

EULER-LAGRANGIAN APPROACH TO 3D STOCHASTIC EULER EQUATIONS

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ABSTRACT. 3D stochastic Euler equations with a special form of multiplicative noise are considered. A Constantin-Iyer type representation in Euler-Lagrangian form is given, based on stochastic characteristics. Local existence and uniqueness of solutions in suitable Hölder spaces is proved from the Euler-Lagrangian formulation.

1. **Introduction.** Recently the vorticity equation (Euler type) with noise

$$d\omega_t + (u_t \cdot \nabla \omega_t - \omega_t \cdot \nabla u_t) dt + \sum_k (\sigma_k \cdot \nabla \omega_t - \omega_t \cdot \nabla \sigma_k) \circ dW_t^k = 0 \quad (1)$$

with $\operatorname{div} u_t = 0$, $\operatorname{curl} u_t = \omega_t$, $\operatorname{div} \sigma_k = 0$, has been studied by Darryl Holm and other authors, see [15, 14, 5]. We assume, to avoid technical difficulties, that there are finitely many smooth and divergence free vector fields $\{\sigma_k\}$. The equation can be written as

$$d\omega_t + \mathcal{L}_{u_t} \omega_t dt + \sum_k \mathcal{L}_{\sigma_k} \omega_t \circ dW_t^k = 0, \quad (2)$$

where $\mathcal{L}_{u_t} \omega_t = [u_t, \omega_t] = u_t \cdot \nabla \omega_t - \omega_t \cdot \nabla u_t$ is the Lie derivative. Equations related to fluid dynamics with multiplicative noise appeared in several other works, see for instance [2, 11, 7, 19, 10] and many others. However, the geometric structure in (1) has special properties, revealed also by the present work.

A first intermediate question we address is finding the noise form when the equation is rewritten in the velocity-pressure variables (u, p) , instead of the velocity-vorticity variables (u, ω) . The dual operators $\mathcal{L}_{\sigma_k}^*$ appear. This is an intermediate step in order to investigate the main topic of this work, namely the Euler-Lagrangian formulation, called also Constantin-Iyer representation after [3, 4]; among related works, see for instance [21, 8]. We prove both the (u, p) -formulation and the Euler-Lagrangian one in Proposition 2.1. The Euler-Lagrangian form is then used to prove

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a local in time existence and uniqueness result for solutions in suitable Hölder spaces, new for equation (1). At the end of the paper we heuristically digress on potential singularities from the viewpoint of this stochastic model and its Euler-Lagrangian formulation, adding some remarks to the discussion of P. Constantin [3, Section 5]. Finally we remark that we restrict ourselves to the three dimensional torus \mathbb{T}^3 for simplicity, though the results remain valid in more general settings under suitable conditions.

2. Equation for the velocity and its representation. Let $\mathcal{L}_{\sigma_k}^*$ be the adjoint operator of \mathcal{L}_{σ_k} with respect to the inner product in $L^2(\mathbb{T}^3, \mathbb{R}^3)$:

$$\langle \mathcal{L}_{\sigma_k}^* v, w \rangle_{L^2} = -\langle v, \mathcal{L}_{\sigma_k} w \rangle_{L^2}.$$

Since σ_k is divergence free, we have

$$\mathcal{L}_{\sigma_k}^* v = \sigma_k \cdot \nabla v + (\nabla \sigma_k)^* v.$$

where, for $i = 1, 2, 3$, $((\nabla \sigma_k)^* v)_i = \sum_{j=1}^3 v_j \partial_i (\sigma_k)_j$. Note that if the vector fields σ_k and v are divergence free, then $\operatorname{div}(\mathcal{L}_{\sigma_k} v) = 0$, but this is not necessarily true for $\mathcal{L}_{\sigma_k}^* v$.

Consider the following equation of characteristics, associated to (1):

$$dX_t = u_t(X_t) dt + \sum_k \sigma_k(X_t) \circ dW_t^k. \quad (3)$$

Assume the random field $u_t(\cdot, \omega) \in C^{l+\alpha}(\mathbb{T}^3, \mathbb{R}^3)$ is defined for t in a random local time interval $[0, \tau(\omega)]$ where τ is a stopping time, $l \geq 1$, $\alpha \in (0, 1]$. We can simply write $u \in C([0, \tau], C^{l+\alpha})$, see the next section for the definition of Hölder spaces. Consider the extended random field $\tilde{u}_t(x, \omega) := u_{t \wedge \tau(\omega)}(x, \omega)$, which is globally defined and let $\tilde{X}_t(x, \omega)$ be the global regular stochastic flow associated to the equation (cf. [18])

$$d\tilde{X}_t = \tilde{u}_t(\tilde{X}_t) dt + \sum_k \sigma_k(\tilde{X}_t) \circ dW_t^k.$$

Define the local stochastic flow $X_t(x, \omega)$ as $\tilde{X}_t(x, \omega)$ restricted to $t \in [0, \tau(\omega)]$. Notice that, for $t \in [0, \tau(\omega)]$, the function $x \mapsto X_t(x, \omega)$ is globally defined on \mathbb{T}^3 and is as smooth as the diffeomorphism $x \mapsto \tilde{X}_t(x, \omega)$. This justifies several passages made below. In the previous definition of local flow we have used a particular stopping time τ ; if we consider two stopping times τ_1, τ_2 and we construct the corresponding local flows, they can be shown to coincide for $t \in [0, \tau_1 \wedge \tau_2]$. Elaborating this argument, we may define the local flow on the maximal time of existence of u .

Proposition 2.1. *Assume $l \geq 2$ and that the pair $(u, p) \in C([0, \tau], C^l \times C^l)$ solves the stochastic Euler equation*

$$du_t + (u_t \cdot \nabla u_t + \nabla p_t) dt + \sum_k \mathcal{L}_{\sigma_k}^* u_t \circ dW_t^k = 0 \quad (4)$$

with $\operatorname{div} u_t = 0$. Then, a.s., the pair (ω_t, u_t) with $\omega_t = \operatorname{curl} u_t$ is a solution to (2) for $t \leq \tau$. Moreover, if $l \geq 3$, then the vector field $u \in C([0, \tau], C^{l+\alpha})$ given below is a solution to (4):

$$\begin{cases} dX_t = \sum_k \sigma_k(X_t) \circ dW_t^k + u_t(X_t) dt, \\ u_t = \mathbf{P}[(\nabla X_t^{-1})^* u_0(X_t^{-1})], \end{cases} \quad t \leq \tau, \quad (5)$$

where \mathbf{P} is the Leray–Hodge projection and $*$ means the transposition of matrices.

Remark 2.2. The stochastic equation (4) is understood as follows: for any smooth divergence free field v on \mathbb{T}^3 , for all $t > 0$,

$$\begin{aligned} \langle u_{t \wedge \tau}, v \rangle_{L^2} &= \langle u_0, v \rangle_{L^2} - \int_0^{t \wedge \tau} \langle u_s \cdot \nabla u_s, v \rangle_{L^2} ds - \sum_k \int_0^{t \wedge \tau} \langle \mathcal{L}_{\sigma_k}^* u_s, v \rangle_{L^2} \circ dW_s^k \\ &= \langle u_0, v \rangle_{L^2} + \int_0^{t \wedge \tau} \langle u_s, u_s \cdot \nabla v \rangle_{L^2} ds + \sum_k \int_0^{t \wedge \tau} \langle u_s, \mathcal{L}_{\sigma_k} v \rangle_{L^2} \circ dW_s^k. \end{aligned}$$

Since $\mathcal{L}_{\sigma_k} v$ is divergence free, we have

$$d \langle u_s, \mathcal{L}_{\sigma_k} v \rangle_{L^2} \cdot dW_s^k = \langle u_s, \mathcal{L}_{\sigma_k}^2 v \rangle_{L^2} ds.$$

Hence, we get

$$\begin{aligned} \langle u_{t \wedge \tau}, v \rangle_{L^2} &= \langle u_0, v \rangle_{L^2} + \sum_k \int_0^{t \wedge \tau} \langle u_s, \mathcal{L}_{\sigma_k} v \rangle_{L^2} dW_s^k \\ &\quad + \frac{1}{2} \sum_k \int_0^{t \wedge \tau} \langle u_s, \mathcal{L}_{\sigma_k}^2 v \rangle_{L^2} ds + \int_0^{t \wedge \tau} \langle u_s, u_s \cdot \nabla v \rangle_{L^2} ds. \end{aligned} \tag{6}$$

Proof of Proposition 2.1. Let us show the first fact. For $t \leq \tau$, using $\text{curl } u_t = \omega_t$ and the fact that $\{\sigma_k\}$ are divergence free, we have the vector identities

$$\omega_t \times u_t = -\frac{1}{2} \nabla (u_t \cdot u_t) + u_t \cdot \nabla u_t$$

and

$$\begin{aligned} \omega_t \times \sigma_k &= -\nabla (\sigma_k \cdot u_t) + (\nabla \sigma_k)^* u_t + \sigma_k \cdot \nabla u_t \\ &= -\nabla (\sigma_k \cdot u_t) + \mathcal{L}_{\sigma_k}^* u_t. \end{aligned}$$

Hence, equation (4) can be rewritten as

$$du_t + (\omega_t \times u_t + \nabla \tilde{p}_t) dt + \sum_k (\omega_t \times \sigma_k + \nabla (\sigma_k \cdot u_t)) \circ dW_t^k = 0,$$

where \tilde{p}_t is a new pressure. Taking curl and using the facts

$$\text{curl}(\omega_t \times u_t) = \mathcal{L}_{u_t} \omega_t, \quad \text{curl}(\nabla \tilde{p}_t) = 0, \quad \text{curl}(\omega_t \times \sigma_k) = \mathcal{L}_{\sigma_k} \omega_t,$$

we get the equation (2) for $t \leq \tau$.

Next we prove the second assertion. Let u be expressed as in (5). For any divergence free vector field v , we have, a.s. for $t \leq \tau$,

$$\begin{aligned} \langle u_t, v \rangle_{L^2} &= \int \langle (\nabla X_t^{-1})^* u_0(X_t^{-1}), v \rangle dx \\ &= \int \langle u_0(X_t^{-1}), (\nabla X_t^{-1}) v \rangle dx \\ &= \int \langle u_0, (\nabla X_t^{-1}(X_t)) v(X_t) \rangle dx, \end{aligned}$$

where in the last step we have used the change of variable formula and the fact that X_t preserves the Lebesgue measure on \mathbb{T}^3 , which is a consequence of the divergence free properties of u_t and σ_k . Recall that $(\nabla X_t^{-1}(X_t)) v(X_t) = (X_t^{-1})_* v$ is the pull-back of v by the diffeomorphism X_t , thus we obtain, a.s.,

$$\langle u_{t \wedge \tau}, v \rangle_{L^2} = \langle u_0, (X_{t \wedge \tau}^{-1})_* v \rangle_{L^2} \quad \text{for all } t \geq 0. \tag{7}$$

This equality holds for any divergence free vector field v .

Now we use Kunita's formula which gives the equation satisfied by the pull-back of vector fields under stochastic flows, see for instance [17, p.265, Theorem 2.1] or [18, Theorem 4.9.1]. Although these formulae were presented for global stochastic flow of diffeomorphisms associated to SDEs, Kunita mentioned that they can be extended to the local case, see the middle of p.202 in [18]. We have

$$\begin{aligned} (X_{t \wedge \tau}^{-1})_* v &= v + \sum_k \int_0^{t \wedge \tau} (X_s^{-1})_* (\mathcal{L}_{\sigma_k} v) dW_s^k \\ &\quad + \int_0^{t \wedge \tau} \left[(X_s^{-1})_* (\mathcal{L}_{u_s} v) + \frac{1}{2} \sum_k (X_s^{-1})_* (\mathcal{L}_{\sigma_k}^2 v) \right] ds. \end{aligned}$$

Substituting this expression into (7) yields

$$\begin{aligned} \langle u_{t \wedge \tau}, v \rangle_{L^2} &= \langle u_0, v \rangle_{L^2} + \sum_k \int_0^{t \wedge \tau} \langle u_0, (X_s^{-1})_* (\mathcal{L}_{\sigma_k} v) \rangle_{L^2} dW_s^k \\ &\quad + \int_0^{t \wedge \tau} \left[\langle u_0, (X_s^{-1})_* (\mathcal{L}_{u_s} v) \rangle_{L^2} + \frac{1}{2} \sum_k \langle u_0, (X_s^{-1})_* (\mathcal{L}_{\sigma_k}^2 v) \rangle_{L^2} \right] ds. \end{aligned}$$

Since the vector fields $\mathcal{L}_{\sigma_k} v$, $\mathcal{L}_{u_s} v$ and $\mathcal{L}_{\sigma_k}^2 v$ are all divergence free, we apply (7) and get

$$\begin{aligned} \langle u_{t \wedge \tau}, v \rangle_{L^2} &= \langle u_0, v \rangle_{L^2} + \sum_k \int_0^{t \wedge \tau} \langle u_s, \mathcal{L}_{\sigma_k} v \rangle_{L^2} dW_s^k \\ &\quad + \int_0^{t \wedge \tau} \left[\langle u_s, \mathcal{L}_{u_s} v \rangle_{L^2} + \frac{1}{2} \sum_k \langle u_s, \mathcal{L}_{\sigma_k}^2 v \rangle_{L^2} \right] ds. \end{aligned}$$

Note that

$$\langle u_s, \mathcal{L}_{u_s} v \rangle_{L^2} = \langle u_s, u_s \cdot \nabla v \rangle_{L^2} - \langle u_s, v \cdot \nabla u_s \rangle_{L^2} = \langle u_s, u_s \cdot \nabla v \rangle_{L^2},$$

therefore, we obtain (6). \square

3. Local existence of the representation (5). In this section we aim at proving the local existence of the system (5), by following the arguments in [3, 16]. First we introduce some notations about Hölder (semi-)norms. For a function or vector field u defined on \mathbb{T}^3 and $\alpha \in (0, 1)$, $l \in \mathbb{N}$,

$$\begin{aligned} |u|_\alpha &= \sup_{x, y \in \mathbb{T}^3} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \\ \|u\|_l &= \sum_{|m| \leq l} \sup_{x \in \mathbb{T}^3} |\partial^m u(x)|, \\ \|u\|_{l, \alpha} &= \|u\|_l + \sum_{|m|=l} |\partial^m u|_\alpha, \end{aligned}$$

where ∂^m denotes the derivative with respect to the multi-index $m \in \mathbb{N}^3$. Note that $\|\cdot\|_0$ is the usual supremum norm. We denote by C^l and $C^{l, \alpha}$ the Hölder spaces with norms $\|u\|_l$ and $\|u\|_{l, \alpha}$, respectively.

We use the idea of [1, 20] to solve the SDE in the system (5). More precisely, we first solve the equation without drift:

$$d\varphi_t = \sum_k \sigma_k(\varphi_t) \circ dW_t^k, \quad \varphi_0 = \mathbf{I}, \quad (8)$$

where \mathbf{I} is the identity diffeomorphism of \mathbb{T}^3 . Since there are only finitely many smooth vector fields $\{\sigma_k\}$, the above equation generates a stochastic flow $\{\varphi_t\}_{t \geq 0}$ of C^∞ -diffeomorphisms on \mathbb{T}^3 .

In this section we denote by ω a generic random element in a probability space Ω ; there will be no confusion with the notation of vorticity, since the latter does not appear in the current section. For a given random vector field $u : \Omega \times [0, \tau] \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$, where τ is a stopping time, we define, for $t \leq \tau$,

$$\tilde{u}_t(\omega, x) = [(\varphi_t(\omega, \cdot)^{-1})_* u_t(\omega, \cdot)](x) \quad (9)$$

which is the pull-back of the field $u_t(\omega, \cdot)$ by the stochastic flow $\{\varphi_t(\omega, \cdot)\}_{t \geq 0}$. If we denote by $K_t(\omega, x) = (\nabla \varphi_t(\omega, x))^{-1}$, i.e., the inverse of the Jacobi matrix, then

$$\tilde{u}_t(\omega, x) = K_t(\omega, x) u_t(\omega, \varphi_t(\omega, x)).$$

From this expression we see that if $u \in C([0, \tau], C^{l+1, \alpha})$ a.s., then one also has a.s. $\tilde{u} \in C([0, \tau], C^{l+1, \alpha})$. Moreover, if the process u is adapted, then so is \tilde{u} . Now for a.s. ω , we consider the random ODE

$$\dot{Y}_t = \tilde{u}_t(\omega, Y_t), \quad Y_0 = \mathbf{I} \quad (10)$$

for $t \leq \tau(\omega)$. Applying the generalized Itô formula, we see that (cf. [1, 20])

$$X_t = \varphi_t \circ Y_t$$

is the flow of $C^{l+1, \alpha}$ -diffeomorphisms associated to the SDE in (5).

Once we have the stochastic flow $\{X_t\}_{t \leq \tau}$, we can use the second formula in (5) to obtain a new random vector field \hat{u} . Our purpose is to show that this series of transforms have a fixed point.

From the above discussions, we see that we can fix a random element $\omega \in \Omega_0$, where Ω_0 is some full measure set, and consider $\varphi_t(\omega, \cdot)$, $u_t(\omega, \cdot)$ and so on as deterministic objects. Hence in Section 3.1 we solve a deterministic fixed-point problem, and apply this result in Section 3.2 to prove the local existence of the system (5).

3.1. Deterministic case. In this section, we assume that we are given a deterministic family of diffeomorphisms $\{\varphi_t\}_{t \in [0, T]}$ of \mathbb{T}^3 satisfying $\varphi \in C([0, T], C^{l+2, \alpha})$ and $\varphi_0 = \mathbf{I}$. For $u \in C([0, T], C^{l+1, \alpha}(\mathbb{T}^3, \mathbb{R}^3))$, we consider the following system:

$$\begin{cases} \tilde{u}_t(x) = [(\varphi_t^{-1})_* u_t](x), \\ \dot{Y}_t = \tilde{u}_t(Y_t), \quad Y_0 = \mathbf{I}, \\ X_t = \varphi_t(Y_t), \\ \hat{u}_t(x) = \mathbf{P}[(\nabla X_t^{-1})_* u_0(X_t^{-1})](x). \end{cases} \quad (11)$$

Following the arguments in [3, Section 4] and [16, Section 4], we shall prove that the map defined by $\hat{u} = \Phi(u)$ has a fixed point in

$$\mathcal{U}_{\tau, U} = \left\{ u \in C([0, \tau], C^{l+1, \alpha}(\mathbb{T}^3, \mathbb{R}^3)) : \sup_{t \leq \tau} \|u_t\|_{l+1, \alpha} \leq U, \operatorname{div}(u_t) = 0, u|_{t=0} = u_0 \right\}$$

for some deterministic small τ and big U . Here is the main result of this section.

Theorem 3.1. *There exists $U = U(l, \|u_0\|_{l+1, \alpha})$ and $\tau = \tau(l, U, \varphi)$ such that the map $\Phi : \mathcal{U}_{\tau, U} \rightarrow \mathcal{U}_{\tau, U}$ has a unique fixed point.*

We need some preparations. The following result is taken from [16, Lemma 4.1].

Lemma 3.2. *If $l \geq 1$ and $\alpha \in (0, 1)$, then there exists $C = C(l, \alpha)$ such that*

$$\begin{aligned} \|f \circ g\|_{l, \alpha} &\leq C \|f\|_{l, \alpha} (1 + \|\nabla g\|_{l-1, \alpha})^{l+1}, \\ \|f_1 \circ g_1 - f_2 \circ g_2\|_{l, \alpha} &\leq C (1 + \|\nabla g_1\|_{l-1, \alpha} + \|\nabla g_2\|_{l-1, \alpha})^{l+1} \\ &\quad \times [\|f_1 - f_2\|_{l, \alpha} + (\|\nabla f_1\|_{l, \alpha} \wedge \|\nabla f_2\|_{l, \alpha}) \|g_1 - g_2\|_{l, \alpha}]. \end{aligned}$$

Lemma 3.3. *Let $u \in C([0, T], C^{l+1, \alpha}(\mathbb{T}^3, \mathbb{R}^3))$ and consider the ODE*

$$\dot{Y}_t = u_t(Y_t), \quad Y_0 = \mathbf{I}.$$

Denote by $U_t = \sup_{s \leq t} \|u_s\|_{l+1, \alpha}$ and $\lambda = Y - \mathbf{I}$, $\ell = Y^{-1} - \mathbf{I}$. Then there exists a continuous function $f_{l, \alpha} : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is increasing in both variables and $f_{l, \alpha}(0, \theta) = 0$ for all $\theta \geq 0$, such that

$$\|\nabla \lambda_t\|_{l, \alpha} \leq f_{l, \alpha}(t, U_t), \quad \|\nabla \ell_t\|_{l, \alpha} \leq f_{l, \alpha}(t, U_t), \quad t \leq T.$$

Proof. We first prove the case for $l = 0$. Using the integral form of the ODE, it is clear that

$$\nabla \lambda_t = \int_0^t (\nabla u_s)(Y_s) (\mathbb{I} + \nabla \lambda_s) ds, \quad (12)$$

where \mathbb{I} is the 3×3 identity matrix. Therefore,

$$\|\nabla \lambda_t\|_0 \leq \int_0^t \|(\nabla u_s)(Y_s)\|_0 \|\mathbb{I} + \nabla \lambda_s\|_0 ds \leq U_t \int_0^t (1 + \|\nabla \lambda_s\|_0) ds.$$

The Gronwall inequality implies that

$$1 + \|\nabla \lambda_t\|_0 \leq e^{tU_t}. \quad (13)$$

Now for $x, y \in \mathbb{T}^3$, $x \neq y$, we deduce from (12) that

$$\begin{aligned} \frac{|\nabla \lambda_t(x) - \nabla \lambda_t(y)|}{|x - y|^\alpha} &\leq \int_0^t \frac{|(\nabla u_s)(Y_s(x)) - (\nabla u_s)(Y_s(y))|}{|x - y|^\alpha} |\mathbb{I} + \nabla \lambda_s(x)| ds \\ &\quad + \int_0^t |(\nabla u_s)(Y_s(x))| \frac{|\nabla \lambda_s(x) - \nabla \lambda_s(y)|}{|x - y|^\alpha} ds \\ &=: J_1 + J_2. \end{aligned} \quad (14)$$

We have

$$\begin{aligned} J_1 &\leq \int_0^t |\nabla u_s|_\alpha \|\nabla Y_s\|_0^\alpha (1 + \|\nabla \lambda_s\|_0) ds \leq \int_0^t |\nabla u_s|_\alpha (1 + \|\nabla \lambda_s\|_0)^{1+\alpha} ds \\ &\leq \int_0^t U_s e^{(1+\alpha)sU_s} ds \leq \frac{1}{1+\alpha} (e^{(1+\alpha)tU_t} - 1), \end{aligned}$$

where the third inequality follows from (13). Moreover,

$$J_2 \leq \int_0^t \|\nabla u_s\|_0 |\nabla \lambda_s|_\alpha ds \leq \int_0^t U_s |\nabla \lambda_s|_\alpha ds.$$

Substituting the estimates J_1 and J_2 into (14), we deduce that

$$|\nabla \lambda_t|_\alpha \leq \frac{1}{1+\alpha} (e^{(1+\alpha)tU_t} - 1) + \int_0^t U_s |\nabla \lambda_s|_\alpha ds.$$

Gronwall's inequality leads to

$$|\nabla \lambda_t|_\alpha \leq \frac{e^{tU_t}}{1+\alpha} (e^{(1+\alpha)tU_t} - 1).$$

Combining this estimate with (13) and using the simple inequality $e^t - 1 \leq te^t$ ($t \geq 0$), we conclude that

$$\|\nabla\lambda_t\|_{0,\alpha} \leq 2tU_t e^{(2+\alpha)tU_t}, \quad t \leq T.$$

Now we prove the general case by induction. Taking the $C^{l,\alpha}$ -norm in (12) and using Lemma 3.2, we obtain

$$\begin{aligned} \|\nabla\lambda_t\|_{l,\alpha} &\leq C_l \int_0^t \|\nabla u_s\|_{l,\alpha} (1 + \|\nabla Y_s\|_{l-1,\alpha})^{l+1} (1 + \|\nabla\lambda_s\|_{l,\alpha}) \, ds \\ &\leq C_l \int_0^t \|\nabla u_s\|_{l,\alpha} (2 + \|\nabla\lambda_s\|_{l-1,\alpha})^{l+1} (1 + \|\nabla\lambda_s\|_{l,\alpha}) \, ds. \end{aligned}$$

Using the induction hypothesis, we have

$$\|\nabla\lambda_t\|_{l,\alpha} \leq C_l \int_0^t U_s (2 + f_{l-1,\alpha}(s, U_s))^{l+1} (1 + \|\nabla\lambda_s\|_{l,\alpha}) \, ds.$$

Again by Gronwall's inequality,

$$1 + \|\nabla\lambda_t\|_{l,\alpha} \leq \exp [C_l t U_t (2 + f_{l-1,\alpha}(t, U_t))^{l+1}].$$

The proof of the first estimate is complete.

To prove the second assertion, note that the inverse flow Y_t^{-1} can be obtained by reversing the time. More precisely, fix any $t \in (0, T]$ and consider

$$\dot{Y}_s^t = -u_{t-s}(Y_s^t), \quad 0 \leq s \leq t, \quad Y_0^t = \mathbf{I}.$$

Then $Y_t^t = Y_t^{-1}$. Similar to the above arguments, we can prove estimates for $\lambda_s^t = Y_s^t - \mathbf{I}$, and hence for $\ell_t = Y_t^{-1} - \mathbf{I} = Y_t^t - \mathbf{I} = \lambda_t^t$, which only depends on the $C^{l+1,\alpha}$ -norm of u_s for $s \in [0, t]$, the latter being dominated by U_t . In this way, we obtain the second result. \square

We need the following key technical result, see [3, Proposition 1] or [16, Corollary 3.1].

Lemma 3.4. *For $l \geq 1$, the operator $\mathbf{W} : (\ell, v) \rightarrow \mathbf{P}[(\mathbf{I} + \nabla\ell)^*v]$ is well defined on $C^{l,\alpha} \times C^{l,\alpha}$ with values in $C^{l,\alpha}$; moreover, there is $C > 0$ depending only on l and α such that*

$$\|\mathbf{W}(\ell, v)\|_{l,\alpha} \leq C(1 + \|\nabla\ell\|_{l-1,\alpha})\|v\|_{l,\alpha}.$$

Recall the map defined by (11). Now we can prove

Proposition 3.5. *There exist $U = U(l, \|u_0\|_{l+1,\alpha}) > 0$ and $\tau_1 = \tau_1(l, U, \varphi) > 0$ such that $\Phi(\mathcal{U}_{\tau_1, U}) \subset \mathcal{U}_{\tau_1, U}$.*

Proof. Let $U > 0$ be a constant which will be determined later. We divide the proof into four steps.

Step 1. Take $u \in \mathcal{U}_{T, U}$. By the definition of \tilde{u} in (11), we have

$$\begin{aligned} \|\tilde{u}_t\|_{l+1,\alpha} &\leq C_l \|(\nabla\varphi_t)^{-1}\|_{l+1,\alpha} \|u_t \circ \varphi_t\|_{l+1,\alpha} \\ &\leq C_l \|(\nabla\varphi_t)^{-1}\|_{l+1,\alpha} \|u_t\|_{l+1,\alpha} (1 + \|\nabla\varphi_t\|_{l,\alpha})^{l+2}. \end{aligned}$$

Note that

$$\|(\nabla\varphi_t)^{-1}\|_{l+1,\alpha} = \|(\nabla\varphi_t^{-1}) \circ \varphi_t\|_{l+1,\alpha} \leq C_l \|\nabla\varphi_t^{-1}\|_{l+1,\alpha} (1 + \|\nabla\varphi_t\|_{l,\alpha})^{l+2}.$$

Therefore,

$$\tilde{U}_t := \sup_{s \leq t} \|\tilde{u}_s\|_{l+1,\alpha} \leq C_{l,\varphi,t} U_t,$$

where

$$C_{l,\varphi,t} = \sup_{s \leq t} (1 + \|\varphi_s\|_{l+2,\alpha} \vee \|\varphi_s^{-1}\|_{l+2,\alpha})^{2l+5}. \quad (15)$$

Step 2. Let Y be the flow generated by \tilde{u} , and denote by $\ell = Y^{-1} - \mathbf{I}$. Then applying Lemma 3.3 with u replaced by \tilde{u} gives us

$$\|\nabla \ell_t\|_{l,\alpha} \leq f_{l,\alpha}(t, \tilde{U}_t) \leq f_{l,\alpha}(t, C_{l,\varphi,t} U_t), \quad t \leq T.$$

Step 3. Let $X_t = \varphi_t \circ Y_t$ and denote by $m_t = \varphi_t^{-1} - \mathbf{I}$, $0 \leq t \leq T$. Then we have

$$X_t^{-1} = Y_t^{-1} \circ \varphi_t^{-1} = \varphi_t^{-1} + \ell_t \circ \varphi_t^{-1} = \mathbf{I} + m_t + \ell_t \circ \varphi_t^{-1}.$$

By Lemma 3.2 and Step 2,

$$\begin{aligned} \|\nabla(\ell_t \circ \varphi_t^{-1})\|_{l,\alpha} &\leq C_l \|(\nabla \ell_t) \circ \varphi_t^{-1}\|_{l,\alpha} \|\nabla \varphi_t^{-1}\|_{l,\alpha} \\ &\leq C_l \|\nabla \ell_t\|_{l,\alpha} (1 + \|\nabla \varphi_t^{-1}\|_{l-1,\alpha})^{l+1} \|\nabla \varphi_t^{-1}\|_{l,\alpha} \\ &\leq C_{l,\varphi,t} f_{l,\alpha}(t, C_{l,\varphi,t} U_t), \end{aligned}$$

where $C_{l,\varphi,t}$ is defined in (15).

Let $\tilde{\ell} = m + \ell \circ \varphi^{-1}$ and $C_{l,m,t} := \sup_{s \leq t} \|\nabla m_s\|_{l,\alpha}$. Then

$$\|\nabla \tilde{\ell}_t\|_{l,\alpha} \leq \|\nabla m_t\|_{l,\alpha} + \|\nabla(\ell_t \circ \varphi_t^{-1})\|_{l,\alpha} \leq C_{l,m,t} + C_{l,\varphi,t} f_{l,\alpha}(t, C_{l,\varphi,t} U_t). \quad (16)$$

Step 4. By the definitions of \hat{u} in (11) and $\tilde{\ell}$ in Step 3, we have

$$\hat{u}_t = \mathbf{P}[(\nabla X_t^{-1})^* u_0(X_t^{-1})] = \mathbf{W}(\tilde{\ell}_t, u_0(X_t^{-1})).$$

Lemmas 3.4 and 3.2 imply that

$$\|\hat{u}_t\|_{l+1,\alpha} \leq C(1 + \|\nabla \tilde{\ell}_t\|_{l,\alpha}) \|u_0 \circ (\mathbf{I} + \tilde{\ell}_t)\|_{l+1,\alpha} \leq C_l \|u_0\|_{l+1,\alpha} (1 + \|\nabla \tilde{\ell}_t\|_{l,\alpha})^{l+3}. \quad (17)$$

Let $U = 2^{l+3} C_l \|u_0\|_{l+1,\alpha}$. Since $\varphi_0 = \mathbf{I}$, one has $m_0 \equiv 0$ which implies that $C_{l,m,t}$ decreases to 0 as $t \rightarrow 0$. Hence, by the definitions of $C_{l,m,t}$ and $f_{l,\alpha}(t, \theta)$, we see that

$$\tau_1 = \inf \{t > 0 : C_{l,m,t} + C_{l,\varphi,t} f_{l,\alpha}(t, C_{l,\varphi,t} U) > 1\} > 0.$$

For $u \in \mathcal{U}_{\tau_1, U}$, one has $U_t = \sup_{s \leq t} \|u_s\|_{l+1,\alpha} \leq U$ for all $t \leq \tau_1$. By (16), $\|\nabla \tilde{\ell}_t\|_{l,\alpha} \leq 1$ for all $t \leq \tau_1$. Thus (17) implies that $\hat{u}|_{t \in [0, \tau_1]} \in \mathcal{U}_{\tau_1, U}$. Now it is clear that one has $\Phi(\mathcal{U}_{\tau_1, U}) \subset \mathcal{U}_{\tau_1, U}$ for U and τ_1 defined above. \square

The next estimate is need for establishing contraction property of Φ .

Lemma 3.6. *Let $u, \bar{u} \in C([0, T], C^{l+1,\alpha}(\mathbb{T}^3, \mathbb{R}^3))$ be satisfying*

$$\sup_{s \leq t} (\|u_s\|_{l+1,\alpha} \vee \|\bar{u}_s\|_{l+1,\alpha}) \leq U_t, \quad t \in [0, T].$$

Let Y, \bar{Y} be the flows generated by u and \bar{u} , respectively. Then there exists a continuous function $\bar{f}_{l,\alpha} : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is increasing in both variables, such that

$$\|Y_t - \bar{Y}_t\|_{l,\alpha} \vee \|Y_t^{-1} - \bar{Y}_t^{-1}\|_{l,\alpha} \leq \bar{f}_{l,\alpha}(t, U_t) \int_0^t \|u_s - \bar{u}_s\|_{l,\alpha} ds.$$

Proof. We have

$$Y_t - \bar{Y}_t = \int_0^t (u_s(Y_s) - \bar{u}_s(\bar{Y}_s)) ds.$$

Therefore, by Lemma 3.2,

$$\begin{aligned} \|Y_t - \bar{Y}_t\|_{l,\alpha} &\leq \int_0^t \|u_s(Y_s) - \bar{u}_s(\bar{Y}_s)\|_{l,\alpha} \, ds \\ &\leq C_l \int_0^t (1 + \|\nabla Y_s\|_{l-1,\alpha} + \|\nabla \bar{Y}_s\|_{l-1,\alpha})^{l+1} \\ &\quad \times [\|u_s - \bar{u}_s\|_{l,\alpha} + (\|\nabla u_s\|_{l,\alpha} \wedge \|\nabla \bar{u}_s\|_{l,\alpha}) \|Y_s - \bar{Y}_s\|_{l,\alpha}] \, ds, \end{aligned}$$

Lemma 3.3 implies that

$$\|\nabla Y_s\|_{l-1,\alpha} = \|\mathbb{I} + \nabla \lambda_s\|_{l-1,\alpha} \leq 1 + f_{l,\alpha}(s, U_s).$$

Similarly, $\|\nabla \bar{Y}_s\|_{l-1,\alpha} \leq 1 + f_{l,\alpha}(s, U_s)$. Substituting these estimates into the above inequality yields

$$\|Y_t - \bar{Y}_t\|_{l,\alpha} \leq 3^{l+1} C_l \int_0^t [1 + f_{l,\alpha}(s, U_s)]^{l+1} [\|u_s - \bar{u}_s\|_{l,\alpha} + U_s \|Y_s - \bar{Y}_s\|_{l,\alpha}] \, ds.$$

From this and Gronwall's inequality we obtain the first assertion. The estimate on the inverse flows follows analogously by reversing the time, see the end of the proof of Lemma 3.3. \square

Before proving that the map Φ is a contraction in a certain space, we introduce the following property of the operator \mathbf{W} defined in Lemma 3.4 (see [16, Proposition 3.1]).

Lemma 3.7. *Let $l \geq 1$ and $\ell_i, v_i \in C^{l,\alpha}$, satisfying*

$$\|\nabla \ell_i\|_{l-1,\alpha} \leq L, \quad \|v_i\|_{l,\alpha} \leq V, \quad i = 1, 2.$$

Then there exists $C = C(l, \alpha, L)$ such that

$$\|\mathbf{W}(\ell_1, v_1) - \mathbf{W}(\ell_2, v_2)\|_{l,\alpha} \leq C(V \|\ell_1 - \ell_2\|_{l,\alpha} + \|v_1 - v_2\|_{l,\alpha}).$$

Proposition 3.8. *Let U be given in Proposition 3.5. There exists $\tau \in (0, \tau_1]$ such that $\Phi : \mathcal{U}_{\tau,U} \rightarrow \mathcal{U}_{\tau,U}$ is a contraction with respect to the weaker norm $\|u\|_{\mathcal{U}_{\tau,U}} = \sup_{t \leq \tau} \|u_t\|_{l,\alpha}$.*

Proof. Step 1. Let $u_1, u_2 \in \mathcal{U}_{\tau_1,U}$. Define \tilde{u}_1, \tilde{u}_2 as in (11). We have by Lemma 3.2 that

$$\begin{aligned} \|\tilde{u}_{1,t} - \tilde{u}_{2,t}\|_{l,\alpha} &\leq C_l \|(\nabla \varphi_t)^{-1}\|_{l,\alpha} \|u_{1,t} \circ \varphi_t - u_{2,t} \circ \varphi_t\|_{l,\alpha} \\ &\leq C_l \|(\nabla \varphi_t)^{-1}\|_{l,\alpha} \|u_{1,t} - u_{2,t}\|_{l,\alpha} (1 + \|\nabla \varphi_t\|_{l-1,\alpha})^{l+1} \\ &\leq C_{l,\varphi,t} \|u_{1,t} - u_{2,t}\|_{l,\alpha}, \end{aligned} \tag{18}$$

where $C_{l,\varphi,t}$ is defined in (15). Recall that by Step 1 in the proof of Proposition 3.5, we have

$$\begin{aligned} \tilde{U}_t &:= \sup_{s \leq t} (\|\tilde{u}_{1,s}\|_{l+1,\alpha} \vee \|\tilde{u}_{2,s}\|_{l+1,\alpha}) \\ &\leq C_{l,\varphi,t} \sup_{s \leq t} (\|u_{1,s}\|_{l+1,\alpha} \vee \|u_{2,s}\|_{l+1,\alpha}) \leq C_{l,\varphi,t} U. \end{aligned} \tag{19}$$

Step 2. Let Y_i be the flow associated to \tilde{u}_i and $\ell_i = Y_i^{-1} - \text{I}$, $i = 1, 2$. By Lemma 3.6,

$$\begin{aligned} \|\ell_{1,t} - \ell_{2,t}\|_{l,\alpha} &= \|Y_{1,t}^{-1} - Y_{2,t}^{-1}\|_{l,\alpha} \leq \bar{f}_{l,\alpha}(t, \tilde{U}_t) \int_0^t \|\tilde{u}_{1,s} - \tilde{u}_{2,s}\|_{l,\alpha} ds \\ &\leq C_{l,\varphi,t} \bar{f}_{l,\alpha}(t, C_{l,\varphi,t} U) \int_0^t \|u_{1,s} - u_{2,s}\|_{l,\alpha} ds, \end{aligned}$$

where the last inequality follows from (18) and (19).

Step 3. Recall that $m_t = \varphi_t^{-1} - \text{I}$. Now for $\tilde{\ell}_{i,t} = m_t + \ell_{i,t} \circ \varphi_t^{-1}$, $i = 1, 2$, one has, by Lemma 3.2 and Step 2,

$$\begin{aligned} \|\tilde{\ell}_{1,t} - \tilde{\ell}_{2,t}\|_{l,\alpha} &= \|\ell_{1,t} \circ \varphi_t^{-1} - \ell_{2,t} \circ \varphi_t^{-1}\|_{l,\alpha} \\ &\leq C_l \|\ell_{1,t} - \ell_{2,t}\|_{l,\alpha} (1 + \|\nabla \varphi_t^{-1}\|_{l-1,\alpha})^{l+1} \\ &\leq C_{l,\varphi,t}^2 \bar{f}_{l,\alpha}(t, C_{l,\varphi,t} U) \int_0^t \|u_{1,s} - u_{2,s}\|_{l,\alpha} ds. \end{aligned} \quad (20)$$

By the definition of τ_1 which depends only on φ and U , we have

$$\|\nabla \tilde{\ell}_{i,t}\|_{l,\alpha} \leq 1, \quad t \leq \tau_1, \quad i = 1, 2. \quad (21)$$

Step 4. Denote by $X_{i,t} = \varphi_t \circ Y_{i,t}$; then $X_{i,t}^{-1} = \text{I} + \tilde{\ell}_{i,t}$ and $\hat{u}_{i,t} = \mathbf{W}(\tilde{\ell}_{i,t}, u_0 \circ X_{i,t}^{-1})$, $i = 1, 2$. By (21) and Lemma 3.7, there exists a constant C depending only on l and α such that

$$\|\hat{u}_{1,t} - \hat{u}_{2,t}\|_{l,\alpha} \leq C(V_t \|\tilde{\ell}_{1,t} - \tilde{\ell}_{2,t}\|_{l,\alpha} + \|u_0 \circ X_{1,t}^{-1} - u_0 \circ X_{2,t}^{-1}\|_{l,\alpha}), \quad (22)$$

where

$$\begin{aligned} V_t &= \max_{i=1,2} \|u_0 \circ X_{i,t}^{-1}\|_{l,\alpha} \leq C \|u_0\|_{l,\alpha} \max_{i=1,2} (1 + \|\nabla X_{i,t}^{-1}\|_{l-1,\alpha})^{l+1} \\ &\leq C \|u_0\|_{l,\alpha} \max_{i=1,2} (1 + \|\nabla \tilde{\ell}_{i,t}\|_{l-1,\alpha})^{l+1} \leq C 2^{l+1} \|u_0\|_{l,\alpha} \leq U, \end{aligned}$$

where the third inequality follows from (21). By Lemma 3.2,

$$\begin{aligned} \|u_0 \circ X_{1,t}^{-1} - u_0 \circ X_{2,t}^{-1}\|_{l,\alpha} &\leq C(1 + \|\nabla X_{1,t}^{-1}\|_{l-1,\alpha} + \|\nabla X_{2,t}^{-1}\|_{l-1,\alpha})^{l+1} \\ &\quad \times \|\nabla u_0\|_{l,\alpha} \|X_{1,t}^{-1} - X_{2,t}^{-1}\|_{l,\alpha} \\ &\leq U \|\tilde{\ell}_{1,t} - \tilde{\ell}_{2,t}\|_{l,\alpha}. \end{aligned}$$

Therefore, substituting this estimate into (22) yields

$$\|\hat{u}_{1,t} - \hat{u}_{2,t}\|_{l,\alpha} \leq CU \|\tilde{\ell}_{1,t} - \tilde{\ell}_{2,t}\|_{l,\alpha} \leq C_{l,\varphi,t}^2 U \bar{f}_{l,\alpha}(t, C_{l,\varphi,t} U) \int_0^t \|u_{1,s} - u_{2,s}\|_{l,\alpha} ds,$$

where the last inequality follows from (20). Consequently,

$$\sup_{s \leq t} \|\hat{u}_{1,s} - \hat{u}_{1,s}\|_{l,\alpha} \leq C_{l,\varphi,t}^2 U \bar{f}_{l,\alpha}(t, C_{l,\varphi,t} U) t \sup_{s \leq t} \|u_{1,s} - u_{1,s}\|_{l,\alpha}.$$

Define

$$\tau := \inf \{t \in (0, \tau_1] : C_{l,\varphi,t}^2 U \bar{f}_{l,\alpha}(t, C_{l,\varphi,t} U) t > 1/2\} \quad (23)$$

with the convention that $\inf \emptyset = \tau_1$. Then it is clear that the map Φ is a contraction on $\mathcal{U}_{\tau,U}$ with respect to the norm $\|u\|_{\mathcal{U}_{\tau,U}} = \sup_{t \leq \tau} \|u_t\|_{l,\alpha}$. \square

Finally we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. With the above preparations, the proof is the same as that in [16]. The existence of a fixed point of Φ follows by successive iteration. We define $u_{n+1} = \Phi(u_n)$. The sequence converges strongly with respect to the $C^{l,\alpha}$ -norm. Since $\mathcal{U}_{\tau,U}$ is closed and convex, and the sequence $\{u_n\}_{n \geq 1}$ is uniformly bounded in the $C^{l+1,\alpha}$ -norm, it must have a weak limit $u \in \mathcal{U}_{\tau,U}$. Since Φ is continuous with respect to the weaker $C^{l,\alpha}$ -norm, this limit must be a fixed point of Φ , and thus a solution of the deterministic system (11). \square

3.2. Local existence of (5). Let $\Omega_0 \subset \Omega$ be a full measure set such that for all $\omega \in \Omega_0$, $\varphi_t(\omega, \cdot)$ is a $C^{l+2,\alpha}$ -diffeomorphism on \mathbb{T}^3 for any $t > 0$. Let Φ_ω be the map in Theorem 3.1 associated to $\{\varphi_t(\omega, \cdot)\}_{t \geq 0}$. By Theorem 3.1, there exists $\tau(\omega) > 0$ and a time-dependent vector field $u_t(\omega) \in \mathcal{U}_{\tau(\omega),U}$ such that $\Phi_\omega(u_t(\omega)) = u_t(\omega)$. It follows from the definition of τ in (23) that it is a stopping time. Then the random vector field $u(\omega) : [0, \tau(\omega)] \times \mathbb{T}^3 \ni (t, x) \rightarrow u_t(\omega, x) \in \mathbb{R}^3$ is a solution to (5).

4. Discussions. The inverse flow $A_t(x) = X_t^{-1}(x)$ is a random vector field, solution of the stochastic transport equation

$$dA_t(x) + u_t(x) \cdot \nabla A_t(x) dt + \sum_k \sigma_k(x) \cdot \nabla A_t(x) \circ dW_t^k = 0.$$

This equation does not contain stretching terms of the form $A_t(x) \cdot \nabla u_t(x) dt$ and $A_t(x) \cdot \nabla \sigma_k(x) \circ dW_t^k$, hence the quantity $A_t(x)$ is only transported. Therefore we do not expect a blow-up of $A_t(x)$ itself. We may however expect, in analogy with shocks appearing in nonlinear transport equations (like Burgers equation), that space derivatives of $A_t(x)$ may blow-up. This is the potential mechanism which could lead to blow-up in the formula

$$u_t(x) = \mathbf{P}[(\nabla A_t)^* u_0 \circ A_t](x)$$

as discussed in [3, Section 5].

The question posed by the presence of noise is: could the noise prevent or mitigate blow-up of ∇A_t ? If $u_t(x)$, in the SPDE above, would be given (passive field A_t) and deterministic, several results of regularization due to noise have been proved for similar equations, for instance the absence of shocks for the scalar transport equation with u of class $L^q(0, T; L^p(\mathbb{R}^d, \mathbb{R}^d))$ with $\frac{d}{p} + \frac{2}{q} < 1$ (see [9]), or the absence of singularity for a passive magnetic field proved in [12, 13] under various assumptions. The intuitive reason is that noise prevents $A_t(x)$ to stretch for too much time around the more singular points of $u_t(x)$, because $A_t(x)$ is continuously randomly displaced. However, no result of this form has been proved until now in the case when $u_t(x)$ is random (see [6] for a related work), as it is in the nonlinear case; the obstruction is not technical but conceptual: the singularities of $u_t(x)$ move accordingly to noise and to $A_t(x)$ itself, hence there is no straightforward reason why noise should displace $A_t(x)$ to avoid those singularities.

Thus the question of singularities remains open, as it is in the deterministic case but here, thanks to the noise, new intuitions may develop.

Remark 4.1. As remarked by an anonymous referee, one could introduce a size parameter, say $\sqrt{2\nu}$, $\nu > 0$, in the vector fields σ_k and try to prove that the results above are better for a large value of the parameter $\nu > 0$, namely for stronger noise. In particular, the Ito-Stratonovich corrector, in special cases, is a Laplacian multiplied by $\nu > 0$, hence the parameter may be intuitively associated to a concept of viscosity. Unfortunately there is no technology nowadays to understand whether

these intuitions correspond to true facts; there are positive results in the linear case, see [9, 10, 12], but not in the nonlinear one. On the contrary, unfortunately, our estimates proved above go in the opposite direction. In our proof, we first solve the Stratonovich equation without drift to get a smooth stochastic flow; then, using the stochastic flow, we define a random vector field and consider the corresponding ODE. The regularity (i.e. the norm in certain Hölder spaces) of the vector field depends on that of the stochastic flow. If ν is larger, then we get worse regularity estimates on the random vector field, which in turn implies existence of solution to the ODE with shorter lifetime.

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