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Research Article

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Geodesics in Asymmetric Metric Spaces

Abstract: In a recent paper [17] we studied *asymmetric metric spaces*; in this context we studied the length of paths, introduced the class of *run-continuous* paths; and noted that there are different definitions of "*length spaces*" (also known as "*path-metric spaces*" or "*intrinsic spaces*"). In this paper we continue the analysis of asymmetric metric spaces. We propose possible definitions of *completeness* and *(local) compactness*. We define the *geodesics* using as admissible paths the class of *run-continuous* paths. We define *midpoints*, *convexity*, and *quasi–midpoints*, but without assuming the space be intrinsic. We distinguish all along those results that need a stronger separation hypothesis. Eventually we discuss how the newly developed theory impacts the most important results, such as the existence of geodesics, and the renowned Hopf–Rinow (or Cohn-Vossen) theorem.

Keywords: asymmetric metric, general metric, quasi metric, ostensible metric, Finsler metric, path metric, length space, geodesic curve, Hopf–Rinow theorem

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1 Introduction

We continue here the analysis of asymmetric metric spaces proposed in [17]. To keep this paper as self contained as possible we will summarize the main definitions of [17] in Section 2. We now start with a few definitions and an informal discussion.

Let M be a non empty set.

Definition 1.1. $b : M \times M \to [0, \infty]$ is an *asymmetric distance* if

- $\forall x \in M$, $b(x, x) = 0$;
- $\forall x, y \in M$, $b(x, y) = b(y, x) = 0$ implies $x = y$,
- $\forall x, y, z \in M$, $b(x, z) \le b(x, y) + b(y, z)$.

The second condition implies that the associated topology (that is defined in Sec. 2) is T_2, so we will call it *separation hypothesis*. The third condition is usually called the *triangle inequality*. If the second condition does not hold, then b is an *asymmetric semidistance*. (A semidistance is also called a "pseudometric".)

We call the pair (M, b) an *asymmetric metric space*.

The setting presented here and in [17] is similar to the approach of Busemann, see *e.g.* [4–6]; it is also similar to the metric part of Finsler Geometry, as presented in [2]. Differences were discussed in the Appendix of [17], and are further highlighted in Appendix A.2 of this paper.

A different point of view is found in the theory of *quasi metrics* (or *ostensible metrics*); the main difference is that in this presentation there is only one topology associated to the space, whereas a quasi-metric

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is associated to *three different topologies*. This brings forth many different and non-equivalent definitions of *"completeness"* and *"compactness"*. We will compare the two fields in Appendix A.4.

In [17] we introduced three different classes of paths in M; this produced three different definitions of "intrinsic space". The larger class was the class $\mathcal{C}_r$ of "*run-continuous paths*", that are paths $\xi : [a, c] \to M$ such that the length

$$\mathrm{Len}^b \left(\xi|_{[a,t]}\right)$$

of the path ξ restricted to $[a, t]$ is a continuous function of t. (Len^b is the length computed using the total variation formula, see eqn. (2.2)). We then presented in [17] some results regarding length structures and induced distances; those results show that this class $\mathcal{C}_r$ seems more natural in the asymmetric case than the usual class $\mathcal{C}_g$ of *continuous paths*.

We state a stronger version of the second condition in 1.1:

$$\forall x, y \in M, \quad b(x, y) = 0 \Rightarrow x = y \,; \tag{1.1}$$

note that this is the "separation hypothesis" used by Busemann in [4, 6] and Zaustinsky [24], and in [16]. We will call *strongly separated* an asymmetric metric space (M, b) for which (1.1) holds. We will see all along this paper that using the weaker or respectively the stronger separation hypothesis has many effects on the theory; whereas the stronger separation hypothesis was unneeded for the results in [17].

In this paper we will continue the analysis of asymmetric metric spaces. We will propose possible definitions of *completeness* and *(local) compactness*. We will define the *geodesics* using as admissible paths the class of *run-continuous* paths. We will define *midpoints*, *convexity*, and *quasi–midpoints*. Eventually we will discuss some classical topics, such as the existence of geodesics, and the Hopf–Rinow (or Cohn-Vossen) theorem.

1.1 Hopf–Rinow like theorem

We will use the notations and definitions used in the books by Gromov [12], or by Burago & Burago & Ivanov [3]. Note that the authors of [12] and [3] were not the first to discover this kind of result; but the axioms and definitions used in previous works such as [5, 6] were different from what we use here. Note also that a first form of Theorem 1.2 is due to Cohn-Vossen [7], according to the introduction of Busemann's [6]. Consider a symmetric metric space (M, d): we can define the length $\mathrm{Len}^d\, \gamma$ of a continuous path γ using the total variation formula (again, see eqn. (2.2)); then we can define a new metric $d^g(x, y)$ as the infimum of $\mathrm{Len}^d(\gamma)$ in the class of all continuous paths connecting x to y. When $d = d^g$ Gromov defines that the space is *"path-metric"*, or *"intrinsic"*; whereas [3] calls such a space a *"length space"*.

In §2.5.3 in [3] we can then find this result (a smaller version is in §1.11 §1.12 in [12]).

Theorem 1.2 (symmetric Hopf–Rinow or Cohn-Vossen theorem). *Suppose that (M, d) is intrinsic and locally compact; then the following facts are equivalent.*

1. *(M, d) is complete;*
2. *closed bounded sets are compact;*
3. *every geodesic $\gamma : [0, 1) \to M$ can be extended to a continuous path $\overline{\gamma} : [0, 1] \to M$.*

The above is the metric counterpart of the theorem of Hopf–Rinow in Riemannian Geometry: indeed, if (M, g) is a finite-dimensional Riemannian manifold, and d is the associated distance, then (M, d) is path-metric and locally compact.

Since there is a Hopf–Rinow theorem in Finsler Geometry, we would expect that there would be a corresponding theorem for "asymmetric metric spaces". Indeed Busemann proved such a result in its theory of *"General Metric Spaces"* (see *e.g.* Chap. 1 in [6]) for the case of intrinsic and locally compact spaces. (Note that in *"General Metric Spaces"* there is only one notion of "intrinsic", as in the symmetric case).

In the following sections we will state "asymmetric definitions", such as "forward ball", "forward local compactness", "forward completeness", "forward boundedness", (and respectively "backward") and so on.

We have moreover discussed in Sec. 3.6.2 in [17] three different definitions of *"intrinsic"* for the asymmetric case (they are recalled in Definition 2.1 here). Eventually we will prove the desired Hopf–Rinow-like result for *asymmetric metric spaces* in Theorem 12.1.

1.2 Outline of the paper

In this paper we start in Sec. 2 by reviewing the definitions from [17]. In the initial sections we will propose the basic definitions for this paper. In Sec. 3 we will propose possible definitions of *(local) compactness*, and of *completeness* in Sec. 4. We will explore the relations between these notions, keeping parallels with the usual theory of symmetric metric spaces. Sec. 5 contains technical lemmas that the casual reader may want to skip on a first reading. We will then encounter in Sec. 6 *quasi-midpoints*, and show (similarly to the symmetric case) that the existence of *quasi-midpoints* is tightly related to the space being *"r–intrinsic"*. We will define in Sec. 7 *geodesics* as length minimizing paths in the class $\mathcal{C}_r$ of *run-continuous paths*. If the space is compact and "strongly separated" then the run-continuous paths are continuous, *i.e.* $\mathcal{C}_r \equiv \mathcal{C}_g$, so the theories of "continuous geodesics" and "run-continuous geodesics" coincide. In general they do not. We will then note in Sec. 8 that, in spaces that are not strongly separated, the concept of *arc-length reparameterization* needs special care; and in particular that the reparameterization of a continuous rectifiable path may fail to be continuous. (All works fine though in the realm $\mathcal{C}_r$ of *run-continuous* paths.) In Sec. 9 we will show results of existence of geodesics when appropriate container sets are compact (similarly to the classical results); both in the class $\mathcal{C}_r$ and in $\mathcal{C}_g$. In Sec. 10 we will talk of *"convexity"*, define *midpoints* and use them to build geodesics (similarly to the classical theory by Menger, but without forcing the space to be "intrinsic" in some sense); we will then note that in the asymmetric case the classical method of Menger builds *run–continuous geodesics*, and not *continuous geodesics*! In Sec. 11 we will see examples and counterexamples. Eventually in Sec. 12 we will prove the renowned Hopf–Rinow (or Cohn-Vossen) theorem. We will conclude the analysis with some remarks on the separation hypotheses in Sec. 13, and the case when $b(x, y) = \infty$ for some points in Sec. 14. In Sec. 15 we will draw some conclusions; in particular we will argue that, in the asymmetric metric spaces, the class of $\mathcal{C}_r$ of run-continuous paths is more "natural" than the class $\mathcal{C}_g$ of continuous paths.

2 Main definitions

We provide a short summary of the main definitions presented in the previous paper [17].

We already defined the *asymmetric distance* b in 1.1, and the *asymmetric metric space* as the pair (M, b). The space (M, b) is endowed with the topology τ generated by the families of *forward* and *backward open balls*

$$B^+(x,\varepsilon) \stackrel{\text{def}}{=} \{y \mid b(x,y) < \varepsilon\}, \quad B^-(x,\varepsilon) \stackrel{\text{def}}{=} \{y \mid b(y,x) < \varepsilon\}$$

for $\varepsilon \in (0,\infty)$; this is also the topology generated by the symmetric distance

$$d(x,y) \stackrel{\text{def}}{=} b(x,y) \vee b(y,x) \ . \tag{2.1}$$

When we will talk of "continuity", "compactness" or of "convergence", we will always use the topology τ on M. Note that a sequence $(x_n)_n \subset M$ converges to x if and only if $d(x, x_n) \to_n 0$; note also that b is continuous. More details are in Sec. 3 in [17].

We also define

$$D^+(x,\varepsilon) \stackrel{\text{def}}{=} \{y \mid b(x,y) \le \varepsilon\} \ , \ D^-(x,\varepsilon) \stackrel{\text{def}}{=} \{y \mid b(y,x) \le \varepsilon\} \ ,$$

for convenience. Note that in general $\overline{B^+} \neq D^+$ (even in the symmetric case).

Given a (semi)distance b and $\xi : I \to M$ with $I \subseteq \mathbb{R}$ an interval, we define from b the *length* Len^b of ξ by using the *total variation*

$$\mathrm{Len}^b(\xi) \stackrel{\text{def}}{=} \sup_T \sum_{i=1}^n b\big(\xi(t_{i-1}), \xi(t_i)\big) \tag{2.2}$$

where the sup is carried out over all finite subsets $T \subset I$ that we enumerate as $T = \{t_0, \cdots, t_n\}$ so that $t_0 < \cdots < t_n$. When $\mathrm{Len}^b(\xi) < \infty$ we say that ξ is *rectifiable*.

Given $\gamma : [a, c] \to M$, we define the *running length*[1] $\ell^\gamma : [a, c] \to \mathbb{R}^+$ of γ to be the length of γ restricted to $[a, t]$, that is

$$\ell^\gamma(t) \stackrel{\text{def}}{=} \mathrm{Len}^b\left(\gamma|_{[a,t]}\right) \quad . \tag{2.3}$$

We will call *run-continuous* a rectifiable $\gamma : [a, c] \to M$ such that ℓ^γ is continuous.

More in general, given an interval $I \subseteq \mathbb{R}$ (possibly unbounded) and a map $\xi : I \to M$, we will say that ξ is *run-continuous* when, for any $a, c \in I$ with $a < c$, we have that ξ restricted to $[a, c]$ is rectifiable and run-continuous. (Note that it may be the case that ξ is not rectifiable — as in the case of a straight line in the Euclidean space).

Note that a *run-continuous path* is not necessarily continuous. Actually we will use the word "path" only to denote a *run-continuous path*; otherwise we will say "map" or "function". See Cor. 5.5 for an equivalent definition of *run-continuous path*.

Let $a \le s \le t \le c$, then the length of γ restricted to $[s, t]$ is $\ell^\gamma(t) - \ell^\gamma(s)$; so by the definition (2.2) we obtain that

$$b\big(\gamma(s), \gamma(t)\big) \le \ell^\gamma(t) - \ell^\gamma(s) \quad . \tag{2.4}$$

We say that a path $\gamma : [a, c] \to M$ *"connects x to y"* when $\gamma(a) = x$, $\gamma(c) = y$.

We define three classes of paths taking values in M.

- $\mathcal{C}_r$ is the class of all run-continuous paths;
- $\mathcal{C}_g$ is the class of all continuous rectifiable paths (that are also run-continuous, by Prop. 3.9 in [17] or Lemma 5.4 here);
- $\mathcal{C}_s$ is the class of all continuous paths such that both γ and $\hat\gamma(t) \stackrel{\text{def}}{=} \gamma(-t)$ are rectifiable. (Note that other equivalent definitions are in Prop. 3.8 in [17]).

We noted in [17] that $\mathcal{C}_r \subseteq \mathcal{C}_g \subseteq \mathcal{C}_s$; in symmetric metric spaces the three classes coincide, but in asymmetric metric spaces they may differ.

These classes induce three new distances. Let then $b^r(x, y)$ (respectively $b^g(x, y)$, $b^s(x, y)$) be the infimum of $\mathrm{Len}^b(\xi)$ for all ξ connecting x, y and $\xi \in \mathcal{C}_r$ (respectively $\xi \in \mathcal{C}_g$, $\xi \in \mathcal{C}_s$). Obviously

$$b \le b^r \le b^g \le b^s \quad . \tag{2.5}$$

Note that $b^r(x, y) < \infty$ if and only if there is a run-continuous rectifiable path that connects x to y; and so on.

We thus proposed this definition.

Definition 2.1. An asymmetric metric space (M, b) is called

- *r–intrinsic* when $b \equiv b^r$,
- *g–intrinsic* when $b \equiv b^g$,
- *s–intrinsic* when $b \equiv b^s$.

(By eqn. (2.5) the third implies the second, the second implies the first). In symmetric metric spaces the three notions coincide, so we simply say *intrinsic*. Theorem 3.15 in [17] shows that the induced metric space (M, b^r) is always r–intrinsic, and (M, b^s) is always s–intrinsic. It may be that (M, b^g) is not g–intrinsic, see Example 4.4 in [17].

Remark 2.2. For any "*forward*" definition in this paper there is a corresponding "*backward*" definition, obtained by exchanging the first and the second argument of b, *i.e.* by using the *conjugate distance* $\overline{b}$ defined by

$$\overline{b}(x, y) = b(y, x) \quad . \tag{2.6}$$

1 Sometimes denoted as "curvilinear abscissa" in kinematics.

For this reason, in this paper we will mostly present the *forward* versions of the theorems, since *backward* results are obtained by replacing b with $\overline{b}$. For any *forward* definition there is also a corresponding *symmetric* definition, obtained by replacing b with d.

Before we end the introduction, we recall the definitions of *Finslerian metric* and of *General Metric Space* for the convenience of the reader.

Definition 2.3. We recall that a *"General Metric Space"*, according to Busemann [4, 6] and Zaustinsky [24], is a strongly separated [2] asymmetric metric space satisfying

$$\forall x \in M, \forall (x_n) \subset M, \quad \lim_{n\to\infty} b(x_n, x) = 0 \text{ iff } \lim_{n\to\infty} b(x, x_n) = 0 \ . \tag{2.7}$$

As already remarked in the appendix of [17], due to the extra hypothesis (2.7), in a *"General metric space"* every *run-continuous* path is also continuous; so the classes $\mathcal{C}_r = \mathcal{C}_g$ and $b^r = b^g$.

The following classical example was already discussed in [17] (see Example 1.3 and section 2.5.3) but again is here reported for convenience of the reader.

Example 2.4. Suppose that M is a differential manifold. Suppose that we are given a Borel function $F : TM \to [0, \infty]$, and that for all fixed $x \in M$, $F(x, \cdot)$ is positively 1-homogeneous. We define the *length* $\operatorname{len}^F(\xi)$ of an absolutely continuous path $\xi : [0, 1] \to M$ as

$$\operatorname{len}^F(\xi) = \int_0^1 F(\xi(s), \dot{\xi}(s))\, \mathrm{d}s \ . \tag{2.8}$$

We then define the asymmetric semidistance function $b^F(x, y)$ on M to be the infimum of this length $\operatorname{len}^F(\xi)$ in the class of all absolutely continuous ξ connecting x to y.

The length len^F is called a *Finslerian Length* in Example 2.2.5 in [3]. So we will call b^F the *Finslerian distance function*.

3 Local compactness

We say that (M, b) is *forward-locally compact* if $\forall x \in M\ \exists \varepsilon > 0$ such that

$$D^+(x, \varepsilon) \stackrel{\text{def}}{=} \{y \mid b(x, y) \le \varepsilon\}$$

is compact. *"Backward"* and *"symmetrical"* definitions are obtained as explained in Remark 2.2. We say that (M, b) is *locally compact* if $\forall x \in M\ \exists \varepsilon > 0$ such that both $D^-(x, \varepsilon)$ and $D^+(x, \varepsilon)$ are compact; that is, if (M, b) is both forward and backward locally compact. The following implications hold.

$$\begin{tikzcd}
\text{locally compact} \arrow[r] \arrow[d] & \text{forward locally compact} \arrow[d] \\
\text{backward locally compact} \arrow[r] & \text{symmetrically locally compact}
\end{tikzcd}$$

The opposite implications do not hold in general, as shown in examples in Section 11.

Other definitions are used in the literature, such as *finitely compact*, see Section A.3 and Section A.1 in [17].

2 Note that if an asymmetric space satisfies condition (2.7), then it has to be strongly separated (indeed, consider the case when $x_n \equiv y$). This useful remark was provided by an anonymous reviewer.

3.1 Properties in strongly separated spaces

This section collects properties valid in (locally) compact spaces that are strongly separated (*i.e.* where (1.1) holds). All may be proved by using this Lemma (that is similar to (2.3),(2.6) in Zaustinsky's [24]).

Lemma 3.1 (modulus of symmetrization)**.** *The space is strongly separated if and only if the following property holds. For any $C \subseteq M$ compact set there exists a monotonic non decreasing continuous function*

$$\omega : [0, \infty) \to [0, \infty)$$

with $\omega(0) = 0$, such that

$$\forall x, y \in C, \quad b(x, y) \leq \omega(b(y, x)) \ . \tag{3.1}$$

Proof. Define

$$f(r) = \sup_{x,y \in C,\ b(x,y) \leq r} b(y, x)$$

and then f is monotone. Since C is compact, then $f < \infty$. Moreover $\lim_{r \to 0} f(r) = 0$; otherwise we may find $\varepsilon > 0$ and x_n, y_n s.t. $b(x_n, y_n) \to 0$ while $b(y_n, x_n) > \varepsilon$; but, extracting converging subsequences, we obtain a contradiction. From f we can define an ω as required, for example $\omega(r) = \frac{1}{r} \int_r^{2r} f(s) ds$ (note that $\omega \geq f$).

Vice versa for any pair $x, y \in M$, let $C = \{x, y\}$, if relation (3.1) holds then $b(y, x) = 0$ implies $b(x, y) = 0$. □

Corollary 3.2. *Suppose that the space is strongly separated. If $(x_n) \subset M$ is a sequence such that $b(x, x_n) \to 0$, and M is forward-locally compact, then $x_n \to x$.*

The lemma may also be used as follows.

Corollary 3.3. *Suppose that the space is strongly separated. If (M, b) is locally compact then $\forall x \in M, \varepsilon > 0$ $\exists r > 0$ s.t.*

$$B^+(x, r) \subseteq B^-(x, \varepsilon), \quad B^-(x, r) \subseteq B^+(x, \varepsilon)$$

and then $\tau = \tau^+ = \tau^-$.

In particular, an asymmetric metric space that is compact and strongly separated, is also a *General Metric Space* as defined by Busemann (see Definition 2.3), and $\mathcal{C}_r = \mathcal{C}_g$ (but $\mathcal{C}_g \neq \mathcal{C}_s$ in Exa. 4.4 in [17]).

The following is another corollary of 3.1 and is, in a sense, a vice versa of Prop. 3.9 in [17].

Corollary 3.4. *Suppose that the space is strongly separated. Let $\gamma : [a, c] \to M$ be a rectifiable path, and ℓ^γ be its running length. Suppose that ℓ^γ is continuous and that the image of γ is compact, then γ is continuous. (Proof follows from lemma 3.1 and eqn. (2.4)).*

Note that, when the space is not strongly separated, then the examples 8.3 and 8.6 provide counterexamples to the above theses.

4 Completeness

Definition 4.1. A sequence $(x_n)_{n \in \mathbb{N}} \subset M$ is called *a forward Cauchy sequence* if

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ such that } \forall n, m, \ m \geq n \geq N \ \Rightarrow \ b(x_n, x_m) < \varepsilon \tag{4.1}$$

Definition 4.2. We say that (M, b) is *forward complete* if any *forward Cauchy sequence* (x_n) converges to a point $x \in M$.

"Backward" and *"symmetrical"* definitions are obtained as explained in Remark 2.2. Note that these definitions agree with those used in Finsler Geometry (see Chapter VI in [2]). In the appendix, in Remark A.4 we will present a different definition. When (M, b) is both "forward" and "backward" complete, we will simply say that it is *complete*.

Some relations hold.

Proposition 4.3. *Let $(x_n) \subset M$ be a sequence. Then the following are equivalent*

- (x_n) *is* forward Cauchy *and* backward Cauchy,
- (x_n) *is* symmetrically Cauchy.

From that we obtain that, if (M, b) is either forward or backward complete, then it is symmetrically complete.

$$\boxed{\text{forward complete}} \Rightarrow \boxed{\text{symmetrically complete}} \Leftarrow \boxed{\text{backward complete}}$$

The second statement cannot be inverted, as shown in Example 4.2 in [17], and 11.3, 11.4 here.

Proof. Suppose that (x_n) is *symmetrically Cauchy*: then $\forall \varepsilon > 0\ \exists N$ such that $\forall n, m > N$, $d(x_n, x_m) < \varepsilon$: then $b(x_n, x_m) < \varepsilon$.

Suppose that (x_n) is *forward Cauchy* and is *backward Cauchy*: $\forall \varepsilon > 0\ \exists N''$ such that $\forall n > m > N''$, $b(x_n, x_m) < \varepsilon$, and $\exists N'$ such that $\forall n > m > N'$, $b(x_m, x_n) < \varepsilon$: then we let $N = N' \vee N''$, and $\forall n, m > N$, $d(x_n, x_m) < \varepsilon$.

Suppose that (M, b) is forward complete; let (x_n) be *symmetrically Cauchy*: then it is *forward Cauchy*, and then, since (M, b) is forward complete, there is an x such that $x_n \to x$. Similarly if (M, b) is backward complete. □

Some important properties that are often used in symmetric metric spaces hold also in the asymmetric case.

Proposition 4.4.
- *If $x_n \to x$ (according to τ) then the sequence (x_n) is a* symmetrically Cauchy sequence *and hence (x_n) is both a* forward Cauchy sequence *and a* backward Cauchy sequence.
- *Suppose that (x_n) is either* forward Cauchy, *or* backward Cauchy, *and there exists a subsequence n_k and a point x such that $\lim_{k\to\infty} x_{n_k} = x$. Then $\lim_{n\to\infty} x_n = x$.*

(Note that this type of result does not hold in "quasi metric spaces", due to the different choice of topology, see Remark A.5).

Proof. The first statement is well–known, since it deals with the symmetric metric space (M, d), the second is Prop. 4.3.
Fix $\varepsilon > 0$; since (x_n) is forward Cauchy, $\exists N$ such that $\forall m, m'$ with $m' \geq m \geq N$, $b(x_m, x_{m'}) \leq \varepsilon$; let H be such that $n_H \geq N$ and $\forall k \geq H$, $d(x, x_{n_k}) \leq \varepsilon$; for $n \geq n_H$,

$$b(x, x_n) \leq b(x, x_{n_H}) + b(x_{n_H}, x_n) \leq d(x, x_{n_H}) + b(x_{n_H}, x_n) \leq 2\varepsilon$$

at the same time, choosing a large $h \geq H$ such that $n_h \geq n$,

$$b(x_n, x) \leq b(x_n, x_{n_h}) + b(x_{n_h}, x) \leq b(x_n, x_{n_h}) + d(x_{n_h}, x) \leq 2\varepsilon$$

so in conclusion $d(x_n, x) \leq 2\varepsilon$. Similarly if (x_n) is *backward Cauchy*. □

A similarly looking property though does not hold.

Remark 4.5. Fix a sequence $(x_n) \subset M$. Suppose that $\forall \varepsilon > 0$ there exists a converging sequence (y_n) such that $\forall n$, $b(y_n, x_n) < \varepsilon$. If b is symmetric and (M, b) is complete, then (x_n) converges. If b is asymmetric and (M, b) is complete, then there is a counter-example in Example 11.5.(8).

This standard property holds, as in the symmetric case.

Proposition 4.6. *Suppose that (M, b) is compact, then it is complete.*

The proof follows from Prop. 4.4. We will see that other properties valid in the symmetric case may fail, though.

Another interesting property links completeness and induced distances.

Proposition 4.7. *Suppose that (M, b) is forward complete, then (M, b^r) and (M, b^g) are forward complete.*

Proof. Let $(x_n)_{n\ge 0}$ be a forward Cauchy sequence in (M, b^r). Up to a subsequence, with no loss of generality (using Prop. 4.4), we assume that $b^r(x_n, x_{n+1}) \le 2^{-n}$. Let $\varepsilon > 1$. We can then build a run-continuous path $\gamma : [0, 1) \to M$ such that $\gamma(1 - 2^{-n}) = x_n$ and the length of $\gamma(t)$ for $t \in [1 - 2^{-n}, 1 - 2^{-n-1}]$ is less than $\varepsilon 2^{-n}$; so γ is rectifiable. Since (M, b) is forward complete, by Lemma 5.8 there exists $z = \lim_{t\to 1-} \gamma(t)$, and we define $\gamma(1) = z$ for convenience. Since $\gamma(t)$ is continuous for $t = 1$, Lemma 5.4 guarantees that $\ell^\gamma(t)$ is continuous at $t = 1$ as well; this implies that γ is run-continuous on all of $[0, 1]$; so by definition of b^r, $\lim_{\tau\to 1-} b^r(\gamma(\tau), z) = 0$, so we conclude using Prop. 4.4 in (M, b^r).

For the case of (M, b^g) we use continuous rectifiable paths. □

The opposite is not true, as shown in this simple (and symmetric) example.

Example 4.8. Let $M \subset \mathbb{R}^2$ be given by the union of segments as follows

$$M = \bigcup_{n\in\mathbf{N}, n\ge 1} \{(x, y) : x \in [0, 1], y = x/n\}$$

and b the Euclidean distance, then M is not closed in $\mathbb{R}^2$, its closure is

$$\overline{M} = M \cup \{(x, 0), x \in [0, 1]\} \;,$$

hence (M, b) is not complete; but (M, b^r) is complete.

If we add the segment $\{(x, 0), x \in [1/2, 1]\}$ to M, we obtain a set $\tilde{M}$ such that $(\tilde{M}, b)$ is connected but $(\tilde{M}, b^r)$ is disconnected.

4.1 Completeness and run-continuous paths

Proposition 4.9. *Suppose that the space is strongly separated. Suppose $\gamma : [a, c] \to M$ is rectifiable and run-continuous. If the space (M, b) is backward complete, then γ is right-continuous; if (M, b) is forward complete, then γ is left-continuous.*

The proof follows from technical Lemmas 5.8, 5.9 and 5.4 (that also detail the rôle played by each of the hypotheses). If the space is not strongly separated, then this result may be false, see the path ψ in Example 8.6.

In Proposition 3.9 in [17] we saw that a rectifiable and continuous path is also run-continuous. The opposite holds in complete strongly separated spaces.

Corollary 4.10. *If (M, b) is complete and strongly separated, then any run-continuous rectifiable path is continuous; hence the classes $\mathcal{C}_r$ and $\mathcal{C}_g$ coincide, and $b^r \equiv b^g$.*

Note that a space may be complete and strongly separated, but still not a *General Metric Space*, as in Example 11.5.

5 Technical lemmas

This section contains some technical lemmas and definitions that are needed in proofs. The reader not interested in the details of the fine properties of run-continuous paths may skip to next section.

Lemma 5.1. *Suppose that $\gamma : [a, c] \to M$ is run-continuous, let $z \in M$, define $\varphi(t) \stackrel{def}{=} b(z, \gamma(t))$, suppose that for all $t \in [a, c]$, $\varphi(t) < \infty$. Then $\forall \tau \in [a, c]$*

$$\text{if } \tau \neq a,\ \liminf_{\theta \to \tau-} \varphi(\theta) \geq \varphi(\tau)\ ;\ \text{ if } \tau \neq c,\ \varphi(\tau) \geq \limsup_{\theta \to \tau+} \varphi(\theta)\ . \tag{5.1}$$

This has some consequences. For any s, t with $a \leq s < t \leq c$ such that $\varphi(s) < \varphi(t)$, the image of φ on $[s, t]$ contains the interval $[\varphi(s), \varphi(t)]$. Moreover there is a $\tilde{t} \in (s, t]$ such that $\varphi(\tilde{t}) = \varphi(t)$ and $\varphi(\tau) < \varphi(t)$ when $s \leq \tau < \tilde{t}$, and again $\varphi([s, \tilde{t}]) \supseteq [\varphi(s), \varphi(t)]$.

The same holds for $\varphi(t) = b^r(z, \gamma(t))$.

Intuitively the above is a Darboux–type condition that holds only when φ increases; and $\tilde{t}$ is the first time when $\varphi(\tilde{t}) = \varphi(t)$. In general $\varphi(t)$ is not continuous (set $z = 1, \gamma(t) = t$ in example 4.6 in [17]).

Proof. For $s, t \in [a, c]$, $b(z, \gamma(s)) + b(\gamma(s), \gamma(t)) \geq b(z, \gamma(t))$ so when $s < t$

$$\varphi(t) - \varphi(s) \leq \ell^\gamma(t) - \ell^\gamma(s) \tag{5.2}$$

and then we can prove (5.1). By eqn. (3.11) in [17] the same holds for b^r. The rest of the proof is based only on (5.2) and is standard.

We first prove that the image of φ on $[s, t]$ contains the interval $[\varphi(s), \varphi(t)]$; that is for any λ with $\varphi(s) \leq \lambda \leq \varphi(t)$ there exists τ with $s \leq \tau \leq t$ such that $\varphi(\tau) = \lambda$.

Assume $\lambda > \varphi(s)$. Consider I_λ to be union of all intervals $[s, a]$ (with $s \leq a \leq t$) such that $[s, a] \subseteq \{\varphi < \lambda\}$; this union is an interval of the form $I_\lambda = [s, \tau)$ or $I_\lambda = [s, \tau]$ with $\tau = \tau(\lambda)$. If $\varphi(\tau) < \lambda$ then $\tau < t$ and by (5.1), $\varphi(\theta) < \lambda$ for $\theta \in [\tau, \tau + \varepsilon]$ with $\varepsilon > 0$ small, then $[s, \tau + \varepsilon] \subseteq \{\varphi < \lambda\}$, contradicting the definition of τ. If $\varphi(\tau) > \lambda$ then $\tau > s$ and by (5.1), $\varphi(\theta) > \lambda$ for $\theta \in [\tau - \varepsilon, \tau]$, contradiction again. So $\varphi(\tau) = \lambda$ and $I_\lambda = [s, \tau)$. To conclude define $\tilde{t} = \tau(t)$ so $[s, \tilde{t}) = I_{\varphi(t)}$; then replace t with $\tilde{t}$ and use the first condition. □

5.1 On length and dense subsets

We introduce a convenient notation. Let $\xi : [a, c] \to M$ be a path. For any $T \subset [a, c]$ finite subset (containing at least two points), we denote by $\Sigma(\xi, T)$ the sum

$$\Sigma(\xi, T) \stackrel{\text{def}}{=} \sum_{i=1}^{n} b\left(\xi(t_{i-1}), \xi(t_i)\right) \tag{5.3}$$

that is used when computing the length (*cf* eqn. (2.2)), where we enumerate $T = \{t_0, \cdots, t_n\}$ so that $t_0 < \cdots < t_n$. The definition in eqn. (2.2) then reads

$$\operatorname{Len}^b(\xi) \stackrel{\text{def}}{=} \sup_{T \in \mathcal{F}, T \subset [a,c]} \Sigma(\xi, T) \tag{5.4}$$

where $\mathcal{F}$ is the family of all finite subsets of $\mathbb{R}$. Note that $\Sigma(\xi, \cdot)$ is monotonically non decreasing *w.r.t.* inclusion (due to the triangle inequality); so the definition (5.4) is also the limit on the directed family $\mathcal{F}$ (ordered by inclusion).

For the purposes of this technical section, we generalize slightly the definitions given in the introduction.

Definition 5.2. Given $D \subseteq I \subseteq \mathbb{R}$, given a map $\xi : I \to M$, we define

$$\operatorname{Len}^b_D(\xi) \stackrel{\text{def}}{=} \sup_{T \in \mathcal{F}, T \subset D} \Sigma(\xi, T)\ . \tag{5.5}$$

Usually in the applications I is an interval and D a set dense in I; or $I = D$ is dense in an interval. We agree that if D contains less than two points, then we set $\operatorname{Len}^b_D(\xi) = 0$.

Similarly given $D \subseteq I \subseteq [a, c]$ and given $\gamma : I \to M$ we define $\ell^\gamma_D : [a, c] \to \mathbb{R}$ as

$$\ell^\gamma_D(t) \stackrel{\text{def}}{=} \operatorname{Len}^b_D\left(\gamma|_{[a,t]}\right) = \sup_{T \in \mathcal{F}, T \subset D \cap [a,t]} \Sigma(\gamma, T)\ . \tag{5.6}$$

When $I = D$ we can omit the subscript "D" in Len^b_D and ℓ^γ_D.

Lemma 5.3. *Take γ as above and $a \le s \le t \le c$. If $s \in D$, or if γ is continuous at s, then the length of γ restricted to $[s, t]$ can be deduced from ℓ^γ_D using*

$$\mathrm{Len}^b_D(\gamma|_{[s,t]}) = \ell^\gamma_D(t) - \ell^\gamma_D(s) \ ; \tag{5.7}$$

otherwise in general the length may be strictly less.

Lemma 5.4. *Let $D \subseteq [a, c]$, with D dense in $[a, c]$. Let $\xi : D \to M$ be a rectifiable map. Let $\tau \in D$. We write ℓ for ℓ^ξ_D for simplicity.*

- *Suppose $\tau > a$. Then*
$$\ell(\tau) - \lim_{t\to\tau-} \ell(t) = \lim_{t\to\tau-} b(\xi(t), \xi(\tau)) \tag{5.8}$$
(and the limit in RHS is guaranteed to exist).
In particular, ℓ is left continuous at τ iff $\lim_{t\to\tau-} b(\xi(t), \xi(\tau)) = 0$.
- *Vice versa, suppose $\tau < c$, then*
$$\lim_{t\to\tau+} \ell(t) - \ell(\tau) = \lim_{t\to\tau+} b(\xi(\tau), \xi(t)) \ . \tag{5.9}$$
In particular, ℓ is right continuous at τ iff $\lim_{t\to\tau+} b(\xi(\tau), \xi(t)) = 0$.

Note that this lemma proves (in a more descriptive way) Prop. 3.9 in [17], namely, "a rectifiable continuous path is run-continuous".

Proof. Let $(s_k)_k \subset [a, \tau] \cap D$ be an increasing sequence with $\lim_k s_k = \tau$; let $x_k = \xi(s_k)$ and $z = \xi(\tau)$ for convenience. Let $\mathcal{F}$ be the family of finite subsets T of $[a, \tau] \cap D$, by definition

$$\ell(\tau) = \sup_{T\in\mathcal{F}} \Sigma(\xi, T) \ .$$

Let $\mathcal{F}_k$ be the subfamily of $T \in \mathcal{F}$ such that $s_k, \tau \in T$ and the last element before τ in T is s_k. Let

$$L_k \stackrel{\text{def}}{=} \sup_{T\in\mathcal{F}_k} \Sigma(\xi, T) \ ,$$

obviously $L_k \le \ell(\tau)$. Expanding the definition of $\Sigma(\xi, T)$ when $T \in \mathcal{F}_k$ we see that

$$L_k = \ell(s_k) + b(x_k, z) \ .$$

For any $T \in \mathcal{F}_k$ we can add s_{k+1} to it and obtain that $(T \cup \{s_{k+1}\}) \in \mathcal{F}_{k+1}$, so we obtain that $L_k \le L_{k+1}$. Since the family $\bigcup_k \mathcal{F}_k$ is cofinal in $\mathcal{F}$ and L_k is monotonic then

$$\ell(\tau) = \sup_k \sup_{T\in\mathcal{F}_k} \Sigma(\xi, T) = \sup_k L_k = \lim_k L_k \ .$$

Since ℓ is monotonic then

$$\ell(\tau) = \lim_k L_k = \lim_k \left(\ell(s_k) + b(x_k, z)\right) = \lim_k \ell(s_k) + \lim_k b(x_k, z) \ .$$

The limit $\lim_k \ell(s_k)$ does not depend on the choice of the sequence, hence the limit $\lim_k b(x_k, z)$ as well.

For the vice versa, let $(s_k)_k \subset [\tau, c]$ be a decreasing sequence with $\lim_k s_k = a$; we now let $\mathcal{G}$ be the family of finite subsets T of $[\tau, c] \cap D$, and $\mathcal{G}_k$ be the subfamily of $T \in \mathcal{G}$ such that $\tau, s_k \in T$ and the first element after τ in T is s_k; reasoning as above

$$\ell(c) - \ell(\tau) = \sup_k \sup_{T\in\mathcal{G}_k} \Sigma(\xi, T) = \lim_k \left(b(z, x_k) + \ell^\xi(c) - \ell(s_k)\right)$$

etcetera. ☐

The above Lemma is the quantitative argument behind this fact.

Corollary 5.5. *Let τ^+ be the topology generated by forward balls, τ^- be the topology generated by backward balls. Let $\xi : [a,c] \to M$ be a rectifiable map. ξ is run-continuous, if and only if ξ is left continuous in the τ^- topology and ξ is right continuous in the τ^+ topology.*

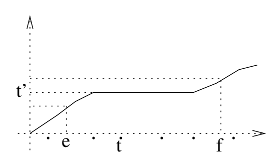
(One implication in this corollary was already announced in Remark 3.7 in [17]).

Lemma 5.6. *Let $D \subseteq I \subseteq [a,c]$. Let $\gamma : I \to M$ be a rectifiable path such that ℓ^γ is continuous on $[a,c]$. Let $L = \ell^\gamma(c) = \mathrm{Len}^b(\gamma)$ be its length. Suppose that $\ell^\gamma(D)$ is dense in $[0,L]$ then*

$$\mathrm{Len}^b(\gamma) = \mathrm{Len}^b_D(\gamma) \ , \ \ \ell^\gamma \equiv \ell^\gamma_D \ .$$

Note that if D is dense in $[a,c]$ then $\ell^\gamma(D)$ is dense in $[0,L]$; but the opposite may be false.

Proof. As a first step, suppose for a moment that $a, c \in D$. We know that $L = \mathrm{Len}^b(\gamma) \geq \mathrm{Len}^b_D(\gamma)$ we wish to prove the converse. Let $D' = \ell^\gamma(D)$. Fix $\varepsilon > 0$. Let $T \subseteq I$ finite such that $\Sigma(\gamma, T) \geq L - \varepsilon$. Let n be the number of points in T. Let $T' = \ell^\gamma(T)$.

1. For any $t' \in T'$ with $t' \notin D'$ (note that then $t' \neq 0, L$), find two nearby points $e, f \in D$ with $e < f$ such that $\ell^\gamma(f) - \ell^\gamma(e) < \varepsilon/n$ and moreover all counterimages t of t' lie in $[e,f]$, but no other points of T lie there; *i.e.* in formulas
$$\forall t \in T, \ell^\gamma(t) = t' \Rightarrow e < t < f$$
and
$$\ell^\gamma\big([e,f] \cap T\big) = \{t'\} \ .$$


(In the picture the points in T are represented as black dots).
Then we add all such points e, f to T; at the end of this step $\Sigma(\gamma, T)$ may have increased.
2. For any $t' \in T'$ with $t' \notin D'$, and then for any t with $\ell^\gamma(t) = t'$, we delete t from T; at the end of this step $\Sigma(\gamma, T)$ may have decreased, but not more than 2ε.

At the end of all steps above, we obtain a $T \subset D$ such that $\Sigma(\gamma, T) \geq L - 3\varepsilon$. So by arbitrariness of ε, we have proved that, when $a, c \in D$, surely $\mathrm{Len}^b(\gamma) = \mathrm{Len}^b_D(\gamma)$.

Now, given $s, t \in D$, $s < t$, the same reasoning may be applied by restricting γ to $I \cap [s,t]$; so by Lemma 5.3 we obtain that

$$\ell^\gamma(t) - \ell^\gamma(s) = \ell^\gamma_D(t) - \ell^\gamma_D(s)$$

since both functions are monotonic and ℓ^γ is continuous and

$$\ell^\gamma(a) = \ell^\gamma_D(a) = 0 \ , \ \ \ell^\gamma(c) \geq \ell^\gamma_D(c)$$

we obtain that $\ell^\gamma \equiv \ell^\gamma_D$. □

In the previous lemma it is not enough to ask that ℓ^γ_D is continuous on $[a,c]$, as can be seen with simple examples of maps $\gamma : [0,1] \to \mathbb{R}$.

Lemma 5.7. *Let $\gamma : I \to M$ be a map; let $D \subseteq I$ be a dense set, that is $I \subseteq \overline{D}$; suppose that for any $t \in I$, $t \notin D$ there exists a sequence $(s_n) \subset D$ with $s_n \to_n t$ and $\gamma(s_n) \to_n \gamma(t)$; then*

$$\mathrm{Len}^b(\gamma) = \mathrm{Len}^b_D(\gamma) \ .$$

Proof. Obviously $\mathrm{Len}^b(\gamma) \geq \mathrm{Len}^b_D(\gamma)$; for $0 \leq l < \mathrm{Len}^b(\gamma)$ consider a $T \subset I$ finite such that $\Sigma(\gamma, T) > l$; we want to build a $S \subset D$ so that $\Sigma(\gamma, S) > l$: indeed using the hypothesis, for any $t \in T$, if $t \in D$ we add t to S; whereas if $t \notin D$ we can add to S a point $d \in D$ that is near enough to t (taking it from the approximating sequence in hypothesis), and use the continuity of b to obtain in the end $\Sigma(\gamma, S) > l$. By arbitrariness of l we conclude. □

This is (an easy adaptation of) a well–known result for functions of bounded variations.

Lemma 5.8. *Let $D \subseteq [a, c]$ be a dense set; let $\gamma : D \to M$ be a rectifiable map. Let $\tau \in [a, c]$.*

- *If $\tau > a$ and (M, b) is forward complete then the limit*
$$\lim_{t \to \tau-} \gamma(t)$$
exists;
- *If $\tau < c$ and (M, b) is backward complete, then the limit*
$$\lim_{t \to \tau+} \gamma(t)$$
exists.

Proof. We write ℓ for ℓ_D^γ for simplicity.

- Consider an increasing sequence $(s_n)_n \subset D$ with $s_n \nearrow_n \tau$; let $x_n = \gamma(s_n)$ then for $\forall n, m,\ m \geq n$
$$b(x_n, x_m) \leq \ell(s_m) - \ell(s_n) \leq \lim_{t \to \tau-} \ell(t) - \ell(s_n)$$
(by eqn. (2.4) and (5.7)) so the sequence $(x_n)_n$ is forward Cauchy, hence it converges to a point x. Given another increasing sequence $(t_n)_n \subset D$ with $t_n \to_n \tau$, suppose for a moment that $\gamma(t_n)$ converges to a point z; the union $(l_n)_n$ of the two sequences $(s_n)_n$ and $(t_n)_n$ is though an increasing sequence, hence the limit of $\gamma(l_n)$ must be both x and y (by Prop. 4.4) so $x = z$. Hence the limit does not depend on the chosen sequence.
- Similar, using a decreasing sequence $(\tilde{s}_n)_n \subset D$ and proving that $\tilde{x}_n = \gamma(\tilde{s}_n)$ is a backward Cauchy sequence.

□

The following Lemma is particularly useful when the space is strongly separated.

Lemma 5.9. *Let $D \subseteq [a, c]$ be a dense set; let $\gamma : D \to M$ be a rectifiable function. Let $\tau \in [a, c]$, and suppose that $\ell^\gamma(t)$ is continuous at τ.*

- *If $\tau \in (a, c)$, if the limits*
$$x = \lim_{s \to \tau-} \gamma(s)\ ,\quad y = \lim_{t \to \tau+} \gamma(t)$$
exist then necessarily $b(x, y) = 0$.
- *If $\tau > a$ and the limit*
$$x = \lim_{s \to \tau-} \gamma(s)$$
exists and $\tau \in D$ then necessarily $b(x, \gamma(\tau)) = 0$.
- *If $\tau < c$ and the limit*
$$y = \lim_{t \to \tau+} \gamma(t)$$
exist and $\tau \in D$ then necessarily $b(\gamma(\tau), y) = 0$.

Proof. We write ℓ for ℓ^γ for simplicity.

- Consider $s < \tau < t$, by triangle inequality
$$b(x, y) \leq b(x, \gamma(s)) + b(\gamma(s), \gamma(t)) + b(\gamma(t), y) \leq b(x, \gamma(s)) + \ell(t) - \ell(s) + b(\gamma(t), y)$$
then let $t \searrow \tau$ and $s \nearrow \tau$ and use the continuity of b and of ℓ at τ.
- If $\tau \in D$ then write
$$b(x, \gamma(\tau)) \leq b(x, \gamma(s)) + b(\gamma(s), \gamma(\tau)) \leq b(x, \gamma(s)) + \ell(\tau) - \ell(s)$$
then let $s \nearrow \tau$.

- If $\tau \in D$ then write

$$b(\gamma(\tau), y) \le b(\gamma(\tau), \gamma(t)) + b(\gamma(t), y) \le \ell(t) - \ell(\tau) + b(\gamma(t), y)$$

then let $t \searrow \tau$.

□

Lemma 5.10. *Let $D \subseteq [a, c]$ be a dense set. Let $\xi : D \to M$ be a map. Suppose that ℓ_D^ξ is continuous on all $[a, c]$, and $L = \mathrm{Len}_D^b(\xi) < \infty$. Suppose that one of these two holds:*

- *(M, b) is forward complete and $a \in D$; or*
- *(M, b) is backward complete and $c \in D$.*

Then there exists a run-continuous path $\gamma : [a, c] \to M$ that extends ξ, and $\ell^\gamma \equiv \ell_D^\xi$.

Proof. Assume that (M, b) is forward complete. For any $t \in [a, c]$ if $t \in D$ we define $\gamma(t) = \xi(t)$; whereas if $t \notin D$, we use Lemma 5.8 and define $\gamma(t) = \lim_{s\to t-} \xi(s)$. We then use Lemma 5.7 on all intervals $[a, t]$ to obtain that $\ell^\gamma(t) = \ell_D^\gamma(t) = \ell_D^\xi(t)$, for all $t \in [a, c]$.
Assume that (M, b) is backward complete. For any $t \in [a, c]$ if $t \in D$ we define $\gamma(t) = \xi(t)$; whereas if $t \notin D$, we use Lemma 5.8 and define $\gamma(t) = \lim_{s\to t+} \xi(s)$. We then use Lemma 5.7 on all intervals $[t, c]$ to obtain that $\mathrm{Len}^b(\gamma|_{[t,c]}) = \mathrm{Len}_D^b(\gamma|_{[t,c]}) = \mathrm{Len}_D^b(\xi|_{[t,c]})$. In particular $\mathrm{Len}^b(\gamma) = \mathrm{Len}_D^b(\gamma) = L$. By equation (5.7) in Lemma 5.3 this implies that $\ell^\gamma(t) = \ell_D^\gamma(t) = \ell_D^\xi(t)$ for all $t \in [a, c] \cap D$. Since D is dense, and ℓ_D^ξ is assumed to be continuous, and both are monotonic, and

$$\ell^\gamma(a) = \ell_D^\gamma(a) = 0 \ , \ \ \ell^\gamma(c) = \ell_D^\gamma(c) = L$$

this implies the result. □

6 Quasi midpoints

In r-intrinsic spaces, for any two points x, y with $b(x, y) < \infty$, for any $\varepsilon > 0$ small, there is always a path ξ_ε joining them with a quasi optimal length, that is, $\mathrm{Len}^b(\xi_\varepsilon) - \varepsilon < b(x, y) = b^r(x, y)$. This is used in the following proposition to find *approximate intermediate points* z between x and y.

Proposition 6.1. *Suppose that the asymmetric metric space (M, b) is r–intrinsic (that is $b \equiv b^r$). Then $\forall\theta \in (0, 1)$, $\forall x, y \in M$ with $b(x, y) < \infty$*

$$\begin{aligned} &\forall\varepsilon > 0 \ \ \exists z \in M \ , \quad \textit{such that} \\ &b(x, z) < \theta b(x, y) + \varepsilon, \qquad b(z, y) < (1 - \theta)b(x, y) + \varepsilon \ . \end{aligned} \tag{6.1}$$

Note that the triangle inequality is almost an equality for the triple x, z, y: indeed summing the above two inequalities we obtain

$$b(x, z) + b(z, y) < b(x, y) + 2\varepsilon \ .$$

The proof follows straightforward from the definition and from the relation eqn. (2.4). When $\theta = 1/2$, the point z is a called *ε-midpoint* in Lemma 2.4.10 in [3].

On the other hand.

Proposition 6.2. *If M is either forward or backward complete, if* $\exists\theta \in (0, 1)$ *such that* $\forall x, y$ *with* $b(x, y) < \infty$ *property* (6.1) *holds, then* (M, b) *is r–intrinsic.*

We just sketch the proof since it is classic. (It is also quite similar to the proof of Prop. 10.3).

Proof. Let $x, y \in M$ with $x \neq y$; if $b(x, y) = 0$ then $b^r(x, y) = 0$ as in the proof of Prop. 7.4. Suppose now $x \neq y$ and $b(x, y) > 0$. Fix $\varepsilon > 0$. We aim to define a run-continuous $\gamma : [0, 1] \to M$ connecting x to y that has $\mathrm{Len}^b(\gamma) \leq b(x, y)(1 + \varepsilon)$. By arbitrariness of ε this will imply that $b(x, y) = b^r(x, y)$. We set $D_0 = \{0, 1\}$; given D_h we define D_{h+1} as

$$D_{h+1} \stackrel{\text{def}}{=} D_h \cup \{t\theta + s(1-\theta) : s, t \in D_h \text{ consecutive points in } D_h, s < t\}$$

that is we interpolate consecutive points in D_h using the parameter θ. We define the dense set $D \subset [0, 1]$ as $D = \cup_h D_h$. Let $\delta > 0$ small so that $1/(1-\delta) < 1 + \varepsilon$. Let

$$\delta_h \stackrel{\text{def}}{=} 1 + \delta^{(2^h)} \quad , \quad c_0 \stackrel{\text{def}}{=} \delta_0 \quad , \quad c_{h+1} \stackrel{\text{def}}{=} c_h \delta_{h+1} \tag{6.2}$$

and note that

$$c_n \stackrel{\text{def}}{=} \prod_{h=0}^{n} \delta_h = \sum_{k=0}^{2^{n+1}-1} \delta^k = \frac{1-\delta^{(2^{n+1})}}{1-\delta} \nearrow_n \frac{1}{1-\delta} \quad . \tag{6.3}$$

Let $L = b(x, y)$. Iterating on h we can define a map $\xi : D \to M$ such that

$$\forall \tau_1, \tau_2 \in D_h, \tau_1 < \tau_2 \quad , \quad b(\xi(\tau_1), \xi(\tau_2)) \leq L c_h (\tau_2 - \tau_1) \quad . \tag{6.4}$$

Indeed $\xi(0) = x$, $\xi(1) = y$ so ξ is defined on D_0 and $b(\xi(0), \xi(1)) \leq L c_0$. Once ξ is defined on D_h, consider $\tau \in D_{h+1} \setminus D_h$; then $\tau = t\theta + s(1-\theta)$ with s, t consecutive points in D_h; we define ξ on τ using (6.1), we set $\xi(\tau)$ to be the point z such that

$$b(\xi(s), z) \leq \theta\, \delta_{h+1}\, b(\xi(s), \xi(t)) \quad , \quad b(z, \xi(t)) \leq (1-\theta)\, \delta_{h+1}\, b(\xi(s), \xi(t)) \quad , \tag{6.5}$$

and then we can prove (using the previous equations and some triangle inequalities) that eqn. (6.4) holds at the next step, namely

$$\forall \tau_1, \tau_2 \in D_{h+1}, \tau_1 < \tau_2 \quad , \quad b(\xi(\tau_1), \xi(\tau_2)) \leq L c_{h+1} (\tau_2 - \tau_1) \quad .$$

By eqn. (6.3) and (6.4) we also obtain, for $s, t \in D$, $s < t$, that $\mathrm{Len}^b(\xi|_{[s,t]}) \leq L(1+\varepsilon)(t-s)$; moreover the space is forward or backward complete; so we can use Lemma 5.10 to extend ξ to a run-continuous path $\gamma : [0, 1] \to M$ with $\mathrm{Len}^b(\gamma) \leq L(1 + \varepsilon)$. ☐

Regarding the completeness hypothesis, see example 11.1.

We now prove a property in Cor. 6.4 that will be used in Lemma 12.5, that is needed for the proof of the Hopf–Rinow–like Theorem. We will use the technical Lemma 5.1.

Lemma 6.3. *If (M, b) is r–intrinsic, $\rho > 0$, $x, z \in M$ with $\rho \leq b(x, z) < \infty$ then*

$$\inf_{y \in D^+(x,\rho)} b(y, z) = b(x, z) - \rho \quad . \tag{6.6}$$

In particular if the above infimum has a minimum point $\tilde{y}$ then $b(x, \tilde{y}) = \rho$ and $b(\tilde{y}, z) = b(x, z) - \rho$.

Proof. Let $\delta \stackrel{\text{def}}{=} \inf_{z \in D^+(x,\rho)} b(y, z)$ be the LHS and $t \stackrel{\text{def}}{=} b(x, z) - \rho$ be the RHS; note that by hypothesis $t \geq 0$. Since $b(y, z) \geq b(x, z) - b(x, y) \geq t$ then to the infimum $\delta \geq t$. For any $\varepsilon > 0$ small there exists a run-continuous $\gamma_\varepsilon : [0, 1] \to M$ connecting $\gamma_\varepsilon(0) = x$ to $\gamma_\varepsilon(1) = z$ and with

$$L_\varepsilon = \mathrm{Len}^b(\gamma_\varepsilon) < b^r(x, z) + \varepsilon = b(x, z) + \varepsilon = \rho + t + \varepsilon \quad ,$$

where we define $L_\varepsilon \stackrel{\text{def}}{=} \mathrm{Len}^b(\gamma_\varepsilon)$ for convenience.

We use Lemma 5.1; we set $\varphi_\varepsilon(t) = b(x, \gamma_\varepsilon(t))$ that is finite due to (2.4), and $s = a = 0$, $t = c = 1$ in the Lemma; by the Lemma we obtain $\tilde{t}_\varepsilon$ such that, setting $y_\varepsilon \stackrel{\text{def}}{=} \gamma_\varepsilon(\tilde{t}_\varepsilon)$, the image of $b(x, \gamma_\varepsilon(\cdot))$ on $[0, \tilde{t}_\varepsilon]$ is $[0, \rho]$ and $b(x, y_\varepsilon) = \rho$, The paths $\gamma_\varepsilon(t)$ are divided into two parts, the part for $t \in [0, \tilde{t}_\varepsilon]$ that connects x to y_ε and has length λ_ε, and the part for $t \in [\tilde{t}_\varepsilon, 1]$ that connects y_ε to to z and has length $L_\varepsilon - \lambda_\varepsilon$. As $\varepsilon \to 0$ the paths get tighter and tighter; since in the first part we have

$$\rho = b(x, y_\varepsilon) = b^r(x, y_\varepsilon) \le \lambda_\varepsilon$$

then for the second part

$$b(y_\varepsilon, z) \le L_\varepsilon - \lambda_\varepsilon \le \rho + t + \varepsilon - \rho = t + \varepsilon \ .$$

So letting $\varepsilon \to 0$ we prove that $\delta = t$.

The last claim follows from $\delta = b(\tilde{y}, z) \ge b(x, z) - b(x, \tilde{y}) \ge b(x, z) - \rho = t$. □

Corollary 6.4. *Let $x \in M, \rho, t > 0$ and*

$$V_t \stackrel{def}{=} \bigcup_{y \in D^+(x,\rho)} D^+(y, t) \ ; \tag{6.7}$$

then (by triangle inequality) $V_t \subseteq D^+(x, \rho+t)$. If (M, b) is r–intrinsic and $D^+(x, \rho)$ is compact then $V_t = D^+(x, \rho+t)$.

Proof. Let $z \in D^+(x, \rho + t)$; if $z \in D^+(x, \rho)$ then $z \in V_t$; suppose $z \notin D^+(x, \rho)$, then we use Lemma 6.3 and since $D^+(x, \rho)$ is compact, we obtain an $\tilde{y} \in D^+(x, \rho)$ such that $b(x, \tilde{y}) = \rho$ and $b(\tilde{y}, z) = b(x, z) - \rho \le t$ so $z \in D^+(\tilde{y}, t)$ and then $z \in V_t$. □

In general we do not have equality in (6.7): consider the Example 10.5 and set $x = (-1, 0), \rho = t = 1$.

The equality $V_t = D^+(x, \rho + t)$ holds also when (M, b) is convex and is r-intrinsic; see 10.6.

As noted in Remark 3.17 in [17], not all properties that are valid in *intrinsic symmetric metric spaces* are also valid in *r–intrinsic asymmetric metric spaces*.

Proposition 6.5. *Suppose that the asymmetric metric space (M, b) is g–intrinsic (that is $b \equiv b^r \equiv b^g$). Let $\rho > 0$ and*

$$S^+(a, \rho) \stackrel{def}{=} \{y \mid b(a, y) = \rho\}$$

$$D^+(a, \rho) \stackrel{def}{=} \{y \mid b(a, y) \le \rho\}$$

(which are closed, since b is continuous) then

$$\overline{B^+(a, \rho)} = D^+(a, \rho), \quad \mathring{S}^+(a, \rho) = \emptyset \ . \tag{6.8}$$

Note that this result may be false in spaces where $b \equiv b^r \neq b^g$ (set $x = -1, \rho = 1$ in Example 4.2).

7 Geodesics

We recall that we will say that a path $\gamma : [a, c] \to M$ *"connects x to y"* when $\gamma(a) = x, \gamma(c) = y$.

Definition 7.1.

1. Given $x, y \in M$, a *"minimizing geodesic connecting x to y"* is a run-continuous rectifiable path γ that attains the minimum of $\mathrm{Len}^b(\xi)$ in the family of all run-continuous paths ξ connecting x to y. In particular $\mathrm{Len}^b(\gamma) = b^r(x, y)$.
2. A run-continuous path $\gamma : I \to M$, with $I \subseteq \mathbb{R}$ interval, is a *global minimizing geodesic* when any part of γ is a minimizing geodesic connecting its endpoints; that is, $\forall s, t \in I$, $s < t$ we have that

$$b^r(\gamma(s), \gamma(t)) = \mathrm{Len}^b(\gamma_{|[s,t]}) \ . \tag{7.1}$$

3. A run-continuous path $\gamma : I \to M$, with $I \subseteq \mathbb{R}$ interval, is a *local geodesic* [3] when it is a minimizing geodesic on short enough sub parts; that is, $\forall t_0 \in I \exists \varepsilon > 0$ such that $\forall s, t \in I$ with
$$t_0 - \varepsilon < s < t < t_0 + \varepsilon$$
we have that
$$b^r(\gamma(s), \gamma(t)) = \mathrm{Len}^b(\gamma_{|[s,t]}) \ .$$

Note that for the minimizing geodesic connecting x to y we must have $\mathrm{Len}^b(\gamma) = b^r(x, y) < \infty$; but a global geodesic may have $\mathrm{Len}^b(\gamma) = \infty$.

As in the symmetric case, there may be none, one, or multiple minimizing geodesics connecting x to y.

There are many different definitions of *"geodesics"* in the literature. A short overview and discussion of merits and caveats is in appendix A.2. The most prominent difference between our definition and the definitions in other texts is that we use the class $\mathcal{C}_r$ of run-continuous paths, instead of the class of continuous paths. We may provide other notions of "geodesic" using the classes $\mathcal{C}_g$ or $\mathcal{C}_s$, but we will skip the definitions for sake of brevity, and just provide some remarks.

Proposition 7.2. *A* "minimizing geodesic" *is also a* "local geodesic".

When $I = [a, c]$ then the "minimizing geodesic γ connecting x to y" *is a* "global minimizing geodesic" *and vice versa; so we will usually just call it minimizing geodesic. The proofs are easy and identical to the symmetric case (see* e.g. *[18]); indeed for $t_0, t_1, t_2 \in I$, $t_0 < t_1 < t_2$ we have that*
$$b^r(\gamma(t_0), \gamma(t_2)) = b^r(\gamma(t_0), \gamma(t_1)) + b^r(\gamma(t_1), \gamma(t_2)) \ . \tag{7.2}$$

The property (7.2) implies that, if we split a piece out of a minimizing geodesic, then this piece is itself a minimizing geodesic. Geodesics can also be joined, as follows.

Lemma 7.3 (joining). *Suppose that $\gamma : [t_0, t_1] \to M$, $\gamma' : [t_1, t_2] \to M$, are minimizing geodesics, of lengths L and L'. Suppose that $\gamma'(t_1) = \gamma(t_1)$: then the two paths may be joined to form a path $\xi : [t_0, t_2] \to M$ (see Definition 2.2 in [17] for details).*

ξ is a minimizing geodesic if and only if
$$b^r\big(\xi(t_1), \xi(t_2)\big) = L + L' \ ,$$
that is, if the triangle inequality is an equality.

(This is a well–known property, see *e.g.* Prop. 2.2.11 in [18]).

We conclude with this Proposition.

Proposition 7.4. *(M, b) is strongly separated iff (M, b^r) is strongly separated.*

Proof. The rightward implication follows from the relation $b \le b^r$.

For the leftward implication consider $x, y \in M, x \neq y$ but such that $b(x, y) = 0$; consider the curve $\gamma : [0, 1] \to M$, $\gamma(1) = y$, $\gamma(t) = x$ for $t \in [0, 1)$. The length of this curve is zero, so γ is run-continuous and it is a minimizing geodesic connecting x to y; moreover we obtain that $b^r(x, y) = 0$. We conclude that, if (M, b) is not strongly separated, then (M, b^r) is not strongly separated as well. ☐

8 Arc parameter

Definition 8.1. We will say that a path $\xi : [a, c] \to M$ is *parameterized by arc parameter* when $\ell^\xi(t) = t - a$.

3 We will not study the properties of *local geodesics* in this paper. This definition is needed to be used as comparison tool in section A.2.

The following is a long–known but very powerful result. It may be found in section I in [4] or in Ch. 1 Sec. 1 in [6]. In the symmetric case, as Theorem 2.5.9 in [3], or as Theorem 4.2.1 in [1], that also discusses the *metric derivative* issue. On the metric derivative, see also Sec. 2.7 in [3], or Sec. 2 in [20].

Lemma 8.2. *Suppose that the space is strongly separated. For any run-continuous rectifiable path γ of length L there exists an unique path $\xi : [0, L] \to M$ such that*

$$\gamma(t) = \xi(\ell^\gamma(t)), \quad \ell^\xi(t) = t, \quad \forall t \in [0, L] \tag{8.1}$$

where ℓ^γ is the running length *of γ and ℓ^ξ of ξ. Moreover if γ is continuous then ξ is continuous.*

ξ is called the *reparameterization to arc parameter* of γ. Note that if γ is a minimizing geodesic then ξ is a minimizing geodesic.

Proof. The definition of ξ by the relation $\gamma(t) = \xi(\ell^\gamma(t))$ is well posed: indeed, if $\ell^\gamma(t) = \ell^\gamma(t')$ then $\gamma(t) = \gamma(t')$ by (2.4) and (1.1); moreover the domain of ξ is $[0, L]$ that is also the image of ℓ^γ. Let $\hat{l} = \ell^\gamma(\hat{t})$. If we restrict γ to $[a, \hat{t}]$ and ξ to $[0, \hat{l}]$, we can write $\gamma = \xi \circ \ell^\gamma$. Since the length of a curve does not change when a curve is reparameterized (see *e.g.* Proposition 2.7 in [17]), we have that $\ell^\gamma(\hat{t}) = \ell^\xi(\hat{l})$, that is $\hat{l} = \ell^\xi(\hat{l})$. The statement on continuity follows from Prop. 3.4. □

8.1 Quasi arc parameter

When the space is not strongly separated, the above Lemma may fail, as in this simple example.

Example 8.3. Let $M = \mathbb{R}$ and

$$b(x, y) = \begin{cases} y - x & \text{if } y \ge x \ , \\ 0 & \text{if } y < x \ . \end{cases}$$

Let $\varepsilon > 0$ and $\gamma : [-\varepsilon, \varepsilon] \to M$ be defined simply as $\gamma(t) = -t$, this path has length zero but is not constant and then it cannot be reparameterized to arc parameter. (Note that (M, b) is neither forward nor backward complete, but see Example 8.6).

The above example is induced by the Finslerian metric $F(x, v) = v^+$ on $TM = \mathbb{R}^2$, (as explained in Example 2.4).

In this general case we may anyway state two results. The first may be seen as the metric generalization of Lemma 2.24 in [17].

Lemma 8.4. *For any run-continuous rectifiable path $\gamma : [a, c] \to M$ of length L, for any $\varepsilon > 0$, there exists an increasing homeomorphism $\varphi : [a, c] \to [a, c]$ such that for the path*

$$\xi : [a, c] \to M \ , \ \xi = \gamma \circ \varphi^{-1} \tag{8.2}$$

we have that ℓ^ξ satisfies

$$\forall s, t \ , \ a \le s \le t \le c \ , \ \ell^\xi(t) - \ell^\xi(s) \le (t - s)(L + \varepsilon)/(c - a) \tag{8.3}$$

i.e. *ℓ^ξ is Lipschitz of constant $(L + \varepsilon)/(c - a)$.*

Note that the length of ξ is again L.

Proof. Let

$$\varphi(t) = a + \frac{(c - a)\ell^\gamma(t) + (t - a)\varepsilon}{L + \varepsilon} \tag{8.4}$$

then $\varphi : [a, c] \to [a, c]$ is an increasing homeomorphism. We now prove (8.3). Setting $\tilde{t} = \varphi(t)$ and $\tilde{s} = \varphi(s)$ we obtain

$$(L + \varepsilon)(\tilde{t} - \tilde{s}) = \big(\ell^\gamma(t) - \ell^\gamma(s)\big)(c - a) + \varepsilon(t - s) \ .$$

From $\xi = \gamma \circ \varphi^{-1}$ we obtain that $\ell^\xi \circ \varphi = \ell^\gamma$ hence eventually

$$\begin{aligned}\big(\ell^\xi(\tilde t) - \ell^\xi(\tilde s)\big)(c-a) &= \big(\ell^\gamma(t) - \ell^\gamma(s)\big)(c-a) = \\ &= (L+\varepsilon)(\tilde t - \tilde s) - \varepsilon(t-s) \le (L+\varepsilon)(\tilde t - \tilde s)\ .\end{aligned}$$

□

This second Lemma is the generalization of Lemma 8.2 to the case of (possibly) non strongly separated spaces.

Lemma 8.5. *For any run-continuous rectifiable path $\gamma : [a, c] \to M$ of length $L > 0$, we define the map ψ : $[0, L] \to [a, c]$ by*

$$\psi(s) = \min\{t \in [a, c]\ ,\ s = \ell^\gamma(t)\}\ ; \tag{8.5}$$

then the paths

$$\theta : [0, L] \to M\ ,\ \theta = \gamma \circ \psi \tag{8.6}$$

and

$$\beta : [0, L] \to M\ ,\ \beta(t) = \begin{cases} \theta(t) & \text{if } t < L \\ \gamma(c) & \text{if } t = L \end{cases}$$

are both parameterized by arc parameter, that is $\ell^\theta(t) = \ell^\beta(t) = t$.

The proof is in Section B.5. Note that the map ψ is not the unique possible map, any right inverse of ℓ^γ will do. Since $\psi(0) = a$, then $\theta(0) = \beta(0) = \gamma(a)$; it may be the case that $\psi(L) < c$ and $\theta(L) \neq \gamma(c)$, hence the definition of β. ψ is monotonically increasing, but may fail to be surjective (it is surjective iff ℓ^γ is injective). So the trace of the paths θ and β are contained in the trace of the path γ; but they may be quite different. In particular if γ is continuous it may be easily the case that θ and β are not continuous (as seen in the following example).

Consider that if we apply the reparameterization (8.6) in the Lemma to the map γ in the Example 8.3 (that has length zero) then the map θ is just $\theta : \{0\} \to \mathbb{R}$ with $\theta(0) = \varepsilon$. (And this is quite unsatisfactory!)

Run continuity of γ is essential in this Lemma. A monotonic map $\gamma : [0, 1] \to \mathbb{R}$ with a jump discontinuity cannot be reparameterized to arc parameter.

We view the two lemmas in action in the example that follows.

Example 8.6. Let $M = \mathbb{R}$ and define the Finslerian metric

$$F(x, v) = \begin{cases} v & \text{if } v \ge 0\ , \\ -v & \text{if } v < 0\ ,\ x \notin [-1, 1] \\ 0 & \text{if } v < 0\ ,\ x \in [-1, 1]. \end{cases}$$

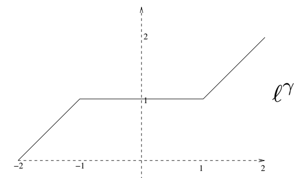
so that induced distance (as by 2.4) is

$$b(x, y) = \begin{cases} y - x & \text{if } x \le y\ , \\ x - y & \text{if } 1 \le y < x \vee y < x < -1\ , \\ x - 1 & \text{if } -1 < y < 1 \le x\ , \\ x - y - 2 & \text{if } y < -1, 1 \le x\ . \\ 0 & \text{if } -1 \le y < x < 1\ , \\ -1 - y & \text{if } y < -1 \le x < 1\ . \end{cases}$$

Note that the associated topology is Euclidean; and all discs $D^+(x, r)$ and $D^-(x, r)$ are compact; so this space (M, b) is complete.

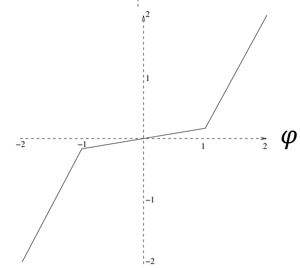
Let $\gamma : [-2, 2] \to M$ be defined simply as $\gamma(t) = -t$; this path is continuous and is a minimizing geodesic connecting 2 to -2; but is not parameterized by arc parameter, indeed

$$\ell^\gamma(t) = 1 + \begin{cases} t+1 & \text{if } t < -1, \\ 0 & \text{if } -1 \le t \le 1, \\ t-1 & \text{if } t \ge 1\,. \end{cases}$$



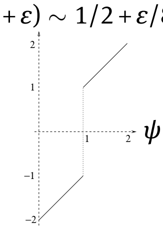
In this case the reparameterization (8.4) is

$$\varphi(t) = -2 + \frac{4\ell^\gamma(t) + (t+2)\varepsilon}{2+\varepsilon} = \frac{4\big(\ell^\gamma(t)-1\big) + t\varepsilon}{2+\varepsilon}$$



so that the "quasi arc parameterized curve" is just $\xi(t) = -\varphi^{-1}(t)$; this curve transverses the segment $[-1, 1]$, where $F = 0$, with Euclidean speed $(2+\varepsilon)/\varepsilon$; and the parts where $F \neq 0$ with speed $(2+\varepsilon)/(4+\varepsilon) \sim 1/2+\varepsilon/8$. The map ψ defined in the Lemma 8.5 is just

$$\psi(s) = \begin{cases} s-2 & \text{if } s \le 1 \\ s & \text{if } s > 1 \end{cases}$$



and $\theta(s) = \beta(s) = -\psi(s)$, that are not continuous.

More in general, given $a \in [-1, 1]$, all maps $\theta_a : [0, 2] \to \mathbb{R}$

$$\theta_a(s) = \begin{cases} 2-s & \text{if } s < 1 \\ a & \text{if } s = 1 \\ -s & \text{if } s > 1 \end{cases}$$

are arc parameterized minimal geodesics connecting 2 to -2.

9 Existence of Geodesics

The following results is similar in spirit to (5),(6),(7) in section I in [6], but is extended to the settings and needs of this paper. We define

$$D^{+r}(x, v) \stackrel{\text{def}}{=} \{y \mid b^r(x, y) \le v\}\ .$$

Theorem 9.1. *Let $\rho > 0$. Fix $x, y \in M$ with $b^r(x, y) \le \rho$. Assume that, for all v with $0 < v < \rho$, $D^{+r}(x, v)$ is contained in a compact set (compact according to the (M, b) topology τ). Then there is an arc-parameterized minimizing geodesic connecting x to y.*

The proof of this theorem is in Sec. B.6. The choice of hypotheses in the above Theorem is different from what is usually seen in texts; see the discussion in Sec. A.2.2. Note that $D^{+r}(x, \rho)$ is not guaranteed to be closed in the (M, b) topology: just consider the set $\tilde{M}$ in Example 4.8 and consider the sequence $(1/2, 1/2n)$ in $D^{+r}\big((0, 0), 1\big)$. Note that, in the above Theorem, we cannot replace D^{+r} with D^+: see in example 4.8 (that is a symmetric metric space!) in [17]. The above Theorem can be applied to Example 8.6, where θ_a are different arc parameterized minimal geodesics connecting 2 to -2 that the proof of the Theorem can construct.

Under the additional hypotheses that the space be strongly separated and $D^{+r}(x, \rho)$ be contained in a compact set, a different proof of the theorem 9.1 is possible, a proof that follows the method used by Busemann for existence of geodesics in *General Metric Spaces*; such proof uses the symmetrization estimate in Lemma 3.1 and the reparameterization Lemma 8.2; see in Sec. B.7.

A possible result in the class $\mathcal{C}_g$ is as follows.

Theorem 9.2. *Suppose that the space is strongly separated. Fix $x, y \in M$. Let $\rho > 0$. We define*

$$D^{+g}(x,\rho) \stackrel{def}{=} \{y \mid b^g(x,y) \le \rho\} \ .$$

Suppose that $D^{+g}(x,\rho)$ is contained in a compact set (compact according to the (M,b) topology). Then for any $y \in D^{+g}(x,\rho)$ (i.e. $b^g(x,y) \le \rho$), there is an arc-parameterized continuous minimizing geodesic connecting x to y.

The proof is in Sec. B.7.

We currently do not know if the hypothesis "strongly separated" can be dropped in this theorem.

As a corollary of the above discussion we obtain what is nowadays known as *Busemann's theorem.*

Theorem 9.3. *Suppose that (M,b) is compact; for any $x, y \in M$ with $b^r(x,y) < \infty$ there is a minimizing geodesic ξ that connects x to y.*

10 Convexity

All of the results in Sec. 6 relate the existence of *approximate intermediate points* to the fact that (M,b) be intrinsic (in some sense). In this section we will deal with *exact intermediate points* z at a prescribed distance b^r from x and y. We present two different definitions. We start with a weaker definition.

Definition 10.1. (M,b) is "*weakly convex*" if given two different points $x, y \in M$ with $0 < b^r(x,y) < \infty$, a third (different from x, y) point z exists such that $b^r(x,z) + b^r(z,y) = b^r(x,y)$.

The weaker definition is similar to the definition of "*Menger convexity*" used in other texts; the main difference is that, in our definition we used b^r, whereas in other papers the definition is stated using b; see the discussion in Section A.3 in appendix.

We propose also a stronger definition.

Definition 10.2. (M,b) is "*strongly convex*" if given any two points $x, y \in M$ with $b^r(x,y) < \infty$, and any $\theta \in (0,1)$,

$$\exists z \in M \quad \text{such that} \quad b^r(x,z) = \theta b^r(x,y), \qquad b^r(z,y) = (1-\theta) b^r(x,y) \ . \tag{10.1}$$

Note that the triangle inequality is an equality for the triple x, z, y: indeed summing the above two equalities we obtain

$$b^r(x,z) + b^r(z,y) = b^r(x,y) \ .$$

By analogy with the theory of quasi-midpoints, we will call *midpoint* a point z such that $b^r(x,z) = b^r(x,y)/2$, $b^r(z,y) = b^r(x,y)/2$.

With additional hypotheses, these definitions imply that any two points may be connected by a minimizing geodesics, as shown below. In turn, if any two points x, y with $b^r(x,y) < \infty$ may be connected by a minimizing geodesics, then it is easily seen that the space is convex (in both senses).

The following result is similar to a classical result, see Proposition A.3 in appendix.

Proposition 10.3. *Suppose M is either forward or backward complete. Suppose that there exists a $\theta \in (0,1)$ such that for any two points x, y with $b^r(x,y) < \infty$ the property* (10.1) *holds. Then for any two points x, y with $b^r(x,y) < \infty$ there exists a minimizing geodesic connecting them.*

(We just sketch the proof; it is also quite similar to the proof of Prop. 6.2).

Proof. Let $x, y \in M$. If $x = y$ then the geodesic is the constant path. If $x \neq y$ but $b^r(x, y) = 0$ then the geodesic is γ as defined in the proof of Prop. 7.4. The last case is when $x \neq y$ and $b^r(x, y) > 0$. Let $L = b^r(x, y)$. We aim to define a $\gamma : [0, 1] \to M$ connecting x to y with $\mathrm{Len}^b(\gamma) = L$; this γ is the sought geodesic. Let D and D_h be defined as in the proof of Prop. 6.2. Iterating on h we can define a map $\xi : D \to M$; indeed $\xi(0) = x$, $\xi(1) = y$ so ξ is defined on D_0; once ξ is defined on D_h, taken $\tau \in D_{h+1} \setminus D_h$ then $\tau = t\theta + s(1-\theta)$ with s, t consecutive points in D_h; we define ξ on τ to be the point z such that

$$b^r\big(\xi(s), z\big) = \theta\, b^r\big(\xi(s), \xi(t)\big) \text{ and } b^r\big(z, \xi(t)\big) = (1-\theta)\, b^r\big(\xi(s), \xi(t)\big)$$

(using strong convexity with the given θ). For $s, t \in D, s < t$ we can prove that $b^r(\xi(s), \xi(t)) = L(t-s)$, again by induction on h, and using some triangle inequalities. Since $b \le b^r$ then $\mathrm{Len}^b(\xi|_{[s,t]}) \le L(t-s)$; moreover the space is forward or backward complete; so we can use Lemma 5.10 to extend ξ to run-continuous path $\gamma : [0, 1] \to M$ connecting x to y and satisfying $\mathrm{Len}^b(\gamma|_{[s,t]}) \le L(t-s)$. Eventually we can prove that $\mathrm{Len}^b(\gamma|_{[s,t]}) = L(t-s)$ using triangle inequalities and the fact that the length of γ is at least L. □

Another result is as follows.

Proposition 10.4. *Suppose M is complete and weakly convex. Then for any two points x, y with $b^r(x, y) < \infty$ there exists a minimizing geodesic connecting them.*

The proof uses Zorn's Lemma to find a maximal $\xi : D \to M$ such that $b^r(\xi(s), \xi(t)) = t-s$; it uses completeness to prove that D is closed, and convexity to prove that $D = [0, 1]$.

Other similar statements are possible; we skip them for sake of brevity.

The above proposition are quite similar to the usual results in symmetric metric theory. The surprising remark, in the setting of this paper, is that the classical method of Menger, in the asymmetric case, builds *run-continuous geodesics*, and not *continuous geodesics*! So the fact that the classical theory of geodesics studies *continuous geodesics* looks just as a byproduct of the symmetry of the theory, and not as a natural necessity.

"Completeness" is fundamental in the above propositions.

Example 10.5. The subset $M = \mathbb{R}^2 \setminus \{(0, 0)\}$ obtained by deleting the origin from 2-space, endowed with the Euclidean distance $b(x, y) = |x-y|$, is a simple example of a space that is intrinsic, it is weakly convex, but not strongly convex. It is also locally compact but not complete. The points $(0, 1)$ and $(0, -1)$ cannot be connected by a minimizing geodesic.

We conclude with this proposition.

Proposition 10.6. *Let $x \in M, \rho, t > 0$ and*

$$V_t \overset{def}{=} \bigcup_{y \in D^+(x,\rho)} D^+(y, t) \ ;$$

as in eqn. (6.7). *Suppose that (M, b) is r–intrinsic and strongly convex, then $V_t = D^+(x, \rho + t)$.*

11 Examples

This subset of $\mathbb{R}^2$ complements the already seen Example 10.5.

Example 11.1. The set $M = (\mathbb{R} \times \mathbb{Q}) \cup (\{0\} \times \mathbb{R})$, equipped with $b(x, y) = |x - y|$, admits quasi–midpoints; and it is weakly convex but not strongly convex; and M is arc connected; but (M, b) is not complete; and (M, b) is not intrinsic, since $b^g(x, y) = |x_1| + |y_1| + |x_2 - y_2|$ when $x_2 \neq y_2$.

Example 11.2. Consider $M \subset \mathbb{R}^n$ to be an open set, and b to be the Euclidean distance; then

- (M, b) is locally compact;
- (M, b) admits minimizing geodesics iff M is convex;

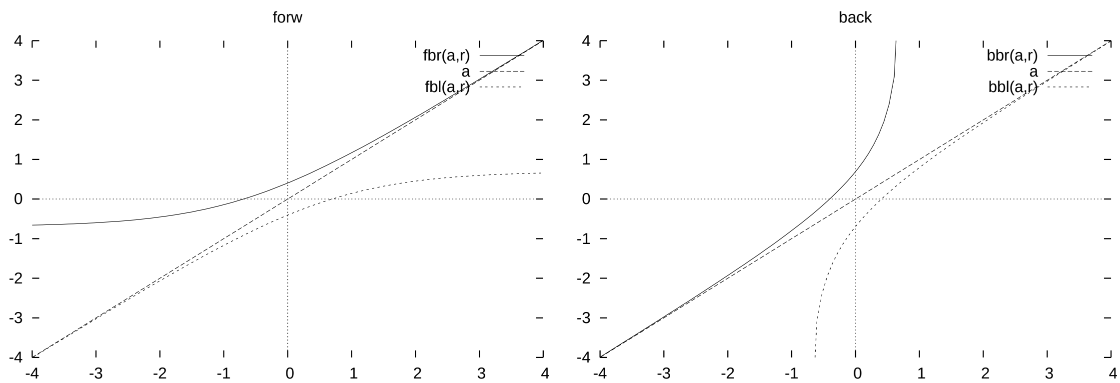


Fig. 1. *f*orward and *b*ackward *b*alls, *l*eft and *r*ight extrema (M is vertical, $r = 1/2$, a in abscissa)

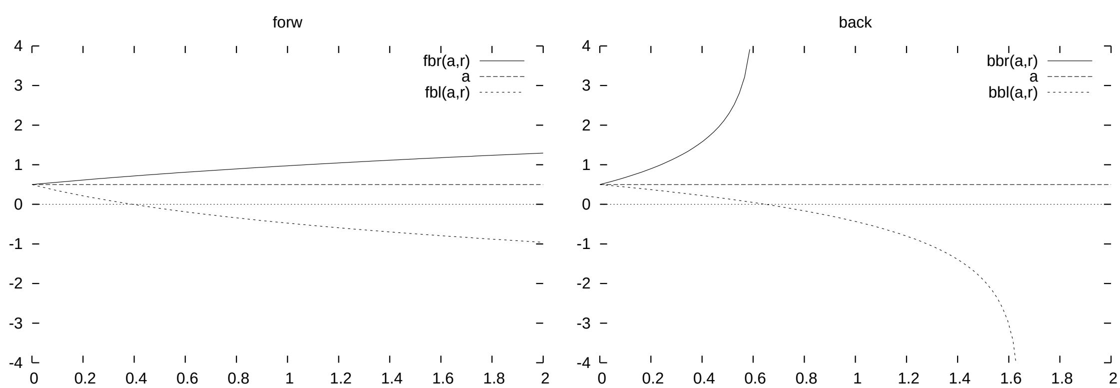


Fig. 2. *f*orward and *b*ackward *b*alls, *l*eft and *r*ight extrema (M is vertical, $a = 1/2$, r in abscissa)

- (M, b) is complete iff $M = \mathbb{R}^n$;
- if moreover M is equal to the interior of the closure of M in $\mathbb{R}^n$, then (M, b) is intrinsic iff M is convex.

Example 11.3. If $M = \mathbb{R}$ and

$$b(x, y) = \begin{cases} e^y - e^x & \text{if } x < y \\ e^{-y} - e^{-x} & \text{if } x > y \end{cases} \tag{11.1}$$

then b generates on $\mathbb{R}$ the usual topology, and (M, b) is locally compact and s–intrinsic. Note that (M, b) is a Finsler manifold, indeed the distance b derives from the metric

$$F(x, v) = \begin{cases} e^x v & v > 0 \\ -e^{-x} v & v < 0 \end{cases} .$$

The balls are the open intervals

$$B^+(a, r) = \{y \mid b(a, y) < r\} = \big(-\log(r + e^{-a}), \log(r + e^a)\big)$$

$$B^-(a, r) = \{x \mid b(x, a) < r\} = \big(\log(e^a - r), -\log(e^{-a} - r)\big)$$

where $\log(z) = -\infty$ if $z \le 0$. (see fig. 1 and 2)

1. We immediately note that $B^-(0, 1) = \mathbb{R}$, that is, M is backward bounded (but is not forward bounded).
2. If $x_n = n$ then for $m < n$, $b(x_n, x_m) = e^{-m} - e^{-n} < e^{-m}$ so this sequence is backward-Cauchy: then M is not backward complete.

3. (M, b) is forward complete (and then is symmetrically complete, by 4.3). Proof: Suppose that (x_n) is an increasing forward-Cauchy sequence: then for $m > n > N$, $b(x_n, x_m) = e^{x_m} - e^{x_n} < \varepsilon$ that implies $x_m \le \log(\varepsilon + e^{x_n})$, so x_m has a limit $x_m \to x$; while if it were a decreasing forward-Cauchy sequence for $m > n > N$, $b(x_n, x_m) = e^{-x_m} - e^{-x_n} < \varepsilon$ that implies $x_m \ge -\log(\varepsilon + e^{-x_n})$, and again x_m has a limit. Suppose that x_n is a generic forward-Cauchy sequence: from any subsequence x_{nk} of x_n we may extract a monotonic sub-sub-sequence $(x_{nkh})_h$: this would be convergent to a point x; this point does not depend on the choice of subsequence: indeed, $b(x_n, x_{nkh}) \to_h b(x_n, x) < \varepsilon$ for n large. □
4. Consider the ball $B^+(0, \rho)$ of extrema $(-R, R)$ with $R(\rho) = \log(\rho + 1)$, and then the two balls

$$B^+(R, r) = \big(-\log(r + e^{-R}), \log(r + e^{R})\big), \quad B^+(-R, r) = \big(-\log(r + e^{R}), \log(r + e^{-R})\big)$$

then if $r \ge \rho/(\rho + 1)$,

$$B^+(0, \rho) \subset B^+(R, r) \cup B^+(-R, r) \ .$$

Proof: indeed the right extrema of $B^+(-R, r)$ is positive when $\log(r + e^{-R}) \ge 0$ that is $(r + e^{-\log(\rho+1)}) \ge 1$ that is $(r + 1/(\rho + 1)) \ge 1$ □

Example 11.4. Consider two copies (M, b) of the above space, join them at the origin, reverse the metric on one: the resulting space $\tilde{M}, \tilde{b}$ is symmetrically complete, but is neither forward nor backward complete.

More precisely, let $M_+ = \mathbb{R} \times \{+\}$, $M_- = \mathbb{R} \times \{-\}$,

$$\tilde{M} = M_+ \cup M_- / \sim$$

where $(0, +) \sim (0, -)$ and then

$$\tilde{b}(x, y) = \begin{cases} b(x, y) & \text{if } x, y \in M_+ \\ b(y, x) & \text{if } x, y \in M_- \\ \tilde{b}(x, [0]) + \tilde{b}([0], y) & \text{otherwise} \end{cases} .$$

Example 11.5. Let M be the disjoint union of segments with glued extrema; more precisely

$$M = (\mathbb{Z} \times [0, 1]) / \sim$$

with $\sim$ identifying $(n, 0) \sim (m, 0)$ and $(n, 1) \sim (m, 1)$ for $n, m \in \mathbb{Z}$, so that all points $(n, 0)$ collapse into a class $[0]$ in M, and all points $(n, 1)$ into a class $[1]$ in M.

On each segment we define a Finslerian distance b_n as follows, if $x = (n, s)$, $y = (n, t)$ then

$$b_n(x, y) = \begin{cases} (s - t)(1 + e^{-n}) & s \ge t \\ (t - s)(1 + e^{n}) & s \le t \end{cases} ;$$

when we glue the segments together, we define b on M "geodesically": first we define $\tilde{b}(x, y)$ for convenience, for $x = (n, s)$ and $y = (m, t)$, as

$$\begin{aligned} \tilde{b}(x, y) = \min\big\{ & b_n(x, [0]) + b_m([0], y), \ \ b_n(x, [1]) + b_m([1], y), \\ & b_n(x, [0]) + 1 + b_m([1], y), \ \ b_n(x, [1]) + 1 + b_m([0], y) \big\} \ ; \end{aligned}$$

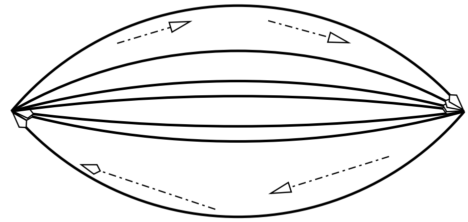
eventually, given any $x, y \in M$ with $x = (n, s)$ and $y = (m, t)$, we define b on M by

$$b(x, y) = \begin{cases} \tilde{b}(x, y) & \text{if } n \ne m \ , \\ \min\{\tilde{b}(x, y), \ b_n(x, y)\} & \text{if } n = m \ . \end{cases}$$

Note that $b([0], [1]) = b([1], [0]) = 1$.

We highlight the following properties

1. (M, b) is a complete space.
2. (M, b) is a s–intrinsic space.
3. It is strongly separated.



4. (M, b) is not locally compact.
5. There is no minimizing geodesic connecting [0] to [1].
6. If $x_n = (n, 1/n)$ then
$$b(x_n, [0]) = b_n(x_n, [0]) = (1 + e^{-n})/n \to 0$$
but (for n large)
$$b([0], x_n) = b([0], [1]) + b_n([1], x_n) + = 1 + (1 + e^{-n})(1 - 1/n) \to 2 \ ;$$
so this space is not "General metric space", it does not satisfy (2.7).
7. Note that the sequence $(x_n)_n$ is neither *forward Cauchy* nor *backward Cauchy* since $\lim_{n\to\infty} b(x_n, x_{n+1}) = \lim_{n\to\infty} b(x_{n+1}, x_n) = 2$.
8. Moreover $\forall \varepsilon > 0$ we can define
$$y_n = \begin{cases} x_n & \text{if } n\varepsilon < 2 \\ [0] & \text{if } n\varepsilon \geq 2 \end{cases}$$
and then $b(x_n, y_n) < \varepsilon$ and $y_n \to [0]$; but nonetheless $x_n \not\to [0]$.

The following example marks a fundamental difference between the cutlocus in Riemannian Manifolds, and in asymmetric metric spaces. It is based on Example 4.6 in [17]. (A similar example using a Finsler structure may be built starting from Example 4.1 in [17].) Unfortunately in 4.6 in [17] we claimed that "$(\mathcal{C}_H, \mathrm{len}_5)$ is a length structure"; this is inexact, since the class $\mathcal{C}_H$ described in 4.6 is not closed under join of paths. The correct class would be the class $\mathcal{C}_P$ described in this example. That inexactness fortunately does not affect the importance of the Example 4.6.

Example 11.6. Given a continuous injective monotonic map $\xi : [a, c] \to \mathbb{R}$ we define the length len_6 as
$$\mathrm{len}_6(\xi) = \xi(c) - \xi(a)$$
if ξ is increasing, while when ξ is decreasing we set
$$\mathrm{len}_6(\xi) = \xi(a) - \xi(c) + \frac{1}{2}\#\Big((\xi(c), \xi(a)] \cap \mathbb{Z}\Big)$$
where $\#A$ is the cardinality of the set A. As in 4.6 the effect is that, when paths run rightwards the space is Euclidean, whereas when they run leftwards, it looks as if there is a gap of width $1/2$ near any integer point ("infinitesimally at the left of" any integer point, in a sense).

We then define $\mathcal{C}_P$ as the class of $\gamma : [a, c] \to \mathbb{R}$ that are continuous and piecewise strictly monotonic, and extend the length len_6 to $\mathcal{C}_P$ by addittivity.

We identify the circle S^1 with the quotient $\mathbb{R}/\mathbb{Z}$. We define a length structure in S^1 by projecting the above structure $(\mathcal{C}_p, \mathrm{len}_6)$. We then induce the distance function b_6 from this length structure. By using Theorem 2.19 in [17], it is possible to prove that $\mathrm{len}_6 \equiv \mathrm{Len}^{b_6}$ in $\mathcal{C}_P$.

Consider now $x = [1/3] \in S^1$ and the paths $\xi, \gamma : [0, 1] \to S^1$ given simply by $\xi(t) = 1/3 - t$ and $\gamma(t) = 1/3 + t$; for these paths we have $\ell^\gamma(t) = t$ but
$$\ell^\xi(t) = \begin{cases} t & \text{if } t \leq 1/3 \\ t + 1/2 & \text{if } t > 1/3 \end{cases} .$$

Let then $\varepsilon \geq 0$ but $\varepsilon < 1/12$; if we want to connect x to $y = [-\varepsilon]$ we can use the path $\gamma(t)$ with $t \in [0, 2/3-\varepsilon]$, or the path $\xi(t)$ with $t \in [0, 1/3+\varepsilon]$. The remarkable fact is that, if $\varepsilon > 0$ then the path γ is the minimizing geodesic, and ξ is not even run-continuous; but for $\varepsilon = 0$ then ξ is the minimizing geodesic, and γ is strictly longer than ξ, so γ is not a minimizing geodesic anymore.

Note that the space in Example 4.6 in [17] is not r–intrinsic, since there is no run continuous path connecting 1 to −1 hence $b(1,-1) = 3 \neq b^r(1,-1) = \infty$. Similarly this space (S^1, b_6) is not r–intrinsic and not compact (indeed (S^1, b_6) is homeomorphic to $[0, 1)$ and not to S^1 with the usual topology).

11.1 Examples from [17]

In the following we refer to the examples in [17].

Consider the space $([-1, 1], b_1)$ in Example 4.1.

- (M, b_1) is forward complete. Indeed, we show that if x_n is forward Cauchy, then it converges. If at a certain point we have that $x_n > 0$ then for all $m > n$, $x_m > 0$ (otherwise, $b(x_n, x_m) = \infty$!). So the sequence is definitively $x_n \leq 0$, or definitively $x_n > 0$. In the first case definitively $b(x_n, x_m) = |x_n - x_m|$ so it converges. In the second case let $\varepsilon > 0$ and then $m \geq n \geq N$ as per definition, then whenever $x_m < x_n$ we have that $b(x_n, x_m) = \log(x_n) - \log(x_m) \leq \varepsilon$ so x_m is bounded from below by $x_N e^{-\varepsilon}$, but also bounded from above by $x_N + \varepsilon$; in this interval b is equivalent to the Euclidean metric.
- (M, b_1) is not backward complete since the sequence $x_n = 1/n$ does not converge to zero, but, for $m \geq n \geq N$, $b_1(x_m, x_n) = 1/n - 1/m \leq 1/n \leq 1/N$.
- We have that $D^+(0, \varepsilon) = [-\varepsilon, \varepsilon]$ while $D^-(0, \varepsilon) = [-\varepsilon, 0]$, so (M, b_1) is backward locally compact but is not forward locally compact.

Consider the space $([-1, 1], b_2)$ in Example 4.2.

- The space $([-1, 1], b_2)$ is symmetrically complete but it is not forward complete and is not backward complete.
- (M, b_2) is symmetrically locally compact, but it is not forward locally compact and not backward locally compact.

The space $([-1, 1], b_3)$ in 4.4 is forward and backward complete; it is compact.

11.2 Randers spaces

We now propose an example based on the *Randers metrics* (that are a classical example of Finsler structures — see in Sec. 1.3C and Chap. XI in [2]).

Consider a Riemannian manifold (M, g); we call len^g the length of absolutely continuous paths in (M, g), $|v|_x = \sqrt{g_x(v, v)}$ the norm of vectors $v \in T_x M$, and δ the Riemannian distance in (M, g). It is well–known that $\mathrm{len}^g = \mathrm{Len}^\delta$. (A possible proof is Prop. 2.25 in [17]).

Suppose moreover that there exists a smooth $f : M \to \mathbb{R}$ such that

$$\forall x \in M \ , \quad |\nabla f(x)|_x \leq 1 \ ; \tag{11.2}$$

this implies that

$$|f(y) - f(x)| \leq \delta(x, y) \ .$$

We now proceed as in Example 2.4. We define

$$F(x, v) = |v|_x + g_x(\nabla f(x), v) \ ,$$

a simple computation shows that

$$\mathrm{len}^F(\gamma) = \mathrm{len}^g(\gamma) + f(y) - f(x)$$

for any γ connecting x to y, and then

$$b^F(x,y) = \delta(x,y) + f(y) - f(x) \ ,$$

and

$$d^F(x,y) = \delta(x,y) + |f(y) - f(x)| \ .$$

It is easy to prove that b^F is always a distance (and not only a semi distance). Moreover the identity map $(M, b^F) \to (M, \delta)$ is continuous, so the topology of (M, b^F) is finer than the topology of (M, δ) (it has more open sets and less compact sets).

If the inequality in (11.2) is always strict, then the space (M, F) is a classical *"Randers space"*. In particular, the space (M, b^F) is strongly separated, and the distances δ and b^F are locally equivalent, so that the topology of (M, b^F) coincides with the topology of M (as a differential manifold) and of (M, δ) (as a metric space).

If instead there is a large enough region in M where (11.2) is an equality, then the space (M, b) is not strongly separated. In the general case we can anyway study the above objects using the methods developed in this paper.

Proposition 11.7. $\mathrm{len}^F = \mathrm{Len}^{b^F}$*, and the space* (M, b) *is r–intrinsic.*

Proof. We recall some results from [17]. Let $(\mathrm{len}, \mathcal{C})$ be a length structure, and b^l be the induced semi distance. In Sec. 2.4 in [17] we noted that Len^{b^l} is the relaxation of len according to an appropriate topology τ_{DF}; in Thm. 2.19 we then concluded that $\mathrm{len}^F = \mathrm{Len}^{b^f}$ iff len^F is lower semi continuous in τ_{DF}.

Let now $\mathcal{C}_{\mathrm{AC}}$ be the class of absolutely continuous paths. $(\mathrm{len}^g, \mathcal{C}_{\mathrm{AC}})$ is a length structure. We already argued that $\mathrm{len}^g = \mathrm{Len}^\delta$, and len^g is lower semi continuous in τ_{DF}.

The quantity $f(y) - f(x)$ is locally constant according to the topology τ_{DF}. So we obtain that len^F is lower semi continuous on $\mathcal{C}_{\mathrm{AC}}$ and that $\mathrm{len}^F = \mathrm{Len}^{b^F}$.

It is also easy to see that $(\mathrm{len}^F, \mathcal{C}_{\mathrm{AC}})$ is a *run-continuous length structure* (as defined in 2.4 in [17]). So by Prop. 3.18 in [17] the space (M, b) is r–intrinsic. ☐

Note that, when we have equality in (11.2) at some points in M, we cannot use Prop. 2.25 in [17] directly.

We conclude by remarking that the above type of reasonings was also one of the main ingredients of [16].

12 Hopf–Rinow Theorem

We now present the asymmetric Hopf–Rinow-like theorem that holds in our settings. We define that A is *forward–bounded* if $A \subseteq B^+(x, r)$ for a choice of $x \in M$, $r > 0$.

Note for example that the image $\gamma([a, c])$ of a run-continuous path $\gamma : [a, c] \to M$ is forward and backward–bounded.

Theorem 12.1 (Hopf–Rinow)**.** *Consider the following three statements.*

1. *Forward–bounded and closed sets are compact.*
2. (M, b) *is forward complete.*
3. *Any rectifiable minimizing geodesic* $\gamma : [a, c) \to M$ *may be completed to a path that is run-continuous on* $[a, c]$ *and continuous at* c.

In general, the implications (1)⇒(2)⇒(3) *hold for any asymmetric metric space* (M, b).

Suppose that (M, b) *is r–intrinsic and forward-locally compact, then the three properties above are equivalent.*

We gladly note that the theorem works as desired in the realm $\mathcal{C}_r$ of *run-continuous* paths; and that *strong separation* is not a necessary condition.

Note that statement *(1)* implies that any two points x, y with $\rho = b^r(x, y) < \infty$ may be connected by a minimizing geodesic (that is also continuous if the space is strongly separated); indeed $D^{+r}(x,\rho) \subseteq D^+(x,\rho)$ so we may apply Thm. 9.1.

Note that statement *(3)* is not saying that the extension of γ is a geodesic. Indeed Example 11.6 suggests that this may be false in general.

The example 2.1.17 in [18] show that, even in the symmetric case, we cannot discard any of the hypotheses in the above theorem.

12.1 Lemmas

In the rest of the section we will mainly prove the Theorem. We extracted from the proof a plethora of lemmas (some of some interest in themselves); some were presented in the section 6 on midpoint properties, some in Sec. 5, the others are here following.

Definition 12.2. Let $D^+(a,\rho) \stackrel{\text{def}}{=} \{y \mid b(a,y) \le \rho\}$. We define the *forward radius of compactness* $R : M \to [0, +\infty]$ as

$$R(x) \stackrel{\text{def}}{=} \sup\{\rho \ge 0 \mid D^+(x,\rho) \text{ is compact }\} \quad .$$

Note that for all ρ with $0 \le \rho < R(x)$ we have that $D^+(x,\rho)$ is compact. (M, b) is *forward-locally compact* iff $R(x) > 0\ \forall x \in M$.

Lemma 12.3.
- *$\forall x, y \in M$ with $b(x, y) < \infty$ we have $R(y) \ge R(x) - b(x, y)$.*
- *Consequently, if $R(y) < \infty$ then $\forall x \in M$ with $b(x, y) < \infty$ we have $R(x) < \infty$.*
- *$d(x, y) \ge |R(x) - R(y)|$ for all $x, y \in M$ for which $R(x), R(y) < \infty$.*

Proof. Indeed, fix $x, y \in M$ with $b(x, y) < \infty$; if $R(x) \le b(x, y)$ there is nothing to be proved; otherwise, for any ρ with $b(x, y) < \rho < R(x)$ we have that $D^+(y, \rho - b(x, y))$ is compact, since

$$D^+(y, \rho - b(x, y)) \subseteq D^+(x, \rho) \quad .$$

This implies that $R(y) \ge \rho - b(x, y)$, and then by arbitrariness of ρ we obtain $R(y) \ge R(x) - b(x, y)$. If $R(y)$ is finite, the above entails

$$b(x, y) \ge R(x) - R(y) \ ;$$

reversing the role of x, y we obtain the second statement. □

In general (even when $0 < R(x) < \infty$) it is possible to find examples where $D^+(x, R(x))$ is compact, and examples where it is not.

Example 12.4. Let $M \subset \mathbb{R}$ be given by

$$I = \left([-7, -6] \cap \mathbb{Q}\right) \cup [-6, -4) \ ;$$

and $M = I \cup \{0\}$; let

$$b(x, y) = \begin{cases} |x - y| & \text{if } x, y \in I, \\ 4 & \text{if } x \in I, y = 0, \\ \infty & \text{if } x = 0, y \neq 0 \end{cases}$$

The topology of (M, b) is the Euclidean topology. We note that

- $R(x) = 0$ when $x \in [-7, -6]$,
- $R(x) = x + 6$ when $x \in [-6, -5)$, and $D^+(x, R(x))$ is compact,
- $R(x) = -x - 4$ when $x \in [-5, -4)$, and $D^+(x, R(x))$ is not compact,
- $R(0) = \infty$ since $D^+(0, \rho) = \{0\}$ is compact for any ρ.

In *r–intrinsic* and *forward-locally compact* spaces, instead, we can precisely describe the behavior of $R(x)$ as follows.

Lemma 12.5. *Suppose that* (M, b) *is* r–intrinsic *and* forward-locally compact. *Choose* $x \in M$ *such that* $R(x) < \infty$ *and fix* $\rho > 0$ *such that* $D^+(x, \rho)$ *is compact; let* $\delta \overset{def}{=} \min_{y \in D^+(x,\rho)} R(y)$. *Then* $\delta > 0$*, and* $R(x) = \rho + \delta$.

Proof. Since $D^+(x, \rho)$ is compact then b is bounded on it, so by the second point of 12.3, $R(y) < \infty$ for all $y \in D^+(x, \rho)$; hence R is continuous and bounded by the third point of 12.3; since $D^+(x, \rho)$ is compact and $R > 0$ we conclude that $\delta > 0$. Choose $0 < t < \delta$; define V_t as in 6.4; we know that $V_t = D^+(x, \rho + t)$. We want to prove that V_t is compact. Indeed choose $(z_n)_n \subset V_t$; then $z_n \in D^+(y_n, t)$ for a choice of $y_n \in D^+(x, \rho)$; choose s so that $t < s < \delta$; up to a subsequence, $y_n \to y \in D^+(x, \rho)$, so that for n large, $d(y, y_n) \leq s - t$ hence $b(y, z_n) \leq s$. We proved that z_n is definitively contained in $D^+(y, s)$; $D^+(y, s)$ is compact since $s < \delta \leq R(y)$; so we can extract a converging subsequence.

"$V_t \supseteq D^+(x, \rho + t)$" and "$V_t$ compact" imply that $R(x) \geq \rho + t$, and we conclude by arbitrariness of t that $R(x) \geq \rho + \delta$. The opposite inequality is easily inferred from first point in the previous lemma 12.3, namely $R(y) \geq R(x) - b(x, y)$, that implies $\delta \geq R(x) - \rho$. □

A corollary of the above lemma is that (in the above hypothesis) $D^+(x, R(x))$ is not compact; so the above lemma is the quantitative version of the argument shown in (8) in section I in [6].

Lemma 12.6. *Suppose that* (M, b) *is r-intrinsic, for simplicity. Suppose that* $0 < R(x) < \infty$*, let* $\rho = R(x)$*; then for any* y *with* $L = b(x, y) \leq \rho$ *there is an arc-parameterized minimizing geodesic* $\gamma : [0, L] \to M$ *connecting* x *to* y.

This Lemma is actually a Corollary of Theorem 9.1. An alternative proof, based on the repeated application of Lemma 6.3 (similarly to Prop. 6.5.1 in [2]) is in Sec. B.8.

Example 4.1 shows that, even if the space is strongly separated, we cannot expect that $\gamma(t)$ be continuous at $t = \rho$ in general: indeed the "identity path" $\gamma : [0, 1] \to [0, 1]$ is a minimizing geodesic but is not continuous at 0.

Lemma 12.7. *Suppose that* (M, b) *is r–intrinsic. Suppose that there is an* x *such that* $0 < R(x) < \infty$*, let* $\rho = R(x)$*. Suppose that any rectifiable minimizing geodesic* $\gamma : [a, c) \to M$*, with* $\gamma(a) = x$ *and* $\mathrm{Len}^b(\gamma) \leq \rho$*, may be extended to a path that is run-continuous on* $[a, c]$ *and continuous at* c*. Then for any sequence* $(y_n)_n \subset D^+(x, \rho)$ *there is an* $y \in D^+(x, \rho)$ *and a subsequence* n_k *such that* $\lim_k b(y, y_{n_k}) = 0$.

Proof. Let $(y_n)_n \subset D^+(x, \rho)$, and let $L_n = b(x, y_n)$. If $\liminf_n L_n < \rho$, we can extract a subsequence n_k s.t. $L_{n_k} \leq t < \rho$, that is, $\{y_{n_k}\} \subset D^+(x, t)$ that is compact: so we can extract a converging subsequence.

Suppose now that $\lim_n L_n = \rho$. For any y_n we use Lemma 12.6 to obtain the minimizing geodesic $\gamma_n : [0, L_n] \to M$ connecting x to y_n; if $L_n < \rho$ we extend γ_n constantly to $[L_n, \rho]$ (the same method is used in eqn. (B.5)).

This part of the proof follows closely the proof of Theorem 9.1 (see Sec. B.6), so we will just sketch these steps. Let $D \subseteq [0, \rho)$ be a countable dense subset. For any $t \in D$, $t < \rho$ we have that $\gamma_n(t) \in D^+(x, t)$ that is compact. Using a diagonal argument, up to a subsequence, we obtain that $\lim_n \gamma_n(t)$ exists, and we define $\xi : D \to M$ by $\xi(t) = \lim_n \gamma_n(t)$. Again up to a subsequence we can assume that ℓ^{γ_n} converges uniformly to an ℓ that is Lipschitz (of constant 1). We imitate the reasoning following eqn. (B.2): any ball $D^+(x, r)$ with $r < \rho$ is compact; so we can extend ξ to a run-continuous $\gamma : [0, \rho) \to M$, and $\ell^\gamma \equiv \ell^\xi$. Moreover γ is a arc-parameterized minimizing geodesic, since for any $t \in D$,

$$\ell^{\xi_n}(t) = b(x, \xi_n(t)) = t$$

for n large (*i.e.* when $L_n \geq t$), and moreover

$$\mathrm{Len}^b \gamma_{|[a,t]} \leq \liminf_n \mathrm{Len}^b(\gamma_n)_{|[a,t]} = \lim_n b(x, \gamma_n(t)) = b(x, \gamma(t)) = t\ ,$$

but $b = b^r$ so we obtain that $\mathrm{Len}^b \gamma_{|[a,t]} = b(x, \gamma(t)) = t$ *i.e.* $\ell^\xi(t) = \ell(t) = t$.

The hypothesis now states that we can further extend $\gamma : [0, \rho] \to M$ so that $\gamma(t)$ is continuous at $t = \rho$; let $y = \gamma(\rho)$. Let now $\varepsilon > 0$; since $\gamma(t)$ is continuous at $t = \rho$, there is a t s.t. $b(y, \gamma(s)) < \varepsilon$ for all s, $t \le s \le \rho$; fix $s > t$, $s > \rho - \varepsilon$, $s \in D$; we apply the triangle inequality to prove that

$$b(y, y_n) \le b(y, \gamma(s)) + b(\gamma(s), \gamma_n(s)) + b(\gamma_n(s), y_n)$$

this is less than 3ε, definitively in n. □

12.2 Proof of Theorem 12.1

We now use the above lemmas to prove 12.1.

Proof. Suppose that forward-bounded closed sets are compact, we prove that the space is forward complete. If x_n is a forward-Cauchy sequence, then there exists N s.t. $b(x_N, x_m) \le 1$ for $m > N$, that is, $x_m \in D^+(x_N, 1)$ that is compact; then we can extract a converging subsequence, and use Prop. 4.4 to obtain the result. Let $\gamma : [a, c) \to M$ be the rectifiable geodesic. Suppose that (M, b) is forward complete, we prove that geodesics may be completed; indeed by Lemma 5.8 the limit $y = \lim_{t\to c-} \gamma(t)$ exists, so we define $\gamma(c) = y$. By Lemma 5.4 this extension is run-continuous. Suppose that the space is r–intrinsic and forward-locally compact, and that any minimizing geodesic $\gamma : I \to M$ defined on $I = [a, c)$ may be extended: we will prove that forward–bounded closed sets are compact (that is, that the *radius of compactness* $R(x) \equiv \infty$). The proof is by contradiction: suppose that there is an x such that $\rho = R(x) < \infty$; Lemma 12.5 implies that $D^+(x, \rho)$ cannot be compact. At the same time, fix any $(y_n)_n \subset D^+(x, \rho)$, by Lemma 12.7 there is $y \in D^+(x, \rho)$ and a subsequence n_k such that $\lim_k b(y, y_{n_k}) = 0$, so y_{n_k} enters definitively in the ball $D^+(y, \varepsilon)$, that is compact when $0 < \varepsilon < R(y)$; so y_{n_k} admits a converging subsequence, hence $D^+(x, \rho)$ would be compact. Summarizing, by contradiction, we deduce that $R(x) = \infty$.

□

We remark that the proof of the above equivalence cannot simply follow from the proof for metric spaces §1.11 in [12], since that proof uses the property described in Remark 4.5; neither it does follow from the proof in Finsler Geometry (see section VI of [2]), since the latter uses the exponential map. The proof is also more involved than in *General metric spaces*, many more extra technical lemmas are needed.

13 On the semidistances and the separation hypotheses

We conclude the paper with some remarks on the rôles of $b = 0$ or $b = \infty$.

Consider a symmetric semidistance d, that is a $d : M \times M \to [0, \infty]$ satisfying

- $d \ge 0$ and $\forall x \in M$, $d(x, x) = 0$;
- $d(x, y) = d(y, x)\ \forall x, y \in M$;
- $d(x, z) \le d(x, y) + d(y, z)\ \forall x, y, z \in M$.

There is a standard procedure to reduce this case to the more usual case of metric spaces. Indeed, the relation $x \sim y$ given by

$$x \sim y \iff d(x, y) = 0$$

is an equivalence relation; if we define $\tilde{M} = M/\sim$ and let $\tilde{d}([x], [y]) = d(x, y)$ then $(\tilde{M}, \tilde{d})$ is a metric space. Many important properties and operations (both metric and topological) can be "projected" from (M, d) to $(\tilde{M}, \tilde{d})$.

Suppose that b is an asymmetric distance. If the space is not strongly separated, it may be the case that, for a pair $x, y \in M$ with $x \neq y$, $b(x, y) = 0$ but $b(y, x) > 0$. When we associate to (M, b) the symmetric distance d using (2.1), we also have that $d(x, y) > 0$. So we cannot address this situation projecting to the quotient, as above. This is the reason why we have to deal with the case of $x, y \in M$, $x \neq y$, $b(x, y) = 0$ in some results of this paper, such as 10.3.

Note that the procedure described at the beginning of the section may be instead used, in the asymmetric case, to project a space (M, b) where b is an asymmetric semidistance to a space $(\tilde{M}, \tilde{b})$ where $\tilde{b}$ is an asymmetric distance; by defining $\tilde{M} = M/\!\sim$ and

$$x \sim y \iff b(x, y) = b(y, x) = 0 \ .$$

14 When $b = \infty$

The attentive reader may have noted that, in our definition of asymmetric metric space, there may be points x, y at infinite distance. There are good reasons at that. In general, even if the distance d is symmetric and $d < \infty$ at all points, it may be the case that two points $x, y \in M$ cannot be connected by a continuous curve, so the induced geodesic distance $d^g(x, y) = \infty$. So we included this possibility in the definition. We propose some remarks on the infinite distance of points.

- Consider a symmetric distance d such that $d : M \times M \to [0, \infty]$, but otherwise satisfying all the usual axioms. Again, there is a standard procedure [4] to reduce this case to the more usual case of metric spaces with $d < \infty$. Indeed, the relation $x \sim y$ given by

  $$x \sim y \iff d(x, y) < \infty$$

  is an equivalence relation; moreover equivalence classes are both open and closed, that is, points in different equivalence classes are in different connected components.
  Usual questions in topology and geometry can be studied by restricting our attention to an equivalence class (or to a connected component); so most texts, when presenting the theory of metric spaces, define the distance d as $d : M \times M \to [0, \infty)$.
- The above method immediately fails if the distance b is not symmetric, since $b(x, y) < \infty$ is not an equivalence relation (it fails to be symmetric, obviously). More in general, we have developed the theory of geodesics using *run-continuous* curves; we have seen in examples that a run-continuous curve can start in a connected component and end in a different connected component: so we cannot study this theory by "restricting to a connected component".
- Another approach is as follows.

  Proposition 14.1. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and concave, $\varphi(x) = 0$ only for $x = 0$. Then φ is subadditive. So, defining $\tilde{b} \overset{def}{=} \varphi \circ b$, $\tilde{b}$ is an asymmetric distance.*

  Define φ by $\varphi(t) = t/(1 + t)$ and $\varphi(\infty) = 1$; then we set $\tilde{b} \overset{\text{def}}{=} \phi \circ b$; so we obtain a space $(M, \tilde{b})$ that is topologically equivalent, and where the distance does not assume the value $+\infty$.
- Unfortunately this last remedy is, in general, only a placebo, when we are interested in intrinsic spaces and/or in studying geodesics: suppose b is intrinsic and we decide to set φ as above and define $\tilde{b} \overset{\text{def}}{=} \phi \circ b$: then $\tilde{b}$ is not intrinsic; so if we try to substitute $\tilde{b}$ by its generated r-intrinsic distance $\tilde{b}^r$ we find out, by prop. 14.2 below, that $\tilde{b}^r = b^r = b$, and we are back to square one.

So it seems that we may sometimes be forced to address the case when $b = \infty$ for some points.

Proposition 14.2. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and concave, $\varphi(x) = 0$ only for $x = 0$. Let $\tilde{b} \overset{def}{=} \phi \circ b$. Suppose moreover the derivative of φ exists and is finite at 0. If γ is run-continuous then*

$$\varphi'(0) \operatorname{Len}^b \gamma = \operatorname{Len}^{\tilde{b}} \gamma$$

4 Compare the idea in Exercise 2.1.3 in [3].

so

$$b^r(x,y)\varphi'(0) = \tilde{b}^r(x,y) \ ;$$

and similarly for b^g and b^s.

Proof. Let $\gamma : [a,c] \to M$. Let $\varepsilon > 0$. In the definition (2.2) of $\mathrm{Len}^b \gamma$ it is not restrictive to use only subsets T of $[a,c]$ such that $b(\gamma(t_i), \gamma(t_{i+i})) \le \varepsilon \ \forall i \in \{1, \dots, n-1\}$ (use uniform continuity of ℓ^γ and (2.4) to prove this fact).

Let now $\delta > 0$; then there exists an $\varepsilon > 0$ such that

$$x(a-\delta) \le \varphi(x) \le x(a+\delta) \quad \forall x \in [0,\varepsilon]$$

with $a = \varphi'(0)$; we obtain that

$$(a-\delta)\,\mathrm{Len}^b \gamma \le \mathrm{Len}^{\tilde{b}} \gamma \le (a+\delta)\,\mathrm{Len}^b \gamma$$

hence the conclusion. □

If $\varphi'(0) = \infty$, wild things may happen: see Example 3.6 in [17], or 1.4.b in [12].

15 Conclusions

In developing this paper some natural definitions and questions have been skipped; for example we did not present a definition of *Lipschitz maps*. (A definition had been there in a draft version...).

In the symmetric case the length of a curve may be seen as the integral of the *metric derivative* along the curve itself; and the *metric derivative* itself is then perused in the developing of *analysis in metric spaces*, a field in current and interesting active development. A definition of *metric derivative* for strongly separated asymmetric distances is discussed in Sec. 2 in [20].

The reader may also have noted that no theory have been presented about the class $\mathcal{C}_s$ defined in the introduction. A theory of geodesics in the class $\mathcal{C}_s$ has yet to be developed (if it will be of any interest).

15.1 On the class $\mathcal{C}_r$

The results in Sec. 6 and Sec. 10 state that, in complete spaces ("complete" in an appropriate sense),

- existence of *quasi-midpoints* is tightly related to the space being *r–intrinsic*;
- existence of *midpoints* is tightly related to existence of geodesics (in the class $\mathcal{C}_r$ of *run-continuous* paths).

Furthermore,

- in Sec. 9, where we presented theorems that ensure the existence of minimizing geodesics, we found out that the results in the class $\mathcal{C}_r$ are more satisfactory than the results in the class $\mathcal{C}_g$; indeed in $\mathcal{C}_g$ (at the state of the art of this paper) we needed to assume that the space is strongly separated;
- in Sec. 8 we noted that the *arc-length reparameterization* of a run-continuous path is always run-continuous, but the reparameterization of a continuous path may fail to be continuous.

Those results further support the idea (already developed in [17]) that, in the asymmetric case, run-continuous paths are more "natural" than continuous paths.

15.2 On Ascoli–Arzelà-type theorems

The proof of Theorem 9.1 (in Sec. B.6) and the proof of Lemma 12.7 contain an argument of this type: *let $I \subset \mathbb{R}$ interval, $D \subseteq I$ dense and countable, and $\gamma_n : I \to M$ run-continuous paths with* $\lim_n \mathrm{Len}^b(\gamma_n) = L < \infty$;

suppose that the traces $\gamma_n(I)$ are contained in a common compact set; up to an appropriate reparameterization $\tilde{\gamma}_n$ of γ_n, there exists a sequence n_k and a run-continuous path $\gamma : I \to M$ with $\mathrm{Len}^b(\gamma) \le L$ and such that $\tilde{\gamma}_{n_k}(t) \to \gamma(t)$ for all $t \in D$.

The above argument is a primitive Ascoli–Arzelà-type theorem, applied only to paths, and where the convergence $\tilde{\gamma}_{n_k} \to \gamma$ is not as strong as the standard Ascoli–Arzelà theorem would suggest — this is due to the fact that the space is not assumed to be strongly separated (and then arc-parameterized geodesics may be not continuous, even when contained in a compact set, *cf*. Cor. 3.4 and Example 8.6).

It would be interesting to study this argument further.

Collins and Zimmer [8] present an asymmetric Ascoli–Arzelà theorem for functions $f_n : M \to N$ between quasi–metric spaces (where the topology is distinguished in a "forward" and a "backward" topology, and hence the continuity of functions).

Rossi *et al* [20] study in Sec. 2 the following setting: let (X, σ) be Hausdorff topological space, and Δ a atrongly separated asymmetric distance that is lower semi continuous according to σ; then they propose in Proposition 4.8 an Ascoli–Arzelà theorem for paths $u_n : [0, T] \to X$.

It may be the case that an adaptation of the arguments in [8] or in [20] would provide an asymmetric Ascoli–Arzelà for an appropriate class of functions between asymmetric metric spaces, so that this novel theorem may be used to write a more concise proof of Theorem 9.1 and of Lemma 12.7. (But, see Example 5.13 in [8] for caveats).

15.3 Hamilton-Jacobi equations

The definitions of *asymmetric metric space*, of *Cauchy sequences* and *completeness* used in this paper were already presented in [16].

The paper [16] dealt with viscosity solutions $u : M \to \mathbb{R}$ of an Hamilton-Jacobi equation $H(x, Du(x)) = 0$; where M is a differentiable manifold. It has long been known that such equation is associated to a (possibly asymmetric) distance b on M. The paper [16] assumed a hypothesis "$\exists \underline{u}$", that says that *there exists a smooth subsolution $\underline{u}$ such that* $H(x, D\underline{u}(x)) < 0$; this hypothesis implied that the asymmetric metric space (M, b) is strongly separated; in particular, inside any compact subset, the space is a *General Metric Space* (see Prop. 3.8 in [16]), hence the results due to Busemann could be applied. Eventually [16] used the Hopf–Rinow theorem in (M, b) to show that, when the space (M, b) is backward complete, results of existence and uniqueness of the solution hold.

There may be interest in weakening the requirement "$\exists \underline{u}$" to: *"there exists a subsolution $\underline{u}$ such that* $H(x, D\underline{u}(x)) \le 0$": see the discussion in Sec. 3.6 in [16]. In this case the associated asymmetric metric space would not (necessarily) be strongly separated. One hope in developing the current paper is that the Hopf–Rinow Theorem 12.1 here presented may be used for a generalization of the results in [16].

A Comparison with related works

A.2 Regarding geodesics

Remark A.1. In all works cited in this section, a path is a continuous mapping. We remark moreover that in all of these works, a run-continuous path is also continuous; either because the metric is symmetric, or because of the extra hypothesis (2.7). Instead in this paper, when studying geodesics, we considered run-continuous paths; and in this paper a run-continuous path is not necessarily continuous. This marks a fundamental difference between the definitions in 7.1 here, and those found in other papers.

A.2.1 Glossary

Unfortunately the word *geodesic* has been associated to different and incompatible definitions in the literature. Let's see other possible names and definitions. Let $I \subseteq \mathbb{R}$ be an interval.

- Consider a *minimizing geodesic connecting x to y* as defined in 7.1.
  It is called *shortest path* in the definition 2.5.15 in [3].
  It is called *shortest join* in [5, 6].
  It is called *geodesic* in the introduction of Chap. 2 in [18]. Curiously, the definition of *geodesic* given in the introduction of Chap. 2 in [18] is different from the Definition 2.2.1 in the same book [18], see (A.2) here.
- Consider a *local geodesic* as defined here in 7.1.
  When the space is intrinsic, it is called *geodesic* in the definition 2.5.27 of [3].
- Consider a path connecting x, y such that

$$b(x, y) = \mathrm{Len}^b(\gamma) \ . \tag{A.1}$$

  It is called *segment* in [4–6, 24].
  A *segment* is necessarily a *minimizing geodesic*; the vice versa is true when the space is intrinsic. Moreover existence of segments does imply that the space be intrinsic, as we will see afterward.
- Consider a distance preserving path $\gamma : I \to M$, *i.e.*

$$\forall s, t \in I, s < t \ \Rightarrow \ t - s = b\big(\gamma(s), \gamma(t)\big) \ . \tag{A.2}$$

  When the space is intrinsic, this is called *minimizing geodesic* in Sec. 1.9 in [12].
  In [5, 6] Busemann calls this a *straight line* (assuming $I = \mathbb{R}$).
  In Definition 2.2.1 in [18] this is called a *geodesic path* (or simply *geodesic*) when $I = [a, b]$, a *geodesic ray* if $I = [0, \infty)$, a *geodesic line* if $I = \mathbb{R}$. A *geodesic segment* in [18] is the image of a geodesic path; a *straight line* is the image of a geodesic line.
  Up to arc reparameterization, this is similar to the definition of *global geodesic* proposed here in, but with an important difference: b^r is replaced with b. That is, if the space is r-intrinsic, then a path satisfying (A.2) is a *arc parameterized global geodesic* (in the language of this paper). See though Remark A.2.
- Consider a locally distance preserving path $\gamma : I \to M$, *i.e.*

$$\forall t_0 \in I,\ \exists \varepsilon > 0,\ \forall s, t \in I,\ t_0 - \varepsilon < s < t < t_0 + \varepsilon \ \Rightarrow \ t - s = b\big(\gamma(s), \gamma(t)\big) \ . \tag{A.3}$$

  In [5, 6] Busemann calls this a *partial geodesic*, or *geodesic* if $I = \mathbb{R}$. (In [4] *geodesics* were equivalence classes of paths; that definition was simplified in later texts.)
  When (M, b) is intrinsic, this is called *geodesic* in Sec. 1.9 in [12].
  It is called *local geodesic* in Definition 2.4.8 in [18].
  In [24] this same definition is called *extremal*.
  This is similar to the definition of *local geodesic* proposed here in, but with an important difference: b^r is replaced with b. That is, if the space is r-intrinsic, then a path satisfying (A.3) is a *arc parameterized local geodesic*. See though Remark A.2.

Remark A.2 (Arc parameter and strong separation). If the space (M, b) is r-intrinsic and strongly separated, then a theory of *continuous partial geodesics* based on the the definition (A.3) would be equivalent to a theory of *continuous local geodesics* based on the definition in 7.1. Indeed we may always reparameterize any local geodesic to arc parameter, so as to satisfy (A.3). Due to the discussion in Sec. 8.1 we understand that, in the general case of asymmetric metrics, when the "strong separation" hypothesis (1.1) does not necessarily hold, the two approaches are not equivalent. This explains why, in Definition 7.1, we used a formulation that does not force geodesics to be arc–parameterized. The same remark holds for the definition in eqn. (A.2) *vs* the definition of *geodesic* here presented.

A.2.2 Non intrinsic spaces

As we see above, another important difference between the theory of geodesics in some texts and the Definition 7.1 here, is that b is used where we instead use b^r.

A first consequence is Prop. 2.4.2 in [18]: *"if in a space any two points can be connected by a segment (*i.e. *a path satisfying* (A.1)*), then the space has to be intrinsic"*. The same is noted in the introduction in [6].

In some sense, this different choice does not lead to a loss of generality. We recall from [17] that $\mathrm{Len}^b \equiv \mathrm{Len}^{b^r}$ and that the space (M, b^r) is r-intrinsic. So a path γ that is a *geodesic* in (M, b) according to the definition in 7.1, is a *geodesic/segment* in (M, b^r) according to the definition (A.1). So we may think that the two approaches are equivalent, up to replacing (M, b) with (M, b^r).

There is though a subtle difference. Indeed the topology of (M, b) and of (M, b^r) may be different (even when the metric is symmetric). In particular if $D \subseteq M$ is compact in (M, b^r) then it is compact in (M, b); the opposite is not true, as shown *e.g.* by example 4.7 in [17]. So the result 9.1 is more general than what may be expressed using the Definition A.1 and/or assuming that the space is intrinsic. This result 9.1 is then quite useful in cases when (M, b) is not intrinsic, the distance b is known and well understood, but b^r is not completely understood, and yet it is possible to prove that D^{+r} is compact, or contained in a compact set, in the (M, b) topology: this is the case *e.g.* in [9].

A.3 Menger convexity

Suppose that, for any $r > 0, x \in M$, $D^+(x, r)$ and $D^-(x, r)$ are compact: such space (M, b) is called *finitely compact* in [4] and other texts. A *finitely compact* space is also *complete*.

In [4] and later works (or also in [18], where though only symmetric metrics are studied) a (possibly asymmetric) metric space (M, b) is called *Menger convex* if given two different points $x, y \in M$, a third point z (different from x, y) exists such that $b(x, z) + b(z, y) = b(x, y)$.

Proposition A.3. *If the* General Metric space (M, b) *is* finitely compact *and* Menger convex *then the space is intrinsic and any two points* x, y *with* $b(x, y) < \infty$ *can be connected with a* minimizing geodesic.

This proposition is adapted to the language of this paper from (1.16) in [4] [5]; [4] attributes to Menger this kind of result. As we see, the classical definition of *Menger convexity* forces the space to be intrinsic.

For the already expressed reasons, we preferred to propose definitions of *convexity* that do not force the space to be intrinsic.

The Example 10.5 shows that "*finitely compact*" cannot be replaced with "*locally compact*".

A.4 Comparison with quasi metric spaces

A.4.1 Topology

As already pointed out, our definition of the *asymmetric metric* is quite similar to the definition of a *quasi metric* that is found in the literature, cf. Wilson [23], Kelly [13], Reilly, Subrahmanyam and Vamanamurthy [19]. Fletcher and Lindgren [10, (pp 176-181)], Künzi [14], and more recently Künzi and Schellekens [15], Collins and Zimmer [8]. [19] provides also a wide discussion of the references on *quasi metrics*. One important difference is that in many texts, quasi–metrics are defined to be strongly separated (an exception being [15]). Another important difference between our theory of *asymmetric metric spaces* and *quasi metric spaces* is in the choice of the associated topology.

5 Or see Theorem 2.6.2 in [18], where the hypothesis "proper" is the same as the hypothesis "finitely compact" in Busemann's works.

Indeed, we have three topologies at hand:

- the topology τ , generated by the families of forward and backward balls, or equivalently by the metric d defined in (3.1);
- the topology τ^+ generated by the families of forward balls;
- the topology τ^- generated by the families of backward balls.

The topology τ is the topology used in this work; it is called *associated symmetric topology* in [15].

In general it may happen that these three topologies are different.

This problem has been studied in [13]: there Kelly introduces the notion of a *bitopological space* (M, τ^+, τ^-), and extends many definition and theorems, (such as the Urysohn lemma, the Tietze's extension theorem, the Baire category theorem[6]) to these spaces. Unfortunately Kelly does not include the topology τ in his work [13].

We have chosen to associate the topology τ to the "asymmetric metric space". τ is a *symmetric kind of object*, as a consequence, we have only one notion of *"open set"*, of *"compact set"*, of "*the sequence* (x_n) *converges to* x", and of "*the functions* $f : N \to M$ *and* $g : M \to N$ *are continuous*". Furthermore this choice saves some results that are familiar in the common symmetric case, such as Prop. 4.4; these results are false in "quasi metric space" where usually the topology τ^+ is used to test convergence (see Remark A.5 following).

A.4.2 Cauchy sequences, and completeness

In papers on *quasi metric spaces*, the quasi-metric space (M, b) is instead usually endowed with the topology τ^+: this entails a different notion of *convergence* and *compactness*, and poses the problem to find a good definition of "*Cauchy sequence*" and "*complete space*".

This problem has been studied in [19], where 7 different notions of "*Cauchy sequence*" are presented.

We remark that the list in [19] includes the three that we defined in Section 4: a "forward Cauchy sequence" (resp. backward) is a "left K-Cauchy sequence" (resp. right); a "symmetrical Cauchy sequence" is a "b-Cauchy sequence" (and a "biCauchy sequence" in [15]).

Combining these 7 definition with the τ^+ topology, [19] presents 7 different definitions of "*complete space*". Actually, by combining 7 "*Cauchy sequences*" with all the above 3 topologies, we may reach a total of 14 (!) different definitions of "*complete space*" (using the $\overline{b}$ instead of b, see eq. (2.6)). To our knowledge, no one has taken the daunting task of examining all of them.

One of the notions of "*Cauchy sequence*" and "*complete space*" from [19] has been further studied by Künzi [14]; we present it here.

Definition A.4. Künzi [14] defines that:

- a sequence $(x_n) \subset M$ is a "*left b-Cauchy sequence*" when $\forall \varepsilon > 0\ \exists x \in M$ and $\exists k \in \mathbb{N}$ such that $b(x, x_m) < \varepsilon$ whenever $m \geq k$;
- (M, b) is a "*left b-sequentially complete space*" if any *left b-Cauchy sequence* converges to a point, according to the topology τ^+.

Remark A.5. It is easy to prove that

1. if $x_n \to x$ according to τ^+ then the sequence (x_n) is a "*left b-Cauchy sequence*".
2. Any "*forward Cauchy sequence*" (as defined in (4.1)) is a "*left b-Cauchy sequence*". *(To prove this, choose $n = N = k$ and $x = x_n$ in the definition of* left b-Cauchy sequence*)*.
3. If $\tau = \tau^+$, then any "*left b-sequentially complete space*" is a "*forward complete metric space*" as defined in this paper.
   In case $\tau \neq \tau^+$, the implication may not hold.

6 Another version of Baire theorem, using a better definition of completeness, is found in Thm 2 in [19].

4. Whereas, if $x_n \to x$ according to τ^+ then the sequence (x_n) may fail to be either a "*forward Cauchy sequence*" or a "*backward Cauchy sequence*". Indeed, Kelly [13] had encountered this problem, which was a motivation of [19]. Such is the case for the sequence in Example 11.5.(6) here.

For those reasons, it is not easy to compare the results and examples in the above papers, with the result and examples here presented.

A.4.3 Other fields of interest

An interest for quasi metric spaces may be found in Theoretical Computer Science; Smyth [21] [22] proposed quasi-uniformities as a generalized framework in denotational semantics; in doing so, he generalized the concept of "completeness" from uniformities to quasi-uniformities. That notion may be related to the notions presented above: indeed, given a quasi metric space (M, b), this is associated to the quasi uniformity generated by the base of sets $\{(x, y) : b(x, y) < \varepsilon\} \subset M \times M$ (see [15]); if this latter is complete (in the sense of Smyth) then any forward Cauchy sequence will converge according to the topology τ^+.

An interest for quasi metric spaces may be also found in Theoretical Physics; in a study aimed at comparing different notions of *causal boundary* for a spacetime, Flores *et al* [11] compare different notions of *boundaries, completions and compactifications* for Riemannian and Finslerian manifolds; in particular they note that the Cauchy completion of a "General Metric Spaces" is not necessarily a "General Metric Spaces", but in general it is a "quasi metric space"; they propose two notions of *(forward) Cauchy sequence*; they show in Theorem 3.29 that the Cauchy completion is "complete" in the sense that any forward Cauchy sequence will converge according to the topology τ^-, and argue that this topology is the natural one for their framework.

B Proofs

B.5 Proof of 8.5

Proof. For $s \in [0, L]$ we define the preimage

$$I_s \stackrel{\text{def}}{=} \{t \in [a, c]\ ,\ s = \ell^\gamma(t)\}\ \ ;$$

since ℓ^γ is continuous then it is surjective, so I_s is a bounded closed interval, never empty, hence $\psi(s)$ is just its leftmost point. Moreover ψ is injective, and ψ is a right inverse of ℓ^γ *i.e.* $\ell^\gamma(\psi(s)) = s$ for all $s \in [a, c]$.

Let D be the image of ψ, that is the family of all left extrema of I_s for $s \in [0, L]$. Obviously $\ell^\gamma(D) = [0, L]$, so by Lemma 5.6 we obtain that $\ell^\gamma \equiv \ell^\gamma_D$.

Fix $\tilde{s} \in [0, L]$. Let $\tilde{t} = \psi(\tilde{s})$ so $\ell^\gamma(\tilde{t}) = \tilde{s}$. Since ψ is injective and its image is D, visual inspection shows that $\ell^\gamma_D(\tilde{t}) = \ell^\theta(\tilde{s})$ so $\ell^\theta(\tilde{s}) = \tilde{s}$.

The proof for β is identical, just replace $\psi(L)$ in D with c. □

B.6 Proof of 9.1

Proof. Fix $x, y \in M, \rho > 0$, as in the statement.

Suppose $y \neq x$ (otherwise $\gamma \equiv x$ is the geodesic) and $b(x, y) > 0$ (otherwise the geodesic can be defined as in the proof of 7.4).

Let $L = b^r(x, y)$; $L \le \rho$ by hypothesis; assuming $b(x, y) > 0$ then $L > 0$. Let $\gamma_n : [0, 1] \to M$ be a sequence of rectifiable run-continuous paths from x to y such that, $L_n \stackrel{\text{def}}{=} \text{Len}^b\, \gamma_n$,

$$L + 1/n \ge L_n \ge L_{n+1} \ge L\ \ ,$$

and γ_n are parameterized using Lemma 8.4 with $\varepsilon = 1/n$; so ℓ^{γ_n} is Lipschitz of constant $L + 2/n$.

Let now $D \subset [0,1]$ be a dense countable set with $0, 1 \in D$; let $\xi_n = \gamma_n|_D$ the restriction; by Lemma 5.6 $\ell^{\gamma_n} \equiv \ell^{\xi_n}$. (Recall that, by the definition in eqn. (5.6), both functions $\ell^{\gamma_n}, \ell^{\xi_n}$ are defined on all $[0,1]$).

Consider $t \in D$, $t \neq 0, 1$; let v such that $tL < v < \rho$; $t(L + 2/n) < v$ for n large, and

$$b^r(x, \xi_n(t)) \le \ell^{\xi_n}(t) \le t(L + 2/n)$$

so we obtain that $\xi_n(t) \in D^{+r}(x, v)$ for n large; hence $\xi_n(t)$ admits a converging subsequence. Using a diagonal argument we can find a subsequence n_k such that $\xi_{n_k}(t)$ converges for all $t \in D$. We define $\xi : D \to M$ as $\xi(t) = \lim_k \xi_{n_k}(t)$. For $t = 0, 1$ we have

$$x = \gamma_n(0) = \xi_n(0) = \xi(0) \ , \quad y = \gamma_n(1) = \xi_n(1) = \xi(1) \ \forall n.$$

Using Ascoli-Arzelá theorem we can also assume that $\ell^{\xi_{n_k}}$ converges to a function ℓ that is monotonic and Lipschitz of constant L. For simplicity we will rename the subsequence n_k to n in the following.

The length Len^b_D is lower semi continuous *w.r.t.* the pointwise convergence, hence

$$\ell^{\xi}(1) = \mathrm{Len}^b(\xi) \le \liminf_n \mathrm{Len}^b(\xi_n) = L \ . \tag{B.1}$$

By applying the above idea to any subinterval $[s, t]$ with $s \in D$ and using Lemma 5.3 we also obtain

$$\ell^{\xi}(t) - \ell^{\xi}(s) \le \liminf_n \left(\ell^{\xi_n}(t) - \ell^{\xi_n}(s)\right) = \ell(t) - \ell(s) \le L(t-s) \ . \tag{B.2}$$

This inequality implies two important properties.

- Since D is dense it implies that ℓ^{ξ} is continuous on $[0,1]$.
- Setting $s = 0$, $t \in [0,1)$, from $\ell^{\xi}(t) \le Lt$ we obtain that $\xi(t) \in D^{+r}(x, Lt)$. But $Lt < L \le \rho$ so $D^{+r}(x, Lt)$ is contained in a compact set.

Hence we can define a map $\gamma : [0,1] \to M$ as follows (similarly to the proof of Lemma 5.10), setting $\gamma = \xi$ on D, while for any $t \in (0,1) \setminus D$ we define $\gamma(t)$ as the limit of a subsequence of $\xi(s_n)$ for $s_n \subseteq D$ with $s_n \nearrow t$. By Lemma 5.7 we obtain that $\ell^{\gamma} \equiv \ell^{\xi}$ on $[0,1]$. So γ is run-continuous. Since $L = b^r(x, y)$ is the infimum of the lengths, by eqn. (B.1) we obtain that actually $\ell^{\gamma}(1) = \mathrm{Len}^b(\gamma) = L$. Eventually exploiting the relation (B.2) and the fact that $\ell^{\gamma}(1) = L$, $\ell^{\gamma}(0) = 0$ we prove that actually $\ell^{\gamma}(t) = \ell(t) = Lt$. By a linear change of parameter we obtain the desired curve. □

B.7 Proof of 9.2

This is instead the proof of 9.2; it is based on the classical “direct method” in Calculus of Variations. By replacing $D^{+g}(x, \rho)$ with $D^{+r}(x, \rho)$, and dropping the request that paths be continuous, [7] it can also be a proof of 9.1 when the space is strongly separated and $D^{+r}(x, \rho)$ is compact.

Proof. Fix $x, y \in M$, $\rho > 0$, as in the statement. We will write “D^{+g}” instead of “$D^{+g}(x, \rho)$” and “len” for “Len^b” for brevity. By Lemma 3.1, let ω be the modulus of symmetrization of D^{+g}; let $\tilde\omega(r) \stackrel{\text{def}}{=} \max\{r, \omega(r)\}$: then

$$d(z, y) \le \tilde\omega\left(b(z, y)\right) \tag{B.3}$$

for any $z, y \in D^{+g}$.

Let $L = b^r(x, y)$; suppose $y \neq x$ (otherwise $\gamma \equiv x$ is the geodesic). Let $\gamma_n : [0,1] \to M$ be a sequence of rectifiable continuous paths from x to y such that

$$L_n \stackrel{\text{def}}{=} \mathrm{len}\,\gamma_n \ , \quad L_n \ge L_{n+1} \to_n L$$

7 But note that, due to Proposition 4.9, run-continuous paths are continuous in this proof!

and moreover γ_n are parameterized using Lemma 8.4 with $\varepsilon = 1/n$. By (B.3) above,

$$d\big(\gamma_n(t), \gamma_n(s)\big) \le \tilde{\omega}\big((L_1 + 1)\,|t - s|\big) \tag{B.4}$$

for all $s, t \in [0, 1]$.

Suppose now that $L = b^r(x, y) < \rho$; then definitively $L_n \le \rho$; by eqn. (2.4), we know that all of γ_n is contained in D^{+g}. Combining this argument and (B.4) we can apply the Ascoli-Arzelà theorem: we know that there is a $\gamma : [0, 1] \to M$ (that again satisfies (B.4)) such that, up to a subsequence, there is uniform convergence of $\gamma_n \to \gamma$; this uniform convergence is *w.r.t.* the distance $d(x, y) = b(x, y) \vee b(y, x)$. The functional $\gamma \mapsto \operatorname{len}\gamma$ is l.s.c. *w.r.t.* uniform convergence so we conclude that γ is a geodesic connecting x to y.

When $L = b^r(x, y) = \rho$, if γ_n is frequently wholly contained in D^{+g}, all works as above; otherwise the proof is obtained by a slight change in the above argument. Let t_n be the last of the times such that $\gamma_n([0, t]) \subseteq D^{+g}$, that is,

$$t_n \stackrel{\text{def}}{=} \inf\{s \in [0, 1] \mid \gamma_n(s) \notin D^{+g}(x, \rho)\} = \inf\{s \in [0, 1] \mid b^r(x, \gamma_n(s)) > \rho\} \ .$$

Since ℓ^{γ_n} is Lipschitz then

$$b^r(x, \gamma_n(s)) \le \ell^{\gamma_n}(s) \le (L_n + 1/n)s$$

so

$$\rho \le (L_n + 1/n)\tilde{t}_n$$

so $\tilde{t}_n \to_n 1$. Define

$$\tilde{\gamma}_n(t) = \begin{cases} \gamma_n(t) & \text{if } t < t_n \\ \gamma_n(t_n) & \text{if } t_n \le t \le 1 \end{cases} \quad ; \tag{B.5}$$

since $\tilde{\gamma}_n$ are wholly contained in D^{+g}, we can apply the above reasoning to say that there is a continuous $\tilde{\gamma}$ such that $\tilde{\gamma}_n \to \tilde{\gamma}$ uniformly. To conclude the proof we need to prove that $\tilde{\gamma}(L) = y$:

$$b(\tilde{\gamma}_n(t_n), y) \le b^r(\tilde{\gamma}_n(t_n), y) = L_n - \rho$$

so by (B.3) again, $\tilde{\gamma}_n(t_n) \to y$: the sequence $\tilde{\gamma}_n$ is uniformly equicontinuous, this implies that $\tilde{\gamma}(1) = y$. □

B.8 Proof of Lemma 12.6

This is an alternative proof of Lemma12.6.

Proof. We will define the sequence $(z_n)_{n\ge 1}$ iteratively. $z_0 = x$ and z_{n+1} is minimum point for the problem

$$\min_{z \in D^+(z_n, 2^{-n-1}\rho)} b(z, y) \ . \tag{B.6}$$

Let $\rho_n = (1 - 2^{-n})\rho$ so $\rho_{n-1} + \rho 2^{-n} = \rho_n$. Iteratively, we assert by induction that

$$b(x, z_n) = \rho_n \ \ , \ \ b(z_n, y) = 2^{-n}\rho \ \ ,$$

then $D^+(z_n, 2^{-n-1}\rho) \subseteq D^+(x, \rho_{n+1})$; since $\rho_{n+1} < \rho$ we obtain that the leftmost is compact, so the above problem (B.6) has a minimum z_{n+1} that satisfies

$$b(z_n, z_{n+1}) = 2^{-n-1}\rho \ \ , \ \ b(z_{n+1}, y) = b(z_n, y) - 2^{-n-1}\rho = 2^{-n-1}\rho$$

by Lemma 6.3, so

$$b(x, z_{n+1}) \le b(x, z_n) + b(z_n, z_{n+1}) \le \rho_n + \rho 2^{-n-1} = \rho_{n+1}$$

but also

$$\rho = b(x, y) \le b(x, z_{n+1}) + b(z_{n+1}, y) \le \rho$$

so $b(x, z_{n+1}) = \rho_{n+1}$ and induction step is concluded.

Using 9.1, there exists a arc–parameterized geodesic $\gamma_n : [\rho_n, \rho_{n+1}] \to M$ connecting z_n to z_{n+1}. We define γ on $[0, \rho)$ as the join of all these paths; since all triangle inequalities above are equalities, then γ is a arc-parameterized geodesic on $[0, \rho)$.

To conclude, we set $\gamma(\rho) = y$; the properties $\gamma(\rho_n) = z_n$, $b(z_n, y) = 2^{-n}\rho$ and Lemma 5.4 imply that γ is rectifiable and $\operatorname{Len}^b(\gamma) = \rho$, and that γ is run-continuous on all of $[0, \rho]$. □

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