

WELL-POSEDNESS OF ROUGH 2D EULER EQUATIONS WITH BOUNDED VORTICITY*

LEONARDO ROVERI[†] AND FRANCESCO TRIGGIANO[‡]

Abstract. We consider the 2D Euler equations with bounded initial vorticity and perturbed by rough transport noise. We show that a unique solution exists, which coincides with the starting condition advected by the Lagrangian flow. Moreover, we prove that the solution map is continuous with respect to the initial vorticity, the advecting vector fields and the rough perturbation. As an immediate corollary, we obtain a Wong-Zakai result for fractional Brownian driving paths.

Key words. 2D Euler equations, rough transport noise, bounded vorticity

MSC codes. 35Q31, 60L50

1. Introduction. The Euler equations describe the motion of an incompressible non-viscous fluid in a domain. Despite being formulated for the first time almost three centuries ago, the interest in some stochastic versions of them is far more recent [8, 5, 6, 42, 10, 9, 15]. Since these first works, well-posedness results have been obtained in several different settings, but it is still an open problem to determine whether a (unique) solution exists in the Yudovich vorticity setting, i.e. if the only hypothesis on the initial datum is boundedness, whenever the random forcing is less regular in time than a Brownian motion.

The growing interest in stochastic fluid dynamics is fundamentally rooted in the necessity to account for the unresolved scales of turbulent flows. As originally advocated in the seminal work by Brzeźniak et al. [8], stochastic partial differential equations provide a rigorous framework to model the complex, unpredictable evolution of small-scale features that cannot be exactly tracked, yet significantly influence the statistical properties of the fluid.

In recent years, this approach has gained further prominence in geophysical fluid dynamics (GFD) and climate science. The demand for robust and computationally efficient stochastic parameterization schemes is driven by the need for reliable uncertainty quantification in weather and climate predictions [4, 25]. In this context, stochastic perturbations are not merely an analytical tool, but a mean to better capture the feedback of small-scale effects on large-scale dynamics. Significant advancements in this direction include the development of variational principles for fluid dynamics on rough paths [38, 17], which provide a geometric foundation for stochastic GFD.

Among the various stochastic formulations, transport-type noise, often derived by modeling small-scale effects as additive noise at the Lagrangian level, represents an extremely promising area of research [9, 16, 23]. Indeed, transport-type noise driven by Brownian motion has yielded significant analytical results such as regularization-by-noise effects, which restore well-posedness in settings where deterministic models may

*

Funding: L.R. was supported by the project *Mathematical methods for climate science*, funded by the Ministry of University and Research (MUR) as part of the PON 2014-2020 “Research and Innovation” resources - Green Action - DM MUR 1061/2022. F.T. is supported by the Istituto Nazionale di Alta Matematica (INdAM) through the GNAMPA 2025 project “Modelli stocastici in Fluidodinamica e Turbolenza” (CUP E5324001950001)

[†]Department of Mathematics, University College London, 25 Gordon Street, London, UK-WC1E 6BT, UK (l.roveri@ucl.ac.uk).

[‡]Scuola Normale Superiore, Piazza dei Cavalieri 7, Pisa, I-56126, Italy (francesco.triggiano@sns.it).

show ill-posedness [23, 14, 30, 24], and, under suitable scaling limits, the appearance of a spare dissipative effect [22, 11].

While all the aforementioned works are restricted to the case of Brownian motion, some results have been identified for the Lagrangian flows even with less regular noise, such as fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$ [13, 12, 28]. Since the latter is not a semimartingale, classical stochastic calculus results are unavailable, and time integrals must be built in the rough integral sense. This suggests that rough path theory [40, 26, 27] may offer the right tools to treat fluid dynamics equations perturbed by transport-type noise when the driving signals are very irregular in time [33, 32].

Lately, the introduction of the “unbounded rough drivers” formalism [2, 19] has led to the recovery of classical tools, like the Grönwall lemma, in the rough path setting, allowing to successfully obtain well-posedness results [36, 37, 35, 16].

In particular, Crisan et al. [16] show local existence and uniqueness for a rough perturbation of d -dimensional Euler equations. Indeed, therein the authors prove local well-posedness whenever the starting velocity is in $H^{\lfloor d/2 \rfloor + 2}$, a Beale-Kato-Majda (BKM) blow-up criterion and a Wong-Zakai result. It is worth mentioning that, in the two-dimensional case, their rough Euler equations coincide exactly with the RPDE under consideration in this paper (see [16, Remark 3.5]). Specifically, by exploiting the active scalar structure of the vorticity and the exact preservation of all L^p norms, their BKM criterion allows them to show global well-posedness in such setting for initial vorticities in H^2 . Given that the embedding $H^2 \hookrightarrow L^\infty$ is strict, well-posedness for 2D Euler equations perturbed by rough transport noise is still lacking in the Yudovich setting [43], namely the vorticity formulation with bounded initial datum.

Our work is concerned exactly with the Yudovich case. Let us formulate the problem as

$$(1.1) \quad \begin{cases} \partial_t w(t, x) + u(t, x) \cdot \nabla w(t, x) = \sum_{j=1}^M \sigma_j(x) \cdot \nabla w(t, x) \dot{Z}_t^j, \\ w(0, x) = w_0(x), \quad u(t, \cdot) = K_{BS} * w(t, \cdot), \end{cases}$$

where $x \in \mathbb{T}^2$, $w_0 \in L^\infty(\mathbb{T}^2)$, $\sigma_j : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ are sufficiently smooth and divergence-free vector fields, Z^j are finite p -variation paths with $p \in [2, 3)$, and K_{BS} denotes the Biot-Savart kernel. We show that there exists a unique weak solution to (1.1) and to the corresponding Lagrangian equation. Moreover, we also obtain that the solution map is continuous with respect to the initial condition, the σ_j s and the drivers Z^j .

Apart from closing the gap between the results in [16] and the Yudovich setting [43], our work generalizes the well-posedness established by Brzeźniak et al. [9] for the 2D Euler equations driven by Stratonovich Brownian transport noise. It is important to note that, in the Brownian setting, Stratonovich integration represents a specific subcase of a geometric rough lift [26, Chapter 3]; therefore, our RPDE covers their SPDE as a particular case. In [9], authors obtain well-posedness in L^∞ for advecting fields $\{\sigma_k\}_k$ that satisfy a lower regularity requirement than ours, but at the cost of imposing the structural constraint $\sum_k \sigma_k(x) \sigma_k(x)^T = cI_2$. Our approach, not requiring such an algebraic constraint on the noise, allows for a more general class of perturbations, although it necessitates higher spatial regularity on the coefficients $\{\sigma_k\}_k$.

Along the way to prove our main results, we highlight a preparatory result that is of interest per se. Indeed, we manage to prove that the solution of a Lagrangian RDE with log-Lipschitz drift is unique (in the sense of controlled rough paths, see [31]) via

a localization procedure coupled with a tailored rough Grönwall Lemma.

Aside from that, the existence proof for the Euler equations follows by an approximation scheme and a compactness result, while the uniqueness one is obtained by showing that any solution is a Lagrangian one, namely it is advected by the associated flow ϕ_t . This Lagrangian approach, a general description of which can be found in [41], is proved via a new application of the doubling variable strategy introduced in [19]. Indeed, instead of using it to obtain control over some norm of the solution, like the kinetic energy, but we apply it in order to reproduce Dobrushin's approach from [21]. In this way we show that any observable of the inverse flow, $h_t = \psi(\phi_t^{-1}(x))$, solves a rough PDE and that the following holds

$$\int_{\mathbb{T}^2} w_t(x) h_t(x) dx = \int_{\mathbb{T}^2} w_0(x) h_0(x) dx.$$

We can informally state our main results as follows.

THEOREM 1.1. *Let $w_0 \in L^\infty(\mathbb{T}^2)$, $\mathbf{Z}$ be a p -geometric rough path with $p \in [2, 3)$ and $\{\sigma_j\}_{j=1}^M \subset C^3(\mathbb{T}^2, \mathbb{R}^2)$ be divergence-free vector fields. Then, the following statements hold:*

1. (Theorem 4.4) *If w is a bounded solution of (1.1), in the sense of Definition 2.14, and $u = K_{BS} * w$, where K_{BS} is the Biot-Savart kernel, then there exists a unique controlled rough path ϕ_t w.r.t. $\mathbf{Z}$ such that*

$$\phi_t = Id_{\mathbb{T}^2} + \int_0^t u(r, \phi_r) dr - \int_0^t \sigma(\phi_r) d\mathbf{Z}_r;$$

2. (Theorems 3.6, 5.7, and 5.8) *There exists a unique weak solution $w \in L_{t,x}^\infty$ (in the sense of Definition 2.14) to (1.1). Moreover, w is advected by the associated flow, as the push-forward*

$$w_t = (\phi_t)_\# w_0,$$

and the solution map $(w_0, \{\sigma_j\}_j, \mathbf{Z}) \mapsto w$ is continuous.

Let us mention that the results presented here have been supplemented by some interesting technical extensions in a subsequent preprint [29]. Therein, the authors extend the well-posedness of rough 2D Euler equations to the unbounded domain $\mathbb{R}^2$ and provide a refinement of the Wong-Zakai convergence in slightly stronger topologies.

Given that the Euler equations are closely related to the Navier-Stokes ones, we conclude this introduction by mentioning that similar techniques were used to prove well-posedness results also for the latter, when perturbed by rough transport noise. In [36] it is proved that there exist weak Leray-Hopf solutions in two and three dimensions and that uniqueness holds in the two-dimensional setting up to requiring the rough forcing to be constant in space, while in [37] the same authors show existence and uniqueness (the latter again in the two-dimensional case) of strong Leray-Hopf solutions in the vorticity formulation. However, proving well-posedness in the Euler setting presents a significative additional difficulty, since in the Navier-Stokes case the diffusive viscosity term grants a priori regularity of the solution via classical energy estimates.

The paper is organized as follows: Section 2 contains the basic notions on unbounded rough drivers, some results on p -variation spaces and the definition of solution of (1.1); in Section 3 we prove that there exists a solution to 2D Euler equations;

Section 4 is about the well-posedness results for the Lagrangian RDE and, at last, in Section 5 we deduce uniqueness for the 2D Euler equations and present a Wong-Zakai result.

2. Preliminaries.

2.1. Localized variation spaces. Since it emerges naturally in rough path theory, let us recall the definition of the term “control”. For a deeper description of its properties, we refer e.g. to [27].

From now on, we are going to denote

$$\Delta_T := \{(s, t) | 0 \leq s \leq t \leq T\}, \quad \Delta_T^2 := \{(s, r, t) | 0 \leq s \leq r \leq t \leq T\},$$

and, for two-index functions $g : \Delta_T \rightarrow V$, $(s, r, t) \in \Delta_T^2$,

$$\delta g_{s,r,t} := g(s, t) - g(s, r) - g(r, t).$$

In addition, throughout our discussion, we are going to use the notation $a \lesssim b$ whenever there is an inequality up to a (positive) constant, i.e. there exists some $C > 0$ such that $a \leq Cb$. When the constant depends on a parameter, $C = C(\text{par})$, we highlight that as follows: $a \lesssim_{\text{par}} b$.

DEFINITION 2.1. *A continuous function $\omega : \Delta_T \rightarrow \mathbb{R}^+$ is a control if*

- $\omega(s, s) = 0$, for all $s \in [0, T]$;
- ω is super-additive, i.e. $\omega(s, u) + \omega(u, t) \leq \omega(s, t)$ for all $(s, u, t) \in \Delta_T^2$.

Moreover, a control ω is said to be strictly positive if, for all $(s, t) \in \Delta_T$,

$$\omega(s, t) = 0 \implies s = t.$$

Another key concept is the p -variation of a path, which can be seen as a generalization of Hölder continuity.

DEFINITION 2.2. *Given a Banach space V , $p \in (0, \infty)$ and a continuous function $g : [0, T] \rightarrow V$, we say that g has finite p -variation, $g \in C^{p\text{-var}}([0, T], V)$, if*

$$\|g\|_p^p := \sup_{\Pi \in \Pi([0, T])} \sum_{[s, t] \in \Pi} |g_t - g_s|^p < \infty,$$

where $\Pi([0, T])$ is the set of partitions of $[0, T]$, $g_t := g(t)$ and $|\cdot| = \|\cdot\|_V$ (we are going to adopt this notation for a norm whenever the space V is clear from the context).

In the case of a two-index function $g : \Delta_T \rightarrow V$, the above notation can be adapted by changing $|g_t - g_s|$ into $|g_{s,t}|$, where $g_{s,t} := g(s, t)$.

In particular, we are going to extensively use partitions whose mesh is bounded from above by some quantity. To this aim, we introduce the concept of *localized p -variation*.

DEFINITION 2.3. *Given a Banach space V , $p \in (0, \infty)$, a control $\bar{\omega}$ and $L > 0$, we denote by $C_{\bar{\omega}, L}^{p\text{-var}, 2}(V)$ the space of 2-index continuous functions $g : \Delta_T \rightarrow V$ such that*

$$(2.1) \quad \|g\|_{p, (\bar{\omega}, L), [0, T]}^p := \sup_{\Pi \in \Pi(\bar{\omega}, L)} \sum_{[s, t] \in \Pi} |g_{s,t}|^p < \infty,$$

where $\Pi(\bar{\omega}, L)$ denotes the space of partitions Π of $[0, T]$ such that $\bar{\omega}(s, t) \leq L$ for all $[s, t] \in \Pi$.

Analogously, $C_{\bar{\omega}, L}^{p\text{-var}}(V)$ denotes the space of 1-index continuous functions $f : [0, T] \rightarrow V$ such that (2.1) holds, with $g_{s,t} := |g_t - g_s|$.

By slightly modifying some results in [27] (see, e.g., Proposition 5.8 and Proposition 5.10), we obtain the following Proposition.

PROPOSITION 2.4. *Let $g \in C_{\bar{\omega}, L}^{p\text{-var}, 2}(V)$ or $g \in C_{\bar{\omega}, L}^{p\text{-var}}(V)$. Then, the function $\omega_g^{loc} : \Delta_T \rightarrow \mathbb{R}^+$ defined by*

$$\omega_g^{loc}(s, t) = \|g\|_{p, (\bar{\omega}, L), [s, t]}^p$$

is a control. In addition, ω_g^{loc} is the smallest control such that

$$(2.2) \quad |g_{s, t}| \leq \omega_g^{loc}(s, t)^{\frac{1}{p}} \text{ for all } (s, t) \in \Delta_T \text{ such that } \bar{\omega}(s, t) \leq L.$$

Namely, if ω is a control which satisfies (2.2), then

$$\omega_g^{loc}(u, v) \leq \omega(u, v),$$

for all $(u, v) \in \Delta_T$.

We conclude this introduction to localized p -variation spaces by showing a result that holds only for 1-index functions. Let us recall that $p \in [1, \infty)$ is the only interesting case for 1-index functions (see [27, Proposition 5.2]).

PROPOSITION 2.5. *Let $g \in C_{\bar{\omega}, L}^{p\text{-var}}(V)$. Then, the following hold:*

1. *$C_{\bar{\omega}, L}^{p\text{-var}} = C^{p\text{-var}}$, i.e. if g has finite localized p -variation with respect to $(\bar{\omega}, L)$, then it has finite p -variation. Moreover, the following inequalities hold*

$$(2.3) \quad \|g\|_{p, (\bar{\omega}, L), [s, t]}^p \leq 2^{p-1} \|g\|_{p, (\bar{\omega}, \frac{L}{2}), [s, t]}^p \quad \forall (s, t) \in \Delta_T;$$

$$(2.4) \quad \|g\|_{p, [0, T]}^p \lesssim \left(\frac{\bar{\omega}(0, T)}{L} \right)^{p-1} \|g\|_{p, [0, T], (\bar{\omega}, L)}^p.$$

2. *If $\bar{\omega}$ is a strictly positive control, then*

$$(2.5) \quad \lim_{L \rightarrow 0} \|g\|_{q, (\bar{\omega}, L), [s, t]} = 0,$$

for all $q > p \geq 1$, $(s, t) \in \Delta_T$.

Proof. 1. Let us prove (2.3). The second inequality, (2.4), can be easily obtained by properly iterating the same strategy.

Given $(s, t) \in \Delta_T$, consider a partition coherent with the localization, namely $P = \{s =: t_0, t_1, \dots, t_N := t\} \in \Pi(\bar{\omega}, L)$. Then, the continuity of $\bar{\omega}$ implies that we can find $s_i \in (t_i, t_{i+1})$ such that $\bar{\omega}(t_i, s_i) \leq \frac{L}{2}$ and $\bar{\omega}(s_i, t_{i+1}) \leq \frac{L}{2}$, for all $i = 0, \dots, N-1$. Therefore, $Q = P \cup \{s_i\}_i$ is a partition of $[s, t]$ that satisfies the localization $(\bar{\omega}, \frac{L}{2})$. At last, the additivity of 1-index functions' variation, i.e. $g_{s, t} = g_{s, u} + g_{u, t}$, implies that

$$\sum_{t_i \in P} |g_{t_i, t_{i+1}}|^p \leq 2^{p-1} \sum_{r_i \in Q} |g_{r_i, r_{i+1}}|^p.$$

The obtained inequality proves the statement.

2. Notice that

$$\sum_{[s, t] \in \Pi} |g_{s, t}|^q \leq \|g\|_p^p \sup_{(s, t) \in \Delta_T, \bar{\omega}(s, t) \leq L} |g_{s, t}|^{q-p},$$

for any partition $\Pi \in \mathbf{\Pi}(\bar{\omega}, L)$.

Now, let us prove by contradiction that $\lim_{L \rightarrow 0} h(L) = 0$, where

$$h(L) = \sup_{(s,t) \in \Delta_T, \bar{\omega}(s,t) \leq L} |g_{s,t}|.$$

Assume that $\lim_{L \rightarrow 0} h(L) = \delta > 0$. Then for all $n \in \mathbb{N}$, there exists $(s_n, t_n) \in \Delta_T$ such that $\bar{\omega}(s_n, t_n) \leq \frac{1}{n}$ and $|g_{s_n, t_n}| \geq \frac{\delta}{2}$. Therefore, up to taking a subsequence, there exists $(s, t) \in \Delta_T$ such that

$$(s_n, t_n) \rightarrow (s, t), \quad \bar{\omega}(s_n, t_n) \rightarrow \bar{\omega}(s, t), \quad |g_{s,t}| \geq \frac{\delta}{2}.$$

This is a contradiction, because $\bar{\omega}$ is strictly positive and $\bar{\omega}(s, t) = 0$, hence $s = t$ but $|g_{s,t}| > 0$. $\square$

Remark 2.6. The assumptions $q > p$ and $\bar{\omega}$ being strictly positive are necessary for (2.5) to be true, and cannot be weakened. For example, any monotone function $g(t)$ shows that the limit does not hold for $q = p = 1$. Instead, $\bar{\omega} \equiv 0$ indicates that the same claim does not hold without requiring strict positiveness.

2.2. Localized controlled rough paths. In this section, we recall the definition of rough paths. We also introduce a generalized version of controlled rough paths as defined in [31].

DEFINITION 2.7. Let $M \in \mathbb{N}$, $p \in [2, 3)$. A p -rough path is a pair

$$\mathbf{Z} = (Z, \mathbb{Z}) \in C^{p\text{-var}}(\mathbb{R}^M) \times C^{\frac{p}{2}\text{-var}, 2}(\mathbb{R}^{M \times M})$$

that satisfies the Chen's relation, namely

$$(2.6) \quad \mathbb{Z}_{s,t} - \mathbb{Z}_{s,u} - \mathbb{Z}_{u,t} = Z_{s,u} \otimes Z_{u,t}, \quad \forall (s, u, t) \in \Delta_T^2.$$

Moreover, a p -rough path $\mathbf{Z}$ is called geometric if there exists a sequence $\{Z^n\}_n \subset C^\infty([0, T], \mathbb{R}^M)$ such that $\mathbf{Z}_{s,t}^n := (Z_{s,t}^n, \int_s^t Z_{s,r}^n \otimes dZ_r^n)$ converges to $\mathbf{Z}$ with respect to the product topology of $C^{p\text{-var}}(\mathbb{R}^M) \times C^{\frac{p}{2}\text{-var}, 2}(\mathbb{R}^{M \times M})$. The space of geometric rough paths will be denoted by $\mathcal{G}_g^{p\text{-var}}(\mathbb{R}^M)$.

Remark 2.8. For any p -rough path, there exists a control $\omega_{\mathbf{Z}}$ such that

$$|Z_{s,t}|^p \leq \omega_{\mathbf{Z}}(s, t), \quad |\mathbb{Z}_{s,t}|^{\frac{p}{2}} \leq \omega_{\mathbf{Z}}(s, t).$$

Let $\mathbf{Z}^1$ and $\mathbf{Z}^2$ be two geometric rough paths. Although the difference $Z^1 - Z^2$ does not necessarily admit a rough path lift, with a slight abuse of notation we denote by $\omega_{\mathbf{Z}^1 - \mathbf{Z}^2}$ any control such that

$$|Z_{s,t}^1 - Z_{s,t}^2|^p \leq \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s, t), \quad |\mathbb{Z}_{s,t}^1 - \mathbb{Z}_{s,t}^2|^{\frac{p}{2}} \leq \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s, t).$$

DEFINITION 2.9. Consider a p -rough path $\mathbf{Z}$, a control $\bar{\omega}$, a positive constant L , a Banach space V and denote the space of continuous and linear operators from $\mathbb{R}^M$ to V by $L(\mathbb{R}^M, V)$.

A continuous function $X \in C^{p\text{-var}}([0, T], V)$ is a localized controlled rough path with respect to $\mathbf{Z}$ and $(\bar{\omega}, L)$ if there exists a pair $(X', R^X) \in C^{p\text{-var}}(L(\mathbb{R}^M, V)) \times C^{\frac{p}{2}\text{-var}, 2}(V)$ such that

$$(2.7) \quad X_{s,t} = X'_s Z_{s,t} + R_{s,t}^X, \quad \forall (s, t) \in \Delta_T.$$

Moreover, if $R^X \in C^{\frac{p}{2}\text{-var}, 2}(V)$, then X will be called controlled rough path w.r.t. $\mathbf{Z}$. The spaces of localized controlled rough paths and of controlled rough paths will be denoted by $\mathcal{D}_{\mathbf{Z}, (\bar{\omega}, L)}^{p\text{-var}}(V)$ and $\mathcal{D}_{\mathbf{Z}}^{p\text{-var}}(V)$, respectively.

A rough integral, i.e. an integral with respect to a rough path, can be built via Lemma 6.2 (the so-called Sewing Lemma). We report below some useful inequalities that can be proved by easily generalizing [26, Theorem 4.10] and [26, Theorem 4.17], respectively.

PROPOSITION 2.10. *Consider two p -rough paths $\mathbf{Z}^1, \mathbf{Z}^2$, and two controlled rough paths $(X, X') \in \mathcal{D}_{\mathbf{Z}^1, (\bar{\omega}_1, L_1)}^{p\text{-var}}(L(\mathbb{R}^M, V))$, $(Y, Y') \in \mathcal{D}_{\mathbf{Z}^2, (\bar{\omega}_2, L_2)}^{p\text{-var}}(L(\mathbb{R}^M, V))$. Then,*

1. $(\int_0^\cdot X_r d\mathbf{Z}_r^1, X_\cdot) \in \mathcal{D}_{\mathbf{Z}^1, (\bar{\omega}_1, L_1)}^{p\text{-var}}$. Moreover, for all $(s, t) \in \Delta_T$ s.t. $\bar{\omega}_1(s, t) \leq L_1$, the following estimate holds

$$\left| \int_s^t X_r d\mathbf{Z}_r - X_s Z_{s,t} - X'_s Z_{s,t} \right| \lesssim \omega_{R^X}^{loc}(s, t)^{\frac{2}{p}} \omega_{\mathbf{Z}}(s, t)^{\frac{1}{p}} + \omega_{X'}^{loc}(s, t)^{\frac{1}{p}} \omega_{\mathbf{Z}}(s, t)^{\frac{2}{p}},$$

where $\omega_{R^X}^{loc}$ and $\omega_{X'}^{loc}$ are the controls introduced at Proposition 2.4.

2. The following inequality holds

$$\begin{aligned} \left\| \int_0^\cdot X_r d\mathbf{Z}_r^1 - \int_0^\cdot Y_r d\mathbf{Z}_r^2 \right\|_{p, [0, T]}^p &\lesssim_{L_1, L_2} |X - Y|_{C_t}^p \omega_{\mathbf{Z}^1}(0, T) \\ &+ |X|_{C_t}^p \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(0, T) + |X'|_{C_t}^p \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(0, T)^2 \\ &+ \omega_{\mathbf{Z}^1}(0, T) \omega_{R^X - R^Y}^{loc}(0, T)^2 + \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(0, T) \omega_{R^Y}^{loc}(0, T)^2 \\ &+ \omega_{Y'}^{loc}(0, T) \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(0, T)^2 + \omega_{\mathbf{Z}^2}(0, T) \omega_{X' - Y'}^{loc}(0, T) \\ &+ |X' - Y'|_{C_t}^p \omega_{\mathbf{Z}^2}(0, T)^2, \end{aligned}$$

where $|f|_{C_t}$ denotes $\sup_{s \in [0, T]} |f(s)|$.

2.3. Unbounded rough drivers and smoothing operators. Following [18, 2, 19, 15, 37], we rewrite the integral against Z as a Taylor expansion up to order two in order to rewrite our equation (1.1) in a more treatable form. In particular, this decomposition produces some sort of “operator-valued” rough path, i.e. linear unbounded operators enjoying rough path properties, and an additional remainder.

Let $\{E^n\}_{n \in \mathbb{N}}$ be a family of Banach spaces such that there is a continuous embedding $E^m \subseteq E^n$ for all $m \geq n$. From now on, we denote their dual spaces by E^{-n} .

The “operator-valued” rough path can be formally defined as follows.

DEFINITION 2.11. *Let $p \in [2, 3)$. A continuous unbounded p -rough driver on the scales $\{E^n\}_{n \in \mathbb{N}}$ is a pair (A^1, A^2) of 2-index maps such that, for all $s, t \in \Delta_T$,*

$$(2.8) \quad \begin{aligned} A_{s,t}^1 &\in L(E^{-n}, E^{-n-1}) \text{ for } n \in \{0, 1, 2\}, \\ A_{s,t}^2 &\in L(E^{-n}, E^{-n-2}) \text{ for } n \in \{0, 1\}. \end{aligned}$$

Moreover, Chen’s relation holds

$$(2.9) \quad A_{s,t}^1 = A_{s,r}^1 + A_{r,t}^1 \quad A_{s,t}^2 = A_{s,r}^2 + A_{r,t}^2 + A_{r,t}^1 A_{s,r}^1, \quad \forall (s, r, t) \in \Delta_T^2$$

and there exists a control ω_A such that

$$(2.10) \quad \|A_{s,t}^1\|_{L(E^{-n}, E^{-n-1})} \leq \omega_A(s, t)^{\frac{1}{p}}, \quad \|A_{s,t}^2\|_{L(E^{-n}, E^{-n-2})} \leq \omega_A(s, t)^{\frac{2}{p}}.$$

In dual form, (2.10) can also be rewritten as

$$(2.11) \quad \|A_{s,t}^{1,*}\|_{L(E^{n+1}, E^n)} \leq \omega_A(s, t)^{\frac{1}{p}}, \quad \|A_{s,t}^{2,*}\|_{L(E^{n+2}, E^n)} \leq \omega_A(s, t)^{\frac{2}{p}}.$$

We also need to introduce the concept of smoothing operators. They are crucial for estimating the time regularity of the remainder.

DEFINITION 2.12. A family of operators $(J^n)_{n \in \mathbb{N}}$ is a family of smoothing operators on the scale $\{E^n\}_{n \in \mathbb{N}}$ if the following inequalities hold

$$(2.12) \quad \begin{aligned} \|J^n \psi\|_{E^{n+m}} &\lesssim \eta^{-m} \|\psi\|_{E^n}, \\ \|(I - J^n) \psi\|_{E^n} &\lesssim \eta^m \|\psi\|_{E^{n+m}}, \end{aligned}$$

for all $\psi \in E^{n+m}$ and $n, m \in \mathbb{N}$.

2.4. Formulation of the equation. Since the driving signal in (1.1) is too irregular to be integrated à la Young, we follow the strategy introduced at [18, 2, 19, 16, 36, 37]: iterating the equation, namely substituting the solution term w in $\sigma_j \cdot \nabla w \dot{Z}^j$. This way, we obtain

$$\begin{aligned} w_{s,t} &= - \int_s^t u_r \cdot \nabla w_r dr + \int_s^t \sigma_j \cdot \nabla w_r dZ_r^j \\ &= - \int_s^t u_r \cdot \nabla w_r dr + \sigma_j \cdot \nabla w_s Z_{s,t}^j - \int_s^t (\sigma_j \cdot \nabla) \int_s^r u_{r_1} \cdot \nabla w_{r_1} dr_1 dZ_r^j \\ &\quad + \int_s^t (\sigma_j \cdot \nabla) \int_s^r \sigma_i \cdot \nabla w_{r_1} dZ_{r_1}^i dZ_r^j \\ &= - \int_s^t u_r \cdot \nabla w_r dr + \sigma_j \cdot \nabla w_s Z_{s,t}^j - \int_s^t (\sigma_j \cdot \nabla) \int_s^r u_{r_1} \cdot \nabla w_{r_1} dr_1 dZ_r^j \\ &\quad + (\sigma_j \cdot \nabla)(\sigma_i \cdot \nabla) w_s \mathbb{Z}_{s,t}^{i,j} - \int_s^t (\sigma_j \cdot \nabla) \int_s^r (\sigma_i \cdot \nabla) \int_s^{r_1} u_{r_2} \cdot \nabla w_{r_2} dr_2 dZ_{r_1}^i dZ_r^j \\ &\quad + \int_s^t (\sigma_j \cdot \nabla) \int_s^r (\sigma_i \cdot \nabla) \int_s^{r_1} \sigma_h \cdot \nabla w_{r_2} dZ_{r_2}^h dZ_{r_1}^i dZ_r^j. \end{aligned}$$

The above can be reformulated as

$$(2.13) \quad w_{s,t} = \mu_{s,t} + A_{s,t}^1 w_s + A_{s,t}^2 w_s + w_{s,t}^{\natural},$$

where, for the sake of notation, we denoted the drift by

$$\mu_{s,t} = - \int_s^t u_r \cdot \nabla w_r dr,$$

the unbounded rough operators by

$$(2.14) \quad \begin{aligned} A_{s,t}^1 \varphi &= \sigma_j \cdot \nabla \varphi Z_{s,t}^j \\ A_{s,t}^2 \varphi &= (\sigma_j \cdot \nabla)(\sigma_i \cdot \nabla) \varphi \mathbb{Z}_{s,t}^{i,j} \end{aligned}$$

and the remainder term by

$$\begin{aligned} w_{s,t}^{\natural} &= - \int_s^t \int_s^r (\sigma_j \cdot \nabla)(u_{r_1} \cdot \nabla) w_{r_1} dr_1 dZ_r^j \\ &\quad - \int_s^t \int_s^r \int_s^{r_1} (\sigma_j \cdot \nabla)(\sigma_i \cdot \nabla)(u_{r_2} \cdot \nabla) w_{r_2} dr_2 dZ_{r_1}^i dZ_r^j \\ &\quad + \int_s^t \int_s^r \int_s^{r_1} (\sigma_j \cdot \nabla)(\sigma_i \cdot \nabla)(\sigma_h \cdot \nabla) w_{r_2} dZ_{r_2}^h dZ_{r_1}^i dZ_r^j. \end{aligned}$$

Before giving the definition of a solution to (2.13), some comments are needed on μ , (A^1, A^2) and $w^\natural$. Notice also that, from now on, we are going to denote by $W^{-n,q}$ the dual of a Sobolev spaces $W^{n,q'}$. In addition, $L_{t,x}^\infty$ and $C_t X$ will stand for $L^\infty([0, T] \times \mathbb{T}^2)$ and $C([0, T], X)$, respectively.

Remark 2.13. Let us emphasise some properties of the terms appearing in (2.13).

1. The drift $\mu_{0,t}$ has finite 1-variation as a $W^{-1,\infty}$ -valued function. This property follows directly from u being divergence-free and from assuming $w \in L_{t,x}^\infty$, which allows for the estimate $\|u_t\|_{L_x^\infty} \leq \|w_t\|_{L_x^\infty}$ thanks to the properties of the Biot-Savart kernel.
2. If $\sigma_j \in C_b^3$, then the pair (A^1, A^2) defined at (2.14) is a continuous unbounded p -rough driver w.r.t. the scale $E^n = W^{n,1}$. Indeed, (2.8) follows from standard computations, (2.9) from (2.6), and (2.10) can be proved starting from ω_Z (see Remark 2.8).
3. Notice that the remainder term $w_{s,t}^\natural$ is defined starting from a solution w , and what one has to verify is that it is in fact negligible, i.e. it has finite $\frac{p}{3}$ -variation.

DEFINITION 2.14. A function $w : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}$ is a weak solution of (2.13) if $w \in C_t W^{-1,\infty} \cap L_{t,x}^\infty$ and there exists a localization $(\bar{\omega}, L)$ s.t. $w^\natural \in C_{2,\bar{\omega},L}^{p/3-\text{var}}(W^{-3,\infty})$, i.e.

$$w_{s,t}^\natural(\psi) := w_{s,t}(\psi) - \mu_{s,t}(\psi) - w_s \left((A_{s,t}^{1,*} + A_{s,t}^{2,*}) \psi \right)$$

has finite $\frac{p}{3}$ -variation with respect to $(\bar{\omega}, L)$ for all $\psi \in W^{3,1}$.

3. Existence of solutions. In this section, we aim to show the existence of solutions in the sense of Definition 2.14.

3.1. A priori estimates. We begin by providing a priori estimates for the solutions and their remainders. The proof's strategy follows standard arguments in the RPDE literature, see [2, 19, 34].

In particular, to prove them we need the following smoothing operators.

LEMMA 3.1. Consider a (symmetric) mollifier ρ with support on the ball of radius equal to 1. Then, $(J^\eta)_{\eta \in [0,1]}$, defined by $J^\eta \psi = \rho_\eta * \psi$, is a family of smoothing operators on the scale $W^{n,1}(\mathbb{T}^2)$ in the sense of Definition 2.12.

The first estimate is on the remainder $w^\natural$, defined in Definition 2.14. We denote $\omega_\natural := \omega_{w^\natural}^{\text{loc}}$.

LEMMA 3.2. Let w be a solution of (2.13), in the sense of Definition 2.14, such that $\omega^\natural \in C_2^{p/3-\text{var}}(W^{-3,\infty})$. Then there exists $\bar{L} > 0$ such that

$$(3.1) \quad \omega_\natural(s, t) \lesssim_p \|w\|_{L_{t,x}^\infty}^{\frac{p}{3}} \omega_A(s, t) + \|w\|_{L_{t,x}^\infty}^{\frac{2p}{3}} |t - s|^{\frac{p}{3}} \left(\omega_A(s, t)^{\frac{1}{3}} + \omega_A(s, t)^{\frac{2}{3}} \right),$$

for all $(s, t) \in \Delta_T$ s.t. $\omega_A(s, t) \leq \bar{L}$.

Proof. Let $(s, r, t) \in \Delta_T^2$. It follows from (2.13) that, for all $\psi \in W^{3,1}$,

$$\begin{aligned} \delta w_{s,r,t}^\natural(\psi) &= -w_s \left([A_{s,t}^{1,*} + A_{s,t}^{2,*}] \psi \right) + w_s \left([A_{s,r}^{1,*} + A_{s,r}^{2,*}] \psi \right) \\ &\quad + w_r \left([A_{r,t}^{1,*} + A_{r,t}^{2,*}] \psi \right) \\ &= w_{s,r} \left(A_{r,t}^{2,*} \psi \right) + (w_{s,r} - w_s A_{s,r}^{1,*}) \left(A_{r,t}^{1,*} \psi \right) \\ &= w_{s,r} \left(A_{r,t}^{2,*} \psi \right) + w_{s,r}^\dagger \left(A_{r,t}^{1,*} \psi \right), \end{aligned}$$

where in particular we have exploited (2.9) and defined

$$(3.2) \quad w_{s,t}^\dagger := w_{s,t} - A_{s,t}^1 w_s = \mu_{s,t} + A_{s,t}^2 w_s + w_{s,t}^\natural.$$

Let now J^η be a smoothing operator, as defined in Lemma 3.1: we are going to apply it to ψ in order to separately analyze a rougher and a smoother part of its duality product against $\delta w^\natural$. Starting with the former,

$$\begin{aligned} |\delta w_{s,r,t}^\natural((I - J^\eta)\psi)| &\leq \left| w_{s,r} \left(A_{r,t}^{2,*} (I - J^\eta)\psi \right) \right| + \left| w_{s,r}^\dagger \left(A_{r,t}^{1,*} (I - J^\eta)\psi \right) \right| \\ &\lesssim \|w\|_{L_{t,x}^\infty} \left[\|A_{r,t}^{2,*} (I - J^\eta)\psi\|_{L^1} + \|A_{r,t}^{1,*} (I - J^\eta)\psi\|_{L^1} \right. \\ &\quad \left. + \|A_{s,r}^{1,*} A_{r,t}^{1,*} (I - J^\eta)\psi\|_{L^1} \right] \\ &\lesssim \|w\|_{L_{t,x}^\infty} \left[\omega_A(s, t)^{\frac{2}{p}} \|(I - J^\eta)\psi\|_{W^{2,1}} \right. \\ &\quad \left. + \omega_A(s, t)^{\frac{1}{p}} \|(I - J^\eta)\psi\|_{W^{1,1}} \right] \\ &\leq \|w\|_{L_{t,x}^\infty} \left[\omega_A(s, t)^{\frac{2}{p}} \eta + \omega_A(s, t)^{\frac{1}{p}} \eta^2 \right] \|\psi\|_{W^{3,1}}. \end{aligned}$$

As for the smoothed part,

$$\begin{aligned} |\delta w_{s,r,t}^\natural(J^\eta\psi)| &\leq \left| w_{s,r} \left(A_{r,t}^{2,*} J^\eta\psi \right) \right| + \left| w_{s,r}^\dagger \left(A_{r,t}^{1,*} J^\eta\psi \right) \right| \\ &\leq \left| \mu_{s,r} \left(A_{r,t}^{2,*} J^\eta\psi \right) \right| + \left| w_s \left(A_{s,r}^{1,*} A_{r,t}^{2,*} J^\eta\psi \right) \right| + \left| w_s \left(A_{s,r}^{2,*} A_{r,t}^{2,*} J^\eta\psi \right) \right| \\ &\quad + \left| w_{s,r}^\natural \left(A_{r,t}^{2,*} J^\eta\psi \right) \right| + \left| \mu_{s,r} \left(A_{r,t}^{1,*} J^\eta\psi \right) \right| + \left| w_s \left(A_{s,r}^{2,*} A_{r,t}^{1,*} J^\eta\psi \right) \right| \\ &\quad + \left| w_{s,r}^\natural \left(A_{r,t}^{1,*} J^\eta\psi \right) \right| \\ &\leq \|w\|_{L_{t,x}^\infty}^2 \|A_{r,t}^{2,*} J^\eta\psi\|_{W^{1,1}} |t - s| + \|w\|_{L_{t,x}^\infty} \|A_{s,r}^{1,*} A_{r,t}^{2,*} J^\eta\psi\|_{L^1} \\ &\quad + \|w\|_{L_{t,x}^\infty} \|A_{s,r}^{2,*} A_{r,t}^{2,*} J^\eta\psi\|_{L^1} + \|w_{s,r}^\natural\|_{W^{-3,\infty}} \|A_{r,t}^{2,*} J^\eta\psi\|_{W^{3,1}} \\ &\quad + \|w\|_{L_{t,x}^\infty}^2 \|A_{r,t}^{1,*} J^\eta\psi\|_{W^{1,1}} |t - s| + \|w\|_{L_{t,x}^\infty} \|A_{s,r}^{2,*} A_{r,t}^{1,*} J^\eta\psi\|_{L^1} \\ &\quad + \|w_{s,r}^\natural\|_{W^{-3,\infty}} \|A_{r,t}^{1,*} J^\eta\psi\|_{W^{3,1}}. \end{aligned}$$

Due to the regularity properties of the unbounded rough drivers, (2.11), and to those of the smoothing operators, (2.12), we can rewrite the estimate as

$$\begin{aligned} |\delta w_{s,r,t}^\natural(J^\eta\psi)| &\leq \|w\|_{L_{t,x}^\infty}^2 \|J^\eta\psi\|_{W^{3,1}} \omega_A(s, t)^{\frac{2}{p}} |t - s| + \|w\|_{L_{t,x}^\infty} \|J^\eta\psi\|_{W^{3,1}} \omega_A(s, t)^{\frac{3}{p}} \\ &\quad + \|w\|_{L_{t,x}^\infty} \|J^\eta\psi\|_{W^{4,1}} \omega_A(s, t)^{\frac{4}{p}} + \omega_\natural(s, t)^{\frac{3}{p}} \|J^\eta\psi\|_{W^{5,1}} \omega_A(s, t)^{\frac{2}{p}} \\ &\quad + \|w\|_{L_{t,x}^\infty}^2 \|J^\eta\psi\|_{W^{2,1}} \omega_A(s, t)^{\frac{1}{p}} |t - s| + \|w\|_{L_{t,x}^\infty} \|J^\eta\psi\|_{W^{3,1}} \omega_A(s, t)^{\frac{3}{p}} \\ &\quad + \omega_\natural(s, t)^{\frac{3}{p}} \|J^\eta\psi\|_{W^{4,1}} \omega_A(s, t)^{\frac{1}{p}} \\ &\leq \|w\|_{L_{t,x}^\infty}^2 \|\psi\|_{W^{3,1}} \omega_A(s, t)^{\frac{2}{p}} |t - s| + \|w\|_{L_{t,x}^\infty} \|\psi\|_{W^{3,1}} \omega_A(s, t)^{\frac{3}{p}} \\ &\quad + \eta^{-1} \|w\|_{L_{t,x}^\infty} \|\psi\|_{W^{3,1}} \omega_A(s, t)^{\frac{4}{p}} + \eta^{-2} \omega_\natural(s, t)^{\frac{3}{p}} \|\psi\|_{W^{3,1}} \omega_A(s, t)^{\frac{2}{p}} \\ &\quad + \|w\|_{L_{t,x}^\infty}^2 \|\psi\|_{W^{3,1}} \omega_A(s, t)^{\frac{1}{p}} |t - s| + \|w\|_{L_{t,x}^\infty} \|\psi\|_{W^{3,1}} \omega_A(s, t)^{\frac{3}{p}} \\ &\quad + \eta^{-1} \omega_\natural(s, t)^{\frac{3}{p}} \|\psi\|_{W^{3,1}} \omega_A(s, t)^{\frac{1}{p}}. \end{aligned}$$

Putting back together $(I - J^\eta)\psi$ and $J^\eta\psi$, we can set $\eta = \omega_A(s, t)^{1/p}\lambda$ for some $\lambda > 0$ constant. This yields

$$\begin{aligned} \|\delta w_{s,r,t}^\natural\|_{W^{-3,\infty}} &\lesssim (\lambda^{-1} + 1 + \lambda + \lambda^2) \|w\|_{L_{t,x}^\infty} \omega_A(s, t)^{\frac{3}{p}} \\ &\quad + \|w\|_{L_{t,x}^\infty}^2 \left(\omega_A(s, t)^{\frac{1}{p}} + \omega_A(s, t)^{\frac{2}{p}} \right) |t - s| + (\lambda^{-2} + \lambda^{-1}) \omega_\natural(s, t)^{\frac{3}{p}}. \end{aligned}$$

Now, Corollary 6.3 implies that

$$\begin{aligned} (3.3) \quad \omega_\natural(s, t) &\leq C_p \left[(\lambda^{-1} + 1 + \lambda + \lambda^2)^{\frac{p}{3}} \|w\|_{L_{t,x}^\infty}^{\frac{p}{3}} \omega_A(s, t) \right. \\ &\quad \left. + \|w\|_{L_{t,x}^\infty}^{\frac{2p}{3}} \left(\omega_A(s, t)^{\frac{1}{3}} + \omega_A(s, t)^{\frac{2}{3}} \right) |t - s|^{\frac{p}{3}} + (\lambda^{-2} + \lambda^{-1})^{\frac{p}{3}} \omega_\natural(s, t) \right]. \end{aligned}$$

It is now sufficient to choose λ in a way such that $C_p(\lambda^{-2} + \lambda^{-1})^{p/3} \leq \frac{1}{2}$ and $\bar{L} > 0$ such that $\eta \leq 1$ to obtain the thesis. $\square$

LEMMA 3.3. *Let w be a solution of (2.13), in the sense of Definition 2.14, such that $\omega^\natural \in C_2^{p/3-\text{var}}(W^{-3,\infty})$. Then $w \in C_T^{p-\text{var}}W^{-1,\infty}$ and there exists $\bar{L} > 0$ such that, given $\omega_w(s, t) := \|w\|_{p-\text{var},[s,t]}^p$,*

$$(3.4) \quad \omega_w(s, t) \lesssim_p \left(1 + \|w\|_{L_{t,x}^\infty} \right)^{2p} (|t - s|^p + \omega_A(s, t) + \omega_\natural(s, t)),$$

for all $(s, t) \in \Delta_T$ s.t. $\omega_A(s, t), \omega_\natural(s, t) \leq \bar{L}$.

Proof. Consider again a family of smoothing operators as in Lemma 3.1. Given any $\psi \in W^{1,1}$ analyzing separately $J^\eta\psi$ and $(I - J^\eta)\psi$ we get

$$|w_{s,t}((I - J^\eta)\psi)| \leq \|w_{s,t}\|_{L_x^\infty} \|(I - J^\eta)\psi\|_{L^1} \leq 2\eta \|w\|_{L_{t,x}^\infty} \|\psi\|_{W^{1,1}}$$

and, exploiting the formulation (2.13),

$$\begin{aligned} |w_{s,t}(J^\eta\psi)| &\leq \|w\|_{L_{t,x}^\infty}^2 \|J^\eta\psi\|_{W^{1,1}} |t - s| + \|w\|_{L_{t,x}^\infty} \omega_A(s, t)^{\frac{1}{p}} \|J^\eta\psi\|_{W^{1,1}} \\ &\quad + \|w\|_{L_{t,x}^\infty} \omega_A(s, t)^{\frac{2}{p}} \|J^\eta\psi\|_{W^{2,1}} + \omega_\natural(s, t)^{\frac{3}{p}} \|J^\eta\psi\|_{W^{3,1}} \\ (3.5) \quad &\leq \|\psi\|_{W^{1,1}} \left(1 + \|w\|_{L_{t,x}^\infty} \right)^2 \\ &\quad \left(|t - s| + \omega_A(s, t)^{\frac{1}{p}} + \eta^{-1} \omega_A(s, t)^{\frac{2}{p}} + \eta^{-2} \omega_\natural(s, t)^{\frac{3}{p}} \right). \end{aligned}$$

This, together with the choice $\eta = \omega_A(s, t)^{1/p} + \omega_\natural(s, t)^{1/p}$, yields the thesis whenever $\omega_A(s, t), \omega_\natural(s, t)$ are less or equal of some $\bar{L}$ such that $\eta \leq 1$. $\square$

Remark 3.4. In Lemmas 3.2 and 3.3, we assume that $\omega^\natural \in C_2^{p/3-\text{var}}(W^{-3,\infty})$ instead of $\omega^\natural \in C_{2,(\bar{\omega},L)}^{p/3-\text{var}}(W^{-3,\infty})$, that coincides with the weaker condition required in Definition 2.14. We stress that Lemmas 3.2 and 3.3 can be easily generalized under the weaker hypothesis $\omega^\natural \in C_{2,(\bar{\omega},L)}^{p/3-\text{var}}(W^{-3,\infty})$, e.g. [36, 37]. Nevertheless, the current version of Lemmas 3.2 and 3.3 will be sufficient to show the existence result, cf. Theorem 3.6.

3.2. Existence. We discuss here how to prove our existence result. The strategy outline is classic: first, we prove the existence of solutions with an additional viscous dissipative term. Then, we prove the convergence of such solutions while letting the

viscosity vanish, recovering a solution of the original problem, in the sense of Definition 2.14.

The proof of the following lemma can be easily obtained by adapting the results in [3, 43].

LEMMA 3.5. *Let $w_0 \in C^\infty(\mathbb{T}^2)$, $\{\sigma_j\}_{j=1}^M \subset C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ be a family of divergence-free vector fields and $Z \in C^\infty([0, T], \mathbb{R}^M)$. Then, there exists a unique solution $w \in C^\infty([0, T] \times \mathbb{T}^2)$ to the following equation*

$$(3.6) \quad \begin{cases} \partial_t w + u \cdot \nabla w - \nu \Delta w = \sum_{j=1}^M \sigma_j \cdot \nabla w \dot{Z}^j, \\ w(0) = w_0, \quad u = K_{BS} * w. \end{cases}$$

In particular, we have the estimate

$$\|w\|_{L_{t,x}^\infty} \leq \|w_0\|_{L_x^\infty},$$

where C is a positive constant that does not depend on the viscosity ν .

Thanks to the above, we can start with a smooth approximation of both solution and noise and let them converge.

THEOREM 3.6. *Let $w_0 \in L^\infty(\mathbb{T}^2)$, $\{\sigma_j\}_{j=1}^M \subset C^3(\mathbb{T}^2, \mathbb{R}^2)$ be divergence-free vector fields and $\mathbf{Z} \in \mathcal{C}_g^{p\text{-var}}$. Then, there exists a solution to (1.1) in the sense of Definition 2.14.*

Proof. Since $\mathbf{Z}$ is a geometric rough path, there exists a sequence of smooth paths, $\{Z^N\}_{N \in \mathbb{N}}$, whose canonical lifts converge to $\mathbf{Z}$ in $\mathcal{C}_g^{p\text{-var}}$.

Let w^N be the solution to (3.6) with initial datum $w_0^N = \rho_{1/N} * w_0$, viscosity $\nu = N^{-1}$, $\sigma_j^N = \sigma_j * \rho_{1/N}$ and (smooth) driving signal Z^N . By iterating the equation, following the procedure already used in Subsection 2.4, we can see that w^N solves an equation of the form

$$(3.7) \quad w_{s,t}^N = \int_s^t \left[\frac{1}{N} \Delta w_r^N - u_r^N \cdot \nabla w_r^N \right] dr + A_{s,t}^{N,1} w_s^N + A_{s,t}^{N,2} w_s^N + w_{s,t}^{N,\natural},$$

where

$$A_{s,t}^{N,1} w_s = \sigma_j^N \cdot \nabla w_s Z_{s,t}^{N,j}, \quad A_{s,t}^{N,1} w_s = (\sigma_j^N \cdot \nabla)(\sigma_i^N \cdot \nabla) w_s Z_{s,t}^{N,i,j}$$

and the remainder $w_{s,t}^{N,\natural}$ is properly defined via a suitable adaptation of (2.15).

Adapting the proofs of Lemma 3.2 and Lemma 3.3, one has to take into account that $\mu_{s,r}$ therein now contains the additional term $\frac{1}{N} \int_s^r \Delta w_u^N du$, which can be treated as follows:

$$\begin{aligned} \left| \left(\int_s^r \frac{1}{N} \Delta w_u^N du \right) \left(A_{r,t}^{2,*} J^\eta \psi \right) \right| &\leq \eta^{-1} \frac{1}{N} \|w^N\|_{L_{t,x}^\infty} \|\psi\|_{W^{3,1} \omega_A(r,t)^{\frac{2}{p}}} |t-s| \\ \left| \left(\int_s^r \frac{1}{N} \Delta w_u^N du \right) \left(A_{r,t}^{1,*} J^\eta \psi \right) \right| &\leq \frac{1}{N} \|w^N\|_{L_{t,x}^\infty} \|\psi\|_{W^{3,1} \omega_A(r,t)^{\frac{1}{p}}} |t-s|. \end{aligned}$$

With the same choice $\eta = \omega_A(r,t)^{\frac{1}{p}} \lambda$ as in Lemma 3.2, one can easily obtain that,

uniformly in $N \in \mathbb{N}$,

$$\begin{aligned} \omega_{N,\natural}(s, t) &\lesssim_p (\lambda^{-1} + 1 + \lambda + \lambda^2)^{\frac{p}{3}} \|w\|_{L_{t,x}^\infty}^{\frac{p}{3}} \omega_A(s, t) \\ &\quad + \|w\|_{L_{t,x}^\infty}^{\frac{2p}{3}} \left(\omega_A(s, t)^{\frac{1}{3}} + \omega_A(s, t)^{\frac{2}{3}} \right) |t - s|^{\frac{p}{3}} + (\lambda^{-2} + \lambda^{-1})^{\frac{p}{3}} \omega_{N,\natural}(s, t) \\ &\quad + (\lambda^{-1} + 1)^{\frac{p}{3}} \|w\|_{L_{t,x}^\infty} \omega_A(s, t)^{\frac{1}{3}} |t - s|^{\frac{p}{3}}. \end{aligned}$$

Since the above is closely related to (3.3), up to a term $(\lambda^{-1} + 1)^{\frac{p}{3}} \|w\|_{L_{t,x}^\infty} \omega_A(s, t)^{\frac{1}{3}} |t - s|^{\frac{p}{3}}$ due to the dissipative effect, we can proceed exactly in the same way. Then, for a suitable λ , one has

$$(3.8) \quad \omega_{N,\natural}(s, t) \lesssim_{p,w_0} \omega_A(s, t) + |t - s|^{\frac{p}{3}} (\omega_A(s, t)^{\frac{1}{3}} + \omega_A(s, t)^{\frac{2}{3}}).$$

As for the control on the p -variation of $\|w^N\|_{W^{-1,\infty}}$, much like in Lemma 3.3, we obtain that

$$\begin{aligned} \|w_{s,t}^N\|_{W^{-1,\infty}} &\leq \left(1 + \|w\|_{L_{t,x}^\infty}\right)^2 \\ &\quad \left(\tilde{\eta} + (1 + \tilde{\eta}^{-1})|t - s| + \omega_A(s, t)^{\frac{1}{p}} + \tilde{\eta}^{-1} \omega_A(s, t)^{\frac{2}{p}} + \tilde{\eta}^{-2} \omega_{\natural}(s, t)^{\frac{3}{p}}\right). \end{aligned}$$

Similarly to (3.5), we can then choose $\tilde{\eta} = \omega_A(s, t)^{1/p} + \omega_{\natural}(s, t)^{1/p} + |t - s|^{1-\frac{1}{p}}$, yielding

$$(3.9) \quad \|w_{s,t}^N\|_{W^{-1,\infty}}^p \lesssim_{p,w_0} |t - s| + |t - s|^p + \omega_A(s, t) + \omega_{\natural}(s, t).$$

The estimates (3.8) and (3.9) hold for all suitable localizations based on L such that both η and $\tilde{\eta}$ are less or equal than 1.

At this point, Lemma 6.1 allows to select a subsequence (still denoted by w^N for the sake of notation) converging to some $w \in C([0, T], W^{-1,\infty})$. Testing (3.7) against any $\phi \in W^{3,1}$, and passing to the limit $w^N \rightarrow w$ and $\mathbf{Z}^N \rightarrow \mathbf{Z}$, we can see that such w is in fact a solution of (1.1). In particular, the convergence of the nonlinear transport term is ensured by the fact that $w^N \rightarrow w$ in $C([0, T], W^{-1,2})$ implies $u^N \rightarrow u$ in $C([0, T], L^2)$.

The proof is then complete once we notice that the remainder, defined by $w_{s,t}^{\natural} := \lim_{N \rightarrow \infty} w_{s,t}^N$, has finite $\frac{p}{3}$ -variation with respect to a suitable localization thanks to the uniform bound (3.8). $\square$

4. Euler Lagrangian flows. This section is focused on the study of two Lagrangian equations associated with the flow of solutions to (1.1). We are first going to prove that

$$\phi_t(x) = x + \int_0^t u(s, \phi_s(x)) dr + \int_0^t \bar{\sigma}(\phi_r(x)) d\mathbf{Z}_r,$$

admits a solution, whenever u is a spatially log-Lipschitz drift like the one obtained by applying the Biot-Savart kernel to a bounded function. In the equation above, and along the course of this section, we denote by $\bar{\sigma}$ the functional $\bar{\sigma} : C(\mathbb{T}^2, \mathbb{R}^2) \rightarrow L(\mathbb{R}^M, C(\mathbb{T}^2, \mathbb{R}^2))$ defined by $\bar{\sigma}(f)(z) := \sum_{j=1}^M \sigma_j(f(x)) z_j$, up to extending the σ_j by periodicity, and by $D\bar{\sigma}$ its Fréchet derivative. Afterwards, we are going to show a uniqueness result for the non-local RDE

$$\phi_t(x) = x + \int_0^t \int_{\mathbb{T}^2} K_{BS}(\phi_r(x) - \phi_r(y)) w_0(y) dy dr + \int_0^t \bar{\sigma}(\phi_r(x)) d\mathbf{Z}_r^j,$$

where K_{BS} is the Biot-Savart kernel and w_0 is a function bounded in space and constant in time.

Let us stress that both the Lagrangian equations present a rough integral, in the sense of Proposition 2.10, where the rough path has finite p -variation for $p \in [2, 3)$.

4.1. Log-Lipschitz RDE. Consider a bounded, measurable function $u : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$ that is log-Lipschitz in the spatial component, namely

$$|u(t, x) - u(t, y)| \leq L_u \gamma(|x - y|) \quad \forall t \in [0, T], x, y \in \mathbb{T}^2,$$

where we set $L_u > 0$ and $\gamma(r) = r(1 - \log r)\mathbf{1}_{(0, 1/e)}(r) + (r + 1/e)\mathbf{1}_{[1/e, \infty)}(r)$ for all $r \geq 0$. We prove that the RDE

$$(4.1) \quad \phi_t(x) = \phi_0(x) + \int_0^t u(r, \phi_r(x)) dr + \int_0^t \bar{\sigma}(\phi_r(x)) d\mathbf{Z}_r,$$

is well-posed by a limiting procedure based on the a priori estimates provided in Propositions 4.2 and 4.3.

DEFINITION 4.1. Let $\phi \in C^{p\text{-var}}([0, T], C(\mathbb{T}^2, \mathbb{R}^2))$. We say that ϕ is a solution to (4.1) if $(\phi, \bar{\sigma}(\phi)) \in \mathcal{D}_{Z, (\bar{\omega}, L)}^{p\text{-var}}(C(\mathbb{T}^2, \mathbb{R}^2))$, in the sense of Definition 2.9, and

$$\phi_t = \phi_0 + \int_0^t u(r, \phi_r) dr + \int_0^t \bar{\sigma}(\phi_r) d\mathbf{Z}_r,$$

where the first integral is a standard Riemann integral and the second one is a rough integral in the sense of Proposition 2.10.

PROPOSITION 4.2. Let $u : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$ be bounded measurable, $\{\sigma_j\}_{j=1}^M$ be $C^3(\mathbb{T}^2, \mathbb{R}^2)$ -vector fields, $\mathbf{Z}$ be a p -rough path and $\phi_0 \in C(\mathbb{T}^2, \mathbb{R}^2)$. Given a solution $(\phi, \bar{\sigma}(\phi)) \in \mathcal{D}_Z^{p\text{-var}}(C(\mathbb{T}^2, \mathbb{R}^2))$ to (4.1), in the sense of the previous definition, it holds that, for $q \in (p, 3)$,

$$(4.2) \quad \|\phi\|_{q\text{-var}} \leq C_1,$$

$$(4.3) \quad \|R^\phi\|_{q\text{-var}, (\bar{\omega}, \mu_0)} \leq C_2,$$

where $\bar{\omega}(s, t) = \omega_{\mathbf{Z}}(s, t) + |t - s|^p$ and C_1, C_2, μ_0 are positive constants depending only on $\|\bar{\sigma}\|_{C^3}$, $\|u\|_{L_{t,x}^\infty}$, q and $\omega_{\mathbf{Z}}(0, T)$.

Proof. Let us stress that in this proof we consider controls associated with the q - and $q/2$ -variations with respect to the localization $(\bar{\omega}, \mu)$ of several 1- and 2-index functions.

If $(s, t) \in \Delta_T$ are such that $\bar{\omega}(s, t) \leq \mu$, then

$$\begin{aligned} |R_{s,t}^\phi| &\leq \left| \int_s^t u(r, \phi_r) dr \right| + \left| \int_s^t \bar{\sigma}(\phi_r) d\mathbf{Z}_r - \bar{\sigma}(\phi_s) Z_{s,t} - D\bar{\sigma}(\phi_s) \bar{\sigma}(\phi_s) \mathbb{Z}_{s,t} \right| \\ &\quad + |D\bar{\sigma}(\phi_s) \bar{\sigma}(\phi_s) \mathbb{Z}_{s,t}| \\ &\lesssim \|u\|_{L_{t,x}^\infty} |t - s| + \omega_{R\bar{\sigma}(\phi)}^{loc}(s, t)^{\frac{2}{q}} \omega_{\mathbf{Z}}(s, t)^{\frac{1}{q}} + \omega_{\bar{\sigma}(\phi)}^{loc}(s, t)^{\frac{1}{q}} \omega_{\mathbf{Z}}(s, t)^{\frac{2}{q}} \\ &\quad + \|\bar{\sigma}\|_{C^1}^2 \omega_{\mathbf{Z}}(s, t)^{\frac{2}{q}} \\ &\lesssim \|u\|_{L_{t,x}^\infty} |t - s| + \|\bar{\sigma}\|_{C^2} (\omega_\phi^{loc}(s, t)^{\frac{2}{q}} + \omega_{R\phi}^{loc}(s, t)^{\frac{2}{q}}) \omega_{\mathbf{Z}}(s, t)^{\frac{1}{q}} \\ &\quad + \|\bar{\sigma}\|_{C^2}^2 \omega_\phi^{loc}(s, t)^{\frac{1}{q}} \omega_{\mathbf{Z}}(s, t)^{\frac{2}{q}} + \|\bar{\sigma}\|_{C^1}^2 \omega_{\mathbf{Z}}(s, t)^{\frac{2}{q}}, \end{aligned}$$

where the second inequality is due to the Proposition 2.10, while the third one follows from standard facts (see [26, Chapter 7]) and the localization $(\bar{\omega}, \mu)$. At this point we can apply Proposition 2.4 and, fixed μ small enough to have $\|\bar{\sigma}\|_{C^2}^{q/2} \omega_{\mathbf{Z}}(s, t)^{1/2} < 1/2$, absorb the control $\omega_{R^\phi}^{loc}$ on the left-hand side:

$$\begin{aligned} \omega_{R^\phi}^{loc}(s, t) &\lesssim_q 2\|u\|_{L_{t,x}^\infty}^{\frac{q}{2}} |t-s|^{\frac{q}{2}} + \omega_\phi^{loc}(s, t) + \|\bar{\sigma}\|_{C^2}^{\frac{q}{2}} \omega_\phi^{loc}(s, t)^{\frac{1}{2}} \omega_{\mathbf{Z}}(s, t)^{\frac{1}{2}} \\ &\quad + 2\|\bar{\sigma}\|_{C^2}^q \omega_{\mathbf{Z}}(s, t) \\ &\lesssim_q \|u\|_{L_{t,x}^\infty}^{\frac{q}{2}} |t-s|^{\frac{q}{2}} + \omega_\phi^{loc}(s, t) + \|\bar{\sigma}\|_{C^2}^q \omega_{\mathbf{Z}}(s, t). \end{aligned}$$

Also, by definition of $(\phi, \bar{\sigma}(\phi)) \in \mathcal{D}_Z^{p\text{-var}}$,

$$\begin{aligned} |\phi_{s,t}| &\leq |\bar{\sigma}(\phi_s)Z_{s,t} + R_{s,t}^\phi| \\ &\leq \|\bar{\sigma}\|_{C^0} \omega_{\mathbf{Z}}(s, t)^{\frac{1}{q}} + \omega_{R^\phi}^{loc}(s, t)^{\frac{2}{q}} \\ &\lesssim_q \|\bar{\sigma}\|_{C^0} \omega_{\mathbf{Z}}(s, t)^{\frac{1}{q}} + \|u\|_{L_{t,x}^\infty} |t-s| + \omega_\phi^{loc}(s, t)^{\frac{2}{q}} + \|\bar{\sigma}\|_{C^2}^2 \omega_{\mathbf{Z}}(s, t)^{\frac{2}{q}}, \end{aligned}$$

for all $(s, t) \in \Delta_T$ such that $\bar{\omega}(s, t) \leq \mu$. Therefore, Proposition 2.4 implies that there exists a positive constant C , depending on q , $\|\bar{\sigma}\|_{C^2}$ and $\|u\|_{L_{t,x}^\infty}$, such that for all $(s, t) \in \Delta_T$

$$\omega_\phi^{loc}(s, t) \leq C\mu(1 + \mu) + C\omega_\phi^{loc}(s, t)^2.$$

Therefore, choosing $\mu_0 \leq \mu$ to have $4C^2\mu_0(1 + \mu_0) \leq 1$, given

$$x_{1,2}^\mu = \frac{1}{2C} \left(1 \pm \sqrt{1 - 4C^2\mu(1 + \mu)} \right), \quad \mu \leq \mu_0,$$

we find that

$$\omega_\phi^{loc}(s, t) \leq x_2^\mu \quad \text{or} \quad \omega_\phi^{loc}(s, t) \geq x_1^\mu.$$

We know by Proposition 2.5 that, for μ very small, the former inequality holds (indeed, the control goes to 0 as μ does). At the cost of possibly re-choosing it, for a small enough $\mu_0 > 0$ we have $x_1^{\mu_0} > \frac{1}{2C}$ and $x_2^{\mu_0} \leq \frac{1}{8C}$. To show that $\|\phi\|_{q,[s,t],(\bar{\omega},\mu)}^q \leq x_2^{\mu_0}$ for all $\mu \leq \mu_0$ it is then sufficient to exclude any jump of size greater than 4. To this aim, (2.3) yields that

$$\|\phi\|_{q,[s,t],(\bar{\omega},\mu)}^q \leq 2^{q-1} \|\phi\|_{q,[s,t],(\bar{\omega},\mu/2)}^q \leq 2^{q-1} \lim_{\bar{\mu} \rightarrow \mu^-} \|\phi\|_{q,[s,t],(\bar{\omega},\bar{\mu})}^q,$$

and since $4x_2^{\mu_0} < x_1^{\mu_0}$ this implies that we can never go from $\omega_\phi^{loc}(s, t) \leq x_2^{\mu_0}$ to $\omega_\phi^{loc}(s, t) \geq x_1^\mu$.

At last, (2.4) allows us to conclude the proof. $\square$

We have proved that the localized variations of both ϕ and its remainder are bounded from above whenever the localization we choose is small enough. Let us now show that a solution to (4.1) is continuous with respect to its initial datum, driver, drift and diffusion.

PROPOSITION 4.3. *Let $\mathbf{Z}^1, \mathbf{Z}^2$ be two p -rough paths, $u_1, u_2 : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$ bounded, measurable and log-Lipschitz in the spatial component, $\bar{\sigma}_1, \bar{\sigma}_2$ be two families of M vector fields in $C^3(\mathbb{T}^2, \mathbb{R}^2)$ and $\phi_0^1, \phi_0^2 \in C(\mathbb{T}^2, \mathbb{R}^2)$. Moreover, assume that $(\phi_t^i, \bar{\sigma}_i(\phi_t^i)) \in \mathcal{D}_{Z^i}^{p\text{-var}}(C(\mathbb{T}^2, \mathbb{R}^2))$ are solutions, in the sense of Definition 4.1, to*

$$\phi_t^i = \phi_0^i + \int_0^t u_i(r, \phi_r^i) dr + \int_0^t \bar{\sigma}_i(\phi_r^i) d\mathbf{Z}_r^i \quad \forall t \in [0, T], i = 1, 2.$$

537 Then, there exists a positive constant C depending on $\|\bar{\sigma}_i\|_{C^3}$, $\|u_i\|_{L_{t,x}^\infty}$, $\omega_{Z^i}(0, T)$ and
 538 the log-Lipschitz constants L_{u_i} such that

$$\begin{aligned}
 539 \quad & \sup_{t \in [0, T]} \|\phi^1 - \phi^2\|_{C(\mathbb{T}^2)} \\
 540 \quad & \leq C \left(\|\phi_0^1 - \phi_0^2\|_{C(\mathbb{T}^2)} + \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^3} + \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(0, T)^{\frac{1}{2q}} \right. \\
 541 \quad & \left. + \|u_1 - u_2\|_{L_{t,x}^\infty} T + \int_0^T \gamma(\|\phi_r^1 - \phi_r^2\|_{C(\mathbb{T}^2)}) dr \right).
 \end{aligned}$$

542 *Proof.* We wish to estimate $\sup_{t \in [0, T]} \|\phi^1 - \phi^2\|_{C(\mathbb{T}^2)}$ by means of a Rough Grön-
 543 wall Lemma, see Lemma 6.4. Notice that what we really need to estimate is the
 544 difference $|R_{s,t}^1 - R_{s,t}^2|$, since

$$\begin{aligned}
 545 \quad (4.4) \quad & |\phi_{s,t}^1 - \phi_{s,t}^2| = |\bar{\sigma}_1(\phi_s^1)Z_{s,t}^1 - \bar{\sigma}_2(\phi_s^2)Z_{s,t}^2 + R_{s,t}^1 - R_{s,t}^2| \\
 & \leq (\|\bar{\sigma}_1\|_{C^1} I_{s,t} + \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^0}) |Z_{s,t}^2| + \|\bar{\sigma}_1\|_{C^0} |Z_{s,t}^1 - Z_{s,t}^2| \\
 & \quad + |R_{s,t}^1 - R_{s,t}^2|,
 \end{aligned}$$

546 where we called $I_{s,t} := \sup_{r \in [s, t]} |\phi_r^1 - \phi_r^2|$ to ease the notation and where R^i is the
 547 Gubinelli remainder of ϕ^i , for $i = 1, 2$.

548 In the following, we always consider controls associated with the q - and $q/2$ -
 549 variations with respect to the localization $(\omega_{\mathbf{Z}^1}(s, t) + \omega_{\mathbf{Z}^2}(s, t) + |t - s|^p, \mu)$ of several
 550 1- and 2-index functions. In particular, adding $|t - s|^p$ results in a strictly positive
 551 control. As in Proposition 4.2, the parameter μ will be properly chosen. For the sake
 552 of readability, we will also denote the controls associated with the Gubinelli derivatives
 553 (respectively, remainders) of $\bar{\sigma}_i(\phi^i)$ by $\omega_{\bar{\sigma}_i}^{loc}$ (respectively, $\omega_{R^{\bar{\sigma}_i}}^{loc}$). Again at the price
 554 of a slight abuse of notation, like in Remark 2.8, we will also denote a control on the
 555 difference of $[\bar{\sigma}_1(\phi^1)]'$, the Gubinelli derivative of $\bar{\sigma}_1(\phi^1)$, and $[\bar{\sigma}_2(\phi^2)]'$, the Gubinelli
 556 derivative of $\bar{\sigma}_2(\phi^2)$ (respectively, between the remainders $R^{\bar{\sigma}_1(\phi^1)}$ and $R^{\bar{\sigma}_2(\phi^2)}$) by
 557 $\omega_{\bar{\sigma}_1' - \bar{\sigma}_2'}^{loc}$ (respectively, $\omega_{R^{\bar{\sigma}_1} - R^{\bar{\sigma}_2}}^{loc}$).

558 For all $(s, t) \in \Delta_T$ such that $\omega_{\mathbf{Z}^1}(s, t) + \omega_{\mathbf{Z}^2}(s, t) + |t - s|^p \leq \mu$,

$$\begin{aligned}
 559 \quad (4.5) \quad & |R_{s,t}^1 - R_{s,t}^2| \leq \left| \int_s^t u_1(r, \phi_r^1) - u_2(r, \phi_r^2) dr \right| \\
 & + \left| \int_s^t \bar{\sigma}_1(\phi_r^1) d\mathbf{Z}_r^1 - \bar{\sigma}_1(\phi_s^1) Z_{s,t}^1 - \int_s^t \bar{\sigma}_1(\phi_r^2) d\mathbf{Z}_r^2 + \bar{\sigma}_2(\phi_s^2) Z_{s,t}^2 \right| \\
 & \lesssim \|u_1 - u_2\|_{L_{t,x}^\infty} |t - s| + \min\{L_{u_1}, L_{u_2}\} \int_s^t \gamma(|\phi_r^1 - \phi_r^2|) dr \\
 & + \left| \int_s^t \bar{\sigma}_1(\phi_r^1) d\mathbf{Z}_r^1 - \bar{\sigma}_1(\phi_s^1) Z_{s,t}^1 - \int_s^t \bar{\sigma}_1(\phi_r^2) d\mathbf{Z}_r^2 + \bar{\sigma}_2(\phi_s^2) Z_{s,t}^2 \right|.
 \end{aligned}$$

560 We are left with a rough integral term, that can be estimated by means of Proposi-

tion 2.10. Indeed,

$$\begin{aligned}
& \left| \int_s^t \bar{\sigma}_1(\phi_r^1) d\mathbf{Z}_r^1 - \bar{\sigma}_1(\phi_s^1) Z_{s,t}^1 - \int_s^t \bar{\sigma}_1(\phi_r^2) d\mathbf{Z}_r^2 + \bar{\sigma}_2(\phi_s^2) Z_{s,t}^2 \right| \\
& \lesssim_{\|\sigma_i\|_{C^3}} \left| D\bar{\sigma}_1(\phi_s^1) \bar{\sigma}_1(\phi_s^1) \mathbf{Z}_{s,t}^1 - D\bar{\sigma}_2(\phi_s^2) \bar{\sigma}_2(\phi_s^2) \mathbf{Z}_{s,t}^2 \right| \\
& \quad + \omega_{R\bar{\sigma}_1}^{loc}(s, t)^{\frac{2}{q}} \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s, t)^{\frac{1}{q}} + \omega_{R\bar{\sigma}_1 - R\bar{\sigma}_2}^{loc}(s, t)^{\frac{2}{q}} \omega_{\mathbf{Z}^1}(s, t)^{\frac{1}{q}} \\
& \quad + \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s, t)^{\frac{2}{q}} \omega_{\bar{\sigma}_2'}^{loc}(s, t)^{\frac{1}{q}} + \omega_{\bar{\sigma}_2' - \bar{\sigma}_1'}^{loc}(s, t)^{\frac{1}{q}} \omega_{\mathbf{Z}^1}(s, t)^{\frac{2}{q}} \\
& \lesssim_{\|\sigma_i\|_{C^3}} \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s, t)^{\frac{2}{q}} + \omega_{\mathbf{Z}^1}(s, t)^{\frac{2}{q}} (I_{s,t} + |\bar{\sigma}_1 - \bar{\sigma}_2|_{C^2}) \\
& \quad + \omega_{R\bar{\sigma}_1}^{loc}(s, t)^{\frac{2}{q}} \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s, t)^{\frac{1}{q}} + \omega_{R\bar{\sigma}_1 - R\bar{\sigma}_2}^{loc}(s, t)^{\frac{2}{q}} \omega_{\mathbf{Z}^1}(s, t)^{\frac{1}{q}} \\
& \quad + \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s, t)^{\frac{2}{q}} \omega_{\bar{\sigma}_2'}^{loc}(s, t)^{\frac{1}{q}} + \omega_{\bar{\sigma}_2' - \bar{\sigma}_1'}^{loc}(s, t)^{\frac{1}{q}} \omega_{\mathbf{Z}^1}(s, t)^{\frac{2}{q}}.
\end{aligned} \tag{4.6}$$

We report below some estimates for those controls above that depend on $\bar{\sigma}_i$. Recall that multiplying a control by any power of $I_{s,t}$ yields a new control (see e.g. [27, Chapter 1]). By combining Proposition 2.4 and a modification of [26, Lemma 7.3] in terms of p -variation, we have that

$$\begin{aligned}
& \omega_{\bar{\sigma}_i'}^{loc}(s, t) \lesssim_{\bar{\sigma}_i, q} \omega_{\phi_i^1}^{loc}(s, t), \\
& \omega_{\bar{\sigma}_2' - \bar{\sigma}_1'}^{loc}(s, t) \lesssim_{\bar{\sigma}_i, q} [\omega_{\phi_1^1}^{loc}(s, t) \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^2}^q + \omega_{\phi_1^1 - \phi_2^1}^{loc}(s, t) \\
& \quad + I_{s,t}^q (\omega_{\phi_1^1}^{loc}(s, t) + \omega_{\phi_2^1}^{loc}(s, t))], \\
& \omega_{R\bar{\sigma}_1}^{loc}(s, t) \lesssim_{\bar{\sigma}_i, q} (\omega_{R^1}^{loc}(s, t) + \omega_{\phi_1^1}^{loc}(s, t)), \\
& \omega_{R\bar{\sigma}_1 - R\bar{\sigma}_2}^{loc}(s, t) \lesssim_{\bar{\sigma}_i, q} \omega_{R^1 - R^2}^{loc}(s, t) + \omega_{R^1}^{loc}(s, t) (\|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^2}^{\frac{q}{2}} + I_{s,t}^{\frac{q}{2}}) \\
& \quad + \omega_{\phi_2^1}^{loc}(s, t) \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^2}^{\frac{q}{2}} + \omega_{\phi_1^1 - \phi_2^1}^{loc}(s, t) \\
& \quad + I_{s,t}^{\frac{q}{2}} \omega_{\phi_1^1}^{loc}(s, t).
\end{aligned} \tag{4.7}$$

Equations (4.5)–(4.7) above imply that there is a positive constant C , depending on $\min\{L_{u_1}, L_{u_2}\}$, $\max\{\|\bar{\sigma}_1\|_{C^3}, \|\bar{\sigma}_2\|_{C^3}\}$ and q , such that

$$\begin{aligned}
& |R_{s,t}^1 - R_{s,t}^2| \lesssim_C \|u_1 - u_2\|_{L_{t,x}^\infty} |t - s| + \int_s^t \gamma(|\phi_r^1 - \phi_r^2|) dr \\
& \quad + \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s, t)^{\frac{2}{q}} + \omega_{\mathbf{Z}^1}(s, t)^{\frac{2}{q}} (I_{s,t} + \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^2}) \\
& \quad + (\omega_{R^1}^{loc}(s, t) + \omega_{\phi_1^1}^{loc}(s, t))^{\frac{2}{q}} \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s, t)^{\frac{1}{q}} + \omega_{\mathbf{Z}^1}(s, t)^{\frac{1}{q}} [\omega_{R^1 - R^2}^{loc}(s, t) \\
& \quad + \omega_{R^1}^{loc}(s, t) (\|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^2}^{\frac{q}{2}} + I_{s,t}^{\frac{q}{2}}) + \omega_{\phi_2^1}^{loc}(s, t) \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^2}^{\frac{q}{2}} \\
& \quad + \omega_{\phi_1^1 - \phi_2^1}^{loc}(s, t) + I_{s,t}^{\frac{q}{2}} \omega_{\phi_1^1}^{loc}(s, t)]^{\frac{2}{q}} + \omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s, t)^{\frac{2}{q}} \omega_{\phi_2^1}^{loc}(s, t)^{\frac{1}{q}} \\
& \quad + \omega_{\mathbf{Z}^1}(s, t)^{\frac{2}{q}} [\omega_{\phi_1^1}^{loc}(s, t) \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^2}^q + \omega_{\phi_1^1 - \phi_2^1}^{loc}(s, t) \\
& \quad + I_{s,t}^q (\omega_{\phi_1^1}^{loc}(s, t) + \omega_{\phi_2^1}^{loc}(s, t))]^{\frac{1}{q}},
\end{aligned}$$

for all $(s, t) \in \Delta_T$ such that $\omega_{\mathbf{Z}^1}(s, t) + \omega_{\mathbf{Z}^2}(s, t) + |t - s| \leq \mu$. At this point one can fix μ small enough for the term $C \omega_{\mathbf{Z}^1}(s, t)^{\frac{1}{q}} \omega_{R^1 - R^2}^{loc}(s, t)^{\frac{2}{q}}$ term to be smaller than

580 $2^{-\frac{2}{q}} \omega_{R^1-R^2}^{loc}(s, t)^{\frac{2}{q}}$. Proposition 2.4 implies that in this case

$$\begin{aligned}
 \omega_{R^1-R^2}^{loc}(s, t) &\lesssim \left[\|u_1 - u_2\|_{L_{t,x}^\infty} |t - s| + \int_s^t \gamma(|\phi_r^1 - \phi_r^2|) dr \right]^{\frac{q}{2}} \\
 &\quad + \omega_{\mathbf{Z}^1-\mathbf{Z}^2}(s, t) + \omega_{\mathbf{Z}^1}(s, t) (I_{s,t} + \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^1})^{\frac{q}{2}} \\
 &\quad + (\omega_{R^1}^{loc}(s, t) + \omega_{\phi^1}^{loc}(s, t)) \omega_{\mathbf{Z}^1-\mathbf{Z}^2}(s, t)^{\frac{1}{2}} \\
 581 \quad (4.8) \quad &\quad + \omega_{\mathbf{Z}^1}(s, t)^{\frac{1}{2}} [\omega_{R^1}^{loc}(s, t) (\|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^2}^{\frac{q}{2}} + I_{s,t}^{\frac{q}{2}}) + I_{s,t}^{\frac{q}{2}} \omega_{\phi^1}^{loc}(s, t) \\
 &\quad + \omega_{\phi^2}^{loc}(s, t) \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^2}^{\frac{q}{2}}] + \omega_{\phi^2}^{loc}(s, t)^{\frac{1}{2}} \omega_{\mathbf{Z}^1-\mathbf{Z}^2}(s, t) \\
 &\quad + \omega_{\mathbf{Z}^1}(s, t) [\omega_{\phi^1}^{loc}(s, t) \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^2}^q \\
 &\quad + (\omega_{\phi^1}^{loc}(s, t) + \omega_{\phi^2}^{loc}(s, t)) I_{s,t}^q]^{\frac{1}{2}} \\
 &\quad + \omega_{\mathbf{Z}^1}(s, t)^{\frac{1}{2}} \omega_{\phi^1-\phi^2}^{loc}(s, t) + \omega_{\mathbf{Z}^1}(s, t) \omega_{\phi^1-\phi^2}^{loc}(s, t)^{\frac{1}{2}}.
 \end{aligned}$$

582 Finally, we can combine (4.4) and (4.8) and Proposition 2.4 to get

$$\begin{aligned}
 583 \quad \omega_{\phi^1-\phi^2}^{loc}(s, t) &\leq \omega_{\mathbf{Z}^2}(s, t) (\|\bar{\sigma}_2\|_{C^1} I_{s,t} + \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^0})^q \\
 584 \quad &\quad + \|\bar{\sigma}_1\|_{C^0}^q \omega_{\mathbf{Z}^1-\mathbf{Z}^2}(s, t) + \omega_{R^1-R^2}^{loc}(s, t)^2 \\
 585 \quad &\leq C [F(s, t) + \mu \omega_{\phi^1-\phi^2}^{loc}(s, t)^2 + \mu^2 \omega_{\phi^1-\phi^2}^{loc}(s, t)],
 \end{aligned}$$

586 where $F(s, t)$ contains all terms apart from the ones involving $\omega_{\phi^1-\phi^2}^{loc}$. The latter
 587 inequality yields that

$$588 \quad \omega_{\phi^1-\phi^2}^{loc}(s, t) \leq x_2^\mu(s, t) \quad \text{or} \quad \omega_{\phi^1-\phi^2}^{loc}(s, t) \geq x_1^\mu(s, t),$$

589 where

$$590 \quad x_{1,2}^\mu(s, t) = \frac{1 - C\mu^2 \pm \sqrt{(1 - C\mu^2)^2 - 4\mu C^2 F(s, t)}}{2\mu C}.$$

591 By Proposition 4.2, $\|F\|_{L^\infty} \leq M$. We can then choose μ sufficiently small to have

$$592 \quad x_1^\mu(s, t) \geq \frac{1 - C\mu^2 + \sqrt{(1 - C\mu^2)^2 - 4\mu C^2 M}}{2\mu C} \geq 3M.$$

593 Notice in particular that the choice of μ only depends on the parameters of the
 594 problem, i.e. $\|\bar{\sigma}_i\|_{C^3}$, $\|u_i\|_{L_{t,x}^\infty}$, L_{u_i} and $\omega_{\mathbf{Z}^i}(0, T)$. Moreover, again Proposition 4.2
 595 implies that

$$\begin{aligned}
 596 \quad (4.9) \quad \omega_{\phi^1-\phi^2}^{loc}(s, t) &\leq x_2^\mu(s, t) = \frac{1 - C\mu^2 - \sqrt{(1 - C\mu^2)^2 - 4\mu C^2 F(s, t)}}{2\mu C} \\
 597 \quad &\leq 2F(s, t), \\
 598 \quad &
 \end{aligned}$$

599 where the last inequality is due, up to shrinking μ again, to

$$600 \quad \sqrt{a} - b \leq \sqrt{a - b} \quad \forall a > \frac{1}{4}, b \in \left[0, a - \frac{1}{4}\right].$$

At last, (4.9) implies that

$$\begin{aligned} |(\phi^1 - \phi^2)_{s,t}| &\lesssim F(s,t)^{\frac{1}{q}} \\ &\leq \|u_1 - u_2\|_{L_{t,x}^\infty} |t - s| + \int_s^t \gamma(|\phi_r^1 - \phi_r^2|) dr + C\omega_{\mathbf{Z}^1 - \mathbf{Z}^2}(s,t)^{\frac{1}{q}} \\ &\quad + C(\omega_{\mathbf{Z}^1}(s,t) + \omega_{\mathbf{Z}^2}(s,t))^{\frac{1}{q}} (I_{s,t} + \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{C^3}) \end{aligned}$$

for all $(s,t) \in \Delta_T$ such that $\omega_{\mathbf{Z}^1}(s,t) + \omega_{\mathbf{Z}^2}(s,t) + |t - s|^p \leq \mu$.

The above satisfies the hypothesis of the Rough Grönwall Lemma (Lemma 6.4), which we can then apply to conclude the proof. $\square$

We are finally ready to state a well-posedness result for (4.1). We also prove that the solution is measure-preserving in space for all times $t \in [0, T]$.

THEOREM 4.4. *Let $\mathbf{Z}$ be a p -geometric rough path, $p \in [2, 3)$, $u : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$ be bounded, measurable and log-Lipschitz in space, $\{\sigma_j\}_{j=1}^M$ be $C^3(\mathbb{T}^2)$ -vector fields and $\phi_0 \in C(\mathbb{T}^2, \mathbb{R}^2)$.*

Then, for $q \in (p, 3)$ there exists a localization $(\bar{\omega}, L)$ and a controlled rough path $(\phi_t, \bar{\sigma}(\phi_t)) \in \mathcal{D}_{Z, (\bar{\omega}, L)}^{q\text{-var}}(C(\mathbb{T}^2, \mathbb{R}^2))$ solving (4.1), namely s.t.

$$\phi_t = \phi_0 + \int_0^t u(r, \phi_r) dr + \int_0^t \bar{\sigma}(\phi_r) d\mathbf{Z}_r.$$

Such $(\phi_t, \bar{\sigma}(\phi_t))$ is unique: if there exists $(\varphi_t, \bar{\sigma}(\varphi_t)) \in \mathcal{D}_{Z, (\bar{\omega}', L')}^{q\text{-var}}(C(\mathbb{T}^2, \mathbb{R}^2))$ which also solves (4.1), then $\varphi_t = \phi_t$ for all t .

In addition, if ϕ_0 is measure preserving (like in the case $\phi_0 = Id_{\mathbb{T}^2}$) and u, σ_j are divergence-free, then ϕ_t is also measure preserving for all $t \in [0, T]$.

Proof. Since $\mathbf{Z}$ is geometric, there is a sequence of functions in $C^\infty([0, T], \mathbb{R}^M)$ whose canonical lifts converge to $\mathbf{Z}$ in $\mathcal{C}_g^{p\text{-var}}$. In particular, we choose a subsequence $\{Z^N\}_{N \in \mathbb{N}}$ such that, $\omega_{\mathbf{Z}^N - \mathbf{Z}}(0, T) < \frac{1}{N}$ for $N \rightarrow \infty$. Without loss of generality [36, Proof of Theorem 4.1], we can also assume that

$$|Z_{s,t}^N| \lesssim \omega_{\mathbf{Z}}(s,t)^{\frac{1}{p}} \text{ and } |Z_{s,t}^N| \lesssim \omega_{\mathbf{Z}}(s,t)^{\frac{2}{p}}.$$

In addition, let ρ be a mollifier (in space) and define, up to properly extending u to $\mathbb{R}^2$ by periodicity, $u^N = u * \rho_{\frac{1}{N}}$. The latter is divergence-free due to u, σ_j being so. Therefore the flow Φ^N of the associated RDE is measure preserving.

Since u^N is sufficiently smooth for all $N \in \mathbb{N}$, standard results (see e.g. [26]) imply the existence of a unique $\phi^N \in C^{p\text{-var}}$, of the form $\phi_t^N = \Phi_t^N \circ \phi_0$, such that $(\phi^N, \bar{\sigma}(\phi^N)) \in \mathcal{D}_{Z^N}^{p\text{-var}}$ solves the RDE

$$\phi_t^N = \phi_0 + \int_0^t u^N(r, \phi_r^N) dr + \int_0^t \bar{\sigma}(\phi_r^N) d\mathbf{Z}_r^N.$$

In particular, being the composition of measure preserving functions, ϕ^N is itself measure preserving. Proposition 4.3 and Osgood's criterion (cf. [7]) imply that

$$(4.10) \quad |\phi_t^N - \tilde{\phi}_t^N| \lesssim z(t, \|u^N - u^{\tilde{N}}\|_{L_{t,x}^\infty} T + \omega_{Z^N - Z^{\tilde{N}}}(0, T)^{\frac{1}{2q}}),$$

where $z(t, z_0)$ is the solution to $\dot{z}_t = \gamma(z_t)$ with z_0 as starting condition. Notice that $z(t, z_0) \leq e z_0^{exp(-t)}$ as long as z_0 is sufficiently small (cf. [9]).

By combining Proposition 4.2, (4.10) and the interpolation inequality

$$\|\phi^N - \phi^{\tilde{N}}\|_{q,[0,T]} \lesssim (\|\phi^N - \phi^{\tilde{N}}\|_{p,[0,T]})^{\frac{p}{q}} (\|\phi^N - \phi^{\tilde{N}}\|_{L^\infty})^{1-\frac{p}{q}},$$

we eventually see that $\{\phi^N\}$ and $\{\bar{\sigma}(\phi^N)\}$ are Cauchy sequences in $C^{q\text{-var}}$. Moreover, we know that $\|u^N\|_{L_{t,x}^\infty} \lesssim \|u\|_{L_{t,x}^\infty}$ and that the sequence u^N converges uniformly in (t, x) to u and is equi-log-Lipschitz, i.e.

$$|u^N(t, x) - u^N(t, y)| \leq L_u \gamma(|x - y|) \quad \forall N \in \mathbb{N}, t \in [0, T] \text{ and } x, y \in \mathbb{R}^2.$$

These properties, Propositions 4.2 and 4.3 and (4.8) imply that there exists $\mu_0 > 0$ such that $\|R^N - R^{\tilde{N}}\|_{\frac{q}{2}, (\bar{\omega}, \mu_0), [0, T]}$ is a Cauchy sequence, with R^N being the remainder of ϕ^N with respect to Z^N and $\bar{\omega}(s, t) = \omega_{\mathbf{Z}}(s, t) + |t - s|^p$.

It follows that, given $\lim_{N \rightarrow \infty} (\phi^N, \bar{\sigma}(\phi^N)) =: (\phi, \bar{\sigma}(\phi)) \in \mathcal{D}_{Z, (\bar{\omega}, \mu_0)}^{q\text{-var}}(C_b(\mathbb{T}^2, \mathbb{R}^2))$, by continuity of rough integrals, i.e. Proposition 2.10, $(\phi, \bar{\sigma}(\phi))$ solves

$$\phi_t = \phi_0 + \int_0^t u(s, \phi_s) dx + \int_0^t \bar{\sigma}(\phi_s) d\mathbf{Z}_s.$$

Uniqueness then comes, once again, from a trivial generalization of Proposition 4.3, which takes into account also the localizations $(\bar{\omega}, L)$ and $(\bar{\omega}', L')$, combined with Osgood's criterion.

As for the measure-preservingness of ϕ_t : since u^N and the σ_j 's are divergence-free, then the ϕ^N 's are measure-preserving as long as ϕ_0 is. Therefore, ϕ_t is also measure-preserving because, thanks to (4.10), it is the limit of the sequence $\{\phi_t^N\}_t$. $\square$

Once the well-posedness for the RDE with log-Lipschitz drift is obtained, we conclude this section by studying the inverse flow map.

THEOREM 4.5. *Let $\mathbf{Z}$, p , u and $\bar{\sigma}$ be as in the previous theorem. Given $\phi_0 = Id_{\mathbb{T}^2}$, if $\phi : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is the solution of*

$$\phi_t = Id_{\mathbb{T}^2} + \int_0^t u(r, \phi_r) dr - \int_0^t \bar{\sigma}(\phi_r) d\mathbf{Z}_r,$$

the following hold:

1. ϕ_t is invertible, for all $t \in [0, T]$;
2. given $\psi \in C^\infty(\mathbb{T}^2, \mathbb{R})$ and $h_t := \psi(\phi_t^{-1})$, there exist a localization $(\bar{\omega}, L)$ and $h^\natural \in C_{2, \bar{\omega}, L}^{p/3\text{-var}}(W^{-3, \infty})$ such that, for all $\gamma \in W^{3, 1}$,

$$(4.11) \quad h_{s,t}(\gamma) = \int_s^t h_r u_r \cdot \nabla \gamma dr + h_s \left(\left[A_{s,t}^{1,*} + A_{s,t}^{2,*} \right] \gamma \right) + h_{s,t}^\natural(\gamma),$$

where (A^1, A^2) is the unbounded rough driver given by (2.14).

Proof. 1. Given $t \in [0, T]$, we can build the inverse of ϕ_t by solving the equation backward in time. Namely, $\phi_t^{-1} = Y_t$, where $(Y_s)_{s \in [0, t]}$ is the solution of

$$Y_s = Id_{\mathbb{T}^2} - \int_0^s u(t-r, Y_r) dr - \int_0^s \bar{\sigma}(Y_r) d\bar{\mathbf{Z}}_r,$$

driven by the geometric rough path associated with $\bar{\mathbf{Z}}_s = Z_{t-s}$ (we refer e.g. to [26, Exercise 2.6] for more details).

2. Similarly to the previous theorem, consider $\{Z^N\}_{N \in \mathbb{N}} \subset C^\infty([0, T], \mathbb{R}^M)$, approximating $\mathbf{Z}$ in $\mathcal{C}_g^{p\text{-var}}$, and $\{u_N\}_{N \in \mathbb{N}} \subset C^\infty([0, T] \times \mathbb{T}^2, \mathbb{R}^2)$, converging to u .
For each N , the corresponding flows ϕ^N are invertible and sufficiently regular.
In particular since $\partial_t[\phi_t^N(\phi_t^{N,-1}(x))] \equiv 0$, one has that

$$\begin{aligned} \partial_t(\phi_t^{N,-1}(x)) &= -[(\nabla^T \phi_t^N)^{-1}(\phi_t^{N,-1}(x))] [(\partial_t \phi_t^N)(\phi_t^{N,-1}(x))] \\ &= -[(\nabla^T \phi_t^N)^{-1}(\phi_t^{N,-1}(x))] [u_t^N(x) - \bar{\sigma}(x) \dot{Z}_t^N]. \end{aligned}$$

Therefore,

$$\partial_t h_t^N(x) = -u_t^N(x) \cdot \nabla h_t^N(x) + \bar{\sigma}(x) \cdot \nabla h_t^N(x) \dot{Z}_t^N,$$

with $h_t^N := \psi(\phi_t^{N,-1})$. The result then follows as in Theorem 3.6, namely we pass through the integral formulation and iterate the equation as we did in Subsection 2.4. After proving, in a similar fashion, that h^N (respectively, $h^{N,\sharp}$) has finite p -variation (respectively, $\frac{p}{3}$ -variation), and thanks to the convergence of $\phi_t^{N,-1}$ to ϕ_t^{-1} (by Proposition 4.3), we can eventually send N to infinity in the equation solved by h^N . $\square$

4.2. RDEs with non-local drift. Let us consider the second Lagrangian RDE, which plays a crucial role in the proof of uniqueness for solutions of the Euler equations (2.13). In particular, we prove a uniqueness result for the following RDE

$$(4.12) \quad \phi_t(x) = x + \int_0^t \int_{\mathbb{T}^2} K_{BS}(\phi_r(x) - \phi_r(y)) w_0(y) dy dr + \int_0^t \bar{\sigma}(\phi_r(x)) d\mathbf{Z}_r,$$

where w_0 is a bounded function.

Since the drift is non-local, let us introduce a slightly different concept of solution.

DEFINITION 4.6. Let $\phi \in C([0, T] \times \mathbb{T}^2, \mathbb{R}^2)$. We say that ϕ is a solution to the non-local RDE if there exists a localization $(\bar{w}, L)$ s.t. $(\phi(x), \bar{\sigma}(\phi(x))) \in \mathcal{D}_{Z,(\bar{w},L)}^{p\text{-var}}(\mathbb{R}^2)$ for all $x \in \mathbb{T}^2$, and (4.12) holds.

Remark 4.7. Notice that if $(\phi, \bar{\sigma}(\phi)) \in \mathcal{D}_{Z,(\bar{w},L)}^{p\text{-var}}(C(\mathbb{T}^2, \mathbb{R}^2))$, then for all $x \in \mathbb{T}^2$, $(\phi(x), \bar{\sigma}(\phi(x))) \in \mathcal{D}_{Z,(\bar{w},L)}^{p\text{-var}}(\mathbb{R}^2)$.

THEOREM 4.8. Let $\mathbf{Z}$ be a p -geometric rough path, $p \in [2, 3)$, $w_0 \in L^\infty(\mathbb{T}^2)$, $\{\sigma_j\}_{j=1}^M$ be $C^3(\mathbb{T}^2)$ -vector fields and $\phi_0 \in C(\mathbb{T}^2, \mathbb{R}^2)$.

If there exist two controls, $\bar{w}_1$ and $\bar{w}_2$, and two positive constants, L_1 and L_2 , such that $(\phi, \bar{\sigma}(\phi)) \in \mathcal{D}_{Z,(\bar{w}_1,L_1)}^{p\text{-var}}(C(\mathbb{T}^2, \mathbb{R}^2))$ and $(\psi, \bar{\sigma}(\psi)) \in \mathcal{D}_{Z,(\bar{w}_2,L_2)}^{p\text{-var}}(C(\mathbb{T}^2, \mathbb{R}^2))$ are measure-preserving solutions to (4.12), in the sense of the previous definition, then

$$\phi_t = \psi_t \quad \forall t \in [0, T].$$

Remark 4.9. An existence result for (4.12) can be easily produced by combining Proposition 4.3, Theorem 4.4 and the iterative procedure proposed in [9]. We avoid replicating such method, as it plays no role in the uniqueness of solutions to (2.13).

Proof. Let $t \in [0, T]$, then for all $x \in \mathbb{T}^2$

$$\begin{aligned} \phi_t(x) - \psi_t(x) &= \int_0^t \int_{\mathbb{T}^2} [K_{BS}(\phi_r(x) - \phi_r(y)) - K_{BS}(\psi_r(x) - \psi_r(y))] w_0(y) dy dr \\ &\quad + \int_0^t \bar{\sigma}(\phi_r(x)) - \bar{\sigma}(\psi_r(x)) d\mathbf{Z}_r. \end{aligned}$$

Up to working with $\mathbb{R}^2$ -valued controlled rough paths, a procedure analogous to the one used to prove Proposition 4.3 yields

$$\begin{aligned} \omega_{\phi_x - \psi_x}^{loc}(s, t) &\lesssim \left[\|w_0\|_{L^\infty} \int_s^t \int_{\mathbb{T}^2} |K_{BS}(\phi_r(x) - \phi_r(y)) - K_{BS}(\psi_r(x) - \psi_r(y))| dy dr \right]^q \\ &\quad + I_{s,t}^q(x) \omega_{\mathbf{Z}}(s, t) [1 + \omega_{R^{\phi_x}}^{loc}(s, t)^2 + \omega_{\phi_x}^{loc}(s, t) + \omega_{\phi_x}^{loc}(s, t)^2 + \omega_{\psi_x}^{loc}(s, t)] \\ &\quad + \omega_{\mathbf{Z}}(s, t) \omega_{\phi_x - \psi_x}^{loc}(s, t)^2 + \omega_{\mathbf{Z}}(s, t)^2 \omega_{\phi_x - \psi_x}^{loc}(s, t), \end{aligned}$$

where $I_{s,t}(x) := \sup_{r \in [s,t]} |\phi_r(x) - \psi_r(x)|$, $\omega_{R^{\phi_x}}^{loc}$, $\omega_{\phi_x}^{loc}$ and $\omega_{\phi_x - \psi_x}^{loc}$ denote the localized controls associated with R^{ϕ_x} , the remainder of $\phi_t(x)$, $\phi_t(x)$ and $\phi_t(x) - \psi_t(x)$. With the same argument used for Proposition 4.3, we then obtain that

$$\begin{aligned} |\phi_{s,t}(x) - \psi_{s,t}(x)| &\leq \|w_0\|_{L^\infty} \\ &\quad \int_s^t \int_{\mathbb{T}^2} |K_{BS}(\phi_r(x) - \phi_r(y)) - (K_{BS}(\psi_r(x) - \psi_r(y))| dy dr \\ &\quad + C \omega_{\mathbf{Z}}(s, t)^{\frac{1}{q}} I_{s,t}(x). \end{aligned}$$

By Lemma 6.4, the above yields

$$\sup_{r \in [0,t]} |\phi_r(x) - \psi_r(x)| \lesssim \int_0^t \int_{\mathbb{T}^2} |K_{BS}(\phi_r(x) - \phi_r(y)) - (K_{BS}(\psi_r(x) - \psi_r(y))| dy dr.$$

Now, recalling (see e.g. [41]) that the Biot-Savart kernel enjoys the property

$$\int_{\mathbb{T}^2} |K_{BS}(x - y) - K_{BS}(x' - y)| dy \lesssim \gamma(|x - x'|), \quad \forall x, x' \in \mathbb{T}^2,$$

we can integrate in space and add and subtract a term $K_{BS}(\psi_r(x) - \phi_r(y))$. Then, applying Fubini-Tonelli's theorem and exploiting the concavity of γ to use Jensen's inequality, we get

$$\int_{\mathbb{T}^2} |\phi_t(x) - \psi_t(x)| dx \lesssim \|w_0\|_{L^\infty} \int_0^t \gamma \left(\int_{\mathbb{T}^2} |\phi_r(x) - \psi_r(x)| dx \right) dr.$$

At this point, as in Theorem 4.4, Osgood's criterion yields the result. $\square$

5. Uniqueness of the solution. In this section we exploit the uniqueness results obtained for the Lagrangian flows to prove that there exists a unique solution for the two-dimensional Euler equations (1.1). To this aim, we show that $w_t = (\phi_t)_\# w_0$ following Dobrushin's strategy [21], i.e. proving that

$$(5.1) \quad \int_{\mathbb{T}^2} w_t(x) h_t(x) dx = \int_{\mathbb{T}^2} w_0(x) h_0(x) dx,$$

where $h_t := \psi(\phi_t^{-1}(x))$.

Since $w^{\natural}$ and $h^{\natural}$ are poorly regular, we cannot test directly the equation for w_t against the one satisfied by h_t . Therefore, we must develop a strategy similar to the one of [37, 19]. There, a doubling variable technique is used to estimate some crucial quantities, such as the energy of the solution. Instead, we need to apply it for any smooth function ψ in order to prove that any solution w is transported by its corresponding flow.

5.1. Doubling of variables. Let $(\sigma_j)_{j=1}^M \subset C^3(\mathbb{T}^2, \mathbb{R}^2)$ be divergence-free vector fields, $w_0 \in L^\infty(\mathbb{T}^2, \mathbb{R})$ and $\psi \in C^\infty(\mathbb{T}^2, \mathbb{R})$. Consider, according to Definition 2.14, a solution w to (2.13) with initial condition w_0 and consider ϕ, h as in Theorem 4.5: ϕ solving

$$(5.2) \quad \phi_t = I_{\mathbb{T}^2} + \int_0^t u(r, \phi_r) dr - \int_0^t \bar{\sigma}(\phi_r) d\mathbf{Z}_r,$$

where $u = K_{BS} * w$, and $h = \psi(\phi^{-1})$.

Let us first state the following (cf. [36, 37, 19]). From now on, we are also going to denote by $W_{\otimes}^{n,2}$ the Sobolev space $W^{n,2}(\mathbb{T}^2 \times \mathbb{T}^2)$.

LEMMA 5.1. *The mapping $g : [0, T] \rightarrow W_{\otimes}^{0,2}$ defined by $g_s(x, y) = w_s(x) \otimes h_s(y)$ satisfies the equation*

$$(5.3) \quad g_{s,t} = - \int_s^t ((u_r \cdot \nabla w_r) \otimes h_r + w_r \otimes (u_r \cdot \nabla h_r)) dr + \Gamma_{s,t}^1 g_s + \Gamma_{s,t}^2 g_s + g_{s,t}^{\natural},$$

where $g^{\natural} \in C_{2,\bar{w},L}^{p/3-\text{var}}(W_{\otimes}^{-3,\infty})$ (notice that $W^{-3,\infty}$ is continuously embedded into $W^{-3,2}$) and (Γ^1, Γ^2) is an unbounded rough driver on the scale $\{W_{\otimes}^{n,2}\}_n$ (see Subsection 2.3) defined by

$$(5.4) \quad \Gamma_{s,t}^1 = A_{s,t}^1 \otimes I + I \otimes A_{s,t}^1, \quad \Gamma_{s,t}^2 = A_{s,t}^2 \otimes I + I \otimes A_{s,t}^2 + A_{s,t}^1 \otimes A_{s,t}^1,$$

with A^i , for $i = 1, 2$, given by (2.14).

Keeping in mind that h solves (4.11), the above statement follows from straightforward algebraic manipulations of the tensor product between w and h and their expression as solutions of RDEs. In particular, it is easy to check that (Γ^1, Γ^2) satisfies properties (2.8)–(2.10) and that the regular-in-time terms appearing in the tensor product can be “hidden” into $g_{s,t}^{\natural}$, see e.g. [36, Lemma 4.2].

Now, we want to test (5.3) against a smooth approximation of the diagonal $\delta_{x=y}$. Following [37], let us denote $x_{\pm} := \frac{x \pm y}{2}$ and define, for $n \in \mathbb{N}$, the space

$$\mathcal{E}^{n,\nabla} := \left\{ \alpha \in W^{n,\infty}(\mathbb{R}^2 \times \mathbb{R}^2) \mid \alpha(x + 2k\pi, y + 2k\pi) = \alpha(x, y) \ \forall k \in \mathbb{Z}^2, \right. \\ \left. |x_-| > 1 \implies \alpha(x, y) = 0 \right\}$$

normed by

$$\|\alpha\|_{n,\nabla} := \max_{k+l \leq n} |\nabla_+^k \nabla_-^l \alpha(x, y)|_{L_{x,y}^\infty},$$

where $\nabla_{\pm} := \frac{1}{2}(\nabla_x \pm \nabla_y)$. We also need to introduce the dual spaces $\mathcal{E}^{-n,\nabla} := (\mathcal{E}^{n,\nabla})^*$, the duality pairing

$$\langle \gamma, \alpha \rangle_{\nabla} := \int_{B(0,1)} \int_{\mathbb{T}^2} \gamma(x_+ + x_-, x_+ - x_-) \alpha(x_+ + x_-, x_+ - x_-) dx_+ dx_-$$

and a blow-up transformation: for $\epsilon > 0$ and $\alpha \in \mathcal{E}^{n,\nabla}$, it is given by the map

$$T_\epsilon \alpha(x, y) := \frac{1}{\epsilon^2} \alpha\left(x_+ + \frac{x_-}{\epsilon}, x_+ - \frac{x_-}{\epsilon}\right),$$

whose adjoint operator is $T_\epsilon^* \gamma(x, y) = \gamma(x_+ + \epsilon x_-, x_+ - \epsilon x_-)$.

At this point, up to extending by periodicity, we can test $g_{s,t}$ against $T_\epsilon \alpha$. Let us start with the drift term.

LEMMA 5.2. 1. Let $\alpha \in \mathcal{E}^{1,\nabla}$ and $\nu_{s,t} \in W_{\otimes}^{-1,2}$ be defined as

$$\nu_{s,t} := - \int_s^t (u_r \cdot \nabla w_r) \otimes h_r + w_r \otimes (u_r \cdot \nabla h_r) dr.$$

Then

$$|\nu_{s,t}(T_\epsilon \alpha)| \leq C \|\alpha\|_{1,\nabla} |t - s|,$$

where C is a constant only depending on $\|w\|_{L_{t,x}^\infty}$ and $\|\psi\|_{L^\infty}$.

2. Let $\alpha(x, y) = \beta(x_+) f(|x_-|^2)$, with $\beta \in C^\infty(\mathbb{T}^2, \mathbb{R})$ and $f \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ such that it has support contained into $[0, 1]$ and $\int_{B(0,1)} f(|x_-|^2) dx_- = 2$. Then,

$$\lim_{\epsilon \rightarrow 0} \nu_{s,t}(T_\epsilon \alpha) = \int_s^t \int_{\mathbb{T}^2} w_r(x) h_r(x) u_r(x) \cdot \nabla \beta(x) dx.$$

Proof. 1. We have that

$$\begin{aligned} \nu_{s,t}(T_\epsilon \alpha) &= \int_s^t dr \int \frac{1}{\epsilon^2} u_r(x) w_r(x) h_r(y) \nabla_x \left[\alpha\left(x_+ + \frac{x_-}{\epsilon}, x_+ - \frac{x_-}{\epsilon}\right) \right] dx dy \\ &+ \int_s^t dr \int \frac{1}{\epsilon^2} u_r(y) w_r(x) h_r(y) \nabla_y \left[\alpha\left(x_+ + \frac{x_-}{\epsilon}, x_+ - \frac{x_-}{\epsilon}\right) \right] dx dy \\ &= \frac{1}{2} \int_s^t dr \int w_r(x_+ + \epsilon x_-) h_r(x_+ - \epsilon x_-) u_r^{\epsilon,+}(x_+, x_-) (\nabla_+ \alpha)(x, y) dx_+ dx_- \\ &+ \frac{1}{2} \int_s^t dr \int w_r(x_+ + \epsilon x_-) h_r(x_+ - \epsilon x_-) \frac{u_r^{\epsilon,-}(x_+, x_-)}{\epsilon} (\nabla_- \alpha)(x, y) dx_+ dx_- \\ &=: \nu_{s,t}(T_\epsilon \alpha)_+ + \nu_{s,t}(T_\epsilon \alpha)_-, \end{aligned}$$

where $u_r^{\epsilon,\pm}(x_+, x_-) := [u_r(x_+ + \epsilon x_-) \pm u_r(x_+ - \epsilon x_-)]$. The first term can be easily estimated uniformly in ϵ . Instead, for the second term

$$\begin{aligned} &\left| \int_s^t dr \int w_r(x_+ + \epsilon x_-) h_r(x_+ - \epsilon x_-) \frac{u_r^{\epsilon,-}(x_+, x_-)}{\epsilon} \cdot (\nabla_- \alpha)(x, y) dx_- dx_+ \right| \\ &\leq \|w\|_{L_{t,x}^\infty} \|\psi\|_{L^\infty} \|\alpha\|_{1,\nabla} \int_s^t dr \int_{B(0,1)} dx_- \int_{\mathbb{T}^2} dx_+ \left| \frac{u_r^{\epsilon,-}(x_+, x_-)}{\epsilon} \right| \\ &\leq \|w\|_{L_{t,x}^\infty} \|\psi\|_{L^\infty} \|\alpha\|_{1,\nabla} \|u\|_{L_t^\infty W_x^{1,1}} |t - s|. \end{aligned}$$

2. It is enough to prove that $\lim_{\epsilon \rightarrow 0} \nu_{s,t}(T_\epsilon \alpha)_- = 0$. In particular, up to approximating the velocity u in $L_T^\infty W^{1,1}$ by smooth functions, let us assume that u is smooth in space. Then,

$$\begin{aligned} \nu_{s,t}(T_\epsilon \alpha)_- &= \int_s^t dr \int_{B(0,1)} dx_- \int_{\mathbb{T}^2} dx_+ w_r(x_+ + \epsilon x_-) h_r(x_+ - \epsilon x_-) \beta(x_+) \\ &\quad \left(\int_0^1 \nabla^T u_r(x_+ - \epsilon x_- + 2\theta \epsilon x_-) x_- d\theta \right) \cdot x_- f'(|x_-|^2), \end{aligned}$$

which, as ϵ goes to 0, converges to

$$\begin{aligned} &\int_s^t dr \int_{\mathbb{T}^2} dx_+ (w_r h_r \beta)(x_+) \sum_{i,j=1}^2 \partial_j u_r^i(x_+) \int_{B(0,1)} x_-^j x_-^i f'(|x_-|^2) dx_- \\ &= \int_s^t dr \int_{\mathbb{T}^2} dx_+ w_r(x_+) h_r(x_+) \beta(x_+) \operatorname{div}(u_r)(x_+) \int_0^1 \pi \rho^3 f'(\rho^2) d\rho = 0 \quad \square \end{aligned}$$

Remark 5.3. Although the setting is different, we point out that the second statement of the above Lemma 5.2, as well as the strategy to prove it, shares technical ideas with DiPerna-Lions' theory of commutators [20, 1].

It is now the turn of the rough terms to be tested against $T_\epsilon \alpha$. The following has already been proved in [19], hence we are only going to give the statement.

LEMMA 5.4. *Consider the unbounded rough drivers $(\Gamma_{s,t}^1, \Gamma_{s,t}^2)$ defined in (5.4): $(\Gamma_{s,t}^{1,\epsilon}, \Gamma_{s,t}^{2,\epsilon})_\epsilon := (T_\epsilon^* \Gamma_{s,t}^1 T_\epsilon^{*, -1}, T_\epsilon^* \Gamma_{s,t}^2 T_\epsilon^{*, -1})_\epsilon$ is a bounded family of unbounded rough drivers on the scale $\mathcal{E}^{n,\nabla}$.*

Moreover, if $\alpha(x, y) = \beta(x_+) f(|x_-|^2)$, with $\beta \in C^\infty(\mathbb{T}^2, \mathbb{R})$ and $f \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ such that it has support contained into $[0, 1]$ and $\int_{B(0,1)} f(|x_-|^2) dx_- = 2$, then

$$\lim_{\epsilon \rightarrow 0} \langle T_\epsilon^* g_s, \Gamma_{s,t}^{i,\epsilon,*} \alpha \rangle_\nabla = \int_{\mathbb{T}^2} w_s(x) h_s(x) (A_{s,t}^{i,*} \beta)(x) dx, \quad i = 1, 2,$$

where A^i is as in (2.14).

We conclude the paragraph by identifying the equation satisfied by $T_\epsilon^* g_s$. In order to estimate the remainder appearing in it, we first report a result from [37].

LEMMA 5.5. *There exists a family of smoothing operators $(\bar{J}^\eta)_{\eta \in [0,1]}$ on the scale $\mathcal{E}^{n,\nabla}$.*

Eventually, the following is what we need.

PROPOSITION 5.6. *For any $\epsilon > 0$, given g as in Lemma 5.1, $g_s^\epsilon := T_\epsilon^* g_s$ satisfies in $\mathcal{E}^{-3,\nabla}$ the equation*

$$(5.7) \quad g_{s,t}^\epsilon = \nu_{s,t}^\epsilon + \Gamma_{s,t}^{1,\epsilon} g_s^\epsilon + \Gamma_{s,t}^{2,\epsilon} g_s^\epsilon + g_{s,t}^{\natural,\epsilon},$$

where $\nu_{s,t}^\epsilon(\alpha) = \nu_{s,t}(T_\epsilon \alpha)$ is defined as in Lemma 5.2. Moreover, the family $(g_{s,t}^{\natural,\epsilon})_{\epsilon > 0}$ is bounded in $C_{2,\bar{\omega},L}^{p/3-\text{var}}(\mathcal{E}^{-3,\nabla})$.

Proof. The equation satisfied by g_s^ϵ can be easily found by testing (5.3) against $T_\epsilon \alpha$ for all $\alpha \in \mathcal{E}^{3,\nabla}$.

Instead, the uniform bound on the localized $\frac{p}{3}$ -variation of the remainders $g^{\natural,\epsilon}$ follows from Lemma 5.2, Lemma 5.4 and Lemma 5.5 with a procedure analogous to the one already showed in Lemma 3.2. Indeed, as shown in [19], such procedure can be replicated once we have, like here, a family of smoothing operators, the drifts $\{\nu_{s,t}^\epsilon\}_{\epsilon > 0}$ uniformly in $C^{1-\text{var}}(\mathcal{E}^{-1,\nabla})$ and $\{(\Gamma_{s,t}^{1,\epsilon}, \Gamma_{s,t}^{2,\epsilon})\}_{\epsilon > 0}$ uniformly unbounded rough drivers. $\square$

5.2. Uniqueness result. We are finally ready to prove the remaining part of Theorem 1.1, namely that the 2D Euler equations, driven by rough transport noise, have a unique solution whenever the initial condition is bounded.

THEOREM 5.7. *Let $w_0 \in L^\infty(\mathbb{T}^2)$, $\mathbf{Z}$ be a p -geometric rough path, $p \in [2, 3)$, and $\{\sigma_j\}_{j=1}^M \subset C^3(\mathbb{T}^2, \mathbb{R}^2)$ be a family of divergence-free vector fields. Given w a solution of (1.1) in the sense of Definition 2.14, then w is unique.*

In particular, it can be written as the advection of w_0 along the corresponding flow,

$$w_t = (\phi_t)_\# w_0,$$

where ϕ solves (5.2).

Proof. We start proving that any solution w in the sense of Definition 2.14 is advected by the associated flow.

To this goal, consider $\psi \in C^\infty(\mathbb{T}^2, \mathbb{R})$, $h_t = \psi(\phi_t^{-1})$ and $g_s^\epsilon = T_\epsilon^*(w_s \otimes h_s)$. Given $\alpha(x, y) = \beta(x_+)f(|x_-|^2)$, with $\beta \in C^\infty(\mathbb{T}^2, \mathbb{R})$ and $f \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ with support contained into $[0, 1]$ and $\int_{B(0,1)} f(|x_-|^2) dx_- = 2$, we can test (5.7) against α and send ϵ to 0. Then, Lemmas 5.2 and 5.4 imply that

$$\begin{aligned} \int_{\mathbb{T}^2} (w_t(x)h_t(x) - w_s(x)h_s(x)) \beta(x) dx &= \int_s^t \int_{\mathbb{T}^2} w_r(x)h_r(x)u_r(x) \cdot \nabla \beta(x) dx dr \\ &+ \int_{\mathbb{T}^2} w_s(x)h_s(x)A_{s,t}^{1,*}\beta(x) dx + \int_{\mathbb{T}^2} w_s(x)h_s(x)A_{s,t}^{2,*}\beta(x) dx + \bar{g}_{s,t}^{\sharp}(\beta), \end{aligned}$$

where $\bar{g}_{s,t}^{\sharp}(\beta) := \lim_{\epsilon \rightarrow 0} g_{s,t}^{\epsilon,\sharp}(\alpha)$ has finite localized $\frac{p}{3}$ -variation in $W^{-3,\infty}$ thanks to Proposition 5.6. In particular, if we choose $\beta \equiv 1$, we get the equality

$$\int_{\mathbb{T}^2} (w_t(x)h_t(x) - w_s(x)h_s(x)) dx = \bar{g}_{s,t}^{\sharp}(1),$$

which yields that $\bar{g}_{s,t}^{\sharp}(1)$ is an additive function with finite $\frac{p}{3}$ -variation, meaning that it is identically 0.

Hence, up to approximation with smooth functions, we can take $\psi = \bar{\psi} \circ \phi_t$ for any $\bar{\psi} \in C^\infty(\mathbb{T}^2, \mathbb{R})$ and obtain that

$$\int_{\mathbb{T}^2} w_t(x)\bar{\psi}(x) dx = \int_{\mathbb{T}^2} w_0(x)\bar{\psi}(\phi_t(x)) dx,$$

which is, w_t is transported along the associated flow.

Consider now two solutions w^1, w^2 to (1.1), both starting at w_0 , driven by $\mathbf{Z}$ and the same vector fields $(\sigma_j)_{j=1}^M$. We have proved that $w_t^i = (\phi_t^i)_\# w_0$, $i = 1, 2$, where ϕ_t^i solves (4.1) with drift $K_{BS} * w_t^i$. Noting that

$$\begin{aligned} \int_0^t (K_{BS} * w_r^i)(\phi_r^i(x)) dr &= \int_0^t \int_{\mathbb{T}^2} K_{BS}(\phi_r^i(x) - y) w_r^i(y) dy dr \\ &= \int_0^t \int_{\mathbb{T}^2} K_{BS}(\phi_r^i(x) - \phi_r^i(y)) w_0(y) dy dr, \end{aligned}$$

hence ϕ_t^1 and ϕ_t^2 are both solutions to (4.12) and by Theorem 4.8 it holds $\phi^1 = \phi^2$, concluding the proof. $\square$

5.3. Continuity of the solution map. We conclude our work by proving that solutions to (2.13) are continuous with respect to initial parameters w_0, σ and $\mathbf{Z}$. This is not only important per se, but also immediately yields a Wong-Zakai type result for our problem.

THEOREM 5.8. *Let $(L^\infty)^{weak,*}$ be L^∞ equipped with the weak-* topology. Then, the solution map*

$$(L_x^\infty)^{weak,*} \times (C^3)^M \times \mathcal{C}_g^{p\text{-var}} \rightarrow (L_{t,x}^\infty)^{weak,*} \cap C_t W^{-1,\infty} \\ (w_0, \sigma, \mathbf{Z}) \mapsto w$$

is continuous.

Moreover, given $\{B^{H,N}\}_{N \in \mathbb{N}}$ a piecewise linear approximation of a fractional Brownian motion B^H , with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$, then the solutions $\{w^N\}$ of (1.1) with $\dot{Z} = \dot{B}^{H,N}$ converge almost surely to the solution w of

$$(5.9) \quad \begin{cases} dw + u \cdot \nabla w dt = \sum_{j=1}^M \sigma_j \cdot \nabla w \circ dB^{H,j}, \\ w(0) = w_0, \quad u = K_{BS} * w. \end{cases}$$

Proof. We start by showing that, given a converging sequence of parameters, the solutions depending on it converge themselves. To this aim, consider $\{w_0^N, \sigma^N, \mathbf{Z}^N\} \subseteq (L^\infty)^{weak,*} \times (C^3)^M \times \mathcal{C}_g^{p\text{-var}}$ and assume it converges to some $\{w_0, \sigma, \mathbf{Z}\}$ in the same space. By well-posedness of (2.13), we know that for each $N \in \mathbb{N}$ there exists a solution w^N associated to the N -th element of the sequence. In particular, since the bounds we found only depend on initial parameters, the sequence of solutions $\{w^N\}$ is uniformly bounded in $L_{t,x}^\infty \cap C_T^{p\text{-var}} W^{-1,\infty}$: this ensures us that there exists a subsequence $\{w^{N_k}\}$ converging strongly to $w \in C_t W^{-1,\infty}$ (cf. Lemma 6.1).

We conclude by noting that the unbounded rough drivers $(A^{N,1}, A^{N,2})$, associated to $(\sigma^N, \mathbf{Z}^N)$, converge, making us able to prove that w solves (2.13) like we did for the existence proof, and that, by uniqueness of solutions, it is the whole sequence $\{w^N\}$ that converges.

At this point, the second statement is trivial: we know (see e.g. [26]) that piecewise linear are one of those approximations of a fractional Brownian motion such that their canonical lift $(B^{H,N}, \int_0^\cdot B^{H,N} \otimes dB^{H,N})$ converges to $(B^H, \mathbb{B}^H)$ in the rough path topology almost surely. $\square$

Remark 5.9. In the case $H = \frac{1}{2}$, B^H is a standard Brownian motion. Then it is well known that piecewise linear approximations and their canonical lifts converge to Stratonovich enhanced Brownian motion, $(B, \mathbb{B}^{\text{Strat}})$. Since Stratonovich rough integration against an enhanced Brownian motion coincides almost surely with standard Stratonovich integration, this yields that the w^N associated to $\mathbf{B}^{H,N}$ converge to a probabilistically strong solution of (5.9). In this case, we end up recovering the result obtained in [9]. Moreover, if $H \in (\frac{1}{2}, 1)$ the result still holds, as the rough lifts of piecewise linear approximations simply converge to a canonical Young integral.

Remark 5.10. The reasoning behind the proof of the Wong-Zakai statement in Theorem 5.8 actually holds in more general cases. For example, it holds for Gaussian rough paths satisfying the hypothesis of [26, Theorem 10.4], for which it can be proven that piecewise linear approximations converge in rough path topology. Another possible generalisation regards instead the type of approximation. As already pointed out in the proof of the theorem, the key for the convergence result is a combination of the continuity for the solution map and of the convergence of an approximating

sequence in rough path topology. This actually allows for other approximations, like by convolution or à la McShane [39].

6. Appendix.

6.1. A compactness argument. We prove here a compactness result, which plays a crucial role in Theorem 3.6.

LEMMA 6.1. *Consider a sequence $\{g_n\}_{n \in \mathbb{N}} \subset L^\infty([0, T] \times \mathbb{T}^2) \cap C^{p\text{-var}}(W^{-1, \infty})$ such that $\|g_n\|_{L_{t,x}^\infty}$ is uniformly bounded and there exist $L > 0$, two controls ω and $\bar{\omega}$ satisfying*

$$\|(g_n)_{s,t}\|_{W^{-1, \infty}} \leq \omega(s, t)^{1/p} \quad \forall n \in \mathbb{N}, (s, t) \in \Delta_T \text{ s.t. } \bar{\omega}(s, t) \leq L.$$

Then, $\{g_n\}_n$ is compact in $C([0, T], W^{-1, \infty})$.

Proof. For all $\alpha \in (0, L]$, define the operator $J_\alpha f(s) = \frac{1}{\alpha} \int_s^{s+\alpha} f(r) dr$, where we impose $f(r) = f(T)$ for all $r > T$. The following hold true:

1. For any fixed α , $J_\alpha g_n : [0, T] \rightarrow W^{-1, \infty}$ are equicontinuous functions;
2. As α vanishes, $\sup_{n \in \mathbb{N}} \|J_\alpha g_n - g_n\|_{C_t W_x^{-1, 1}} \rightarrow 0$;
3. For all $s \in [0, T]$ and $\alpha \in (0, L]$, $\{J_\alpha g_n(s)\}_n$ is relatively compact in $W^{-1, \infty}$;
4. There exists a subsequence g_{n_k} that converges weakly-* to some $g \in L_{t,x}^\infty$.

The first and second properties follow from the hypothesis, the third one from $W^{1, 1}$ being compactly embedded into L^1 , and the last one from standard results in functional analysis.

Let now α be fixed: Ascoli-Arzelà theorem yields a subsequence, denoted again by $\{J_\alpha g_{n_k}\}$, such that $J_\alpha g_{n_k}$ converges to some h in $C_t W_x^{-1, \infty}$. Then, point 4 above and the continuity of J_α imply that $h = J_\alpha g$. In particular, any subsequence of $J_\alpha g_{n_k}$ has a sub-subsequence that converges to h ; it follows that the full sequence $J_\alpha g_{n_k}$ must converge to h .

Eventually, the result follows from the inequality

$$\begin{aligned} \|g_{n_k} - g_{n_l}\|_{C_t W^{-1, \infty}} &\leq \|g_{n_k} - J_\alpha g_{n_k}\|_{C_t W^{-1, \infty}} \\ &\quad + \|J_\alpha g_{n_k} - J_\alpha g_{n_l}\|_{C_t W_x^{-1, \infty}} + \|J_\alpha g_{n_l} - g_{n_l}\|_{C_t W_x^{-1, \infty}}. \quad \square \end{aligned}$$

6.2. Sewing Lemma and Rough Grönwall Lemma. The Sewing Lemma is crucial in rough paths theory since it allows the construction rough integrals. We report a version of it, whose proof is a straightforward modification of [19, Lemma 2.2].

LEMMA 6.2. *Let $\xi \in (0, 1)$, $L > 0$ and V be a Banach space. Consider two controls $\omega, \bar{\omega}$ and a 2-index function $h : \Delta_T \rightarrow V$ such that*

$$|\delta h_{s,u,t}| := |h_{s,t} - h_{s,u} - h_{u,t}| \leq \omega(s, t)^{\frac{1}{\xi}},$$

for all $(s, u, t) \in \Delta_T^2$ with $\bar{\omega}(s, t) \leq L$. Then, there exists a unique path $I(h) : [0, T] \rightarrow V$ with $I(h)_0 = 0$ such that the 2-index function $\Lambda_{s,t} := I(h)_{s,t} - h_{s,t} \in C_{\bar{\omega}, L}^{\xi\text{-var}, 2}$. Moreover, there exists $C_\xi > 0$ such that

$$|\Lambda_{s,t}| \leq C_\xi \omega(s, t)^{\frac{1}{\xi}},$$

for all $(s, t) \in \Delta_T$ with $\bar{\omega}(s, t) \leq L$.

In addition, if $h \in C_{\bar{\omega}, L}^{p\text{-var}, 2}(V)$ for some $p \geq \xi$, then $I(h) \in C^{p\text{-var}}(V)$.

The following corollary can be found at [36, Corollary B.2]

COROLLARY 6.3. *Let $\xi \in (0, 1)$. Consider $h : \Delta_T \rightarrow V$ such that $h \in C_{(\bar{\omega}, L)}^{\xi\text{-var}, 2}(V)$ and it satisfies the assumptions of Lemma 6.2. Then,*

$$|h_{s,t}| \leq C_\xi \omega(s, t)^{\frac{1}{\xi}},$$

for all $(s, t) \in \Delta_T$ s.t. $\bar{\omega}(s, t) \leq L$.

The second reported result is a generalization of the Rough Grönwall Lemma first proved in [19].

LEMMA 6.4. *Consider a function $G : [0, T] \rightarrow \mathbb{R}^+$, three controls $\omega_1, \omega_2, \omega_3$ and some constants $L, C' > 0$, $C \geq 1$, $k > k' \geq 1$.*

Assume that the following conditions hold:

1. $\omega_2(s, t) \leq \omega_1(s, t)$ for all $(s, t) \in \Delta_T$;
2. For all $(s, t) \in \Delta_T$ with $\omega_1(s, t) \leq L$,

$$G_{s,t} \leq C \left(\sup_{r \in [0, T]} G_r + C' \right) \omega_1(s, t)^{\frac{1}{k}} + \omega_2(s, t)^{\frac{1}{k'}} + \omega_3(s, t).$$

Then,

$$\begin{aligned} \sup_{t \in [0, T]} G_t &\leq 2 \exp \left(\frac{\omega_1(0, T)}{\alpha L} \right) \left\{ G_0 + \sup_{t \in [0, T]} \left(\omega_3(0, t) \exp \left(\frac{-\omega_1(0, t)}{\alpha L} \right) \right) \right. \\ &\quad \left. + \sup_{t \in [0, T]} \left(\omega_2(0, t)^{\frac{1-\theta}{k'}} \exp \left(\frac{-\omega_1(0, t)}{\alpha L} \right) \right) + C' \right\}, \end{aligned}$$

where $\theta = \frac{k'}{k}$ and $\alpha = \min \left(1, \frac{1}{L(2Ce^2)^k} \right)$.

Proof. For any $t \in [0, T]$, consider the following functions:

$$G_{\leq t} := \sup_{s \in [0, t]} G_s,$$

$$H_t := \exp \left(- \frac{\omega_1(0, t)}{\alpha L} \right) \left(G_{\leq t} + \omega_2(0, t)^{\frac{1-\theta}{k'}} + C' \right),$$

$$H_{\leq t} := \sup_{s \in [0, t]} H_s.$$

Let $K \in \mathbb{N}^*$ s.t. $\omega_1(0, T) \in (\alpha L(K-1), \alpha LK]$, then there exists a partition $P = \{t_0 := 0, \dots, t_K := T\}$ such that $\omega_1(0, t_j) = \alpha j L$ for all $j \in \{1, \dots, K-1\}$. Since ω_1

is super-additive, then $\omega_1(t_{j-1}, t_j) \leq \alpha L$. Consider $t \in [t_{i-1}, t_i]$, then

$$\begin{aligned}
 G_{0,t} &= \sum_{j=0}^{i-2} G_{t_j, t_{j+1}} + G_{t_{i-1}, t} \\
 &\leq \sum_{j=0}^{i-2} \left[C(G_{\leq t_{j+1}} + C') \omega_1(t_j, t_{j+1})^{\frac{1}{k}} + \omega_2(t_j, t_{j+1})^{\frac{1}{k'}} + \omega_3(t_j, t_{j+1}) \right] \\
 &\quad + C(G_{\leq t} + C') \omega_1(t_{i-1}, t)^{\frac{1}{k}} + \omega_2(t_{i-1}, t)^{\frac{1}{k'}} + \omega_3(t_{i-1}, t) \\
 &\leq \sum_{j=0}^{i-2} \left[C(\alpha L)^{\frac{1}{k}} (G_{\leq t_{j+1}} + C') + \omega_2(0, t_{j+1})^{\frac{1-\theta}{k'}} \omega_1(t_j, t_{j+1})^{\frac{1}{k}} \right] \\
 &\quad + C(G_{\leq t_i} + C') (\alpha L)^{\frac{1}{k}} + \omega_2(0, t_i)^{\frac{1-\theta}{k'}} \omega_1(t_{i-1}, t)^{\frac{1}{k}} + \omega_3(0, t) \\
 &\leq C(\alpha L)^{\frac{1}{k}} \sum_{j=0}^{i-1} \left[G_{\leq t_{j+1}} + \omega_2(0, t_{j+1})^{\frac{1-\theta}{k'}} + C' \right] + \omega_3(0, t),
 \end{aligned}$$

where the first and second inequalities are due to the hypothesis. Now,

$$\begin{aligned}
 \sum_{j=0}^{i-1} \left[G_{\leq t_{j+1}} + \omega_2(0, t_{j+1})^{\frac{1-\theta}{k'}} + C' \right] &= \sum_{j=0}^{i-1} H_{t_{j+1}} \exp\left(\frac{\omega_1(0, t_{j+1})}{\alpha L}\right) \\
 &\leq H_{\leq T} \sum_{j=0}^{i-1} \exp(j+1) \\
 &\leq H_{\leq T} \exp(i+1).
 \end{aligned}$$

Therefore,

$$(6.1) \quad G_t \leq G_0 + \omega_3(0, t) + C(\alpha L)^{\frac{1}{k}} H_{\leq T} \exp(i+1).$$

We exploit (6.1) to estimate $H_{\leq T}$. Indeed, given $t \in [t_{i-1}, t_i]$, then

$$\begin{aligned}
 H_t &= \exp\left(-\frac{\omega_1(0, t)}{\alpha L}\right) \left(G_{\leq t} + \omega_2(0, t)^{\frac{1-\theta}{k'}} + C'\right) \\
 &\leq \exp\left(-\frac{\omega_1(0, t)}{\alpha L}\right) \\
 &\quad \left[G_0 + \omega_3(0, t) + C(\alpha L)^{1/k} H_{\leq T} \exp(i+1) + \omega_2(0, t)^{\frac{1-\theta}{k'}} + C'\right] \\
 &\leq G_0 + \sup_{r \in [0, t]} \left(\omega_3(0, r) \exp\left(-\frac{\omega_1(0, r)}{\alpha L}\right)\right) + \\
 &\quad \sup_{r \in [0, t]} \left(\omega_2(0, r)^{\frac{1-\theta}{k'}} \exp\left(-\frac{\omega_1(0, r)}{\alpha L}\right)\right) + C(\alpha L)^{\frac{1}{k}} e^2 H_{\leq T} + C'.
 \end{aligned}$$

Therefore, taking the supremum in the time variable t and $C(\alpha L)^{1/k} e^2 \leq 1/2$ imply that

$$\begin{aligned}
 (6.2) \quad H_{\leq T} &\leq 2G_0 + 2 \sup_{r \in [0, T]} \left(\omega_3(0, r) \exp\left(-\frac{\omega_1(0, r)}{\alpha L}\right)\right) \\
 &\quad + 2 \sup_{r \in [0, T]} \left(\omega_2(0, r)^{\frac{1-\theta}{k'}} \exp\left(-\frac{\omega_1(0, r)}{\alpha L}\right)\right) + 2C'.
 \end{aligned}$$

At last, notice that

$$G_{\leq T} \leq H_T \exp\left(\frac{\omega_1(0, T)}{\alpha L}\right) \leq H_{\leq T} \exp\left(\frac{\omega_1(0, T)}{\alpha L}\right),$$

which proves the result once it is combined with (6.2). $\square$

Acknowledgments. The authors wish to thank Mario Maurelli, Marco Romito and Franco Flandoli for the insightful discussions and their kind suggestions. They are also grateful to the reviewers for their observations and questions, which allowed to clarify in greater detail some key passages of the paper.

REFERENCES

- [1] L. AMBROSIO, *Transport equation and Cauchy problem for BV vector fields.*, Inventiones mathematicae, 158 (2004).
- [2] I. BAILLEUL AND M. GUBINELLI, *Unbounded rough drivers*, in Annales de la Faculté des Sciences de Toulouse: Mathématiques, vol. 26, 2017, pp. 795–830.
- [3] M. BEN-ARTZI, *Global solutions of two-dimensional Navier-Stokes and Euler equations*, Archive for Rational Mechanics and Analysis, 128 (1994), pp. 329–358.
- [4] J. BERNER, U. ACHATZ, L. BATTE, L. BENGTSOON, A. D. L. CÁMARA, H. M. CHRISTENSEN, M. COLANGELI, D. R. COLEMAN, D. CROMMELIN, S. I. DOLAPTCHEV, ET AL., *Stochastic parameterization: Toward a new view of weather and climate models*, Bulletin of the American Meteorological Society, 98 (2017), pp. 565–588.
- [5] H. BESSAIH, *Martingale solutions for stochastic Euler equations*, Stochastic Anal. Appl., 17 (1999), pp. 713–725, <https://doi.org/10.1080/07362999908809631>.
- [6] H. BESSAIH AND F. FLANDOLI, *2-D Euler equation perturbed by noise*, NoDEA Nonlinear Differential Equations Appl., 6 (1999), pp. 35–54, <https://doi.org/10.1007/s000300050063>.
- [7] I. BIHARI, *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta Mathematica Hungarica, 7 (1956), pp. 81–94.
- [8] Z. BRZEŹNIAK, M. CAPIŃSKI, AND F. FLANDOLI, *Stochastic partial differential equations and turbulence*, Mathematical Models and Methods in Applied Sciences, 1 (1991), pp. 41–59.
- [9] Z. BRZEŹNIAK, F. FLANDOLI, AND M. MAURELLI, *Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity*, Archive for Rational Mechanics and Analysis, 221 (2016), <https://doi.org/10.1007/s00205-015-0957-8>.
- [10] Z. BRZEŹNIAK AND S. PESZAT, *Stochastic two dimensional Euler equations*, Ann. Probab., 29 (2001), pp. 1796–1832, <https://doi.org/10.1214/aop/1015345773>.
- [11] F. BUTORI AND E. LUONGO, *Mean-Field Magnetohydrodynamics models as scaling limits of stochastic induction equations*, arXiv preprint arXiv:2406.07206, (2024).
- [12] R. CATELLIER, *Rough linear transport equation with an irregular drift*, Stochastics and Partial Differential Equations: Analysis and Computations, 4 (2016), pp. 477–534.
- [13] R. CATELLIER AND M. GUBINELLI, *Averaging along irregular curves and regularisation of ODEs*, Stochastic Processes and Their Applications, 126 (2016), pp. 2323–2366.
- [14] M. COGHI AND M. MAURELLI, *Existence and uniqueness by Kraichnan noise for 2D Euler equations with unbounded vorticity*, arXiv preprint arXiv:2308.03216, (2023).
- [15] D. CRISAN, F. FLANDOLI, AND D. D. HOLM, *Solution properties of a 3D stochastic Euler fluid equation*, Journal of Nonlinear Science, 29 (2019), pp. 813–870.
- [16] D. CRISAN, D. D. HOLM, J.-M. LEAHY, AND T. NILSEN, *Solution properties of the incompressible Euler system with rough path advection*, J. Funct. Anal., 283 (2022), pp. Paper No. 109632, 51, <https://doi.org/10.1016/j.jfa.2022.109632>, <https://doi.org/10.1016/j.jfa.2022.109632>.
- [17] D. CRISAN, D. D. HOLM, J.-M. LEAHY, AND T. NILSEN, *Variational principles for fluid dynamics on rough paths*, Adv. Math., 404 (2022), pp. Paper No. 108409, 75, <https://doi.org/10.1016/j.aim.2022.108409>, <https://doi.org/10.1016/j.aim.2022.108409>.
- [18] A. M. DAVIE, *Differential equations driven by rough paths: an approach via discrete approximation*, Applied Mathematics Research eXpress, 2008 (2008), p. abm009.
- [19] A. DEYA, M. GUBINELLI, M. HOFMANOVÁ, AND S. TINDEL, *A priori estimates for rough PDEs with application to rough conservation laws*, Journal of Functional Analysis, 276 (2019), pp. 3577–3645.
- [20] R. J. DiPerna and P.-L. LIONS, *Ordinary differential equations, transport theory and Sobolev spaces*, Inventiones mathematicae, 98 (1989), pp. 511–547.

- [21] R. L. DOBRUSHIN, *Vlasov equations*, Functional Analysis and Its Applications, 13 (1979), pp. 115–123.
- [22] F. FLANDOLI, L. GALEATI, AND D. LUO, *Scaling limit of stochastic 2D Euler equations with transport noises to the deterministic Navier–Stokes equations*, Journal of Evolution Equations, 21 (2021), pp. 567–600.
- [23] F. FLANDOLI, M. GUBINELLI, AND E. PRIOLA, *Well-posedness of the transport equation by stochastic perturbation*, Inventiones Mathematicae, 180 (2010), pp. 1–53.
- [24] F. FLANDOLI AND D. LUO, *High mode transport noise improves vorticity blow-up control in 3D Navier–Stokes equations*, Probability Theory and Related Fields, 180 (2021), pp. 309–363.
- [25] C. L. FRANZKE, T. J. O’KANE, J. BERNER, P. D. WILLIAMS, AND V. LUCARINI, *Stochastic climate theory and modeling*, Wiley Interdisciplinary Reviews: Climate Change, 6 (2015), pp. 63–78.
- [26] P. K. FRIZ AND M. HAIRER, *A course on rough paths*, (2020), <https://doi.org/10.1007/978-3-030-41556-3>.
- [27] P. K. FRIZ AND N. B. VICTOIR, *Multidimensional stochastic processes as rough paths: theory and applications*, 120 (2010).
- [28] L. GALEATI, F. A. HARANG, AND A. MAYORCAS, *Distribution dependent SDEs driven by additive fractional Brownian motion*, Probability Theory and Related Fields, 185 (2023), pp. 251–309.
- [29] L. GALEATI, J.-M. LEAHY, AND T. NILSSEN, *On the well-posedness of (nonlinear) rough continuity equations*, Journal of Differential Equations, 462 (2026), p. 114124, <https://doi.org/10.1016/j.jde.2026.114124>.
- [30] L. GALEATI AND D. LUO, *Weak well-posedness by transport noise for a class of 2d fluid dynamics equations*, Journal of Functional Analysis, 289 (2025), p. 111158, <https://doi.org/10.1016/j.jfa.2025.111158>.
- [31] M. GUBINELLI, *Controlling rough paths*, Journal of Functional Analysis, 216 (2004), pp. 86–140.
- [32] M. GUBINELLI AND S. TINDEL, *Rough evolution equations*, The Annals of Probability, 38 (2010), pp. 1 – 75, <https://doi.org/10.1214/08-AOP437>.
- [33] M. HAIRER, *Rough stochastic PDEs*, Communications on Pure and Applied Mathematics, 64 (2011), pp. 1547–1585.
- [34] A. HOCQUET AND M. HOFMANOVÁ, *An energy method for rough partial differential equations*, Journal of Differential Equations, 265 (2018), pp. 1407–1466.
- [35] A. HOCQUET, T. NILSSEN, AND W. STANNAT, *Generalized Burgers equation with rough transport noise*, Stochastic Processes and their Applications, 130 (2020), pp. 2159–2184.
- [36] M. HOFMANOVÁ, J.-M. LEAHY, AND T. NILSSEN, *On the Navier-Stokes equation perturbed by rough transport noise*, J. Evol. Equ., 19 (2019), pp. 203–247, <https://doi.org/10.1007/s00028-018-0473-z>.
- [37] M. HOFMANOVÁ, J.-M. LEAHY, AND T. NILSSEN, *On a rough perturbation of the Navier-Stokes system and its vorticity formulation*, Ann. Appl. Probab., 31 (2021), pp. 736–777, <https://doi.org/10.1214/20-aap1603>, <https://doi.org/10.1214/20-aap1603>.
- [38] D. D. HOLM, *Variational principles for stochastic fluid dynamics*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 471 (2015), p. 20140963.
- [39] N. IKEDA AND S. WATANABE, *Stochastic differential equations and diffusion processes*, vol. 24, Elsevier, 2014.
- [40] T. J. LYONS, *Differential equations driven by rough signals*, Revista Matemática Iberoamericana, 14 (1998), pp. 215–310.
- [41] C. MARCHIORO AND M. PULVIRENTI, *Mathematical theory of incompressible nonviscous fluids*, vol. 96, Springer Science & Business Media, 2012.
- [42] R. MIKULEVICIUS AND G. VALIUKEVICIUS, *On stochastic Euler equation in $\mathbf{R}^d$* , Electron. J. Probab., 5 (2000), pp. no. 6, 20, <https://doi.org/10.1214/EJP.v5-62>.
- [43] V. I. YUDOVICH, *Non-stationary flow of an ideal incompressible liquid*, USSR Computational Mathematics and Mathematical Physics, 3 (1963), pp. 1407–1456.