-primitive* extra-logical rule of the form:

$$\frac{\{\Theta_{\vec{j}} \overset{i}{\vdash} \Lambda_{\vec{j}} \mid \mathcal{R}_{\vec{j}}\}_{\vec{j} \in \mathbf{J}}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta \mid \mathcal{R}'' \mid \mathcal{S}}$$

with $\mathcal{R}'' = \{\mathcal{R}_{\vec{j}}\}_{\vec{j} \in \mathbf{J}}$ and $\mathcal{S} = \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m$.

Condition (vi) ensures that extra-logical rules are closed under reasoning by cases - namely, that there exists a rule from $B_1 \vee \cdots \vee B_m$ to $D_1 \vee \cdots \vee D_m$ whenver there are rules from B_i to D_i, for any $1 \leq i \leq m$.

In [97, pp. 403-404], no derivation of $\mathsf{deriv}_{2,4}$ features applications of (OR) followed by instances of (WO). In our framework, this restriction is implemented through the distinction between *primitive* and *non-primitive* extra-logical rules. As a matter of fact, any instance of (WO) corresponds to the application of a finite number of rules generated in accordance with (v), whereas any application of (OR) corresponds to the application of a finite number of rules generated in accordance with (vi).

EXAMPLE 7.6. *Let $\langle \mathcal{G}, \mathcal{A}, \mathcal{C} \rangle$ be defined as in Example 7.5. The following are non-primitive extra-logical rules of the $\mathsf{HG4}^2_\mathsf{c}$ calculus for $\langle \mathcal{G}, \mathcal{A}, \mathcal{C} \rangle$:*

$$\dfrac{\overset{i}{\vdash} p, q \mid \mathcal{R}}{\Gamma, r \overset{o}{\vdash} r, s, s, \Delta \mid \mathcal{R} \mid \overset{o}{\dashv} \neg C} \qquad \dfrac{\overset{i}{\vdash} p, q \mid \mathcal{R}}{\Gamma, r \overset{o}{\vdash} r, s, \Delta \mid \mathcal{R} \mid \overset{o}{\dashv} \neg C}$$

$$\dfrac{\overset{i}{\vdash} s, p \mid \mathcal{R}_1 \qquad \overset{i}{\vdash} s, q \mid \mathcal{R}_2}{\Gamma \overset{o}{\vdash} t, s, s, \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \overset{o}{\dashv} \neg C} \qquad \dfrac{\overset{i}{\vdash} u, v \mid \mathcal{R}}{\Gamma \overset{o}{\vdash} v, u, \Delta \mid \mathcal{R} \mid \overset{o}{\dashv} \neg C}$$

DEFINITION 7.3. The hypersequent calculus $\mathsf{HG4}^3_\mathsf{c}$ for $\bigcup \mathsf{outfamily}_3(\mathcal{G}, \mathcal{A}, \mathcal{C})$ is defined by the rules in Figure 1, provided that the conditions (i) – (v) above, as well as condition (vii) below, are met.

(vii) For any primitive extra-logical rule of the form:

$$\dfrac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}\ \delta$$

there exists a primitive extra-logical rule

$$\dfrac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{i}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}\ \delta$$

The condition (vii) ensures that primitive extra-logical rules are closed under iterated application – i.e., reuse of conclusions as premises.

EXAMPLE 7.7. *Let $\langle \mathcal{G}, \mathcal{A}, \mathcal{C} \rangle$ be defined as in Example 7.5. The following are primitive extra-logical rules of the $\mathsf{HG4}^3_\mathsf{c}$ calculus for $\langle \mathcal{G}, \mathcal{A}, \mathcal{C} \rangle$:*

$$\dfrac{\overset{i}{\vdash} p \mid \mathcal{R}}{\Gamma \overset{i}{\vdash} r, s, \Delta \mid \mathcal{R} \mid \overset{o}{\dashv} \neg C} \qquad \dfrac{\overset{i}{\vdash} p \mid \mathcal{R}_1 \qquad \overset{i}{\vdash} q \mid \mathcal{R}_2}{\Gamma \overset{i}{\vdash} s, \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \overset{o}{\dashv} \neg C}$$

Here is an example of $\mathsf{HG4}^3_\mathsf{c}$-derivation π:

$$\dfrac{\dfrac{ax^i\ \dfrac{}{\overset{i}{\vdash} p \mid \mathcal{R}_1} \qquad \dfrac{}{\overset{i}{\vdash} q \mid \mathcal{R}_2}\ ax^i}{\overset{i}{\vdash} s \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \overset{o}{\dashv} \neg C}\ \delta}{\overset{o}{\vdash} t, z \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \overset{o}{\dashv} \neg C}\ \delta$$

Definition 7.4. The hypersequent calculus $\mathsf{HG4}^4_\mathsf{c}$ for $\bigcup \mathsf{outfamily}_4(\mathcal{G}, \mathcal{A})$ is defined by the rules in Figure 1, provided that the conditions $(i) - (vi)$, as well as the following condition, are fulfilled.

$(viii)$ For any primitive extra-logical rule of the form:

$$\dfrac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}\ \delta$$

there exists an extra-logical rule

$$\dfrac{\{\Gamma, \Phi, \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{i}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}\ \delta$$

Condition $(viii)$ guarantees that the materializations of the generators in $\mathcal{G}$ are input formulas (cf. Theorem 2.5).

Example 7.8. *Let* $\langle \mathcal{G}, \mathcal{A}, \mathcal{C} \rangle$ *be defined as in Example 7.5. The following are primitive extra-logical rules of the* $\mathsf{HG4}^3_\mathsf{c}$ *calculus for* $\langle \mathcal{G}, \mathcal{A}, \mathcal{C} \rangle$*:*

$$\dfrac{\Gamma \overset{i}{\vdash} p, r, s, \Delta \mid \mathcal{R}}{\Gamma \overset{i}{\vdash} r, s, \Delta \mid \mathcal{R} \mid \overset{o}{\dashv} \neg C} \qquad \dfrac{\Gamma \overset{i}{\vdash} p, s, \Delta \mid \mathcal{R}_1 \qquad \Gamma \overset{i}{\vdash} q, s, \Delta \mid \mathcal{R}_2}{\Gamma \overset{i}{\vdash} s, \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \overset{o}{\dashv} \neg C}$$

Here is an example of $\mathsf{HG4}^3_\mathsf{c}$*-derivation:*

$$\dfrac{\dfrac{ax^i\ \dfrac{}{\Gamma \overset{i}{\vdash} p, s, t, \Delta \mid \mathcal{R}_1} \qquad \dfrac{}{\Gamma \overset{i}{\vdash} q, s, t, \Delta \mid \mathcal{R}_2}\ ax^i}{\Gamma \overset{i}{\vdash} s, t, \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \overset{o}{\dashv} \neg C}\ \delta}{\Gamma \overset{o}{\vdash} t, \Delta \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \overset{o}{\dashv} \neg C}\ \delta$$

Definition 7.5. Let $\mathsf{i} = 1, 3$ and A be the conjunction of the formulas in $\mathcal{A}$. The hypersequent calculus $\mathsf{HG4}^{\mathsf{i}+}_\mathsf{c}$ for $\bigcup \mathsf{outfamily}^+_\mathsf{i}(\mathcal{G}, \mathcal{A}, \mathcal{C})$ is defined by the rules in Figure 1, provided that the following conditions are fulfilled.

(i_t) If $\Theta \overset{*}{\vdash} \Lambda$ belongs to $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\vdash} A)$, there exists an instance of ax^o of the form $\Gamma, \Theta \overset{o}{\vdash} \Lambda, \Delta \mid \mathcal{R}$.

(ii_t) For any primitive extra-logical rule of the form:

$$\dfrac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}$$

if the atomic (anti)sequent $\Phi' \overset{*}{\vdash} \Psi'$ occurs in $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\vdash} A)$ without belonging to $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\vdash} A)$, then there exists a primitive extra-logical rule of the form:

$$\dfrac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi' \overset{o}{\vdash} \Psi', \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}$$

Conditions $(i_t) - (ii_t)$ ensure that the input formulas are also output formulas, and that the set of output formulas (including input ones) is closed under Contraction and Cut.

Example 7.9. *Let* $\langle \mathcal{G}, \mathcal{A}, \mathcal{C} \rangle$ *be defined as in Example 7.5. If* $\mathsf{i} = \mathsf{1}, \mathsf{3}$*, the* $\mathsf{HG4}^{\mathsf{i}+}_{\mathsf{c}}$ *calculus features the following extra-logical rule:*

$$\frac{\overset{i}{\vdash} s \mid \mathcal{R}}{\Gamma \overset{o}{\vdash} \Delta \mid \mathcal{R} \;\; \overset{o}{\dashv} \neg C}$$

Definition 7.6. Let $\mathsf{i} = \mathsf{2}, \mathsf{4}$ and A be the conjunction of the formulas in $\mathcal{A}$. The hypersequent calculus $\mathsf{HG4}^{\mathsf{i}+}_{\mathsf{c}}$ for $\bigcup \mathsf{outfamily}^{+}_{\mathsf{i}}(\mathcal{G}, \mathcal{A}, \mathcal{C})$ is defined by the axiomatic and logical rules in Figure 1, provided that conditions (i) – (iii), (i_t) as well as the following conditions are fulfilled.

(iii_t) For any conditional obligation (B, D) in $\mathcal{G}$, if $\mathsf{top}_{\mathsf{c}}(\overset{*}{\vdash} B) = \{\Theta_1 \overset{*}{\vdash} \Lambda_1, \ldots, \Theta_m \overset{*}{\vdash} \Lambda_m\}$ and $\Phi \overset{*}{\vdash} \Psi$ occurs in $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} D)$, then there exists an extra-logical rule of the form:

$$\frac{\{\Gamma, \Phi, \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C} \;\delta$$

with C being the conjunction of formulas in $\mathcal{C}$.

(iv_t) For any extra-logical rule of the form:

$$\frac{\{\Gamma, \Phi, \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}$$

if the atomic (anti)sequent $\Phi' \overset{*}{\vdash} \Psi'$ occurs in $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} A)$ without belonging to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} A)$, then there exists a primitive extra-logical rule of the form:

$$\frac{\{\Gamma, \Phi', \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi', \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi' \overset{o}{\vdash} \Psi', \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}$$

(v_t) For any pair of extra-logical rules

$$\frac{\{\Gamma, \Phi, \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C} \qquad \frac{\{\Gamma, \Phi', \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi', \Delta \mid \mathcal{R}_i\}_{m+1 \leq i \leq n}}{\Gamma, \Phi' \overset{o}{\vdash} \Psi', \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C}$$

if the atomic (anti)sequent $\Xi \overset{*}{\vdash} \Omega$ occurs in $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} (\bigwedge \Phi \to \bigvee \Psi) \wedge (\bigwedge \Phi' \to \bigvee \Psi'))$ without belonging to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} (\bigwedge \Phi \to \bigvee \Psi))$ or $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} (\bigwedge \Phi' \to \bigvee \Psi'))$, then there exists an extra-logical rule of the form:

$$\frac{\{\Gamma, \Xi, \Theta_i \overset{i}{\vdash} \Lambda_i, \Omega, \Delta \mid \mathcal{R}_i\}_{1 \leq n}}{\Gamma, \Xi \overset{o}{\vdash} \Omega, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n \mid \dashv \neg C}$$

(vi_t) For any extra-logical rule of the form:

$$\frac{\{\Gamma, \Phi, \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C} \;\delta$$

there exists an extra-logical rule

$$\frac{\{\Gamma, \Phi, \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{i}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C} \;\delta$$

Conditions $(i_t), (iv_t)$ ensure that the input formulas are also output formulas, whereas condition (v_t) guarantees that the set of output formulas is closed under Contraction and Cut. Moreover, condition (vi_t) allows for iterated applications of (primitive) extra-logical rules.

The formulation of extra-logical rules in $\mathsf{HG4}_{\mathsf{c}}^{2,4+}$ calculi, as well as the extra-logical rules in $\mathsf{HG4}_{\mathsf{c}}^{4}$ calculi which are generated in accordance with $(viii)$, differs from the formulation of extra-logical rules in other $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ calculi for the fact that arbitrary contexts are allowed in the premises. It is worth noticing that the unlabelled versions of the extra-logical rules of $\mathsf{HG4}_{\mathsf{c}}^{2,4+}$ are equivalent to the rules employed to incorporate extra-logical axioms in the sequent calculi detailed in [109].

Finally, for each $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ calculus for $\bigcup \mathsf{outfamily}_{\mathsf{i}}^{(+)}(\mathcal{G}, \mathcal{A}, \mathcal{C})$, we obtain the $\mathsf{HG4}^{\mathsf{i}(+)}$ calculus for $\mathsf{out}_{\mathsf{i}}^{(+)}(\mathcal{G}, \mathcal{A})$ by considering empty refutational parts:

Definition 7.7. Let $\mathsf{i} = 1, 2, 3, 4$. The (hyper)sequent calculus $\mathsf{HG4}^{\mathsf{i}(+)}$ for $\mathsf{out}_{\mathsf{i}}^{(+)}(\mathcal{G}, \mathcal{A})$ is obtained from $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ by taking all instances of ax^i with an empty refutational part and removing $\overset{o}{\dashv} \neg C$ from the conclusion of extra-logical rules.

It is easy to find cases where an $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-derivation of a hypersequent $\Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}$ exists, but $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is not in $\bigcup \mathsf{outfamily}_{\mathsf{i}}^{(+)}(\mathcal{G}, \mathcal{A})$ – unlike what happens with $\mathsf{HG4}^{\mathsf{i}(+)}$-derivations and formulas in $\mathsf{out}_{\mathsf{i}}^{(+)}(\mathcal{G}, \mathcal{A})$ (take e.g. the derivation π in Example 7.7 with $\mathcal{C} = \mathcal{A}$).

Consequently, we must establish a criterion to differentiate $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-derivations that produce elements of $\bigcup \mathsf{outfamily}_{\mathsf{i}}^{(+)}(\mathcal{G}, \mathcal{A})$ from those that do not. To achieve this, we introduce additional terminology and conceptual refinements.

For any $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ calculus and each of its extra-logical rules δ, we define a label $\delta_{\mathcal{G}'}$ to track the conditional obligations encoded in δ:

- $(\star)$ if δ is generated according to point (iv), then $\mathcal{G}' = \{(B, D)\}$;
- $(\star\star)$ if δ comes from extra-logical rules $\delta_1, \ldots, \delta_m$ with labels $\delta_{\mathcal{G}_1}, \ldots, \delta_{\mathcal{G}_m}$, respectively, according to points $(v) - (viii)$ and $(ii_t) - (vi_t)$, then $\mathcal{G}' = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$.

For each $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-derivation π, we write $\mathcal{G}_\pi$ to denote the set of generators from $\mathcal{G}$ occurring in the labels of extra-logical rules in π (if any), whenever $\mathsf{i}(+) = \mathsf{1}(+), \mathsf{2}(+), \mathsf{3}(+), \mathsf{4}+$, and the set of generators from $\mathcal{G}$ occurring in the labels of extra-logical rules in π with o-labelled conclusions (if any), if $\mathsf{i}(+) = \mathsf{4}$. Let us define $(\mathcal{G}_\pi)^{or}$ to be the smallest set including $\mathcal{G}_\pi$ which fulfills the condition that whenever $(B_1, D_1), \ldots, (B_m, D_m)$ belong to $\mathcal{G}_\pi$, the pair $(B_1 \vee \cdots \vee B_m, D_1 \vee \cdots \vee D_m)$ belongs to $(\mathcal{G}_\pi)^{or}$. Moreover, we say that $(\mathcal{G}_\pi)^{mat}$ is the smallest set which satisfies the condition that whenever (B_1, D_1) occurs in $\mathcal{G}_\pi$, the pair $(\top, B_1 \rightarrow D_1)$ occurs in $(\mathcal{G}_\pi)^{mat}$. We take $\mathcal{H}_\pi$ to denote

(A) the rules in $\mathcal{G}_\pi$, if $\mathsf{i}(+) = \mathsf{1}(+), \mathsf{3}(+)$;

(B) the rules in $(\mathcal{G}_\pi)^{or}$ whose body belongs to $Cn(\mathcal{A})$, if $\mathsf{i} = 2$;

(C) the rules in $(\mathcal{G}_\pi)^{or}$ whose body belongs to $Cn(\mathcal{A} \cup m(\mathcal{G}))$, if $\mathsf{i} = 4$;

(D) the rules in $(\mathcal{G}_\pi)^{mat}$, if $\mathsf{i}+ = \mathsf{2}+, \mathsf{4}+$.

Finally, we use H_π to denote the conjunction of the heads of the elements in $\mathcal{H}_\pi$.

Definition 7.8. Let $\mathsf{i} = \mathsf{1}, \mathsf{2}, \mathsf{3}, \mathsf{4}$ and A being the conjunction of the formulas in $\mathcal{A}$. An $\mathsf{HG4}_\mathsf{c}^{\mathsf{i}(+)}$-derivation π of $\Gamma \stackrel{\times}{\vdash} \Delta \mid \ \Pi_1 \stackrel{o}{\dashv} \Sigma_1 \mid \cdots \mid \Pi_n \stackrel{o}{\dashv} \Sigma_n$ is a *proof* whenever $\overline{\overline{\mathsf{G4}}}$ refutes $(A,)H_\pi \vdash \bigwedge \Pi_h \rightarrow \bigvee \Sigma_h$ for any $1 \leq h \leq n$ – and an $\mathsf{HG4}_\mathsf{c}^{\mathsf{i}}$-*paraproof*, otherwise.

Example 7.10. *Let* $\langle \mathcal{G}, \mathcal{A}, \mathcal{A} \rangle$ *be defined as in Example 7.5. Take the* $\mathsf{HG4}_\mathsf{c}^{\mathsf{3}}$*-derivation* π *in Example 7.7:*

$$\dfrac{\dfrac{ax^i\ \dfrac{}{\stackrel{i}{\vdash} p \mid \mathcal{R}_1} \qquad \dfrac{}{\stackrel{i}{\vdash} q \mid \mathcal{R}_2}\ ax^i}{\stackrel{i}{\vdash} s \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \stackrel{o}{\dashv} \neg A}\ \delta_{\{(p, r\vee s), (q, \neg r \vee s)\}}}{\stackrel{o}{\vdash} t, z \mid \mathcal{R}_1 \mid \mathcal{R}_2 \mid \stackrel{o}{\dashv} \neg A}\ \delta_{\{(s,t)\}}$$

We have that $\mathcal{G}_\pi = \{(p, r \vee s), (q, \neg r \vee s), (s, t)\}$, $A = p \wedge q \wedge (u \vee v) \wedge \neg t$ *and* $H_\pi = (r \vee s) \wedge (\neg r \vee s) \wedge t$. *Since* $\overline{\overline{\mathsf{G4}}}$ *proves* $A, H_\pi \vdash \neg A$, *the derivation* π *is an* $\mathsf{HG4}_\mathsf{c}^{\mathsf{3}}$*-paraproof. On the other hand, it is easy to verify that its immediate subderivation is an* $\mathsf{HG4}_\mathsf{c}^{\mathsf{3}}$*-proof.*

7.3. Proof-theoretic results

Let us begin by stating that a rule of the form

$$\frac{\Gamma_1 \stackrel{\times}{\vdash} \Delta_1 \mid \mathcal{R}_1 \qquad \cdots \qquad \Gamma_n \stackrel{\times}{\vdash} \Delta_n \mid \mathcal{R}_n}{\Gamma \stackrel{\times}{\vdash} \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}$$

is *admissible* in $\mathsf{HG4}_\mathsf{c}^{\mathsf{i}(+)}$ if the hypersequent $\Gamma \stackrel{\times}{\vdash} \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n$ is provable whenever the hypersequents $\{\Gamma_i \stackrel{\times}{\vdash} \Delta_i \mid \mathcal{R}_i\}_{1 \leq i \leq n}$ are provable.

Lemma 7.5. *The rules of Left and Right Weakening*

$$\frac{\Gamma \stackrel{\times}{\vdash} \Delta \mid \mathcal{R}}{B, \Gamma \stackrel{\times}{\vdash} \Delta \mid \mathcal{R}} \qquad \frac{\Gamma \stackrel{\times}{\vdash} \Delta \mid \mathcal{R}}{\Gamma \stackrel{\times}{\vdash} \Delta, B \mid \mathcal{R}}$$

are height-preserving admissible in $\mathsf{HG4}_\mathsf{c}^{\mathsf{i}(+)}$.

Proof. By induction on the height of a $\mathsf{HG4}_\mathsf{c}^{\mathsf{i}(+)}$-proof π of the premise, where height is the number of nodes in the longest branch. If $h(\pi) > 1$ and the last rule applied is an extra-logical rule of the form:

$$\dfrac{\begin{array}{c}\vdots \\ \{\Theta_i \stackrel{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}\end{array}}{\Gamma, \Phi \stackrel{\times}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \stackrel{o}{\dashv} \neg C}\ \delta_{\mathcal{G}}$$

it suffices to consider one the following proofs to get the conclusion:

$$\delta_{\mathcal{G}}\ \dfrac{\begin{array}{c}\vdots\\ \{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1\leq i\leq m}\end{array}}{B,\Gamma,\Phi \overset{\times}{\vdash} \Psi,\Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C} \qquad \dfrac{\begin{array}{c}\vdots\\ \{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1\leq i\leq m}\end{array}}{\Gamma,\Phi \overset{\times}{\vdash} \Psi,\Delta,B \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}\ \delta_{\mathcal{G}}$$

☐

If an $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-derivation π ends with an External Weakening application (i.e., the addition of an antisequent to the refutational part of the premise), and its immediate subderivation π_1 is an $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof, the External Weakening application is *safe* precisely when π is also an $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof.

LEMMA 7.6. *The rule of safe External Weakening*

$$\dfrac{\Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}}{\Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R} \mid \Pi \overset{o}{\dashv} \Sigma}$$

is height-preserving admissible in $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$.

PROOF. By induction on the height of a $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof of the premise. If $h(\pi) = 1$, we exploit condition (i) in the definition of $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$. If $h(\pi) > 1$, we rely on the inductive hypothesis to complete the proof. ☐

We say that a rule of the form:

$$\dfrac{\Gamma_1 \overset{\times}{\vdash} \Delta_1 \mid \mathcal{R}_1 \qquad \cdots \qquad \Gamma_n \overset{\times}{\vdash} \Delta_n \mid \mathcal{R}_n}{\Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}$$

is *invertible* if and only if a rule of the form

$$\dfrac{\Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}{\Gamma_i \overset{\times}{\vdash} \Delta_i \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n}$$

is admissible in $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$, for any $1 \leq i \leq n$.

LEMMA 7.7. *Any logical rule is height-preserving invertible in* $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$.

PROOF. By induction on the height of the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof of the premise. For instance, let π be a $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof of $B \to D, \Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}$, with $h(\pi) > 1$ and where the last rule is an extra-logical one of the form:

$$\dfrac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1\leq i\leq m}}{B \to D, \Gamma', \Phi \overset{\times}{\vdash} \Psi, \Delta' \mid \mathcal{R}}$$

with $\mathcal{R} = \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C$. It is sufficient to consider applications of the same rule yielding $\Gamma', \Phi \overset{\times}{\vdash} \Psi, \Delta' \mid \mathcal{R}, B$ and $D, \Gamma', \Phi \overset{\times}{\vdash} \Psi, \Delta' \mid \mathcal{R}$ to get the conclusion. Notice that the height of the proofs π', π'' of $\Gamma', \Phi \overset{\times}{\vdash} \Psi, \Delta' \mid \mathcal{R}, B$ and $D, \Gamma', \Phi \overset{\times}{\vdash} \Psi, \Delta' \mid \mathcal{R}$, respectively, does not surpass the height of π. ☐

LEMMA 7.8. *The rules of Left and Right Contraction*

$$\frac{B, B, \Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}}{B, \Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}} \qquad \frac{\Gamma \overset{\times}{\vdash} \Delta, B, B \mid \mathcal{R}}{\Gamma \overset{\times}{\vdash} \Delta, B \mid \mathcal{R}}$$

are height-preserving admissible in $\mathsf{HG4}^{\mathsf{i}(+)}_{\mathsf{c}}$.

Proof. We focus on Left Contraction, reasoning by induction on the height of an $\mathsf{HG4}^{\mathsf{i}(+)}_{\mathsf{c}}$-proof π of $B, B, \Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}$.

If $h(\pi) = 1$ and both occurrences of B are principal in the last rule applied, the hypersequent $B, B, \Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}$ has the form $B, B, \Theta, \Gamma' \overset{\times}{\vdash} \Delta', \Lambda \mid \mathcal{R}$, with B, B, Θ, Λ being principal in the ax application. Since $B, B, \Theta \overset{*}{\vdash} \Lambda$ belongs to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} A)$ and $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} A)$ is closed under Contraction, there exists an (anti)sequent $B, \Theta \overset{*}{\vdash} \Lambda$ in $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} A)$: by conditions (i), (ii) and (i_t) in the definition $\mathsf{HG4}^{\mathsf{i}(+)}_{\mathsf{c}}$ there exists an instance of ax yielding $B, \Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}$. By Definition 7.8, if $B, B, \Theta, \Gamma' \overset{\times}{\vdash} \Delta', \Lambda \mid \mathcal{R}$ is provable, then also $B, \Theta, \Gamma' \overset{\times}{\vdash} \Delta', \Lambda \mid \mathcal{R}$ is provable.

If $h(\pi) > 1$, we reason by cases over the last rule applied in π. If the last rule applied is logical, we exploit the inductive hypothesis and Lemma 7.7 to get the conclusion. Suppose that the last rule applied is an extra-logical rule $\delta_{\mathcal{G}'}$ of the form:

$$\frac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{B, B, \Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}}$$

with $\mathcal{R} = \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C$. If B is atomic, we distinguish three subcases:

(1) both occurrences of B are principal;

(2) only one occurrence of B is principal;

(3) no occurrence of B is principal.

In (1), the hypersequent $B, B, \Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}$ has the form $B, B, \Theta, \Gamma' \overset{\times}{\vdash} \Delta', \Lambda \mid \mathcal{R}$, with B, B, Θ, Λ being principal in the $\delta_{\mathcal{G}'}$ application. By conditions $(iv) - (vii)$ and (ii_t) in the definition of $\mathsf{HG4}^{\mathsf{i}(+)}_{\mathsf{c}}$, there exists an $\mathsf{HG4}^{\mathsf{i}(+)}_{\mathsf{c}}$-proof π' ending with an application of $\delta_{\mathcal{G}'}$ yielding $B, \Theta, \Gamma' \overset{\times}{\vdash} \Delta', \Lambda \mid \mathcal{R}$. In (2), the premise has the form $B, B, \Theta, \Gamma' \overset{\times}{\vdash} \Delta', \Lambda \mid \mathcal{R}$, with B, Θ, Λ principal in the $\delta_{\mathcal{G}'}$ application. We replace this $\delta_{\mathcal{G}'}$ application with one yielding $B, \Theta, \Gamma' \overset{\times}{\vdash} \Delta', \Lambda \mid \mathcal{R}$. In (3), we proceed as in (2) to complete the proof.

On the other hand, suppose that the last rule applied is an extra-logical rule $\delta_{\mathcal{G}'}$ of the form:

$$\frac{\{\Gamma', B, \Theta, B, \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma', B, \Theta, B \overset{\times}{\vdash} \Delta \mid \mathcal{R}}$$

with $\mathcal{R} = \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C$, B atomic and (2) only one occurrence of B being principal in $\delta_{\mathcal{G}'}$. We apply the inductive hypothesis to infer that $\mathsf{HG4}^{\mathsf{i}(+)}_{\mathsf{c}}$ proves the hypersequents $\{\Gamma', B, \Theta, \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}$. Hence, there exists an $\mathsf{HG4}^{\mathsf{i}(+)}_{\mathsf{c}}$-proof π' ending with an application of $\delta_{\mathcal{G}'}$ yielding $\Gamma', B, \Theta \overset{\times}{\vdash} \Delta \mid \mathcal{R}$. For the subcase (1), we leverage the conditions

$(viii)$ and $(iii_t) - (vi_t)$ to obtain the conclusion. As to the subcase (3), we simply replace the $\delta_{\mathcal{G}'}$ application with one yielding $\Gamma', B, \Theta \overset{\times}{\vdash} \Delta \mid \mathcal{R}$. $\square$

If a $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-derivation π ends with a Cut application, and its immediate subderivations π_1 and π_2 are $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proofs, then we say that the Cut application is *safe* exactly when π is a $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof.

EXAMPLE 7.11. *Let $\langle \mathcal{G}, \mathcal{A}, \mathcal{A} \rangle$ be defined as in Example 7.5. The $\mathsf{HG4}_{\mathsf{c}}^{1}$-derivation above features a safe Cut application; the $\mathsf{HG4}_{\mathsf{c}}^{3+}$ below features a Cut application which is not safe.*

$$\dfrac{\delta_{\{(p,r\vee s)\}}\,\dfrac{ax_i\,\dfrac{}{\overset{i}{\vdash} p}}{\overset{o}{\vdash} r, s \mid \overset{o}{\dashv} \neg A} \qquad \dfrac{\dfrac{}{\overset{i}{\vdash} q}\,ax^i}{r \overset{o}{\vdash} s \mid \overset{o}{\dashv} \neg A}\,\delta_{\{(q,\neg r\vee s)\}}}{\overset{o}{\vdash} s, s \mid \overset{o}{\dashv} \neg A}\,\mathit{cut}$$

$$\dfrac{\delta_{\{(s,t)\}}\,\dfrac{\delta_{\{(p,r\vee s),(q,\neg r\vee s)\}}\,\dfrac{ax^i\,\dfrac{}{\overset{i}{\vdash} p} \qquad \dfrac{}{\overset{i}{\vdash} q}\,ax^i}{\overset{i}{\vdash} s}}{\Gamma \overset{o}{\vdash} t, \Delta} \qquad \dfrac{}{\Pi, t \overset{o}{\vdash} \Sigma}\,ax^o}{\Pi, \Gamma \overset{o}{\vdash} \Delta, \Sigma}\,\mathit{cut}$$

THEOREM 7.2. *The rule of safe Cut*

$$\dfrac{\Gamma \overset{\times}{\vdash} \Delta, B \mid \mathcal{R}' \qquad B, \Pi \overset{\times}{\vdash} \Sigma \mid \mathcal{R}''}{\Pi, \Gamma \overset{\times}{\vdash} \Delta, \Sigma \mid \mathcal{R}' \mid \mathcal{R}''}$$

is admissible in $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$.

PROOF. We analyze the topmost safe Cut application using primary induction on the logical complexity of the Cut formula B and secondary induction on the sum of the heights of the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proofs of the premises. The process of elimination of the topmost safe Cut application continues until both premises reduce to initial hypersequents when $\times = i$, and to initial hypersequents or conclusions of extra-logical rules when $\times = o$. We then show that initial i-labelled hypersequents, as well as initial o-labelled hypersequents and conclusions of extra-logical rules, are closed under Cut. The remainder of the proof focuses on configurations arising from extra-logical rule applications.

Suppose that the last rule in the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof of $\Gamma \overset{o}{\vdash} \Delta, B \mid \mathcal{R}'$ is a primitive extra-logical one. If B is principal in $\delta_{\mathcal{G}'}$, the derivation has the following form

$$\dfrac{\delta_{\mathcal{G}'}\,\dfrac{\begin{array}{c}\vdots\\ \{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1\leq i\leq m}\end{array}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta, B \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C} \qquad \begin{array}{c}\vdots\\ B, \Pi \overset{o}{\vdash} \Sigma \mid \mathcal{R}'\end{array}}{\Pi, \Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta, \Sigma \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C \mid \mathcal{R}'}\,\mathit{cut}$$

We reason by cases over the last rule applied in the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof of $B, \Pi \overset{o}{\vdash} \Sigma \mid \mathcal{R}'$ to reach the conclusion. Here, we detail the most meaningful cases, leaving the others to the reader.

(*a*) If $B, \Pi \overset{o}{\vdash} \Sigma \mid \mathcal{R}'$ is an initial hypersequent, $B, \Pi \vdash \Sigma$ is not an identity sequent and A is principal, then $B, \Pi \vdash \Sigma$ is of the form $\Pi', B, \Phi' \vdash \Psi', \Sigma'$ for (at least) one $B, \Phi' \overset{*}{\vdash} \Psi'$ in $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} A)$. If $\overline{\overline{\mathsf{G4}}}$ refutes $\Phi', \Phi \vdash \Psi, \Psi'$ and $\Phi', \Phi \overset{*}{\vdash} \Psi, \Psi'$ does not belong to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} A)$, we rely on condition (ii_t) in the definition of $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}+}$ to infer the existence of an extra-logical rule $\delta_{\mathcal{G}''}$ of the form:

$$\dfrac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma', \Phi', \Phi \overset{o}{\vdash} \Psi, \Psi', \Delta' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}\;\delta_{\mathcal{G}''}$$

Hence, we replace the application of $\delta_{\mathcal{G}'}$ with an application of $\delta_{\mathcal{G}''}$ and exploit Lemma 7.6 to obtain the conclusion.

(*b*) If the last rule applied in the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof of $B, \Pi \overset{o}{\vdash} \Sigma \mid \mathcal{R}'$ is a primitive extra-logical one and B is principal in it, then the derivation of $B, \Pi \overset{o}{\vdash} \Sigma \mid \mathcal{R}'$ has the following form:

$$\dfrac{\begin{array}{c}\vdots\\ \{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{m+1 \leq i \leq n}\end{array}}{\Pi, B, \Phi' \overset{o}{\vdash} \Psi', \Sigma \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \overset{o}{\dashv} \neg C}\;\delta_{\mathcal{G}'''}$$

If $\overline{\overline{\mathsf{G4}}}$ refutes $\Phi', \Phi \vdash \Psi, \Psi'$ and $\Phi', \Phi \overset{*}{\vdash} \Psi, \Psi'$ belongs neither to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} \bigwedge \Phi \to (\bigvee \Psi \vee B))$ nor to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} (\bigwedge \Phi \wedge B) \to \bigvee \Psi)$, we rely on the condition (v) in the definition of the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ calculus to infer the existence of an extra-logical $\delta_{\mathcal{G}' \cup \mathcal{G}'''}$ of the following form:

$$\dfrac{\{\Theta_j \overset{i}{\vdash} \Lambda_j \mid \mathcal{R}_j\}_{1 \leq j \leq n}}{\Gamma'', \Phi', \Phi \overset{o}{\vdash} \Psi, \Psi', \Delta'' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n \mid \overset{o}{\dashv} \neg C}\;\delta_{\mathcal{G}' \cup \mathcal{G}'''}$$

Hence, we replace the $\delta_{\mathcal{G}'}$ and $\delta_{\mathcal{G}'''}$ applications with an application of $\delta_{\mathcal{G}' \cup \mathcal{G}'''}$ to obtain the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof

$$\dfrac{\begin{array}{c}\vdots\\ \{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq n}\end{array}}{\Pi, \Gamma, \Phi', \Phi \overset{o}{\vdash} \Psi, \Psi', \Delta, \Sigma \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_n \mid \overset{o}{\dashv} \neg C}\;\delta_{\mathcal{G}' \cup \mathcal{G}'''}$$

If B is not principal in the last rule applied in the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof of $B, \Pi \overset{o}{\vdash} \Sigma \mid \mathcal{R}'$, then the derivation has the following form

$$\dfrac{\delta_{\mathcal{G}'}\;\dfrac{\begin{array}{c}\vdots\\ \{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}\end{array}}{\Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta, B \mid \mathcal{R}'} \qquad \dfrac{\begin{array}{c}\vdots\\ \{\Theta_j \overset{i}{\vdash} \Lambda_j \mid \mathcal{R}_j\}_{m+1 \leq j \leq n}\end{array}}{B, \Pi, \Phi' \overset{o}{\vdash} \Psi', \Sigma \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \overset{o}{\dashv} \neg C}\;\delta_{\mathcal{G}'''}}{\Pi, \Phi', \Gamma, \Phi \overset{o}{\vdash} \Psi, \Delta, \Psi', \Sigma \mid \mathcal{R}' \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \mathcal{R}'_{m+1} \mid \cdots \mid \mathcal{R}'_n \mid \overset{o}{\dashv} \neg C}\;cut$$

It suffices to consider an $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof of the form:

$$\dfrac{\begin{array}{c}\vdots\\ \{\Theta_j \overset{i}{\vdash} \Lambda_j \mid \mathcal{R}_j\}_{m+1 \leq j \leq n}\end{array}}{\Pi, \Phi, \Phi' \overset{o}{\vdash} \Psi', \Sigma, \Psi \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \overset{o}{\dashv} \neg C}\ \delta_{\mathcal{G}'''}$$

(c) Suppose that the last rule applied in the HG4-proof of $B, \Pi \overset{o}{\vdash} \Sigma \mid \mathcal{R}'$ is a non-primitive extra-logical one $\delta_{\mathcal{H}}$ and B is principal in it. Moreover, suppose that $\delta_{\mathcal{H}}$ is generated from the following primitive rules:

$$\delta_{\mathcal{H}_1}\ \dfrac{\{\Theta_{j_1} \overset{i}{\vdash} \Lambda_{j_1} \mid \mathcal{R}_{j_1}\}_{1 \leq j \leq n}}{\Pi', B, \Phi_1 \overset{o}{\vdash} \Psi_1, \Sigma' \mid \mathcal{R}_{1_1} \mid \cdots \mid \mathcal{R}_{n_1} \mid \mathcal{S}_1} \qquad \dfrac{\{\Theta_{j_2} \overset{i}{\vdash} \Lambda_{j_2} \mid \mathcal{R}_{j_2}\}_{1 \leq j \leq n}}{\Pi', \Phi_2 \overset{o}{\vdash} \Psi_2, \Sigma' \mid \mathcal{R}_{1_2} \mid \cdots \mid \mathcal{R}_{n_2} \mid \mathcal{S}_2}\ \delta_{\mathcal{H}_2}$$

with $\mathcal{S}_1 \neq \mathcal{S}_2$ (so as to cover cases arising with hypersequent calculi for disjunctive default logic - cf. Section 7.5). We take the derivation of $B, \Pi \overset{o}{\vdash} \Sigma \mid \mathcal{R}'$ to have the following form:

$$\dfrac{\begin{array}{c}\vdots\\ \{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{m+1 \leq i \leq n}\end{array}}{\Pi', B, \Xi \overset{o}{\vdash} \Omega, \Sigma' \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \mathcal{S}_1 \mid \mathcal{S}_2}\ \delta_{\mathcal{H}}$$

with each $\Theta_i \overset{*}{\vdash} \Lambda_i$ being a distinct $\Theta_{k_2}, \Theta_{j_1} \overset{*}{\vdash} \Lambda_{j_1}, \Lambda_{k_2}$ with $1 \leq j, k \leq n$, $\mathcal{R}_i = \mathcal{R}_{j_1} \mid \mathcal{R}_{k_2}$, the (anti)sequent $B, \Xi \overset{*}{\vdash} \Omega$ within the closure under Contraction of $\Phi_2, B, \Phi_1 \overset{*}{\vdash} \Psi_1, \Psi_2$ and $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$. We must distinguish two sub-cases, according to whether (c_1) $B, \Xi \overset{*}{\vdash} \Omega$ results from Contraction of $B \in \Phi_2$ or (c_2) not.

If (c_2) holds, we exploit condition (v) in the definition of $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ to infer the existence of a primitive extra-logical rule of the form:

$$\dfrac{\{\Theta_h \overset{i}{\vdash} \Lambda_h \mid \mathcal{R}_h\}_{h \in H}}{\Pi', \Gamma, \Phi_1, \Phi \overset{o}{\vdash} \Psi, \Psi_1, \Delta, \Sigma' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_{n_1} \mid \overset{o}{\dashv} \neg C \mid \mathcal{S}_1}\ \delta_{\mathcal{G}' \cup \mathcal{H}_1}$$

with $H = \{1, \ldots, m, 1_1, \ldots, n_1\}$. Moreover, condition ($vi$) in the definition of $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ guarantees the existence of a non-primitive extra-logical rule of the form:

$$\dfrac{\{\Theta_{k_2}, \Theta_h \overset{i}{\vdash} \Lambda_h, \Lambda_{k_2} \mid \mathcal{R}_h \mid \mathcal{R}_k\}_{h \in H, 1 \leq k \leq n}}{\Pi', \Gamma, \Xi, \Phi \overset{o}{\vdash} \Psi, \Omega, \Delta, \Sigma' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_{n_1} \mid \mathcal{R}_{1_2} \mid \cdots \mid \mathcal{R}_{n_2} \mid \mathcal{S}_1 \mid \mathcal{S}_2 \mid \overset{o}{\dashv} \neg C}\ \delta_{\mathcal{G}' \cup \mathcal{H}}$$

Hence, we apply Lemmas 7.5 and 7.6 to obtain the following $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-derivation (we used dashed (doubled) lines to denote (possibly, iterated) applications of admissible rules):

$$\begin{array}{cl}
\vdots & \\
\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m} & \\
\hdashline \hdashline
\{\Theta_{k_2}, \Theta_{j_1}, \Theta_i \overset{i}{\vdash} \Lambda_i, \Lambda_{j_1}, \Lambda_{k_2} \mid \mathcal{R}_i\}_{1 \leq i \leq m, 1 \leq j, k \leq n} & wk \\
\hdashline \hdashline
\{\Theta_{k_2}, \Theta_{j_1}, \Theta_i \overset{i}{\vdash} \Lambda_i, \Lambda_{j_1}, \Lambda_{k_2} \mid \mathcal{R}_i \mid \mathcal{R}_{j_1} \mid \mathcal{R}_{k_2}\}_{1 \leq i \leq m, 1 \leq j, k \leq n} & ewk \\
\hline
\Pi', \Gamma, \Xi, \Phi \overset{o}{\vdash} \Psi, \Omega, \Delta, \Sigma' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_{n_1} \mid \mathcal{R}_{1_2} \mid \cdots \mid \mathcal{R}_{n_2} \mid \mathcal{S}_1 \mid \mathcal{S}_2 \mid \overset{o}{\dashv} \neg C & \delta_{\mathcal{G}' \cup \mathcal{H}}
\end{array}$$

By Definition 7.8, this derivation is an $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof.

If (c_1) is the case, we have that $\delta_{\mathcal{H}_2}$ has the form:

$$\dfrac{\{\Theta_{j_2} \overset{i}{\vdash} \Lambda_{j_2} \mid \mathcal{R}_{j_2}\}_{1\leq j\leq n}}{\Pi', B, \Phi_2' \overset{o}{\vdash} \Psi_2, \Sigma' \mid \mathcal{R}_{1_2} \mid \cdots \mid \mathcal{R}_{n_2} \mid \mathcal{S}_2}\;\delta_{\mathcal{H}_2}$$

with $\Phi_2 = B, \Phi_2'$. Since $\delta_{\mathcal{H}_2}$ is a primitive rule, we rely on the condition (v) in the definition of $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ to infer the existence of a primitive extra-logical rule of the following form:

$$\dfrac{\{\Theta_l \overset{i}{\vdash} \Lambda_l \mid \mathcal{R}_l\}_{l\in L}}{\Pi', \Gamma, \Phi_2, \Phi \overset{o}{\vdash} \Psi, \Psi_2, \Delta, \Sigma' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_{n_2} \mid \overset{o}{\dashv} \neg C \mid \mathcal{S}_2}\;\delta_{\mathcal{G}'\cup\mathcal{H}_2}$$

with $L = \{1, \ldots, m, 1_2, \ldots, n_2\}$. Since both $\delta_{\mathcal{G}'\cup\mathcal{H}_1}$ and $\delta_{\mathcal{G}'\cup\mathcal{H}_2}$ are primitive extra-logical rules, condition (vi) in the definition of $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ ensures the existence of a non-primitive extra-logical rule of the form:

$$\dfrac{\{\Theta_h, \Theta_l \overset{i}{\vdash} \Lambda_l, \Lambda_h \mid \mathcal{R}_h \mid \mathcal{R}_l\}_{h\in H, l\in L}}{\Pi', \Gamma, \Xi, \Phi \overset{o}{\vdash} \Psi, \Omega, \Delta, \Sigma' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \cdots \mid \mathcal{R}_{1_1} \mid \cdots \mid \mathcal{R}_{n_2} \mid \mathcal{S}_1 \mid \mathcal{S}_2 \mid \overset{o}{\dashv} \neg C}\;\delta_{\mathcal{G}'\cup\mathcal{H}_1\cup\mathcal{H}_2}$$

Hence, we apply Lemmas 7.5 and 7.6 to obtain the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-derivation

$$\begin{array}{cl}
\vdots & \\
\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1\leq i\leq m} & \\
\hdashline
\{\Theta_{k_2}, \Theta_{j_1}, \Theta_i \overset{i}{\vdash} \Lambda_i, \Lambda_{j_1}, \Lambda_{k_2} \mid \mathcal{R}_i\}_{1\leq i\leq m, 1\leq j,k\leq n} & wk \\
\hdashline
\{\Theta_{k_2}, \Theta_{j_1}, \Theta_i \overset{i}{\vdash} \Lambda_i, \Lambda_{j_1}, \Lambda_{k_2} \mid \mathcal{R}_i \mid \mathcal{R}_{j_1} \mid \mathcal{R}_{k_2}\}_{1\leq i\leq m, 1\leq j,k\leq n} & ewk \\
\hline
\Pi', \Gamma, \Xi, \Phi \overset{o}{\vdash} \Psi, \Omega, \Delta, \Sigma' \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \cdots \mid \mathcal{R}_{1_1} \mid \cdots \mid \mathcal{R}_{n_2} \mid \mathcal{S}_1 \mid \mathcal{S}_2 \mid \overset{o}{\dashv} \neg C & \delta_{\mathcal{G}'\cup\mathcal{H}_1\cup\mathcal{H}_2}
\end{array}$$

By Definition 7.8, this derivation is an $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof.

(d) If the last rule applied in the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof of $B, \Pi \overset{o}{\vdash} \Sigma \mid \mathcal{R}''$ is a logical one, say $L\rightarrow$, then the derivation has the form:

$$\dfrac{\begin{array}{c}\vdots\\ B, \Pi' \overset{o}{\vdash} \Sigma, D \mid \mathcal{S}\end{array} \qquad \begin{array}{c}\vdots\\ B, E, \Pi' \overset{o}{\vdash} \Sigma \mid \mathcal{S}'\end{array}}{B, D \rightarrow E, \Pi' \overset{o}{\vdash} \Sigma \mid \mathcal{S} \mid \mathcal{S}'}\;L\rightarrow$$

with $\Pi = \Pi' \cup [D \rightarrow E]$ and $\mathcal{R}'' = \mathcal{S} \mid \mathcal{S}'$. By hypothesis, the $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proofs π and π' of $B, D, \Pi' \overset{o}{\vdash} \Sigma \mid \mathcal{S}'$ and $B, \Pi' \overset{o}{\vdash} \Sigma, E \mid \mathcal{S}$, respectively, are cut-free. We reason by cases, according to whether π and π' lack extra-logical rules or not. In both scenarios, we reduce ourselves to cases where B is introduced either by an initial hypersequent or an extra-logical rule.

If the last rule applied in the proof of each premise is non-primitive, and the Cut formula is principal in both, we argue as in (c) above. Lastly, consider the case where the last rule applied in the proof of (at least) one premise is an extra-logical one of the form:

$$\dfrac{\{\Gamma, \Phi, \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1\leq i\leq m}}{\Gamma, \Phi \overset{\times}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}$$

We rely on the same argument put forward by [109]. □

Theorem 7.2 guarantees that the set $\mathsf{out}_{\mathsf{i}}^{(+)}(\mathcal{G}', \mathcal{A})$ for each $\mathcal{G}' \in \mathsf{maxfamily}_{\mathsf{i}}^{(+)}(\mathcal{G}, \mathcal{A}, \mathcal{C})$ is closed under *modus ponens*. The failure of unrestricted Cut reflects the fact that the closure under *modus ponens* is not preserved *across* different elements of $\mathsf{maxfamily}_{\mathsf{i}}^{(+)}$ (cf. Example 7.11).

The $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ hypersequent calculi enjoy a weakened form of analyticity: let us emphasize that the same weakened version of the subformula property is enjoyed by the sequent calculi for axiomatic extensions of classical propositional logic detailed in [109].

THEOREM 7.3. *If there exists an* $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$*-proof* π *of* $\Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}$*, then for any hypersequent* $\Gamma' \overset{\times}{\vdash} \Delta' \mid \mathcal{R}'$ *in* π *the formulas in* Γ', Δ' *are either subformulas of formulas in* Γ, Δ *or atoms.*

PROOF. By induction on the height of π. ☐

7.4. Completeness

In this section, we capitalize on the structural properties of $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ calculi analyzed in the previous section to establish that $\mathsf{HG4}^{\mathsf{i}(+)}$ calculi are sound and weakly complete for $\mathsf{out}_{\mathsf{i}}^{(+)}$ (Theorems 7.4 and Corollary 7.5), and that $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$ calculi for $\langle \mathcal{G}, \mathcal{A}, \mathcal{C} \rangle$ are sound and weakly complete for $\bigcup \mathsf{outfamily}_{\mathsf{i}}^{(+)}(\mathcal{G}, \mathcal{A}, \mathcal{C})$ (Theorem 7.6). To this aim, we first prove some preliminary lemmas.

LEMMA 7.9. *Let* $\mathsf{top}(\overset{*}{\vdash} B_1) = \{\Theta_{1_1} \overset{*}{\vdash} \Lambda_{1_1}, \ldots, \Theta_{n_1} \overset{*}{\vdash} \Lambda_{n_1}\}$ *and* $\mathsf{top}(\overset{*}{\vdash} B_2) = \{\Theta_{1_2} \overset{*}{\vdash} \Lambda_{1_2}, \ldots, \Theta_{n_2} \overset{*}{\vdash} \Lambda_{n_2}\}$*.* $\mathsf{HG4}^{\mathsf{i}(+)}$ *proves* $\overset{\times}{\vdash} B_1 \vee B_2$ *if and only if it proves* $\Theta_{i_1}, \Theta_{j_2} \overset{\times}{\vdash} \Lambda_{j_2}, \Lambda_{i_1}$*, with* $1 \leq i, j \leq n$*.*

PROOF. We proceed by induction on $n_1 + n_2$. If $n_1 + n_2 = 2$, we use Lemma 7.7 to infer that $\mathsf{HG4}^{\mathsf{i}(+)}$ proves $\Theta_{1_2}, \Theta_{1_1} \overset{\times}{\vdash} \Lambda_{1_1}, \Lambda_{1_2}$ if and only if it proves $\overset{\times}{\vdash} (\bigwedge \Theta_{1_2} \to \bigvee \Lambda_{1_2}), (\bigwedge \Theta_{1_1} \to \bigvee \Lambda_{1_1})$: by Theorem 7.2, this suffices to the conclusion. If $n_1 + n_2 > 2$ and (say) $n_1 > 1$, we have that $B_1 = B_1' \wedge (\bigwedge \Theta_{n_1} \to \bigvee \Lambda_{n_1})$: we apply the inductive hypothesis to infer that $\mathsf{HG4}^{\mathsf{i}(+)}$ proves $\overset{\times}{\vdash} B_1' \vee B_2$ if and only if it proves $\Theta_{i_1}, \Theta_{j_2} \overset{\times}{\vdash} \Lambda_{j_2}, \Lambda_{i_1}$, for any $1 \leq i \leq n-1$ and $1 \leq j \leq n$. On the other hand, consider the formula $(\bigwedge \Theta_{n_1} \to \bigvee \Lambda_{n_1}) \vee B_2$: by inductive hypothesis, $\mathsf{HG4}^{\mathsf{i}(+)}$ proves $\overset{\times}{\vdash} (\bigwedge \Theta_{n_1} \to \bigvee \Lambda_{n_1}) \vee B_2$ iff it proves $\Theta_{n_1}, \Theta_{j_2} \overset{\times}{\vdash} \Lambda_{j_2}, \Lambda_{n_1}$, for any $1 \leq j \leq n$. We get the conclusion by noticing that Lemma 7.7 and Theorem 7.2 ensure that $\mathsf{HG4}^{\mathsf{i}(+)}$ proves $\overset{\times}{\vdash} B_1 \vee B_2$ iff it proves $\overset{\times}{\vdash} B_1' \vee B_2$ and $\overset{\times}{\vdash} (\bigwedge \Theta_{n_1} \to \bigvee \Lambda_{n_1}) \vee B_2$. ☐

LEMMA 7.10. *Let* $\mathsf{i} = 1, 2$ *and* A *being the conjunction of the formulas in* $\mathcal{A}$*. Then,* $\mathsf{HG4}^{\mathsf{i}(+)}$ *proves* $\Gamma \overset{i}{\vdash} \Delta$ *if and only if* $\bigwedge \Gamma \to \bigvee \Delta$ *belongs to* $Cn(\{A\})$*.*

PROOF. We argue as [129, pp. 10-11]. ☐

LEMMA 7.11. *Let* $\mathsf{i} = 3$ *and* $\mathcal{A}_0', \mathcal{A}_1', \ldots$ *be defined as follows:* $\mathcal{A}_0' = Cn(\mathcal{A})$*,* $\mathcal{A}_{n+1}' = Cn(\mathcal{A}_n' \cup \mathcal{G}(\mathcal{A}_n'))$*. Then,* $\mathsf{HG4}^{\mathsf{i}(+)}$ *proves* $\Gamma \overset{i}{\vdash} \Delta$ *if and only if* $\bigwedge \Gamma \to \bigvee \Delta$ *belongs to some* $\mathcal{A}_i'$*.*

PROOF. ($\Rightarrow$) We reason by induction on the height of a $\mathsf{HG4}^{\mathsf{3}(+)}$-proof π of $\Gamma \overset{i}{\vdash} \Delta$. If $h(\pi) = 1$, condition (ii) ensures that $\bigwedge \Gamma \rightarrow \bigvee \Delta$ belongs to $Cn(\mathcal{A})$. If $h(\pi) > 1$ and the last rule applied is an extra-logical rule δ, we apply the inductive hypothesis to infer that the formula translations $\{\bigwedge \Gamma_i \rightarrow \bigvee \Delta_i\}_{1 \leq i \leq m}$ of the premises belong to $\mathcal{A}'_n$, for some n. If δ is generated according to $(iv) - (v)$ and (vii), then $\bigwedge \Gamma \rightarrow \bigvee \Delta$ belongs to $Cn(\mathcal{G}(\mathcal{A}'_n))$. On the other hand, if δ is generated according to (ii_t) and (vii), then $\bigwedge \Gamma \rightarrow \bigvee \Delta$ belongs to $Cn(\mathcal{A} \cup \mathcal{G}(\mathcal{A}'_n))$. It is immediate to verify that $Cn(\mathcal{G}(\mathcal{A}'_n)), Cn(\mathcal{A} \cup \mathcal{G}(\mathcal{A}'_n)) \subseteq \mathcal{A}'_{n+1}$.

($\Leftarrow$) We reason by induction on i. If $i = 0$, the conclusion is immediate. If $i = j + 1$, by compactness of Cn there exist formulas $A_1, \ldots, A_m$ belonging to $\mathcal{A}'_j$ and $\bigwedge \Gamma \rightarrow \bigvee \Delta$ occurs in $Cn(A_1, \ldots, A_m)$. As a result, $\mathsf{G4}$ proves $A_1, \ldots, A_m \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$: it is easy to verify that $\mathsf{HG}^{\mathsf{3}(+)}$ proves $A_1, \ldots, A_m \overset{i}{\vdash} \bigwedge \Gamma \rightarrow \bigvee \Delta$. By inductive hypothesis, $\mathsf{HG}^{\mathsf{3}(+)}$ proves $\overset{i}{\vdash} A_k$ for any $1 \leq k \leq m$: by Theorem 7.2, we infer that $\mathsf{HG}^{\mathsf{3}(+)}$ proves $\overset{i}{\vdash} \bigwedge \Gamma \rightarrow \bigvee \Delta$. Lemma 7.7 ensures that $\mathsf{HG}^{\mathsf{3}(+)}$ proves $\Gamma \overset{i}{\vdash} \Delta$. ☐

LEMMA 7.12. *Let* $\mathsf{i} = 4$. *Then,* $\mathsf{HG4}^{\mathsf{i}(+)}$ *proves* $\Gamma \overset{i}{\vdash} \Delta$ *if and only if* $\bigwedge \Gamma \rightarrow \bigvee \Delta$ *belongs to some* $Cn(\mathcal{A} \cup m(\mathcal{G}))$.

PROOF. We argue as [129, pp. 10-11], exploiting Lemma 7.8 and Theorem 2.5. It should be noticed that the argument yields a proof-theoretic reformulation of the corollary to [97, obs. 7]. ☐

We are ready to prove the first adequacy result of this section: $\mathsf{HG4}^{\mathsf{i}}$ calculi are sound and (strongly) complete with respect to $\mathsf{out}_{\mathsf{i}}(\mathcal{G}, \mathcal{A})$.

THEOREM 7.4. *Let* $\mathsf{i} = 1, 2, 3, 4$. *Then, there exists an* $\mathsf{HG4}^{\mathsf{i}}$*-proof* π *of* $\Gamma \overset{o}{\vdash} \Delta$ *if and only if* $\bigwedge \Gamma \rightarrow \bigvee \Delta$ *belongs to* $\mathsf{out}_{\mathsf{i}}(\mathcal{G}, \mathcal{A})$.

PROOF. For the cases where $\mathsf{i} = 1, 2, 3$, we leverage Lemmas 7.10, 7.11 and 7.1 to obtain the conclusion. Here, we focus on the case where $\mathsf{i} = 4$, leaving the others to the reader.

$\mathsf{i} = 4$ ($\Rightarrow$) We reason by induction on the height of π. The most meaningful case occurs when $h(\pi) > 1$ and the last rule applied is a non-primitive extra-logical rule $\delta_{\mathcal{G}'}$. Let $\delta_{\mathcal{G}_1}, \ldots, \delta_{\mathcal{G}_m}$ be the n primitive extra-logical rules from which $\delta_{\mathcal{G}'}$ is generated. Moreover, let $\Theta_{1_i} \overset{i}{\vdash} \Lambda_{1_i}, \ldots, \Theta_{n_i} \overset{i}{\vdash} \Lambda_{n_i}$ be the premises of $\delta_{\mathcal{G}_i}$, $\Theta_1 \overset{i}{\vdash} \Lambda_1, \ldots, \Theta_n \overset{i}{\vdash} \Lambda_n$ be the premises of $\delta_{\mathcal{G}'}$, $\Phi_i \overset{o}{\vdash} \Psi_i$ and $\Phi \overset{o}{\vdash} \Psi$ be the conclusions (with empty contexts) of $\delta_{\mathcal{G}_i}$ and $\delta_{\mathcal{G}'}$ respectively – for any $1 \leq i \leq m$.

By Lemma 7.7, $\mathsf{HG4}^{\mathsf{4}}$ proves $\{\Theta_i \overset{i}{\vdash} \Lambda_i\}_{1 \leq i \leq m}$ if and only if it proves $\overset{i}{\vdash} (\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1) \wedge \cdots \wedge (\bigwedge \Theta_n \rightarrow \bigvee \Lambda_n)$. Lemma 7.9 ensures that $\mathsf{HG4}^{\mathsf{4}}$ proves the latter sequent if and only if it proves $\overset{i}{\vdash} \bigvee_{i=1}^{m}((\bigwedge \Theta_{1_i} \rightarrow \bigvee \Lambda_{1_i}) \wedge \cdots \wedge (\bigwedge \Theta_{n_i} \rightarrow \bigvee \Lambda_{n_i}))$. Hence, Lemma 7.12 guarantees that the formula $\bigvee_{i=1}^{m}((\bigwedge \Theta_{1_i} \rightarrow \bigvee \Lambda_{1_i}) \wedge \cdots \wedge (\bigwedge \Theta_{n_i} \rightarrow \bigvee \Lambda_{n_i}))$ belongs to $Cn(\mathcal{A} \cup m(\mathcal{G}))$. As a result, the latter formula belongs to any mce $\mathcal{B}$ of $\mathcal{A} \cup m(\mathcal{G})$. This implies that, for any such $\mathcal{B}$, there exists (at least) one disjunct $(\bigwedge \Theta_{1_i} \to \bigvee \Lambda_{1_i}) \wedge \cdots \wedge (\bigwedge \Theta_{n_i} \to \bigvee \Lambda_{n_i})$ which belongs to $\mathcal{B}$.

If $(\bigwedge \Theta_{1_i} \to \bigvee \Lambda_{1_i}) \wedge \cdots \wedge (\bigwedge \Theta_{n_i} \to \bigvee \Lambda_{n_i})$ which belongs to $\mathcal{B}$, then $\bigwedge \Phi_i \to \bigvee \Psi_i$ belongs to $Cn(\mathcal{G}(\mathcal{B}))$. Since for any $\mathcal{B}$ there exists (at least) one $(\bigwedge \Theta_{1_i} \to \bigvee \Lambda_{1_i}) \wedge \cdots \wedge (\bigwedge \Theta_{n_i} \to \bigvee \Lambda_{n_i})$ occurring in $\mathcal{B}$, we have that $\bigvee_{i=1}^{m}(\bigwedge \Phi_i \to \bigvee \Psi_i)$ belongs to $\mathsf{out}_2(\mathcal{G}, \mathcal{A} \cup m(\mathcal{G}))$: by Lemma 7.2, it belongs to $\mathsf{out}_4(\mathcal{G}, \mathcal{A})$. It is easy to show that $\bigvee_{i=1}^{m}(\bigwedge \Phi_i \to \bigvee \Psi_i)$ is logically equivalent to $\bigwedge \Phi \to \bigvee \Psi$: since $\bigwedge \Gamma \to \bigvee \Delta \in Cn(\{\bigwedge \Phi \to \bigvee \Psi\})$, this suffices to the conclusion.

$\mathsf{i} = 4$ $(\Leftarrow)$ Suppose that $\mathcal{A}$ is consistent, $\bigwedge \Gamma \to \bigvee \Delta \in \mathsf{out}_4(\mathcal{G}, \mathcal{A})$ and $\bigwedge \Gamma \to \bigvee \Delta \notin \mathsf{out}_3(\mathcal{G}, \mathcal{A})$ (the non-trivial case). We have that:

(1) for any mce $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B} \supseteq \mathcal{G}(\mathcal{B})$, the formula $\bigwedge \Gamma \to \bigvee \Delta$ belongs to $Cn(\mathcal{G}(\mathcal{B}))$;
(2) there exists (at least) one logically closed superset $\mathcal{B}'$ of $\mathcal{A}$ such that $\mathcal{B}' \supseteq \mathcal{G}(\mathcal{B}')$ and $\bigwedge \Gamma \to \bigvee \Delta$ does not belong to $Cn(\mathcal{G}(\mathcal{B}'))$.

From (1) and (2) we infer the existence of some $\mathcal{G}' \subseteq \mathcal{G}$ such that $\mathcal{B}' = Cn(\mathcal{A} \cup m(\mathcal{G}'))$, and that the formula $\bigwedge \Gamma \to \bigvee \Delta$ does not belong to $Cn(\mathcal{G}(Cn(\mathcal{A} \cup m(\mathcal{G}'))))$, while it does occur in $Cn(\mathcal{G}(\mathcal{B}))$ for any mce $\mathcal{B}$ of $\mathcal{A} \cup m(\mathcal{G})$. We can distinguish two subcases:

(a) $\bigwedge \Gamma \to \bigvee \Delta$ belongs to $Cn(\mathcal{G}(Cn(\mathcal{A} \cup m(\mathcal{G}))))$;
(b) $\bigwedge \Gamma \to \bigvee \Delta$ does not occur in $Cn(\mathcal{G}(Cn(\mathcal{A} \cup m(\mathcal{G}))))$.

If (a) is the case, $\bigwedge \Gamma \to \bigvee \Delta$ belongs to $Cn(\mathcal{G}(Cn(\mathcal{A} \cup m(\mathcal{G})))) \setminus Cn(\mathcal{G}(Cn(\mathcal{A} \cup m(\mathcal{G}'))))$. By compactness of classical consequence, there must exist $B_1, \ldots, B_m$ such that $\{B_1, \ldots, B_m\} \subseteq \mathcal{G}(Cn(\mathcal{A} \cup m(\mathcal{G})))$ and $\bigwedge \Gamma \to \bigvee \Delta \in Cn(\{B_1, \ldots, B_m\})$. Since $\{B_1, \ldots, B_m\} \not\subseteq \mathcal{G}(Cn(\mathcal{A} \cup m(\mathcal{G}')))$, there must be pairs $(C_1, D_1), \ldots, (C_n, D_n)$ in $\mathcal{G}$ such that the B's are logical consequence of the D's, $\{C_1, \ldots, C_n\} \subseteq Cn(\mathcal{A} \cup m(\mathcal{G}))$ and $\{C_1, \ldots, C_n\} \not\subseteq Cn(\mathcal{A} \cup m(\mathcal{G}'))$. By Lemmas 7.12 and 7.7, $\mathsf{HG4}^4$ proves any sequent of the form $\Theta \mathrel{|\!\overset{i}{-}} \Lambda$, with $\Theta \mathrel{|\!\overset{*}{-}} \Lambda$ in $\mathsf{top}_\mathsf{c}(\mathrel{|\!\overset{*}{-}} C_1 \wedge \cdots \wedge C_n)$. Conditions $(iv) - (v)$ in the definition of $\mathsf{HG4}^4$ ensure that there exists $\mathsf{HG4}^4$-proofs of $\Phi \mathrel{|\!\overset{o}{-}} \Psi$, for any $\Phi \mathrel{|\!\overset{*}{-}} \Psi$ in $\mathsf{top}^\star_\mathsf{c}(\mathrel{|\!\overset{*}{-}} D_1 \wedge \cdots \wedge D_n)$ – and this suffices to the conclusion.

If (b) is the case, there must exist finite sets of formulas $\mathcal{B}_1, \ldots, \mathcal{B}_m$ for which the following holds: (3) for any mce $\mathcal{B}$ of $\mathcal{A} \cup m(\mathcal{G})$ there exists a $\mathcal{B}_i \subseteq \mathcal{G}(\mathcal{B})$ such that $\bigwedge \Gamma \to \bigvee \Delta \in \bigcap_{i=1}^{m} Cn(\mathcal{B}_i)$ and $\mathcal{B}_i \not\subseteq \mathcal{G}(Cn(\mathcal{A} \cup m(\mathcal{G})))$. Hence, for any $\mathcal{B}_i$ there must exist pairs $(C_{1_i}, D_{1_i}), \ldots, (C_{n_i}, D_{n_i})$ from $\mathcal{G}$ such that $\{C_{1_i}, \ldots, C_{n_i}\} \subseteq \mathcal{B}$, $\{C_{1_i}, \ldots, C_{n_i}\} \not\subseteq Cn(\mathcal{A} \cup m(\mathcal{G}))$ and $\bigwedge \Gamma \to \bigvee \Delta \in Cn(\{D_{1_i}, \ldots, D_{n_i}\})$ for any i. From (3) we infer that $\bigvee_{i=1}^{m}(C_{1_i} \wedge \cdots \wedge C_{n_i})$ occurs in $Cn(\mathcal{A} \cup m(\mathcal{G}))$. By Lemma 7.12, $\mathsf{HG4}^4$ proves $\mathrel{|\!\overset{i}{-}} \bigvee_{i=1}^{m}(C_{1_i} \wedge \cdots \wedge C_{n_i})$: Lemma 7.9 guarantees that $\mathsf{HG4}^4$ proves $\Theta \mathrel{|\!\overset{i}{-}} \Lambda$, with $\Theta \mathrel{|\!\overset{*}{-}} \Lambda$ having the form $\Theta_1, \ldots, \Theta_m \mathrel{|\!\overset{*}{-}} \Lambda_m, \ldots, \Lambda_1$ and $\Theta_i \mathrel{|\!\overset{*}{-}} \Lambda_i$ being any (anti)sequent in $\mathsf{top}_\mathsf{c}(\mathrel{|\!\overset{*}{-}} C_{1_i} \wedge \cdots \wedge C_{n_i})$.

Condition (vi) in the definition of $\mathsf{HG4}^{\mathsf{4}}$ ensures the existence of $\mathsf{HG4}^{\mathsf{4}}$-proof of $\Phi \overset{\circ}{\vdash} \Psi$, for any $\Phi \overset{*}{\vdash} \Psi$ in $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\vdash} \bigvee_{i=1}^{m}(D_{1_i} \vee \cdots \vee D_{n_i}))$ – and this suffices to the conclusion. $\square$

Next, we exploit this result to establish that $\mathsf{HG4}^{\mathsf{i}+}$ calculi are sound and (strongly) complete with respect to $\mathsf{out}_{\mathsf{i}}^{+}(\mathcal{G},\mathcal{A})$.

Corollary 7.5. *Let* $\mathsf{i} = 1, 2, 3, 4$. *Then, there exists an* $\mathsf{HG4}^{\mathsf{i}+}$*-proof* π *of* $\Gamma \overset{\circ}{\vdash} \Delta$ *if and only if* $\bigwedge \Gamma \to \bigvee \Delta$ *belongs to* $\mathsf{out}_{\mathsf{i}}^{+}(\mathcal{G},\mathcal{A})$.

Proof. If $\mathsf{i} = 1, 3$, the proof follows from Definition 7.5, arguing as in [129, pp. 10-11]. If $\mathsf{i} = 2, 4$, an analogous proof follows from Definition 7.6 and Theorem 2.5. It should be noticed that in the case where $\mathsf{i} = 2, 4$, the argument we produce is a proof-theoretic reformulation of [97, obs. 16]. $\square$

We end this section with the proof that $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}+}$ calculi are sound and (weakly) complete with respect to $\bigcup \mathsf{outfamily}_{\mathsf{i}}^{+}(\mathcal{G},\mathcal{A},\mathcal{C})$.

Theorem 7.6. *Let* $\mathsf{i} = 1, 2, 3, 4$. *Then, there exists an* $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$*-proof* π *of* $\Gamma \overset{\circ}{\vdash} \Delta \mid \overset{\circ}{\dashv} \neg C$ *if and only if* $\bigwedge \Gamma \to \bigvee \Delta$ *belongs to* $\bigcup \mathsf{outfamily}_{\mathsf{i}}^{(+)}(\mathcal{G},\mathcal{A},\mathcal{C})$.

Proof. $(\Rightarrow)$ If there exists an $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof π of $\Gamma \overset{\circ}{\vdash} \Delta \mid \overset{\circ}{\dashv} \neg C$, there exists an $\mathsf{HG4}^{\mathsf{i}(+)}$-proof π' of $\overset{\circ}{\vdash} B$ which is identical to π except for the fact that all refutational parts in π' are empty. Theorem 7.4 entails that $\bigwedge \Gamma \to \bigvee \Delta$ belongs to $\mathsf{out}_{\mathsf{i}}^{(+)}(\mathcal{H}_{\pi},\mathcal{A})$. On the other hand, Definition 7.8 ensures that $\overline{\overline{\overline{\mathsf{G4}}}}$ refutes $(A,)H_{\pi} \vdash \neg C$. The formula H_{π} is the conjunction of the heads of all the rules in $\mathcal{H}_{\pi}$: it is easy to see that $\overline{\overline{\overline{\mathsf{G4}}}}$ proves $(A,)H_{\pi} \vdash D$ for any $D \in \mathsf{out}_{\mathsf{i}}^{(+)}(\mathcal{H}_{\pi},\mathcal{A})$. Hence, $\mathsf{out}_{\mathsf{i}}^{(+)}(\mathcal{H}_{\pi},\mathcal{A})$ is consistent with $\mathcal{C}$: by Lemma 7.4 we get the conclusion.

$(\Leftarrow)$ If $\bigwedge \Gamma \to \bigvee \Delta$ belongs to $\bigcup \mathsf{outfamily}_{\mathsf{i}}^{(+)}(\mathcal{G},\mathcal{A},\mathcal{C})$, Lemma 7.4 ensures the existence of an $\mathcal{H} \subseteq \mathcal{G}$ such that $\bigwedge \Gamma \to \bigvee \Delta \in \mathsf{out}_{\mathsf{i}}^{(+)}(\mathcal{H},\mathcal{A})$ and $\mathsf{out}_{\mathsf{i}}^{(+)}(\mathcal{H},\mathcal{A})$ is consistent with $\mathcal{C}$. By Theorem 7.4, there must exist a derivation π of $\overset{\circ}{\vdash} \bigwedge \Gamma \to \bigvee \Delta$ in the $\mathsf{HG4}^{\mathsf{i}(+)}$ calculus for $(\mathcal{H},\mathcal{A})$. It is easy to see that $H_{\pi} \in \mathsf{out}_{\mathsf{i}}^{(+)}(\mathcal{H},\mathcal{A})$: as a result, we have that $\overline{\overline{\overline{\mathsf{G4}}}}$ refutes $(A,)H_{\pi} \vdash \neg C$. Hence, there must exist an $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-derivation π' of $\overset{\circ}{\vdash} \bigwedge \Gamma \to \bigvee \Delta \mid \overset{\circ}{\dashv} \neg C$ which is identical to π except for the fact any application of an extra-logical rule in π' introduces $\overset{\circ}{\dashv} \neg C$. By Definition 7.8, π' is a $\mathsf{HG4}_{\mathsf{c}}^{\mathsf{i}(+)}$-proof: Lemma 7.7 suffices to the conclusion. $\square$

7.5. Constrained I/O logics and default logics

For any default theory $\langle \mathcal{W}, \mathcal{D} \rangle$ we introduced the hypersequent calculus $\mathsf{HG4c}$, which is sound and (weakly) complete for m-credulous consequences of $\mathcal{W}$. Let us recall here the definition of the $\mathsf{HG4c}$ calculus.

Definition 7.9. Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be any default theory. The hypersequent calculus $\mathsf{HG4c}$ for $\langle \mathcal{W}, \mathcal{D} \rangle$ can be defined by adopting unlabelled versions of the rules in Figure 1, provided that the conditions $(i') - (v')$ below are fulfilled.

(i') For each instance of ax, either $\Theta \cap \Lambda \neq \varnothing$ or $\Theta \overset{*}{\models} \Lambda$ is a complementary (anti)sequent in $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\models} W)$, with W being the conjunction of the extra-logical rules in $\mathcal{W}$.

(ii') For each instance of ax, the refutational part of the conclusion is arbitrary.

(iii') For any default rule of the form $\dfrac{B : C_1, \ldots, C_k}{D}$ in $\mathcal{D}$, if $\mathsf{top}_\mathsf{c}(\overset{*}{\models} B) = \{\Theta_1 \overset{*}{\models} \Lambda_1, \ldots, \Theta_m \overset{*}{\models} \Lambda_m\}$ and $\Phi \overset{*}{\models} \Psi$ belongs to $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\models} D)$, then there exists an extra-logical rule of the following form[3].

$$\frac{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k} \, \delta_{\{\frac{B : C_1, \ldots, C_k}{D}\}}$$

(iv') For any extra-logical rule

$$\frac{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k} \, \delta_{\mathcal{D}'}$$

if the atomic (anti)sequent $\Phi' \overset{*}{\models} \Psi'$ occurs in $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\models} (\bigwedge \Phi \to \bigvee \Psi) \wedge W)$ without belonging to $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\models} W)$, then there exists an extra-logical rule of the following form:

$$\frac{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi' \vdash \Psi', \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k} \, \delta_{\mathcal{D}'}$$

(v') For any pair of extra-logical rules

$$\frac{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k} \, \delta_{\mathcal{D}'}$$

$$\frac{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}_i\}_{m+1 \leq i \leq n}}{\Gamma, \Phi' \vdash \Psi', \Delta \mid \mathcal{R}_{m+1} \mid \cdots \mid \mathcal{R}_n \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_{k'}} \, \delta_{\mathcal{D}''}$$

if the atomic (anti)sequent $\Xi \overset{*}{\models} \Omega$ occurs in $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\models} (\bigwedge \Phi \to \bigvee \Psi) \wedge (\bigwedge \Phi' \to \bigvee \Psi'))$ without belonging to $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\models} (\bigwedge \Phi \to \bigvee \Psi))$, $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\models} (\bigwedge \Phi' \to \bigvee \Psi'))$ or $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\models} W)$, then there exists an extra-logical rule of the following form:

$$\frac{\{\Theta_j \vdash \Lambda_j \mid \mathcal{R}_j\}_{1 \leq j \leq n}}{\Gamma, \Xi \vdash \Omega, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_n \mid \dashv \neg C_1 \mid \cdots \mid \dashv \neg C_k \mid \dashv \neg C'_1 \mid \cdots \mid \dashv \neg C'_{k'}} \, \delta_{\mathcal{D}' \cup \mathcal{D}''}$$

In this section, we present proof-theoretic translations between hypersequent calculi for constrained $\mathsf{out}^+_\mathsf{i}$, with $\mathsf{i} = 3, 4$, and the hypersequent calculus $\mathsf{HG4c}$ for default logics. We begin by introducing two unlabelled calculi for constrained $\mathsf{out}^+_\mathsf{i}$, which are equivalent to the corresponding labelled calculi $\mathsf{HG4}^{\mathsf{i}(+)}_\mathsf{c}$ (cf. Theorem 7.7). These unlabelled systems preserve

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[3] In Chapter 5, extra-logical rules feature hypersequents of the form $\{\Gamma, \Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}$ as additional premises, in order to allow the permutation of logical rules over extra-logical rules. Given the structural properties of the calculus, the two formulations are equivalent: in this chapter, we stick to the simpler one.

the same structural properties and serve as natural intermediaries in the translation to $\mathsf{HG4c}$ calculi for default logics.

DEFINITION 7.10. Let $\mathcal{A} = \mathcal{C}$. The hypersequent calculus $\mathsf{HG4}^{\mathsf{3+d}}_{\mathsf{c}}$ for $\bigcup \mathsf{outfamily}^{+}_{3}(\mathcal{G}, \mathcal{A}, \mathcal{C})$ is identical to the hypersequent calculus $\mathsf{HG4}^{\mathsf{3+}}_{\mathsf{c}}$ for $\bigcup \mathsf{outfamily}^{+}_{3}(\mathcal{G}, \mathcal{A}, \mathcal{C})$, except for the fact that the following conditions are satisfied.

(i_d) Any hypersequent $\Gamma \overset{\times}{\vdash} \Delta \mid \Pi_1 \overset{o}{\dashv} \Sigma_1 \mid \cdots \mid \Pi_m \overset{o}{\dashv} \Sigma_m$ is replaced by an hypersequent of the form $\Gamma \vdash \Delta \mid \Pi_1 \dashv \Sigma_1 \mid \cdots \mid \Pi_m \dashv \Sigma_m$.

(ii_d) Let $\mathcal{G}' = \{(B_1, D_1), \ldots, (B_n, D_n)\}$. $\mathsf{HG4}^{\mathsf{3+}}_{\mathsf{c}}$ features a primitive extra-logical rule $\delta_{\mathcal{G}'}$ of the form:

$$\frac{\{\Theta_i \overset{i}{\vdash} \Lambda_i \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{\times}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}$$

if and only if $\mathsf{HG4}^{\mathsf{3+d}}_{\mathsf{c}}$ features a primitive extra-logical rule $\delta'_{\mathcal{G}'}$ of the form:

$$\frac{\{\Theta_i \vdash \Lambda_i \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg D_1 \mid \cdots \mid \dashv \neg D_n}$$

with $\mathcal{R}'_i$ being the unlabelled version of $\mathcal{R}_i$, for any $1 \leq i \leq m$.

The notion of $\mathsf{HG4}^{\mathsf{3+d}}_{\mathsf{c}}$-proof is analogous to the notion of $\mathsf{HG4}^{\mathsf{3+}}_{\mathsf{c}}$-proof (Definition 7.8).

DEFINITION 7.11. Let $\mathcal{A} = \mathcal{C}$. The hypersequent calculus $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$ for $\bigcup \mathsf{outfamily}^{+}_{4}(\mathcal{G}, \mathcal{A}, \mathcal{C})$ is identical to the hypersequent calculus $\mathsf{HG4}^{\mathsf{4+}}_{\mathsf{c}}$ for $\bigcup \mathsf{outfamily}^{+}_{4}(\mathcal{G}, \mathcal{A}, \mathcal{C})$, except for the fact that (i_d) and the following condition are satisfied.

(iii_d) Let $\mathcal{G}' = \{(B_1, D_1), \ldots, (B_n, D_n)\}$. $\mathsf{HG4}^{\mathsf{4+}}_{\mathsf{c}}$ features an extra-logical rule $\delta_{\mathcal{G}'}$ of the form:

$$\frac{\{\Gamma, \Phi, \Theta_i \overset{i}{\vdash} \Lambda_i, \Psi, \Delta \mid \mathcal{R}_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \overset{\times}{\vdash} \Psi, \Delta \mid \mathcal{R}_1 \mid \cdots \mid \mathcal{R}_m \mid \overset{o}{\dashv} \neg C}$$

if and only if $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$ features an extra-logical rule $\delta'_{\mathcal{G}'}$ of the form:

$$\frac{\{\Gamma, \Phi, \Theta_i \vdash \Lambda_i, \Psi, \Delta \mid \mathcal{R}'_i\}_{1 \leq i \leq m}}{\Gamma, \Phi \vdash \Psi, \Delta \mid \mathcal{R}'_1 \mid \cdots \mid \mathcal{R}'_m \mid \dashv \neg(B_1 \to D_1) \mid \cdots \mid \dashv \neg(B_1 \to D_n)}$$

with $\mathcal{R}'_i$ being the unlabelled version of $\mathcal{R}_i$, for any $1 \leq i \leq m$.

The notion of $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$-proof is analogous to the notion of $\mathsf{HG4}^{\mathsf{4+}}_{\mathsf{c}}$-proof (Definition 7.8).

THEOREM 7.7. *Let* $\mathsf{i} = 3, 4$ *and* $\mathcal{R}'$ *be the unlabelled version of* $\mathcal{R}$. *Then,* $\mathsf{HG4}^{\mathsf{i+}}_{\mathsf{c}}$ *proves* $\Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}$ *if and only if* $\mathsf{HG4}^{\mathsf{i+d}}_{\mathsf{c}}$ *proves* $\Gamma \vdash \Delta \mid \mathcal{R}'$.

PROOF. We prove the statement for $\mathsf{i} = 3$: the argument in the case $\mathsf{i} = 4$ is analogous.

($\Rightarrow$) If $\mathsf{HG4}^{\mathsf{3+}}_{\mathsf{c}}$ proves $\Gamma \overset{\times}{\vdash} \Delta \mid \mathcal{R}$, Definition 7.8 ensures that $\overline{\overline{\overline{\mathsf{G4}}}}$ refutes $A, H_\pi \vdash \neg C$ and $A, H_\pi \vdash \neg(\bigwedge \Pi \to \bigvee \Sigma)$, for any antisequent $\Pi \overset{o}{\dashv} \Sigma$ in $\mathcal{R}$ different from $\overset{o}{\dashv} \neg C$. Let ρ be the $\mathsf{HG4}^{\mathsf{3+d}}_{\mathsf{c}}$-proof obtained from π by deleting labels from turnstiles and replacing the primitive extra-logical rules from $\mathsf{HG4}_{\mathsf{c}}^{3+}$ (if any) with the primitive extra-logical rules from $\mathsf{HG4}_{\mathsf{c}}^{3+\mathsf{d}}$. It is easy to verify that $H_\pi = H_\rho$. Hence, $\overline{\overline{\mathsf{G4}}}$ refutes $A, H_\rho \vdash \neg C$ and $A, H_\pi \vdash \neg(\bigwedge \Pi \to \bigvee \Sigma)$, for any $\Pi \overset{\circ}{\dashv} \Sigma$. If $\overline{\overline{\mathsf{G4}}}$ refutes $A, H_\rho \vdash \neg C$, it refutes $A, H_\rho, D \vdash$, for any $D \in \mathcal{H}_\pi$: as a result, it refutes also $A, H_\rho \vdash \neg D$ for any $D \in \mathcal{H}_\pi$. It suffices to notice that the primitive extra-logical rules from $\mathsf{HG4}_{\mathsf{c}}^{3+\mathsf{d}}$ introduce antisequents of the form $\overset{\circ}{\dashv} \neg D$ to get the result.

($\Leftarrow$) If $\mathsf{HG4}_{\mathsf{c}}^{3+\mathsf{d}}$ proves $\Gamma \vdash \Delta \mid \mathcal{R}'$, Definition 7.8 guarantees that $\overline{\overline{\mathsf{G4}}}$ refutes $A, H_\pi \vdash \neg D$ and $A, H_\pi \vdash \neg(\bigwedge \Pi \to \bigvee \Sigma)$, for any D being the head of a rule in $\mathcal{H}_\pi$ and any antisequent $\Pi \overset{\circ}{\dashv} \Sigma$ in $\mathcal{R}$ different from $\overset{\circ}{\dashv} \neg C$. Let ρ be the $\mathsf{HG4}_{\mathsf{c}}^{3+}$-proof obtained from π by replacing turnstiles with labelled ones and replacing the primitive extra-logical rules from $\mathsf{HG4}_{\mathsf{c}}^{3+\mathsf{d}}$ (if any) with the primitive extra-logical rules from $\mathsf{HG4}_{\mathsf{c}}^{3+}$. It is trivial to show that $H_\pi = H_\rho$: hence, $\overline{\overline{\mathsf{G4}}}$ refutes $A, H_\rho \vdash \neg D$ for any D. This entails that $\overline{\overline{\mathsf{G4}}}$ refutes $A, H_\rho \vdash$ and $A, A, H_\rho \vdash$: as a result, it refutes also $A, H_\rho \vdash \neg A$. Since $\mathcal{A} = \mathcal{C}$, we are done. □

We are ready to present the main results of this section. First, we infer the correspondence between hypersequent calculi for normal default theories $\langle \mathcal{W}, \mathcal{D} \rangle$ and the (unlabelled) hypersequent calculi for $\mathsf{out}_3^+(red(\mathcal{D}), \mathcal{W})$ with input formulas as constraints.

Theorem 7.8. *Let* $\langle \mathcal{W}, \mathcal{D} \rangle$ *be any normal default theory. The* $\mathsf{HG4c}$ *calculus for* $\langle \mathcal{W}, \mathcal{D} \rangle$ *proves* $\Gamma \vdash \Delta \mid \mathcal{R}$ *exactly when the* $\mathsf{HG4}_{\mathsf{c}}^{3+\mathsf{d}}$ *calculus for* $\bigcup \mathsf{outfamily}_3^+(red(\mathcal{D}), \mathcal{W}, \mathcal{W})$ *proves* $\Gamma \vdash \Delta \mid \mathcal{R}$.

Proof. Straightforward from Definition 7.9. □

The following result demonstrates that out_4^+ with input formulas as constraints corresponds to the Besnard-Quiniou-Quinton disjunctive translation of a normal default theory $\langle \mathcal{W}, \mathcal{D} \rangle$.

Theorem 7.9. *Let* $\langle \mathcal{W}, \mathcal{D} \rangle$ *be any normal default theory. The* $\mathsf{HG4c}$ *calculus for* $\langle \mathcal{W}, \mathcal{D}^{bqq} \rangle$ *proves* $\Gamma \vdash \Delta \mid \mathcal{R}$ *exactly when the* $\mathsf{HG4}_{\mathsf{c}}^{4+\mathsf{d}}$ *calculus for* $\bigcup \mathsf{outfamily}_4^+(red(\mathcal{D}), \mathcal{W}, \mathcal{W})$ *proves* $\Gamma \vdash \Delta \mid \mathcal{R}$.

Proof. The $\mathsf{HG4}_{\mathsf{c}}^{4+\mathsf{d}}$ calculus proves $\Gamma \vdash \Delta \mid \mathcal{R}$ exactly when $\bigwedge \Gamma \to \bigvee \Delta$ belongs to $\bigcup \mathsf{outfamily}_4^+(red(\mathcal{D}), \mathcal{A}, \mathcal{A})$. We leverage Lemma 7.3 to infer that $\bigwedge \Gamma \to \bigvee \Delta$ belongs to $\bigcup \mathsf{outfamily}_4^+(red(\mathcal{D}), \mathcal{A}, \mathcal{A})$ if and only if $\bigwedge \Gamma \to \bigvee \Delta$ is a *bqq*-disjunctive consequence of $\langle \mathcal{W}, \mathcal{D} \rangle$. □

Constrained I/O logic $\bigcup \mathsf{outfamily}_4^+(\mathcal{G}, \mathcal{A}, \mathcal{A})$ corresponds to brave disjunctive reasoning. To capture less brave kinds of disjunctive reasoning (like those formalized by Konolige or Moinard disjunctive translations), we can modify the $\mathsf{HG4}_{\mathsf{c}}^{4+\mathsf{d}}$ calculus by injecting specific antisequents into $\mathsf{HG4}_{\mathsf{c}}^{4+\mathsf{d}}$-derivations, so as to turn $\mathsf{HG4}_{\mathsf{c}}^{4+\mathsf{d}}$-proofs generating undesired conclusions into paraproofs. Such antisequents can be introduced by *ax* applications or by applications of extra-logical rules. A notable illustration of this modular approach to disjunctive default and normative reasoning is provided by the following definition[4].

Definition 7.12. The hypersequent calculus $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$ for *cautious* $\bigcup\mathsf{outfamily}^{+}_{\mathsf{4}}(\mathcal{G},\mathcal{A},\mathcal{A})$ is obtained from the $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$ calculus for $\bigcup\mathsf{outfamily}^{+}_{\mathsf{4}}(\mathcal{G},\mathcal{A},\mathcal{A})$ by replacing any extra-logical rules of the form:

$$\frac{\{\Gamma,\Phi,\Theta_i\vdash\Lambda_i,\Psi,\Delta\mid\mathcal{R}_i\}_{1\leq i\leq m}}{\Gamma,\Phi\vdash\Psi,\Delta\mid\mathcal{R}_1\mid\cdots\mid\mathcal{R}_m\mid\dashv\neg(B_1\to D_1)\mid\cdots\mid\dashv\neg(B_1\to D_n)}$$

with an extra-logical rule of the form:

$$\frac{\{\Gamma,\Phi,\Theta_i\vdash\Lambda_i,\Psi,\Delta\mid\mathcal{R}_i\}_{1\leq i\leq m}}{\Gamma,\Phi\vdash\Psi,\Delta\mid\mathcal{R}_1\mid\cdots\mid\mathcal{R}_m\mid\dashv\neg(B_1\wedge D_1)\mid\cdots\mid\dashv\neg(B_1\wedge D_n)}$$

The notion of $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$-proof is analogous to the notion of $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$-proof (Definition 5.2).

Notice that the $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$ calculus can be obtained from $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$ by imposing that each extra-logical rules in $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$ introduces both antisequents of the form $\dashv\neg(B_i\to D_i)$ and antisequents of the form $\dashv\neg(B_i\wedge D_i)$ (rather than only antisequents of the latter form). As a matter of fact, if π is an $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$-derivation, $\overline{\overline{\overline{\mathsf{G4}}}}$ refutes $A,H_\pi\vdash\neg(B_i\to D_i)$ whenever it refutes $A,H_\pi\vdash\neg(B_i\wedge D_i)$: it suffices to extend the $\overline{\overline{\overline{\mathsf{G4}}}}$ derivation of $A,H_\pi\dashv\neg(B_i\wedge D_i)$ with an Anticut application.

The $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$-calculus is sound and (weakly) complete with respect to a less brave fragment of constrained $\mathsf{out}^{+}_{\mathsf{4}}$, where e.g. implicit instances of contraposition are disallowed.

Example 7.12. *Let* $\langle\mathcal{G},\mathcal{A},\mathcal{C}\rangle$ *be defined as follows:* $\mathcal{G}=\{(p,s),(q,u),(p\vee q,\neg s)\}$, $\mathcal{A}=\{p\vee q\}$, $\mathcal{C}=\mathcal{A}$ *(cf.* [170, p. 320]*). Take the following* $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$*-derivation:*

$$\dfrac{\dfrac{ax\,\dfrac{}{\vdash p,q,u}\qquad\dfrac{}{\vdash p,q,q,u}\,ax}{\vdash q,u\mid\dashv\neg(p\to s)\mid\dashv\neg((p\vee q)\to\neg s)}\,\delta_{\{(p,s),(p\vee q,\neg s)\}}}{\vdash u\mid\dashv\neg(p\to s)\mid\dashv\neg((p\vee q)\to\neg s)\mid\dashv\neg(q\to u)}\,\delta_{\{(q,u)\}}$$

We have that $H_\pi=(p\to s)\wedge((p\vee q)\to\neg s)\wedge(q\to u)$*. Hence,* $\overline{\overline{\overline{\mathsf{G4}}}}$ *refutes the sequents* $A,H_\pi\vdash\neg(p\to s)$*,* $A,H_\pi\vdash\neg((p\vee q)\to\neg s)$ *and* $A,H_\pi\vdash\neg(q\to u)$*. As a result,* π *is an* $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$*-proof. This implies that* $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$ *permits implicit instances of contraposition: in the scenario represented by* $\langle\mathcal{G},\mathcal{A},\mathcal{C}\rangle$*, inferring* u *when reasoning by cases over* $p\vee q$ *amounts to applying* (q,u) *under the implicit assumption of reasoning contrapositively from* $\neg s$ *to* $\neg p$ *and then inferring* q *from* $p\vee q$ *by disjunctive syllogism. Now, consider the following* $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$*-derivation* ρ*:*

---

[4] A modular approach to disjunctive default reasoning which is closed in spirit to the one detailed here is presented in [171].

$$\dfrac{\dfrac{\dfrac{}{\vdash p, q, u}\,{\scriptstyle ax} \qquad \dfrac{}{\vdash p, q, q, u}\,{\scriptstyle ax}}{\vdash q, u \mid \dashv \neg(p \wedge s) \mid \dashv \neg((p \vee q) \wedge \neg s)}\,\delta'_{\{(p,s),(p\vee q,\neg s)\}}}{\vdash u \mid \dashv \neg(p \wedge s) \mid \dashv \neg((p \vee q) \wedge \neg s) \mid \dashv \neg(q \wedge u)}\,\delta'_{\{(q,u)\}}$$

We have that $H_\rho = H_\pi$. *Hence,* $\overline{\overline{\mathsf{G4}}}$ *proves* $A, H_\pi \vdash \neg(p \wedge s)$: *the* $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$*-derivation* ρ *is a paraproof. This means that* $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$ *disallows the implicit instance of contraposition underlying the derivation of* u *by application of* (q, u).

The following result establishes that the fragment of constrained out^+_4 captured by $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$ corresponds to the Moinard disjunctive translation of a normal default theory $\langle \mathcal{W}, \mathcal{D}\rangle$.

Theorem 7.10. *Let* $\langle \mathcal{W}, \mathcal{D}\rangle$ *be any normal default theory.The* $\mathsf{HG4c}$ *calculus for* $\langle \mathcal{W}, \mathcal{D}^m\rangle$ *proves* $\Gamma \vdash \Delta \mid \mathcal{R}$ *exactly when the* $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$ *calculus proves* $\Gamma \vdash \Delta \mid \mathcal{R}$.

Proof. We argue as in the proof of Theorem 7.9. □

The kind of disjunctive default reasoning embodied by $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$ is notably cautious. However, the cautiousness of $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$ is not so restrictive as to preclude $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$-proofs that yield desirable conclusions – such as those derived via disjunctive syllogism on prerequisites. This contrasts with more restrictive frameworks, like the classical system described in [51]. We conclude this section by illustrating how the $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{cc}}$ calculus supports such inferences.

Example 7.13. *Let* $\langle \mathcal{G}, \mathcal{A}, \mathcal{C}\rangle$ *be defined as follows:* $\mathcal{G} = \{(p, r \vee s), (r, \neg t), (s, t), (q, \neg r), (q, t)\}$, $\mathcal{A} = \{p, q\}$ *and* $\mathcal{C} = \mathcal{A}$. *Take the following* $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$*-derivation:*

$$\dfrac{\dfrac{\dfrac{}{\vdash p, s, t}\,{\scriptstyle ax} \qquad \dfrac{}{\vdash q, s, t}\,{\scriptstyle ax}}{\vdash s, t \mid \dashv \neg(p \to (r \vee s)) \mid \dashv \neg(q \to \neg r)}\,\delta_{\{(p,r\vee s),(q,\neg r)\}}}{\vdash t \mid \dashv \neg(p \to (r \vee s)) \mid \dashv \neg(q \to \neg r) \mid \dashv \neg(q \to t)}\,\delta_{\{(s,t)\}}$$

We have that $H_\pi = (p \to (r \vee s)) \wedge (q \to \neg r) \wedge (s \to t)$. *Hence,* $\overline{\overline{\mathsf{G4}}}$ *refutes the sequents* $A, H_\pi \vdash \neg(p \to (r \vee s))$, $A, H_\pi \vdash \neg(q \to \neg r)$ *and* $A, H_\pi \vdash \neg(s \to t)$: π *is an* $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$*-proof. This indicates that* $\mathsf{HG4}^{\mathsf{4+d}}_{\mathsf{c}}$ *supports instances of disjunctive syllogism over prerequisites. In the scenario represented by* $\langle \mathcal{G}, \mathcal{A}, \mathcal{C}\rangle$, *the conclusion* t *is warranted not only by the provability of* q, *but also by that of* s, *which is derived from* $r \vee s$ *via disjunctive syllogism using* $\neg r$, *itself inferred from* q. *Now, consider the following* $\mathsf{HG4}^{\mathsf{4+p}}_{\mathsf{cc}}$*-derivation* ρ:

$$\dfrac{\dfrac{\dfrac{}{\vdash p, s, t}\,{\scriptstyle ax} \qquad \dfrac{}{\vdash q, s, t}\,{\scriptstyle ax}}{\vdash s, t \mid \dashv \neg(p \wedge (r \vee s)) \mid \dashv \neg(q \wedge \neg r)}\,\delta'_{\{(p,r\vee s),(q,\neg r)\}}}{\vdash t \mid \dashv \neg(p \wedge (r \vee s)) \mid \dashv \neg(q \wedge \neg r) \mid \dashv \neg(q \wedge t)}\,\delta'_{\{(s,t)\}}$$

$H_\rho = H_\pi$: *as a result,* $\overline{\overline{\mathsf{G4}}}$ *refutes* $A, H_\rho \vdash \neg(p \wedge (r \vee s))$, $A, H_\rho \vdash \neg(q \wedge \neg r)$ *and* $A, H_\rho \vdash \neg(q \wedge t)$. *Hence,* ρ *is an* $\mathsf{HG4}^{\mathsf{4+p}}_{\mathsf{cc}}$*-proof.*

CHAPTER 8

Controlled sequents for defaults and norms

Norms – whether expressed as parental recommendations, military orders, traffic regulations, religious commandments, workplace policies, or self-imposed codes of conduct – present a logical puzzle. On one hand, their observance or violation is a binary matter, naturally modeled by the semantics of classical propositional logic. On the other hand, norms are often unstable and context-dependent: the introduction of a new norm can override or invalidate previous ones, which places their behavior squarely within the realm of non-monotonic logic. To take an example from the first book of Plato's *Republic*, the norm "One ought to return what one has borrowed" may be overridden in exceptional circumstances – for instance, if the person who lent a weapon has since gone mad. The norm is not monotonic: it applies by default, but can be defeated by relevant contextual changes, just as the generalization "dogs bark" is overridden by the more specific fact that "Basenjis do not bark". Finally, norms interact in complex ways, sometimes generating conflicts, even though – being commands rather than declarative statements – they do not possess truth values in the traditional logical sense.

Not only the formalization of normative systems, but also the resolution of conflicts between multiple norms are tasks of growing relevance in the field of artificial intelligence. For AI systems to interact effectively and responsibly with humans, they must be capable of understanding, representing, and reasoning about the intricate web of norms that structure human societies. Accordingly, growing attention has been devoted to developing logical tools that equip artificial agents to manage such normative complexity.

As is well known, from a logical standpoint, the most traditional approach to normative reasoning is provided by the family of deontic logics, which extend classical modal logic by introducing specific operators that enable the formal representation of norms, obligations and permissions within a Hilbert-style system, sequent calculus, or one of their variants [**175**, **68**, **114**, **30**]. However, standard deontic logics suffer from several well-documented limitations, including difficulties in addressing deontic paradoxes (such as the Good Samaritan paradox [**136**] or contrary-to-duty obligations [**27**, **48**, **172**]), and limited flexibility in accommodating context-sensitive, specificity-based or dynamic normative structures. By contrast, about 25 years ago a different proposal was developed in terms of the so-called Input/Output Logics (I/O logics) [**97**, **98**, **99**]. Such systems offer a non-modal alternative by treating conditional obligations as inference rules, explicitly separating input conditions (I) from outputs (O) expressing what is obliged, prohibited or permitted under those conditions. This framework emphasizes what ought to be done in a given situation, without treating obligations as modal truths. However – while more adaptable and better suited for modeling conditional and defeasible norms – I/O logics remain at a relatively early stage in the development of robust and comprehensive proof systems.

The novel perspective proposed in this chapter differs significantly from both of these paradigms: it is modular, computationally more efficient, and provides fine-grained control over inference steps. These features allow for more transparent, scalable, and adaptive normative reasoning, particularly in complex or multi-layered settings. Our aim is to develop a general and non-modal proof-theoretic framework for norms from the standpoint of a rational or artificial agent navigating a dynamic environment. This environment is stable enough to support expected obligations and prohibitions while remaining open and adaptable to new, potentially conflicting norms. As evidence of such flexibility, we will consider a wide range of concrete deontic scenario.

Moreover, the framework introduced articulates the interplay between normative reasoning and default reasoning. The latter is a type of non-monotonic reasoning that enables the derivation of plausible conclusions in contexts of incomplete information and absent explicit contradictory evidence. Such conclusions are defeasible: they can be retracted when challenged by new information [142, 101, 7]. Default reasoning can be formalized by extending classical logic with a collection of *extra-logical axioms* – which represent the propositional contents of an ideal reasoner's beliefs – and a set of *default rules*, which encode the inferential pathways that lead to defeasible yet consistent conclusions. In adopting a *credulous* approach, the ideal reasoner accepts as many individually consistent beliefs as possible. When reasoning with norms, such credulity is not merely permissible but necessary: a skeptical stance would unduly inhibit normative inference by prematurely excluding viable obligations.

Technically, our approach is based on the notion of *controlled* classical proof-system: a system built upon classical sequents augmented with a layer of extra-logical information, whose propagation is governed by the very structure of derivations. In prior work, controlled classical sequents – written as $\Gamma \vdash_{\mathbf{S}} \Delta$ – were introduced to capture non-monotonic and paraconsistent phenomena within a classical setting [122]. The idea was that the component $\mathbf{S}$ – referred to as the *control set* – encodes extra-logical information that can selectively block the derivability of Δ from Γ [20, 34, 21, 122, 130]. In this chapter, instead, controlled sequents are syntactic constructs specifically designed to handle defaults and norms on a classical base. We adopt specific extra-logical rules to formalize defaults, as well as (un)conditional obligations and permissions: each sequent in the calculus is annotated with a *control pair* $\langle \mathbf{T}, \mathbf{S} \rangle$ (written: $\Gamma \,\substack{\vdash_{\mathbf{T}} \\ \vdash_{\mathbf{S}}}\, \Delta$), where $\mathbf{T}$ and $\mathbf{S}$ are sets of formulas that specify prescriptions on what should or should not be entailed by the formulas in the antecedent in order for extra-logical rules to be applied. To give the flavor of the approach developed in this chapter, we begin with an informal example.

Example 8.1. *Let the extra-logical rule*

$$\frac{\Gamma \,|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\, p \qquad \Gamma \,|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\, q}{\Gamma \,|\!\frac{\mathbf{T}}{\mathbf{S}}\, r}$$

stand for the normative statement: 'If one prays and he is a male, one ought to wear Tefillin' – where $\mathbf{S} = \mathbf{S_1} \cup \mathbf{S_2} \cup \{\{p, r\}\}$*. Let* s *stand for the (factual) statement 'it is nighttime'. Consider the following derivation:*

$$\frac{\dfrac{\Gamma \,|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\, p \qquad \Gamma \,|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\, q}{\Gamma \,|\!\frac{\mathbf{T}}{\mathbf{S}}\, r}}{\Gamma \,|\!\frac{\mathbf{T'}}{\mathbf{S'}}\, r}$$

where $\mathbf{T'} = \mathbf{T}$ *and* $\mathbf{S'} = \mathbf{S} \cup \{\{s\}\}$*. The conclusion of such derivation expresses the normative statement: 'if one prays, then one ought to wear the Tefillin provided that one is male, unless it is nighttime'. This formulation captures the rule for wearing Tefillin during prayer, accounting for gender and time constraints.*

The chapter is structured as follows. Section 8.1 contains proof-theoretic preliminaries, as well as notions and results concerning constrained obligations and permissions. In particular, we introduce a notion of *deontic extension* for systems of obligations and permissions, analogous to the notion of Łukasiewicz extension for defaults, which generalizes the notion of *outfamily* in I/O logics. In Section 8.2 we introduce a uniform proof-theoretic platform for defaults, obligations and permissions based on controlled sequents. Controlled sequent calculi enjoy admissibility of contraction and non-analytic cut, and proofs without non-analytic cuts exhibit a weakened form of analyticity. We establish that controlled sequent calculi are strongly complete with respect to credulous consequence based on Łukasiewicz extensions (for defaults) and deontic extensions (for obligations and permissions). Hence, we show that controlled sequent calculi enable the formalization of weak versions of cumulative transitivity and cautious monotony for the underlying credulous consequence relations. Lastly, Section 8.4 gathers a number of examples of deontic scenarios, observed through the *lens* of controlled calculi.

8.1. Preliminary notions and results

8.1.1. The $\overline{\overline{\mathsf{G4^{pn}}}}$ calculus. In this chapter, we employ a standard propositional language, consisting of a denumerable set of atoms $p, q, r, \ldots$, the unary connective $\neg$ and the binary connectives $\wedge, \vee$. We use capital Greek letters $\Gamma, \Delta, \Pi, \Sigma, \ldots$ to denote finite multisets of formulas, and $\Theta, \Lambda, \ldots$ to denote finite multisets of *literals* – namely, atomic formulas and

AXIOMS

$$\frac{}{\Gamma, p \vdash p, \Delta}\,ax \qquad \frac{}{\Gamma, p, \neg p \vdash \Delta}\,ax \qquad \frac{}{\Gamma \vdash p, \neg p, \Delta}\,ax \qquad \frac{}{\Gamma, \neg p \vdash \neg p, \Delta}\,ax$$

$$\frac{}{\Theta \dashv \Lambda}\,\overline{ax}$$

LOGICAL RULES

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,L\wedge \qquad \frac{\Gamma, A, B \dashv \Delta}{\Gamma, A \wedge B \dashv \Delta}\,L'\wedge$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,R\wedge \qquad \frac{\Gamma \dashv \Delta, A_i}{\Gamma \dashv \Delta, A_1 \wedge A_2}\,R'_i\wedge$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\,L\vee \qquad \frac{A_i, \Gamma \dashv \Delta}{A_1 \vee A_2, \Gamma \dashv \Delta}\,L'_i\vee$$

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\,R\vee \qquad \frac{\Gamma \dashv \Delta, A, B}{\Gamma \dashv \Delta, A \vee B}\,R'_i\vee$$

$$\frac{\Gamma, \neg A \vdash \Delta \qquad \Gamma, \neg B \vdash \Delta}{\Gamma, \neg(A \wedge B) \vdash \Delta}\,L\neg\wedge \qquad \frac{\Gamma, \neg A_i \dashv \Delta}{\Gamma, \neg(A_1 \wedge A_2) \dashv \Delta}\,L'_i\neg\wedge$$

$$\frac{\Gamma \vdash \Delta, \neg A, \neg B}{\Gamma \vdash \Delta, \neg(A \wedge B)}\,R\neg\wedge \qquad \frac{\Gamma \dashv \Delta, \neg A, \neg B}{\Gamma \dashv \Delta, \neg(A \wedge B)}\,R'\neg\wedge$$

$$\frac{\neg A, \neg B, \Gamma \vdash \Delta}{\neg(A \vee B), \Gamma \vdash \Delta}\,L\neg\vee \qquad \frac{\neg A, \neg B, \Gamma \dashv \Delta}{\neg(A \vee B), \Gamma \dashv \Delta}\,L'\neg\vee$$

$$\frac{\Gamma \vdash \Delta, \neg A \qquad \Gamma \vdash \Delta, \neg B}{\Gamma \vdash \Delta, \neg(A \vee B)}\,R\neg\vee \qquad \frac{\Gamma \dashv \Delta, \neg A_i}{\Gamma \dashv \Delta, \neg(A_1 \vee A_2)}\,R'_i\neg\vee$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \neg\neg A \vdash \Delta}\,L\neg\neg \qquad \frac{\Gamma, A \dashv \Delta}{\Gamma, \neg\neg A \dashv \Delta}\,L'\neg\neg$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, \neg\neg A}\,R\neg\neg \qquad \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, \neg\neg A}\,R'\neg\neg$$

FIGURE 1. $\mathsf{G4pn}$ and $\overline{\overline{\mathsf{G4}^{\mathsf{pn}}}}$ sequent calculi

their negations. For any contexts Θ and Γ we shall be adopting the following convention: if $\Theta = \{p_1, \dots, p_k, \neg q_1, \dots, \neg q_m\}$ and $\Gamma = \{A_1, \dots, A_n\}$, then

$$\Theta^{\perp} = \{\neg p_1, \dots, \neg p_k, q_1, \dots, q_m\} \qquad \bigwedge \Gamma = A_1 \wedge A_2 \wedge \cdots \wedge A_n \qquad \bigvee \Gamma = A_1 \vee A_2 \vee \cdots \vee A_n$$

For $\Theta = \{A\}$, we use $A^{\perp}$ for $\Theta^{\perp}$. Moreover, for $\Gamma = \emptyset$, we set $\Gamma^{\perp} = \Gamma$, $\bigwedge \Gamma = \top$, and $\bigvee \Gamma = \bot$, where $\top$ and $\bot$ stand for an arbitrarily chosen tautology and contradiction, respectively. We take the *logical complexity* $\mathrm{C}(A)$ of a formula A is 1 if A is a literal, $\mathrm{C}(B) + \mathrm{C}(C) + 1$ if A is of the form $B \otimes C$ and $\mathrm{C}(\neg B) + \mathrm{C}(\neg C) + 1$ if A is of the form $\neg(B \otimes C)$, with $\otimes \in \{\wedge, \vee\}$.

The system $\overline{\overline{\mathsf{G4^{pn}}}}$ for classical propositional logic is a modification of the system $\overline{\overline{\mathsf{G4}}}$, which absorbs the rules for $\neg$ in $\overline{\overline{\mathsf{G4}}}$ *via* a larger class of *ax* instances and the rules for $\neg\wedge$, $\neg\vee$ and $\neg\neg$. A $\overline{\overline{\mathsf{G4^{pn}}}}$ derivation π may end either in a sequent $\Gamma \vdash \Delta$ or in an antisequent $\Gamma \dashv \Delta$: in the first case, we say that π is a *proof* for $\Gamma \vdash \Delta$; in the second, π qualifies as a *refutation* for $\Gamma \vdash \Delta$.

We can decompose any (anti)sequent $\Gamma \overset{*}{\vdash} \Delta$ into a set of atomic (anti)sequents, using bottom-up the rules $L\wedge, R\wedge, L\vee, R\vee, L\neg\wedge, R\neg\wedge, L\neg\vee, R\neg\vee, L\neg\neg, R\neg\neg$ in Figure 1, with $\overset{*}{\vdash}$ in place of $\vdash$, until each leaf of the resulting tree ends with an atomic (anti)sequent – i.e., an (anti)sequent featuring only literals.

We spell out two crucial features of the $\overline{\overline{\mathsf{G4^{pn}}}}$ proof system:

Proposition 8.1. $\overline{\overline{\mathsf{G4^{pn}}}}$ proves (refutes) $\Gamma \vdash \Delta$ if and only if the formula $\neg \bigwedge \Gamma \vee \bigvee \Delta$ is classically valid (*in*valid).

Proposition 8.2. Maximal $\overline{\overline{\mathsf{G4^{pn}}}}$-decomposition yields a unique set of atomic (anti)sequents.

Proof. We argue as in the proof of the same property for $\overline{\overline{\mathsf{G4}}}$ in [14, 123]. ☐

Theorem 8.2 allows us to directly refer to a unique set of atomic (anti)sequents associated with a certain (anti)sequent $\Gamma \overset{*}{\vdash} \Delta$, being such a decomposition independent of the specific derivation delivering it. For any set of atomic (anti)sequents $\mathcal{C}$, we say that $\mathcal{C}$ is *closed under Negation* whenever $\Theta_1, \Lambda_2^{\perp} \overset{*}{\vdash} \Theta_2^{\perp}, \Lambda_1$ belongs to $\mathcal{C}$ if $\Theta_1, \Theta_2 \overset{*}{\vdash} \Lambda_2, \Lambda_1$ belongs to $\mathcal{C}$. We write $\mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$ to indicate the closure under Negation of the set of atomic (anti)sequents associated with $\Gamma \overset{*}{\vdash} \Delta$, and $\mathsf{top_c}(\Gamma \overset{*}{\vdash} \Delta)$ to indicate the set of the atomic (anti)sequents $\Theta \overset{*}{\vdash} \Lambda$ in $\mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$ such that $\Theta \cap \Lambda = \varnothing$ and Λ does not contain an atom and its negation – where the subscript c stands for 'complementary'. Moreover, we use $\mathsf{top_r}(\Gamma \overset{*}{\vdash} \Delta)$ to denote the set of complementary (anti)sequents of the form $\overset{*}{\vdash} \Lambda$ in $\mathsf{top_c}(\Gamma \overset{*}{\vdash} \Delta)$ – where the subscript r stands for 'one-sided'.

For any set $\mathcal{C}$ of atomic (anti)sequents, we say that $\mathcal{C}$ is closed under Cut whenever $\Phi, \Theta \overset{*}{\vdash} \Lambda, \Psi$ belongs to $\mathcal{C}$ if either $\Theta \overset{*}{\vdash} \Lambda, A$ and $A, \Phi \overset{*}{\vdash} \Psi$, or $\Theta \overset{*}{\vdash} \Lambda, A^{\perp}$ and $\Phi \overset{*}{\vdash} \Psi, A$, or $A, \Theta \overset{*}{\vdash} \Lambda$ and $A^{\perp}, \Phi \overset{*}{\vdash} \Psi, A$ belong to $\mathcal{C}$. Furthermore, we say that $\mathcal{C}$ is closed under Contraction whenever $A, \Theta \overset{*}{\vdash} \Lambda$ belongs to $\mathcal{C}$ if either $A, A, \Theta \overset{*}{\vdash} \Lambda$ or $A, \Theta \overset{*}{\vdash} \Lambda, A^{\perp}$ belongs to $\mathcal{C}$, and $\Theta \overset{*}{\vdash} \Lambda, A$ belongs to $\mathcal{C}$ if either $\Theta \overset{*}{\vdash} \Lambda, A, A$ or $A^{\perp}, \Theta \overset{*}{\vdash} \Lambda, A$ belongs to $\mathcal{C}$. We write $\mathsf{top}^{\star}(\Gamma \overset{*}{\vdash} \Delta)$ to refer to the set of atomic (anti)sequents which is obtained from $\mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$ by maximal application of the following steps (cf. [129, p. 9]):

(i) start with $\mathcal{C}_0 = \mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$;

(ii) take the closure under Contraction of $\mathcal{C}_n$;

(iii) if either $\Theta \overset{*}{\vdash} \Lambda, A$ and $A, \Phi \overset{*}{\vdash} \Psi$, or $A^{\perp}, \Theta \overset{*}{\vdash} \Lambda$ and $A, \Phi \overset{*}{\vdash} \Psi$, or $A, \Theta \overset{*}{\vdash} \Lambda$ and $A^{\perp}, \Phi \overset{*}{\vdash} \Psi$ belong to $\mathcal{C}_n$, and $\Phi, \Theta \overset{*}{\vdash} \Lambda, \Psi$ does not belong to $\mathcal{C}_n$, then take $\mathcal{C}_{n+1} = \mathcal{C}_n \cup \{\Phi, \Theta \overset{*}{\vdash} \Lambda, \Psi\}$.

We use $\mathsf{top}^{\star}_{\mathsf{c}}(\Gamma \overset{*}{\vdash} \Delta)$ to denote the set of complementary (anti)sequents in $\mathsf{top}^{\star}(\Gamma \overset{*}{\vdash} \Delta)$. Finally, we employ $\mathsf{top}^{\star}_{\mathsf{r}}(\Gamma \overset{*}{\vdash} \Delta)$ to denote the smallest set containing all the complementary (anti)sequents in $\mathsf{top}^{\star}(\Gamma \overset{*}{\vdash} \Delta)$ whose left-hand side is empty.

8.1.2. Constrained obligations and permissions. We consider a *constrained conditional obligation* (permission) as a domain-specific inference rule of the form

$$\frac{B : C_1, \ldots, C_n}{D} \tag{8.1.1}$$

where B is the *conditions*, the non-contradictory formulas $C_1, \ldots, C_n$ are the *constraints* and the non-tautological formula D is the *conclusion* of the constrained conditional obligation (permission, respectively). Its interpretation is that if B is/ought to be (is/is permitted to be) the case, then D ought to be (is permitted to be) the case, so long as $C_1, \ldots, C_n$ ought not (are not permitted to, respectively) be the case. If $n = 1$, we say that a constrained conditional obligation (permission) of the form (8.1.1) is *normal* whenever $C_1 = D$. If B is logically equivalent to $\top$, a rule of the form 8.1.1 is a constrained *unconditional* obligation. We say that an obligation (permission) is *triggered* whenever its condition is proved.

An *obligation system* is a pair $\langle \mathcal{W}^{\mathsf{o}}, \mathcal{O} \rangle$, where $\mathcal{W}^{\mathsf{o}}$ is a finite, consistent set of extra-logical axioms – expressing unconstrained, unconditional obligations – and $\mathcal{O}$ is a finite, non-empty set of constrained (un)conditional obligations. On the other hand, a *permission system* is a pair $\langle \mathcal{W}^{\mathsf{p}}, \mathcal{P} \rangle$, where $\mathcal{W}^{\mathsf{p}}$ is a finite, consistent set of extra-logical axioms – expressing unconstrained, unconditional permissions – and $\mathcal{P}$ is a finite, non-empty set of constrained (un)conditional permissions.

A *normative system* $\mathcal{N}$ is a pair $\langle \langle \mathcal{W}^{\mathsf{o}}, \mathcal{O} \rangle, \langle \mathcal{W}^{\mathsf{p}}, \mathcal{P} \rangle \rangle$, where ($i$) $\langle \mathcal{W}^{\mathsf{o}}, \mathcal{O} \rangle$ is an obligation system, (ii) $\langle \mathcal{W}^{\mathsf{p}}, \mathcal{P} \rangle$ is a permission system and (iii) $\mathcal{W}^{\mathsf{o}} \subseteq \mathcal{W}^{\mathsf{p}}$ and $\mathcal{O} \subseteq \mathcal{P}$. A normative system $\langle \langle \mathcal{W}^{\mathsf{o}}, \mathcal{O} \rangle, \langle \mathcal{W}^{\mathsf{p}}, \mathcal{P} \rangle \rangle$ is said to be normal if and only if all the permissions in $\mathcal{P}$ are normal. We shall use $cond(\mathcal{Q}')$, $constr(\mathcal{Q}')$ and $concl(\mathcal{Q}')$ to refer to the set of conditions, constraints and conclusions, respectively, of the obligations (permissions) in any $\mathcal{O}' \subseteq \mathcal{O}$ ($\mathcal{P}' \subseteq \mathcal{P}$, respectively). If $\mathcal{Q} \in \{\mathcal{O}, \mathcal{P}\}$, we write $red(\mathcal{Q})$ to denote the set of inference rules obtained from $\mathcal{Q}$ by deleting their constraints.

Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be a default theory and $\mathcal{N}$ a normative system. Given a modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W}, \mathcal{D} \rangle$, an *obligation* (*permission*) *extension* is a set of formulas derivable from $\mathcal{E}, \mathcal{W}^{o}$ and $\mathcal{W}^{p}$ by classical logic and maximal application of the obligations (permissions, respectively) in $\mathcal{N}$ whose consistency condition holds (both before and after their being triggered) relatively to a simultanously defined support set. Specifically, we say that $\langle \mathcal{S}_{\mathcal{E}}, \mathcal{T}_{\mathcal{E}} \rangle$ is an obligation extension of $\langle \langle \mathcal{W}, \mathcal{D} \rangle, \mathcal{N} \rangle$ if and only if $\mathcal{S}_{\mathcal{E}}$ and $\mathcal{T}_{\mathcal{E}}$ are quasi-inductively defined as follows:

$$\mathcal{S}^{0}_{\mathcal{E}} = \mathcal{W}^{o} \qquad \mathcal{T}^{0}_{\mathcal{E}} = \varnothing$$

$$\mathcal{S}^{k+1}_{\mathcal{E}} = Cn(\mathcal{S}^{k}_{\mathcal{E}}) \cup \{concl(\mathcal{O}') \mid \mathcal{O}' \subseteq \mathcal{O}, req(\mathcal{O}') \subseteq \mathcal{S}^{k}_{\mathcal{E}} \cup \mathcal{E}, A \in (\mathcal{T}_{\mathcal{E}} \cup just(\mathcal{O}')) \Rightarrow \neg A \notin Cn(\mathcal{S}_{\mathcal{E}} \cup concl(\mathcal{O}'))\}$$

$$\mathcal{T}^{k+1}_{\mathcal{E}} = \mathcal{T}^{k}_{\mathcal{E}} \cup \{just(\mathcal{O}') \mid \mathcal{O}' \subseteq \mathcal{O}, req(\mathcal{O}') \subseteq \mathcal{S}^{k}_{\mathcal{E}} \cup \mathcal{E}, A \in (\mathcal{T}_{\mathcal{E}} \cup just(\mathcal{O}')) \Rightarrow \neg A \notin Cn(\mathcal{S}_{\mathcal{E}} \cup concl(\mathcal{O}'))\}$$

$$\mathcal{S}_{\mathcal{E}} = \bigcup_{i=0}^{\omega} \mathcal{S}^{i}_{\mathcal{E}} \qquad \mathcal{T}_{\mathcal{E}} = \bigcup_{i=0}^{\omega} \mathcal{T}^{i}_{\mathcal{E}}$$

A permission extension $\langle \mathcal{S}_{\mathcal{E}}, \mathcal{T}_{\mathcal{E}} \rangle$ of $\langle \langle \mathcal{W}, \mathcal{D} \rangle, \mathcal{N} \rangle$ is defined as an obligation extension of $\langle \langle \mathcal{W}, \mathcal{D} \rangle, \mathcal{N} \rangle$, except for the fact that we replace $\mathcal{W}^{o}$ with $\mathcal{W}^{\mathsf{p}}$ and $\mathcal{O}', \mathcal{O}$ with $\mathcal{P}', \mathcal{P}$. We say that $\langle \mathcal{S}_{\mathcal{E}}, \mathcal{T}_{\mathcal{E}} \rangle$ is a deontic extension of $\langle \langle \mathcal{W}, \mathcal{D} \rangle, \mathcal{N} \rangle$ if $\langle \mathcal{S}_{\mathcal{E}}, \mathcal{T}_{\mathcal{E}} \rangle$ is an obligation extension or a permission extension of $\langle \langle \mathcal{W}, \mathcal{D} \rangle, \mathcal{N} \rangle$.

If $\langle \langle \mathcal{W}, \mathcal{D} \rangle, \mathcal{N} \rangle$ is a default theory associated with a normative system, then A is a *deontic credulous consequence* (in short, a d-credulous consequence) of $\mathcal{W} \cup \mathcal{W}^{o} \cup \mathcal{W}^{\mathsf{p}}$ if and only if A belongs to at least one deontic extension of $\langle \langle \mathcal{W}, \mathcal{D} \rangle, \mathcal{N} \rangle$. Intuitively, the credulous agent commits herself to as many individually consistent obligations and permissions as possible.

PROPOSITION 8.3. Let $\langle \mathcal{E}, \mathcal{F} \rangle$ be a modified extension of a default theory $\langle \mathcal{W}, \mathcal{D} \rangle$, $\mathcal{O}, \mathcal{P}$ be the obligation and the permission systems of a normative system $\mathcal{N}$, $\mathcal{Q} \in \{\mathcal{O}, \mathcal{P}\}$, $\mathcal{Q} \subseteq \mathcal{Q}'$ and $\mathcal{N}'$ obtained from $\mathcal{N}$ replacing $\mathcal{Q}$ with $\mathcal{Q}'$. For any obligation/permission extension $\langle \mathcal{S}^{\mathcal{E}}_{1}, \mathcal{T}^{\mathcal{E}}_{1} \rangle$ of $\langle \langle \mathcal{W}, \mathcal{D} \rangle, \mathcal{N} \rangle$, there exists (at least) one obligation/permission extension $\langle \mathcal{S}^{\mathcal{E}}_{2}, \mathcal{T}^{\mathcal{E}}_{2} \rangle$ of $\langle \langle \mathcal{W}, \mathcal{D} \rangle, \mathcal{N}' \rangle$ such that $\mathcal{S}^{\mathcal{E}}_{1} \subseteq \mathcal{S}^{\mathcal{E}}_{2}$ and $\mathcal{T}^{\mathcal{E}}_{1} \subseteq \mathcal{T}^{\mathcal{E}}_{2}$.

PROOF. We argue as in the proof of Proposition 5.5. □

8.2. Controlled sequent calculi for default logics

Informally, a *control set* is a set of sets of formulas that are supposed to overturn a certain proof within a given proof system. To design controlled sequent calculi for deontic default reasoning, we build on the notion of control set, introducing the notion of *control pair*.

8.2.1. Control pairs. In this subsection, we shall use capital Greek letters $\Gamma, \Delta, \Theta, \ldots$ to denote multisets as well as *sets* of formulas (we will make explicit the difference whenever needed in order to avoid confusion). Boldface capital letters $\mathbf{S}, \mathbf{S_1}, \ldots \ldots$ stand for finite sets of finite sets of formulas.

A *control pair* is a pair of sets $\langle \mathbf{T}, \mathbf{S} \rangle$, where $\mathbf{T}$ collects sets of *conditions*, and $\mathbf{S}$ collects sets of *constraints*. For their part, conditions and constraints are formulas, possibly labelled with specific superscripts $\mathsf{f}, \mathsf{o}, \ldots$ to denote their role as facts, obligations, etc. – respectively. We decorate the turnstile $\vdash$ with a control pair: intuitively, this decoration expresses the fact that the derivation of a sequent $\Gamma \vdash \Delta$ must fulfill the conditions contained in any element of $\mathbf{T}$ and respect the constraints contained in any element of $\mathbf{S}$. In particular, we want Γ (possibly strengthened with other formulas, cf. Definition 8.4) to be compatible with the sets of conditions in $\mathbf{T}$ and the sets of constraints from $\mathbf{S}$ in the following sense.

Definition 8.1. Given a multiset Γ and a control pair $\langle \mathbf{T}, \mathbf{S} \rangle$, Γ is *compatible* with $\langle \mathbf{T}, \mathbf{S} \rangle$ (in symbols $\Gamma \,||\, \langle \mathbf{T}, \mathbf{S} \rangle$) if and only if for every formula A in $\bigcup \mathbf{T}$ and every formula B in $\bigcup \mathbf{S}$, $\overline{\overline{\mathsf{G4^{pn}}}}$ proves $\Gamma \vdash A$ and refutes $\Gamma \vdash B$.

Lemma 8.1. *Let* $\mathbf{S} \subseteq \mathbf{S}'$, $\mathbf{T} \subseteq \mathbf{T}'$ *and the sequent* $\Gamma \vdash \bigwedge \Delta$ *be provable in* $\overline{\overline{\mathsf{G4^{pn}}}}$. *The following statements about compatibility hold:*

(1) $\Gamma \,||\, \langle \mathbf{T}', \mathbf{S}' \rangle$ *implies* $\Gamma \,||\, \langle \mathbf{T}, \mathbf{S} \rangle$*;*

(2) $\Delta \,||\, \langle \mathbf{T}, \varnothing \rangle$ *implies* $\Gamma \,||\, \langle \mathbf{T}, \varnothing \rangle$*;*

(3) $\Gamma \,||\, \langle \varnothing, \mathbf{S} \rangle$ *implies* $\Delta \,||\, \langle \varnothing, \mathbf{S} \rangle$*.*

Proof. Immediate from Definition 8.1. □

Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be any default theory and $\mathcal{N}$ be any normative system associated with $\langle \mathcal{W}, \mathcal{D} \rangle$. We exploit control pairs to define controlled sequent calculi for m-credulous consequences of $\mathcal{W}$ and d-credulous consequences of $\mathcal{W} \cup \mathcal{W}^{\mathsf{o}} \cup \mathcal{W}^{\mathsf{p}}$.

To distinguish the roles of derivable formulas, we use labelled turnstiles in sequents:

- ${}_{\mathsf{F}}\!\vdash$ for sequents tracking m-credulous consequences,
- ${}_{\mathsf{O}}\!\vdash$ for sequents tracking d-credulous consequences which correspond to obligations, and
- ${}_{\mathsf{P}}\!\vdash$ for sequents tracking d-credulous consequences which correspond to permissions.

When generalizing over a subset of $\{{}_{\mathsf{F}}\!\vdash, {}_{\mathsf{O}}\!\vdash, {}_{\mathsf{P}}\!\vdash\}$, we use the labelled turnstile ${}_{\times}\!\vdash$.

8.2.2. Control pairs for defaults. We are ready to present controlled sequent calculi for m-credulous consequence.

Definition 8.2. Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be any default logic. The controlled sequent calculus $\mathsf{G4}^{\delta}$ for $\langle \mathcal{W}, \mathcal{D} \rangle$ can be defined by adopting the rules in Figure 2, provided that the conditions $(i) - (vii)$ below are fulfilled.

AXIOMS AND STRUCTURAL RULES

$$\frac{}{\Theta \;{}_{\mathsf{F}}\Big|\frac{\varnothing}{\varnothing}\; \Lambda}\,ax$$

$$\frac{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}{A, \Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}\,LW \qquad \frac{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta, A}\,RW$$

$$\frac{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_1}}{\mathbf{S_1}}\; \Delta, A \qquad \Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_2}}{\mathbf{S_2}}\; \Delta, \neg A}{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}\,cut_{asa} \qquad \frac{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S'}}\; \Delta}\,\sigma$$

LOGICAL RULES

$$\frac{\Gamma, A, B \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}{\Gamma, A \wedge B \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}\,L\wedge \qquad \frac{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_1}}{\mathbf{S_1}}\; \Delta, A \qquad \Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_2}}{\mathbf{S_2}}\; \Delta, B}{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta, A \wedge B}\,R\wedge$$

$$\frac{\Gamma, A \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_1}}{\mathbf{S_1}}\; \Delta \qquad \Gamma, B \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_2}}{\mathbf{S_2}}\; \Delta}{\Gamma, A \vee B \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}\,L\vee \qquad \frac{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta, A, B}{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta, A \vee B}\,R\vee$$

$$\frac{\Gamma, \neg A \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_1}}{\mathbf{S_1}}\; \Delta \qquad \Gamma, \neg B \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_2}}{\mathbf{S_2}}\; \Delta}{\Gamma, \neg(A \wedge B) \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}\,L\neg\wedge \qquad \frac{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta, \neg A, \neg B}{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta, \neg(A \wedge B)}\,R\neg\wedge$$

$$\frac{\Gamma, \neg A, \neg B \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}{\Gamma, \neg(A \vee B) \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}\,L\neg\vee \qquad \frac{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_1}}{\mathbf{S_1}}\; \Delta, \neg A \qquad \Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_2}}{\mathbf{S_2}}\; \Delta, \neg B}{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta, \neg(A \vee B)}\,R\neg\vee$$

$$\frac{A, \Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}{\neg\neg A, \Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}\,L\neg\neg \qquad \frac{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta, A}{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Delta, \neg\neg A}\,R\neg\neg$$

EXTRA-LOGICAL RULES

$$\frac{\{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T_i}}{\mathbf{S_i}}\; \Theta_i\}_{0 \leq i \leq m}}{\Gamma \;{}_{\mathsf{F}}\Big|\frac{\mathbf{T}}{\mathbf{S}}\; \Phi}\,\delta$$

Figure 2. Controlled sequent calculus for a default theory $\langle \mathcal{W}, \mathcal{D} \rangle$

(i) For each instance of ax, one of the following conditions holds: $\Theta = \Lambda = \{A\}$ for some literal A, $\Theta = \{A, \neg A\}$ or $\Lambda = \{A, \neg A\}$ for some atom A, $\Theta \overset{*}{\mid\!\sim} \Lambda$ belongs to $\mathsf{top}^{\star}_{\mathsf{c}}(\overset{*}{\mid\!\sim} W)$, with W being the conjunction of the formulas in $\mathcal{W}$.

(ii) For each instance of binary structural and logical rules, $\mathbf{T} = \mathbf{T_1} \cup \mathbf{T_2}$ and $\mathbf{S} = \mathbf{S_1} \cup \mathbf{S_2}$.

(iii) For each instance of cut_{asa}, A is an atom occurring in some formula in $\Gamma \cup \mathcal{W}$.

(iv) For each instance of σ, $\mathbf{S} \subseteq \mathbf{S}'$.

(v) For any default rule of the form $\dfrac{B : C_1, \ldots, C_k}{D}$ in $\mathcal{D}$, if $\mathsf{top}_{\mathsf{r}}(\overset{*}{\mid\!\sim} B) = \{\overset{*}{\mid\!\sim} \Theta_1, \ldots, \overset{*}{\mid\!\sim} \Theta_m\}$ with $m > 0$ and $\overset{*}{\mid\!\sim} \Phi$ belongs to $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\mid\!\sim} D)$, then there exists an extra-logical rule of the following form:

$$\dfrac{\{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T_i}}_{\mathbf{S_i}} \Theta_i\}_{1\leq i\leq m}}{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T}}_{\mathbf{S}} \Phi}\,\delta$$

with $\mathbf{T} = \bigcup_{i=1}^{m} \mathbf{T_i} \cup \{\{(\bigwedge_{i=1}^{m}(\bigvee \Theta_i))^{\mathsf{f}}\}\}$ and $\mathbf{S} = \bigcup\limits_{i=1}^{m} \mathbf{S_i} \cup \{\{\neg C_1^{\mathsf{f}}, \ldots, \neg C_k^{\mathsf{f}}\}\}$.

If $\mathsf{top}_{\mathsf{r}}(\overset{*}{\mid\!\sim} B) = \varnothing$ and $\overset{*}{\mid\!\sim} \Phi$ belongs to $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\mid\!\sim} D)$, then there exists an extra-logical rule of the following form:

$$\dfrac{}{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T}}_{\mathbf{S}} \Phi}\,\delta$$

with $\mathbf{T}$ being $\{\{\top^{\mathsf{f}}\}\}$ and $\mathbf{S}$ being $\{\{\neg C_1^{\mathsf{f}}, \ldots, \neg C_k^{\mathsf{f}}\}\}$.

(vi) For any extra-logical rule

$$\dfrac{\{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T_i}}_{\mathbf{S_i}} \Theta_i\}_{1\leq i\leq m}}{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T}}_{\mathbf{S}} \Phi}\,\delta$$

with $m \geq 0$, if $\overset{*}{\mid\!\sim} \Phi'$ occurs in $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\mid\!\sim} (\bigvee \Phi) \wedge W)$ without belonging to $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\mid\!\sim} W)$, then there is an extra-logical rule of the following form:

$$\dfrac{\{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T_i}}_{\mathbf{S_i}} \Theta_i\}_{1\leq i\leq m}}{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T}}_{\mathbf{S}} \Phi'}\,\delta$$

(vii) For any pair of extra-logical rules

$$\dfrac{\{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T'_i}}_{\mathbf{S'_i}} \Theta_i\}_{1\leq i\leq m}}{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T_1}}_{\mathbf{S_1}} \Phi}\,\delta \qquad \dfrac{\{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T'_i}}_{\mathbf{S'_i}} \Theta_i\}_{m+1\leq i\leq n}}{\Gamma \ {}_{\mathsf{F}}\!\mid\!\sim^{\mathbf{T_2}}_{\mathbf{S_2}} \Phi'}\,\delta$$

with $m + n \geq 0$, if $\overset{*}{\mid\!\sim} \Psi$ occurs in $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\mid\!\sim} (\bigvee \Phi) \wedge (\bigvee \Phi'))$ without belonging to $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\mid\!\sim} (\bigvee \Phi))$, $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\mid\!\sim} (\bigvee \Phi'))$ or $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\mid\!\sim} W)$, then there exists an extra-logical rule of the following form:

$$\frac{\{\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_i'}}{\mathbf{S_i'}}\right. \Theta_i\}_{1\leq i\leq n}}{\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Psi}\,\delta$$

with $\mathbf{S} = \mathbf{S_1} \cup \mathbf{S_2}$ and $\mathbf{T} = \mathbf{T_1} \cup \mathbf{T_2}$.

The condition (i) ensures that the $\mathsf{G4}^\delta$ calculus incorporates the extra-logical axioms from $\mathcal{W}$, whereas the condition (v) guarantees that $\mathsf{G4}^\delta$ calculi feature extra-logical rules corresponding to the default rules occurring in $\mathcal{D}$. On the other hand, the closure conditions (vi) and (vii) are needed in order to ensure that the conclusions of extra-logical rules are closed under Contraction and Cut (cf. Lemmas 8.3 and 8.8, Theorems 8.1, 8.4, 8.5 and 8.6).

It is easy to find cases where there is (at least) one $\mathsf{G4}^\delta$-derivation of a controlled sequent $\Gamma \left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta$ such that $\bigvee \Delta$ is not an m-credulous consequence of $\mathcal{W} \cup \Gamma$ (analogously to what happens with $\mathsf{HG4c}$ derivations, cf. Chapter 5). As a result, we need to offer a criterion to single out $\mathsf{G4}^\delta$-derivations which deliver m-credulous consequences of $\mathcal{W} \cup \Gamma$ from those which do not. To this aim, we make some additions to our terminological and conceptual apparatus.

For any extra-logical rule δ in $\mathsf{G4}^\delta$, we define the label $\delta_{\mathcal{D}'}$ as follows:

(a) if δ is generated in accordance with point (iii) in Definition 8.2, then

$$\mathcal{D}' = \left\{\frac{B : C_1, \ldots, C_k}{D}\right\}$$

(b) if δ is generated from extra-logical rules δ' and δ'' with labels $\delta_{\mathcal{D}_1}$ and $\delta_{\mathcal{D}_2}$, respectively, in accordance with points $(iv) - (v)$ in Definition 8.2, then $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2$.

For each $\mathsf{G4}$-derivation π we say that a default rule $\dfrac{B : C_1, \ldots, C_k}{D}$ *belongs to* $def(\pi)$ if and only if there is (at least) one extra-logical rule labelled $\delta_{\mathcal{D}'}$ which is applied in π and such that $\dfrac{B : C_1, \ldots, C_k}{D}$ belongs to $\mathcal{D}'$. Moreover, we employ $def'(\pi)$ to denote $def(\pi_1) \cup \cdots \cup def(\pi_m)$ whenever $\pi_1, \ldots, \pi_m$ are the immediate subderivations yielding the premises of the lowermost extra-logical rules applications in π.

We shall use D_π to refer to the conjunction of the formulas in $concl(def(\pi))$, and E_π to denote the conjunction of the formulas in $concl(def'(\pi))$.

Definition 8.3. Let π be a $\mathsf{G4}^\delta$-derivation of $\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta$. The sequent $\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta$ is

(i) *sound under conditions* if and only if $([W, E_\rho] \cup \Gamma) \,||\, \langle \mathbf{V}, \varnothing \rangle$ for any subderivation ρ of π with $\Pi \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{V}}{\mathbf{U}}\right. \Sigma$ as conclusion;

(ii) *sound under constraints* if and only if $([W, D_\pi] \cup \Gamma) \,||\, \langle \varnothing, \mathbf{S} \rangle$;

(iii) *sound* if and only if it is sound under conditions and sound under constraints.

Definition 8.4. Let π be a $\mathsf{G4}^{\delta}$-derivation of $\Gamma \,\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\, \Delta$. Then, π is a *proof* of $\Gamma \,\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\, \Delta$ if and only if any sequent $\Pi \,\Big|\!\frac{\mathbf{T}'}{\mathbf{S}'}\, \Sigma$ occurring in π is sound – and a *paraproof*, otherwise.

Before proceeding to the structural analysis of $\mathsf{G4}^{\delta}$ calculi, let us gather some further remarks about the conditions in Definition 8.2.

(A) For any extra-logical rule, the formulas which are principal in any of its applications occur only on the right-hand side of the sequent. If we replace $\mathsf{top}^{\star}_{\mathsf{r}}$ with $\mathsf{top}^{\star}_{\mathsf{c}}$, we embark unsound formalizations of default rules (cf. Definition 8.3). Notice that the availability of the $\mathsf{top}^{\star}_{\mathsf{r}}$ is guaranteed by the adoption of the $\overline{\overline{\mathsf{G4^{pn}}}}$ calculus for classical logic, where negation is primitive and no rule shifts a formula from one side to the other.

(B) Let us employ the expression *semi-analytic Cut* to denote a Cut applications where (a) both Cut formulas occur on the right-hand side, and (b) the Cut formula is a subformula of the formulas occurring on the left-hand side of the conclusion or a subformula of the extra-logical axioms. The condition (iii) ensures that $\mathsf{G4}^{\delta}$ calculi feature atomic, semi-analytic Cut as a primitive rule. This is needed is needed in order to achieve strong adequacy (cf. Theorem 8.8): as a matter of fact, $\mathsf{G4}^{\delta}$ calculi without cut_{asa} are not closed under cut_{asa} applications. Consider e.g. the $\mathsf{G4}^{\delta}$ calculus for $\langle \mathcal{W}, \mathcal{D} \rangle$, with

$$\mathcal{W} = \varnothing$$
$$\mathcal{D} = \left\{ \frac{\top : p}{p}, \frac{p \vee q : \neg p}{\neg p} \right\}$$

Now, take the following $\mathsf{G4}^{\delta}$-proof π:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \,\Big|\!\frac{\varnothing}{\varnothing}\, p}\,ax}{p \,\Big|\!\frac{\varnothing}{\varnothing}\, p,q}\,RW \qquad \dfrac{\dfrac{}{q \,\Big|\!\frac{\varnothing}{\varnothing}\, q}\,ax}{q \,\Big|\!\frac{\varnothing}{\varnothing}\, p,q}\,RW}{p \vee q \,\Big|\!\frac{\varnothing}{\varnothing}\, p,q}\,L\vee \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p \,\Big|\!\frac{\varnothing}{\varnothing}\, p}\,ax}{p \,\Big|\!\frac{\varnothing}{\varnothing}\, p,q}\,RW \qquad \dfrac{\dfrac{}{q \,\Big|\!\frac{\varnothing}{\varnothing}\, q}\,ax}{q \,\Big|\!\frac{\varnothing}{\varnothing}\, p,q}\,RW}{p \vee q \,\Big|\!\frac{\varnothing}{\varnothing}\, p,q}\,L\vee}{p \vee q \,\Big|\!\frac{\{p\vee q\}}{\{\neg\neg p\}}\, \neg p}\,\delta_{\mathcal{D}''}}{p \vee q \,\Big|\!\frac{\{p\vee q\}}{\{\neg\neg p\}}\, q}\,cut_{asa}$$

It is easy to see that closure conditions (vi) – (vii) do not suffice to absorb this Cut application without losing soundness.

(C) Let us employ the expression *analytic Cut* to denote a semi-analytic Cut where the Cut formula is a subformula of the formulas occurring on the left-hand side of the conclusion. If we replace $\mathsf{top}^{\star}_{\mathsf{c}}$ with $\mathsf{top}^{\star}_{\mathsf{r}}$ in condition (i), we can prove that all the instances of cut_{asa} which are not analytic can be eliminated. In view of Theorems 8.3, 8.4 and 8.5, this implies that $\mathsf{G4}^{\delta}$ calculi are non-analytic cut-free complete.

8.2.3. Structural properties. In this subsection, we make a structural analysis of $\mathsf{G4}^\delta$ calculi for default theories to prove strong adequacy with respect to m-credulous consequence. To this aim, let us introduce some terminology. A rule application is *safe* (under the empty set of constraints) whenever the existence of proofs of the premises entails the existence of proofs of the conclusion, the set of conditions of the conclusion is included in the union of the sets of conditions of the premises, and the set of constraints of the premises is the same as the set of constraints of the conclusion (the empty set of constraints, respectively). We state that a rule is *admissible* (under the empty set of constraints) whenever each application of the rule is safe (under the empty set of constraints). Moreover, we say that a rule is *invertible* (under the empty set of constraints) whenever any inverse of the rule is admissible (under the empty set of constraints).

Lemma 8.2. *The rules $R\wedge, R\vee, R\neg\wedge, R\neg\vee$ and $R\neg\neg$ are invertible in $\mathsf{G4}^\delta$.*

Proof. We reason by induction on the height of a $\mathsf{G4}^\delta$-proof π of the premise(s). Let us focus on the rule $R\wedge$, since the others are analogous. Moreover, let's take $B \wedge C$ to be the active formula in the premise of the inverse of $R\wedge$.

If $h(\pi) = 1$, the conclusion holds vacuously. Otherwise, we reason by cases over the last rule applied in π. If the last rule in π is $R\wedge$ and $B \wedge C$ is principal in it, we take the $\mathsf{G4}^\delta$-proofs of the premises of the latter, possibly followed by σ applications. Notice that the height of the resulting $\mathsf{G4}^\delta$-derivation may surpass the height of the $\mathsf{G4}^\delta$-proof of the premise of the inverse rule, due to the σ-applications. Since such σ-applications are safe, the resulting $\mathsf{G4}^\delta$-derivation is a proof. On the other hand, if the last rule in π is RW and $B \wedge C$ is principal in it, we replace such Weakening application with another one yielding B or C to reach the conclusion. If $B \wedge C$ is not principal in the last rule in π, we apply the inductive hypothesis to get $\mathsf{G4}^\delta$-derivations of the conclusions of the inverse rules. Remark that the application of the inductive hypothesis to any sequent above the conclusion in π does not revoke its soundness: as a result, such $\mathsf{G4}^\delta$-derivations are proofs – as desired. $\square$

Lemma 8.3. *The rule of Right Contraction*

$$\frac{\Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta, A, A}{\Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T}'}{\mathbf{S}}\; \Delta, A}\, RC$$

is admissible in $\mathsf{G4}^\delta$.

Proof. We reason by primary induction over the height of a $\mathsf{G4}^\delta$-proof π of the premise and secondary induction over the logical complexity of A. If $h(\pi) = 1$, we leverage conditions (i) and (v) of Definition 8.2 to reach the conclusion: as a matter of fact, $\mathsf{top}^\star_\mathsf{c}(\overset{*}{\vdash} B)$ is closed under Contraction for any formula B. Otherwise, we reason by cases over the last rule applied in π. If A is atomic and both occurrences are principal in the last rule , we leverage conditions $(v)-(vii)$ of Definition 8.2. If one occurrence of atomic A is principal in the last rule and the other not, the former occurrence of A is introduced by RW: it suffices to take the premise of the RW instance to conclude. If no occurrence of A is principal in the last rule , we apply the inductive hypothesis to the premises to permute upwards RC: notice that this can be done also in presence of cut_{asa}, due to its context-sharing formulation. Moreover, the application of the inductive hypothesis does always preserve soundness, since nothing on the left-hand side of involved sequents is modified. If A is not atomic, the only non-trivial cases arise whenever one occurrence of A is principal in the last rule . To deal with these configurations, we leverage Lemma 8.2: since the invertibility of right rules does not preserve height, we are forced to apply the secondary inductive hypothesis to obtain the conclusion. ☐

Lemma 8.4. *The rules $L\wedge, L\neg\vee$ and $L\neg\neg$ are invertible in $\mathsf{G4}^{\delta}$.*

Proof. We reason by induction on the height of a $\mathsf{G4}^{\delta}$-proof π of the premise. We argue as in the proof of Lemma 8.2. ☐

Lemma 8.5. *The logical rules of $\mathsf{G4}^{\delta}$ are height-preserving invertible under the empty set of constraints.*

Proof. We reason by induction on the height of a $\mathsf{G4}^{\delta}$-proof π of the premise. We argue as in the proof of Lemma 8.2: since we deal with empty sets of constraints and conditions, the height of the $\mathsf{G4}^{\delta}$-proof of the conclusion of the inverse rule does not surpass the height of the $\mathsf{G4}^{\delta}$-proofs of the premise. ☐

Lemma 8.6. *The rules of Left Contraction*

$$LC\,\frac{A, A, \Gamma\ {}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\ \Delta}{A, \Gamma\ {}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\ \Delta}$$

is admissible in $\mathsf{G4}^{\delta}$ under the empty set of constraints.

Proof. We reason by induction on the height of the $\mathsf{G4}^{\delta}$-proof π of the premise. As usual, we exploit Lemma 8.5 whenever one of the two occurrences of the contracted formula is principal in the last rule in π and the latter is a logical rule. Notice that we do not need secondary induction over the logical complexity of A, due to the fact invertibility is height-preserving under the empty sets of constraints. ☐

Theorem 8.1. *The rule of Cut*

$$\frac{\Gamma\ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right.\ A, \Delta \qquad A, \Pi\ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right.\ \Sigma}{\Pi, \Gamma\ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right.\ \Delta, \Sigma}\,Cut$$

is admissible in $\mathsf{G4}^{\delta}$.

Proof. We argue as in [129]. ☐

Proposition 8.4. The Left and Right Negation rules

$$L\neg\,\dfrac{\Gamma \ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right. \Delta, A}{\neg A, \Gamma \ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right. \Delta} \qquad \dfrac{A, \Gamma \ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right. \Delta}{\Gamma \ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right. \Delta, \neg A}\,R\neg$$

are admissible and invertible in $\mathsf{G4}^{\delta}$.

Proof. We leverage Theorem 8.1 as follows:

$$Cut\,\dfrac{\Gamma \ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right. \Delta, A \qquad \begin{matrix}\vdots\\ A, \neg A \ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right.\end{matrix}}{\neg A, \Gamma \ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right. \Delta} \qquad \dfrac{\begin{matrix}\vdots\\ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right. A, \neg A\end{matrix} \qquad A, \Gamma \ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right. \Delta}{\Gamma \ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right. \Delta, \neg A}\,Cut$$

☐

Theorem 8.2. *$\mathsf{G4}^{\delta}$ proves $\Gamma \ {}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right. \Delta$ if and only if $\bigvee \Delta$ belongs to $Cn(\mathcal{W} \cup \Gamma)$.*

Proof. We argue as in [129]. ☐

Example 8.2. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be defined as follows:*

$$\mathcal{W} = \varnothing$$

$$\mathcal{D} = \left\{ \frac{\top : p}{p}, \frac{p \vee q : \neg p}{\neg p} \right\}$$

Consider the following $\mathsf{G4}^{\delta}$-derivation π:

$$\dfrac{LW\,\dfrac{\delta_{\mathcal{D}'}\,\dfrac{}{\left|\frac{\top}{\{\neg p\}}\right. p,q}}{p\vee q \left|\frac{\top}{\{\neg p\}}\right. p,q} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p \left|\frac{\varnothing}{\varnothing}\right. p}\,ax}{p \left|\frac{\varnothing}{\varnothing}\right. p,q}\,RW \qquad \dfrac{\dfrac{}{q \left|\frac{\varnothing}{\varnothing}\right. q}\,ax}{q \left|\frac{\varnothing}{\varnothing}\right. p,q}\,RW}{p\vee q \left|\frac{\varnothing}{\varnothing}\right. p,q}\,L\vee}{p\vee q \left|\frac{\{p\vee q\}}{\{\neg\neg p\}}\right. \neg p}\,\delta_{\mathcal{D}''}}{p\vee q \left|\frac{\{\top,\ p\vee q\}}{\{\neg p, \neg\neg p\}}\right. q}\,cut_{asa}$$

By Definition 8.4, π is a $\mathsf{G4}^{\delta}$-paraproof: the conclusion is sound with respect to conditions but unsound with respect to constraints. Hence, the cut_{asa} application in π is unsafe. On the other hand, consider the following $\mathsf{G4}^{\delta}$-derivation ρ:

$$\dfrac{L\vee\,\dfrac{RW\,\dfrac{ax\,\dfrac{}{p \left|\frac{\varnothing}{\varnothing}\right. p}}{p \left|\frac{\varnothing}{\varnothing}\right. p,q} \qquad RW\,\dfrac{ax\,\dfrac{}{q \left|\frac{\varnothing}{\varnothing}\right. q}}{q \left|\frac{\varnothing}{\varnothing}\right. p,q}}{p\vee q \left|\frac{\varnothing}{\varnothing}\right. p,q} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p \left|\frac{\varnothing}{\varnothing}\right. p}\,ax}{p \left|\frac{\varnothing}{\varnothing}\right. p,q}\,RW \qquad \dfrac{\dfrac{}{q \left|\frac{\varnothing}{\varnothing}\right. q}\,ax}{q \left|\frac{\varnothing}{\varnothing}\right. p,q}\,RW}{p\vee q \left|\frac{\varnothing}{\varnothing}\right. p,q}\,L\vee}{p\vee q \left|\frac{\{p\vee q\}}{\{\neg\neg p\}}\right. \neg p}\,\delta_{\mathcal{D}''}}{p\vee q \left|\frac{\{p\vee q\}}{\{\neg\neg p\}}\right. q}\,cut_{asa}$$

By Definition 8.4, ρ is a $\mathsf{G4}^{\delta}$-proof. Unlike the cut_a application in π, the cut_{asa} application in ρ is safe.

THEOREM 8.3. *Let A be an atom occurring in some formula in $\Gamma, \Pi, \mathcal{W}$. The rule of safe, atomic semi-analytic Cut*

$$\dfrac{\Gamma\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T_1}}{\mathbf{S_1}}\ \Delta, A \qquad \Pi\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T_2}}{\mathbf{S_2}}\ \Sigma, \neg A}{\Pi, \Gamma\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T}}{\mathbf{S}}\ \Delta, \Sigma}\ cut_a$$

is admissible in $\mathsf{G4}^{\delta}$.

PROOF. For any such Cut application, consider the following $\mathsf{G4}^{\delta}$-derivation:

$$\dfrac{RW\,\dfrac{LW\,\dfrac{\Gamma\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T_1}}{\mathbf{S_1}}\ \Delta, A}{\Pi, \Gamma\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T_1}}{\mathbf{S_1}}\ \Delta, A}}{\Pi, \Gamma\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T_1}}{\mathbf{S_1}}\ \Delta, \Sigma, A} \qquad \dfrac{\dfrac{\Pi\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T_2}}{\mathbf{S_2}}\ \Sigma, \neg A}{\Pi, \Gamma\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T_2}}{\mathbf{S_2}}\ \Sigma, \neg A}\,LW}{\Pi, \Gamma\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T_2}}{\mathbf{S_2}}\ \Delta, \Sigma, \neg A}\,RW}{\Pi, \Gamma\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T}}{\mathbf{S}}\ \Delta, \Sigma}\ cut_{asa}$$

By hypothesis, all *LW* applications are safe: hence, such $\mathsf{G4}^{\delta}$-derivation is a proof. ☐

DEFINITION 8.5. Let π be a $\mathsf{G4}^{\delta}$-derivation of $\Gamma\ \Big|\!\tfrac{\mathbf{T}}{\mathbf{S}}\ \Delta$ and A be a formula occurrence in Γ (Δ). The *predecessors tree* of A in π, in symbols $\mathcal{T}_{\pi}(A)$ is a labelled tree generated according to the following procedure.

- (i) The root of $\mathcal{T}_{\pi}(A)$ is labelled by the occurrence of A in Γ (Δ).
- (ii) Let ν be a node of $\mathcal{T}_{\pi}(A)$ is labelled by an occurrence of A in the antecedent Π (succedent Σ, respectively) of a sequent $\Pi\ \Big|\!\tfrac{\mathbf{T'}}{\mathbf{S'}}\ \Sigma$ occurring in π, and † be the last rule in the $\mathsf{G4}^{\delta}$-subderivation of π concluding $\Pi\ \Big|\!\tfrac{\mathbf{T'}}{\mathbf{S'}}\ \Sigma$.
  - $(ii.i)$ If the occurrence of A is principal in †, ν has no immediate children.
  - $(ii.ii)$ If the occurrence of A is not principal in †, ν has (at most) two immediate children, and their labels are the instances of A occurring in the antecedents (succedents, respectively) of the premises of † which correspond to the occurrence of A.

THEOREM 8.4. *Let A be an atom which does not occur in any formula in $\Gamma, \Pi, \mathcal{W}$. The rule of safe, atomic non-semi-analytic Cut*

$$\dfrac{\Gamma\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T_1}}{\mathbf{S_1}}\ \Delta, A \qquad \Pi\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T_2}}{\mathbf{S_2}}\ \Sigma, \neg A}{\Pi, \Gamma\ {}_{\mathsf{F}}\Big|\!\tfrac{\mathbf{T}}{\mathbf{S}}\ \Delta, \Sigma}\ cut_a$$

is admissible in $\mathsf{G4}^{\delta}$.

Proof. We focus on the topmost safe, atomic non-semi-analytic Cut application, reasoning by induction on the sum of the heights of the $\mathsf{G4}^{\delta}$-proofs π_1, π_2 of the left and right premise, respectively. In the base case, $h(\pi_1) = h(\pi_2) = 1$: since A does not occur in any formula in $\mathcal{W}$, it cannot be the case that last rule in π_1, π_2 is an extra-logical instance of ax. Moreover, if the last rule in π_1 or π_2 is a logical instance of ax, the conclusion is immediate. Hence, the only non-trivial scenario arises whenever the last rules applied in π_1 and π_2 are extra-logical ones: we leverage condition (vii) of Definition 8.2 to conclude. To prove the induction step, we reason by cases over the last rule applied in π_1.

If A is principal in the last rule in π_1, the only non-trivial cases arise whenever the last rule in π_1 is extra-logical. To complete the proof, we must distinguish subcases according to the last rule in π_2.

(a) If $\neg A$ is principal in the last rule in π_2, the latter is either RW or an extra-logical rule δ. In the first scenario, we apply LW, RW and σ to obtain a $\mathsf{G4}^{\delta}$-derivation of $\Pi, \Gamma \; {}_{\mathsf{F}}\!\left|\frac{\mathbf{T_2}}{\mathbf{S}}\right. \Delta, \Sigma$: it is easy to check that this derivation is a $\mathsf{G4}^{\delta}$-proof. In the second scenario, we leverage condition (vii) of Definition 8.2 to reach the conclusion: again, the application of δ is safe by construction.

(b) If $\neg A$ is not principal in the last rule in π_2 and latter is neither $L\vee$ nor $L\neg\wedge$, we simply apply the inductive hypothesis to its premises to permute upwards the Cut application: notice that soundness under constraints is always preserved. If the last rule in π_2 is either $L\vee$ or $L\neg\wedge$, the same upwards permutation of Cut may fail, due to the fact that soundness under constraints may not be preserved. For instance, consider the following configuration – where B^f belongs to $\bigcup \mathbf{S_1}$, $\Pi = \Pi', B \vee C$, $\mathbf{T_2} = \mathbf{V_1} \cup \mathbf{V_2}$ and $\mathbf{S_2} = \mathbf{U_1} \cup \mathbf{U_2}$:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \; {}_{\mathsf{F}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. \Delta, A\end{array} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ B, \Pi' \; {}_{\mathsf{F}}\!\left|\frac{\mathbf{V_1}}{\mathbf{U_1}}\right. \Sigma, \neg A & C, \Pi' \; {}_{\mathsf{F}}\!\left|\frac{\mathbf{V_2}}{\mathbf{U_2}}\right. \Sigma, \neg A\end{array}}{B \vee C, \Pi' \; {}_{\mathsf{F}}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \Sigma, \neg A}\, L\vee}{B \vee C, \Pi', \Gamma \; {}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta, \Sigma}\, cut_a$$

To reach the conclusion, we consider $\mathcal{T}_{\pi_2}(\neg A)$. If the label of any leaf in $\mathcal{T}_{\pi_2}(\neg A)$ is introduced by an RW application, we remove such RW applications altogether, thus obtaining a $\mathsf{G4}^{\delta}$-proof π_2' of $\Pi \; {}_{\mathsf{F}}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \Sigma$: we apply LW, RW and σ to conclude. Otherwise, there must exist one extra-logical rule applied in π_2. By contradiction, suppose this is not the case: from the fact that A does not occur in $\mathcal{W}$ we infer the existence of (at least) one leaf in $\mathcal{T}_{\pi_2}(\neg A)$ labelled by an occurrence of $\neg A$ introduced by a logical instance of ax. This implies that $\neg A$ occurs in Π – contrary to the hypothesis.

In the scenario where there exists (at least) one extra-logical rule applied in π_2, we globally rewrite π_2 and proceed by secondary induction over the number of logical rules applied below the topmost applications extra-logical rules in order to complete the proof. Starting from the topmost applications of extra-logical rules in π_2, soundness under conditions of $\Pi, \Gamma \ {}_{\mathsf{F}}|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Delta, \Sigma$ ensures that $\overline{\overline{\mathsf{G4^{pn}}}}$ proves $W, \Pi \vdash B$, with B being any of the premises of the corresponding default rules. Hence, B belongs to $Cn(\mathcal{W} \cup \Pi)$: Theorem 8.2 guarantees that $\mathsf{G4}^{\delta}$ proves $\Pi \ {}_{\mathsf{F}}|\!\frac{\varnothing}{\varnothing}\ \Theta_i$, for any $|\!\frac{*}{\;}\ \Theta_i$ belonging to $\mathsf{top_r}(|\!\frac{*}{\;}\ B)$. Now, we apply the extra-logical rule to the premises $\Pi \ {}_{\mathsf{F}}|\!\frac{\varnothing}{\varnothing}\ \Theta_i$: soundness under constraints of $\Pi, \Gamma \ {}_{\mathsf{F}}|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Delta, \Sigma$ ensures that the application is safe. Subsequently, we apply all the right rules that are applied below such extra-logical rule application in π_2: notice that instances of cut_{asa} are preserved, and that any sequent thus inferred is sound. The same procedure is applied to any successive extra-logical rule application in π_2: the $\mathsf{G4}^{\delta}$-proof π_2' thus obtained infers $\Pi \ {}_{\mathsf{F}}|\!\frac{\mathbf{T_2'}}{\mathbf{S_2}}\ \Sigma, \neg A$ with a lower number of logical rules under the topmost extra-logical rules. At this stage, we apply the secondary inductive hypothesis – and we are done.

If A is not principal in the last rule in π_1 and latter is neither $L\vee$ nor $L\neg\wedge$, we apply (at most, twice) the inductive hypothesis to conclude. If the last rule in π_1 is (say) $L\vee$, we focus on $\mathcal{T}_{\pi_1}(A)$. If each leaf of $\mathcal{T}_{\pi_1}(A)$ is labelled by an occurrence of A introduced by RW, we remove such RW applications to get a $\mathsf{G4}^{\delta}$-proof of $\Gamma \ {}_{\mathsf{F}}|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\ \Delta$: we apply LW, RW and σ to obtain the conclusion. Otherwise, we reason by contradiction (as we did in (b) with π_2) to infer the existence of (at least) one extra-logical rule applied in π_1. We globally rewrite π_1 (again, as in (b) above) to lower the number of logical rules applied below the topmost applications extra-logical rules: we proceed by secondary induction over the number of such rules to complete the proof. □

Lemma 8.7. *Let $i = 1, 2$. The rules*

$$E\vee\ \frac{A_i, A_1 \vee A_2, \Gamma \ {}_{\mathsf{F}}|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Delta}{A_i, A_i, \Gamma \ {}_{\mathsf{F}}|\!\frac{\mathbf{T'}}{\mathbf{S}}\ \Delta} \qquad \frac{\neg A_i, \neg(A_1 \wedge A_2), \Gamma \ {}_{\mathsf{F}}|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Delta}{\neg A_i, \neg A_i, \Gamma \ {}_{\mathsf{F}}|\!\frac{\mathbf{T'}}{\mathbf{S}}\ \Delta}\ E\neg\wedge$$

are admissible in $\mathsf{G4}^{\delta}$.

Proof. We focus on $E\vee$, since the argument for $E\neg\wedge$ is completely analogous. We reason by induction on the height of the $\mathsf{G4}^{\delta}$-proof π of the premise. We assume that $\mathbf{S}, \mathbf{T} \neq \varnothing$: if $\mathbf{S} = \mathbf{T} = \varnothing$, we apply Lemma 8.5 to get the conclusion.

If $h(\pi) = 1$, the last rule is a 0-ary extra-logical rule: we simply replace $A_1 \vee A_2$ with A_i to obtain the conclusion. If $h(\pi) > 1$, we reason by cases over the last rule applied in π. If the latter is $L\vee$ and $A_1 \vee A_2$ is principal in it, we perform the following transformation on π:

$$L\vee\ \begin{array}{cc} \vdots & \vdots \\ A_i, A_i, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\; \Delta & A_i, A_{3-i}, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\; \Delta \\ \hline \multicolumn{2}{c}{A_i, A_1 \vee A_2, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta} \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \vdots \\ A_i, A_i, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\; \Delta \\ \hline\hline A_i, A_i, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S}}\; \Delta \end{array}\ \sigma$$

If the last rule in π is LW and $A_1 \vee A_2$ is principal in it, we perform the following transformation:

$$LW\ \begin{array}{c} \vdots \\ A_i, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta \\ \hline A_i, A_1 \vee A_2, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \vdots \\ A_i, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta \\ \hline A_i, A_i, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta \end{array}\ LW$$

If a formula in Γ or Δ is principal in the last rule in π, we apply the inductive hypothesis to the premises of the latter to conclude.

If the last rule in π is cut_{asa}, the application of the inductive hypothesis to the premises may not be possible: if the atomic cut formula is a subformula of A_{3-i} without occurring in any formula in $\Gamma \cup [A_i] \cup \mathcal{W}$, the side condition for the applicability of cut_{asa} is no more satisfied. In this scenario, we perform the upwards permutation of $E\vee$: next, we replace the cut_{asa} application with an instance of cut_a. It suffices to exploit Theorem 8.4 to reach the conclusion.

If A_i is principal in the last rule in π, we cannot apply the inductive hypothesis to reach the conclusion: instead, we follow another argument, based on the fact that the premise $A_i, A_1 \vee A_2, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta$ is sound under conditions. Definition 8.3 ensures that $([W, E_\rho] \cup [A_i] \cup [A_1 \vee A_2] \cup \Gamma) \mid\mid \langle \mathbf{V}, \varnothing \rangle$ for any subderivation ρ of π with $\Pi \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{V}}{\mathbf{U}}\; \Sigma$ as conclusion. Starting from the topmost applications of extra-logical rules in π, if Θ is the succedent of any premise of such applications, we leverage Theorem 8.2 to infer that $\mathsf{G4}^\delta$ proves $A_i, A_i, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\varnothing}{\varnothing}\; \Theta$. Hence, we apply the corresponding extra-logical rules to these sequents: since $([W, D_\pi] \cup [A_i] \cup [A_1 \vee A_2] \cup \Gamma) \mid\mid \langle \varnothing, \mathbf{S} \rangle$, Lemma 8.1 ensures that such applications are safe. Next, we apply all the rules applied in π below such extra-logical rules instances, with the exception of left (logical and structural) rules. Theorem 8.4 ensures that any application of cut_{asa} which is not preserved by the deletion of A_{3-i} can be replaced by an instance of cut_a. Moreover, the fact that $([W, D_\pi] \cup [A_i] \cup [A_1 \vee A_2] \cup \Gamma) \mid\mid \langle \varnothing, \mathbf{S} \rangle$, together with Lemma 8.1, guarantees that all σ applications remain safe. Such global rewriting of π yields a $\mathsf{G4}^\delta$-proof of $A_i, A_i, \Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta$. $\square$

LEMMA 8.8. *The rule of Left Contraction*

$$LC\ \begin{array}{c} A, A, \Gamma \;\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta \\ \hline A, \Gamma \;\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta \end{array}$$

is admissible in $\mathsf{G4}^\delta$.

Proof. We focus on Left Contraction, arguing by primary induction on the height of the $\mathsf{G4}^{\delta}$-proof π of the premise and secondary induction on the logical complexity of A. If $h(\pi) = 1$, we leverage conditions (i) and (v) in Definition 8.2 to reach the conclusion. If $h(\pi) > 1$, we reason by cases over the last rule applied in π. If no occurrence of A is principal in the last rule , we simply apply the inductive hypothesis to the premises. If one occurrence of A is principal in the last rule , we proceed by cases over A's principal connective. If A is a literal, then A is introduced by an LW application: it suffices to remove the latter to conclude. If A has either the form $B \wedge C$, or the form $\neg(B \vee C)$, or the form $\neg\neg B$, we leverage Lemma 8.4 to apply the secondary inductive hypothesis and reach the conclusion. On the other hand, if A has either the form $B \vee C$ or the form $\neg(B \wedge C)$, we exploit Lemma 8.7 to apply the secondary inductive hypothesis and complete the proof. ☐

Theorem 8.5. *The rule of safe Cut*

$$\frac{\Gamma \mathrel{\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}} \Delta, A \qquad \Pi \mathrel{\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}} \Sigma, \neg A}{\Gamma, \Pi \mathrel{\Big|\!\frac{\mathbf{T}}{\mathbf{S}}} \Delta, \Sigma}\, cut$$

is admissible in $\mathsf{G4}^{\delta}$.

Proof. We reason by induction over the complexity of A. If A is a literal, Theorems 8.3 and 8.4 ensure the conclusion. If A is not a literal, we reason by cases over A's principal connective. We leverage Lemma 8.2 to lower the complexity of the Cut formula: to reach the conclusion, we apply Lemmas 8.3 and 8.8. ☐

Proposition 8.5. The rules of Left Negated Contraction

$$LNC\, \frac{A, \Gamma \mathrel{\Big|\!\frac{\mathbf{T}}{\mathbf{S}}} \Delta, \neg A}{A, \Gamma \mathrel{\Big|\!\frac{\mathbf{T}}{\mathbf{S}}} \Delta} \qquad \frac{\neg A, \Gamma \mathrel{\Big|\!\frac{\mathbf{T}}{\mathbf{S}}} \Delta, A}{\neg A, \Gamma \mathrel{\Big|\!\frac{\mathbf{T}}{\mathbf{S}}} \Delta}\, LNC$$

are admissible in $\mathsf{G4}^{\delta}$.

Proof. Immediate from Theorem 8.5 and Lemma 8.8. ☐

Before proceeding to the proof of strong adequacy for $\mathsf{G4}^{\delta}$ calculi, we show that the latter are expressive enough to formalize weak cumulativity properties satisfied by modified credulous consequence. First, let us consider the following Weak Cumulative Cut rule – a special case of the standard Cut rule:

$$\frac{\Gamma \;{}_{\mathsf{F}}\!\mathrel{\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}} A \qquad A, \Pi \;{}_{\mathsf{F}}\!\mathrel{\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}} \Sigma}{\Pi, \Gamma \;{}_{\mathsf{F}}\!\mathrel{\Big|\!\frac{\mathbf{T}}{\mathbf{S}}} \Sigma}\, wcCut$$

Theorem 8.6. *The rule of safe Weak Cumulative Cut is admissible in* $\mathsf{G4}^{\delta}$.

Proof. We focus on a topmost Weak Cumulative Cut application, reasoning by primary induction on the logical complexity of the Cut formula A and secondary induction on the sum of the heights of the $\mathsf{G4}^{\delta}$-proofs π_1, π_2 of the left and right premise, respectively. In view of Theorem 8.1, we assume that either $\mathbf{S} \neq \varnothing$ or $\mathbf{T} \neq \varnothing$. If A is a literal, we reason by cases over the last rule in π_1. Here, we focus on the most meaningful cases

- (a) If the last rule in π_1 is ax, then by hypothesis, the last rule in π_2 is not ax. If the last rule in π_2 is extra-logical, apply the secondary inductive hypothesis m times to the premises to obtain the conclusion: if $m = 0$, replace the δ-application concluding $A, \Pi \;{}_{\mathsf{F}}\big|^{\mathbf{T_2}}_{\mathbf{S_2}}\; \Sigma$ with one concluding $\Pi, \Gamma \;{}_{\mathsf{F}}\big|^{\mathbf{T_2}}_{\mathbf{S_2}}\; \Sigma$, (possibly) followed by σ-applications yielding $\Pi, \Gamma \;{}_{\mathsf{F}}\big|^{\mathbf{T_2}}_{\mathbf{S}}\; \Sigma$. It is easy to see that all rule applications in the $\mathsf{G4}^{\delta}$-derivation thus obtained are safe. If the last rule in π_2 is LW and A principal in it, we replace the Weakening application with LW applications adding the formulas in Γ (if any), plus σ applications to add sets in $\mathbf{S}$ not occurring in $\mathbf{S_2}$ (if any). If the last rule in π_2 is cut_{asa}, we apply the secondary inductive hypothesis to its premises: if the side condition for the applicability of cut_{asa} is not fulfilled, we leverage Theorem 8.4 to reach the conclusion. If the last rule in π_2 is either $L\vee$ or $L\neg\wedge$, we may not be able to apply the secondary inductive hypothesis without losing soundness under constraints. If no extra-logical rule is applied in π_2, we remove any σ application from π_2 and apply Lemma 8.4 to infer the existence of a $\mathsf{G4}^{\delta}$-proof of $\Pi \;{}_{\mathsf{F}}\big|^{\varnothing}_{\varnothing}\; \Sigma, \neg A$. We apply σ to obtain a $\mathsf{G4}^{\delta}$-derivation of $\Pi \;{}_{\mathsf{F}}\big|^{\varnothing}_{\mathbf{S_2}}\; \Sigma, \neg A$: it is immediate to verify that these σ-application are safe, and thus that such derivation is a proof. If there exists (at least) one extra-logical rule applied in π_2, we globally rewrite π_2 to obtain a $\mathsf{G4}^{\delta}$-proof of $\Pi, \Gamma \;{}_{\mathsf{F}}\big|^{\mathbf{T_2}}_{\mathbf{S_2}}\; \Sigma$ (following the same procedure as in the proof of Theorem 8.4, point (b)).
- (b) If the last rule in π_1 is extra-logical and the last rule in π_2 is ax, we apply Lemma 8.4 and Theorem 8.5 to conclude. All other possible scenarios are treated as in (a) above.
- (c) If the last rule in π_1 is either $L\vee$ or $L\neg\wedge$, we may not be able to apply the secondary inductive hypothesis to both premises to reach the conclusion. Instead, we globally rewrite π_2 to obtain a $\mathsf{G4}^{\delta}$-proof of $\Pi, \Gamma \;{}_{\mathsf{F}}\big|^{\mathbf{T_2}}_{\mathbf{S_2}}\; \Sigma$ (again, following the same procedure as in the proof of Theorem 8.4, point (b))

If A is not a literal, we reason by cases over A's principal connective. If A is either $B \wedge C$ or $\neg(B \vee C)$ or $\neg\neg B$, we leverage Lemmas 8.2 and 8.4 to apply (at most, twice) the primary inductive hypothesis, and then Lemmas 8.3 and 8.8 to reach the conclusion. If A is either $B \vee C$ or $\neg(B \wedge C)$, and π_2 does not contain applications of extra-logical rules, we remove all σ applications (if any) and then apply Lemma 8.4 and Theorem 8.5 to obtain a $\mathsf{G4}^{\delta}$-proof of $\Pi, \Gamma \;{}_{\mathsf{F}}\!\vert\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\; \Sigma$: we employ safe σ applications to prove $\Pi, \Gamma \;{}_{\mathsf{F}}\!\vert\!\frac{\mathbf{T_1}}{\mathbf{S}}\; \Sigma$. On the other hand, if π_2 contains applications of extra-logical rules, we globally rewrite π_2 exploiting soundness under conditions to obtain a $\mathsf{G4}^\delta$-proof of $\Pi, \Gamma \;{}_{\mathsf{F}}\!\vert\!\frac{\mathbf{T_2}}{\mathbf{S}}\; \Sigma$ (again, following the same procedure as in the proof of Theorem 8.4, point (b)). □

PROPOSITION 8.6. The rules of Right Negated Contraction

$$RNC\,\dfrac{A, \Gamma \vert\!\frac{\varnothing}{\varnothing}\; \Delta, \neg A}{\Gamma \vert\!\frac{\varnothing}{\varnothing}\; \Delta, \neg A} \qquad \dfrac{\neg A, \Gamma \vert\!\frac{\varnothing}{\varnothing}\; \Delta, A}{\Gamma \vert\!\frac{\varnothing}{\varnothing}\; \Delta, A}\,RNC$$

are admissible in $\mathsf{G4}^\delta$.

PROOF. We exploit Theorem 8.6 as follows (the other case is completely analogous):

$$\dfrac{\begin{array}{c}\vdots\\ {}_{\mathsf{F}}\!\vert\!\frac{\varnothing}{\varnothing}\; A \vee \neg A\end{array} \qquad \dfrac{A, \Gamma \;{}_{\mathsf{F}}\!\vert\!\frac{\varnothing}{\varnothing}\; \Delta, \neg A \qquad \begin{array}{c}\vdots\\ \neg A, \Gamma \;{}_{\mathsf{F}}\!\vert\!\frac{\varnothing}{\varnothing}\; \Delta, \neg A\end{array}}{A \vee \neg A, \Gamma \;{}_{\mathsf{F}}\!\vert\!\frac{\varnothing}{\varnothing}\; \Delta, \neg A}\,L\vee}{\Gamma \;{}_{\mathsf{F}}\!\vert\!\frac{\varnothing}{\varnothing}\; \Delta, \neg A}\,wcCut$$

□

Next, let us consider the following, weak version of the principle of Cautious Monotony:

THEOREM 8.7. *The rule of safe Weak Cautious Monotony*

$$\dfrac{\Gamma \;{}_{\mathsf{F}}\!\vert\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\; \Delta_1 \qquad \Pi \;{}_{\mathsf{F}}\!\vert\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\; \Delta_2}{\Gamma, \bigvee \Delta_1 \vee \bigvee \Delta_2, \Pi \;{}_{\mathsf{F}}\!\vert\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta_2}\,wcMon$$

is admissible in $\mathsf{G4}^\delta$.

PROOF. We reason by induction over the sum of the heights of the $\mathsf{G4}^\delta$-proofs π_1, π_2 of the left and right premise, respectively. If $h(\pi_1) = h(\pi_2) = 1$, the only non-trivial case arise when (say) the last rule in π_1 is an extra-logical rule δ: we reason by cases over the last rule in π_2 to conclude. If the last rule in π_2 is ax, it suffices to apply LW and σ to obtain a $\mathsf{G4}^\delta$-derivation of $\bigvee \Delta_1 \vee \bigvee \Delta_2, \Gamma, \Pi \;{}_{\mathsf{F}}\!\vert\!\frac{\varnothing}{\mathbf{S}}\; \Delta_2$: it is easy to verify that such derivation is a proof. On the other hand, if the last rule in π_2 is an extra-logical rule δ', we replace the application of the latter with another one, whose conclusion has the formulas $\bigvee \Delta_1 \vee \bigvee \Delta_2, \Gamma, \Pi$ in the antecedent. Hence, we apply σ to get the sequent $\bigvee \Delta_1 \vee \bigvee \Delta_2, \Gamma, \Pi \;{}_{\mathsf{F}}\!\vert\!\frac{\varnothing}{\mathbf{S}}\; \Delta_2$: soundness under conditions is obviously preserved; Lemma 8.1 ensures that also soundness under constraints is preserved. If $h(\pi_1) + h(\pi_2) > 2$, we reason by cases over the last rule in π_2. If the latter is such that a formula in Π is principal, it suffices to apply the inductive hypothesis to the premises. We argue in the same way whenever the last rule in π_2 is cut_{asa}. If the last rule is such that (at least) one formula in Δ_2 is principal in the last rule , the only non-trivial case arises when the latter is an extra-logical rule δ. In this scenario, we apply LW and σ to the premises of δ, and then apply δ: Lemma 8.1 preserves soundness under constraints – and thus the resulting $\mathsf{G4}^{\delta}$-derivation is a proof. □

Now, we are ready to present the proof of the main result of this section:

Theorem 8.8. *Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be a default theory. Then there exists a $\mathsf{G4}^{\delta}$-proof π of $\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta$, for some $\mathbf{S}$ and $\mathbf{T}$, iff $\bigvee \Delta$ is an m-credulous consequence of $\mathcal{W} \cup \Gamma$.*

Proof. ($\Rightarrow$) We reason by induction on the height of π to establish the existence of a modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W} \cup \Gamma, def(\pi) \rangle$ such that $\bigvee \Delta$ belongs to $\mathcal{E}$: by Proposition 5.5, this suffices to the conclusion.

If $h(\pi) = 1$, the only non-trivial case arises whenever the last rule is an extra-logical rule $\delta_{\mathcal{D}'}$. By conditions $(v) - (vii)$ in Definition 8.2, $\bigvee \Delta$ belongs to $Cn(D)$, where D is the conclusion of the single default rule in $\mathcal{D}'$. From the fact that $\Gamma \left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta$ is sound under constraints we infer that $\bigvee \Delta$ belongs to a modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W} \cup \Gamma, \mathcal{D}' \rangle$. If $h(\pi) > 1$, we reason by cases over the last rule in π. If the latter is RW, σ or a unary logical rule, we apply the inductive hypothesis to the premises to reach the conclusion: for left logical rules, we leverage the fact that any modified extension of $\langle \mathcal{W}', def(\pi) \rangle$ is a modified extension of $\langle \mathcal{W}'', def(\pi) \rangle$ and vice versa, whenever $\mathcal{W}'$ is classically equivalent to $\mathcal{W}''$. If the last rule in π is LW, the formula A is principal in it and $\Gamma = \Gamma' \cup [A]$, we apply the inductive hypothesis to infer the existence of a modified extension $\langle \mathcal{E}, \mathcal{F} \rangle$ of $\langle \mathcal{W} \cup \Gamma', def(\pi) \rangle$: from the fact that $A, \Gamma' \left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta$ is sound under constraints we obtain the conclusion. If the last rule in π is an extra-logical rule $\delta_{\mathcal{D}'}$ and $\delta_{\mathcal{D}'}$ is generated according to point (v) in Definition 8.2, π has the following form:

$$\dfrac{\begin{array}{c}\vdots_{\pi_1} \\ \Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. \Theta_1\end{array} \qquad \cdots \qquad \begin{array}{c}\vdots_{\pi_m} \\ \Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_m}}{\mathbf{S_m}}\right. \Theta_m\end{array}}{\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Phi}\,\delta_{\mathcal{D}'}$$

with $m > 1$. By Definitions 8.1 and 8.3, the fact that $\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Phi$ is sound under conditions entails that $\overline{\overline{\mathsf{G4}^{\mathsf{pn}}}}$ proves $W, D'_{\pi}, \Gamma \vdash \bigwedge_{i=1}^{m}(\bigvee \Theta_i)$: by definition, this means that $\overline{\overline{\mathsf{G4}^{\mathsf{pn}}}}$ proves $W, \bigwedge concl(def(\pi_1) \cup \cdots \cup def(\pi_m)), \Gamma \vdash \bigwedge_{i=1}^{m}(\bigvee \Theta_i)$. On the other hand, $\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Phi$ is sound under constraints: Definitions 8.1, 8.3 and Lemma 8.1 ensure the existence of a modified extension $\langle \mathcal{E}', \mathcal{F}' \rangle$ of $\langle \mathcal{W} \cup \Gamma, def(\pi_1) \cup \cdots \cup def(\pi_m) \rangle$ such that $\mathcal{E}' = Cn(\mathcal{W} \cup concl(def(\pi_1) \cup \cdots \cup def(\pi_m)))$. As a result, we have that $\bigwedge_{i=1}^{m}(\bigvee \Theta_i)$ belongs to $\mathcal{E}'$: hence, the default rule in $\mathcal{D}'$ is triggered in $\mathcal{E}'$. The soundness of $\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Phi$ under constraints guarantees the existence of a modified extension $\langle \mathcal{E}'', \mathcal{F}'' \rangle$ of $\langle \mathcal{W} \cup \Gamma, def(\pi) \rangle$ such that the conclusion of the default rule belongs to $\mathcal{E}''$: by condition (v) of Definition 8.2 and closure under *modus ponens* of $\mathcal{E}''$ we infer that $\bigvee \Phi$ belongs to $\mathcal{E}''$. The arguments employed in the scenario where $\delta_{\mathcal{D}'}$ is generated according to points $(vi) - (vii)$ is analogous. If the last rule in π is a binary right logical rule or cut_{asa}, we apply twice the inductive hypothesis and leverage the soundness of $\Gamma \;{}_{\mathsf{F}}\!\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta$ to obtain the conclusion. If the last rule in π is $L\vee$ or $L\neg\wedge$, then π has the following form:

$$\dfrac{\begin{matrix}\vdots \\ B, \Gamma \;{}_{\mathsf{F}}\!\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\; \Delta\end{matrix} \qquad \begin{matrix}\vdots \\ C, \Gamma \;{}_{\mathsf{F}}\!\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\; \Delta\end{matrix}}{B \vee C, \Gamma \;{}_{\mathsf{F}}\!\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; \Delta}\; L\vee$$

If no extra-logical rule is applied in π, Theorem 8.2 ensures that $\bigvee \Delta$ belongs to $Cn(\mathcal{W} \cup \Gamma)$, and thus to any modified extension of $\langle \mathcal{W} \cup \Gamma, \mathcal{D} \rangle$. If there exists (at least) one extra-logical rule applied in π, we globally rewrite π so as to permute $L\vee$ above any extra-logical rule application, thus obtaining a $\mathsf{G4}^{\delta}$-derivation ρ. Then, we proceed by induction on the number of instances of extra-logical rules in ρ to conclude.

($\Leftarrow$) If $\langle \mathcal{E}, \mathcal{F} \rangle$ is a modified extension of $\langle \mathcal{W}, \mathcal{D} \rangle$ and $A \in \mathcal{E}$, then there exists a k such that $A \in \mathcal{E}^k$. We show by (transfinite) induction on k that there exists (at least) one $\mathsf{G4}^{\delta}$-proof π of $\Gamma \;{}_{\mathsf{F}}\!\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; A$, for some $\mathbf{S}, \mathbf{T}$, such that $D_\pi \in \mathcal{E}$. If $k = j + 1$ and A can be obtained from formulas in $\mathcal{E}^j$ *via* suitable applications of *modus ponens*, we leverage Theorem 8.5 to reach the conclusion. The most meaningful case arises when $k = j + 1$, A is not a classical consequence of formulas in $\mathcal{E}^j$ and there exists a default rule of the form $\dfrac{B : C_1, \ldots, C_k}{A}$ such that $B \in \mathcal{E}^j$. If $A \in \mathcal{E}$ and $\mathcal{D}'$ represents the maximal set of applicable defaults triggered by formulas in $\mathcal{E}^j$, then $C_h \in (\mathcal{F} \cup just(\mathcal{D}'))$, and so $\neg C_h \notin Cn(\mathcal{E} \cup concl(\mathcal{D}'))$, for any $1 \leq h \leq k$. By inductive hypothesis, there exists (at least) one $\mathsf{G4}^{\delta}$-proof ρ of $\Gamma \;{}_{\mathsf{F}}\!\Big|\!\frac{\mathbf{V}}{\mathbf{U}}\; B$, for some $\mathbf{U}, \mathbf{V}$, such that $D_\rho \in \mathcal{E}$. Theorem 8.5 and Lemma 8.3 guarantee the existence of a $\mathsf{G4}^{\delta}$-proof ρ_i of $\Gamma \;{}_{\mathsf{F}}\!\Big|\!\frac{\mathbf{V'_i}}{\mathbf{U}}\; \Theta_i$, for any $\overset{*}{\vdash} \Theta_i$ occurring in $\mathsf{top_r}(\overset{*}{\vdash} B) = \{\overset{*}{\vdash} \Theta_i\}_{1 \leq i \leq m}$, such that $D_{\rho_i} \in \mathcal{E}$. Hence, for any $\overset{*}{\vdash} \Phi$ in $\mathsf{top_r}(\overset{*}{\vdash} A)$ consider the following $\mathsf{G4}^{\delta}$-derivation π':

$$\dfrac{\begin{matrix}\vdots_{\rho_i} \\ \Gamma \;{}_{\mathsf{F}}\!\Big|\!\frac{\mathbf{V'_i}}{\mathbf{U}}\; \Theta_i\end{matrix}}{\Gamma \;{}_{\mathsf{F}}\!\Big|\!\frac{\mathbf{V'}}{\mathbf{U'}}\; \Phi}\; \delta_{\mathcal{D}'}$$

where $\mathbf{U'} = \mathbf{U} \cup \{\{\neg C_1^f, \ldots, \neg C_k^f\}\}$. If $\neg C_h \notin Cn(\mathcal{E} \cup concl(\mathcal{D}'))$, then $\neg C_h \notin Cn(\mathcal{W} \cup \{D_{\rho_i}\}_{1 \leq i \leq m} \cup \{A\})$: this implies that π' is a $\mathsf{G4}^{\delta}$-proof, as desired. $\square$

8.3. Controlled sequent calculi for normative systems

In this subsection, we extend $\mathsf{G4}^{\delta}$ calculi for m-credulous consequence with rules for O-labelled and P-labelled sequents, in order to obtain strongly complete $\mathsf{G4}^{\delta}$ calculi for m- and d-credulous consequence.

In $\mathsf{G4}^{\delta}$ calculi for d-credulous consequence, factual detachment is formalized *via* specific extra-logical rules, which have F-labelled premises and $\times$-labelled conclusions, with $\times \in \{\mathsf{O}, \mathsf{P}\}$. To keep trace of the *factual assumptions* on the left-hand side of the F-labelled premises, we enrich the structure of each controlled $\times$-labelled sequent with an additional context, a *repository*. Formally, a *deontic controlled sequent* is a controlled sequent $\Gamma \mathrel{{}_{\times}|\!\frac{\mathbf{T}}{\mathbf{S}}} \Delta$ with attached a context Π, as follows: $\Pi \mid \Gamma \mathrel{{}_{\times}|\!\frac{\mathbf{T}}{\mathbf{S}}} \Delta$. The rules of $\mathsf{G4}^{\delta}$ transmit repositories along derivations in the same way as they transmit control pairs (see the rules in Figure 3).

Repositories were originally introduced in [122] to keep trace of the formulas shifted by the rules from the left-hand side of the sequent to the right-hand side, in order to ensure Cut elimination. In [130], the authors show that repositories can be dropped *via* a polarity-based definition of soundness. In the $\mathsf{G4}^{\delta}$ calculi presented here, no rule licenses shifts of formulas from the left-hand side of the sequent to the right-hand side: as a result, there is no need of repositories to store negative formulas, as in [122], and no need of a polarity-based definition of soundness, like in [130]. Rather, we employ repositories with the purpose of representing factual assumptions in parallel with deontic assumptions – i.e., the formulas on the left-hand side of $\times$-labelled sequents.

The provability of a sequent of the form (say) $\Pi \mid \Gamma \mathrel{{}_{\mathsf{O}}|\!\frac{\mathbf{T}}{\mathbf{S}}} \Delta$ corresponds to the truth of the statement 'in (at least) one modified extension of $\langle \mathcal{W} \cup \Pi, \mathcal{D} \rangle$, $\bigvee \Delta$ is obligatory according to the normative system $\langle \langle \mathcal{W}^{\mathsf{o}} \cup \Gamma, \mathcal{O} \rangle, \langle \mathcal{W}^{\mathsf{p}}, \mathcal{P} \rangle \rangle$'[1]. Let us remark that labeling turnstiles rather than the formulas separated by the turnstile corresponds to the idea that the content of a formula is the same, regardless of whether the formula occurs in a factual or a normative statement. We take the same formula to play different roles in different games (the 'game' of facts, the 'game' of obligations, the 'game' of permissions), just like the same card from the same deck plays different roles in different games[2].

Definition 8.6. Let $\langle \mathcal{W}, \mathcal{D} \rangle$, $\langle \mathcal{W}^{\mathsf{o}}, \mathcal{O} \rangle$ and $\langle \mathcal{W}^{\mathsf{p}}, \mathcal{P} \rangle$ be a default theory, a system of obligations and a system of static permissions, respectively. The controlled sequent calculus $\mathsf{G4}^{\delta}$ for $\langle \mathcal{W}, \mathcal{D} \rangle$, $\langle \mathcal{W}^{\mathsf{o}}, \mathcal{O} \rangle$ and $\langle \mathcal{W}^{\mathsf{p}}, \mathcal{P} \rangle$ is an extension of the $\mathsf{G4}^{\delta}$ calculus for $\langle \mathcal{W}, \mathcal{D} \rangle$ with rules in Figure 3, provided that the conditions below are fulfilled.

(i_{op}) For each instance of $ax^{\times}$, one of the following conditions holds: $\Theta = \Lambda = \{A\}$ for some literal A, $\Theta = \{A, \neg A\}$ or $\Lambda = \{A, \neg A\}$ for some atom A, $\Theta \mathrel{|\!\overset{*}{-}} \Lambda$ belongs to $\mathsf{top}^{\star}_{\mathsf{c}}(\mathrel{|\!\overset{*}{-}} W^{\times})$, with $W^{\times}$ being the conjunction of the formulas in $\mathcal{W}^{\times}$.

---

[1] We can see a modified extension of $\langle \mathcal{W} \cup \Pi, \mathcal{D} \rangle$ as an incomplete possible world, and $\mathcal{N}$ as an incomplete normative system – i.e., a normative system such that there exists (at least) one formula which is neither explicitly required, nor explicitly permitted, nor explicitly prohibited by the system. Under this view, provability of $\Pi \mid \Gamma \mathrel{{}_{\mathsf{O}}|\!\frac{\mathbf{T}}{\mathbf{S}}} \Delta$ would correspond to truth in *incomplete* versions of Gibbardian factual-normative worlds [53].

[2] Formulas occurring in control pairs are labeled just for notational convenience.

AXIOMS AND STRUCTURAL RULES

$$\dfrac{}{\varnothing \mid \Theta \; {}_{\times}\!\left|\frac{\varnothing}{\varnothing}\right. \Lambda}\, ax^{\times}$$

$$\dfrac{\Pi \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}{\Pi \mid A, \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}\, LW^{\times} \qquad \dfrac{\Pi \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}{\Pi \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta, A}\, RW^{\times}$$

$$\dfrac{\Pi_1 \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. \Delta, A \qquad \Pi_2 \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \Delta, \neg A}{\Pi_1, \Pi_2 \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}\, cut^{\times}_{asa} \qquad \dfrac{\Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}{\Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S'}}\right. \Delta}\, \sigma^{\times}$$

LOGICAL RULES

$$\dfrac{\Pi \mid \Gamma, A, B \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}{\Pi \mid \Gamma, A \wedge B \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}\, L\wedge^{\times} \qquad \dfrac{\Pi_1 \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. \Delta, A \qquad \Pi_2 \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \Delta, B}{\Pi_1, \Pi_2 \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta, A \wedge B}\, R\wedge^{\times}$$

$$\dfrac{\Pi_1 \mid \Gamma, A \; {}_{\times}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. \Delta \qquad \Pi_2 \mid \Gamma, B \; {}_{\times}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \Delta}{\Pi_1, \Pi_2 \mid \Gamma, A \vee B \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}\, L\vee^{\times} \qquad \dfrac{\Pi \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta, A, B}{\Pi \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta, A \vee B}\, R\vee^{\times}$$

$$\dfrac{\Pi_1 \mid \Gamma, \neg A \; {}_{\times}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. \Delta \qquad \Pi_2 \mid \Gamma, \neg B \; {}_{\times}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \Delta}{\Pi_1, \Pi_2 \mid \Gamma, \neg(A \wedge B) \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}\, L\neg\wedge^{\times} \qquad \dfrac{\Pi \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta, \neg A, \neg B}{\Pi \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta, \neg(A \wedge B)}\, R\neg\wedge^{\times}$$

$$\dfrac{\Pi \mid \Gamma, \neg A, \neg B \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}{\Pi \mid \Gamma, \neg(A \vee B) \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}\, L\neg\vee^{\times} \qquad \dfrac{\Pi_1 \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. \Delta, \neg A \qquad \Pi_2 \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \Delta, \neg B}{\Pi_1, \Pi_2 \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta, \neg(A \vee B)}\, R\neg\vee^{\times}$$

$$\dfrac{\Pi \mid A, \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}{\Pi \mid \neg\neg A, \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta}\, L\neg\neg^{\times} \qquad \dfrac{\Pi \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta, A}{\Pi \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Delta, \neg\neg A}\, R\neg\neg^{\times}$$

EXTRA-LOGICAL RULES

$$\dfrac{\{\Gamma \; {}_{\mathsf{F}}\!\left|\frac{\mathbf{T_i}}{\mathbf{S_i}}\right. \Theta_i\}_{0 \leq i \leq m}}{\Gamma \mid \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Phi}\, \delta \qquad \dfrac{\{\Pi_i \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T_i}}{\mathbf{S_i}}\right. \Theta_i\}_{0 \leq i \leq m}}{\Pi \mid \Gamma \; {}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \Phi}\, \delta$$

FIGURE 3. Controlled sequent rules for a normative system $\langle\langle \mathcal{W}^{\circ}, \mathcal{O}\rangle, \langle \mathcal{W}^{\mathsf{p}}, \mathcal{P}\rangle\rangle$

(ii_{op}) For each structural and logical rule, the left subscript of the turnstile $\vdash$ and the right superscript of the rule name, $\times$, is either o or p.

(iii_{op}) For any constrained conditional obligation of the form $\dfrac{B : C_1, \ldots, C_k}{D}$ in $\mathcal{O}$, if $\mathsf{top_r}(\vdash^{*} B) = \{\vdash^{*} \Theta_1, \ldots, \vdash^{*} \Theta_m\}$ with $m > 0$ and $\vdash^{*} \Phi$ belongs to $\mathsf{top}^{\star}_{\mathsf{r}}(\vdash^{*} D)$, then there exist extra-logical rules of the following form:

$$\frac{\{\Gamma\ {}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_i}}{\mathbf{S_i}}\ \Theta_i\}_{1\leq i\leq m}}{\Gamma \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Phi}\,\delta \qquad \frac{\{\Pi_i \mid \Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_i}}{\mathbf{S_i}}\ \Theta_i\}_{1\leq i\leq m}}{\Pi \mid \Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T'}}{\mathbf{S'}}\ \Phi}\,\delta$$

with $\mathbf{T} = \bigcup_{i=1}^{m} \mathbf{T_i} \cup \{\{(\bigwedge_{i=1}^{m}(\bigvee \Theta_i))^{\mathsf{f}}\}\}$, $\mathbf{S} = \bigcup_{i=1}^{m} \mathbf{S_i} \cup \{\{\neg C_1^{\mathsf{o}}, \ldots, \neg C_k^{\mathsf{o}}\}, \{\neg C_1^{\mathsf{p}}, \ldots, \neg C_k^{\mathsf{p}}\}\}$, $\mathbf{T'} = \bigcup_{i=1}^{m} \mathbf{T_i} \cup \{\{(\bigwedge_{i=1}^{m}(\bigvee \Theta_i))^{\mathsf{o}}\}, \{(\bigwedge_{i=1}^{m}(\bigvee \Theta_i))^{\mathsf{p}}\}\}$, $\mathbf{S'} = \mathbf{S}$ and $\Pi = \Pi_1, \ldots, \Pi_m$. On the other hand, if $\mathsf{top_r}(\vdash^{*} B) = \varnothing$, then there exists an extra-logical rule of the following form:

$$\frac{}{\Pi \mid \Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Phi}\,\delta$$

with $\mathbf{T} = \{\{\top^{\mathsf{o}}\}, \{\top^{\mathsf{p}}\}\}$ and $\mathbf{S} = \bigcup_{i=1}^{m} \mathbf{S_i} \cup \{\{\neg C_1^{\mathsf{o}}, \ldots, \neg C_k^{\mathsf{o}}\}, \{\neg C_1^{\mathsf{p}}, \ldots, \neg C_k^{\mathsf{p}}\}\}$. The same condition applies to any constrained conditional permission of the form $\dfrac{B : C_1, \ldots, C_k}{D}$ in $\mathcal{P}$, with ${}_{\mathsf{P}}\vdash$ replacing ${}_{\mathsf{O}}\vdash$ and no o-labelled formula being introduced in the control pair.

(iv_{op}) For any pair of extra-logical rules

$$\frac{\{\Gamma\ {}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_i}}{\mathbf{S_i}}\ \Theta_i\}_{1\leq i\leq m}}{\Gamma \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Phi}\,\delta \qquad \frac{\{\Pi_i \mid \Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_i}}{\mathbf{S_i}}\ \Theta_i\}_{1\leq i\leq m}}{\Pi \mid \Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Phi}\,\delta$$

with $m \geq 0$ and $\Pi = \Pi_1, \ldots, \Pi_m$, if $\vdash^{*} \Phi'$ occurs in $\mathsf{top}^{\star}_{\mathsf{r}}(\vdash^{*} (\bigvee \Phi) \wedge W)$ without occurring in $\mathsf{top}^{\star}_{\mathsf{r}}(\vdash^{*} W^{\mathsf{o}})$, there exists a pair of extra-logical rules of the form

$$\frac{\{\Gamma\ {}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_i}}{\mathbf{S_i}}\ \Theta_i\}_{1\leq i\leq m}}{\Gamma \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Phi'}\,\delta \qquad \frac{\{\Pi_i \mid \Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_i}}{\mathbf{S_i}}\ \Theta_i\}_{1\leq i\leq m}}{\Pi \mid \Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Phi'}\,\delta$$

The same condition applies to any pair of extra-logical rules of the same form, with ${}_{\mathsf{P}}\vdash$ replacing ${}_{\mathsf{O}}\vdash$ and no o-labelled formula being introduced in the control pair.

(v_{op}) For any pair of extra-logical rules

$$\frac{\{\Gamma\ {}_{\mathsf{F}}\Big|\!\frac{\mathbf{T'_i}}{\mathbf{S'_i}}\ \Theta_i\}_{1\leq i\leq m}}{\Gamma \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\ \Phi}\,\delta \qquad \frac{\{\Gamma\ {}_{\mathsf{F}}\Big|\!\frac{\mathbf{T'_i}}{\mathbf{S'_i}}\ \Theta_i\}_{m+1\leq i\leq n}}{\Gamma \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\ \Phi'}\,\delta$$

with $m + n \geq 0$, if $\vdash^{*} \Psi$ occurs in $\mathsf{top}^{\star}_{\mathsf{r}}(\vdash^{*} (\bigvee \Phi) \wedge (\bigvee \Phi'))$ without belonging to $\mathsf{top}^{\star}_{\mathsf{r}}(\vdash^{*} (\bigvee \Phi))$, $\mathsf{top}^{\star}_{\mathsf{r}}(\vdash^{*} (\bigvee \Phi'))$ or $\mathsf{top}^{\star}_{\mathsf{r}}(\vdash^{*} W^{\mathsf{o}})$, then there exists an extra-logical rule of the following form:

$$\frac{\{\Gamma\ {}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_i'}}{\mathbf{S_i'}}\ \Theta_i\}_{1\leq i\leq n}}{\Gamma\mid{}_{\mathsf{O}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Psi}\,\delta$$

with $\mathbf{S} = \mathbf{S_1} \cup \mathbf{S_2}$ and $\mathbf{T} = \mathbf{T_1} \cup \mathbf{T_2}$. The same condition applies to any pair of extra-logical rules of the same form, with ${}_{\mathsf{P}}\vdash$ replacing ${}_{\mathsf{O}}\vdash$ and no o-labelled formula being introduced in the control pair.

(vi_{op}) For any pair of extra-logical rules

$$\frac{\{\Pi_i\mid\Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_i'}}{\mathbf{S_i'}}\ \Theta_i\}_{1\leq i\leq m}}{\Pi'\mid\Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\ \Phi}\,\delta \qquad \frac{\{\Pi_i\mid\Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_i'}}{\mathbf{S_i'}}\ \Theta_i\}_{m+1\leq i\leq n}}{\Pi''\mid\Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\ \Phi'}\,\delta$$

with $m + n \geq 0$, $\Pi' = \Pi_1, \ldots, \Pi_m$ and $\Pi'' = \Pi_{m+1}, \ldots, \Pi_n$, if $\overset{*}{\vdash}\ \Psi$ occurs in $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\vdash}\ (\rightarrow \bigvee \Phi) \wedge (\bigvee \Phi'))$ without belonging to $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\vdash}\ (\bigvee \Phi))$, $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\vdash}\ (\bigvee \Phi'))$ or $\mathsf{top}^{\star}_{\mathsf{r}}(\overset{*}{\vdash}\ W^{\circ})$, then there exists an extra-logical rule of the following form:

$$\frac{\{\Pi_i\mid\Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_i'}}{\mathbf{S_i'}}\ \Theta_i\}_{1\leq i\leq n}}{\Pi\mid\Gamma\ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\ \Psi}\,\delta$$

with $\mathbf{S} = \mathbf{S_1} \cup \mathbf{S_2}$, $\mathbf{T} = \mathbf{T_1} \cup \mathbf{T_2}$ and $\Pi = \Pi', \Pi''$. The same condition applies to any pair of extra-logical rules of the same form, with ${}_{\mathsf{P}}\vdash$ replacing ${}_{\mathsf{O}}\vdash$ and no o-labelled formula being introduced in the control pair.

For any extra-logical rule δ in $\mathsf{G4}^{\delta}$ whose conclusion is O-labelled (P-labelled), we define the label $\delta_{\mathcal{O}'}$ ($\delta_{\mathcal{P}'}$) as follows:

(a_{op}) if δ is generated in accordance with point (iii_{op}) in Definition 8.6, then $\mathcal{O}'$ ($\mathcal{P}'$, respectively) is $\left\{\dfrac{B : C_1, \ldots, C_k}{D}\right\}$;

(b_{op}) if δ is generated from extra-logical rules δ' and δ'' with labels $\delta_{\mathcal{O}_1}$ and $\delta_{\mathcal{O}_2}$ ($\delta_{\mathcal{P}_1}$ and $\delta_{\mathcal{P}_2}$) in accordance with points (iv_{op}) – (vi_{op}) in Definition 8.6, then $\mathcal{O}'$ is $\mathcal{O}_1 \cup \mathcal{O}_2$ ($\mathcal{P}'$ is $\mathcal{P}_1 \cup \mathcal{P}_2$, respectively).

For each $\mathsf{G4}$-derivation π we say that a constrained conditional obligation $\dfrac{B : C_1, \ldots, C_k}{D}$ *belongs to obl*(π) if and only if there is (at least) one extra-logical rule labelled $\delta_{\mathcal{O}'}$ which is applied in π and such that $\dfrac{B : C_1, \ldots, C_k}{D}$ belongs to $\mathcal{O}'$.

On the other hand, we say that a rule $\dfrac{B : C_1, \ldots, C_k}{D}$ *belongs to perm*(π) if and only if there is (at least) one extra-logical rule labelled $\delta_{\mathcal{P}'}$ which is applied in π and such that $\dfrac{B : C_1, \ldots, C_k}{D}$ belongs to $\mathcal{P}'$, or it belongs to *obl*(π).

We employ *obl*′(π) (*perm*′(π)) to denote *obl*(π_1)$\cup \cdots \cup$*obl*(π_m) (*perm*(π_1)$\cup \cdots \cup$*perm*(π_m), respectively) whenever $\pi_1, \ldots, \pi_m$ are the immediate subderivations yielding the premises of the lowermost extra-logical rules applications in π. We shall use

- D^{o}_{π} to refer to the conjunction of the formulas in *concl*(*obl*(π)), and E_{π} to denote the conjunction of the formulas in *concl*(*obl*′(π));
- D^{p}_{π} to refer to the conjunction of the formulas in *concl*(*perm*(π)), and E_{π} to denote the conjunction of the formulas in *concl*(*perm*′(π)).

Definition 8.7. Let π be a $\mathsf{G4}^{\delta}$-derivation of $\Pi \mid \Gamma \;{}_{\mathsf{O}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\; \Delta$ and $\mathbf{T}^{\mathsf{o}}, \mathbf{S}^{\mathsf{o}}$ be the subsets of $\mathbf{T}, \mathbf{S}$, respectively, containing only sets of o-labelled formulas. The sequent $\Pi \mid \Gamma \;{}_{\mathsf{O}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\; \Delta$ is

(i) *sound under conditions* if and only if $([W^{\mathsf{o}}, E^{\mathsf{o}}_{\rho}] \cup \Gamma') \,||\, \langle \mathbf{V}^{\mathsf{o}}, \varnothing \rangle$ for any subderivation ρ of π with $\Pi' \mid \Gamma' \;{}_{\mathsf{O}}\!\left|\frac{\mathbf{V}}{\mathbf{U}}\right.\; \Delta'$ as conclusion;

(ii) *sound under constraints* if and only if $([W^{\mathsf{o}}, D^{\mathsf{o}}_{\pi}] \cup \Gamma) \,||\, \langle \varnothing, \mathbf{S}^{\mathsf{o}} \rangle$;

(iii) *sound* if and only if it is sound under conditions and sound under constraints.

Definition 8.8. Let π be a $\mathsf{G4}^{\delta}$-derivation of $\Gamma \;{}_{\mathsf{P}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\; \Delta$ and $\mathbf{T}^{\mathsf{p}}, \mathbf{S}^{\mathsf{p}}$ be the subsets of $\mathbf{T}, \mathbf{S}$, respectively, containing only sets of o- and p-labelled formulas. The sequent $\Gamma \;{}_{\mathsf{P}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\; \Delta$ is

(i) *sound under conditions* if and only if $([W^{\mathsf{p}}, E^{\mathsf{p}}_{\rho}] \cup \Gamma) \,||\, \langle \mathbf{V}^{\mathsf{p}}, \varnothing \rangle$ for any subderivation ρ of π with $\Pi \;{}_{\times}\!\left|\frac{\mathbf{V}}{\mathbf{U}}\right.\; \Sigma$ as conclusion;

(ii) *sound under constraints* if and only if $([W^{\mathsf{p}}, D^{\mathsf{p}}_{\pi}] \cup \Gamma) \,||\, \langle \varnothing, \mathbf{S}^{\mathsf{p}} \rangle$;

(iii) *sound* if and only if it is sound under conditions and sound under constraints.

Definition 8.9. Let $\times \in \{\mathsf{O}, \mathsf{P}\}$ and π be a $\mathsf{G4}^{\delta}$-derivation of $\Pi \mid \Gamma \;{}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\; \Delta$. Then, π is a *proof* of $\Pi \mid \Gamma \;{}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\; \Delta$ if and only if any sequent $\Pi' \mid \Gamma' \;{}_{\times}\!\left|\frac{\mathbf{T}'}{\mathbf{S}'}\right.\; \Delta'$ and any sequent $\Gamma'' \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T}''}{\mathbf{S}''}\right.\; \Delta''$ occurring in π is sound – and a *paraproof*, otherwise.

Lemma 8.9. *Let* $\times \in \{\mathsf{O}, \mathsf{P}\}$. *The rule of safe External Weakening*

$$\frac{\Pi \mid \Gamma \;{}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\; \Delta}{A, \Pi \mid \Gamma \;{}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\; \Delta}\, EW$$

is admissible in $\mathsf{G4}^{\delta}$.

Proof. We reason by induction on the height of the proof π of $\Pi \mid \Gamma \;{}_{\times}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right.\; \Delta$. If $h(\pi) = 1$, the conclusion is immediate. Otherwise, we reason by cases over the last rule applied. If the latter is a structural or a logical rule, we simply apply the inductive hypothesis to one of the premises to conclude. The same argument holds whenever the last rule is extra-logical and the premises are $\times$-labelled sequents. The only non-trivial case arises if the last rule is an extra-logical rule δ and the premises are f-labelled sequents. To reach the conclusion, we apply safe LW to each premise and then apply δ. ☐

Lemma 8.10. *The rule of safe Deontic Implication*

$$\frac{\Pi \mid \Gamma \; {}_{\mathsf{O}}|\!\frac{\mathbf{T}}{\mathbf{S}} \; \Delta}{\Pi \mid \Gamma \; {}_{\mathsf{P}}|\!\frac{\mathbf{T}}{\mathbf{S}} \; \Delta}\, op$$

is admissible in $\mathsf{G4}^{\delta}$.

PROOF. By routine induction on the height of the proof of the premise. ☐

LEMMA 8.11. *Let* $\times \in \{\mathsf{O}, \mathsf{P}\}$ *and* $\dagger \in \{\wedge, \vee, \neg\}$. $\mathsf{G4}^{\delta}$ *calculi enjoy the following structural properties.*

(i) *The rules* $R\dagger^{\times}$, $R\neg\dagger^{\times}$, $L\wedge$, $L\neg\vee$ *and* $L\neg\neg$ *are invertible in* $\mathsf{G4}^{\delta}$.

(ii) *The rules* $E\vee^{\times}$ *and* $E\neg\wedge^{\times}$ *are admissible in* $\mathsf{G4}^{\delta}$.

(iii) *The* $\times$*-labelled versions of Left and Right Contraction are admissible in* $\mathsf{G4}^{\delta}$.

PROOF. (i) We argue as in the proofs of Lemmas 8.2 and 8.4. (ii) We proceed as in the proof of Lemma 8.7. (iii) We argue as in the proofs of Lemmas 8.3 and 8.8. ☐

THEOREM 8.9. *The rule of safe Cut is admissible in* $\mathsf{G4}^{\delta}$.

PROOF. We proceed as in the proof of Theorem 8.5, exploiting Lemma 8.11. ☐

THEOREM 8.10. *Let* $\langle \mathcal{W}, \mathcal{D} \rangle$ *be a default theory and* $\langle \langle \mathcal{W}^{\mathsf{o}}, \mathcal{O} \rangle, \langle \mathcal{W}^{\mathsf{p}}, \mathcal{P} \rangle \rangle$ *be a normative system. Then the following statements hold.*

(i) $\mathsf{G4}^{\delta}$ *proves* $\Pi \mid \Gamma \; {}_{\mathsf{O}}|\!\frac{\mathbf{T}}{\mathbf{S}} \; A$ *if and only if* A *belongs to (at least) one d-extension of* $\langle \langle \mathcal{W} \cup \Pi, \mathcal{D} \rangle, \langle \langle \mathcal{W}^{\mathsf{o}} \cup \Gamma, \mathcal{O} \rangle, \langle \mathcal{W}^{\mathsf{p}}, \mathcal{P} \rangle \rangle \rangle$.

(ii) $\mathsf{G4}^{\delta}$ *proves* $\Pi \mid \Gamma \; {}_{\mathsf{P}}|\!\frac{\mathbf{T}}{\mathbf{S}} \; A$ *if and only if* A *belongs to (at least) one d-extension of* $\langle \langle \mathcal{W} \cup \Pi, \mathcal{D} \rangle, \langle \langle \mathcal{W}^{\mathsf{o}}, \mathcal{O} \rangle, \langle \mathcal{W}^{\mathsf{p}} \cup \Gamma, \mathcal{P} \rangle \rangle \rangle$.

PROOF. For the proof of each statement, we argue as in the proof of Theorem 8.8. ☐

8.4. Navigating doxastic and normative conflicts

In this section, we analyze a wide range of deontic scenarios with the proof-theoretic tools introduced so far. We show that controlled sequent calculi are flexible enough to formalize deontic notions beyond plain obligations and permissions, as well as to handle paradoxical or dilemmatic cases of conflicting obligations and permissions. In most cases, $\mathsf{G4}^{\delta}$ calculi are extended with suitable extra-logical rules generating conclusion which are aligned to our intuitive assessments of problematic scenarios.

To maintain focus and conciseness, all the examples below lack explicit definitions of the underlying default theories and normative systems (we leave such details to the reader). Moreover, we assume that constrained obligations and permissions are normal, and thus that consistency with extra-logical axioms is always preserved. In constrained I/O logics, this assumption corresponds to taking the set of constraints as identical to the input set.

Example 8.3 (Typicality-based obligations). *Let the extra-logical rules*

$$\delta_{\mathcal{D}_1}\,\dfrac{\Gamma\;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right.\,p}{\Gamma\;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T'_1}}{\mathbf{S'_1}}\right.\,q}\qquad\dfrac{\Gamma\;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right.\,q}{\Gamma\mid\;{}_{\mathsf{O}}\!\left|\frac{\mathbf{T'_2}}{\mathbf{S'_2}}\right.\,s}\,\delta_{\mathcal{O}_1}\qquad\dfrac{\Gamma\;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_3}}{\mathbf{S_3}}\right.\,r}{\Gamma\mid\;{}_{\mathsf{O}}\!\left|\frac{\mathbf{T'_3}}{\mathbf{S'_3}}\right.\,t}\,\delta_{\mathcal{O}_2}$$

stand for the default rule 'If Mary is an adult, then she is employed, unless she is a university student', the conditional obligation 'If Mary is employed, she ought to fill in an annual income-tax form' and the conditional obligation 'If Mary is a university student, she ought to pay the tuition fee', respectively – where

(*i*) $\mathbf{T'_1} = \mathbf{T_1} \cup \{\{p^{\mathsf{f}}\}\}$ *and* $\mathbf{S'_1} = \mathbf{S_1} \cup \{\{(\neg p \vee r)^{\mathsf{f}}\}\}$;

(*ii*) $\mathbf{T'_2} = \mathbf{T_2} \cup \{\{q^{\mathsf{f}}\}\}$ *and* $\mathbf{S'_2} = \mathbf{S_2} \cup \{\{\neg s^{\mathsf{o}}\}\}$;

(*iii*) $\mathbf{T'_3} = \mathbf{T_3} \cup \{\{r^{\mathsf{f}}\}\}$ *and* $\mathbf{S'_3} = \mathbf{S_3} \cup \{\{\neg t^{\mathsf{o}}\}\}$.

Consider the following derivations:

$$\delta_{\mathcal{O}_1}\,\dfrac{\delta_{\mathcal{D}_1}\,\dfrac{ax\,\dfrac{}{p\;{}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right.\,p}}{p\;{}_{\mathsf{F}}\!\left|\frac{\mathbf{V_1}}{\mathbf{U_1}}\right.\,q}}{p\mid\;{}_{\mathsf{O}}\!\left|\frac{\mathbf{V'_1}}{\mathbf{U'_1}}\right.\,s}\qquad\dfrac{\dfrac{\dfrac{\dfrac{}{p\;{}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right.\,p}\,ax}{r,p\;{}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right.\,p}\,LW}{r,p\;{}_{\mathsf{F}}\!\left|\frac{\mathbf{V_1}}{\mathbf{U_1}}\right.\,q}\,\delta_{\mathcal{D}_1}}{r,p\mid\;{}_{\mathsf{O}}\!\left|\frac{\mathbf{V'_1}}{\mathbf{U'_1}}\right.\,s}\,\delta_{\mathcal{O}_1}\qquad\dfrac{\dfrac{\dfrac{}{r\;{}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right.\,r}\,ax}{p,r\;{}_{\mathsf{F}}\!\left|\frac{\varnothing}{\varnothing}\right.\,r}\,LW}{p,r\mid\;{}_{\mathsf{O}}\!\left|\frac{\mathbf{V_2}}{\mathbf{U_2}}\right.\,t}\,\delta_{\mathcal{O}_2}$$

where $\mathbf{V_1} = \{\{p^{\mathsf{p}}\}\}$, $\mathbf{U_1} = \{\{(\neg p \vee r)^{\mathsf{f}}\}\}$, $\mathbf{V'_1} = \mathbf{V_1} \cup \{\{q^{\mathsf{f}}\}\}$, $\mathbf{U'_1} = \mathbf{U_1} \cup \{\{\neg s^{\mathsf{o}}\}\}$, $\mathbf{V_2} = \{\{r^{\mathsf{f}}\}\}$ *and* $\mathbf{U_2} = \{\{\neg t^{\mathsf{o}}\}\}$. *The leftmost derivation is a proof: if Mary is an adult, she is employed by default – and then she ought to fill in an annual income-tax form. The rightmost derivation is a proof too: if Mary is a university student, she ought to pay the tuition fee. On the other hand, the derivation in the middle is a paraproof: accordingly, if Mary is an adult and a university student, there is no obligation for her to fill in an annual income-tax form.*

Example 8.4 (Chisholm's paradox). *Let the extra-logical rules*

$$\delta_{\mathcal{O}_1}\,\dfrac{}{\Pi\mid\Gamma\;{}_{\mathsf{O}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right.\,p}\qquad\dfrac{\Gamma\;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right.\,p}{\Gamma\mid\;{}_{\mathsf{O}}\!\left|\frac{\mathbf{T'_2}}{\mathbf{S'_2}}\right.\,q}\,\delta_{\mathcal{O}_2}\qquad\dfrac{\Gamma\;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_3}}{\mathbf{S_3}}\right.\,\neg p}{\Gamma\mid\;{}_{\mathsf{O}}\!\left|\frac{\mathbf{T_3}}{\mathbf{S'_3}}\right.\,\neg q}\,\delta_{\mathcal{O}_3}$$

stand for the (un)conditional obligations: 'The manager ought to review the budget', 'If the manager reviews the budget, then she ought to submit it for approval', and 'If the manager does not review the budget, then she ought not submit it for approval', respectively [27] *– where*

(*i*) $\mathbf{T_1} = \{\{\top^{\mathsf{o}}\}\}$ *and* $\mathbf{S_1} = \varnothing$;

(*ii*) $\mathbf{T'_2} = \mathbf{T_2} \cup \{\{p^{\mathsf{f}}\}\}$ *and* $\mathbf{S'_2} = \mathbf{S_2} \cup \{\{\neg q^{\mathsf{o}}\}\}$;

(*iii*) $\mathbf{T'_3} = \mathbf{T_3} \cup \{\{\neg p^{\mathsf{f}}\}\}$ *and* $\mathbf{S'_3} = \mathbf{S_3} \cup \{\{\neg\neg q^{\mathsf{o}}\}\}$.

There exists an unconditional obligation to p, and the latter is equivalent to the negation of the condition of the norm captured by the rule – namely, $\neg p$. This corresponds to the scenario where there exists a primary obligation and a violation of the latter triggers a secondary obligation. The intuitive assessment is that the secondary obligation should hold, in spite of the fact that the primary obligation is violated.

Consider the following application of a suitable inference rule ctd:

$$\dfrac{\delta_{\mathcal{O}_1}\,\dfrac{}{\varnothing \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. p} \qquad \dfrac{\dfrac{}{\neg p \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_3}}{\mathbf{S_3}}\right. \neg p}\,ax}{\neg p \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{T_3'}}{\mathbf{S_3'}}\right. \neg q}\,\delta_{\mathcal{O}_3}}{\neg p \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. p \wedge \neg q}\,ctd$$

with $\mathbf{T_3} = \mathbf{S_3} = \varnothing$, $\mathbf{T} = \mathbf{T_1} \cup \mathbf{T_3'}$, $\mathbf{S} = \mathbf{S_1} \cup \mathbf{S_3''} \cup \{\{p^\circ\}\}$. Such derivation is a paraproof, whereas its immediate subderivations are proofs. Accordingly, if the manager does not review the budget, then she ought not submit it for approval – in spite of the existence of an unconditional obligation to reviewing the budget.

Example 8.5 (Forrester's paradox). *Let the extra-logical rules*

$$\delta_{\mathcal{O}_1}\,\dfrac{}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. \neg p} \qquad \dfrac{\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. p}{\Gamma \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{T_2'}}{\mathbf{S_2'}}\right. q}\,\delta_{\mathcal{O}_2}$$

stand for the (un)conditional obligations 'One ought not kill' and 'If one kills someone, she ought to kill her gently', respectively [48] *– where*

(*i*) $\mathbf{T_1} = \{\{\top^\circ\}\}$ *and* $\mathbf{S_1} = \{\{\neg\neg p^o\}\}$;

(*ii*) $\mathbf{T_2'} = \mathbf{T_2} \cup \{\{p^{\mathsf{f}}\}\}$ *and* $\mathbf{S_2'} = \mathbf{S_2} \cup \{\{\neg q^o\}\}$.

Consider the following derivation:

$$\dfrac{\delta_{\mathcal{O}_1}\,\dfrac{}{\varnothing \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. \neg p} \qquad \dfrac{\dfrac{}{p \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. p}\,ax}{p \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{T_2'}}{\mathbf{S_2'}}\right. q}\,\delta_{\mathcal{O}_2}}{p \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. \neg p \wedge q}\,ctd$$

with $\mathbf{T_2} = \mathbf{S_2} = \varnothing$, $\mathbf{T} = \mathbf{T_1'} \cup \mathbf{T_2'}$, $\mathbf{S} = \mathbf{S_1} \cup \mathbf{S_2'} \cup \{\{\neg p^\circ\}\}$. Such derivation is a paraproof. This corresponds to the fact that if one kills someone, she ought to kill them gently – in spite of the existence of an unconditional obligation not to kill anyone.

Example 8.6 (Specificity principle). *Let the extra-logical rules*

$$\delta_{\mathcal{O}_1}\,\dfrac{\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. p}{\Gamma \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{T_1'}}{\mathbf{S_1'}}\right. \neg q} \qquad \dfrac{\Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. p \qquad \Gamma \;{}_{\mathsf{F}}\!\left|\frac{\mathbf{T_2'}}{\mathbf{S_2'}}\right. r}{\Gamma \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{T_2''}}{\mathbf{S_2''}}\right. q}\,\delta_{\mathcal{O}_2}$$

stand for the conditional obligations 'If Fido is a dog, you ought not kill Fido' and 'If Fido is a dog and Fido is attacking a child, you ought to kill Fido', respectively – where

(*i*) $\mathbf{T_1'} = \mathbf{T_1} \cup \{\{p^{\mathsf{f}}\}\}$ *and* $\mathbf{S_1'} = \mathbf{S_1} \cup \{\{\neg\neg q^{\circ}\}\}$;

(*ii*) $\mathbf{T_2''} = \mathbf{T_2} \cup \mathbf{T_2'} \cup \{\{p^{\mathsf{f}}, r^{\mathsf{f}}\}\}$ *and* $\mathbf{S_2''} = \mathbf{S_2} \cup \mathbf{S_2'} \cup \{\{\neg q^{\circ}\}\}$.

Consider the following application of a suitable inference rule $(\delta_{\mathcal{O}_1})^{sp}$*:*

$$\dfrac{\dfrac{\dfrac{}{p \;{}_{\mathsf{F}}\left|\frac{\varnothing}{\varnothing}\right.\; p}\,ax}{r, p \;{}_{\mathsf{F}}\left|\frac{\varnothing}{\varnothing}\right.\; p}\,LW}{r, p \mid {}_{\mathsf{O}}\left|\frac{\mathbf{V}}{\mathbf{U}}\right.\; \neg q}\,(\delta_{\mathcal{O}_1})^{sp}$$

with $\mathbf{V} = \{\{p^{\mathsf{f}}\}\}$ *and* $\mathbf{U} = \{\{\neg\neg q^{\circ}\}, \{r^{\mathsf{f}}\}\}$. *Such derivation is a paraproof. This corresponds to the fact that if Fido is a dog and is attacking a child, one ought to kill him – and this obligation overrides the obligation not to kill him under the weaker condition that he is just a dog.*

Example 8.7 (Extended Forrester's paradox). *Let the extra-logical rules*

$$\delta_{\mathcal{O}_1}\,\dfrac{}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right.\; \neg p} \qquad \dfrac{\Gamma \;{}_{\mathsf{F}}\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right.\; p}{\Gamma \mid {}_{\mathsf{O}}\left|\frac{\mathbf{T_2'}}{\mathbf{S_2'}}\right.\; p}\,\delta'_{\mathcal{O}_2} \qquad \dfrac{\Gamma \;{}_{\mathsf{F}}\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right.\; p}{\Gamma \mid {}_{\mathsf{O}}\left|\frac{\mathbf{T_2'}}{\mathbf{S_2'}}\right.\; q}\,\delta''_{\mathcal{O}_2}$$

$$\dfrac{\Gamma \;{}_{\mathsf{F}}\left|\frac{\mathbf{T_3}}{\mathbf{S_3}}\right.\; r}{\Gamma \mid {}_{\mathsf{O}}\left|\frac{\mathbf{T_3'}}{\mathbf{S_3'}}\right.\; p}\,\delta'_{\mathcal{O}_3} \qquad \dfrac{\Gamma \;{}_{\mathsf{F}}\left|\frac{\mathbf{T_3}}{\mathbf{S_3}}\right.\; r}{\Gamma \mid {}_{\mathsf{O}}\left|\frac{\mathbf{T_3'}}{\mathbf{S_3'}}\right.\; q}\,\delta''_{\mathcal{O}_3}$$

stand for the (un)conditional obligations 'There ought not be a fence around your house', 'If there is a fence around your house, there ought be a white fence around your house' and 'If you have a dog, there ought be a white fence around your house', respectively [172] *– where*

(*i*) $\mathbf{T_1} = \{\{\top^{\circ}\}\}$ *and* $\mathbf{S_1} = \{\{\neg\neg p^{\circ}\}\}$;

(*ii*) $\mathbf{T_2'} = \mathbf{T_2} \cup \{\{p^{\mathsf{f}}\}\}$, $\mathbf{S_2'} = \mathbf{S_2} \cup \{\{\neg(p \wedge q)^{\circ}\}\}$;

(*iii*) $\mathbf{T_3'} = \mathbf{T_3} \cup \{\{r^{\mathsf{f}}\}\}$, $\mathbf{S_3'} = \mathbf{S_3} \cup \{\{\neg(p \wedge q)^{\circ}\}\}$.

Consider the following derivations:

$$(\delta_{\mathcal{O}_1})^{sp}\,\dfrac{}{r \mid {}_{\mathsf{O}}\left|\frac{\mathbf{T_1}}{\mathbf{S_1'}}\right.\; \neg p} \qquad \dfrac{\dfrac{}{\varnothing \mid {}_{\mathsf{O}}\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right.\; \neg p} \qquad \dfrac{\dfrac{\dfrac{}{p \;{}_{\mathsf{F}}\left|\frac{\varnothing}{\varnothing}\right.\; p}\,ax}{p \mid {}_{\mathsf{O}}\left|\frac{\mathbf{V_1}}{\mathbf{U_1}}\right.\; p}\,\delta'_{\mathcal{O}_2} \qquad \dfrac{\dfrac{}{p \;{}_{\mathsf{F}}\left|\frac{\varnothing}{\varnothing}\right.\; p}\,ax}{p \mid {}_{\mathsf{O}}\left|\frac{\mathbf{V_1}}{\mathbf{U_1}}\right.\; q}\,\delta''_{\mathcal{O}_2}}{p \mid {}_{\mathsf{O}}\left|\frac{\mathbf{V_1}}{\mathbf{U_1}}\right.\; p \wedge q}}{p \mid {}_{\mathsf{O}}\left|\frac{\mathbf{V}}{\mathbf{U}}\right.\; \neg p \wedge (p \wedge q)}\,ctd$$

where $\mathbf{S_1'} = \mathbf{S_1} \cup \{\{r^{\mathsf{f}}\}\}$, $\mathbf{V_1} = \{\{p^{\mathsf{f}}\}\}$, $\mathbf{U_1} = \{\{\neg(p \wedge q)^{\circ}\}\}$, $\mathbf{V} = \mathbf{T_1} \cup \mathbf{V_1}$ *and* $\mathbf{U} = \mathbf{S_1} \cup \mathbf{U_1} \cup \{\{\neg p^{\circ}\}\}$. *It is easy to verify that both derivations are paraproofs. This corresponds to the fact that (a) the unconditional prohibition of having a fence around your house is overridden by the obligation of having a fence in case you possess a dog, and that (b) when the unconditional prohibition of having a fence around your house is violated, a secondary obligation is triggered for you to have a white one.*

Example 8.8 (Sartre's dilemma). *Let the extra-logical rules*

$$\delta_{\mathcal{O}_1}\;\dfrac{\Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\; p}{\Gamma \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_1'}}{\mathbf{S_1'}}\; q} \qquad \dfrac{\Gamma \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\; r}{\Gamma \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_2'}}{\mathbf{S_2'}}\; \neg q}\;\delta_{\mathcal{O}_2}$$

stand for the conditional obligations: 'Any man ought to stay home whenever his mother is ill' and 'Any man ought not stay home whenever his country is attacked', respectively [148] *– where*

(*i*) $\mathbf{T_1'} = \mathbf{T_1} \cup \{\{p^{\mathsf{f}}\}\}$ *and* $\mathbf{S_1'} = \mathbf{S_1} \cup \{\{\neg q^{\circ}\}\}$;

(*ii*) $\mathbf{T_2'} = \mathbf{T_2} \cup \{\{r^{\mathsf{f}}\}\}$ *and* $\mathbf{S_2'} = \mathbf{S_2} \cup \{\{\neg\neg q^{\circ}\}\}$.

Consider the following derivation:

$$\dfrac{\delta_{\mathcal{O}_1}\;\dfrac{ax\;\dfrac{}{p \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\; p}}{p \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_1'}}{\mathbf{S_1'}}\; q} \qquad \dfrac{\dfrac{}{r \;{}_{\mathsf{F}}\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\; r}\;ax}{r \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_2'}}{\mathbf{S_2'}}\; \neg q}\;\delta_{\mathcal{O}_2}}{p, r \mid \Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; q \wedge \neg q}\;R\wedge$$

with $\mathbf{T_2} = \mathbf{S_2} = \varnothing$, $\mathbf{T} = \mathbf{T_1'} \cup \mathbf{T_2'}$ *and* $\mathbf{S} = \mathbf{S_1'} \cup \mathbf{S_2'}$. *Such derivation is a paraproof. This corresponds to the fact that no man ought and ought not stay home whenever his mother is ill and his country is attacked.*

Example 8.9 (Euthyphro's dilemma). *Euthyphro is prosecuting his father for murder, despite the traditional belief that it is impious to prosecute or dishonor a parent. Let the extra-logical rules*

$$\delta_{\mathcal{O}_1}\;\dfrac{}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\; p} \qquad \dfrac{}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\; \neg q}\;\delta_{\mathcal{O}_2}$$

stand for the unconditional obligations: 'One ought to prosecute her father for murder' and 'One ought not to dishonor her father', respectively – where $\mathbf{T_1} = \mathbf{T_2} = \{\{\top^{\circ}\}\}$, $\mathbf{S_1} = \{\{\neg p^{\circ}\}\}$ *and* $\mathbf{S_2} = \{\{\neg\neg q^{\circ}\}\}$. *Consider the following derivation:*

$$\dfrac{\delta_{\mathcal{O}_1}\;\dfrac{}{\varnothing \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\; p} \qquad \dfrac{}{\varnothing \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\; \neg q}\;\delta_{\mathcal{O}_2}}{\varnothing \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\; p \wedge \neg q}\;R\wedge$$

with $\mathbf{T} = \mathbf{T_1} \cup \mathbf{T_2}$ *and* $\mathbf{S} = \mathbf{S_1} \cup \mathbf{S_2}$. *Such derivation is a paraproof. This corresponds to the fact that Eutyphro cannot be obliged to prosecute his father and not to dishonor him at the same time.*

Example 8.10 (Disjunctive response). *Let the extra-logical rules*

$$\delta_{\mathbb{O}_1}\,\dfrac{}{\Pi \mid \Gamma \ {}_{\mathsf{O}}\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. p} \qquad \dfrac{}{\Pi \mid \Gamma \ {}_{\mathsf{O}}\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \neg q}\,\delta_{\mathbb{O}_2}$$

be as in Example 8.9. Consider the following derivations:

$$R\lor\,\dfrac{RW\,\dfrac{\delta_{\mathbb{O}_1}\,\dfrac{}{\Pi \mid \ {}_{\mathsf{O}}\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. p}}{\Pi \mid \ {}_{\mathsf{O}}\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. p, \neg q}}{\Pi \mid \ {}_{\mathsf{O}}\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. p \lor \neg q} \qquad \dfrac{\dfrac{\dfrac{}{\Pi \mid \ {}_{\mathsf{O}}\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \neg q}\,\delta_{\mathbb{O}_2}}{\Pi \mid \ {}_{\mathsf{O}}\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \neg q, p}\,RW}{\Pi \mid \ {}_{\mathsf{O}}\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \neg q \lor p}\,R\lor$$

These derivations are proofs. This corresponds to the fact that if one ought to prosecute her father and ought not dishonor her father, then one ought to either prosecute or not dishonor her father [55].

The following examples show that controlled calculi are expressive enough to handle deontic notions and principles which cannot be formalized in the standard formulation of constrained I/O logic.

Example 8.11 (Obligations with exceptions). *Let the extra-logical rule*

$$\dfrac{\Gamma \ {}_{\mathsf{F}}\left|\frac{\mathbf{T}}{\mathbf{S}}\right. p}{\Gamma \mid \ {}_{\mathsf{O}}\left|\frac{\mathbf{T'}}{\mathbf{S'}}\right. \neg q}\,\delta_{\mathbb{O}'}$$

stand for the conditional obligation 'If one is served a meal, one ought not eat with fingers – provided the meal is not asparagus' [173], *where* $\mathbf{T'} = \mathbf{T} \cup \{\{p^{\mathsf{f}}\}\}$ *and* $\mathbf{S'} = \mathbf{S} \cup \{\{\neg\neg q^{\circ}\}, \{\neg r^{\mathsf{f}}\}\}$. *Consider the following derivation:*

$$\dfrac{\dfrac{\dfrac{}{p \ {}_{\mathsf{F}}\left|\frac{\varnothing}{\varnothing}\right. p}\,ax}{r, p \ {}_{\mathsf{F}}\left|\frac{\varnothing}{\varnothing}\right. p}\,LW}{r, p \mid \ {}_{\mathsf{O}}\left|\frac{\mathbf{V}}{\mathbf{U}}\right. \neg q}\,\delta_{\mathbb{O}'}$$

with $\mathbf{V} = \{\{p^{\mathsf{f}}\}\}$ *and* $\mathbf{U} = \{\{\neg\neg q^{\circ}\}, \{\neg r^{\mathsf{f}}\}\}$. *Such derivation is a paraproof. This corresponds to the fact that if one is served asparagus, the obligation not to eat with fingers ceases to apply.*

Example 8.12 (Guarded free choice permission). *Let the extra-logical axiom and rule*

$$ax^{\mathsf{P}}\,\dfrac{}{\Pi \mid \ {}_{\mathsf{P}}\left|\frac{\varnothing}{\varnothing}\right. p, q} \qquad \dfrac{}{\Pi \mid \Gamma \ {}_{\mathsf{O}}\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. r}\,\delta_{\mathbb{O}'}$$

stand for the unconditional permission 'One is permitted to work or relax' and the unconditional obligation 'One ought to pay the bill', respectively – where $\mathbf{T_1} = \{\{\top^{\circ}\}\}$ *and* $\mathbf{S_1} = \{\{\neg r^{\circ}\}\}$.

The free choice principle allows to infer that a non-contradictory formula A_i *is permitted whenever* $A_1 \vee \cdots \vee A_m$ *is permitted, for any* $1 \leq i \leq m$ [175, p. 21]. *The following extra-logical rule formalizes such principle:*

$$\dfrac{\Pi \mid \Gamma \ {}_{\mathsf{P}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\ A_1, \ldots, A_m}{\Pi \mid \Gamma \ {}_{\mathsf{P}}\Big|\!\frac{\mathbf{T}}{\mathbf{S}}\ A_i}\, fcp$$

Unrestricted applications of the free choice principle lead to permission explosion [12]. *For instance, if one is permitted to work or relax or not pay the bill, then one is not permitted to pay the bill – whereas the obligation to pay the bill entails the permission to pay the bill. Now, consider the following application of a suitable rule* $fcp^{\star}$*:*

$$\dfrac{RW\,\dfrac{ax^{\mathsf{p}}\,\dfrac{}{\varnothing \mid {}_{\mathsf{P}}\Big|\!\frac{\varnothing}{\varnothing}\ p, q}}{\varnothing \mid {}_{\mathsf{P}}\Big|\!\frac{\varnothing}{\varnothing}\ p, q, \neg r} \qquad \dfrac{\dfrac{}{\varnothing \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\ r}\,\delta_{\mathcal{O}'}}{\varnothing \mid {}_{\mathsf{P}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\ r}\,op}{\varnothing \mid {}_{\mathsf{P}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S'_1}}\ \neg r}\, fcp^{\star}$$

where $\mathbf{S'_1} = \mathbf{S_1} \cup \{\{\neg r^{\mathsf{p}}\}\}$. *Such derivation is a paraproof. This corresponds to the application of a guarded version of the free choice permission, according to which you can infer* A_i *provided that there is no permission to the contrary.*

There exist interesting deontic notions arising from the non-trivial interaction between the notions of fact, (un)conditioned (constrained) obligation and permission. In the following examples we deal with these non-primitive deontic notions through the *lens* of controlled calculi. In some cases, we enable control pairs to *decrease* their size along derivations – contrary to what happens with applications of standard extra-logical rules in $\mathsf{G4}^{\delta}$ calculi.

EXAMPLE 8.13 (Violations and sanctions). *Let the extra-logical rule*

$$\delta_{\mathcal{O}_1}\,\dfrac{}{\Pi \mid {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_1}}{\mathbf{S_1}}\ \neg p} \qquad \dfrac{\Pi \ {}_{\mathsf{F}}\Big|\!\frac{\mathbf{T'_2}}{\mathbf{S'_2}}\ p \qquad \Pi \mid \Gamma \ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T''_2}}{\mathbf{S''_2}}\ \neg p}{\Pi \mid \Gamma \ {}_{\mathsf{O}}\Big|\!\frac{\mathbf{T_2}}{\mathbf{S_2}}\ q}\,\delta_{\mathcal{O}_2}$$

stand for 'One ought not double-park' and 'If one double-parks and she ought not double-park, she ought to pay a fine', respectively – where

(*i*) $\mathbf{T_1} = \{\{\top^{\circ}\}\}$ *and* $\mathbf{S_1} = \varnothing$;

(*ii*) $\mathbf{T_2} = \mathbf{T'_2} \cup \mathbf{T''_2} \cup \{\{p^{\mathsf{f}}\}, \{\neg p^{\circ}\}\}$ *and* $\mathbf{S_2} = \mathbf{S'_2} \cup \mathbf{S''_2} \cup \{\{\neg q^{\circ}\}\}$.

Consider the following derivation:

$$\dfrac{\begin{array}{c}\vdots\\ \Pi\ {}_{\mathsf{F}}\Big|^{\mathbf{T}'_2}_{\mathbf{S}'_2}\ p\end{array} \qquad \dfrac{}{\Pi \mid {}_{\mathsf{O}}\Big|^{\mathbf{T_1}}_{\mathbf{S_1}}\ \neg p}\,\delta_{\mathcal{O}_1}}{\Pi \mid \Gamma\ {}_{\mathsf{O}}\Big|^{\mathbf{T}}_{\mathbf{S}}\ q}\,\delta_{\mathcal{O}_2}$$

with $\mathbf{T} = \mathbf{T'_2} \cup \mathbf{T_1} \cup \{\{p^{\mathsf{f}}\}, \{\neg p^{\circ}\}\}$ *and* $\mathbf{S} = \mathbf{S'_2} \cup \mathbf{S_1} \cup \{\{\neg q^{\circ}\}\}$. *Such derivation is a proof. This corresponds to the fact that if the obligation of not double-parking is actual, then the obligation of paying a fine is actual.*

EXAMPLE 8.14 (Permissions as exceptions). *Let the extra-logical rule*

$$\dfrac{}{\Pi \mid \Gamma\ {}_{\mathsf{O}}\Big|^{\mathbf{T}}_{\mathbf{S}}\ \neg p}\,\delta_{\mathcal{O}'}$$

stand for the unconditional obligation 'One ought not kill herself' – where $\mathbf{T} = \{\{\top^{\circ}\}\}$ *and* $\mathbf{S} = \{\{\neg\neg p^{\circ}\}\}$. *The Torah contains such obligation as a commandment (cf. Genesis 9:5). However, some Rabbinic scholars state that suicide (or surrender to death) is permissible when one is threatened with conversion (as in the case of King Saul in 2 Samuel, 1:5-10). Let the extra-logical rule*

$$\dfrac{}{\Pi \mid \Gamma\ {}_{\mathsf{O}}\Big|^{\mathbf{T}}_{\mathbf{S}'}\ \neg p}\,\delta'_{\mathcal{O}'}$$

stand for the unconditional obligation with exceptions 'One ought not kill herself, provided that she is not threatened with conversion' – where $\mathbf{S}' = \mathbf{S} \cup \{\{q^{\mathsf{f}}\}\}$. *To infer the positive permission to kill oneself, we can apply the following rule pe:*

$$\dfrac{\dfrac{}{\varnothing \mid {}_{\mathsf{O}}\Big|^{\mathbf{T}}_{\mathbf{S}'}\ \neg p}\,\delta'_{\mathcal{O}'}}{q \mid {}_{\mathsf{P}}\Big|^{\mathbf{T}}_{\mathbf{S}}\ p}\,pe$$

EXAMPLE 8.15 (Dynamic positive permissions). *The notion of dynamic positive permission is introduced in* [99]*: B is dynamically permitted under a condition A if prohibiting B in the context of A would block the exercise of some explicit (static) permission, thereby generating incoherence. Consider e.g. the following scenario* [161]*:*

> *Freedom of expression [...] is recognized as a human right under Article 19 of the Universal Declaration of Human Rights [...]. An example that comes to mind is the Jyllands-Posten incident of 2005, when Muslim organizations led a complaint with the Danish police, following the publication of twelve cartoons depicting the Islamic prophet Mohammad. The investigation was discontinued by the Regional Prosecutor in Viborg, who concluded that Jyllands-Posten must be reckoned protected by the freedom of expression. [...] One may say, therefore, that the printing of the cartoons was deemed [dynamically] permitted by the Danish authorities.*

Let the extra-logical rule

$$\delta_{\mathcal{P}_1}\,\dfrac{}{\Pi \mid \Gamma \;{}_{\mathsf{P}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. p} \qquad \dfrac{\Pi \mid \Gamma \;{}_{\mathsf{O}}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. \neg q}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\!\left|\frac{\mathbf{T'_2}}{\mathbf{S'_2}}\right. \neg p}\,\delta_{\mathcal{O}_1}$$

stand for the unconditional permission 'One is permitted to express herself freely' and the conditional obligation 'If one ought not print cartoons depicting Mohammed, then one ought not express herself freely', respectively – where

(*i*) $\mathbf{T_1} = \{\{\top^{\mathsf{p}}\}\}$ *and* $\mathbf{S_1} = \{\{\neg p^{\mathsf{p}}\}\}$*;*

(*ii*) $\mathbf{T'_2} = \mathbf{T_2} \cup \{\{\neg q^{\mathsf{o}}\}\}$ *and* $\mathbf{S'_2} = \mathbf{S_2} \cup \{\{\neg\neg p^{\mathsf{o}}\}\}$*.*

Suppose one adds to $\mathsf{G4}^{\delta}$ *the extra-logical rule*

$$\dfrac{}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\!\left|\frac{\mathbf{V}}{\mathbf{U}}\right. \neg q}\,\delta_{\mathcal{O}_2}$$

standing for the unconditional obligation 'One ought not print the cartoons depicting Mohammed' – where $\mathbf{V} = \{\{\top^{\mathsf{f}}\}, \{\top^{\mathsf{o}}\}\}$ *and* $\mathbf{U} = \{\{\neg\neg q^{\mathsf{o}}\}\}$*. The derivation of the dynamic permission to print the cartoons depicting Mohammed corresponds to the following application of a suitable inference rule dp:*

$$\dfrac{\delta_{\mathcal{P}_1}\,\dfrac{}{\varnothing \mid {}_{\mathsf{P}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. p} \qquad \dfrac{\dfrac{}{\varnothing \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{V}}{\mathbf{U}}\right. \neg q}\,\delta_{\mathcal{O}_2}}{\varnothing \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{V'}}{\mathbf{U'}}\right. \neg p}\,\delta_{\mathcal{O}_1}}{\varnothing \mid {}_{\mathsf{P}}\!\left|\frac{\mathbf{T}}{\mathbf{S}}\right. q}\,dp$$

with $\mathbf{V'} = \mathbf{V} \cup \{\{\neg q^{\mathsf{o}}\}\}$, $\mathbf{U'} = \mathbf{U} \cup \{\{\neg\neg p^{\mathsf{o}}\}\}$, $\mathbf{T} = \mathbf{T_1} \cup \mathbf{V}$ *and* $\mathbf{S} = \mathbf{S_1}$*. Notice that neither* $\mathbf{T}$ *is* $\mathbf{T_1} \cup \mathbf{V'}$*, nor* $\mathbf{S}$ *is* $\mathbf{S_1} \cup \mathbf{U'}$*. In other words, the dp application causes the removal of sets of formulas from* $\mathbf{T_1} \cup \mathbf{V'}$ *and* $\mathbf{S_1} \cup \mathbf{U'}$*.*

Example 8.16 (Talmudic *Qal wa-ḥomer*). *Let the extra-logical rules*

$$\delta_{\mathcal{O}_1}\,\dfrac{}{\Pi \mid {}_{\mathsf{O}}\!\left|\frac{\mathbf{T_1}}{\mathbf{S_1}}\right. \neg p} \qquad \dfrac{}{\Pi \mid {}_{\mathsf{P}}\!\left|\frac{\mathbf{T_2}}{\mathbf{S_2}}\right. q}\,\delta_{\mathcal{P}_1} \qquad \dfrac{\Pi \mid \Gamma \;{}_{\mathsf{P}}\!\left|\frac{\mathbf{T_3}}{\mathbf{S_3}}\right. q}{\Pi \mid \Gamma \;{}_{\mathsf{P}}\!\left|\frac{\mathbf{T'_3}}{\mathbf{S'_3}}\right. p}\,\delta_{\mathcal{P}_2}$$

stand for the unconditional obligation 'One ought not marry the daughter of his daughter', the unconditional permission 'One is allowed to marry one's daughter' and the conditional permission 'If one is allowed to marry one's daughter, one is allowed to marry the daughter of his daughter', respectively – where

(*i*) $\mathbf{T_1} = \{\{\top^{\mathsf{o}}\}\}$ *and* $\mathbf{S_1} = \{\{\neg\neg p^{\mathsf{o}}\}\}$*;*

(*ii*) $\mathbf{T_2} = \{\{\top^{\mathsf{p}}\}\}$ *and* $\mathbf{S_2} = \{\{\neg q^{\mathsf{p}}\}\}$*;*

(*iii*) $\mathbf{T'_3} = \mathbf{T_3} \cup \{\{q^{\mathsf{p}}\}\}$ *and* $\mathbf{S'_3} = \mathbf{S_3} \cup \{\{\neg p^{\mathsf{p}}\}\}$*.*

The Talmudic principle of Qal wa-ḥomer *licenses the inference to the unconditional obligation 'One ought not marry his daughter'* [158, p. 223]*. This corresponds to the following application of a suitable inference rule qw:*

$$\dfrac{\delta_{\mathcal{O}_1}\,\dfrac{}{\varnothing \mid {}_{\mathsf{O}}\Big|\tfrac{\mathbf{T_1}}{\mathbf{S_1}}\ \neg p} \qquad \dfrac{\dfrac{}{\varnothing \mid {}_{\mathsf{P}}\Big|\tfrac{\mathbf{T_2}}{\mathbf{S_2}}\ q}\,\delta_{\mathcal{P}_1}}{\varnothing \mid {}_{\mathsf{P}}\Big|\tfrac{\mathbf{T'_2}}{\mathbf{S'_2}}\ p}\,\delta_{\mathcal{P}_2}}{\varnothing \mid {}_{\mathsf{O}}\Big|\tfrac{\mathbf{T}}{\mathbf{S}}\ \neg q}\,qw$$

with $\mathbf{T'_2} = \mathbf{T_2} \cup \{\{q^{\mathsf{p}}\}\}$, $\mathbf{S'_2} = \mathbf{S_2} \cup \{\{\neg p^{\mathsf{p}}\}\}$, $\mathbf{T} = \mathbf{T_1}$ *and* $\mathbf{S} = \{\{\neg\neg q^{\mathsf{o}}\}\}$.

Here is an example of an extra-logical rule application which simultaneously adds and removes formulas from the control pairs of the premises.

Example 8.17 (Deontic sure-thing principle). *Let the extra-logical rules*

$$\delta_{\mathcal{O}_1}\,\dfrac{\Gamma \ {}_{\mathsf{F}}\Big|\tfrac{\mathbf{T_1}}{\mathbf{S_1}}\ p}{\Gamma \mid {}_{\mathsf{O}}\Big|\tfrac{\mathbf{T'_1}}{\mathbf{S'_1}}\ q} \qquad \dfrac{\Gamma \ {}_{\mathsf{F}}\Big|\tfrac{\mathbf{T_2}}{\mathbf{S_2}}\ \neg p}{\Gamma \mid {}_{\mathsf{O}}\Big|\tfrac{\mathbf{T'_2}}{\mathbf{S'_2}}\ q}\,\delta_{\mathcal{O}_1}$$

stand for the conditional obligations 'If there will be a nuclear war, we ought to be disarmed' and 'If there will not be a nuclear war, we ought to be disarmed', respectively – where

(*i*) $\mathbf{T'_1} = \mathbf{T_1} \cup \{\{p^{\mathsf{f}}\}\}$ *and* $\mathbf{S'_1} = \mathbf{S_1} \cup \{\{\neg q^{\mathsf{o}}\}\}$*;*

(*ii*) $\mathbf{T'_2} = \mathbf{T_2} \cup \{\{\neg p^{\mathsf{f}}\}\}$ *and* $\mathbf{S'_2} = \mathbf{S_2} \cup \{\{\neg q^{\mathsf{o}}\}\}$.

The deontic sure-thing principle licenses the inference to the unconditional obligation 'One ought to be disarmed' [75]*. This corresponds to the following application of a suitable inference rule st:*

$$\dfrac{\delta_{\mathcal{O}_1}\,\dfrac{ax\,\dfrac{}{p \ {}_{\mathsf{F}}\Big|\tfrac{\varnothing}{\varnothing}\ p}}{p \mid {}_{\mathsf{O}}\Big|\tfrac{\mathbf{T'}}{\mathbf{S'}}\ q} \qquad \dfrac{\dfrac{}{\neg p \ {}_{\mathsf{F}}\Big|\tfrac{\varnothing}{\varnothing}\ \neg p}\,ax}{\neg p \mid {}_{\mathsf{O}}\Big|\tfrac{\mathbf{T''}}{\mathbf{S''}}\ q}\,\delta_{\mathcal{O}_2}}{p \vee \neg p \mid {}_{\mathsf{O}}\Big|\tfrac{\mathbf{T}}{\mathbf{S}}\ q}\,st$$

with $\mathbf{T'} = \{\{p^{\mathsf{f}}\}\}$, $\mathbf{S'} = \mathbf{S''} = \mathbf{S} = \{\{\neg q^{\mathsf{o}}\}\}$, $\mathbf{T''} = \{\{\neg p^{\mathsf{f}}\}\}$ *and* $\mathbf{T} = \{\{p \vee \neg p^{\mathsf{f}}\}, \{p \vee \neg p^{\mathsf{o}}\}\}$. *Let us mention another example of a scenario where the deontic sure-thing principle is applied: "a priestess enjoined upon her son not to take to public speaking: 'For', she said, 'if you say what is right, men will hate you; if you say what is wrong, the gods will hate you' " (Aristotle,* Rhetoric*, 1399a).*

Control pairs can be refined so as to handle problematic scenarios involving unrestricted weakening of obligations. A *symmetric control pair* is a pair of the form $\langle, \mathbf{T}, \langle \mathbf{S}, \mathbf{S'} \rangle$, where $\mathbf{T}$ collects sets of conditions, $\mathbf{S}$ collects sets of constraints for formulas in antecedent position and $\mathbf{S'}$ collects sets of constraints for formulas in succedent position. Let us adapt the notions of compatibility and soundness to symmetric control pairs as follows.

Definition 8.10. Given a multiset Δ and a symmetric control pair $\langle \mathbf{T}, \langle \mathbf{S}, \mathbf{S'} \rangle\rangle$, Δ is *dually compatible* with $\langle \mathbf{T}, \langle \mathbf{S}, \mathbf{S'} \rangle\rangle$ (in symbols $\Delta \,||^{d}\, \langle \mathbf{T}, \langle \mathbf{S}, \mathbf{S'} \rangle\rangle$) if and only if for every formula A in $\bigcup \mathbf{S'}$, $\overline{\overline{\mathsf{G4^{pn}}}}$ refutes $A \vdash \bigvee \Delta$.

Definition 8.11. Let $\mathfrak{S}$ be the pair $\langle \mathbf{S_1}, \mathbf{S_2} \rangle$ and π be a $\mathsf{G4}^{\delta}$-derivation of $\Gamma \;{}_{\mathsf{O}}\big|^{\mathbf{T}}_{\mathfrak{S}}\; \Delta$ and $\mathbf{T}^{\circ}, \mathbf{S}^{\circ}_{\mathbf{1}}, \mathbf{S}^{\circ}_{\mathbf{2}}$ be the subsets of $\mathbf{T}, \mathbf{S_1}, \mathbf{S_2}$, respectively, containing only sets of o-labelled formulas. The sequent $\Gamma \;{}_{\mathsf{O}}\big|^{\mathbf{T}}_{\mathfrak{S}}\; \Delta$ is *dually sound* if and only if $\Gamma \;{}_{\mathsf{O}}\big|^{\mathbf{T}}_{\mathbf{S_1}}\; \Delta$ is sound and $\Delta \,||^{d}\, \langle \varnothing, \langle \varnothing, \mathbf{S_2} \rangle \rangle$.

We end this section with the following applications of symmetric control pairs.

Example 8.18 (Ross's paradox). *Let the extra-logical axiom and rules*

$$\delta_{\mathcal{O}_1}\,\frac{}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\big|^{\mathbf{T_1}}_{\mathbf{S_1}}\; p} \qquad \delta_{\mathcal{O}_1}\,\frac{}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\big|^{\mathbf{T_2}}_{\mathbf{S_2}}\; \neg q} \qquad \frac{}{r \;{}_{\mathsf{F}}\big|^{\varnothing}_{\varnothing}\; q}\,ax$$

stand for the unconditional obligations 'One ought to mail the letter', 'One ought not destroy the letter' and the factual statement 'If one burns the letter, she destroys it', respectively – where $\mathbf{T_1} = \mathbf{T_2} = \{\{\top^{\circ}\}\}$, $\mathbf{S_1} = \{\{\neg p^{\circ}\}\}$ and $\mathbf{S_2} = \{\{\neg\neg q^{\circ}\}\}$.

Since 'One mails the letter or one burns the letter' can be inferred from 'One mails the letter' by classical logic, closure of obligations under classical consequence yields the unconditional obligation 'One ought to post the letter or burn it'. This implies that the obligation of mailing the letter entails an obligation which can be fulfilled by burning it – something which seems undesirable [145].

To avoid this scenario, we may block the instance of Right Weakening via *the following application of a suitable rule rrw:*

$$\frac{\delta_{\mathcal{O}_1}\,\dfrac{}{\varnothing \mid \;{}_{\mathsf{O}}\big|^{\mathbf{T_1}}_{\mathbf{S_1}}\; p} \qquad \dfrac{}{\varnothing \mid \;{}_{\mathsf{O}}\big|^{\mathbf{T_2}}_{\mathbf{S_2}}\; \neg q}\,\delta_{\mathcal{O}_1} \qquad \dfrac{}{r \;{}_{\mathsf{F}}\big|^{\varnothing}_{\varnothing}\; q}\,ax}{\varnothing \mid \;{}_{\mathsf{O}}\big|^{\mathbf{T}}_{\mathfrak{S}}\; p \vee q}\,rrw$$

where $\mathfrak{S} = \langle \mathbf{S_1} \cup \mathbf{S_2}, \{\{p \vee q^{\circ}\}\} \rangle$.

Example 8.19 (Good Samaritan paradox). *Let the following extra-logical rules*

$$\delta'_{\mathcal{O}_1}\,\frac{}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\big|^{\mathbf{T_1}}_{\mathbf{S_1}}\; p} \qquad \frac{}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\big|^{\mathbf{T_1}}_{\mathbf{S_1}}\; q}\,\delta''_{\mathcal{O}_1}$$

stand for the unconditional obligation 'It ought to be that the Samaritan helps Jones whom Smith has robbed', and the extra-logical rule

$$\frac{}{\Pi \mid \Gamma \;{}_{\mathsf{O}}\big|^{\mathbf{T_2}}_{\mathbf{S_2}}\; \neg q}\,\delta_{\mathcal{O}_2}$$

stand for the unconditional obligation 'Smith ought not rob Jones' [141] *– where $\mathbf{T_1} = \mathbf{T_2} = \{\{\top^{\circ}\}\}$, $\mathbf{S_1} = \{\{\neg(p \wedge q)^{\circ}\}\}$ and $\mathbf{S_2} = \{\{\neg\neg q^{\circ}\}\}$. Consider the following application of rrw:*

$$\frac{\dfrac{\delta'_{\mathcal{O}_1}\,\dfrac{}{\varnothing \mid \;{}_{\mathsf{O}}\big|^{\mathbf{T_1}}_{\mathbf{S_1}}\; p} \qquad \delta''_{\mathcal{O}_1}\,\dfrac{}{\varnothing \mid \;{}_{\mathsf{O}}\big|^{\mathbf{T_1}}_{\mathbf{S_1}}\; q}}{\varnothing \mid \;{}_{\mathsf{O}}\big|^{\mathbf{T_1}}_{\mathbf{S_1}}\; p \wedge q} \qquad \dfrac{}{\varnothing \mid \;{}_{\mathsf{O}}\big|^{\mathbf{T_2}}_{\mathbf{S_2}}\; \neg q}\,\delta_{\mathcal{O}_2} \qquad \begin{matrix} \vdots \\ p \wedge q \;{}_{\mathsf{F}}\big|^{\varnothing}_{\varnothing}\; q \end{matrix}}{\varnothing \mid \;{}_{\mathsf{O}}\big|^{\mathbf{T}}_{\mathfrak{S}}\; q}\,rrw$$

where $\mathbf{T} = \mathbf{T_1} \cup \mathbf{T_2}$ *and* $\mathfrak{S} = \langle \mathbf{S_1} \cup \mathbf{S_2}, \{\{q^\circ\}\}\rangle$*. Such derivation is a paraproof. This corresponds to the fact that the obligation to help victims does not imply that the existence of victims is itself obligatory. Notice that such application of rrw corresponds to the intuitive assessment of the paradox even when consistency constraints are dropped – and thus* $\mathbf{S_1} = \mathbf{S_2} = \varnothing$*.*

CHAPTER 9

Concluding remarks and open problems

This dissertation discusses a uniform proof-theoretic approach to different kinds of non-monotonic and normative reasoning based on the based on combinations of proofs and refutations framed in suitable Gentzen-style calculi. In this chapter, we shall discuss further applications of our proof-theoretic framework, sketching some lines for future research.

9.1. On anticut rules

In Chapter 3, we conducted a proof-theoretic investigation of D-R calculi for classical and non-classical logics, such as **FDE** and intuitionistic logic. Specifically, we have proved that anticut rules cannot be eliminated from hybrid sequent calculi without compromising Ł-completeness. Moreover, when anticut rules are combined with an appropriate set of rules, the resulting system attains refutational completeness. Finally, we have explored the role of anticut rules in unmixed sequent calculi, showing that anticut rules can be constructively eliminated by leveraging Ł-completeness.

From a methodological standpoint, combining anticut rules with an appropriate set of additional refutational principles proves to be a flexible and robust approach, adaptable to a wide range of logics. For instance, this method is equally applicable to various modal logics [56, 76] and intermediate logics [154, 59]. The minimal hybrid calculi developed for these logics can serve as effective frameworks for constructing new unmixed calculi tailored to the same logics, with **S4** being a notable example [71, 46].

From a broader perspective, it would be worthwhile to investigate whether there exist logics whose antitheorems can be recursively axiomatized only through hybrid, rather than unmixed, D-R systems. An especially intriguing case study in this context is Medvedev's logic of finite frames, given the long-standing open problem concerning its decidability [26, 154].

Finally, a further promising line of research appears to be the connection between anticut rules and the formulation of abductive problems viewed as deductive arguments in reverse (see Chapter 4).

9.2. On abductive reasoning

In Chapter 4, we presented a proof-theoretic framework to analyze abductive reasoning in classical propositional logic by reading abduction as an *enthymematic* deductive argument in reverse. We assumed the minimal set of logical conditions A1-A3 for abductive explanations,

$$(1)\ \cfrac{\cfrac{\cfrac{[p]^1 \qquad \cfrac{p \to q \qquad [p]^1}{q}\,\to \mathcal{E}}{p \wedge q}\,\wedge\mathcal{I} \qquad (p \wedge q) \to r}{r}\,\to \mathcal{E}}{p \to r}\,\to \mathcal{I}$$

$$(1)\ \cfrac{\cfrac{\cfrac{p \to q \qquad [p]^1}{q}\,\to \mathcal{E} \qquad q \to r}{r}\,\to \mathcal{E}}{p \to r}\,\to \mathcal{I}$$

FIGURE 1. LCH and EH from the natural deduction point of view

though we acknowledge that the literature suggests additional conditions [157, 49] that could be explored in combination with the ones we focussed on in these pages. We also highlighted certain discrepancies between the deductively minimal solution and the expected solution. This led us to design an effective procedure (Procedure 3) which recovers what seems to better approximate the reasoner's expectations by pruning the leaves of the deduction-tree from the redundant information.

It should be noted that when presented in a standard natural deduction calculus, achieving deductive saturation through an expected hypothesis often requires fewer steps than achieving it through the minimal hypothesis. This suggests that a better understanding of the notion of expected explanation could be gained by aiming for minimality in terms of derivation length. As shown in Figure 1, consider the abductive problem $p \to q, ? \vdash p \to r$. It can be observed that inserting the expected hypothesis $q \to r$ results in a simpler derivation compared to assuming the deductively minimal formula $(p \wedge q) \to r$. However, such a characterization is inherently arbitrary because the complexity of a derivation depends on the specific formalism used as a measuring device.

We believe it would be valuable to broaden the application of our proof-theoretic framework to include conservative extensions of classical propositional logic, such as modal logics [104, 111], supraclassical logics [93, 121], non-monotonic logics [143, 4, 34] and a logic for exception and typicality [130]. Additionally, it appears that modifications of this framework could work for other non-classical logics. Moreover, the refutation-based approach presented in our work can, in theory, be extended to decidable fragments of predicate logic, with monadic first-order logic presenting an interesting case study, particularly in relation to the traditional topic of *inventio medii* (see e.g. [100]). A broader perspective could involve taming full first-order logic by utilizing an appropriate notion of *approximated* refutation and *approximated* deductive saturation.

Procedure 3 provides a proof-theoretic account of the process whereby a rational agent produces an optimal *analytic* solution for a given abductive problem. However, there has been an increasing emphasis among philosophers of science on cases of *creative* abduction, that is situations in which the reasoner formulates abductive hypotheses by incorporating pieces of information not deducible from the original problem [150]. By its very nature, deductive logic cannot anticipate the specific non-analytic information that a rational agent will utilize to solve the abductive problem. Nonetheless, the technical results presented in Section 5 offer a comprehensive approach to effectively distinguish analytic components from non-analytic ones within any non-analytical solution. This methodical treatment of non-analytic solutions seems to suggest that *supraclassical* analytic calculi may offer the appropriate proof-theoretic framework for tackling the challenge of creative abduction (cf. [121]).

Another topic for future investigation is the application of our proof-theoretic account to *contrastive* explanations [87] – i.e., explanations whereby one clarifies why a particular outcome occurs rather than a different, alternative outcome, by focusing on some difference between the two. More broadly, it would be intriguing to investigate whether the proof-theoretic apparatus employed in the present thesis can be applied to Mill's methods for eliminative induction [155, pp. 69-109].

Finally, it is widely accepted that the best *explanans* should be chosen based on its higher degree of truthlikeness or verisimilitude [39, p. 48], [25, 112]. It would be interesting to examine our approach for identifying candidates for the best *explanans* in relation to the definitions of truthlikeness proposed in the literature [113], and explore the possibility of using a fractional approach [123] to further refine our method.

9.3. On hybrid hypersequents for default logics

Let us discuss further applications of the proof-theoretic framework for default logics detailed in Chapter 5, sketching some lines for future research.

9.3.1. Reiter extensions. We introduced HG4c calculi which are sound and (weakly) complete with respect to m-credulous consequence: crucially, Theorem 3.5 capitalizes on the property of semi-monotonicity, enabling the inductive approximation of modified extensions (cf. Proposition 5.5). Extending our framework to encompass Reiter's *credulous consequence* would be appealing. However, the main challenge lies in the occurrence of pathological cycles in default applicability: in such scenarios, a finite set of rules can be activated and must be simultaneously withheld (cf. Examples 5.3 and 5.19).

Many strategies aimed at preventing the occurrence of such cycles necessitate restricting to specific classes of default theories [44, 115, 28]. One particularly promising approach is detailed in [86], where the issue is addressed by transitioning from *global* consistency checks to *local* ones through the introduction of auxiliary concepts such as *blocking sets* and *supporting sets* for a default rule [86, pp. 39 and 48]. It's worth noting that these concepts can be reformulated within our hypersequent-based framework:

DEFINITION 9.1. Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be a default theory and $\delta \in \mathcal{D}$ with $just(\{\delta\}) = C_1, \dots, C_k$. We say that an $\mathsf{HG4}$-derivation π is *orthogonal to* δ if $\overline{\overline{\mathsf{G4}}}$ proves $W, D_\pi \vdash \neg C_h$ with $1 \leq h \leq k$.

- (i) $\mathcal{D}' \subseteq \mathcal{D}$ is a *blocking set for* δ if the following conditions hold:
  - (a) there exists an $\mathsf{HG4}$-derivation π orthogonal to δ such that $\mathcal{D}' = def(\pi)$;
  - (b) for any $\delta' \in \mathcal{D}'$, if there exists an $\mathsf{HG4}$-proof π' such that $(\mathcal{D}' \setminus \{\delta'\}) = def(\pi')$, then π' is not orthogonal to δ.
- (ii) $\mathcal{D}' \subseteq \mathcal{D}$ is a *supporting set for* δ if $\mathcal{D}'$ is a blocking set for δ', and $\delta' \in \mathcal{D}''$ for some blocking set $\mathcal{D}''$ for δ.

We envisage that Definition 5.2 can be suitably strengthened so as to get $\mathsf{HG4}$ calculi which are sound and (weakly) complete with respect to credulous consequence.

9.3.2. Cumulative default logics. We contend that our proof-theoretic framework encompasses other default logics, wherein the consistency checks for default applicability are *stronger* than those mandated by Łukaszewicz's interpretation [22, 36, 149, 37]. Since skeptical consequence based on some of these variants enjoys both Cautious Monotony and Cumulative Cut (cf. Propositions 5.3 and 5.4), we claim that $\mathsf{HG4}$ calculi can be formulated in such a manner as to accommodate *cumulative* default logics.

9.3.3. Exclusionary default reasoning. Let $\langle \mathcal{W}, \mathcal{D} \rangle$ be a normal default theory. If δ is a default rule of the form

$$\frac{B : D}{D}$$

then any rule δ' of the form

$$\frac{C : \neg D}{\neg D}$$

is a *rebutting defeater* for δ whenever B and C are proved. We can pursue the approach in [70] and augment the language with a constant out_δ for the default rule δ: this move enables us to express a rule δ'' of the form

$$\frac{E : out_\delta}{out_\delta}$$

δ'' is an *undercutting defeater* for δ [133] whenever B and E are proved. The intuitive interpretation of δ'' is that if E is proved, then δ cannot be applied, so long as $\neg out_\delta$ is not provable – even if B is proved and $\neg D$ is not.

Defining $\mathsf{HG4}$ calculi for exclusionary default reasoning presents notable challenges, particularly without resorting to the explicit adoption of out_δ. If we consider $\mathsf{top_c}(\mid\!\overset{*}{-}\, E) = \Theta \mid\!\overset{*}{-}\, \Lambda$, one may be tempted to associate with δ'' an extra-logical rule of the following form:

$$\frac{\Gamma, \Theta \vdash \Lambda, \Delta \mid \mathcal{R} \qquad \Theta \vdash \Lambda \mid \mathcal{R}'}{\Gamma, \Theta \vdash \Lambda, \Delta \mid \mathcal{R} \mid \mathcal{R}' \mid \dashv D}\ \delta''$$

Investigating whether this formalization, perhaps augmented with additional syntactic apparatus, is sufficiently expressive to encapsulate the inferential dynamics of exclusionary default reasoning could provide deeper insights into the structure of defeasible reasoning systems.

9.3.4. Non-classical default logics. The realm of default logics built upon a non-classical base remains largely unexplored – with a few exceptions [41, 84, 102, 132]. We posit that our proof-theoretic framework offers significant potential for generating new insights in this area. For instance, HG4 calculi can be adapted to default logics based on the first degree fragment **FDE** of relevance logic **E**, building on the sequent calculus for **FDE** in [140, pp. 446-447]: it would be interesting to compare such a proof-theoretic treatment with the one detailed in [18]. Moreover, it is intriguing to note the conceptual resonance between HG4-paraproof and incorrect *proof structure* in the proof-theory of linear logic [54]: in the case of Reiter extensions, we may observe that the former shares with the latter a fundamental graph-theoretic character [86].

9.3.5. Decision procedure for skeptical m-consequence. In this paper, we have introduced a decision method for skeptical m-consequence based on the notion of orthogonal HG4-paraproof, proving its soundness and completeness (cf. Theorem 5.5). Nonetheless, several related areas warrant further exploration.

One important direction for future work is conducting a complexity analysis of Procedure 4.2 and comparing it with other decision methods, such as the enhanced sequent calculus for skeptical consequence proposed in [19]. Another avenue of research could focus on refining Procedure 6 to handle specific classes of m-skeptical consequence more efficiently. Moreover, tackling the issue of floating conclusions [96] from a proof-theoretic perspective may offer valuable insights. Such investigations could deepen our understanding of the structural and computational properties inherent to skeptical reasoning systems.

9.3.6. Default reasoning and abduction. HG4 calculi provide fresh insight into the interplay between default reasoning and abductive reasoning. Similar to non-monotonic reasoning, abductive reasoning operates under uncertainty in searching for the missing premise of a deductively invalid argument.

Let $\overline{\overline{\mathsf{G4s}}}$ be the hybrid sequent calculus for $\mathcal{W}$. By *abductive problem* we mean any expression of the form $\Gamma, \textcircled{?} \vdash G$, with $\Gamma \neq \varnothing$ and such that $\overline{\overline{\mathsf{G4s}}}$ refutes $\Gamma \vdash G$. Furthermore, an *abductive hypothesis for G given* Γ refers to any formula H such that $\Gamma, H \vdash G$ is provable in $\overline{\overline{\mathsf{G4s}}}$. If π is an HG4-proof of $\Gamma \vdash \Delta \mid \Pi_1 \dashv \Sigma_1 \mid \cdots \mid \Pi_m \dashv \Sigma_m$ with minimal refutational part, then the conclusion encodes abductive problems of the form $D_\pi, \textcircled{?} \vdash \bigwedge \Pi_h \to \bigvee \Sigma_h$, for any $1 \leq h \leq m$. Moreover, if π' is an HG4-paraproof orthogonal to π and π'' its immediate subderivation distinct from π, then there exists $\Pi \dashv \Sigma$ in $\Pi_1 \dashv \Sigma_1 \mid \cdots \mid \Pi_m \dashv \Sigma_m \mid \mathcal{R}_{\pi''}$ such that (at least) one of the following cases holds:

(i) $D_{\pi''}$ is an abductive hypothesis for $\bigwedge \Pi \to \bigvee \Sigma$ given D_π;

(*ii*) D_π is an abductive hypothesis for $\bigwedge \Pi \to \bigvee \Sigma$ given $D_{\pi''}$.

We thus notice that Procedure 6 assesses m-skeptical provability *via* finite sequences of abductive inferences against background assumptions in $\mathcal{W}$. Moreover, if $\langle \mathcal{W}, \mathcal{D} \rangle$ is a supernormal default theory, then Theorem 5.5 is equivalent to a result proved in [134, pp. 99-100].

Our proof-theoretic framework shows promise in offering a logical account of the notion of the "best explanation" in a defeasible setting, aligning with the analysis of this concept as presented in Chapter 4. This could provide fresh insight into the relationship between the two [43, 159].

9.3.7. A general approach to non-monotonic reasoning. Hybrid hypersequent calculi may offer a uniform and modular proof-theoretic platform for various logics designed to formalize defeasible reasoning. It would be interesting to compare the expressive power of HG4 calculi with the scope of application displayed by adaptive logics [162] and annotated sequent calculi [8, 9, 168].

9.4. On base-generated belief revision

In Chapter 6, we leverage a syntactic account of maximal consistent subsets of sets of clauses to define hybrid hypersequent calculi for a refined version of AGM belief revision. Specifically, we exploit the parallel composition of sequents and antisequents built into hybrid hypersequents to express contrary updating on the provability of initial sequents: this feature allows us to give an adequate formalization of base-generated revision – equivalently, a weak version of system R.

For future work, we believe that a hypersequent-based approach to iterated AGM belief revision would be promising, especially in view of the largely open problem of finding adequate operators satisfying DP postulates [35]. Moreover, we deem that hypersequent calculi for AGM belief revision can be easily adapted to multiple bases revision, with possible applications to belief merging [60].

Moreover, it would be valuable to extend the scope of application of the hypersequent-based approach to include KLM logics [82]. If we confine ourselves to KLM logics formulated in finite propositional languages, it appears that sound and (weakly) complete HG4 calculi can be obtained by endowing hybrid hypersequents with a tree-like structure: the binary relation $<$ on states would be codified by the refutational part of initial hypersequents, whereas formal conditions on $<$ would be captured by suitable structural rules.

9.5. On hybrid hypersequents for I/O logics

Our hypersequent-based approach to normative reasoning (detailed in Chapter 7) provides a unified framework that encompasses both unconstrained and constrained I/O logics, while remaining aligned with the original meta-level treatment of conditional obligations proposed in [97]. We proved that hybrid hypersequent calculi for I/O logics display desirable structural properties, including a constrained form of Cut elimination. Moreover, we detailed simple proof-theoretic translations between hypersequent calculi for $\bigcup \mathsf{outfamily}_3^+$ and $\bigcup \mathsf{outfamily}_4^+$ with input formulas as constraints and hypersequent calculi for normal default logic and a brave version of disjunctive default logic, respectively. These translations lay the groundwork for a modular, proof-theoretic treatment of both disjunctive default logics and disjunctive normative reasoning.

A promising research direction involves developing hybrid *nested* sequent calculi to address priority relations among conflicting obligations [116], prioritized default reasoning [23, 70], and reasoning based on specificity [38]. Furthermore, it would be valuable to extend the scope of application of hybrid hypersequents beyond the notion of conditional obligation. For instance, we deem that violations and sanctions can be captured via extra-logical rules integrating i-labelled and o-labelled premises [163]. Similarly, obligations with exceptions can be modeled using extra-logical rules where o-labelled sequents are composed in parallel with i-labelled antisequents. Finally, it would be interesting to embed I/O logics and default logics within a unified proof-theoretic framework using controlled sequent calculi [122, 130].

9.6. On controlled sequents for defaults and norms

In Chapter 8, we introduced a uniform proof-theoretic framework for defaults, obligations, and permissions based on controlled sequent calculi. We defined deontic extensions, paralleling Łukasiewicz extensions for defaults and generalizing outfamilies in I/O logics. Controlled calculi were shown to admit contraction and non-analytic cut, to exhibit a weak form of analyticity, and to be strongly complete with respect to credulous consequence grounded on Łukasiewicz and deontic extensions. Moreover, they support weak versions of cumulative transitivity and cautious monotony, thus providing a unified and robust basis for deontic and nonmonotonic reasoning.

Several perspectives open up. We have treated prohibitions as obligations to the contrary. However, certain contexts call for a more refined treatment of prohibitions – for instance, within Talmudic deontic logic [1]. It would therefore be of particular interest to integrate explicit prohibitions into the very definition of normative system, introducing a dedicated labelled turnstile ${}_{\mathsf{PR}}\!\vdash$ to develop strongly complete $\mathsf{G4}^\delta$ calculi. Additional extra-logical rules involving ${}_{\mathsf{PR}}\!\vdash$ could then capture the intricate interactions between prohibitions and other deontic primitives, such as obligations and permissions.

Moreover, it would be worthwhile to examine whether $\mathsf{G4}^\delta$ calculi can be employed to address prioritized default and normative reasoning [23, 69, 139]. Since prioritized m-

and d-extensions lack the key property of semimonotonicity, defining a sound notion of $\mathsf{G4}^{\delta}$-proof capable of tracking prioritized credulous consequence may require a suitable notion of *hypothetical soundness* – i.e., soundness under the assumption that some formulas are not provable.

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