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Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity

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Abstract. We deal with a class on nonlinear Schrödinger equations (NLS) with potentials $V(x) \sim |x|^{-\alpha}$, $0 < \alpha < 2$, and $K(x) \sim |x|^{-\beta}$, $\beta > 0$. Working in weighted Sobolev spaces, the existence of ground states v_ε belonging to $W^{1,2}(\mathbb{R}^N)$ is proved under the assumption that $\sigma < p < (N+2)/(N-2)$ for some $\sigma = \sigma_{N,\alpha,\beta}$. Furthermore, it is shown that v_ε are *spikes* concentrating at a minimum point of $\mathcal{A} = V^\theta K^{-2/(p-1)}$, where $\theta = (p+1)/(p-1) - 1/2$.

Keywords. Nonlinear Schrödinger equations, weighted Sobolev spaces

1. Introduction

This paper deals with existence of ground state solutions of stationary nonlinear Schrödinger equations of the form

$$\begin{cases} -\varepsilon^2 \Delta v + V(x) v = K(x) v^p, & x \in \mathbb{R}^N, \\ v \in W^{1,2}(\mathbb{R}^N), \quad v(x) > 0, & \lim\limits_{|x| \to \infty} v(x) = 0. \end{cases} \tag{NLS}$$

Here and below, $N \geq 3$ and $1 < p < (N+2)/(N-2)$. A solution of (NLS) is called a *ground state* if it is a Mountain-Pass critical point of the corresponding Euler functional, and hence its Morse index is 1. If u is a solution of (NLS), then

$$\psi(x,t) = \exp(i\lambda\varepsilon^{-1}t) u(x)$$

represents a standing wave of the nonlinear Schrödinger equation

$$i\varepsilon \frac{\partial \psi}{\partial t} = -\varepsilon^2 \Delta \psi + (V(x) - \lambda)\psi - K(x)|\psi|^{p-1}\psi, \tag{1}$$

where ε $(= \hbar)$ is the Planck constant and i is the imaginary unit.

One of the main purposes of this paper is to look for solutions v_ε of (NLS) which have the following properties:

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(i) $v_\varepsilon \in W^{1,2}(\mathbb{R}^N)$;
(ii) v_ε is a ground state.

As for (i), let us point out that standing waves v which have finite L^2 norm are the most relevant from the physical point of view since they correspond to *bound states*. Moreover, if $v \in W^{1,2}(\mathbb{R}^N)$, one can prove that $\lim_{|x|\to\infty} v(x) = 0$ (see the proof of Theorem 16), which implies that solutions are well localized in space.

On the other hand, concerning (ii), the interest in searching ground states relies on the fact that a standing wave is possibly orbitally stable provided it corresponds to a ground state of (NLS), in the sense specified in the literature (see e.g. [14, 17]).

A lot of work has been devoted to the existence of solutions of (NLS), both for $\varepsilon = 1$ and for ε tending to zero. In the latter case, as a specific feature of the nonlinear (focusing) model, solutions concentrate at points with a *soliton* profile. We limit ourselves to citing a few recent papers [5, 6, 8, 9, 11, 12, 15, 16], referring to their bibliography for a broader list of works, although still not exhaustive.

However, to our knowledge, it is everywhere assumed (with the only exception of [18, 20]) that $\liminf_{|x|\to\infty} V(x) > 0$. The main new feature of the present paper is that we will be concerned with potentials V such that $\lim_{|x|\to\infty} V = 0$.

Our main results are Theorems 1 and 3. The former deals with existence of ground states of (NLS), the latter with concentration.

Roughly, (NLS) has a ground state which concentrates at a global minimum point of the auxiliary potential $\mathcal{A} := V^\theta K^{-2/(p-1)}$, where $\theta = (p+1)/(p-1) - N/2$, provided

(i) $V(x) \sim |x|^{-\alpha}$ with $0 < \alpha < 2$,
(ii) $K(x) \sim |x|^{-\beta}$ with $\beta > 0$,
(iii) $\sigma < p < (N+2)/(N-2)$, where σ is a number depending upon α and β, and defined in (2) below.

Some comments on the proof and the preceding assumptions are in order. If we deal with a potential V which decays to zero at infinity, the methods used in the preceding papers cannot be employed. First of all, variational theory in $W^{1,2}(\mathbb{R}^N)$, as in [11, 12], cannot be used. Nor can one apply perturbation methods, as in [6, 16], since the spectrum of the linear operator $-\Delta + V$ is $[0, \infty)$ (see [10]).

To overcome this difficulty, we frame our problem in a class of weighted Sobolev spaces $\mathcal{H}_\varepsilon$, discussed in [19], consisting of the functions u on $\mathbb{R}^N$ for which

$$\int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u(x)|^2 + V(x) u^2(x))\, dx < \infty.$$

In these spaces the nonlinear term $\int_{\mathbb{R}^N} K|u|^{p+1}\, dx$ is well defined if (i)–(iii) hold. Moreover, under these conditions the Euler functional satisfies the Palais–Smale compactness condition on $\mathcal{H}_\varepsilon$, and this allows us to find in a straightforward way a positive Mountain-Pass solution $v_\varepsilon \in \mathcal{H}_\varepsilon$ (see Theorem 13). It is worth pointing out that for such a result it suffices to assume that (i) holds with $0 < \alpha \le 2$.

However, we are interested in solutions which belong to $W^{1,2}$ and which decay to zero at infinity. To achieve these conditions we first prove some careful integral estimates for solutions in $\mathcal{H}_\varepsilon$. The proof of the concentration phenomenon also relies on some sharp pointwise decay estimates and on appropriate bounds of the energy of the Mountain-Pass solutions v_ε, uniformly with respect to ε. These estimates require α to be smaller than 2 and represent one of the main novelties of the present paper.

As for the assumptions, we point out that if $V(x) \sim |x|^{-\alpha}$ with $0 < \alpha \leq 2$, then (iii) cannot be eliminated if we want to find *ground states*. For more details concerning this claim, we refer to Proposition 15 in Section 4. Concerning assumption (ii), see also Remark 14(i).

As already pointed out, the only papers dealing with equations on $\mathbb{R}^N$ with potentials vanishing at infinity are [18] and [20]. The former deals with an *eigenvalue problem* in the radial case. In the latter, weighted Sobolev spaces have also been used. For more details, see Remark 14(ii)–(iii) later on. However, in both the aforementioned papers neither results concerning the fact that the solutions belong to $W^{1,2}(\mathbb{R}^N)$ are given, nor concentration is proved.

The rest of the paper is organized as follows. Section 2 contains our assumptions and main results. Section 3 is devoted to discussing the weighted spaces $\mathcal{H}_\varepsilon$ (including an embedding theorem from [19]), as well as to proving some uniform integral estimates that are used in what follows. In Section 4 we deal with the main existence result, Theorem 1. We first prove (see Theorem 13) that in $\mathcal{H}_\varepsilon$ the Mountain-Pass Theorem applies in a direct way for any $0 < \alpha \leq 2$; next, we assume that $0 < \alpha < 2$ and prove some exponential decay for the above Mountain-Pass critical points, which allows us to show that they give rise to ground states of (NLS); see Theorem 16. Finally, in Section 5 we prove that these ground states are *spikes* concentrating at a minimum point of $\mathcal{A}$. This result is achieved by using the preceding decay estimates, jointly with a uniform bound on the energy of the Mountain-Pass critical points found before.

Notation. Throughout the paper we will use the following notation:

- B_R is the ball $\{x \in \mathbb{R}^N : |x| < R\}$;
- $W^{k,p}(\Omega)$, $W^{k,p}(\mathbb{R}^N)$ are the usual Sobolev spaces;
- $L^p(\Omega)$, $L^p(\mathbb{R}^N)$ are the usual Lebesgue spaces;
- $c, c_1, \ldots, C, C_1, \ldots$ denote possibly different constants;
- $h_1 \sim h_2$ means that h_1 and h_2 are of the same order as $\varepsilon \to 0$.

2. Assumptions and main results

In order to find solutions of (NLS) we will make the following assumptions on V and K:

(V) $V : \mathbb{R}^N \to \mathbb{R}$ is smooth and there exist $\alpha, a, A > 0$ such that

$$\frac{a}{1 + |x|^\alpha} \leq V(x) \leq A,$$

(K) $K : \mathbb{R}^N \to \mathbb{R}$ is smooth and there exist $\beta, k > 0$ such that

$$0 < K(x) \leq \frac{k}{1 + |x|^\beta}.$$

In order to prove existence of ground states of (NLS) as well as their concentration properties we assume a suitable bound on p involving α and β. Let

$$\sigma = \sigma_{N,\alpha,\beta} = \begin{cases} \dfrac{N+2}{N-2} - \dfrac{4\beta}{\alpha(N-2)} & \text{if } 0 < \beta < \alpha, \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

Our main existence result is the following.

Theorem 1. *Let (V), (K) hold, with $0 < \alpha < 2$ and $\beta > 0$, respectively, and suppose p satisfies*

$$\sigma < p < \frac{N+2}{N-2}. \tag{3}$$

Then for every $\varepsilon > 0$ equation (NLS) *has a positive classical solution $v_\varepsilon \in W^{1,2}(\mathbb{R}^N)$. Moreover, v_ε is a ground state of the energy functional corresponding to* (NLS).

Remark 2. (i) The ground state found above is obtained as Mountain-Pass of the energy functional associated to (19) or, equivalently, it realizes the following supremum:

$$\sup_{u \in \mathcal{H}_\varepsilon \setminus \{0\}} \frac{\int_{\mathbb{R}^N} K|u|^{p+1}}{\int_{\mathbb{R}^N} [\varepsilon^2 |\nabla u|^2 + V u^2]^{(p+1)/2}},$$

where $\mathcal{H}_\varepsilon$ is a suitable weighted Sobolev space defined in Section 3. Such a supremum is $+\infty$ if $p < \sigma$ as well as if $p > (N+2)/(N-2)$. For more details we refer to Proposition 15.

(ii) If $0 < \beta < \alpha$, then $\sigma > 1$ and the range of p in (3) is smaller than the usual one $1 < p < (N+2)/(N-2)$. If $\beta = 0$, we would have $\sigma = (N+2)/(N-2)$ and the interval of admissible p would be empty.

(iii) When $\alpha = 2$ we can still find a solution in $\mathcal{H}_\varepsilon$ but not in $W^{1,2}(\mathbb{R}^N)$ (see Theorem 13). □

Concerning semiclassical states of (NLS) we show the following concentration behavior.

Theorem 3. *Let the assumptions of Theorem 1 hold. Then the ground state v_ε concentrates at a global minimum point x^* of $\mathcal{A} = V^\theta K^{-2/(p-1)}$ with $\theta = (p+1)/(p-1) - N/2$. More precisely, v_ε has a unique maximum point x_ε with $x_\varepsilon \to x^*$ as $\varepsilon \to 0$, and*

$$v_\varepsilon(x) = U^*\left(\frac{x - x_\varepsilon}{\varepsilon}\right) + \omega_\varepsilon(x) \quad \text{as } \varepsilon \to 0,$$

where $\omega_\varepsilon \to 0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ and in $L^\infty(\mathbb{R}^N)$ as $\varepsilon \to 0$, and U^* is the unique positive radial solution of

$$-\Delta U^* + V(x^*)U^* = K(x^*)(U^*)^p.$$

The proofs of the above two theorems will be carried out in the rest of the paper.

3. Some weighted Sobolev spaces

As anticipated in the Introduction, we will work in a class of weighted Sobolev spaces. Precisely, let us set, for all $\varepsilon > 0$,

$$\mathcal{H}_\varepsilon = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} [\varepsilon^2 |\nabla u(x)|^2 + V(x) u^2(x)]\, dx < \infty \right\}.$$

$\mathcal{H}_\varepsilon$ is a Hilbert space with scalar product and norm, respectively,

$$\|u\|_\varepsilon^2 = \int_{\mathbb{R}^N} [\varepsilon^2 |\nabla u(x)|^2 + V(x) u^2(x)]\, dx,$$

$$(u|v)_\varepsilon = \int_{\mathbb{R}^N} [\varepsilon^2 \nabla u(x) \cdot \nabla v(x) + V(x) u(x) v(x)]\, dx.$$

Set $\mathcal{H} = \mathcal{H}_1$ with norm $\| \cdot \|_{\mathcal{H}}$.

Remark 4. Since V is positive and uniformly bounded, it follows that $W^{1,2}(\mathbb{R}^N) \subset \mathcal{H}_\varepsilon$ for all $\varepsilon > 0$. □

Denote by L_K^q the weighted space of measurable $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$|u|_{q,K} = \left[\int_{\mathbb{R}^N} K(x) |u(x)|^q \, dx \right]^{1/q} < \infty.$$

$\mathcal{H}_\varepsilon$ and L_K^q are particular cases of weighted spaces discussed in [19], where the following result is proved.

Theorem 5. *Let $N \geq 3$ and suppose that (V), (K) hold with $\alpha \in (0, 2]$ and $\beta > 0$, respectively. Then for all $\varepsilon > 0$, $\mathcal{H}_\varepsilon \subset L_K^{p+1}$ provided*

$$\sigma \leq p \leq \frac{N+2}{N-2},$$

and there is $C_\varepsilon > 0$ such that

$$|u|_{q,K} \leq C_\varepsilon \|u\|_\varepsilon, \qquad \forall u \in \mathcal{H}_\varepsilon. \tag{4}$$

Furthermore, the embedding of $\mathcal{H}_\varepsilon$ into L_K^{p+1} is compact if (3) *holds.*

In view of this theorem we will assume in what follows that p, α and β always satisfy (3).

Remark 6. (i) If $a \leq V(x) \leq A$, that is, when $\alpha = 0$, we have $\mathcal{H}_\varepsilon = W^{1,2}(\mathbb{R}^N)$ and Theorem 5 implies that $W^{1,2}(\mathbb{R}^N)$ is compactly embedded in L_K^{p+1} provided ($\beta > 0$ and) (3) holds.

(ii) If $V(x) \sim (1 + |x|^\alpha)^{-1}$ and $K(x) \sim (1 + |x|^\beta)^{-1}$, with $0 < \alpha \leq 2$ and $\beta > 0$, it is proved in [19] that the growth restriction $\sigma \leq p \leq (N+2)/(N-2)$ is a necessary condition for $\mathcal{H}_\varepsilon$ to be embedded into $L_K^{p+1}(\mathbb{R}^N)$ (see also Proposition 15). □

In the rest of this section we will prove some integral estimates for functions in $\mathcal{H}_\varepsilon$, uniform with respect to ε. We anticipate that, as a byproduct, we will deduce a proof of the embedding result stated in Theorem 5 (see Remark 10 below).

Proposition 7. *Let* $0 < \alpha < 2$ *and let* p *satisfy* (3)*. Then for all* $\delta > 0$ *there exists* $\overline{R} > 0$ *such that, for all* $R \geq \overline{R}$ *and all* $u \in \mathcal{H}_\varepsilon$ *with* $\operatorname{supp}(u) \cap B_R = \emptyset$*, one has*

$$\int_{\mathbb{R}^N} K|u|^{p+1} \leq \delta \varepsilon^{-(p-1)N/2} \|u\|_\varepsilon^{p+1}. \tag{5}$$

Proof. The proof is carried out in several steps. First, let us introduce some quantities we need in the proof:

(i) the sequence of radii $R_{n,\varepsilon}$ defined by

$$R_{n,\varepsilon} = \varepsilon R_n, \quad R_n = \left[\frac{2-\alpha}{a^{1/2}} n\right]^{2/(2-\alpha)};$$

(ii) the sequence of continuous functions $\psi_{n,\varepsilon} : \mathbb{R}_+ \to [0, 1]$ satisfying

$$\psi_{1,\varepsilon}(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq R_{n-1,\varepsilon}, \\ -\dfrac{r - R_{1,\varepsilon}}{R_{2,\varepsilon} - R_{1,\varepsilon}} + 1 & \text{if } R_{1,\varepsilon} \leq r \leq R_{2,\varepsilon}, \\ 0 & \text{if } r \geq R_{2,\varepsilon}, \end{cases}$$

and for $n \geq 2$,

$$\psi_{n,\varepsilon}(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq R_{n-1,\varepsilon}, \\ \dfrac{r - R_{n,\varepsilon}}{R_{n,\varepsilon} - R_{n-1,\varepsilon}} + 1 & \text{if } R_{n-1,\varepsilon} \leq r \leq R_{n,\varepsilon}, \\ -\dfrac{r - R_{n,\varepsilon}}{R_{n+1,\varepsilon} - R_{n,\varepsilon}} + 1 & \text{if } R_{n,\varepsilon} \leq r \leq R_{n+1,\varepsilon}, \\ 0 & \text{if } r \geq R_{n+1,\varepsilon}; \end{cases}$$

(iii) the sequence of sets

$$A_{n,\varepsilon} = \{x \in \mathbb{R}^N : R_{n-1,\varepsilon} \leq |x| \leq R_{n+1,\varepsilon}\}.$$

Note that $A_{1,\varepsilon}$ is a ball, $A_{n,\varepsilon}$ is an annulus for $n \geq 2$, and the $\psi_{n,\varepsilon}$'s have been chosen in such a way that

$$u(x) = \sum_n \psi_{n,\varepsilon}(|x|)u(x).$$

These cut-off functions are useful to estimate integrals over $\mathbb{R}^N$ by means of a discrete sum of integrals on the annuli $A_{n,\varepsilon}$.

Lemma 8. *There exists $c > 0$ such that*

$$\int_{\mathbb{R}^N} K|u|^{p+1} \leq c \sum_{n=1}^{\infty} \frac{1}{1+R_{n,\varepsilon}^{\beta}} \int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1}, \qquad \forall u \in \mathcal{H}_\varepsilon,\ \forall \varepsilon \in (0,1].$$

Proof. On $A_{n,\varepsilon}$ one has $u = \psi_{n-1,\varepsilon}u + \psi_{n,\varepsilon}u + \psi_{n+1,\varepsilon}u$ (with abuse of notation we are taking $A_{0,\varepsilon} = \emptyset$), which implies

$$\int_{A_{n,\varepsilon}} K|u|^{p+1} \leq \sup_{A_{n,\varepsilon}} K \left(\int_{A_{n-1,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1} + \int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1} + \int_{A_{n+1,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1} \right).$$

Since the width of $A_{n,\varepsilon}$ is small with respect to $R_{n,\varepsilon}$ there exists $c_2 = c_2(K)$ such that

$$\sup_{A_{n,\varepsilon}} K \leq c_2 \frac{1}{1+R_{n+1,\varepsilon}^{\beta}} \leq c_2 \frac{1}{1+R_{n,\varepsilon}^{\beta}} \leq c_2 \frac{1}{1+R_{n-1,\varepsilon}^{\beta}}. \tag{6}$$

The last two formulas imply

$$\int_{A_{n,\varepsilon}} K|u|^{p+1} \leq c_2 \left(\frac{1}{1+R_{n-1,\varepsilon}^{\beta}} \int_{A_{n-1,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1} + \frac{1}{1+R_{n,\varepsilon}^{\beta}} \int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1} + \frac{1}{1+R_{n+1,\varepsilon}^{\beta}} \int_{A_{n+1,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1} \right).$$

Summing over all integers n completes the proof. □

Next, we estimate each term $\int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1}$. Let γ satisfy $\gamma(2^* - 2) = p - 1$, or equivalently $2^*\gamma = (p-1)N/2$. Then

$$\int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1} = \int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{2^*\gamma + 2 - 2\gamma}.$$

Using the Hölder inequality we find that

$$\int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1} \leq c_1 \left[\int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{2^*} \right]^{\gamma} \left[\int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{2} \right]^{1-\gamma}.$$

From the embedding of $\mathcal{D}^{1,2}$ into L^{2^*} we infer that

$$\int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{p+1} \leq c_2 \left[\int_{A_{n,\varepsilon}} |\nabla(\psi_{n,\varepsilon} u)|^{2} \right]^{2^*\gamma/2} \left[\int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{2} \right]^{1-\gamma}. \tag{7}$$

From (6) and (7) we get

$$\int_{A_{n,\varepsilon}} K|u|^{p+1} \leq c_3 \frac{1}{1+R_{n,\varepsilon}^{\beta}} \left[\int_{A_{n,\varepsilon}} |\nabla(\psi_{n,\varepsilon} u)|^{2} \right]^{2^*\gamma/2} \left[\int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^{2} \right]^{1-\gamma}. \tag{8}$$

We now show

Lemma 9. *We have*

$$\int_{A_{n,\varepsilon}} |\nabla(\psi_{n,\varepsilon} u)|^2 \le c_4 \int_{A_{n,\varepsilon}} [|\nabla u|^2 + \varepsilon^{-2} V u^2].$$

Proof. First we estimate

$$|\nabla(\psi_{n,\varepsilon} u)|^2 = |u\nabla\psi_{n,\varepsilon} + \psi_{n,\varepsilon}\nabla u|^2 \le 2u^2|\nabla\psi_{n,\varepsilon}|^2 + 2|\nabla u|^2. \tag{9}$$

From the definition of $R_{n,\varepsilon}$ we get

$$|R_{n+1,\varepsilon} - R_{n,\varepsilon}|^2 = \varepsilon^2 |R_{n+1} - R_n|^2 \ge c\varepsilon^2 R^{\alpha}_{n+1,\varepsilon}.$$

As above, $R^{-\alpha}_{n+1,\varepsilon} \le c_5 \inf_{A_{n,\varepsilon}} V$, and we deduce that

$$|\nabla\psi_{n,\varepsilon}|^2 \le c_6 |R_{n+1,\varepsilon} - R_{n,\varepsilon}|^{-2} \le c_7 \varepsilon^{-2} V, \qquad x \in A_{n,\varepsilon}.$$

Substituting in (9) and integrating over $A_{n,\varepsilon}$ proves the lemma. □

Proof of Proposition 7 completed. Lemma 9 together with (8) yields

$$\int_{A_{n,\varepsilon}} K|u|^{p+1} \le c \frac{1}{1 + R^{\beta}_{n,\varepsilon}} \left[\int_{A_{n,\varepsilon}} [|\nabla u|^2 + \varepsilon^{-2} V u^2]\right]^{2^*\gamma/2} \left[\int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^2\right]^{1-\gamma}. \tag{10}$$

Let $M, s > 0$ and let θ, θ' be any pair of conjugate exponents (M, s, θ, θ' will be fixed appropriately later). For brevity, set

$$S = S_{n,\varepsilon} = \int_{A_{n,\varepsilon}} [|\nabla u|^2 + \varepsilon^{-2} V u^2], \qquad T = T_{n,\varepsilon} = \int_{A_{n,\varepsilon}} |\psi_{n,\varepsilon} u|^2$$

so that (10) becomes

$$\int_{A_{n,\varepsilon}} K|u|^{p+1} \le c \frac{1}{1 + R^{\beta}_{n,\varepsilon}} S^{2^*\gamma/2} \cdot T^{1-\gamma}.$$

Since

$$\begin{aligned} S^{2^*\gamma/2} \cdot \frac{T^{1-\gamma}}{1 + R^{\beta}_{n,\varepsilon}} &= M\varepsilon^s S^{2^*\gamma/2} \cdot M^{-1}\varepsilon^{-s} \frac{T^{1-\gamma}}{1 + R^{\beta}_{n,\varepsilon}} \\ &\le \frac{1}{\theta} M^{\theta} \varepsilon^{s\theta} S^{2^*\gamma\theta/2} + \frac{1}{\theta'} M^{-\theta'} \varepsilon^{-s\theta'} \frac{T^{(1-\gamma)\theta'}}{(1 + R^{\beta}_{n,\varepsilon})^{\theta'}}, \end{aligned}$$

we get

$$\int_{A_{n,\varepsilon}} K|u|^{p+1} \le c_1 M^{\theta} \varepsilon^{s\theta} S^{2^*\gamma\theta/2} + c_2 M^{-\theta'} \varepsilon^{-s\theta'} \frac{T^{(1-\gamma)\theta'}}{(1 + R^{\beta}_{n,\varepsilon})^{\theta'}}. \tag{11}$$

Now we choose s, θ satisfying

$$2^*\gamma\theta = p + 1, \qquad \theta(s - 2^*\gamma) = -s\theta'.$$

Then

$$S^{2^*\gamma\theta/2} = \left[\int_{A_{n,\varepsilon}} [|\nabla u|^2 + \varepsilon^{-2} V u^2]\right]^{2^*\gamma\theta/2} = \varepsilon^{-(p+1)}\left[\int_{A_{n,\varepsilon}} [\varepsilon^2|\nabla u|^2 + V u^2]\right]^{(p+1)/2},$$

and hence

$$M^\theta \varepsilon^{s\theta} S^{2^*\gamma\theta/2} = M^\theta \varepsilon^{-(p-1)N/2}\left[\int_{A_{n,\varepsilon}} [\varepsilon^2|\nabla u|^2 + V u^2]\right]^{(p+1)/2}. \tag{12}$$

On the other hand, we also have

$$\begin{aligned}
\frac{T^{(1-\gamma)\theta'}}{(1+R_{n,\varepsilon}^\beta)^{\theta'}} &\le \frac{1}{(1+R_{n,\varepsilon}^\beta)^{\theta'}}\left[\int_{A_{n,\varepsilon}} u^2\right]^{(p+1)/2}\\
&= \frac{(1+R_{n,\varepsilon}^\alpha)^{(p+1)/2}}{(1+R_{n,\varepsilon}^\beta)^{\theta'}}\left[\int_{A_{n,\varepsilon}} \frac{u^2}{1+R_{n,\varepsilon}^\alpha}\right]^{(p+1)/2}\\
&\le \frac{(1+R_{n,\varepsilon}^\alpha)^{(p+1)/2}}{(1+R_{n,\varepsilon}^\beta)^{\theta'}}\left[\int_{A_{n,\varepsilon}} V u^2\right]^{(p+1)/2}.
\end{aligned}$$

Inserting the above inequality and (12) into (11), and taking into account that $-s\theta' = \theta(s - 2^*\gamma) = -(p-1)N/2$, we infer that

$$\begin{aligned}
&\int_{A_{n,\varepsilon}} K|u|^{p+1} \le c_3 \varepsilon^{-(p-1)N/2}\\
&\times\left(M^\theta\left[\int_{A_{n,\varepsilon}} [|\varepsilon^2\nabla u|^2 + V u^2]\right]^{(p+1)/2} + M^{-\theta'}\frac{(1+R_{n,\varepsilon}^\alpha)^{(p+1)/2}}{(1+R_{n,\varepsilon}^\beta)^{\theta'}}\cdot\left[\int_{A_{n,\varepsilon}} V u^2\right]^{(p+1)/2}\right).
\end{aligned}$$

Now, let us remark that

$$\frac{(1+R_{n,\varepsilon}^\alpha)^{(p+1)/2}}{(1+R_{n,\varepsilon}^\beta)^{\theta'}} \sim R_{n,\varepsilon}^{-\beta\theta' + \alpha(p+1)/2} \to 0 \quad (R_{n,\varepsilon} \to \infty),$$

since $p > \sigma$ implies that $-\beta\theta' + \alpha(p+1)/2 < 0$. Then, given $\delta > 0$, we can choose $M, \overline{R} > 0$ such that

$$M^\theta < \frac{\delta}{2c_3}, \quad \text{and} \quad M^{-\theta'} R^{-\beta\theta' + \alpha(p+1)/2} < \frac{\delta}{2c_3} \quad \text{for } R \ge \overline{R},$$

yielding

$$\int_{A_{n,\varepsilon}} K|u|^{p+1} \le \delta \varepsilon^{-(p-1)N/2}\left[\int_{A_{n,\varepsilon}} [\varepsilon^2|\nabla u|^2 + V u^2]\right]^{(p+1)/2},$$

provided that $R_{n-1,\varepsilon} > \overline{R}$. Summing over these annuli $A_{n,\varepsilon}$ and using the fact that $\operatorname{supp}(u) \cap B_R = \emptyset$ for all $R \geq \overline{R}$ we get

$$\int_{|x|>R} K|u|^{p+1} \leq \delta\varepsilon^{-(p-1)N/2} \sum_n \left[\int_{A_{n,\varepsilon}} [\varepsilon^2|\nabla u|^2 + Vu^2]\right]^{(p+1)/2}.$$

Setting

$$a_n = a_{n,\varepsilon} = \int_{A_{n,\varepsilon}} [\varepsilon^2|\nabla u|^2 + Vu^2],$$

one has $\sum a_n < \|u\|_\varepsilon^2 < \infty$. Letting $\widetilde{\alpha}_n = a_n / \sum a_n$, we have $0 < \widetilde{\alpha}_n \leq 1$ for all n and hence $\widetilde{\alpha}_n^{(p+1)/2} \leq \widetilde{\alpha}_n$, that is, $a_n^{(p+1)/2} \leq (\sum a_n)^{(p+1)/2-1} a_n$. Summing over all n, it follows that $\sum a_n^{(p+1)/2} \leq (\sum a_n)^{(p+1)/2} < \infty$. This implies

$$\int_{|x|>R} K|u|^{p+1} \leq \delta\varepsilon^{-(p-1)N/2}\left[\sum \int_{A_{n,\varepsilon}} [\varepsilon^2|\nabla u|^2 + Vu^2]\right]^{(p+1)/2}$$
$$\leq \delta\varepsilon^{-(p-1)N/2}\|u\|_\varepsilon^{p+1},$$

completing the proof of Proposition 7.

Remark 10. For $\varepsilon = 1$ the preceding arguments give an alternative proof of the embedding result stated in Theorem 5. To see this, let us write $u = \chi_R u + (1 - \chi_R)u$, where χ_R is a cut-off function such that $\chi_R \equiv 0$ on B_R, $\chi_R \equiv 1$ for $|x| \geq R+1$, and χ_R is linear on $R < |x| < R+1$. For $\sigma < p < (N+2)/(N-2)$ we can use inequality (5) to estimate $\int_{\mathbb{R}^N} K|\chi_R u|^{p+1}$, while the integral $\int_{|x|\leq R+1} K|(1-\chi_R)u|^{p+1}$ can be bounded by using the standard Sobolev embedding theorem. If $\sigma \leq p \leq (N+2)/N-2)$, the above method shows that there exist $C > 0$ and $R \gg 1$ for which

$$\int_{|x|>R} K|u|^{p+1} \leq C\|u\|_{\mathcal{H}}^{(p+1)/2}, \qquad \forall u \in \mathcal{H}.$$

Moreover, modifying the definition of $R_{n,\varepsilon}$ (with a logarithmic dependence on n) we could also recover the embedding in the case $\alpha = 2$. □

Proposition 11. *Let* $0 < \alpha < 2$ *and let* p *satisfy* (3). *Then for all* $\delta > 0$ *there exists* $\overline{R} > 0$ *such that for all* $R \geq \overline{R}$,

$$\int_{|x|>R} K(x)|u(x)|^{p+1}\,dx$$
$$\leq \delta\varepsilon^{-(p-1)N/2}\left(\int_{|x|>R} [\varepsilon^2|\nabla u(x)|^2 + V(x)u^2(x)]\,dx\right)^{(p+1)/2}, \qquad \forall u \in \mathcal{H}_\varepsilon. \tag{13}$$

Proof. Let $\widetilde{\psi}_{R,\varepsilon} : \mathbb{R}_+ \to [0,1]$ be a smooth non-decreasing function such that

$$\widetilde{\psi}_{R,\varepsilon}(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq R - \varepsilon R^{\alpha/2}, \\ 1 & \text{if } r \geq R, \end{cases}$$

satisfying $|\widetilde{\psi}'_{R,\varepsilon}(r)| \le 2\varepsilon^{-1}R^{-\alpha/2}$. Define, in polar coordinates $(r,\vartheta) \in \mathbb{R}_+ \times S^{N-1}$,

$$\widetilde{u}_{R,\varepsilon}(r,\vartheta) = \begin{cases} \widetilde{\psi}_{R,\varepsilon}(r)u(2R-r,\vartheta) & \text{if } R-\varepsilon R^{\alpha/2} \le r \le R, \\ u(r,\vartheta) & \text{if } r > R. \end{cases}$$

In the annulus $A_{R,\varepsilon} = \{R-\varepsilon R^{\alpha/2} < |x| < R\}$ we have (subscripts denote partial derivatives)

$$\nabla\widetilde{u}_{R,\varepsilon} = -\widetilde{\psi}_{R,\varepsilon}(r)u_r(2R-r,\vartheta)\mathbf{e}_r + \frac{1}{r}\widetilde{\psi}_{R,\varepsilon}(r)u_\vartheta(2R-r,\vartheta)\mathbf{e}_\vartheta + \widetilde{\psi}'_{R,\varepsilon}(r)u(2R-r,\vartheta)\mathbf{e}_r,$$

where $\mathbf{e}_r = x/|x|$ and $\mathbf{e}_\vartheta$ is a unit vector tangent to the unit sphere $\{|x|=1\}$. Thus, in $A_{R,\varepsilon}$ one finds that

$$|\nabla\widetilde{u}_{R,\varepsilon}|^2 \le c_1|\nabla u(2R-r,\vartheta)|^2 + c_2\varepsilon^{-2}R^{-\alpha}u^2(2R-r,\vartheta).$$

Let us explicitly point out that here and below the constants c_i do not depend upon R,ε. Integrating in $A_{R,\varepsilon}$ and performing the change of variable $(r,\vartheta)\mapsto(2R-r,\vartheta)$ we get

$$\begin{aligned}\int_{A_{R,\varepsilon}} |\nabla\widetilde{u}_{R,\varepsilon}|^2 &\le c_3\int_{R<|x|<R+\varepsilon R^{\alpha/2}} [|\nabla u|^2 + \varepsilon^{-2}R^{-\alpha}u^2] \\ &\le c_4\varepsilon^{-2}\int_{R<|x|<R+\varepsilon R^{\alpha/2}} [\varepsilon^2|\nabla u|^2 + V(x)u^2]. \qquad (14)\end{aligned}$$

Here we have taken into account that $\widetilde{u}_{R,\varepsilon} \equiv u$ for $|x| > R$. From (14) we infer that

$$\int_{A_{R,\varepsilon}} |\nabla\widetilde{u}_{R,\varepsilon}|^2 \le c_5\varepsilon^{-2}\int_{|x|>R} [\varepsilon^2|\nabla u|^2 + V(x)u^2]. \qquad (15)$$

Moreover, similar arguments yield

$$\int_{A_{R,\varepsilon}} V(x)\widetilde{u}^2_{R,\varepsilon} \le c_6\int_{R<|x|<R+\varepsilon R^{\alpha/2}} V(x)\widetilde{u}^2_{R,\varepsilon} \le c_6\int_{|x|>R} V(x)u^2. \qquad (16)$$

From (15) and (16) we deduce that

$$\int_{A_{R,\varepsilon}} [\varepsilon^2|\nabla\widetilde{u}_{R,\varepsilon}|^2 + V(x)\widetilde{u}^2_{R,\varepsilon}] \le c_7\int_{|x|>R} [\varepsilon^2|\nabla u|^2 + V(x)u^2]. \qquad (17)$$

From the embedding (4) and since $\widetilde{u}_{R,\varepsilon} = u$ for $r \ge R$, we get

$$\int_{|x|>R} K(x)|u|^{p+1} \le \int_{\mathbb{R}^N} K(x)|\widetilde{u}_{R,\varepsilon}|^{p+1}.$$

From Proposition 7 we have

$$\begin{aligned}\int_{\mathbb{R}^N} K|\widetilde{u}_{R,\varepsilon}|^{p+1} &\le \delta\varepsilon^{-(p-1)N/2}\left(\int_{\mathbb{R}^N}[\varepsilon^2|\nabla\widetilde{u}_{R,\varepsilon}|^2 + V\widetilde{u}^2_{R,\varepsilon}]\right)^{(p+1)/2} \\ &\le \delta\varepsilon^{-(p-1)N/2}\left(\int_{A_{R,\varepsilon}}[\varepsilon^2|\nabla\widetilde{u}_{R,\varepsilon}|^2 + V\widetilde{u}^2_{R,\varepsilon}] + \int_{|x|>R}[\varepsilon^2|\nabla\widetilde{u}_{R,\varepsilon}|^2 + V\widetilde{u}^2_{R,\varepsilon}]\right)^{(p+1)/2}.\end{aligned}$$

From this and (17) we finally find (13). □

4. Proof of the existence results

This section is devoted to the proof of Theorem 1, which is divided into two parts. First, we show the existence of a least-energy solution in $\mathcal{H}_\varepsilon$ (see Theorem 13 below); in the second part of the section we prove that such a ground state belongs indeed to $W^{1,2}(\mathbb{R}^N)$.

Let us start by introducing the functional set up. If (3) holds, then Theorem 5 applies, yielding

$$\int_{\mathbb{R}^N} K(x)|u(x)|^{p+1}\,dx < \infty, \qquad \forall u \in \mathcal{H}_\varepsilon. \tag{18}$$

Define

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \varepsilon^2|\nabla u(x)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2(x)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(x)|u(x)|^{p+1}\,dx \\ &= \frac{1}{2}\|u\|_\varepsilon^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)|u(x)|^{p+1}\,dx. \end{aligned}$$

From (18) and (V) it follows that I_ε is well defined on $\mathcal{H}_\varepsilon$ for all $\varepsilon > 0$. Moreover, I_ε is of class C^1 and

$$\begin{aligned} &(I'_\varepsilon(u)|v) \\ &= \int_{\mathbb{R}^N} [\varepsilon^2\nabla u(x)\cdot\nabla v(x) + V(x)u(x)v(x) - K(x)|u(x)|^{p-1}u(x)v(x)]\,dx, \quad \forall v \in \mathcal{H}_\varepsilon. \end{aligned}$$

Hence any critical point $u_\varepsilon \in \mathcal{H}_\varepsilon$ of I_ε is a weak solution of (NLS).

Remark 12. By Remark 6(ii), if $V(x) \sim (1+|x|^\alpha)^{-1}$ and $K(x) \sim (1+|x|^\beta)^{-1}$, with $0 < \alpha \le 2$ and $\beta > 0$, then the growth restriction $\sigma \le p \le (N+2)/(N-2)$ is necessary in order to work in $\mathcal{H}_\varepsilon$ with the functional I_ε. □

Critical points of I_ε can be found by the Mountain-Pass Theorem in a straightforward way.

Theorem 13. *Let (V), (K) hold with $0 < \alpha \le 2$, $\beta > 0$, respectively, and suppose that p satisfies* (3). *Then*

$$b_\varepsilon = \inf_{u\in\mathcal{H}_\varepsilon\setminus\{0\}} \max_{t\ge 0} I_\varepsilon(tu)$$

is a critical level of I_ε. Hence for all $\varepsilon > 0$ the equation

$$-\varepsilon^2\Delta v + V(x)v = K(x)v^p, \quad x \in \mathbb{R}^N, \tag{19}$$

has a positive (classical) solution $v_\varepsilon \in \mathcal{H}_\varepsilon$. Moreover, there exists $C > 0$ such that

$$\|v_\varepsilon\|_\varepsilon^2 \le Cb_\varepsilon. \tag{20}$$

Proof. Let ϕ be a smooth positive function with compact support in $\mathbb{R}^N$. Then (recall that $p > 1$) one has $I_\varepsilon(t\phi) \to -\infty$ as $t \to +\infty$. Hence I_ε has the M-P geometry. Since $\mathcal{H}_\varepsilon$ is compactly embedded into L_K^{p+1}, standard arguments imply that b_ε is a M-P critical level carrying a critical point $v_\varepsilon \in \mathcal{H}_\varepsilon$ of I_ε which is a weak solution of (NLS). Since V and K are smooth, local regularity implies that v_ε is in fact a classical solution. It is also standard to see that $v_\varepsilon > 0$. From

$$-\varepsilon^2 \Delta v_\varepsilon + V(x) v_\varepsilon = K(x) v_\varepsilon^p$$

we infer that

$$\int_{\mathbb{R}^N} [\varepsilon^2 |\nabla v_\varepsilon|^2 + V(x) v_\varepsilon^2]\, dx = \int_{\mathbb{R}^N} K(x) v_\varepsilon^{p+1}\, dx.$$

Thus

$$b_\varepsilon = I_\varepsilon(v_\varepsilon) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} [\varepsilon^2 |\nabla v_\varepsilon|^2 + V(x) v_\varepsilon^2]\, dx = \left(\frac{1}{2} - \frac{1}{p+1}\right) \|v_\varepsilon\|_\varepsilon^2.$$

This concludes the proof. □

Remark 14. (i) Assumption (V) includes potentials which are bounded away from zero (that is, $0 < \inf_{\mathbb{R}^N} V \le \sup_{\mathbb{R}^N} V < \infty$). In this case, the space $\mathcal{H}_\varepsilon$ is nothing but $W^{1,2}(\mathbb{R}^N)$ and in order to recover compactness our approach requires $\beta > 0$ (see Remark 6(i)). Let us recall that when, in addition, also K is bounded away from zero (that is, $0 < \inf_{\mathbb{R}^N} K \le \sup_{\mathbb{R}^N} K < \infty$), proving the existence of solutions to (19) requires appropriate assumptions on V and/or K (see the papers cited in the Introduction and [4]). On the other hand, it is well known that if $\beta = 0$ a necessary condition for (NLS) to have a solution is that $\int_{\mathbb{R}^N} \partial_{x_i} V(x) u^2(x)\, dx = 0$. Moreover, if (V) holds with $0 < \alpha \le 2$ and K is bounded away from zero, then the critical level b_ε (or the supremum considered in the statement of Proposition 15) is clearly equal to ∞. Of course, it is a different story if we look for solutions that are not ground states. For example, it is proved in [6, 24] that if $0 < \inf_{\mathbb{R}^N} V \le \sup_{\mathbb{R}^N} V < \infty$ and $0 < \inf_{\mathbb{R}^N} K \le \sup_{\mathbb{R}^N} K < \infty$, then a solution exists provided ε is *sufficiently small* and the auxiliary potential $\mathcal{A}$ has a *stable* stationary point.

(ii) In [18, Thm. 2] the authors consider the *eigenvalue problem*

$$\begin{cases} -\Delta u + V(|x|)u = \lambda K(|x|) u^p, & x \in \mathbb{R}^N, \\ u \in C^2_{\mathrm{loc}}(\mathbb{R}^N), & \lim\limits_{|x| \to \infty} u(x) = 0, \end{cases}$$

proving the existence of positive solutions (λ, u_λ), $u_\lambda \in L^{2^*}(\mathbb{R}^N)$ and $u_\lambda = O(r^{(2-n)/2})$, $r = |x|$. It is assumed that $V(r) \ge 0$ and $K(r) = O(r^{-\beta})$, $\beta > 0$. In addition, if $\beta < 2$, it is required that $p > (N + 2 - 2\beta)/(N - 2)$. Let us point out that the last condition is stronger than ours ($p > \sigma$).

(iii) Theorem 13 follows from [20, Thm. 3.1] combined with Theorem 5. Moreover, the case in which $p = \sigma$ or $p = (N + 2)/(N - 2)$ is also studied in [20, Thm. 3.2], under some further restrictions on V and K. □

It is also worth pointing out that if $\sigma < p$ (here we take $0 < \beta < \alpha$, otherwise $\sigma = 1$ and p satisfies the usual growth assumption), I_ε has no Mountain-Pass solution.

Proposition 15. *If either $p < \sigma$ or $p > (N+2)/(N-2)$, then*

$$\sup_{u\in\mathcal{H}_\varepsilon\setminus\{0\}} \frac{\int_{\mathbb{R}^N} K|u|^{p+1}}{(\int_{\mathbb{R}^N}[\varepsilon^2|\nabla u|^2 + Vu^2])^{(p+1)/2}} = \infty.$$

Proof. We can assume for simplicity that $\varepsilon = 1$. Let us consider a function Ψ with compact support, and let

$$u_\xi(x) = \Psi(\lambda(x-\xi)), \qquad |\xi| \gg 1, \lambda = |\xi|^{-\alpha/2}. \tag{21}$$

From the definition of u_ξ and the conditions on λ, ξ (see (21)), we easily find that

$$\begin{aligned} &\int |\nabla u_\xi|^2 = \lambda^{2-N}\int |\nabla\Psi|^2, \qquad \int Vu_\xi^2 \geq \frac{C^{-1}}{|\xi|^\alpha\lambda^N}\int \Psi^2,\\ &\int K|u|^{p+1} \leq \frac{C}{|\xi|^\beta\lambda^N}\int |\Psi|^{p+1}. \end{aligned} \tag{22}$$

Hence it follows that

$$\frac{\int_{\mathbb{R}^N} Ku_\xi^{p+1}}{(\int_{\mathbb{R}^N}[|\nabla u_\xi|^2 + Vu_\xi^2])^{(p+1)/2}} \leq C\frac{|\xi|^{-\beta}\lambda^{-N}}{(\lambda^{-N}|\xi|^{-\alpha})^{(p+1)/2}} = C|\xi|^{\frac{\alpha}{4}[(N+2)-p(N-2)]-\beta} \to \infty$$

as $|\xi| \to \infty$, because $p < \sigma$.

On the other hand, also in the case $p > (N+2)/(N-2)$ it is standard to see that the above supremum is ∞. It is sufficient for example to consider the family of functions $u_\lambda(x) = \Psi(\lambda x)$, with $\lambda \to +\infty$. □

In the second part of this section we will show that the Mountain-Pass solutions of (NLS) found above belong indeed to $W^{1,2}(\mathbb{R}^N)$, provided $0 < \alpha < 2$.

Theorem 16. *Let (V), (K) hold with $0 < \alpha < 2$, $\beta > 0$, respectively, and suppose that p satisfies* (3). *Then the Mountain-Pass solution v_ε found in Theorem* 13 *is a ground state of* (NLS). *In particular, $v_\varepsilon \in W^{1,2}(\mathbb{R}^N)$, $v_\varepsilon \in C^2(\mathbb{R}^N)$, $v_\varepsilon(x) > 0$ and $\lim_{|x|\to\infty} v_\varepsilon(x) = 0$.*

The proof of Theorem 16 requires some preliminary decay estimates, based upon the results discussed in Section 3. Let us point out that to establish the concentration phenomena discussed in Section 5, the decay is proved with estimates which are uniform in ε.

In these lemmas it is always understood that the assumptions of Theorem 16 hold true.

Lemma 17. *Let v_ε be solutions of* (19) *and suppose there exists $\Gamma > 0$ such that*

$$\|v_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \leq \Gamma\varepsilon^N. \tag{23}$$

Then there exists $R_\Gamma > 0$ *such that for all* $R \geq R_\Gamma$ *and all* $\Omega_{n,\varepsilon} \subseteq \mathbb{R}^N \setminus B_R$,

$$\int_{\Omega_{n+1,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2]\,dx \leq \frac{3}{4}\int_{\Omega_{n,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2]\,dx,$$

where $\Omega_{n,\varepsilon} = \mathbb{R}^N \setminus B_{R_{n,\varepsilon}}$ *and* $R_{n,\varepsilon} = \varepsilon n^{2/(2-\alpha)}$.

Proof. Let $R_{n,\varepsilon}$ be as in the statement, and let $\chi_{n,\varepsilon}(r)$ be piecewise affine functions such that

$$\chi_{n,\varepsilon}(r) \equiv 0, \quad \forall r \leq R_{n,\varepsilon}, \quad \chi_{n,\varepsilon}(r) \equiv 1, \quad \forall r \geq R_{n+1,\varepsilon}.$$

By the definition of $R_{n,\varepsilon}$ it follows that

$$|R_{n+1,\varepsilon} - R_{n,\varepsilon}| \geq C^{-1}\varepsilon^{(2-\alpha)/2}R_{n+1,\varepsilon}^{\alpha/2} \geq C^{-1}\varepsilon R_{n+1,\varepsilon}^{\alpha/2}.$$

Then

$$\varepsilon^2|R_{n+1,\varepsilon} - R_{n,\varepsilon}|^{-2} \leq c_1 R_{n+1,\varepsilon}^{-\alpha} \leq c_2 \inf\{V(x) : R_{n,\varepsilon} \leq |x| \leq R_{n+1,\varepsilon}\},$$

and hence

$$\varepsilon^2|\nabla\chi_{n,\varepsilon}(x)|^2 \leq V(x), \qquad \forall x \in \mathbb{R}^N. \tag{24}$$

Let us test (19) on $\chi_{n,\varepsilon}v_\varepsilon$. Recalling that $\chi_{n,\varepsilon} = 0$ on $B_{R_{n,\varepsilon}}$ and that $\chi_{n,\varepsilon} \leq 1$, we get

$$\begin{aligned}\int_{\Omega_{n,\varepsilon}} \chi_{n,\varepsilon}[\varepsilon^2|\nabla v_\varepsilon|^2 + Vv_\varepsilon^2] &= \int_{\Omega_{n,\varepsilon}} \chi_{n,\varepsilon}Kv_\varepsilon^{p+1} - \varepsilon^2\int_{\Omega_{n,\varepsilon}} \nabla v_\varepsilon \cdot \nabla\chi_{n,\varepsilon}v_\varepsilon \\ &\leq \int_{\Omega_{n,\varepsilon}} Kv_\varepsilon^{p+1} + \frac{1}{2}\varepsilon^2\int_{\Omega_{n,\varepsilon}} [|\nabla v_\varepsilon|^2 + |\nabla\chi_{n,\varepsilon}|^2v_\varepsilon^2].\end{aligned}$$

Using (24) we infer that

$$\varepsilon^2\int_{\Omega_{n,\varepsilon}} [|\nabla v_\varepsilon|^2 + |\nabla\chi_{n,\varepsilon}|^2v_\varepsilon^2] \leq \int_{\Omega_{n,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2].$$

From the last two estimates, it follows that

$$\begin{aligned}\int_{\Omega_{n+1,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + Vv_\varepsilon^2]\,dx &\leq \int_{\Omega_{n,\varepsilon}} \chi_{n,\varepsilon}[\varepsilon^2|\nabla v_\varepsilon|^2 + Vv_\varepsilon^2] \\ &\leq \int_{\Omega_{n,\varepsilon}} Kv_\varepsilon^{p+1} + \frac{1}{2}\int_{\Omega_{n,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2].\end{aligned}$$

Then, from Proposition 11, if $\delta > 0$ is given and R is sufficiently large we deduce that

$$\begin{aligned}&\int_{\Omega_{n+1,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + Vv_\varepsilon^2]\,dx \\ &\leq \delta\varepsilon^{-(p-1)N/2}\left(\int_{\Omega_{n,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2]\,dx\right)^{(p+1)/2} + \frac{1}{2}\int_{\Omega_{n,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2].\end{aligned}$$

Now we write

$$\left(\int_{\Omega_{n,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2]\,dx\right)^{(p+1)/2} = \left(\int_{\Omega_{n,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2]\,dx\right)^{(p-1)/2} \int_{\Omega_{n,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2]\,dx.$$

From (23) and the last two formulas it follows that

$$\int_{\Omega_{n+1,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V v_\varepsilon^2]\,dx \le \left(\frac{1}{2} + \delta\Gamma^{(p-1)/2}\right)\int_{\Omega_{n,\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2]\,dx.$$

Choosing δ sufficiently small (and hence for R large) we obtain the assertion. □

Lemma 18. *Let v_ε be solutions of* (19)*, and let Γ, R_Γ be as above. Then, for all $\rho \ge 2R_\Gamma$,*

$$\int_{|x|>\rho} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2]\,dx \le \overline{C}_\Gamma \varepsilon^N \exp\left\{-\frac{1}{2}\left|\log\frac{3}{4}\right|\varepsilon^{-1}(\rho^{(2-\alpha)/2} - R_\Gamma^{(2-\alpha)/2})\right\}, \qquad (25)$$

for some constant $\overline{C}_\Gamma$ depending only on Γ.

Proof. Given $\rho > 2R_\Gamma$, let $\tilde{n} > \overline{n}$ be positive integers such that

$$R_{\overline{n},\varepsilon} \le R_\Gamma \le R_{\overline{n}+1,\varepsilon}, \qquad R_{\tilde{n}-1,\varepsilon} \le \rho \le R_{\tilde{n},\varepsilon}.$$

From (17), we deduce that

$$\int_{|x|>\rho} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2] \le \int_{|x|>R_{\tilde{n},\varepsilon}} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2] \le \left(\frac{3}{4}\right)^{\tilde{n}-\overline{n}} \int_{|x|>R_\Gamma} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2].$$

Then (23) implies

$$\int_{|x|>\rho} [\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2] \le \left(\frac{3}{4}\right)^{\tilde{n}-\overline{n}} \Gamma\varepsilon^N. \qquad (26)$$

By our choices of $\overline{n}$, $\tilde{n}$,

$$\rho \sim \varepsilon^{2/(2-\alpha)}\tilde{n}^{2/(2-\alpha)}, \qquad R_\Gamma \sim \varepsilon^{2/(2-\alpha)}\overline{n}^{2/(2-\alpha)},$$

which implies

$$\tilde{n} - \overline{n} \ge \frac{1}{2}\varepsilon^{-1}(\rho^{(2-\alpha)/2} - R_\Gamma^{(2-\alpha)/2}).$$

The estimate in (26) and the last formula conclude the proof. □

Proof of Theorem 16. Here $\varepsilon > 0$ is fixed and can be taken equal to 1 to simplify the notation. Let $v \in \mathcal{H}$ be any solution of (19) (with $\varepsilon = 1$) and let $y \in \mathbb{R}^N$ be such that $|y| > 2$. Then

$$\int_{B_1(y)} v^2 = \int_{B_1(y)} V(x)v^2 \cdot \frac{1}{V(x)} \leq c_1|y|^\alpha \int_{B_1(y)} V(x)v^2.$$

For $R = \frac{1}{2}|y|$ we have

$$\int_{B_1(y)} V(x)v^2 \leq \int_{\mathbb{R}^N \setminus B_R} V(x)v^2.$$

From the preceding two estimates and Lemma 18 we get

$$\int_{B_1(y)} v^2 \leq C_3|y|^\alpha \exp\{-C_4|y|^{1-\alpha/2}\}, \qquad \forall |y| \gg 1. \tag{27}$$

Let $m \in \mathbb{N}$ and $y_i \in \mathbb{R}^N$, $i = 1, \dots, m$, be such that $B_5 \setminus B_2 \subset \bigcup_{i=1}^m B_1(y_i)$, and let $y_{i,k} = 2^k y_i$. Then we get

$$\int_{\mathbb{R}^N \setminus B_2} v^2 \leq \sum_{k=0}^\infty \int_{2^k(B_5 \setminus B_2)} v^2 \leq \sum_{i,k} \int_{B_{2^k}(y_{i,k})} v^2.$$

To estimate the right hand side, we use (27) for $k \gg 1$, which yields

$$\int_{\mathbb{R}^N \setminus B_2} v^2 \leq C_3 \sum_{i,k} |y_{i,k}|^\alpha \exp\{-C_4|y_{i,k}|^{1-\alpha/2}\} < \infty,$$

since $0 < \alpha < 2$. This shows that $v \in L^2(\mathbb{R}^N)$, whence $v \in W^{1,2}(\mathbb{R}^N)$. As already pointed out in Theorem 13, $v \in C^2(\mathbb{R}^N)$ and $v > 0$. Finally, standard arguments show that $\lim_{|x| \to \infty} v(x) = 0$ (see for example [22]). □

5. Semiclassical limits for (NLS)

In this section we study the behavior of some solutions of (NLS) as ε tends to 0, and in particular of those obtained in Theorem 13. We always assume that (V), (K) hold true with $0 < \alpha < 2$ and $\beta > 0$, and that p satisfies (3). However some results, as Lemma 19 below, hold even if $0 < \alpha \leq 2$.

The next lemma provides an upper bound for the critical values b_ε in terms of the auxiliary functional $\mathcal{A} = V^\theta K^{-2/(p-1)}$ introduced in Theorem 3. It is worth pointing out explicitly that, since $p > \sigma$, $\mathcal{A}(x) \to \infty$ as $|x| \to \infty$, and therefore $\mathcal{A}$ has a global minimum on all of $\mathbb{R}^N$.

Lemma 19. *There exists $C_0 > 0$ such that for all $\xi \in \mathbb{R}^N$ and all ε sufficiently small,*

$$\varepsilon^{-N} b_\varepsilon = \varepsilon^{-N} I_\varepsilon(v_\varepsilon) \leq C_0\mathcal{A}(\xi) + o(1) \quad \text{as } \varepsilon \to 0^+. \tag{28}$$

In particular there exists $C^* > 0$ such that $b_\varepsilon \leq C^*\varepsilon^N$.

Proof. For any $\xi \in \mathbb{R}^N$, let us define the functional F_ξ on $W^{1,2}(\mathbb{R}^N)$ by setting

$$F_\xi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}V(\xi)\int_{\mathbb{R}^N} u^2 - \frac{1}{p+1}K(\xi)\int_{\mathbb{R}^N} |u|^{p+1}.$$

Let $f(\xi)$ denote the Mountain-Pass critical level of F_ξ. It is well known that

$$f(\xi) = \inf_{u\in\mathcal{N}_\xi} F_\xi(u),$$

where $\mathcal{N}_\xi$ is the Nehari manifold

$$\mathcal{N}_\xi = \left\{u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u|^2 + V(\xi)\int_{\mathbb{R}^N} u^2 = K(\xi)\int_{\mathbb{R}^N} |u|^{p+1}\right\}.$$

Let us point out that $u \in \mathcal{N}_\xi$ if and only if

$$\hat{u}(y) := K^{1/(p-1)}(\xi)V^{-1/(p-1)}(\xi)u(V^{-1/2}(\xi)y) \in \mathcal{N},$$

where $\mathcal{N} = \{u \in W^{1,2}(\mathbb{R}^N) : u \neq 0$ and $\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) = \int_{\mathbb{R}^N} |u|^{p+1}\}$. Hence, with a direct calculation we find

$$\begin{aligned} f(\xi) = \inf_{\mathcal{N}_\xi} F_\xi &= \left(\frac{1}{2} - \frac{1}{p+1}\right)K(\xi) \inf_{u\in\mathcal{N}_\xi}\int_{\mathbb{R}^N} |u|^{p+1}\,dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)K^{-2/(p-1)}(\xi)V^{(p+1)/(p-1)-N/2}(\xi) \inf_{v\in\mathcal{N}}\int_{\mathbb{R}^N} |v|^{p+1}\,dy. \end{aligned}$$

Let $\bar{U}$ denote the unique positive radial solution in $W^{1,2}(\mathbb{R}^N)$ of

$$-\Delta\bar{U} + \bar{U} = \bar{U}^p \quad \text{in } \mathbb{R}^N.$$

Since $\inf_{v\in\mathcal{N}}\int_{\mathbb{R}^N} |v|^{p+1}\,dy$ is achieved at $\bar{U}$, we get

$$\begin{aligned} f(\xi) &= [K(\xi)]^{-2/(p-1)}[V(\xi)]^{(p+1)/(p-1)-N/2}\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} |\bar{U}|^{p+1}\,dx \\ &= C_0\mathcal{A}(\xi). \end{aligned} \tag{29}$$

Since $f(\xi)$ is a Mountain-Pass level of F_ξ, for all $\nu > 0$ there exists $w \in W^{1,2}(\mathbb{R}^N)$ such that

$$f(\xi) \leq \max_{t>0} F_\xi(tw) \leq f(\xi) + \nu.$$

Let $\varphi \in C^2(\mathbb{R}^N)$ be a cut-off function such that $\varphi \equiv 1$ in a neighborhood of ξ and define, for any $\varepsilon > 0$, $w_\varepsilon \in W^{1,2}(\mathbb{R}^N)$ by

$$w_\varepsilon(x) = \varphi(x)w\left(\frac{x-\xi}{\varepsilon}\right).$$

Since $W^{1,2}(\mathbb{R}^N) \subset \mathcal{H}_\varepsilon$, we have $w_\varepsilon \in \mathcal{H}_\varepsilon$ for any ε; in particular it makes sense to compute $I_\varepsilon(tw_\varepsilon)$, which yields

$$I_\varepsilon(tw_\varepsilon) = \frac{t^2}{2}\|w_\varepsilon\|_\varepsilon^2 - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N} K(x)|w_\varepsilon|^{p+1}\,dx.$$

By the change of variable $y = (x-\xi)/\varepsilon$, we get

$$\begin{aligned}\varepsilon^{-N}\|w_\varepsilon\|_\varepsilon^2 = \varepsilon^2 &\int_{\mathbb{R}^N} |\nabla\varphi(\varepsilon y+\xi)|^2 w^2(y)\,dy\\ &+ \varepsilon\int_{\mathbb{R}^N} \nabla w(y)\cdot\nabla\varphi(\varepsilon y+\xi)w(y)\varphi(\varepsilon y+\xi)\,dy\\ &+ \int_{\mathbb{R}^N}\varphi^2(\varepsilon y+\xi)|\nabla w_\varepsilon(y)|^2\,dy + \int_{\mathbb{R}^N} V(\varepsilon y+\xi)\varphi^2(\varepsilon y+\xi)w^2(y)\,dy,\end{aligned}$$

as well as

$$\varepsilon^{-N}\int_{\mathbb{R}^N} K(x)|w_\varepsilon(x)|^{p+1}\,dx = \int_{\mathbb{R}^N} K(\varepsilon y+\xi)|\varphi(\varepsilon y+\xi)w(y)|^{p+1}\,dy.$$

Putting together the preceding equations we deduce that

$$\varepsilon^{-N} I_\varepsilon(tw_\varepsilon) = F_\xi(tw) + o(1) \quad \text{as } \varepsilon \to 0.$$

Hence

$$\begin{aligned}\varepsilon^{-N} I_\varepsilon(v_\varepsilon) &= \inf_{v\in\mathcal{H}_\varepsilon\setminus\{0\}}\ \max_{t>0} \varepsilon^{-N} I_\varepsilon(tv)\\ &\le \max_{t>0} I_\varepsilon(tw_\varepsilon) \le \max_{t>0} F_\xi(tw) + o(1)\\ &\le f(\xi) + \nu + o(1) = C_0\mathcal{A}(\xi) + \nu + o(1).\end{aligned}$$

Since $\nu > 0$ is arbitrary, the estimate in (28) is proved. The last statement follows from the fact that $\mathcal{A}$ has a global minimum on $\mathbb{R}^N$ since $p > \sigma$. □

Remark 20. To prove that $b_\varepsilon \le C^*\varepsilon^N$ one could also argue as follows. Consider the functionals $\widetilde{I}_\varepsilon, \widehat{I}_\varepsilon : W^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\begin{aligned}\widetilde{I}_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^N}[\varepsilon^2|\nabla u|^2 + Au^2]\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(x)|u|^{p+1}\,dx,\\ \widehat{I}_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla u|^2 + Au^2]\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)|u|^{p+1}\,dx.\end{aligned}$$

Clearly, $\widehat{u}_\varepsilon(x)$ is a critical point of $\widehat{I}_\varepsilon$ iff $\widetilde{u}_\varepsilon(x) := \widehat{u}_\varepsilon(x/\varepsilon)$ is a critical point of $\widetilde{I}_\varepsilon$; moreover, $\widetilde{I}_\varepsilon(\widetilde{u}_\varepsilon) = \varepsilon^N\widehat{I}_\varepsilon(\widehat{u}_\varepsilon)$. Let $\widetilde{b}_\varepsilon$, resp. $\widehat{b}_\varepsilon$, denote the Mountain-Pass critical level of $\widetilde{I}_\varepsilon$, resp. $\widehat{I}_\varepsilon$. Since $\sup V \le A$ and $W^{1,2}(\mathbb{R}^N) \subset \mathcal{H}_\varepsilon$, one easily deduces that $b_\varepsilon \le \widetilde{b}_\varepsilon = \varepsilon^N\widehat{b}_\varepsilon$.

On the other hand, critical points of $\widehat{I_\varepsilon}$ can be found near those of the *unperturbed* functional $\widehat{I_0} \equiv \widehat{I_{\varepsilon=0}}$. Up to translation, we can assume that $K(0) = \max K$. Let U be the unique positive radial solution of

$$-\Delta U + AU = K(0)U^p, \quad U \in W^{1,2}(\mathbb{R}^N).$$

Then, using [1], one infers that $\widehat{I_\varepsilon}$ has, for $\varepsilon > 0$ small, a critical point u_ε such that $\widehat{u_\varepsilon} \to U$ as $\varepsilon \to 0$. In particular, from $\widehat{I_\varepsilon}(\widehat{u_\varepsilon}) \to \widehat{I_0}(U)$ it follows that there exists $C > 0$ such that $\widehat{I_\varepsilon}(\widehat{u_\varepsilon}) \leq C$ for all $\varepsilon > 0$ small enough. Moreover, since U is a Mountain-Pass critical point of $\widehat{I_0}$, the same holds for u_ε. This implies that $\widehat{b_\varepsilon} \leq \widehat{I_\varepsilon}(\widehat{u_\varepsilon})$, and the result follows. ◻

Lemma 19 and (20) yield

Corollary 21. *For ε small there exists $\Gamma > 0$ such that*

$$\|v_\varepsilon\|_\varepsilon^2 \leq \Gamma \varepsilon^N,$$

where v_ε is given by Theorem 13.

The next lemma provides pointwise uniform decay estimates for the solutions v_ε. Here $0 < \alpha < 2$ is needed. We follow closely the method illustrated in [21, Appendix B].

Lemma 22. *Let Γ, R_Γ and v_ε be as in Lemma* 17. *Then there exists a constant C, depending only on Γ, p and N, and a positive number $d > 0$, depending on N, p, α and β, such that*

$$|v_\varepsilon(x)| \leq C|x|^d \varepsilon^{-d} \exp\left\{-\frac{1}{4}\left|\log\frac{3}{4}\right|\varepsilon^{-1}(|x|^{(2-\alpha)/2} - R_\Gamma^{(2-\alpha)/2})\right\} \ \text{for } |x| \geq 2R_\Gamma + C. \tag{30}$$

Proof. The functions v_ε satisfy the equation

$$-\varepsilon^2 \Delta v_\varepsilon + V(x) v_\varepsilon = K(x) v_\varepsilon^p. \tag{31}$$

Given $x_0 \in \mathbb{R}^N$ with $|x_0| \geq 2R_\Gamma + 2$, we consider a smooth cut-off function η satisfying

$$\eta(x) = \begin{cases} 1 & \text{for } x \in B_1(x_0), \\ 0 & \text{for } x \in \mathbb{R}^N \setminus B_2(x_0), \end{cases} \qquad |\nabla \eta| \leq 2. \tag{32}$$

Letting for simplicity $v = v_\varepsilon$, given $L > 0$ and $s \geq 0$, we also define the function $\phi = \phi_{s,L} \equiv v \min\{|v|^{2s}, L^2\}\eta^2$. Testing (31) on ϕ we obtain

$$\begin{aligned} \varepsilon^2 \int |\nabla v|^2 \min\{|v|^{2s}, L^2\}\eta^2 + \frac{s}{2}\varepsilon^2 \int_{\{|v|^s \leq L\}} |\nabla(|v|^2)|^2 v^{2s-2}\eta^2 + \int V(x) v^2 \eta^2 \min\{|v|^{2s}, L^2\} \\ \leq -2\varepsilon^2 \int v\eta \min\{|v|^{2s}, L^2\} \nabla v \cdot \nabla \eta + \int K v^{p+1} \eta^2 \min\{|v|^{2s}, L^2\} \\ \leq \frac{1}{2}\varepsilon^2 \int |\nabla v|^2 \min\{|v|^{2s}, L^2\}\eta^2 + C\varepsilon^2 \int v^2 \min\{|v|^{2s}, L^2\} |\nabla \eta|^2 \\ + \int K v^{p+1} \eta^2 \min\{|v|^{2s}, L^2\}. \end{aligned}$$

Hence, if we set

$$w = \eta v \min\{|v|^s, L\}, \tag{33}$$

from the above inequality we get

$$\int \varepsilon^2 |\nabla w|^2 + V(x) w^2 \le C\varepsilon^2 \int v^2 \min\{|v|^{2s}, L^2\} + \int K v^{p+1} \eta^2 \min\{|v|^{2s}, L^2\}. \tag{34}$$

Next, given $M > 0$, we divide the last integral into the two regions $\{v \le M\}$ and $\{v > M\}$ to obtain

$$\begin{aligned}\int K v^{p+1}\eta^2 \min\{|v|^{2s}, L^2\} \\ \le M^{p-1} \int K\eta^2 v^2 \min\{|v|^{2s}, L^2\} + \int_{\{v>M\}\cap B_2(x_0)} K v^{p-1} \eta^2 v^2 \min\{|v|^{2s}, L^2\}.\end{aligned}$$

By the Hölder and Sobolev inequalities we can write

$$\begin{aligned}\int_{\{v>M\}\cap B_2(x_0)} K v^{p-1}\eta^2 v^2 \min\{|v|^{2s}, L^2\} &\le \left(\int_{\{v>M\}\cap B_2(x_0)} (K v^{p-1})^{N/2}\right)^{2/N} \|v\|^2_{L^{2^*}} \\ &\le C\left(\int_{\{v>M\}\cap B_2(x_0)} (K v^{p-1})^{N/2}\right)^{2/N} \int |\nabla v|^2 \\ &= C\varepsilon^{-2}\left(\int_{\{v>M\}\cap B_2(x_0)} (K v^{p-1})^{N/2}\right)^{2/N} \varepsilon^2 \int |\nabla v|^2.\end{aligned}$$

If we can make $\int (K v^{p-1})^{N/2}$ sufficiently small, then we can bring this term on the left-hand side of (34). We note that, since $\|v\|^2_{L^{2^*}(\mathbb{R}^N)} \le C\varepsilon^N$ (by our assumptions), we have

$$M^{2N/(N-2)} |\{v > M\}| \le \int v^{2^*} \le C\varepsilon^N, \quad \text{and so} \quad |\{v > M\}| \le C\varepsilon^N M^{-2N/(N-2)}.$$

Next, from the Hölder inequality we get

$$\int_{\{v>M\}\cap B_2(x_0)} (K v^{p-1})^{N/2} \le \left(\int_{\{v>M\}} v^{(p-1)Nq/2}\right)^{1/q} \left(\int_{\{v>M\}} K^{Nq'/2}\right)^{1/q'}.$$

If we choose q in such a way that $(p-1)Nq/2 = 2N/(N-2)$, that is, if

$$q' = \frac{4}{4 - (p-1)(N-2)},$$

from the above estimates it follows that

$$\begin{aligned}&\int \varepsilon^2 |\nabla w|^2 + V(x) w^2 \\ &\le C\varepsilon^2 \int_{B_2(x_0)} v^{2s+2} + M^{p-1} \int_{B_2(x_0)} K\eta^2 v^{2s+2} + C\varepsilon^{-2} |x_0|^{-\beta} M^{-4/(N-2)q'} \varepsilon^2 \int |\nabla v|^2 \\ &\le C(\varepsilon^2 + M^{p-1}|x_0|^{-\beta}) \int_{B_2(x_0)} v^{2s+2} + C\varepsilon^{-2} |x_0|^{-\beta} M^{-\frac{4-(p-1)(N-2)}{N-2}} \varepsilon^2 \int |\nabla v|^2.\end{aligned}$$

Now we choose M in such a way that $\varepsilon^{-2}|x_0|^{-\beta} M^{-\frac{4-(p-1)(N-2)}{N-2}}$ is a small constant, namely we take

$$M = C(\varepsilon^{-2}|x_0|^{-\beta})^{\frac{N-2}{4-(N-2)(p-1)}}$$

with C sufficiently large. In this way, choosing s in such a way that $2s+2 = p+1$, we get

$$\begin{aligned} \int [\varepsilon^2|\nabla w|^2 + V(x)w^2] &\le C(\varepsilon^2 + |x_0|^{-\beta} M^{p-1}) \int_{B_2(x_0)} v^{2s+2} \\ &\le C\varepsilon^{-\frac{2(p-1)(N-2)}{4-(N-2)(p-1)}} \int_{B_2(x_0)} v^{p+1}. \end{aligned} \tag{35}$$

From our assumptions on the functions $v = v_\varepsilon$ and the Hölder inequality it follows that

$$\int_{B_2(x_0)} v^{p+1} \le \left(\int_{B_2(x_0)} v^2 \right)^{\omega} + \left(\int_{B_2(x_0)} v^{2^*} \right)^{1-\omega} \le C\varepsilon^{N(1-\omega)} \left(|x_0|^{\alpha} \int_{B_2(x_0)} V v^2 \right)^{\omega}$$

for some $\omega \in (0,1)$. By Lemma 18 and the last two estimates, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} [\varepsilon^2|\nabla w|^2 + V(x)w^2] \\ &\le C|x_0|^{d_1}\varepsilon^{-d_1} \exp\left\{ -\frac{p+1}{4} \left| \log \frac{3}{4} \right| \varepsilon^{-1} (|x_0|^{(2-\alpha)/2} - R_\Gamma^{(2-\alpha)/2}) \right\}, \qquad |x_0| \ge 2R_\Gamma + 2, \end{aligned}$$

for some constant C depending on Γ, p and N, and some positive number $d_1 > 0$, depending on N, p, α and β.

We note at this point that the last estimate is independent of the number L in the definition of w. This implies that $|v|^{s+1}$ belongs to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, with some quantitative estimates on the integrals, which are given in the last formula. Then the Sobolev embedding theorem implies $v \in L^{(s+1)2^*}_{\mathrm{loc}}$.

Finally, proceeding in this way and using a bootstrap argument, we obtain the result after a finite number of steps. □

Remark 23. Although we already proved that $\lim_{|x|\to\infty} v_\varepsilon(x) = 0$ for any fixed $\varepsilon > 0$, the preceding lemma is needed since it gives a pointwise decay *uniform* in ε. □

Lemma 24. *Let v_ε be solutions of* (19) *satisfying* (23). *Let x_ε denote any maximum point of v_ε. Then there exists a constant $C > 0$, $C = C(\Gamma)$, such that $|x_\varepsilon| \le C$ for every ε sufficiently small.*

Proof. Since x_ε is a maximum point of v_ε, one has $\Delta v_\varepsilon(x_\varepsilon) \le 0$. Therefore, from (19) it follows that

$$V(x_\varepsilon)K^{-1}(x_\varepsilon) \le v_\varepsilon^{p-1}(x_\varepsilon). \tag{36}$$

From (V) and (K) it follows that there exists $c > 0$ such that

$$c|x_\varepsilon|^{\beta-\alpha} \le V(x_\varepsilon)K^{-1}(x_\varepsilon). \tag{37}$$

From (36), (37), and (30) we deduce that if $|x_\varepsilon| \geq 2R_\Gamma$, then

$$c|x_\varepsilon|^{\beta-\alpha} \leq |x_\varepsilon|^{d(p-1)}\varepsilon^{-d(p-1)} \exp\left\{-\frac{1}{4}(p-1)\left|\log\frac{3}{4}\right|\varepsilon^{-(2-\alpha)/a}(|x_\varepsilon|^{(2-\alpha)/2} - R_\Gamma^{(1-\alpha)/2})\right\}. \quad (38)$$

This immediately implies that $|x_\varepsilon|$ stays bounded as $\varepsilon \to 0$. Lemma 24 is thereby proved. ◻

Lemma 25. *Let v_ε be as in Lemma 24. Then there exists a constant $C > 0$ such that $\|v_\varepsilon\|_{L^\infty} \geq C^{-1}$ for all ε sufficiently small.*

Proof. From (19) we get

$$\|v_\varepsilon\|_\varepsilon^2 = \int_{\mathbb{R}^N}[\varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2] = \int_{\mathbb{R}^N} K(x)v_\varepsilon^{p+1}. \quad (39)$$

Let us fix $\delta < \Gamma^{-(p-1)/2}$. Then from Proposition 11 there exists R such that

$$\int_{|x|>R} K(x)v_\varepsilon^{p+1}\,dx \leq \delta\varepsilon^{-N(p-1)/2}\|v_\varepsilon\|_\varepsilon^{p+1}. \quad (40)$$

From (V) and (K) we have

$$K(x) \leq \frac{k}{a}\frac{1+|x|^\alpha}{1+|x|^\beta}V(x) \leq \frac{k}{a}(1+R^\alpha)V(x) \quad \text{for any } |x| \leq R,$$

hence

$$\int_{|x|\leq R} K(x)v_\varepsilon^2 \leq \frac{k}{a}(1+R^\alpha)\int_{|x|\leq R} V(x)v_\varepsilon^2 \leq \frac{k}{a}(1+R^\alpha)\|v_\varepsilon\|_\varepsilon^2.$$

From this it follows that

$$\int_{|x|\leq R} K(x)v_\varepsilon^{p+1} \leq \|v_\varepsilon^{p-1}\|_{L^\infty}\int_{|x|\leq R} K(x)v_\varepsilon^2 \leq \frac{k}{a}(1+R^\alpha)\|v_\varepsilon^{p-1}\|_{L^\infty}\|v_\varepsilon\|_\varepsilon^2. \quad (41)$$

From (39), (40), and (41) we get

$$\|v_\varepsilon\|_\varepsilon^2 = \int_{\mathbb{R}^N} K(x)v_\varepsilon^{p+1} \leq \delta\varepsilon^{-N(p-1)/2}\|v_\varepsilon\|_\varepsilon^{p+1} + \frac{k}{a}(1+R^\alpha)\|v_\varepsilon^{p-1}\|_{L^\infty}\|v_\varepsilon\|_\varepsilon^2,$$

which yields

$$1 \leq \delta\varepsilon^{-N(p-1)/2}\|v_\varepsilon\|_\varepsilon^{p-1} + \frac{k}{a}(1+R^\alpha)\|v_\varepsilon^{p-1}\|_{L^\infty}. \quad (42)$$

Since $\|v_\varepsilon\|_\varepsilon^{p-1} \leq \Gamma^{(p-1)/2}\varepsilon^{N(p-1)/2}$, which follows from (23), the estimate (42) implies

$$1 \leq \delta\widetilde{C}^{(p-1)/2} + \frac{k}{a}(1+R^\alpha)\|v_\varepsilon^{p-1}\|_{L^\infty},$$

hence, for our choice of δ, we deduce that

$$\|v_\varepsilon\|_{L^\infty}^{p-1} \geq (1-\delta\widetilde{C}^{(p-1)/2})\frac{a}{k(1+R^\alpha)} > 0,$$

which proves the lemma. ◻

We are now in a position to characterize the ground states when ε tends to 0.

Theorem 26. *Let the assumptions of Theorem* 16 *hold. Then* (NLS) *has a (classical, positive) ground state* v_ε *concentrating, as* $\varepsilon \to 0$, *at a global minimum point* x^* *of* $\mathcal{A} = V^\theta K^{-2/(p-1)}$, *where* $\theta = (p+1)/(p-1) - N/2$. *More precisely,* v_ε *has a unique maximum point* x_ε *such that* $x_\varepsilon \to x^*$ *as* $\varepsilon \to 0$, *and*

$$v_\varepsilon(x) = U^*\left(\frac{x - x_\varepsilon}{\varepsilon}\right) + \omega_\varepsilon(x) \quad \text{as } \varepsilon \to 0,$$

where $\omega_\varepsilon \to 0$ *in* $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ *and in* $L^\infty(\mathbb{R}^N)$ *as* $\varepsilon \to 0$, *and* U^* *is the unique positive radial solution of*

$$-\Delta U^* + V(x^*)U^* = K(x^*)(U^*)^p.$$

Proof. The proof is based upon the preceding lemmas and is rather standard (see e.g. [12, 24]). However, to keep the paper as self-contained as possible, we will carry out the arguments in detail. Let x_ε denote a global maximum point of v_ε (such a maximum exists since $v_\varepsilon(x) \to 0$ as $|x| \to \infty$). From Lemma 24, we know that, up to a subsequence, $x_\varepsilon \to x^*$ for some $x^* \in \mathbb{R}^N$. Set

$$\psi_\varepsilon(x) := v_\varepsilon(\varepsilon x + x_\varepsilon).$$

Since v_ε solves (19), ψ_ε satisfies

$$-\Delta\psi_\varepsilon(x) + V(\varepsilon x + x_\varepsilon)\psi_\varepsilon(x) = K(\varepsilon x + x_\varepsilon)\psi_\varepsilon^p(x), \quad x \in \mathbb{R}^N. \tag{43}$$

From Corollary 21 and assumption (V) it follows that

$$\begin{aligned}
\Gamma \geq \varepsilon^{-N}\|v_\varepsilon\|_\varepsilon^2 &= \varepsilon^{-N}\int_{\mathbb{R}^N}[\varepsilon^2|\nabla v_\varepsilon(x)|^2 + V(x)v_\varepsilon^2(x)]\,dx \\
&\geq \varepsilon^{-N}\int_{\mathbb{R}^N}\left[\varepsilon^2|\nabla v_\varepsilon(x)|^2 + \frac{a}{1+|x|^\alpha}v_\varepsilon^2(x)\right]dx \\
&= \int_{\mathbb{R}^N}\left[|\nabla\psi_\varepsilon(y)|^2 + \frac{a}{1+|\varepsilon y + x_\varepsilon|^\alpha}\psi_\varepsilon^2(y)\right]dy.
\end{aligned}$$

From Lemma 24 we infer that $|\varepsilon y + x_\varepsilon| \leq C(1+|y|)$ and therefore

$$\int_{\mathbb{R}^N}\left[|\nabla\psi_\varepsilon(y)|^2 + \frac{a}{1+|y|^\alpha}\psi_\varepsilon^2(y)\right]dy \leq C',$$

where C' is independent of ε. In particular $\{\psi_\varepsilon\}_\varepsilon$ is bounded in C^∞_{loc}, uniformly with respect to ε, and we deduce that ψ_ε converges in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ to some $U^* \in C^2_{\mathrm{loc}}(\mathbb{R}^N)$. Furthermore, using arguments similar to those carried out in the proof of Lemma 22, one infers that $\psi_\varepsilon \to U^*$ also in $L^\infty(\mathbb{R}^N)$. Passing to the limit in equation (43), we find that $U^* \geq 0$ is a classical solution to

$$-\Delta U^*(x) + V(x^*)U^*(x) = K(x^*)(U^*)^p(x), \quad x \in \mathbb{R}^N. \tag{44}$$

Moreover, since ψ_ε attains its maximum at 0, so does U^*. Furthermore, Lemma 25 shows that $\psi_\varepsilon(0) = v_\varepsilon(x_\varepsilon) = \|v_\varepsilon\|_{L^\infty} \geq C^{-1}$ for some positive constant C, and thus $\max U^* = U^*(0) \geq C^{-1} > 0$. In particular, $U^* \not\equiv 0$ (hence $U^* > 0$ by the maximum principle) and is a radial function according to the Gidas–Ni–Nirenberg result [13]. Using again

Corollary 21 we get, for any sequence $R_n \to \infty$,

$$\int_{\overline{B}_{R_n}} [|\nabla \psi_\varepsilon(x)|^2 + V(\varepsilon x + x_\varepsilon)\psi_\varepsilon^2(x)]\,dx \le \varepsilon^{-N}\|v_\varepsilon\|_\varepsilon^2 \le \Gamma. \tag{45}$$

Since $\psi_\varepsilon \to U^*$ in $C^1(\overline{B}_{R_n})$, the Dominated Convergence Theorem allows us to pass to the limit in (45) as $\varepsilon \to 0$ to obtain

$$\int_{\overline{B}_{R_n}} [|\nabla U^*(x)|^2 + V(x^*)(U^*)^2(x)]\,dx \le \Gamma.$$

Letting now $R_n \to \infty$, we infer that $U^* \in W^{1,2}(\mathbb{R}^N)$.

To complete the proof of Theorem 26, a further lemma is in order, which provides a lower bound for b_ε in terms of U^* and x^*.

Lemma 27. *Let F_ξ be as in the proof of Lemma* 19. *Then*

$$F_{x^*}(U^*) \le \liminf_{\varepsilon\to 0} \varepsilon^{-N} I_\varepsilon(v_\varepsilon) = \liminf_{\varepsilon\to 0} \varepsilon^{-N} b_\varepsilon. \tag{46}$$

Proof. One has $I_\varepsilon(v_\varepsilon) = \varepsilon^N \int_{\mathbb{R}^N} h_\varepsilon(x)\,dx$, where

$$h_\varepsilon(x) = \frac{1}{2}|\nabla\psi_\varepsilon(x)|^2 + \frac{1}{2}V(\varepsilon x + x_\varepsilon)\psi_\varepsilon^2(x) - \frac{1}{p+1}K(\varepsilon x + x_\varepsilon)\psi_\varepsilon^{p+1}(x). \tag{47}$$

Let $R > 0$ to be chosen later. In view of the C^1-convergence of ψ_ε to U^* over the compact sets of $\mathbb{R}^N$ we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\overline{B}_R} h_\varepsilon\,dx\\ &\quad= \frac{1}{2}\int_{\overline{B}_R} |\nabla U^*|^2\,dx + \frac{1}{2}V(x^*)\int_{\overline{B}_R}(U^*)^2\,dx - \frac{1}{p+1}K(x^*)\int_{\overline{B}_R}(U^*)^{p+1}\,dx.\end{aligned}$$

Since $U^* \in W^{1,2}(\mathbb{R}^N)$, for any $\nu > 0$ we can choose $R > 0$ large enough such that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\overline{B}_R} h_\varepsilon\,dx &\ge \int_{\mathbb{R}^N}\left[\frac{1}{2}|\nabla U^*|^2 + \frac{1}{2}V(x^*)(U^*)^2 - \frac{1}{p+1}K(x^*)(U^*)^{p+1}\right]dx - \nu\\ &= F_{x^*}(U^*) - \nu.\end{aligned} \tag{48}$$

Let now η_R be a cut-off function such that $\eta_R = 0$ in B_{R-1}, $\eta_R = 1$ in $\mathbb{R}^N \setminus B_R$, $0 \le \eta_R \le 1$, $|\nabla\eta_R| \le C$, with C independent of R. Testing (43) on $\eta_R\psi_\varepsilon$ we obtain

$$2\int_{\mathbb{R}^N\setminus B_R} h_\varepsilon\,dx + \left(\frac{2}{p+1} - 1\right)\int_{\mathbb{R}^N\setminus B_R} K(\varepsilon x + x_\varepsilon)\psi_\varepsilon^{p+1}\,dx + E_\varepsilon = 0,$$

where

$$E_\varepsilon = \int_{B_R\setminus B_{R-1}} [\nabla\psi_\varepsilon\cdot\nabla(\eta_R\psi_\varepsilon) + V(\varepsilon x + x_\varepsilon)\eta_R\psi_\varepsilon^2 - K(\varepsilon x + x_\varepsilon)\eta_R\psi_\varepsilon^{p+1}]\,dx.$$

Hence $\int_{\mathbb{R}^N \setminus B_R} h_\varepsilon\, dx \geq -E_\varepsilon/2$. Again by the convergence of ψ_ε in C^1_{loc} to $U^* \in W^{1,2}(\mathbb{R}^N)$, we deduce that for R large enough $\lim_{\varepsilon\to 0} |E_\varepsilon| \leq \nu$ and hence

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N \setminus B_R} h_\varepsilon\, dx \geq -\frac{\nu}{2}. \tag{49}$$

From (48) and (49) we conclude that

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} h_\varepsilon\, dx \geq F_{x^*}(U^*) - \frac{3}{2}\nu$$

for any $\nu > 0$, and (46) follows. □

Proof of Theorem 26 completed. Let us first prove that x^* is a minimum point of the function $f(\xi) = C_0\mathcal{A}(\xi)$. Arguing by contradiction, we assume that there exists $\xi^* \in \mathbb{R}^N$ such that $f(x^*) > f(\xi^*)$. From (46) and (28), it follows that

$$F_{x^*}(U^*) \leq \liminf_{\varepsilon\to 0} \varepsilon^{-N} I_\varepsilon(v_\varepsilon) \leq C_0\mathcal{A}(\xi), \qquad \forall \xi \in \mathbb{R}^N.$$

On the other hand, since U^* solves equation (44),

$$F_{x^*}(U^*) \geq \inf_{u\in\mathcal{N}_{x^*}} F_{x^*}(u) = f(x^*) > f(\xi^*) = C_0\mathcal{A}(\xi^*).$$

which yields a contradiction.

It remains to show that v_ε has at most one maximum point. The proofs relies on the arguments carried out above and so we will be sketchy. By contradiction, assume that, up to a subsequence, v_ε has two distinct maxima $x_\varepsilon, z_\varepsilon$. From Lemma 24 it follows that there exist $x^*, z^* \in \mathbb{R}^N$ such that $x_\varepsilon \to x^*$ and $z_\varepsilon \to z^*$. Let ψ_ε and U^* be as above. The convergence of ψ_ε to U^* in C^2_{loc} and the properties of U^* readily imply that there exists $r > 0$ such that $\psi''_\varepsilon(x) < \text{const} < 0$ for $x \in B_r$ provided ε is small enough. Since $\varepsilon^{-1}(z_\varepsilon - x_\varepsilon)$ is a maximum point of ψ_ε, two cases can occur.

Case 1: $\varepsilon^{-1}(z_\varepsilon - x_\varepsilon)$ is bounded and hence, up to a subsequence, it converges to some $P \in \mathbb{R}^N$. Since $\psi_\varepsilon(\varepsilon^{-1}(z_\varepsilon - x_\varepsilon)) = \max \psi_\varepsilon$ converges to $\max U^* = U^*(0)$, we conclude that $P = 0$. Therefore $\varepsilon^{-1}(z_\varepsilon - x_\varepsilon) \in B_r$ for ε sufficiently small, which is impossible since 0 is the only critical point of ψ_ε in B_r.

Case 2: $\varepsilon^{-1}(z_\varepsilon - x_\varepsilon)$ is unbounded, and hence it tends to ∞, up to a subsequence. As above, one shows that $\widetilde{\psi}_\varepsilon$ C^2_{loc}-converges to $\tilde{U}^*$, where $\widetilde{\psi}_\varepsilon := v_\varepsilon(\varepsilon x + z_\varepsilon)$ and $\tilde{U}^*$ is the unique positive radial solution in $W^{1,2}(\mathbb{R}^N)$ of

$$-\Delta\tilde{U}^*(x) + V(z^*)\tilde{U}^*(x) = K(z^*)(\tilde{U}^*)^p(x), \quad x \in \mathbb{R}^N.$$

Let us remark that, since $|\varepsilon^{-1}(z_\varepsilon - x_\varepsilon)| \to \infty$, for any R the balls $\overline{B}_R$ and $\overline{B}^\varepsilon := \overline{B}_R(\varepsilon^{-1}(z_\varepsilon - x_\varepsilon))$ are disjoint provided ε is small enough. Using this fact and repeating the arguments carried out above, we readily find that for any $\nu > 0$ it is possible to choose $R > 0$ large enough such that

$$\lim_{\varepsilon \to 0} \int_{\overline{B}^\varepsilon} h_\varepsilon \geq F_{z^*}(\tilde{U}^*) - \nu, \tag{50}$$

as well as

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus (B_R \cup B^\varepsilon)} h_\varepsilon \geq -\nu. \tag{51}$$

From (48), (50) and (51) we conclude that

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^N} h_\varepsilon \geq F_{x^*}(U^*) + F_{z^*}(\tilde{U}^*) - 3\nu.$$

Since ν is arbitrary we find that

$$\liminf_{\varepsilon \to 0} \varepsilon^{-N} b_\varepsilon \geq F_{x^*}(U^*) + F_{z^*}(\tilde{U}^*). \tag{52}$$

From (28) and (52) it follows that $F_{x^*}(U^*) + F_{z^*}(\tilde{U}^*) \leq f(x^*)$. Since x^* and z^* are both global minimum points of f, we have $f(x^*) = f(z^*)$ and hence, using the definition of f, we deduce that

$$F_{x^*}(U^*) + F_{z^*}(\tilde{U}^*) \leq \frac{1}{2}(f(x^*) + f(z^*)) \leq \frac{1}{2}(F_{x^*}(U^*) + F_{z^*}(\tilde{U}^*)),$$

which is not possible. The proof is now complete. □

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