Lecture Notes on Optimal Transport Problems *

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Introduction

These notes are devoted to the Monge–Kantorovich optimal transport problem. This problem, in the original formulation of Monge, can be stated as

*Lectures given in Madeira (PT), Euro Summer School "Mathematical aspects of evolving interfaces", 2–9 July 2000 follows: given two distributions with equal masses of a given material $g_0(x)$, $g_1(x)$ (corresponding for instance to an embankment and an excavation), find a transport map ψ which carries the first distribution into the second and minimizes the transport cost

$$C(\psi) := \int_X |x - \psi(x)| g_0(x)\, dx.$$

The condition that the first distribution of mass is carried into the second can be written as

$$\int_{\psi^{-1}(B)} g_0(x)\, dx = \int_B g_1(y)\, dy \qquad \forall B \subset X \text{ Borel} \tag{1}$$

or, by the change of variables formula, as

$$g_1(\psi(x))\, |\det \nabla \psi(x)| = g_0(x) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in B$$

if ψ is one to one and sufficiently regular.

More generally one can replace the functions g_0, g_1 by positive measures f_0, f_1 with equal mass, so that (1) reads $f_1 = \psi_\# f_0$, and replace the euclidean distance by a generic cost function $c(x,y)$, studying the problem

$$\min_{\psi_\# f_0 = f_1} \int_X c(x, \psi(x))\, df_0(x). \tag{2}$$

The infimum of the transport problem leads also to a c-dependent distance between measures with equal mass, known as Kantorovich–Wasserstein distance.

The optimal transport problem and the Kantorovich–Wasserstein distance have a very broad range of applications: Fluid Mechanics [9], [10]; Partial Differential Equations [31, 28]; Optimization [13], [14] to quote just a few examples. Moreover, the 1-Wasserstein distance (corresponding to $c(x,y) = |x-y|$ in (2)) is related to the so-called flat distance in Geometric Measure Theory, which plays an important role in its development (see [6], [24], [29], [27], [43]). However, rather than showing specific applications (for which we mainly refer to the Evans survey [21] or to the introduction of [9]), the main aim of the notes is to present the different formulations of the optimal transport problem and to compare them, focussing mainly on the linear case $c(x,y) = |x-y|$. The main sources for the preparation of the notes have been the papers by Bouchitté–Buttazzo [13, 14], Caffarelli–Feldman–McCann [15], Evans–Gangbo [22], Gangbo–McCann [26], Sudakov [41] and Evans [21].

The notes are organized as follows. In Section 1 we discuss some basic examples and in Section 2 we discuss Kantorovich's generalized solutions, i.e. the transport plans, pointing out the connection between them and the transport maps. Section 3 is entirely devoted to the one dimensional case: in this situation the order structure plays an important role and considerably simplifies the theory. Sections 4 and 5 are devoted to the ODE and PDE based formulations of the optimal transport problem (respectively due to Brenier and Evans–Gangbo); we discuss in particular the role of the so-called transport density and the equivalence of its different representations. Namely, we prove that any transport density μ can be represented as $\int_0^1 \pi_{t\#}(|y-x|\gamma)\,dt$, where γ is an optimal planning, as $\int_0^1 |E_t|\,dt$, where E_t is the "velocity field" in the ODE formulation, or as the solution of the PDE $\mathrm{div}(\nabla_\mu u\mu) = f_1 - f_0$, with no regularity assumption on f_1, f_0. Moreover, in the same generality we prove convergence of the p-laplacian approximation.

In Section 6 we discuss the existence of the optimal transport map, following essentially the original Sudakov approach and filling a gap in his original proof (see also [15, 42]). Section 7 deals with recent results, related to those obtained in [25], on the regularity and the uniqueness of the transport density. Section 8 is devoted to the connection between the optimal transport problem and the so-called mass optimization problem. Finally, Section 9 contains a self contained list of the measure theoretic results needed in the development of the theory.

Main notation

X	a compact convex subset of an Euclidean space $\mathbb{R}^n$
$\mathcal{B}(X)$	Borel σ-algebra of X
$\mathcal{L}^n$	Lebesgue measure in $\mathbb{R}^n$
$\mathcal{H}^k$	Hausdorff k-dimensional measure in $\mathbb{R}^n$
$\mathrm{Lip}(X)$	real valued Lipschitz functions defined on X
$\mathrm{Lip}_1(X)$	functions in $\mathrm{Lip}(X)$ with Lipschitz constant not greater than 1
Σ_u	the set of points where u is not differentiable
π_t	projections $(x,y) \mapsto x + t(y-x)$, $t \in [0,1]$
$\mathcal{S}_o(X)$	open segments $]\!]x,y[\![$ with x, $y \in X$
$\mathcal{S}_c(X)$	closed segments $[\![x,y]\!]$ with x, $y \in X$, $x \neq y$

$\mathcal{M}(X)$	signed Radon measures with finite total variation in X
$\mathcal{M}_+(X)$	positive and finite Radon measures in X
$\mathcal{M}_1(X)$	probability measures in X
$\lvert\mu\rvert$	total variation of $\mu \in [\mathcal{M}(X)]^n$
μ^+, μ^-	positive and negative part of $\mu \in \mathcal{M}(X)$
$f_\# \mu$	push forward of μ by f

1 Some elementary examples

In this section we discuss some elementary examples that illustrate the kind of phenomena (non existence, non uniqueness) which can occur. The first one shows that optimal transport maps need not exist if the first measure f_0 has atoms.

Example 1.1 (Non existence) Let $f_0 = \delta_0$ and $f_1 = (\delta_1 + \delta_{-1})/2$. In this case the optimal transport problem has no solution simply because there is no map ψ such that $\psi_\# f_0 = f_1$.

The following two examples deal with the case when the cost function c in $X \times X$ is $|x-y|$, i.e. the euclidean distance between x and y. In this case we will use as a test for optimality the fact that the infimum of the transport problem is always greater than

$$\sup\left\{\int_X u\, d(f_1 - f_0) :\ u \in \mathrm{Lip}_1(X)\right\}. \tag{3}$$

Indeed,

$$\int_X u\, d(f_1 - f_0) = \int_X u(\psi(x)) - u(x)\, df_0(x) \le \int_X |\psi(x) - x|\, df_0(x)$$

for any admissible transport ψ. Actually we will prove this lower bound is sharp if f_0 has no atom (see (6) and (13)).

Our second example shows that in general the solution of the optimal transport problem is not unique. In the one-dimensional case we will obtain (see Theorem 3.1), uniqueness (and existence) in the class of nondecreasing maps.

Example 1.2 (Book shifting) Let $n \ge 1$ be an integer and $f_0 = \chi_{[0,n]}\mathcal{L}^1$ and $f_1 = \chi_{[1,n+1]}\mathcal{L}^1$. Then the map $\psi(t) = t + 1$ is optimal. Indeed, the cost relative to ψ is n and, choosing the 1-Lipschitz function $u(t) = -t$ in (3), we obtain that the supremum is at least n, whence the optimality of ψ follows. But since the minimal cost is n, if $n > 1$ another optimal map ψ is given by

$$\psi(t) = \begin{cases} t + n & \text{on } [0,1] \\ t & \text{on } [1,n]. \end{cases}$$

In the previous example the two transport maps coincide when $n = 1$; however in this case there is one more (and actually infinitely many) optimal transport map.

Example 1.3 Let $f_0 = \chi_{[0,1]}\mathcal{L}^1$ and $f_1 = \chi_{[1,2]}\mathcal{L}^1$ (i.e. $n = 1$ in the previous example). We have already seen that $\psi(t) = t + 1$ is optimal. But in this case also the map $\psi(t) = 2 - t$ is optimal as well.

In all the previous examples the optimal transport maps ψ satisfy the condition $\psi(t) \geq t$. However is is easy to find examples where this does not happen.

Example 1.4 Let $f_0 = \chi_{[-1,1]}\mathcal{L}^1$ and $f_1 = (\delta_{-1} + \delta_1)/2$. In this case the optimal transport map ψ is unique (modulo $\mathcal{L}^1$-negligible sets); it is identically equal to -1 on $[-1, 0)$ and identically equal to 1 on $(0, 1]$. The verification is left to the reader as an exercise.

We conclude this section with some two dimensional examples.

Example 1.5 Assume that $2f_0$ is the sum of the unit Dirac masses at $(1,1)$ and $(0,0)$, and that $2f_1$ is the sum of the unit Dirac masses at $(1,0)$ and $(0,1)$. Then the "horizontal" transport and the "vertical" transport are both optimal. Indeed, the cost of these transports is 1 and choosing $u(x_1, x_2) = x_1$ in (3) we obtain that the infimum of the transport problem is at least 1.

Example 1.6 Assume that f_1 is the sum of two Dirac masses at A, $B \in \mathbb{R}^2$ and assume that f_0 is supported on the middle axis between them. Then

$$\int_X |x - \psi(x)|\, df_0(x) = \int_X |x - A|\, df_0(x)$$

whenever $\psi(x) \in \{A, B\}$, hence *any* admissible transport is optimal.

2 Optimal transport plans: existence and regularity

In this section we discuss Kantorovich's approach to the optimal transport problem. His idea has been to look for optimal transport "plans" , i.e. probability measures γ in the product space $X \times X$, rather than optimal transport maps. We will see that this more general viewpoint can be used in several situations to prove that actually optimal transport maps exist (this intermediate passage through a weak formulation of the problems is quite common in PDE and Calculus of Variations).

(MK) Let $f_0,\ f_1 \in \mathcal{M}_1(X)$. We say that a probability measure γ in $\mathcal{M}_1(X \times X)$ is *admissible* if its marginals are f_0 and f_1, i.e.

$$\pi_{0\#}\gamma = f_0, \qquad \pi_{1\#}\gamma = f_1.$$

Then, given a Borel cost function $c : X \times X \to [0, \infty]$, we minimize

$$I(\gamma) := \int_{X\times X} c(x, y)\, d\gamma(x, y)$$

among all admissible γ and we denote by $\mathcal{F}_c(f_0, f_1)$ the value of the infimum.

We also call an admissible γ a *transport plan*. Notice also that in Kantorovich's setting no restriction on f_0 or f_1 is necessary to produce admissible transport plans: the product measure $f_0 \times f_1$ is always admissible. In particular the following definition is well posed and produces a family of distances in $\mathcal{M}_1(X)$.

Definition 2.1 (Kantorovich–Wasserstein distances) *Let $p \geq 1$ and $f_0,\ f_1 \in \mathcal{M}_1(X)$. We define the p-Wasserstein distance between f_0 and f_1 by*

$$\mathcal{F}_p(f_0, f_1) := \left(\min_{\pi_{0\#}\gamma = f_0,\, \pi_{1\#}\gamma = f_1} \int_{X\times X} |x - y|^p\, d\gamma \right)^{1/p}. \tag{4}$$

The difference between transport maps and transport plans can be better understood with the following proposition.

Proposition 2.1 (Transport plans versus transport maps) *Any Borel transport map $\psi : X \to X$ induces a transport plan γ_ψ defined by*

$$\gamma_\psi := (Id \times \psi)_\# f_0. \tag{5}$$

Conversely, a transport plan γ is induced by a transport map if γ is concentrated on a γ-measurable graph Γ.

Proof. Let ψ be a transport map. Since $\pi_0 \circ (Id \times \psi) = Id$ and $\pi_1 \circ (Id \times \psi) = \psi$ we obtain immediately that $\pi_{0\#}\gamma_\psi = f_0$ and $\pi_{1\#}\gamma_\psi = \psi_\# f_0 = f_1$. Notice also that, by Lusin's theorem, the graph of ψ is γ_ψ-measurable.

Conversely, let $\Gamma \subset X \times X$ be a γ-measurable graph on which γ is concentrated and write

$$\Gamma = \{(x, \phi(x)) : \ x \in \pi_0(\Gamma)\}$$

for some function $\phi : \pi_0(\Gamma) \to X$. Let (K_h) be an increasing sequence of compact subsets of Γ such that $\gamma(\Gamma \setminus K_h) \to 0$ and notice that

$$f_0\left(\pi_0(K_h)\right) = \gamma\left(\pi_0^{-1}(\pi_0(K_h))\right) \geq \gamma(K_h) \to 1.$$

Hence, $\pi_0(\Gamma) \supset \cup_h \pi_0(K_h)$ is f_0-measurable and with full measure in X. Moreover, representing γ as $\gamma_x \otimes f_0$ as in (58) we get

$$\begin{aligned} 0 &= \lim_{h\to\infty} \gamma(X \times X \setminus K_h) = \int_X \gamma_x\left(\{y : \ (x,y) \notin \cup_h K_h\}\right) df_0(x) \\ &\geq \int^*_{\pi_0(\Gamma)} \gamma_x(X \setminus \{\phi(x)\})\, df_0(x) \end{aligned}$$

(here $\int^*$ denotes the outer integral). Hence γ_x is the unit Dirac mass at $\phi(x)$ for f_0-a.e. $x \in X$. Since

$$x \mapsto \psi(x) := \int_X y\, d\gamma_x(y)$$

is a Borel map coinciding with ϕ f_0-a.e., we obtain that γ_x is the unit Dirac mass at $\psi(x)$ for f_0-a.e. x. For A, $B \in \mathcal{B}(X)$ we get

$$\gamma(A \times B) = \int_A \gamma_x(B)\, df_0(x) = f_0\left(\{x : \ (x, \psi(x)) \in A \times B\}\right) = \gamma_\psi(A \times B)$$

and therefore $\gamma = \gamma_\psi$. $\square$

The existence of optimal transport plans is a straightforward consequence of the w^*-compactness of probability measures and of the lower semicontinuity of I.

Theorem 2.1 (Existence of optimal plans) *Assume that c is lower semicontinuous in $X \times X$. Then there exists $\gamma \in \mathcal{M}_1(X \times X)$ solving (MK). Moreover, if c is continuous and real valued we have*

$$\min \textit{(MK)} = \inf_{\psi_\# f_0 = f_1} \int_X c(x, \psi(x))\, df_0(x) \tag{6}$$

provided f_0 has no atom.

Proof. Clearly the set of admissible γ's is closed, bounded and w^*-compact for the w^*-convergence of measures (i.e. in the duality with continuous functions in $X \times X$). Hence, it suffices to prove that

$$I(\gamma) \le \liminf_{h \to \infty} I(\gamma_h)$$

whenever γ_h w^*-converge to γ. This lower semicontinuity property follows by the fact that c can be approximated from below by an increasing sequence of continuous and real valued functions c_h (this is a well known fact: see for instance Lemma 1.61 in [4]). The functionals I_h induced by c_h converge monotonically to I, whence the lower semicontinuity of I follows.

In order to prove the last part we need to show the existence, for any $\gamma \in \mathcal{M}_+(X \times X)$ with $\pi_{0\#}\gamma = f_0$ and $\pi_{1\#}\gamma = f_1$, of Borel maps $\psi_h : X \to X$ such that $\psi_{h\#} f_0 = f_1$ and $\delta_{\psi_h(x)} \otimes f_0$ weakly converge to γ in $\mathcal{M}(X \times X)$. The approximation Theorem 9.3 provides us, on the other hand, with a Borel map $\varphi : X \to X$ such that $\varphi_\# f_0$ has no atom, is arbitrarily close to f_1 and $\delta_{\varphi(x)} \otimes f_0$ is arbitrarily close (with respect to the weak topology) to γ. We will build ψ_h by an iterated application of this result.

By a standard approximation argument we can assume that $L = \mathrm{Lip}(c)$ is finite and that

$$\delta |x - y| \le c(x, y) \qquad\qquad \forall x,\, y \in X$$

for some $\delta > 0$. Possibly replacing c by c/δ we assume $\delta = 1$.

Fix now an integer h, set $f_0^0 = f_0$ and choose $\varphi_0 : X \to X$ such that $\varphi_{0\#} f_0^0$ has no atom and

$$\mathcal{F}_1(f_1, \varphi_{0\#} f_0^0) < 2^{-h} \qquad \text{and} \qquad \int_X c(\varphi_0(x), x)\, df_0^0(x) < \mathcal{F}_c(f_1, f_0^0) + 2^{-h}.$$

Then, we set $f_0^1 = \varphi_{0\#} f_0^0$, find $\varphi_1 : X \to X$ such that $\varphi_{1\#} f_0^1$ has no atom and

$$\mathcal{F}_1(f_1, \varphi_{1\#} f_0^1) < 2^{-1-h} \qquad \text{and} \qquad \int_X c(\varphi_1(x), x)\, df_0^1(x) < \mathcal{F}_c(f_1, f_0^1) + 2^{-1-h}.$$

Proceeding inductively and setting $f_0^k = \varphi_{(k-1)\#} f_0^{k-1}$ we find φ_k such that

$$\mathcal{F}_1(f_1, \varphi_{k\#} f_0^k) < 2^{-k-h} \qquad \text{and} \qquad \int_X c(\varphi_k(x), x)\, df_0^k(x) < \mathcal{F}_c(f_1, f_0^k) + 2^{-k-h}$$

and $\varphi_{k\#} f_0^k$ has no atom. Then, we set $\phi_0(x) = x$ and $\phi_k = \varphi_{k-1} \circ \cdots \circ \varphi_0$ for $k \geq 1$, so that $f_0^k = \phi_{k\#} f_0$. We claim that (ϕ_k) is a Cauchy sequence in $L^1(X, f_0; X)$. Indeed,

$$\begin{aligned} \sum_{k=0}^{\infty} \int_X |\phi_{k+1}(x) - \phi_k(x)|\, df_0(x) &= \sum_{k=0}^{\infty} \int_X |\varphi_k(y) - y|\, df_0^k(y) \\ &\leq 2^{1-h} + \sum_{k=0}^{\infty} \mathcal{F}_1(f_1, f_0^k) < \infty. \end{aligned}$$

Denoting by ψ_h the limit of ϕ_k, passing to the limit as $k \to \infty$ we obtain $\psi_{h\#} f_0 = f_1$; moreover, we have

$$\begin{aligned} \int_X c(\phi_k(x), x)\, df_0(x) &\leq \int_X c(\varphi_0(x), x)\, df_0(x) + L \sum_{i=1}^{k} \int_X |\phi_i(x) - \phi_{i-1}(x))|\, df_0(x) \\ &\leq \mathcal{F}_c(f_1, f_0) + 2^{-h} + L \sum_{i=1}^{k} \int_X |\varphi_i(y) - y|\, df_0^i(y) \leq \mathcal{F}_c(f_1, f_0) + 2^{-h}(1 + 2L). \end{aligned}$$

Passing to the limit as $k \to \infty$ we obtain

$$\int_X c(\psi_h(x), x)\, df_0(x) \leq \mathcal{F}_c(f_1, f_0) + 2^{-h}(1 + 2L)$$

and the proof is achieved. $\square$

For instance in the case of Example 1.1 (where transport maps do not exist at all) it is easy to check that the unique optimal transport plan is given by

$$\frac{1}{2}\delta_0 \times \delta_{-1} + \frac{1}{2}\delta_0 \times \delta_1.$$

In general, however, uniqueness fails because of the linearity of I and of the convexity of the class of admissible plans γ. In Example 1.2, for instance, any measure

$$t(Id \times \psi_1)_\# f_0 + (1-t)(Id \times \psi_2)_\# f_0$$

is optimal, with $t \in [0,1]$ and ψ_1, ψ_2 optimal transport maps.

In order to understand the regularity properties of optimal plans γ we introduce, following [26, 35], the concept of cyclical monotonicity.

Definition 2.2 (Cyclical monotonicity) *Let $\Gamma \subset X \times X$. We say that Γ is c-ciclically monotone if*

$$\sum_{i=1}^{n} c(x_{i+1}, y_i) \geq \sum_{i=1}^{n} c(x_i, y_i) \tag{7}$$

whenever $n \geq 2$ and $(x_i, y_i) \in \Gamma$ for $1 \leq 1 \leq n$, with $x_{n+1} = x_1$.

The cyclical monotonicity property can also be stated in a (apparently) stronger form:

$$\sum_{i=1}^{n} c(x_{\sigma(i)}, y_i) \geq \sum_{i=1}^{n} c(x_i, y_i) \tag{8}$$

for any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. The equivalence can be proved either directly (reducing to the case when σ has no nontrivial invariant set) or verifying, as we will soon do, that any cyclically monotone set is contained in the c-superdifferential of a c-concave function and then checking that the superdifferential fulfils (8).

Theorem 2.2 (Regularity of optimal plans) *Assume that c is continuous and real valued. Then, for any optimal γ the set $spt\gamma$ is c-ciclically monotone. Moreover, the union of $spt\gamma$ as γ range among all optimal plans is c-ciclically monotone.*

Proof. Assume by contradiction that there exist an integer $n \geq 2$ and points $(x_i, y_i) \in \mathrm{spt}\gamma$, $i = 1, \ldots, n$, such that

$$f\left((x_i),(y_i)\right) := \sum_{i=1}^{n} c(x_{i+1}, y_i) - c(x_i, y_i) < 0$$

with $x_{n+1} = x_1$. For $1 \leq i \leq n$, let U_i, V_i be compact neighbourhoods of x_i and y_i respectively such that $\gamma(U_i \times V_i) > 0$ and $f\left((u_i),(v_i)\right) < 0$ whenever $u_i \in U_i$ and $v_i \in V_i$.

Set now $\lambda = \min_i \gamma(U_i \times V_i)$ and denote by $\gamma_i \in \mathcal{M}_1(U_i \times V_i)$ the normalized restriction of γ to $U_i \times V_i$. We can find a compact space Y, a probability measure σ in Y and Borel maps $\eta_i = u_i \times v_i : X \to U_i \times V_i$ such that $\gamma_i = \eta_{i\#}\sigma$ for $i = 1, \ldots, n$ (it suffices for instance to define Y as the product of $U_i \times V_i$, so that η_i are the projections on the i-coordinate) and define

$$\gamma' := \gamma + \frac{\lambda}{n} \sum_{i=1}^{n} (u_{i+1} \times v_i)_{\#}\sigma - (u_i \times v_i)_{\#}\sigma.$$

Since $\lambda \eta_{i\#}\sigma = \lambda \gamma_i \leq \gamma$ we obtain that $\gamma' \in \mathcal{M}_+(X \times X)$; moreover, it is easy to check that $\pi_{0\#}\gamma' = f_0$ and $\pi_{1\#}\gamma' = f_1$. This leads to a contradiction because

$$I(\gamma') - I(\gamma) = \frac{\lambda}{n} \int_Y \sum_{i=1}^{n} c(u_{i+1}, v_i) - c(u_i, v_i)\, d\sigma < 0.$$

In order to show the last part of the statement we notice that the collection of optimal transport plans is w^*-closed and compact. If $(\gamma_h)_{h \geq 1}$ is a countable dense set, then

$$\bigcup_{i=1}^{h} \mathrm{spt}\gamma_i = \mathrm{spt}\left(\sum_{i=1}^{h} \frac{1}{h}\gamma_i\right)$$

is c-ciclically monotone for any $h \geq 1$. Passing to the limit as $h \to \infty$ we obtain that the closure of the union of $\mathrm{spt}\gamma_h$ is c-ciclically monotone. By the density of (γ_h), this closure contains $\mathrm{spt}\gamma$ for any optimal plan γ. $\square$

Next, we relate the c-cyclical monotonicity to suitable concepts (adapted to c) of concavity and superdifferential.

Definition 2.3 (c-concavity) *We say that a function $u : X \to \mathbb{R}$ is c-concave if it can be represented as the infimum of a family (u_i) of functions given by*

$$u_i(x) := c(x, y_i) + t_i$$

for suitable $y_i \in X$ and $t_i \in \mathbb{R}$.

Remark 2.1 [Linear and quadratic case] In the case when $c(x, y)$ is a symmetric function satisfying the triangle inequality, the notion of c-concavity is equivalent to 1-Lipschitz continuity with respect to the metric d_c induced by c. Indeed, given $u \in \mathrm{Lip}_1(X, d_c)$, the family of functions whose infimum is u is simply

$$\{c(x, y) + u(y) : \ y \in X\} .$$

In the quadratic case $c(x, y) = |x - y|^2/2$ a function u is c-concave if and only if $u - |x|^2/2$ is concave. Indeed, $u = \inf_i c(\cdot, y_i) + t_i$ implies

$$u(x) - \frac{1}{2}|x|^2 = \inf_i \langle x, -y_i\rangle + \frac{1}{2}|y_i|^2 + t_i$$

and therefore the concavity of $u - |x|^2/2$. Conversely, if $v = u - |x|^2/2$ is concave, from the well known formula

$$v(x) = \inf_{y,\, p\in\partial^+ v(y)} v(y) + \langle p, x - y\rangle$$

(here $\partial^+ v$ is the superdifferential of v in the sense of convex analysis) we obtain

$$u(x) = \inf_{y,\, -p\in\partial^+ v(y)} \frac{1}{2}|p - x|^2 + c(p, y).$$

□

Definition 2.4 (c-superdifferential) *Let $u : X \to \mathbb{R}$ be a function. The c-superdifferential $\partial_c u(x)$ of u at $x \in X$ is defined by*

$$\partial_c u(x) := \{y : \ u(z) \le u(x) + c(z, y) - c(x, y) \ \forall z \in X\} . \tag{9}$$

The following theorem ([39], [40], [35]) shows that the graphs of superdifferentials of c-concave functions are maximal (with respect to set inclusion) c-cyclically monotone sets. It may considered as the extension of the well known result of Rockafellar to this setting.

Theorem 2.3 *Any c-ciclically monotone set Γ is contained in the graph of the c-superdifferential of a c-concave function. Conversely, the graph of the c-superdifferential of a c-concave function is c-ciclically monotone.*

Proof. This proof is taken from [35]. We fix $(x_0, y_0) \in \Gamma$ and define

$$u(x) := \inf c(x, y_n) - c(x_n, y_n) + \cdots + c(x_1, y_0) - c(x_0, y_0) \qquad \forall x \in X$$

where the infimum runs among all collections $(x_i, y_i) \in \Gamma$ with $1 \leq i \leq n$ and $n \geq 1$. Then u is c-concave by construction and the cyclical monotonicity of Γ gives $u(x_0) = 0$ (the minimum is achieved with $n = 1$ and $(x_1, y_1) = (x_0, y_0)$).

We will prove the inequality

$$u(x) \leq u(x') + c(x, y') - c(x', y') \tag{10}$$

for any $x \in X$ and $(x', y') \in \Gamma$. In particular (choosing $x = x_0$) this implies that $u(x') > -\infty$ and that $y' \in \partial_c u(x')$. In order to prove (10) we fix $\lambda > u(x')$ and find $(x_i, y_i) \in \Gamma$, $1 \leq i \leq n$, such that

$$c(x', y_n) - c(x_n, y_n) + \cdots + c(x_1, y_0) - c(x_0, y_0) < \lambda.$$

Then, setting $(x_{n+1}, y_{n+1}) = (x', y')$ we find

$$\begin{aligned} u(x) &\leq c(x, y_{n+1}) - c(x_{n+1}, y_{n+1}) + c(x_{n+1}, y_n) - c(x_n, y_n) \\ &+ \cdots + c(x_1, y_0) - c(x_0, y_0) \leq c(x, y') - c(x', y') + \lambda. \end{aligned}$$

Since λ is arbitrary (10) follows.

Finally, if v is c-concave, $y_i \in \partial_c v(x_i)$ for $1 \leq i \leq n$ and σ is a permutation we can add the inequalities

$$v(x_{\sigma(i)}) - v(x_i) \leq c(x_{\sigma(i)}, y_i) - c(x_i, y_i)$$

to obtain (8). $\square$

In the following corollary we assume that the cost function is symmetric, continuous and satisfies the triangle inequality, so that c-concavity reduces to 1-Lipschitz continuity with respect to the distance induced by c.

Corollary 2.1 (Linear case) *Let $\gamma \in \mathcal{M}_1(X \times X)$ with $\pi_{0\#}\gamma = f_0$ and $\pi_{1\#}\gamma = f_1$. Then γ is optimal for (MK) if and only if there exists $u : X \to \mathbb{R}$ such that*

$$|u(x) - u(y)| \leq c(x, y) \qquad \forall (x, y) \in X \times X \tag{11}$$

$$u(x) - u(y) = c(x,y) \qquad \textit{for } (x,y) \in spt\gamma. \tag{12}$$

In addition, there exists u satisfying (11) such that (12) holds for any optimal planning γ. We will call any function u with these properties a maximal Kantorovich potential.

Proof. (Sufficiency) Let γ' be any admissible transport plan; by applying (11) first and then (12) we get

$$\begin{aligned} I(\gamma') &\geq \int_{X\times X} u(x) - u(y)\, d\gamma' = \int_X u\, df_0 - \int_X u\, df_1 \\ &= \int_X u(x) - u(y)\, d\gamma = I(\gamma). \end{aligned}$$

(Necessity) Let Γ be the closure of the union of sptγ' as γ' varies among all optimal plans for (MK). Then we know that Γ is c-cyclically monotone, hence there exists a c-concave function u such that $\Gamma \subset \partial_c u$. Then, (11) follows by the c-convexity of u, while the inclusion $\Gamma \subset \partial_c u$ implies

$$u(y) - u(x) \leq c(y,y) - c(x,y) = -c(x,y)$$

for any $(x,y) \in \mathrm{spt}\gamma \subset \Gamma$. This, taking into account (11), proves (12). $\square$

A direct consequence of the proof of sufficiency is the identity

$$\min\,(\mathrm{MK}) = \max\left\{\int ud(f_0 - f_1):\ u \in \mathrm{Lip}_1(X, d_c)\right\} \tag{13}$$

(where d_c is the distance in X induced by c) and the maximum on the right is achieved precisely whenever u satisfies (12).

In the following corollary, instead, we consider the case when $c(x,y) = |x-y|^2/2$. The result below, taken from [26], was proved first by Brenier in [7, 8] under more restrictive assumptions on f_0, f_1 (see also [26] for the general case $c(x,y) = h(|x-y|)$). Before stating the result we recall that the set Σ_v of points of nondifferentiability of a real valued concave function v (i.e. the set of points x such that $\partial^+ v(x)$ is not a singleton) is countably (CC) regular (see [44] and also [1]). This means that Σ_v can be covered with a countable family of (CC) hypersurface, i.e. graphs of differences of convex functions of $n-1$ variables. This property is stronger than the canonical $\mathcal{H}^{n-1}$-rectifiability: it implies that $\mathcal{H}^{n-1}$-almost all of Σ_v can be covered by a sequence of C^2 hypersurfaces.

Corollary 2.2 (Quadratic case) *Assume that any (CC) hypersurface is f_0-negligible. Then the optimal planning γ is unique and is induced by an optimal transport map ψ. Moreover ψ is the gradient of a convex function.*

Proof. Let $u : X \to \mathbb{R}$ be a c-concave function such that the graph Γ of its superdifferential contains the support of any optimal planning γ. As $c(x,y) = |x-y|^2/2$, an elementary computation shows that $(x_0, y_0) \in \Gamma$ if and only if

$$-y_0 \in \partial^+ v(x_0)$$

where $v(x) = u(x) - |x|^2/2$ is the concave function already considered in Remark 2.1. Then, by the above mentioned results on differentiability of concave functions, the set of points where v is not differentiable is f_0-negligible, hence for f_0-a.e. $x \in X$ there is a unique $y_0 = -\nabla v(x_0) \in X$ such that $(x_0, y_0) \in \Gamma$. As $\mathrm{spt}\gamma \subset \Gamma$, by Proposition 2.1 we infer that

$$\gamma = (Id \times \psi)_\# f_0$$

for any optimal planning γ, with $\psi = -\nabla v$. $\square$

Example 2.1 (Brenier polar factorization theorem) A remarkable consequence of Corollary 2.2 is the following result, known as polar factorization theorem. A vector field $r : X \to X$ such that

$$\mathcal{L}^n(r^{-1}(B)) = 0 \qquad \text{whenever} \qquad \mathcal{L}^n(B) = 0 \tag{14}$$

can be written as $\nabla u \circ \eta$, with u convex and η measure preserving.

It suffices to apply the Corollary to the measure $f_0 = r_\#(\mathcal{L}^n \llcorner X)$ (absolutely continuous, due to (14)) and $f_1 = \mathcal{L}^n \llcorner X$; we have then

$$\mathcal{L}^n \llcorner X = (\nabla v)_\#(r_\#(\mathcal{L}^n \llcorner X)) = (\nabla v \circ r)_\#(\mathcal{L}^n \llcorner X)$$

for a suitable convex function v, hence $\eta = (\nabla v) \circ r$ is measure preserving. The desired representation follows with $u = v^*$, since $\nabla u = (\nabla v)^{-1}$.

Let us assume that $f_0 = \mathcal{L}^n \llcorner X$ and let $f_1 \in \mathcal{M}_+(X)$ be any other measure such that $f_1(X) = \mathcal{L}^n(X)$; in general the problem of mapping f_0 into f_1 through a Lipschitz map has no solution (it suffices to consider, for instance, the case when $X = \overline{B}_1$ and $f_1 = \frac{1}{n}\mathcal{H}^1 \llcorner \partial B_1$). However, another remarkable consequence of Corollary 2.2 is that the problem has solution if we require the transport map to be only a function of bounded variation: indeed, bounded monotone functions (in particular gradients of Lipschitz convex functions) are functions of bounded variation (see for instance Proposition 5.1 of [2]). Moreover, we can give a sharp quantitative estimate of the error made in the approximation by Lipschitz transport maps.

Theorem 2.4 *There exists a constant $C = C(n, X)$ with the following property: for any $\mu \in \mathcal{M}_+(X)$ with $\mu(X) = \mathcal{L}^n(X)$ and any $M > 0$ there exist a Lipschitz function $\phi : X \to X$ and $B \in \mathcal{B}(X)$ such that* $\mathrm{Lip}(\phi) \leq M$, *$\mathcal{L}^n(X \setminus B) \leq C/M$ and*

$$\mu = \phi_\#(\mathcal{L}^n \llcorner B) + \mu^s$$

with $\mu^s \in \mathcal{M}_+(X)$ and $\mu^s(X) \leq \mathcal{L}^n(X \setminus B)$.

Proof. By Corollary 2.2 we can represent $\mu = \psi_\#(\mathcal{L}^n \llcorner X)$ with $\psi : X \to X$ equal to the gradient of a convex function. Let Ω be the interior of X; by applying Proposition 5.1 of [2] (valid, more generally, for monotone operators) we obtain that the total variation $|D\psi|(\Omega)$ can be estimated with a suitable constant C depending only on n and X. Therefore, by applying Theorem 5.34 of [4] we can find a Borel set $B \subset X$ (it is a suitable sublevel set of the maximal function of $|D\psi|$) such that $\mathcal{L}^n(X \setminus B) \leq c(n)|D\psi|(\Omega)/M$ and the restriction of ψ to B is a M-Lipschitz function, i.e. with Lipschitz constant not greater than M. By Kirszbraun theorem (see for instance [24]) we can extend $\psi|_B$ to a M-Lipschitz function $\phi : X \to X$. Setting $B^c = X \setminus B$ we have then

$$\mu = \psi_\#(\mathcal{L}^n \llcorner B) + \psi_\#(\mathcal{L}^n \llcorner B^c) = \phi_\#(\mathcal{L}^n \llcorner B) + \psi_\#(\mathcal{L}^n \llcorner B^c)$$

and setting $\mu^s = \psi_\#(\mathcal{L}^n \llcorner B^c)$ the proof is achieved. $\square$

3 The one dimensional case

In this section we assume that $X = I$ is a closed interval of the real line; we also assume for simplicity that the transport cost is $c(x, y) = |x - y|^p$ for some $p \geq 1$.

Theorem 3.1 (Existence and uniqueness) *Assume that f_0 is a diffuse measure, i.e. $f_0(\{t\}) = 0$ for any $t \in I$. Then* *(i) there exists a unique (modulo countable sets) nondecreasing function* $\psi : spt f_0 \to X$ *such that* $\psi_\# f_0 = f_1$*;*

(ii) the function ψ *in (i) is an optimal transport, and if* $p > 1$ *is the unique optimal transport.*

In the one-dimensional case these results are sharp: we have already seen that transport maps need not exist if f_0 has atoms (Example 1.1) and that, without the monotonicity constraint, are not necessarily unique when $p = 1$.

Proof. (i) Let $m = \min I$ and define

$$\psi(s) := \sup\{t \in I : \ f_1([m,t]) \leq f_0([m,s])\}\,. \tag{15}$$

It is easy to check that the following properties hold:

(a) ψ is non decreasing;

(b) $\overline{\psi(I)} \supset \mathrm{spt} f_1$;

(c) if $\psi(s)$ is not an atom of f_1 we have

$$f_1([m,\psi(s)]) = f_0([m,s]). \tag{16}$$

Let T be the at most countable set made by the atoms of f_1 and by the points $t \in I$ such that $\psi^{-1}(t)$ contains more than one point; then ψ^{-1} is well defined on $\psi(I)\backslash T$ and $\psi^{-1}([t,t']) = [\psi^{-1}(t), \psi^{-1}(t')]$ whenever $t,\ t' \in \psi(I)\backslash T$ with $t < t'$. Then (c) gives

$$\begin{aligned} f_1([t,t']) &= f_1([m,t']) - f_1([m,t]) = f_0([m,\psi^{-1}(t')]) - f_0([m,\psi^{-1}(t)]) \\ &= f_0([\psi^{-1}(t), \psi^{-1}(t')]) = f_0(\psi^{-1}([t,t'])) \end{aligned}$$

(notice that only here we use the fact that f_0 is diffuse). By (b) the closed intervals whose endpoints belong to $\psi(I) \setminus T$ generate the Borel σ-algebra of $\mathrm{spt} f_1$, and this proves that $\psi_\# f_0 = f_1$.

Let ϕ be any nondecreasing function such that $\phi_\# f_0 = f_1$ and assume, possibly modifying ϕ on a countable set, that ϕ is right continuous. Let

$$T := \{s \in \mathrm{spt} f_0 : \ (s,s') \cap \mathrm{spt} f_0 = \emptyset \text{ for some } s' > s\ \}\,.$$

and notice that T is at most countable (since we can index with T a family of pairwise disjoint open intervals). We claim that $\phi \geq \psi$ on $\mathrm{spt} f_0 \setminus T$; indeed, for $s \in \mathrm{spt} f_0 \setminus T$ and $s' > s$ we have the inequalities

$$f_1([m, \phi(s')]) = f_0(\phi^{-1}([m, \phi(s')])) \geq f_0([m, s']) > f_0([m, s])$$

and, by the definition of ψ, the inequality follows letting $s' \downarrow s$. In particular

$$\int_I \phi - \psi \, df_0 = \int_I \phi \, df_0 - \int_I \psi \, df_0 = \int_I 1 \, df_1 - \int_I 1 \, df_1 = 0$$

whence $\phi = \psi$ f_0-a.e. in I. It follows that $\phi(s) = \psi(s)$ at any continuity point $s \in \mathrm{spt} f_0$ of ϕ and ψ.

(ii) By a continuity argument it suffices to prove that ψ is the unique solution of the transport problem for any $p > 1$ (see also [26, 15]). Let γ be an optimal planning and notice that the cyclical monotonicity proved in Theorem 2.2 gives

$$|x - y'|^p + |x' - y|^p \geq |x - y|^p + |x' - y'|^p$$

whenever (x, y), $(x', y') \in \mathrm{spt}\gamma$. If $x < x'$, this condition implies that $y \leq y'$ (this is a simple analytic calculation that we omit, and here the fact that $p > 1$ plays a role). This means that the set

$$T := \{x \in \mathrm{spt} f_0 : \ \mathrm{card}(\{y : \ (x, y) \in \mathrm{spt}\gamma\}) > 1\}$$

is at most countable (since we can index with T a family of pairwise disjoint open intervals) hence f_0-negligible. Therefore for f_0-a.e. $x \in I$ there exists a unique $y = \tilde{\psi}(x) \in I$ such that $(x, y) \in \mathrm{spt}\gamma$ (the existence of at least one y follows by the fact that the projection of $\mathrm{spt}\gamma$ on the first factor is $\mathrm{spt} f_0$). Notice also that $\tilde{\psi}$ is nondecreasing in its domain.

Arguing as in Proposition 2.1 we obtain that

$$\gamma = (Id \times \tilde{\psi})_{\#} f_0.$$

In particular

$$f_1 = \pi_{1\#}\gamma = \pi_{1\#}\left((Id \times \tilde{\psi})_{\#} f_0\right) = \tilde{\psi}_{\#} f_0$$

and since $\tilde{\psi}$ is non decreasing it follows that $\tilde{\psi} = \psi$ (up to countable sets) on $\mathrm{spt} f_0$. $\square$

4 The ODE version of the optimal transport problem

In this and in the next section we rephrase the optimal transport problem in differential terms. In the following we consider a fixed auxiliary open set Ω containing X; we assume that Ω is sufficiently large, namely that the open r-neighbourhood of X is contained in Ω, with $r > \mathrm{diam}(X)$ (the necessity of this condition will be discussed later on).

The first idea, due to Brenier ([10, 9] and also [11]) is to look for all the paths f_t in $\mathcal{M}_+(X)$ connecting f_0 to f_1. In the simplest case when $f_t = \delta_{x(t)}$, it turns out that the velocity field $E_t = \dot{x}(t)\delta_{x(t)}$ is related to f_t by the equation

$$\dot{f}_t + \nabla \cdot E_t = 0 \qquad \text{in } (0,1) \times \Omega \tag{17}$$

in the distribution sense. Indeed, given $\varphi \in C_c^\infty(0,1)$ and $\phi \in C_c^\infty(\Omega)$, it suffices to take $\varphi(t)\phi(x)$ as test function in (17) and to use the definitions of f_t and E_t to obtain

$$\begin{aligned}
(\dot{f}_t + \nabla \cdot E_t)(\varphi\phi) &= -\int_0^1 \dot{\varphi}(t)\phi(x(t)) + \varphi(t)\langle \nabla\phi(x(t)), \dot{x}(t)\rangle \, dt \\
&= -\int_0^1 \frac{d}{dt}\left[\varphi(t)\phi(x(t))\right] dt = 0.
\end{aligned}$$

More generally, regardless of any assumption on $(f_t, E_t) \in \mathcal{M}_+(X) \times [\mathcal{M}(\Omega)]^n$, it is easy to check that (17) holds in the distribution sense if and only if

$$\dot{f}_t(\phi) = \nabla\phi \cdot E_t \quad \text{in (0,1)} \qquad \forall \phi \in C_c^\infty(\Omega) \tag{18}$$

in the distribution sense. We will use both interpretations in the following.

One more interpretation of (17) is given in the following proposition. Recall that a map f defined in $(0,1)$ with values in a metric space (E,d) is said to be absolutely continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_i (y_i - x_i) < \delta \qquad \Longrightarrow \qquad \sum_i d\left(f(y_i), f(x_i)\right) < \varepsilon$$

for any family of pairwise disjoint intervals $(x_i, y_i) \subset (0,1)$.

Proposition 4.1 *If some family $(f_t) \subset \mathcal{M}_1(X)$ fulfils (17) for suitable measures $E_t \in [\mathcal{M}(\Omega)]^n$ satisfying $\int_0^1 |E_t|(\Omega)\,dt < \infty$ then f is an absolutely continuous map between $(0,1)$ and $\mathcal{M}_1(X)$, endowed with the 1-Wasserstein distance (4) and*

$$\lim_{h\to 0} \frac{\mathcal{F}_1(f_{t+h}, f_t)}{|h|} \le |E_t|(\Omega) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,1). \tag{19}$$

Conversely, if f_t is an absolutely continuous map we can choose E_t so that equality holds in (19).

Proof. In (13) we can obviosuly restrict to test functions u such that $|u| \le r = \mathrm{diam}(X)$ on X; by our assumption on Ω any of these function can be extended to $\mathbb{R}^n$ in such a way that the Lipschitz constant is still less than 1 and $u \equiv 0$ in a neighbourhood of $\mathbb{R}^n \setminus \Omega$. In particular, choosing an optimal u and setting $u_\varepsilon = u * \rho_\varepsilon$ we get

$$\begin{aligned} \mathcal{F}_1(f_s, f_t) &= \lim_{\varepsilon\to 0^+} \int_X u_\varepsilon \, d(f_t - f_s) = \lim_{\varepsilon\to 0^+} \int_s^t \nabla \cdot E_\tau(u_\varepsilon)\, d\tau \\ &= -\lim_{\varepsilon\to 0^+} \int_s^t E_\tau \cdot \nabla u_\varepsilon \, d\tau \le \int_s^t |E_\tau|(\Omega)\, d\tau \end{aligned} \tag{20}$$

whenever $0 \le s \le t \le 1$ and this easily leads to (19).

In the proof of the converse implication we can assume with no loss of generality (up to a reparameterization by arclength) that f is a Lipschitz map. We can consider $\mathcal{M}_1(X)$ as a subset of the dual $Y = G^*$, where

$$G := \left\{\phi \in C^1(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n) : \ \phi \equiv 0 \ \text{ on } \mathbb{R}^n \setminus \Omega\right\}$$

endowed with the norm $\|\phi\| = \mathrm{Lip}(\phi)$. By using convolutions and (13) it is easy to check that $\mathcal{F}_1(\mu,\nu) = \|\mu - \nu\|_Y$, so that $\mathcal{M}_1(X)$ is isometrically embedded in Y. Moreover, using Hahn–Banach theorem it is easy to check that any $y \in Y$ is representable as the divergence of a measure $E \in [\mathcal{M}(\Omega)]^n$ with $\|y\| = |E|(\Omega)$ (E is not unique, of course).

By a general result proved in [5], valid also for more than one independent variable, any Lipschitz map f from $(0,1)$ into the dual Y of a separable Banach space is weakly*-differentiable for $\mathcal{L}^1$-almost every t, i.e.

$$\exists w^* - \lim_{h\to 0} \frac{f_{t+h} - f_t}{h} =: \dot f(t)$$

and $f_t - f_s = \int_s^t \dot f(\tau)\,d\tau$ for $s, t \in (0,1)$. In addition, the map is also metrically differentiable for $\mathcal{L}^1$-almost every t, i.e.

$$\exists \lim_{h\to 0} \frac{\|f_{t+h} - f_t\|}{|h|} =: \mathrm{m}\dot f(t)$$

Although $\dot f$ is only a w^*-limit of the difference quotients, it turns out (see [5]) that the metric derivative $\mathrm{m}\dot f$ is $\mathcal{L}^1$-a.e. *equal* to $\|\dot f\|$.

Putting together these informations the conclusion follows. $\square$

According to Brenier, we can formulate the optimal transport problem as follows.

(ODE) Let f_0, $f_1 \in \mathcal{M}_1(X)$ be given probability measures. Minimize

$$J(E) := \int_0^1 |E_t|(\Omega)\,dt \tag{21}$$

among all Borel maps $f_t : [0,1] \to \mathcal{M}_+(X)$ and $E_t : [0,1] \to [\mathcal{M}(\Omega)]^n$ such that (17) holds.

Example 4.1 In the case considered in Example 1.3 the measures

$$f_t = \chi_{[t,t+1]}\mathcal{L}^1, \qquad E_t = \chi_{[t,t+1]}\mathcal{L}^1$$

provide an admissible and optimal flow, obviously related to the optimal transport map $x \mapsto x+1$. But, quite surprisingly, we can also define an optimal flow by

$$f_t = \begin{cases} \frac{1}{1-2t}\chi_{[2t,1]}\mathcal{L}^1 & \text{if } 0 \le t < 1/2 \\ \delta_1 & \text{if } t = 1/2 \\ \frac{1}{2t-1}\chi_{[1,2t]}\mathcal{L}^1 & \text{if } 1/2 < t \le 1 \end{cases}$$

whose "velocity field" is

$$E_t = \begin{cases} \frac{2(1-x)}{(1-2t)^2}\chi_{[2t,1]}\mathcal{L}^1 & \text{if } 0 \le t < 1/2 \\ \frac{2(x-1)}{(2t-1)^2}\chi_{[1,2t]}\mathcal{L}^1 & \text{if } 1/2 < t \le 1. \end{cases}$$

It is easy to check that $\dot f_t = \nabla \cdot E_t = 0$ and that $|E_t|$ are probability measures for any $t \neq 1/2$, hence $J(E) = 1$. The relation of this new flow with the optimal transport map $x \mapsto 2 - x$ will be seen in the following (see Remark 4.1(3)).

In order to relate solutions of (ODE) to solutions of (MK) we will need an uniqueness theorem, under regularity assumptions in the space variable, for the ODE $\dot{f}_t + \nabla \cdot (g_t f_t) = 0$. If f_t, g_t are smooth (say C^2) with respect to both the space and time variables, uniqueness is a consequence of the classical method of characteristics (see for instance §3.2 of [20]), which provides the representation

$$f_t(x_t) = f_0(x) \exp\left(-\int_0^t c_s(x_s)\, ds\right)$$

where $c_t = \nabla \cdot g_t$ and x_t solves the ODE

$$\dot{x}_t = g_t(x_t), \quad x_0 = x, \qquad\qquad t \in (0,1).$$

See also [32] for more general uniqueness and representation results in a weak setting.

Theorem 4.1 *Assume that*

$$\dot{f}_t + \nabla \cdot (g_t f_t) = 0 \qquad \text{in } (0,1) \times \mathbb{R}^n \tag{22}$$

where $\int_0^1 |f_t|(\mathbb{R}^n)\, dt < \infty$ *and* $|g_t| + \mathrm{Lip}(g_t) \leq C$, *with* C *independent of* t *and* $f_0 = 0$. *Then* $f_t = 0$ *for any* $t \in (0,1)$.

Proof. Let g_t^ε be obtained from g_t by a mollification with respect to the space and time variables and define $X^\varepsilon(s,t,x)$ as the solution of the ODE $\dot{x} = g_s^\varepsilon(x)$ (with s as independent variable) such that $X^\varepsilon(s,s,x) = x$. Define, for $\psi \in C_c^\infty((0,1) \times \mathbb{R}^n)$ fixed,

$$\varphi^\varepsilon(t,x) := -\int_t^1 \psi\left(s, X^\varepsilon(s,t,x)\right)\, ds.$$

Since $X^\varepsilon\left(s,t,X^\varepsilon(t,0,x)\right) = X^\varepsilon(s,0,x)$ we have

$$\varphi^\varepsilon\left(t, X^\varepsilon(0,t,x)\right) = -\int_t^1 \psi\left(s, X^\varepsilon(s,0,x)\right)\, ds$$

and, differentiating both sides, we infer

$$\left[\frac{\partial \varphi^\varepsilon}{\partial t} + g_t^\varepsilon \cdot \nabla \varphi^\varepsilon\right]\left(t, X^\varepsilon(t,0,x)\right) = \psi\left(t, X^\varepsilon(t,0,x)\right)$$

whence $\psi = (\partial\varphi^\varepsilon/\partial t + g_t^\varepsilon \cdot \nabla\varphi^\varepsilon)$ in $(0,1)\times\mathbb{R}^n$.

Insert now the test function φ^ε in (22) and take into account that $f_0 = 0$ to obtain

$$0 = \int_0^1 \int_{\mathbb{R}^n} f_t(\frac{\partial\varphi^\varepsilon}{\partial t} + g_t \cdot \nabla\varphi^\varepsilon)\,dxdt = \int_0^1 \int_{\mathbb{R}^n} f_t\psi + f_t(g_t - g_t^\varepsilon)\cdot\nabla\varphi^\varepsilon\,dxdt.$$

The proof is finished letting $\varepsilon \to 0^+$ and noticing that $|g_t^\varepsilon| + |\nabla\varphi^\varepsilon| \le C$, with C independent of ε. $\square$

In the following theorem we show that (MK) and (ODE) are basically equivalent. Here the assumption that Ω is large enough plays a role: indeed, if for instance $X = [\![x_0, x_1]\!]$, $f_0 = \delta_{x_0}$ and $f_1 = \delta_{x_1}$, the infimum of (ODE) is easily seen to be less than

$$\mathrm{dist}(x_0, \partial\Omega) + \mathrm{dist}(x_1, \partial\Omega).$$

Therefore, in the general case when f_0, f_1 are arbitrary measures in X, we require that $\mathrm{dist}(\partial\Omega, X) > \mathrm{diam}(X)$.

Theorem 4.2 ((MK) versus (ODE)) *The problem (ODE) has at least one solution and* $\min$ *(ODE)* $=$ $\min$ *(MK). Moreover, for any optimal planning $\gamma \in \mathcal{M}_1(X\times X)$ for (MK) the measures*

$$f_t := \pi_{t\#}\gamma, \quad E_t := \pi_{t\#}\left((y-x)\gamma\right) \qquad t\in[0,1] \tag{23}$$

with $\pi_t(x,t) = x + t(y-x)$ solve (ODE).

Proof. Let (f_t, E_t) be defined by (23). For any $\phi \in C^\infty(\mathbb{R}^n)$ we compute

$$\begin{aligned} \frac{d}{dt}\int_X \phi\,df_t &= \frac{d}{dt}\int_{X\times X} \phi\left(x + t(y-x)\right)\,d\gamma \\ &= \int_{X\times X} \nabla\phi\left(x+t(y-x)\right)\cdot(y-x)\,d\gamma \\ &= \sum_{i=1}^n \int_X \nabla_i\phi\,dE_{t,i} = -\nabla\cdot E_t(\phi) \end{aligned}$$

hence the ODE (17) is satisfied. Then, we simply evaluate the energy $J(E)$ in (21) by

$$\begin{aligned} J(E) &= \int_0^1 |\pi_{t\#}((y-x)\gamma)|(X)\,dt \le \int_0^1 \pi_{t\#}(|y-x|\gamma)(X)\,dt \qquad (24) \\ &= \int_0^1\int_{X\times X} |x-y|\,d\gamma dt = I(\gamma). \end{aligned}$$

This shows that inf (ODE) $\leq$ min (MK).

In order to prove the opposite inequality we first use Proposition 4.1 and then we present a different strategy, which provides more geometric informations.

By Proposition 4.1 we have

$$\mathcal{F}_1(f_0, f_1) \quad \leq \quad \int_0^1 \frac{d}{dt}\mathcal{F}_1(f_t, f_0)\, dt \leq \int_0^1 |E_t|(\Omega)\, dt$$

for any admissible flow (f_t, E_t). This proves that min (MK) $\leq$ inf (ODE).

For given (and admissible) (f_t, E_t), and assuming that

$$\mathrm{spt} \int_0^1 |E_t|\, dt \subset\subset \Omega$$

we exhibit an optimal planning γ with $I(\gamma) \leq J(E)$.

To this aim we fix a cut-off function $\theta \in C_c^\infty(\Omega)$ with $0 \leq \theta \leq 1$ and $\theta \equiv 1$ on $X \cup \mathrm{spt} \int_0^1 |E_t|\, dt$; then, we define

$$f_t^\varepsilon := f_t * \rho_\varepsilon + \varepsilon\theta \quad \text{and} \quad E_t^\varepsilon := E_t * \rho_\varepsilon$$

where ρ is any convolution kernel with compact support and $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$. Notice that f_t^ε are strictly positive on $\mathrm{spt}E_t$; moreover (17) still holds, $\mathrm{spt}E_t^\varepsilon \subset \Omega$ for ε small enough and $\mathcal{L}^1$-a.e. t and by (53) we have

$$\int_{\mathbb{R}^n} |E_t^\varepsilon|(y)\, dy \leq |E_t|(\Omega) \qquad \forall t \in [0, 1]. \tag{25}$$

We define $g_t^\varepsilon = E_t^\varepsilon / f_t^\varepsilon$ and denote by $\psi_t^\varepsilon(x)$ the semigroup in $[0, 1]$ associated to the ODE

$$\dot{\psi}_t^\varepsilon(x) = g_t^\varepsilon\left(\psi_t^\varepsilon(x)\right), \qquad \psi_0^\varepsilon(x) = x. \tag{26}$$

Notice that this flow for ε small enough maps Ω into itself and leaves $\mathbb{R}^n \setminus \Omega$ fixed. Now, we claim that $f_t^\varepsilon = \psi_{t\#}^\varepsilon(f_0^\varepsilon \mathcal{L}^n)$ for any $t \in [0, 1]$. Indeed, since by definition

$$\dot{f}_t^\varepsilon + \nabla \cdot (g_t^\varepsilon f_t^\varepsilon) = \dot{f}_t^\varepsilon + \nabla \cdot E_t^\varepsilon = 0$$

by Theorem 4.1 and the linearity of the equation we need only to check that also $\nu_t^\varepsilon = \psi_{t\#}^\varepsilon(f_0^\varepsilon \mathcal{L}^n)$ satisfies the ODE $\dot{\nu}_t^\varepsilon + \nabla \cdot (g^\varepsilon \nu_t^\varepsilon) = 0$. This is a straightforward computation based on (26).

Using this representation, we can view ψ_1^ε as approximate solutions of the optimal transport problem and define

$$\gamma^\varepsilon := (Id \times \psi_1^\varepsilon)_\#(f_0^\varepsilon \mathcal{L}^n),$$

i.e.

$$\int \varphi(x,y)\, d\gamma^\varepsilon(x) = \int_{\mathbb{R}^n} \varphi\left(x, \psi_1^\varepsilon(x)\right) f_0^\varepsilon(x)\, dx$$

for any bounded Borel function φ in $\mathbb{R}^n \times \mathbb{R}^n$.

In order to evaluate $I(\gamma^\varepsilon)$, we notice that

$$\mathrm{Length}(\psi_t^\varepsilon(x)) = \int_0^1 |g_t^\varepsilon|(\psi_t^\varepsilon(x))\, dt$$

hence, by multiplying by f_0^ε and integrating we get

$$\begin{aligned} \int_{\mathbb{R}^n} \mathrm{Length}(\psi_t^\varepsilon(x)) f_0^\varepsilon(x)\, dx &= \int_0^1 \int_{\mathbb{R}^n} |g_t^\varepsilon(y)| f_t^\varepsilon(y)\, dy dt \\ &= \int_0^1 \int_{\mathbb{R}^n} |E_t^\varepsilon|(y)\, dy dt \le \int_0^1 |E_t|(\Omega)\, dt. \end{aligned} \tag{27}$$

As

$$I(\gamma^\varepsilon) = \int_{\mathbb{R}^n} |\psi_1^\varepsilon(x) - x| f_0^\varepsilon(x)\, dx \le \int_{\mathbb{R}^n} \mathrm{Length}(\psi_t^\varepsilon(x))\, df_0(x),$$

this proves that $I(\gamma^\varepsilon) \le J(E)$. Passing to the limit as $\varepsilon \downarrow 0$ and noticing that

$$\pi_{0\#}\gamma^\varepsilon = f_0^\varepsilon \mathcal{L}^n, \qquad \pi_{1\#}\gamma^\varepsilon = f_1^\varepsilon \mathcal{L}^n$$

we obtain, possibly passing to a subsequence, an admissible planning $\gamma \in \mathcal{M}_1(\mathbb{R}^n \times \mathbb{R}^n)$ for f_0 and f_1 such that $I(\gamma) \le J(E)$. We can turn γ in a planning supported in $X \times X$ simply replacing γ by $(\pi \times \pi)_\#\gamma$, where $\pi : \mathbb{R}^n \to X$ is the orthogonal projection. $\square$

Remark 4.1 (1) Starting from (f_t, E_t), the (possibly multivalued) operation leading to γ and then to the new flow

$$\tilde{f}_t := \pi_{t\#}\gamma, \qquad \tilde{E}_t := \pi_{t\#}\left((y-x)\gamma\right)$$

can be understood as a sort of "arclength reparameterization" of (f_t, E_t). However, since this operation is local in space (consider for instance the case of two line paths, one parameterized by arclength, one not), in general there is no function $\varphi(t)$ such that

$$(\tilde f_t, \tilde E_t) = (f_{\varphi(t)}, E_{\varphi(t)}).$$

(2) The solutions (f_t, E_t) in Example 4.1 are built as $\pi_{t\#}\gamma$ and $\pi_{t\#}((y-x)\gamma)$ where $\gamma = (Id \times \psi)_\# f_0$ and $\psi(x) = x+1$, $\psi(x) = 2-x$. In particular, in the second example, $f_{1/2} = \delta_1$ because 1 is the midpoint of *any* transport ray.
(3) By making a regularization first in space and then in time of (f_t, E_t) one can avoid the use of Theorem 4.1, using only the classical representation of solutions of (22) with characteristics. □

Remark 4.2 (Optimality conditions) (1) If (f_t, E_t) is optimal, then $\operatorname{spt} \int_0^1 |E_t|\, dt \subset\subset \Omega$. Indeed, we can find $\Omega' \subset\subset \Omega$ which still has the property that the open r-neighbourhood of X is contained in Ω', with $r = \operatorname{diam}(X)$, hence

$$\int_0^1 |E_t|(\Omega)\, dt = \min\,(\mathrm{MK}) = \int_0^1 |E_t|(\Omega')\, dt.$$

(2) Since the two sides in the chain of inequalities (24) are equal when γ is optimal, we infer

$$|\pi_{t\#}((y-x)\gamma)| = \pi_{t\#}(|y-x|\gamma) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,1). \tag{28}$$

Analogously, we have

$$\left| \int_0^1 E_t\, dt \right| = \int_0^1 |E_t|\, dt \tag{29}$$

whenever (f_t, E_t) is optimal. If the strict inequality $<$ holds, then

$$\hat f_t = f_0 + t(f_1 - f_0) \quad \text{and} \quad \hat E_t = \int_0^1 E_\tau\, d\tau$$

provide an admissible pair for (ODE) with strictly less energy. □

Remark 4.3 (Nonlinear cost) When the cost function $c(x,y)$ is $|x-y|^p$ for some $p > 1$ the corresponding problem (MK) is still equivalent to (ODE), provided we minimize, instead of J, the energy

$$J_p(f, E) := \int_0^1 \Phi_p(f_t, E_t)\, dt$$

where

$$\Phi_p(\sigma,\nu) = \begin{cases} \displaystyle\int_X \left|\frac{\nu}{\sigma}\right|^p d\sigma & \text{if } |\nu| << \sigma; \\ \\ +\infty & \text{otherwise.} \end{cases}$$

The proof of the equivalence is quite similar to the one given in Theorem 4.2. Again the essential ingredient is the inequality

$$\Phi_p(\nu * \rho_\varepsilon, \sigma * \rho_\varepsilon) \leq \Phi_p(\nu,\sigma)$$

The latter follows by Jensen's inequality and the convexity of the map $(z,t) \mapsto |z|^p t^{1-p}$ in $\mathbb{R}^n \times (0,\infty)$, which provide the pointwise estimate

$$\left|\frac{\nu * \rho_\varepsilon}{\sigma * \rho_\varepsilon}\right|^p \sigma * \rho_\varepsilon \leq \left(\left|\frac{\nu}{\sigma}\right|^p \sigma\right) * \rho_\varepsilon.$$

In this case, under the same assumptions on f_0 made in Corollary 2.2, due to the uniqueness of the optimal planning $\gamma = (Id \times \psi_\#) f_0$, we obtain that, for optimal (f_t, E_t), $\gamma^\varepsilon = (Id \times \psi_1^\varepsilon)_\# f_0^\varepsilon$ converge to γ as $\varepsilon \to 0^+$. As a byproduct, the maps ψ_1^ε converge to ψ as Young measures. If $f_0 << \mathcal{L}^n$ it follows that ψ_h converge to ψ in $[L^r(f_0)]^n$ for any $r \in [1,\infty)$ (see Lemma 9.1 and the remark following it).

See §2.6 of [4] for a systematic analysis of the continuity and semicontinuity properties of the functional $(\sigma,\nu) \mapsto \Phi_p(\sigma,\nu)$ with respect to weak convergence of measures. □

In the previous proof I don't know whether it is actually possible to show full convergence of γ^ε as $\varepsilon \to 0^+$. However, using a more refined estimate and a geometric lemma (see Lemma 4.1 below) we will prove that any limit measure γ satisfies the condition

$$\int_0^1 \pi_{t\#}(|y-x|\gamma)\, dt = \int_0^1 |E_t|\, dt. \tag{30}$$

We call *transport density* any of the measures in (30). This double representation will be relevant in Section 7, where under suitable assumptions we will obtain the uniqueness of the transport density.

Corollary 4.1 *Let (f_t, E_t) be optimal for (ODE). Then there exists an optimal planning γ such that (30) holds. In particular spt $\int_0^1 |E_t|\, dt \subset X$.*

Proof. Let $m = \min(\mathrm{MK}) = \min(\mathrm{ODE})$ and recall that, by Remark 4.2(1), the measure $\int_0^1 |E_t|\,dt$ has compact support in Ω. It suffices to build an optimal planning γ such that

$$\int_0^1 \pi_{t\#}((y-x)\gamma)\,dt = \int_0^1 E_t\,dt. \tag{31}$$

Indeed, by (29) we get

$$\int_0^1 \pi_{t\#}(|y-x|\gamma) \geq \int_0^1 |E_t|\,dt$$

and the two measures coincide, having both mass equal to m.

Keeping the same notation used in the final part of the proof of Theorem 4.2, we define γ_t^ε as $(Id \times \psi_t^\varepsilon)_\# f_0^\varepsilon$ and we compute

$$I(\gamma^\varepsilon) = \int_0^1 \frac{d}{dt} I(\gamma_t^\varepsilon)\,dt = \int_0^1 \int_{B_R} \frac{\psi_t^\varepsilon(x) - x}{|\psi_t^\varepsilon(x) - x|} \cdot \dot{\psi}_t^\varepsilon(x) f_0^\varepsilon(x)\,dxdt.$$

Since $I(\gamma^\varepsilon) \to m$ as $\varepsilon \to 0^+$ and since (by (25))

$$\int_0^1 \int_{B_R} |\dot{\psi}_t^\varepsilon(x)| f_0^\varepsilon(x)\,dxdt = \int_0^1 \int_{B_R} |E_t^\varepsilon|\,dxdt \leq \int_0^1 |E_t|(\Omega)\,dt = m$$

we infer

$$\lim_{\varepsilon \to 0^+} \int_{B_R} \int_0^1 \left| \frac{\dot{\psi}_t^\varepsilon(x)}{|\dot{\psi}_t^\varepsilon(x)|} - \frac{\psi_t^\varepsilon(x) - x}{|\psi_t^\varepsilon(x) - x|} \right|^2 |\dot{\psi}_t^\varepsilon(x)| f_0^\varepsilon(x)\,dtdx = 0. \tag{32}$$

Now using Lemma 4.1 and the Young inequality $2ab \leq \delta a^2 + b^2/\delta$, for any $\phi \in C_c^\infty(\mathbb{R}^n)$ we obtain

$$\begin{aligned}
& \left| \int_0^1 [\pi_{t\#}((y-x)\gamma^\varepsilon) - E_t^\varepsilon](\phi)\,dt \right| \\
= & \left| \int_X \int_0^1 [(\psi_1^\varepsilon(x) - x)\phi(t(\psi_1^\varepsilon(x) - x)) - \dot{\psi}_t^\varepsilon(x)\phi(\psi_t^\varepsilon(x))] f_0^\varepsilon(x)\,dtdx \right| \\
\leq & \; R\mathrm{Lip}(\phi)\delta \int_X \mathrm{Length}(\psi_t^\varepsilon) f_0^\varepsilon(x)\,dx \\
+ & \; \frac{R\mathrm{Lip}(\phi)}{\delta} \int_X \int_0^1 \left| \frac{\dot{\psi}_t^\varepsilon(x)}{|\dot{\psi}_t^\varepsilon(x)|} - \frac{\psi_t^\varepsilon(x) - x}{|\psi_t^\varepsilon(x) - x|} \right|^2 |\dot{\psi}_t^\varepsilon(x)| f_0^\varepsilon(x)\,dtdx.
\end{aligned}$$

with $R = \mathrm{diam}(\Omega)$. Passing to the limit first as $\varepsilon \to 0^+$, taking (32) into account and then passing to the limit as $\delta \to 0^+$ we obtain

$$\int_0^1 \pi_{t\#}((y-x)\gamma)(\phi)\,dt = \int_0^1 E_t(\phi)\,dt.$$

□

Lemma 4.1 *Let $\psi \in \mathrm{Lip}([0,1],\mathbb{R}^n)$ with $\psi(0)=0$ and $\phi \in \mathrm{Lip}(\mathbb{R}^n)$. Then*

$$\left|\int_0^1 \psi(1)\phi(t\psi(1)) - \dot\psi(t)\phi(\psi(t))\,dt\right| \le LR\int_0^1 \left|\frac{\dot\psi(t)}{|\dot\psi(t)|} - \frac{\psi(t)}{|\psi(t)|}\right| |\dot\psi(t)|\,dt$$

with $L = \mathrm{Lip}(\phi)$, $R = \sup|\psi|$.

Proof. We start from the elementary identity

$$\begin{aligned} \frac{d}{dt}\left[\phi(s\psi(t))\psi_i(t)\right] \quad &- \quad \frac{d}{ds}\left[s\phi(s\psi(t))\dot\psi_i(t)\right] \\ &= \quad s\sum_{j=1}^n \frac{\partial\phi}{\partial x_j}(s\psi(t))\left(\psi_i(t)\dot\psi_j(t) - \psi_j(t)\dot\psi_i(t)\right) \end{aligned} \tag{33}$$

and integrate both sides in $[0,1]\times[0,1]$. Then, the left side becomes exactly the integral that we need to estimate from above. The right side, up to the multiplicative constant $\mathrm{Lip}(\phi)$, can be estimated with

$$\begin{aligned} \int_0^1 |\psi(t)\wedge\dot\psi(t)|\,dt \quad &= \quad \int_0^1 |\psi(t)|\left|\dot\psi(t)\wedge\left(\frac{\dot\psi(t)}{|\dot\psi(t)|} - \frac{\psi(t)}{|\psi(t)|}\right)\right|\,dt \\ &\le \quad \sup|\psi|\int_0^1 \left|\frac{\dot\psi(t)}{|\dot\psi(t)|} - \frac{\psi(t)}{|\psi(t)|}\right| |\dot\psi(t)|\,dt. \end{aligned}$$

□

Remark 4.4 The geometric meaning of the proof above is the following: the integral to be estimated is the action on the 1-form ϕdx_i of the closed and rectifiable current T associated to the closed path starting at 0, arriving at

$\psi(1)$ following the curve $\psi(t)$, and then going back to 0 through the segment $[\![0, \psi(1)]\!]$; for any 2-dimensional current G such that $\partial G = T$, as

$$T(\phi dx_i) = G(d\phi \wedge dx_i), \tag{34}$$

this action can be estimated by the mass of G times $\mathrm{Lip}(\phi)$. The cone construction provides a current G with $\partial G = T$, whose mass can be estimated by

$$\int_0^1 \int_0^1 s|\psi(t) \wedge \dot{\psi}(t)|\, dsdt.$$

With this choice of G the identity (34) corresponds to (33). $\square$

In order to relate also (f_t, E_t) to the Kantorovich potential, we define the transport rays and the transport set and we prove the differentiability of the potential on the transport set.

Definition 4.1 (Transport rays and transport set) *Let $u \in \mathrm{Lip}_1(X)$. We say that a segment $]\!]x, y[\![\subset X$ is a* transport ray *if it is a maximal open oriented segment whose endpoints x, y satisfy the condition*

$$u(x) - u(y) = |x - y|. \tag{35}$$

The transport set *$\mathcal{T}_u$ is defined as the union of all transport rays. We also define $\mathcal{T}_u^e$ as the union of the closures of all transport rays.*

Denoting by $\mathcal{F}$ the compact collection of pairs (x, y) with $x \neq y$ such that (35) holds, the transport set is also given by

$$\mathcal{T}_u = \bigcup_{t \in (0,1)} \bigcup_{(x,y) \in \mathcal{F}} \{x + t(y - x)\}$$

and therefore is a Borel set (precisely a countable union of closed sets).

We can now easily prove that the u is differentiable at any point in $\mathcal{T}_u$.

Proposition 4.2 (Differentiability of the potential) *Let $u \in \mathrm{Lip}_1(X)$. Then u is differentiable at any point $z \in \mathcal{T}_u$. Moreover $-\nabla u(z)$ is the unit vector parallel to the transport ray containing z.*

Proof. Let x, $y \in X$ be such that (35) holds. By the triangle inequality and the 1-Lipschitz continuity of u we get

$$u(x) - u(x + t(y - x)) = t|y - x| \qquad \forall t \in [0, 1].$$

This implies that, setting $\nu = (y-x)/|y-x|$, the partial derivative of u along ν is equal to -1 for any internal point z of the segment. For any unit vector ξ perpendicular to ν we have

$$\begin{aligned} u(z+h\xi) - u(z) &= u(z+h\xi) - u(z+\sqrt{|h|}\nu) + u(z+\sqrt{|h|}\nu) - u(z) \\ &\leq \sqrt{|h|^2+|h|} - \sqrt{|h|} = O(|h|^{3/2}) = o(|h|). \end{aligned}$$

A similar argument also proves that $u(z+h\xi) - u(z) \geq o(|h|)$. This proves the differentiability of u at z and the identity $\nabla u(z) = -\nu$. $\square$

In Section 6 we will need a mild Lipschitz property of the potential ∇u on $\mathcal{T}_u^e$ (see also [15, 42] for the case of general strictly convex norms). We will use this property to prove in Corollary 6.1 that $\mathcal{T}_u^e \setminus \mathcal{T}_u$ is Lebesgue negligible. This property can also be used to prove that ∇u is approximately differentiable $\mathcal{L}^n$-a.e. on $\mathcal{T}_u$ (see for instance Theorem 3.1.9 of [24]).

Theorem 4.3 (Countable Lipschitz property) *Let $u \in \mathrm{Lip}_1(X)$. There exists a sequence of Borel sets T_h covering $\mathcal{L}^n$-almost all of $\mathcal{T}_u^e$ such that ∇u restricted to T_h is a Lipschitz function.*

Proof. Given a direction $\nu \in \mathbf{S}^{n-1}$ and $a \in \mathbb{R}$, let R be the union of the half closed transport rays $[\![x, y[\![$ with $\langle y-x, \nu\rangle \geq 0$ and $\langle y, \nu\rangle \geq a$. It suffices to prove that the restriction of ∇u to

$$T := R \cap \{x : \ x\cdot\nu < a\} \setminus \Sigma_u$$

has the countable Lipschitz property stated in the theorem. To this aim, since BV_{loc} funtions have this property (see for instance Theorem 5.34 of [4] or [24]), it suffices to prove that ∇u coincides $\mathcal{L}^n$-a.e. in T with a suitable function $w \in [BV_{\mathrm{loc}}(S)]^n$, where $S_a = \{x : \ x\cdot\nu < a\}$. To this aim we define

$$\tilde{u}(x) := \min\{u(y) + |x-y| : \ y \in Y_a\}$$

where Y_a is the collection of all right endpoints of transport rays with $y\cdot\nu \geq a$. By construction $\tilde{u} \geq u$ and equality holds on $R \supset T$.

We claim that, for $b < a$, $\tilde{u} - C|x|^2$ is concave in S_b for $C = C(b)$ large enough. Indeed, since $|x-y| \geq a-b > 0$ for any $y \in Y_a$ and any $x \in S_b$, the functions

$$u(y) + |x-y| - C|x|^2, \qquad y \in Y_a$$

are all concave in H for C large enough depending on $a-b$. In particular, as gradients of real valued concave functions are $BV_{\rm loc}$ (see for instance [2]), we obtain that

$$w := \nabla\tilde u = \nabla(u - C|x|^2) + 2Cx$$

is a $BV_{\rm loc}$ function in S_a. Since $\nabla u = w$ $\mathcal{L}^n$-a.e. in T the proof is achieved.

By a similar argument, using semi-convexity in place of semi-concavity, one can take into account the right extreme points of the transport rays. □

As a byproduct of the equivalence between (MK) and (ODE) we can prove that (ODE) has "regular" solutions, related to the transport set and to the gradient of any maximal Kantorovich potential u.

Theorem 4.4 (Regularity of (f_t, E_t)) *For any solution (f_t, E_t) of (ODE) representable as in Theorem 4.2 for a suitable optimal planning γ there exists a 1-Lipschitz function u such that*

(i) f_t is concentrated on the transport set $\mathcal{T}_u$ and $|E_t| \le Cf_t$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$, with $C = \mathrm{diam}(X)$;

(ii) $E_t = -\nabla u|E_t|$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$;

(iii) for any convolution kernel ρ we have

$$\lim_{\varepsilon\to 0^+} \int_0^1 \int_X |\nabla u - \nabla u * \rho_\varepsilon|^2 \, d|E_t| \, dt = 0; \qquad (36)$$

(iv) $|E_t|(X) = \int_0^1 |E_\tau|(X)\, d\tau$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$.

Proof. (i) Let u be any maximal Kantorovich potential and let $\mathcal{T} = \mathcal{T}_u$ be the associated transport set. For any $t \in (0,1)$ we have

$$f_t(X \setminus \mathcal{T}) = \int_{X\times X} \chi_{\{(x,y):\ x+t(y-x)\notin\mathcal{T}\}}\, d\gamma = 0$$

because, by (12), any segment $]\!]x,y[\![$ with $(x,y) \in \mathrm{spt}\gamma$ is contained in a transport ray, hence contained in $\mathcal{T}$. The inequality $|E_t| \le Cf_t$ simply follows by the fact that $|x-y| \le C$ on $\mathrm{spt}\gamma$.

(ii) Choosing t satisfying (28) and taking into account Proposition 4.2, for any $\phi \in C(X)$ we obtain

$$\begin{aligned}\int_X \phi \nabla u \, d|E_t| &= \int_{X\times X} \phi(\pi_t)\nabla u(\pi_t)|y-x| \, d\gamma \\ &= -\int_{X\times X} \phi(\pi_t)(y-x)\, d\gamma = -E_t(\phi).\end{aligned}$$

(iii) Let $m = J(E) = \min(\mathrm{MK})$. By (13) we get

$$\begin{aligned} m &= \int_X u\, d(f_0 - f_1) = \lim_{\varepsilon\to 0^+}\int_X u*\rho_\varepsilon \, d(f_1-f_0) \\ &= \lim_{\varepsilon\to 0^+}\int_X\int_0^1 u*\rho_\varepsilon\, d\dot{f}_t\, dt = \lim_{\varepsilon\to 0^+}\int_0^1 \nabla u*\rho_\varepsilon\cdot E_t\, dt \\ &\leq \int_0^1 |E_t|(X)\, dt = m. \end{aligned}$$

This proves that

$$\lim_{\varepsilon\to 0^+}\int_0^1\left[|E_t|(X) - \int_X \langle \nabla u*\rho_\varepsilon, \nabla u\rangle\, d|E_t|\right] dt = 0$$

whence, taking into account that $|\nabla u| = 1$ and $|\nabla u * \rho_\varepsilon| \leq 1$, (36) follows.
(iv) The inequality $|E_t|(X) \leq I(\gamma) = J(E)$ has been estabilished during the proof of Theorem 4.2. By minimality equality holds for $\mathcal{L}^1$-a.e. $t \in (0,1)$. □

Remark 4.5 (1) It is easy to produce examples of optimal flows (f_t, E_t) satisfying (i), (ii), (iii), (iv) which are not representable as in Theorem 4.2 (again it suffices to consider the sum of two paths, with constant sum of velocities). It is not clear which conditions must be added in order to obtain this representation property.
(2) Notice that condition (i) actually implies that f is a Lipschitz map from $[0,1]$ into $\mathcal{M}_1(X)$ endowed with the 1-Wasserstein metric. □

5 The PDE version of the optimal transport problem and the p-laplacian approximation

In this section we see how, in the case when the cost function is linear, the optimal transport problem can be rephrased using a PDE. As in the previous section we consider an auxiliary bounded open set Ω such that the open r-neighbourhood of X is contained in Ω, with $r > \mathrm{diam}(X)$; we assume also that Ω has a Lipschitz boundary.

Specifically, we are going to consider the following problem and its connections with (MK) and (ODE).

(PDE) Let $f \in \mathcal{M}(X)$ be a measure with $f(X) = 0$. Find $\mu \in \mathcal{M}_+(\Omega)$ and $u \in \mathrm{Lip}_1(\Omega)$ such that:

(i) there exist smooth functions u_h uniformly converging to u on X, equal to 0 on $\partial\Omega$ and such that the functions ∇u_h converge in $[L^2(\mu)]^n$ to a function $\nabla_\mu u$ satisfying

$$|\nabla_\mu u| = 1 \qquad \mu\text{-a.e. in } \Omega; \tag{37}$$

(ii) the following PDE is satisfied in the sense of distributions

$$-\nabla \cdot (\nabla_\mu u\, \mu) = f \qquad \text{in } \Omega. \tag{38}$$

Given (μ, u) admissible for (PDE), we call μ the *transport density* (the reason for that soon will be clear) and $\nabla_\mu u$ the *tangential gradient* of u.

Remark 5.1 (1) Choosing u_h as test function in (38) gives

$$\begin{aligned} \mu(\Omega) &= \int_\Omega |\nabla_\mu u|^2 \, d\mu = \lim_{h\to\infty} \int_\Omega \langle \nabla_\mu u, \nabla u_h \rangle \, d\mu \\ &= \lim_{h\to\infty} f(u_h) = f(u), \end{aligned} \tag{39}$$

so that the total mass $\mu(\Omega)$ depends only on f and u and that μ is concentrated on X. We will prove in Theorem 5.1 that actually $\mu(\Omega)$ depends only on f.

(2) It is easy to prove that, given (μ, u), there is at most one tangential gradient $\nabla_\mu u$: indeed if $\nabla u_h \to \nabla_\mu u$ and $\nabla \tilde{u}_h \to \tilde{\nabla}_\mu u$ we have

$$\int_\Omega \nabla_\mu u \cdot \tilde{\nabla}_\mu u \, d\mu = \lim_{h\to\infty} \int_\Omega \nabla_\mu u \cdot \nabla \tilde{u}_h \, d\mu = \lim_{h\to\infty} f(\tilde{u}_h) = f(u) = \mu(\Omega)$$

whence $\nabla_\mu u = \tilde{\nabla}_\mu u$ μ-a.e. On the other hand, Example 5.1 below shows that to some u there could correspond more than one measures μ solving (PDE). We will obtain uniqueness in Section 7 under absolute continuity assumptions on f^+ or f^-. $\square$

We will need the following lemma, in which the assumption that Ω is "large enough" plays a role.

Lemma 5.1 *If (μ, u) solves (PDE) then $spt\,\mu \subset\subset \Omega$.*

Proof. Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ be such that $|x-y| > r = \mathrm{diam}(X)$ for any $x \in X$ and any $y \in \mathbb{R}^n \setminus \Omega'$. We choose a minimum point x_0 for the restriction of u to X and define

$$w(x) := \min\left\{[u(x) - u(x_0)]^+, \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega')\right\}.$$

Then, it is easy to check that $w(x) = u(x) - u(x_0)$ on X and that $w \equiv 0$ on $\mathbb{R}^n \setminus \Omega'$. Since

$$\begin{aligned} \mu(\Omega) &= f(u) = f(u - u(x_0)) = \lim_{\epsilon \to 0^+} f(w * \rho_\varepsilon) \\ &= \lim_{\epsilon \to 0^+} \int_\Omega \nabla_\mu u \cdot \nabla w * \rho_\varepsilon \, dx \le \mu(\Omega'') \end{aligned}$$

we conclude that $\mu(\Omega \setminus \Omega'') = 0$. $\square$

In the following theorem we build a solution of (PDE) choosing μ as a transport density of (MK) with $f_0 = f^+$ and $f_1 = f^-$. As a byproduct, by Corollary 4.1 we obtain that the support of μ is contained in X and that μ is representable as the right side in (30).

Theorem 5.1 ((ODE) versus (PDE)) *Assume that (f_t, E_t) with $spt \int_0^1 |E_t|\, dt \subset\subset \Omega$ solves (ODE) and let u be a maximal Kantorovich potential relative to f_0, f_1. Then, setting $f = f_0 - f_1$, the pair (μ, u) with $\mu = \int_0^1 |E_t|\, dt$ solves (PDE) with $\nabla_\mu u = \nabla u$ and, in particular, $spt\mu \subset X$.*

Conversely, if (μ, u) solves (PDE), setting $f_0 = f^+$ and $f_1 = f^-$, the measures (f_t, E_t) defined by

$$f_t := f_0 + t(f_1 - f_0), \qquad E_t := -\nabla_\mu u \, \mu$$

solve (ODE) and $spt\,\mu \subset X$. In particular

$$\mu(X) = \min \textit{(ODE)} = \min \textit{(MK)}.$$

Proof. By Corollary 4.1 we can assume that μ is representable as in (30) for a suitable optimal planning γ. Then, condition (i) in (PDE) with $\nabla_\mu u = \nabla u$ follows by (36). By (17) and condition (ii) of Theorem 4.4 we get

$$\nabla \cdot (\nabla_\mu u |F_t|) = \nabla \cdot (\nabla u |F_t|) = \dot{f}_t$$

with $F_t = \pi_{t\#}((y-x)\gamma)$. Integrating in time and taking into account the definition of μ, we obtain (38). Notice also that $\mu(X) = J(E)$.

If (f_t, E_t) are defined as above, we get

$$\dot{f}_t + \nabla \cdot E_t = f_1 - f_0 - \nabla \cdot (\nabla_\mu u \mu) = f_1 - f_0 + f = 0.$$

Also in this case $J(E) = \mu(X)$. $\square$

Example 5.1 (Non uniqueness of μ) In general, given f, the measure which solves (PDE) for a given $u \in \mathrm{Lip}_1(X)$ is not unique. As an example one can consider the situation in Example 1.5, where μ can be concentrated either on the horizontal sides of the square or on the vertical sides of the square.

As shown in [22], an alternative construction can be obtained by solving the problem "$-\Delta_\infty u = f$", i.e. studying the following problems

$$\begin{cases} -\nabla \cdot (|\nabla u_p|^{p-2} \nabla u_p) = f & \text{in } \Omega \\ u_p = 0 & \text{on } \partial\Omega \end{cases} \tag{40}$$

as $p \to \infty$. In this case μ is the limit, up to subsequences, of $|\nabla u_p|^{p-1}\mathcal{L}^n$. Under suitable regularity assumptions, Evans and Gangbo prove that $\mu = a\mathcal{L}^n$ for some $a \in L^1(\Omega)$ and use (a, u) to build an optimal transport ψ; their construction is based on a careful regularization corresponding to the one used in Theorem 4.2 in the special case $E_t = a\mathcal{L}^n$ and $f_t = f_0 + t(f_1 - f_0)$. However, since the goal in Theorem 4.2 is to build only an optimal planning, and not an optimal transport, our proof is much simpler and works in a much greater generality (i.e. no special assumption on f).

Now we also prove that the limiting procedure of Evans and Gangbo leads to solutions of (PDE), regardless of any assumption on the data f_0, f_1.

Theorem 5.2 *Let u_p be solutions of (40) with $p \geq n+1$. Then*

(i) the measures $H_p = |\nabla u_p|^{p-2} \nabla u_p \mathcal{L}^n \llcorner \Omega$ are equi-bounded in Ω;

(ii) (u_p) *are equibounded and equicontinuous in* Ω*;*

(iii) If $(H,u)=\lim_j(H_{p_j},u_{p_j})$ *with* $p_j\to\infty$*, then* (μ,u) *solves (PDE) with* $\mu=|H|$*.*

Proof. (i) Using u_p as test function and the Sobolev embedding theorem we get

$$\int_\Omega |\nabla u_p|^p\,dx = f(u_p)\le |f|(X)\|u_p\|_\infty \le C\left(\int_\Omega |\nabla u_{p_0}|^{p_0}\,dx\right)^{1/p_0}$$

with $p_0=n+1$. Using Hölder inequality we infer

$$\int_\Omega |\nabla u_p|^p\,dx \le C^{\frac{p}{p-1}}\mathcal{L}^n(\Omega)^{\frac{p-p_0}{p_0(p-1)}}.$$

(ii) Follows by (i) and the Sobolev embedding theorem.
(iii) Clearly $-\nabla\cdot H=f$ in Ω. By (i) and Hölder inequality we infer

$$\sup_{q\ge n+1}\int_\Omega |\nabla u|^q\,dx<\infty$$

whence $|\nabla u|\le 1$ $\mathcal{L}^n$-a.e. in Ω.

We notice first that the functional

$$\nu\mapsto\int_\Omega\left|\frac{\nu}{|\nu|}-w\right|^2\phi\,d|\nu|$$

is lower semicontinuous with respect to the weak convergence of measures for any nonnegative $\phi\in C_c(\Omega)$, $w\in[C(\Omega)]^n$. The verification of this fact is straightforward: it suffices to expand the squares. Second, we notice that

$$\lim_{\epsilon\to 0^+}\limsup_{p\to\infty}\int_\Omega\left|\frac{H_p}{|H_p|}-\nabla u_\varepsilon\right|^2\,d|H_p|=0$$

whenever $u_\varepsilon\in\mathrm{Lip}_1(\mathbb{R}^n)$ are smooth functions uniformly converging to u in X and equal to 0 on $\partial\Omega$. Indeed, as $|\nabla u_\varepsilon|\le 1$, we have

$$\begin{aligned}\int_\Omega\left|\frac{H_p}{|H_p|}-\nabla u_\varepsilon\right|^2\,d|H_p| &\le 2\int_\Omega|\nabla u_p|^{p-1}\left(1-\frac{\nabla u_\varepsilon\cdot\nabla u_p}{|\nabla u_p|}\right)\,dx\\ &\le 2\int_\Omega|\nabla u_p|^{p-2}(|\nabla u_p|^2-\nabla u_\varepsilon\cdot\nabla u_p)\,dx+\omega_p\\ &= 2f(u_p)-2f(u_\varepsilon)+\omega_p\end{aligned}$$

where $\omega_p = \sup_{t\geq 0} t^{p-1} - t^p$ tends to 0 as $p \to \infty$.

Taking into account these two remarks, setting $p = p_j$ and passing to the limit as $j \to \infty$ we obtain

$$\lim_{\epsilon \to 0^+} \int_\Omega \left| \frac{H}{|H|} - \nabla u_\varepsilon \right|^2 \phi \, d|H| = 0.$$

for any nonnegative $\phi \in C_c(\Omega)$. This implies that ∇u_ε converge in $L^2_{\rm loc}(|H|)$ to the Radon–Nikodým derivative $H/|H|$. Since $|\nabla u_\varepsilon| \leq 1$ we have also convergence in $[L^2(|H|)]^n$.

We now set $\mu = |H|$ and $\nabla_\mu u = \lim_\epsilon \nabla u_\epsilon$, so that $H = \nabla_\mu u \mu$ and (38) is satisfied. $\square$

6 Existence of optimal transport maps

In this section we prove the existence of optimal transport maps in the case when $c(x, y) = |x - y|$ and f_0 is absolutely continuous with respect to $\mathcal{L}^n$, following essentially the original Sudakov approach and filling a gap in its original proof (see the comments after Theorem 6.1). Using a maximal Kantorovich potential we decompose almost all of X in transport rays and we build an optimal transport maps by gluing the 1-dimensional transport maps obtained in each ray. The assumption $f_0 << \mathcal{L}^n$ is used to prove that the conditional measures f_{0C} within any transport ray C are non atomic (and even absolutely continuous with respect to $\mathcal{H}^1 \llcorner C$), so that Theorem 3.1 is applicable. Therefore the proof depends on the following two results whose proof is based on the countable Lipschitz property of ∇u stated in Theorem 4.3.

Theorem 6.1 *Let $B \in \mathcal{B}(X)$ and let $\pi : B \to \mathcal{S}_c(X)$ be a Borel map satisfying the conditions*

(i) $\pi(x) \cap \pi(x') = \emptyset$ whenever $\pi(x) \neq \pi(x')$;

(ii) $x \in \pi(x)$ for any $x \in B$;

(iii) the direction $\nu(x)$ of $\pi(x)$ is a $\mathbf{S}^{n-1}$-valued countably Lipschitz map on B.

Then, for any measure $\lambda \in \mathcal{M}_+(X)$ absolutely continuous with respect to $\mathcal{L}^n \llcorner B$, setting $\mu = \pi_\# \lambda \in \mathcal{M}_+(\mathcal{S}_c(X))$, the measures λ_C of Theorem 9.1 are absolutely continuous with respect to $\mathcal{H}^1 \llcorner C$ for μ-a.e. $C \in \mathcal{S}_c(X)$.

Proof. Being the property stated stable under countable disjoint unions we may assume that

(a) there exists a unit vector ν such that $\nu(x) \cdot \nu \geq \frac{1}{2}$ for any $x \in B$;

(b) $\nu(x)$ is a Lipschitz map on B;

(c) B is contained in a strip

$$\{x : \ a - b \leq x \cdot \nu \leq a\}$$

with $b > 0$ sufficiently small (depending only on the Lipschitz constant of ν) and $\pi(x)$ intersects the hyperplane $\{x : \ x \cdot \nu = a\}$.

Assuming with no loss of generality $\nu = e_n$ and $a = 0$, we write $x = (y, z)$ with $y \in \mathbb{R}^{n-1}$ and $z < 0$. Under assumption (a), the map $T : \pi(B) \to \mathbb{R}^{n-1}$ which associates to any segment $\pi(x)$ the vector $y \in \mathbb{R}^{n-1}$ such that $(y, 0) \in \pi(x)$ is well defined. Moreover, by condition (i), T is one to one. Hence, setting $f = T \circ \pi : B \to \mathbb{R}^{n-1}$,

$$\nu = T_\# \mu = f_\# \lambda, \qquad\qquad C(y) = T^{-1}(y) \supset f^{-1}(y)$$

and representing $\lambda = \eta_y \otimes \nu$ with $\eta_y = \lambda_{C(y)} \in \mathcal{M}_1(f^{-1}(y))$, we need only to prove that $\eta_y << \mathcal{H}^1 \llcorner C(y)$ for ν-a.e. y.

To this aim we examine the Jacobian, in the y variables, of the map $f(y,t)$. Writing $\nu = (\nu_y, \nu_t)$, we have

$$f(y,t) = y + \tau(y,t)\nu_y(y,t) \qquad \text{with} \qquad \tau(y,t) = -\frac{t}{\nu_t(y,t)}.$$

Since $\nu_t \geq 1/2$ and $\tau \leq 2b$ on B we have

$$\det\left(\nabla_y f(y,t)\right) = \det\left(Id + \tau \nabla_y \nu_y + \frac{t}{\nu_t^2} \nabla_y \nu_t \otimes \nu_y\right) > 0$$

if b is small enough, depending only on $\mathrm{Lip}(\nu)$.

Therefore, the coarea factor

$$\mathbf{C}f := \sqrt{\sum_A \det{}^2 A}$$

(where the sum runs on all $(n-1) \times (n-1)$ minors A of ∇f) of f is strictly positive on B and, writing $\lambda = g\mathcal{L}^n$ with $g = 0$ out of B, Federer's coarea formula (see for instance [4], [37]) gives

$$\lambda = \frac{g}{\mathbf{C}f}\mathbf{C}f\mathcal{L}^n = \frac{g}{\mathbf{C}f}\,\mathcal{H}^1 \llcorner f^{-1}(y) \otimes \mathcal{L}^{n-1} = \eta'_y \otimes \nu'$$

with $B = \{y \in \mathbb{R}^{n-1} :\ \mathcal{H}^1(f^{-1}(y)) > 0\}$ and

$$\eta'_y := \frac{\frac{g}{\mathbf{C}f}\,\mathcal{H}^1 \llcorner f^{-1}(y)}{\int_{f^{-1}(y)} g/\mathbf{C}f\, d\mathcal{H}^1}\,, \qquad \nu' = \left(\int_{f^{-1}(y)} \frac{g}{\mathbf{C}f}\, d\mathcal{H}^1\right) \mathcal{L}^{n-1} \llcorner B.$$

By Theorem 9.2 we obtain $\nu = \nu'$ and $\eta_y = \eta'_y$ for ν-a.e. y, and this concludes the proof. $\square$

Remark 6.1 (1) In [41] V.N.Sudakov stated the theorem above (see Proposition 78 therein) for maps $\pi : B \to \mathcal{S}_o(X)$ without the countable Lipschitz assumption (iii) and also in a greater generality, i.e. for a generic "Borel decomposition" of the space in open affine regions, even of different dimensions. However, it turns out that the assumption (iii) is *essential* even if we restrict to 1-dimensional decompositions. Indeed, G.Alberti, B.Kirchheim and D.Preiss [3] have recently found an example of a compact family of open and pairwise disjoint segments in $\mathbb{R}^3$ such that the collection B of the midpoints of the segments has strictly positive Lebesgue measure. In this case, of course, if $\lambda = \mathcal{L}^n \llcorner B$ the conditional measures λ_C are unit Dirac masses concentrated at the midpoint of C, so that the conclusion of Theorem 6.1 fails.

If $n = 2$ it is not hard to prove that actually condition (iii) follows by (ii), so that the counterexample mentioned above is optimal.

(2) Since any open segment can be approximated from inside by closed segments, by a simple approximation argument one can prove that Theorem 6.1 still holds as well for maps $\pi : B \to \mathcal{S}_o(X)$ or for maps with values into half open $[\![x, y[\![$ segments.

Also the assumption (i) that the segments do not intersect can be relaxed. For instance we can assume that the segments can intersect only at their extreme points, provided the collection of these extreme points is Lebesgue negligible. This is precisely the content of the next corollary. □

Corollary 6.1 (Negligible extreme points) *Let $u \in \mathrm{Lip}_1(X)$. Then the collection of the extreme points of the transport rays is Lebesgue negligible.*

Proof. We prove that the collection L of all left extreme points is negligible, the proof for the right ones being similar. Let $B = L \setminus \Sigma_u$ and set

$$\pi(x) := [\![x, x - \frac{r(x)}{2}\nabla u(x)]\!]$$

where $r(x)$ is the length of the transport ray emanating from x (this ray is unique due to the differentiability of u at x). By Theorem 4.3 the map π has the countable Lipschitz property on B and, by construction, $\pi(x) \cap \pi(x') = \emptyset$ whenever $x \neq x'$. By Theorem 6.1 we obtain

$$\lambda = \int_{\mathcal{S}_c(X)} \lambda_C \, d\nu$$

with $\lambda = \mathcal{L}^n \llcorner B$, $\nu = \pi_\# \lambda$ and $\lambda_C << \mathcal{H}^1 \llcorner C$ probability measures concentrated on $\pi^{-1}(C)$ for ν-a.e. C. Since $\pi^{-1}(C)$ contains only one point for any $C \in \pi(B)$ we obtain $\lambda_C = 0$ for ν-a.e. C, whence $\lambda(B) = \nu(\mathcal{S}_c(X)) = 0$. □

Theorem 6.2 (Sudakov) *Let f_0, $f_1 \in \mathcal{M}_1(X)$ and assume that $f_0 << \mathcal{L}^n$. Then there exists an optimal transport ψ mapping f_0 to f_1. Moreover, if $f_1 << \mathcal{L}^n$ we can choose ψ so that ψ^{-1} is well defined f_1-a.e. and $\psi^{-1}_\# f_1 = f_0$.*

Proof. Let γ be an optimal planning. In the first two steps we assume that

$$\text{for } f_0\text{-a.e. } x \text{ there exists } y \neq x \text{ such that } (x, y) \in \mathrm{spt}\gamma. \tag{41}$$

This condition holds for instance if $f_0 \wedge f_1 = 0$ because in this case any optimal planning γ does not charge the diagonal Δ of $X \times X$, i.e. $\gamma(\Delta) = 0$ (otherwise $h = \pi_{0\#}(\chi_\Delta \gamma) = \pi_{1\#}(\chi_\Delta \gamma)$ would be a nonzero measure less than f_0 and f_1).

Let $u \in \mathrm{Lip}_1(X)$ be a maximal Kantorovich potential given by Corollary 2.1, i.e. a function satisfying

$$u(x) - u(y) = |x - y| \qquad \gamma\text{-a.e. in } X \times X \tag{42}$$

for any optimal planning γ and let $\mathcal{T}_u$ be the transport set relative to u (see Definition 4.1). In the following we set

$$\widetilde{X} = \{x \in X \setminus \Sigma_u : \ \exists y \neq x \text{ s.t. } (x, y) \in \mathrm{spt}\gamma\}.$$

By (41) and the absolute continuity assumption we know that f_0 is concentrated on $\widetilde{X}$. Notice also that for any $x \in \widetilde{X}$ there exists a unique closed transport ray containing x: this follows by the fact that any $x \in \widetilde{X}$ is a differentiability point of u and by Proposition 4.2.

Step 1. We define $r : X \times X \to \mathcal{S}_c(X)$ as the map which associates to any pair (x, y) the closed transport ray containing $[\![x, y]\!]$. By (42) the map r is well defined γ-a.e. out of the diagonal Δ; moreover, being f_0 concentrated on $\widetilde{X}$, r is also well defined γ-a.e. on Δ. Hence, according to Theorem 9.1 we can represent

$$\gamma = \gamma_C \otimes \nu \qquad \text{with} \qquad \nu := r_{\#}\gamma$$

and (42) gives

$$u(x) - u(y) = |x - y| \qquad \gamma_C\text{-a.e. in } X \times X \tag{43}$$

for ν-a.e. $C \in \mathcal{S}_c(X)$. By the sufficiency part in Corollary 2.1 we infer that γ_C is an optimal planning relative to the probability measures

$$f_{0C} := \pi_{0\#}\gamma_C, \qquad f_{1C} := \pi_{1\#}\gamma_C$$

for ν-a.e. $C \in \mathcal{S}_c(X)$. By (56) we infer

$$\pi_{t\#}\gamma(B) = \int_{\mathcal{S}_c(X)} \pi_{t\#}\gamma_C(B)\, d\nu(C) \qquad \forall t \in [0, 1], \ B \in \mathcal{B}(X). \tag{44}$$

Notice also that

$$\mathcal{F}_1(f_0, f_1) = \int_{\mathcal{S}_c(X)} \mathcal{F}_1(f_{0C}, f_{1C})\, d\nu(C). \tag{45}$$

Indeed,

$$I(\gamma) = \int_{X\times X} |x-y|\, d\gamma = \int_{\mathcal{S}_c(X)} \int_{X\times X} |x-y|\, d\gamma_C\, d\nu(C) = \int_{\mathcal{S}_c(X)} I(\gamma_C)\, d\nu(C)$$

and we know by (43) that γ_C is optimal for ν-a.e. $C \in \mathcal{S}_c(X)$.
Step 2. We denote by $\pi : \widetilde{X} \to \mathcal{S}_c(X)$ the natural map, so that $\pi(x)$ is the closed transport ray containing x. Since $r = \pi \circ \pi_0$ on $\widetilde{X} \times X$ we obtain

$$\nu = r_\# \gamma = \pi_\# (\pi_{0\#} \gamma) = \pi_\# f_0.$$

Moreover (44) gives

$$f_0(B) = \int_{\mathcal{S}_c(X)} f_{0C}(B)\, d\nu(C) \qquad \forall B \in \mathcal{B}(X). \tag{46}$$

Notice that the segments $\pi(x)$, $x \in \widetilde{X}$, can intersect only at their right extreme point and that, by Corollary 6.1, the collection of these extreme points is Lebesgue negligible. As a consequence of (46) with $\nu = \pi_\# f_0$ and Remark 6.1(2), the measures f_{0C} are absolutely continuous with respect to $\mathcal{H}^1 \llcorner C$ for μ-a.e. $C \in \mathcal{S}_c(X)$. Hence, by Theorem 3.1, for μ-a.e. $C \in \mathcal{S}_c(X)$ we can find a nondecreasing map $\psi_C : C \to C$ (this notion makes sense, since C is oriented) such that $\psi_{C\#} f_{0C} = f_{1C}$. Notice also that ψ^{-1} is well defined f_{1C}-a.e., if also $f_{1C} << \mathcal{H}^1 \llcorner C$.

Taking into account that the closed transport rays in $\pi(\widetilde{X})$ are pairwise disjoint in $\widetilde{X}$, we can glue all the maps ψ_C to produce a single Borel map $\psi : \widetilde{X} \to X$. The map ψ is Borel because we have been able to exhibit the one dimensional transport map constructively and because of the Borel property of the maps $C \mapsto f_{iC}$ (see (15) and (54)); the simple but boring details are left to the reader.

Since $\psi_\# f_{0C} = \psi_{C\#} f_{0C} = f_{1C}$ for μ-a.e. $C \in \mathcal{S}_c(X)$, taking (44) into account we infer

$$\psi_\# f_0 = \int_{\mathcal{S}_c(X)} \psi_\# f_{0C}\, d\nu(C) = \int_{\mathcal{S}_c(X)} f_{1C}\, d\nu(C) = f_1.$$

Finally ψ is an optimal transport because (45) holds and any ψ_C is an optimal transport.
Step 3. In this step we show how the assumption (41) can be removed. We define

$$X' := \{x \in X :\ (x,x) \in \mathrm{spt}\gamma \text{ and } (x,y) \notin \mathrm{spt}\gamma\ \forall y \neq x\}$$

and $L = \{(x,x) : \ x \in X'\}$. Then, we set $f_0' = f_0 \llcorner X'$ and $f_0'' = f_0 \llcorner X''$, with $X'' = X \setminus X'$, and

$$f_1' := \pi_{1\#}(\gamma \llcorner L), \qquad f_1'' := f_1 - f_1'.$$

Since $L \subset \Delta$, we have $f_1' = \pi_{0\#}(\gamma \llcorner L) = f_0'$, hence we can choose on X' the map $\psi_1 = Id$ as transport map to obtain $(\psi_1)_\# f_0' = f_1'$. Since f_0'' is concentrated on X'' the condition (41) is satisfied with f_0'' in place of f_0 and we can find an optimal transport map $\psi_2 : X'' \to X$ such that $(\psi_2)_\# f_0'' = f_1''$. Gluing these two transport maps we obtain a transport map ψ such that $\psi_\# f_0 = f_1$; since, by construction,

$$u(x) - u(\psi(x)) = |x - \psi(x)| \qquad f_0\text{-a.e. in } X$$

we infer that ψ is optimal. $\square$

This proof strongly depends on the strict convexity of the euclidean distance, which provides the first and second order differentiability properties of the potential u on the transport set $\mathcal{T}_u$. Notice also that if $c(x,y) = \|x-y\|$ and the norm $\|\cdot\|$ is not strictly convex, then the "transport rays" need not be one-dimensional and, to our knowledge, the existence of an optimal transport map is an open problem in this situation. Indeed, this existence result is stated by Sudakov in [41] but his proof is faulty, for the reasons outlined in Remark 6.1(1).

Under special assumptions on the data f_0, f_1 (absolute continuity, separated supports, Lipschitz densities) Evans and Gangbo provided in [22] a proof based on differential methods of the existence of optimal transport maps. For stricly convex norms, the first fully rigorous proofs of the existence of an optimal transport map under the only assumption that $f_0 << \mathcal{L}^n$ have been given in [15] and [42]. As in Theorem 6.2 the proof is strongly based on the differentiability of the directions of transport rays.

7 Regularity and uniqueness of the transport density

In this section we investigate the regularity and the uniqueness properties of the transport density μ arising in (PDE). Recall that, as Theorem 5.1 shows, any such measure μ can also be represented as $\int_0^1 |E_t|\,dt$ for a suitable optimal pair (f_t, E_t) for (ODE) (even with E_t independent of t). In turn, by Corollary 4.1, any optimal measure $\mu = \int_0^1 |E_t|\,dt$ can be represented as

$$\mu = \int_0^1 \pi_{t\#}(|y-x|\gamma)\,dt \tag{47}$$

for a suitable optimal planning γ. For this reason, in the following we restrict our attention to the representation (47), valid for any optimal measure μ for (PDE).

A more manageable formula for μ, first considered by G.Bouchitté and G.Buttazzo in [14], is given in the following elementary lemma.

Lemma 7.1 *Let μ be as in (47). Then*

$$\mu(B) = \int_{X\times X} \mathrm{Length}\,(]\!]x,y[\![\cap B)\ d\gamma(x,y) \qquad \forall B \in \mathcal{B}(X). \tag{48}$$

Proof. For any Borel set B we have

$$\begin{aligned}
\mu(B) &= \int_0^1 \int_{\pi_t^{-1}(B)} |y-x|\,d\gamma(x,y)\,dt = \int_{X\times X} |y-x| \int_0^1 \chi_{\pi_t^{-1}(B)}\,dt\,d\gamma(x,y) \\
&= \int_{X\times X} \mathrm{Length}\,(]\!]x,y[\![\cap B)\ d\gamma(x,y).
\end{aligned}$$

□

A direct consequence of (48) is that μ is concentrated on the transport set (since γ-a.e. segment $]\!]x,y[\![$ is contained in a transport ray); another consequence is the density estimate

$$\frac{\mu(B_r(x))}{2r} \le f_0(X) f_1(X) \tag{49}$$

because $\mathrm{Length}(]\!]x,y[\![\cap B_r(x)) \le 2r$ for any ball $B_r(x)$ and any segment $]\!]x,y[\![$. The first results on μ that we state, proved by A.Pratelli in [33] (see also [19]), relate the dimensions of f_0 and f_1 to the dimension of μ and show that necessarily μ is absolutely continuous with respect to the Lebesgue measure if f_0 (or, by symmetry, f_1) has this property.

Theorem 7.1 *Assume that for some $k > 0$ we have*

$$\sup_{r\in(0,1)} \frac{f_0(B_r(x))}{r^k} < \infty \quad f_0\text{-a.e. in } X.$$

Then μ has the same property. In particular $\mu << \mathcal{L}^n$ if $f_0 << \mathcal{L}^n$.

The proof of Theorem 7.1 is based on (48); the density estimate on μ is achieved by a careful analysis of the transport rays crossing a generic ball $B_r(x)$. A similar analysis proved the following summability estimate (see [19]).

Theorem 7.2 *Assume that $f_0 = g_0\mathcal{L}^n$ and $f_1 = g_1\mathcal{L}^n$ with $g_0,\, g_1 \in L^p(X)$, $p > 1$. Then $\mu = h\mathcal{L}^n$ with $h \in L^\infty(X)$ if $p = \infty$, $h \in L^q(X)$ for any $q < p$ if $p < \infty$.*

It is not known whether $g_0,\, g_1 \in L^p$ implies $h \in L^p$ for $p < \infty$.

Definition 7.1 (Hausdorff dimension of a measure) *Let $\mu \in \mathcal{M}_+(X)$. The Hausdorff dimension $\mathcal{H}\text{-dim}(\mu)$ is the supremum of all $k \geq 0$ such that $\mu << \mathcal{H}^k$.*

In other words $\mu(B) = 0$ whenever $\mathcal{H}^k(B) = 0$ for some $k < \mathcal{H}\text{-dim}(\mu)$ and for any $k > \mathcal{H}\text{-dim}(\mu)$ there exists a Borel set B with $\mu(B) > 0$ and $\mathcal{H}^k(B) = 0$. Notice that if μ is made of pieces of different dimensions, then $\mathcal{H}\text{-dim}(\mu)$ is the smallest of these dimensions.

Using the density estimates and the implications (see for instance Theorem 2.56 of [4] or [37]; here $t > 0$ and $k > 0$)

$$\limsup_{r\to 0^+} \frac{\mu(B_r(x))}{\omega_k r^k} \geq t \ \ \forall x \in B \quad \Longrightarrow \quad \mu(B) \geq t\,\mathcal{H}^k(B) \tag{50}$$

$$\limsup_{r\to 0^+} \frac{\mu(B_r(x))}{\omega_k r^k} \leq t \ \ \forall x \in B \quad \Longrightarrow \quad \mu(B) \leq 2^k t\,\mathcal{H}^k(B) \tag{51}$$

we can prove a natural lower bound on the Hausdorff dimension of μ.

Corollary 7.1 *Let μ be a transport density. Then*

$$\mathcal{H}\text{-dim}(\mu) \geq \max\left\{1, \mathcal{H}\text{-dim}(f_0), \mathcal{H}\text{-dim}(f_1)\right\}.$$

Proof. By (49) we infer that μ has finite (and even bounded) 1-dimensional density at any point. In particular (51) gives $\mu(B)=0$ whenever $\mathcal{H}^1(B)=0$, so that $\mathcal{H}\text{-dim}(\mu)\geq 1$.

Let $k=\mathcal{H}\text{-dim}(f_0)$ and $k'<k$; then f_0 has finite k'-dimensional density f_0-a.e., otherwise by (50) the set where the density is not finite would be $\mathcal{H}^{k'}$-negligible and with strictly positive f_0-measure. By Theorem 7.1 we infer that μ has finite k'-dimensional density μ-a.e. By the same argument used before with $k'=1$ we obtain that $\mathcal{H}\text{-dim}(\mu)\geq k'$ and therefore, since $k'<k$ is arbitrary, $\mathcal{H}\text{-dim}(\mu)\geq k$. A symmetric argument proves that $\mathcal{H}\text{-dim}(\mu)\geq\mathcal{H}\text{-dim}(f_1)$. $\square$

We conclude this section proving the uniqueness of the transport density, under the assumption that either $f_0<<\mathcal{L}^n$ or $f_1<<\mathcal{L}^n$ (see Example 5.1 for a nonuniqueness example if neither f_0 nor f_1 are absolutely continuous). Similar results have been first announced by Feldman and McCann (see [25]).

We first deal with the one dimensional case, where this absolute continuity assumption is not needed.

Lemma 7.2 *Let μ be a transport density in $X\subset\mathbb{R}$ and assume that the interior (a,b) of X is a transport ray. Then $\mu=h\mathcal{L}^1$ in (a,b) with*

$$h(t)=\sigma f\left((a,t)\right)+c\quad \mathcal{L}^1\text{-a.e. in } (a,b)$$

for some constant c, where $f=f_1-f_0$ and $\sigma=1$ if $u'=1$ in (a,b), $\sigma=-1$ if $u'=-1$ in (a,b). Moreover

$$c=\frac{\mathcal{F}_1(f_0,f_1)-\sigma\int_{(a,b)}(b-t)\,df(t)}{b-a}.$$

Proof. The proof of the first statement is based on the equation $\mu'=\sigma f$ and on a smoothing argument. By integrating both sides we get

$$\begin{aligned} c(b-a) &= \mu(X)-\sigma\int_a^b f\left((a,t)\right)\,dt=\mu(X)-\sigma\int_a^b\int_{(a,t)}1\,df(\tau)\,dt\\ &= \mu(X)-\sigma\int_{(a,b)}\int_\tau^b 1\,dt\,df(\tau)=\mu(X)-\sigma\int_{(a,b)}(b-\tau)\,df(\tau). \end{aligned}$$

The conclusion is achieved taking into account that $\mu(X)=\min(\mathrm{MK})=\mathcal{F}_1(f_0,f_1)$. $\square$

Theorem 7.3 (Uniqueness) *Assume that either f_0 or f_1 are absolutely continuous with respect to $\mathcal{L}^n$. Then the transport density is absolutely continuous and unique.*

Proof. We already know from Theorem 7.1 that any transport density is absolutely continuous and we can assume that $f_0 << \mathcal{L}^n$. By Corollary 4.1 we know that the class of transport densities relative to (f_0, f_1) is equal to the class of transport densities relative to $(f_0 - h, f_1 - h)$ where h is any measure in $\mathcal{M}_+(X)$ such that $h \leq f_0 \wedge f_1$ (indeed, this subtraction does not change the velocity field E_t). Hence, it is not restrictive to assume that $f_0 \wedge f_1 = 0$.

We can assume that μ is representable as in (30) for a suitable optimal planning γ. Adopting the same notation of the proof of Theorem 6.1, we have

$$\gamma = \gamma_C \otimes \nu$$

where $\nu = \pi_\# f_0$ does not depend on γ (and here the absolute continuity assumption on f_0 plays a crucial role) and γ_C is an optimal planning relative to the measures $f_{0C} = \pi_{0\#}\gamma_C$, $f_{1C} = \pi_{1\#}\gamma_C$ for ν-a.e. C. In particular

$$\mu = \int_0^1 \pi_{t\#}\left(|y-x|\gamma_C \otimes \nu\right)\, dt = \int_0^1 \pi_{t\#}(|y-x|\gamma_C)\, dt \otimes \nu.$$

Hence, it suffices to show that the measures

$$\mu_C := \int_0^1 \pi_{t\#}(|y-x|\gamma_C)\, dt$$

do not depend on γ (up to ν-negligible sets of course). To this aim, taking into account Lemma 7.2, it suffices to show that f_{0C} and f_{1C} do not depend on γ.

Indeed, we already know from (46) and Theorem 9.2 that f_{0C} do not depend on γ.

The argument for f_{1C} is more involved. First, since $\gamma(\Delta) = 0$ (due to the assumption $f_0 \wedge f_1 = 0$), we have $|x-y| > 0$ γ-a.e., and therefore $|x-y| > 0$ γ_C-a.e. for ν-a.e. $C \in \mathcal{S}_c(X)$. As a consequence, setting $C = [\![x_C, y_C]\!]$, we obtain that $f_{1C}(x_C) = 0$ for ν-a.e. C. Second, we examine the restriction f'_{1C} of f_{1C} to the relative interior of C noticing that (44) gives

$$f_1 \llcorner \mathcal{T}_u = \int_{\mathcal{S}_c(X)} f_{1C} \llcorner \mathcal{T}_u \, d\nu(C) = \int_{\mathcal{S}_c(X)} f'_{1C}\, d\nu(C)$$

because $\mathcal{T}_u \cap C$ is the relative interior of C for any $C \in \pi(\tilde{X})$. By Theorem 9.2 we obtain that f'_{1C} depend only on f_1, $\mathcal{T}_u$ and ν.

In conclusion, since

$$f_{1C} = f'_{1C} + (1 - f'_{1C}(X))\,\delta_{y_C}$$

we obtain that f_{1C} do not depend on γ as well. $\square$

8 The Bouchitté–Buttazzo mass optimization problem

In [13, 14] Bouchitté and Buttazzo consider the following problem. Given $f \in \mathcal{M}(X)$ with $f(X) = 0$, they define

$$\mathcal{E}(\mu) := \inf\left\{\int_X \frac{1}{2}|\nabla v|^2\, d\mu - f(v) :\ v \in C^\infty(X)\right\}$$

for any $\mu \in \mathcal{M}_+(X)$. Then, they raised the following *mass optimization problem*.

(BB) Given $m > 0$, maximize $\mathcal{E}(\mu)$ among all measures $\mu \in \mathcal{M}_+(X)$ with $\mu(X) = m$.

A possible physical interpretation of this problem is the following: we may imagine that μ represents the conductivity of some material, thinking that the conductivity (i.e. the inverse resistivity) is zero out of sptμ; accordingly we may imagine that $f = f^+ - f^-$ is a balanced density of positive and negative charges. Then $-\mathcal{E}(\mu)$ represents the heating corresponding to the given conductivity, so that there is an obvious interest in maximizing $\mathcal{E}(\mu)$ and, for a minimizer u, the (formal) first variation of the energy

$$-\nabla \cdot (\nabla u\, \mu) = f^+ - f^-$$

corresponds to Ohm's law.

More generally, problems of this sort appear in Shape Optimization and Linear Elasticity. In these cases u is no longer real valued and general energies of the form

$$\mathcal{F}(u) = \int_X f\left(x, \nabla u(x)\right)\, d\mu(x) \tag{52}$$

must be considered in order to take into account *light structures*, corresponding to mass distributions not absolutely continuous with respect to $\mathcal{L}^n$. This was the main motivation for Bouchitté, Buttazzo and Seppecher [12] in their development of a general differential calculus with measures. This calculus, based on a suitable concept of tangent space to a measure, enables the study of the relaxation of functionals (52) and provides an explicit formula for their lower semicontinuous envelope.

Coming back to the scalar problem (BB), a remarkable fact discovered in [13, 14] is its connection with the Monge–Kantorovich optimal transport problem, in the case when $c(x,y) = |x-y|$. It turns out that the solutions of (BB) are in one to one correspondence with constant multiples of transport densities for (PDE) (or, equivalently, for (MK) with $f_0 = f^+$ and $f_1 = f^-$).

As a byproduct we have that $\mu << \mathcal{L}^n$ if either f_0 or f_1 are absolutely continuous with respect to $\mathcal{L}^n$ and μ is unique if both f_0 and f_1 are absolutely continuous with respect to $\mathcal{L}^n$.

Theorem 8.1 ((PDE) versus (BB)) *Let (μ, u) be a solution of (PDE) and set $\tilde{m} = \mu(X)$. Then $\tilde{\mu} = \frac{m}{\tilde{m}}\mu$ solves (BB) and any solution of (BB) is representable in this way. Moreover*

$$\max(BB) = -\frac{1}{2}\frac{\tilde{m}^2}{m}.$$

Proof. Setting $v = \lambda u_h$ (with u_h as in the definition of (PDE)), for any $\nu \in \mathcal{M}_+(X)$ with $\nu(X) = m$ we estimate

$$\mathcal{E}(\nu) \leq \frac{m}{2}\lambda^2 - \lambda f(u_h)$$

so that, letting $h \to \infty$ and taking into account (39), we obtain

$$\mathcal{E}(\nu) \leq \frac{m}{2}\lambda^2 - \tilde{m}\lambda.$$

By minimizing with respect to λ we obtain $\mathcal{E}(\nu) \leq -\tilde{m}^2/(2m)$.

On the other hand, using Young inequality and choosing λ so that $\lambda\tilde{m} = m$ we can estimate

$$\begin{aligned}
\int_X \frac{1}{2}|\nabla v|^2\, d\tilde{\mu} - f(v) &\geq \int_X \langle \nabla v, \lambda \nabla_\mu u\rangle\, d\tilde{\mu} - \lambda^2 \frac{m}{2} - f(v) \\
&= -\nabla\cdot(\nabla_\mu u\, \mu)(v) - f(v) - \frac{1}{2}\frac{\tilde{m}^2}{m} \\
&= -\frac{1}{2}\frac{\tilde{m}^2}{m}
\end{aligned}$$

for any $v \in C^\infty(X)$. This proves that $\tilde{\mu}$ is optimal for (BB).

Conversely, it has been proved in [13, 14] that for any solution σ of (BB) there exists $v \in \mathrm{Lip}(X)$ such that $|\nabla_\sigma v| = m/\tilde{m}$ σ-a.e. and

$$-\nabla\cdot(\nabla_\sigma v\sigma) = f,$$

where $\nabla_\sigma v$ is understood in the Bouchitté–Buttazzo sense. But since (see [14] again)

$$\int_X |\nabla_\sigma v|^2\, d\sigma = \inf\left\{\liminf_{h\to\infty}\int_X |\nabla v_h|^2\, d\sigma :\ v_h \to v \text{ uniformly},\ v_h \in C^\infty(X)\right\}$$

we obtain a sequence of smooth functions v_h uniformly converging to v such that $\nabla v_h \to \nabla_\sigma v$ in $[L^2(\sigma)]^n$. Setting $u = \frac{\tilde{m}}{m} v$ and $\mu = \frac{m}{\tilde{m}}\sigma$, this proves that (σ, u) solves (PDE). $\square$

9 Appendix: some measure theoretic results

In this section we list all the measure theoretic results used in the previous section; reference books for the content of this section are [36], [18] and [4].

Let us begin with some terminology.

• **(Measures)** Let X be a locally compact and separable metric space. We denote by $[\mathcal{M}(X)]^m$ the space of Radon measures with values in $\mathbb{R}^m$ and with finite total variation in X. We recall that the total variation measure of $\mu = (\mu_1, \ldots, \mu_m) \in [\mathcal{M}(X)]^m$ is defined by

$$|\mu|(B) := \sup\left\{\sum_{i=1}^\infty |\mu(B_i)| :\ B = \bigsqcup_{i=1}^\infty B_i,\ B_i \in \mathcal{B}(X)\right\}$$

and belongs to $\mathcal{M}_+(X)$. By Riesz theorem the space $[\mathcal{M}(X)]^m$ endowed with the norm $\|\mu\| = |\mu|(X)$ is isometric to the dual of $[C(X)]^m$. The duality is given by the integral, i.e.

$$\langle \mu, u \rangle := \sum_{i=1}^{m} \int_X u_i \, d\mu_i.$$

Recall also that, for $\mu \in \mathcal{M}_+(X)$ and $f \in [L^1(X,\mu)]^m$, the measure $f\mu \in [\mathcal{M}(X)]^m$ is defined by

$$f\mu(B) := \int_B f \, d\mu \qquad\qquad \forall B \in \mathcal{B}(X)$$

and $|f\mu| = |f|\mu$.

• **(Push forward of measures)** Let $\mu \in [\mathcal{M}(X)]^m$, let Y be another metric space and let $f : X \to Y$ be a Borel map. Then the push forward measure $f_\#\mu \in [\mathcal{M}(Y)]^m$ is defined by

$$f_\#\mu(B) := \mu\left(f^{-1}(B)\right) \qquad \forall B \in \mathcal{B}(Y)$$

and satisfies the more general property

$$\int_Y u \, df_\#\mu = \int_X u \circ f \, d\mu \qquad \text{for any bounded Borel function } u : Y \to \mathbb{R}.$$

It is easy to check that $|f_\#\mu| \le f_\#|\mu|$.

• **(Support)** We say that $\mu \in [\mathcal{M}(X)]^m$ is concentrated on a Borel set B if $|\mu|(X \setminus B) = 0$ and we denote by $\mathrm{spt}\mu$ the smallest closed set on which μ is concentrated (the existence of a smallest set follows by the separability of X, precisely by the Lindelöf property). The support is also given by the formula (sometimes taken as the definition)

$$\mathrm{spt}\mu = \{x \in X : \ |\mu|(B_\varrho(x)) > 0 \ \forall \varrho > 0\}.$$

• **(Convolution)** If $X \subset \mathbb{R}^n$, $\mu \in [\mathcal{M}(X)]^m$ and $\rho \in C_c^\infty(\mathbb{R}^n)$ (for instance a convolution kernel); we define

$$\mu * \rho(x) := \int_X \rho(x-y) \, d\mu(y) \qquad\qquad x \in \mathbb{R}^n.$$

Notice that $\mu * \rho \in [C^\infty(\mathbb{R}^n)]^m$ because $D^\alpha(\mu * \rho) = (D^\alpha \rho) * \mu$ for any multiindex α and

$$\sup |D^\alpha(\mu * \rho)| \leq \sup |D^\alpha \rho| \|\mu\|.$$

Moreover, using Jensen's inequality it is easy to check (see for instance Theorem 2.2(ii) of [4]) that

$$\int_C |\mu * \rho| \, dx \leq |\mu|(C_\rho) \quad \text{where} \quad C_\rho := \{x : \ \mathrm{dist}(x, C) \leq \mathrm{diam}(\mathrm{spt}\rho)\} \tag{53}$$

for any closed set $C \subset \mathbb{R}^n$.

• **(Weak convergence)** Assume that X is a compact metric space. We say that a family of measures $(\mu_h) \subset \mathcal{M}(X)$ weakly converges to $\mu \in \mathcal{M}(X)$ if

$$\lim_{h \to \infty} \int_X u \, d\mu_h = \int_X u \, d\mu \qquad \forall u \in C(X)$$

i.e., if μ_h weakly* converge to μ as elements of the dual of $C(X)$. In the case X is locally compact and separable, the concept is analogous, simply replacing $C(X)$ the the subspace $C_c(X)$ of functions with compact support. It is easy to check that $\mu * \rho_\varepsilon \mathcal{L}^n$ weakly converge to μ in $\mathbb{R}^n$ whenever ρ is a convolution kernel.

We mention also the following criterion for the weak convergence of positive measures (see for instance Proposition 1.80 of [4]): if μ_h, $\mu \in \mathcal{M}_+(X)$ satisfy

$$\liminf_{h \to \infty} \mu_h(A) \geq \mu(A) \quad \forall A \subset X \text{ open}$$

and

$$\limsup_{h \to \infty} \mu_h(X) \leq \mu(X)$$

then $\int_X \phi \, d\mu_h \to \int_X \phi \, d\mu$ for any bounded continuous function $\phi : X \to \mathbb{R}$.

• **(Measure valued maps)** Let X, Y be locally compact and separable metric spaces. Let $y \mapsto \lambda_y$ be a map which assigns to any $y \in Y$ a measure $\lambda_y \in [\mathcal{M}(X)]^m$. We say that λ_y is a Borel map if $y \mapsto \lambda_y(A)$ is a real valued Borel map for any open set $A \subset X$. By a monotone class argument it can be proved that $y \mapsto \lambda_y$ is a Borel map if and only if

$$y \mapsto \lambda_y \left(\{x : \ (y, x) \in B\}\right) \quad \text{is a Borel map for any } B \in \mathcal{B}(Y \times X). \tag{54}$$

Moreover $y \mapsto |\lambda_y|$ is a Borel map whenever $y \mapsto \lambda_y$ is Borel (detailed proofs are in §2.5 of [4]). If $\mu \in \mathcal{M}_+(Y)$, analogous statements hold for

$\mathcal{B}(Y)_\mu$-measurable measure valued maps, where $\mathcal{B}(Y)_\mu$ is the σ-algebra of μ-measurable sets (in this case one has to replace $\mathcal{B}(Y \times X)$ by $\mathcal{B}(Y)_\mu \otimes \mathcal{B}(X)$ in (54)).

• **(Decomposition of measures)** The following result plays a fundamental role in these notes; it is also known as disintegration theorem.

Theorem 9.1 (Decomposition of measures) *Let X, Y be locally compact and separable metric spaces and let $\pi : X \to Y$ be a Borel map. Let $\lambda \in [\mathcal{M}(X)]^m$ and set $\mu = \pi_\# |\lambda| \in \mathcal{M}_+(Y)$. Then there exist measures $\lambda_y \in [\mathcal{M}(X)]^m$ such that*

(i) $y \mapsto \lambda_y$ is a Borel map and $|\lambda_y|$ is a probability measure in X for μ-a.e. $y \in Y$;

(ii) $\lambda = \lambda_y \otimes \mu$, i.e.

$$\lambda(A) = \int_Y \lambda_y(A)\, d\mu(y) \qquad \forall A \in \mathcal{B}(X); \tag{55}$$

(iii) $|\lambda_y| \left(X \setminus \pi^{-1}(y)\right) = 0$ for μ-a.e. $y \in Y$.

The representation provided by Theorem 9.1 of λ can be used sometimes to compute the push forward of λ. Indeed,

$$f_\#(\lambda_y \otimes \mu) = f_\# \lambda_y \otimes \mu \tag{56}$$

for any Borel map $f : X \to Z$, where Z is any other compact metric space. Notice also that

$$|\lambda| = |\lambda_y| \otimes \mu. \tag{57}$$

Indeed, the inequality $\le$ is trivial and the opposite one follows by evaluating both measures at $B = X$, using the fact that $|\lambda_y|(X) = 1$ for μ-a.e. y.

In the case when $m = 1$ and $\lambda \in \mathcal{M}_+(X)$ the proof of Theorem 9.1 is available in many textbooks of measure theory or probability (in this case λ_y are the the so-called conditional probabilities induced by the random variable π, see for instance [18]); in the vector valued case one can argue component by component, but the fact that $|\lambda_y|$ are probability measures is not straightforward.

In Theorem 2.28 of [4] the decomposition theorem is proved in the case when $X = Y \times Z$ is a product space and $\pi(y, z) = y$ is the projection on the first variable; in this situation, since λ_y are concentrated on $\pi^{-1}(y) = \{y\} \times Z$, it is sometimes convenient to consider them as measures on Z, rather than measures on X, writing (55) in the form

$$\lambda(B) = \int_Y \lambda_y \left(\{z : \ (y, z) \in B\}\right) \, d\mu(y) \qquad \forall B \in \mathcal{B}(X). \tag{58}$$

Once the decomposition theorem is known in the special case $X = Y \times Z$ and $\pi(y, z) = z$ the general case can be easily recovered: it suffices to embed X into the product $Y \times X$ through the map $f(x) = (\pi(x), x)$ and to apply the decomposition theorem to $\tilde{\lambda} = f_\#\lambda$.

Now we discuss the uniqueness of λ_x and μ in the representation $\lambda = \lambda_x \otimes \mu$. For simplicity we discuss only the case of positive measures.

Theorem 9.2 *Let X, Y and π be as in Theorem 9.1; let $\lambda \in \mathcal{M}_+(X)$, $\mu \in \mathcal{M}_+(Y)$ and let $y \mapsto \eta_y$ be a Borel $\mathcal{M}_+(X)$-valued map defined on Y such that*

(i) $\lambda = \eta_y \otimes \mu$, i.e. $\lambda(A) = \int_Y \eta_y(A) \, d\mu(y)$ for any $A \in \mathcal{B}(X)$;

(ii) $\eta_y(X \setminus \pi^{-1}(y)) = 0$ for μ-a.e. $y \in Y$.

Then η_y are uniquely determined μ-a.e. in Y by (i), (ii) and, setting $B = \{y : \ \eta_y(X) > 0\}$, the measure $\mu \llcorner B$ is absolutely continuous with respect to $\pi_\#\lambda$. In particular

$$\frac{\mu \llcorner B}{\pi_\#\lambda} \eta_y = \lambda_y \qquad \text{for } \pi_\#\lambda\text{-a.e. } y \in Y \tag{59}$$

where λ_y are as in Theorem 9.1.

Proof. Let η_y, η'_y be satisfying (i), (ii). We have to show that $\eta_y = \eta'_y$ for μ-a.e. y. Let (A_n) be a sequence of open sets stable by finite intersection which generates the Borel σ-algebra of X. Choosing $A = A_n \cap \pi^{-1}(B)$, with $B \in \mathcal{B}(Y)$, in (i) gives

$$\int_B \eta_y(A_n) \, d\mu(y) = \int_B \eta'_y(A_n) \, d\mu(y).$$

Being B arbitrary, we infer that $\eta_y(A_n) = \eta'_y(A_n)$ for μ-a.e. y, and therefore there exists a μ-negligible set N such that $\eta_y(A_n) = \eta'_y(A_n)$ for any $n \in \mathbb{N}$ and any $y \in Y \setminus N$. By Proposition 1.8 of [4] we obtain that $\eta_y = \eta'_y$ for any $y \in Y \setminus N$.

Let $B' \subset B$ be any $\pi_\#\lambda$-negligible set; then $\pi^{-1}(B')$ is λ-negligible and therefore (ii) gives

$$0 = \int_Y \eta_y\left(\pi^{-1}(B')\right) d\mu(y) = \int_{B'} \eta_y(X)\, d\mu(y).$$

As $\eta_y(X) > 0$ on $B \supset B'$ this implies that $\mu(B') = 0$. Writing $\mu \llcorner B = h\pi_\#\lambda$ we obtain $\lambda = h\eta_y \otimes \pi_\#\lambda$ and $\lambda = \lambda_y \otimes \pi_\#\lambda$. As a consequence (59) holds. $\square$

• **(Young measures)** These measures, introduced by L.C.Young, arise in a natural way in the study of oscillatory phenomena and in the analysis of weak limits of nonlinear quantities (see [30] for a comprehensive introduction to this wide topic).

Specifically, assume that we are given compact metric spaces X, Y and a sequence of Borel maps $\psi_h : X \to Y$; in order to understand the limit behaviour of ψ_h we associate to them the measures

$$\gamma_{\psi_h} = (Id \times \psi_h)_\#\mu = \int \delta_{\psi_h(x)}\, d\mu(x)$$

and we study their limit in $\mathcal{M}(X \times Y)$. Assuming, possibly passing to a subsequence, that $\gamma_{\psi_h} \to \gamma$, due to the fact that $\pi_{0\#}(\gamma_{\psi_h}) = \mu$ for any h we obtain that $\pi_{0\#}\gamma = \mu$, hence according to Theorem 9.1 we can represent γ as

$$\gamma = \gamma_x \otimes \mu$$

for suitable probability measures γ_x in Y, with $x \mapsto \gamma_x$ Borel. The family of measures γ_x is called *Young limit* of the sequence (ψ_h); once the Young limit is known, we can compute the w^*-limit of $\gamma(\psi_h)$ in $L^\infty(X, \mu)$ for *any* $\gamma \in C(Y)$: indeed, using test function of the form $\phi(x)\gamma(y)$, we easily obtain that the limit (in the dual of $C(X)$ and therefore in $L^\infty(X, \mu)$) is given by

$$L_\gamma(x) := \int_Y \gamma(y)\, d\gamma_x(y).$$

We will use the following two well known results of the theory of Young measures.

Theorem 9.3 (Approximation theorem) *Let $\gamma \in \mathcal{M}_+(X \times Y)$ and write $\gamma = \gamma_x \otimes \mu$ with $\mu = \pi_{0\#}\gamma$. Then, if μ has no atom, there exists a sequence of Borel maps $\psi_h : X \to Y$ such that*

$$\gamma_x \otimes \mu = \lim_{h\to\infty} \delta_{\psi_h(x)} \otimes \mu.$$

Moreover, we can choose ψ_h in such a way that the measures $\psi_{h\#}\mu$ have no atom as well.

Proof. (Sketch) We assume first that $\gamma_x = \gamma$ is independent of x. By approximation we can also assume that

$$\gamma = \sum_{i=1}^{p} p_i \delta_{y_i}$$

for suitable $y_i \in Y$ and $p_i \in [0,1]$. Let Q_h, $h \in \mathbb{Z}^n$, be a partition of $\mathbb{R}^n$ in cubes with side length $1/h$; since μ has no atom, by Lyapunov theorem we can find a partition $X_h^1, \dots, X_h^p$ of $X \cap Q_h$ such that $\mu(X_h^i) = p_i \mu(X \cap Q_h)$. Then we define

$$\psi_h = p_i \quad \text{on } X_h^i.$$

For any $\phi \in C(X)$ and $\varphi \in C(Y)$ we have

$$\begin{aligned}
\int_{X\times Y} \phi\varphi \, d(\delta_{\psi_h} \otimes \mu) &= \sum_{h\in\mathbb{Z}^n} \sum_{i=1}^{p} \varphi(y_i) \int_{X_h^i} \phi \, d\mu \\
&\sim \sum_{h\in\mathbb{Z}^n} \sum_{i=1}^{p} p_i \varphi(y_i) \int_{X\cap Q_h} \phi \, d\mu = \int_{X\times Y} \phi\varphi \, d\mu \times \gamma
\end{aligned}$$

and this proves that

$$\mu \times \gamma = \lim_{h\to\infty} \delta_{\psi_h} \otimes \mu.$$

If we want $\psi_{h\#}\mu$ to be non atomic, the above construction needs to be modified only slightly: it suffices to take small balls B_i centered at y_i and to define ψ_h equal to ϕ_i on X_h^i, where $\phi_i : \mathbb{R}^n \to B_i$ is any Borel and one to one map.

If γ_x is piecewise constant (say in a canonical subdivision of X induced by a partition in cubes) then we can repeat the local construction above in each region where γ_x is constant. Moreover we can approximate Lipschitz functions γ_x (with respect to the 1-Wasserstein distance) with piecewise constant ones. Finally, any Borel map γ_x can be approximated by Lipschitz ones through a convolution. $\square$

We will also need the following result.

Lemma 9.1 *Let ψ_h, $\psi : X \to Y$ be Borel maps and $\mu \in \mathcal{M}_1(X)$. Then $\psi_h \to \psi$ μ-a.e. if and only if*

$$\gamma_h := \delta_{\psi_h(x)} \otimes \mu(x) \to \gamma := \delta_{\psi(x)} \otimes \mu(x)$$

weakly in $\mathcal{M}(X \times Y)$.

Proof. Assume that $\psi_h \to \psi$ μ-a.e. Since

$$\int_{X\times Y} \varphi \, d\gamma_\varphi = \int_X \varphi(x, \phi(x)) \, d\mu(x) \qquad \forall \varphi \in C(X \times Y)$$

the dominated convergence theorem gives that $\gamma_h \to \gamma$. To prove the opposite implication, fix $\varepsilon > 0$ and a compact set K such that $\psi|_K$ is continuous and $\mu(X \setminus K) < \varepsilon$. We use as test function

$$\varphi(x, y) = \chi_K(x)\gamma\left(y - \psi(x)\right)$$

with $\gamma(t) = 1 \wedge |t|/\varepsilon$ (by approximation, although not continuous in $X \times Y$, this is an admissible test function) to obtain

$$\lim_{h\to\infty} \mu\left(\{x \in K : \ |\psi_h(x) - \psi(x)| \geq \epsilon\}\right) = 0.$$

The conclusion follows letting $\varepsilon \to 0^+$. $\square$

With a similar proof one can obtain a slightly more general result, namely

$$\gamma_h := \delta_{\psi_h(x)} \otimes \mu_h \to \gamma := \delta_{\psi(x)} \otimes \mu.$$

implies $\psi_h \to \psi$ μ-a.e. provided $|\mu_h - \mu|(X) \to 0$.

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