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Cohomology of finite and affine type Artin groups over abelian representations

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Contents

Introduction

Background

The subject of this thesis is the study of the cohomology of some families of Artin groups. The classical braid group $\mathrm{Br}(n)$ was first defined by Artin in [Art25]. Given a manifold $\mathbf{M}$ of dimension $d \geq 2$ we define the space $F(\mathbf{M}, n)$ as the configuration space of *ordered* n-tuples of distinct points in M and the space $C(\mathbf{M}, n)$ as the configuration space of *unordered* n-tuples of distinct points in M. Artin gives a geometric description of the braid group $\mathrm{Br}(n)$. If [FN62] Fox and Neuwirth proved that the braid group defined by Artin is the fundamental group of the configuration space $C(\mathbf{E}, n)$ of the Euclidean plane $\mathbf{E}$. The definition given by Artin leaded to many interesting generalization.

The classical theory of braids is deeply connected with the theory of reflection groups or Coxeter groups. Loosely speaking the classical braid group can be obtained from the permutation group $\mathfrak{S}_n$ dropping the torsion relations $s^2 = e$. We have the same kind of relation between Artin groups and reflection groups. In particular for each Coxeter group W we can associate an Artin group G_W. The classical Artin braid group $\mathrm{Br}(n)$ is the Artin group associated to the Coxeter group of type A_{n-1}, that is the group of permutations of n elements $\mathfrak{S}_n$. For each reflection group we can associate the arrangement of reflection hyperplanes $\mathcal{A}_W$. The Coxeter group W acts on the complement of this arrangement. In [Bri71] Brieskorn proved that the fundamental group of the quotient $\mathbf{X}_W$ of the complement $\mathbf{Y}_W$ of the complexified arrangement with respect to the action of the group W is the corresponding Artin group G_W.

Actually, for finite reflection groups, the orbit space $\mathbf{X}_W$ of the complement $\mathbf{Y}_W$ of the complex arrangement is a classifying space of the corresponding Artin group. This was proved by Fox and Neuwirth ([FN62]) for the Artin braid group. This result had also been proved for Coxeter groups of type A_n, C_n, D_n, G_2, F_4 and $I_2(p)$ by easy direct methods by Brieskorn [Bri73a]. In [Del72] Deligne gives a general proof of this result for all finite Coxeter groups (see also [Par93]). It turns out that the braid group and all the other Artin groups associated to finite Coxeter groups are torsion free; for an infinite Coxeter group W the corresponding Artin group G_W is known to be torsion free only when the space $\mathbf{X}_W$ is known to be a $K(\pi, 1)$ (see further for examples), but no much is known in general. Recently Bessis proved ([Bes06]) that the complement $\mathbf{Y}_W$ is a $K(\pi, 1)$ space for all finite *complex* reflection groups.

The fact that for a finite Coxeter group W the spaces $\mathbf{X}_W$ and $\mathbf{Y}_W$ are of type $K(\pi, 1)$ is used by Brieskorn ([Bri73a]), Arnol′d ([Arn69]), Fuks ([Fuk70]), Vaĭnšteĭn ([Vaĭ78]), Gorjunov ([Gor78]) and others to compute the cohomology of the corresponding fundamental groups. Independently in [Coh76], Cohen compute the homology of the braid group using the theory of iterated loop spaces.

For a Coxeter group W of type A_k, D_k, E_6, E_7 and E_8, the space $\mathbf{X}_W$ is homeomorphic to the complement of the discriminant locus in the base space of the versal deformation of the corresponding rational singularity (see [Bri73b]). It is not known whether these complements are $K(\pi, 1)$'s for all singularities.

When W is finite, the space $\mathbf{X}_W$ is an affine variety, the complement of a singular hypersurface $\mathbf{V}$ associated to the group W in an affine space $\mathbf{E}$. In order to study the singularity associated to a Coxeter group W, one can consider the corresponding Milnor fibration. In [Mil68], Milnor studies the local behaviour of a complex hypersurface $\mathbf{V}$ in an euclidean space $\mathbf{E}$ around a singular point z_0. If S is a sphere of sufficiently small radius about z_0 and $K = S \cap V$, then $S - K$ is a smooth fiber bundle over the circle S^1. In fact if $f(z) = 0$ defines V, f a complex polynomial, then $z \to f(z)/|f(z)|$ defines the fibration. Many results are given in case of an isolated singularity and the fiber $\mathbf{F}$ turns out to be, up to homotopy, a wedge of spheres. The singularities associated to Coxeter groups are not isolated. It is still possible to study the topology of the fiber, computing the cohomology groups and the monodromy action of the fibrations. It turns out that this data provide much more information that the cohomology of the complement $\mathbf{X}_W$ of the singularity. Actually the Milnor fiber $\mathbf{F}$ is homotopy equivalent to an infinite cyclic covering of the complement $\mathbf{X}_W$ and the cohomology of the fiber $\mathbf{F}$, together with the action induced by the monodromy, is equivalent to the cohomology of the complement $\mathbf{X}_W$ in a local system $\mathcal{L}$ defined over the ring of Laurent polynomials in one variable $\mathbb{Q}[q^{\pm 1}]$. The result follows easily from the Shapiro's Lemma for homology, while in cohomology to provide another result (Theorem 2.7.2) in order to switch from Laurent series to Laurent polynomials. In the case of the classical braid group this local system corresponds to the determinant of the Burau representation ([Bur35]) and the singularity associated to W is the discriminant. The theory of hypergeometric functions (described in [Gel86], see also [OT01, Var95]) provides further motivations to the study of twisted coefficients cohomology for Artin groups.

As we have seen, the theory of Artin groups is connected with the study of hyperplane arrangements associated to Coxeter groups. For each hyperplane arrangement $\mathcal{A}$ in an affine space $\mathbf{E}$ we can associate a combinatorial data, namely the stratification induced by the arrangement on the space $\mathbf{E}$. From the topological viewpoint it is interesting to consider the complement $\mathbf{Y}_{\mathcal{A}}$ of the arrangement. In general one can ask how much the combinatorial data determines the topology. In [Sal87] Salvetti introduced a CW-complex $\mathbf{C}$ associated to a real arrangement $\mathcal{A}$ and determined by the stratification data. He proved that this complex is homotopy equivalent to the complement of the complexified arrangement. Moreover if the arrangement $\mathcal{A}$ is associated to a reflection group W, the group W acts on the complex $\mathbf{C}$ and the quotient complex is homotopy equivalent to the space $\mathbf{X}_W$ (see [Sal94], [DCS96]). An extension of the results for an oriented matroid can be found in [GR89]. For a general complex arrangement, in [BZ92] Björner and Ziegler construct a finite regular cell complex of the homotopy type of the complement of the arrangement

Using the purely combinatorial description of the Salvetti complex, it is possible to study the local system cohomology of the space $\mathbf{X}_W$. We recall that since for a finite Coxeter group W the space $\mathbf{X}_W$ is of type $K(\pi, 1)$, the cohomology of $\mathbf{X}_W$ (eventually with a local system of coefficients) is equivalent to the cohomology of the corresponding Artin group G_W. For a finite Coxeter group W an equivalent complex was discovered by different methods by Squier in [Squ94]. The cohomology of the space $\mathbf{X}_W$ over the local system $\mathcal{L}$ given by the ring of Laurent polynomials with rational coefficients $\mathbb{Q}[q^{\pm 1}]$ was computed for all finite irreducible Coxeter groups W by De Concini, Procesi, Salvetti and Stumbo ([DCPS01, DCPSS99]). Independently Frenkel$'$ ([Fre88]) and Markaryan ([Mar96]) get the same result, with different methods, for the Coxeter group of type A_n.

A natural and useful generalization of these computations is the study of the local system over the ring of Laurent polynomials with integer coefficients $\mathbb{Z}[q^{\pm 1}]$. For Artin groups associated to exceptional Coxeter groups the computations are given in [CS04]. In this thesis we give a complete description in cases A_n and B_n of the homology of the Artin groups, with coefficients in the ring $\mathbb{Z}[q^{\pm 1}]$ of Laurent polynomials. We recall that this is equivalent to the computation of the homology of the corresponding Milnor fiber with integer coefficients and of the associated monodromy action. We do this generalizing some ideas of Markaryan and studying a spectral sequence induced by a natural filtration on the Salvetti complex. We computes the homology of these groups, instead of the cohomology, for technical reasons and simplicity of computations. The final results stated in Theorem 4.4.1 and Theorem 5.3.1 and are completely equivalent to cohomology computations. The main problem with this computations is that the ring $\mathbb{Z}[q^{\pm 1}]$ is not a PID and so a deep study of the spectral sequence is needed. The natural embedding of Coxeter groups $W_n \hookrightarrow W_{n+1}$, where W_n is a Coxeter group of type A_n or B_n induces analogous maps for Artin groups and hence homology maps. This allows us to define the limit group G_∞, that turns out to be infinitely generated. In case A_n we have the braid group $\mathrm{Br}(\infty)$ on infinitely many strands. In Corollary 4.4.2 and Theorem 5.3.2 we give a description of the homology of this groups as the stable homology of the corresponding finitely generated Artin groups. Our results generalize the computations obtained by different methods by Cohen and Pakianathan in [CP07] in the case of the braid group $\mathrm{Br}(\infty)$ on infinitely many strands on the same local system with coefficients in a field.

Another natural generalization of the results of [DCPS01, DCPSS99] is the computation of the cohomology of Artin groups associated to infinite Coxeter groups. In this case many complications occurs. First of all in general it is only conjectured that the space $\mathbf{X}_W$ is of type $K(\pi, 1)$. The conjecture is proved only in special or *generic* cases. Okonek in [Oko79] proves that the space $\mathbf{X}_W$ is of type $K(\pi, 1)$ for the affine Coxeter group of type $\widetilde{A}_n$ (see also [CP03]) and $\widetilde{C}_n$. Other results are given by Hendriks ([Hen85]) for Coxeter groups of *large type* and by Charney and Davis ([CD95b]) for Coxeter groups satisfying the FC condition and for other infinite Coxeter groups that are in some sense *generic* (see Section 1.4). Moreover in the case of infinite Coxeter groups we can construct a Milnor fibration only up to homotopy.

We compute the cohomology, with coefficients in the local system given by the ring of Laurent polynomials $\mathbb{Q}[q^{\pm 1}]$, for the Artin group of affine type $\widetilde{A}_n$. Some interesting group embedding show that this cohomology is equivalent to the cohomology of the classical braid group $\mathrm{Br}(n)$ with coefficients in a non-abelian representation introduced by Tits in [Tit66] and *re-discovered* later by Tong, Yang and Ma in [TYM96]. Another group embedding relates the homology of the affine Artin group of type $\widetilde{A}_n$ to the cohomology of the Artin group of type B_n with coefficients in the Laurent polynomial in two indeterminates, $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$, that turns out to be the $\mathbb{Q}$-algebra of the abelianization of the Artin group of type B_n. Hence from this cohomology we can recover any other cohomology of this group with coefficients in an abelian representation. The result is given in Theorem 6.1.1. We compute this cohomology using a filtration and an induced spectral sequence for the Salvetti complex for B_n. Some applications are given (see Theorem 6.3.3 and Proposition 6.4.5).

Moreover in Theorem 7.2.1 we prove that the complement $\mathbf{Y}_{\mathcal{A}}$ of the complex arrangement of type $\widetilde{B}_n$ is a $K(\pi, 1)$ space. This fact allows us to compute the cohomology of the Artin group of affine type $\widetilde{B}_n$ with coefficient in the ring of Laurent polynomials $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ (Theorem 7.3.1). In order to perform this computation we use the Salvetti complex associated to the Coxeter group $\widetilde{B}_n$. Since this group is infinite, some modification are necessary respect to the usual Salvetti complex. Moreover some technical restrictions are needed in order to use the Laurent polynomials local system to describe the cohomology of the Milnor fiber (see Proposition 7.3.2). Also in this case we are considering the most general abelian representation for this group, hence as a corollary we can recover the cohomology with constant coefficients for the space $\mathbf{X}_W$ associated to the affine Coxeter group of type $\widetilde{B}_n$ (Theorem 7.3.7).

Throughout all this thesis, we always want to keep in mind, as a fixed picture, the classical Artin braid group. Most of our work is a study around the topological constructions related to this group with some possible generalizations. We will try to stress this, in particular in the first chapters, with examples that show how general constructions, when applied to the case of classical braids, give rise to familiar objects.

Overview

The first two chapters of this thesis are a short survey of some standard facts about Coxeter groups, Artin groups and group cohomology.

Chapter 1

We briefly resume the theory of Coxeter groups, their classifications and their invariant polynomial algebras. We shows how one can associate to a Coxeter group W a real hyperplane arrangement and to this arrangement a space $\mathbf{X}_W$ with a fundamental group that is an Artin group. We also present the problem about whether or not the space $\mathbf{X}_W$ is a $K(\pi, 1)$ space and we see how this problem has a positive solution when the Coxeter group W is finite. Some technical statements about Poincaré series for Coxeter group are given. Finally we present the construction of a Milnor fibration applied to a singularity associated to a Coxeter group.

Chapter 2

We give the basic definitions about group cohomology and local systems. We state the classical results about the cohomology of Artin groups and in particular the Shapiro's Lemma that relates the homology (or cohomology) of two groups H, G, in case of group a extension $H < G$. Hence we introduce the local system of Laurent polynomials associated to the Milnor fibration described in the previous Chapter and we state the known results about the cohomology of Artin groups with coefficients in this local system.

Chapter 3

We present the construction of the Salvetti complex for a general arrangement and for the special case of an arrangement of reflection hyperplanes associated to a Coxeter group. We explain how this gives a finite complex that compute the cohomology of the Artin groups. Hence we give a general idea of the filtration and spectral sequence techniques that we use to study this complex. In Section 3.4 and 3.5 we give two independent arguments in order to see how, using Shapiro's Lemma, the cohomology of the Milnor fiber is in some cases equivalent to the cohomology of the Artin group with local coefficients in the ring of Laurent polynomials.

Chapter 4

In this Chapter we compute the homology of the braid groups, with coefficients in the module $\mathbb{Z}[q^{\pm 1}]$ given by the ring of Laurent polynomials with integer coefficients. The action of the braid group is defined by mapping each generator of the standard presentation to multiplication by $-q$. The main tool for our computation is the study of the cohomology of the *algebra of q-divided polynomials*. The homology is endowed with a natural ring structure. Through the Chapter we prove some technical results about cyclotomic polynomials, needed for computations.

Chapter 5

In this Chapter we compute the homology of the Artin groups of type B_n, with coefficients in the module $\mathbb{Z}[q^{\pm 1}]$ given by the ring of Laurent polynomials with integer coefficients. This homology turns out to be a module on the homology computed in the previous Chapter.

Chapter 6

The result of this Chapter is the determination of the cohomology of Artin groups of type A_n, B_n and $\tilde{A}_n$ with non-trivial local coefficients. The main result (Theorem 6.1.1) is an explicit computation of the cohomology of the Artin group of type B_n with coefficients over the module $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$. Here the first $(n-1)$ standard generators of the group act by $(-q)-$multiplication, while the last one acts by $(-t)-$multiplication.

Chapter 7

In this Chapter we prove that the complement to the affine complex arrangement of type $\widetilde{B}_n$ is a $K(\pi, 1)$ space. We also compute the cohomology of the affine Artin group G of type $\widetilde{B}_n$ with coefficients in some local systems. In particular, we consider the module $Q[q^{\pm 1}, t^{\pm 1}]$, where the first n-standard generators of G act by $(-q)$-multiplication while the last generator acts by $(-t)$-multiplication. The cohomology of G with trivial coefficients is derived from the previous one.

The results presented in the sections 3.4 and 3.5 and in the chapters 4, 5, 6 and 7 are original. The results presented in Section 3.5 are published in [Cal05]. The results of Chapter 4 are published in [Cal06]. The results of the chapters 6 and 7 are obtained in collaboration with Davide Moroni and Mario Salvetti; the results of Chapter 6 appear in the work [CMS06a] accepted for publication; the results of Chapter 7 appear in [CMS06c] and are submitted for publication.

Chapter 1

Coxeter groups and arrangements of hyperplanes

In this Chapter we introduce some definitions and constructions related to Coxeter groups. For the results presented in sections 1.1, 1.2, 1.3 and 1.5 we mainly refer to [Bou68] and [Hum90].

1.1 Coxeter groups

We denote by W a group (with multiplicative notation) and by S a system of generators of W such that $S = S^{-1}$, $e \notin S$.

Definition 1.1.1. Let $w \in W$. The smallest integer $l \geq 0$ such that w can be written as a product of l elements of S is called *length* of w with respect to S and is written $l_S(w)$.

Moreover, now we suppose that the set S is made of elements of order 2.

Definition 1.1.2. The couple (W, S) is called *Coxeter system* if, for all $s, s' \in S$, the following holds: let $m(s, s')$ be the order or the product ss' and let I be the set of couples (s, s') such that $m(s, s')$ is finite, then the set S with the relations $(ss')^{m(s,s')} = e$ for all (s, s') in I is a presentation for the group W, that is

$$W =< s, s' \mid (ss')^{m(s,s')} = e > .$$

If (W, S) is a Coxeter system we also call W a *Coxeter group*. Let $X \subseteq S$, we write W_X for the subgroup of W generated by the elements of X. A subgroup like this and any subgroup conjugated to it by an element of W is called *parabolic subgroup*.

Proposition 1.1.3. *Let (W, S) be a Coxeter system and let $X \subseteq S$. The following properties hold:*

(i) (W_X, X) *is a Coxeter system, for any* $X \subset S$*;*

(ii) *If we consider* W_X *as a subgroup of* W *with length function* l_X*, then we have the equality* $l_X = l_S$ *for all the elements in the subgroup* W_X*;*

(iii) *Let us define* $W^X = \{w \in W \mid l(ws) > l(w) \forall s \in X\}$*. Given* $w \in W$ *there exist a unique element* $u \in W^X$ *and a unique element* $v \in W_X$ *such that* $w = uv$*. Moreover* $l(w) = l(u) + l(v)$ *and* u *is the unique element of shortest length in the coset* wW_X*.*

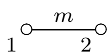
$\square$

We can associate to a Coxeter system (W, S) a symmetric matrix with values in $\mathbb{N} \cup \{+\infty\}$; this matrix is called *Coxeter matrix* ad is defined as follows: $M = (m(s, s'))_{s,s' \in S}$, where $m(s, s')$ is the order of the element ss'. We can also associate to the Coxeter system (W, S) a graph Γ, called *Coxeter graph*, with labelling function f, as follows: the vertexes of the graph Γ correspond to the elements of S and the edges of Γ correspond to the pairs $\{s, s'\}$ of distinct elements of S such that $m(s, s') \geq 3$. The function f associates to the edge $\{s, s'\}$ the value $m(s, s')$. It is clear that the Coxeter matrix completely determines the Coxeter graph and also the converse is true.

Definition 1.1.4. We call a Coxeter system (W, S) *irreducible* if its graph Γ is connected.

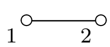
In the picture of a Coxeter graph it is convenient to write above each edge e the value $f(e)$. In order to simplify the reading we usually omit to write the value $f(e)$ when it is equal to 3.

Example 1.1.5. We can easily see that the dihedral group of order $2m$ is a Coxeter group with two generators. Its Coxeter matrix is $\begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}$, and its Coxeter graph is:



if $m \geq 3$

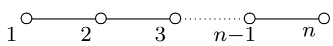
or simply



if $m = 3$

if $m = 2$

Analogously the symmetric group $\mathfrak{S}_{n+1}$ can be represented by the graph:



with n vertexes.

1.2 Coxeter groups and arrangements

Let $\mathbf{E}$ be an affine real space of finite dimension and suppose we have a set of hyperplanes $\mathfrak{H} \subset \mathbf{E}$. We call the collection $\mathfrak{H}$ an *arrangement* of hyperplanes of $\mathbf{E}$. We can define the following equivalent relation between points in $\mathbf{E}$: we say that two points x, y are in the same equivalence class if and only if for all $\mathbf{H} \in \mathfrak{H}$ one of the following hold:

(i) $x \in \mathbf{H}$ and $y \in \mathbf{H}$;

(ii) x and y are in the same open half-space determined by the hyperplane $\mathbf{H}$.

Definition 1.2.1. We call *facet* of $\mathbf{E}$ with respect to the arrangement $\mathfrak{H}$ each one of the equivalence classes for the relation defined above.

Definition 1.2.2. We call *support* of a facet the intersection of all the hyperplanes that contain the facet.

Definition 1.2.3. We call *chamber* of $\mathbf{E}$ (with respect to the arrangement $\mathfrak{H}$) a facet the is not contained in any hyperplane of $\mathfrak{H}$.

Definition 1.2.4. Let C be a chamber of $\mathbf{E}$. We call *faces* of C all the facets that are contained in the closure $\overline{C}$ and whose support is an hyperplane of the arrangement $\mathfrak{H}$. Moreover we say that an hyperplane $\mathbf{H}$ is a *wall* of C if it is the support of a face of C

Now we suppose that the space $\mathbf{E}$ is endowed with a scalar product (i.e. a bilinear, symmetric, positively defined, non degenerate form). Given an hyperplane $\mathbf{H}$ we write $s_{\mathbf{H}}$ for the orthogonal reflection with respect to $\mathbf{H}$. Let $\mathfrak{H}$ an arrangement of hyperplanes of $\mathbf{E}$ and let W be the group of isometries generated by the orthogonal reflections with respect to these hyperplanes. We call W *reflection group*. Clearly W acts in a natural way on the affine subspaces of $\mathbf{E}$. Let us suppose that the following conditions are satisfied:

(i) The arrangement $\mathfrak{H}$ is closed for the action of W;

(ii) Given two compact subsets $\mathbf{K}$ and $\mathbf{L}$ of $\mathbf{E}$, the set $\{w \in W \mid w(\mathbf{K}) \cap \mathbf{L} \neq \emptyset\}$ is finite.

Then the set $\mathfrak{H}$ is locally finite and we have the following:

Theorem 1.2.5. *Let C be a chamber and S the set of orthogonal reflections with respect to the walls of C.*

(i) The couple (W, S) is a Coxeter system;

(ii) Let $w \in W$ and $\mathbf{H}$ be a wall of C. The relation $l(s_{\mathbf{H}}w) > l(w)$ imply that the chambers C and $w(C)$ are on the same side with respect to to the hyperplane H;

(iii) For every chamber C' there exist a unique element $w \in W$ such that $w(C) = C'$;

(iv) $\mathfrak{H}$ is the set of hyperplanes $\mathbf{H}$ such that $s_{\mathbf{H}} \in W$.

□

Let (W, S) be a Coxeter system. We can represent W as a group generated by orthogonal reflections in an euclidean space. Consider the real vector space $\mathbf{V}$ with dimension $n = |S|$ and with basis $\{\alpha_s \mid s \in S\}$. We endow the space $\mathbf{V}$ with the bilinear symmetric form so defined:

$$B(\alpha_s, \alpha_{s'}) = \begin{cases} -\cos\frac{\pi}{m(s,s')} & \text{if } m(s,s') \in \mathbb{Z}, \\ -1 & \text{if } m(s,s') = \infty. \end{cases}$$

For each $s \in S$ we associate the reflection σ_s given by

$$\sigma_s(x) = x - 2B(\alpha_s, x)\alpha_s.$$

By means of the reflections σ_s we can define a unique homomorphism $\sigma : W \to \mathbf{GL}(\mathbf{V})$, which is called *Tit's representation* of W.

Theorem 1.2.6. *Let W be a Coxeter group. The homomorphism σ defined above is injective. Moreover the following statements are equivalent:*

(i) W is finite;

(ii) The form B is positive, non degenerate;

(iii) W is a finite orthogonal reflection group.

□

Remark 1.2.7. It follows from the injectivity of σ that a reflection group is completely determined by its Coxeter graph.

1.3 Finite and affine Coxeter groups

Let $\mathbf{V}$ be a vector space of dimension n over a generic field K.

Definition 1.3.1. We call *pseudo-reflection* an endomorphism s of $\mathbf{V}$ such that $\mathrm{Id} - s$ has rank 1. We call s *reflection* if $s^2 = \mathrm{Id}$.

Now let we suppose that G is a subgroup of $\mathbf{GL}(\mathbf{V})$ generated by pseudo-reflections. Moreover we suppose that the characteristic of K does not divide the order of the group G (e.g. when $\mathrm{char}(K)$ is 0). We fix a basis $\{v_1, \ldots, v_n\}$ of $\mathbf{V}$ and we consider the symmetric algebra $\mathcal{S}$ of $\mathbf{V}$. This algebra is canonically isomorphic to the polynomial algebra $\mathcal{A} = K[x_1, \ldots, x_n]$. We can consider the *subalgebra* $\mathcal{R} = \mathcal{A}^G$ *of the G-invariant polynomials*. We have :

Theorem 1.3.2 (Chevalley). *The subalgebra $\mathcal{R}$ is generated, as a K-algebra, by n homogeneous algebraically independent elements $f_1, \dots, f_n$ of positive degree. In particular $\mathcal{R}$ is a graded polynomial K-algebra with trascendancy degree n over K.* □

Clearly the family of generators is not uniquely determined, but we have the following uniqueness result:

Theorem 1.3.3. *Let $f_1, \dots, f_n$ and $g_1, \dots, g_n$ be two set of homogeneous, algebraically independent polynomials generating $\mathcal{R}$. We write d_i (resp. e_i) for the degree of f_i (resp. g_i). Then, up to reordering the two sets, for all indexes i we have that $d_i = e_i$.* □

Definition 1.3.4. We call the integers d_i *characteristic degrees* (or *characteristic exponents*) of G.

We can now state the following result for finite Coxeter groups:

Theorem 1.3.5. *Let (W, S) be a finite irreducible Coxeter system. Its Coxeter graph is one of the following listed in Table 1.1 and its characteristic degrees are indicated in the Table.* □

Example 1.3.6. In the case A_n the Coxeter group W is isomorphic to the group of permutations $\mathfrak{S}_{\mathfrak{n}+\mathfrak{1}}$ with set of generators $S = \{\sigma_{i,i+1} \mid i = 1, \dots, n\}$. The group $\mathfrak{S}_{\mathfrak{n}+\mathfrak{1}}$ acts over $\mathbb{R}^{n+1}$ permuting the coordinates and the subspace of W-invariant vectors is given by the line $\lambda = \{x_1 = \cdots = x_{n+1}\}$. We can chose as a set of invariant polynomials the symmetric polynomials in $n+1$ variables. If we consider the orthogonal space $\lambda^\perp$ we get an n-dimensional vector space where W acts as a group of orthogonal transformations. Moreover the action of W on the space $\lambda^\perp$ is essential, that is the space of W-invariant vectors is $\{0\}$. Finally the W-invariant polynomials algebra is generated by n algebraically independent elements, that is the symmetric polynomials of degree $2, \dots, n+1$ in the variables $x_1, \dots, x_{n+1}$.

Now let $\mathbf{V}$ be a real vector space endowed with a scalar product $(\,\cdot\,,\,\cdot\,)$. We call *lattice* of $\mathbf{V}$ the $\mathbb{Z}$-span of a basis of $\mathbf{V}$. A subgroup G of $\mathbf{GL}(\mathbf{V})$is said to be *crystallographic* if it stabilizes a lattice L in $\mathbf{V}$, that is $gL \subset L$ for all $g \in G$. It turns out that *most* finite reflection group are crystallographic. We have the following condition:

Proposition 1.3.7. *If a Coxeter group W is crystallographic, then each integer $m(s, s')$ must be $2, 3, 4$ or 6 when $s \neq s'$ in S.* □

This criterion rules out the groups of type H_3, H_4 and the groups $I_2(m)$ for m different from $2, 3, 4, 6$. For all the remaining cases it is possible to construct a W-stable lattice.

Now we need to introduce the definition of root system. Given a nonzero vector $\alpha \in \mathbf{V}$ we write $\mathbf{H}_\alpha$ for the hyperplane orthogonal to α and s_α for the orthogonal reflection that fixes $\mathbf{H}_\alpha$ pointwise and maps α to $-\alpha$.

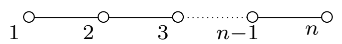
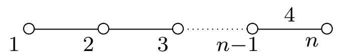
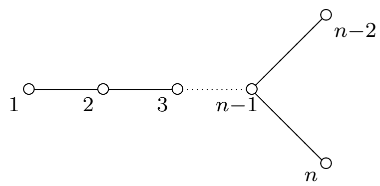
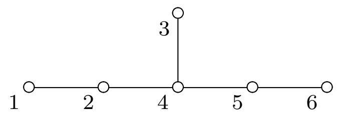
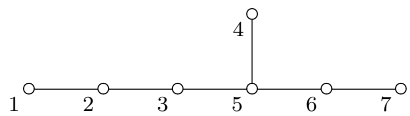
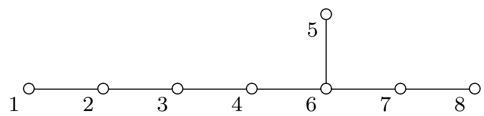
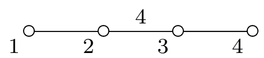
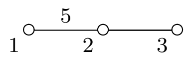
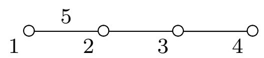
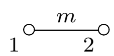
A_n		$2, 3, \ldots, n+1$
B_n		$2, 4, 6, \ldots, 2n$
D_n		$2, 4, 6, \ldots, 2n-2, n$
E_6		$2, 5, 6, 8, 9, 12$
E_7		$2, 6, 8, 10, 12, 14, 18$
E_8		$2, 8, 12, 14, 18, 20, 24, 30$
F_4		$2, 6, 8, 12$
H_3		$2, 6, 10$
H_4		$2, 12, 20, 30$
$I_2(m)$		$2, m$

Table 1.1: Coxeter graphs for finite Coxeter groups

Definition 1.3.8. A finite subset Φ of nonzero vectors in $\mathbf{V}$ is called *root system* if it satisfies the following conditions:

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$;

(R2) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$;

(R3) $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Definition 1.3.9. Setting $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$, the set $\Phi^\vee$ is the set of all *coroots* $\alpha^\vee$ for $\alpha \in \Phi$. It is also a crystallographic root system called inverse or *dual root system*.

Definition 1.3.10. The $\mathbb{Z}$-span $L(\Phi)$ of Φ in $\mathbf{V}$ is called *root lattice*. Similarly we define the *coroot lattice* $L(\Phi^\vee)$.

The group W generated by the reflections s_α for $\alpha \in \Phi$ is known as *Weyl group* of Φ and it turns out that Weyl groups are the same thing as crystallographic Coxeter group. However there are two distinct root systems B_n and C_n dual to each other, each giving as Weyl group the group previously labelled as B_n.

We want to consider not just orthogonal reflections on $\mathbf{V}$, but also affine reflections relative to hyperplanes that do not necessarily pass through the origin. Hence we introduce the *affine group* $\mathrm{Aff}(\mathbf{V})$, which is the semidirect product of $\mathbf{GL}(\mathbf{V})$ and the group of translations by elements of $\mathbf{V}$. It is easy to see that the group of translations is normalized by $\mathbf{GL}(\mathbf{V})$.

Given a root system Φ, for each root α and each integer k define the affine hyperplane

$$H_{\alpha,k} = \{v \in V \mid (v, \alpha) = k\}.$$

Note that $H_{\alpha,0} = H_\alpha$ and that $H_{\alpha,k}$ can be obtained by translating H_α by $\frac{k}{2}\alpha^\vee$. We define the corresponding affine reflection as follows:

$$s_{\alpha,k}(v) = v - ((v, \alpha) - k)\alpha^\vee.$$

Definition 1.3.11. We define the *affine Weyl group* W_a to be the subgroup of $\mathrm{Aff}(V)$ generated by all affine reflections $s_{\alpha,k}$, where $\alpha \in \Phi, k \in \mathbb{Z}$.

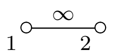
The structure of W_a is more clear considering the following:

Proposition 1.3.12. *The group W_a is the semidirect product of W and the translation group corresponding to the coroot lattice $L = L(\Phi^\vee)$.* □

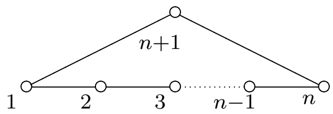
It turns out that any affine Weyl group can be represented as a reflection group. We will call these group affine Coxeter group. The following classification result is the affine analogous of Theorem 1.3.5.

Theorem 1.3.13. *The Coxeter group for which the associated bilinear form B is positive semidefinite and not positive definite are precisely the affine Weyl groups. The Coxeter graphs of the irreducible affine Weyl groups are those listed in the following Table 1.2.* □

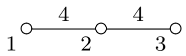
$\widetilde{A}_1$



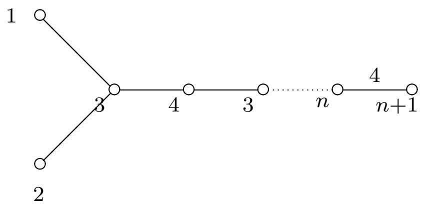
$\widetilde{A}_n (n \geq 2)$



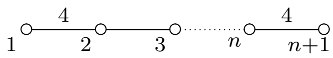
$\widetilde{B}_2 = \widetilde{C}_2$



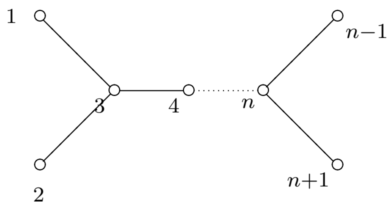
$\widetilde{B}_n (n \geq 3)$



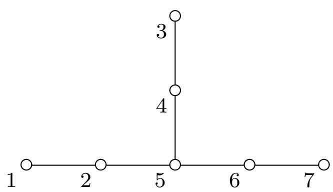
$\widetilde{C}_n (n \geq 2)$



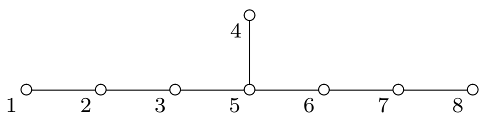
$\widetilde{D}_n (n \geq 4)$



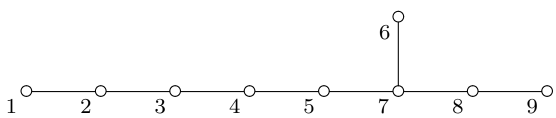
$\widetilde{E}_6$



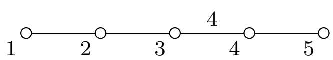
$\widetilde{E}_7$



$\widetilde{E}_8$



$\widetilde{F}_4$



$\widetilde{G}_2$

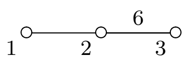


Table 1.2: Coxeter graphs for affine Coxeter groups

1.4 $K(\pi, 1)$ spaces, arrangements and Artin groups

Definition 1.4.1. Let $\mathbf{X}$ be a topological space and let π be a discrete group. We call $\mathbf{X}$ *classifying space* for the group π, or simply we say that $\mathbf{X}$ is a $K(\pi, 1)$ space, if the following conditions hold:

(i) $\mathbf{X}$ is path connected;

(ii) $\pi_1(\mathbf{X}) = \pi$;

(iii) $\pi_n(\mathbf{X}) = 0 \quad \forall n > 1$.

Equivalently $\mathbf{X}$ is a path connected space with a contractible universal cover.

For any discrete group G it is possible to realize a classifying space as a CW-complex. This complex can be obtained starting from a presentation of G and defining the complex skeleton by skeleton. We start with a single point x_0 and we attach to x_0 as many 1-cells as the number of generators of G. Then for each relation in the presentation of G we can attach a 2-cell, gluing the boundary along the loop given by the relation, in order to *add* the relation to the fundamental group of the CW-complex. This construction give us a space with G as fundamental group. Hence we add, skeleton by skeleton, higher dimension cells in order to *kill* all the higher homotopy groups. In general it is not possible to get a $K(\pi, 1)$ space as a finite CW-complex. For example if the group G is finite and the CW-complex $\mathbf{X}$ is a classifying space for G, then $\mathbf{X}$ must be infinite dimensional. We'll see that in the case of some of the groups that we are going to consider there exist classifying spaces that are finite CW-complexes. This will give us an easier way to perform explicit computations in cohomology.

Let V be a finite dimensional vector space and let

$$\mathcal{A} = \{\mathbf{H}_1, \ldots, \mathbf{H}_n\}$$

be a finite *arrangement of hyperplanes* of $\mathbf{V}$. If all the hyperplanes of the arrangement $\mathcal{A}$ pass through the origin we say that the arrangement is *central*. If $\mathbf{V}$ is a complex vector space, a deep topological problem is to understand how the combinatorial properties of the arrangement $\mathcal{A}$ determine the topology of the complement

$$\mathbf{Y}_{\mathcal{A}} = \mathbf{V} \setminus \bigcup_{\mathbf{H} \in \mathcal{A}} \mathbf{H}.$$

An important problem is to understand when the space $\mathbf{Y}_{\mathcal{A}}$ is a $K(\pi, 1)$ space. Given a real arrangement $\mathcal{A}_{\mathbb{R}}$ in the real space $\mathbf{V}_{\mathbb{R}}$, the complement $\mathbf{Y}_{\mathcal{A}_{\mathbb{R}}}$ is the disjoint union of open chambers. If the arrangement is central, these chambers are cones on the origin. We say that a central arrangement $\mathcal{A}_{\mathbb{R}}$ is *simplicial* if each chamber of the complement $\mathbf{Y}_{\mathcal{A}_{\mathbb{R}}}$ is a simplicial cone, that is it is a cone over the standard simplex.

A special family of complex arrangement is given by arrangements of hyperplanes defined by real equations. It means that we can get such an arrangement starting from a real arrangement $\mathcal{A}_{\mathbb{R}}$ in a real vector space $\mathbb{R}$ and complexifying our space and our arrangement:

$$V = \mathbf{V}_{\mathbb{R}} \otimes_{\mathbb{R}} C$$

$$\mathcal{A} = \{\mathbf{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}, \text{ for } \mathbf{H}_{\mathbb{R}} \in \mathcal{A}_{\mathbb{R}}\}.$$

An important property of simplicial arrangements is given by the following

Theorem 1.4.2 ([Del72]). *Let $\mathcal{A}_{\mathbb{R}}$ be a real finite central arrangement and let $\mathbf{Y}_{\mathcal{A}}$ be the complement of its complexification. If $\mathcal{A}_{\mathbb{R}}$ is simplicial, then $\mathbf{Y}_{\mathcal{A}}$ is a $K(\pi, 1)$ space.* □

With more generality one can consider affine arrangements and infinite arrangements.

Let W be a Coxeter group, with presentation

$$< s, s' \mid (ss')^{m(s,s')} = e > .$$

We can give the following:

Definition 1.4.3. We call *Artin group* of type W the group G_W given by the following presentation:

$$< g_s \mid s \in S, \overbrace{g_s g_{s'} g_s g_{s'} \cdots}^{m(s,s')\text{factors}} = \overbrace{g_{s'} g_s g_{s'} g_s \cdots}^{m(s,s')\text{factors}} \text{ per } s \neq s', m(s,s') \neq +\infty > .$$

Loosely speaking the group G_W is the group obtained from a presentation of W dropping the relation $s^2 = e$ for $s \in S$.

Example 1.4.4. In the case of the symmetric group $W = \mathfrak{S}_{n+1}$ with generators $s_1, \ldots, s_n$ we get the commuting relations

$$s_i s_j = s_j s_i$$

for $\mid i - j \mid > 2$ and

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

for $i = 1, \ldots, n-1$. The last ones are called braid relations. In this example the group G_W associated to W is called *braid group* on n strands and is denoted by $\mathrm{Br}(n)$. We can think to an element of $\mathrm{Br}(n)$ (a braid) as a collection of n paths in $\mathbb{R}^3$ that do not intersect each other. Moreover each path connect a point of a n-tuple, for example we can think to the set of points $(i, 0, 0), i = 1, \ldots, n$, with a point in a second n-tuple, for example the points $(i, 1, 0), i = 1, \ldots, n$. Finally we require that each path (each strand) is parametrized such that the second coordinates is an increasing function. Two braids are identified if they are homotopic. We can endow the set of equivalence class of homotopic braids with a group structure as follows. The product of two braids is given gluing the second braid to the end of the first braid and rescaling the new braid. If we consider the space $\mathbb{C}(n)$ of unordered n-tuples of distinct points in the space $\mathbb{C}$, we can define the braid group as $\mathrm{Br}(n) = \pi_1(\mathbb{C}(n))$. Notice that $\mathbb{C}(n)$ can be obtained taking the complement of the hyperplanes of equation $z_i = z_j$ in $\mathbb{C}^n$, that is taking the complement of the arrangement of reflection hyperplanes associated to $\mathfrak{S}_n$ and then, taking the quotient with respect to the action of $\mathfrak{S}_n$.

The same construction can be done for a finite Coxeter group W, we can realize W as a group of linear reflections of $\mathbb{R}^m$ (as explained in Section 1.2) and let $\mathfrak{H}$ be the arrangement of hyperplanes associated to W. The group W acts on the space

$$\mathbf{Y}_W = \mathbb{C}^m \setminus \bigcup_{\mathbf{H} \in \mathfrak{H}} \mathbf{H}_{\mathbb{C}}$$

and a result of Brieskorn (see [Bri71]) states that $G_W = \pi_1(\mathbf{Y}_W/W)$. If there is no ambiguity we simply write $\mathbf{Y}$ instead of $\mathbf{Y}_W$, with the group W understood.

With more generality, let W be a (finitely generated) Coxeter group, which we realize through the Tits representation as a group of (in general, non orthogonal) reflections in $\mathbb{R}^n$, where the base-chamber C_0 is the positive octant and S is the set of reflections with respect to the coordinate hyperplanes. (It is possible to consider more general representations; see [Vin71]). Let $U := W.\overline{C}_0$ be the orbit of the closure of the base chamber (the *Tits cone*). Recall that ([Vin71]):

(i) U is a convex cone in $\mathbb{R}^n$ with vertex 0.

(ii) $U = \mathbb{R}^n$ if and only if W is finite.

(iii) $U^0 := int(U)$ is open in $\mathbb{R}^n$ and a (relative open) facet $\mathbf{F} \subset \overline{C}_0$ is contained in U^0 if and only if the stabilizer $W_{\mathbf{F}}$ is finite.

Let $\mathcal{A}$ be the arrangement of reflection hyperplanes of W. Set

$$\mathbf{Y}_W := [U^0 + i\mathbb{R}^n] \setminus \bigcup_{\mathbf{H} \in \mathcal{A}} \mathbf{H}_{\mathbb{C}}$$

as the complement of the complexified arrangement. Notice that the group W acts freely on $\mathbf{Y}_W$ so we can consider the *orbit space*

$$\mathbf{X}_W := \mathbf{Y}_W/W$$

.

Theorem 1.4.5 ([Bri73a, Dun83, vdL83]). *The Artin group G_W is the fundamental group of the orbit space $\mathbf{X}_W$.* □

We can also consider the fundamental group of $\mathbf{Y}_W$:

Definition 1.4.6. We call *pure Artin group* associated to W the group

$$PA_W = \pi_1(\mathbf{Y}_W).$$

Since we have the covering

$$W \hookrightarrow \mathbf{Y}_W \to \mathbf{X}_W$$

the group PA_W is simply the kernel of the natural homomorphism $G_W \to W$ and we have the short exact sequence that we obtain from the homotopy long exact sequence of the covering:

$$0 \to PA_W \overset{i}{\hookrightarrow} G_W \overset{\pi}{\to} W \to 0.$$

where every standard generator of the Artin group G_W maps to the corresponding standard generator of the Coxeter group W:

$$\pi : g_s \mapsto s \quad \forall s \in S.$$

Given an element $w \in W$ we can take a reduced expression of w, that is $w = s_{i_1} \cdots s_{1_l}$ where $l = l(w)$ is the length of w. In this way we can lift w to the element $\psi(w) = g_{s_{i_1}} \cdots g_{s_{1_l}}$ that lies in G_W (actually $\psi(w)$ lies also in the positive monoid of G_W). We have the following Theorem (see. e.g. [GP00]):

Theorem 1.4.7 (Matsumoto's Theorem). *Let (W, S) be a Coxeter system and $(M, \dot{})$ a monoid. Suppose that is given a map $f : S \to M$ such that:*

$$\overbrace{f(s)f(t)f(s)\cdots}^{m(s,t) \ factors} = \overbrace{f(t)f(s)f(t)\cdots}^{m(s,t) \ factors}$$

where both the terms have $m(s,t)$ factors. Then there exist a unique extension $\widehat{f} : W \to M$, such that $\widehat{f}(w) = f(s_{i_1}) \cdots f(s_{1_l})$ whenever $w = s_{i_1} \cdots s_{1_l}$ is a reduced expression. □

As a consequence of this Theorem, the element $\psi(w)$ does not depend on the choice of the reduced expression of w, so we have a set-theoretic section $\psi : W \to G_W$ for the map $\pi : G_W \to W$.

In the case of the braid group described above, we have that the kernel of the projection $\pi : Br(n) \to \mathfrak{S}_n$ is the *pure braid group* on n strands $\mathrm{PBr}(n)$ that is the subgroup of $\mathrm{Br}(n)$ given by those braids that do not permute the ending points, that is those collections of paths $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$ such that $\gamma_i(0) = (i, 0, 0)$ and $\gamma_i(1) = (i, 1, 0)$.

In the case we are considering (type A_n, $W = \mathfrak{S}_{n+1}$) it is easy to see that the space $\mathbf{Y}_W$ is a $K(\pi, 1)$ by induction on n (see [Bri73a]). In fact the projection on the first n coordinates gives a fibration $p : \mathbf{Y}_{A_n} \to \mathbf{Y}_{A_{n-1}}$ with fiber over the point $x = (x_1, \dots, x_n)$ given by the space $\mathbb{C} \setminus \{x_1, \dots, x_n\}$. This is a $K(\pi, 1)$, hence the thesis follows from the homotopy long exact sequence of the fibration. As a consequence $\mathbf{Y}_W/W$ is a $K(\pi, 1)$ because it is covered by $\mathbf{Y}_W$.

In general since any real arrangement associate to a finite Coxeter group is simplicial, there is the following important consequence of Theorem [Del72]:

Theorem 1.4.8 ([Del72])**.** *Let W be a finite Coxeter group, then the space $\mathbf{Y}_W$ is a $K(\pi, 1)$ space.* □

For a general Coxeter group W it is only conjectured that $\mathbf{Y}_W$ is a $K(\pi, 1)$ space. For affine Coxeter groups of type $\widetilde{A}_n$ and $\widetilde{C}_n$ Okonek proved that the space $\mathbf{Y}$ is a $K(\pi, 1)$ (see [Oko79]). In [Hen85] Hendriks prove the conjecture for Coxeter groups of *large type*, that is groups with Coxeter matrix m with $m(s, t) \neq 2$ for all s, t.

Accordingly to [CD95b], we say that a Coxeter system satisfies the FC condition if for every subset $T \subset S$ such that every pair of elements $t, t' \in T$ generate a finite group, the T generate a finite group.

In [CD95b] Charney and Davis gave a proof of the conjecture for Coxeter group satisfying the FC condition and for those Coxeter groups that are in some sense *generic*, that is those for which each parabolic subgroup generated by 3 elements in infinite.

In a complex vector space $\mathbf{V}$ it is possible to consider also *complex reflections*, that is pseudoreflections of finite order, possibly different from 2. A group generated by complex reflections is called *complex reflection group*. The theory of this kind of groups seems much more complicated than the theory of usual Coxeter groups. It is possible to consider the complement $\mathbf{Y}$ of the arrangement of the hyperplanes fixed by complex reflections of a complex reflection group. Recently Bessis proved ([Bes06]) that for any finite complex reflection group, the complement $\mathbf{Y}$ is a $K(\pi, 1)$ space.

In Section 7.2 we prove that

Theorem 1.4.9. *For the affine Coxeter group W of type $\widetilde{B}_n$ the space $\mathbf{Y}_W$ is a $K(\pi, 1)$.* □

1.5 Poincaré polynomials and Poincaré series

Let us fix a Coxeter system (W, S). For every subset $X \subset W$ we can define the (finite or infinite) sum

$$X(t) = \sum_{w \in X} t^{l(w)}.$$

In case $X = W$ we get, by definition, the *Poincaré series* $W(q)$ of W. Moreover it holds

$$W(t) = \sum_{w\in W} t^{l(w)} = \sum_{n\geq 0} a_n t^n$$

where $a_n = \#\{w \in W \mid l(w) = n\}$. If $I \subset S$, then $W_I(t)$ coincides with the Poincaré series of W_I, as follows from (ii) in Proposition 1.1.3. As a consequence of (iii) of the same Proposition we have

$$W(t) = W_I(t)W^I(t).$$

For finite Coxeter groups the Poincaré series (actually polynomials) are well known. We mention a factorization formula that will be useful later:

Theorem 1.5.1. *Let W be a finite Coxeter group. We have:*

$$W(t) = \prod_{i=1}^{n} \frac{t^{d_i} - 1}{t - 1}$$

where $d_1, \dots, d_n$ are the characteristic degrees of W. □

In order to recall closed formulas for Poincaré series, we first fix some notations that will be adopted throughout all the following. We define the *q-analog* of a positive integer m to be the polynomial

$$[m]_q := 1 + q + \cdots q^{m-1} = \frac{q^m - 1}{q - 1}$$

Let $\varphi_i(q)$ be the i-th cyclotomic polynomial in the variable q. It is easy to see that $[m] = \prod_{i|m} \varphi_i(q)$. Moreover we define the *q-factorial* and *double factorial* inductively as:

$$[m]_q! := [m]_q \cdot [m-1]_q!$$
$$[m]_q!! := [m]_q \cdot [m-2]_q!!$$

where it is understood that $[1]! = [1]!! = [1]$ and $[2]!! = [2]$. A *q-analog of the binomial* $\binom{m}{i}$ is given by the polynomial

$$\begin{bmatrix} m \\ i \end{bmatrix}_q := \frac{[m]_q!}{[i]_q![m-i]_q!}$$

We can also define the *(q,t)-analog* of an even number

$$[2m]_{q,t} := [m]_q(1 + tq^{m-1})$$

and of the double factorial

$$[2m]_{q,t}!! := \prod_{i=1}^{m} [2i]_{q,t} \;=\; [m]_q! \prod_{i=0}^{m-1} (1 + tq^i).$$

Note that specializing t to q, we recover the q-analog of an even number and of its double factorial. Finally, we define the polynomial

$$\begin{bmatrix} m \\ i \end{bmatrix}'_{q,t} := \frac{[2m]_{q,t}!!}{[2i]_{q,t}!![m-i]_q!} = \begin{bmatrix} m \\ i \end{bmatrix}_q \prod_{j=i}^{m-1} (1+tq^j). \tag{1.5.1}$$

With this notation the ordinary Poincaré series for A_n, B_n and D_n may be written as

$$A_n(q) := \sum_{w \in A_n} q^{\ell(w)} = [n+1]_q! \tag{1.5.2}$$

$$B_n(q) := \sum_{w \in B_n} q^{\ell(w)} = [2n]_q!! \tag{1.5.3}$$

$$D_n(q) := \sum_{w \in D_n} q^{\ell(w)} = [2(n-1)]_q!!. \cdot [n]_q \tag{1.5.4}$$

For future use in cohomology computations, we are interested in a (q,t)-analog of the usual Poincaré series for B_n, that is an analog of the Poincaré series with coefficients in the ring $R = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ of Laurent polynomials. This result and similar ones are studied in [Rei93], to which we refer for details.

Consider the Coxeter group W of type B_n with its standard generating reflections $s_1, s_2, \dots, s_n$. For $w \in W$, let $n(w)$ be the number of times s_n appears in a reduced expression for w. By Matsumoto Lemma, the number $n(w)$ is well-defined for a reduced expression of w.
We define the (q,t)-weighted Poincaré series for the Coxeter group of type B_n as the sum

$$W(q,t) = \sum_{w \in W} q^{\ell(w)-n(w)} t^{n(w)},$$

where ℓ is the length function.

Proposition 1.5.2 ([Rei93]).

$$W(q,t) = [2n]_{q,t}!!.$$

Proof. Consider the parabolic subgroup W_I associated to the subset of reflections $I = \{s_1, \dots, s_{n-1}\}$. Notice that W_I is isomorphic to the symmetric group on n letters A_{n-1} and that it has index 2^n in B_n. Let W^I be the set of minimal coset representatives for W/W_I. Then, by multiplicative properties on reduced expressions:

$$\begin{aligned} W(q,t) &= \sum_{w \in W} q^{\ell(w)-n(w)} t^{n(w)} \\ &= \Big(\sum_{w' \in W^I} q^{\ell(w')-n(w')} t^{n(w')} \Big) \cdot \Big(\sum_{w'' \in W_I} q^{\ell(w'')-n(w'')} t^{n(w'')} \Big). \end{aligned} \tag{1.5.5}$$

Clearly, for elements $w'' \in W_I$, we have $n(w'') = 0$; so the second factor in (1.5.5) reduces to the well-known Poincaré series for A_{n-1}:

$$\sum_{w'' \in W_I} q^{\ell(w'')-n(w'')} t^{n(w'')} = [n]_q!.$$

To deal with the first factor, instead, we explicitly enumerate the elements of W^I. Let $p_i = s_i s_{i+1} \cdots s_n$ for $1 \leq i \leq n$. Then, it can be easily verified that $W^I = \{p_{i_r} p_{i_{r-1}} \cdots p_{i_2} p_{i_1} \,|\, i_1 < i_2 < \cdots < i_{r-1} < i_r\}$. Notice that $n(p_{i_r} p_{i_{r-1}} \cdots p_{i_2} p_{i_1}) = r$ and $\ell(p_{i_r} p_{i_{r-1}} \cdots p_{i_2} p_{i_1}) = \sum_{j=1}^{r} \ell(p_{i_j}) = \sum_{j=1}^{r} (n + 1 - i_j)$. Thus,

$$\sum_{w' \in W^I} q^{\ell(w')-n(w')} t^{n(w')} = \prod_{i=0}^{n-1} (1 + tq^i).$$

Finally,

$$W(q,t) = \Big(\prod_{i=0}^{n-1} (1 + tq^i)\Big) [n]_q! = [2n]_{q,t}!!.$$

$\square$

1.6 The Milnor fibration for arrangements

We now give a short resume of some results due to Milnor (see [Mil68] as a main reference).

Let $f(x_1, \ldots, x_m)$ be an analytic function on m complex variables defined on a neighborhood of the origin that maps to $\mathbb{C}$ and is null in the origin. We denote by $\mathbf{Z}$ the set $\{x \mid f(x) = 0\}$ and by $\mathbf{K}$ the intersection of $\mathbf{Z}$ with $S_\epsilon = \{z \in \mathbb{C}^n \mid \|z\| = \epsilon\}$. We can map the complement $S_\epsilon \setminus \mathbf{K}$ to the circle S^1 with the map

$$\phi(z) = \frac{f(z)}{\mid f(z) \mid}.$$

The following Theorem holds:

Theorem 1.6.1 (Fibration Theorem)**.** *There exists an $\epsilon_0 > 0$ such that for $\epsilon \leq \epsilon_0$ the space $S_\epsilon \setminus \mathbf{K}$ is a C^∞ fibre bundle over S^1, with projection map $\phi(z) = \frac{f(z)}{|f(z)|}$.* $\square$

It follows from this Theorem that each fiber

$$\mathbf{G}_\theta = \phi^{-1}(e^{i\theta})$$

is a C^∞ manifold of real dimension $2(m-1)$. We call the fiber $\mathbf{G} = \mathbf{G}_0$ the *Milnor fiber* of f. Moreover we have this result:

Theorem 1.6.2. *Each fiber $\mathbf{G}_\theta$ is parallelizable and it has the homotopy type of a finite CW-complex of dimension $m - 1$.* $\square$

Definition 1.6.3. Let us chose a smooth one-parameter set of homomorphisms

$$h_t : \mathbf{G}_0 \to \mathbf{G}_t$$

for $0 \le t \le 2\pi$, where h_0 is the identity. We call the homomorphism $h = h_{2\pi}$ the *characteristic homomorphism* of the fibration f.

Note that h depends on the choice that we have made for the 1-parameter set of maps h_t's, but its homotopy class is uniquely determined.

Definition 1.6.4. We say that a polynomial $f(z_1, \dots, z_m)$ is a *weighted homogeneous polynomial* of type $(a_1, \dots, a_m)$ if it can be written as a linear combination of monomials $z_1^{i_1} \cdots z_m^{i_m}$ such that

$$\frac{i_1}{a_1} + \cdots + \frac{i_m}{a_m} = 1$$

This is equivalent to ask that

$$f(e^{\frac{c}{a_1}} z_1, \dots, e^{\frac{c}{a_m}} z_m) = e^c f(z_1, \dots, z_m) \qquad \forall c \in \mathbb{C}.$$

Every homogeneous polynomial is an analytic function that is zero at the origin. So we have:

Proposition 1.6.5. *Let f be a weighted homogeneous polynomial. The Milnor fiber $\mathbf{G}$ given by f is diffeomorphic to the nonsingular hypersurface*

$$\mathbf{F} = \{z \in \mathbb{C}^m \mid f(z) = 1\}.$$

We can choose as a characteristic homomorphism $h : \mathbf{G} \to \mathbf{G}$ (or $h : \mathbf{F} \to \mathbf{F}$) the unitary transformation

$$h(z_1, \dots, z_m) = (e^{2\pi i/a_1} z_1, \dots, e^{2\pi i/a_m} z_m).$$

□

Example 1.6.6. Given an hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^n$, for each hyperplane $\mathbf{H} \in \mathcal{A}$ let $l_{\mathbf{H}}$ be a linear functional such that $\ker l_{\mathbf{H}} = \mathbf{H}$. Moreover we can associate to each hyperplane $\mathbf{H}$ a nonnegative integer $a(\mathbf{H})$. Now we can consider the homogeneous polynomial given by the product

$$f(z) = \prod_{\mathbf{H} \in \mathcal{A}} l_{\mathbf{H}}^{a(\mathbf{H})}.$$

This gives a Milnor fibration $f : \mathbf{Y}_{\mathcal{A}} \to \mathbb{C}^*$. It is a standard fact in the theory of hyperplane arrangements that $\mathbf{Y}_{\mathcal{A}}$ has torsion-free homology (see [OT92]). If $n = 2$, it is known that $H_*(\mathbf{F}; \mathbb{Z})$ is torsion-free, as $\mathbf{F}$ is homotopic to a bouquet of $n - 1$-spheres. In [CDS03] it is proved that for each prime p and integer $n > 2$ there exist an arrangement $\mathcal{A}$ in $\mathbb{C}^n$ and integers $a(\mathbf{H})$

for $\mathbf{H} \in \mathcal{A}$ for which $H_1(\mathbf{F}, \mathbb{Z})$ has p-torsion. An important feature of the polynomials f constructed in [CDS03] is the fact that they are not reduced, that is $a(\mathbf{H}) > 1$ for some $\mathbf{H}$. The possibility of finding torsion homology in the Milnor fiber for a reduced f remains open. We note also that if $[\gamma_{\mathbf{H}}]$ is the homotopy class of a complete loop around the hyperplane $\mathbf{H}$ in the fundamental group $\pi_1(\mathbf{Y}_{\mathcal{A}})$, then the map $f_\sharp$ induced by f sends each element $[\gamma_{\mathbf{H}}]$ to $a(\mathbf{H}) \in \mathbb{Z} = \pi_1(\mathbb{C}^*)$ and clearly the fundamental group of the Milnor fiber is the kernel of the map:

$$f_\sharp : \pi_1(\mathbf{Y}_{\mathcal{A}}) \to \mathbb{Z}.$$

Now consider a finite Coxeter group W acting on the real vector space $\mathbf{V}$ as a reflection group. Let $\mathbf{V}_{\mathbb{C}} = \mathbf{V} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified space of $\mathbf{V}$. The group W acts in a natural way on the space $\mathbf{V}_{\mathbb{C}}$. Since the polynomials f_i of Theorem 1.3.2 are W-invariant, the map $f : \mathbf{V}_{\mathbb{C}} \to \mathbb{C}^n$ defined by $f : (x_1, \ldots, x_n) \mapsto (f_1, \ldots, f_n)$ passes to the quotient and induces a map $f' : \mathbf{V}_{\mathbb{C}}/W \to \mathbb{C}^n$. We get that the quotient space $\mathbf{V}_{\mathbb{C}}/W$ is an affine variety isomorphic to the affine complex space $\mathbb{C}^n$. In fact the following Theorem holds (see [Che55, ST54, Loo84]):

Proposition 1.6.7. *The map f' is a proper homeomorphism of the space $\mathbf{V}_{\mathbb{C}}/W$ onto a normal subvariety of $\mathbb{C}^n$, whose algebra of regular functions correspond through the map f to the algebra $\mathcal{R}$ of W-invariant polynomials.*

□

In particular, since $\mathbf{V}_{\mathbb{C}}$ is irreducible and has complex dimension n, its image must be an open set of $\mathbb{C}^n$ containing the origin and it follows easily that the map f' must be surjective.

We denote by $\mathfrak{H}$ the arrangement of hyperplanes whose reflections $s_{\mathbf{H}}$ belong to W

$$\mathfrak{H} = \{\mathbf{H} \subset \mathbf{V} \mid \sigma_{\mathbf{H}} \in W\}.$$

Again we consider the map f. It determines a ramified covering $f : \mathbf{V}_{\mathbb{C}} \to \mathbb{C}^n$ and we have the following

Proposition 1.6.8. *Let us fix a set $f_1, \ldots, f_n$ of homogeneous, algebraically independent generators of the algebra of W-invariant polynomials $\mathcal{R}$. Set $J = \det(\frac{\partial f_j}{\partial x_j})$ to be the corresponding Jacobian. Moreover let we set $D = \prod_{\mathbf{H} \in \mathfrak{H}} l_{\mathbf{H}}$ to be the defining polynomial of the arrangement.*

(a) $J = kD$ for some nonzero constant k depending on the choice of the polynomials f_i.

(b) A polynomial g is W-alternating if and only if it can be written as the product of J times an invariant polynomial.

□

Hence given $\Delta = \bigcup_{\mathbf{H}\in\mathfrak{H}} \mathbf{H}$ and let Δ' be the image of Δ through f. If we consider the restriction of the map f to the sets:

$$\mathbf{Y}_W = \mathbf{V}_{\mathbb{C}} \setminus \Delta$$

$$\mathbf{X}_W = \mathbb{C}^n \setminus \Delta'$$

we get the covering

$$\mathbf{Y}_W \xmapsto{f} \mathbf{X}_W$$

where the group W is the group of deck transformations of the covering and the space $\mathbf{X}_W$ is isomorphic to the space $\mathbf{Y}_W/W$.

Now let $W, \mathbf{V}, D$ be as in Proposition 1.6.8 and set $\delta = D^2$. It is clear that δ is a W-invariant homogeneous polynomial. Hence δ can be written as a polynomial on the f_i's, that is:

$$\delta(z_1, \dots, z_m) = \delta'(f_1, \dots, f_m).$$

Let $\mathbf{F}$ be the Milnor fiber for the map $\delta : \mathbf{V}_{\mathbb{C}} \to \mathbb{C}$. Each element of W maps the fiber $\mathbf{F}$ to itself. If we consider the quotient map $\delta' : \mathbf{V}_{\mathbb{C}}/W \to \mathbb{C}$ (where $\mathbf{V}_{\mathbb{C}}/W$ is considered as an affine space) and the corresponding Milnor fiber $\mathbf{F}'$, it is clear that $\mathbf{F}' = \mathbf{F}/W$. Moreover, if we denote by N the degree of δ, it turns out that δ is a weighted homogeneous polynomial of type $(1/N, \dots, 1/N)$, while δ' is weighted homogeneous of type $(d_1/N, \dots, d_m/N)$, where d_i is the degree of f_i.

We recall that since W is finite the space $\mathbf{X}_W$ is a classifying space for the Artin group G_W and the map $\delta'_\sharp$ induced by δ' on the fundamental groups sends each standard generator g_s, $s \in S$ of the Artin group to $1 \in \mathbb{Z} = \pi_1(\mathbb{C}^*)$. If we set $H_W = \pi_1(\mathbf{F}')$ we get the short exact sequence

$$0 \to H_W \lhd G_W \xrightarrow{\delta'_\sharp} \mathbb{Z} \to 0.$$

Hence the Milnor fiber $\mathbf{F}'$ is a classifying space for the group H_W.

Remark 1.6.9. The description of the map $f_\#$ given in Example 1.6.6 implies that each standard generator g_s of the group G_W corresponds to a loop around the singularity in the space $\mathbf{X}_W$ and lifts to an half loop in the space $\mathbf{Y}_W$. Moreover each standard generator g_s maps to $1 \in \mathbb{Z}$.

Remark 1.6.10. If W is a finite irreducible Coxeter group different from $B_n, F_4, I_2(2m)$, then the group H_W is the commutator subgroup of the Artin group G_W.

Example 1.6.11. Again we consider the special case of W of type A_{n-1} and so the Artin group G_W is the braid group $\mathrm{Br}(n)$. In this case the space $\mathbf{Y}$ is the complement of the braid arrangement

$$\mathbb{C}^n \setminus \bigcup_{i<j} \{z_i = z_j\}$$

that is the space $\mathbf{F}(\mathbb{C}, n)$ of ordered configurations of n distinct point in $\mathbb{C}$. The quotient $\mathbf{X} = \mathbf{Y}/W$ is the space $C(\mathbb{C}, n) = \mathbb{C}(n)$ of unordered configurations of n distinct points in $\mathbb{C}$. The map δ is given by the product

$$\prod_{i<j}(z_i - z_j)^2.$$

We can regard the space $\mathbf{X}$ as the space of monic polynomials of degree n with n distinct roots and, for a polynomial $p \in \mathbf{X}$, the map $\delta'(p)$ is the discriminant $d(p)$, that is the resultant of p and its derivative p'. So the Milnor fiber is given by

$$\mathbf{F} = \{p \mid \mathrm{res}(p, p') = 1\}$$

where $\mathrm{res}(p, p')$ is the resultant of the polynomial p and its derivative p' (see [AGZV88].

In this work we want to study the cohomology (and the homology) of the fiber $\mathbf{F}'$ for some irreducible Coxeter groups. We will also try to generalize, up to homotopy, the fibrations described here to fibrations over $(\mathbb{C}^*)^2$ and to fibrations for infinite Coxeter groups and we will study the cohomology of the generalized homotopy Milnor fiber.

Chapter 2

Group cohomology and local systems

Sections 2.1 and 2.2 contains standard definitions and constructions of group cohomology. We refer for most of the proofs and for any missing detail to [AM94] (see also [Ste51]).

2.1 Principal bundles and $K(\pi, 1)$ spaces

Let $\mathbf{F} \to \mathbf{E} \xrightarrow{p} \mathbf{B}$ be a locally trivial fiber bundle with fiber $\mathbf{F}$ and with projection map p. Given another space $\mathbf{X}$ and a map $f : \mathbf{X} \to \mathbf{B}$ there exist an *induced bundle* $f^\sharp(\mathbf{E})$ that is the subspace of the product $\mathbf{X} \times \mathbf{E}$ given by the couples (x, y) such that $f(x) = p(y)$. The bundle $f^\sharp(\mathbf{E})$ is also called pull-back bundle. We have the following commuting diagram:

$$\begin{tikzcd}
\mathbf{F} \arrow[r, equal] \arrow[d] & \mathbf{F} \arrow[d] \\
f^\sharp(\mathbf{E}) \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{X} \arrow[r, "f"] & \mathbf{B}
\end{tikzcd}$$

Definition 2.1.1. Let G be a topological group. A fiber bundle

$$\mathbf{F} \to \mathbf{E} \xrightarrow{p} \mathbf{B}$$

is called *principal G-bundle* if the action of the group G on the total space $\mathbf{E}$ is effective (that is $gx = x$ if and only if $x = e$) and each fiber $\mathbf{F}$ is exactly an orbit of the action of G.

It is clear from the definition that the fiber $\mathbf{F}$ of a principal G-bundle is homeomorphic to the group G. If G has the homotopy type of a CW-complex there is a special principal G-bundle such that all other G-bundles can be obtained from this as pull back. We have the following Theorem due to Milnor

Theorem 2.1.2. *For a given group G there exists a space $\mathbf{B}_G$ and a principal G-bundle with total space $\mathbf{E}_G$,*

$$G \to \mathbf{E}_G \xrightarrow{p} \mathbf{B}_G,$$

such that for any principal G-bundle $G \to \mathbf{E} \to \mathbf{B}$ there exists a unique homotopy class of maps $f : \mathbf{X} \to \mathbf{B}_G$ such that $f^\sharp(\mathbf{E}_G) = \mathbf{E}$. □

It follows that $\mathbf{B}_G$ is unique, up to homotopy equivalence. In fact if we have two such spaces, say $\mathbf{B}_G, \mathbf{B}'_G$, we should have two maps $f : \mathbf{B}_G \to \mathbf{B}'_G$, $g : \mathbf{B}'_G \to \mathbf{B}_G$ that induce respectively the fiber bundle $\mathbf{E}_G$ and $\mathbf{E}'_G$. It follows that the compositions $fg : \mathbf{B}'_G \to \mathbf{B}'_G$, $gf : \mathbf{B}_G \to \mathbf{B}_G$ induce respectively the fiber bundle $\mathbf{E}'_G$ and $\mathbf{E}_G$. Hence the two compositions must be homotopic to the identity, respectively on the space $\mathbf{B}_G, \mathbf{B}'_G$.

Definition 2.1.3. A principal G-bundle $G \to \mathbf{E}_G \xrightarrow{p} \mathbf{B}_G$ as in Theorem 2.1.2 is called *universal* principal G-bundle.

We have the following result:

Theorem 2.1.4. *A principal G-bundle $G \to \mathbf{E} \xrightarrow{p} \mathbf{B}$ is universal if and only if the total space is contractible.* □

As a consequence of this Theorem we have:

Corollary 2.1.5. *If π is a discrete group and $\mathbf{B}$ is a $K(\pi, 1)$ space, the universal covering $G \to \mathbf{E} \xrightarrow{p} \mathbf{B}$ is a universal π-bundle.* □

This justify the following generalized definition of classifying space:

Definition 2.1.6. If G is a topological group and $G \to \mathbf{E}_G \xrightarrow{p} \mathbf{B}_G$ is a universal principal G-bundle, the space $\mathbf{B}_G$ is called *classifying space* for G.

2.2 Homology and cohomology of groups

Using the notion of classifying space we can give the following definition:

Definition 2.2.1. Let A be an abelian group, we call the groups

$$H^*(G; A) = H^*(\mathbf{B}_G; A)$$

the *cohomology groups* of the group G with (untwisted) coefficients in the group A.

It is possible to give a purely algebraic definition of the cohomology of a group G and this allows us to generalize this concept. First suppose that a group G acts on the group of coefficients A, that is suppose that A is a G-module. We can define the *group ring* $\mathbb{Z}[G]$ as the ring of finite linear combinations of elements of G with coefficients in $\mathbb{Z}$, with the product defined as follows:

$$\left(\sum m_i g_i\right) \cdot \left(\sum m_i' g_i'\right) = \sum (m_i m_i')(g_i g_i').$$

With this definition the group A is a natural $\mathbb{Z}[G]$-module and the structure of G-module corresponds to the structure of $\mathbb{Z}[G]$-module.

Definition 2.2.2. Let M be a $\mathbb{Z}[G]$-module, we call *resolution* of M over G a long exact sequence of $\mathbb{Z}[G]$-modules C_i with maps ∂_i,

$$0 \leftarrow M \overset{\partial_0}{\leftarrow} C_0 \overset{\partial_1}{\leftarrow} C_1 \overset{\partial_2}{\leftarrow} C_2 \leftarrow \cdots \overset{\partial_i}{\leftarrow} C_i \leftarrow \cdots .$$

If each module C_i is a free $\mathbb{Z}[G]$-module, we have a *free resolution.*

It is easy to see that a free resolution of M always exists.

Proposition 2.2.3. *Let $\phi : M \to N$ be a momorphism of $\mathbb{Z}[G]$-modules and let*

$$0 \leftarrow M \overset{\partial_0}{\leftarrow} C_0 \overset{\partial_1}{\leftarrow} C_1 \overset{\partial_2}{\leftarrow} C_2 \leftarrow \cdots$$

$$0 \leftarrow N \overset{\partial_0}{\leftarrow} D_0 \overset{\partial_1}{\leftarrow} D_1 \overset{\partial_2}{\leftarrow} D_2 \leftarrow \cdots$$

be two free resolutions. Then there exist morphisms $\phi_i : C_i \to D_i$, $0 \leq i < \infty$, such that the diagram

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "\phi"] & C_0 \arrow[l, "\partial_0"'] \arrow[d, "\phi_0"] & C_1 \arrow[l, "\partial_1"'] \arrow[d, "\phi_1"] & C_2 \arrow[l, "\partial_2"'] \arrow[d, "\phi_2"] & \cdots \arrow[l] \\
0 & N \arrow[l] & D_0 \arrow[l, "\partial_0"] & D_1 \arrow[l, "\partial_1"] & D_2 \arrow[l, "\partial_2"] & \cdots \arrow[l]
\end{tikzcd}$$

commutes. Moreover, given another choice of morphisms $\phi_0', \phi_1', \ldots$ such that the diagram commutes, there exist maps $\mu_i : C_i \to D_i, 0 \leq i < \infty$ such that $\partial_{i+1}\mu_i + \mu_{i-1}\partial_i = \phi_i' - \phi_i$. □

The maps μ_i give a *chain homotopy* of the two resolutions. Again we consider A a $\mathbb{Z}[G]$-module and we define the Ext functor

$$\mathrm{Ext}^i_{\mathbb{Z}[G]}(M; A)$$

to be the i-th cohomology group of the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}[G]}(C_j, A) = C^j(M, A)$$

for a given free resolution of M. It follows from Proposition 2.2.3 that any two free resolutions are chain homotopic, hence the groups $\mathrm{Ext}^i_{\mathbb{Z}[G]}(M;A)$ do not depend on the chosen resolution.

In case that $\mathbb{Z}$ and A are trivial $\mathbb{Z}[G]$-modules, than we get the previously defined untwisted cohomology of G:

$$\mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z};A) = H^i(G;A).$$

In fact we can consider a CW-complex structure on B_G and the induced CW-complex structure in E_G: since E_G is contractible, the associated cellular chain complex is a free resolution of $\mathbb{Z}$ with $\mathbb{Z}[G]$-modules (the structure is given by the action of G on the fiber of each cell of B_G); moreover

$$\mathrm{Hom}_{\mathbb{Z}[G]}(C_*(E_G),A) = \mathrm{Hom}_{\mathbb{Z}}(C_*(B_G),A).$$

Hence the groups Ext give a generalizations of the cohomology groups of G.

If the group ring $\mathbb{Z}[G]$ acts non trivially on the group A, we will write $H^*_t(G;A)$ in place of

$$\mathrm{Ext}_{\mathbb{Z}[G]}(\mathbb{Z};A).$$

Definition 2.2.4. The groups $\mathrm{Ext}_{\mathbb{Z}[G]}(\mathbb{Z};A) = H^*_t(G;A)$ are called *cohomology groups* of G. If the action of G is nontrivial we talk of twisted coefficients or simply we say that these groups are the cohomology groups of G with coefficients in the module M if the action of G is understood (in this case we can simply write $H^*(G;A)$).

In an analogous way we can define the Tor functor

$$\mathrm{Tor}^{\mathbb{Z}[G]}_*(M,A) = H_*(C_* \otimes_G A)$$

for a free resolution C_* of M with G-modules and we can give the following

Definition 2.2.5. The groups $\mathrm{Tor}^{\mathbb{Z}[G]}_*(\mathbb{Z},A) = H_*(G;A)$ are the groups of *homology of G* with coefficients in the module M.

As for the cohomology, the algebraic definition generalizes a geometric construction and we have that

$$H_*(G;A) = H_*(B_G;A)$$

.

The case of twisted cohomology (resp. homology) corresponds to the topological definition of cohomology (resp. homology) of a space X with a local system of coefficients.

Theorem 2.2.6. *Let G be a discrete group, A a $\mathbb{Z}[G]$-module and $\mathbf{X}$ a $k(G,1)$ space. The map $G \to \mathrm{Aut}(A)$ determines a local system of groups $\mathcal{L}_A$ on the space $\mathbf{X}$. Let $\widetilde{\mathbf{X}}$ be the universal cover of $\mathbf{X}$. We have the following isomorphism, of complexes:*

$$C^*(\mathbf{X}, \mathcal{L}_A) = \mathrm{Hom}_{\mathbb{Z}[G]}(C_*(\widetilde{\mathbf{X}}), A)$$

$$C_*(\mathbf{X}, \mathcal{L}_A) = C_*(\widetilde{X}) \otimes_{\mathbb{Z}[G]} A.$$

In particular the twisted cohomology (resp. homology) of G with coefficients in the module A is isomorphic to the cohomology (resp. homology) of the space $\mathbf{X}$ with coefficients in the local system $\mathcal{L}_A$:

$$H^*(G, A) \simeq H^*(\mathbf{X}, \mathcal{L}_A)$$

$$H_*(G, A) \simeq H_*(\mathbf{X}, \mathcal{L}_A).$$

□

2.3 Cohomology of Artin groups

The study of the cohomology with constant (or untwisted) coefficients for braid groups and the other infinite families of finite type Artin groups (i.e. B_n and D_n) has been started by Arnold in the 70's (see [Arn69, Arn70a, Arn70b]) and has been terminated with the contribution of many people.

A first important computations, due to Fuks, is the cohomology of the braid groups $\mathrm{Br}(n)$ with coefficients in $\mathbb{Z}_2$:

Theorem 2.3.1 ([Fuk70])**.** *The generators of the group $H^k(\mathrm{Br}(n), \mathbb{Z}_2)$ can be identified with the partitions of n as a sum of $n-k$ powers of 2. We can denote this generators by $< 2^{l_1}, \ldots, 2^{l_{n-k}} >, l_1 \geq \cdots \geq l_{n-k}$.*

The multiplicative ring $H^*(\mathrm{Br}(n), \mathbb{Z}_2)$ is generated by the elements

$$a_{r,k} = \overbrace{< 2^r, \ldots, 2^r >}^{2^k \text{ elements}} \qquad (r \geq 1, k \geq 0)$$

with $\dim a_{r,k} = 2^k(2^r - 1)$, and with relations

$$< \overbrace{2^m, \ldots, 2^m}^{k_m}, \ldots, \overbrace{2, \ldots, 2}^{k_1} >< \overbrace{2^m, \ldots, 2^m}^{l_m}, \ldots, \overbrace{2, \ldots, 2}^{l_1} >=$$

$$= \binom{k_m + l_m}{k_m} \cdots \binom{k_1 + l_1}{k_1} < \overbrace{2^m, \ldots, 2^m}^{k_m + l_m}, \ldots, \overbrace{2, \ldots, 2}^{k_1 + l_1} >$$

$$a_{r,k}^2 = 0$$

$$a_{r_1,k_1} \cdots a_{r_q,k_q} = 0 \qquad \text{if } 2^{r_1+k_1+\cdots+r_q+k_q} > m.$$

□

The computation of the cohomology is done using a decomposition in cells of the Alexandroff's compactification of the unordered configuration space $\mathbb{C}(n)$. To obtain such a decomposition it is considered the point at infinity and the cells of type $e(m_1, \ldots, m_k)$, where $e(m_1, \ldots, m_k)$ is the subset of $\mathbb{C}(n)$ given by the points $\{z_1, \ldots, z_n\} \in \mathbb{C}(n)$ such that the points $z_1, \ldots, z_n \in \mathbb{C}$ lie exactly on k vertical real lines and moreover, numbering the lines from left to right, we have that exactly m_i points lie on the ith line.

After this result, the cohomology of braid groups has been computed with coefficients in $\mathbb{Z}_p$ for all other primes. Several independent results have been obtained by Cohen ([Coh76]), Segal ([Seg73]) and Vainshtein ([Vaĭ78]).

In [Coh76] Cohen establishes a general theory of homology operations in n-fold loop spaces. This leads to the computation of the homology of $\Omega^n\Sigma^n\mathbf{X}$. When $n = 2$ and $\mathbf{X} = S^0$ this gives the homology of classical braid groups. The key point for the computation of the homology of braid groups is the study of the map

$$C_n(\mathbb{R}^2) \hookrightarrow \Omega^2\Sigma^2 S^2$$

where $C_n(\mathbb{R}^2)$ is the space of unordered configurations of n points in $\mathbb{R}^2$.

In [Vaĭ78] Vainshtein use the same decomposition introduced by Fuks. Here we rewrite the result in the formulation given by Vainshtein.

First note that the braid group $\mathrm{Br}(n)$ naturally embeds in the group $\mathrm{Br}(n+1)$ and we can define the limit group

$$\mathrm{Br}(\infty) = \varinjlim_n \mathrm{Br}(n).$$

Theorem 2.3.2 ([Vaĭ78])**.** *For a prime $p \neq 2$ the ring $H*(\mathrm{Br}(\infty); \mathbb{Z}_p)$ is the tensor product of the polynomial algebras $\mathbb{Z}_p[x_i]$, $\dim x_i = 2p^{i+1} - 2, i \geq 0$ and the exterior algebra with generators $y_j, \dim y_j = 2p^j - 1, j \geq 0$. There is a natural map*

$$H^*(\mathrm{Br}(\infty); \mathbb{Z}_p) \to H^*(\mathrm{Br}(n); \mathbb{Z}_p)$$

that is surjective. Its kernel is generated by monomials $x_{r_1} \cdots x_{r_s} y_{l_1} \cdots y_{l_t}$ with $2(p^{r_1+1} + \cdots + p^{r_s+1} + p^{l_1} + \cdots + p^{l_t}) > n$. Let β_p be the Bockstein homomorphism associated to the short exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0,$$

then it holds that $\beta_p x_i = y_{i+1}, \beta_p y_i = 0.$ □

Theorem 2.3.3 ([Vaĭ78])**.** *We have the following isomorphisms:*

$$H^0(\mathrm{Br}(n); \mathbb{Z}) = H^1(\mathrm{Br}(n); \mathbb{Z}) = \mathbb{Z}$$

$$H^q(\mathrm{Br}(n); \mathbb{Z}) = \bigoplus_p \beta_p H^{q-1}(\mathrm{Br}(n); \mathbb{Z}_p) \qquad \text{for } q \geq 2$$

where the sum is understood over all primes p. □

The cohomology with integer coefficients for Artin groups of families B_n and D_n has been computed by Gorjunov:

Theorem 2.3.4 ([Gor78, Gor81]).

$$\begin{aligned}
H^q(G_{B_n};\mathbb{Z}) &= \bigoplus_{i=0}^{\infty} H^{q-i}(\mathrm{Br}(n-i);\mathbb{Z});\\
H^q(G_{D_n};\mathbb{Z}) &= H^q(\mathrm{Br}(n);\mathbb{Z}) \oplus\\
&\oplus \left[\bigoplus_{j=0}^{\infty} H^{q-2j}(\mathrm{Br}(n-2j);\mathbb{Z})/H^{q-2j}(\mathrm{Br}(n-2j-1);\mathbb{Z})\right] \oplus\\
&\oplus \left[\bigoplus_{k=0}^{\infty} H^{q-2k-3}(\mathrm{Br}(n-2k-3);\mathbb{Z}_2)\right].
\end{aligned}$$

The multiplicative structure of $H^*(G_{B_n};\mathbb{Z})$ and $H^*(G_{D_n};\mathbb{Z})$ is induced by multiplication in the cohomology of braids. □

For Artin group of type $E_6, E_7, E_8, F_4, H_3, H_4, I_2(m)$ (exceptional cases) the cohomology has been computed by Salvetti in [Sal94]. The result is showed in Table 2.1.

W	$I_2(2s)$	$I_2(2s+1)$	H_3	H_4	F_4	E_6	E_7	E_8
H^0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H^1	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H^2	$\mathbb{Z}$		$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	0	0
H^3			$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
H^4				$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
H^5						$\mathbb{Z}_6$	$\mathbb{Z}_6^2$	$\mathbb{Z}_2\times\mathbb{Z}_6$
H^6						$\mathbb{Z}_3$	$\mathbb{Z}_3\times\mathbb{Z}_6\times\mathbb{Z}$	$\mathbb{Z}_3\times\mathbb{Z}_6$
H^7							$\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_6\times\mathbb{Z}$
H^8								$\mathbb{Z}$

Table 2.1: Cohomology with constant coefficients: exceptional cases

2.4 Fibrations for Artin groups

Given an Artin group G_W we consider again the space $\mathbf{X}_W$ in order to study the Milnor fibration

$$\mathbf{F}' \hookrightarrow \mathbf{X}_W \to \mathbb{C}^*.$$

If the group W is finite, we recall that it acts on the complex vector space $\mathbf{V}$ with orthogonal reflections. The product

$$\delta = \prod l_H^2$$

where $\mathbf{H}$ are the hyperplanes such that the corresponding reflection is in W, determines a W-invariant map $\mathbf{V} \to \mathbb{C}$ and hence induce a map

$$\delta' : \mathbf{V}/W \to \mathbb{C}$$

and the fiber $\mathbf{F}'$ is the Milnor fiber of the non isolated singularity $\mathbf{F}'_0 = \delta'^{-1}(0) = \delta^{-1}(0)/W$. Moreover the map

$$\delta'_\sharp : G_W = \pi_1(\mathbf{X}_W) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$$

maps each generator g_s to 1 and the Milnor fiber $\mathbf{F}'$ is a classifying space for the $\ker \delta'_\sharp$.

If W is an affine Coxeter group, which acts by affine orthogonal reflections on the complex affine space $\mathbf{E}$, we can consider for each root α the complex function $f_\alpha : \mathbf{E} \to \mathbb{C}$ so defined:

$$f_\alpha(v) = (e^{2\pi\imath(v,\alpha)} - 1)$$

and we define the map δ as

$$\prod_{\alpha\in\Phi} f_\alpha.$$

Again it is clear that the map δ is W-invariant and δ is zero exactly on the union of the hyperplanes

$$\mathbf{F}_0 = \bigcup_{\alpha\in\Phi, k\in\mathbb{Z}} H_{\alpha,k}.$$

In general δ (and so also δ') is not a fibration, but we can take homotopy equivalent spaces $\widetilde{\mathbf{X}_W}$ and $\widetilde{\mathbb{C}^*}$ and a fibration $\widetilde{\delta'} : \widetilde{\mathbf{X}_W} \to \widetilde{\mathbb{C}^*}$ homotopy equivalent to δ'. Call $\mathbf{F}'$ the generic fiber of $\widetilde{\delta'}$.

Hence $\mathbf{F}'$ is the homotopy Milnor fiber of the nonisolated singularity $\mathbf{F}'_0 = \mathbf{F}_0/W$ of the analytic map δ' induced by δ.

In order to understand the topology of this singularity, we want to study the cohomology (or homology) of the fiber $\mathbf{F}'$.

2.5 A topological construction

Let $\mathbf{V} = \mathbb{C}^m$ a complex vector space and $f : \mathbf{V} \to \mathbb{C}$ a weighted homogeneous polynomial of type $(a_1, \ldots, a_m)$ as in Section 1.6. The singular fiber is $\mathbf{F}_0 = \{v \in \mathbf{V} \mid f(v) = 0\}$ and we have the Milnor fibration

$$\mathbf{F} \hookrightarrow \mathbf{X} \xrightarrow{f} \mathbb{C}^*$$

where $\mathbf{X}$ is the complement $\mathbf{V} \backslash \mathbf{F}_0$. Fix 1 as base point of $\mathbb{C}^*$ and an arbitrary point x_0 as base point of $\mathbf{F}$. By standard homotopy theory we can extend the fibration sequence to the left, to obtain, up to homotopy, a new fibration:

$$\Omega\mathbb{C}^* \hookrightarrow \mathbf{F} \to \mathbf{X}$$

where $\Omega\mathbb{C}^*$ is the space of maps $S^1 \to \mathbb{C}^*$, with base point the constant map 1 and it is homotopy equivalent to the integers $\mathbb{Z}$. Since the inclusion $\mathbf{F} \hookrightarrow \mathbf{X}$ is not a fibration, we have to replace $\mathbf{F}$ by a homotopy equivalent space $\overline{\mathbf{F}}$ to get an actual fibration over $\mathbf{X}$:

$$\Omega\mathbb{C}^* \hookrightarrow \overline{\mathbf{F}} \xrightarrow{\pi} X. \tag{2.5.1}$$

The space $\overline{\mathbf{F}}$ is given by:

$$\overline{\mathbf{F}} = \{(x, \gamma) : x \in \mathbf{X}, \gamma : [0,1] \to \mathbb{C}^*, \gamma(0) = 1, \gamma(1) = f(x)\}.$$

The map π is given by the projection on the first coordinate. We can fix an homotopy equivalence between $\mathbf{F}$ and $\overline{\mathbf{F}}$ as follows. First define the inclusion $i : \mathbf{F} \hookrightarrow \overline{\mathbf{F}}$, mapping a point $x \in \mathbf{F}$ to the couple $(x, 1)$ where 1 is the constant path. We define how to lift a point $(x, \gamma) \in \overline{\mathbf{F}}$ to a path $\widetilde{\gamma} : [0,1] \to \mathbf{X}$ by

$$\widetilde{\gamma}(t) = (x_1[\gamma(1-t)]^{-\frac{1}{a_1}}, \ldots, x_m[\gamma(1-t)]^{-\frac{1}{a_m}}).$$

Finally, we define a retraction $r : \overline{\mathbf{F}} \to \mathbf{F}$ by $r(x, \gamma) = \widetilde{\gamma}(0)$. It is easy to prove that the maps i and r define an homotopy equivalence between $\mathbf{F}$ and $\overline{\mathbf{F}}$.

This means that the Milnor fiber is homotopy equivalent to an infinite *cyclic cover* of $\mathbf{X}$. We can count the sheets of the cover by the numbers of loop in $\mathbb{C}^*$. Moreover the deck transformations of $\overline{\mathbf{F}}$ coincide with the monodromy group of $\mathbf{F}$ generated by the characteristic homomorphism h. Since the inclusion $i : \mathbf{F} \hookrightarrow \overline{\mathbf{F}}$ is an homotopy equivalence we can identify the cohomology groups $H^*(\mathbf{F}; R) \simeq H^*(\overline{\mathbf{F}}; R)$ for a ring R. Set $a := \prod_i a_i$. Since $h^a = \mathrm{Id}$, the action of $\mathbb{Z} = \pi_1(\mathbb{C}^*)$ on $H^*(\mathbf{F}; R)$ factors through $\mathbb{Z}/a\mathbb{Z}$. So we can consider $H^*(\mathbf{F}; R)$ as a $R[q^{\pm}]$-module, where multiplication by q acts as h.

Associated to the fibration (2.5.1) the have a Leray-Serre spectral sequence that converges to $H^*(\mathbf{F}, R)$. The E_2-term is

$$E_2^{p,q} = H^p(\mathbf{X}; H^q(\Omega\mathbb{C}^*; R))$$

where the action of the fundamental group of $\mathbf{X}$ on $H^q(\Omega\mathbb{C}^*; R)$ is the monodromy action. Since $\Omega\mathbb{C}^*$ is homotopy equivalent to $\mathbb{Z}$, its cohomology is concentrated in degree 0 and we can write it multiplicatively as the module of Laurent series $R[[t^{\pm}]]$. If we consider the subgroup $\mathbb{Z} \subset \mathrm{Aut}_{R[q^{\pm}]}(R[[q^{\pm}]])$ generated by the multiplication by q, we have that

$$H^*(\mathbf{F}, R) = H^*(\mathbf{X}; R[[q^{\pm}]])$$

where we consider the local system on the space $\mathbf{X}$ given by the module $R[[t^{\pm}]]$ with action defined through the map

$$\pi_1(\mathbf{X}) \xrightarrow{f} \pi_1(\mathbb{C}^*) = \mathbb{Z} \subset \mathrm{Aut}_{R[q^{\pm}]}(R[[q^{\pm}]]).$$

The dual result in homology gives

$$H_*(\mathbf{F}, R) = H_*(\mathbf{X}; R[q^{\pm}])$$

with the analogous local system defined with the module $R[q^{\pm}]$ on the space $\mathbf{X}$. Since $h^a = \mathrm{Id}$ we finally observe that $1 - q^a$ annihilates $H^*(\mathbf{F}, R)$ and $H_*(\mathbf{F}, R)$.

2.6 The Shapiro's Lemma

Let $H < G$ be two groups and let M be a $\mathbb{Z}[H]$-module. By means of the ring inclusion $\mathbb{Z}[H] \hookrightarrow \mathbb{Z}[G]$ we can define the induced and coinduced modules of H in G. We set:

$$\mathrm{Ind}_H^G M = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$$

and

$$\mathrm{CoInd}_H^G M = \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M).$$

We have the following result (see [Bro82]):

Lemma 2.6.1 (Shapiro's Lemma). *If $H < G$ and M is a $\mathbb{Z}[H]$-module, then*

$$H_*(H; M) \simeq H_*(G, \mathrm{Ind}_H^G M)$$

and

$$H^*(H; M) \simeq H^*(G; \mathrm{CoInd}_H^G M).$$

Proof. Let C be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$. We can regard C as a resolution of $\mathbb{Z}$ with free $\mathbb{Z}[H]$ modules, hence we have the isomorphism

$$H_*(H; M) \simeq H_*(C \otimes_{\mathbb{Z}[H]} M).$$

Since $C \otimes_{\mathbb{Z}[H]} M \simeq C \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M) \simeq C \otimes_{\mathbb{Z}[G]} (\mathrm{Ind}_H^G M)$, we get the first equality.

The second equality follows from the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}[H]}(C, M) \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(C, \mathrm{CoInd}_H^G M)$$

and this is a consequence of the following general fact: let $\alpha : R \to S$ be a homomorphism of unitary rings, given a S-module N, a R-module M and a morphism of R-modules $f : N \to M$, there exists a unique morphism of S-modules $g : N \to \mathrm{Hom}_R(S, M)$ such that $\pi g = f$:

$$\begin{tikzcd} & \mathrm{Hom}_R(S, M) \arrow[d, "\pi"] \\ N \arrow[ur, "g"] \arrow[r, "f"'] & M \end{tikzcd}$$

where π is the valuation induced on the element 1. Hence:

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) \simeq \mathrm{Hom}_R(N, M).$$

In order to prove the last statement we note that g must verify the relation $sg(n) = g(sn)$ for all $s \in S, n \in N$; valuating both the terms of the equality in 1 we find that $g(n)(s) = g(sn)(1) = \pi(g(sn)) = f(sn)$; then the existence and uniqueness of such a morphism g follow easily. So the Lemma is proved. □

2.7 Topology of the Milnor fiber for Artin groups

Recalling the previous observations, let W be a Coxeter group. We consider the fibration

$$\mathbf{F} \hookrightarrow \mathbf{Y}_W \xrightarrow{\delta} \mathbb{C}^*$$

$$\mathbf{F}' \hookrightarrow \mathbf{X}_W \xrightarrow{\delta'} \mathbb{C}^*$$

described in Section 2.4. We have that $\pi_1(\mathbf{X}_W) = G_W$ and $\pi_1(\mathbf{Y}_W) = PA_W$ and if we set $\pi_1(\mathbf{F}') = H_W$ we have the following short exact sequence of homotopy groups:

$$0 \to H_W \lhd G_W \xrightarrow{\pi} \mathbb{Z} \to 0$$

that is $\mathbb{Z} \simeq G_W/H_W$. Recall that π maps each standard generator g_s to 1.

If the group W is finite, then the map δ' is given by a weighted homogeneous polynomial and we can use the observations of Section 2.5 to show that

$$H^*(\mathbf{F}'; R) = H^*(\mathbf{X}_W; R[[q, q^{-1}]])$$

and

$$H_*(\mathbf{F}'; R) = H_*(\mathbf{X}_W; R[q, q^{-1}])$$

where G_W acts on $R[q, q^{-1}]$ and $R[q, q^{-1}]$ mapping each g_s to the multiplication by q.

By means of the Shapiro's Lemma we can state the result with more generality. Consider any Coxeter group W such that $\mathbf{Y}_W$ is a $K(\pi, 1)$ space. If W is infinite, let $\mathbf{F}'$ be the homotopy Milnor fiber of the map δ'. We have the following result (see [Fre88]):

Theorem 2.7.1. *Let W be a Coxeter group such that $\mathbf{Y}_W$ is a $K(\pi, 1)$ space and let*

$$\mathbf{F}' \hookrightarrow \mathbf{X}_W \xrightarrow{\delta'} \mathbb{C}^*$$

be the associated Milnor fibration. The fiber $\mathbf{F}'$ is a $K(\pi, 1)$ space and we have the isomorphisms

$$H_*(\mathbf{F}'; R) = H_*(H_W; R) = H_*(G_W; R[q, q^{-1}]) = H_*(\mathbf{X}_W; R[q, q^{-1}]),$$

$$H^*(\mathbf{F}'; R) = H^*(H_W; R) H^*(G_W; R[[q, q^{-1}]]) = H^*(\mathbf{X}_W; R[[q, q^{-1}]]).$$

□

The argument is the same as that used in [CS98] for the homology of arrangements of hyperplanes.

One can also prove (as we will see in Section 3.4 and 3.5) that for finite W the cohomology of G_W computed over the Laurent series module $R[[q,q^{-1}]]$ is equivalent to the cohomology computed with coefficients over the module of Laurent polynomial $R[q,q^{-1}]$, with a degree shift:

Theorem 2.7.2. *If W is a finite Coxeter group*

$$H^i(G_W; R[[q,q^{-1}]]) \simeq H^{i+1}(G_W; R[q,q^{-1}]).$$

□

In the case $R = \mathbb{Q}$ this result has already been observed by C. De Concini using the Universal Coefficients Theorem. In Section 3.4 and Section 3.5 two independent proofs of Theorem 2.7.2 are given. In the case of an infinite Coxeter group, a partial result can be given in order to compute the cohomology $H^*(G_W; R[[q,q^{-1}]])$ in terms of $H^*(G_W; R[q,q^{-1}])$. We will see an application of this partial generalisation in Section 7.3.2.

2.8 Cohomology of the Milnor fiber: known results

The cohomology of braid group with twisted coefficients over the Laurent polynomial ring $\mathbb{Q}[q,q^{-1}]$ was computed in [DCPS01]. This result has been obtained using a natural filtration on the complex C^* associated to the braid group. Later in [DCPSS99] the computations for the cohomology over the same ring $\mathbb{Q}[q,q^{-1}]$ have been performed with similar methods for Artin groups of type B_n and D_n.

We recall the notation φ_m for the m-th cyclotomic polynomial. The results are the following:

Theorem 2.8.1 ([DCPS01])**.** *Let R_q be the local system over the ring $R = \mathbb{Q}[q,q^{-1}]$ given by mapping each standard generator of* $\mathrm{Br}(m+1)$ *to the multiplication by $-q$. We denote by $\{h\}$ the quotient module $R/(\varphi_h)$. If $hi = m+1$ or if $hi = m$ then*

$$H^{i(h-2)+1}(\mathrm{Br}(m+1), R_q) = \{h\}.$$

All the other cohomology groups are zero. □

The same result was independently found by Frenkel in [Fre88] and Markaryan in [Mar96].

With the same notation, for the cases B_n and D_n we have:

Theorem 2.8.2 ([DCPSS99])**.**

$$H^n(G_{B_n}, R_q) = \bigoplus_{r|n} \{2r\}$$

and, for $s > 0$,

$$H^{n-2s+1}(G_{B_n}, R_q) = \bigoplus_{r \leq \frac{n}{2s}, r|n} \{2r\}$$

$$H^{n-2s}(G_{B_n}, R_q) = \bigoplus_{r \leq \frac{n-1}{2s}, r|n} \{2r\}.$$

□

Theorem 2.8.3 ([DCPSS99])**.** *For* $n \in \mathbb{N}$*, let* $S_n = \{k \in \mathbb{N} : k \mid n$ *or* $k \mid 2(n-1)$ *but* $k \nmid n-1\}$*. We use the convention that* $\{h\} = R/(\varphi_h)$ *if* h *is an integer, otherwise* $\{h\} = 0$*, we have that:*

$$H^n(G_{D_n}, R_q) = \bigoplus_{2h \in S-n} \{2h\}$$

and, for $s > 0$,

$$H^{n-2s}(G_{D_n}, R_q) = \bigoplus_{1<h\leq \frac{n-2}{2s}, 2h \in S_n} \{2h\} \oplus \left\{\frac{n-1}{s}\right\}$$

$$H^{n-2s+1}(G_{D_n}, R_q) = \begin{cases} \displaystyle\bigoplus_{1<h\leq \frac{n-2}{2s}, 2h \in S_n} \{2h\} \oplus \left\{\frac{n}{s}\right\}^{\oplus 2} & \textit{if } \frac{n}{s} \textit{is an integer, even}, n > 2 \\ \displaystyle\bigoplus_{1<h\leq \frac{n-2}{2s}, 2h \in S_n} \{2h\} \oplus \left\{\frac{n}{s}\right\} & \textit{otherwise.} \end{cases}$$

□

For Artin groups of type $I_2(m)$, H_3, H_4, F_4, E_6, E_7, E_8, analogous computations over the ring of Laurent polynomials with rational coefficients $\mathbb{Q}[q^{\pm 1}]$ can be found in [DCPSS99]. The same computations have been extended to integer coefficients, $\mathbb{Z}[q^{\pm 1}]$ in [CS04]. We report the result in Table 2.2. We use the convention of writing R for the ring $\mathbb{Z}[q^{\pm 1}]$ and the number m for the module $R/(\varphi_m)$. Moreover we define the following ideals:

$$I_4 = (\varphi_2, 2)[60]/\varphi_{60}; \qquad J_4 = [24]/\varphi_{24};$$
$$I_6 = (\varphi_3\varphi_6\varphi_9\varphi_{12}); \qquad I_7 = (\varphi_2\varphi_6\varphi_{14}\varphi_{18});$$
$$I_8 = (\varphi_2, \varphi_4)\varphi_2 0[24][30]/[6].$$

W	$I_2(m)$	H_3	H_4	F_4
H^0	0	0	0	0
H^1	2	2	2	2
H^2	$R/[m]$	0	0	2
H^3		$R/(\varphi_2\varphi_6\varphi_{10})$	0	$R/(\varphi_2\varphi_3\varphi_6)$
H^4			R/I_4	R/J_4

W	E_6	E_7	E_8
H^0	0	0	0
H^1	2	2	2
H^2	0	0	0
H^3	0	0	0
H^4	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
H^5	$6 \oplus 8$	$6 \oplus \mathbb{Z}/2$	4
H^6	R/I_6	$\mathbb{Z}/3 \oplus (\varphi_2, 3)R/\varphi_6(\varphi_2, 2)$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
H^7		R/I_7	$8 \oplus 12 \oplus \mathbb{Z}/3$
H^8			R/I_8

Table 2.2: Cohomology with coefficients in the ring $\mathbb{Z}[q^{\pm 1}]$: exceptional cases

Chapter 3

Topology of arrangements

3.1 The Salvetti complex

Now we present a cell complex that has the same homotopy type of a the complement $\mathbf{Y}_{\mathcal{A}}$ of the complexification of a real arrangement $\mathcal{A}_{\mathbb{R}}$. If the arrangement $\mathcal{A}_{\mathbb{R}}$ is the reflection arrangement of a Coxeter group W, then the complex we are going to show is W-invariant and taking the quotient respect to the action of W we get a complex with the same homotopy type of the space $\mathbf{X}_W$. The construction we are going to present can be found in [Sal87], [Sal94] and [DCS96], where we refer for all proofs.

Let $\mathcal{A}_{\mathbb{R}}$ be a finite real arrangement in $\mathbb{R}^n$. The arrangement induces a stratification of the space $\mathbb{R}^n$ in facets. We call $\mathcal{S}$ the set of all facets and we partially order $\mathcal{S}$ saying that $F > F'$ if and only if $\overline{F} \supset F'$. Let $\mathbf{Q}$ the dual cell complex of $\mathcal{S}$. We can realize $\mathbf{Q}$ inside $\mathbb{R}^n$ associating to each facet F^j of codimension j the point $v(F^j) \in F^j$ and considering the simplexes

$$\sigma(F^{i_0}, \dots, F^{i_j}) = \{\sum_{k=0}^{j} \lambda_k v(F^{j_k}) \mid \sum \lambda_k = 1, \lambda_k \in [0,1]\}$$

where $F^{i_k} > F^{i_{k+1}}$ for $k = 0, \dots, j-1$. We define the j-cell $e^j(\widetilde{F}^j)$, dual to $\widetilde{F}^j$, as the union $\bigcup \sigma(F^0, \dots, F^{j-1}, \widetilde{F}^j)$, over all the chains $F^0 > \cdots > \widetilde{F}^j$. Hence we have $\mathbf{Q} = \bigcup e^j(F^j)$, where the union is taken over all facets in $\mathcal{S}$.

We can think to the 1-skeleton $\mathbf{Q}_1$ as a graph and we define the combinatorial distance between two vertexes v, v' of $\mathbf{Q}$ as the minimum number of edges in an edge-path connecting v to v'. For each cell e^j we indicate by $V(e^j) = \mathbf{Q}_0 \cap e^j$ the 0-skeleton of e^j. We have:

Lemma 3.1.1. *For every vertex $v \in \mathbf{Q}_0$ and for every cell $e^i \in \mathbf{Q}$, there exists a unique vertex $\underline{w}(v, e^i) \in V(e^i)$ of minimal distance from v, that is:*

$$d(v, \underline{w}(v, e^i)) < d(v, v') \qquad \forall v' \in V(e^i) \setminus \{\underline{w}(v, e^i)\}$$

If $e^i \subset e^j$ then $\underline{w}(v, e^j) = \underline{w}(\underline{w}(v, e^i), e^j)$. □

Take a cell $e^j = e^j(F^j) = \bigcup \sigma(F^0, \dots, F^{j-1}, F^j)$ of Q and let $v \in V(e^j)$. We can map the simplex $s(F^0, \dots, F^j)$ in $\mathbb{C}^n$ by the application

$$\phi_{v,e^j}(\sum \lambda_k v(F^k)) = \sum \lambda_k v(F^k) + i \sum \lambda_l(\underline{w}(v, e^k) - v(F^k)).$$

One can prove that ϕ_{v,e^j} gives an embedding of e^j in $\mathbf{Y}_{\mathcal{A}}$. We call $E^j(e^j, v)$ the image of the map ϕ_{v,e^j}, and we define the *Salvetti complex* for the arrangement $\mathcal{A}$ as the union

$$\mathbf{M} = \bigcup E^j(e^j, v),$$

where the union is taken over all e^j and v.

Theorem 3.1.2 ([Sal87]). *The CW-complex* $\mathbf{M}$ *is homotopy equivalent to the complement* $\mathbf{Y}_{\mathcal{A}}$. □

We remark that the fact that the maps ϕ_{v,e^j} glue together in the proper way is a consequence of Lemma 3.1.1.

Suppose that the arrangement $\mathcal{A}$ is the reflection arrangement associated to a finite Coxeter group W. We have that the complex $\mathbf{M}$ is invariant for the action of W and we call

$$\mathbf{C} = \mathbf{M}/W$$

its quotient. We will give a description of this quotient complex.

Let us fix a chamber C_0 in the complement of the real arrangement $\mathcal{A}_{\mathbb{R}}$ and let v_0 be the vertex of $\mathbf{Q}$ contained in C_0. We denote by $\mathcal{S}_0$ the system of facets of C_0 (that is, the set of all facets included in the closure of C_0) and by $\mathcal{Q}_0$ the set of all cells that are in $\mathbf{Q}$ and are dual to a facet in $\mathcal{F}_0$, that is $\mathcal{Q}_0$ is the set of all cells that intersect the chamber C_0. Note that $\mathcal{Q}_0$ is not a CW-complex. The following result holds:

Proposition 3.1.3. *For every facet $F \in \mathcal{S}$, there exists a unique facet $F_0 \in \mathcal{S}_0$ that belongs to the W-orbit of F. For every cell $e \in Q$, there exists a unique cell $e_0 \in \mathcal{Q}_0$ that belongs to the W-orbit of e. The elements $\gamma \in W$ such that $\gamma(e_0) = e$ give a left coset of the stabilizer W_{F^0} of F^0, where F^0 is the dual of e_0* □

As a consequence of Proposition 1.1.3 it follows that there exists a unique element of minimal length in each coset of W_{F^0}. We denote by $\gamma_{(e)}$ the element of minimal length with respect to the Coxeter system associated to the chamber C_0 and which maps e_0 to e. Now let $\mathbf{Y}_{\mathcal{A}}$ be as before. We have the following result:

Theorem 3.1.4. *The space $\mathbf{Y}/W$ has the same homotopy type of the CW-complex $\mathbf{C}$ obtained as a quotient of $\mathbf{Q}$ identifying two cells e, e' if and only if they are in the same W-orbit, using the identification map induced by the element $\gamma_{(e)}\gamma_{(e')}^{-1}$.* □

Definition 3.1.5. We call the complex $\mathbf{C} = \mathbf{C}_W$ the Salvetti complex for the group W and we denote by $\pi_W : \mathbf{Q} \to \mathbf{C}_W$ the projection map.

Corollary 3.1.6. *Let W be a finite essential Coxeter group, the number of i-cells in $\mathbf{C}_W$ is* $\binom{n}{i}$. □

We can use the complex $\mathbf{C}_W$ to compute the cohomology and homology of the group G_W. First suppose that the action of W is essential. Let $H_1, \dots, H_n$ be the walls of the fundamental chamber C_0, in correspondence with the reflections of the set of generators of W $\{s_1, \dots, s_n\} = S$. Let v_i be the chosen point in $H_i \cap \overline{C_0}$. Each facet $F \in \mathcal{S}_0$ corresponds to a unique intersection $H_{i_1} \cap \cdots \cap H_{i_k}$, $k = \operatorname{codim}(F)$, where the H_{i_j} are the walls of C_0 that contain F and $i_1 < \cdots < i_k$. We endow the dual cell $e(F)$ with the orientation induced by the order of $v_0, v_{i_1}, \dots, v_{i_k}$. Moreover we orient each cell $e \in \mathbf{Q}$ requiring the map $\gamma_{(e)}$ to be orientation-preserving. In this way the incidence number $[e : e'] \in \{0, 1, -1\}$ between cells of $\mathbf{M}$ is well defined and passes to the quotient in $\mathbf{X}_W$.

To each cell in $\mathbf{C}_W$ there corresponds a unique cell in $\mathcal{Q}_0$ and a unique facet in $\mathcal{S}_0$. As a consequence, for every cell $F \in \mathcal{F}_0$ there is a unique subset $\Gamma = \Gamma(F) \subset I_n = \{1, \dots, n\}$ such that $F = \cap_{j \in \Gamma} H_j$.

Lemma 3.1.7. *Let $F \in \mathcal{S}_0$ corresponds to the set Γ. Let $G' \in \mathcal{S}$ a facet W-equivalent to $F' \in \mathcal{S}_0$, with $\Gamma' = \Gamma(F') \subset \Gamma, |\Gamma| = |\Gamma'| + 1$, and such that $F' \subset \overline{G'}$ (hence G' is W_F-equivalent to F'). Then $e(G') \subset e(F)$ and the following holds:*

$$[e(F) : e(G')] = (-1)^{l(G')}[e(F) : e(F')]$$

where $l(G')$ is the minimum length $l(g)$ of an element $g \in W$ mapping F' to G'. □

We give a description of the algebraic complex C^* that compute the cohomology of $\mathbf{C}_W$, that is the *algebraic Salvetti complex.*

Let us define C^k as the free $\mathbb{Z}$-module generated by k-cells of $\mathbf{C}_W, k = 0, \dots, n$:

$$C^k = \{\sum a_\Gamma \Gamma \mid a_\Gamma \in \mathbb{Z}, \Gamma \subset I_n, |\Gamma| = k\}.$$

Theorem 3.1.8. *We have*

$$H^*(\mathbf{C}_W; \mathbb{Z}) \simeq H^*(C^*)$$

where the coboundary $\delta^k : C^k \to C^{k+1}$ is defined by:

$$\delta^k(\Gamma) = \sum_{j \in I_n \setminus \Gamma} (-1)^{\sigma(j,\Gamma)+1)} (\sum_{\underline{h} \in W_{\Gamma \cup \{j\}}/W_\Gamma} (-1)^{l(\underline{h})}) (\Gamma \cup \{j\})$$

We define $\sigma(j, \Gamma) = |\{i \in \Gamma \mid i < j\}|$ and $W_{\Gamma(F)} = W_F$. $l(\underline{h})$ is the minimum of the length of an element h in the coset $\underline{h}$. □

More generally, we can consider the cohomology with coefficients in a local system. We use the following result:

Theorem 3.1.9. *Let W be a finite Coxeter group and $\mathbf{C}_W$ the Salvetti complex for W. Let M be a G_W-module and let $\psi : W \to G_W$ be the section defined in Theorem 1.4.7. Suppose we have a local system $\mathcal{L} = \mathcal{L}(\mathbf{C}_W; M)$ over $\mathbf{C}_W$ with coefficients in the module M given by the action of G_W on M. Then*

$$H^*(\mathbf{C}_W; \mathcal{L}) \simeq H^*(C^*)$$

where the complex $C^*(q)$ is given by

$$C^k = \{\sum a_\Gamma e(\Gamma) \mid a_\Gamma \in M, \Gamma \subset I_n, |\Gamma| = k\}$$

and the coboundary map is

$$\delta^k(a_\Gamma e(\Gamma)) = \sum_{j \in I_n \setminus \Gamma} \sum_{w \in W^{\Gamma}_{\Gamma \cup \{j\}}} (-1)^{\sigma(j,\Gamma)} (-1)^{l(w)} \psi(w) a_\Gamma e(\Gamma \cup \{j\}).$$

□

In the special case where R is a unitary commutative ring with q a unit of R and M is an R-module, consider the local system $\mathcal{L}_q = \mathcal{L}_q(\mathbf{C}_W; M)$ over $\mathbf{C}_W$ with coefficients in the module M given by mapping each standard generator of $G_W = \pi_1(\mathbf{C}_W)$, represented by a 1-cell $\pi_W(e), e \in \mathcal{Q}_0 \cap Q_1$, to the automorphism of M given by multiplication by $-q$. Then

$$H^*(\mathbf{C}_W; \mathcal{L}_q) \simeq H^*(C^*(q))$$

where the complex $C^*(q)$ is given by

$$C^k(q) = \{\sum a_\Gamma \Gamma \mid a_\Gamma \in M, \Gamma \subset I_n, |\Gamma| = k\}$$

and the coboundary map is

$$\delta^k(q)(\Gamma) = \sum_{j \in I_n \setminus \Gamma} (-1)^{\sigma(j,\Gamma)} \frac{W_{\Gamma \cup \{j\}}(q)}{W_\Gamma(q)} (\Gamma \cup \{j\}).$$

The cohomology of exceptional Artin groups was computed in [Sal94] using this special case of the complex described in Theorem 3.1.8.

Clearly one can use the Salvetti complex also to compute the homology. The homology version of the complex of Theorem 3.1.9 is:

$$C_k = \{\sum a_\Gamma e(\Gamma) \mid a_\Gamma \in M, \Gamma \subset I_n, |\Gamma| = k\}$$

and the boundary map is

$$\partial_k(a_\Gamma e(\Gamma)) = \sum_{j \in} \sum_{w \in W_\Gamma^{\Gamma \setminus \{j\}}} (-1)^{\sigma(j,\Gamma)} (-1)^{l(w)} \psi(w) a_\Gamma e(\Gamma \setminus \{j\}).$$

For the local system $\mathcal{L}_q = \mathcal{L}_q(\mathbf{C}_W; M)$ defined as before, the complex $C_* = C_*(q)$ for the homology is the following

$$C_k(q) = \{\sum a_\Gamma \Gamma \mid a_\Gamma \in M, \Gamma \subset I_n, |\Gamma| = k\}$$

and the boundary map is

$$\partial_k(q)(\Gamma) = \sum_{j \in \Gamma} (-1)^{\sigma(j,\Gamma)} \frac{W_\Gamma(q)}{W_{\Gamma \setminus \{j\}}(q)} (\Gamma \setminus \{j\}).$$

We remark that although the choice of the multiplication by q would be equivalent, we use $-q$, which seems more natural to us, and also for coherence with [CP07], [DCPS01], [DCPSS99].

The construction of the Salvetti complex can be extended to any infinite Coxeter group W.

Take one point x_0 in the fundamental chamber C_0; for any subset $J \subset S$ such that the parabolic subgroup W_J is finite, construct a $(|J| - 1)$-cell in U^0 as the "convex hull" of the W_J-orbit of x_0 in $\mathbb{R}^n$.

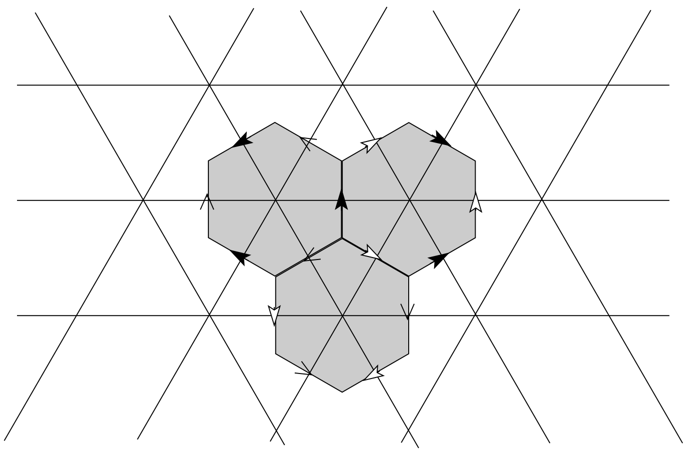


Figure 3.1: *the space $K(G_{\tilde{A}_2}, 1)$ is given as union of* 3 *hexagons with edges glued according to the arrows (there are:* 1 *0-cell,* 3 *1-cells,* 3 *2-cells in the quotient).*

So, we obtain a finite cell complex (see Figure 3.1) which is the union of (in general, different dimensional) polyhedra, corresponding to the maximal subsets J such that W_J is finite. Now take identifications on the faces of these polyhedra, the same as described for the finite case (they are shown in Figure 3.1 for the case $\tilde{A}_2$). We obtain a finite CW-complex $\mathbf{X}_W$: it has a $|J|$-cell for each $J \subset S$ such that W_J is finite.

We obtain as in [Sal94]:

Theorem 3.1.10. $\mathbf{C}_W$ *is a deformation retract of the orbit space.* □

The algebraic complex we get, for a local system over a G_W-module M is

$$C_k = \{\sum a_\Gamma e(\Gamma) \mid a_\Gamma \in M, \Gamma \subset I_n, |\Gamma| = k, |W_\Gamma| < \infty\}.$$

The boundary formula is exactly the same. For the coboundary, just take the sum over the j's such that $|W_{\Gamma\cup\{j\}}| < \infty$.

Remark 3.1.11. The standard presentation for G_W is quite easy to derive from the topological description of $\mathbf{C}_W$; we may thus recover Van der Lek's result [vdL83].

Proposition 3.1.12. *Let* $K_W^{fin} := \{J \subset S \ : \ |W_J| < \infty\}$ *with the natural structure of simplicial complex. Then the Euler characteristic of the orbit space (so, of the group* G_W *when such space is of type* $K(\pi, 1)$*)) equals*

$$\chi(K_W^{fin}).$$

In particular, if W *is affine of rank* $n+1$ *we have*

$$\chi(\mathbf{X}_W) \ = \ \chi(K_W^{fin}) = \ 1 - \chi(S^{n-1}) \ = \ (-1)^n$$

Proof. Last statement follows from the fact that K_W^{fin} contains all proper subsets of S; thus:

$$H_*(K_W^{fin}) \ = \ \tilde{H}_{*-1}(S^{n-1}).$$

□

Remark 3.1.13. In our computations in the next chapters, sometimes we will study cohomology and some others homology. The different choice has no special meaning and is done just in order to get easier computations. In fact the module structure of homology and cohomology give us the same information, since they are related by the Universal Coefficient Theorem.

3.2 Filtrations on the Salvetti complex: an example

Let us now consider again our main example of Coxeter group, that is the group of permutations on $n+1$ elements $W = A_n$. We want to compute the cohomology of the associated Artin group G_W, that is the Artin braid group $\mathrm{Br}(n+1)$ with coefficients in a local system over a G_W-module M. The generators of the complex C_n^* are associated to subsets of the set of generators, that is the nodes of the Coxeter diagram for A_n. We can denote each generator by a subset Γ of the set $\{1, \dots, n\}$. We can identify each subset Γ with its characteristic functions, that is with a sequence of 0's and 1's of length n. A symbol 0 in the i-th position of the string means that $i \notin \Gamma$, while a symbol 1 means that $i \in \Gamma$. If A and B are two strings, we write AB for their concatenation. The degree $|A|$ corresponds to the number of 1's that appear in A, and so it is the dimension of the corresponding cell. Hence $C_n^{(k)}$ is the subcomplex generated by strings of length n and degree k.

Using the notations of q-analog, for a string A we write $A! = \prod_i [k_i + 1]!$, where the numbers k_i are the length of the maximal substrings of A containing only 1's. Hence, applying Theorem 3.1.9 and the computations of Section 1.5 the coboundary δ is defined by the following rules:

$$\delta(A00B) = \delta(A0)0B + (-1)^{|A|}A0\delta(0A),$$

$$\delta(A101B) = \delta(A1)01B + (-1)^{|A|+1}A10\delta(0B) + (-1)^{|A|+1}\frac{A111B!}{A101B!}A111B,$$

$$\delta 1^k = 0,$$

$$\delta 01^k = [k+2]1^{k+1},$$

$$\delta 1^k 0 = (-1)^k[k+2]1^{k+1},$$

$$\delta 01^k 0 = [k+2](1^{k+1}0 + (-1)^k 01^{k+1}).$$

We can endow the complex (C_n^*, δ) with a filtration F defined as follows: F^sC_n is the subcomplex of C_n^* generated by strings of kind $A1^s$. We have the following isomorphisms of complexes:

$$(F^sC_n/F^{s+1}C_n) \simeq C_{n-s-1}[s+\ 1],$$

where the notations $[s+1]$ in square brackets means a shifting in the degree by $s+1$ that is $(F^sC_n/F^{s+1}C_n)^t = C_{n-s-1}^{(t-s-1)}$, where the isomorphism is given by the mapping:

$$A01^s \mapsto A$$

$$A11^s \mapsto 0$$

We finally define a spectral sequence associated to C_n and to the filtration $(F^sC_n)_s$. We set

$$E_0^{s,t} = (F^sC_n/F^{s+1}C_n)^{s+t}$$

and let the 0-differential $d_0 : E_0^{s,t} \to E_0^{s,t+1}$ be the map induced by δ.

Note that the quotient $F^{n-1}C_n/F^nC_n$ is isomorphic to the module M generated by the string 01^{n-1} and $F^nC_n/F^{n+1}C_n = M$ with generator 1^n. We can give the following definition (see [Spa66]):

$$Z_r^s = \{c \in F^sC_n \mid \delta c \in F^{s+r}C_n\}$$

$$Z_\infty^s = \{c \in F^sC_n \mid \delta c = 0\}$$

where it is understood the natural graduation induced by the graduation of C_n.

Moreover we set:

$$E_r^s = Z_r^s/(Z_{r-1}^{s+1} + \delta Z_{r-1}^{s-r+1})$$

and we define the differential $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$ to be the homomorphism naturally induced by the map $\delta : Z_r^s :\to Z_r^{s+r}$ and by $\delta : Z_{r-1}^{s+1} + \delta Z_{r-1}^{s-r+1} \to \delta Z_{r-1}^{s+1}$.

In particular we get

$$E_1^{s,t} = H^{s+t}(F^s c_n/F^{s+1}C_n) = H^t(C_{n-s-1})$$

and the differential d_1 is the boundary operator associated to the short exact sequence

$$0 \to F^{s+1}C_n/F^{s+2}C_n \to F^sC_n/F^{s+2}C_n \to F^sC_n/F^{s+1}C_n \to 0.$$

Hence the i-th columns of the term E_1 of spectral sequence for the cohomology of C_n is given by the cohomology of C_{n-i-1}.

Using the filtration associate to the numbering of the Coxeter graph we can give analogous spectral sequences for the other Artin groups.

3.3 Filtrations and spectral sequences

We want to give a more general description of what we explained with the example in the previous Section, in order to understand how we can use spectral sequences to compute the cohomology of Artin groups.

Suppose we have fixed a finitely generated Coxeter group $W = W_\Gamma$ and let Γ be the Coxeter graph and $G = G_W$ the associated Artin group. Let n be the number of vertexes of Γ, that is the number of standard generator of W. We can suppose to have a total order on the set of vertexes of Γ (in what follows, if not specified, for finite and affine irreducible Coxeter groups we always use the order given by the numbering in Tables 1.1 and Table 1.2). Let C^* be the algebraic Salvetti complex associated to G. We use the numbering on the vertexes of Γ in order to write the generators of C^* as strings of 0's and 1's.

Hence we filter the complex C^* as follows: F^sC is the complex generated by strings of type $A1^s$. The quotients $G^sC = F^sC/F^{s+1}C$ are isomorphic to simpler complexes (with less generators), so we can assume we already know their cohomology. The starting page of the associated spectral sequence is

$$E_0^{s,t} = G^sC^{s+t}$$

and, since the differential d_0 is just the map induced by the coboundary of C^* on the quotients $G^sC = F^sC/F^{s+1}C$, the next term is

$$E_1^{s,t} = H^{s+t}(G^sC).$$

This means that the columns of the complex E_1 corresponds to the cohomology of the complexes G^sC. We can choose as generators of the module $E_1^{s,t}$ some combinations $\sum m_i A_i$ where the strings A_i are in the form $A_i'01^s$. The differential d_n is the map induced by the coboundary δ of the complex C^*, hence

$$d_n : E_n^{s,t} \to E_n^{s+n,t-n+1}$$

maps an element $a = \sum m_i A_i$ to the sum of terms in δa corresponding to strings terminating with 01^{s+n}. For example, in case $W = A_n$, if $a = \sum m_i A_i 01^{n-1}01^s + \sum l_i B_i \in E_n^{s,t}$, where the strings B_i's do not end with the sequence $01^{n-1}01^s$, the differential $d_n a$ is given by the equivalence class of

$$\begin{bmatrix} s+n+1 \\ n \end{bmatrix} \sum m_i A_i 01^{s+n}.$$

In particular we can consider the special case when the complex G^sC is isomorphic to the complex $\widetilde{C}$ associated to a Coxeter group $\widetilde{G}$ with $n-s-1$ generators, with a graph $\widetilde{\Gamma} \subset \Gamma$. If the sum $\widetilde{a} = \sum m_i A_i$ is a generator of $H^t(\widetilde{C})$, $a = \sum m_i A_i 01^s$ is the corresponding generator of $H^{s+t}(G^sC)$.

In this way, it is possible to perform a computation of the spectral sequence up to the term E_∞. To compute the cohomology $H^*(C^*)$ in general we need to lift the generators of E_∞ to generators of the cohomology C^* in order to solve, by means of direct computations on the complex C^*, the indeterminacy about lifting of modules.

3.4 The Novikov homology and degree shift

Let $\mathbf{X}$ be a connected differentiable manifold and let $h : \mathbf{X} \to \mathbf{X}$ be an automorphism o $\mathbf{X}$. We can consider the manifold $\mathbf{Y}$ defined as follows:

$$\mathbf{Y} = \mathbf{X} \times_h S^1 = (\mathbf{X} \times \mathbb{R})/\sim$$

where $(x,t) \sim (x',t')$ if $t' = t+n$ and $x = h^n(x')$ $(n \in \mathbb{Z})$. Let G be the fundamental group of $\mathbf{Y}$. The natural fibration

$$f : \mathbf{Y} \to S^1$$

induces a surjective homomorphism for the fundamental groups:

$$f_\# : G \to \mathbb{Z}.$$

We need to fix some notation. Let A be a principal ideal domain (PID). We write R for the ring of Laurent polynomials with coefficients in A, $R = A[q,q^{-1}]$. Moreover we write M for the R-module of Laurent series with coefficients in A, $M = A[[q,q^{-1}]]$. We denote by N_+ the ring of Laurent series $N_+ = A[[q]][q^{-1}]$ and analogously $N_- = A[q][[q^{-1}]]$. Note that the two rings N_+ and N_- are PID (see [Far04], Lemma 1.10). Finally consider the quotients $M_+ = M/N_-$ e $M_- = M/N_+$.

All these rings and modules have a natural structure of R-module and so, by means of the map $f_\#$ they have an induced structure of module over the group-ring $\mathbb{Z}[G]$, where the action is given by mapping an element $g \in G$ to the multiplication by $q^{f_\#(g)}$.

Hence we can define on the space Y the local systems $\mathcal{R}$, $\mathcal{M}$, $\mathcal{M}_+$, $\mathcal{M}_-$, $\mathcal{N}_+$, $\mathcal{N}_-$ associated respectively to the modules R, M, M_+, M_-, N_+, N_-, using the action induced by $f_\#$. We note that, using the map h, the cohomology of the space $\mathbf{X}$ with coefficients in the ring A is endowed with a natural structure of R-module.

Lemma 3.4.1. *The following isomorphisms of R-modules hold:*

$$H_*(\mathbf{X}, A) = H_*(\mathbf{Y}, \mathcal{R}).$$

$$H^*(\mathbf{X}, A) = H^*(\mathbf{Y}, \mathcal{M}).$$

Proof. We have a fiber bundle of the space $\mathbf{X} \times \mathbb{R}$ over the space $\mathbf{Y}$, with fiber $\mathbb{Z}$:

$$\mathbb{Z} \hookrightarrow \mathbf{X} \times \mathbb{R} \to \mathbf{Y}.$$

We have homology and cohomology spectral sequences associated to the fiber bundle and since the fiber is 0-dimensional we get that:

$$H_*(\mathbf{X}, A) = H_*(\mathbf{Y}, H_*(\mathbb{Z}, A)).$$

$$H^*(\mathbf{X}, A) = H^*(\mathbf{Y}, H^*(\mathbb{Z}, A)).$$

Finally we observe that the fundamental group of the base $\mathbf{Y}$ acts on $R \simeq H_*(\mathbb{Z}, A)$ (resp. on $M \simeq H^*(\mathbb{Z}, A)$) by means of the natural structure of G-modulo of R (resp. M). □

In the early '80 Novikov presents a generalization of Morse theory of multivalued functions and closed 1-forms. In [Nov81] he considers the problem of finding relations between critical points and the topology of the manifold. Given a closed 1-form ω on a compact manifold $\mathbf{M}$, he considers a covering

$$\pi : \mathbf{N} \to \mathbf{M}$$

(for a function $g : \mathbf{M} \to S^1$ take $\omega = dg$) where the pull back $\pi^*(\omega) = dh$ is an exact form: hence a Morse function arises on the non compact manifold $\mathbf{N}$. Novikov defines a complex, $(C^k_{\mathfrak{n}}, \delta)$, similar to the Morse complex, with generators in 1-1 correspondence with the critical points of the form ω and with a boundary defined, as in Morse complex, counting flow lines. He proves also that the homology of the complex $(C^k_{\mathfrak{n}}, \delta)$ is isomorphic to the homology of the manifold $\mathbf{N}$ with non compact "semi-open" support, that is with support in a closed set $\mathbf{V} \subset \{x \in \mathbf{N} \mid h(x) > K\}$ for a certain constant K. We refer to [Nov81, Nov82] for a more detailed construction.

Lemma 3.4.2. *The cohomology $H^*(\mathbf{Y}, \mathcal{N}_+)$ is the Novikov cohomology of the space $\mathbf{Y}$ associated to the map f. If the manifold $\mathbf{X}$ is compact or if the automorphism h has finite order $h^N = \mathrm{Id}$, this cohomology is zero. Analogously the cohomology with coefficients in the module $\mathcal{N}_-$ is also zero.*

Proof. The Novikov homology of the space $\mathbf{Y}$ corresponds (see [Nov81]) to the homology of $\mathbf{X}$ with coefficients in the local system $\mathcal{N}_+$. To prove this we can take a CW-decomposition of $\mathbf{Y}$. Let C_* be the associated algebraic complex. Using the covering f we can lift the decomposition of $\mathbf{X}$ to a CW-decomposition of $\mathbf{X} \times \mathbb{R}$, which has an associated complex that is homotopy equivalent (with a chain-homotopy) to $C_* \otimes_G R$ where the action of G on R is just the action induced by $f_\#$. This complex computes (see Lemma 3.4.1) the homology of the space $\mathbf{X}$.

Using [Nov81, Lemma 1] we have that the Novikov homology of $\mathbf{Y}$ associated to f is equivalent to the homology of the complex $(C_* \otimes_G R) \otimes_G N_+ = C_* \otimes_G N_+$, that is the homology of the space $\mathbf{Y}$ with coefficients in the local system $\mathcal{N}_+$.

If $\mathbf{M}$ is compact, the fact that the Novikov homology is actually zero follows easily from the observation that map f is a fibration and hence there are no critical points. If h has finite order, note that the action of $h \times \mathrm{Id}$ on $\mathbf{Y}$ induces in the homology $H_*(\mathbf{Y}, \mathcal{N}_+)$ the multiplication by q and since if A is a field then the ring N_+ is also a field and $c = q^N - 1$ is different from 0, we have that the Novikov homology must be zero, because multiplication by a nonzero element gives the null map. Finally applying the Universal Coefficient Theorem for local coefficients (see for example [God73], Theorem 5.4.2, p. 101) it follows that the cohomology $H^*(\mathbf{Y}, \mathcal{N}_+)$ is zero. The proof for $\mathcal{N}_-$ is completely equivalent. □

Theorem 3.4.3. *Under the hypotheses of Lemma 3.4.2 we have the following isomorphism of R-modules:*

$$H^*(\mathbf{X}, A) = H^{*+1}(\mathbf{Y}, \mathcal{R}).$$

Proof. Since we can apply Lemma 3.4.2, we need just to prove that

$$H^*(\mathbf{Y}, \mathcal{M}) = H^{*+1}(\mathbf{Y}, \mathcal{R}).$$

It is easy to see that we have the following exact sequence:

$$0 \to R \hookrightarrow N_+ \to M_+ \to 0. \tag{3.4.1}$$

Since $H^*(\mathbf{Y}, \mathcal{N}_+) = 0$ the associated cohomology long exact sequence for the space $\mathbf{Y}$ becomes:

$$H^*(\mathbf{Y}, \mathcal{M}_+) = H^{*+1}(\mathbf{Y}, \mathcal{R}). \tag{3.4.2}$$

In the same way we have an isomorphism with $\mathcal{M}_-$ in place of $\mathcal{M}_+$. Denote respectively with δ_+ and δ_- these isomorphisms.

Let us now consider the exact sequence

$$0 \to R \hookrightarrow M \to M_+ \oplus M_- \to 0. \tag{3.4.3}$$

The cohomology of the space $\mathbf{Y}$ with coefficients in $\mathcal{M}_+ \oplus \mathcal{M}_-$ splits into the direct sum of two pieces, namely the cohomology with coefficients in $\mathcal{M}_+$ and the cohomology with coefficients in $\mathcal{M}_-$. Hence in long exact sequence associated to Equation (3.4.3) we have the map

$$\delta' : H^*(\mathbf{Y}, \mathcal{M}_+) \oplus H^*(\mathbf{Y}, \mathcal{M}_-) \to H^{*+1}(\mathbf{Y}, \mathcal{R})$$

where $\delta' = (\delta'_+, \delta'_-)$. We have that the map δ_+ (resp. δ_-) is equal to the map δ'_+ (resp. δ'_-). To see this consider the following exact diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & R \arrow[r, hook] \arrow[d, equal] & N_+ \arrow[r] \arrow[d, hook] & M_+ \arrow[r] \arrow[d, equal] & 0 & \\
0 \arrow[r] & R \arrow[r, hook] \arrow[d] & M \arrow[r] \arrow[d] & M_+ \quad \oplus & M_- \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & M_- \arrow[rr, equal] \arrow[d] & & M_- \arrow[r] \arrow[d] & 0 \\
 & & 0 & & 0 &
\end{tikzcd}$$

In particular, looking at the cohomology long exact sequences associated to the rows, we have:

$$\begin{tikzcd}
H^*(\mathbf{Y}, \mathcal{M}_+) \arrow[rr, "\delta_+"] \arrow[d, equal] & & H^{*+1}(\mathbf{Y}, \mathcal{R}) \arrow[d, equal] \\
H^*(\mathbf{Y}, \mathcal{M}_+) \quad \oplus & H^*(\mathbf{Y}, \mathcal{M}_-) \arrow[r, "\delta'"] & H^{*+1}(\mathbf{Y}, \mathcal{R})
\end{tikzcd}$$

and so it follows that $\delta_+ = \delta'_+$. In the same way one can shows that $\delta_- = \delta'_-$.

In particular it turns out that the map δ' is surjective, so we have the following short exact sequence:

$$0 \to H^*(\mathbf{Y}, \mathcal{M}) \overset{\alpha}{\hookrightarrow} H^*(\mathbf{Y}, \mathcal{M}_+) \oplus H^*(\mathbf{Y}, \mathcal{M}_-) \xrightarrow{\delta'} H^{*+1}(\mathbf{Y}, \mathcal{R}) \to 0 \tag{3.4.4}$$

where $\alpha = (\alpha_+, \alpha_-)$. Using Equation (3.4.4) one can deduce that the map α_+ is an isomorphism (and analogously α_-). The Theorem follows from this last observation together with Equation (3.4.2). □

We note that, since in the case of a Milnor fibration given by a weighted homogeneous polynomial the characteristic automorphism h has finite order, the previous result applies to the case of a fibration $\mathbf{X}_W \to \mathbb{C}^*$ associated to a finite Coxeter group W, hence provides a proof of Theorem 2.7.2:

Corollary 3.4.4. *If W is a finite Coxeter group*

$$H^i(G_W; R[[q, q^{-1}]]) \simeq H^{i+1}(G_W; R[q, q^{-1}]).$$

□

3.5 Well filtered complexes and degree shift

3.5.1 Main theorem

Let (C_1, d) be a graduated complex and let $C_3 \subset C_2 \subset C_1$ be inclusions of graduate complexes. Denote by $d_{ij} : C_i/C_j \to C_i/C_j$ the induced coboundary on the quotient complex $(1 \leq i < j \leq 3)$. There is an obvious exact sequence of complexes:

$$0 \to C_2/C_3 \hookrightarrow C_1/C_3 \xrightarrow{\pi} C_1/C_2 \to 0$$

When d_{12} and d_{23} vanish (for example if the complexes are trivial in all degrees except exactly one) we get that $H^*(C_1/C_2) = C_1/C_2$ and $H^*(C_2/C_3)$ $= C_2/C_3$, so the differential $H^*(C_1/C_2) \to H^*(C_2/C_3)$ of the long exact sequence associated to the above sequence gives a map

$$\overline{d} : C_1/C_2 \to C_2/C_3.$$

In the following we call this map *induced differential.*

Let A be a commutative unitary ring. In this Section we indicate by $R = A[q, q^{-1}]$, the ring of Laurent polynomials with coefficients in A and by M the R-module $A[[q, q^{-1}]]$. Let (C^*, d^*) be a graduate cochain complex, with C^* an R-module and d^* an R-linear map. We give the following recursive definition:

Definition 3.5.1. The complex (C^*, d^*) is called *well filtered* if C^* is a free finitely generated R-module, $C^* \neq R$ and moreover, if $C^* \neq 0$, the following conditions are satisfied:

(a) C^* is a filtered complex with a decreasing filtration F which is compatible with the coboundary map d^* and such that $F_0C = C^*$ and $F_{n+1}C = \{0\}$ for an integer $n > 0$;

(b) $F_nC = (F_nC)^n \simeq F_{n-1}C/F_nC = (F_{n-1}C/F_nC)^{n-1} \simeq R$;

(c) the induced differential $\overline{d} : F_{n-1}C/F_nC \to F_nC/F_{n+1}C$ (following from condition (b)) corresponds to the multiplication by a non-zero polynomial $p \in R$ with first and last non-zero coefficients invertible in A;

(d) for all integer $i \neq n-1, n$ the induced complex $((F_iC/F_{i+1}C)^*, d_i^*)$ is a *well filtered* complex.

In the following when we consider a well filtered complex we always suppose to have also a filtration F as above. We write (C_M^*, d_M^*) for the complex $C^* \otimes_R M$ with the natural induced graduation and coboundary.

Theorem 3.5.2. *Let (C^*, d^*) be a well filtered complex. We have the following isomorphism:*
$$H^{*+1}(C^*) \simeq H^*(C^*_M).$$

In order to proof this fact we need two preliminary lemmas.

As a first step let us consider the natural inclusion of R-modules $R \hookrightarrow M$. We have the short exact sequence of R-modules:

$$0 \to R \hookrightarrow M \to M' \to 0$$

where $M' = M/R$. We indicate by C'^* the complex $C^* \otimes_R M'$ and we consider the complexes C^*, C^*_M, C'^*. In a similar way we have the following short exact sequence of R-modules:

$$0 \to C^* \overset{i}{\hookrightarrow} C^*_M \overset{\pi}{\to} C'^* \to 0.$$

Since the maps i and π commute with the coboundary maps, we actually have a short exact sequence of complexes. So we obtain the following long exact sequence:

$$\begin{array}{ccc} & \cdots \xrightarrow{\pi^*} H^{i-1}(C'^*) \xrightarrow{\delta^*} & \\ \xrightarrow{\delta^*} & H^i(C^*) \xrightarrow{i^*} H^i(C^*_M) \xrightarrow{\pi^*} H^i(C'^*) & \xrightarrow{\delta^*} \\ \xrightarrow{\delta^*} H^{i+1}(C^*) \xrightarrow{i^*} \cdots & & \end{array} \tag{3.5.1}$$

Lemma 3.5.3. *Let (C^*, d^*) be a well filtered complex. With the notation given above we have:*

$$H^i(C'^*) \simeq H^i(C^*_M) \oplus H^i(C^*_M)$$

Proof. The R-module M' splits into the sum of two modules in the following way:
$$M' = M'_+ \oplus M'_-$$
where $M'_+ = M/(A[q][[q^{-1}]]), M'_- = M/(A[q^{-1}][[q]])$. In a similar way we get the splitting
$$C'^* = C'^*_+ \oplus C'^*_-.$$
Moreover C'^*_+ and C'^*_- are invariant for the coboundary induced by d^*, so the cohomology also splits:

$$H^*(C'^*) = H^*(C'^*_+) \oplus H^*(C'^*_-).$$

We want to show that the quotient projection $\pi_+ : C^*_M \to C'^*_+$ induces an isomorphism π^*_+ in cohomology. We will prove this by induction on the number of generators of C^* as a free R-module.

If $C^* = \{0\}$ the assertion is obvious. Suppose that C^* has m generators, with $m > 1$. Then the complexes $((F_iC/F_{i+1}C)^*, d^*_i)$ have a smaller number of generators and for $i \neq n-1, n$ they are well filtered. Therefore we can suppose by induction that the map π_{i+}, defined analogously to π_+, induces an isomorphism in cohomology for all the complexes $((F_iC/F_{i+1}C)^*, d_i^*)$, $i \neq n-1, n$, that is the map

$$\pi_{i+}^* : H^*((F_iC/F_{i+1}C)^* \otimes_R M) \to H^*((F_iC/F_{i+1}C)^* \otimes_R M'_+)$$

is an isomorphism for such i.

The filtration F on C^* induces filtrations on C_M^* and $C_+'^*$ in the following way: $F_iC_M = F_iC \otimes_R M$, $F_iC'_+ = F_iC \otimes_R M'_+$. We have the following natural isomorphisms:

$$(F_iC/F_{i+1}C)^* \otimes_R M \simeq (F_iC_M/F_{i+1}C_M)^*$$

$$(F_iC/F_{i+1}C)^* \otimes_R M'_+ \simeq (F_iC'_+/F_{i+1}C'_+)^*.$$

Through these isomorphisms the maps

$$(F_iC_M/F_{i+1}C_M)^* \to (F_iC'_+/F_{i+1}C'_+)^*$$

induced by π_+ correspond to π_{i+} and hence induce an isomorphism in cohomology for $i \neq n-1, n$.

Let us consider the spectral sequences $E_r^{i,j}$ and $\overline{E}_r^{i,j}$ associated to the complexes C_M^* and C'^*_+ with the respective filtrations. We write π_+^* also for the spectral sequences homomorphism induced by π_+. By the definition of the filtration F we have that $E_r^{i,j} = \overline{E}_r^{i,j} = 0$ if $i > n$ or if $i = n, n-1$ and $j \neq 0$. It is also clear that $E_1^{n-1,0} \simeq E_1^{n,0} = M$ and $\overline{E}_1^{n,0} \simeq \overline{E}_1^{n-1,0} = M'_+$. For $0 \leq i < n-1$ we get that $E_1^{i,j} \simeq H^{i+j}(F_iC_M^*/F_{i+1}C_M^*)$ and $\overline{E}_1^{i,j} \simeq H^{i+j}(F_iC'^*_+/F_{i+1}C'^*_+)$ therefore the inductive hypothesis gives that $E_1^{i,j} \simeq \overline{E}_1^{i,j}$ and the isomorphism between the terms of the spectral sequences is given by π_+^*. Now consider the maps $d_1^{n-1,0} : M \to M$ and $\overline{d}_1^{n-1,0} : M'_+ \to M'_+$. By condition (c) we have that these maps correspond to the multiplication by a non-zero polynomial $p = \sum_{i=s}^t b_i q^i$ with b_s, b_t invertible elements of the ring A. We can rewrite p as follows:

$$p = q^s b_s(1 + qp') = q^t b_t(1 + q^{-1}p'')$$

with $p' \in A[q]$, $p'' \in A[q^{-1}]$. Now we can look at these elements in M:

$$p_+^{-1} = q^{-s} b_s^{-1} \sum_{i=0}^{\infty} (-qp')^i$$

$$p_-^{-1} = q^{-t} b_t^{-1} \sum_{i=0}^{\infty} (-q^{-1}p'')^i.$$

Let $m \in M$, $m = \sum_{i\in\mathbb{Z}} a_i q^i$, we can write $m = m_+ + m_-$, with $m_+ = \sum_{i=0}^{\infty} a_i q^i$ and $m_- = m - m_+$. Notice that the products $p_+^{-1}m_+$ and $p_-^{-1}m_-$ are well defined and the following equality holds:

$$m = p(p_+^{-1}m_+ + p_-^{-1}m_-).$$

It turns out that the map $d_1^{n-1,0} : M \to M$ is surjective and the same holds, when passing to the quotient, for the map $\overline{d}_1^{n-1,0} : M_+ \to M_+$.

Let us suppose that an element $m = \sum_{i\in\mathbb{Z}} a_i q^i$ is in the kernel of $d_1^{n-1,0}$. This means that $pm = 0$, that is for all integers k we have:

$$\sum_{i=s}^{t} b_i a_{k-i} = 0$$

and so we obtain:

$$a_k = -b_s^{-1} \sum_{i=1}^{t-s} b_{s+i} a_{k-i} \tag{3.5.2}$$

$$a_k = -b_t^{-1} \sum_{i=1}^{t-s} b_{t-i} a_{k+i}. \tag{3.5.3}$$

Therefore if we know a sequence of $t-s$ consecutive coefficients of an element m sent to zero by the multiplication by p we can use (3.5.2) and (3.5.3) to calculate recursively all the other coefficients, determining m completely. So we find a bijection between $\ker d_1^{n-1,0}$ and $\ker \overline{d}_1^{n-1,0}$. In fact, if $m \in M$ is such that $pm = 0$, then trivially also $p[m]_+ = 0$ (we write $[m]_+$ for the equivalence class of m in M'_+). Conversely if $p[m]_+ = 0$ then we have $pm = z$, with $z \in A[q][[q^{-1}]]$, that is $z = \sum_{i\in\mathbb{Z}} v_i q^i$ with $v_i \in A$ and there exists an integer l such that $v_i = 0$ for all $i > l$. We can define recursively, for $j \geq 0$, the following elements:

$$\widetilde{a}[-1]_i = a_i$$

$$\widetilde{a}[j]_i = \begin{cases} \widetilde{a}[j-1]_i & \text{if} \quad i \neq l-t-j \\ -b_t^{-1} \sum_{k=1}^{t-s} b_{t-k} \widetilde{a}[j-1]_{i+k} & \text{if} \quad i = l-t-j \end{cases}$$

and

$$\widetilde{a}_i = \begin{cases} a_i & \text{if} \quad i > l-t \\ \widetilde{a}[l-t-i]_i & \text{if} \quad i \leq l-t \end{cases}$$

Notice that the coefficients v_i for $i > h$ depend only on the coefficients a_i for $i > h-t$, so if we write $\widetilde{m} = \sum_{i\in\mathbb{Z}} \widetilde{a}_i q^i$ we have that $p\widetilde{m} = 0$ and $[m]_+ = [\widetilde{m}]_+$.

To sum up we have that the map π_+^* gives an isomorphism between the terms $E_1^{i,j}$ and $\overline{E}_1^{i,j}$ for $i < n-1$ and between $\ker d_1^{n-1,0}$ and $\ker \overline{d}_1^{n-1,0}$.

Moreover $E_2^{i,j} = \overline{E}_2^{i,j} = 0$ for $i = n-1$ and $j \neq 0$ and for $i > n-1$; π_+^* commutes with the differentials in the spectral sequences (i. e. $\pi_+^* d_i = \overline{d}_i \pi_+^*$). We remark that im $d_1^{n-2,0} \subset \ker d_1^{n-1,0}$ and im $\overline{d}_1^{n-2,0} \subset \ker \overline{d}_1^{n-1,0}$ and so π_+^* induces an isomorphism between im $d_1^{n-2,0}$ and im $\overline{d}_1^{n-2,0}$. This implies that π_+^* gives the isomorphisms $E_2^{n-2,0} \simeq \overline{E}_2^{n-2,0}$ and $E_2^{n-1,0} \simeq \overline{E}_2^{n-1,0}$. Then we have a complete isomorphism between E_2 and $\overline{E}_2$ and so between E_∞ and $\overline{E}_\infty$. It follows that π_+^* induces an isomorphism in cohomology.

It is clear that the same fact holds for the map $\pi_- : C_M^* \to C'^*_-$ and so the Lemma is proved. □

We write Φ for the isomorphism built in the proof of the previous Lemma.

Lemma 3.5.4. *In the exact sequence (3.5.1) the map π^* composed with the isomorphism Φ corresponds to the diagonal map Σ:*

$$H^i(C_M^*) \overset{\Sigma}{\hookrightarrow} H^i(C_M^*) \oplus H^i(C_M^*).$$

Proof. It is enough to notice that, making the identification $H^*(C'^*) = H^*(C'^*_+) \oplus H^*(C'^*_-)$, we have that $\pi^* = (\pi_+^*, \pi_-^*)$ and so the statement follows immediately. □

Proof. [Proof (of Theorem 3.5.2).] First of all we notice that, being π^* injective, i^* turns out to be the null map and δ^* is surjective. We call $p_1 : H^i(C_M^*) \oplus H^i(C_M^*) \to H^i(C_M^*)$ the projection on the first component, p_2 the projection on the second component and $i_1 : H^i(C_M^*) \hookrightarrow H^i(C_M^*) \oplus H^i(C_M^*)$ the inclusion defined by $i_1 : b \mapsto (b, 0)$. Finally we define $\alpha = \delta^* \circ \Phi^{-1} \circ i_1$. We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^i(C_M^*) \arrow[r, hook, "\Sigma"] \arrow[d, equal] & H^i(C_M^*) \oplus H^i(C_M^*) \arrow[r, shift left, "p_1 - p_2"] \arrow[d, "\Phi", "\wr"'] & H^i(C_M^*) \arrow[l, shift left, "i_1"] \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & H^i(C_M^*) \arrow[r, "\pi^*"] & H^i(C'^*_M) \arrow[r, "\delta^*"] & H^{i+1}(C^*) \arrow[r] & 0
\end{tikzcd}$$

Clearly both the lines are exact. We want to show that the diagram commutes. The commutativity for the first square follows by Lemma 3.5.4, so it remains to prove that the second square commutes. A pair $(a, b) \in H^i(C_M^*) \oplus H^i(C_M^*)$ is sent, by the multiplication by $p_1 - p_2$, into the element $a - b \in H^i(C_M^*)$. Then we have $i_1(a-b) = (a-b, 0)$ and the difference $(a, b) - (a-b, 0) = (b, b)$ is in the image of the map Σ. Therefore, because of the commutativity of the first square, the images of the pairs (a, b) and of $(a-b, 0)$ in $H^i(C'^*_M)$ are taken into the same element by the map δ^*. So we get the commutativity of the diagram. The Theorem follows from the five lemma. □

3.5.2 Applications

Let us consider a finite set Γ endowed with a fixed total ordering. We will indicate by Δ a generic subset of Γ. We also set again $R = A[q, q^{-1}]$, with A a commutative unitary ring. For every pair (Δ, w) with $\Delta \subset \Gamma$, $w \in \Gamma \setminus \Delta$ we associate a polynomial $p_{\Delta,w}(q, q^{-1}) \in R \setminus \{0\}$ such that the first and the last non-zero coefficients are invertible in A. Let also suppose that for every pair (w, w') with $w \neq w'$ and $w, w' \in \Gamma \setminus \Delta$ the following equation holds:

$$p_{\Delta,w}(q,q^{-1})p_{\Delta\cup\{w\},w'}(q,q^{-1}) + p_{\Delta,w'}(q,q^{-1})p_{\Delta\cup\{w'\},w}(q,q^{-1}) = 0 \quad (3.5.4)$$

Then we can consider the complex (C^*_Γ, d^*) defined as follows:

$$C^*_\Gamma = \bigoplus_{\Delta \subset \Gamma} R.e_\Delta$$

$$d^* e_\Delta = \sum_{w \in \Gamma \setminus \Delta} p_{\Delta,w}(q,q^{-1}) e_{\Delta\cup\{w\}}.$$

We remark that the relation (3.5.4) gives $d^{*2} = 0$. We can also give a natural graduation to C^*_Γ by defining the degree of an element e_Δ as the cardinality of Δ, so we get a cochain complex.

Without loss of generality we can think $\Gamma = \{1, \ldots, n\}$. We introduce the following notation: indicate by Γ_i and Δ_i respectively the subsets $\{1, \ldots, n-i-1\}$ and $\{n-i+1, \ldots, n\}$. We can filter the complex C^*_Γ in the following way (see also [DCPS01]): let $F_i C_\Gamma$ be the subcomplex generated by the elements e_Δ, with $\Delta_i \subset \Delta$.

We have the following result:

Theorem 3.5.5. *With the filtration defined above the complex (C^*_Γ, d^*) is well filtered.*

Proof. We can prove this by induction on the cardinality of Γ. If Γ is empty the Theorem is obvious. Therefore let us suppose that the Theorem holds for all the complexes made up from a set with less than n elements and we prove it for a complex C^*_Γ, with $\Gamma = \{1, \ldots, n\}$.

It is straightforward to see that $F_0 C_\Gamma = C^*_\Gamma$ and $F_{n+1} C_\Gamma = \{0\}$. Moreover $F_n C_\Gamma$ and $F_{n-1} C_\Gamma / F_n C_\Gamma$ are generated respectively by the elements e_Γ and $e_{\Delta_{n-1}}$ and they are both isomorphic to R. The induced differential

$$\overline{d} : F_{n-1} C_\Gamma / F_n C_\Gamma \to F_n C_\Gamma / F_{n+1} C_\Gamma$$

corresponds to the multiplication by the polynomial $p_{\Delta_{n-1},1}(q, q^{-1})$.

Finally the complex $((F_i C_\Gamma / F_{i+1} C_\Gamma)^*, d_i^*)$ is isomorphic to the complex $C^*_{\Gamma_i}$, where the coboundary is defined by the polynomials

$$\overline{p}_{\Delta,j}(q,q^{-1}) := p_{\Delta\cup\Delta_i,j}(q,q^{-1}) \qquad \text{for } \Delta \subset \Gamma_i, j \in \Gamma_i \setminus \Delta$$

and so it is well filtered by induction. $\square$

Now we apply last result and Theorem 3.5.2 to the cohomology with local coefficients of Artin groups. Consider the Salvetti complex C^* defined in Theorem 3.1.9 for the local system $\mathcal{L}_q = \mathcal{L}_q(\mathbf{X}_W; M)$.

Proposition 3.5.6. *Let $R = A[q, q^{-1}]$ and $M = R$ and let W be a finite Coxeter group. Then the complex C^* in Theorem 3.1.9 is well filtered.*

Proof. In fact the polynomial $W_\Gamma(q)$ divides $W_{\Gamma'}(q)$ when $\Gamma \subset \Gamma'$. Moreover the polynomials $W_\Gamma(q)$ are products of cyclotomic polynomials (see [Bou68]), so they have first and last non-zero coefficients equal to 1. By using Theorem 3.5.5 we can easily see that C^* is well filtered. □

Hence we find another proof of Theorem 2.7.2 announced in Section 2.7:

Corollary 3.5.7. *Let W be a finite irreducible Coxeter group and let*

$$\mathbf{F}_W \hookrightarrow \mathbf{X}_W \xrightarrow{\delta'} \mathbb{C}^*$$

be the fibration defined in Section 2.7. Let $R = A[q, q^{-1}]$ be considered as a G_W-module with the action defined before. Then the following equality holds:

$$H^*(\mathbf{F}_W; A) = H^{*+1}(\mathbf{X}_W; R)$$

□

Chapter 4

The integral homology of the Milnor fiber for Artin groups of type A_n

4.1 Notations and definitions

In [Mar96] Markaryan used the isomorphism between the standard resolution of a certain algebra and the algebraic complex associated to the classifying spaces for braid groups to compute the homology of braid groups with coefficients in the local system $\mathbb{Q}[q^{\pm 1}]$. In this Chapter we extend the use of this resolution in order to compute the homology of braid groups with coefficients in the local system $K[q^{\pm 1}]$ for a generic field K.

Let R be a ring with identity and let q be an element of R. Following [Mar96] we define the *algebra of q-divided polynomials* $\Gamma_R(t, q)$ as the graded algebra over R with generators t_i ($i \in \mathbb{N}$, $\deg t_i = i$) and relations

$$t_i t_j = \left[\begin{array}{c} i+j \\ i \end{array} \right] t_{i+j}.$$

We recall that if q commutes with a and b and $ba = qab$, then

$$(a+b)^n = \sum_{i=0}^{n} \left[\begin{array}{c} n \\ i \end{array} \right] a^i b^{n-i}.$$

Now we want to study the homology and cohomology (as defined in [CE56]) of the algebra $\Gamma_R(t, q)$. We can consider the *normalized standard complex* (see [CE56] for a general definition) that calculates the homology of the algebra $\Gamma_R(t, q)$. The complex is given as follows:

$$0 \leftarrow R = C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} \cdots,$$

where the R-module C_n is freely generated by all the monomials of the form $a \otimes t_{i_1} \otimes \cdots \otimes t_{i_n}$, with $a \in R$ and the boundary formula is:

$$\begin{aligned} & \partial(a \otimes t_{i_1} \otimes \cdots \otimes t_{i_n}) = \\ = & \sum_{j=1}^{n-1} (-1)^{j+1} a \otimes t_{i_1} \otimes \cdots \otimes t_{i_j} t_{i_{j+1}} \otimes \cdots \otimes t_{i_n} = \\ = & \sum_{j=1}^{n-1} (-1)^{j+1} \left[\begin{array}{c} i_j + i_{j+1} \\ i_j \end{array} \right] a \otimes t_{i_1} \otimes \cdots \otimes t_{i_j + i_{j+1}} \otimes \cdots \otimes t_{i_n}. \end{aligned}$$

By means of the grading of the algebra $\Gamma_R(t,q)$, the module C is decomposed into the direct sum of complexes of different degrees,

$$C = \bigoplus_{i=0}^{\infty} C^{(i)},$$

where $\deg(a \otimes t_{i_1} \otimes \cdots \otimes t_{i_n}) = i_1 + \cdots + i_n$ and for $c \in C_k^{(n)}$ we set $\deg c = n$ and $\dim c = k$.

The dual complex C^*, given by the modules $C^n = \mathrm{Hom}(C_n, R)$ and with coboundary map the transposed map of ∂, computes the cohomology ring of the algebra $\Gamma_R(t,q)$. The multiplication is defined on representatives as follows: if m_1^* and m_2^* are the dual classes of the monomials m_1 and m_2, respectively, then the product $m_1^* m_2^*$ is the dual class of the monomial $m_1 \otimes m_2$.

Given a space $\mathbf{X}$ such that $\pi_1(\mathbf{X}) = \mathrm{Br}(n)$, we can define a local system R on $\mathbf{X}$. Over a point $x \in \mathbf{X}$ we have the ring R; the system of coefficients is twisted and the action is given by sending each standard generator of the group $\mathrm{Br}(n)$ to multiplication by $-q$. This action corresponds to the determinant of the Burau representation for the braid group $\mathrm{Br}(n)$ (see, for example, [CP07]). We remark that although the choice of the multiplication by q would be equivalent, we use $-q$, which seems more natural to us, and also for coherence with [CP07], [DCPS01], [DCPSS99].

The complex $C_{n-*}^{(n)}$ coincides with the complex (defined in [Sal94]) that computes the cohomology of the group $\mathrm{Br}(n)$ with local coefficients R.

By the module $H_*(\mathrm{Br}(*), R)$ we mean the bigraded module (the gradings are the degree deg and the dimension dim) whose component of degree n and dimension l is $H_l(\mathrm{Br}(n), R)$. We can think of $H_*(\mathrm{Br}(*), R)$ as a ring using the multiplication induced by the standard homomorphism (obtained by juxtaposing braids)

$$\mu_{ij} : \mathrm{Br}(i) \times \mathrm{Br}(j) \to \mathrm{Br}(i+j).$$

We have:

Theorem 4.1.1. ([Mar96]) *The ring $H_*(\mathrm{Br}(*), R)$ coincides, modulo a change of indexes, with the cohomology ring of the algebra $\Gamma_R(t,q)$:*

$$H_l(\mathrm{Br}(n), R) \simeq H^{n-l}(\Gamma_R(t,q))_{(\deg = n)}$$

and the product structure in $H_*(\mathrm{Br}(*), R)$ coincides with the cohomological multiplication of the ring $H^*(\Gamma_R(t,q))$. □

4.2 The Milnor fiber and some lemmas

We recall a construction from the previous chapters. Let $V = \mathbb{C}^n$ be a finite-dimensional complex vector space. The symmetric group on n elements $\mathfrak{S}_n$ acts on this space by permuting the coordinates. Let l_{ij} be the linear functional $z_i - z_j$ and let $\mathcal{H}_{ij}$ be the hyperplane $\{l_{ij} = 0\}$. The complement of the union of the hyperplanes

$$\mathbf{Y}_n = V \setminus \bigcup_{i<j} \mathcal{H}_{ij}$$

is a classifying space for the pure braid group on n strands. If we consider the quotient of $\mathbf{Y}_n$ with respect to the action of $\mathfrak{S}_n$

$$\mathbf{X}_n = \mathbf{Y}_n / \mathfrak{S}_n$$

we get a classifying space for the braid group $\mathrm{Br}(n)$. Consider the product $\delta = \prod_{i<j} l_{ij}^2$. The polynomial δ is invariant with respect to the action of $\mathfrak{S}_n$ and so it induces a map

$$\delta' : \mathbf{X}_n \to \mathbb{C}^*.$$

The fiber $\mathbf{F}_1(n) = \delta'^{-1}(1)$ is the Milnor fiber of the discriminant singularity $\mathbf{F}_0(n) = \bigcup \mathcal{H}_{ij}/\mathfrak{S}_n$ in the affine variety $V/\mathfrak{S}_n$ (which is isomorphic to the complex space $\mathbb{C}^n$). The complement of $\mathbf{F}_0(n)$ in $V/\mathfrak{S}_n$ can also be regarded as the set of polynomials with distinct roots in the space of all monic polynomials of degree n with complex coefficients. Moreover, the fiber $\mathbf{F}_1(n)$ is a classifying space for the commutator subgroup $\mathrm{Br}(n)'$ of the braid group $\mathrm{Br}(n)$ and, as we have seen before, we have:

$$H_*(\mathrm{Br}(n)', \mathbb{Z}) \simeq H_*(\mathbf{F}_1(n), \mathbb{Z}) \simeq H_*(\mathrm{Br}(n), \mathbb{Z}[q^{\pm 1}])$$

and

$$H^*(\mathrm{Br}(n)', \mathbb{Z}) \simeq H^*(\mathbf{F}_1(n), \mathbb{Z}) \simeq H^{*+1}(\mathrm{Br}(n), \mathbb{Z}[q^{\pm 1}]),$$

with the usual q-action.

In what follows K is a field and p always refers to the characteristic of the field K ($p = 0$ or p a prime). Cyclotomic polynomials are usually defined over a field of characteristic 0, by saying that the m-th cyclotomic polynomial is the monic polynomial whose roots are all simple roots and are all the m-th primitive roots of unity. Over a generic field K we can define by induction the m-th cyclotomic polynomial φ_m, by saying that $\varphi_1 = q-1$ and $q^m - 1 = \prod_{i|m} \varphi_i$. For each positive integer m we define the ring $K(m) = K[q]/\varphi_m$.

We have the following technical lemmas:

Lemma 4.2.1. ([Cal03]) *Let $m < n$ be two positive integers. In the ring $\mathbb{Z}[q\pm 1]$ we have:*

$$(\varphi_m, \varphi_n) = \begin{cases} (\varphi_m, p) & \text{if } n = mp^i, i \geq 1, \text{for a prime } p \\ (1) & \text{otherwise.} \end{cases}$$

Proof. First of all, notice that the polynomials φ_m are irreducible for all $m \in \mathbb{N}$; hence, the quotient rings $\mathbb{Z}[q]/(\varphi_m)$ are integral domains.

(i) First suppose that $m \nmid n$ and let $l = \operatorname{lcm}(m, n)$. Moreover we set $m' = \frac{l}{m}$, $n' = \frac{l}{n}$. We have that $\varphi_n \mid \frac{[l]}{[m]}$ and $\varphi_m \mid \frac{[l]}{[n]}$. Furthermore $\frac{[l]}{[m]} \equiv m' \pmod{\varphi_m}$ and $\frac{[l]}{[n]} \equiv n' \pmod{\varphi_n}$. Since we have $(m', n') = (1)$ it follows that $(\varphi_m, \varphi_n) = (1)$. Hence the polynomial φ_n is invertible in $\mathbb{Z}[q]/(\varphi_m)$ (and φ_m is invertible in $\mathbb{Z}[q]/(\varphi_n)$).

(ii) Now we suppose that $m \mid n$. For a fixed m we want to prove by induction on n that, modulo the multiplication by an invertible element in $\mathbb{Z}[q]/(\varphi_m)$, the following holds:

$$\begin{aligned} \varphi_n &\equiv p \quad \text{if } n = mp^i, i \geq 1 \\ \varphi_n &\equiv 1 \qquad \text{otherwise.} \end{aligned}$$

If $n = mp$ we have that

$$\frac{[n]}{[m]} \equiv p \pmod{\varphi_m}$$

and so we can write

$$\frac{[n]}{[m]} = \varphi_n \prod_{\substack{m' \mid m, m' < m \\ p \nmid m/m'}} \varphi_{pm'}$$

Since all the factors in the product are invertible, it follows that, modulo multiplication by invertible elements in $\mathbb{Z}[q]/(\varphi_m)$, we get $\varphi_n \equiv p$.

If $n = mp^i$, in a similar way the next equality holds:

$$\frac{[n]}{[m]} = \varphi_n \prod_{\substack{1 \leq j < i \\ m' \mid m, m'p^j \neq n \\ p \nmid m/m'}} \varphi_{m'p^j} \equiv p^i \pmod{\varphi_m}$$

In the product there are exactly $i-1$ factors congruent to p and all the others are invertible, so modulo invertible elements we have that $\varphi_n \equiv p$.

Finally we consider the case $n = mp_1^{i_1} \cdots p_k^{i_k}$. Let us define the set

$$\mathcal{I} = \left\{ (m', j_1, \ldots, j_k) \in \mathbb{N}^{k+1} \;\middle|\; \begin{array}{c} m' \mid m, \\ 0 \leq j_s \leq i_s \forall s, \\ (j_1, \ldots, j_k) \neq (0, \ldots, 0), \\ m' \neq m \text{ if } j_s = i_s \forall s, \\ p_s \nmid (m/m') \quad \forall s \text{ s.t. } j_s \neq 0 \end{array} \right\}.$$

We have that:

$$\begin{aligned} \frac{[n]}{[m]} &= \prod_{n' \mid n, n' \nmid m} \varphi_{n'} = \\ &= \prod_{I \in \mathcal{I}} \varphi_{m' p_1^{j_1} \cdots p_k^{j_k}} \equiv \\ &\equiv p_1^{i_1} \cdots p_k^{i_k} \pmod{\varphi_m} \end{aligned}$$

and, by the inductive hypothesis, in the product, for all s there are exactly i_s factors congruent to p_s; hence all the other factors are invertible and φ_n must be invertible, too. So the Lemma is proved. □

As an easy consequence of this Lemma we obtain the following Corollary, whose proof is left to the reader:

Corollary 4.2.2. *Let $i < j$ be two positive integers. Then we can write*

$$\varphi_{p^j} = \varphi_{p^i} \omega + p\psi \tag{4.2.1}$$

where $\omega, \psi \in \mathbb{Z}[q^{\pm 1}]$ and ψ is invertible $\operatorname{mod}(\varphi_{p^i})$*;*

$$\varphi_{mp^j} = \varphi_{mp^i} \omega + p\psi \tag{4.2.2}$$

where $\omega, \psi \in \mathbb{Z}[q^{\pm 1}]$ and ψ is invertible $\operatorname{mod}(\varphi_{mp^i})$*.* □

We can fix once and for all polynomials ω_{p^j,p^i}, ω_{mp^j,mp^i}, ψ_{p^j,p^i}, ψ_{mp^j,mp^i} satisfying the equations (4.2.1) and (4.2.2).

Lemma 4.2.3. *([Gue68]) Let m be an integer and p a prime. We have:*

$$\varphi_{p^i} \equiv \varphi_p^{p^{i-1}} \operatorname{mod}(p); \tag{4.2.3}$$

if we suppose that $p \nmid m$, then:

$$\varphi_{mp^i} \equiv \varphi_m^{\phi(p^i)} \operatorname{mod}(p) \tag{4.2.4}$$

where ϕ denotes the Euler ϕ-function. □

Now we consider again the algebra of q-divided polynomials $\Gamma_R(t,q)$ in the case $R = K(m)$.

Lemma 4.2.4. *The following decompositions hold:*

(a) if $p = 0$ *(see also [Mar96]):*

$$\Gamma_{K(m)}(t,q) \simeq K(m)[u_m] \otimes K(m)[u_1]/(u_1^m); \tag{4.2.5}$$

(b) if $p \neq 0$*:*

$$\Gamma_{K(p)}(t,q) \simeq \bigotimes_{i=0}^{\infty} K(p)[u_{p^i}]/(u_{p^i}^p); \tag{4.2.6}$$

(c) if $p \neq 0$ *and* $p \nmid m$*:*

$$\Gamma_{K(m)}(t,q) \simeq K(m)[u_1]/(u_1^m) \otimes \bigotimes_{i=0}^{\infty} K(m)[u_{p^i m}]/(u_{p^i m}^p); \tag{4.2.7}$$

with $\deg u_j = j$.

Proof. The proof of a) is given in [Mar96].

For b) let $[i]!_{\varphi_p}$ be the exponent of the biggest power of φ_p that divides $[i]!$. The isomorphism is given as follows:

$$t_i \mapsto \frac{u_1^{k_0} u_p^{k_1} \cdots u_{p^r}^{k_r}}{[i]!/\varphi_p^{[i]!_{\varphi_p}}} \tag{4.2.8}$$

where $k_r \cdots k_0$ is the expression of i in base p.

For c) we have the isomorphism given by

$$t_i \mapsto \frac{u_1^{k} u_m^{k_0} u_{mp}^{k_1} \cdots u_{mp^r}^{k_r}}{[i]!/\varphi_m^{[i]!_{\varphi_m}}} \tag{4.2.9}$$

where k is the remainder of the division of i by m and $k_r \cdots k_0$ is the expression of $(i-k)/m$ in base p.

The Lemma follows from the next key observation: if $k_r \cdots k_0$ is the expression of i in the base p and $k_r' \cdots k_0'$ is the expression for j (resp. k, $k_r \cdots k_0$ and k', $k_r' \cdots k_0'$ are the numbers associated to i and j as in (4.2.9)), then the polynomial φ_p (resp. φ_m) does not divide $\begin{bmatrix} i+j \\ i \end{bmatrix}$ if and only if the expression for $i+j$ in base p is given by $h_r \cdots h_0$, with $h_l = k_l + k_l'$ for $l = 0, \ldots, r$ (resp. the numbers associated to $i+j$ are h, $h_r \cdots h_0$, with $h = k + k'$, $h_l = k_l + k_l'$ for $l = 0, \ldots, r$). □

The cohomology rings of $R[u]$ and $R[u]/(u^i)$ are already known. In fact we have:

Lemma 4.2.5. ([Mar96])

$$H^*(R[u]) \simeq \Lambda[x], \quad \deg(x) = \deg(u), \dim(x) = 1;$$

$$H^*(R[u]/(u^n)) \simeq \begin{cases} R[x] & \text{for } n = 2 \\ R[y] \otimes \Lambda[x] & \text{for } n > 2 \end{cases}$$

where $\deg(x) = \deg(u)$, $\deg(y) = n\deg(u)$, $\dim(x) = 1$, $\dim(y) = 2$ *and* $\Lambda[x]$ *is the exterior algebra over the ring* R *in the variable* x. □

We remark that generators of the rings in the Lemma can be given as follows: a representative x is given by the dual class of u. Moreover in characteristic $p = 0$, a representative of y is given by

$$\sum_{i=1}^{n-1} \binom{n}{i} (u^i \otimes u^{n-i})^*$$

and with $p \neq 0$, if n is a power of p, we can choose as a representative

$$\frac{1}{p} \sum_{i=1}^{n-1} \binom{n}{i} (u^i \otimes u^{n-i})^*,$$

where the notation $(u^i \otimes u^{n-i})^*$ means the dual class of $(u^i \otimes u^{n-i})$.

4.3 Computations and results

Now we can calculate the cohomology of $\Gamma_{K_p}(t, q)$ (and so the homology of $\mathrm{Br}(*)$ with coefficients in the local system $K(m)$) applying the fact that the cohomology of a tensor product of algebras is the tensor product of the cohomology of the factors.

Applying Lemma 4.2.5 we have the following straightforward results:

Theorem 4.3.1. ([Mar96]) *If* $p = 0$ *and* $m = 2$ *then*

$$H_*(\mathrm{Br}(*); K(2)) \simeq \Lambda[x_2] \otimes K(2)[x_1];$$

for $m > 2$*:*

$$H_*(\mathrm{Br}(*); K(m)) \simeq \Lambda[x_m] \otimes K(m)[y_m] \otimes \Lambda[x_1];$$

with $\deg x_i = i$, $\dim x_i = i - 1$, $\deg y_m = m$, $\dim y_m = m - 2$. □

Theorem 4.3.2. *Let* p *be a prime and* m *be a positive integer, such that* $p \nmid m$. *We have the following cases:*

(a) if $p = 2$*:*

$$H_*(\mathrm{Br}(*); K(2)) \simeq \bigotimes_{i=0}^{\infty} K(2)[x_{2^i}];$$

$$H_*(\mathrm{Br}(*); K(m)) \simeq K(m)[y_m] \otimes \Lambda[x_1] \otimes \bigotimes_{i=0}^{\infty} K(m)[x_{m2^i}];$$

(b) if $p > 2$ *and* $m = 2$*:*

$$H_*(\mathrm{Br}(*); K(p)) \simeq \bigotimes_{i=0}^{\infty} (K(p)[y_{p^{i+1}}] \otimes \Lambda[x_{p^i}]);$$

$$H_*(\mathrm{Br}(*); K(2)) \simeq K(2)[x_1] \otimes \bigotimes_{i=0}^{\infty} (K(2)[y_{2p^{i+1}}] \otimes \Lambda[x_{2p^i}]);$$

(c) if $p > 2$ *and* $m > 2$*:*

$$H_*(\mathrm{Br}(*); K(m)) \simeq K(m)[y_m] \otimes \Lambda[x_1] \bigotimes_{i=0}^{\infty} (K(m)[y_{p^{i+1}m}] \otimes \Lambda[x_{p^i m}]);$$

where $\deg x_i = \deg y_i = i$*,* $\dim x_i = i - 1$*,* $\dim y_i = i - 2$*.* $\square$

We want to use these results to compute the homology of $\mathrm{Br}(*)$ with coefficients in the local system A over the ring $A = K[q^{\pm 1}]$ with the same twisting defined as in Section 4.1.

The exact sequence

$$1 \to \mathrm{Br}(n)' \hookrightarrow \mathrm{Br}(n) \to \mathbb{Z} \to 1,$$

tells us that the homology $H_*(\mathrm{Br}(n); A)$ is $H_*(\mathrm{Br}(n)'; K)$ as an A-module (see for example [Mar96], [CS98] or [Cal05]); since for $n \neq 3, 4$, $\mathrm{Br}(n)'' = \mathrm{Br}(n)'$ (see [GL69] for a proof of this), we have that $H_0(\mathrm{Br}(n)'; K) = K$, $H_1(\mathrm{Br}(n)'; K) = 0$. Moreover the A-action on H_0 is trivial and so

$$H_0(\mathrm{Br}(n); A) = A/(q + 1)$$

as an A-module. Moreover we have:

Lemma 4.3.3. ([Mar96]) *The* R*-modules* $H_l(\mathrm{Br}(n), A)(n > 1, l > 0)$ *are annihilated by multiplication by* $[n]!$*.* $\square$

Let us consider a polynomial $a \in A$. We can consider the set S_a of all elements $b \in A$ that are prime with a. It is clear that S_a is a multiplicatively closed set. We write $A_{(a)}$ for the localization A_{S_a} of the ring A respect to the set S_a.

It follows from Lemma 4.2.1 that for $p = 0$, φ_m is invertible in $A_{(\varphi_n)}$ if and only if $m \neq n$; for $p \neq 0$, φ_m is invertible in $A_{(\varphi_n)}$ if and only if $n \neq mp^i$ and $m \neq np^i$, $\forall i \geq 1$.

The following decompositions hold for the homology of $\mathrm{Br}(n)$ with coefficients in the local system A:

Lemma 4.3.4. *Let $n > 1$. For $p = 0$ we have:*

$$H_*(\mathrm{Br}(n); A) \simeq \bigoplus_{m=2}^{\infty} H_*(\mathrm{Br}(n); A_{(\varphi_m)});$$

for $p \neq 0$:

$$H_*(\mathrm{Br}(n); A) \simeq \bigoplus_{p \nmid m \text{ or } m=p} H_*(\mathrm{Br}(n); A_{(\varphi_m)}).$$

Proof. Consider the homomorphism

$$i_{m*} : H_*(\mathrm{Br}(n); A) \to H_*(\mathrm{Br}(n); A_{(\varphi_m)})$$

induced by the injection $i_m : A \hookrightarrow A_{(\varphi_m)}$. We extend in a natural way the map i_{m*} through the tensor product with $A_{(\varphi_m)}$ and we get the new map

$$\widetilde{i_m} : H_*(\mathrm{Br}(n); A) \otimes_A A_{(\varphi_m)} \to H_*(\mathrm{Br}(n); A_{(\varphi_m)}). \tag{4.3.1}$$

Using Lemmas 4.2.1, 4.2.3 and 4.3.3 it is easy to see that in order to prove Lemma 4.3.4 it is enough to show that the map $\widetilde{i_m}$ is an isomorphism.

First we prove the injectivity of $\widetilde{i_m}$. Let α be a representative of an element v in $H_*(\mathrm{Br}(n); A) \otimes_A A_{(\varphi_m)}$. If the corresponding class of $\widetilde{i_m}v$ is zero, and so $\widetilde{i_m}\alpha$ is a boundary, then there exists an element β such that $d\beta = \widetilde{i_m}\alpha$. Multiplying β by an appropriate polynomial ψ prime with φ_m, we get an element $\beta' = \psi\beta$ that belongs to the resolution of $\mathrm{Br}(n)$ over A, so $d\beta' = \psi\alpha$. This means that $\psi\alpha$ belongs to the zero class in $H_*(\mathrm{Br}(n); A)$ and, since ψ is invertible in $A_{(\varphi_m)}$, α belongs to the zero class in $H_*(\mathrm{Br}(n); A) \otimes_A A_{(\varphi_m)}$. This proves the injectivity of $\widetilde{i_m}$.

To prove the surjectivity of $\widetilde{i_m}$ we consider a class w in $H_*(\mathrm{Br}(n); A_{(\varphi_m)})$ and we choose a representative β for w. Multiplying β by an appropriate polynomial θ prime with φ_m we get an element $\beta' = \theta\beta$ in the resolution for $H_*(\mathrm{Br}(n); A)$ and we have that

$$\widetilde{i_m}(\beta' \otimes \theta^{-1}) = \beta.$$

This completes the proof. $\square$

The next step is to compute $H_*(\mathrm{Br}(n); A_{(\varphi_m)})$. To do this, consider the following short exact sequence:

$$0 \to A_{(\varphi_m)} \overset{\varphi_m}{\hookrightarrow} A_{(\varphi_m)} \overset{\pi}{\to} K(m) \to 0$$

where the first map is multiplication by φ_m. We want to study the corresponding homology long exact sequence:

$$\begin{array}{cccc} & \cdots & \xrightarrow{\pi_*} H_{l+1}(\mathrm{Br}(*);K(m)) \xrightarrow{\beta} \\ \xrightarrow{\beta} H_l(\mathrm{Br}(*);A_{(\varphi_m)}) \xrightarrow{(\varphi_m)_*} & H_l(\mathrm{Br}(*);A_{(\varphi_m)}) & \xrightarrow{\pi_*} H_l(\mathrm{Br}(*);K(m)) \xrightarrow{\beta} \\ \xrightarrow{\beta} H_{l-1}(\mathrm{Br}(*);A_{(\varphi_m)}) \xrightarrow{(\varphi_m)_*} & \cdots & \end{array}$$

We can decompose $H_l(\mathrm{Br}(*);A_{(\varphi_m)})$ as a direct sum of terms $A/(\psi^i)$, where ψ is a prime factor of φ_m. So, if $H_l(\mathrm{Br}(*);A_{(\varphi_m)})$ has a direct summand $A/(\psi^i)$, generated by an element v, it follows that $H_{l+1}(\mathrm{Br}(*);K(m))$ and $H_l(\mathrm{Br}(*);K(m))$ have as direct summand a copy of $A/(\psi)$ generate respectively by w and w' and we have that

$$\beta w = \psi^{i-1}v$$

and

$$\pi_* v = w'.$$

In Theorem 4.3.1 (case $p=0$) we have these maps (see also [Mar96]):

$$\beta x_m = \widetilde{y_m}, \pi_*\widetilde{y_m} = y_m, \beta x_1 = 0.$$

while in Theorem 4.3.2 (case $p \neq 0$), the homomorphisms act as follows:

$$\beta y_i = 0, \qquad \beta x_1 = 0,$$

$$\beta x_{2^i} = \varphi_2^{2^{i-1}-1}\widetilde{x_{2^{i-1}}}^2; \qquad \beta x_{p^i} = \varphi_p^{p^{i-1}-1}\widetilde{y_{p^i}} \quad \text{for } p>2;$$

$$\beta x_{mp^i} = \varphi_m^{\phi(p^i)-1}\widetilde{y_{mp^i}} \quad \text{for } i>0 \text{ or } m>2;$$

where $\widetilde{x_i}$ and $\widetilde{y_i}$ are defined such that:

$$\pi_*\widetilde{x_i} = x_i; \pi_*\widetilde{y_i} = y_i.$$

We can now state the following result:

Proposition 4.3.5. *For $p=0$ we have, for $m=2$:*

$$H_*(\mathrm{Br}(*);A_{(\varphi_2)}) \simeq A_{(\varphi_2)}[x_1]/(\varphi_2 x_1^2);$$

and for $m>2$:

$$H_*(\mathrm{Br}(*);A_{(\varphi_m)}) \simeq A_{(\varphi_m)}[x_1,y_m]/(x_1^2,\varphi_m y_m);$$

if $p \neq 0$ and $p \nmid m$ we have the following cases:

(a) for $p = 2$*:* $H_*(\mathrm{Br}(*); A_{(\varphi_2)}) \simeq A_{(\varphi_2)} \left[\begin{array}{c} x_1, x_{2^j}^2, \\ x_{2^i}^2 x_{2^{i_1}} \cdots x_{2^{i_h}} \end{array} \right] \Big/ \Big(\ \varphi_2^{2^i} x_{2^i}^2 \ \Big)$
with $0 \leq i,\ i+1 < i_1 < \cdots < i_h,\ 0 < j$*;*

$$H_*(\mathrm{Br}(*); A_{(\varphi_m)}) \simeq A_{(\varphi_m)} \left[\begin{array}{c} x_1, y_m, x_{m2^i}^2, \\ x_{m2^i}^2 x_{m2^{i_1}} \cdots x_{m2^{i_h}}, \\ y_m x_{m2^{j_1}} \cdots x_{m2^{j_h}} \end{array} \right] \Big/ \left(\begin{array}{c} x_1^2, \varphi_m y_m, \\ \varphi_m^{2^i} x_{m2^i}^2 \end{array} \right)$$

with $0 \leq i,\ i+1 < i_1 < \cdots < i_h,\ 0 < j_1 < \cdots < j_h$*;*

(b) for $p > 2$ *and* $m = 2$*:*

$$H_*(\mathrm{Br}(*); A_{(\varphi_p)}) \simeq A_{(\varphi_p)} \left[\begin{array}{c} x_1, y_{p^i}, \\ y_{p^i} x_{p^{i_1}} \cdots x_{p^{i_h}} \end{array} \right] \Big/ \left(\begin{array}{c} x_{p^i}^2, \\ \varphi_p^{p^{i-1}} y_{p^i} \end{array} \right)$$

with $0 < i < i_1 < \cdots < i_h$*;*

$$H_*(\mathrm{Br}(*); A_{(\varphi_2)}) \simeq A_{(\varphi_2)} \left[\begin{array}{c} x_1, y_{2p^i}, \\ x_1^2 x_{2p^{j_1}} \cdots x_{2p^{j_h}} \\ y_{2p^i} x_{2p^{i_1}} \cdots x_{2p^{i_h}} \end{array} \right] \Big/ \left(\begin{array}{c} \varphi_2 x_1^2, x_{2p^i}^2, \\ \varphi_2^{\phi(p^i)} y_{2p^i} \end{array} \right)$$

with $0 < i < i_1 < \cdots < i_h,\ 0 < j_1 < \cdots < j_h$*;*

(c) for $p > 2$ *and* $m > 2$*:*

$$H_*(\mathrm{Br}(*); A_{(\varphi_m)}) \simeq A_{(\varphi_m)} \left[\begin{array}{c} x_1, y_{mp^i}, \\ y_{mp^i} x_{mp^{i_1}} \cdots x_{mp^{i_h}} \end{array} \right] \Big/ \left(\begin{array}{c} x_1^2, x_{mp^i}^2, \\ \varphi_m^{\phi(p^i)} y_{mp^i} \end{array} \right)$$

with $0 \leq i < i_1 < \cdots < i_h$*;*

in all cases $\deg x_i = \deg y_i = i, \dim x_i = i-1,\ \dim y_i = i-2.$ □

In order to get $H_*(\mathrm{Br}(n); \mathbb{Z}[q^{\pm 1}])$ we still have to compute the Bockstein homomorphism β_p associated to the short exact sequence

$$0 \to \mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0.$$

We'll see that, as in the case of trivial coefficients (see [Coh76] or [Vaĭ78]), there is no p^2-torsion in the homology of braid groups.

In case *a)* of Proposition 4.3.5 the Bockstein acts as follows (the coefficients $\psi_{i,j}$ are those defined in Corollary 4.2.2):

$$\beta_2 x_1 = 0, \quad \beta_2 x_{2^i}^2 = 0,$$

$$\beta_2 x_{2^i}^2 x_{2^{i_1}} \cdots x_{2^{i_h}} = \sum_{s=1}^{h} \left(\psi_{2^{i_s}, 2^i} x_{2^i}^2 x_{2^{i_s - 1}}^2 \prod_{t \neq s} x_{2^{i_t}} \right);$$

$$\beta_2 y_m = 0, \beta_2 x_{m2^i}^2 = 0,$$

$$\beta_2 x_{m2^i}^2 x_{m2^{i_1}} \cdots x_{m2^{i_h}} = \sum_{s=1}^{h} \left(\psi_{m2^{i_s-1}, m2^i} x_{m2^i}^2 x_{m2^{i_s-1}}^2 \prod_{t \neq s} x_{m2^{i_t}} \right),$$

$$\beta_2 y_m x_{m2^{i_1}} \cdots x_{m2^{i_h}} = \sum_{s=1}^{h} \left(\psi_{m2^{i_s-1}, m} y_m x_{m2^{i_s-1}}^2 \prod_{s \neq t} x_{m2^{i_t}} \right);$$

in case *b)* we have:

$$\beta_p x_1 = 0, \beta_p y_{p^i} = 0,$$

$$\beta_p y_{p^i} x_{p^{i_1}} \cdots x_{p^{i_h}} = -\sum_{s=1}^{h} \left(\psi_{p^{i_s}, p^i} y_{p^i} y_{p^{i_s-1}} \prod_{s \neq t} x_{p^{i_t}} \right);$$

$$\beta_p x_1^2 x_{2p^{i_1}} \cdots x_{2p^{i_h}} = -\sum_{s=1}^{h} \left(\psi_{2p^{i_s}, 2p^i} x_1^2 y_{2p^{i_s-1}} \prod_{s \neq t} x_{2p^{i_t}} \right);$$

$$\beta_p y_{2p^i} = 0,$$

$$\beta_p y_{2p^i} x_{2p^{i_1}} \cdots x_{2p^{i_h}} = -\sum_{s=1}^{h} \left(\psi_{2p^{i_s}, 2p^i} y_{2p^i} y_{2p^{i_s-1}} \prod_{s \neq t} x_{2p^{i_t}} \right);$$

finally in case *c)* the map is:

$$\beta_p x_1 = 0, \beta_p y_{mp^i} = 0,$$

$$\beta_p y_{mp^i} x_{mp^{i_1}} \cdots x_{mp^{i_h}} = -\sum_{s=1}^{h} \left(\psi_{mp^{i_s}, mp^i} y_{mp^i} y_{mp^{i_s-1}} \prod_{s \neq t} x_{mp^{i_t}} \right).$$

Lemma 4.3.6. *The homology groups $H_*(\mathrm{Br}(*); \mathbb{Z}[q^{\pm 1}])$ have no p^2-torsion.*

Proof. Notice that the monomials $1, x_1$ generate the groups

$$H_0(\mathrm{Br}(0), \mathbb{Z}[q^{\pm 1}]), \quad H_0(\mathrm{Br}(1), \mathbb{Z}[q^{\pm 1}])$$

and that both these modules are equal to $\mathbb{Z}[q^{\pm 1}]$. For $i \geq 2$ the groups $H_0(\mathrm{Br}(i), \mathbb{Z}[q^{\pm 1}]) = \mathbb{Z}$ are generated by the monomials x_1^i. Now consider the following monomials:

$$y_m^i, \quad y_m^i x_1 \qquad \text{(case } p = 0\text{)} \tag{4.3.2}$$

$$\left.\begin{matrix} x_{2^j}^{2i}, & x_{2^j}^{2i} x_1, \\ x_{m2^j}^{2i}, & x_{m2^j}^{2i} x_1 \end{matrix}\right\} \qquad \text{(case } p = 2\text{)} \tag{4.3.3}$$

$$\left.\begin{matrix} y_{p^j}^i, & y_{p^j}^i x_1, \\ y_{2p^j}^i, & y_{2p^j}^i x_1, \\ y_{mp^j}^i, & y_{mp^j}^i x_1 \end{matrix}\right\} \qquad \text{(case } p > 2\text{)} \tag{4.3.4}$$

Because of the computations over $\mathbb{Q}[q^{\pm 1}]$ ([DCPS01], [Mar96]), their lifting generate a free $\mathbb{Z}$-module of type $\mathbb{Z}[q^{\pm 1}]/(\varphi_h)$ in dimension d in the homology of $\mathrm{Br}(n)$ whenever $n = kh$ or $n = kh+1$ and $d = k(h-2)$ and the Bockstein is zero for all these monomials.

All the other monomials lift to torsion classes and all these classes don't have p^2-torsion for any prime p. To prove this it is enough to show that in the submodule $M_p \subset H_*(\mathrm{Br}(*), \mathbb{Z}_p[q^{\pm 1}])$ generated by all the monomials different from the ones in (4.3.3) or (4.3.4), we have that

$$\ker \beta_p = \mathrm{im}\ \beta_p.$$

Let us consider the set S of the monomials that appear in the polynomial rings of part *a)*, *b)* and *c)* of Proposition 4.3.5 and different from these in (4.3.3) and (4.3.4).

Let us say that a monomial w *rises up* to a monomial w' is w appears as a summand in $\beta_p w'$. We call w a *basic* monomial if it doesn't appear as a summand in $\beta_p w'$ for any monomial w'. We also say that a monomial w is a *child* of w' if w' is basic and we can rise up from w to w' in a finite number of steps. We notice that in general a basic polynomial can be of the following form:

$$w = x_{i_1}^{2a_1} \cdots x_{i_k}^{2a_k} y_{j_1}^{b_1} \cdots y_{j_h}^{b_k} x_{l_1} \cdots x_{l_s}.$$

Let Δ_w be the set of all monomials that are children of w (including w itself). It is easy to see that Δ_w is in bijection with the set of the parts of $\{1, \ldots, s\}$ (with $s \geq 1$) if $l_1, \ldots, l_s$ are all different from 1, or with the set of the parts of $\{1, \ldots, s-1\}$ (with $s \geq 2$) if one of $l_1, \ldots, l_s$ is 1.

Let us say that a monomial w has φ-torsion (over the ring $\mathbb{Z}_p[q^{\pm 1}]$) if it generates a module isomorphic to $\mathbb{Z}_p[q^{\pm 1}]/(\varphi)$. If a monomial w has φ-torsion over $\mathbb{Z}_p[q^{\pm 1}]$ then all the other monomials children of w have the same torsion. Moreover consider the algebraic complex (M_w, β_p) given by the module M_w generated (over $\mathbb{Z}_p[q^{\pm 1}]$) by all the monomials in Δ_w and with the restriction of β_p to M_w as a boundary map: we have that (M_w, β_p) is isomorphic to the algebraic complex that computes the reduced homology of the $(s-1)$-dimensional simplex with constant coefficients, over the ring $\mathbb{Z}_p[q^{\pm 1}]/(\varphi)$ and so $\ker \beta_p = \mathrm{im}\ \beta_p$ on M_w.

One can check that for every monomial w in S there exists one and only one basic monomial w' such that w is a child of w'. This implies that the family of all different sets Δ_w gives a partition of S and so $\ker \beta_p = \mathrm{im}\ \beta_p$ on all the module M. The Lemma follows. □

4.4 Main Result

As a consequence of the last Lemma and of the previous computations, we can now state our main Theorem. Recall that the ring $H_*(\mathrm{Br}(*); R[q^{\pm 1}])$ is the bigraded direct sum of the modules $H_i(\mathrm{Br}(n); R[q^{\pm 1}]) = H_i(\mathbf{F}_1(n), R)$, where $\mathbf{F}_1(n)$ is the Milnor fiber of the discriminant singularity for $\mathrm{Br}(n)$.

Theorem 4.4.1. *Set* $\deg x_i = \deg y_i = i, \dim x_1 = 0, \dim x_i = i-1$, $\dim y_i = i-2$. *Then:*

$$H_*(\mathrm{Br}(*); \mathbb{Q}[q^{\pm 1}]) \simeq \mathbb{Q}[q^{\pm 1}] \left[\ x_1, y_m, m > 2\ \right] / \left(\ \varphi_2 x_1^2, \varphi_m y_m\ \right);$$

$$H_*(\mathrm{Br}(*); \mathbb{Z}_2[q^{\pm 1}]) \simeq \mathbb{Z}_2[q^{\pm 1}] \left[\begin{array}{c} x_1, y_m, x_{2^{i+1}}^2, x_{m2^i}^2, \\ x_{2^i}^2 x_{2^{i_1}} \cdots x_{2^{i_h}}, \\ x_{m2^i}^2 x_{m2^{i_1}} \cdots x_{m2^{i_h}}, \\ y_m x_{m2^{j_1}} \cdots x_{m2^{j_h}}, \\ m \geq 2, 2 \nmid m \end{array} \right] \Bigg/ \left(\begin{array}{c} \varphi_2^{2^i} x_{2^i}^2, \\ \varphi_m y_m, \\ \varphi_m^{2^i} x_{m2^i}^2 \end{array} \right)$$

with $0 \leq i$, $i+1 < i_1 < \cdots < i_h$, $0 < j_1 < \cdots < j_h$;
for $p > 2$:

$$H_*(\mathrm{Br}(*); \mathbb{Z}_p[q^{\pm 1}]) \simeq \mathbb{Z}_p[q^{\pm 1}] \left[\begin{array}{c} x_1, y_{p^i}, y_{mp^j}, y_{2p^i} \\ y_{p^i} x_{p^{i_1}} \cdots x_{p^{i_h}}, \\ x_1^2 x_{2p^{j_1}} \cdots x_{2p^{j_h}} \\ y_{2p^i} x_{2p^{i_1}} \cdots x_{2p^{i_h}}, \\ y_{mp^j} x_{mp j_1} \cdots x_{mp^{j_h}} \\ m > 2, p \nmid m \end{array} \right] \Bigg/ \left(\begin{array}{c} \varphi_2 x_1^2, x_{2p^i}^2, \\ x_{p^i}^2, x_{mp^j}^2, \\ \varphi_p^{p^{i-1}} y_{p^i}, \\ \varphi_2^{\phi(p^i)} y_{2p^i}, \\ \varphi_m^{\phi(p^j)} y_{mp^j} \end{array} \right)$$

with $0 < i < i_1 < \cdots < i_h$, $0 \leq j < j_1 < \cdots < j_h$. *Finally, using the notation of the proof of Lemma 4.3.6, we have:*

$$\begin{gathered} H_*(\mathrm{Br}(*); \mathbb{Z}[q^{\pm 1}]) \simeq \\ \simeq \mathbb{Z}[q^{\pm 1}] \left[\ x_1, y_m, m > 2\ \right] / \left(\ \varphi_2 x_1^2, \varphi_m y_m\ \right) \oplus \bigoplus_{p \geq 2} \beta_p M_p. \end{gathered}$$

□

In tables 4.1, 4.2, 4.3 and 4.4 we give the explicit computations for some cases. The results in Table 4.3 correspond to those in [DCPS01] for cohomology. We use the notation φ_h^i for the module $\mathbb{Z}_p[q]/(\varphi_h^i)$ or $\mathbb{Q}[q]/(\varphi_h^i)$ (note that $R[q^{\pm 1}]/(\varphi_h^i) = R[q]/(\varphi_h^i)$). In Table 4.4 we describe the additive structure of the integral homology of the fiber $\mathbf{F}_1(n)$.

Now consider the natural embeddings $j_n : \mathrm{Br}(n) \hookrightarrow \mathrm{Br}(n+1)$. By definition the direct limit $\varinjlim_n \mathrm{Br}(n)$ is the *braid group on infinitely many strands* $\mathrm{Br}(\infty)$. Notice that the first p-torsion class in the groups $H_*(\mathrm{Br}(n); \mathbb{Z}[q^{\pm 1}])$ appears for $n = 2p+2$ in dimension $2p-2$ and is stable; the corresponding generator is $x_1^2 x_2^2$ for $p = 2$ or $x_1^2 y_{2p}$ for $p > 2$. An equivalent result for the cohomology was proved in [Cal03].

In [CP07] Cohen and Pakianathan compute the homology of the braid group on infinitely many strands $\mathrm{Br}(\infty)$ with coefficients in the local system

n	2	3	4	5	6	7	8	9	10
H_0	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2
H_1		φ_3	φ_3						
H_2			φ_2^2	φ_2^2	$\varphi_2 \oplus \varphi_3$	$\varphi_2 \oplus \varphi_3$	φ_2	φ_2	φ_2
H_3				φ_5	$\varphi_2 \oplus \varphi_5$	φ_2	φ_2	$\varphi_2 \oplus \varphi_3$	$\varphi_2 \oplus \varphi_3$
H_4					φ_3	φ_3	φ_2^2	φ_2^2	φ_2
H_5						φ_7	φ_7	φ_3	$\varphi_2 \oplus \varphi_3$
H_6							φ_2^4	$\varphi_2^4 \oplus \varphi_3$	$\varphi_2 \oplus \varphi_3 \oplus \varphi_5$
H_7								φ_9	$\varphi_2 \oplus \varphi_9$
H_8									φ_5

Table 4.1: $H_*(\mathrm{Br}(n); \mathbb{Z}_2[q^{\pm 1}])$

n	2	3	4	5	6	7	8	9	10
H_0	φ_2	φ_2	2	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2
H_1		φ_3	φ_3						
H_2			φ_4	φ_4	φ_3	φ_3			
H_3				φ_5	φ_5			φ_3	φ_3
H_4					φ_2^2	φ_2^2	$\varphi_2 \oplus \varphi_4$	$\varphi_2 \oplus \varphi_4$	φ_2
H_5						φ_7	$\varphi_2 \oplus \varphi_7$	φ_2	φ_2
H_6							φ_8	φ_8	φ_5
H_7								φ_3^2	φ_3^2
H_8									φ_{10}

Table 4.2: $H_*(\mathrm{Br}(n); \mathbb{Z}_3[q^{\pm 1}])$

$K[q^{\pm 1}]$ for any field K: this is the stable part of the homology of $\mathrm{Br}(n)$ (with coefficients the same local system) with respect to the embeddings $j_n : \mathrm{Br}(n) \hookrightarrow \mathrm{Br}(n+1)$. We obtain the same result; moreover we are able to compute the Bockstein operator, hence we give a presentation of the homology of $\mathrm{Br}(\infty)$ with coefficients in the local system $\mathbb{Z}[q^{\pm 1}]$.

Corollary 4.4.2. *We have that*

$$H_*(\mathrm{Br}(\infty); \mathbb{Q}[q^{\pm 1}]) = \mathbb{Q},$$

concentrated in dimension 0*;*

$$H_*(\mathrm{Br}(\infty); \mathbb{Z}_2[q^{\pm 1}]) = \mathbb{Z}_2[x_2^2, x_{2^i}, i > 1]$$

and for a prime $p > 2$

$$H_*(\mathrm{Br}(\infty); \mathbb{Z}_p[q^{\pm 1}]) = \mathbb{Z}_p[y_{2p^i}, x_{2p^i}, i > 0]/(x_{2p^i}^2)$$

n	2	3	4	5	6	7	8	9	10
H_0	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2
H_1		φ_3	φ_3						
H_2			φ_4	φ_4	φ_3	φ_3			
H_3				φ_5	φ_5			φ_3	φ_3
H_4					φ_6	φ_6	φ_4	φ_4	
H_5						φ_7	φ_7		
H_6							φ_8	φ_8	φ_5
H_7								φ_9	φ_9
H_8									φ_{10}

Table 4.3: $H_*(\mathrm{Br}(n); \mathbb{Q}[q^{\pm 1}])$

n	2	3	4	5	6	7	8	9	10
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1		$\mathbb{Z}^2$	$\mathbb{Z}^2$						
H_2			$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
H_3				$\mathbb{Z}^4$	$\mathbb{Z}^4$			$\mathbb{Z}^2$	$\mathbb{Z}^2$
H_4					$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}_3 \oplus \mathbb{Z}^2$	$\mathbb{Z}_3 \oplus \mathbb{Z}^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$
H_5						$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
H_6							$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}_2 \oplus \mathbb{Z}^4$
H_7								$\mathbb{Z}^6$	$\mathbb{Z}^6$
H_8									$\mathbb{Z}^4$

Table 4.4: $H_*(\mathbf{F}_1(n); \mathbb{Z})$

with $\dim x_i = i - 1$, $\dim y_i = i - 2$. *The Bockstein operator acts as follows:*

$$\beta_2 x_{2^i} = x_{2^{i-1}}^2;$$

$$\beta_p y_i = 0; \quad \beta_p x_i = y_i \qquad (\textit{for } p > 2).$$

The homology $H_*(\mathrm{Br}(\infty); \mathbb{Z}[q^{\pm 1}])$ *has no* p^2*-torsion for any prime* p. *A presentation of* $H_*(\mathrm{Br}(\infty); \mathbb{Z}[q^{\pm 1}])$ *is given by*

$$\mathbb{Z}\left[\begin{array}{c} y_{2p^i}, x_{2^j}^2, \\ x_{2^i}^2 x_{2^{i_1}} \cdots x_{2^{i_h}} \\ y_{2p^j} x_{2p^{j_1}} \cdots x_{2p^{j_h}} \end{array}\right] \Big/ \Big(\; 2x_{2^i}, p y_{2p^j}, x_{2p^j}^2 \; \Big)$$

with indexes running as follows: $0 < i, i+1 < i_1 < \cdots < i_h$, $0 < j < j_1 < \cdots < j_h$ *and* p *in the set of odd primes. The structure of* $\mathbb{Z}[q^{\pm 1}]$*-module is trivial and so the action of* q *corresponds to multiplication by* -1. $\square$

Chapter 5

The integral homology of the Milnor fiber for Artin groups of type B_n

5.1 The filtration and the homology spectral sequence for B_n

In this Chapter we compute the homology of the Artin groups G_{B_n} with coefficients in the local system given by the ring $R = \mathbb{Z}[q^{\pm 1}]$, where each standard generator maps to $(-q)$-multiplication. As we explained in Section 3.3, we use a filtration F on the algebraic Salvetti complex $C(n) = C_*(B_n)$, induced by the order of the vertexes of the Coxeter graph of B_n given in Table 1.1. Hence the last vertex, connected with the special edge with label 4, is in position n.

Using the notation of Section 3.2, we write the generator of the complex as strings of 0's and 1's. We write $\mathrm{l}(X)$ for the length of the string X and degree $\deg(X) = |X|$ for the number of 1's in the string X, that is equal to the dimension of the corresponding cell. We define the subcomplex

$$F_sC(n) = F_sC_*(B_n)$$

to be the subcomplex generated by the strings of type XY, where $\mathrm{l}(X) = n-i-1$, $\mathrm{l}(Y) = i+1$, $\deg Y \leq i$. We have the inclusions

$$F_0C(n) =< X0 >\subset F_1C(n) \subset \cdots \subset F_{n+1}C(n) = C(n).$$

We note that we have the following isomorphism for the quotients:

$$F_iC(n)/F_{i-1}C(n) \simeq C_{A_{n-i-1}}[i]$$

generated by the strings of the form $X01^i$ that corresponds to the string X in the complex $C_{A_{n-i-1}}[i]$ (recall that the index i between square brackets means a graduation shifted by i). In this isomorphism the boundary map of ∂ of the complex $C_{A_{n-i-1}}$ corresponds to the map induced on the quotient by the boundary of the complex $C(n)$.

We can use the homology of C_{A_j}, computed in Chapter 4, in order to describe the spectral sequence for $C(n)$. In fact we have:

Proposition 5.1.1. *Given a ring R with a unit q, we can consider R as a G_{B_n} module mapping each standard generator to multiplication by $(-q)$. The filtration F on the complex $C(n)$ induces a spectral sequence E^* that converges to $H_*(G_{B_n}, R)$. The first term or the spectral sequence is given by:*

$$E^1_{s,t} = H_s(\mathrm{Br}(t), R)$$

where the ring R is considered as a module over the group $\mathrm{Br}(n)$ *with the usual action, as in Chapter 4.*

As we did in Chapter 4 for the homology of braid groups, we start computing the homology of G_{B_n} with coefficient over the ring $R = K(m) = K[q]/(\varphi_m)$ where K is a field of characteristic p (for simplicity we can fix $K = \mathbb{Q}$ if $p = 0$ and $K = \mathbb{F}_p$ for $p \neq 0$). In what follows p will always refer to the characteristic of the field K. Moreover we denote by $\Lambda[x]$ the exterior algebra over the ring R in the variable x.

We recall some results for the homology of the braid groups, from Chapter 4. They are the starting point for the description of the spectral sequence E^*.

In case $p = 0$, $m > 2$:

$$H_*(\mathrm{Br}(*), K(2)_q) \simeq \Lambda(2)[x_2] \otimes K(2)[x_1]$$

$$H_*(\mathrm{Br}(*), K(m)_q) \simeq \Lambda(m)[x_m] \otimes K(m)[y_m] \otimes \Lambda(m)[x_1].$$

In case $p = 2$, $2 \nmid m$:

$$H_*(\mathrm{Br}(*), K(2)_q) \simeq \bigotimes_{i=0}^{\infty} K(2)[x_{2^i}]$$

$$H_*(\mathrm{Br}(*), K(m)_q) \simeq \bigotimes_{i=0}^{\infty} K(m)[x_{m2^i}] \otimes K(m)[y_m] \otimes \Lambda(m)[x_1].$$

In case $p > 2$, $p \nmid m$:

$$H_*(\mathrm{Br}(*), K(p)_q) \simeq \bigotimes_{i=0}^{\infty} \left(K(p)[y_{p^{i+1}}] \otimes \Lambda(p)[x_{p^i}]\right)$$

$$H_*(\mathrm{Br}(*), K(2)_q) \simeq K(2)[x_1] \otimes \bigotimes_{i=0}^{\infty} \left(K(2)[y_{2p^{i+1}}] \otimes \Lambda(2)[x_{2p^i}]\right)$$

$$\begin{aligned} H_*(\mathrm{Br}(*),&K(m)_q) \simeq \\ &\simeq \Lambda(m)[x_1] \otimes K(m)[y_m] \otimes \bigotimes_{i=0}^{\infty} \left(K(m)[y_{mp^{i+1}}] \otimes \Lambda(m)[x_{mp^i}]\right) \end{aligned}$$

We can give representatives for the x_i and y_i terms:

$$x_i = 1^{i-1}0,$$

$$y_{mp^i} = \frac{d(1^{mp^i-1}0)}{\varphi_m^a}$$

where a is the biggest power of φ_m that divides all the terms in $d(1^{mp^i-1}0)$. The dimension and degree of the generators are the following: $\deg x_i = \deg y_i = i$, $\dim x_i = i-1$, $\dim y_i = i-2$. Moreover, in the following sections we introduce a generator z_i, where we set $\deg z_i = \dim z_i = i$. The generator z_i in the homology of G_{B_n} is represented by the string $1^{i-1}\bar{1}$, where $\bar{1}$ is the special vertex in the n-th position in the Coxeter graph of B_n.

5.2 Computations

In Chapter 4 we considered the sum

$$\bigoplus_n H_*(\mathrm{Br}(n), R) = H_*(\mathrm{Br}(*), R)$$

as a ring with the product structure induced by the map $\mathrm{Br}(i) \times \mathrm{Br}(j) \to \mathrm{Br}(i+j)$. Now it is useful to consider the sum

$$\bigoplus_n H_*(C(n), R) = H_*(C(*), R)$$

where $H_*(C(*), R)_{\deg = n} = H_*(C(n), R)$. It turns out that the description of the complexes allow to define for the group $H_*(C(*), R)$ a structure of module over the ring $H_*(\mathrm{Br}(*), R)$. Hence in our description of the homology groups $H_*(C(n), R)$ we can simply give the set of generators of $H_*(C(*), R)$ as a $H_*(\mathrm{Br}(*), R)$-module, with the nontrivial relations.

We omit the details of the study of the spectral sequences E^* given in the previous Section and we simply state the results. The computations are straightforward, even if not completely trivial, and involve only standard techniques.

We can resume the first step of the spectral sequence computations in the next statement:

Theorem 5.2.1. *The homology groups of the complex $C(n)$ with coefficients over the ring $K(m)$ are the following: Case $p=0$, $m=2$:*

$$H_*(C(n), K(2)_q) \simeq$$
$$\simeq \big(\Lambda(2)[x_2] \otimes K(2)[x_1] \otimes K(2)[z_1, \dots, z_n]/(z_i^2, z_i z_j)\big)_{\deg=n}.$$

Case $p=0$, $m>2$, $2 \nmid m$:

$$H_*(C(n), K(m)_q) \simeq 0.$$

Case $p=0$, $m>2$, $2 \mid m$:

$$H_*(C(n), K(m)_q) \simeq$$
$$\simeq \Big(\Lambda(m)[x_m] \otimes K(m)[y_m] \otimes K(m)[x_1 z_{k\frac{m}{2}-1}, z_{k\frac{m}{2}}]/(z_i^2, z_i z_j)\Big)_{\deg=n}.$$

Case $p=2$, $m=2$:

$$H_*(C(n), K(2)_q) \simeq \left(\left(\bigotimes_{i=0}^{\infty} K(2)[x_{2^i}]\right) \otimes K(2)[z_1, \dots, z_n]/(z_i^2, z_i z_j)\right)_{\deg=n}.$$

Case $p=2$, $m>2$ $(2 \nmid m)$:

$$H_*(C(n), K(m)_q) \simeq$$
$$\simeq \left(\left(\bigotimes_{i=0}^{\infty} K(m)[x_{m2^i}]\right) \otimes K(m)[z_m, z_{2m}, \dots]/(z_i^2, z_i z_j)\right)_{\deg=n}.$$

Case $p>2$, $m=p$:

$$H_*(C(n), K(p)_q) \simeq 0.$$

Case $p>2$, $m=2$:

$$H_*(C(n), K(2)_q) \simeq$$
$$\simeq \left(K(2)[x_1] \otimes \bigotimes_{i=0}^{\infty} \big(K(2)[y_{2p^{i+1}}] \otimes \Lambda[x_{2p^i}]\big) \otimes K(2)[z_1, \dots, z_n]\right)_{\deg=n}.$$

Case $p>2$, $m>2$, $p \nmid m$, $2 \nmid m$:

$$H_*(C(n), K(m)_q) \simeq 0.$$

Case $p>2$, $m>2$, $p \nmid m$, $2 \mid m$:

$$H_*(C(n), K(m)_q) \simeq$$
$$\simeq \left(\bigotimes_{i=0}^{\infty} \big(K(m)[y_{mp^i}] \otimes \Lambda(m)[x_{mp^i}]\big) \otimes K(m)[x_1 z_{k\frac{m}{2}-1}, z_{k\frac{m}{2}}]/(z_i^2, z_i z_j)\right)_{\deg=n}.$$

$\square$

Let $A = K[q^{\pm 1}]$. The next step is the computation of the homology of $C(n)$ with coefficients in the ring A. We define the localization $A_{(a)}$ as in Section 4.3. The analogous of Lemma 4.3.3 and Lemma 4.3.4 holds for the homology of G_{B_n}:

Lemma 5.2.2. ([Mar96]) *The R-modules $H_l(C(n), A)(n > 0, l \geq 0)$ are annihilated by multiplication by* $[2n]!!$. □

Lemma 5.2.3. *Let $n > 1$. For $p = 0$ we have:*

$$H_*(C(n), A) \simeq \bigoplus_{m=2}^{\infty} H_*(C(n), A_{(\varphi_m)});$$

for $p \neq 0$:

$$H_*(C(n); A) \simeq \bigoplus_{p \nmid m \text{ or } m=p} H_*(C(n), A_{(\varphi_m)}).$$

□

The proofs are straightforward generalizations of the proofs of the corresponding statements for the homology of $\mathrm{Br}(n)$ and we omit them.

By Lemma 5.2.3 in order to understand $H_*(C(n), A)$ we have to compute the modules $H_*(C(n), A_{(\varphi_m)})$. To do this we can consider the following short exact sequence:

$$0 \to A_{(\varphi_m)} \overset{\varphi_m}{\hookrightarrow} A_{(\varphi_m)} \xrightarrow{\pi} K(m) \to 0$$

where the first map is multiplication by φ_m. We want to study the corresponding homology long exact sequence:

$$\begin{array}{llll} & \cdots & \xrightarrow{\pi_*} H_{l+1}(C(n), K(m)) \xrightarrow{\beta} \\ \xrightarrow{\beta} H_l(C(n), A_{(\varphi_m)}) \xrightarrow{(\varphi_m)_*} & H_l(C(n), A_{(\varphi_m)}) & \xrightarrow{\pi_*} H_l(C(n), K(m)) \xrightarrow{\beta} \\ \xrightarrow{\beta} H_{l-1}(C(n), A_{(\varphi_m)}) \xrightarrow{(\varphi_m)_*} & \cdots & \end{array}$$

Once we compute the map β, we can recover the description of the module $H_*(C(n), A_{(\varphi_m)})$ using the same argument of Section 4.3. In the computation of the map β we can omit some terms that are not significant for the computation of $H_*(C(n), A_{(\varphi_m)})$: in general when we write

$$\beta a = \varphi_m^i b \quad \text{we mean} \quad \beta a = \varphi_m^i b + (\text{other terms multiplied by } \varphi_m^{i+1}).$$

A case by case computation gives the following data. In the next propositions we give the generators of the homology of the complex $C(*)$ as a module over the corresponding homology ring of braid groups. The usual relations of the homology ring of braids hold. We specify the other relations. In all cases we set:

$$\pi_* \widetilde{y}_i = y_i, \pi_* \widetilde{z}_i = z_i, \pi_* \widetilde{x}_i = x_i.$$

Case $p = 0$**,** $m = 2$

$$\pi_*\beta x_1 = 0, \pi_*\beta x_2 = x_1^2,$$

$$\pi_*\beta z_i = x_1 z_{i-1} + x_2 z_{i-2}.$$

Proposition 5.2.4 ($p = 0$, $m = 2$)**.** *The module $H_*(C(*), A_{(\varphi_2)})$ over the homology ring of braids is generated by*

$$1,$$

$$x_1 z_i + x_2 z_{i-1}$$

with φ_2-torsion.

Case $p = 0$**,** $m > 2$**,** $2 \mid m$

$$\beta x_1 = 0, \pi_*\beta x_m = y_m,$$

$$\beta y_m = 0,$$

$$\beta x_1 z_{k\frac{m}{2}-1} = 0,$$

$$\pi_*\beta z_{k\frac{m}{2}} = x_1 z_{k\frac{m}{2}-1}.$$

Proposition 5.2.5 ($p = 0$, $m > 2$, $2 \mid m$)**.** *The module $H_*(C(*), A_{(\varphi_m)})$ over the homology ring of braids is generated by*

$$1,$$

$$x_1 z_{k\frac{m}{2}-1}$$

with φ_m-torsion,

$$y_m z_{k\frac{m}{2}} + x_1 x_m z_{k\frac{m}{2}-1}$$

with φ_m-torsion.

Case $p = 2$**,** $m = 2$

If $2 \nmid a$:

$$\beta x_{2^i} = \varphi_2^{2^{i-1}-1} \widetilde{x}_{2^{i-1}}^2,$$

$$\beta z_{2^i a} = \varphi_2^{2^{i+1}-2} (\widetilde{x}_1 \widetilde{z}_{2^i a-1} + \widetilde{x}_2 \widetilde{z}_{2^i a-2}),$$

$$\beta z_{2b+1} = x_1 z_{2b}.$$

Proposition 5.2.6 ($p = 2$, $m = 2$). *The module $H_*(C(*), A_{(\varphi_2)})$ over the homology ring of braids is generated by the terms:*

$$1,$$

$$x_1 x_{2^{i_1}} \cdots x_{2^{i_h}} z_{2b}$$

($i_1 > 1$) with φ_2-torsion (the indexes $i_1, \ldots, i_h$ are always intended $i_1 < \cdots < i_h$),

$$x_1 x_{2^{i_1}} \cdots x_{2^{i_h}} z_{2^i a - 1} + x_2 x_{2^{i_1}} \cdots x_{2^{i_h}} z_{2^i a - 2}$$

($i_1 > i+1$) with ${\varphi_2}^{2^{i+1}-1}$-torsion,

$$x_{2^{i_1-1}}^2 x_{2^{i_2}} \cdots x_{2^{i_h}} z_{2^i a}$$

($i_1 \leq i+1$), with $\varphi_2^{2^{i_1-1}}$-torsion. We have the relations

$$\varphi_2(x_1^2 z_{2k-1} + x_1 x_2 z_{2k-2}) = 0.$$

Case $p = 2$**,** $m > 2$**,** $2 \nmid m$

$$\beta x_{m2^i} = \varphi_m^{2^{i-1}-1} \widetilde{x}_{m2^{i-1}}^2,$$

if $2 \nmid a$ (let $z_0 = 1$)

$$\pi_* \beta z_{am} = x_m z_{(a-1)m},$$

for $i > 0$

$$\beta z_{2^i am} = \varphi_m^{2^{i+1}-2} (\widetilde{x}_m \widetilde{z}_{(2^i a-1)m} + \widetilde{x}_{2m} \widetilde{z}_{(2^i a-2)m}).$$

Proposition 5.2.7 ($p = 2$, $m > 2$, $2 \nmid m$). *The module $H_*(C(*), A_{(\varphi_m)})$ over the homology ring of braids is generated by the terms:*

$$1,$$

$$x_m x_{m2^{i_1}} \cdots x_{m2^{i_h}} z_{2^i am}$$

($i_1 > 1$, $i \geq 1$, $2 \nmid a$) with φ_m-torsion,

$$x_m x_{m2^{i_1}} \cdots x_{m2^{i_h}} z_{(2^i a-1)m} + x_{2m} x_{m2^{i_1}} \cdots x_{m2^{i_h}} z_{(2^i a-2)m}$$

($i_1 > i+1$, $i > 0$, $2 \nmid a$) with $\varphi_m^{2^{i+1}-1}$-torsion

$$x_{m2^{i_1-1}}^2 x_{m2^{i_2}} \cdots x_{m2^{i_h}} z_{2^i am}$$

($i \geq i_1 - 1$, $i_1 > 0$) with $\varphi_m^{2^{i_1-1}}$-torsion. The action of the elements x_1, y_m of the homology ring of braids is trivial (i. e. multiplication by these elements maps to zero).

Case $p > 2$**,** $m = 2$

$$\pi_* \beta x_2 = x_1^2$$

and for $i > 0$

$$\beta x_{2p^i} = \varphi_2^{(p-1)p^{i-1}-1} \widetilde{y}_{2p^i},$$

if $p \nmid a$, $i > 0$ (let $z_0 = 1$)

$$\beta z_{ap^i} = \varphi_2^{p^i-2}(\widetilde{x}_1 \widetilde{z}_{ap^i-1} + \widetilde{x}_2 \widetilde{z}_{ap^i-2}),$$

$$\pi_* \beta z_{ap^i+1} = x_1 z_{ap^i},$$

$$\pi_* \beta z_{ap^i+b} = x_1 z_{ap^i+b-1} + x_2 z_{ap^i+b-2}$$

if $b = 2, \ldots, p-1$.

Proposition 5.2.8 ($p > 2$, $m = 2$)**.** *The module $H_*(C(*), A_{(\varphi_2)})$ over the homology ring of braids is generated by the terms (let $p \nmid a$):*

$$1,$$

$$x_1 x_{2p^{i_1}} \cdots x_{2p^{i_h}} z_{p^i a-1} + x_2 x_{2p^{i_1}} \cdots x_{2p^{i_h}} z_{p^i a-2}$$

($i_1 > i$) with $\varphi_2^{p^i}$-torsion,

$$x_1 x_{2p^{i_1}} \cdots x_{2p^{i_h}} z_{p^i a}$$

with φ_2-torsion,

$$x_1 x_{2p^{i_1}} \cdots x_{2p^{i_h}} z_{p^i a+b-1} + x_2 x_{2p^{i_1}} \cdots x_{2p^{i_h}} z_{p^i a+b-2}$$

($b = 2, \ldots, p-1$) with φ_2-torsion,

$$y_{2p^{i_1}} x_{2p^{i_2}} \cdots x_{2p^{i_h}} z_{p^i a}$$

($i \leq i_1$) with $\varphi_2^{(p-1)p^{i-1}}$-torsion. Moreover we have the relations:

$$\varphi_2(x_1^2 z_{ap^i-1} + x_1 x_2 z_{ap^i-2}) = 0$$

for $b = 1, \ldots, p-1$.

Case $p > 2$**,** $m > 2$**,** $2 \mid m$**,** $p \nmid m$

This is the last non-trivial case that we have to investigate.

$$\beta y_{mp^i} = 0,$$

$$\beta x_{mp^i} = \varphi_m^{(p-1)p^{i-1}-1} \widetilde{y}_{mp^i},$$

if $p \nmid a$

$$\beta z_{ap^i \frac{m}{2}} = \varphi_m^{p^i-1} \widetilde{x}_1 \widetilde{z}_{ap^i \frac{m}{2}-1},$$

$$\beta x_1 z_{ap^i \frac{m}{2}-1} = 0.$$

Proposition 5.2.9 $(p>2, m>2, 2 \mid m, p \nmid m)$**.** *The module $H_*(C(*),A_{(\varphi_m)})$ over the homology ring of braids is generated by the terms (let $p \nmid a$):*

$$1,$$

$$y_{mp^{i_1}} x_{mp^{i_2}} \cdots x_{mp^{i_h}} z_{ap^i \frac{m}{2}}$$

$(i_1 < i)$ with $\varphi_m^{(p-1)p^{i_1-1}}$-torsion,

$$x_1 x_{mp^{i_1}} \cdots x_{mp^{i_h}} z_{ap^i \frac{m}{2}-1}$$

$(i \leq i_1)$ with $\varphi_m^{p^i}$-torsion. The action of the element x_1 of the homology ring of braids is trivial (i. e. multiplication maps to zero)

Bockstein homomorphism

In order to compute the homology with coefficients in the ring $R = \mathbb{Z}[q^{\pm 1}]$ we need also to compute the Bockstein operator associated to the short exact sequence

$$0 \to \mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0.$$

The Bockstein operator β_2 acts as follows:

$$\beta_2 x_1 = 0,$$

$$\beta_2 x_{2^j}^2 = 0,$$

$$\beta_2 x_1 z_{2k} = 0,$$

$$\beta_2 (x_1 z_{2k-1} + x_2 z_{2k-2}) = 0,$$

$$\beta_2 x_1 x_{2^{i_1}} \cdots x_{2^{i_h}} z_{2k} = \sum_{j=1}^{h} \left(\alpha_j x_1 x_{2^{i_j-1}}^2 z_{2k} \prod_{t \neq j} x_{2^{i_t}} \right),$$

$$\beta_2 x_{2^{i_1}} \cdots x_{2^{i_h}} (x_1 z_{2^i a-1} + x_2 z_{2^i a-2}) =$$

$$= \sum_{j=1}^{h} \left(\alpha_j x_{2^{i_j-1}}^2 (x_1 z_{2^i a-1} + x_2 z_{2^i a-2}) \prod_{t \neq j} x_{2^{i_t}} \right)$$

with $2 \nmid a$ $(i_1 > 1)$,

$$\beta_2 x_{2^{i_1}-1}^2 x_{2^{i_2}} \cdots x_{2^{i_h}} z_{2^i a} =$$

$$= \sum_{j=2}^{h} \alpha_j x_{2^{i_1}-1}^2 x_{2^{i_j}-1}^2 \prod_{t \neq j} x_{2^{i_t}} z_{2^i a}$$

$(i_1 \leq i+1)$ where the coefficients α_j in the last equations are invertible modulo φ_2.

$$\beta_2 x_m = 0,$$

$$\beta_2 x_{m2^i}^2 = 0,$$

$$\beta_2 x_m z_{2^i am} = 0,$$

$$\beta_2 (x_m z_{(2^i a-1)m} + x_{2m} z_{(2^i a-2)m}) = 0,$$

$$\beta_2 x_m x_{m2^{i_1}} \cdots x_{m2^{i_h}} z_{2^i am} = \sum_{j=1}^{h} \left(\alpha_j x_m x_{m2^{i_j-1}}^2 z_{2^i am} \prod_{t \neq j} x_{m2^{i_t}} \right)$$

with $i_1 > 1$, $i \geq 1$, $2 \nmid a$,

$$\beta_2 x_{m2^{i_1}} \cdots x_{m2^{i_h}} (x_m z_{(2^i a-1)m} + x_{2m} z_{(2^i a-2)m}) =$$

$$= \sum_{j=1}^{h} \left(\alpha_j x_{m2^{i_j-1}}^2 (x_m z_{(2^i a-1)m} + x_{2m} z_{(2^i a-2)m}) \prod_{t \neq j} x_{2^{i_t}} \right)$$

with $i_1 > i+1$, $i > 0$, $2 \nmid a$,

$$\beta_2 x_{m2^{i_1-1}}^2 x_{m2^{i_2}} \cdots x_{m2^{i_h}} z_{2^i am} =$$

$$= \sum_{j=2}^{h} \alpha_j x_{m2^{i_1-1}}^2 x_{m2^{i_j-1}}^2 \prod_{t \neq j} x_{m2^{i_t}} z_{2^i am}$$

with $i \geq i_1 - 1$, $i_1 > 0$, where the coefficients α_j are invertible modulo φ_m. The computations for the Bockstein operator β_p give:

$$\beta_p x_1 = 0,$$

$$\beta_p y_{2p^i} = 0,$$

$$\beta_p (x_1 z_{a-1}) = 0,$$

$$\beta_p x_1 x_{2p^{i_1}} \cdots x_{2p^{i_h}} z_{p^i a-1} + x_2 x_{2p^{i_1}} \cdots x_{2p^{i_h}} z_{p^i a-2} =$$

$$= \sum_{j=1}^{h} \alpha_j \left(x_1 z_{p^i a-1} + x_2 z_{p^i a-2} \right) y_{2p^j} \prod_{t \neq j} x_{2p^t},$$

$$\beta_p x_1 x_{2p^{i_1}} \cdots x_{2p^{i_h}} z_{p^i a} =$$

$$= \sum_{j=1}^{h} \alpha_p x_1 y_{2p^j} \prod_{t \neq j} x_{2p^t} z_{p^i a}$$

$$\beta_p x_1 x_{2p^{i_1}} \cdots x_{2p^{i_h}} z_{p^i a+b-1} + x_2 x_{2p^{i_1}} \cdots x_{2p^{i_h}} z_{p^i a+b-2} =$$

$$= \sum_{j=1}^{h} \alpha_j \left(x_1 z_{p^i a+b-1} + x_2 z_{p^i a+b-2} \right) y_{2p^j} \prod_{t \neq j} x_{2p^t},$$

$$\beta_p y_{2p^{i_1}} x_{2p^{i_2}} \cdots x_{2p^{i_h}} z_{p^i a} =$$

$$= y_{2p^{i_1}} \sum_{j=2}^{h} \alpha_j y_{2p^j} \prod_{t \neq j} x_{2p^{i_t}} z_{p^i a}$$

where the coefficients α_j are invertible modulo φ_m.

$$\beta_p y_{mp^i} = 0,$$

$$\beta_p y_{mp^{i_1}} x_{mp^{i_2}} \cdots x_{mp^{i_h}} z_{ap^i \frac{m}{2}} =$$

$$= y_{mp^{i_1}} \sum_{j=2}^{h} \alpha_j y_{mp^j} \prod_{t \neq j} x_{mp^t} z_{ap^i \frac{m}{2}}$$

$(i_1 < i)$,

$$\beta_p x_1 x_{mp^{i_1}} \cdots x_{mp^{i_h}} z_{ap^i \frac{m}{2} - 1} =$$

$$= x_1 \sum_{j=1}^{h} \alpha_j y_{mp^j} \prod_{t \neq j} x_{mp^t} z_{ap^i \frac{m}{2} - 1}$$

$(i \leq i_1)$. The same argument used in Lemma 4.3.6 for braid groups homology gives:

Lemma 5.2.10. *The homology groups $H_*(C(*), \mathbb{Z}[q^{\pm 1}])$ have no p^2-torsion for any prime p.* □

5.3 Main Theorem

Finally we can state the result:

Theorem 5.3.1. *Set* $\deg x_i = \deg y_i = i, \dim x_i = i - 1$, $\dim y_i = i - 2$, $\deg z_i = \dim z_i = i$. *Let $C(n)$ be the algebraic complex that compute the homology of the Artin group G_{B_n}. Let*

$$C(*) = \bigoplus_n C(n).$$

We have the following description for the homology of $C(*)$: the module $H_*(C(*), \mathbb{Q}[q^{\pm 1}])$ is generated by the terms

$$< 1, x_1 z_i + z_2 z_{i-1}, x_1 z_{km-1}, y_{2m} z_{km} + x_1 x_{2m} z_{km-1} >$$

with relations

$$\varphi_2 z_i = \varphi_{2m} z_{km} = \varphi_{2m} z_{km-1} = 0$$

over the ring

$$H_*(\mathrm{Br}(*), \mathbb{Q}[q^{\pm 1}]) \simeq \mathbb{Q}[q^{\pm 1}] \left[\ x_1, y_m, m > 2\ \right] / \left(\ \varphi_2 x_1^2, \varphi_m y_m\ \right).$$

For $p = 2$:

$$H_*(C(*), \mathbb{Z}_2[q\text{`}pmu) = H_*(C(*), A_{(\varphi_2)}) \oplus \bigoplus_{2\nmid m} H_*(C(*), A_{(\varphi_m)}).$$

For $p > 2$:

$$H_*(C(*), \mathbb{Z}_2[q\text{`}pmu) = H_*(C(*), A_{(\varphi_2)}) \oplus \bigoplus_{\substack{2|m \\ p\nmid m}} H_*(C(*), A_{(\varphi_m)}).$$

The module $H_*(C(*), \mathbb{Z}[q^{\pm 1}])$ is the direct sum of the free $\mathbb{Z}$-module generated by the terms

$$< 1, x_1 z_i + z_2 z_{i-1}, x_1 z_{km-1}, y_{2m} z_{km} + x_1 x_{2m} z_{km-1} >$$

with relations

$$\varphi_2 z_i = \varphi_{2m} z_{km} = \varphi_{2m} z_{km-1} = 0$$

over the ring

$$H_*(\mathrm{Br}(*), \mathbb{Z}[q^{\pm 1}]) \simeq \mathbb{Z}[q^{\pm 1}] \left[\ x_1, y_m, m > 2\ \right] / \left(\ \varphi_2 x_1^2, \varphi_m y_m\ \right).$$

and of the sum of torsion modules

$$\bigoplus_p \beta_p H_*(C(*), \mathbb{Z}_p[q^{\pm 1}]).$$

□

We can consider the embedding $j_n : G_{B_n} \hookrightarrow G_{B_{n+1}}$. We define the group G_{B_∞} as the limit group $\varinjlim_n G_{B_n}$. The maps $j_{n\#}$ induced in homology give a description of the homology of the limit group G_{B_∞} as the limit $\varinjlim_n H_*(G_{B_n})$, that is the stable homology of G_{B_n}. We obtain the following result.

Theorem 5.3.2. *Set $\mathfrak{M}_p$ (resp. $\mathfrak{M}_\mathbb{Z}$) be the graduated K-module given by an infinite direct sum of a copy of K (resp. a copy of $\mathbb{Z}$) for each nonnegative dimension ($p = \mathrm{char} K$). We have:*

$$H_*(G_{B_\infty}, \mathbb{Q}[q^{\pm 1}]) = \mathfrak{M}_0;$$

for $p = 2$

$$H_*(G_{B_\infty}, K[q^{\pm 1}]) = \mathbb{Z}_2[x_2^2, x_{2^i}, i > 1] \otimes \mathfrak{M}_2;$$

for a prime $p > 2$

$$H_*(G_{B_\infty}, K[q^{\pm 1}]) = \mathbb{Z}_p[y_{2p^i}, x_{2p^i}, i > 0]/(x_{2p^i}^2) \otimes \mathfrak{M}_p.$$

The Bockstein operator acts as follows:

$$\beta_2 x_{2^i} = x_{2^{i-1}}^2;$$

$$\beta_p y_i = 0; \quad \beta_p x_i = y_i \qquad (\text{for } p > 2),$$

$$\beta_p \mathfrak{M}_p = 0 \qquad \forall p \text{ prime.}$$

The homology $H_*(G_{B_\infty}; \mathbb{Z}[q^{\pm 1}])$ has no p^2-torsion for any prime p. A presentation of $H_*(G_{B_\infty}; \mathbb{Z}[q^{\pm 1}])$ is given by

$$\mathbb{Z}\left[\begin{array}{c} y_{2p^i}, x_{2^j}^2, \\ x_{2^i}^2 x_{2^{i_1}} \cdots x_{2^{i_h}} \\ y_{2p^j} x_{2p^{j_1}} \cdots x_{2p^{j_h}} \end{array}\right] \Big/ \Big(\; 2x_{2^i}, py_{2p^j}, x_{2p^j}^2 \; \Big) \otimes \mathfrak{M}_{\mathbb{Z}}$$

with indexes running as follows: $0 < i, i+1 < i_1 < \cdots < i_h$, $0 < j < j_1 < \cdots < j_h$ *and p in the set of odd primes. The structure of $\mathbb{Z}[q^{\pm 1}]$-module is trivial and so the action of q corresponds to multiplication by -1.*

Moreover the homology of G_{B_∞} has a natural structure of module over the homology ring of $\mathrm{Br}(\infty)$. □

Chapter 6

The case of the affine arrangements of type $\widetilde{A}_n$

6.1 Introduction

In this Chapter we give a detailed calculation of the cohomology of some Artin groups with non-trivial local coefficients. Let $R := \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ be the ring of two-parameters Laurent polynomials. The main result (Theorem 6.1.1) is the cohomology of the Artin group G_{B_n} (of type B_n) with coefficients in the module $R_{q,t}$. The latter is the ring R with the module structure defined as follows: the generators associated to the first $n-1$ nodes of the Coxeter diagram of B_n act by $(-q)-$multiplication; the one associated to the last node acts by $(-t)-$multiplication.

Let $\varphi_m(q)$ be the m-th cyclotomic polynomial in the variable q. Define the R-modules $(m > 1,\ i \geq 0)$

$$\{m\}_i = R/(\varphi_m(q), q^i t + 1).$$

and for $m = 1$ set:

$$\{1\}_i = R/(q^i t + 1).$$

Notice that the modules $\{m\}_i$ are all non isomorphic as R-modules. $\{m\}_i$ and $\{m'\}_{i'}$ are isomorphic as $\mathbb{Q}[q^{\pm 1}]$-modules if and only if $m = m'$ and are isomorphic as $\mathbb{Q}[t^{\pm 1}]$-modules if and only if $\phi(m) = \phi(m')$ (ϕ is the Euler function) and $\frac{m}{(m,i)} = \frac{m'}{(m,i')}$.

Our main result is the following

Theorem 6.1.1.

$$H^i(G_{B_n}, R_{q,t}) = \begin{cases} \bigoplus_{d \mid n,\, 0 \leq k \leq d-2} \{d\}_k \oplus \{1\}_{n-1} & \textit{if } i = n \\ \bigoplus_{d \mid n,\, 0 \leq k \leq d-2,\, d \leq \frac{n}{j+1}} \{d\}_k & \textit{if } i = n-2j \\ \bigoplus_{d \nmid n,\, d \leq \frac{n}{j+1}} \{d\}_{n-1} & \textit{if } i = n-2j-1. \end{cases}$$

□

Notice also the geometrical meaning of the two-parameters cohomology of G_{B_n} : similar to the one-parameter case, it is equivalent to the trivial cohomology of the "homotopy-Milnor fibre" associated to the natural map of the orbit space onto a two-dimensional torus.

6.2 Inclusions of Artin groups

In this Section, we are primarily interested in Artin groups associated to Coxeter graph of type A_n, B_n and $\tilde{A}_{n-1}$ (see Figure 6.1).

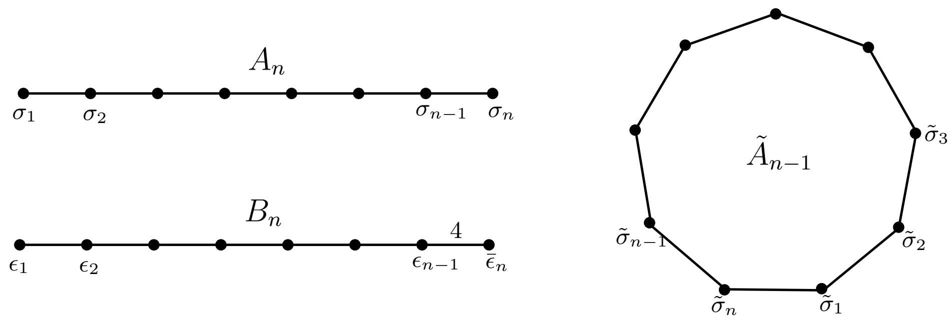


Figure 6.1: Coxeter graph of type A_n, B_n ($n \geq 2$) and $\tilde{A}_{n-1}$ ($n \geq 3$). Labels equal to 3, as usual, are not shown. Moreover, to fix notation, every vertex is labelled with the corresponding generator in the Artin group.

Let $\mathrm{Br}_{n+1} := G_{A_n}$ be the braid group on $n+1$ strands and $\mathrm{Br}_{n+1}^{n+1} < \mathrm{Br}_{n+1}$ be the subgroup of braids fixing the $(n+1)$-st strand. The group Br_{n+1}^{n+1} is called the annular braid group. Let $K_{n+1} = \{p_1, \ldots, p_{n+1}\}$ be a set of $n+1$ distinct points in $\mathbb{C}$. The classical braid group $\mathrm{Br}_{n+1} = G_{A_n}$ can be realized as the fundamental group of the space of unordered configurations of $n+1$ points in $\mathbb{C}$ with base point K_{n+1} (see the left part of Figure 6.2), with $K_6 = \{1, \ldots, 6\}$). We can now think to the subgroup $\mathrm{Br}_{n+1}^{n+1} < \mathrm{Br}_{n+1}$ as the fundamental group of the space of unordered configurations of n points in $\mathbb{C}^*$: in fact if we take $p_{n+1} = 0$ and $p_i \in S^1 \subset \mathbb{C}$ for $i \in 1, \ldots, n$, since for a braid $\beta \in \mathrm{Br}_{n+1}^{n+1}$ the orbit of the $(n+1)$-st point can be thought constant, up to homotopy, we can think to β as a braid with n strands in the annulus (see the right part of Figure 6.2).

It is well known that the annular braid group is isomorphic to the Artin group G_{B_n} of type B_n. For a proof of the following Theorem see [Cri99] or [Lam94].

Theorem 6.2.1. *Let $\sigma_1, \ldots, \sigma_n$ and $\epsilon_1, \ldots, \epsilon_{n-1}, \bar{\epsilon}_n$ be respectively the stan-*

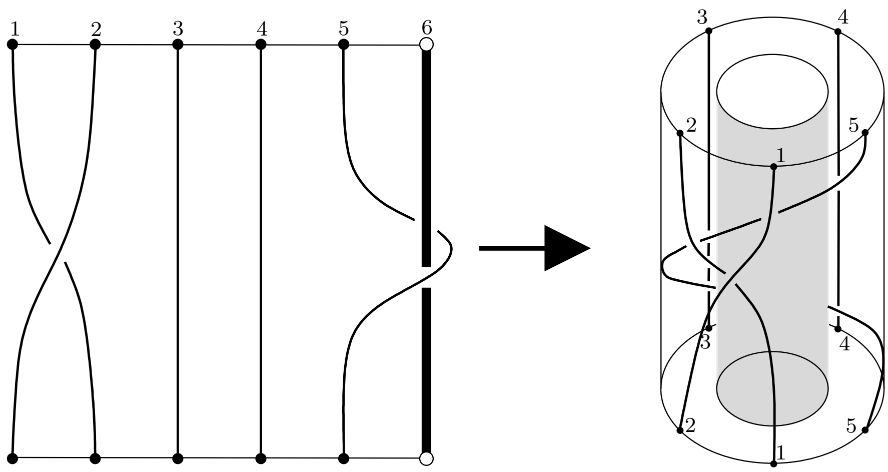


Figure 6.2: A braid in Br_6^6 represented as an annular braid on 5 strands.

dard generators for G_{A_n} and G_{B_n}. Then, the map

$$\begin{aligned} G_{B_n} &\to \mathrm{Br}_{n+1}^{n+1} < \mathrm{Br}_{n+1} \\ \epsilon_i &\mapsto \sigma_i \qquad \textit{for } 1 \le i \le n-1 \\ \bar{\epsilon}_n &\mapsto \sigma_n^2 \end{aligned}$$

is an isomorphism. □

Using the suggestion given by the identification with the annular braid group, a new interesting presentation for G_{B_n} can be worked out. Let $\tau = \bar{\epsilon}_n \epsilon_{n-1} \cdots \epsilon_2 \epsilon_1$. It is easy to verify that:

$$\tau^{-1} \epsilon_i \tau = \epsilon_{i+1} \quad \text{for } 1 \le i < n-1$$

i.e. conjugation by τ shifts forward the first $n-2$ standard generators. By analogy, let $\epsilon_n = \tau^{-1} \epsilon_{n-1} \tau$. We have the following

Theorem 6.2.2 ([CP03]). *The group G_{B_n} has presentation $\langle \mathcal{G} | \mathcal{R} \rangle$ where*

$$\begin{aligned} \mathcal{G} =& \{\tau, \epsilon_1, \epsilon_2, \ldots, \epsilon_n\} \\ \mathcal{R} =& \{\epsilon_i \epsilon_j = \epsilon_j \epsilon_i \quad \textit{for } i \ne j-1, j+1\} \cup \\ & \{\epsilon_i \epsilon_{i+1} \epsilon_i = \epsilon_{i+1} \epsilon_i \epsilon_{i+1}\} \cup \\ & \{\tau^{-1} \epsilon_i \tau = \epsilon_{i+1}\} \end{aligned}$$

where are all indexes should be considered modulo n. □

Letting $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_n$ be the standard generator of the Artin group of type $\tilde{A}_{n-1}$, we have the following straightforward corollary:

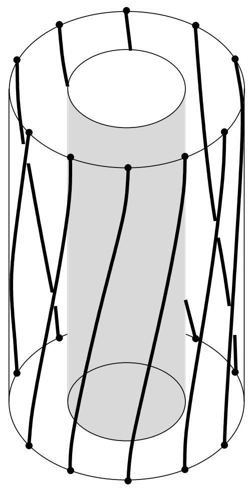


Figure 6.3: As an annular braid the element τ is obtained turning the bottom annulus by a rotation of $2\pi/n$.

Corollary 6.2.3 ([CP03]). *The map*

$$G_{\tilde{A}_{n-1}} \ni \tilde{\sigma}_i \mapsto \epsilon_i \in G_{B_n}$$

gives an isomorphism between the group $G_{\tilde{A}_{n-1}}$ and the subgroup of G_{B_n} generated by $\epsilon_1, \ldots, \epsilon_n$. Moreover, we have a semidirect product decomposition $G_{B_n} \simeq G_{\tilde{A}_{n-1}} \rtimes \langle\tau\rangle$. □

We have thus a "curious" inclusion of the Artin group of infinite type $\tilde{A}_{n-1}$ into the Artin group of finite type B_n.

Remark 6.2.4. The proof of Theorem 6.2.2 presented in the original paper is algebraic and based on Tietze moves; a somewhat more concise proof can however be obtained by standard topological constructions. Indeed, one can exhibit an explicit infinite cyclic covering $K(G_{\tilde{A}_{n-1}}, 1) \to K(G_{B_n}, 1)$ (see [All02]).

6.3 The cohomology of G_{B_n}

6.3.1 Proof of the Main Theorem

In this Section we prove Theorem 6.1.1 enunciated in the introduction. We use the notations given in the Introduction.

To perform our computation we will use the algebraic Salvetti complex introduced in 3.1 and the spectral sequence induced by a natural filtration.

The complex that computes the cohomology of G_{B_n} over $R_{q,t}$ is given as follows (see [Sal94]):

$$C_n^* = \bigoplus_{\Gamma \subset I_n} R.\Gamma$$

where I_n denote the set $\{1, \dots, n\}$ and the graduation is given by $\mid \Gamma \mid$.

The set I_n corresponds to the set of nodes of the Coxeter diagram of B_n and in particular the last element, n, corresponds to the last node.

It is useful to consider also the complex $\overline{C}_n^*$ for the cohomology of G_{A_n} on the local system $R_{q,t}$. In this case the action associated to a standard generator is always the $(-q)$-multiplication and so the complex $\overline{C}_n^*$ and its cohomology are free as $\mathbb{Q}[t^{\pm}]$-modules. The complex $\overline{C}_n^*$ is isomorphic to C_n^* as a R-module. In both complexes the coboundary map is

$$\delta(q,t)(\Gamma) = \sum_{j \in I_n \setminus \Gamma} (-1)^{\sigma(j,\Gamma)} \frac{W_{\Gamma \cup \{j\}}(q,t)}{W_\Gamma(q,t)} (\Gamma \cup \{j\}) \tag{6.3.1}$$

where $\sigma(j, \Gamma)$ is the number of elements of Γ that are less than j. In the case A_n $W_\Gamma(q,t)$ is the Poincaré polynomial of the parabolic subgroup $W_\Gamma \subset A_n$ generated by the elements in the set Γ, with weight $-q$ for each standard generator, while in the case B_n $W_\Gamma(q,t)$ is the Poincaré polynomial of the parabolic subgroup $W_\Gamma \subset B_n$ generated by the elements in the set Γ, with weight $-q$ for the first $n-1$ generators and $-t$ for the last generator.

Using Proposition 1.5.2 we can give an explicit computation of the coefficients $\frac{W_{\Gamma \cup \{j\}}(q,t)}{W_\Gamma(q,t)}$. For any $\Gamma \subset I_n$, let $\overline{\Gamma}$ be the subgraph of the Coxeter diagram B_n which is spanned by Γ. Recall that if $\overline{\Gamma}$ is a connected component of the Coxeter diagram of B_n without the last element, then

$$W_\Gamma(q,t) = [m+1]_q!,$$

where $m =\mid \Gamma \mid$. If $\overline{\Gamma}$ is connected and contains the last element of B_n, then

$$W_\Gamma(q,t) = [2m]_{q,t}!!,$$

where $m =\mid \Gamma \mid$.

If $\overline{\Gamma}$ is the union of several connected components of the Coxeter diagram, $\overline{\Gamma} = \overline{\Gamma}_1 \cup \cdots \cup \overline{\Gamma}_k$, then $W_\Gamma(q,t)$ is the product

$$\prod_{i=1}^{k} W_{\Gamma_i}(q,t)$$

of the factors corresponding to the different components.

If $j \notin \Gamma$ we can write $\overline{\Gamma}(j)$ for the connected component of $\overline{\Gamma \cup \{j\}}$ containing j. Suppose that $m =\mid \Gamma(j) \mid$ and i is the number of elements in $\Gamma(j)$ greater than j. Then, if $n \in \Gamma(j)$ we have

$$\frac{W_{\Gamma \cup \{j\}}(q,t)}{W_\Gamma(q,t)} = \begin{bmatrix} m \\ i \end{bmatrix}'_{q,t}$$

and

$$\frac{W_{\Gamma\cup\{j\}}(q,t)}{W_\Gamma(q,t)} = \begin{bmatrix} m+1 \\ i+1 \end{bmatrix}_q$$

otherwise.

It is convenient to represent generators $\Gamma \subset I_n$ by their characteristic functions $I_n \to \{0,1\}$ so, simply by strings of 0s and 1s of length n.

We define a decreasing filtration F on the complex (C_n^*, δ): F^sC_n is the subcomplex generated by the strings of type $A1^s$ (ending with a string of s 1's) and we have the inclusions

$$C_n = F^0C_n \supset F^1C_n \supset \cdots \supset F^nC_n = R.1^n \supset F^{n+1}C_n = 0.$$

We have the following isomorphism of complexes:

$$(F^sC_n/F^{s+1}C_n) \simeq \overline{C}_{n-s-1}[s] \tag{6.3.2}$$

where $\overline{C}_{n-s-1}$ is the complex for $G_{A_{n-s-1}}$ and the notation $[s]$ means that the degree is shifted by s. Let E_* be the spectral sequence associated to the filtration F. The equality 6.3.2 tells us how the E_1 term of the spectral sequence looks like. In fact for $0 \leq s \leq n-2$ we have

$$E_1^{s,r} = H^r(G_{A_{n-s-1}}, R_{q,t}) = H^r(G_{A_{n-s-1}}, \mathbb{Q}[q^{\pm 1}]_q)[t^{\pm 1}] \tag{6.3.3}$$

since the t-action is trivial. For $s = n-1$ and $s = n$ the only non trivial elements in the spectral sequence are

$$E_1^{n-1,0} = E_1^{n,0} = R. \tag{6.3.4}$$

In order to prove Theorem 6.1.1 we need to state the following lemmas.

Lemma 6.3.1. *Let $I(n,k)$ be the ideal generated by the polynomials*

$$\begin{bmatrix} n \\ n-d \end{bmatrix}_{q,t}' \qquad \textit{for } d \mid n \textit{ and } d \leq k$$

If $k \mid n$ the map

$$\alpha_{n,k} : R/(\varphi_k(q)) \to R/I(n,k-1)$$

induced by the multiplication by $\begin{bmatrix} n \\ n-k \end{bmatrix}_{q,t}'$ is well defined and is injective.

Remark. The fact that this map is well defined will follow automatically from the general theory of spectral sequences, as it is clear from the proof of Theorem 6.1.1. However, below we prove it by other means.

Proof. Let d, k be positive integers such that $d \mid n$ and $k \mid n$. We can observe that $\varphi_d(q) \mid \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$ if and only if $d \nmid k$. Moreover each factor φ_d appears in $\begin{bmatrix} n \\ k \end{bmatrix}_q$ at most with exponent 1.

Let $J(n,k)$ be the ideal generated by the polynomials $\begin{bmatrix} n \\ n-d \end{bmatrix}_q$ for $d \mid n$ and $d \leq k$. It is easy to see that we have the following inclusion:

$$\prod_{i=n-k}^{n-1} (1+tq^i) J(n,k) \subset I(n,k).$$

Moreover $J(n,k)$ is a principal ideal and is generated by the product

$$p_{n,k}(q) = \prod_{d \mid n, k<d} \varphi_d(q).$$

It follows that

$$\begin{bmatrix} n \\ n-k \end{bmatrix}_q \varphi_k(q) \in J(n, k-1)$$

and so

$$\begin{bmatrix} n \\ n-k \end{bmatrix}_{q,t}^{\prime} \varphi_k(q) \in I(n, k-1).$$

This proves that the map $\alpha_{n,k}$ is well defined.

Now we notice that the factor $\varphi_k(q)$ divides each generator of $I(n, k-1)$, but does not divide $\begin{bmatrix} n \\ n-k \end{bmatrix}_{q,t}^{\prime}$. This imply that $\alpha_{n,k}$ is not the zero map and that every polynomial in the kernel of $\alpha_{n,k}$ must be a multiple of $\varphi_k(q)$, hence the map must be injective. □

Lemma 6.3.2. *Let $I(n)$ be the ideal generated by the polynomials*

$$\begin{bmatrix} n \\ n-d \end{bmatrix}_{q,t}^{\prime} \qquad \text{for } d \mid n$$

Then $I(n)$ is the direct product of the ideals $I_{i,d} = (\varphi_d(q), q^i t + 1)$ for $d \mid n$ and $0 \leq i \leq d-2$ and of the ideal $I_{n-1} = (q^{n-1}t + 1)$. Moreover the ideals $I_{i,d}$ and I_{n-1} are pairwise co-prime.

Proof: Notice that the polynomial $(1 + tq^{n-1})$ divides each generator of the ideal $I(n)$, so we can write

$$I(n) = (1 + tq^{n-1})\widetilde{I}(n)$$

where $\widetilde{I}(n)$ is the ideal generated by the polynomials

$$\widetilde{\begin{bmatrix} n \\ n-d \end{bmatrix}}'_{q,t} := \begin{bmatrix} n \\ n-d \end{bmatrix}'_{q,t} /(1+tq^{n-1})$$

Let $n = d_1 > \cdots > d_h = 1$ be the list of all the divisors of n in decreasing order. If we set

$$\begin{aligned} P_i :=& \varphi_{d_i}(q) \quad \text{and} \\ Q_i :=& \prod_{j=d_{i+1}+1}^{d_i} (1+tq^{n-j}) \end{aligned}$$

we can rewrite our ideal as

$$\begin{aligned} \widetilde{I}(n) = \Bigg(& \begin{bmatrix} n \\ n-d_h \end{bmatrix}, \begin{bmatrix} n \\ n-d_{h-1} \end{bmatrix} Q_{h-1}, \begin{bmatrix} n \\ n-d_{h-2} \end{bmatrix} Q_{h-2}Q_{h-1}, \ldots \\ & \ldots, \begin{bmatrix} n \\ n-d_2 \end{bmatrix} Q_2 \cdots Q_{h-1}, Q_1 \cdots Q_{h-1} \Bigg) \end{aligned} \tag{6.3.5}$$

We claim that we can reduce to the following set of generators:

$$\begin{aligned} \widetilde{I}(n) = (& P_1 \cdots P_{h-1}, P_1 \cdots P_{h-2}Q_{h-1}, P_1 \cdots P_{h-3}Q_{h-2}Q_{h-1} \ldots \\ & \ldots, P_1 Q_2 \cdots Q_{h-1}, Q_1 \cdots Q_{h-1}) \end{aligned} \tag{6.3.6}$$

The first generator is the same in both equations and the j-th generator in Equation (6.3.6) divides the corresponding generator in Equation (6.3.5). Now suppose that a factor $\varphi_m(q)$ divides $\begin{bmatrix} n \\ n-d_j \end{bmatrix}$ but does not divide $P_1 \cdots P_{j-1}$. We may distinguish two cases:

(i) Suppose that $m \nmid n$. Then we can get rid of the factor $\varphi_m(q)$ in $\begin{bmatrix} n \\ n-d_j \end{bmatrix}$ with an opportune combination with the polynomial

$$P_1 \cdots P_{h-1}$$

(ii) Suppose $m \mid n$. Then $m = d_l$ for some $l > j$ and we can get rid of $\varphi_m(q)$ using a suitable combination with the polynomial

$$P_1 \cdots P_{l-1} Q_l \cdots Q_{h-1}$$

We may now proceed inductively. Supposing we have already reduced the first $j-1$ terms, we can reduce the j-th term of the ideal in Equation (6.3.5) to the corresponding term in Equation (6.3.6).

Now we observe that if J, I_1, I_2 are ideals and $I_1 + I_2 = (1)$, then $(J, I_1 I_2) = (J, I_1)(J, I_2)$. Since the polynomials P_i are all co-prime, we can apply this fact to the ideal $\widetilde{I}(n)$ $h-2$ times. At the i-th step we set

$$I_1 = (P_i),$$

$$I_2 = (P_{i+1} \cdots P_{h-1}, P_{i+1} \cdots P_{h-2} Q_{h-1}, \dots, Q_{i+1} \cdots Q_{h-1}),$$

$$J = (Q_i \cdots Q_{h-1}).$$

So we can factor $\widetilde{I}(n)$ as

$$(P_1, Q_1 \cdots Q_{h-1})(P_2 \cdots P_{h-1}, P_2 \cdots P_{h-2} Q_{h-1}, Q_2 \cdots Q_{h-1}) = \cdots$$

$$\cdots = (P_1, Q_1 \cdots Q_{h-1})(P_2, Q_2 \cdots Q_{h-1}) \cdots (P_{h-1}, Q_{h-1}).$$

Finally we can split $(P_s, Q_s \cdots Q_{h-1})$ as the product

$$(P_s, 1 + tq^{n-d_s}) \cdots (P_s, 1 + tq^{n-d_h-1}).$$

So we have reduced the ideal $I(n)$ in the product stated in the Lemma and it is easy to check that all the ideals of the splitting are co-prime. $\square$

Proof of Theorem 6.1.1. We can now prove our Theorem using the spectral sequence described in the Equations (6.3.3) and (6.3.4).

We introduce, as in [DCPS01], the following notation for the generators of the spectral sequence:

$$\begin{aligned}
w_h &= 01^{h-2}0 \\
z_h &= 1^{h-1}0 + (-1)^h 01^{h-1} \\
b_h &= 01^{h-2} \\
c_h &= 1^{h-1} \\
z_h(i) &= \sum_{j=0}^{i-1} (-1)^{hj} w_h^j z_h w_h^{i-j-1} \\
v_h(i) &= \sum_{j=0}^{i-2} (-1)^{hj} w_h^j z_h w_h^{i-j-2} b_h + (-1)^{h(i-1)} w_h^{i-1} c_h
\end{aligned}$$

We write $\{m\}[t^{\pm 1}]$ for the module $R/(\varphi_m(q))$. The E_1-term of the spectral sequence has a module $\{m\}[t^{\pm 1}]$ in position (s, r) if and only if one of the following condition is satisfied:

(a) $m \mid n - s - 1$ and $r = n - s - 2\frac{n-s-1}{m}$;

(b) $m \mid n - s$ and $r = n - s + 1 - 2(\frac{n-s}{m})$.

Moreover we have modules R in position $(n-1,0)$ and $(n,0)$. We now look at the d_1 map between these two modules. Notice that $E_1^{n-1,0}$ is generated by the string 01^{n-1} and $E_1^{n,0}$ is generated by the string 1^n. Furthermore the map

$$d_1^{n-1,0} : E_1^{n-1,0} \to E_1^{n,0}$$

is given by the multiplication by $\begin{bmatrix} n \\ n-1 \end{bmatrix}'_{q,t} = [n]_q(1+tq^{n-1})$ and is injective. It turns out that $E_2^{n-1,0} = 0$ and $E_2^{n,0} = R/([n]_q(1+tq^{n-1}))$. Moreover all the following terms $E_j^{n,0}$ are quotient of $E_2^{n,0}$.

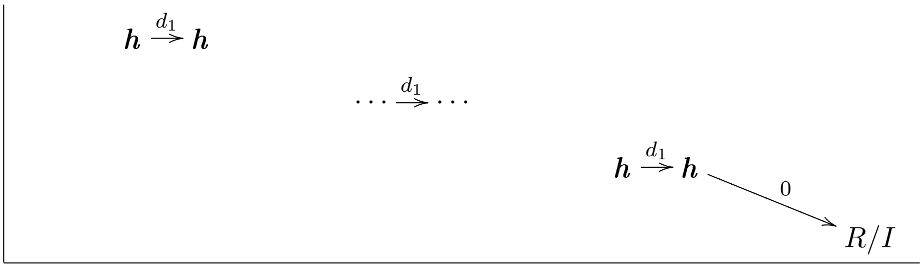
Notice that every map between modules of kind $\{m\}[t^{\pm 1}]$ and $\{m'\}[t^{\pm 1}]$ must be zero if $m \neq m'$. So we can study our spectral sequence considering only maps between the same kind of modules.

First let us consider an integer m that doesn't divide n. Say that $m \mid n+c$ with $1 \leq c < m$ and set $i = \frac{n+c}{m}$. The modules of type $\{m\}[t^{\pm 1}]$ are:

$$\begin{array}{ll} E_1^{\lambda m-c-1,n+c-\lambda(m-2)-2i+1} & \text{generated by } z_m(i-\lambda)01^{\lambda m-c-1} \\ E_1^{\lambda m-c,n+c-\lambda(m-2)-2i+1} & \text{generated by } v_m(i-\lambda)01^{\lambda m-c} \end{array}$$

for $\lambda = 1, \ldots, i-1$.

Here is a diagram for this case (we use the notation $\boldsymbol{h}$ for $\{m\}[t^{\pm 1}]$):



The map

$$d_1 : E_1^{\lambda m-c-1,n+c-\lambda(m-2)-2i+1} \to E_1^{\lambda m-c,n+c-\lambda(m-2)-2i+1}$$

is given by the multiplication by $\begin{bmatrix} \lambda m-c \\ \lambda m-c-1 \end{bmatrix}'_{q,t} = [\lambda m-c]_q(1+tq^{\lambda m-c-1})$. Since $\varphi_m(q) \nmid [\lambda m-c]_q$ the map is injective and in the E_2-term we have:

$$\begin{array}{lcl} E_2^{\lambda m-c-1,n+c-\lambda(m-2)-2i+1} & = & 0 \\ E_2^{\lambda m-c,n+c-\lambda(m-2)-2i+1} & = & \{m\}_{\lambda m-c-1} = \{m\}_{m-c-1} \end{array}$$

for $\lambda = 1, \ldots, i-1$.

The other map we have to consider is

$$d_m^{n-m,m-1} : E_m^{n-m,m-1} \to E_m^{n,0}.$$

The module $E_m^{n-m,m-1} = \{m\}_{m-c-1}$ is generated by $1^{m-1}01^{n-m}$ and so the map is the multiplication by $\begin{bmatrix} n \\ n-m \end{bmatrix}'_{q,t}$. Since $(1+tq^{n-1})$ divides the coefficient $\begin{bmatrix} n \\ n-m \end{bmatrix}'_{q,t}$, the image of the map $d_m^{n-m,m-1}$ must be contained in the submodule

$$(1+tq^{n-1})E_m^{n,0} = (1+tq^{n-1})R/([n]_q(1+tq^{n-1}))$$

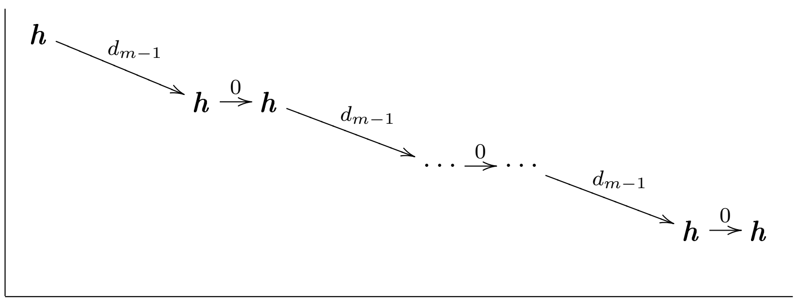
that is in the quotient $R/([n]_q)$. Since $(\varphi_m(q),[n]_q) = (1)$ (recall that m does not divide n) there can be no nontrivial map between the modules $\{m\}_{m-c-1}$ and $R/([n]_q)$. It follows that the differential $d_m^{n-m,m-1}$ must be zero.

As a consequence the E_2 part described before collapses to E_∞ and we have a copy of $\{m\}_{m-c-1}$ as a direct summand of $H^{n-2j-1}(C_n)$ for $j = 0,\dots,i-2$, that is for $m \le \frac{n}{j+1}$.

Now we consider an integer m that divides n and let $i = \frac{n}{m}$. The modules of type $\{m\}[t^{\pm1}]$ are:

$E_1^{\lambda m-1,n-\lambda(m-2)-2i+1}$	generated by	$z_m(i-\lambda)01^{\lambda m-1}$ for $1 \le \lambda \le i-1$
$E_1^{\lambda m,n-\lambda(m-2)-2i+1}$	generated by	$v_m(i-\lambda)01^{\lambda m}$ for $0 \le \lambda \le i-1$.

The situation is shown in the next diagram ($\boldsymbol{h} = \{m\}[t^{\pm1}]$):



The map

$$d_1 : E_1^{\lambda m-1,n-\lambda(m-2)-2i+1} \to E_1^{\lambda m,n-\lambda(m-2)-2i+1}$$

is given by the multiplication by $\begin{bmatrix} \lambda m \\ \lambda m-1 \end{bmatrix}'_{q,t} = [\lambda m]_q(1+tq^{\lambda m-1})$, but in this case the coefficient is zero in the module $\{m\}[t^{\pm1}]$ because $\varphi_m(q) \mid [\lambda m]_q$ and so we have that $E_1 = \cdots = E_{m-1}$. So we have to consider the map

$$d_{m-1}^{\lambda m,n-\lambda(m-2)-2i+1} : E_{m-1}^{\lambda m,n-\lambda(m-2)-2i+1} \to E_1^{(\lambda+1)m-1,n-(\lambda+1)(m-2)-2i+1}$$

for $\lambda = 0,\dots,i-2$.

This map corresponds to the multiplication by

$$\begin{bmatrix} (\lambda+1)m-1 \\ \lambda m \end{bmatrix}'_{q,t} = \begin{bmatrix} (\lambda+1)m-1 \\ \lambda m \end{bmatrix}_q \prod_{j=\lambda m+1}^{(\lambda+1)m-1} (1+tq^{j-1}).$$

It is easy to see that the polynomial $\begin{bmatrix} (\lambda+1)m-1 \\ \lambda m \end{bmatrix}_q$ is prime with the torsion $\varphi_m(q)$ and so the map $d_{m-1}^{\lambda m, n-\lambda(m-2)-2i+1}$ is injective and the cokernel is isomorphic to

$$R \Big/ \left(\varphi_m(q), \prod_{j=\lambda m+1}^{(\lambda+1)m-1} (1+tq^{j-1}) \right) \simeq \bigoplus_{0\leq k\leq m-2} \{m\}_k.$$

As a consequence we have that

$$\begin{array}{lclc} E_m^{\lambda m-1, n-\lambda(m-2)-2i+1} & = & \bigoplus_{0\leq k\leq m-2}\{m\}_k & \text{for } 1\leq\lambda\leq i-1 \\ E_m^{\lambda m, n-\lambda(m-2)-2i+1} & = & 0 & \text{for } 0\leq\lambda\leq i-2. \end{array}$$

and all these modules collapse to E_∞. This means that we can find $\varphi_m(q)$-torsion only in $H^{n-2j}(C_n)$ and for $j\geq 1$ the summand is given by

$$\bigoplus_{0\leq k\leq m-2} \{m\}_k$$

for $d \leq \frac{n}{j+1}$.

We still have to consider all the terms $E_m^{n-m,m-1} = \{m\}[t^{\pm 1}]$ for $m \mid n$. Here the maps we have to look at are the following:

$$d_m^{n-m,m-1} : E_m^{n-m,m-1} \to E_m^{n,0}.$$

These maps correspond to multiplication by the polynomials $\begin{bmatrix} n \\ n-m \end{bmatrix}'_{q,t}$. Moreover recall that

$$E_1^{n,0} = R \Big/ \left(\begin{bmatrix} n \\ n-1 \end{bmatrix}'_{q,t} \right).$$

We can now use Lemma 6.3.1 to say that all the maps $d_m^{n-m,m-1}$ are injective and Lemma 6.3.2 to say that

$$E_{n+1}^{n,0} = E_\infty^{n,0} = \bigoplus_{m\mid n, 0\leq k\leq d-2} \{m\}_k \oplus \{1\}_{n-1}.$$

Since $E_\infty^{n,0} = H^n(C_n)$, this complete the proof of the Theorem. □

6.3.2 Other computations

We may also consider the cohomology of G_{B_n} over the module $\mathbb{Q}[t^{\pm 1}]$, where the action is trivial for the generators $\epsilon_1, \ldots, \epsilon_{n-1}$ and $(-t)$-multiplication for the last generator $\overline{\epsilon}_n$. This cohomology is computed by the complex C_n^* of Section 3 where we specialize q to -1. So we may use similar filtration and associated spectral sequence. We used this argument in [CMS06b]. Here we briefly indicate a different and more concise method, using the results of Theorem 6.1.1. We have:

Theorem 6.3.3.

$$\begin{array}{rcll} H^k(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) & = & \mathbb{Q}[t^{\pm 1}]/(1+t) & 1 \leq k \leq n-1 \\ H^n(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) & = & \mathbb{Q}[t^{\pm 1}]/(1+t) & \textit{for odd } n \\ H^n(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) & = & \mathbb{Q}[t^{\pm 1}]/(1-t^2) & \textit{for even } n. \end{array}$$

Sketch of proof. Consider the short exact sequence:

$$0 \to \mathbb{Q}[q^{\pm 1}, t^{\pm 1}] \xrightarrow{1+q} \mathbb{Q}[q^{\pm 1}, t^{\pm 1}] \to \mathbb{Q}[t^{\pm 1}] \to 0$$

and the induced long exact sequence for cohomology

$$\begin{array}{r} \cdots \to H^i(G_{B_n}, \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]) \xrightarrow{1+q} H^i(G_{B_n}, \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]) \to \\ \to H^i(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) \to \cdots . \end{array}$$

The result is now a straightforward consequence of Theorem 6.1.1. $\square$

6.4 More consequences

By means of Shapiro's Lemma, the inclusions introduced in Section 6.2 can be exploited to link the cohomology of the Artin group of type $\tilde{A}_{n-1}$, A_n to the cohomology of G_{B_n}.

6.4.1 Cohomology of $G_{\tilde{A}_{n-1}}$

Let M be any domain and let q be a unit of M. We indicate by M_q the ring M with the $G_{\tilde{A}_{n-1}}$-module structure where the action of the standard generators is given by $(-q)$-multiplication.

Proposition 6.4.1. *We have*

$$\begin{aligned} H_*(G_{\tilde{A}_{n-1}}, M_q) &\simeq H_*(G_{B_n}, M[t^{\pm 1}]_{q,t}) \\ H^*(G_{\tilde{A}_{n-1}}, M_q) &\simeq H^*(G_{B_n}, M[[t^{\pm 1}]]_{q,t}) \end{aligned}$$

where the action of G_{B_n} on $M[t^{\pm 1}]_{q,t}$ (and on $M[[t^{\pm 1}]]_{q,t}$) is given by $(-q)$-multiplication for the generators $\epsilon_1, \dots, \epsilon_{n-1}$ and $(-t)$-multiplication for the last generator $\bar{\epsilon}_n$.

Proof. Applying Shapiro's lemma to the inclusion $\tilde{A}_{n-1} < G_{B_n}$, one obtains:

$$\begin{aligned} H_*(G_{\tilde{A}_{n-1}}, M_q) \simeq & H_*(G_{B_n}, \mathrm{Ind}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}} M_q) \\ H^*(G_{\tilde{A}_{n-1}}, M_q) \simeq & H^*(G_{B_n}, \mathrm{Coind}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}} M_q). \end{aligned}$$

By Corollary 6.2.3, any element of $\mathrm{Ind}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}} M_q := \mathbb{Z}[G_{B_n}] \otimes_{G_{\tilde{A}_{n-1}}} M_q$ can be represented as a sum of elements of the form $\tau^\alpha \otimes q^m$. Now, we have an isomorphism of $\mathbb{Z}[G_{B_n}]$-modules

$$\mathbb{Z}[G_{B_n}] \otimes_{G_{\tilde{A}_{n-1}}} M_q \to M[t^{\pm 1}]_{q,t}$$

defined by sending $\tau^\alpha \otimes q^m \mapsto (-1)^{n\alpha} t^\alpha q^{(n-1)\alpha+m}$ and the result follows.

In cohomology we have similarly:

$$\mathrm{Coind}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}} M_q := \mathrm{Hom}_{G_{\tilde{A}_{n-1}}}(\mathbb{Z}[G_{B_n}], M_q) \simeq M[[t^{\pm 1}]]_{q,t}.$$

□

By Propositions 6.4.1, in order to determine the cohomology groups

$$H^*(G_{\tilde{A}_{n-1}}, M_q)$$

it is necessary to know the cohomology of G_{B_n} with values in the module $M[[t^{\pm 1}]]$ of Laurent series in the variable t. The latter is linked to the cohomology with values in the module of Laurent polynomials by:

Proposition 6.4.2 (Degree shift)**.**

$$H^*(G_{B_n}, M[[t^{\pm 1}]]_{q,t}) \simeq H^{*+1}(G_{B_n}, M[t^{\pm 1}]_{q,t}).$$

□

This result was obtained in [Cal05] in a slightly weaker form, but it is possible to extend it to our case with little effort.

Let from now on $M = \mathbb{Q}[q^{\pm 1}]$. In this case we have $M[t^{\pm 1}]_{q,t} = R_{q,t}$, so we obtain the cohomology of the Artin group of affine type $\tilde{A}_{n-1}$ with M_q-coefficients by means of Theorem 6.1.1.

In a similar way we get the rational cohomology of $G_{\tilde{A}_{n-1}}$:

Proposition 6.4.3. *We have*

$$\begin{aligned} H_*(G_{\tilde{A}_{n-1}}, \mathbb{Q}) \simeq & H_*(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) \\ H^*(G_{\tilde{A}_{n-1}}, \mathbb{Q}) \simeq & H^*(G_{B_n}, \mathbb{Q}[[t^{\pm 1}]]) \end{aligned}$$

where the action of G_{B_n} on $\mathbb{Q}[t^{\pm 1}]$ (and on $\mathbb{Q}[[t^{\pm 1}]]$) is trivial for the generators $\epsilon_1, \dots, \epsilon_{n-1}$ and $(-t)$-multiplication for the last generator $\bar{\epsilon}_n$.

To obtain the rational cohomology of $G_{\tilde{A}_{n-1}}$ we may apply Proposition 6.4.2 together with Theorem 6.3.3.

6.4.2 Cohomology of G_{A_n} with coefficient in the Tong-Yang-Ma representation

The Tong-Yang-Ma representation is an $(n+1)$-dimensional representation of the classical braid group G_{A_n} discovered in [TYM96]. Below we just recall it, referring to [Sys01] for a discussion of its relevance in braid group representation theory.

Definition 6.4.4. Let V be the free $\mathbb{Q}[u^{\pm 1}]$-module of rank $n+1$. The *Tong-Yang-Ma representation* is the representation

$$\rho : G_{A_n} \to \mathrm{GL}_{\mathbb{Q}[u^{\pm 1}]}(V)$$

defined w.r.t. the basis $e_1, \ldots, e_{n+1}$ of V by:

$$\rho(\sigma_i) = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & u & 0 & \\ & & & I_{n-i} \end{pmatrix}$$

where I_j denote the j-dimensional identity matrix and all other entries are zero.

Notice that the image of the pure braid group under the Tong-Yang-Ma representation is abelian; hence this representation factors through the *extended Coxeter group* presented in [Tit66].

Proposition 6.4.5. *We have*

$$H_*(G_{B_n}, M[t^{\pm 1}]_{q,t}) \simeq H_*(G_{A_n}, M_q \otimes V)$$

$$H^*(G_{B_n}, M[t^{\pm 1}]_{q,t}) \simeq H^*(G_{A_n}, M_q \otimes V)$$

where each generator of G_{A_n} acts on $M_q \otimes V$ by $(-q)$-multiplication on the first factor and by the Tong-Yang-Ma representation the second factor.

Sketch of proof. For the statement in homology, by Shapiro's lemma, it is enough to show that $\mathrm{Ind}_{G_{B_n}}^{G_{A_n}} M[t^{\pm 1}]_{q,t} \simeq M_q \otimes V$.
Notice that $[G_{A_n} : G_{B_n}] = n+1$ and let choose as coset representatives for G_{A_n}/G_{B_n} the elements $\alpha_i = (\sigma_i \sigma_{i+1} \cdots \sigma_{n-1}) \sigma_n (\sigma_i \sigma_{i+1} \cdots \sigma_{n-1})^{-1}$ for $1 \leq i \leq n-1$, $\alpha_n = \sigma_n$, $\alpha_{n+1} = e$.
Then by definition of induced representation, there is an isomorphism of left G_{A_n}-modules,

$$\mathrm{Ind}_{G_{B_n}}^{G_{A_n}} M[t^{\pm 1}]_{q,t} = \bigoplus_{i=1}^{n+1} M[t^{\pm 1}] e_i$$

where the action is on the r.h.s. is as follows. For an element $x \in G_{A_n}$, write $x\alpha_k = \alpha_{k'}x'$ with $x' \in G_{B_n}$. Then x acts on an element $r \cdot e_k \in \bigoplus_{i=1}^{n+1} M[t^{\pm 1}]e_i$ as $x(r \cdot e_k) = (x'r) \cdot e_{k'}$.
Computing explicitly this action for the standard generators of G_{A_n}, we can write the representation in the following matrix form:

$$\sigma_i \mapsto \begin{pmatrix} -qI_{i-1} & & & \\ & 0 & -q & \\ & q^{-1}t & 0 & \\ & & & -qI_{n-i} \end{pmatrix}$$

for $1 \leq i \leq n-1$, whereas

$$\sigma_n \mapsto \begin{pmatrix} -qI_{n-1} & & \\ & 0 & 1 \\ & -t & 0 \end{pmatrix}.$$

Conjugating by $U = \mathrm{Diag}(1, 1, \ldots, 1, -q^{-1})$ and setting $u = -q^{-2}t$, one obtains the desired result.
Finally, since $[G_{A_n} : G_{B_n}] = n+1 < \infty$, the induced and coinduced representation are isomorphic; so the analogous statement in cohomology follows. □

In particular the cohomology of G_{B_n} determined in Theorem 6.1.1 is isomorphic to the cohomology of G_{A_n} with coefficient in the Tong-Yang-Ma representation twisted by an abelian representation.

By means of Shapiro's lemma, we may as well determine the cohomology of G_{A_n} with coefficient in the Tong-Yang-Ma representation. Indeed:

Proposition 6.4.6. *We have*

$$H_*(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) \simeq H_*(G_{A_n}, V)$$
$$H^*(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) \simeq H^*(G_{A_n}, V)$$

where V is the representation of G_{A_n} defined in 6.4.4.

As a consequence we have

Corollary 6.4.7. *Let V be the $(n+1)$-dimensional representation of the braid group Br_{n+1} defined in 6.4.4. Then the cohomology*

$$H^*(\mathrm{Br}_{n+1},\ V)$$

is given as in Theorem 6.3.3.

Remark 6.4.8. In particular the homology of $G_{\tilde{A}_{n-1}}$ with trivial coefficients is isomorphic to the homology of G_{A_n} with coefficients in the Tong-Yang-Ma representation.

Chapter 7

The case of the affine arrangements of type $\widetilde{B}_n$

7.1 Preliminary constructions

In this Chapter, we are primarily interested in Artin braid groups associated to Coxeter graphs of type B_n, $\tilde{B}_n$ and D_n (see Table 7.1).

The associated Coxeter groups can be described as reflection groups with respect to an arrangement of hyperplanes (or mirrors). Let $x_1, \ldots, x_n$ be the standard coordinates in $\mathbb{R}^n$. Consider the linear hyperplanes:

$$\mathbf{H}_k = \{x_k = 0\} \qquad \mathbf{L}_{ij}^{\pm} = \{x_i = \pm x_j\}$$

and, for an integer $a \in \mathbb{Z}$, their affine translates:

$$\mathbf{H}_k(a) = \{x_k = a\} \qquad \mathbf{L}_{ij}^{\pm}(a) = \{x_i = \pm x_j + a\}$$

The Coxeter group B_n is identified with the group of reflections with respect to the mirrors in the arrangement

$$\mathcal{A}(B_n) := \{\mathbf{H}_k \mid 1 \leq k \leq n\} \cup \{\mathbf{L}_{ij}^{\pm} \mid 1 \leq i < j \leq n\}.$$

As such it is the group of signed permutations of the coordinates in $\mathbb{R}^n$. Notice that B_n is generated by n basic reflections $s_1, \ldots, s_n$ having respectively as mirrors the $n-1$ hyperplanes $\mathbf{L}_{i,i+1}^{+}$ $(1 \leq i \leq n-1)$ and the hyperplane $\mathbf{H}_n$. This numbering of the reflections is consistent with the numbering of the vertexes of the Coxeter graph for B_n shown in Table 7.1.

The affine Coxeter group $\tilde{B}_n$ is the semidirect product of the Coxeter group B_n and the coroot lattice, consisting of integer vectors whose coordinates add up to an even number. The arrangement of mirrors is then the affine hyperplane arrangement:

$$\mathcal{A}(\tilde{B}_n) := \{\mathbf{H}_k(a) \mid 1 \leq k \leq n,\, a \in \mathbb{Z}\} \cup \{\mathbf{L}_{ij}^{\pm}(a) \mid 1 \leq i < j \leq n,\, a \in \mathbb{Z}\}. \tag{7.1.1}$$

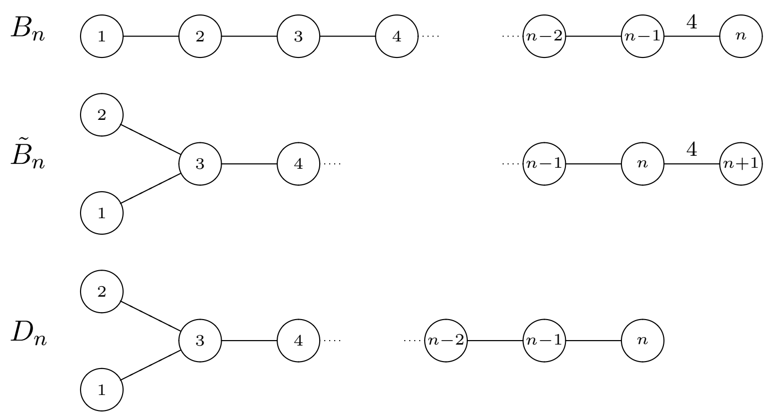


Table 7.1: Coxeter graphs of type B_n, $\tilde{B}_n$, D_n.

It is generated by the basic reflections for B_n plus an extra affine reflection $\tilde{s}$ having $\mathbf{L}_{12}^{-}(1)$ as mirror. The latter commutes with all the basic reflections of B_n but s_2, for which $(\tilde{s}s_2)^3 = 1$. This accounts for the Coxeter graph of type $\tilde{B}_n$ in the table, where, however, we chose by our convenience a somewhat unusual vertex numbering.

Finally the group D_n has reflection arrangement:

$$\mathcal{A}(D_n) := \{\mathbf{L}_{ij}^{\pm} \,|\, 1 \leq i < j \leq n\}$$

and it can be regarded as the group of signed permutations of the coordinates which involve an even number of sign changes. In particular D_n is a subgroup of index 2 in B_n. The group is generated by n basic reflections w.r.t. the hyperplanes $\mathbf{L}_{12}^{-}$ and $\mathbf{L}_{i,i+1}^{+}$ $(1 \leq i \leq n-1)$.

7.2 The $K(\pi, 1)$ problem for the affine Artin group of type $\tilde{B}_n$

Recall that infinite type Artin groups are represented as groups of linear, not necessarily orthogonal, reflections w.r.t. the walls of a polyhedral cone C of maximal dimension in $\mathbf{V} = \mathbb{R}^n$. It can be shown that the union $U = \bigcup_{w \in W} wC$ of W-translates of C is a convex cone and that W acts properly on the interior U^0 of U. We may now rephrase the construction used in the finite case as follows. Let $\mathcal{A}$ be the complexified arrangement of the mirrors of the reflections in W and consider $I := \{v \in \mathbf{V} \otimes \mathbb{C} \,|\, \Re(v) \in U^0\}$. Then W acts freely on $\mathbf{Y} = I \setminus \bigcup_{\mathbf{H} \in \mathcal{A}} \mathbf{H}$ and we can form the orbit space $\mathbf{X} := \mathbf{Y}/W$. It is known ([vdL83]; see also [Sal94]) that G_W is indeed the fundamental group of $\mathbf{X}$, but in general it is only conjectured that $\mathbf{X}$ is a $K(\pi, 1)$. In this Section, we extend this result to the affine Artin group of type $\widetilde{B}_n$, showing:

Theorem 7.2.1. $\mathbf{Y}(\tilde{B}_n)$ *and, hence,* $\mathbf{X}(\tilde{B}_n)$ *are* $K(\pi,1)$ *spaces.*

The idea of proof can be described in few words: up to a $\mathbb{C}^*$ factor, the orbit space is presented (through the exponential map) as a covering of the complement to a finite simplicial arrangement, so we apply Theorem 1.4.2.

We just digress a bit on the peculiarity of affine Artin groups. In this case the associated Coxeter group is an affine Weyl group W_a and, as such, it can be geometrically represented as a group generated by affine (orthogonal) reflections in a real vector space. This geometric representation and that given by the Tits cone are linked in a precise manner; indeed it turns out that U_0 for an affine Weyl group is an open half space in $\mathbf{V}$ and that W_a acts as a group of affine orthogonal reflections on a hyperplane section E of U_0. The representation on E coincides with the geometric representation and $\mathbf{Y}(W_a)$ is homotopic to the complement of the complexified affine reflection arrangement.

Using the explicit description of the reflection mirrors in Equation (7.1.1), the complement of the complexified affine reflection arrangement of type $\tilde{B}_n$ is given by:

$$\mathbf{Y} := \mathbf{Y}(\tilde{B}_n) = \{x \in \mathbb{C}^n \mid x_i \pm x_j \notin \mathbb{Z} \text{ for all } i \neq j,\ x_k \notin \mathbb{Z} \text{ for all } k\}$$

On $\mathbf{Y}$ we have, by standard facts, a free action by translations of the coweight lattice Λ, identified with the standard lattice $\mathbb{Z}^n \subset \mathbb{C}^n$.

Proof of Theorem 7.2.1 We first explicitly describe the covering $\mathbf{Y} \to \mathbf{Y}/\Lambda$ applying the exponential map $y = \exp(2\pi \mathrm{i} x)$ componentwise to $\mathbf{Y}$:

$$\mathbf{Y} \xrightarrow{\quad\pi\quad} \mathbf{Y}/\Lambda \simeq \{y \in \mathbb{C}^n \mid y_i \neq y_j^{\pm 1},\ y_k \neq 0, 1\}$$

$$(x_1, \ldots, x_n) \longmapsto \big(\exp(2\pi \mathrm{i} x_1), \ldots, \exp(2\pi \mathrm{i} x_n)\big)$$

Notice now that the function

$$\mathbb{C} \setminus \{0,1\} y \mapsto g(y) = \frac{1+y}{1-y} \in \mathbb{C} \setminus \{\pm 1\}$$

satisfies $g(y^{-1}) = -g(y)$. Further g is invertible, its inverse being given by $z \mapsto \frac{z-1}{z+1}$. Therefore applying g componentwise to $\mathbf{Y}/\Lambda$, we have:

$$\mathbf{Y}/\Lambda \simeq \{z \in \mathbb{C}^n \mid z_i \neq \pm z_j,\ z_k \neq \pm 1\}$$

Consider now the arrangement $\mathcal{A}$ in $\mathbb{R}^{n+1}$ consisting of the hyperplanes $\mathbf{L}_{ij}^{\pm}$ for $1 \leq i < j \leq n+1$ and $\mathbf{H}_1$ and let $\mathbf{Y}(\mathcal{A})$ be the complement of its complexification.

We have an homeomorphism

$$\eta : \mathbb{C}^* \times \mathbf{Y}/\Lambda \to \mathbf{Y}(\mathcal{A})$$

defined by

$$\eta\big(\lambda, (z_1, \dots, z_n)\big) = (\lambda, \lambda z_1, \dots, \lambda z_n)$$

To show that $\mathbf{Y}/\Lambda$ is a $K(\pi, 1)$, it is then sufficient to show that $\mathbf{Y}(\mathcal{A})$ is a $K(\pi, 1)$. We will show in Lemma 7.2.2 below that $\mathcal{A}$ is simplicial, and therefore the result follows from Deligne's Theorem 1.4.2. □

Remark By the same exponential argument one may recover the results of [Oko79] for the affine Artin group of type $\tilde{A}_n, \tilde{C}_n$ (for further applications we refer to [All02]).

Lemma 7.2.2. *Let $\mathcal{A}$ be the real arrangement in $\mathbb{R}^{n+1}$ consisting of the hyperplanes $\mathbf{L}_{ij}^{\pm}$ for $1 \le i < j \le n+1$ and $\mathbf{H}_1$. Then $\mathcal{A}$ is simplicial.*

Proof. Notice that $\mathcal{A}$ is the union of the reflection arrangement $\mathcal{A}(D_{n+1})$ of type D_{n+1} and the hyperplane $\mathbf{H}_1 = \{x_1 = 0\}$. Hence we study how the chambers of $\mathcal{A}(D_{n+1})$ are cut by the hyperplane $\mathbf{H}_1$. Since the Coxeter group D_{n+1} acts transitively on the collection of chambers, it is enough to consider how the fundamental chamber $\mathbf{C}_0$ of $\mathcal{A}(D_{n+1})$ is cut by the D_{n+1}-translates of the hyperplane $\mathbf{H}_1$, i.e. by the coordinate hyperplanes $\mathbf{H}_k$ for $k = 1, 2, \dots, n+1$.
We may choose

$$\mathbf{C}_0 = \{-x_2 < x_1 < x_2 < \ldots < x_n < x_{n+1}\}$$

as fundamental chamber. Of course, this is a simplicial cone. Notice that the coordinate of a point in $\mathbf{C}_0$ are all positive except (possibly) the first. Thus it is clear that for $k \ge 2$ the hyperplanes $\mathbf{H}_k$ do not cut $\mathbf{C}_0$.
A quick check shows instead that $\mathbf{H}_1$ cuts $\mathbf{C}_0$ into two simplicial cones $\mathbf{C}_1$, $\mathbf{C}_2$ given precisely by:

$$\mathbf{C}_1 = \{0 < x_1 < x_2 < \ldots < x_n < x_{n+1}\}$$
$$\mathbf{C}_2 = \{0 < -x_1 < x_2 < \ldots < x_n < x_{n+1}\}$$

□

□

7.3 Cohomology

The second main result of this Chapter is the computation of the cohomology of the group $G_{\tilde{B}_n}$ (so, by Theorem 7.2.1), of $\mathbf{X}(\tilde{B}_n)$) with local coefficients. We consider the 2-parameters representation of $G_{\tilde{B}_n}$ over the ring $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ and over the module $\mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]]$ defined by sending the standard generator corresponding to the last node of the Coxeter diagram to $(-t)$−multiplication and the other standard generators to $(-q)$−multiplication (minus sign is only for technical reasons). Such representations are quite natural to be considered: they generalize the analog 1-parameter representations that (for finite type) correspond to considering the structure of bundle over the complement of the discriminant hypersurface in the orbit space and the monodromy action on the cohomology of the associated Milnor fibre (see for example [Fre88], [CS98]). We explain in Section 7.3.2 various relations between these cohomologies and the cohomology of the commutator subgroup of $G_{\tilde{B}_n}$.

The main tool to perform computations is the algebraic Salvetti complex. The cohomology factorizes into two parts (see also [DCPSS99]) : the *invariant* part reduces to that of the Artin group of finite type B_n, whose 2-parameters cohomology was computed in [CMS06a]; for the *anti-invariant* part we use suitable filtrations and the associated spectral sequences.

Let φ_d be the d-th cyclotomic polynomial in the variable q. We define the quotient rings

$$\{1\}_i = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]/(1 + tq^i)$$
$$\{d\}_i = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]/(\varphi_d, 1 + tq^i)$$
$$\{\{d\}\}_j = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]/(\varphi_d, \prod_{i=o}^{d-1} 1 + tq^i)^j.$$

The final result is the following one:

Theorem 7.3.1. *The cohomology $H^{n-s}(G_{\widetilde{B}_n}, \mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]])$ is given by*

$$\mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]] \qquad \textit{for} \quad s = 0$$

$$\bigoplus_{h>0} \{\{2h\}\}_{f(n,h)} \qquad \textit{for} \quad s = 1$$

$$\bigoplus_{\substack{h>2 \\ i \in I(n,h)}} \{2h\}_i^{c(n,h,s)} \oplus \bigoplus_{\substack{d \mid n \\ 0 \leq i \leq d-2}} \{d\}_i \oplus \{1\}_{n-1} \qquad \textit{for} \quad s = 2$$

$$\bigoplus_{\substack{h>2 \\ i \in I(n,h)}} \{2h\}_i^{c(n,h,s)} \oplus \bigoplus_{\substack{d \mid n \\ 0 \leq i \leq d-2 \\ d \leq \frac{n}{j+1}}} \{d\}_i \qquad \textit{for} \quad s = 2 + 2j$$

$$\bigoplus_{\substack{h>2 \\ i \in I(n,h)}} \{2h\}_i^{c(n,h,s)} \oplus \bigoplus_{\substack{d \nmid n \\ d \leq \frac{n}{j+1}}} \{d\}_{n-1} \qquad \textit{for} \quad s = 3 + 2j$$

where $c(n, h, s) = max(0, \lfloor \frac{n}{2h} \rfloor - s)$, $f(n, h) = \lfloor \frac{n+h-1}{2h} \rfloor$ and $I(n, h) = \{n, \ldots, n+h-2\}$ if $n \equiv 0, 1, \ldots, h \operatorname{mod}(2h)$ and $I(n, h) = \{n+h-1, \ldots, n+2h-1\}$ if $n \equiv h+1, h+2, \ldots, 2h-1 \operatorname{mod}(2h)$.

As a corollary we also derive the cohomology with trivial coefficients of $G_{\tilde{B}_n}$ (Theorem 7.3.7)

We use a suitable filtration of the algebraic complex, reducing computation of the cohomology mainly to:

- calculation of generators of certain subcomplexes for the Artin group of type D_n (whose cohomology was known from [DCPSS99], but we need explicit suitable generators);
- analysis of the associated spectral sequence to deduce the cohomology of $\tilde{B}_n$ with local coefficients;
- use of some exact sequences for the cohomology with constant coefficients.

7.3.1 Algebraic complexes for Artin groups

As a main tool for cohomological computations we use the algebraic Salvetti complex (see Section 3.1), which provides an effective way to determine the cohomology of the orbit space $\mathbf{X}(W)$ with values in an arbitrary G_W-module. When $\mathbf{X}(W)$ is a $K(\pi, 1)$ space, of course, we get the cohomology of the group G_W.

For sake of simplicity, we restrict ourself to the abelian representations considered in Section 1.5. Let (W, S) be a Coxeter system. We recall that, given a a representation $\eta : G_W \to R^*$, let M_η be the induced structure of G_W-module on the R-module M. We may describe a cochain complex $C^*(W)$ for the cohomology $H^*(X(W); M_\eta)$ as follows. The cochains in dimension k consist in the free R-module indexed by the finite parabolic subgroup of W:

$$C^k(W) := \bigoplus_{\substack{\Gamma:|\Gamma|=k \\ |W_\Gamma|<\infty}} M.e_\Gamma \tag{7.3.1}$$

and the coboundary map are completely described by the formula:

$$\mathrm{d}(e_\Gamma) = \sum_{\substack{\Gamma' \supset \Gamma \\ |\Gamma'|=|\Gamma|+1 \\ |W_{\Gamma'}|<\infty}} (-1)^{\alpha(\Gamma,\Gamma')} \frac{W_{\Gamma'}(\eta)}{W_\Gamma(\eta)} e_{\Gamma'} \tag{7.3.2}$$

where $W_\Gamma(\eta)$ is the η-Poincaré series of the parabolic subgroup W_Γ and $\alpha(\Gamma, \Gamma')$ is an incidence index depending on a fixed linear order of S. For $\Gamma' \setminus \Gamma = \{s'\}$ it is defined as

$$\alpha(\Gamma, \Gamma') := |\{s \in \Gamma \ : \ s < s'\}|$$

We identify (consistently with Table 7.1) the generating reflections set S for $\tilde{B}_n$ with the set $\{1, 2, \ldots, n+1\}$. It is useful to represent a subset $\Gamma \subset S$ with its characteristic function. For example the subset $\{1, 3, 5, 6\}$ for $\tilde{B}_6$ may be represented as the binary string:

$$\begin{matrix} 0 \\ 1 \end{matrix}\ 10110$$

To determine the cohomology of $G_{\tilde{B}_n}$, it will be necessary to give a close look to the cohomology of G_{D_n}. It is convenient to number the vertex of D_n as in table 7.1 and to regard parabolic subgroups as binary strings as before.

7.3.2 A generalized shift

Let R be the ring of Laurent polynomials $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ and M be the R-module of Laurent series $\mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]]$ and let $R_{q,t}$, $M_{q,t}$ be the corresponding local systems, with action $\eta_{q,t}$. Our main interest is to compute the cohomology with trivial rational coefficient of the group

$$Z_{\widetilde{B}_n} = \ker\left(G_{\widetilde{B}_n} \to \mathbb{Z}^2\right)$$

that is the commutator subgroup of $G_{\widetilde{B}_n}$. By Shapiro Lemma (see [Bro82]) we have the following equivalence:

$$H^*(Z_{\widetilde{B}_n}, \mathbb{Q}) \simeq H^*(G_{\widetilde{B}_n}, M_{q,t})$$

and the second term of the equality is computed by the Salvetti complex $C^*(\widetilde{B}_n)$ over the module $M_{q,t}$. Notice that the finite parabolic subgroups of $W_{\widetilde{B}_n}$ are in $1-1$ correspondence with the proper subsets of the set of simple roots S. the index set $S = \{1, \ldots, n+1\}$ except that for the whole set S of all simple roots of $\widetilde{B}_n$.

We can define an *augmented* Salvetti complex $\widehat{C}^*(\widetilde{B}_n)$ as follows:

$$\widehat{C}^*(\widetilde{B}_n) = C^*(\widetilde{B}_n) \oplus (M_{q,t}).e_S.$$

We need to define the boundary map for the n-dimensional generators. Let us first define a quasi-Poincaré polynomial for $G_{\widetilde{B}_n}$. We set

$$\widehat{W_S}(q,t) = \widehat{W}_{\widetilde{B}_n}(q,t) = [2(n-1)]!!\ [n] \prod_{i=0}^{n-1} (1 + tq^i).$$

It is easy to verify that $\widehat{W}_{\widetilde{B}_n}(q,t)$ is the least common multiple of all $W_\Gamma(q,t)$, for $\Gamma \subset S$ with $|\Gamma| = n$. This allows us to define the boundary map for the generators e_Γ, with $|\Gamma| = n$:

$$d(e_\Gamma) = (-1)^{\alpha(\Gamma,S)} \frac{\widehat{W}_{\widetilde{B}_n}(q,t)}{W_\Gamma(q,t)} e_S$$

and it is straightforward to verify that $\widehat{C}^*(\widetilde{B}_n)$ is still a chain complex. Moreover we have the following relations between the cohomologies of $C^*(\widetilde{B}_n)$ and $\widehat{C}^*(\widetilde{B}_n)$:

$$H^i(C^*(\widetilde{B}_n)) = H^i(\widehat{C}^*(\widetilde{B}_n))$$

for $i \neq n, n+1$ and we have the short exact sequence

$$0 \to H^n(\widehat{C}^*(\widetilde{B}_n), M_{q,t}) \to H^n(C^*(\widetilde{B}_n), M_{q,t}) \to M_{q,t} \to 0.$$

Finally one can prove that the complex $\widehat{C}^*(\widetilde{B}_n)$ with coefficients in the local system $R_{q,t}$ is *well filtered* (as defined in [Cal05]) with respect to the variable t and so it gives the same cohomology, modulo an index shifting, of the complex with coefficients over the module $\mathbb{Q}[t^{\pm 1}][[q^{\pm 1}]]$. Another index shifting can be proved with a slight improvement of the results in [Cal05], allowing to pass to the module M. Hence we have the following

Proposition 7.3.2.

$$H^i(Z_{\widetilde{B}_n}, \mathbb{Q}) \simeq H^i(\widehat{C}^*(\widetilde{B}_n), M_{q,t}) \simeq H^{i+2}(\widehat{C}^*(\widetilde{B}_n), R_{q,t}) \simeq H^{i+2}(G_{\widetilde{B}_n}, R_{q,t})$$

for $i \neq n, n+1$ and

$$H^n(Z_{\widetilde{B}_n}, \mathbb{Q}) \simeq H^n(G_{\widetilde{B}_n}, M_{q,t}) \simeq M$$

$$H^{n+1}(Z_{\widetilde{B}_n}, \mathbb{Q}) \simeq H^{n+1}(G_{\widetilde{B}_n}, M_{q,t}) \simeq 0.$$

□

From now on we deal only with the complex $\widehat{C}^*(\widetilde{B}_n)$ with coefficients in the local system $R_{q,t}$.

7.3.3 A splitting of complexes

For Coxeter groups of type $W = D_n$, $\tilde{B}_n$ the Salvetti's complex C^*W exhibits an involution σ defined by:

$$\begin{array}{c}0\\0\end{array} A \xrightarrow{\sigma} \begin{array}{c}0\\0\end{array} A \qquad\qquad \begin{array}{c}1\\1\end{array} A \xrightarrow{\sigma} - \begin{array}{c}1\\1\end{array} A$$

$$\begin{array}{c}0\\1\end{array} A \xrightarrow{\sigma} \begin{array}{c}1\\0\end{array} A \qquad\qquad \begin{array}{c}1\\0\end{array} A \xrightarrow{\sigma} \begin{array}{c}0\\1\end{array} A.$$

Let I^*W be the module of σ-invariants and K^*W the module of σ-anti-invariants. We may then split the complex into:

$$C^*W = I^*W \oplus K^*W.$$

In particular the computation of the cohomology of C^*W may be performed analyzing separately the two subcomplexes.

7.3.4 Cohomology of K^*D_n

The cohomology of the anti-invariant subcomplex for D_n was completely determined in [DCPSS99]. However we will need for our purposes generators for the cohomology groups which are not easily deduced from the argument in the original paper. So we briefly recall this result.

Let G_n^1 be the subcomplex of $C(D_n)$ generated by the strings of type $\begin{smallmatrix}0\\1\end{smallmatrix}A$ and $\begin{smallmatrix}1\\1\end{smallmatrix}A$. It is easy to see that G_n^1 is isomorphic (as a complex) to $K(D_n)$.

Define the set

$$S_n = \{h \in \mathbb{N} \text{ s. t. } 2h|n \text{ or } h|n-1 \text{ and } 2h \nmid (n-1)\}$$

Note that h appears in S_n if and only if $n = 2\lambda h$ (i.e. n is an even multiple of h) or $n = (2\lambda+1)h+1$ (n is an odd multiple of h incremented by 1).

Proposition 7.3.3 ([DCPSS99]). *The top-cohomology of G_n^1 is:*

$$H^n G_n^1 = \bigoplus_{h \in S_n} \{2h\}$$

whereas for $s > 0$ one has:

$$H^{n-2s} G_n^1 = \bigoplus_{\substack{h \in S_n \\ 1 < h < \frac{n}{2s}}} \{2h\}$$

$$H^{n-2s+1} G_n^1 = \bigoplus_{\substack{h \in S_n \\ 1 < h \leq \frac{n}{2s}}} \{2h\}.$$

□

We need a description of the generators for these modules.
First we define the following basic binary strings:

$$o_\mu[h] = \begin{cases} \begin{smallmatrix}0\\1\end{smallmatrix} 1^{h-1} & \text{for } \mu = 0 \\ \\ \begin{smallmatrix}1\\1\end{smallmatrix} 1^{2\mu h-2}01^h & \text{for } \mu \geq 1 \end{cases}$$

$$e_\mu[h] = \begin{smallmatrix}1\\1\end{smallmatrix} 1^{(2\mu-1)h-1}01^{h-2} \text{ for } \mu \geq 1$$

$$s_h = 01^{h-2} \qquad\qquad l_h = 01^h.$$

A set of candidate cohomology generators is given by the following cocycles:

$$\begin{aligned}
o_{\mu,2i}[h] &= \frac{1}{\varphi_{2h}} d(o_\mu[h](s_h l_h)^i) \\
o_{\mu,2i+1}[h] &= \frac{1}{\varphi_{2h}} d(o_\mu[h](s_h l_h)^i s_h) \\
e_{\mu,2i}[h] &= \frac{1}{\varphi_{2h}} d(e_\mu[h](l_h s_h)^i) \\
e_{\mu,2i+1}[h] &= \frac{1}{\varphi_{2h}} d(e_\mu[h](l_h s_h)^i l_h).
\end{aligned}$$

Indeed these cocycles account for all the generators:

Proposition 7.3.4.
1. *Let $n = 2\lambda h$. Then for $0 \leq s < \lambda$ the summand of $H^{n-2s}(G_n^1)$ isomorphic to $\{2h\}$ is generated by $e_{\lambda-s,2s}[h]$. Similarly for $0 \leq s < \lambda$ the summand of $H^{n-2s-1}(G_n^1)$ is generated by $o_{\lambda-s-1,2s+1}[h]$.*
2. *Let $n = (2\lambda+1)h+1$. Then for $0 \leq s \leq \lambda$ the summand of $H^{n-2s}(G_n^1)$ isomorphic to $\{2h\}$ is generated by $o_{\lambda-s,2s}[h]$. For $0 \leq s < \lambda$ the summand of $H^{n-2s-1}(G_n^1)$ is generated by $e_{\lambda-s,2s+1}[h]$.*

Proposition 7.3.4 is best proved by induction on n, recovering in particular the quoted result from [DCPSS99].

Proof. We filter the complex G_n^1 from the right and use the associated spectral sequence. Let:

$$F_k G_n^1 = \langle A1^k \rangle$$

be the subcomplex generated by binary strings ending with at least k ones. We have a filtration

$$G_n^1 = F_0 G_n^1 \supset F_1 G_n^1 \supset \ldots \supset F_{n-2} G_n^1 \supset F_{n-1} G_{n-1}^1 \supset 0$$

in which the subsequent quotients for $k = 1, 2, \ldots, n-3$

$$\frac{F_k G_n^1}{F_{k+1} G_n^1} = \langle A01^k \rangle \simeq G_{n-k-1}^1[k]$$

are isomorphic to the complex for G_{n-k-1}^1 shifted in degree by k, while

$$\frac{F_{n-2} G_n^1}{F_{n-1} G_n^1} = \left\langle \begin{matrix} 0 \\ 1 \end{matrix} \, 1^{n-2} \right\rangle \simeq R[n-1] \quad F_{n-1} G_n^1 = \left\langle \begin{matrix} 1 \\ 1 \end{matrix} \, 1^{n-2} \right\rangle \simeq R[n].$$

Therefore the columns of the E_1 term of the spectral sequence are either the module R or are given by the cohomology of $G_{n'}^1$ with $n' < n$. Reasoning by induction, we may thus suppose that their cohomology has the generators prescribed by the proposition. Since there can be no non-zero maps between

the module $\{2h\}$, $\{2h'\}$ for $h \neq h'$, we may separately detect the φ_{2h}-torsion in the cohomology.
Fix an integer $h > 1$. Then the relevant modules for the φ_{2h}-torsion in the E_1 term are suggested in Table 7.2. We will call a column *even* if it is relative to $G^1_{2\mu h}$ and *odd* if it is relative to $G^1_{(2\mu+1)h+1}$ for some μ. The

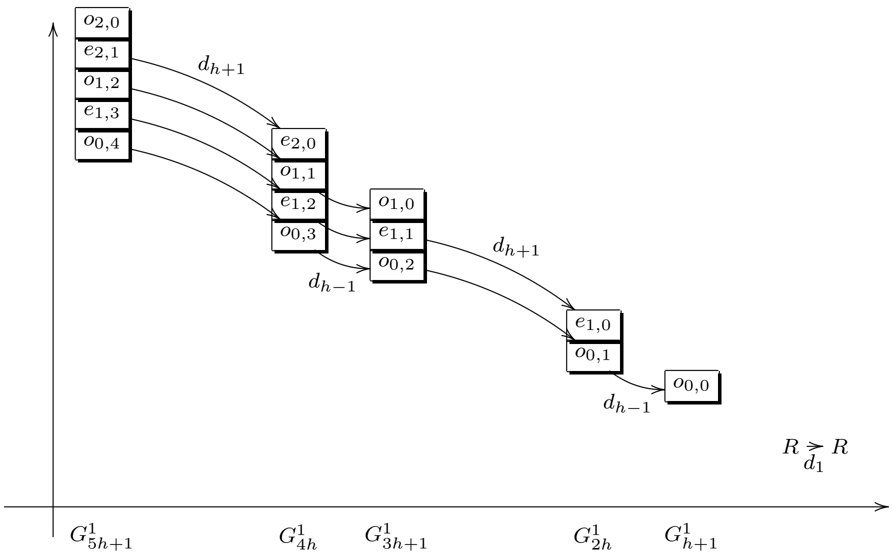


Table 7.2: Spectral sequence for G^1_n

differential d_1 is zero everywhere but $d_1 : E_1^{(n-2,1)} \to E_1^{(n-1,1)}$ where it is given by multiplication by $[2(n-1)]!!/[n-1]!$. Thus the E_2 term differs from the E_1 only in positions $(n-2,1)$ and $(n-1,1)$, where:

$$E_2^{(n-2,1)} = 0 \qquad\qquad E_2^{(n-1,1)} = \frac{R}{[2(n-1)]!!/[n-1]!}$$

Then all other differentials are zero up to d_{h-2}.
It is now useful to distinguish among 4 cases according to the remainder of $n \bmod (2h)$:

a) $n = 2\lambda h + c$ for $1 \leq c \leq h$

b) $n = (2\lambda+1)h + 1$

c) $n = (2\lambda+1)h + 1 + c$ for $1 \leq c \leq h-2$

d) $n = 2\lambda h$-

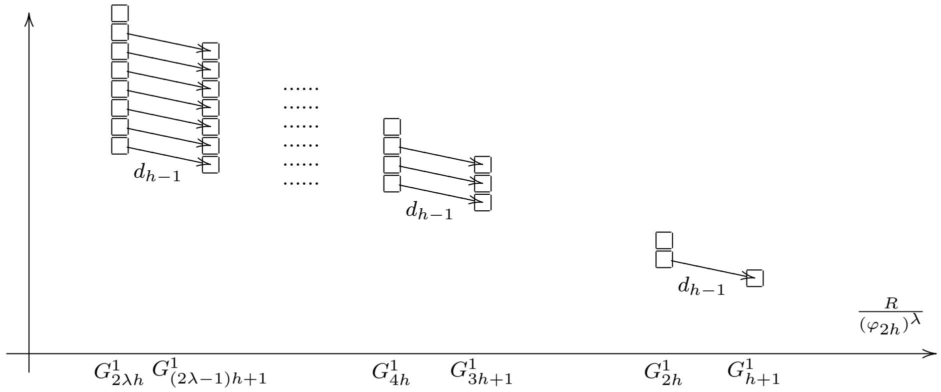


Table 7.3: E_{h-1}-term of the spectral sequence for G^1_n in case a)

In case a), note the first column relevant for φ_{2h}-torsion is even (see also Table 7.3).

The differential d_{h-1} maps the modules of positive codimension of an even column $G^1_{2\mu h}$ $(1 \le \mu \le \lambda)$ to those in the odd column $G^1_{(2\mu-1)h+1}$. Using the suitable generators of type $e_{\cdot,\cdot}[h], o_{\cdot,\cdot}[h]$, the map d_{h-1} may be identified with the multiplication by

$$\begin{bmatrix} n-(2\mu-1)h-1 \\ h-1 \end{bmatrix} = \begin{bmatrix} 2(\lambda-\mu)+c+h-1 \\ h-1 \end{bmatrix} \tag{7.3.3}$$

Since this polynomial is non-divisible by φ_{2h}, the restriction of d_{h-1} to positive codimension elements in even columns is injective. It follows that in the E_h-term the only survivors are in positions $(c+2(\lambda-\mu)h-1, 2\mu h)$, generated by $e_{\mu,0}[h]$ and

$$E_h^{(n-1,1)} \simeq E_2^{(n-1,1)} = \frac{R}{[2(n-1)]!!/[n-1]!}.$$

Note that in $E_h^{(n-1,1)}$ the only torsion of type φ^l_{2h} is given by the summand:

$$\frac{R}{(\varphi_{2h})^\lambda}$$

The setup is summarized in Table 7.4. In the Table the survivors are in dark grey boxes while annihilated terms are in light grey.

Further, using the generators and up to an invertible, we may identify the differential $d_{2\mu h} : E_{2\mu h}^{(c+2(\lambda-\mu)h-1, 2\mu h)} \to E_{2\mu h}^{n-1,1}$ with the multiplication by $\varphi_{2h}^{\lambda-\mu}$ $(1 \le \mu \le \lambda)$. Thus, for example, in the E_{2h+1} term the module in position $(c+2(\lambda-1)h-1, 2h)$ vanishes and the φ_{2h}-torsion in $E_{2h+1}^{(n-1,1)}$ is reduced to $R/(\varphi_{2h})^{\lambda-1}$. Continuing in this way, all φ_{2h}-torsion vanishes. In summary there is no φ_{2h}-torsion in the cohomology of G^1_n; this ends case a).

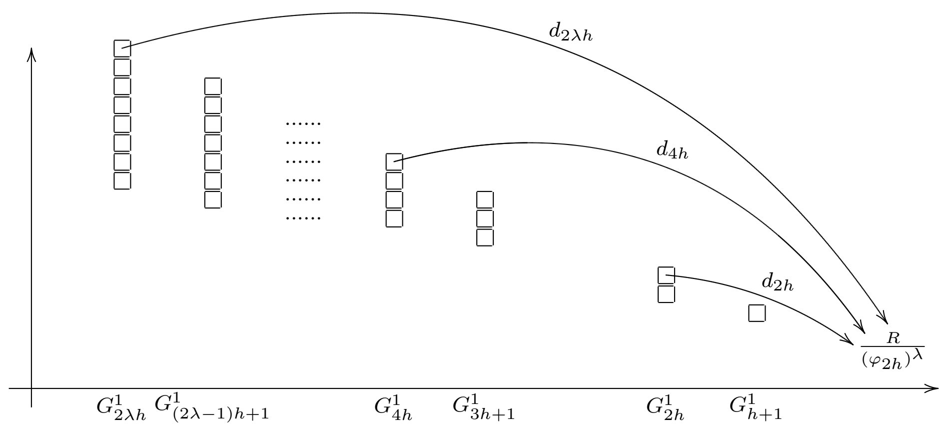


Table 7.4: Setup for the higher degree terms in the spectral sequence for G^1_n in case a)

For case b), the first column in the spectral sequence relevant for φ_{2h} is still even. The differential d_{h-1} may be identified again as multiplication as in formula 7.3.3, but now it vanishes, since the polynomial is divisible by φ_{2h}.

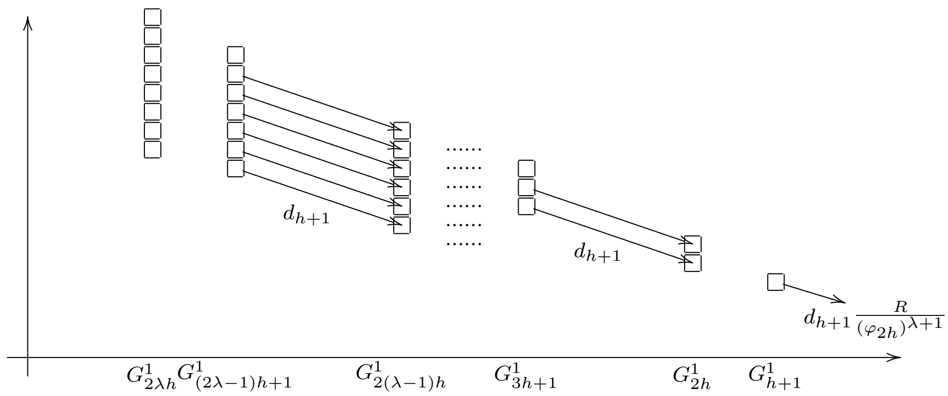


Table 7.5: E_{h-1}-term of the spectral sequence for G^1_n in case b)

The next non-vanishing differential is d_{h+1}. See Table 7.5. It takes the module in positive codimension in an odd column $G^1_{(2\mu+1)h+1}$ to the elements in the even column $G^1_{2\mu h}$ (for $1 \leq \mu \leq \lambda - 1$). Via generators, it may be identified with the multiplication by

$$\begin{bmatrix} n-2\mu h \\ h+1 \end{bmatrix} = \begin{bmatrix} 2(\lambda-\mu)h+h+1 \\ h+1 \end{bmatrix} \tag{7.3.4}$$

and it is therefore injective when restricted to modules in positive codimension in odd columns. Further d_{h+1} is also non-zero as a map $E_{h+1}^{(2\lambda h-1,h+1)} \to E_{h+1}^{(n-1,1)}$. Actually the term

$$E_{h+1}^{(n-1,1)} \simeq E_2^{(n-1,1)} \simeq \frac{R}{[2(n-1)]!!/[n-1]!}$$

has $R/(\varphi_{2h})^{\lambda+1}$ as the only summand with torsion of type φ_{2h}^l. It is easy to check that the relative map can be identified with the multiplication by φ_{2h}^λ.

Thus, the only survivors in the E_{2h} term are the first even column, the top modules in the odd columns, generated in positions $(2(\lambda-\mu)h-1,(2\mu+1)h+1)$ by $o_{\mu,0}$ for $1 \leq \mu \leq \lambda-1$, as well as $E_{2h}^{(n-1,1)}$ which has $R/(\varphi_{2h})^\lambda$ as summand.

Note that the higher differentials vanish when restricted to the first even column. Actually we may lift the generators of type $e_{\lambda-s,2s}[h]$ to global generators $e_{\lambda-s,2s+1}[h]$ for $0 \leq s < \lambda$. Similarly for $0 \leq s < \lambda$ we may lift $o_{\lambda-s-1,2s+1}[h]$ to the global generator $o_{\lambda-s-1,2s+2}[h]$. Finally, as in case a),

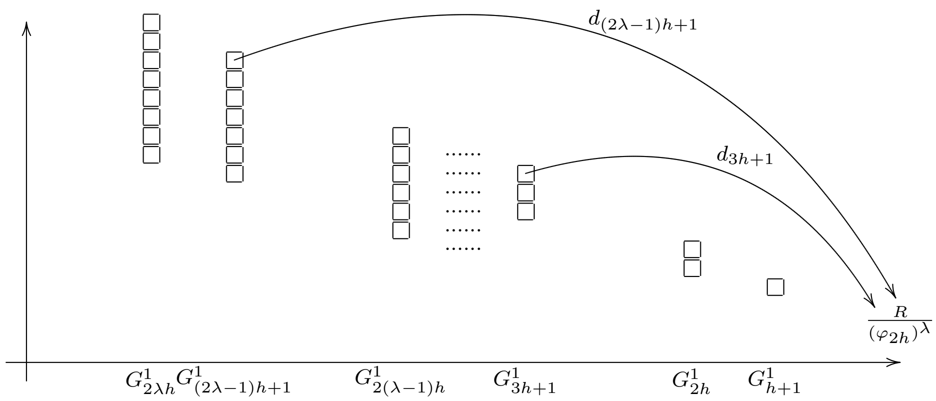


Table 7.6: Setup for the higher degree terms in the spectral sequence for G^1_n in case b)

the module in positions $(2(\lambda-\mu)h-1,(2\mu+1)h+1)$ for $1 \leq \mu \leq \lambda-1$ vanish in the higher terms of the spectral sequence while the module in position

$(n-1,1)$ has eventually as summand R/φ_{2h}. Clearly the coboundary $o_{\lambda,0}[h]$ projects onto a generator of the latter.

Case c) and d) present no new complications and are omitted. □

□

7.3.5 Spectral sequence for $G_{\tilde{B}_n}$

We can now compute the cohomology $H^*(G_{\widetilde{B}_n}, R_{q,t})$. We will do this by means of the Salvetti complex $\widehat{C^*}\widetilde{B}_n$.

As in Section [7.3.3], let $\widehat{I}\widetilde{B}_n$ be the module of the σ-invariant elements and $\widehat{K}\widetilde{B}_n$ the module of the σ-anti-invariant elements. We can split our module $\widehat{C^*}\widetilde{B}_n$ into the direct sum:

$$\widehat{C^*}\widetilde{B}_n = \widehat{I}\widetilde{B}_n \oplus \widehat{K}\widetilde{B}_n.$$

Using the map $\beta : C^*B_n \to \widehat{C^*}\widetilde{B}_n$ so defined:

$$\beta : 0A \mapsto \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} A$$

$$\beta : 1A \mapsto \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} A + \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} A$$

one can see that the submodule $\widehat{I}\widetilde{B}_n$ is isomorphic (as a differential complex) to C^*B_n. Its cohomology has been computed in [CMS06a]. We recall the result:

Theorem 7.3.5 ([CMS06a])**.**

$$H^i(G_{B_n}, R_{q,t}) = \begin{cases} \bigoplus_{d|n, 0\le i\le d-2}\{d\}_i \oplus \{1\}_{n-1} & \text{if } i = n \\ \bigoplus_{d|n, 0\le i\le d-2, d\le \frac{n}{j+1}}\{d\}_i & \text{if } i = n-2j \\ \bigoplus_{d\nmid n, d\le \frac{n}{j+1}}\{d\}_{n-1} & \text{if } i = n-2j-1. \end{cases}$$

□

Hence we only need to compute the cohomology of $\widehat{K}\widetilde{B}_n$. In order to do this we make use of the results presented in Section 7.3.4. First consider the subcomplex of $\widehat{C^*}\widetilde{B}_n$ defined as

$$L_n^1 = < \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} A,\ \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} A > .$$

We define the map $\kappa : L_n^1 \to \widehat{K}\widetilde{B}_n$ by

$$\kappa : \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} A \mapsto \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} A - \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} A$$

$$\kappa : \begin{array}{c}1\\1\end{array} A \mapsto 2 \begin{array}{c}1\\1\end{array} A.$$

It is easy to check that κ gives an isomorphism of differential complex. Now we define a filtration $\mathcal{F}$ on the complex L_n^1:

$$\mathcal{F}_i L_n^1 =< \begin{array}{c}0\\1\end{array} A1^i, \begin{array}{c}1\\1\end{array} A1^i > .$$

The quotient $\mathcal{F}_i L_n^1 / \mathcal{F}_{i+1} L_n^1$ is isomorphic to the complex $\left(G_{n-i}^1[t^{\pm 1}]\right)[i]$ (see Proposition 7.3.3) with trivial action on the variable t. Hence we use the spectral sequence defined by the filtration $\mathcal{F}$ to compute the cohomology of the complex L_n^1.

The E_0-term of the spectral sequence is given by

$$\begin{aligned} E_0^{i,j} =& \frac{\left(\mathcal{F}_i L_n^1\right)^{(i+j)}}{(\mathcal{F}_{i+1} L_n^1)^{(i+j)}} \\ =& \left((G_{n-i}^1)^{(i+j)}[t^{\pm 1}]\right)[i] \\ =& (G_{n-i}^1)^j [t^{\pm 1}] \end{aligned}$$

for $0 \leq i \leq n-2$. Finally:

$$E_0^{n-1,1} = R \qquad\qquad E_0^{n,1} = R$$

and all the other terms are zero. The differential $d_0 : E_0^{i,j} \to E_0^{i,j+1}$ corresponds to the differential on the complex G_{n-i}^1.It follows that the E^1-term is given by the cohomology of the complexes G_{n-i}^1:

$$E_{i,j}^1 = H^j(G_{n-i}^1)[t^{\pm 1}]$$

for $0 \leq i \leq n-2$ and

$$E_1^{n-1,1} = R, \quad E_1^{n,1} = R.$$

As in Section 7.3.4, we can separately consider in the spectral sequence E_* the modules with torsion of type φ_{2h}^l for an integer $h \geq 1$.

For a fixed integer $h > 0$, let $c \in \{0, \ldots, 2h-1\}$ be the congruency class of $n \mod(2h)$ and let λ be an integer such that $n = c + 2\lambda h$. We consider the two cases:

a) $0 \leq c \leq h$;

b) $h + 1 \leq c \leq 2h - 1$.

In case a) the modules of φ_{2h}-torsion are:
with $0 \leq \mu \leq \lambda - 1, 0 \leq i \leq \lambda - \mu - 1$

$$E_1^{c+2\mu h, 2(\lambda-\mu)h-2i} \simeq \{2h\}[t^{\pm 1}]$$

generated by $e_{\lambda-\mu-i,2i}[h]01^{c+2\mu h}$;
with $0 \leq \mu \leq \lambda-1, 0 \leq i \leq \lambda-\mu-1$

$$E_1^{c+2\mu h,2(\lambda-\mu)h-2i-1} \simeq \{2h\}[t^{\pm 1}]$$

generated by $o_{\lambda-\mu-i-1,2i+1}[h]01^{c+2\mu h}$;
with $0 \leq \mu \leq \lambda-1, 0 \leq i \leq \lambda-\mu-1$

$$E_1^{c+2\mu h+h-1,2(\lambda-\mu)h-h+1-2i} \simeq \{2h\}[t^{\pm 1}]$$

generated by $o_{\lambda-\mu-i-1,2i}[h]01^{c+2\mu h+h-1}$;
with $0 \leq \mu \leq \lambda-2, 0 \leq i \leq \lambda-\mu-2$

$$E_1^{c+2\mu h+h-1,2(\lambda-\mu)h-h+1-2i-1} \simeq \{2h\}[t^{\pm 1}]$$

generated by $e_{\lambda-\mu-i-1,2i+1}[h]01^{c+2\mu h+h-1}$.

In case b) the modules of φ_{2h}-torsion are:
with $0 \leq \mu \leq \lambda-1, 0 \leq i \leq \lambda-\mu-1$

$$E_1^{c+2\mu h,2(\lambda-\mu)h-2i} \simeq \{2h\}[t^{\pm 1}]$$

generated by $e_{\lambda-\mu-i,2i}[h]01^{c+2\mu h}$;
with $0 \leq \mu \leq \lambda-1, 0 \leq i \leq \lambda-\mu-1$

$$E_1^{c+2\mu h,2(\lambda-\mu)h-2i-1} \simeq \{2h\}[t^{\pm 1}]$$

generated by $o_{\lambda-\mu-i-1,2i+1}[h]01^{c+2\mu h}$;
with $0 \leq \mu \leq \lambda, 0 \leq i \leq \lambda-\mu$

$$E_1^{c+2\mu h-h-1,2(\lambda-\mu)h+h+1-2i} \simeq \{2h\}[t^{\pm 1}]$$

generated by $o_{\lambda-\mu-i,2i}[h]01^{c+2\mu h-h-1}$;
with $0 \leq \mu \leq \lambda-1, 0 \leq i \leq \lambda-\mu-1$

$$E_1^{c+2\mu h-h-1,2(\lambda-\mu)h+h+1-2i-1} \simeq \{2h\}[t^{\pm 1}]$$

generated by $e_{\lambda-\mu-i,2i+1}[h]01^{c+2\mu h-h-1}$.

In the E_1-term of the spectral sequence, the only non-trivial map is the map $d_1 : E_1^{n-1,1} \to E_1^{n,1}$, that corresponds to the multiplication by the polynomial

$$\frac{\widehat{W}_{\widetilde{B}_n}[q,t]}{W_{B_n}[q,t]} = \prod_{i=1}^{n-1}(1+q^i) = \prod_{h \leq n} \varphi_{2h}^{\lfloor \frac{n-1}{h} \rfloor - \lfloor \frac{n-1}{2h} \rfloor}.$$

Then in E_2 we have:

$$E_2^{n-1,1} = 0$$

and

$$E_2^{n,1} = \bigoplus R/(\varphi_{2h}^{\lfloor \frac{n-1}{h} \rfloor - \lfloor \frac{n-1}{2h} \rfloor}).$$

Notice that the integer $f(n,h) = \lfloor \frac{n-1}{h} \rfloor - \lfloor \frac{n-1}{2h} \rfloor$ corresponds to λ in case a) and to $\lambda + 1$ in case b).

Now we consider the higher differentials in the spectral sequence. The first possibly non-trivial maps are d_{h-1} and d_{h+1}. In case a) the map d_{h-1} is given by the multiplication by

$$\prod_{i=n}^{n+h-2} (1+tq^i)$$

and the map d_{h+1} is the null map. The maps

$$d_{2(\lambda-\mu)h} : \{2h\}[t^{\pm 1}] = E_{2(\lambda-\mu)h}^{c+2\mu h, 2(\lambda-\mu)h} \to E_{2(\lambda-\mu)h}^{n,1}$$

where μ goes from $\lambda - 1$ to 0, correspond, up to invertible, modulo φ_{2h},to multiplication by

$$\varphi_{2h}^{\mu}(\prod_{i=0}^{2h-1} (1+tq^i))^{\lambda-\mu}.$$

Moreover they are all injective and the term $E_{2(\lambda)h+1}^{n,1}$ is given by the quotient

$$R/(\varphi_{2h}^{\lambda}, \varphi_{2h}^{\lambda-1} \prod_{i=0}^{2h-1} (1+tq^i), \ldots, (\prod_{i=0}^{2h-1} (1+tq^i))^{\lambda}) =$$

$$= R/(\varphi_{2h}, \prod_{i=0}^{2h-1} (1+tq^i))^{\lambda}.$$

In case b) the map d_{h-1} is null and the map d_{h+1} is the multiplication by the polynomial

$$\prod_{i=n+h-1}^{n+2h-1} (1+tq^i).$$

The maps

$$d_{2(\lambda-\mu)h+h+1} : \{2h\}[t^{\pm 1}] = E_{2(\lambda-\mu)h+h+1}^{c+2\mu h+h-1, 2(\lambda-\mu)h-h} \to E_{2(\lambda-\mu)h+h+1}^{1,n}$$

where μ goes from λ to 0, correspond, up to invertible, modulo φ_{2h},to multiplication by

$$\varphi_{2h}^{\mu}(\prod_{i=0}^{2h-1} (1+tq^i))^{\lambda-\mu+1}.$$

Hence they are all injective and the term $E_{2(\lambda)h+h+2}^{n,1}$ is given by the quotient

$$R/(\varphi_{2h}, \prod_{i=0}^{2h-1} (1+tq^i))^{\lambda+1}.$$

Since all the generators lift to global cocycles, it turns out that all the other differentials are null. Hence we proved the following:

Theorem 7.3.6.
$$H^{n+1}(\widehat{K}\widetilde{B}_n) \simeq \bigoplus_{h>0} \{\{2h\}\}_{f(n,h)}$$

and, for $s \geq 0$:

$$H^{n-s}(\widehat{K}\widetilde{B}_n) \simeq \bigoplus_{\substack{h>2 \\ i\in I(n,h)}} \{2h\}_i^{\oplus max(0,\lfloor \frac{n}{2h} \rfloor - s)}$$

with $I(n,h) = \{n,\ldots,n+h-2\}$ if $n \simeq 0,1,\ldots,h \bmod(2h)$, $f(n,h) = \lfloor \frac{n+h-1}{2h} \rfloor$ and $I(n,h) = \{n+h-1,\ldots,n+2h-1\}$ if $n \simeq h+1, h+2, \ldots, 2h-1 \bmod(2h)$. □

Putting together the results of Theorem 7.3.5 and 7.3.6, we get Theorem 7.3.1.

As a corollary, we use the long exact sequences associated to

$$0 \longrightarrow \mathbb{Q}[[t^{\pm 1}]] \xrightarrow{m(q)} M \xrightarrow{1+q} M \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Q} \xrightarrow{m(t)} \mathbb{Q}[[t^{\pm 1}]] \xrightarrow{1+t} \mathbb{Q}[[t^{\pm 1}]] \longrightarrow 0$$

to get the constant coefficients cohomology for $G_{\widetilde{B}_n}$. Here $m(x)$ is the multiplication by the series

$$\sum_{i\in\mathbb{Z}} (-x)^i.$$

We give only the result, omitting details which come from non difficult analysis of the above mentioned sequences and recalling that the Euler characteristic of the complex is 1, for n even, and -1, for n odd.

Theorem 7.3.7.

$$H^i(G_{\widetilde{B}_n}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if} \quad i = 0 \\ \mathbb{Q}^2 & \text{if} \quad 1 \leq i \leq n-2 \\ \mathbb{Q}^{2+\lfloor \frac{n}{2} \rfloor} & \text{if} \quad i = n-1, n \end{cases}$$

where the t and q actions correspond to the multiplication by -1. □

Open problems

We want to list some of the many open problems related to our work and that are left to forthcoming research.

As a generalization of the computations of Chapter 4 and Chapter 5 one can study the homology, with integer coefficients in the ring $\mathbb{Z}[q^{\pm 1}]$ for the Artin groups of type D_n. Moreover the computations could be extended to the ring of coefficients $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$, for the cases where the abelianization of the Artin group is $\mathbb{Z}^2$, that is B_n, F_4, $I_2(2m)$. In Chapter 6 we compute the cohomology of the Artin group of type B_n with coefficients over the ring $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$. The extension of the computations to the ring $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ seems to be affordable, even if much more complicated.

A natural generalization of the results in Chapter 6 and Chapter 7 is the computation of the cohomology for the other affine type Artin groups. This leads to another problem. The computations in Chapter 7 were made possible using the fact that the space $\mathbf{X}_W$ in case $\widetilde{B}_n$ is of type $K(\pi, 1)$. It is an open conjecture that the space $\mathbf{X}_W$ is a $K(\pi, 1)$ for a any Coxeter group W. Even in affine cases, we know the conjecture holds only for $\widetilde{A}_n$ and $\widetilde{C}_n$ ([Oko79]), $\widetilde{B}_n$ (Theorem 7.2.1) and $\widetilde{G}_2$ (as a consequence of the results in [CD95b]). If this conjecture holds, it is possible to compute the cohomology for all Artin groups using the Salvetti complex. Moreover, the conjecture implies that all the Artin groups are torsion free, since the Salvetti complex provides a finite resolution.

Similar to Chapter 6, Proposition 6.4.5, one can also study the cohomology of Artin groups over non abelian representations.

Index

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