SOBOLEV SUBSPACES OF NOWHERE BOUNDED FUNCTIONS

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ABSTRACT. We prove that in any Sobolev space which is subcritical with respect to the Sobolev Embedding Theorem there exists a closed infinite dimensional linear subspace whose non zero elements are nowhere bounded functions. We also prove the existence of a closed infinite dimensional linear subspace whose non zero elements are nowhere L^q functions for suitable values of q larger than the Sobolev exponent.

1. INTRODUCTION

Given $l \in \mathbb{N}$, $p \in [1, \infty]$ and an open set Ω in $\mathbb{R}^N$, the Sobolev space $W^{l,p}(\Omega)$ is defined as the space of those real valued functions in $L^p(\Omega)$ with distributional derivatives in $L^p(\Omega)$, up to order l endowed with the norm defined by $\|v\|_{W^{l,p}(\Omega)} = \sum_{0 \le |\alpha| \le l} \|D^\alpha v\|_{L^p(\Omega)}$ for all $v \in W^{l,p}(\Omega)$.

Sobolev spaces play a prominent role in modern Mathematical Analysis and applications to partial differential equations. In particular, they provide a natural setting for the study of fundamental problems from Mathematical Physics, in which case the so-called energy spaces are often identified with suitable closed subspaces of $W^{l,2}(\Omega)$ containing $C_c^\infty(\Omega)$, see, e.g., [11, 15].

One of the main features of Sobolev spaces is their completeness. In fact, for $p < \infty$, they can be defined as the completion of the space of smooth functions with respect to the norm above, which clearly allows Sobolev spaces to posses badly behaved functions.

Nevertheless, the celebrated Sobolev Embedding Theorem gives sharp information on the intrinsic regularity of the functions in Sobolev spaces, see [5, Chapter 4] for instance. In particular, such theorem states that if Ω is a sufficiently regular open set (say, Ω satisfies the cone condition) and $pl > N$ for $p \ne 1$, or $pl \ge N$ for $p = 1$, then $W^{l,p}(\Omega)$ is continuously embedded into $C_b(\Omega)$, where $C_b(\Omega)$ denotes the space of real valued, bounded, continuous functions on Ω endowed with the usual sup-norm.

It is well-known that, in the subcritical case $pl < N$, the Sobolev space $W^{l,p}(\Omega)$ contains unbounded functions, as well as functions which are nowhere bounded in Ω. If Ω is bounded, an example of a nowhere bounded function v in $W^{l,p}(\Omega)$ can be easily provided by considering a numerable dense subset $\{x_n\}_{n \in \mathbb{N}}$ of Ω and setting $v(x) = \sum_{n=1}^\infty |x -$

$x_n|^\mu/2^n$ for almost all $x \in \Omega$, where $\mu \in]l - N/p, 0[$ (see also, e.g., [9, Example 4, p. 247]; see Section 2 below for the general case).

The aim of the present paper is to study such unbounded functions in the frame of a comparatively new field of investigation devoted to the analysis of spaces of pathological functions. In his seminal paper [10], Vladimir I. Gurariy proved the existence of a closed infinite dimensional linear subspace of $C([0,1])$ whose non zero elements are nowhere differentiable functions. Following [10], a number of authors have addressed analogous problems concerning that or other counterintuitive properties of functions, see e.g., [3,8] . We refer to the recent monograph [1] for an extensive discussion of old and new results in this topic as well as for references. We also refer to [7] for some historical and pedagogical remarks in the realm of functions with strange properties.

In this paper, we prove that every Sobolev space $W^{l,p}(\Omega)$ with $pl \leq N$ if $p \neq 1$, and $pl < N$ if $p = 1$, contains a closed infinite dimensional linear subspace whose non zero elements are nowhere bounded functions. Actually, if $pl < N$ with $p \geq 1$, we shall prove even more. Indeed recall that, in the case $pl < N$, the Sobolev Embedding Theorem provides some additional integrability properties for the functions in $W^{l,p}(\Omega)$. More precisely, such theorem states that if Ω is a sufficiently regular open set as above and $pl < N$, then $W^{l,p}(\Omega)$ is continuously embedded into $L^{q^*}(\Omega)$, where $q^* = Np/(N - pl)$ is the celebrated Sobolev exponent. It is an exercise to prove that the Sobolev critical exponent q^* cannot be improved, that is, if $pl < N$ and $W^{l,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for some $q \in [1, \infty[$, then $q \leq q^*$. We plan to prove that this is true in a stronger way.

Following [12], we say that a real valued function v defined on Ω is nowhere L^q for some $q \in [1, \infty]$ if, for any non empty open subset U of Ω, $v|_U$ is not in $L^q(U)$ (the case $q = \infty$ is exactly the case of nowhere bounded functions mentioned above).

With this terminology, our main result is the following

Theorem 1. *Let Ω be a non empty open set in $\mathbb{R}^N$ and let $l \in \mathbb{N}$, $p \in [1, \infty[$. If $pl < N$ then, for every $r \in]lq^*, \infty]$ fixed, the space $W^{l,p}(\Omega)$ contains a closed infinite dimensional linear subspace whose non zero elements are nowhere L^r functions. If $pl = N$ with $p \neq 1$, then the space $W^{l,p}(\Omega)$ contains a closed infinite dimensional linear subspace whose non zero elements are nowhere bounded functions.*

Theorem 1 is proved in Section 2 where the required closed infinite dimensional linear subspace of $W^{l,p}(\Omega)$ is defined as the image of a closed infinite dimensional linear subspace of $W^{l,p}(\Omega) \cap W^{1,pl}(\Omega)$ via a suitable compact perturbation of the identity. Such subspace turns out to be a closed subspace also of $W^{l,p}(\Omega) \cap W^{1,pl}(\Omega)$, see Remark 1. By the Sobolev Embedding Theorem the space $W^{1,pl}(\Omega)$ is embedded into $L_{loc}^{lq^*}(\Omega)$ and this explains why, in Theorem 1, we require that r belongs to $]lq^*, \infty]$, which is smaller than the interval $]q^*, \infty]$ when $l \geq 2$. We also note that our compact operator is defined by means of a suitable composition operator and that the condition $r \in]lq^*, \infty]$ is needed to make it well-defined when $l \geq 2$.

2. Proof of Theorem 1

In this section, we always assume that N, l, p and r are fixed and are as in Theorem 1. Note that $N \geq 2$. Moreover, we need to fix a number a in $]0, 1[$ as follows. If $pl < N$ and

$r < \infty$, we take $a \in]0, 1[$ such that $ar > lq^*$; if $pl = N$ or $r = \infty$, a is any number in the interval $]0, 1[$.

We begin with some preliminaries. Let $f \in W^{l,p}(\mathbb{R}^N)$ be a function with compact support, continuous in $\mathbb{R}^N \setminus \{0\}$, that does not change sign and such that $|f(x)| \to \infty$ as $x \to 0$. For reasons that will be clear later, in the case $l \geq 2$ we also require in addition that $f \in W^{1,pl}(\mathbb{R}^N)$. In the case $pl < N$ and $r < \infty$, we also require the extra condition $f \notin L^q(\mathbb{R}^N)$ for any $q \in [ar, \infty]$. The existence of such functions is well-known, see [5, Example 8, p. 32].

Let $\{x_n\}_{n \in \mathbb{N}}$ be a numerable dense set in $\mathbb{R}^N$. Let u be the real valued function defined in $\mathbb{R}^N$ by

$$(2.1) \qquad u(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - x_n),$$

for all $x \in \mathbb{R}^N$. It is an exercise to prove that the series in (2.1) is convergent in $W^{l,p}(\mathbb{R}^N)$, as well as in $W^{1,pl}(\mathbb{R}^N)$ if $l \geq 2$. Moreover, such series also converges almost everywhere in $\mathbb{R}^N$. Since f does not change sign, by (2.1) it follows that

$$(2.2) \qquad |u(x)| \geq \frac{1}{2^n} |f(x - x_n)|,$$

for all $n \in \mathbb{N}$ almost everywhere on $\mathbb{R}$. Thus u is nowhere bounded in $\mathbb{R}^N$ and belongs to $W^{l,p}(\mathbb{R}^N)$, and also to $W^{1,pl}(\mathbb{R}^N)$ if $l \geq 2$. Moreover, in the case $pl < N$, u is also nowhere L^{ar} in $\mathbb{R}^N$.

Let $(a_n)_{n \in \mathbb{N}} \subset]a, 1[$ be a strictly decreasing sequence. Let us now define a sequence of real valued functions $\{q_n\}_{n \in \mathbb{N}}$ on $\mathbb{R}$ by setting $q_n(t) = |t|^{a_n}$ for all $t \in \mathbb{R}$, $n \in \mathbb{N}$. Note that, for all $n, m \in \mathbb{N}$,

$$(2.3) \qquad \left| \frac{d^m}{dt^m} q_n(t) \right| \leq m! \, |t|^{a_n - m}, \quad \forall t \neq 0.$$

Let $\psi \in C^\infty(\mathbb{R})$ be fixed in such a way that $\psi(t) = 0$ for all $|t| \leq 1$ and $\psi(t) = 1$ for all $|t| \geq 2$. We set $Q_n = \psi q_n$ for all $n \in \mathbb{N}$. Clearly $Q_n \in C^\infty(\mathbb{R})$ and by the Leibniz rule it easily follows that, for every $m \in \mathbb{N}$, there exists $k_m > 0$ independent of n such that

$$\left| \frac{d^m}{dt^m} Q_n(t) \right| \leq k_m,$$

for all $t \in \mathbb{R}$, $n \in \mathbb{N}$.

Thus, for any sequence $c = (c_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{N})$, the function defined by

$$g_c(t) = \sum_{n=1}^{\infty} c_n Q_n(t), \quad t \in \mathbb{R},$$

belongs to $C^\infty(\mathbb{R})$ and for every $m \in \mathbb{N}$ we have

$$\left| \frac{d^m}{dt^m} g_c(t) \right| = \left| \sum_{n=1}^{\infty} c_n \frac{d^m}{dt^m} Q_n(t) \right| \leq k_m \|c\|_{\ell^1(\mathbb{N})},$$

for all $t \in \mathbb{R}$. Moreover, for any $c \in \ell^1(\mathbb{N}) \setminus \{0\}$, we have that

$$(2.4) \qquad |g_c(t)| = \left| \sum_{n \geq \bar{n}} c_n |t|^{a_n} \right| = |t|^{a_{\bar{n}}} \left| \sum_{n \geq \bar{n}} c_n |t|^{a_n - a_{\bar{n}}} \right| \geq \frac{|c_{\bar{n}}|}{2} |t|^a,$$

for any $|t|$ sufficiently big, where $\bar{n} = \min\{n \in \mathbb{N} : c_n \neq 0\}$.

We can now prove the following result, where $W^{l,p}(\mathbb{R}^N) \cap W^{1,pl}(\mathbb{R}^N)$ is endowed with the usual norm obtained by summing the norms of $W^{l,p}(\mathbb{R}^N)$ and $W^{1,pl}(\mathbb{R}^N)$. We set $bc = (b_n c_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{N})$ for all $b, c \in \ell^2(\mathbb{N})$ and we note that $\|bc\|_{\ell^1(\mathbb{N})} \leq \|b\|_{\ell^2(\mathbb{N})} \|c\|_{\ell^2(\mathbb{N})}$.

Lemma 2. *Let u be as in (2.1) and let $b = (b_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ be fixed with $b_n \neq 0$ for all $n \in \mathbb{N}$. The linear operator $\mathcal{T}_u$ from $\ell^2(\mathbb{N})$ to $W^{l,p}(\mathbb{R}^N) \cap W^{1,pl}(\mathbb{R}^N)$ defined by*

$$\mathcal{T}_u(c) = g_{bc} \circ u = \left(\sum_{n=1}^{\infty} b_n c_n Q_n \right) \circ u,$$

for all $c = (c_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$, is continuous, injective and compact. Moreover, $\mathcal{T}_u(c)$ is a nowhere bounded function in $\mathbb{R}^N$ for all $c \in \ell^2(\mathbb{N}) \setminus \{0\}$ and, in the case $pl < N$, $\mathcal{T}_u(c)$ is a nowhere L^r function in $\mathbb{R}^N$ for all $c \in \ell^2(\mathbb{N}) \setminus \{0\}$.

Proof. Since $g_{bc} \in C^\infty(\mathbb{R})$ has bounded derivatives, $g_{bc}(0) = 0$, $u \in W^{l,p}(\mathbb{R}^N)$ if $l \geq 1$ and $u \in W^{l,p}(\mathbb{R}^N) \cap W^{1,pl}(\mathbb{R}^N)$ if $l \geq 2$, the function $\mathcal{T}_u(c)$ belongs to $W^{l,p}(\mathbb{R}^N) \cap W^{1,pl}(\mathbb{R}^N)$. Indeed, the case $l = 1$ is a direct application of the chain rule, which also allows to easily prove that

$$(2.5) \qquad \|\mathcal{T}_u(c)\|_{W^{1,p}(\mathbb{R}^N)} \leq k_1 \|b\|_{\ell^2(\mathbb{N})} \|c\|_{\ell^2(\mathbb{N})} \|u\|_{W^{1,p}(\mathbb{R}^N)}.$$

In the case $l \geq 2$, one needs to use the condition $u \in W^{l,p}(\mathbb{R}^N) \cap W^{1,pl}(\mathbb{R}^N)$ in a substantial way, in order to avoid the possible appearance of the so called *Dahlberg degeneracy phenomenon*, which prevents non trivial composition operators to preserve Sobolev spaces $W^{l,p}$ when $1 + 1/p < l < N/p$ (including those with fractional order of smoothness). For further details and discussions, we refer to [16, §5.2.5] and [6]. In particular, by estimate (2) in [16, §5.2.5], we can also immediately deduce that there exists a constant $K > 0$, independent of u, b, and c, such that

$$(2.6) \qquad \|\mathcal{T}_u(c)\|_{W^{l,p}(\mathbb{R}^N)} \leq K \max_{i=1,\dots,l} k_i \, \|b\|_{\ell^2(\mathbb{N})} \|c\|_{\ell^2(\mathbb{N})} \left(\|u\|_{W^{l,p}(\mathbb{R}^N)} + \|u\|_{W^{1,pl}(\mathbb{R}^N)}^l \right).$$

When $l \geq 2$, similarly to (2.5), we also have

$$(2.7) \qquad \|\mathcal{T}_u(c)\|_{W^{1,pl}(\mathbb{R}^N)} \leq k_1 \|b\|_{\ell^2(\mathbb{N})} \|c\|_{\ell^2(\mathbb{N})} \|u\|_{W^{1,pl}(\mathbb{R}^N)}.$$

By (2.5), (2.6) and (2.7), we can conclude that the operator $\mathcal{T}_u$ is well defined and continuous.

By (2.2) combined with (2.4), we have that for any $c \in \ell^2(\mathbb{N}) \setminus \{0\}$ and $n \in \mathbb{N}$ there exists an open neighbourhood U_n of x_n and $\alpha_n > 0$ such that

$$(2.8) \qquad |\mathcal{T}_u(c)| \geq \alpha_n |u|^a \text{ on } U_n.$$

By (2.8), it follows that $\mathcal{T}_u(c)$ is nowhere bounded and, if $pl < N$, $\mathcal{T}_u(c)$ is also nowhere L^r for all $c \in \ell^2(\mathbb{N}) \setminus \{0\}$. In particular, the operator $\mathcal{T}_u$ is injective.

It remains to prove that $\mathcal{T}_u$ is compact. For any $k \in \mathbb{N}$, consider the linear continuous operator $\mathcal{T}_u^{(k)}$ from $\ell^2(\mathbb{N})$ to $W^{l,p}(\mathbb{R}^N) \cap W^{1,pl}(\mathbb{R}^N)$ defined by

$$\mathcal{T}_u^{(k)}(c) = \left(\sum_{n=1}^{k} b_n c_n Q_n \right) \circ u,$$

for all $c \in \ell^2(\mathbb{N})$. Note that $\mathcal{T}_u^{(k)}$ is a finite rank operator. Then, by estimates analogous to (2.5), (2.6) and (2.7) and by the Cauchy–Schwarz inequality, we get

$$\|\mathcal{T}_u(c) - \mathcal{T}_u^{(k)}(c)\|_{W^{l,p}(\mathbb{R}^N) \cap W^{1,pl}(\mathbb{R}^N)} \leq B \left(\sum_{n=k+1}^{+\infty} b_n^2 \right)^{\frac{1}{2}} \|c\|_{\ell^2(\mathbb{N})},$$

where $B > 0$ is a constant independent of c. Hence

$$\lim_{k \to +\infty} \|\mathcal{T}_u - \mathcal{T}_u^{(k)}\|_{\ell^2(\mathbb{N}) \to W^{l,p}(\mathbb{R}^N) \cap W^{1,pl}(\mathbb{R}^N)} = 0,$$

where in the left-hand side we are using the standard operator norm of $\mathcal{T}_u - \mathcal{T}_u^{(k)}$. Thus $\mathcal{T}_u$ is approximated in norm by finite rank operators, hence it is compact. This concludes the proof of Lemma 2. $\qquad\square$

We can now prove our main result.

Proof of Theorem 1. Let $\bar{x} \in \Omega$ and let $R > 0$ be such that $B(\bar{x}, R) \subset \Omega$. Let A be the open annulus defined by $A = \{x \in \mathbb{R}^N : R/2 < |x - \bar{x}| < R\}$. We denote by $W_0^{l,p}(A)$ the standard Sobolev space defined as the closure of $C_c^\infty(A)$ in $W^{l,p}(A)$. We set

(2.9) $\qquad\qquad X = \{v \in W_0^{l,p}(A) : \ v \text{ is radial with respect to } \bar{x}\},$

where it is meant that a function v is radial with respect to $\bar{x}$ if the value of $v(x)$ depends only on $|x - \bar{x}|$ for all $x \in A$. The space X can be naturally seen as a subspace of $W^{l,p}(\Omega)$ by extending functions by zero outside A. Moreover, it is straightforward that X is a closed subspace of $W^{l,p}(\Omega)$ with infinite dimension.

Since any function v in X is of the form $v(x) = g_v(|x - \bar{x}|)$ for a suitable function g_v and is zero outside A, by the Radial Lemma (see [14, Lemma II.1]) it is easy to see that X is continuously embedded into the space $C_b(R/2, R)$ via the embedding $v \mapsto g_v$. In particular, any function in X is bounded and continuous. Since $C_b(R/2, R)$ is continuously embedded into $L^2(R/2, R)$, which is a Hilbert space isometric to $\ell^2(\mathbb{N})$, we conclude that there exists a continuous embedding $\mathcal{J}$ of X into $\ell^2(\mathbb{N})$.

Let $\mathcal{R}$ be the restriction operator from $W^{l,p}(\mathbb{R}^N)$ to $W^{l,p}(\Omega)$ and let $\mathcal{T}_u$ be the operator defined in Lemma 2 considered as an operator from $\ell^2(\mathbb{N})$ to $W^{l,p}(\mathbb{R}^N)$. It is obvious that the operator $\mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}$ is injective because all the non zero functions in the image of the operator $\mathcal{T}_u \circ \mathcal{J}$ are nowhere bounded. Thus $\mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}$ is a compact embedding of X into $W^{l,p}(\Omega)$. We now consider the operator $\mathcal{I} - \mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}$ from X to $W^{l,p}(\Omega)$, where $\mathcal{I}$ denotes the identity operator of $W^{l,p}(\Omega)$.

We note that $\ker(\mathcal{I} - \mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}) = \{0\}$. Indeed, if $v \in X$ is such that $v = \mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}(v)$, then $\mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}(v)$ is a bounded continuous function; but this implies that $\mathcal{J}(v) = 0$, hence $v = 0$ (in fact, otherwise, if $v \neq 0$ then $\mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}(v)$ would be nowhere bounded).

Let Y be the subspace of $W^{l,p}(\Omega)$ defined by

(2.10) $\qquad\qquad Y = (\mathcal{I} - \mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J})(X).$

Since $\mathcal{I} - \mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}$ is a compact perturbation of the identity, by the Fredholm Alternative Theorem (see, e.g., [4, Exercise 6.9, (4)]) it follows that Y is a closed subspace of $W^{l,p}(\Omega)$. In particular, since $\mathcal{I} - \mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}$ is injective, $\dim Y = \dim X = \infty$.

Finally, we observe that any function in $Y \setminus \{0\}$ is nowhere bounded. Indeed, as observed above, any function $v \in X$ is bounded, while $\mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}(v)$ is nowhere bounded if $v \neq 0$. Hence $(\mathcal{I} - \mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J})(v)$ is nowhere bounded for all $v \in X \setminus \{0\}$. Moreover, in the case $pl < N$, $(\mathcal{I} - \mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J})(v)$ is nowhere L^r for all $v \in X \setminus \{0\}$, because $\mathcal{R} \circ \mathcal{T}_u \circ \mathcal{J}(v)$ is nowhere L^r for all $v \in X$. This concludes the proof. $\qquad\square$

Remark 1. Assume that $l \geq 2$. In the proof of Theorem 1 one can replace the space $W^{l,p}$ by $W^{l,p} \cap W^{1,pl}$. Indeed, a simple argument allows to see that the space X in (2.9) is a closed subspace also of $W^{l,p}(\Omega) \cap W^{1,pl}(\Omega)$. Thus, by Lemma 2, one can consider the operator $\mathcal{T}$ in the proof of Theorem 1 as an operator from X (considered as a closed subspace of $W^{l,p}(\Omega) \cap W^{1,pl}(\Omega)$) to $W^{l,p}(\Omega) \cap W^{1,pl}(\Omega)$, and conclude that the space Y in (2.10) is also a closed subspace also of $W^{l,p}(\Omega) \cap W^{1,pl}(\Omega)$.

Remark 2. We briefly observe that the subcritical Sobolev space $W^{l,p}(\Omega)$ contains also a $\mathfrak{c}$-dimensional linear subspace and a countably generated algebra whose non zero elements are nowhere bounded functions.

The existence of such a $\mathfrak{c}$-dimensional linear subspace follows immediately from the fact that any infinite dimensional Banach space has Hamel dimension at least $\mathfrak{c}$, see [13]. Thus, in Theorem 1, it is possible to replace 'closed infinite dimensional linear subspace' with '$\mathfrak{c}$-dimensional linear subspace'. For a simpler construction, consider the vector space generated by the functions $h_\mu \circ u$, where h_μ is a real valued function defined on $\mathbb{R}$ such that $h_\mu \in C^l(\mathbb{R})$, $h_\mu(0) = 0$ and $h_\mu(t) = |t|^\mu$ for $|t| \geq 1$ for any $\mu \in [a, 1]$, where u is the function defined in (2.1) and $a \in\,]0, 1[$ is chosen as at the beginning of Section 2.

To prove the existence of a countably generated algebra of nowhere bounded functions, it is enough to consider the algebra generated by the functions $l_n \circ u$, where l_n is a real valued function defined on $\mathbb{R}$ such that $l_n \in C^l(\mathbb{R})$, $l_n(0) = 0$ and $l_n(t) = \log(1 + l_{n-1}(t))$, $l_0(t) = t$, for $|t| \geq 1$ and $n \in \mathbb{N}$. Therefore, in the case $pl \leq N$ for $p \neq 1$, or $pl < N$ for $p = 1$, $W^{l,p}(\Omega)$ contains a countably generated algebra whose non zero elements are nowhere bounded functions.

References

[1] R. M. Aron, L. Bernal-González, D. M. Pellegrino, and J. B. Seoane-Sepúlveda, *Lineability: The Search for Linearity in Mathematics*, Monographs and Research Notes in Mathematics, vol. 14, CRC Press, 2015.

[2] R. M. Aron, J. A. Conejero, A. Peris, and J. B. Seoane-Sepúlveda, *Uncountably Generated Algebras of Everywhere Surjective Functions*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), no. 3, 571-575.

[3] R. M. Aron, V. I. Gurariy, and J. B. Seoane-Sepúlveda, *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc. **133** (2004), no. 3, 795-803.

[4] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011.

[5] V. I. Burenkov, *Sobolev spaces on domains*, Vol. 137, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1998.

[6] B. E. J Dahlberg, *A note on Sobolev spaces. Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978)*, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, R.I., Part 1 (1979), 183-185.

[7] G. Drago, P. D. Lamberti, and P. Toni, *A "bouquet" of discontinuous functions for beginners in mathematical analysis*, Americ. Math. Monthly **118** (2011), no. 9, 799-811.

[8] P. H. Enflo, V. I. Gurariy, and J. B. Seoane-Sepúlveda, *Some results and open questions on spaceability in function spaces*, Trans. Amer. Math. Soc. **366** (2014), no. 2, 611-625.

[9] L. C. Evans, *Partial Differential Equations*, first ed., Graduate Studies in Mathematics, vol. 19, Amer. Math. Soc., 1998.

[10] V. I. Gurariy, *Linear spaces composed of everywhere non differentiable functions (Russian)*, C. R. Acad. Bulgare Sci. **44** (1991), no. 5, 13-16.

[11] B. Helffer, *Spectral Theory and its Applications*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2013.

[12] P. L. Kaufmann and L. Pellegrini, *Spaceability of sets of nowhere L^q functions* (2011), available at `http://arxiv.org/abs/1110.5774`.

[13] E. H. Lacey, *The Hamel dimension of any infinite dimensional separable Banach space is* $\mathfrak{c}$, Amer. Math. Monthly **80** (1973), 298.

[14] P. L. Lions, *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal. **49** (1982), 315-334.

[15] J. Necas, *Direct Methods in the Theory of Elliptic Equations*, Springer Monographs in Mathematics, Springer, 2012.

[16] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, De Gruyter series in nonlinear analysis and applications, W. de Gruyter, 1996.

[17] J. B. Seoane-Sepúlveda, *Chaos and lineability of pathological phenomena in Analysis*, Ph.D. Thesis, Kent State University, 2006.

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