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Heisenberg–Pauli–Weyl uncertainty inequalities and polynomial volume growth

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Abstract

In its simpler form, the Heisenberg–Pauli–Weyl inequality says that

$$\|f\|_2^4 \leqslant C\left(\int_{\mathbb{R}^n} |x|^2 \left|f(x)\right|^2 dx\right)\left(\int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{1}{2}} f(x)\right|^2 dx\right).$$

In this paper, we extend this inequality to positive self-adjoint operators L on measure spaces with a "gauge function" such that (a) measures of balls are controlled by powers of the radius (possibly different powers for large and small balls); (b) the semigroup generated by L satisfies ultracontractive estimates with polynomial bounds of the same type. We give examples of applications of this result to sub-Laplacians on groups of polynomial volume growth and to certain higher-order left-invariant hypoelliptic operators on nilpotent groups. We finally show that these estimates also imply generalized forms of local uncertainty inequalities.

Keywords: Lie groups; Fourier analysis; Uncertainty principles

1. Introduction

It is a well-known fact in classical Fourier analysis that a function f and its Fourier transform $\hat{f}$ cannot both be compactly supported, unless $f = 0$ a.e. This is the simplest qualitative form of

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uncertainty principle. The most common quantitative formulation of the uncertainty principle is the Heisenberg–Pauli–Weyl inequality. It says that, if $f \in L^2(\mathbb{R}^n)$ and $\alpha > 0$,

$$\|f\|_2^4 \leqslant C_\alpha \left(\int_{\mathbb{R}^n} |x|^{2\alpha} \left| f(x) \right|^2 dx \right) \left(\int_{\mathbb{R}^n} |\xi|^{2\alpha} \left| \hat{f}(\xi) \right|^2 d\xi \right). \tag{1}$$

We refer to [8] for an overview of the history and the relevance of this inequality, as well as to its generalizations allowing other L^p-norms and different powers of $|x|$ and $|\xi|$.

An equivalent formulation of (1) is

$$\|f\|_2^4 \leqslant C_\alpha \left(\int_{\mathbb{R}^n} |x|^{2\alpha} \left| f(x) \right|^2 dx \right) \left(\int_{\mathbb{R}^n} \left| (-\Delta)^{\frac{\alpha}{2}} f(x) \right|^2 dx \right), \tag{2}$$

where Δ denotes the Laplacian on $\mathbb{R}^n$.

This form of the Heisenberg–Pauli–Weyl inequality is better suited for extensions to other contexts, with the Laplacian replaced by a positive self-adjoint operator, and $|x|$ by a distance function. The interpretation of uncertainty inequalities as spectral properties of differential operators is widely present in the literature [6,7,15].

We are motivated in particular by extensions to the context of Lie groups and left-invariant differential operators on them. The direction we have in mind is that explored in [20], where the analogue of (2) is obtained on the Heisenberg group H_n, involving the $U(n)$-invariant sub-Laplacian and the natural homogeneous norm (we mention that, by a refinement of the same approach, an extension to general sub-Laplacians on step-two nilpotent groups is given in [2], and that other forms of the uncertainty principle on the Heisenberg group are in [10,19]).

In this paper we prove a general form of the Heisenberg–Pauli–Weyl inequality that requires very little structure, basically a measure space with a "gauge function" such that measures of balls are controlled by powers of the radius (possibly different powers for large and small balls), and a contraction semigroup on L^2 satisfying ultracontractive estimates with polynomial bounds of the same type.

Concerning Lie groups, the general setting that we are going to introduce includes at least two cases, which are presented in this paper.

The first situation is that of a group G of polynomial growth, endowed with a sub-Laplacian $L = -\sum_{j=1}^{k} X_j^2$, where the X_j are left-invariant vector fields generating the Lie algebra $\mathfrak{g}$. In this case, the norm $|x|$ in (1) is replaced by the distance of x from the identity in the control metric induced from the X_j.

The second situation is that of a nilpotent group G and a left-invariant, self-adjoint, positive differential operator $P(X_1, \ldots, X_k)$ admitting a homogeneous, hypoelliptic lifting to the free nilpotent group $\tilde{G}$ generated by $X_1, \ldots, X_k$ and of sufficiently high step. This includes, for instance, operators of the form

$$L = \sum_{j=1}^{k} (iX_j)^{2\nu_j},$$

where the ν_j are integers and the X_j generate the full Lie algebra of G. Here, $|x|$ is replaced by the distance of x from the identity element, in the quotient metric on G induced by a homogeneous norm on $\tilde{G}$.

We remark that our results for higher-order operators give new inequalities also in $\mathbb{R}^n$.

In the last part of the paper (Section 4), we comment on the "local uncertainty inequalities," which appear as crucial preliminary results in previous proofs of the Heisenberg–Pauli–Weyl inequality [5,20].

In the classical context, the local uncertainty inequality says that, for a test function f on $\mathbb{R}^n$ and a measurable set E with Lebesgue measure $|E| < \infty$,

$$\int_E \left|\hat{f}(y)\right|^2 dy \leqslant C_\alpha |E|^{\frac{2\alpha}{n}} \int_{\mathbb{R}^n} \left|f(x)\right|^2 |x|^{2\alpha}\, dx, \tag{3}$$

if $0 \leqslant \alpha < \frac{n}{2}$.

The left-hand side of (3) can be interpreted as the L^2-norm squared of $P_E f$, where P_E is the translation-invariant orthogonal projection on L^2 whose Fourier multiplier is the characteristic function χ_E of E. Denoting by w_E the inverse Fourier transform of χ_E, so that $P_E f = f * w_E$, (3) can be expressed as a weighted L^2-estimate for P_E,

$$\|P_E f\|_2 \leqslant C_\alpha \|w_E\|_2^{\frac{2\alpha}{n}} \left\| |x|^\alpha f \right\|_2. \tag{4}$$

We prove that analogues of (4) hold on general Lie groups.

2. The general theorem

We consider a locally compact space X endowed with a positive Borel measure m and:

(i) a non-negative "gauge" $\rho(x, y)$ continuous on $X \times X$, such that

$$\rho(x, y) = 0 \quad \Leftrightarrow \quad x = y, \qquad \rho(x, y) = \rho(y, x), \tag{5}$$

and with the property that the "balls" $B(x, r) = \{y\colon \rho(x, y) < r\}$ satisfy the volume growth conditions

$$m\big(B(x, r)\big) \lesssim \begin{cases} r^{d_0} & \text{for } r \to 0, \\ r^{d_\infty} & \text{for } r \to \infty, \end{cases} \tag{6}$$

for some $d_0, d_\infty > 0$ and uniformly in x;

(ii) a positive self-adjoint operator L on $L^2(X)$ generating an ultracontractive semigroup that satisfies

$$\left\|e^{-tL}\right\|_{1\to\infty} \lesssim \begin{cases} t^{-\frac{d_0}{k}} & \text{for } t \to 0, \\ t^{-\frac{d_\infty}{k}} & \text{for } t \to \infty, \end{cases} \tag{7}$$

for some $k > 0$.

The main example of this kind of situation that we have in mind is a Lie group of polynomial volume growth with a given sub-Laplacian on it.

A non-compact connected Lie group G of polynomial volume growth [12] has the property that the (left and right) Haar measure of large balls, with respect to any left-invariant distance, grows like the d_∞-power of the radius, for some positive integer d_∞ not depending on the distance.

Let $\{X_1, \ldots, X_m\}$ be a family of left-invariant vector fields generating the full Lie algebra of G. The left-invariant Carnot–Carathéodory distance on G induced by this family has the property that the Haar measure of small balls is comparable to the d_0-power of the radius, for some positive integer d_0.

By Hörmander's theorem, the sub-Laplacian

$$L = -\sum_{j=1}^{m} X_j^2$$

is hypoelliptic and generates a semigroup satisfying (7) with $k = 2$ [21].

Other examples in which conditions (6) and (7) are satisfied occur in the context of Riemannian manifolds of non-negative Ricci curvature [4,11] and fractals [1].

We have the following extension of the Heisenberg–Pauli–Weyl inequality.

Theorem 2.1. *Assume that X, m, ρ, L satisfy* (6) *and* (7)*. Let $\alpha, \beta > 0$ and $x_0 \in X$. If f is in the domain of $L^{\beta/k}$,*

$$\|f\|_2 \leqslant C_{\alpha,\beta} \left\|\rho_0^\alpha f\right\|_2^{\frac{\beta}{\alpha+\beta}} \left\|L^{\frac{\beta}{k}} f\right\|_2^{\frac{\alpha}{\alpha+\beta}}, \tag{8}$$

with ρ_0 denoting the function $\rho_0(x) = \rho(x_0, x)$.

Proof. Assume that $\alpha < \frac{d}{2}$, where $d = \min\{d_0, d_\infty\}$. We claim that

$$\left\|e^{-tL} f\right\|_2 \leqslant C_\alpha t^{-\frac{\alpha}{k}} \left\|\rho_0^\alpha f\right\|_2. \tag{9}$$

For $r > 0$, let $B_r = B(x_0, r)$. We set $f_r = f\chi_{B_r}$, $f^r = f - f_r$. Then, since $f^r \leqslant r^{-\alpha}\rho_0^\alpha f$ and e^{-tL} is a semigroup of contractions,

$$\left\|e^{-tL} f^r\right\|_2 \leqslant \left\|f^r\right\|_2 \leqslant r^{-\alpha} \left\|\rho_0^\alpha f\right\|_2.$$

On the other hand, we have

$$\left\|e^{-tL} f_r\right\|_2 \leqslant \left\|e^{-tL}\right\|_{1\to 2} \|f_r\|_1 \leqslant \left\|e^{-2tL}\right\|_{1\to\infty}^{\frac{1}{2}} \biggl(\int\limits_{\rho(x,x_0)<r} \rho_0(x)^{-2\alpha}\, dx\biggr)^{1/2} \left\|\rho_0^\alpha f\right\|_2.$$

It easily follows from (6) and the assumption that $\alpha < \frac{d}{2}$ that

$$\int\limits_{\rho(x,x_0)<r} \rho_0(x)^{-2\alpha}\, dx = \sum_{j\geqslant 0} \int\limits_{\rho(x,x_0)\sim r2^{-j}} \rho_0(x)^{-2\alpha}\, dx \leqslant C r^{-2\alpha} \varphi(r), \tag{10}$$

with $\varphi(r) = r^{d_0}$ for $r < 1$ and $\varphi(r) = r^{d_\infty}$ for $r > 1$, and with C depending on α. Therefore,

$$\left\|e^{-tL} f_r\right\|_2 \leqslant C r^{-\alpha} \varphi(t)^{-\frac{1}{2k}} \varphi(r)^{\frac{1}{2}} \left\|\rho_0^\alpha f\right\|_2.$$

Hence,

$$\left\|e^{-tL} f\right\|_2 \leqslant \left\|e^{-tL} f_r\right\|_2 + \left\|e^{-tL} f^r\right\|_2 \leqslant C r^{-\alpha}\left(\varphi(t)^{-\frac{1}{2k}} \varphi(r)^{\frac{1}{2}} + 1\right)\left\|\rho_0^\alpha f\right\|_2.$$

Choosing $r = t^{1/k}$ we obtain (9).

To prove the HPW inequality we initially assume that $\alpha < \frac{d}{2}$ and $\beta \leqslant k$. By (9),

$$\begin{aligned} \|f\|_2 &\leqslant \left\|e^{-tL} f\right\|_2 + \left\|\left(1 - e^{-tL}\right) f\right\|_2 \\ &\leqslant C t^{-\frac{\alpha}{k}} \left\|\rho_0^\alpha f\right\|_2 + \left\|\left(1 - e^{-tL}\right)(tL)^{-\frac{\beta}{k}} (tL)^{\frac{\beta}{k}} f\right\|_2. \end{aligned}$$

By the spectral theorem, the last term is controlled by $t^{\beta/k}\|L^{\beta/k} f\|_2$, since $(1 - e^{-\lambda})/\lambda^{\beta/k}$ is bounded for $\lambda \geqslant 0$ if $\beta \leqslant k$. Hence, we obtain

$$\|f\|_2 \leqslant C\left(t^{-\frac{\alpha}{k}} \left\|\rho_0^\alpha f\right\|_2 + t^{\frac{\beta}{k}} \left\|L^{\frac{\beta}{k}} f\right\|_2\right),$$

from which, optimizing in t, we obtain (8) for $\alpha < \frac{d}{2}$ and $\beta \leqslant k$.

If $\alpha \geqslant \frac{d}{2}$, let $\alpha' < \frac{d}{2}$. Then for all $\epsilon > 0$,

$$\frac{\rho_0^{\alpha'}}{\epsilon^{\alpha'}} \leqslant 1 + \frac{\rho_0^\alpha}{\epsilon^\alpha}, \tag{11}$$

from which it follows that

$$\left\|\rho_0^{\alpha'} f\right\|_2 \leqslant \epsilon^{\alpha'} \|f\|_2 + \epsilon^{\alpha' - \alpha} \left\|\rho_0^\alpha f\right\|_2.$$

Optimizing in ϵ, we obtain

$$\left\|\rho_0^{\alpha'} f\right\|_2 \leqslant C \left\|\rho_0^\alpha f\right\|_2^{\frac{\alpha'}{\alpha}} \|f\|_2^{1 - \frac{\alpha'}{\alpha}}.$$

Similarly, if $\beta > k$, we start from (8) with β replaced by some $\beta' \leqslant k$. Then, using (11) and the spectral theorem, we obtain the Landau–Kolmogorov inequality

$$\left\|L^{\frac{\beta'}{2}} f\right\|_2 \leqslant C \|f\|_2^{\frac{\beta - \beta'}{\beta}} \left\|L^{\frac{\beta}{2}} f\right\|_2^{\frac{\beta'}{\beta}}.$$

Plugging this into (8) with α replaced by α' and β by β', we get the result. □

3. Higher order operators on nilpotent groups

Let G be a simply connected nilpotent group. The differential operators that we consider are assumed to be expressible as homogeneous “non-commutative polynomials” of some left-invariant vector fields on G.

We follow the notation of [14], and call non-commutative polynomial in some indeterminates $x_1, \ldots, x_n$ an element $P(x_1, \ldots, x_n)$ of the tensor algebra $\mathcal{T}$ generated by $x_1, \ldots, x_n$. Given a n-tuple $a = (a_1, \ldots, a_n)$ of positive integers, we say that P is a-homogeneous of degree m if every monomial $x_{i_1} x_{i_2} \cdots x_{i_p}$ appearing in P satisfies $a_{i_1} + a_{i_2} + \cdots + a_{i_p} = m$.

We say that a left-invariant differential operator L on G is admissible if there is a n-tuple $\mathcal{X} = (X_1, \dots, X_n)$ of vector fields in the Lie algebra $\mathfrak{g}$, a n-tuple a and an a-homogeneous non-commutative polynomial P in n indeterminates, homogeneous of some degree m, such that $L = P(X_1, \dots, X_n)$. We are not imposing that the X_j are linearly independent, not even that they are all distinct. We require, however, that they generate $\mathfrak{g}$.

If ℓ is the length of $\mathfrak{g}$, let also $\tilde{\mathfrak{g}}$ be the free nilpotent algebra with n generators $\tilde{X}_1, \dots, \tilde{X}_n$ and of length ℓ. Then the map sending each $\tilde{X}_j$ into the corresponding X_j extends to a homomorphism π of the universal enveloping algebra $\mathfrak{U}(\tilde{\mathfrak{g}})$ onto $\mathfrak{U}(\mathfrak{g})$. If $\tilde{L} = P(\tilde{X}_1, \dots, \tilde{X}_n)$, then $\tilde{L}$ is homogeneous of degree m with respect to the dilations $\{\delta_r^a\}_{r>0}$ of $\tilde{\mathfrak{g}}$ such that $\delta_r^a \tilde{X}_j = r^{a_j} \tilde{X}_j$ for every j, and $\pi(\tilde{L}) = L$. The corresponding dilations on the simply connected group $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{g}}$ will also be denoted by δ_r^a. We call $\tilde{d}$ the homogeneous dimension of $\tilde{G}$ with respect to these dilations.

We say that P is symmetric if it is invariant under the conjugate-linear anti-automorphism of $\mathcal{T}$ mapping each x_i into $-x_i$. By [16], L and $\tilde{L}$, initially defined on $\mathcal{D}(G)$ and $\mathcal{D}(\tilde{G})$, respectively, are essentially self-adjoint. We keep the same symbols L and $\tilde{L}$ for their closures. We shall also require that L and $\tilde{L}$ are positive and hypoelliptic. It is proved in [14, Theorem 1], that the hypoellipticity of $\tilde{L}$ implies that of L. Notable examples of symmetric polynomials P for which all the conditions above are satisfied are

$$P(x) = \sum_{j=1}^{n} (i x_j)^{2\nu_j},$$

with the ν_j integers (and with $a_j = \prod_{i \neq j} \nu_i$, $m = 2 \prod_{i=1}^{n} \nu_i$) [13].

The same symbol π introduced before at the Lie algebra level will be used to denote also the quotient map from $\tilde{G}$ onto G. We set $H = \ker \pi$.

Lemma 3.1. *Let $P \in \mathcal{T}$ be symmetric and a-homogeneous of order m, and assume that $\tilde{L} = P(\tilde{X}_1, \dots, \tilde{X}_n)$ is hypoelliptic and positive on $\tilde{G}$. Then, for $t > 0$,*

$$e^{-t\tilde{L}} f(x) = f * h_t(x),$$

where $h_t \in \mathcal{S}(\tilde{G})$ and $h_t(x) = t^{-\frac{\tilde{d}}{m}} h_1(\delta^a_{t^{-1/m}} x)$.

The proof is in [9].

Lemma 3.2. *Let $P \in \mathcal{T}$ be symmetric and a-homogeneous of degree m, and assume that $\tilde{L}$ is hypoelliptic and positive on $\tilde{G}$. Then $e^{-tL} f = f * h^\flat(t)$, where*

$$h_t^\flat(x) = \int_H h_t(\tilde{x} y)\, dy = \int_H h_t(y \tilde{x})\, dy, \tag{12}$$

$\tilde{x}$ being any element in $\pi^{-1}(x)$, and dy an appropriate Haar measure on H.

Proof. Define the functions $h_t^\flat$ by (12). They obviously form a one-parameter semigroup under convolution. Since convolution by h_t is a contraction on $L^2(\tilde{G})$, it follows by transference (see [3]) that $T_t f = f * h_t^\flat$ is a contraction on $L^2(G)$.

If f is a function on G, set $\tilde f = f \circ \pi$ on $\tilde G$. Take now $f \in \mathcal{D}(G)$. Then $\tilde f * h_t$ and $\tilde L \tilde f$ are constant on the cosets of H and

$$\tilde f * h_t = \widetilde{f * h_t^\flat}, \qquad \tilde L \tilde f = \widetilde{Lf}.$$

Therefore,

$$\frac{d}{dt}\bigg|_{t=0} \widetilde{f * h_t^\flat} = \frac{d}{dt}\bigg|_{t=0} \tilde f * h_t = -\tilde L \tilde f = -\widetilde{Lf},$$

so that

$$\frac{d}{dt}\bigg|_{t=0} f * h_t^\flat = -Lf. \tag{13}$$

This shows that the infinitesimal generator of the semigroup $\{T_t\}$ contains L, hence it coincides with its closure. □

We must recall at this point some results from [14].

Given $\mathfrak{g}$, $\mathcal{X}$ and a as above, one can define two graded Lie algebra structures, $\mathfrak{g}_0 = \mathfrak{g}_0(\mathcal{X}, a)$ and $\mathfrak{g}_\infty = \mathfrak{g}_\infty(\mathcal{X}, a)$, over the same underlying vector space as $\mathfrak{g}$, in terms of which one can describe the behaviour of fundamental solutions of operators $L = P(X_1, \dots, X_n)$, when P is a-homogeneous and the corresponding $\tilde L$ is hypoelliptic on $\tilde G$.

As graded algebras, $\mathfrak{g}_0$ and $\mathfrak{g}_\infty$ possess natural dilations, those mapping X to $r^j X$ if X is in the jth step of the gradation. Denoting by d_0 and d_∞ the homogeneous dimensions of $\mathfrak{g}_0$ and $\mathfrak{g}_\infty$, respectively, one always has $d_0 \leqslant d_\infty$.

Choose homogeneous norms $\rho_0(x)$ and $\rho_\infty(x)$ on G_0 and G_∞, respectively (which we shall look at as functions on G), and choose a homogeneous norm $\tilde\rho(\tilde x)$ on $\tilde G$. This induces a (non-homogeneous) quotient norm

$$\rho(x) = \inf\{\tilde\rho(\tilde x):\ \pi(\tilde x) = x\} \tag{14}$$

on G. It can be shown that

$$\rho(x) \approx \begin{cases} \rho_0(x) & \text{for } x \text{ near } 0, \\ \rho_\infty(x) & \text{for } x \text{ away from } 0. \end{cases} \tag{15}$$

The part of Theorem 1 in [14] that we need is the following.

Lemma 3.3. *Suppose that P is a-homogeneous of degree $m < d_0$ and that $\tilde L$ is hypoelliptic on $\tilde G$. If $K(\tilde x)$ is the homogeneous fundamental solution of $\tilde L$, then*

$$K^\flat(x) = \int_H K(\tilde x y)\, dy \tag{16}$$

is the unique fundamental solution of L vanishing at infinity. Moreover,

$$\left|K^\flat(x)\right| \lesssim \begin{cases} \rho(x)^{-d_0+m} \approx \rho_0(x)^{-d_0+m} & \text{for } x \text{ near } 0, \\ \rho(x)^{-d_\infty+m} \approx \rho_\infty(x)^{-d_\infty+m} & \text{for } x \text{ away from } 0. \end{cases} \tag{17}$$

Proposition 3.1. *The heat kernel $h_t^\flat$ satisfies the following bounds*:

$$h_t^\flat(0) \lesssim \begin{cases} t^{-\frac{d_0}{m}} & \text{for } t \text{ small}, \\ t^{-\frac{d_\infty}{m}} & \text{for } t \text{ large}. \end{cases}$$

Proof. We apply Lemma 3.3 to the heat operators $L' = \partial_t + L$ and $\tilde{L}' = \partial_t + \tilde{L}$ on the product groups $G' = \mathbb{R} \times G$ and $\tilde{G}' = \mathbb{R} \times \tilde{G}$, respectively. If x_0 is an indeterminate standing for $X_0 = \tilde{X}_0 = \partial_t$, the corresponding non commutative polynomial is $P'(x_0, \ldots, x_n) = x_0 + P(x_1, \ldots, x_n)$. This is a'-homogeneous of degree m for $a' = (m, a_1, \ldots, a_n)$.

As shown in [9], $\tilde{L}'$ is hypoelliptic, and its homogeneous fundamental solution is

$$K(t, \tilde{x}) = \begin{cases} h_t(\tilde{x}) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

The construction in [14] assigning to the Lie algebra $\mathfrak{g}' = \mathbb{R} + \mathfrak{g}$ of G' the two graded subalgebras $\mathfrak{g}'_0$ and g'_∞ is such that $\mathfrak{g}'_0 = \mathbb{R} + \mathfrak{g}_0$ and $\mathfrak{g}'_\infty = \mathbb{R} + \mathfrak{g}_\infty$, with X_0 in the mth step of the gradation. In particular, the two homogeneous dimensions are $d'_0 = d_0 + m$ and $d'_\infty = d_\infty + m$, and if ρ'_0, ρ'_∞ are the corresponding homogeneous norms,

$$\rho_0(t, 0) \approx \rho_\infty(t, 0) \approx |t|^{\frac{1}{m}}. \tag{18}$$

Since $m < d'_0$, we can apply (17) to obtain that

$$K^\flat(t, x) = \int_H K(t, \tilde{x}y)\, dy = \begin{cases} h_t^\flat(x) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0, \end{cases}$$

is a fundamental solution of L', and the conclusion follows from (17) and (18). $\square$

Lemma 3.1 and (14) show that (6) and (7) are satisfied with $k = m$.

This application of Theorem 2.1 can be used to obtain estimates for the Fourier transform in $\mathbb{R}^n$ involving non-homogeneous "norms" on both sides.

Corollary 3.1. *Let $v_1, \ldots, v_N$, $N \geqslant n$, a generating system of vectors in $\mathbb{R}^n$, and let $P(t_1, \ldots, t_N)$ be a polynomial in N variables, a-homogeneous of degree m with respect some N-tuple of exponents $a = (a_1, \ldots, a_N)$, and strictly positive away from the origin.*

Set $\eta(x) = \min\{P(t)\colon \sum_{j=1}^N t_j v_j = x\}$ and $Q(\xi) = P(\xi \cdot v_1, \ldots, \xi \cdot v_N)$ for $x, \xi \in \mathbb{R}^n$. Then, for $f \in L^2(\mathbb{R}^n)$,

$$\|f\|_2 \leqslant C_{\alpha,\beta} \left\|\eta^\alpha f\right\|_2^{\frac{\beta}{\alpha+\beta}} \left\|Q^\beta \hat{f}\right\|_2^{\frac{\alpha}{\alpha+\beta}}.$$

Proof. In this case we take $G = \mathbb{R}^n$, $\tilde{G} = \mathbb{R}^N$, and $\pi : \mathbb{R}^N \to \mathbb{R}^n$ given by $\pi(t) = \sum_{j=1}^N t_j v_j$. If we choose $P(t)^{1/m}$ as the homogeneous norm on $\mathbb{R}^N$, the induced norm on $\mathbb{R}^n$ is $\eta^{1/m}$. At the same time, the operators $L = Q(-i\partial_x)$ on $\mathbb{R}^n$ and $\tilde{L} = P(-i\partial_t)$ on $\mathbb{R}^N$ are hypoelliptic and $\pi(\tilde{L}) = L$. The conclusion follows from Theorem 2.1 and the Plancherel formula. $\square$

4. On local uncertainty estimates

Our method of derivation of the Heisenberg–Pauli–Weyl inequality from polynomial estimates for heat semigroups and volumes of balls is modeled on the rather classical proof based on the local uncertainty estimate (3) on $\mathbb{R}^n$.

In this section we intend to comment that a general version of (3) can be formulated, and that its use is disguised in our previous arguments.

First of all, we rewrite (3) as

$$\|f * w_E\|_2 \leqslant C_\alpha \|w_E\|_2^{\frac{2\alpha}{n}} \left\||x|^\alpha f\right\|_2,$$

where $w_E = \mathcal{F}^{-1}\chi_E \in L^2(\mathbb{R}^n)$. Convolution by w_E is an orthogonal projection on $L^2(X,m)$ and the L^2-norm of w_E can also be interpreted as the $(L^2 \to L^\infty)$-norm of the projection.

The following can then be seen as a natural extension of (3).

Proposition 4.1. *Suppose that X, m, ρ satisfy* (6) *and let P be an orthogonal projection on $L^2(X,m)$ mapping L^2 into L^∞ continuously. If $\alpha < \min\{\frac{d_0}{2}, \frac{d_\infty}{2}\}$ and ρ_0 is as in Theorem* 2.1, *then, for every $f \in L^2(X,m)$,*

$$\|Pf\|_2 \leqslant \begin{cases} C_\alpha \|P\|_{2\to\infty}^{\frac{2\alpha}{d_\infty}} \|\rho_0^\alpha f\|_2 & \text{if } \|P\|_{2\to\infty} < 1, \\ C_\alpha \|P\|_{2\to\infty}^{\frac{2\alpha}{d_0}} \|\rho_0^\alpha f\|_2 & \text{if } \|P\|_{2\to\infty} \geqslant 1. \end{cases} \tag{19}$$

Proof. Decompose f as $f_s + f^s$, where $f_s = f\chi_{\{\rho_0(x)<s\}}$, and use the estimates

$$\|Pf_s\|_2 \leqslant \|P\|_{2\to\infty}\|f_s\|_1, \qquad \left\|Pf^s\right\|_2 \leqslant \|f^s\|_2.$$

As in the proof of Theorem 2.1,

$$\|f_s\|_1 \leqslant \left(\int\limits_{\rho_0(x)<s} \rho_0(x)^{-2\alpha}\,dx\right)^{1/2} \left\|\rho_0^\alpha f\right\|_2,$$

and the proof continues like the proof of (9). □

Taking $L = \int_0^\infty \lambda\,dP(\lambda)$ satisfying (7), then (19) for the spectral projections $P_{[0,\lambda]}$ turns out to be equivalent to (9) [18].

The analogy between (3) and (19) for left-invariant projections on Lie groups can be seen as follows.

Suppose that G is unimodular and type I, as in the cases considered in other parts of this article. The condition that P maps L^2 into L^∞ continuously is equivalent to the fact that $Pf = f * w$ with $w \in L^2$, $w = w^* = w * w$, and $\|w\|_2 = \|P\|_{2\to\infty}$.

In terms of the group Fourier transform of w, this implies that, for a.e. $\pi \in \hat{G}$, $\pi(w)$ is a finite-dimensional orthogonal projection. If $\pi(w) \neq 0$, let $\{e_1^\pi, \dots, e_{n_\pi}^\pi\}$ be an orthonormal basis of the range of $\pi(w)$. Then the factor $\|w\|_2^2$ in (19) equals $\sigma(E)$, where $E = \{(\pi, j)\colon \pi(w) \neq 0, 1 \leqslant j \leqslant n_\pi\} \subset \hat{G} \times \mathbb{N}$ and σ is the product of the Plancherel measure on $\hat{G}$ and the counting measure on $\mathbb{N}$. In particular, Proposition 4.1 contains the local uncertainty inequality in [20].

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References

[1] M. Barlow, E. Perkins, Brownian motion on the Sierpiński gasket, Probab. Theory Related Fields 79 (1988) 543–623.
[2] P. Ciatti, F. Ricci, M. Sundari, A local uncertainty inequality for step-two nilpotent Lie groups, in: Proc. Internat. Conf. in Harmonic Analysis and Quantum Groups, Bull. Kerala Math. Ass. 3 (2006) 53–72.
[3] R. Coifman, G. Weiss, Transference Methods in Analysis, CBMS Reg. Conf. Ser. Math., vol. 31, Amer. Math. Soc., Providence, RI, 1977.
[4] T. Coulhon, Heat kernel and isoperimetry on non-compact Riemannian manifolds, in: Contemp. Math., vol. 338, Amer. Math. Soc., Providence, RI, 2003, pp. 65–99.
[5] W. Faris, Inequalities and uncertainty principles, J. Math. Phys. 19 (1978) 461–466.
[6] C. Fefferman, The uncertainty principle, Bull. Amer. Math. Soc. 9 (1983) 129–206.
[7] C. Fefferman, D.H. Phong, The uncertainty principle and sharp Gårding inequality, Comm. Pure Appl. Math. 34 (1981) 285–331.
[8] G. Folland, A. Sitaram, The uncertainty principle: A mathematical survey, J. Fourier Anal. Appl. 3 (1997) 207–238.
[9] G. Folland, E.M. Stein, Hardy Spaces on Homogeneous Groups, Math. Notes, vol. 28, Princeton Univ. Press, Princeton, NJ, 1982.
[10] N. Garofalo, E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, Ann. Inst. Fourier (Grenoble) 40 (1990) 313–356.
[11] A. Grigor'yan, Estimates of heat kernels on Riemannian manifolds, in: Spectral Theory and Geometry, ICMS Instructional Conference, in: London Math. Soc. Lecture Note Ser., vol. 273, Cambridge Univ. Press, Cambridge, 1999, pp. 140–225.
[12] Y. Guivarc'h, Croissance polynomiale et périodes des fonctions harmoniques, Bull. Soc. Math. France 101 (1973) 333–379.
[13] B. Helffer, J. Nourrigat, Caractérisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué, Comm. Partial Differential Equations 4 (1979) 899–958.
[14] A. Nagel, F. Ricci, E.M. Stein, Harmonic analysis and fundamental solutions on nilpotent Lie groups, in: C. Sadosky (Ed.), Analysis and Partial Differential Equations, in: Lecture Notes in Pure and Appl. Math., vol. 122, Dekker, New York, 1990.
[15] A. Nahmod, Generalized uncertainty principles on spaces of homogeneous type, J. Funct. Anal. 119 (1994) 171–209.
[16] E. Nelson, W. Stinespring, Representation of elliptic operators in an enveloping algebra, Amer. J. Math. 81 (1959) 547–560.
[17] K. Okoudjou, L. Saloff-Coste, A. Templyaev, Weak uncertainty principle for fractals, graphs and metric measure spaces, preprint.
[18] F. Ricci, Uncertainty inequalities on spaces with polynomial volume growth, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) 28 (2006).
[19] A. Sitaram, M. Sundari, S. Thangavelu, Uncertainty principles on certain Lie groups, Proc. Indian Acad. Sci. Math. Sci. 105 (1995) 135–151.
[20] S. Thangavelu, Some uncertainty inequalities, Proc. Indian Acad. Sci. Math. Sci. 100 (1990) 137–145.
[21] N. Varopoulos, L. Saloff-Coste, T. Coulhon, Analysis and Geometry on Groups, Cambridge Univ. Press, Cambridge, 1992.