FRACTIONAL SEMANTICS FOR CLASSICAL LOGIC

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Abstract. This article presents a new (multivalued) semantics for classical propositional logic. We begin by maximally extending the space of sequent proofs so as to admit proofs for *any* logical formula; then, we extract the new semantics by focusing on the axiomatic structure of proofs. In particular, the interpretation of a formula is given by the ratio between the number of identity axioms out of the total number of axioms occurring in any of its proofs. The outcome is an informational refinement of traditional Boolean semantics, obtained by breaking the symmetry between tautologies and contradictions.

§1. Introduction. The purpose of this article is to design and explore an alternative semantics for classical propositional logic set by pure proof-theoretic considerations. We call the new (multivalued) semantics "fractional" to refer to the fact that truth-values become elements of the set of rational numbers $\mathbb{Q}$ within the interval $[0, 1]$. The basic idea is that truth-values exhaust their function in decorating axioms and rules of a suitable classical sequent calculus: they keep record along proofs of the number of occurrences of *identity axioms*, which encode the primitive logical fact that anything implies itself. Since (*i*) the set of formulas interpreted by the value 1 coincides with the set of classical tautologies and (*ii*) we refer exclusively to a proof system which proves to be sound and complete with respect to the set of classically valid sequents, we think it is legitimate to present our fractional semantics as a semantics for *classical* logic.

In particular, we focus on the sequent system $\mathsf{GS4}$ which is the right-sided version of Kleene's $\mathsf{G4}$ (see Figure 1) [10, 11, 20]. The choice of concentrating on right-sided sequents is in the interests of simplicity and clarity to the extent that our results can be straightforwardly applied to $\mathsf{G4}$. The $\mathsf{GS4}$ calculus is here maximally extended to the system $\overline{\overline{\mathsf{GS4}}}$ by adding the complementary axiom schema which enables the introduction of whatsoever clause $\vdash \Delta$, provided that no pair of dual literals occurs in Δ [3, 9, 17, 19, 21]. The system $\overline{\overline{\mathsf{GS4}}}$ is deductively trivial as it proves any sequent and it satisfies cut-elimination à la Gentzen to the extent that its axioms introduce only clauses.

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Axiom:

$$\dfrac{}{\vdash \Gamma, p, p^{\perp}}\, ax.$$

Logical rules:

$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\, \vee \qquad \dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\, \wedge$$

Fig. 1. The $\mathsf{GS4}$ sequent calculus

Like $\mathsf{GS4}$, the system $\overline{\overline{\mathsf{GS4}}}$ is characterized by the invertibility of its logical rules: for any logical rule, the provability of the conclusion implies the provability of (each one of) its premise(s). This property yields two specific features of the calculus which prove crucial for our discourse:

(i) any two cut-free proofs of the same sequent display the *same* top-sequents (cfr. Theorem 2.9);

(ii) if $\vdash \Gamma_1, \vdash \Gamma_2, \ldots, \vdash \Gamma_n$ are the top-sequents of a $\overline{\overline{\mathsf{GS4}}}$ proof ending in $\vdash A$, then $\bigvee \Gamma_1 \wedge \bigvee \Gamma_2 \wedge \cdots \wedge \bigvee \Gamma_n$ is equivalent to A. In other words, since $\overline{\overline{\mathsf{GS4}}}$ proves anything, it provides a general algorithm to decompose any logical formula into an equivalent formula in conjunctive normal form.

Specifically, we are in position to interpret any formula A as the ratio between the number of *identity* top-sequents—i.e., sequents introduced by the standard axiom and so displaying a pair of dual axioms—out of the *total* number of top-sequents in any of its $\overline{\overline{\mathsf{GS4}}}$ proofs. The upshot is a multivalued interpretation function $[\![\,\cdot\,]\!]$ whose range is the set of rational numbers comprised in the interval [0, 1].

EXAMPLE 1.1. *Take the* $\overline{\overline{\mathsf{GS4}}}$*-proof of the truth-functional contingency* $(r \wedge q) \vee (q^{\perp} \wedge t)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash r, q^{\perp}}\,\overline{ax}. \qquad \dfrac{}{\vdash q, q^{\perp}}\, ax.}{\vdash r \wedge q, q^{\perp}}\,\wedge \qquad \dfrac{\dfrac{}{\vdash r, t}\,\overline{ax}. \qquad \dfrac{}{\vdash q, t}\,\overline{ax}.}{\vdash r \wedge q, t}\,\wedge}{\vdash r \wedge q, q^{\perp} \wedge t}\,\wedge}{\vdash (r \wedge q) \vee (q^{\perp} \wedge t)}\,\vee$$

The proof contains one identity axiom out of four axioms in total and so $[\![\,(r \wedge q) \vee (q^{\perp} \wedge t)\,]\!] = \dfrac{1}{4} = 0.25.$

What exactly do we want to accomplish? The beginner in logic, very early on, learns that a deductive system is designed to be sound and complete with respect to a well-determined set of true formulas. Which formulas have to count as true are set by a semantic structure fixed in advance. In this article, we explore the merits and potentialities of a reverse methodology, according to which proofs gain the status of semantic sources by themselves as the truth-values of formulas are established through derivations. More pointedly, our approach may be articulated in three stages:

(i) we extend the space of sequent proofs so as to include proofs for *any* sequent;
(ii) we extract a *multivalued* interpretation function by surveying the axiomatic structure of proofs;
(iii) we establish that our interpretation function is semantically adequate by showing that any two proofs of the *same* formula A confer to A the *same* interpretation.

The lesson, then, is this: the deductive engine of classical logic in the backward construction of proofs can be regimented in such a way as to measure the quantity of identities in a logical formula. The locution "quantity of identities" carries here a precise logical meaning: if $[\![A]\!] = q$, then

- there are $m, n \in \mathbb{N}$ such that $\dfrac{m}{n} = q$;
- the *conjunctive normal form* of A—unfolded by means a $\overline{\overline{\mathsf{GS4}}}$-proof of A—contains m tautological conjuncts out of n conjuncts in total. In the previous example,
$$(r \vee q^{\perp}) \wedge (q \vee q^{\perp}) \wedge (r \vee t) \wedge (q \vee t)$$
exhibits one tautological conjunct out of four in total.

Fractional semantics may be understood as an informational refinement with respect to the standard Boolean view. It tracks the intuition that the distinction between tautologies and nontautologies (or, if you like, between theorems and non-theorems) is too rigid and it can be relaxed in a conservative way. As already remarked, for any formula A:

$$[\![A]\!] = 1 \text{ if and only if } A \text{ is a tautology.}$$

Yet, the symmetry between tautologies and contradictions is broken, as two contradictions may now receive two different semantic values.

EXAMPLE 1.2. *The contradictions $p \wedge p^{\perp}$ and $(p \wedge p^{\perp}) \wedge (p \vee p^{\perp})$ in the system $\overline{\overline{\mathsf{GS4}}}$ (see Definition 2.1) have these proofs*:

$$\dfrac{\dfrac{}{\vdash p}\,\overline{ax}. \qquad \dfrac{}{\vdash p^{\perp}}\,\overline{ax}.}{\vdash p \wedge p^{\perp}}\,\wedge \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash p}\,\overline{ax}. \quad \dfrac{}{\vdash p^{\perp}}\,\overline{ax}.}{\vdash p \wedge p^{\perp}}\,\wedge \qquad \dfrac{\dfrac{}{\vdash p, p^{\perp}}\,ax.}{\vdash p \vee p^{\perp}}\,\vee}{\vdash (p \wedge p^{\perp}) \wedge (p \vee p^{\perp})}\,\wedge$$

In terms of Boolean semantics, the foregoing contradictions are described as identifying the same *logical entity, since they get the same value* 0 *for any valuation. Nonetheless, in the contradiction on the right one conjunct out of three is a tautology, so there is an intuitive pull that such a contradiction as a whole is "less false" (or, if you prefer, "more true") than the contradiction on the left, in which no tautology is present. In our setting, such a difference is captured by $[\![p \wedge p^{\perp}]\!] = 0$ and $[\![(p \wedge p^{\perp}) \wedge (p \vee p^{\perp})]\!] = 0.\overline{3}$.*

This is the roadmap. In §2, we present the logical system $\mathsf{GS4}$ as our *ur*-calculus as well as its extension $\overline{\overline{\mathsf{GS4}}}$ with a further axiom introducing any (consistent) multiset of atoms. We show that the calculus $\overline{\overline{\mathsf{GS4}}}$, though inconsistent, is a useful device for decomposing logical formulas into an equivalent set of disjunctive clauses. §3 describes the multivalued system $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$ driven by proof-theoretic considerations *modulo* $\overline{\overline{\mathsf{GS4}}}$. The system $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$ serves as a decorated version of $\overline{\overline{\mathsf{GS4}}}$'s sequent calculus, whereby decorations keep track of the number of identity axioms in the proof, out of the total number of top-sequents. In §4, we map out a new class of supraclassical logics—that we term *bounded supraclassical logics*—achieved by restricting and constraining the truth-values of their theorems. Moreover, we prove that bounded consequence relations satisfy reflexivity, monotonicity, and structurality (although not always transitivity). Finally, §6 contains some concluding remarks about our perspective, providing less "endogenous" motivations for its adoption. We sketch how it may be regarded as a slide to the theory of belief revision, the project of a proof-theoretic semantics for classical logic, as well as a gainful handling of truth-theoretic paradoxes.

§2. Decomposing by proving.

2.1. Notational preliminaries. We shall be concerned with the system GS4 as displayed in Figure 1 [10, 13, 20]. In GS4 the two structural rules weakening and contraction do not appear explicitly, due to their absorption in the other rules [14]. More pointedly, weakening rule is operative via the generalized formulation of the axiom, whereas contraction rule is implicit in the additive, context-sharing version of the conjunction rule.

Typically, in right-sided sequent systems, negation $(\cdot)^{\perp}$ is attached to atomic sentences and it extends to compound formulas by De Morgan duality. This means that we shall be dealing with a language made up of an extended set of atoms $\mathcal{AT} = \{p, q, \ldots\} \cup \{p^{\perp}, q^{\perp}, \ldots\}$ and only two binary operators $\wedge$ and $\vee$; the implication operator can be recovered by defining it as usual, i.e., $A \to B ::= A^{\perp} \vee B$. In what follows $\mathcal{F}$ denotes the set of all formulas.

Lowercase Greek letters $\pi, \rho, \ldots$ stand for sequent proofs, while uppercase Greek letters $\Gamma, \Delta, \ldots$ indicate logical contexts, alias finite multisets of formulas $[A_1, A_2, \ldots, A_n]$. To lighten notation, we write Γ, A to mean the multiset $\Gamma \uplus [A]$. Moreover, for any context $\Gamma = A_1, A_2, \ldots, A_n$, we denote by $\bigwedge \Gamma$ and $\bigvee \Gamma$ the conjunction of n arguments $A_1 \wedge A_2 \wedge \cdots \wedge A_n$ and the disjunction of n arguments $A_1 \vee A_2 \vee \cdots \vee A_n$, respectively. The negation of a whole context Γ is the multiset $\Gamma^{\perp} = [A^{\perp} \mid A \in \Gamma]$.

Finally, we demand that the axiom rule introduces exclusively *clauses*, namely sequents involving only atomic sentences. The reason for this restriction and for the very adoption of GS4 as logical framework will emerge clearly in the next section.

***2.2. From* GS4 *to* $\overline{\overline{\mathsf{GS4}}}$.** The system GS4 can be maximally extended into the trivial system $\overline{\overline{\mathsf{GS4}}}$ defined as follows.

DEFINITION 2.1 (The sequent system $\overline{\overline{\mathsf{GS4}}}$). *The calculus* $\overline{\overline{\mathsf{GS4}}}$ *is obtained from* GS4 *by adding the* complementary axiom $\dfrac{}{\vdash \Delta}\,\overline{ax}$. *which allows one to introduce any multiset whatsoever of atoms* Δ *with the proviso that* $A^{\perp} \notin \Delta$, *if* $A \in \Delta$.

REMARK 2.2. *Any sequent turns out to be provable in* $\overline{\overline{\mathsf{GS4}}}$ *by dint of the application of the logical rules in their bottom-up reading till the leaves of the proof-tree are all clauses.*

EXAMPLE 2.3. *The truth-functional contingency* $(r \wedge q) \vee (q^{\perp} \wedge t)$ *is derivable in* $\overline{\overline{\mathsf{GS4}}}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash r, q^{\perp}}\,ax. \qquad \dfrac{}{\vdash q, q^{\perp}}\,ax.}{\vdash r \wedge q, q^{\perp}}\,\wedge \qquad \dfrac{\dfrac{}{\vdash r, t}\,ax. \qquad \dfrac{}{\vdash q, t}\,ax.}{\vdash r \wedge q, t}\,\wedge}{\vdash r \wedge q, q^{\perp} \wedge t}\,\wedge}{\vdash (r \wedge q) \vee (q^{\perp} \wedge t)}\,\vee$$

DEFINITION 2.4 (*Identity* and *complementary* top-sequents). *The top-sequents of a* $\overline{\overline{\mathsf{GS4}}}$*-proof* π *are the leaves of the proof-tree, i.e., those sequents directly introduced as instances of the axiom rules.* $\mathsf{Top}^1(\pi)$ (*resp.* $\mathsf{top}^0(\pi)$) *denotes the multiset of all and only* π*'s top-sequents introduced by an identity* (*resp. complementary*) *axiom. Moreover,* $\mathsf{top}(\pi)$ *refers to the whole multiset of* π*'s top-sequents, i.e.,* $\mathsf{top}(\pi) = \mathsf{top}^1(\pi) \uplus \mathsf{top}^0(\pi)$.

From Definition 2.4, we immediately get the following remark:

REMARK 2.5. *Let* π *be a* $\overline{\overline{\mathsf{GS4}}}$*-proof ending with the sequent* $\vdash \Gamma$ *and such that* $\mathsf{top}(\pi) = [\vdash \Gamma_1, \ldots, \vdash \Gamma_n]$. *One readily sees that the two formulas* $\bigvee \Gamma$ *and* $\bigvee \Gamma_1 \wedge \cdots \wedge \bigvee \Gamma_n$ *are logically equivalent. Such an equivalence is an immediate consequence of the invertibility of* $\overline{\overline{\mathsf{GS4}}}$*'s logical rules, the property telling that the provability of their conclusion implies the provability of their premise*(*s*) [1].

EXAMPLE 2.6. *Let* π *be the proof in Example 2.3. In this case* $\mathsf{top}^0(\pi) = [\vdash r, q^{\perp}\,;\, \vdash r, t\,;\, \vdash q, t]$, $\mathsf{top}^1(\pi) = [\vdash q, q^{\perp}]$, *thus* $\mathsf{top}(\pi) = [\vdash r, q^{\perp}\,;\, \vdash r, t\,;\, \vdash q, t\,;\, \vdash q, q^{\perp}]$. *In accordance with the previous remark, we have that the formula* $(r \wedge q) \vee (q^{\perp} \wedge t)$ *is actually equivalent to* $(r \vee q^{\perp}) \wedge (r \vee t) \wedge (q \vee t) \wedge (q \vee q^{\perp})$.

Remark 2.5 tells us that $\overline{\overline{\mathsf{GS4}}}$ can be employed and justified as a syntactic device in spite of its deductive triviality. Indeed, this formalism serves the purpose of framing an algorithm for decomposing each sequent into an equivalent set of clauses or, equivalently, for turning any formula into its conjunctive normal form.

REMARK 2.7. *Standardly, the* height $h(\pi)$ *of a tree-like sequent proof* π *is taken to be the number of sequents featuring in one of its maximally long branches. It is easy to see, then, that the height* $h(\pi)$ *of any* $\overline{\overline{\mathsf{GS4}}}$*-proof* π *equals the number of occurrences of connectives in its end-sequent, plus one.*

LEMMA 2.8. *For any* $\overline{\overline{\mathsf{GS4}}}$*-proof* π *of* $\vdash \Gamma, A@B$, *with* $@ \in \{\wedge, \vee\}$, *there exists a proof* ρ *of the same sequent, and such that*:

- ρ*'s last rule is the specific application of the* $@$*-rule introducing the formula* $A@B$,
- $\mathsf{top}(\pi) = \mathsf{top}(\rho)$.

Proof. By induction over the height of π. It suffices to show that it is possibile to commute downwards the specific application of the $@$-rule for $A@B$ till it becomes the very last rule. □

To our knowledge, the next theorem has been claimed for the first time in [13] with respect to $\mathsf{G3cp}$, alias the two-sided version of $\mathsf{GS4}$. The proof provided there consists of a single phrase ("by nothing that successive application of any two logical rules in $\mathsf{G3cp}$ commutes" [13, p. 51]). Such a proof, as it stands, is too evasive to exempt us from supplying an alternative demonstration, which can be straightforwardly applied to $\mathsf{G3cp}$ as well.

THEOREM 2.9 (unicity). *If π and ρ are two $\overline{\overline{\mathsf{GS4}}}$-proofs with the same end-sequent, then* $\mathsf{top}(\pi) = \mathsf{top}(\rho)$.

Proof. We proceed by induction over $h(\pi)$. By Remark 2.7, we know that $h(\pi) = h(\rho)$. *Base case*: If $h(\pi) = h(\rho) = 1$, then π and ρ are straightforward instances of one of the two kinds of axioms and so they are the very same proof.
Inductive step: we distinguish two cases depending on π's last rule.

- ($\vee$-rule) Let π be the proof ending with $\vdash \Gamma, A \vee B$ whose last rule is the $\vee$-application forming $A \vee B$:

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A, B\end{array}}{\vdash \Gamma, A \vee B}\ \vee.$$

  Consider now the proof ρ also ending with $\vdash \Gamma, A \vee B$. By Lemma 2.8, it can be rewritten into a proof ρ':

$$\dfrac{\begin{array}{c}\rho'\\ \vdots\\ \vdash \Gamma, A, B\end{array}}{\vdash \Gamma, A \vee B}\ \vee$$

  such that $\mathsf{top}(\rho') = \mathsf{top}(\rho)$. Let ρ'_1 (resp. π_1) the subproof of ρ' (resp. π) ending with $\vdash \Gamma, A, B$. By Remark 2.7, we have that $h(\pi_1) = h(\rho'_1)$ and so, by inductive hypothesis, $\mathsf{top}(\pi') = \mathsf{top}(\rho'_1)$, from which it follows that $\mathsf{top}(\pi) = \mathsf{top}(\rho') = \mathsf{top}(\rho)$.
- Similar to the previous case. □

Theorem 2.9 enables us to broaden quite painlessly our notation to apply the functions $\mathsf{top}^1(\cdot)$, $\mathsf{top}^0(\cdot)$, and $\mathsf{top}(\cdot)$ directly to logical contexts. We shall directly write $\mathsf{top}^1(\Gamma)$ (resp. $\mathsf{top}^0(\Gamma)$) to indicate the identity (resp. complementary) top-sequents occurring in *any* $\overline{\overline{\mathsf{GS4}}}$-proof of $\vdash \Gamma$. According to Definition 2.4, $\mathsf{top}(\Gamma) = \mathsf{top}^1(\Gamma) \uplus \mathsf{top}^0(\Gamma)$.

THEOREM 2.10. *For any context Γ and any two formulas A and B*:

(*i*) $\mathsf{top}(\Gamma, A \vee B) = \mathsf{top}(\Gamma, A, B)$,
(*ii*) $\mathsf{top}(\Gamma, A \wedge B) = \mathsf{top}(\Gamma, A) \uplus \mathsf{top}(\Gamma, B)$.

Proof. Easy, by combining Lemma 2.8 and Theorem 2.9. □

§3. The multi-valued system mv-$\overline{\overline{\mathsf{GS4}}}$. The concern is now to investigate the multi-valued semantics induced by $\overline{\overline{\mathsf{GS4}}}$ through simple proof-theoretic considerations. According to this semantic account, the interpretation of a formula A, relative to one of its proofs π, is determined by the ratio between π's identity axioms and the totality of π's axioms. By Theorem 2.9, the litmus test of our semantics lies in its ability to provide an interpretation *invariant* for *any* proof π of A.

Let us denote with $\#\mathsf{top}(\Gamma)$, $\#\mathsf{top}^1(\Gamma)$, and $\#\mathsf{top}^0(\Gamma)$, the cardinality of the multisets $\mathsf{top}(\Gamma)$, $\mathsf{top}^1(\Gamma)$, and $\mathsf{top}^0(\Gamma)$, respectively. Moreover, we stipulate that $\mathbb{Q}^* = [0, 1] \cap \mathbb{Q}$, i.e., $\mathbb{Q}^*$ is the set of the rational numbers comprised in the interval $[0, 1]$. The next definition captures the desired semantic property more formally.

DEFINITION 3.1 (Fractional interpretation). *For any formula $A \in \mathcal{F}$, the interpretation mapping $[\![\cdot]\!] : \mathcal{F} \mapsto \mathbb{Q}^*$ is defined as* $[\![A]\!] = \dfrac{\#\mathsf{top}^1(A)}{\#\mathsf{top}(A)}$.

The mv-$\overline{\overline{\mathsf{GS4}}}$ sequent system comes as a decorated version of $\overline{\overline{\mathsf{GS4}}}$ specifically designed to keep record, along proofs, of the number of identity axioms out of the total number of top-sequents (cfr. Figure 2). In this manner, mv-$\overline{\overline{\mathsf{GS4}}}$ sequents come indexed by *ordered pairs of natural numbers*. Specifically:

- identity axioms are indexed with the ordered pair $\langle 1, 1\rangle$ to mean that there is 1 identity axiom out of 1 axiom in total;
- complementary axioms are instead indexed with $\langle 0, 1\rangle$ to express the presence of 0 identity axioms out of 1 axiom in total.

Accordingly, if the sequent $\vdash^{m}_{n} \Gamma$ is provable in mv-$\overline{\overline{\mathsf{GS4}}}$, then:

- any $\overline{\overline{\mathsf{GS4}}}$-proof π of $\vdash \Gamma$ is such that $\#\mathsf{top}^1(\pi) = m$ and $\#\mathsf{top}(\pi) = n$;
- $[\![\bigvee \Gamma]\!] = \dfrac{m}{n} \in \mathbb{Q}^*$.

EXAMPLE 3.2. *We report below the* mv-$\overline{\overline{\mathsf{GS4}}}$ *version of the* $\overline{\overline{\mathsf{GS4}}}$*-proof in earlier Example 2.3.*

Axioms:

$$\dfrac{}{\vdash^{1}_{1} \Gamma, p, p^{\perp}}\, ax. \qquad \dfrac{}{\vdash^{0}_{1} \Delta}\, \overline{ax}.$$

Logical rules:

$$\dfrac{\vdash^{m}_{n} \Gamma, A, B}{\vdash^{m}_{n} \Gamma, A \vee B}\,\vee \qquad \dfrac{\vdash^{m_1}_{n_1} \Gamma, A \qquad \vdash^{m_2}_{n_2} \Gamma, B}{\vdash^{m_1+m_2}_{n_1+n_2} \Gamma, A \wedge B}\,\wedge$$

Fig. 2. The mv-$\overline{\overline{\mathsf{GS4}}}$ sequent calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} r, q^{\perp}}\,\overline{ax}. \qquad \dfrac{}{\vdash^{1}_{1} q, q^{\perp}}\,ax.}{\vdash^{1}_{2} r \wedge q, q^{\perp}}\,\wedge \qquad \dfrac{\dfrac{}{\vdash^{0}_{1} r, t}\,\overline{ax}. \qquad \dfrac{}{\vdash^{0}_{1} q, t}\,\overline{ax}.}{\vdash^{0}_{2} r \wedge q, t}\,\wedge}{\vdash^{1}_{4} r \wedge q, q^{\perp} \wedge t}\,\wedge}{\vdash^{1}_{4} (r \wedge q) \vee (q^{\perp} \wedge t)}\,\vee$$

According to Definition 3.1, $[\![(r \wedge q) \vee (q^{\perp} \wedge t)]\!] = 0.25$.

LEMMA 3.3 (Basic semantic properties). *Let* $@ \in \{\wedge, \vee\}$. *For any three formulas* A, B, *and* C, *the fractional interpretation mapping* $[\![\cdot]\!]$ *satisfies these properties*:

(*i*) $[\![A@B]\!] = [\![B@A]\!]$

(*ii*) $[\![(A@B)@C]\!] = [\![A@(B@C)]\!]$

(*iii*) $[\![A \vee (B \wedge C)]\!] = [\![(A \vee B) \wedge (A \vee C)]\!]$.

Proof. Properties (*i*) and (*ii*) come straightforwardly. Property (*iii*) is established by observing that the two proofs below convey the same semantic evaluation, and so applying Theorem 2.9.

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \vdash^{m_1}_{n_1} A, B\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \vdash^{m_2}_{n_2} A, C\end{matrix}}{\vdash^{m_1+m_2}_{n_1+n_2} A, B \wedge C}\,\wedge}{\vdash^{m_1+m_2}_{n_1+n_2} A \vee (B \wedge C)}\,\vee \qquad\qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \vdash^{m_1}_{n_1} A, B\end{matrix}}{\vdash^{m_1}_{n_1} A \vee B}\,\vee \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \vdash^{m_2}_{n_2} A, C\end{matrix}}{\vdash^{m_2}_{n_2} A \vee C}\,\vee}{\vdash^{m_1+m_2}_{n_1+n_2} (A \vee B) \wedge (A \vee C)}\,\wedge$$

□

REMARK 3.4. *Semantically speaking, conjunction doesn't necessarily distribute over disjunction. Example*: $[\![\, p \wedge (q \vee q^{\perp})\,]\!] \neq [\![\,(p \wedge q) \vee (p \wedge q^{\perp})\,]\!]$.

For $q \in \mathbb{Q}^*$, let $\mathcal{F}_q ::= \{A \mid [\![A]\!] = q\}$. Clearly, $\mathcal{F}_1$ exactly coincides with the set of tautologies, albeit the set $\mathcal{F}_0$ doesn't coincide with the set of contradictions. Indeed, from Definition 1 it follows immediately that two classical contradictions are amenable to different fractional interpretations. Roughly speaking, such a semantic divergence happens because within a contradiction of the form $A \wedge A^{\perp}$ either the conjunct A or its dual $A^{\perp}$ may "contain" some identities.

EXAMPLE 3.5. *Take* $p \wedge p^{\perp}$ *and* $(p \wedge p^{\perp}) \wedge (p \vee p^{\perp})$.

- $[\![\, p \wedge p^{\perp}\,]\!] = 0$

$$\dfrac{\dfrac{}{\vdash^{0}_{1} p}\,\overline{ax}. \qquad \dfrac{}{\vdash^{0}_{1} p^{\perp}}\,\overline{ax}.}{\vdash^{0}_{2} p \wedge p^{\perp}}\,\wedge$$

- $[\![\,(p \wedge p^{\perp}) \wedge (p \vee p^{\perp})\,]\!] = 0.\overline{3}$

$$\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} p}\,\overline{ax}. \qquad \dfrac{}{\vdash^{0}_{1} p^{\perp}}\,\overline{ax}.}{\vdash^{0}_{2} p \wedge p^{\perp}}\,\wedge \qquad \dfrac{\dfrac{}{\vdash^{1}_{1} p, p^{\perp}}\,ax.}{\vdash^{1}_{1} p \vee p^{\perp}}\,\vee}{\vdash^{1}_{3} (p \wedge p^{\perp}) \wedge (p \vee p^{\perp})}\,\wedge$$

Unlike the identity-free formula $p \wedge p^{\perp}$, the formula $(p \wedge p^{\perp}) \wedge (p \vee p^{\perp})$ contains the identity $p \vee p^{\perp}$ along with two nonidentical components p and $p^{\perp}$. Therefore, the final sequent comes decorated with $\langle 1, 3 \rangle$ as meaning that this formula contains one identical component out of three components in total, and so the assigned semantic value turns out to be $0.\overline{3}$.

REMARK 3.6. *In this article, we take the collection of axioms* $\mathsf{top}(A)$ *as a multiset so that its cardinality function* $\#\mathsf{top}(A)$ *counts possible repetitions of the same axiom. This entails, for instance, the fact that*

$$[\![(p \vee p^{\perp}) \wedge q]\!] \neq [\![(p \vee p^{\perp}) \wedge q \wedge q]\!].$$

One could object that, in the domain of classical logic, repetitions should not matter and soundness and completeness should refer to optimal *conjunctive normal forms, i.e., those with a minimal number of conjuncts. Of course, such a requirement can be met by taking* $\mathsf{top}(A)$ *to be a set, instead of a multiset, because this variant does not affect our achievements. However, the objection suffers from circularity inasmuch as it refers back to Boolean semantics, whereas our aim here is to manufacture a semantics based on proofs and independent of any prior semantical intuition.*

THEOREM 3.7. *For any two contexts* Γ, A *and* Γ, B:

$$[\![\Gamma, A \wedge B]\!] = \frac{\#\mathsf{top}^1(\Gamma, A) + \#\mathsf{top}^1(\Gamma, B)}{\#\mathsf{top}(\Gamma, A) + \#\mathsf{top}(\Gamma, B)}.$$

Proof. Straightforwardly by Theorem 2.10(*ii*). □

REMARK 3.8 (Non-truth functionality). *In the system* $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$*, disjunction and conjunction aren't truth-functional connectives.*

- (Disjunction). *Consider* p, $p^{\perp}$ *and* $p \wedge p^{\perp}$. *We have* $[\![p]\!] = [\![p^{\perp}]\!] = [\![p \wedge p^{\perp}]\!] = 0$, *but* $[\![p \vee p^{\perp}]\!] = 1$ *and* $[\![p \vee (p \wedge p^{\perp})]\!] = 0.5$.
- (Conjunction). *By Theorem 3.7, non-truth functionality follows by considering two formulas* A *and* B *such that* $[\![A]\!] = [\![B]\!]$, *but* $\#\mathsf{top}(A) \neq \#\mathsf{top}(B)$. *Such a divergence is due to the fact that the* $\frac{m_1}{n_1} \oplus \frac{m_2}{n_2} ::= \frac{m_1 + m_2}{n_1 + n_2}$ *is not an operation on* $\mathbb{Q}$. *To take an easy example,* $[\![p]\!] = [\![p \wedge p]\!]$, *but* $[\![p \wedge (p \vee p^{\perp})]\!] \neq [\![(p \wedge p) \wedge (p \vee p^{\perp})]\!]$.

As we said, negation $(\cdot)^{\perp}$ *is not a connective in our language. When it is treated as a meta-connective, its non-truth functionality comes to the fore by taking* p *and* $p \wedge p^{\perp}$, *for instance. Although* $[\![p]\!] = [\![p \wedge p^{\perp}]\!] = 0$, *the interpretation of their negations diverges since* $[\![p^{\perp}]\!] = 0$ *and* $[\![(p \wedge p^{\perp})^{\perp}]\!] = [\![p^{\perp} \vee p]\!] = 1$.

DEFINITION 3.9 (Formulas p^n and $\top_p^n$). *For each atom* p *and any* $n \in \mathbb{N} \backslash \{0\}$ *the formulas* p^n *and* $\top_p^n$ *are recursively defined as follows*:

- $p^1 \equiv p$ *and* $p^n \equiv p^{n-1} \wedge p$.
- $\top_p^1 \equiv p \vee p^{\perp}$ *and* $\top_p^n \equiv \top_p^{n-1} \wedge \top_p^1$.

EXAMPLE 3.10.

- $p^3 \equiv p \wedge p \wedge p$
- $\top^3_p \equiv (p \vee p^\perp) \wedge (p \vee p^\perp) \wedge (p \vee p^\perp)$

The next theorem reveals that any semantic value comprised in $\mathbb{Q}^*$ is the interpretation of some formulas of the language.

THEOREM 3.11. *For any $q \in \mathbb{Q}^*$, there is a formula A such that $[\![A]\!] = q$.*

Proof. The proof consists in showing that for any $m, n \in \mathbb{N}$ such that $n \geqslant m$ and $n \neq 0$ there is a formula A such that the sequent $\vdash^{m}_{n} A$ is provable in mv-$\overline{\overline{\mathsf{GS4}}}$. To this aim, notice that, for each atom p:

$$\#\mathsf{top}^1(\top^n_p) = \#\mathsf{top}(\top^n_p) = n \quad \text{and} \quad \#\mathsf{top}^0(p^n) = \#\mathsf{top}(p^n) = n.$$

Now $\mathsf{top}^1(\top^m_p \wedge p^{n-m}) = m$ and $\mathsf{top}(\top^m_p \wedge p^{n-m}) = n$, hence $[\![\top^m_p \wedge p^{n-m}]\!] = \dfrac{m}{n}$. □

EXAMPLE 3.12. *Take the formula $\top^1_p \wedge p^2 \equiv (p \vee p^\perp) \wedge (p \wedge p)$. The proof below establishes that $[\![\top^1_p \wedge p^2]\!] = 0.\overline{3}$.*

$$\dfrac{\dfrac{\dfrac{}{\vdash^1_1 p, p^\perp}\,ax.}{\vdash^1_1 p \vee p^\perp}\,\vee \qquad \dfrac{\dfrac{}{\vdash^0_1 p}\,\overline{ax}. \qquad \dfrac{}{\vdash^0_1 p}\,\overline{ax}.}{\vdash^0_2 p \wedge p}\,\wedge}{\vdash^1_3 (p \vee p^\perp) \wedge (p \wedge p)}\,\wedge$$

***3.1. Why* $\mathsf{GS4}$.** The reader can now better understand why the sequent system $\mathsf{GS4}$ is the best possible one for our purposes. There are indeed two main reasons. The first one is technical to the extent that a sequent formulation with a multiplicative conjunction rule would not allow us to prove the unicity of interpretations as guaranteed by Theorem 2.9. For instance, these two cut-free proofs of the same formula yield two different interpretations:

$$\dfrac{\dfrac{\dfrac{}{\vdash^0_1 q^\perp, p}\,\overline{ax}. \qquad \dfrac{}{\vdash^0_1 q}\,\overline{ax}.}{\vdash^0_2 q^\perp, p \wedge q}\,(\text{mult.} \wedge)}{\vdash^0_2 q^\perp \vee (p \wedge q)}\,\vee \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash^0_1 p}\,\overline{ax}. \qquad \dfrac{}{\vdash^1_1 q, q^\perp}\,ax.}{\vdash^1_2 q^\perp, p \wedge q}\,(\text{mult.} \wedge)}{\vdash^1_2 q^\perp \vee (p \wedge q)}\,\vee$$

The second reason for adopting $\mathsf{GS4}$ is that its generalized axiom and conjunction rule are so formulated as to be able to *maximize* the number of atoms in axiomatic clauses and hence the number of identity top-sequents in $\overline{\overline{\mathsf{GS4}}}$-proofs. This is to say, for any formula A the value $[\![A]\!]$ which $\overline{\overline{\mathsf{GS4}}}$ assigns to A is always the maximal one with respect to that assigned via proofs of alternative sequent formulations such as Gentzen's original LK [7]. This specific feature of $\mathsf{GS4}$ is required for the *correctness* of interpretations. As illustrative example, we may look at these two proofs:

$$\dfrac{\dfrac{\dfrac{}{\vdash^0_1 q^\perp, p}\,\overline{ax}. \qquad \dfrac{}{\vdash^0_1 q}\,\overline{ax}.}{\vdash^0_2 q^\perp, p \wedge q}\,(\text{mult.} \wedge)}{\vdash^0_2 q^\perp \vee (p \wedge q)}\,\vee \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash^0_1 q^\perp, p}\,\overline{ax}. \qquad \dfrac{}{\vdash^1_1 q, q^\perp}\,ax.}{\vdash^1_2 q^\perp, p \wedge q}\,(\text{add.} \wedge)}{\vdash^1_2 q^\perp \vee (p \wedge q)}\,\vee$$

The decorated proof on the left resorts to the multiplicative (i.e., context-mixing) version of the conjunction rule and it displays 0 identity top-sequents out of 2 top-sequents in total. The proof on the right is instead a $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$-proof with 1 identity top-sequent out of 2 top-sequents in total. Clearly, the *correct* semantic interpretation of $q^{\perp} \vee (p \wedge q)$ — whose conjunctive normal form is $(q^{\perp} \vee p) \wedge (q \vee q^{\perp})$ — is not expected to be 0, but 0.5. In summary, then: Gentzen systems including a multiplicative version of the conjunction rule may produce different interpretations of the same formulas and, obviously, at most only one of them can be the correct one. Finally, it is worth observing that $\mathsf{G4}$'s logical rules (and so $\overline{\overline{\mathsf{GS4}}}$'s rules) when taken in their backward reading, i.e., as decomposition rules, corresponds to tableau rules for classical logic [5].

§4. Bounded supraclassical logics. Supraclassical logics are extensions of classical propositional logic via proper axioms. They have been introduced by David Makinson with the purpose of "bridging the gap" between classical and nonmonotonic reasoning through a "logical continuous" [12, 15]. In this section we direct attention to an alternative way of deductively expanding classical logic, whereby one considers proper subsystems of $\overline{\overline{\mathsf{GS4}}}$ obtained by establishing a lower bound to the fractional truth-values of logical formulas. We term these systems *bounded supraclassical logics*.

DEFINITION 4.1 (Bounded supraclassical systems). *For any $q \in \mathbb{Q}^*$, we indicate with $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_q$ the system whose set of theorems is $\{A \mid [\![A]\!] \geqslant q\}$.*

EXAMPLE 4.2. *From Definition 4.1, $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_{0.\overline{3}}$'s theorems are all those formulas A such that $[\![A]\!] \geqslant 0.\overline{3}$. For instance, the formula $(p \vee p^{\perp}) \wedge q \wedge r$ is a theorem of $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_{0.\overline{3}}$, whilst $(p \vee p^{\perp}) \wedge q \wedge r \wedge s$ is not because $[\![(p \vee p^{\perp}) \wedge q \wedge r \wedge s]\!] = 0.25$.*

REMARK 4.3. *By Theorem 3.11, if $q, q' \in \mathbb{Q}^*$ and $q' < q$, then $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_q \subset \mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_{q'}$. This means that bounded supraclassical logics form a dense hierarchy of supraclassical systems whose extremes are $\mathsf{GS4}$ (all and only the classical tautologies are provable) and $\overline{\overline{\mathsf{GS4}}}$ (anything is provable) (see Figure 3).*

From now on, $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_q$ will refer to a *generic* bounded supraclassical system. For the sake of simplicity, by saying that the sequent $\vdash \Gamma$ is valid in $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_q$ we mean the provability in $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$ of the decorated sequent $\vdash^{m}_{n} \Gamma$ with $\dfrac{m}{n} \geqslant q$.

4.1. Weakening admissibility.

LEMMA 4.4. *For any two clauses $\vdash \Gamma$ and $\vdash \Delta$, $[\![\bigvee \Gamma \vee \bigvee \Delta]\!] \geqslant [\![\bigvee \Gamma]\!]$.*

Proof. To begin with, observe that, for any clause $\vdash \Gamma$, either $[\![\bigvee \Gamma]\!] = 1$ or $[\![\bigvee \Gamma]\!] = 0$. Then, consider the following three cases.

- $\vdash \Gamma$ is an identity clause. Then $\vdash \Gamma, \Delta$ is an identity clause as well, and so $[\![\bigvee \Gamma \vee \bigvee \Delta]\!] = [\![\bigvee \Gamma]\!]$.
- $\vdash \Delta$ is an identity clause, and so is $\vdash \Gamma, \Delta$. Thence, $[\![\bigvee \Gamma \vee \bigvee \Delta]\!] \geqslant [\![\bigvee \Gamma]\!]$.

$$
\begin{array}{ccc}
\text{mv-}\overline{\overline{\mathsf{GS4}}}_1 & = & \mathsf{GS4} \\
\cap & & \\
\vdots & & \\
\cap & & \\
\text{mv-}\overline{\overline{\mathsf{GS4}}}_{0.5} & & \\
\cap & & \\
\vdots & & \\
\cap & & \\
\text{mv-}\overline{\overline{\mathsf{GS4}}}_{0.\overline{3}} & & \\
\cap & & \\
\vdots & & \\
\cap & & \\
\text{mv-}\overline{\overline{\mathsf{GS4}}}_0 & = & \overline{\overline{\mathsf{GS4}}}
\end{array}
$$

Fig. 3

- If neither $\vdash \Gamma$ nor $\vdash \Delta$ is an identity clause, it suffices to observe that the inclusion of new atoms in a complementary clause may transform it into an identity clause. This means that it will be either $[\![\bigvee \Gamma \vee \bigvee \Delta]\!] = 0$ or $[\![\bigvee \Gamma \vee \bigvee \Delta]\!] = 1$. In both cases: $[\![\bigvee \Gamma \vee \bigvee \Delta]\!] \geqslant [\![\bigvee \Gamma]\!]$. □

The following is a consequence.

THEOREM 4.5. *For any context* $\Gamma, A : [\![\bigvee \Gamma \vee A]\!] \geqslant [\![\bigvee \Gamma]\!]$.

Proof. We prove by induction on the complexity of the formula A that, for *any* context Γ, $[\![\bigvee \Gamma \vee A]\!] \geqslant [\![\bigvee \Gamma]\!]$.

Base case: Let $\mathsf{top}(\Gamma) = \big[\vdash \Gamma_1, \ldots, \vdash \Gamma_n \big]$ and $\ell \in \mathcal{AT}$. If $A \equiv \ell$, then $\mathsf{top}(\Gamma, \ell) = \big[\vdash \Gamma_1, \ell, \ldots, \vdash \Gamma_n, \ell \big]$, By Lemma 4.4, for any $1 \leqslant i \leqslant n$, $[\![\bigvee \Gamma_i \vee \ell]\!] \geqslant [\![\bigvee \Gamma_i]\!]$, and so $[\![\bigvee \Gamma \vee \ell]\!] \geqslant [\![\bigvee \Gamma]\!]$ comes straightforwardly.

Inductive step: we distinguish two cases.

- ($A \equiv B \wedge C$). By Theorem 2.10 (*ii*), $\mathsf{top}(\Gamma, B \wedge C) = \mathsf{top}(\Gamma, B) \uplus \mathsf{top}(\Gamma, C)$. By inductive hypothesis, $[\![\bigvee \Gamma \vee B]\!] \geqslant [\![\bigvee \Gamma]\!]$ and $[\![\bigvee \Gamma \vee C]\!] \geqslant [\![\bigvee \Gamma]\!]$. By Theorem 3.7, $[\![\bigvee \Gamma \vee (B \wedge C)]\!] \geqslant [\![\bigvee \Gamma \vee B]\!]$ or $[\![\bigvee \Gamma \vee (B \wedge C)]\!] \geqslant [\![\bigvee \Gamma \vee C]\!]$ and so $[\![\bigvee \Gamma \vee (B \wedge C)]\!] \geqslant [\![\bigvee \Gamma]\!]$.
- ($A \equiv B \vee C$). By Theorem 2.10 (*i*), $\mathsf{top}(\Gamma, A \vee B) = \mathsf{top}(\Gamma, A, B)$. By inductive hypothesis, for *any* context Γ, $[\![\bigvee \Gamma]\!] \leqslant [\![\bigvee \Gamma \vee B]\!]$. By replacing Γ with Γ, A, we finally get the inequality $[\![\bigvee \Gamma \vee A]\!] \leqslant [\![(\bigvee \Gamma \vee A) \vee B]\!] = [\![\bigvee \Gamma \vee (A \vee B)]\!]$. □

COROLLARY 4.6. *For any two formulas A and B*:

(*i*) *if A is a* $\text{mv-}\overline{\overline{\mathsf{GS4}}}_q$*'s theorem, so is* $A \vee B$,

(*ii*) *if A and B are both* $\text{mv-}\overline{\overline{\mathsf{GS4}}}_q$*'s theorems,* $A \wedge B$ *is also a theorem.*

Proof. Immediate, by Theorem 4.5. □

REMARK 4.7. *The disjunction and conjunction rules are clearly invertible in* $\mathsf{GS4}$: *if the lower sequent is provable, so does the upper sequent(s). However, it is easy to check that conjunction does not admit the same in bounded systems* $\mathsf{mv}\text{-}\overline{\overline{\overline{\mathsf{GS4}}}}_q$. *Example*: *the formula* $(p \wedge p^{\perp}) \wedge (p \vee p^{\perp})$ *is a theorem of* $\mathsf{mv}\text{-}\overline{\overline{\overline{\mathsf{GS4}}}}_{0.5}$, *whereas* $p \wedge p^{\perp}$ *is not.*

4.2. Cut-Elimination. As Arnon Avron noted [1], Jean-Yves Girard was the first pointing out that Gentzen's standard cut-elimination algorithm can be easily upgraded in the propositional case so as to cover stronger systems with a set of complementary clauses as proper axioms (provided that the set is closed under cut-applications) [8]. The following theorem is nothing else but an adaption of Girard's observation.

THEOREM 4.8 (Strong cut-elimination). *The cut rule* $\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, A^{\perp}}{\vdash \Gamma}$ *is redundant when added to* $\overline{\overline{\overline{\mathsf{GS4}}}}$.

Proof. This clearly follows from the fact that $\overline{\overline{\overline{\mathsf{GS4}}}}$ proves anything.

In computational terms, recall that Gentzen's cut-elimination algorithm pushes the applications of the cut rule upwards along the proof-tree. Whenever one of the cut-formulas is an atom within a clause introduced by a complementary axiom, the cut is immediately reducible as follows:

$$\dfrac{\begin{matrix}\vdots \\ \vdash \Delta, A\end{matrix} \qquad \dfrac{}{\vdash \Delta, A^{\perp}}\,\overline{ax}.}{\vdash \Delta}\,cut \quad \longrightarrow \quad \dfrac{}{\vdash \Delta}\,\overline{ax}.$$

□

In the context of the multivalued system $\mathsf{mv}\text{-}\overline{\overline{\overline{\mathsf{GS4}}}}$, and for mere combinatorial reasons, the cut rule has to be decorated in the following manner:

$$\dfrac{\models^{m}_{n} \Gamma, A \qquad \models^{m'}_{n'} \Gamma, A^{\perp}}{\models^{m+m'}_{n+n'} \Gamma}\,cut,$$

and, then, the foregoing cut reduction becomes:

$$\dfrac{\begin{matrix}\vdots \\ \models^{m}_{n} \Delta, A\end{matrix} \qquad \dfrac{}{\models^{0}_{1} \Delta, A^{\perp}}\,\overline{ax}.}{\models^{m}_{n+1} \Delta}\,cut \quad \longrightarrow \quad \dfrac{}{\models^{0}_{1} \Delta}\,\overline{ax}.$$

Let's call $\mathsf{mv}\text{-}\overline{\overline{\overline{\mathsf{GS4}}}}^{+}$ (resp. $\mathsf{mv}\text{-}\overline{\overline{\overline{\mathsf{GS4}}}}_q^{+}$) the system obtained by adding the cut rule to $\mathsf{mv}\text{-}\overline{\overline{\overline{\mathsf{GS4}}}}$ (resp. $\mathsf{mv}\text{-}\overline{\overline{\overline{\mathsf{GS4}}}}_q$). The point to stress here is that applications of the cut rule may distort the correct semantic evaluation of proofs by artificially increasing or decreasing it. Inspect for instance the following proof:

$$\dfrac{\dfrac{}{\models^{1}_{1} p, p^{\perp}}\,ax. \qquad \dfrac{}{\models^{0}_{1} p, p}\,\overline{ax}.}{\models^{1}_{2} p}\,cut$$

$$\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} p, q^{\perp}}\,\overline{ax}. \qquad \dfrac{}{\vdash^{1}_{1} p, p^{\perp}}\,ax.}{\vdash^{1}_{2} p, q^{\perp} \wedge p^{\perp}}\,\wedge \qquad \dfrac{\dfrac{}{\vdash^{0}_{1} q, p, p}\,\overline{ax}.}{\vdash^{0}_{1} q \vee p, p}\,\vee}{\vdash^{1}_{3} p}\,cut$$

$$\downarrow$$

$$\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} p, q^{\perp}, p}\,\overline{ax}. \qquad \dfrac{}{\vdash^{0}_{1} q, p, p}\,\overline{ax}.}{\vdash^{0}_{2} p, p}\,cut \qquad \dfrac{}{\vdash^{1}_{1} p, p^{\perp}}\,ax.}{\vdash^{1}_{3} p}\,cut$$

$$\downarrow$$

$$\dfrac{\dfrac{}{\vdash^{0}_{1} p, p}\,\overline{ax}. \qquad \dfrac{}{\vdash^{1}_{1} p, p^{\perp}}\,ax.}{\vdash^{1}_{2} p}\,cut$$

$$\downarrow$$

$$\dfrac{}{\vdash^{0}_{1} p}\,\overline{ax}.$$

Fig. 4

Although this proof attaches to p the value 0.5, the right interpretation is obviously $[\![p]\!] = 0$. An example of proof where the cut rule decreases the semantic value is the following:

$$\dfrac{\dfrac{\dfrac{}{\vdash^{1}_{1} p, p^{\perp}}\,ax.}{\vdash^{1}_{1} p \vee p^{\perp}}\,\vee \qquad \dfrac{\dfrac{}{\vdash^{0}_{1} p}\,\overline{ax}. \qquad \dfrac{\dfrac{}{\vdash^{0}_{1} p, q}\,\overline{ax}. \qquad \dfrac{}{\vdash^{0}_{1} p, \overline{q}}\,\overline{ax}.}{\vdash^{0}_{2} p}\,cut}{\vdash^{0}_{3} p \wedge p}\,\wedge}{\vdash^{1}_{4} (p \vee p^{\perp}) \wedge (p \wedge p)}\,\wedge$$

This is an indication that cut-elimination algorithm in $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}^{+}$ is the key to computing the semantic value of logical formulas to the extent that only cut-free proofs are capable of leading to the correct semantic value.

EXAMPLE 4.9. *In Figure 4 we show how to calculate the value* $[\![p]\!] = 0$ *by implementing the cut-elimination algorithm on a nonanalytic* $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}^{+}$ *proof ending with the sequent* $\vdash^{1}_{3} p$.

REMARK 4.10 (Non-admissibility of the cut rule). *The cut rule is not admissible in bounded systems. To take an example, the two sequents* $\vdash^{2}_{3} p, (p \vee p^{\perp}) \wedge (p \wedge p^{\perp})$ *and* $\vdash^{1}_{1} p, ((p \vee p^{\perp}) \wedge (p \wedge p^{\perp}))^{\perp}$ *are both valid in* $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_{0.7}$, *whereas* $\vdash^{0}_{1} p$ *is not.*

4.3. Structurality. The property of *structurality* of a logical system refers to the closure of provability under uniform substitution: if a formula A is provable, it is also provable the formula $A[^B/_\ell]$, where ℓ is a literal (i.e., $\ell \in \mathcal{AT}$), achieved by replacing each occurrence of ℓ (resp. $\ell^\perp$) in A with the formula B (resp. $B^\perp$) [22]. Since classical propositional logic is Post-complete, no consequence relation extending the classical one can be both Tarskian and satisfy structurality (but the trivial one) (see [4, 18]). While we know that supraclassical consequence relations *à la* Makinson are Tarskian at the expense of structurality, our proof-theoretic approach reverses the outcome. Indeed, the consequence relation induced by bounded systems $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_q$ is non-Tarskian to the extent that it satisfies structurality at the expenses of transitivity.

LEMMA 4.11. *Let* $\Gamma[^B/_\ell] = \{A[^B/_\ell] \,|\, A \in \Gamma\}$. *For any clause* $\vdash \Gamma$, *we have that* $[\![\, \Gamma[^B/_\ell] \,]\!] \geqslant [\![\, \bigvee \Gamma \,]\!]$.

Proof. We distinguish two cases.

- Neither $\ell \in \Gamma$ nor $\ell^\perp \in \Gamma$. In this case $\Gamma[^B/_\ell] = \Gamma$ and so $[\![\, \bigvee \Gamma[^B/_\ell] \,]\!] = [\![\, \bigvee \Gamma \,]\!]$.
- $\ell \in \Gamma$ or $\ell^\perp \in \Gamma$. Two subcases.
  - $\vdash \Gamma$ is an identity clause. Two further subcases.
    - ⋆ If $\Gamma = \Gamma', \ell, \ell^\perp$, then the sequent $\vdash \Gamma'[^B/_\ell], B, B^\perp$ is valid as well and so $[\![\, \bigvee \Gamma'[^B/_\ell] \vee B \vee B^\perp \,]\!] = 1$. Thence: $[\![\, \bigvee \Gamma'[^B/_\ell] \vee B \vee B^\perp \,]\!] = [\![\, \bigvee \Gamma' \vee p \vee p^\perp \,]\!]$.
    - ⋆ If $\Gamma = \Gamma', \hat{\ell}, \hat{\ell}^\perp$, with $\hat{\ell} \neq \ell$, then $\Gamma[^B/_\ell] = \Gamma'[^B/_\ell], \hat{\ell}, \hat{\ell}^\perp$. Thence, as in the previous point, $[\![\, \bigvee \Gamma \,]\!] = [\![\, \bigvee \Gamma'[^B/_\ell] \vee \hat{\ell} \vee \hat{\ell}^\perp \,]\!] = 1$.
  - $\vdash \Gamma$ is a complementary clause. In this case $[\![\, \bigvee \Gamma \,]\!] = 0$ and so, trivially, $[\![\, \bigvee \Gamma[^B/_\ell] \,]\!] \geqslant [\![\, \bigvee \Gamma \,]\!]$. □

In order to prove structurality, it is necessary to define the concept of *open-proof* which further expands that of $\overline{\overline{\mathsf{GS4}}}$-proof.

DEFINITION 4.12 (Open-proof, closed open-proof). *The notion of open-proof expands the set of* $\overline{\overline{\mathsf{GS4}}}$*-proofs by including deductions* with whatsoever top-sequents, not only clauses. *Notationally, we shall distinguish open-proofs from "closed" ones through the superscript "*$\uparrow$*" (e.g.,* $\pi^\uparrow, \rho^\uparrow, \ldots$*). Given an open-proof* $\pi^\uparrow$*, its closure* $\pi^{\uparrow\downarrow}$ *is any* $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$*-proof constructed by:* (*i*) *upwards extending* $\pi^\uparrow$ *by means of* $\overline{\overline{\mathsf{GS4}}}$*'s logical rules in such a way as to afford a complete* $\overline{\overline{\mathsf{GS4}}}$*-proof, and* (*ii*) *decorating all the sequents in this proof, from top-sequents to end-sequent, on the basis of the* $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$ *rules.*

DEFINITION 4.13 (Substitution on proofs). *Let* π *be a* $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$*-proof ending with* $\left|\frac{m}{n}\right. \Gamma$. *We indicate with* $\pi[^B/_\ell]^{\uparrow\downarrow}$ *the closed open-proof ending with* $\left|\frac{m'}{n'}\right. \Gamma[^B/_\ell]$ *which results from the closure of the open-proof obtained by replacing in* π *each occurrence of the literal* ℓ (*resp.* $\ell^\perp$) *by the formula B* (*resp.* $B^\perp$).

EXAMPLE 4.14. *In Figure 5 we consider a* $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$*-proof* π *ending with the sequent* $\left|\frac{0}{2}\right. q^\perp \vee (p \wedge q)$ *and we show how to compute the proof* $\pi[^{p \vee p^\perp}/_p]^{\uparrow\downarrow}$. *The five steps in Figure 5 are commented below.*

STEP 1: *remove decorations,*
STEP 2: *replace everywhere p by $p \vee p^{\perp}$;*
STEP 3: *complete the proof,*
STEP 4: *attach new decorations.*

THEOREM 4.15. *For all formulas A, B and all atoms ℓ, $[\![A[^{B}/_{\ell}]]\!] \geqslant [\![A]\!]$.*

Proof. We attend to two $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$-proofs π and $\pi\,[^{B}/_{\ell}]^{\uparrow\downarrow}$ ending with the sequents $\vdash^{m}_{n} A$ and $\vdash^{m'}_{n'} A[^{B}/_{\ell}]$, respectively. We need assurance that $\dfrac{m'}{n'} \geqslant \dfrac{m}{n}$. Consider then the two multiset of sequents $\mathsf{top}(\pi) = [\vdash \Gamma_1, \ldots, \vdash \Gamma_n]$ and $\mathsf{top}(\pi\,[^{B}/_{\ell}]^{\uparrow}) = [\vdash \Gamma'_1, \ldots, \vdash \Gamma'_n]$. By Lemma 4.11, for each $1 \leqslant i \leqslant n$, $[\![\bigvee \Gamma'_i]\!] \geqslant [\![\bigvee \Gamma_i]\!]$. Thence, we easily get the claim of the theorem. □

From this theorem, structurality follows as an easy corollary.

COROLLARY 4.16. *If the formula A is a theorem of $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_q$, then $A[^{B}/_{\ell}]$ is also a $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}_q$ theorem.*

4.4. Bounded consequence relation. We conclude this section by framing its results in terms of the *bounded consequence relation* denoted with "$\mid\!\sim_q$" and defined as follows.

DEFINITION 4.17. *Let $q \in \mathbb{Q}^*$, we say that the relation $\Gamma \mid\!\sim_q A$ holds whenever $\mathsf{mv}\text{-}\overline{\overline{\mathsf{GS4}}}$ proves the sequent $\vdash^{m}_{n} \Gamma^{\perp}, A$ and $\dfrac{m}{n} \geqslant q$ or, equivalently, $[\![\bigvee \Gamma^{\perp} \vee A]\!] \geqslant q$.*

THEOREM 4.18. *For any $q \in \mathbb{Q}^*$, the bounded consequence relation $\mid\!\sim_q$ is* reflexive, monotonic, *and* structural.

$$\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} q^{\perp}, p}\,\overline{ax}. \qquad \dfrac{}{\vdash^{1}_{1} q^{\perp}, q}\,ax.}{\vdash^{1}_{2} q^{\perp}, p \wedge q}\,\wedge}{\vdash^{1}_{2} q^{\perp} \vee (p \wedge q)}\,\vee \quad \xrightarrow{(1)} \quad \dfrac{\dfrac{\dfrac{}{\vdash q^{\perp}, p}\,\overline{ax}. \qquad \dfrac{}{\vdash q^{\perp}, q}\,ax.}{\vdash q^{\perp}, p \wedge q}\,\wedge}{\vdash q^{\perp} \vee (p \wedge q)}\,\vee \quad \xrightarrow{(2)}$$

$$\dfrac{\dfrac{\vdash q^{\perp}, p \vee p^{\perp} \qquad \dfrac{}{\vdash q^{\perp}, q}\,ax.}{\vdash q^{\perp}, (p \vee p^{\perp}) \wedge q}\,\wedge}{\vdash q^{\perp} \vee ((p \vee p^{\perp}) \wedge q)}\,\vee \quad \xrightarrow{(3)} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash q^{\perp}, p, p^{\perp}}\,ax.}{\vdash q^{\perp}, p \vee p^{\perp}}\,\vee \qquad \dfrac{}{\vdash q^{\perp}, q}\,ax.}{\vdash q^{\perp}, (p \vee p^{\perp}) \wedge q}\,\wedge}{\vdash q^{\perp} \vee ((p \vee p^{\perp}) \wedge q)}\,\vee \quad \xrightarrow{(4)}$$

$$\xrightarrow{(4)} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^{1}_{1} q^{\perp}, p, p^{\perp}}\,ax.}{\vdash^{1}_{1} q^{\perp}, p \vee p^{\perp}}\,\vee \qquad \dfrac{}{\vdash^{1}_{1} q^{\perp}, q}\,ax.}{\vdash^{2}_{2} q^{\perp}, (p \vee p^{\perp}) \wedge q}\,\wedge}{\vdash^{2}_{2} q^{\perp} \vee ((p \vee p^{\perp}) \wedge q)}\,\vee$$

Fig. 5

Proof. We consider each one of the listed properties separately.

- REFLEXIVITY. Immediately by the fact that, for any context Γ, A, there is a $m \in \mathbb{N}$ such that mv-$\overline{\overline{\mathsf{GS4}}}$ proves $\left|\frac{m}{m}\right. \Gamma^{\perp}, A^{\perp}, A$.
- MONOTONICITY. If $\Gamma \mathrel{|\!\sim_q} A$, then there are $m, n \in \mathbb{N}$ such that mv-$\overline{\overline{\mathsf{GS4}}}$ proves $\left|\frac{m}{n}\right. \Gamma^{\perp}, A$ with $\dfrac{m}{n} \geqslant q$. By Theorem 4.5, there are $m', n' \in \mathbb{N}$ such that mv-$\overline{\overline{\mathsf{GS4}}}$ proves $\left|\frac{m'}{n'}\right. \Gamma^{\perp}, A, B^{\perp}$ with $\dfrac{m'}{n'} \geqslant \dfrac{m}{n}$. Thence, $\dfrac{m'}{n'} \geqslant q$ and so $\Gamma, B \mathrel{|\!\sim_q} A$.
- STRUCTURALITY. If $\Gamma \mathrel{|\!\sim_q} A$, then there are $m, n \in \mathbb{N}$ such that mv-$\overline{\overline{\mathsf{GS4}}}$ proves $\left|\frac{m}{n}\right. \Gamma^{\perp}, A$ and $\dfrac{m}{n} \geqslant q$. By Theorem 4.15, $[\![\bigvee \Gamma^{\perp} \vee A[^B/_\ell]]\!] \geqslant [\![\bigvee \Gamma^{\perp} \vee A]\!]$ and so $\Gamma[^B/_\ell] \mathrel{|\!\sim_q} A[^B/_\ell]$. □

REMARK 4.19. *The consequence relation is transitive when the special cases* $|\!\sim_1$ *and* $|\!\sim_0$ *are taken into account. When* $q \in \mathbb{Q}^*\backslash\{0, 1\}$, *transitivity is lost. To take a very easy example, whereas* $p \mathrel{|\!\sim_{0.5}} p \wedge q$ *and* $p \wedge q \mathrel{|\!\sim_{0.5}} q$ *hold, it is not the case that* $p \mathrel{|\!\sim_{0.5}} q$ *because* $[\![q \vee p^{\perp}]\!] = 0$.

§5. Concluding thoughts. We would like to conclude by sketching three themes for further research. For one thing, we suggest that our proof-theoretic approach may be applied to traditional issues in the theory of belief revision. The logical machinery developed in [15] enables to encode a set of propositions $\mathcal{B}$ (believed by an ideal doxastic agent) into a set of complementary clauses $\mathscr{C}_{\mathcal{B}} = \{\vdash \Gamma_1, \ldots, \vdash \Gamma_k\}$. The next stage is to define an "extended" multivalued system mv-$\overline{\overline{\mathsf{GS4}}}_{\mathcal{B}}$ such that $\left|\frac{1}{1}\right. \Gamma_i$ for each $\Gamma_i \in \mathscr{C}_{\mathcal{B}}$; the semantic value of the logical formulas can be now recalculated accordingly, so as to express the set of propositions $\mathcal{B}$ believed by the agent at a given time. If $[\![A]\!]_{\mathcal{B}}$ is the value attached to A by the system mv-$\overline{\overline{\mathsf{GS4}}}_{\mathcal{B}}$, then it is easy to see that $[\![A]\!]_{\mathcal{B}} \geqslant [\![A]\!]$; specifically, we have that $\{A \mid [\![A]\!] = 1\} \subset \{A \mid [\![A]\!]_{\mathcal{B}} = 1\}$.

EXAMPLE 5.1. *Suppose we supplement* $\mathsf{GS4}$ *with the formula* $p \to (q \wedge r)$ *as a proper axiom. According to the method in* [15], *the extended system* $\mathsf{GS4} + \{\vdash p \to (q \wedge r)\}$ *is equivalent to the system* $\mathsf{GS4} + \{\vdash p^{\perp}, q\} + \{\vdash p^{\perp}, r\}$. *Our semantics can be now upgraded by decorating the added clauses as identity axioms — i.e.,* $\left|\frac{1}{1}\right. p^{\perp}, q$ *and* $\left|\frac{1}{1}\right. p^{\perp}, r$. *For instance, whereas* $[\![p^{\perp} \vee (q \wedge t)]\!] = 0$, *it is easy to check that* $[\![p^{\perp} \vee (q \wedge t)]\!]_{\{p \to (q \wedge r)\}} = 0.5$.

Secondly, we support the idea that fractional semantics can be also regarded as a full-fledged proof-theoretic semantics for classical logic, alternative to the proposals presented, for instance, in [6, 16]. According to proof-theoretic semantics, logical rules in their top-down reading, from premises to the conclusion, are what confer the meaning to logical operators, whilst semantical definitions via truth-tables are just subsidiary devices. In short, logical formulas receive their meaning through their derivations, i.e., via finite sequence of meaning-transmitting inferential steps. Of course, the fact that any proof-theoretic semantics is a semantics in terms of proofs does not imply the converse. Anyway, our semantics seems to fit the bill by taking the rules of classical calculus as vehicle for meaning in their *bottom-up* reading, namely by considering them as decomposition rules. This backwards reading of the rules discloses the axiomatic structure of proofs providing the semantic interpretation of logical formulas.

Finally, another application of bounded supraclassical logics, not being transitive, concerns the possibility they offer of accommodating truth-theoretic paradoxes in an alternative way. Examine the mv-$\overline{\overline{\mathsf{GS4}}}$ version of the usual derivation of the empty sequent $\vdash \varnothing$ (i.e., the limiting case of the complementary axiom of $\overline{\overline{\mathsf{GS4}}}$) in a system enriched with the Tarskian truth predicate T with a fixed point λ. In short, $T[\lambda]$ (resp. $T[\lambda]^{\perp}$) and $\lambda^{\perp}$ (resp. λ) are mutually interchangeable in sequent proofs [2]:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^1_1 T[\lambda], \overline{T[\lambda]}}}{\vdash^1_1 \lambda, \overline{T[\lambda]}}}{\vdash^1_1 \lambda, \lambda} \quad \dfrac{}{\vdash^1_1 \lambda, \overline{\lambda}}}{\vdash^2_2 \lambda}\,\mathit{cut} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^1_1 T[\lambda], \overline{T[\lambda]}}}{\vdash^1_1 \lambda, \overline{T[\lambda]}}}{\vdash^1_1 \lambda, \lambda} \quad \dfrac{}{\vdash^1_1 \lambda, \overline{\lambda}}}{\vdash^2_2 \lambda}\,\mathit{cut}}{\vdash^2_2 T[\lambda]}}{\vdash^2_2 \overline{\lambda}}}{\vdash^4_4 \varnothing}\,\mathit{cut}
$$

The point is that $[\![\bigvee \varnothing]\!] \neq 1$, but $[\![\bigvee \varnothing]\!] = 0$. The lesson here seems to be that any bounded system mv-$\overline{\overline{\mathsf{GS4}}}_q$ with $q > 0$ furnishes a natural way to block the derivation of the absurd in logical contexts of this kind.

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