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**Quantum Mechanics and Stochastic Mechanics  
for compatible observables at different times**

**Abstract.** Bohm Mechanics and Nelson Stochastic Mechanics are confronted with Quantum Mechanics in presence of non-interacting subsystems. In both cases, it is shown that correlations at different times of compatible position observables on stationary states agree with Quantum Mechanics only in the case of product wave functions. By appropriate Bell-like inequalities it is shown that no classical theory, in particular no stochastic process, can reproduce the quantum mechanical correlations of position variables of non interacting systems at different times.

## 1. Introduction.

Bohm Mechanics [1] [2] and Nelson's Stochastic Mechanics [3] [4] [5] have been extensively studied as classical dynamical systems associating classical trajectories, respectively deterministic and stochastic, to the Schrödinger equation.

In both theories, particle positions evolve according to equations which depend on the Schrödinger wave function  $\psi(x_1, \dots, x_n, t)$  in such a way that the particle density  $\rho(x_1, \dots, x_n, t)$  coincides with  $|\psi(\mathbf{x}, t)|^2$  at all times if so does at some (initial) time.

Both theories have been advocated [6] as perfectly adequate alternative descriptions of Schrödinger Quantum Mechanics. In particular, it is believed that both theories reproduce all the predictions of non-relativistic Quantum Mechanics of spinless particles (QM) which can be expressed in terms of observations of particle positions [6].

As a matter of fact, coincidence of densities at all times amounts to coincidence of all (probabilistic) predictions for the observation of position variables at any given time. It is a fact that *for generic time evolutions*, there are no further quantum mechanical predictions for position observables, since position variables at different times do not commute in general, so that a common probabilistic interpretation is excluded. Moreover, for spinless Schrödinger particles, there are no additional observables which are compatible with all position variables at a given time, so that no additional prediction can be derived in Quantum Mechanics.

It is therefore concluded that *if only position observables are considered, Bohm Mechanics and Nelson Stochastic Mechanics give exactly the same predictions as Schrödinger Quantum Mechanics.*

However, even restricting the attention to position observables of spinless Schrödinger particles, the above conclusion is not as general as it may appear. In fact, position observables of particles belonging to non-interacting subsystems clearly commute also *at different times*, so that quantum mechanical predictions for their common values are perfectly defined and can be confronted with the corresponding predictions of Bohm and Nelson mechanics. Such a comparison is clearly relevant also in view of the difficulty of arguing against the measurement of positions of non-interacting particles at different times, possibly in very distant regions. On the contrary, it would be hard to assume that exact simultaneity of measurements can be achieved.

Difficulties in the physical interpretation of trajectories, especially in the presence of non interacting subsystems, have been in fact pointed out and discussed by Nelson [7] in terms of problems with "separability", i.e. independence of the stochastic

equations of a subsystem from changes in the interactions within different, non interacting, possibly very distant subsystems. Blanchard et al. showed in ref. [8] that discrepancies between QM and Nelson Mechanics disappear if the effects of measurements on Nelson equations are taken into account, through a mechanism analogous to a wave function “collapse”. As a result of their analysis, however, the dynamical equations of a subsystem depend on measurements performed on different, non interacting, possibly very distant subsystems, and Nelson’s criticism applies.

In the case of Bohm mechanics, to avoid disagreement with QM, Bell proposed to include the measurement apparatus in the system [9], and to consider only position measurements of pointers, *all at the same “final” time*. Clearly, on one side this results in a totally unrealistic constraint on measurements, on the other it forbids any interpretation of trajectories, when the main motivation for Bohm theory was to reformulate and possibly reinterpret Quantum Mechanics in terms of them. A discussion of examples of disagreement between Bohm Mechanics and QM has been given recently in refs. [10] [11].

The purpose of the present paper is to take seriously Bohm and Nelson Mechanics as classical probabilistic theories about trajectories and to clarify the extent of disagreement with the predictions of QM for compatible position observables of non interacting subsystems. We will show in complete generality that both Bohm Mechanics and Nelson Stochastic Mechanics agree with QM, for a stationary state, if and only if the state wave function is a product over the subsystems. Moreover, we will show that such a situation is not special to Bohm and Nelson theories, proving that *no classical probability theory* can reproduce the QM predictions for compatible position observables at different times of a large class of systems. This implies in particular that the proposal of [8] cannot be interpreted as a conditioning procedure on a classical probability theory.

In Sect. 2 the framework of our arguments will be fixed. We will recall that the existence of classical theories reproducing all quantum mechanical predictions for position observables at equal times follows immediately from the spectral theorem and that, more generally, any collection of probabilistic predictions for *disjoint* sets of observables can be reproduced by classical theories. We will conclude that position observables of non interacting particles at different times give the only relevant test for a trajectory description of Schrödinger QM.

In Sect. 3 we will discuss the predictions of Bohm Mechanics and Nelson Stochastic Mechanics for the joint distribution of compatible position observables at different

times, for any stationary state. In both cases they turn out to be different from the corresponding Quantum Mechanical predictions, with the only exception of product wave functions. In the Appendix, Nelson probability distributions will be shown to be computable in terms of Quantum Mechanical evolution at imaginary times, with Dirichlet boundary condition on the nodes of the wave function. An explicit computation will be performed for a system of two harmonic oscillators.

In Sect. 4, by constructing appropriate Bell–Clauser–Horn inequalities, we will show that no classical theory can reproduce the quantum mechanical predictions for joint distribution of position observables of non interacting systems at different times, even in the case of stationary states of very elementary systems.

From the above results we conclude that no attempt for a classical description of Schrödinger Quantum Mechanics in terms of particle trajectories can go beyond the reproduction of equal time correlations and that it is precisely the restriction to disjoint sets of compatible observables, rather than the use of positions variables, that allows for a classical description.

## 2. Comparison of theories and systems of observables.

Comparison between theories clearly depends in general on the class of experiments, or “observables” which are considered. Moreover, since in QM not all pairs of observables can be measured together, the description of sets of observables must include a notion of *joint measurability*. For the sake of clarity, it is convenient to partially formalize such notions through the following definitions (see e.g. refs. [12] [13]).

**Definition 2.1** *A set of observables is a set  $X$  with a relation  $\mathcal{R}$  satisfying reflexivity,  $ARA \forall A \in X$  and symmetry,  $ARB \Rightarrow BRA$ . If  $ARB$ ,  $A$  and  $B$  are called compatible.*

The elements  $A \in X$  are thought to be associated to experimental devices, providing, in case the experiment is performed, a real number  $x_A(\eta)$  as “the result of the measurement of  $A$  in the experiment  $\eta$ ”. Compatible experiments can be performed together; given a subset  $Y$  of  $X$  consisting of compatible observables, and  $A$  in  $Y$ , the corresponding experiments  $\eta_Y$  give rise to functions  $f_A : E_Y \mapsto \mathbf{R}$ .

**Definition 2.2** *An experimental arrangement is a subset  $Y$  of  $X$  consisting of compatible elements, i.e.  $ARB \forall A, B \in Y$ . For each experimental arrangement  $Y$  there is a set  $E_Y$  of experiments and, for all  $A \in Y$ , a function  $f_A : E_Y \mapsto \mathbf{R}$ .*

The functions  $f_A(\eta_Y)$ ,  $\eta_Y \in E_Y$  contain all the information about relations between the results of compatible experiments, usually interpreted as relations between observables. The same information is contained in the  $C^*$  algebras they generate, and in fact it is usually given, e.g. in QM, in terms of commutative  $C^*$  algebras of compatible observables. Such algebras arise automatically if the observables are assumed to form a (non-commutative)  $C^*$  algebra, as in the Haag–Kastler approach to QM [14] [15]. In particular (see below), the above relations can be taken as the basis for the discussion of the problems of the interpretation of QM. We introduce therefore the following notion:

**Definition 2.3** *A system of observables consists of a set  $X$ , a compatibility relation  $\mathcal{R}$ , a collection of sets  $E_Y$  indexed by all experimental arrangements and a collection of functions  $f_A(\eta_Y)$ ,  $A \in Y$ .*

Given a system of observables, a theory consists in general of a set of probability assignments for the values of the functions  $f_A$ , and can be formalized as follows:

**Definition 2.4** *A system of predictions for a system of observables  $(X, \mathcal{R}, \{E_Y\}, \{f_A\})$  is an assignment of probability measures  $d\mu_Y$  on the spaces  $E_Y$  so that all the functions  $f_A(\eta_Y)$ ,  $A \in Y$ , are measurable.*

The predictions for the mean value of the observable  $A$ , measured with the experimental arrangement  $Y$ , is therefore

$$\int_{E_Y} f_A(\eta_Y) d\mu_Y(\eta_Y) \quad .$$

Given a theory, different systems of observables can be introduced. In particular, the choice of a subset  $X_1$  of a system  $X$  of observables amounts to restricting the interpretation of the theory to the system of predictions indexed by the experimental arrangements contained in  $X_1$ .

In the ordinary interpretation of QM it is assumed that observables correspond to selfadjoint operators in a Hilbert space  $\mathcal{H}$ . Moreover, it is assumed

- i) *two observables are compatible if and only if the corresponding operators commute,*
- ii) *all selfadjoint operators define observables.*

Experimental arrangements are therefore indexed by all commutative  $C^*$  subalgebras of  $\mathcal{B}(\mathcal{H})$ , the spaces  $E_Y$  being given by their spectra and  $f_A$  by the corresponding Gelfand representations.

The above notion of system of predictions is close to the generalized notion of state introduced by Bell [16]. It is well known that, by Gleason's theorem [17], no system of predictions on the system of observables defined by i) and ii) can be reproduced by any classical theory. i.e. by the identification of the functions  $f_A(\eta_Y)$  with a set of (measurable) functions  $F_A(\xi)$  on a *common* measure space, if the dimension of  $\mathcal{H}$  is greater than two. It also follows from the Bell–Kochen–Specker theorem [18] that no classical representation exists for any system of predictions on the collection of the commutative subalgebras of the algebra generated by the angular momentum operators in the representation with  $l = 1$ . Similarly, Bell inequalities imply that no classical representation exists for the system of predictions defined by suitable quantum mechanical states on the system of observables defined by appropriate spin observables of two spin 1/2 particles.

However, as stressed by Bohm and Bell [9], assumption ii) may be too general, in particular, angular momentum and spin observables may be in some sense not admissible, and the choice of a physically motivated subset of observables may lead to a better motivated and possibly classical interpretation of QM. This possibility is exploited by Bohm's and Nelson's trajectory description of Schrödinger Quantum Mechanics. In terms of the above discussion, Bohm's and Nelson's theories are compared with QM by substituting assumption ii) above with

iiB) *only position operators define observables.*

In the following we will assume iiB) and consider, for a system of  $N$  particles, only observables consisting of measurable, bounded functions with compact support  $f(x_i(t))$ ,  $i = 1 \dots N$ ,  $t \in \mathbf{R}$ .

To discuss the rôle of compatibility relations, we first notice that the predictions of Bohm and Nelson Mechanics are usually compared to those of QM for position observables *at the same time*. In the framework discussed above, such a restriction amounts to assuming as compatibility relation between position observables the fact that they are measured at the same time.

There are cases in which such a choice is *forced by quantum mechanical compatibility*, i.e. only functions of position at the same time define commuting operators. This is in fact the case of a free particle in one space dimension:

**Proposition 2.5** *Let  $f$  and  $g$  be bounded measurable functions with compact support. If, for some  $t \neq s$ ,  $[f(x + pt/m), g(x + ps/m)] = 0$ , then  $f = 0$  or  $g = 0$ .*

Proof. Let  $\tau \equiv (s - t)/m$ .  $[f(x + pt/m), g(x + ps/m)] = 0$  is equivalent to  $[f(x), g(x +$

$p\tau)] = 0$ . The operator

$$g(x + p\tau + a) = e^{-iax/\tau} g(x + p\tau) e^{iax/\tau} \quad (2.1)$$

then commutes with  $f(x)$  for all  $a \in \mathbf{R}$  and therefore  $g(x + p\tau)$  commutes with  $f(x - a)$  for all  $a$ . If  $f$  is not a constant, the Von Neumann algebra  $F$  generated by  $f(x - a)$ ,  $a \in \mathbf{R}$  contains a characteristic function  $\chi_I(x)$  of a bounded measurable set  $I$ , and therefore all  $\chi_I(x - a)$ . It is easy to see that, e.g.,  $\psi(x) = \exp -x^2$  is a cyclic vector for  $F$  in  $L^2(\mathbf{R})$ ; in fact, for  $\phi(x)$  in  $L^2(\mathbf{R})$ ,

$$0 = (\chi_I(x - a) \psi, \phi) = \int dk e^{ika} \overline{\tilde{\chi}_I}(\tilde{\psi}\phi) \quad \forall a$$

$\tilde{f}$  denoting the Fourier transform of  $f$ , implies

$$\overline{\tilde{\chi}_I}(\tilde{\psi}\phi) = 0$$

and therefore, by analyticity of  $\tilde{\chi}_I$ ,  $(\tilde{\psi}\phi) = 0$ ,  $\psi\phi = 0$  and  $\phi = 0$ . It follows that, since  $g(x + p\tau)$  commutes with  $F$ , it is a multiplication operator,  $g(x + p\tau) = h(x)$  and therefore, by eq.(2.1),  $g$  is a constant.

The above result does not however extend to more than one dimension, since, e.g., different components of the position of a free particle commute and are therefore quantum mechanically compatible also at different times. More generally, for particle systems composed of non interacting subsystems, functions of positions of particles belonging to different subsystems correspond to commuting operators for all times, and therefore they are compatible observables according to QM. For such systems the choice of equal times as the compatibility relation between position observables has no clear motivation. Common probability distributions for such positions are predicted by QM and can be compared with the prediction of Bohm and Nelson Mechanics.

Moreover, such a comparison is in a sense the only relevant test for the above theories. In fact, a compatibility relation consisting of “being observed at the same time” is clearly a transitive relation, giving rise to the partition of observables into disjoint classes; the following Proposition recalls that this fact implies by itself the existence of a common classical description of any set of probability distributions for all admitted experimental arrangements, independently from any other assumption on the observables, in particular from the above assumption iiB). In other terms, if the possibility of measuring observables at different times is discarded, quantum mechanical predictions for any commuting set of observables at each time a priori admit a classical representation.

**Proposition 2.6** *Let  $X, \mathcal{R}, \{E_Y\}, \{f_A(\eta_Y)\}$  be a system of observables, with  $E_Y$  compact Hausdorff spaces, and  $\{d\mu_Y\}$  a system of predictions for it consisting of Borel measures. If  $\mathcal{R}$  is transitive, there is a measure space  $\Xi, d\nu$  and for each  $A$  in  $X$  a measurable functions  $F_A(\xi)$  such that, for all finite sequences of observables  $A^i$  in an experimental arrangement  $Y$  and real intervals  $I^i$ ,*

$$\mu_Y(\{\eta_Y : f_{A^i}(\eta_Y) \in I^i\}) = \int_{\Xi} d\nu \prod_{i=1}^n \chi_{I^i}(F_{A^i}(\xi)) \quad , \quad (2.2)$$

$\chi_I$  denoting the characteristic functions of the interval  $I$ .

Proof. Let  $\Xi$  be the topological product of the spaces  $E_Y$  with the product measure  $d\nu = \prod_Y d\mu_Y$ . Transitivity of  $\mathcal{R}$  implies that experimental arrangements are disjoint sets. Each observable  $A$  belongs therefore to a unique experimental arrangement  $Y_A$ . Denoting points of  $\Xi$  as  $\xi \equiv \{\xi_Y\}$ , the functions  $F_A(\xi) \equiv f_A(\xi_{Y_A})$  are therefore well defined and

$$\begin{aligned} \mu_Y(\{\eta_Y : f_{A^i}(\eta_Y) \in I^i\}) &= \int_{E_Y} d\mu_Y \prod_{i=1}^n \chi_{I^i}(f_{A^i}(\xi_Y)) = \\ &= \int_{\Xi} d\nu \prod_{i=1}^n \chi_{I^i}(F_{A^i}(\xi)) \quad . \end{aligned} \quad (2.3)$$

Since any quantum mechanical state defines, through the spectral theorem, a system of predictions, in the sense of Def. 2.4, on any collection of sets of commuting operators, it follows from Proposition 2.6 that any quantum system has a classical description, in terms of the classical probability space  $\Xi$ , if the experimental arrangements admitted by the choice of the compatibility relation are disjoint.

In particular, if positions observables are assumed to be compatible only at the same time, all the corresponding quantum mechanical predictions are represented, by Proposition 2.6, in terms of a probability measure in the space of trajectories  $\{x_t\}$ . Clearly, the measure  $d\nu$  constructed in Proposition 2.6 is not unique, and in fact, for quantum mechanical position observables, both Bohm and Nelson Mechanics provide valid alternatives. The problem of the abundance of such alternatives has in fact being addressed [19], and clearly no solution with observational content exists if correlations at different times are not taken into account.

We conclude that it is very relevant to extend the comparison of Quantum Mechanics and Stochastic Mechanics to cases in which positions at different times can be taken as compatible observables.

### 3. Bohm's and Nelson's stochastic predictions for compatible positions at different times.

We will compare the predictions of Quantum Mechanics, Bohm Mechanics and Nelson Stochastic Mechanics, in the case of stationary states, for position variables of non interacting subsystem at different times. The stationary case turns out to be sufficient for a clear comparison with Quantum Mechanics, since completely different predictions will result already for stationary states.

Quantum mechanical predictions can be characterized in very simple terms for the following class of systems: let

$$H = \sum_{i=1}^n H_i(x_i, p_i) \quad , \quad (3.1)$$

$x_i, p_i$  denoting positions and momenta of  $n$  clusters of Schrödinger particles. Let

$$H_i = H_i^0 + V_i(x_i)$$

with  $H_i^0$  the kinetic energy operator of the  $i$  cluster and  $V_i(x_i)$  its interaction potential, which we assume small with respect to  $H_i^0$  in the sense of bilinear forms [24], so that  $H_i$  and  $H$  are uniquely defined as self-adjoint operators. All  $H_i$  are then bounded below and their point spectrum is assumed to include non degenerate eigenvalues  $\lambda_i^k$ , with eigenvectors  $\psi_i^k$ . Any eigenstate of  $H$ , with eigenvalue  $E$  is then of the form

$$\psi_E = \sum_{s=1}^N c_s \prod_{i=1}^n \psi_i^{k(i,s)} \quad (3.2)$$

with

$$\sum_{i=1}^n \lambda_i^{k(i,s)} = E \quad \forall s = 1 \dots N \quad (3.3)$$

The QM correlation functions of position observables on such states are immediately calculated as

$$\begin{aligned} & (\psi_E, f_1(x_1(t_1)) \cdots f_n(x_n(t_n)) \psi_E) = \\ & = (\psi_E, e^{iH_1 t_1} f_1(x_1) e^{-iH_1 t_1} \cdots e^{iH_n t_n} f_n(x_n) e^{-iH_n t_n} \psi_E) = \\ & = \sum_{s,p} \bar{c}_s c_p \prod_{i=1}^n (\psi_i^{k(i,s)}, f_i(x_i) \psi_i^{k(i,p)}) e^{i(\lambda_i^{k(i,s)} - \lambda_i^{k(i,p)}) t_i} \end{aligned} \quad (3.4)$$

and are therefore finite combinations of products of trigonometric functions of  $t_1 \dots t_n$ . In particular, for  $n = 2$ , and  $c_s$  real, they are of the form

$$(\psi_E, f_1(x_1(t_1))f_2(x_2(t_2))\psi_E) = \sum_{s=1}^N C_s \cos \omega_s(t_1 - t_2) \quad (3.5)$$

with  $C_s$  real.

The derivation of correlation functions for position variables at different times is almost trivial for Bohm Mechanics, as a consequence of the stationarity assumption. For  $N$  particles, Bohm mechanics is in fact defined by a probability distribution at  $t = 0$

$$\rho(x_1, \dots, x_N) = |\psi(x_1, \dots, x_N)|^2 \quad (3.6)$$

and by evolution equations

$$\frac{dx_i(t)}{dt} = \frac{1}{m_i} \text{Im} \frac{\partial}{\partial x_i} \log \psi(x_1, \dots, x_N, t) \quad , \quad (3.7)$$

$\psi(x_1, \dots, x_N, t)$  denoting a solution of the Schrödinger equation

$$i \frac{\partial}{\partial t} \psi = \sum_i -\frac{1}{2m_i} \Delta_i \psi + V(x_1, \dots, x_N) \psi \quad (3.8)$$

In the case of a stationary state, since  $\psi$  can be taken to be real, the velocity field on the r.h.s. of eq.(3.1) vanishes identically and therefore the trajectories  $x_i(t)$  are simply constant. The probability distribution for position observables  $x_1(t_1), \dots, x_k(t_k)$  is therefore the same as the distribution for  $x_1(0), \dots, x_k(0)$ . From eq.(3.4) we have therefore

**Proposition 3.1** *The Bohm probability distribution for position observables of non interacting subsystems at different times, for a stationary state  $\psi$  of a system with Hamiltonian of the form (3.1) coincide with the quantum mechanical result, eq.(3.4) if and only if  $\psi$  is a product,*

$$\psi(x_1, \dots, x_n) = \psi_1^{k_1}(x_1) \cdot \dots \cdot \psi_n^{k_n}(x_n) \quad (3.9)$$

E.g., for a system of two independent harmonic oscillators, with Hamiltonian

$$H \equiv \frac{p_1^2}{2} + \frac{\omega^2 x_1^2}{2} + \frac{p_2^2}{2} + \frac{\omega^2 x_2^2}{2}$$

in the stationary state

$$\psi \equiv \frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)) \quad ,$$

$\psi_0$  denoting the ground state and  $\psi_1$  the first excited state of  $H_0$ , the Bohm probability distribution for  $x_1(t), x_2(s)$  is independent of times; the quantum mechanical expectation of their product is however

$$(\psi, x_1(t) x_2(s) \psi) = \frac{\cos \omega(t-s)}{2\omega} \quad .$$

In the case of Nelson Mechanics, probability distributions for positions at different times are obtained from solutions of stochastic equations. For a system of  $N$  particles, with positions  $x_i(t)$ , Nelson Mechanics is defined [5] by the forward stochastic equations

$$dx_i = b_i dt + dw_i \quad (3.10)$$

with  $dw_i$  independent white noises,

$$b_i(x_1, \dots, x_N, t) = \frac{1}{m_i} (Re + Im) \frac{\partial}{\partial x_i} \log \psi(x_1, \dots, x_N, t) \quad (3.11)$$

and  $\psi$  a solutions of the Schrödinger equation, eq.(3.8). As in Bohm Mechanics, the probability density is assumed to coincide with  $|\psi|^2$  at some time, and therefore at all times, as a consequence of eqs. (3.10), (3.11) [5].

For non interacting clusters of particles, i.e. for  $V(x_1, \dots, x_n) = \sum_i V_i(x_i)$ ,  $x_i$  denoting the position vector of a cluster of particles, if  $\psi(x_1, \dots, x_n, 0)$  is a product of wave functions, the same holds for all times,

$$\psi(x_1, \dots, x_n, t) = \prod_i \psi_i(x_i, t) \quad (3.12)$$

It follows that the drift for each cluster of particle only depends on position variables within the cluster,

$$b_i = b_i(x_i) \quad (3.13)$$

so that equations (3.10) split into subsets. Each of them gives rise to the Schrödinger evolution for the particle density in the corresponding cluster and therefore the solution of eqs.(3.10) (see below for its existence and uniqueness) gives a joint probability

density for  $x_1(t_1), \dots, x_n(t_n)$

$$\rho(x_1(t_1), \dots, x_n(t_n)) = \prod_i |\psi_i(x_i, t)|^2 \quad (3.14)$$

It follows that for product wave functions the predictions of Nelson Mechanics coincide with those of Quantum Mechanics, also for positions at different times.

The situation is completely different for wave functions which are not a product. In this case, even for stationary states, eqs.(3.10) are not independent and their solution must be examined more closely. Let us consider a (normalized) stationary state  $\psi_E(x_1, \dots, x_N)$  of a Hamiltonian of the form (3.1). The Nelson forward stochastic equations are

$$dx_i = \frac{1}{m_i} \frac{\partial}{\partial x_i} \log \psi_E(x_1, \dots, x_N) dt + dw_i \quad (3.15)$$

and coincide, by reality of  $\psi$ , with the backward equations.

If  $\psi_E$  is not a ground state, it vanishes on surfaces of codimension 1, so that  $\psi_E^{-1} \frac{\partial}{\partial x_i} \psi_E$  is singular on such surfaces. Existence and uniqueness of the solution of eqs. (3.15), i.e. of the stochastic processes associated to eqs. (3.15) does not therefore follow from standard results [20]. A (unique) solution has been constructed by Carlen [21] for a large class of Schrödinger wave functions  $\psi_E(x_1 \dots x_N, t)$ , by approximating singular drifts with regular ones.

Carlen's construction is very general and solves Nelson's equations also in the case of time dependent wave functions. In the case of stationary states, the solution has been discussed in refs. [22] [23] and shown to be given by the unique Markov process with generator extending the Fokker–Planck operator [20] associated to eqs. (3.15). Such a Markov process is well defined, and actually generated [22] by the Friedrichs extension [24] of the Fokker–Planck operator for all  $\psi(x_1, \dots, x_N)$  in  $L^2(\mathbf{R}^N)$  with  $L^2$  derivatives.

It is easy to show that, as a consequence of the general structure of the semigroup associated to a Markov process, invariant under time reflection as required by coincidence of forward and backward drifts, with stationary measure  $|\psi_E|^2 dx_1 \dots dx_n$ , the corresponding predictions for correlations of non interacting subsystems at different times are not compatible with the Quantum Mechanical result, eq. (3.4), unless  $\psi(x_1, \dots, x_n)$  is a product, eq. (3.9). For completeness, in the Appendix the general form of correlations at different times will be given in terms of QM Hamiltonians; moreover, the Carlen construction will be explicitly performed for a system of two harmonic oscillators, resulting in disagreement with the QM predictions.

Let us therefore consider the structure of the unique solution of eqs. (3.15). It is given [20] [23] by a positivity preserving semigroup  $A(t)$ ,  $t \in [0, \infty)$  acting in  $L^2(\mathbb{R}^N, |\psi_E|^2 dx_1 \dots dx_N)$  such that

$$\int d\mu f(x(t)) g(x) = \int f A(t)g |\psi_E|^2 dx_1 \dots dx_n \quad . \quad (3.16)$$

More generally, for  $t_k > t_{k-1} > \dots > t_1$ ,

$$\begin{aligned} & \int d\mu f_1(x(t_1)) \dots f_k(x(t_k)) = \\ & = \int f_k(x) A(t_k - t_{k-1})(f_{k-1}(x) \dots A(t_2 - t_1)f_1(x) \dots) |\psi_E|^2 dx_1 \dots dx_n \quad . \quad (3.17) \end{aligned}$$

Stationarity of  $|\psi_E|^2 dx_1 \dots dx_n$  and time reflection symmetry imply  $A(t)1 = 1$  and hermiticity of  $A(t)$ , so that

$$A(t) = \exp -tT \quad (3.18)$$

with  $T$  selfadjoint. From the spectral representation of the r.h.s. of eq.(3.18) it follows that  $T$  is non negative; in fact, positivity of  $A(t)$  implies boundedness in  $t$  of  $\|A(t)f\|$  for  $f$  in a dense subspace since, for bounded  $f$ ,

$$\begin{aligned} \|A(t)f\|_{L^2(|\psi_E|^2 dx_1 \dots dx_n)}^2 &= (f, A(2t)f) \leq (|f|, A(2t)|f|) \leq \\ &\leq \sup_x |f| (1, A(2t)|f|) \leq \sup_x |f| \|f\| \quad . \end{aligned}$$

Restricting the attention to position variables of different non interacting clusters of particles we have therefore

**Proposition 3.2** *For any stationary state  $\psi_E$  of a system with Hamiltonian of the form (3.1), the solution of the Nelson equations (3.15) defines probability distributions, for position observables of non interacting subsystems at different times, of the form (3.17), (3.18). In particular, the two-particle correlations are given by*

$$\begin{aligned} \int d\mu f(x_1(t)) g(x_2) &= \int f(x_1) \exp -tT g(x_2) |\psi_E(x_1, x_2)|^2 dx_1 dx_2 = \\ &= \int_0^\infty \exp -\lambda t d\nu_{f,g}(\lambda) \quad , \quad (3.19) \end{aligned}$$

with  $d\nu_{f,g}$  a complex measure. They coincide with the QM result, eq. (3.4), if and only if  $\psi$  is a product.

**Proof.** The spectral representation of  $T$  gives rise to the complex measure

$$(f, dE_T(\lambda)g) = d\nu_{f,g}(\lambda)$$

and to the representation (3.19). If a function  $F(t)$  is of the form (3.19), then

$$\lim_{t \rightarrow \infty} F(t) = \nu(\{0\}) \quad ;$$

if  $F(t)$  is also of the form (3.4), it must be a constant and therefore  $\psi_E$  is a product, as in Proposition 3.1.

#### 4. No classical theory can reproduce QM correlation functions of compatible observables at different times.

We will show that the above discrepancy between Stochastic Mechanics and QM is not special to Bohm's and Nelson's theories, by proving that no classical probability model can reproduce the QM correlation functions of compatible position observables at different times for suitable, even very elementary, systems.

Consider a system composed of two identical subsystems, with Hamiltonian

$$H = K(x_1, p_1) + K(x_2, p_2) \equiv K_1 + K_2 \quad (4.1)$$

and let  $E_0, E_1$  be two eigenvalues of  $K$ , with eigenvectors  $\psi_0, \psi_1$ . Let  $f(x)$  be a measurable function, taking values 1 and  $-1$ , and consider the correlation functions

$$(\psi, f(x_1, t) f(x_2, s) \psi) \quad (4.2)$$

with

$$\psi = 1/\sqrt{2} (\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)) \quad (4.3)$$

and  $f(x_i, t)$  the QM observable  $f(x_i)$  at time  $t$ :

$$f(x_i, t) = e^{itH} f(x_i) e^{-itH} = e^{itK_i} f(x_i) e^{-itK_i} \quad (4.3)$$

All the observables  $f(x_i, t)$ ,  $i = 1, 2$ ,  $t \in \mathbf{R}$ , take values in  $\{-1, 1\}$ . Given  $N$  such observables, the most general classical probabilistic theory consists therefore

of a probability measure on the space  $\{-1, 1\}^N$ . Following the discussion of Bell inequalities in ref. [25], we will prove the following:

**Proposition 4.1** *There exist times  $t_i$ ,  $s_i$ ,  $i = 1, 2$  and QM models such that the set of QM predictions for the correlation functions*

$$(\psi, f(x_1, t_i) f(x_2, s_j) \psi)$$

*cannot be reproduced by any probability assignment, i.e. there does not exist a non negative probability distribution  $p$ , on variables  $\sigma_k$  and  $\tau_l$  taking values in  $\{-1, 1\}$ ,*

$$\sum_{\sigma_k=\pm 1} \sum_{\tau_l=\pm 1} p(\sigma_1, \sigma_2, \tau_1, \tau_2) = 1 \quad (4.4)$$

*such that*

$$(\psi, f(x_1, t_i) f(x_2, s_j) \psi) = \sum_{\sigma_k=\pm 1} \sum_{\tau_l=\pm 1} p(\sigma_1, \sigma_2, \tau_1, \tau_2) \sigma_i \tau_j \equiv \langle \sigma_i \tau_j \rangle \quad . \quad (4.5)$$

**Proof.** Following [25], it is immediate to check that the function

$$\sigma_1 \tau_1 + \sigma_2 \tau_2 + \sigma_2 \tau_1 - \sigma_1 \tau_2$$

takes values in  $[-2, 2]$ , so that, for any probability distribution,

$$| \langle \sigma_1 \tau_1 + \sigma_2 \tau_2 + \sigma_2 \tau_1 - \sigma_1 \tau_2 \rangle | \leq 2 \quad (4.6)$$

Eq. (4.5) cannot therefore hold for QM models such that

$$\begin{aligned} & (\psi, f(x_1, t_1) f(x_2, s_1) \psi) + (\psi, f(x_1, t_2) f(x_2, s_2) \psi) + \\ & + (\psi, f(x_1, t_2) f(x_2, s_1) \psi) - (\psi, f(x_1, t_1) f(x_2, s_2) \psi) < -2 \end{aligned} \quad (4.7)$$

In order to construct such models, let

$$f(x) \equiv \text{sign}(x) \quad ,$$

$K$  a selfadjoint operator with

$$K \psi_i = E_i \psi_i \quad , \quad i = 0, 1 \quad (4.8)$$

and  $\psi_i(x)$  real functions satisfying

$$\psi_0(x) = \psi_0(-x) \quad , \quad \psi_1(x) = -\psi_1(-x) \quad , \quad \int dx |\psi_i(x)|^2 = 1$$

It then follows

$$(\psi_0, f(x)\psi_0) = (\psi_1, f(x)\psi_1) = 0 \quad (4.9)$$

$$(\psi_0, f(x)\psi_1) \equiv \alpha \quad , \quad -1 \leq \alpha \leq 1 \quad (4.10)$$

Let  $\omega \equiv E_1 - E_0$ . By eqs (4.3), (4.8), (4.9), (4.10), the above expectation values are given by

$$\begin{aligned} (\psi, f(x_1, t) f(x_2, s)\psi) &= (\psi, e^{iK_1 t} e^{iK_2 s} f(x_1) f(x_2) e^{-iK_1 t} e^{-iK_2 s} \psi) = \\ &= -\text{Re}(\psi_0(x_1)\psi_1(x_2), e^{iK_1 t} e^{iK_2 s} f(x_1) f(x_2) e^{-iK_1 t} e^{-iK_2 s} \psi_1(x_1)\psi_0(x_2)) = \\ &= -\alpha^2 \cos \omega(t - s) \end{aligned}$$

Choosing  $t_1 = 0$ ,  $t_2 = \pi/2\omega$ ,  $s_1 = \pi/4\omega$ ,  $s_2 = 3\pi/4\omega$ , the result for the l.h.s. of eq.(4.7) is

$$-4 \cos(\pi/4) \alpha^2 = -2\sqrt{2} \alpha^2$$

and satisfies inequality (4.7) if

$$\alpha^2 > \sqrt{2}/2 \quad (4.11)$$

The construction of models satisfying inequality (4.11) is straightforward; e.g., for a free particle on an interval  $[-L, L]$ , with Dirichlet boundary conditions, taking  $\psi_0$  and  $\psi_1$  as the lowest energy states, one obtains  $\alpha^2 = 8/(3\pi) > \sqrt{2}/2$ . By taking the lowest energy states of double well potentials one can obtain  $\alpha = 1 - \varepsilon$ , with  $\varepsilon > 0$  arbitrarily small.  $\alpha = 1$  would correspond to the maximal violation of the Bell inequalities (4.6) by QM states [25].

The above derivation strictly reproduces the discussion of Bell inequalities for spin variables,  $f(x_i, t)$  playing the rôle of spin variables and eq. (4.3) corresponding to a state with zero total angular momentum. It shows that the existence of a classical probabilistic representation has nothing to do with the choice of position variables and rather depends, as discussed in Proposition 2.6, on the exclusion of observations at different times, giving rise to a transitive compatibility relation.

## Appendix

Explicit formulas will be given here for the correlation functions at different times on stationary states in Nelson Stochastic Mechanics, based on the general results of [22]. Moreover, the Carlen construction will be performed directly, by elementary methods, for a system of two harmonic oscillators and the result will be compared with the corresponding QM predictions.

**A1.** Let us consider a Hamiltonian

$$H = - \sum_{i=1}^N \frac{\Delta_i}{2m_i} + V(x_1 \dots x_N) \quad (\text{A.1})$$

with  $V = V_1 + V_2$ ,  $V_1$  small in the sense of forms with respect to the kinetic term,  $V_2 \in C^\infty$  and bounded below, and  $\psi(x_1, \dots, x_N)$  a regular ( $C^2$ ) eigenfunction of  $H$ . The assumptions on  $V$  imply that  $\psi$  is in the domain of the form defined by the Laplacean, so that it has  $L^2$  derivatives. It follows [22] that the solution of eqs. (3.15) is given by the semigroup generated by the Friedrichs extension [24] of the Fokker–Planck operator

$$F \equiv \sum_{i=1}^N \frac{\Delta_i}{2m_i} + \sum_{i=1}^N \frac{\partial}{\partial x_i} \log \psi(x) \frac{\partial}{\partial x_i} \quad , \quad (\text{A.2})$$

defined in  $L^2(\mathbf{R}^{3N}, \psi^2 dx_1 \dots dx_N)$  on the domain  $\mathcal{D}_0$  of  $C^2$  functions with compact support, vanishing in a neighbourhood of the set of zeros of  $\psi$ . A more explicit representation can be obtained by using the isometry

$$W : L^2(\mathbf{R}^{3N}, \psi^2 dx_1 \dots dx_N) \longrightarrow L^2(\mathbf{R}^{3N}, dx_1 \dots dx_N)$$

defined by

$$W f(x) \equiv |\psi(x)| f(x) \quad . \quad (\text{A.3})$$

In fact, for every  $f$  in  $\mathcal{D}_0$ ,

$$\begin{aligned} -W^{-1} \left( - \sum_{i=1}^N \frac{\Delta_i}{2m_i} + V(x) \right) W f(x) &= |\psi|^{-1} \sum_i \left( \frac{\Delta_i}{2m_i} |\psi| \right) f + \sum_i \frac{1}{2m_i} \Delta_i f + \\ &+ |\psi|^{-1} \sum_i \frac{\partial |\psi|}{\partial x_i} \frac{\partial f}{\partial x_i} - V f = (F - E) f(x) \end{aligned}$$

where we have used the eigenvalue equation  $H\psi = E\psi$ , so that, on  $\mathcal{D}_0$ ,

$$F = -W^{-1}(H - E)W \quad (\text{A.4})$$

Selfadjoint extensions of  $F$  in  $L^2(\mathbf{R}^{3N}, \psi^2 dx_1 \dots dx_N)$  correspond therefore to selfadjoint extensions of  $H$  from the domain  $W\mathcal{D}_0$ :

**Proposition A.1** *Every selfadjoint extension  $\tilde{F}$  in  $L^2(\mathbf{R}^{3N}, \psi^2 dx_1 \dots dx_N)$  of the operator  $F$  is of the form*

$$\tilde{F} = -W^{-1}(\tilde{H} - E)W \quad (\text{A.5})$$

with  $\tilde{H}$  a self-adjoint extension of  $H$ , defined on  $W\mathcal{D}_0$  in  $L^2(\mathbf{R}^{3N}, dx_1 \dots dx_N)$ ,  $W$  the isometry given by eq. (A.3). The Friedrichs extension of  $F$  is obtained, through eq. (A.5), from the Friedrichs extension  $H^F$  of  $H$ .  $H^F$  is essentially selfadjoint on the domain of  $C^\infty$  functions with compact support vanishing on the zeros of  $\psi$ . The semigroup  $A(t)$ , eq. (3.18) can therefore be written

$$A(t) = W^{-1} e^{-t(-(\sum_i \Delta_i/2m_i)_0 + V - E)} W \quad , \quad (\text{A.6})$$

$(\sum_i \Delta_i/2m_i)_0$  denoting the quantum mechanical kinetic energy operator on  $C^\infty$  functions with compact support vanishing on the zeros of  $\psi$ .

**Proof.** The isometry  $W$  maps the deficiency subspaces of the hermitean operator  $F$ , on  $\mathcal{D}_0$ , into those of  $H$ , on  $W\mathcal{D}_0$ .  $W$  also maps form domains into form domains. By regularity of  $\psi$ , the domain  $W\mathcal{D}_0$  consists of all  $C^2$  functions with compact support not intersecting the zeros of  $\psi$ . The Friedrichs extension of the kinetic term  $H_0$  gives therefore the sum of Laplaceans with Dirichlet boundary conditions and, by an elementary computation of deficiency spaces, this operator is essentially selfadjoint on the domain of  $C^\infty$  functions with compact support vanishing on the zeros of  $\psi$ . Since  $V_2$  is regular and bounded below,  $H_0 + V_2$  remains essentially selfadjoint on the same domain (see [26], Theorems X.27, X.28), and coincides there with its Friedrichs extension from  $W\mathcal{D}_0$ .  $V_1$  is form-small with respect to  $H_0$  and therefore with respect to  $H_0 + V_2$ , as a consequence of  $V_2$  being bounded from below;  $H_0 + V_1 + V_2$  is therefore essentially selfadjoint on the same domain and coincides there with the Friedrichs extension of  $H$  from  $W\mathcal{D}_0$ .

**A2.** We shall now rederive explicitly eq. (A.6) in a simple case. i.e for the first excited state of a harmonic oscillator, by performing the Carlen construction by direct and

elementary methods. The result will turn out to be enough for the discussion of compatible position observables of a pair of harmonic oscillators.

Let  $H$  the Hamiltonian of a one dimensional harmonic oscillator,

$$H = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \omega^2 x^2\right) \quad . \quad (A.7)$$

The Nelson equations corresponding to the first excited state

$$\psi_1(x) = \left(\frac{4\omega^3}{\pi}\right)^{1/4} x e^{-\frac{\omega}{2}x^2} \quad , \quad H\psi_1 = E_1\psi_1 \quad , \quad (A.8)$$

define a Fokker–Planck operator  $F$  of the form

$$F = \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \log \psi_1(x) \frac{\partial}{\partial x} \quad . \quad (A.9)$$

The drift term in eq.(A.9) is singular at the origin and a possible domain  $\mathcal{D} \subset L^2(\mathbf{R}, \psi_1^2 dx)$  of  $F$  consists of (smooth) functions vanishing at the origin. It is easy to see that it is a domain of hermiticity but not of (essential) selfadjointness for  $F$ .

We will perform explicitly Carlen's construction [21], which give the stochastic processes associated to singular drifts in terms of a sequence of regular approximants. The result will be given, as in Proposition (A.1), by the semigroup generated by the Hamiltonian operator (A.7), with Dirichlet boundary conditions at 0, the node of  $\psi_1$ . Following Carlen, we introduce strictly positive regular functions  $\psi_0^\varepsilon(x)$  approximating  $\psi_1$  and discuss convergence of the corresponding Nelson processes. Let  $\psi_0^\varepsilon(x) \equiv |\psi_1(x)|$ , for  $|x| > \varepsilon$ , and  $\psi_0^\varepsilon(x) \equiv g^\varepsilon(x)$ , for  $|x| \leq \varepsilon$ ,  $g^\varepsilon(x)$  a positive even function in  $C^2$ .  $g^\varepsilon$  and its derivative are assumed to coincide with  $|\psi_1|$  at  $x = \pm\varepsilon$ , and to satisfy

$$c_1 \varepsilon^{-2} g^\varepsilon(x) \leq d^2/dx^2 g^\varepsilon(x) \leq c_2 \varepsilon^{-2} g^\varepsilon(x) \quad . \quad (A.10)$$

E.g.,  $g^\varepsilon$  can be taken of the form

$$g^\varepsilon(x) = a^\varepsilon \cosh(b^\varepsilon x)$$

with  $a^\varepsilon = O(\varepsilon)$ ,  $b^\varepsilon = O(1/\varepsilon)$ . Introducing

$$\delta V^\varepsilon(x) \equiv \frac{\partial_x^2 g^\varepsilon(x)}{2g^\varepsilon(x)} - \frac{\omega^2 x^2}{2} + E_1 \quad , \quad |x| \leq \varepsilon$$

and zero otherwise, we have

$$H^\varepsilon \psi_0^\varepsilon = E_1 \psi_0^\varepsilon \quad , \quad (A.11)$$

with

$$H^\varepsilon \equiv H + \delta V^\varepsilon(x) \quad (\text{A.12})$$

Being positive,  $\psi_0^\varepsilon$  is the ground state of  $H^\varepsilon$ . The drifts

$$b^\varepsilon(x) = \frac{\partial}{\partial x} \log \psi_0^\varepsilon(x) \quad (\text{A.13})$$

are regular, so that the Fokker–Planck operator  $F^\varepsilon = \Delta/2 + b^\varepsilon(x)\partial_x$  is essentially self-adjoint on the domain of smooth functions with compact support in  $L^2(\mathbf{R}, (\psi_0^\varepsilon)^2 dx)$  and generates the semigroup  $A^\varepsilon(t)$  associated to the Nelson stochastic processes with drift  $b^\varepsilon$ . As in Proposition A.1, we have the representation

$$A^\varepsilon(t) = \frac{1}{\psi_0^\varepsilon} e^{-(H^\varepsilon - E_1)t} \psi_0^\varepsilon \quad (\text{A.14})$$

Following Carlen’s construction, we will discuss the convergence of  $A^\varepsilon(t)$  for  $\varepsilon \rightarrow 0$ .

**Proposition A.2** *There are sequences  $\varepsilon_k$  such that the operators  $e^{-H^\varepsilon t}$  converge in norm in  $L^2(\mathbf{R}, dx)$ . The limit is unique and given by  $e^{-\tilde{H}t}$ , with  $\tilde{H}$  the Hamiltonian (A.7) with Dirichlet boundary conditions at  $x = 0$ . The corresponding operators  $A^{\varepsilon_k}(t)$  converge strongly on a dense domain in  $L^2(\mathbf{R}, \psi_1^2 dx)$  to*

$$\frac{1}{|\psi_1|} e^{-(\tilde{H} - E_1)t} |\psi_1| = \frac{1}{\psi_1} e^{-(\tilde{H} - E_1)t} \psi_1 \quad (\text{A.15})$$

**Proof.** By eq.(A.10), for  $\varepsilon$  small,

$$0 \leq \delta V^\varepsilon(x) \leq c_2/x^2$$

and therefore

$$H \leq H^\varepsilon \leq H + c_2/x^2 \equiv H^> \quad . \quad (\text{A.16})$$

The operator  $H^>$  is hermitian on the domain  $\mathcal{D}^> = \{f \in C_0^\infty, f(0) = f'(0) = 0\}$  and its Friedrichs extension is greater than  $H^\varepsilon$ , so that, by the minimax principle, the eigenvalues  $E_n^\varepsilon$ ,  $n \geq 0$ , of the operators in eq. (A.16) satisfy

$$E_n \leq E_n^\varepsilon \leq E_n^>$$

There are therefore sequences  $\{\varepsilon_k^i\}$ ,  $\{\varepsilon_k^i\} \subset \{\varepsilon_k^j\}$  if  $i > j$ , such that  $E_k^{\varepsilon_k^i}$  converge as  $k \rightarrow \infty$ . For  $\{\varepsilon_k\} \equiv \{\varepsilon_k^k\}$ , we have convergence of all the eigenvalues for  $k \rightarrow \infty$ . By

Lemma A.3 below, the eigenfunctions  $\psi_n^k$  of  $H^{\varepsilon_k}$  converge to the eigenfunctions  $\tilde{\psi}_n$  of  $\tilde{H}$ . Moreover,

$$e^{-H^{\varepsilon_k} t} = \sum_{n=0}^N e^{-E_n^{\varepsilon_k} t} |\psi_n^k\rangle \langle \psi_n^k| + R_N^k \quad (\text{A.17})$$

with  $\|R_N^k\| \leq e^{-E_{N+1} t}$  as a consequence of eq.(A.16). The operators  $e^{-H^{\varepsilon_k} t}$  converge therefore in norm, for  $k \rightarrow \infty$ , to  $e^{-\tilde{H} t}$ . The ground state wave functions  $\psi_0^k$  are positive and, by Lemma A.3, converge to  $|\psi_1|$  uniformly. The operators

$$A^{\varepsilon_k} = \frac{1}{\psi_0^k} e^{-H^{\varepsilon_k} t} \psi_0^k = \frac{1}{|\psi_1|} \left( \frac{|\psi_1|}{\psi_0^k} e^{-H^{\varepsilon_k} t} \frac{\psi_0^k}{|\psi_1|} \right) |\psi_1| \quad (\text{A.18})$$

therefore converge, on the dense domain of regular functions with compact support excluding 0 in  $L^2(\mathbf{R}, \psi_1^2 dx)$ , to

$$\frac{1}{|\psi_1|} e^{-\tilde{H} t} |\psi_1| = \frac{1}{\psi_1} e^{-\tilde{H} t} \psi_1 \quad ,$$

the last equation following from the fact that  $L^2((0, \infty))$  and  $L^2((-\infty, 0))$  are left invariant by  $e^{-\tilde{H} t}$ .

**Lemma A3** *If a sequence of eigenvalues  $E_n^{\varepsilon_j}$  of the Hamiltonians defined by eq.(A.12),  $n$  fixed, converges for  $j \rightarrow \infty$ , then the corresponding normalized eigenfunctions  $\psi_n^{\varepsilon_j}$  converge to the eigenfunctions of  $\tilde{\psi}_n$  of  $\tilde{H}$ .*

**Proof.** Since  $V^\varepsilon$  is even, the eigenfunctions  $\psi^j \equiv \psi_n^{\varepsilon_j}$  ( $n$  fixed) have definite parity, so that the analysis can be restricted to  $x \in [0, \infty)$  and we can assume  $\psi^j(x) > 0$  for  $x > 0$  sufficiently small. For  $\varepsilon_j$  small and for  $x$  small in  $(0, \varepsilon_j)$ , the Schrödinger equation gives

$$\frac{1}{2} \frac{d^2}{dx^2} \psi^j(x) = \left( \frac{d^2 g^\varepsilon(x)}{dx^2} (2g^\varepsilon(x))^{-1} - E_n^{\varepsilon_j} \right) \psi^j(x) \geq \frac{c}{2\varepsilon_j^2} \psi^j(x) \quad (\text{A.19})$$

for  $c < c_1$  and  $\varepsilon_j$  small.  $d/dx \psi^j(x)$  is therefore positive for  $x > 0$  small, Eq. (A.19) extends to all  $x$  in  $(0, \varepsilon_j)$ , so that  $d/dx \psi^j(x)$  is increasing in  $(0, \varepsilon)$  and we can assume the normalization

$$d/dx \psi^j(\varepsilon_j) = 1 \quad . \quad (\text{A.20})$$

Moreover, eq.(A.19) implies

$$d/dx \psi^j(x) \geq \frac{c}{\varepsilon_j^2} x \psi^j(0)$$

so that

$$\psi^j(0) \leq \varepsilon_j/c \quad d/dx \psi^j(\varepsilon_j) = \varepsilon_j/c \quad (\text{A.21})$$

and therefore

$$0 \leq \psi^j(\varepsilon_j) \leq (c^{-1} + 1)\varepsilon_j \quad (\text{A.22})$$

For  $x > \varepsilon_j$ ,  $\psi^j(x)$  satisfies the equation

$$-1/2 \, d^2/dx^2 \psi^j(x) + \omega^2/2 \, x^2 \psi^j(x) = E_n^{\varepsilon_j} \psi^j(x) \quad , \quad (\text{A.23})$$

with the boundary conditions (A.20), (A.22) at  $x = \varepsilon_j$ . Eqs. (A.20), (A.22), (A.23) imply pointwise convergence of  $\psi^j(x + \varepsilon_j)$ ,  $x \in [0, \infty)$ , uniform on compact sets. Since for  $x$  large we have the bound

$$|\psi^j(x)| < a e^{-bx} \quad , \quad (\text{A.24})$$

this implies convergence of  $\psi^j(x)$  in  $L^2([0, \infty))$ , to a limit  $\psi_n$  satisfying eq.(A.23) with  $E_n \equiv \lim_j E_n^{\varepsilon_j}$  and  $\psi_n(0) = 0$ . By counting the number of zeros, it follows that all the eigenfunctions of  $\tilde{H}$ , the Hamiltonian with Dirichlet boundary conditions in  $x = 0$ , are obtained as limits of  $\psi_n^j$ . In order to prove eq.(A.24), since the wave functions  $\psi^j(x)$  are real and have a finite number of zeros, we can assume  $\psi^j(x) > 0$  for all  $j$ , for  $x$  sufficiently large; the Schrödinger equation (A.23) then implies, for  $x > L$ ,  $L$  sufficiently large

$$d^2/dx^2 \psi^j(x) \geq M \psi^j(x) > 0 \quad (\text{A.25})$$

with  $M = \inf_{j, x > L} (\omega x^2 - 2E_n^{\varepsilon_j})$ . Eq.(A.25) implies  $d/dx \psi^j(x) < 0$  for  $x > L$  and, by multiplication with  $d/dx \psi^j(x)$ ,

$$d/dx (\psi^j)^2 \leq M (\psi^j)^2$$

so that

$$\log \psi^j(x) < \log \psi^j(L) - M/2 \, x$$

and the result follows from convergence of  $\psi^j(L)$  for  $j \rightarrow \infty$ .

**A3.** We will consider now a system of two independent one dimensional harmonic oscillator in the state

$$\psi = \frac{1}{\sqrt{2}} (\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)) \quad ,$$

$\psi_0$  and  $\psi_1$  denoting respectively the ground state and the first excited state of the operator  $1/2(p^2 + \omega^2 x^2)$ . With the change of coordinates

$$x \equiv \frac{x_1 + x_2}{\sqrt{2}} \quad y \equiv \frac{x_1 - x_2}{\sqrt{2}} \quad (\text{A.26})$$

the Hamiltonian  $H = \frac{1}{2}(p_1^2 + p_2^2 + \omega x_1^2 + \omega x_2^2)$  becomes

$$H = -\frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{2} \frac{\partial^2}{\partial y^2} + \frac{1}{2} \omega^2 (x^2 + y^2) \equiv H_x + H_y$$

and the above state

$$\psi(x, y) = \frac{\omega}{\sqrt{\pi}} x e^{-\frac{\omega}{2}(x^2 + y^2)} = \psi_1(x) \psi_0(y) \quad (\text{A.27})$$

Since, in the new variables, the wave function is a product, we can solve separately the stochastic processes associated to  $\psi_1(x)$  and  $\psi_0(y)$  respectively. The  $y$  subsystem is in the ground state and therefore its evolution semigroup is  $A_y(t) = \psi_0(y)^{-1} e^{-(H_y - \omega/2)t} \psi_0(y)$ . On the other hand, the  $H_x$  subsystem is in the first excited state, so that Proposition A.2 applies and the associated semigroup can be written

$$A_x(t) = \frac{1}{\psi_1(x)} e^{-(\tilde{H}_x - E_1)t} \psi_1(x)$$

with  $\tilde{H}_x$  the Hamiltonian with Dirichlet boundary condition at  $x = 0$ . The semigroup associated to the stochastic process of the whole system is then

$$A(t) = \frac{1}{\psi(x, y)} e^{-(\tilde{H}_x - E_1)t} e^{-(H_y - \omega/2)t} \psi(x, y) \quad (\text{A.28})$$

The expectation of the position observable  $x_1(t)x_2(0)$  in Nelson Stochastic Mechanics can therefore be calculated as

$$(\psi, x_1(t) x_2(0) \psi)_{NM} = \frac{1}{2} \left( \psi(x, y), (x + y) e^{-(\tilde{H}_x - \tilde{E}_0)t} e^{-(H_y - \omega/2)t} (x - y) \psi(x, y) \right) \quad (\text{A.29})$$

The only non trivial term in the r.h.s. of eq.(A.29) is

$$\begin{aligned} & \left( \psi(x, y), x e^{-(\tilde{H}_x - \tilde{E}_0)t} e^{-(H_y - \omega/2)t} x \psi(x, y) \right) = \left( \psi_1(x), x e^{-(\tilde{H}_x - \tilde{E}_0)t} x \psi_1(x) \right) = \\ & = \sum_{n \text{ odd}} |c_n|^2 e^{-(n-1)\omega t} \quad , \end{aligned}$$

$c_n$  denoting the coefficients in the expansion of  $x\psi_1(x)$  in terms of the eigenfunctions of  $\tilde{H}_x$ , which can be written

$$c_n = \int_{-\infty}^{\infty} \tilde{\psi}_n^+(x) x \psi_1(x) dx = 2 \int_0^{\infty} x \psi_n(x) \psi_1(x) dx$$

where the eigenfunctions  $\tilde{\psi}_n^+(x)$ ,  $n \in \mathbf{N}$  odd, with eigenvalues  $\tilde{E}_n = (n+1/2)\omega$  are the even continuation in  $(-\infty, 0)$  of the eigenfunctions  $\psi_n(x)$  of the harmonic oscillator in  $x \in (0, \infty)$  (odd continuations also define eigenfunctions, with the same eigenvalues, but they are irrelevant for the above expansion). The other terms are easily calculated and the result is

$$(\psi, x_1(t)x_2(0)\psi)_{NM} = \frac{1}{2} \left( -\frac{e^{-\omega t}}{2\omega} + \sum_{n \text{ odd}} |c_n|^2 e^{-(n-1)\omega t} \right) . \quad (A.30)$$

The same expectation in QM is given by (see Section 3)

$$(\psi, x_1(t)x_2(0)\psi)_{QM} = \frac{\cos \omega t}{2\omega} .$$

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