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Smooth Geometric Evolutions of Hypersurfaces and Singular Approximation of Mean Curvature Flow

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Contents

Citations to Previously Published Works

Several results presented in this thesis appeared in the following papers:

- L. Ambrosio and C. Mantegazza, *Curvature and distance function from a manifold. Dedicated to the memory of Fred Almgren*. J. Geom. Anal. **8** (1998), no. 5, 723–744.
- C. Mantegazza, *Smooth geometric evolutions of hypersurfaces*. Geom. Funct. Anal. **12** (2002), no. 1, 138–182.
- C. Mantegazza and A. C. Mennucci, *Hamilton–Jacobi equations and distance functions on Riemannian manifolds*. Appl. Math. Opt. **47** (2003), 1–25.
- M. Eminenti and C. Mantegazza, *Some properties of the distance function and a conjecture of De Giorgi*. J. Geom. Anal. **14** (2004), no. 2, 267–279.
- G. Bellettini, C. Mantegazza and M. Novaga, *Singular perturbations of mean curvature flow*. J. Diff. Geom. **75** (2007), no. 3, 403–431.
- C. Mantegazza and L. Martinazzi, *A note on quasilinear parabolic equations on manifolds*. Ann. Sc. Norm. Super. Pisa **9** (2012), 857–874.

Foreword

In the last years, a large interest has grown in connection with geometric evolutions of submanifolds, also with motivations coming from mathematical physics (phase transitions, Stefan problem). A model problem is the evolution of surfaces by mean curvature, which can be considered as the gradient flow of the Area functional. Indeed, if M is a compact n–manifold embedded in $\mathbb{R}^N$ without boundary and if Φ_t is a family of diffeomorphisms of $\mathbb{R}^n$ such that Φ_0 is the identity, then

$$\frac{d}{dt}\left[\mathcal{H}^n\big(\Phi_t(M)\big)\right]_{t=0} = -\int_M \langle \mathrm{H}, X\rangle \, d\mathcal{H}^n$$

where $X = [\Phi_t]'_{t=0}$ is the infinitesimal generator of Φ_t, $\mathcal{H}^n$ is the n–dimensional Hausdorff measure and H is the mean curvature vector of M.
This mathematical problem is intriguing because the appearance of singularities during the flow (with the exceptions of planar Jordan curves, convex shapes, codimension one graphs) makes it necessary a weak approach to get a global (in time) solution of the evolution problem. Starting from the pioneering work of Brakke [**18**], a large literature is by now available on this subject (see for instance Chen–Giga–Goto [**25**], Evans–Spruck [**40**], Huisken [**54**], Ilmanen [**62**] and the references therein). The weak formulations are mainly based either on geometric measure theory (currents, varifolds), or on the theory of viscosity solutions (the level set approach of Chen–Giga–Goto [**25**], Evans–Spruck [**40**]). In the latter approach, a crucial role (see for instance Ambrosio–Soner [**7**], Evans–Soner-Souganidis [**39**] and Soner [**89**]) is played by the analytical properties of the distance function $d^M(x)$ from the manifold (see also Delfour–Zolésio [**33, 34**]). For instance, in the codimension one case, it turns out that the boundary $M_t = \partial U_t$ of a family of domains U_t flows by mean curvature if and only if

$$\partial_t \, d(x,t) = \Delta d(x,t) \qquad \text{for every } x \in M_t$$

where $d(x,t)$ is equal to the signed distance function from M_t, that is, $d(x,t) = -d^{M_t}(x)$ if $x \in U_t$ and $d(x,t) = d^{M_t}(x)$ if $x \notin U_t$. Since the signed distance function makes no sense in higher codimension problems, De Giorgi suggested in [**30**], [**31**] and [**32**] to work with the squared distance function $\eta^M(x) = [d^M(x)]^2/2$. Setting $\eta(x,t) = \eta^{M_t}(x)$, it turns out that (see Ambrosio–Soner [**7**]) the mean curvature flow is characterized by the equation

$$\partial_t \, \nabla\eta(x,t) = \Delta\nabla\eta(x,t) \qquad \text{for } x \in M_t$$

because $-\nabla\eta(x,t)$ represents the displacement of $x \in M_t$ and $-\Delta\nabla\eta(x,t)$ is the mean curvature vector of M_t at x.

One of the parts of this work is a systematic study of the connections between the analytical properties of η^M and the geometric invariants of the manifold M. In particular, in Chapter 1 we will prove that $\nabla^3\eta^M(x)$ and the second fundamental form B_x of M are mutually connected for any $x \in M$ by simple linear relations, moreover, for any unit normal vector ν the eigenvalues of $\langle \mathrm{B}_x, \nu\rangle$ on the tangent space (in some sense, the "principal curvatures" in the direction ν) are linked to the eigenvalues of $\nabla^2\eta^M(x_s)$ for any point x_s on the normal line $x + s\nu$.

Our motivations are also related to the analysis of the general class of geometric functionals, including the Area functional and the Willmore functional (see Chen [**24**], Simon [**87**], Weiner [**91**], Willmore [**93**])

$$\mathcal{H}_2(M) = \int_M |\mathrm{H}|^2 \, d\mathcal{H}^n \, ,$$

depending on the second fundamental form of M, which have been widely investigated in the literature (see Langer **[68]**, Reilly **[82]**, Rund **[83]** and Voss **[90]**).

We will see that in principle any autonomous "geometric" functional can be written as

$$\mathcal{F}(M) = \int_M f\left(\nabla^2_{i_1 i_2}\eta^M, \dots, \nabla^\gamma_{j_1 \dots j_\gamma}\eta^M\right) d\mathcal{H}^n$$

for some function f depending on the standard derivatives in $\mathbb{R}^N$ of η^M up to a given order γ. In this setting, the Area functional and the Willmore functional respectively correspond to

$$\frac{1}{N-n}\int_M \sum_{i,j} |\nabla^2_{ij}\eta^M|^2\, d\mathcal{H}^n \qquad \text{and} \qquad \int_M \sum_{i,k} |\nabla^3_{ikk}\eta^M|^2\, d\mathcal{H}^n.$$

One of the results of this work is a constructive algorithm for the computation of the first variation of the functional $\mathcal{F}$. Specifically, under smoothness assumptions on f we prove that there exists a unique normal vector field $E_\mathcal{F}$ such that

$$\frac{d}{dt}\mathcal{F}\big(\Phi_t(M)\big)\Big|_{t=0} = \int_M \langle E_\mathcal{F} | X\rangle\, d\mathcal{H}^n$$

for any family of diffeomorphisms Φ_t whose infinitesimal generator is X. In general, $E_\mathcal{F}$ depends on the derivatives of η^M up to the order $(2\gamma - 1)$ and, if f is a polynomial, then the same is true for $E_\mathcal{F}$.

We will anyway carry out an explicit computation for the generalization of the Willmore functional

$$\mathcal{H}_p(M) = \int_M |\mathrm{H}|^p\, d\mathcal{H}^n.$$

In particular, in the codimension one case, we recover some of the results found by the above mentioned authors (see Reilly **[82]**, Voss **[90]**).

The advantages of this approach are its full generality and its independence by the dimension and the codimension. However, it should be said that assumptions like $n = 1$ or $n = (N-1)$ are very often important to get a manageable expression for $E_\mathcal{F}$. Another difficulty is related to the fact that, even in the codimension one case, any symmetric functions of the principal curvatures is in principle representable as above, but this representation is in practice not quite easy, with the notable exceptions of $|\mathrm{H}|^p$ and $|\mathrm{B}|^p$.

In Chapter 2 we will consider and compute the Euler equations of the functionals on hypersurfaces of $\mathbb{R}^{n+1}$,

$$\mathcal{G}_\gamma(M) = \int_M |\nabla^\gamma \eta^M|^2\, d\mathcal{H}^n\,,$$

where ∇^γ is the standard (iterated) derivative in $\mathbb{R}^{n+1}$, and

$$\mathcal{C}_m(\varphi) = \int_M |\nabla^m \nu|^2\, d\mu\,,$$

representing hypersurfaces in $\mathbb{R}^{n+1}$ as immersions $\varphi : M \to \mathbb{R}^{n+1}$. Here μ and ∇ are respectively the canonical measure and the Levi–Civita connection on the Riemannian manifold (M, g), where the metric g is obtained by pulling back on M the usual metric of $\mathbb{R}^{n+1}$ via the map φ. The symbol ∇^m denotes the m–th iterated covariant derivative and ν a unit normal local vector field to the hypersurface. Finally, B and H are respectively the second fundamental form and the mean curvature of the hypersurface.

The functionals $\mathcal{C}_m$ are strictly related to the $\mathcal{G}_\gamma$ since, roughly speaking, the derivative of the unit normal vector is the curvature of M.

We notice that $\mathcal{G}_2$ is a multiple of the Area functional. When instead $\gamma = 3$, the function inside the integral above is equal to $3|\mathrm{B}|^2$ and, if we also assume $n = 2$, the functional $\mathcal{G}_3$ coincides, up to multiplicative and additive constants, depending on the genus of M (see the discussion at the beginning of Section 2.3), with the Willmore functional.

These two functionals have very similar leading terms in their first variation, that is,

$$2\gamma(-1)^{\gamma-1}\Big(\overbrace{\Delta\Delta\dots\Delta}^{(\gamma-2)\text{–times}}H\Big)\nu \qquad \text{and} \qquad 2(-1)^{m+1}\Big(\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}H\Big)\nu$$

where Δ is the *Laplace–Beltrami* operator of the hypersurface and H is the (scalar) mean curvature of M. Notice that such leading terms actually coincide, up to the constant $m+2$, when $\gamma = m+2$, hence our analysis in Chapters 3, 4 and 5 will be in parallel for the two functionals.

In one of his last papers Ennio De Giorgi stated the following conjecture [**31**, Sect. 5, Conj. 2] – see [**32**] for an English translation.

CONJECTURE (Ennio De Giorgi). *Any compact, n–dimensional, smooth submanifold M of $\mathbb{R}^{n+m}$ without boundary, moving by the gradient of the functional*

$$\mathcal{DG}_k(M) = \int_M 1 + |\nabla^k \eta^M|^2 \, d\mathcal{H}^n \,,$$

where η^M is the square of the distance function from M and $\mathcal{H}^n$ is the n–dimensional Hausdorff measure in $\mathbb{R}^{n+m}$, does not develop singularities, if $k > n+1$.

This result is central in his program to approximate singular geometric flows with sequences of smooth ones that we will discuss in Chapter 6.
We will restrict our attention to the codimension one case, that is, the proof of this conjecture for hypersurfaces. Moreover, instead of dealing directly with the functionals $\mathcal{DG}_k$, we will analyze the gradient flow associated to the functionals

$$\mathcal{F}_m(\varphi) = \int_M 1 + |\nabla^m \nu|^2 \, d\mu$$

and we will then deduce the same conclusion for the original functionals of De Giorgi, thanks to their close connection. Moreover, these functionals can also play the same role in the approximation process in Chapter 6 that De Giorgi suggested.

The first step will be to obtain the following result (Theorem 3.1.3).

THEOREM. *If the differentiation order m is strictly larger than $[n/2]$, then the flows by the gradient of $\mathcal{DG}_{m+2}$ and $\mathcal{F}_m$ of any initial, smooth, compact, n–dimensional, immersed hypersurface of $\mathbb{R}^{n+1}$ exist, are unique and smooth for every positive time ($[n/2]$ means the integer part of $n/2$).*
Moreover, as $t \to +\infty$, the evolving hypersurface φ_t sub–converges (up to reparametrization and translation) to a smooth critical point of the respective functional.

Notice that the hypothesis $m > [n/2]$ in general is weaker than the original one in De Giorgi's conjecture.

The simplest case $n = 1$ and $m = 1$ of this theorem is concerned with curves in the plane evolving by the gradient flow of

$$\mathcal{F}_1(\gamma) = \int_{\mathbb{S}^1} 1 + \kappa^2 \, ds$$

since the curvature κ of a curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ satisfies $\kappa^2 = |\nabla \nu|^2$. The global regularity in such case was showed by Polden in the papers [**79, 80**] which have been a starting point for our work. Wen in [**92**] found results similar to Polden's ones, in considering the flow for $\int_{\mathbb{S}^1} k^2 \, ds$ of curves with a fixed length.

In Chapter 3 we will state and discuss the evolution problems which turn out to be quasilinear parabolic systems of PDE's on the manifold M. Moreover, we will deal with the very first step of the analysis, that is, showing the short time existence and uniqueness of a smooth flow. This is a particular case of a very general result of Polden proven in [**57, 80**].
Then, the long time existence is guaranteed as soon as we have suitable a priori estimates on the flow, which are discussed in Chapter 4.

In the study of the *mean curvature flow* of a hypersurface, which gives rise to a second order quasilinear parabolic system of PDE's, via techniques such as varifolds, level sets, viscosity solutions (see [**4, 7, 18, 40, 62**]), the maximum principle is the key tool to get comparison results and estimates on solutions. In our case, even if $m = 1$, the first variation and hence the corresponding parabolic problem turns out to be of order higher than two, precisely of order $2m + 2$, so we have to deal with equations of fourth order at least. This fact has the relevant consequence that we cannot employ the maximum principle to get pointwise estimates and to compare two solutions, thus losing a whole bunch of geometric results holding for the mean curvature flow. In particular, we cannot expect that an initially embedded hypersurface remains embedded during the flow, since self–intersections could develop (an example is given by Giga and Ito in [**50**]). By these reasons, techniques based on the description of the hypersurfaces as level sets of functions seems of difficult application in this case and therefore we adopt a parametric approach as in the work of Huisken [**54**].

Despite the large number of papers on the mean curvature flow, the literature on fourth or even higher order flows is quite limited. Our work borrows from [**26, 79, 80, 81**] the basic idea of using interpolation inequalities as a tool to get a priori estimates.
We want to remark here that a strong motivation for the study of these flows is actually the fact that, in general, regularity is not shared by second order flows, with the notable exceptions of the evolution by mean curvature of embedded curves in the plane (see [**46, 52, 56**]) and of convex hypersurfaces (see [**54**]). So our result opens the possibility to canonically approximate (possibly) singular flows with smooth ones by perturbation arguments (see [**31, 32**, Sect. 5] and the beginning of Chapter 6).

In order to show regularity, a good substitute of the pointwise estimates coming from the maximum principle, are suitable estimates on the second fundamental form in Sobolev spaces, using Gagliardo–Nirenberg interpolation type inequalities for tensors. Since the constants involved in these inequalities depends on the Sobolev constants and these latter on the geometry of the hypersurface where the tensors are defined, we absolutely need some uniform control independent of time on these constants. In [**79**] these controls are obvious as the constants depend only on the length, on the contrary, much more work is needed here because of the richer geometry of the hypersurfaces.

We will see in Chapter 4 that if m is large enough, the functionals $\mathcal{DG}_{m+2}$ and $\mathcal{F}_m$, which decrease during the flow, control the L^p norm of the second fundamental form for some exponent p larger than the dimension. This fact, combined with a universal Sobolev type inequality due to Michael and Simon [**76**], where the dependence of the constants on the curvature is made explicit, allows us to get an uniform bound on the Sobolev constants of the evolving hypersurfaces and then to obtain, in Chapter 5, time–independent estimates on curvature and all its derivatives in L^2. These bounds will imply in turn the desired pointwise estimates and the long time existence and regularity of the flows.

Pushing a little the analysis, it also follows that if we consider a general, positive, geometric functional

$$\mathcal{G}(\varphi) = \int_M f(\varphi, g, \mathrm{B}, \nu, \, \dots \, , \nabla^s \mathrm{B}, \nabla^l \nu) \, d\mathcal{H}^n \, ,$$

such that the function f is smooth and has polynomial growth, choosing an integer m large enough, the gradient flow of the "perturbed" functional, for any $\varepsilon > 0$,

$$\mathcal{G}^\varepsilon_m(\varphi) = \mathcal{G}(\varphi) + \varepsilon \mathcal{F}_m(\varphi)$$

does not develop singularities (the same holds if we perturb the functional $\mathcal{G}$ with $\varepsilon\mathcal{DG}_{m+2}$).
We then say that $\mathcal{F}_m$ and $\mathcal{DG}_{m+2}$ are *smoothing terms* for the functional $\mathcal{G}$, that possibly does not admit a gradient flow even for short time starting from a generic initial, smooth, compact, immersed hypersurface.
It is then natural to investigate what happens when the constant $\varepsilon > 0$ in front of these smoothing terms goes to zero.

This program, suggested by De Giorgi in [**31, 32**, Sect. 5], can be described as follows: given a geometric functional $\mathcal{G}$ defined on submanifolds of the Euclidean space (or a more general ambient space),

- find a functional $\mathcal{F}$ such that the perturbed functionals $\mathcal{G}^\varepsilon = \mathcal{G} + \varepsilon\mathcal{F}$ give rise to globally smooth flows for every $\varepsilon > 0$;

- study what happens when $\varepsilon \to 0$, in particular, the existence of a limit flow and in this case its relation with the gradient flow of $\mathcal{G}$ (if it exists, smooth or singular).

If proved successful, this scheme would give a *singular approximation* procedure of the gradient flow of $\mathcal{G}$ with a family of globally smooth flows.

Our work shows that the functionals $\mathcal{F}_m$ and $\mathcal{DG}_{m+2}$ satisfy the first point for any geometric functional $\mathcal{G}$ defined on hypersurfaces in $\mathbb{R}^{n+1}$ with polynomial growth, provided we choose an order m large enough (depending on $\mathcal{G}$).

About the second point, the very first case is concerned with the possible limits when $\varepsilon \to 0$ of the gradient flows of $\int_M 1 + \varepsilon|\nabla^m \nu|^2 \, d\mu$ when $m > [n/2]$ and their relation with the mean curvature flow, which is the gradient flow of the Area functional, obtained by letting $\varepsilon = 0$.

Ennio De Giorgi, in the same paper [**31**, Sect. 5, Cong. 3 and Oss. 2/3] cited above (see also [**32**, Sect. 5, Conj. 3 and Rem. 2/3]), *essentially* stated the following conjecture.

CONJECTURE (Ennio De Giorgi). *Let $m > [n/2]$, if the parameter $\varepsilon > 0$ goes to zero, the flows φ_t^ε associated to the functionals*

$$\mathcal{DG}_m^\varepsilon(M) = \int_M 1 + \varepsilon|\nabla^{m+2}\eta^M|^2 \, d\mathcal{H}^n$$

and starting from a common initial, smooth, compact, immersed hypersurface $\varphi_0 : M \to \mathbb{R}^{n+1}$, converge in some sense *to the mean curvature flow of φ_0,*

$$\frac{\partial \varphi}{\partial t} = H\nu$$

(which is the gradient flow associated to the limit Area functional, as $\varepsilon \to 0$).

De Giorgi proposed this conjecture in general codimension, in the following we will discuss only the case of evolving hypersurfaces, see anyway Remark 5.2.4 and Remark 6.3.3. Clearly, an analogous conjecture can be stated for the ε–parametrized family of functionals

$$\mathcal{F}_m^\varepsilon(M) = \int_M 1 + \varepsilon|\nabla^m \nu|^2 \, d\mu \, .$$

The goal of Chapter 6 will be to show the following theorem (Theorem 6.0.5), related to the above conjecture.

THEOREM. *Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth, compact, n–dimensional, immersed submanifold of $\mathbb{R}^{n+1}$. Let $T_{\mathrm{sing}} > 0$ be the first singularity time of the mean curvature flow $\varphi : M \times [0, T_{\mathrm{sing}}) \to \mathbb{R}^{n+1}$ of M. For any $\varepsilon > 0$ let $\varphi^\varepsilon : M \times [0, +\infty) \to \mathbb{R}^{n+1}$ be the flows associated to the functionals $\mathcal{DG}_m^\varepsilon$ (or $\mathcal{F}_m^\varepsilon$) with $m > [n/2]$, that is,*

$$\frac{\partial \varphi^\varepsilon}{\partial t} = H\nu + 2\varepsilon(m+2)(-1)^m \Big(\overbrace{\Delta^{M_t}\Delta^{M_t}\dots\Delta^{M_t}}^{m\text{–times}} H\Big)\,\nu + \varepsilon \mathrm{LOT}\,\nu \, ,$$

where LOT *denotes terms of lower order in the curvature and its derivatives, all starting from the same initial immersion φ_0.*
Then, the maps φ^ε converge locally in $C^\infty(M \times [0, T_{\mathrm{sing}}))$ to the map φ, as $\varepsilon \to 0$.

It is well known that a smooth compact submanifold of the Euclidean space, flowing by mean curvature, develops singularities in finite time. This is a common aspect of geometric evolutions, and motivates the study of the flow past singularities. Concerning the mean curvature motion, several notions of weak solutions have been proposed, after the pioneering work of Brakke [**18**], see for instance [**4, 7, 8, 13, 15, 25, 30, 40, 59, 60, 61, 62, 89**]. We recall that some of these solutions may differ, in particular in presence of the so–called fattening phenomenon (see for instance [**11**]). The above regularization of mean curvature flow with a singular perturbation of higher order could lead to a new definition of generalized solution in any dimension and codimension.

At the moment we are not able to show the existence or characterize the limits of the approximating flows after the first singularity, as the proof of the above theorem relies heavily on the smoothness of the mean curvature flow in the time interval of existence. Our future goal would be to show the existence of some limit flow defined *for all times*, thus providing a new definition of weak solution in any dimension and codimension.

As an example, we mention the simplest open problem in defining a limit flow after the first singularity. It is well known (Gage–Hamilton [**45, 46**] and Huisken [**54**]) that a convex curve in the plane (or hypersurface in $\mathbb{R}^{n+1}$) moving by mean curvature shrinks to a point in finite time, becoming exponentially round. In this case we expect that the approximating flows converge (in a way to be made precise) to such a point at every time after the extinction.

We remark here (but we will not discuss such an extension in this work) that our method works in general for any geometric evolution of submanifolds in a Riemannian manifold till the first singularity time, even when the equations are of high order (like, for instance, the Willmore flow, see [**66, 67, 93**]), choosing an appropriate regularizing term (of *higher order*).

Finally, it should be noted, comparing the evolution equations above with the one of the mean curvature flow, that these perturbations could be considered, in the framework of geometric evolution problems, as an analogue of the so–called *vanishing viscosity method* for PDE's. Indeed, we perturb the mean curvature flow equation with a regularizing higher order term multiplied by a small parameter $\varepsilon > 0$. The lower order terms, denoted by LOT, which appear, are due to the fact that we actually perturb the Area functional and not directly the evolution equation. However, the analogy with the classical viscosity method cannot be pushed too far. For instance, because of the condition $k > [n/2] + 2$, our regularized equations are of order not less than four (precisely at least four for evolving curves, at least six for evolving surfaces). Moreover, as the Laplacians appearing in the evolution equation are relative to the induced metric, the system of PDE's is actually quasilinear and the lower order terms are nonlinear (polynomial).

CHAPTER 1

Geometry of Submanifolds and Distance Functions

In this chapter we introduce the basic notations and we discuss the geometry of Riemannian submanifolds of the Euclidean space. Moreover, we analyze in detail the properties of the distance function from such submanifolds.

1.1. Geometry of Submanifolds

The main objects we will consider are n–dimensional, complete submanifolds, immersed in $\mathbb{R}^{n+m}$, that is, pairs (M, φ) where M is an n–dimensional smooth manifold, compact, connected with empty boundary, and a smooth map $\varphi : M \to \mathbb{R}^{n+m}$ such that the *rank* of $d\varphi$ is everywhere equal to n.
Good references for this section are [**36, 47**] (consider also [**63, 64**]).

The manifold M gets in a natural way a metric tensor g turning it in a Riemannian manifold (M, g), by pulling back the standard scalar product of $\mathbb{R}^{n+m}$ with the immersion map φ.

Taking local coordinates around $p \in M$ given by a chart $F : \mathbb{R}^n \supset U \to M$, we identify the map φ with its expression in coordinates $\varphi \circ F : \mathbb{R}^n \supset U \to \mathbb{R}^{n+m}$, then we have local basis of T_pM and T_p^*M, respectively given by vectors $\left\{\frac{\partial}{\partial x_i}\right\}$ and covectors $\{dx_j\}$.

We will denote vectors on M by $X = X^i$, which means $X = X^i \frac{\partial}{\partial x_i}$, covectors by $Y = Y_j$, that is, $Y = Y_j dx_j$ and a general mixed tensor with $T = T^{i_1\dots i_k}_{j_1\dots j_l}$, where the indices refer to the local basis.

In all the formulas the convention to sum over repeated indices will be adopted.

The tangent space at the point $p \in M$ can be clearly identified with the vector subspace $d\varphi_p(T_pM)$ of $T_{\varphi(p)}\mathbb{R}^{n+m} \approx \mathbb{R}^{n+m}$. Then, we define its m–dimensional orthogonal complement N_pM to be the *normal space* to M at p. Clearly the trivial vector bundle $T\mathbb{R}^{n+m}$ decomposes as $T\mathbb{R}^{n+m} = TM \oplus^{\perp} NM$, that is, the orthogonal direct sum of the tangent bundle and the normal bundle of M.

As the metric tensor g is induced by the scalar product of $\mathbb{R}^{n+m}$, which will be denoted with $\langle \cdot \,|\, \cdot \rangle$, we have

$$g_{ij}(x) = \left\langle \frac{\partial \varphi(x)}{\partial x_i} \,\middle|\, \frac{\partial \varphi(x)}{\partial x_j} \right\rangle .$$

The metric g extends canonically to tensors as follows,

$$g(T, S) = g_{i_1 s_1} \dots g_{i_k s_k} g^{j_1 z_1} \dots g^{j_l z_l} T^{i_1\dots i_k}_{j_1\dots j_l} S^{s_1\dots s_k}_{z_1\dots z_l}$$

where g^{ij} is the inverse of the matrix of the coefficients g^{ij}. Then we define the norm of a tensor T as

$$|T| = \sqrt{g(T, T)} \,.$$

By means of the scalar product of $\mathbb{R}^{n+m}$ we also define a metric tensor on the normal bundle and, as above, on all the tensors acting or with values in NM.

The canonical measure induced by the metric g is given by $\mu = \sqrt{G}\,\mathcal{L}^n$ where $G = \det(g_{ij})$ and $\mathcal{L}^n$ is the standard Lebesgue measure on $\mathbb{R}^n$.

The induced covariant derivatives on (M, g) of a tangent vector field X or of a 1–form ω are given by

$$\nabla^M_i X^j = \frac{\partial}{\partial x_i} X^j + \Gamma^j_{ik} X^k \qquad \text{and} \qquad \nabla^M_i \omega_j = \frac{\partial}{\partial x_i}\omega_j - \Gamma^k_{ij}\omega_k \,,$$

where the Christoffel symbols $\Gamma = \Gamma^k_{ij}$ are expressed by the following formula,

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{ij} - \frac{\partial}{\partial x_l} g_{ij} \right) .$$

It is well know that, for a pair of tangent vector fields X and Y on M, we have

$$\nabla^M_X Y = \left(\nabla^{\mathbb{R}^{n+m}}_X Y \right)^M$$

where the symbol ^{M} denotes the orthogonal projection on the tangent space of M.
Here, $\nabla^{\mathbb{R}^{n+m}}_X Y$ at a point $p \in M$ denotes the covariant derivative of $\mathbb{R}^{n+m}$ acting on some local extensions of the fields X and Y in an open subset of $\mathbb{R}^{n+m}$, once considered M (actually it is sufficient only a local embedding of M around p) as a subset of $\mathbb{R}^{n+m}$. This is a well defined expression, indeed, once identified any T_pM as a vector subspace of $\mathbb{R}^{n+m}$, the extensions of the vector fields X and Y are vector fields in the ambient space $\mathbb{R}^{n+m}$ and it is easy to check that $(\nabla^{\mathbb{R}^{n+m}}_X Y)(p)$ depends only on the values of the two fields on M in the embedded neighborhood of p, by the properties of the covariant derivative.

The covariant derivative $\nabla^M T$ of a tensor $T = T^{i_1 \dots i_k}_{j_1 \dots j_l}$ will be denoted by $\nabla^M_s T^{i_1 \dots i_k}_{j_1 \dots j_l} = (\nabla^M T)^{i_1 \dots i_k}_{s j_1 \dots j_l}$ and with $\nabla^k T$ we will mean the k–th iterated covariant derivative.

The gradient $\nabla^M f$ of a function and the divergence $\operatorname{div} X$ of a tangent vector field are defined respectively by

$$g(\nabla^M f, v) = df_p(v) \qquad \forall v \in TM$$

and

$$\operatorname{div} X = \operatorname{tr} \nabla^M X = \nabla^M_i X^i = \frac{\partial}{\partial x_i} X^i + \Gamma^i_{ik} X^k .$$

The Laplacian $\Delta^M T$ of a tensor T is

$$\Delta^M T = g^{ij} \nabla^M_i \nabla^M_j T .$$

Using the notion of connection and covariant derivative on fiber bundles (for instance, see **[63, 64]**), one can check that the following definition is actually the covariant derivative associated to the metric g on the normal bundle of M.
For any normal vector field ν on M and a tangent vector field X, we set

$$\nabla^\perp_X \nu = \left(\nabla^{\mathbb{R}^{n+m}}_X \nu \right)^\perp$$

where the symbol $^\perp$ denotes the orthogonal projection on the normal space of M.
Then, we can consider from now on the following definition of covariant derivative of any vector field (tangent or not) Y along M as follows

$$\nabla_X Y = \nabla^M_X Y^M + \nabla^\perp_X Y^\perp = \left(\nabla^{\mathbb{R}^{n+m}}_X Y^M \right)^M + \left(\nabla^{\mathbb{R}^{n+m}}_X Y^\perp \right)^\perp ,$$

where Y^M and $Y^\perp$ are respectively the tangent and normal components of the vector field Y.

We extend this covariant derivative also to "mixed" tensors, that is, tensors acting also on the normal bundle of M, not only on the tangent bundle.
For instance, if T "acts" on $(k+l)$–uple of fields along M such that the first k are tangent and the other l are normal, we have

$$\begin{aligned}
\nabla_X T(X_1, \dots, X_k, \nu_1, \dots, \nu_l) &= \nabla_X (T(X_1, \dots, X_k, \nu_1, \dots, \nu_l)) \\
&\quad - T(\nabla^M_X X_1, \dots, X_k, \nu_1, \dots, \nu_l) - \dots - T(X_1, \dots, \nabla^M_X X_k, \nu_1, \dots, \nu_l) \\
&\quad - T(X_1, \dots, X_k, \nabla^\perp_X \nu_1, \dots, \nu_l) - \dots - T(X_1, \dots, X_k, \nu_1, \dots, \nabla^\perp_X \nu_l)
\end{aligned}$$

where ∇_X immediately after the equality "works" according to the "target" bundle of T.

Associated to the connection $\nabla^\perp$ we have also a notion of curvature, called *normal curvature*, defined in the standard way.
For a pair of tangent vector fields X, Y and any normal vector field ν, we set

$$\mathrm{R}^\perp(X,Y)\nu = \nabla^\perp_Y \nabla^\perp_X \nu - \nabla^\perp_X \nabla^\perp_Y \nu - \nabla^\perp_{[Y,X]} \nu$$

and an associated $(0,4)$–curvature tensor $\mathrm{R}^\perp(X,Y,\nu,\xi) = g(\mathrm{R}^\perp(X,Y)\nu,\xi)$ which plays the same role of the Riemann tensor in exchanging the covariant derivatives in the normal bundle.
If ξ_α is a local basis of the normal bundle (which is locally trivial) and $\nu = \nu^\alpha\xi_\alpha$, we have

$$(\nabla^\perp)^2_{ij}\nu^\alpha - (\nabla^\perp)^2_{ji}\nu^\alpha = \mathrm{R}^\perp_{ij\beta\gamma}g^{\beta\alpha}\nu^\gamma\,.$$

It is then natural to consider the following couple of tensors (their tensor nature can be easily checked).
For a pair of tangent vector fields X and Y, the form

$$\mathrm{B}(X,Y) = \left(\nabla^{\mathbb{R}^{n+m}}_X Y\right)^\perp$$

measures the difference between the covariant derivative of (M,g) and the one of the ambient space $\mathbb{R}^{n+m}$, indeed

$$\nabla^M_X Y = \left(\nabla^{\mathbb{R}^{n+m}}_X Y\right)^M = \nabla^{\mathbb{R}^{n+m}}_X Y - \mathrm{B}(X,Y)\,. \tag{1.1.1}$$

For a tangent vector field X and a normal one ν,

$$\mathrm{S}(X,\nu) = -\left(\nabla^{\mathbb{R}^{n+m}}_X \nu\right)^M$$

which clearly satisfies

$$\nabla^\perp_X\nu = \left(\nabla^{\mathbb{R}^{n+m}}_X \nu\right)^\perp = \nabla^{\mathbb{R}^{n+m}}_X \nu + \mathrm{S}(X,\nu)\,.$$

The form B is called *second fundamental form* and it is a symmetric bilinear form with values in the normal bundle NM. Its symmetry can be seen easily as the two connections have no torsion,

$$\mathrm{B}(X,Y) - \mathrm{B}(Y,X) = \nabla^M_Y X - \nabla^M_X Y - \nabla^{\mathbb{R}^{n+m}}_Y X + \nabla^{\mathbb{R}^{n+m}}_X Y = [X,Y]_{\mathbb{R}^{n+m}} - [X,Y]_M = 0$$

and $d\varphi([X,Y]_M) = [d\varphi(X), d\varphi(Y)]_{\mathbb{R}^{n+m}}$.

The bilinear form S, with values in TM, can be seen as an operator $\mathrm{S}(\cdot,\nu) : TM \to TM$ (for every fixed normal vector field $\nu \in NM$) called *shape operator*. Actually, S is self–adjoint and B is the associated quadratic form, if X, Y are tangent vector fields and ν is a normal one, we have

$$\begin{aligned} g(Y,\mathrm{S}(X,\nu)) &= -\,g\Big(Y, \left(\nabla^{\mathbb{R}^{n+m}}_X \nu\right)^M\Big) = -g(Y, \nabla^{\mathbb{R}^{n+m}}_X \nu) \\ &= g(\nabla^{\mathbb{R}^{n+m}}_X Y, \nu) = g\Big(\left(\nabla^{\mathbb{R}^{n+m}}_X Y\right)^\perp, \nu\Big) \\ &= g(\mathrm{B}(X,Y),\nu)\,, \end{aligned} \tag{1.1.2}$$

hence, B and S can be recovered each other.
By the symmetry of B it follows that

$$g(Y,\mathrm{S}(X,\nu)) = g(X,\mathrm{S}(Y,Z))$$

hence, $\mathrm{S}(\cdot,\nu)$ is self–adjoint.
Finally, it is easy to check that $|\mathrm{B}|^2 = |\mathrm{S}|^2$ and also $|\nabla^k\mathrm{B}|^2 = |\nabla^k\mathrm{S}|^2$ for every $k\in\mathbb{N}$.

We extend the forms B and S to any vector field along M as follows

$$\begin{aligned} \mathrm{B}(X,Y) &= \mathrm{B}(X^M,Y^M)\,, \\ \mathrm{S}(X,Y) &= \mathrm{S}(X^M,Y^\perp)\,, \end{aligned} \tag{1.1.3}$$

and, for any normal vector field ν we set

$$\begin{aligned} \mathrm{B}^\nu(X,Y) &= \langle \nu \,|\, \mathrm{B}(X^M,Y^M)\rangle\,, \\ \mathrm{S}_\nu(X) &= \mathrm{S}(X^M,\nu)\,. \end{aligned}$$

Clearly, by equation (1.1.2), it follows $g(Y,\mathrm{S}_\nu(X)) = \mathrm{B}^\nu(X,Y)$.

Choosing a local coordinate basis in M, we have

$$\mathrm{B}_{ij} = \mathrm{B}(\partial_{x_i},\partial_{x_j}) = \left(\nabla^{\mathbb{R}^{n+m}}_{\partial_{x_i}}\partial_{x_j}\right)^\perp = \left(\frac{\partial}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\right)^\perp = \left(\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right)^\perp$$

and

$$\mathrm{B}^\nu_{ij} = \left\langle \nu \,\Big|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle.$$

$$(\mathrm{S}_\nu)_i = \mathrm{S}(\partial_{x_i}, \nu) = -\left(\frac{\partial \nu}{\partial x_i}\right)^M.$$

which are the more familiar definition of second fundamental form and of the shape operator.
The *mean curvature vector* H is the trace (with the induced metric) of the second fundamental form,

$$\mathrm{H} = g^{ij}\mathrm{B}_{ij}\,,$$

by this definition, clearly $\mathrm{H} \in NM$. We also define $\mathrm{H}^\nu = g^{ij}\mathrm{B}^\nu_{ij}$.

Making explicit equation (1.1.1) and using identity (1.1.2) we have the so called *Gauss–Weingarten relations*,

$$\frac{\partial^2\varphi}{\partial x_i \partial x_j} = \Gamma^k_{ij}\frac{\partial \varphi}{\partial x_k} + \mathrm{B}_{ij} \qquad \left(\frac{\partial \nu}{\partial x_i}\right)^M = -\mathrm{B}^\nu_{ik} g^{kj}\frac{\partial\varphi}{\partial x_j}$$

for every normal vector field ν along M.
Notice that the first relation implies

$$\Delta^M \varphi = g^{ij}\nabla^2_{ij}\varphi = g^{ij}\left(\frac{\partial^2\varphi}{\partial x_i\partial x_j} - \Gamma^k_{ij}\frac{\partial\varphi}{\partial x_k}\right) = g^{ij}\mathrm{B}_{ij} = \mathrm{H}\,,$$

component by component.

The second fundamental form B embodies all information on the curvature properties of M, this is expressed by the following relations with the Riemann curvature tensor of (M, g),

$$\begin{aligned}
\mathrm{R}_{ijkl} &= g(\nabla^2_{ji}\partial_{x_k} - \nabla^2_{ij}\partial_{x_k}, \partial_{x_l}) = \langle \mathrm{B}_{ik} \,|\, \mathrm{B}_{jl}\rangle - \langle \mathrm{B}_{il} \,|\, \mathrm{B}_{jk}\rangle\,,\\
\mathrm{R}_{ij} &= g^{kl}\mathrm{R}_{ikjl} = \langle \mathrm{H} \,|\, \mathrm{B}_{ij}\rangle - g^{kl}\langle \mathrm{B}_{il} \,|\, \mathrm{B}_{kj}\rangle\,,\\
\mathrm{R} &= g^{ij}\mathrm{Ric}_{ij} = |\mathrm{H}|^2 - |\mathrm{B}|^2\,,
\end{aligned}$$

where the scalar products are meant in the normal space to M.

REMARK 1.1.1. These equations are often called *Gauss equations* by the connection with his *Theorema Egregium* about the invariance by isometry of the Gaussian curvature G of a surface in $\mathbb{R}^3$, which is actually expressed by the third equation, once we rewrite it as $\mathrm{R} = 2\mathrm{G}$.
We recall that the Gaussian curvature of a surface is the product of the principal eigenvalues of B (in codimension one, B can be seen as a *real valued* bilinear form, as we will see in a while). Equivalently, $\mathrm{G} = \det \mathrm{S}_\nu$ where ν is a local unit normal vector field.

Then, the formulas for the interchange of covariant derivatives, which involve the Riemann tensor, become

$$\begin{aligned}
\nabla^M_i\nabla^M_j X^s - \nabla^M_j\nabla^M_i X^s &= \mathrm{R}_{ijkl}g^{ks}X^l = (\langle \mathrm{B}_{ik} \,|\, \mathrm{B}_{jl}\rangle - \langle \mathrm{B}_{il} \,|\, \mathrm{B}_{jk}\rangle)\, g^{ks}X^l\,,\\
\nabla^M_i\nabla^M_j \omega_k - \nabla^M_j\nabla^M_i \omega_k &= \mathrm{R}_{ijkl}g^{ls}\omega_s = (\langle \mathrm{B}_{ik} \,|\, \mathrm{B}_{jl}\rangle - \langle \mathrm{B}_{il} \,|\, \mathrm{B}_{jk}\rangle)\, g^{ls}\omega_s\,.
\end{aligned} \tag{1.1.4}$$

About the normal curvature, the analogues of Gauss equations are called *Ricci equations*. If ξ_α is a local basis of the normal bundle we have,

$$\mathrm{R}^\perp_{ij\alpha\beta} = -g([\mathrm{S}_\alpha, \mathrm{S}_\beta]\partial_{x_i}, \partial_{x_j})$$

where S_α and S_β are respectively the operators S_{ξ_α} and S_{ξ_β} and $[\mathrm{S}_\alpha, \mathrm{S}_\beta]$ denotes the commutator operator $\mathrm{S}_\alpha\mathrm{S}_\beta - \mathrm{S}_\beta\mathrm{S}_\alpha : TM \to TM$.
Hence, the formula for the interchange of derivatives on the normal bundle become

$$\nabla^\perp_i\nabla^\perp_j\nu^\alpha - \nabla^\perp_j\nabla^\perp_i\nu^\alpha = \mathrm{R}^\perp_{ij\beta\gamma}g^{\beta\alpha}\nu^\gamma = g([\mathrm{S}_\gamma, \mathrm{S}_\beta]\partial_{x_i}, \partial_{x_j})g^{\beta\alpha}\nu^\gamma\,,$$

for every normal vector field $\nu = \nu^\alpha\xi_\alpha$.

Finally, the following *Codazzi equations* hold

$$(\nabla_X \mathrm{B})(Y, Z, \nu) = (\nabla_Y \mathrm{B})(X, Z, \nu) \tag{1.1.5}$$

for every three tangent vector fields X, Y, Z and $\nu \in NM$.
These equation are sometimes also called *Codazzi–Mainardi equations* as Delfino Codazzi **[27]** and

Gaspare Mainardi [**72**] independently derived them (actually, they were discovered earlier by Karl M. Peterson [**78**]).
They can be seen as an analogue of the II Bianchi identity satisfied by the Riemann tensor.

The importance of the Gauss, Ricci and Codazzi equations is that they are the analogues of the Frenet equations for space curves. They determine, up to isometry of the ambient space, the immersed submanifold, as it is expressed by the following fundamental theorem (first proved for surfaces in $\mathbb{R}^3$ by Pierre Ossian Bonnet [**16, 17**]), see [**14**, Chap. 2].

THEOREM 1.1.2. *Let (M, g) be an n–dimensional Riemannian manifold with a Riemannian vector bundle NM of rank m. Let $\nabla^\perp$ a metric connection on NM and B a symmetric bilinear form with values in NM. Define the operator $\mathrm{S}(\cdot,\nu) : TM \to TM$ by $g(Y, \mathrm{S}_\nu(X)) = \langle \nu \,|\, \mathrm{B}(X, Y)\rangle$ and suppose that the equations of Gauss, Ricci and Codazzi are satisfied by these tensors.*
Then, around any point $p \in M$ there exists an open neighborhood $U \subset M$ and an isometric immersion $\varphi : U \to \mathbb{R}^{n+m}$ such that B coincides with the second fundamental form of the immersion φ and NM is isomorphic to the normal bundle.
The immersion is unique up to an isometry of $\mathbb{R}^{n+m}$, moreover, if two immersions have the same second fundamental form and normal connection, they locally coincide up to an isometry of $\mathbb{R}^{n+m}$.

A consequence of Codazzi equations is the following computation of the difference between $\Delta\mathrm{B}$ and $\nabla^2\mathrm{H}$,

$$
\begin{aligned}
\Delta \mathrm{B}^\alpha_{ij} - \nabla^2_{ij}\mathrm{H}^\alpha &= g^{pq}\left\{\nabla^2_{pq}\mathrm{B}^\alpha_{ij} - \nabla^2_{ij}\mathrm{B}^\alpha_{pq}\right\} \\
&= g^{pq}\left\{\nabla^2_{pi}\mathrm{B}^\alpha_{qj} - \nabla^2_{ij}\mathrm{B}^\alpha_{pq}\right\} \\
&= g^{pq}\left\{\nabla^2_{ip}\mathrm{B}^\alpha_{qj} - \nabla^2_{ij}\mathrm{B}^\alpha_{pq}\right\} \\
&\quad + g^{pq}\left(\langle \mathrm{B}_{pq} \,|\, \mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{iq}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sj} \\
&\quad + g^{pq}\left(\langle \mathrm{B}_{pj} \,|\, \mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{ij}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sq} \\
&\quad + g^{pq} g([\mathrm{S}_\gamma, \mathrm{S}_\beta]\partial_{x_p}, \partial_{x_i}) g^{\beta\alpha}\mathrm{B}^\gamma_{qj} \\
&= \left(\langle \mathrm{H} \,|\, \mathrm{B}_{il}\rangle - g^{pq}\langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{iq}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sj} \\
&\quad + g^{pq}\left(\langle \mathrm{B}_{pj} \,|\, \mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{ij}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sq} \\
&\quad + g^{pq}\left[g(\mathrm{S}_\beta(\partial_{x_p}), \mathrm{S}_\gamma(\partial_{x_i})) - g(\mathrm{S}_\beta(\partial_{x_p}), \mathrm{S}_\gamma(\partial_{x_i}))\right] g^{\beta\alpha}\mathrm{B}^\gamma_{qj} \\
&= \left(\langle \mathrm{H} \,|\, \mathrm{B}_{il}\rangle - g^{pq}\langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{iq}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sj} \\
&\quad + g^{pq}\left(\langle \mathrm{B}_{pj} \,|\, \mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{ij}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sq} \\
&\quad + g^{pq}\left(\mathrm{B}^\beta_{pk} g^{kl}\mathrm{B}^\gamma_{il} - \mathrm{B}^\gamma_{pk} g^{kl}\mathrm{B}^\beta_{il}\right) g^{\beta\alpha}\mathrm{B}^\gamma_{qj} \\
&= \left(\langle \mathrm{H} \,|\, \mathrm{B}_{il}\rangle - g^{pq}\langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{iq}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sj} \\
&\quad + \left(\langle \mathrm{B}_{pj} \,|\, \mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{ij}\rangle\right) g^{pq} g^{ls}\mathrm{B}^\alpha_{sq} \\
&\quad + \langle \mathrm{B}_{il} \,|\, \mathrm{B}_{qj}\rangle g^{pq} g^{kl}\mathrm{B}^\alpha_{pk} - \langle \mathrm{B}_{pk} \,|\, \mathrm{B}_{qj}\rangle g^{pq} g^{kl}\mathrm{B}^\alpha_{il} \\
&= \langle \mathrm{H} \,|\, \mathrm{B}_{il}\rangle g^{ls}\mathrm{B}^\alpha_{sj} - \langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{iq}\rangle g^{pq} g^{ls}\mathrm{B}^\alpha_{sj} - \langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{jq}\rangle g^{pq} g^{ls}\mathrm{B}^\alpha_{si} \\
&\quad + \left(2\langle \mathrm{B}_{pj} \,|\, \mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl} \,|\, \mathrm{B}_{ij}\rangle\right) g^{pq} g^{ls}\mathrm{B}^\alpha_{sq}\,.
\end{aligned}
\tag{1.1.6}
$$

Hence, such a difference is a third order homogeneous polynomial in B.

All the relations we discussed in this section are valid in the Euclidean ambient space. When the ambient space is a general Riemannian manifolds all the formulas need a correction term due to its curvature. See [**36**, Chap. 6] *and* [**14**, Chap. 2].

1.1.1. The Codimension One Case. When the codimension is one, the normal space is one–dimensional, so at least locally we can define up to a sign (sometimes we will have to deal with this ambiguity) a smooth unit local normal vector field to M.
Actually, if the hypersurface M is orientable, this choice can be done globally.

In the case the hypersurface M is compact and embedded (hence, it is also orientable), we will always consider ν to be the unit *inner* normal.

The second fundamental form B then coincides with $\mathrm{B}^\nu\nu$, hence in this case we can actually consider the $\mathbb{R}$–valued bilinear form B^ν that, for sake of simplicity, we still call B, for all this section.

We will denote with H the mean curvature *function* $\mathrm{H}^\nu = g^{ij}\mathrm{B}^\nu_{ij}$ and with S the shape operator $\mathrm{S}_\nu = \mathrm{S}(\cdot,\nu) : TM \to TM$.
Notice that B, S and H are defined up to the sign of ν (with the conventional choice above, the second fundamental form of a convex hypersurface is nonnegative definite).

In the codimension one case are commonly defined the so called *principal curvatures* of M at a point p, as the eigenvalues of the form B (defined up to a sign).
The relative eigenvectors in T_pM are called *principal directions*.

In this case, many of the previous formula simplifies, as every derivative of ν must be a tangent field, hence, in particular $\nabla^\perp\nu = 0$,

$$\begin{aligned}
&\nabla^M_X Y = \nabla^{\mathbb{R}^{n+m}}_X Y - \mathrm{B}(X,Y)\nu\\
&\nabla^{\mathbb{R}^{n+m}}_X \nu = -\mathrm{S}(X)\\
&g(Y,\mathrm{S}(X)) = \mathrm{B}(X,Y)\\
&\mathrm{B}_{ij} = \left\langle \nu \,\middle|\, \frac{\partial^2\varphi}{\partial x_i\partial x_j}\right\rangle
\end{aligned}$$

The Gauss–Weingarten relations become

$$\frac{\partial^2\varphi}{\partial x_i\partial x_j} = \Gamma^k_{ij}\frac{\partial\varphi}{\partial x_k} + \mathrm{B}_{ij}\nu \qquad \frac{\partial\nu}{\partial x_i} = -\mathrm{B}_{ik}g^{kj}\frac{\partial\varphi}{\partial x_j}\,. \tag{1.1.7}$$

The Riemann curvature tensor of (M,g) is given by,

$$\begin{aligned}
\mathrm{R}_{ijkl} &= \mathrm{B}_{ik}\mathrm{B}_{jl} - \mathrm{B}_{il}\mathrm{B}_{jk}\,,\\
\mathrm{R}_{ij} &= \mathrm{H}\mathrm{B}_{ij} - g^{lk}\mathrm{B}_{il}\mathrm{B}_{kj}\,,\\
\mathrm{R} &= |\mathrm{H}|^2 - |\mathrm{B}|^2\,.
\end{aligned}$$

Notice that in these last formulas the ambiguity of the definition up to a sign of B and H vanishes.
The Ricci equations are in this case trivial, the Codazzi equations get the simple form

$$\nabla^M_i \mathrm{B}_{jk} = \nabla^M_j \mathrm{B}_{ik}$$

and imply the following *Simons' identity* **[88]**

$$\Delta^M \mathrm{B}_{ij} = \nabla^2_{ij}\mathrm{H} + \mathrm{H}\,\mathrm{B}_{il}g^{ls}\mathrm{B}_{sj} - |\mathrm{B}|^2\mathrm{B}_{ij}\,. \tag{1.1.8}$$

Indeed, recalling the computation (1.1.6), as the normal space is one–dimensional, we have

$$\begin{aligned}
\Delta^M \mathrm{B}_{ij} - \nabla^2_{ij}\mathrm{H} =& \,\mathrm{H}\mathrm{B}_{il}g^{ls}\mathrm{B}_{sj} - \mathrm{B}_{pl}\mathrm{B}_{iq}g^{pq}g^{ls}\mathrm{B}_{sj} - \mathrm{B}_{pl}\mathrm{B}_{jq}g^{pq}g^{ls}\mathrm{B}_{si}\\
&+ (2\mathrm{B}_{pj}\mathrm{B}_{il} - \mathrm{B}_{pl}\mathrm{B}_{ij})\,g^{pq}g^{ls}\mathrm{B}_{sq}\\
=& \,\mathrm{H}\mathrm{B}_{il}g^{ls}\mathrm{B}_{sj} - |\mathrm{B}|^2\mathrm{B}_{ij}\,.
\end{aligned}$$

1.1.2. Example 1. Curves in the Plane. Let $\gamma : (0,1) \to \mathbb{R}^2$ be a smooth curve in the plane, suppose parametrized by the arclength s.
The metric is simply by ds^2, we define the unit tangent vector $\tau = \gamma_s$ and we choose as unit normal vector $\nu = \mathrm{R}\tau$ where R is the counterclockwise rotation in $\mathbb{R}^2$.
The second fundamental form is given by

$$\mathrm{B}_{ss} = \mathrm{B}(\tau,\tau) = \left(\nabla^{\mathbb{R}^{n+m}}_\tau \tau\right)^\perp = (\partial_\tau\gamma_s)^\perp = \gamma^\perp_{ss} = \gamma_{ss}$$

as γ_{ss} is a normal vector.
In the case the curve is not parametrized by arclength, the metric tensor is given by $g_{ss} = |\gamma_s|^2ds^2$

and

$$\mathrm{B}_{ss} = \mathrm{B}(\tau,\tau) = \left(\nabla^{\mathbb{R}^{n+m}}_\tau \tau\right)^\perp = (\partial_\tau \gamma_s)^\perp = \gamma_{ss}^\perp = \gamma_{ss} - \frac{\langle \gamma_{ss} \,|\, \gamma_s \rangle \gamma_s}{|\gamma_s|^2}\,.$$

The mean curvature vector H is then

$$\mathrm{H} = g^{ss}\mathrm{B}_{ss} = \frac{\gamma_{ss}}{|\gamma_s|^2} - \frac{\langle \gamma_{ss} \,|\, \gamma_s \rangle \gamma_s}{|\gamma_s|^4} = \mathrm{k}\nu\,.$$

The mean curvature function k, which is defined up to the sign, is called by simplicity the *curvature* of γ.

1.1.3. Example 2. Curves in $\mathbb{R}^n$. Let $\gamma : (0,1) \to \mathbb{R}^n$ be a smooth curve in the space, parametrized by the arclength s.
The metric is again given by ds^2, and we still define the unit tangent vector $\tau = \gamma_s$ but now we do not have an easy way to choose a unit normal vector as in the previous situation.
The second fundamental form is given by

$$\mathrm{B}_{ss} = \mathrm{B}(\tau,\tau) = \left(\nabla^{\mathbb{R}^{n+m}}_\tau \tau\right)^\perp = (\partial_\tau \gamma_s)^\perp = \gamma_{ss}^\perp = \gamma_{ss}$$

as γ_{ss} is a normal vector. If $\gamma_{ss} \neq 0$ we define $|\gamma_{ss}| = \mathrm{k} \neq 0$ and call *unit normal of* γ the vector $\nu = \gamma_{ss}/|\gamma_{ss}|$, that is, $\gamma_{ss} = \mathrm{k}\nu$ and k is the (mean) curvature of γ which is defined up to the sign.

1.2. Some Extra Conventions

Now we introduce some non standard notation which will be useful in the computations of the following chapters.

We will write $T * S$, following Hamilton [**53**], to denote a tensor formed by contraction on some indices of the tensors T and S using the coefficients g^{ij}.
Abusing a little the notation, if $T_1, \dots, T_l$ is a finite family of tensors (here l is not an index of the tensor T), with the symbol

$$\mathop{\circledast}_{i=1}^{l} T_i$$

we will mean $T_1 * T_2 * \cdots * T_l$.

We will use the symbol $\mathfrak{p}_s(T_1, \dots, T_l)$ for a polynomial in the tensors $T_1, \dots, T_l$ and their iterated covariant derivatives with the $*$ product like

$$\mathfrak{p}_s(T_1, \dots, T_l) = \sum_{i_1+\cdots+i_l=s} c_{i_1\dots i_l}\, \nabla^{i_1} T_1 * \cdots * \nabla^{i_l} T_l\,,$$

where the $c_{i_1\dots i_l}$ are some real constants.
Notice that every tensor T_i must be present in every additive term of $\mathfrak{p}_s(T_1, \dots, T_l)$ and there are not repetitions.
We will use instead the symbol $\mathfrak{q}^s$ when we are in the codimension one case, a unit normal vector field is defined (at least locally) and the tensors involved are only B and $\nabla\nu$. Moreover, repetitions are allowed in $\mathfrak{q}^s$ and in every additive term there must be present every argument of $\mathfrak{q}^s$, for instance,

$$\mathfrak{q}^s(\nabla\nu, \mathrm{B}) = \sum \left(\mathop{\circledast}_{k=1}^{N} \nabla^{i_k}(\nabla\nu) \mathop{\circledast}_{l=1}^{M} \nabla^{j_l}\mathrm{B} \right) \qquad \text{with } N, M \geq 1.$$

The order s denotes the sum

$$s = \sum_{k=1}^{N}(i_k+1) + \sum_{l=1}^{M}(j_l+1)\,.$$

REMARK 1.2.1. Supposing that $\mathfrak{q}^s$ is completely contracted, that is, there are no free indices and we get a function, then the order s has the following strong geometric meaning, if we consider the family of homothetic immersions $\lambda\varphi : M \to \mathbb{R}^{n+1}$ for $\lambda > 0$, they have associated normal ν^λ, metric g^λ, connection ∇^λ and second form B^λ satisfying the following rescaling equations,

$$(\nabla^\lambda)^i \nu^\lambda = \nabla^i \nu \qquad (\nabla^\lambda)^j \mathrm{B}^\lambda = \lambda \nabla^j \mathrm{B}\,,$$

$$(g^\lambda)_{ij} = \lambda^2 g_{ij} \qquad (g^\lambda)^{ij} = \lambda^{-2} g^{ij}\,.$$

Then every completely contracted polynomial $\mathfrak{q}^s$ in $\nabla\nu$ and B will have the form

$$\sum (\nabla^{i_1}\nabla\nu)\dots(\nabla^{i_k}\nabla\nu)\dots(\nabla^{i_N}\nabla\nu)\,\nabla^{j_1}\mathrm{B}\dots\nabla^{j_l}\mathrm{B}\dots\nabla^{j_M}\mathrm{B}\,g^{w_1z_1}\dots g^{w_tz_t}$$

with

$$s = \sum_{k=1}^{N}(i_k+1)\ +\ \sum_{l=1}^{M}(j_l+1)$$

and since the contraction is total it must be

$$t = \frac{1}{2}\left(\sum_{k=1}^{N}(i_k+1) + \sum_{l=1}^{M}(j_l+2)\right)$$

as the sum between the large brackets give the number of covariant indices in the product above. By this argument and the rescaling equations above, we see that $\mathfrak{q}^s$ rescales as

$$\begin{aligned}\mathfrak{q}^s(\nabla^\lambda\nu^\lambda,\dots,\mathrm{B}^\lambda)\ &= \lambda^{M-2t}\mathfrak{q}^s(\nabla\nu,\dots,\mathrm{B})\\ &= \lambda^{-\left(\sum_{k=1}^{N}(i_k+1)+\sum_{l=1}^{M}(j_l+1)\right)}\mathfrak{q}^s(\nabla\nu,\dots,\mathrm{B})\\ &= \lambda^{-s}\mathfrak{q}^s(\nabla\nu,\dots,\mathrm{B})\,.\end{aligned}$$

By this reason, with a little misuse of language, also when $\mathfrak{q}^s$ is not completely contracted, we will say that s is the *rescaling order* of $\mathfrak{q}^s$.

In most of the computations only the rescaling order and the arguments of the polynomials involved will be important, so we will avoid to make explicit their inner structure.
An example in this spirit, are the following substitutions that we will often apply

$$\nabla\mathfrak{p}_s(T_1,\dots,T_l) = \mathfrak{p}_{s+1}(T_1,\dots,T_l) \quad \text{and} \quad \nabla\mathfrak{q}^z(\nabla\nu,\dots,\mathrm{B}) = \mathfrak{q}^{z+1}(\nabla\nu,\dots,\mathrm{B})\,.$$

We advise the reader that the polynomials $\mathfrak{p}_s$ and $\mathfrak{q}^z$ could vary from a line to another in a computation by addition of terms with the same rescaling order. Moreover, also the constants could vary between different formulas and from a line to another.

1.3. Tangential Calculus

We consider now M as an actual subset of $\mathbb{R}^{n+m}$, in order to use the coordinates of the ambient space $\mathbb{R}^{n+m}$, we can always do it at least locally as every immersion is locally an embedding. At every point $x \in M$ we have, as before, the n–dimensional tangent space $T_xM \subset \mathbb{R}^{n+m}$ with an associated linear map $P(x) : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ which is the orthogonal projection on T_xM. Then clearly, the map $(I - P(x)) : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, where I is the identity of $\mathbb{R}^{n+m}$, is instead the orthogonal projection on the m–dimensional normal space NM at x which is the orthogonal complement of T_xM in $\mathbb{R}^{n+m}$.

In this setting, the canonical measure $\mu = \sqrt{G}\,\mathcal{L}^n$ coincides with the n–dimensional Hausdorff measure *counting multiplicities* $\widetilde{\mathcal{H}}^n \llcorner M$.
If M is actually embedded (or the self–intersections have zero measure), we have $\mu = \mathcal{H}^n \llcorner M$ with $\mathcal{H}^n$ the n–dimensional Hausdorff measure of $\mathbb{R}^{n+m}$.

We call *tangential gradient* $\nabla^M f(x)$ of a C^1 function defined in a neighborhood $U \subset \mathbb{R}^{n+m}$ of a point $x \in M$ as the projection of $\nabla^{\mathbb{R}^{n+m}} f(x)$ on T_xM.
It is easy to check that $\nabla^M f$ depends only on the restriction of f to $M \cap U$. Moreover, an extension argument shows that $\nabla^M f$ can also be defined for functions initially defined only on $M \cap U$.
If P_{ij} is the matrix of orthogonal projection $P : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ on the tangent space (here the indices refer to the coordinates of $\mathbb{R}^{n+m}$), we have $\nabla^M_i f(x) = P_{ij}(x)\nabla^j f(x)$.
Notice that $P_{ij}(x) = \nabla^M_i x_j$ for any $x \in M$.

We also define the *tangential derivative* of a vector field $Y = Y^i e_i$ in $\mathbb{R}^{n+m}$ along M, in the direction of a tangent vector $X \in T_x M$ as

$$\nabla^M_X Y(x) = \sum_{i=1}^{n+m} \langle X \,|\, \nabla^M Y^i \rangle e_i$$

where $e_1, \dots, e_{n+m}$ is the standard basis of $\mathbb{R}^{n+m}$.

In a similar way we can define the *tangential divergence* of a vector field X and the *tangential Laplacian* of a function,

$$\operatorname{div}^M X = \sum_{i=1}^{n+m} \nabla^M_i X^i \,, \qquad\qquad \Delta^M f = \operatorname{div}^M \nabla^M f$$

(here again the indices refer to the coordinates of $\mathbb{R}^{n+m}$).
By a straightforward computation one can check that all these tangential operators (if the field X is tangent to M) coincide with the intrinsic ones considering (M, g) as an abstract Riemannian manifold.

In several occasions we will consider the second fundamental form and the shape operator acting on vector fields in $\mathbb{R}^{n+m}$ as defined in formulas (1.1.3), that is, if $e_1, \dots, e_{n+m}$ is the standard basis of $\mathbb{R}^{n+m}$ we have

$$\mathrm{B}^k_{ij} = \langle \mathrm{B}(e_i, e_j) \,|\, e_k \rangle = \langle \mathrm{B}(e^M_i, e^M_j) \,|\, e^\perp_k \rangle \,.$$

It is then easy to see that

$$\mathrm{H}^i = \sum_{j=1}^{n+m} \mathrm{B}^i_{jj}$$

and, by means of the above tangential derivative operator, we can compute the second fundamental form as

$$\mathrm{B}(X, Y) = -\sum_{\alpha=1}^{m} \langle X | \nabla^M_Y \nu^\alpha \rangle \nu^\alpha \qquad\qquad \forall X\, Y \in T_x M \,, \tag{1.3.1}$$

where $\{\nu^\alpha\}$ is any local smooth orthonormal basis of the normal space to M.

For a general smooth map $\Phi : M \to \mathbb{R}^k$ we can consider the *tangential Jacobian*,

$$J^M \Phi(x) = \left[\det\left(d^M \Phi^*_x \circ d^M \Phi_x\right)\right]^{1/2}$$

where $d^M \Phi_x : T_x M \to \mathbb{R}^k$ is the linear map induced by the the tangential gradient and $\left(d^M \Phi_x\right)^* : \mathbb{R}^k \to T_x M$ is the adjoint map.

THEOREM 1.3.1 (Area Formula). *If Φ is a smooth injective map from M to $\mathbb{R}^k$, then we have*

$$\int_{\Phi(M)} f(y) \, d\mathcal{H}^n(y) = \int_M f(\Phi(x)) \, J^M \Phi(x) \, d\mathcal{H}^n(x) \tag{1.3.2}$$

for every $f \in C^0_c(\mathbb{R}^k)$.

If $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^{n+m}$ such that $e_1, \dots, e_n$ is a basis of $T_x M$, we can express the divergence of a tangent vector field X at the point $x \in M$ as

$$\begin{aligned}
\operatorname{div} X(x) &= \sum_{i=1}^{n} g(e_i, \nabla_{e_i} X(x)) = \sum_{i=1}^{n} \langle e_i \,|\, \nabla^{\mathbb{R}^{n+m}}_{e_i} X(x) \rangle = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \langle e_i \,|\, X \rangle(x) \\
&= \sum_{i=1}^{n+m} \nabla^M_i \langle e_i \,|\, X \rangle(x) \,.
\end{aligned}$$

It is not difficult to see that the last term is actually independent of the orthonormal basis $\{e_i\}$, even if $e_1, \dots, e_n$ is not a basis of $T_x M$. Then, we use this last expression (for any arbitrary orthonormal basis $\{e_i\}$ of $\mathbb{R}^{n+m}$) to define the tangential divergence $\operatorname{div}^M X$ of a general, not necessarily tangent, vector field X along M.

Such definition is useful in view of the following *tangential divergence formula* (see [**86**, Chap. 2, Sect. 7]),

$$\int_M \operatorname{div}^M X \, d\mu = -\int_M \langle X \,|\, \mathrm{H} \rangle \, d\mu \tag{1.3.3}$$

holding for every vector field X along M.
If X is a tangent vector field we recover the usual *divergence theorem*,

$$\int_M \operatorname{div} X \, d\mu = 0\,. \tag{1.3.4}$$

For detailed discussions and proofs of these results we address the reader to the books of Federer [**43**] *and of Simon* [**86**].

1.4. Distance Functions

In all this section, $e_1, \dots, e_{n+m}$ is the canonical basis of $\mathbb{R}^{n+m}$, M is a smooth, complete, n–dimensional manifold without boundary, embedded in $\mathbb{R}^{n+m}$ and T_xM, N_xM are respectively the tangent space and the normal space to M at $x \in M \subset \mathbb{R}^{n+m}$.

The *distance function* $d^M : \mathbb{R}^{n+m} \to \mathbb{R}$ and the *squared distance function* $\eta^M : \mathbb{R}^{n+m} \to \mathbb{R}$ are respectively defined by

$$d^M(x) = \operatorname{dist}(x, M) = \min_{y \in M} |x - y|, \qquad \eta^M(x) = \frac{1}{2}[d^M(x)]^2$$

for any $x \in \mathbb{R}^{n+m}$ (we will often drop the superscript M). In this and the next sections we analyze the differentiability properties of d and η and the connection between the derivatives of these functions and the geometric properties of M.

Immediately by its definition, being the minimum of a family of Lipschitz functions with Lipschitz constant 1, the same property holds also for d (the function η is instead only locally Lipschitz). In particular, both functions are differentiable almost everywhere in $\mathbb{R}^{n+m}$, by Rademacher's theorem, moreover, at any differentiability point $x \in \mathbb{R}^{n+m}$ of d there exists a *unique* minimizing point $y \in M$ such that $d(x) = |x - y|$ and

$$\nabla d(x) = \frac{x - y}{|x - y|}$$

for such $y \in M$.
Viceversa, if the point in M of minimum distance from $x \in \mathbb{R}^{n+m} \setminus M$ is unique, the function d is differentiable at x, see Section 1.6.
We have also easily

$$|\nabla d(x)| = 1 \qquad \text{and} \qquad |\nabla \eta(x)|^2 = 2\eta(x) \tag{1.4.1}$$

at any differentiability point of d.

These properties are true even if M is merely a closed set (the relation between the regularity properties of d^M and M is analyzed in detail in [**42, 44**]), as we will see in Section 1.6, but on the second derivatives of d^M and η^M only one side estimates are available, in general. These are actually based on the convexity of the function $A^M(x) = |x|^2/2 - \eta(x)$ which can be expressed as

$$A^M(x) = \max_{y \in M} \ \langle x | y \rangle - \frac{1}{2}|y|^2\,.$$

However, as it is natural to expect, higher regularity of M leads to higher regularity of d^M and η^M as we will see in Section 1.6 (see also [**7**], for instance).

PROPOSITION 1.4.1. *For every point $x \in M$, there exists an open neighborhood of x in $\mathbb{R}^{n+m}$ and a constant $\sigma > 0$ such that η is smooth in the region*

$$\Omega = \big\{ y \in U \,|\, d(y) < \sigma \big\}\,.$$

REMARK 1.4.2. If M is compact we can actually choose $U = \mathbb{R}^{n+m}$ and a uniform constant $\sigma > 0$. Moreover, since we will be mainly interested in *local* geometric properties of M and since every immersion is locally an embedding, all the differential relations that we are going to discuss hold also for submanifolds with self–intersections. We simply have to consider such local embedding in a open set of $\mathbb{R}^{n+m}$ and the distance function only from this piece of M, in a neighborhood, instead than from the whole M.

By the above discussion, in such set Ω it is defined the *projection map* $\pi^M : \Omega \to M$ associating to any point $x \in \Omega$ the unique minimizer in M of the distance from x (again we will often drop the superscript M). This minimizer point is characterized by

$$\pi^M(x) = x - d^M(x)\nabla d^M(x) = x - \nabla\eta^M(x)\,.$$

It should be remarked that $d(x) = \sqrt{2\eta(x)}$ is smooth on $\Omega \setminus M$ but it is not smooth up to M. In the codimension one case this difficulty can be amended by considering the *signed* distance function

$$d^*(x) = \begin{cases} d(x) & \text{if } x \notin E \\ -d(x) & \text{if } x \in E \end{cases}$$

as M is the boundary of a bounded subset E of $\mathbb{R}^{n+m}$.

In higher codimension, the function η is a good substitute of $d^*(x)$ in many situations, see **[7]** for an example of application to the motion by mean curvature.

The following result is concerned with the Hessian matrix of η.

PROPOSITION 1.4.3. *For any $x \in M$ the Hessian matrix $\nabla^2\eta(x)$ is the (matrix of) orthogonal projection onto the normal space N_xM.*
Moreover, for any $x \in M$, letting p to be a unit *vector orthogonal to M at x and defining*

$$\Lambda(s) = \nabla^2\eta(x + sp)$$

for any $s \in [0, \sigma]$ such that the segment $[x, x + \sigma p]$ is contained in Ω, the matrices $\Lambda(s)$ are all diagonal in a common orthonormal basis $\{e_1, \dots, e_{n+m}\}$ such that $\langle e_{n+1}, \dots, e_{n+m}\rangle = N_xM$ and, denoting by $\lambda_1(s), \dots, \lambda_{n+m}(s)$ their eigenvalues in increasing order, we have

$$\lambda_{n+1}(s) = \lambda_{n+2}(s) = \cdots = \lambda_{n+m}(s) = 1 \qquad \forall s \in [0, d(x)]\,.$$

The remaining eigenvalues are strictly less than 1 and satisfy the ODE

$$\lambda_i'(s) = \frac{\lambda_i(s)(1 - \lambda_i(s))}{s} \qquad \forall s \in (0, d(x)]$$

for $i = 1, \dots, n$. Finally, the quotients $\lambda_i(s)/s$ are bounded in $(0, d(x)]$.

PROOF. We follow **[7**, Thm. 3.2**]**. Fixing $x \in M$ and representing locally M as a graph of a smooth function on the tangent space at x, it is easy to see, by an elementary geometric argument, that

$$\eta(x + y) = \frac{|Ny|^2}{2} + o(|y|^2) = \frac{1}{2}\langle Ny \,|\, y\rangle + o(|y|^2),$$

where N is the orthogonal projection on the normal space to M at the point x and $o(t)$ is a real function satisfying $|o(t)|/t \to 0$ as $t \to 0$. By differentiating twice with respect to y and evaluating at $y = 0$, we find $\eta_{ij}(x) = N_{ij}$.

Since the distance function d is smooth in $\Omega \setminus M$, differentiating the equality $|\nabla d|^2 = 1$, we get

$$d_{ij}d_j = 0\,, \qquad d_{ijk}d_j + d_{ij}d_{jk} = 0\,,$$

in $\Omega \setminus M$ and

$$\eta_j\eta_j = 2\eta\,, \qquad \eta_{ij}\eta_j = \eta_i\,, \qquad \eta_{ijk}\eta_j + \eta_{ij}\eta_{jk} = \eta_{ik}\,, \tag{1.4.2}$$

in the whole Ω.
Using the fact that $\nabla\eta(x+sp) = ps$ and the third identity in (1.4.2) we obtain,

$$\begin{aligned}\frac{d}{ds}\,\Lambda_{ij}(s) &= \frac{\partial\eta_{ij}}{\partial x_k}(x+sp)p^k \\ &= \eta_{ijk}(x+sp)\eta_k(x)/s \\ &= \frac{\Lambda_{ij}(s) - \Lambda_{ik}(s)\Lambda_{kj}(s)}{s}\end{aligned} \tag{1.4.3}$$

for every $s \in [0, \sigma]$.
Let $e_1, \ldots, e_{n+m}$ be any basis such that $\Lambda(\sigma)$ is diagonal with associated eigenvalues $\lambda_i(\sigma)$, we consider the unique solution $\mu_i(t)$ of the ODE

$$\frac{d}{ds}\,\mu_i(s) = \frac{\mu_i(s)(1-\mu_i(s))}{s}\,, \qquad \forall s \in (0, \sigma]$$

satisfying $\mu_i(\sigma) = \lambda_i(\sigma)$, for $i = 1, \ldots, n+m$.
Then the matrices

$$\widehat{\Lambda}(s) = \sum_{i=1}^{n+m} \mu_i(s) e_i \otimes e_i\,,$$

solve the differential equation (1.4.3) and satisfy $\widehat{\Lambda}(\sigma) = \Lambda(\sigma)$. Hence, by the uniqueness of solutions to system (1.4.3), we conclude $\Lambda = \widehat{\Lambda}$. Consequently the eigenvectors of $\Lambda(s)$ are equal to e_i for every $s \in (0, \sigma]$ and the eigenvalues $\lambda_i(s)$ solve,

$$\frac{d}{ds}\,\lambda_i(s) = \frac{\lambda_i(s)(1-\lambda_i(s))}{s}\,. \tag{1.4.4}$$

In view of the fact that $\Lambda(s)$ must converge, as $s \to 0^+$, to the matrix of orthogonal projection on the normal space to M at the point x, the conclusion of the proposition follows.

Finally, we show that the quotients $\lambda_i(s)/s$ are bounded as $s \to 0^+$, when $i = 1, \ldots, n$. Solving the differential equation (1.4.4), we find

$$\frac{\lambda_i(s)}{s} = \frac{\lambda_i(\sigma)}{\sigma + (s-\sigma)\lambda_i(\sigma)}\,, \qquad \forall s \in (0, \sigma]\,.$$

Therefore, if $\lambda_i(\sigma) < 0$, then $\lambda_i(s) < 0$ for all s and

$$\left|\frac{\lambda_i(s)}{s}\right| \leq \left|\frac{\lambda_i(\sigma)}{\sigma}\right|\,, \qquad \forall s \in (0, \sigma]\,.$$

If, $\lambda_i(\sigma) > 0$ and $i = 1, \ldots, n$, then $\lambda_i(s) \in [0, 1)$ for all s and

$$\left|\frac{\lambda_i(s)}{s}\right| \leq \frac{\lambda_i(\sigma)}{\sigma(1-\lambda_i(\sigma))}\,, \qquad \forall s \in (0, \sigma]\,.$$

So finally, for all $s \in (0, \sigma]$ and $i = 1, \ldots, n$, we have,

$$\left|\frac{\lambda_i(s)}{s}\right| \leq \max\Big\{\frac{|\lambda|}{\sigma[1 \wedge (1-\lambda)]}\,\Big|\,\lambda < 1 \text{ eigenvalue of } \nabla^2\eta(x) \text{ with } d(x) = \sigma\Big\}$$

and we are done. □

As for every $x \in \Omega$ the gradient $\nabla d(x)$ is a unit vector belonging to $N_{\pi(x)}M$ and constant along the segment $\pi(x) + s(x - \pi(x))$, by using the identity

$$\nabla^2\eta = d\nabla^2 d + \nabla d \otimes \nabla d,$$

it follows that also $\nabla^2 d(\pi(x) + s(x-\pi(x)))$ is diagonal in the same basis above, diagonalizing $\nabla^2\eta(\pi(x))$. Moreover, the eigenvalue associated to the eigenvector $\nabla d(x)$ is zero, $(m-1)$ eigenvalues are equal to $1/s$ and the n remaining ones $\beta_1(s), \ldots, \beta_n(s)$ are bounded and satisfy

$$\beta_i'(s) = -\beta_i^2(s) \qquad\qquad \forall s \in (0, d(x)] \tag{1.4.5}$$

as $\beta_i(s) = \lambda_i(s)/s$, for $i = 1, \ldots, n$.

A straightforward consequence of Proposition 1.4.3 is the following result.

COROLLARY 1.4.4. *Let $x \in \Omega$ and let $\mathrm{K}_x : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}$ be the symmetric 3–linear form induced by $\nabla^3\eta(x)$. Then,*

$$\mathrm{K}_x(u, v, w) = 0$$

if at least two of the vectors u, v and w belong to $N_{\pi(x)}M$.

We discuss now a while the geometric meaning of the eigenvalues $\lambda_i(s)$ in Proposition 1.4.3. We let $x_s = x + sp$ (p is a unit vector orthogonal to T_xM) and we consider the eigenvalues $\lambda_1(s), \dots, \lambda_n(s)$ of $\nabla^2\eta(x_s)$ strictly less than 1 with $e_1, \dots, e_n$ the corresponding eigenvectors (independent of s) spanning T_xM.

PROPOSITION 1.4.5. *For any $i = 1, \dots, n$ we have*

$$\lim_{s\to 0^+} \frac{\lambda_i(s)}{s} = \lambda_i$$

and the values λ_i are the eigenvalues of the symmetric bilinear form

$$-\big\langle \mathrm{B}(x)(u, v) \,|\, p\big\rangle \qquad\qquad u, v \in T_xM$$

with associated eigenvectors $\{e_i\}$.

PROOF. By the remark following the proof of Proposition 1.4.3, $\lambda_i(s)/s$ are the eigenvalues $\beta_i(s)$ of $\nabla^2 d(x_s)$, then the existence of the limits is immediate as the quotients $\lambda_i(s)/s = \beta_i(s)$ are bounded and monotone, by (1.4.5), as $s \to 0^+$.
Let L be the affine $(n+1)$–dimensional space generated by T_xM and p, passing through x. Moreover, let $\Sigma \subset L$ be the smooth n–dimensional manifold obtained projecting $U \cap M$ on L, for a suitable neighborhood U of x, and let $\overline{\mathrm{B}}(x)$ be the second fundamental form of Σ at x, viewing Σ as a surface of codimension one in L. We denote (see Section 1.1.1) by $\lambda_1, \dots, \lambda_n$ the principal curvatures at x of Σ (with the orientation induced near x by p), defined as the eigenvalues of the symmetric bilinear form

$$\langle \overline{\mathrm{B}}(x)(u, v) \,|\, p\rangle \qquad\qquad u, v \in T_x\Sigma = T_xM\,.$$

Under the assumption $m = 1$, we clearly have $\Sigma = M$ and the property is a straightforward consequence of the well known formula (see for instance [**51**, Lemma 14.17])

$$\beta_i(s) = \frac{-\lambda_i}{1 - s\lambda_i} \qquad\qquad \forall s \in (0, d(x)]$$

for the eigenvalues $\beta_i(s)$ of $\nabla^2 d^\Sigma(x_s)$ corresponding to eigenvectors in L (see also [**41**]).
In the general case, we notice that, by Proposition 1.4.1, the function η^Σ is smooth near x and

$$\limsup_{y\to x,\, y\in L} \frac{|\eta^M(y) - \eta^\Sigma(y)|}{|y - x|^4} < +\infty \tag{1.4.6}$$

since Σ is obtained projecting M on the space L, containing $x + T_xM$. By this limit we infer

$$\lim_{s\to 0^+} \frac{\nabla^2\eta^M(x_s) - \nabla^2\eta^\Sigma(x_s)}{s} = 0\,.$$

As all the matrices are diagonal in the same basis, denoting by $\overline{\lambda}_i(s)$ the eigenvalues of $\nabla^2\eta^\Sigma(x_s)$ corresponding to the directions $\{e_i\}$, the quotients $\lambda_i(s)/s$ converge to the same limit of $\overline{\lambda}_i(s)/s$, that is, λ_i.

Finally, by (1.4.6) we have

$$\nabla^3\eta^M(x)(u, v, p) = \nabla^3\eta^\Sigma(x)(u, v, p) \qquad\qquad \forall u, v \in T_xM = T_x\Sigma\,,$$

hence, the relations in Proposition 1.4.9, that we will discuss in a while, yield

$$\big\langle \mathrm{B}(x)(u, v) \,|\, p\big\rangle = \big\langle \overline{\mathrm{B}}(x)(u, v) \,|\, p\big\rangle \qquad\qquad \forall u, v \in T_xM$$

as $p \in N_xM \cap N_x\Sigma$.
This shows that λ_i are the eigenvalues of $-\langle \mathrm{B}(x) \,|\, p\rangle$ and that $\{e_i\}$ are the corresponding eigenvectors. □

REMARK 1.4.6. In particular, the sum of the eigenvalues $\beta_i(s) = \lambda_i(s)/s$ of $\nabla^2 d(x_s)$ converges as $s \to 0^+$ to the quantity $-\langle \mathrm{H}(x) | \, p \rangle$. This property has been used in **[7]** to extend the level set approach (see **[25, 40, 77]**) to the evolution by mean curvature of surfaces of any codimension.

For $x \in M$, we defined $P_{ij}(x)$ as the matrix of orthogonal projection $P : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ on the tangent space and we saw that $P_{ij}(x) = \nabla_i^M x_j$. Actually, by Proposition 1.4.3, we have

$$P_{ij}(x) = (\delta_{ij} - \eta_{ij}(x)) \, ,$$

since $\eta_{ij}(x)$ is the matrix of orthogonal projection on $N_x M$. Notice that such formula defining $P_{ij}(x)$ makes sense in the whole Ω, in this case, Proposition 1.4.3 implies

$$P(x)(T_{\pi(x)}M) = T_{\pi(x)}M, \qquad \text{and} \qquad \operatorname{Ker} P(x) = N_{\pi(x)}M \, .$$

However, we advise the reader that in general $P(x)$ is not the identity on $T_{\pi(x)}M$ ($\nabla^2 eta$ is the identity on $N_{\pi(x)}M$).

We now define the 3–tensor C with components (in the canonical basis)

$$C_{ijk}(x) = \nabla_i^M P_{jk}(x) = \nabla_i^M \nabla_j^M x_k \, ,$$

which is clearly symmetric in the last two indices.
Since for any $x \in M$ the matrix $P(x)$ is the orthogonal projection on $T_x M$, we can expect that the tensor $C(x)$ (encoding the "change" in the tangent plane) contains all information on the curvature of M. In the following three proposition we will see that $\nabla^3 \eta(x)$, the tensor $C(x)$ and the second fundamental form $\mathrm{B}(x)$ are mutually connected by simple linear relations.

PROPOSITION 1.4.7. *The second fundamental form tensors* $\mathrm{B}(x)$ *and the tensor* $C(x)$ *are related for any* $x \in M$ *by the identities*

$$\mathrm{B}_{ij}^k(x) = P_{is}(x) C_{jsk}(x) = P_{js}(x) C_{isk}(x) \, , \qquad\qquad C_{ijk}(x) = \mathrm{B}_{ij}^k(x) + \mathrm{B}_{ik}^j(x) \, . \tag{1.4.7}$$

Moreover, the mean curvature vector $\mathrm{H}(x)$ *of* M *is given by*

$$\mathrm{H}^k(x) = \sum_{s=1}^{n+m} C_{sks}(x) \, . \tag{1.4.8}$$

PROOF. We follow **[58]**. Let $x \in M$, $u = e_i$, $v = e_j$ and let $u' = P(x)e_i$, $v' = P(x)e_j$ be the projections of u and v on $T_x M$. We have then, at the point $x \in M$,

$$\begin{aligned}
\mathrm{B}_{ij}^k &= \frac{\partial [Pe_i]^s}{\partial v'} (\delta_{sk} - P_{sk}) = \frac{\partial P_{is}}{\partial v'} (\delta_{sk} - P_{sk}) = \nabla_l P_{is} P_{lj} (\delta_{sk} - P_{sk}) \\
&= \nabla_j^M P_{is} (\delta_{sk} - P_{sk}) = \nabla_j^M P_{ik} - \nabla_j^M (P_{is} P_{sk}) + P_{is} \nabla_j^M P_{sk} \\
&= \nabla_j^M P_{ik} - \nabla_j^M P_{ik} + + P_{is} \nabla_j^M P_{sk} \\
&= P_{is} \nabla_j^M P_{sk} = P_{is} C_{jsk}
\end{aligned}$$

where we used the fact that $P^2 = P$ on M. The other relation follows by the symmetry of B.

Now we prove the second identity in (1.4.7). Using the first identity and the symmetry of P we get

$$\begin{aligned}
\mathrm{B}_{ij}^k + \mathrm{B}_{ik}^j &= P_{js} C_{isk} + P_{ks} C_{isj} \\
&= P_{js} \nabla_i^M P_{sk} + P_{ks} \nabla_i^M P_{sj} \\
&= \nabla_i^M (P_{js} P_{sk}) \\
&= \nabla_i^M P_{jk} \\
&= C_{ijk} \, .
\end{aligned}$$

Finally, we prove (1.4.8),

$$\mathrm{H}^k = \mathrm{B}_{ii}^k = P_{is} C_{isk} = P_{is} \nabla_i^M P_{sk} = \nabla_s^M P_{sk} = \sum_{s=1}^{n+m} C_{sks} \, .$$

□

PROPOSITION 1.4.8. *The tensor $C(x)$ and $\nabla^3\eta(x)$ are related for any $x \in M$ by the identities*

$$C_{ijk}(x) = -P_{il}(x)\eta_{ljk}(x)\,, \qquad \eta_{ijk}(x) = -\frac{1}{2}\big\{C_{ijk}(x) + C_{jki}(x) + C_{kij}(x)\big\}\,. \tag{1.4.9}$$

PROOF. The first identity is an easy consequence of the fact that $\nabla^2\eta(x)$ is the orthogonal projection on N_xM. To prove the second one, we write (omitting the dependence on x)

$$\begin{aligned}
\eta_{ijk} =& -C_{ijk} + (\delta_{is} - P_{is})\eta_{sjk} \\
=& -C_{ijk} + (\delta_{is} - P_{is})(-C_{jsk} + (\delta_{jt} - P_{jt})\eta_{stk}) \\
=& -C_{ijk} + (\delta_{is} - P_{is})(-C_{jsk} + (\delta_{jt} - P_{jt})(-C_{kst} + (\delta_{kl} - P_{kl})\eta_{stl})) \\
=& -C_{ijk} - C_{jsk}(\delta_{is} - P_{is}) - C_{kst}(\delta_{is} - P_{is})(\delta_{jt} - P_{jt}) \\
&+ (\delta_{is} - P_{is})(\delta_{jt} - P_{jt})(\delta_{kl} - P_{kl})\eta_{stl}\,.
\end{aligned}$$

By Corollary 1.4.4, the last term is zero, so that (1.4.7) yields

$$\begin{aligned}
\eta_{ijk} =& -C_{ijk} - C_{jki} + C_{jsk}P_{is} - C_{kij} + C_{kit}P_{jt} + C_{ksj}P_{si} - C_{kst}P_{is}P_{jt} \\
=& -C_{ijk} - C_{jki} - C_{kij} + \mathrm{B}^k_{ij} + \mathrm{B}^j_{ki} + \mathrm{B}^i_{jk} - P_{jt}\mathrm{B}^t_{ik}\,.
\end{aligned}$$

Since $\mathrm{B}(e_i, e_k) \in N_xM$ we have $P_{jt}\mathrm{B}^t_{ik} = 0$, then exchanging the indices i and j in the above formula, averaging and using the second identity in (1.4.7) we eventually get

$$\begin{aligned}
\eta_{ijk} =& -C_{ijk} - C_{jki} - C_{kij} + \frac{1}{2}\big\{\mathrm{B}^k_{ij} + \mathrm{B}^j_{ki} + \mathrm{B}^i_{jk} + \mathrm{B}^k_{ji} + \mathrm{B}^i_{kj} + \mathrm{B}^j_{ik}\big\} \\
=& -\frac{1}{2}\big\{C_{ijk} + C_{jki} + C_{kij}\big\}\,.
\end{aligned}$$

□

PROPOSITION 1.4.9. *The second fundamental form $\mathrm{B}(x)$ and $\nabla^3\eta(x)$ are related for any $x \in M$ by the identities*

$$\mathrm{B}^k_{ij}(x) = \nabla_k(\eta_{is}\eta_{sj} - \eta_{ij})(x)\,, \qquad \eta_{ijk}(x) = -\mathrm{B}^k_{ij}(x) - \mathrm{B}^i_{jk}(x) - \mathrm{B}^j_{ki}(x)\,. \tag{1.4.10}$$

Moreover, the mean curvature vector $\mathrm{H}(x)$ of M is given by

$$\mathrm{H}(x) = -\Delta(\nabla\eta)(x)\,.$$

PROOF. By Using relations (1.4.7) and (1.4.9) we can write each component B^k_{ij} of the second fundamental form as a function of $\nabla^3\eta$ as follows,

$$\begin{aligned}
\mathrm{B}^k_{ij} &= P_{js}C_{iks} \\
&= -P_{js}P_{il}\eta_{lks} \\
&= -(\delta_{js} - \eta_{js})(\delta_{il} - \eta_{il})\eta_{lks} \\
&= -\eta_{ijk} + \eta_{sj}\eta_{kis} + \eta_{li}\eta_{kjl} - \eta_{js}\eta_{il}\eta_{lks} \\
&= -\eta_{ijk} + \eta_{sj}\eta_{kis} + \eta_{si}\eta_{kjs} \\
&= \nabla_k(\eta_{is}\eta_{sj} - \eta_{ij})\,.
\end{aligned} \tag{1.4.11}$$

Conversely, by the second identities in (1.4.9) and (1.4.7) we get

$$\begin{aligned}
\eta_{ijk} =& -\frac{1}{2}\big\{C_{ijk} + C_{jki} + C_{kij}\big\} \\
=& -\frac{1}{2}\big\{\mathrm{B}^k_{ij} + \mathrm{B}^j_{ik} + \mathrm{B}^i_{jk} + \mathrm{B}^k_{ji} + \mathrm{B}^j_{ki} + \mathrm{B}^i_{kj}\big\} \\
=& -\mathrm{B}^k_{ij} + \mathrm{B}^i_{jk} + \mathrm{B}^j_{ki}\,.
\end{aligned}$$

By the first formula, we have

$$\mathrm{H}^k = -\eta_{kii} + \nabla_k\Big(\sum_{i,s=1}^{n+m}\eta^2_{is}\Big)$$

for every index $k = 1, \dots, n+m$. Since $\nabla^2\eta(x)$ is symmetric, $\sum_{i,s=1}^{n+m} \eta_{is}^2(x)$ coincides with the sum of the squares of the eigenvalues of $\nabla^2\eta(x)$. By Proposition 1.4.3, this quantity is equal to $n + o(|x - x^0|)$ near every point $x^0 \in M$, hence $\nabla_k\big(\sum_{i,s=1}^{n+m} \eta_{is}^2\big)(x)$ vanishes on M. It follows that

$$\mathrm{H}(x) = -\Delta(\nabla\eta)(x) \qquad\qquad \forall x \in M\,. \tag{1.4.12}$$

□

COROLLARY 1.4.10. *Let $x \in M$ and let $\mathrm{K}_x : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}$ be the symmetric 3–linear form induced by $\nabla^3\eta(x)$. Then,*

$$\mathrm{K}_x(u, v, w) = 0$$

if all the three vectors u, v and w belong to $T_{\pi(x)}M$.

PROOF. It follows by the second relation in (1.4.10), as the second fundamental form takes values in the normal space to M at x. □

From now on, instead of dealing with the squared distance function we will consider the function

$$A^M(x) = \frac{|x|^2 - [d^M(x)]^2}{2}\,,$$

clearly smooth as η^M in the neighborhood Ω of M. We set

$$A^M_{i_1\dots i_k}(x) = \frac{\partial^k A^M(x)}{\partial x_{i_1}\dots\partial x_{i_k}}$$

for the derivatives of A^M in Ω.
We define the k–derivative symmetric tensor $A^k(x)$ working on the k–uple of vectors $v_i \in \mathbb{R}^{n+m}$, where $v_i = v_i^j e_j$, as follows

$$A^k(x)(v_1, \dots, v_k) = A^M_{i_1\dots i_k}(x) v_1^{i_1}\dots v_k^{i_k}\,.$$

By sake of simplicity, we dropped the superscript M on A^k, by the same reason, we will also often avoid to indicate the point $x \in M$ in the sequel.

The greater convenience of A^M can be explained noticing that $\nabla^2 A^M(x)$, for $x \in M$, is the projection matrix on T_xM and this quantity often appears in the computation of tangential gradients.

We reformulate now the previous formulas in terms of A^M.

PROPOSITION 1.4.11. *The following properties of A^M hold,*

(a) *for any $x \in \Omega$, the vector $\nabla A^M(x)$ coincide with the projection point $\pi^M(x)$ of x on M. Moreover, $\nabla^2 A^M(x)$ is zero on $N_{\pi(x)}M$ and maps $T_{\pi(x)}M$ onto $T_{\pi(x)}M$. If $x \in M$, then $\nabla^2 A^M(x)$ is the matrix P of orthogonal projection on T_xM;*

(b) *for any $x \in \Omega$, the 3–linear form $\mathrm{K}_x : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ given by*

$$\mathrm{K}_x(u, v, w) = \sum_{i,j,k=1}^{n+m} A^M_{ijk}(x) u^i v^j w^k$$

is equal to zero if at least two of the 3 vectors u, v, w, are normal to M at $\pi(x) = \nabla A^M(x)$ or if $x \in M$ and the three vectors are all tangent;

(c) *for $x \in M$, the second fundamental form $\mathrm{B}(x)$ and the mean curvature vector $\mathrm{H}(x)$ are related to the derivatives of $A^M(x)$ by*

$$\mathrm{B}^k_{ij}(x) = A^M_{js}(x) A^M_{il}(x) A^M_{slk}(x) = \big(\delta_{kl} - A^M_{kl}(x)\big) A^M_{ijl}(x), \tag{1.4.13}$$

$$\mathrm{H}^k(x) = \sum_{j=1}^{n+m} A^M_{jkj}(x), \tag{1.4.14}$$

$$\nabla^M_i A^M_{jk}(x) = \mathrm{B}^k_{ij}(x) + \mathrm{B}^j_{ik}(x). \tag{1.4.15}$$

PROOF. The first statement follows by Proposition 1.4.3 and the second one by Corollary 1.4.4. The first equality in (1.4.13) and (1.4.14) follow by relations (1.4.12) and (1.4.11). The second equality in (1.4.13) can be obtained multiplying both sides of the second relation in (1.4.10) by the normal projection $(I - \nabla^2 A^M)$. Finally (1.4.15) is a restatement of the second equality in (1.4.7). $\square$

By means of the relations in Propositions 1.4.7, 1.4.8, 1.4.9 we have the following estimates.

COROLLARY 1.4.12. *At every point of M we have,*

$$|C|^2 \leq |\nabla^3 A^M|^2 = 3|\mathrm{B}|^2 \leq 3|C|^2 \,.$$

PROOF. We have only to show the identity $|\nabla^3 A^M|^2 = 3|\mathrm{B}|^2$, the other inequalities are immediate as the projection P is a 1–Lipschitz map.
We compute in a orthonormal basis $\{e_i\}$ such that $\langle e_1, \dots, e_n\rangle = T_xM$, by means of the second relation in (1.4.10), and keeping in mind that the second fundamental form B takes values in the normal space N_xM,

$$\begin{aligned}
|\nabla^3 A^M|^2 = & \sum_{i,j,k=1}^{n} |\eta_{ijk}|^2 \\
= & \sum_{\substack{n+1\leq i\leq n+m\\ 1\leq j,k\leq n}} |\eta_{ijk}|^2 + \sum_{\substack{n+1\leq j\leq n+m\\ 1\leq i,k\leq n}} |\eta_{ijk}|^2 + \sum_{\substack{n+1\leq k\leq n+m\\ 1\leq i,j\leq n}} |\eta_{ijk}|^2 \\
= & \sum_{\substack{n+1\leq i\leq n+m\\ 1\leq j,k\leq n}} |\mathrm{B}^k_{ij} + \mathrm{B}^i_{jk} + \mathrm{B}^j_{ki}|^2 + \sum_{\substack{n+1\leq j\leq n+m\\ 1\leq i,k\leq n}} |\mathrm{B}^k_{ij} + \mathrm{B}^i_{jk} + \mathrm{B}^j_{ki}|^2 \\
& + \sum_{\substack{n+1\leq k\leq n+m\\ 1\leq i,j\leq n}} |\mathrm{B}^k_{ij} + \mathrm{B}^i_{jk} + \mathrm{B}^j_{ki}|^2 \\
= & \sum_{\substack{n+1\leq i\leq n+m\\ 1\leq j,k\leq n}} |\mathrm{B}^i_{jk}|^2 + \sum_{\substack{n+1\leq j\leq n+m\\ 1\leq i,k\leq n}} |\mathrm{B}^j_{ki}|^2 + \sum_{\substack{n+1\leq k\leq n+m\\ 1\leq i,j\leq n}} |\mathrm{B}^k_{ij}|^2 \\
= & \, 3 \sum_{\substack{n+1\leq i\leq n+m\\ 1\leq j,k\leq n}} |\mathrm{B}^i_{jk}|^2 \\
= & \, 3|\mathrm{B}|^2 \,.
\end{aligned}$$

$\square$

1.5. Higher Order Relations

In this section we work out some properties of the higher derivatives of the square of the distance function from a submanifold, in particular, the relations with the covariant derivatives of the second fundamental form. The main result here is a recurrence formula for A^k (Proposition 1.5.1), that is, the tensor of k–derivatives of the squared distance function from M, once its action is split on tangent and normal vectors. Such formula is crucial to get "structure information" and estimates on the tensors A^k (Corollary 1.5.3 and Proposition 1.5.6).

PROPOSITION 1.5.1. *For every $k \geq 2$ and $s \in \{0, \dots k\}$ there exists a family $p^{k,s}_{j_1 \dots j_{k-s}}$ of symmetric polynomial tensors of type $(s, 0)$ on M, where $j_1, \dots, j_{k-s} \in \{1, \dots, n+m\}$, which are contractions of the second fundamental form B and its covariant derivatives with the metric tensor g, such that*

$$A^k(X_1, \dots, X_s, N_1, \dots, N_{k-s}) = p^{k,s}_{j_1 \dots j_{k-s}}(X_1, \dots, X_s) N_1^{j_1} \dots N_{k-s}^{j_{k-s}}$$

for every s–uple of tangent vectors X_h and $(k-s)$–uple of normal vectors N_h in $\mathbb{R}^{n+m}$ (with the obvious interpretation if $s = 0$ or $s = k$, that is, for instance in this latter case the symbols indexed by $1, \dots, k-s$ are not present in the formulas).
Moreover, the tensors $p^{k,s}_{j_1 \dots j_{k-s}}$ are invariant by exchange of the j–indices and the maximum order of differentiation of B which appears in every $p^{k,s}_{j_1 \dots j_{k-s}}$ is at most $k-3$, when $k \geq 3$. Considering the

tangent plane at any point $x \in M$ also as a subset of $\mathbb{R}^{n+m}$, the polynomial tensors $p^{k,s}_{j_1\dots j_{k-s}}$ are expressed in the coordinate basis of the Euclidean space as follows

$$p^{k,s}_{j_1\dots j_{k-s}}(X_1,\dots,X_s)N_1^{j_1}\dots N_{k-s}^{j_{k-s}} = p^{k,s}_{j_1\dots j_{k-s},i_1\dots i_s}X_1^{i_1}\dots X_s^{i_s}N_1^{j_1}\dots N_{k-s}^{j_{k-s}}\,.$$

Then, a family of tensors satisfying the above properties can be defined recursively according to the following formulas

$$p^{2,0}_{j_1j_2} = p^{2,1}_{j_1,i_1} = 0\,,\qquad p^{2,2}_{i_1i_2} = \delta_{i_1i_2} \tag{1.5.1}$$

$$p^{k,0}_{j_1\dots j_k} = p^{k,1}_{j_1\dots j_{k-1},i_1} = 0 \qquad \textit{for every } k\geq 2 \tag{1.5.2}$$

$$\begin{aligned}p^{k+1,s}_{j_1\dots j_{k-s+1},i_0i_1\dots i_{s-1}} &= (\nabla p^{k,s-1}_{j_1\dots j_{k-s+1}})_{i_0i_1\dots i_{s-1}}\\ &\quad-\sum_{h=1}^{k-s+1} p^{k,s-1}_{j_1\dots j_{h-1}rj_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}}\mathrm{B}^{j_h}_{ri_0}\\ &\quad-\sum_{h=1}^{s-1} p^{k,s-2}_{rj_1\dots j_{k-s+1},i_1\dots i_{h-1}i_{h+1}\dots i_{s-1}}\mathrm{B}^{r}_{i_0i_h}\\ &\quad+\sum_{h=1}^{k-s+1} p^{k,s}_{j_1\dots j_{h-1}j_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}r}\mathrm{B}^{j_h}_{ri_0}\end{aligned} \qquad \textit{if } 2\leq s<k+1 \tag{1.5.3}$$

$$p^{k+1,k+1}_{i_0i_1\dots i_{k+1}} = \nabla p^{k,k}_{i_0i_1\dots i_k} - \sum_{h=1}^{k} p^{k,k-1}_{r,i_1\dots i_{h-1}i_{h+1}\dots i_k}\mathrm{B}^r_{i_0i_h}\,. \tag{1.5.4}$$

PROOF. If $k=2$ we have immediately

$$A^2(N_1,N_2)=0,\qquad A^2(X_1,N_1)=0,\qquad A^2(X_1,X_2)=X_1^iX_2^i=\delta_{i_1i_2}X_1^{i_1}X_2^{i_2}$$

since X_1 and X_2 are tangent and A^2 is the projection on the tangent space. Hence, formula (1.5.1) follows.

We argue now by induction on $k\geq 2$. When $s=0$ the value $A^k(N_1,\dots,N_k)(x)$ depends only on the function A^M on the m–dimensional normal subspace to M at x, and on this subspace A^M is identically zero, hence the first equality in (1.5.2) is proved.

Suppose now that $s\in\{1,\dots,k+1\}$, we extend the vectors $X_h\in T_xM$ and $N_h\in N_xM$ to a family of local vector fields, respectively tangent and normal to M, then

$$\begin{aligned}A^{k+1}(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1}) &= \frac{\partial}{\partial X_0}\Big(A^k(X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\Big)\\ &\quad-\sum_{h=1}^{s-1}A^k\Big(X_1,\dots X_{h-1},\frac{\partial X_h}{\partial X_0},X_{h+1},\dots,X_{s-1},N_1,\dots,N_{k-s+1}\Big)\\ &\quad-\sum_{h=1}^{k-s+1}A^k\Big(X_1,\dots,X_{s-1},N_1,\dots,\frac{\partial N_h}{\partial X_0},\dots,N_{k-s+1}\Big)\end{aligned}$$

where the last line is not present in the special case $s=k+1$ and the second line is not present if $s=1$. In this last case, we have

$$A^{k+1}(X_0,N_1,\dots,N_k)=\frac{\partial}{\partial X_0}\Big(A^k(N_1,\dots,N_k)\Big)-\sum_{h=1}^{k}A^k\Big(N_1,\dots,\frac{\partial N_h}{\partial X_0},\dots,N_k\Big)=0$$

since the first term of the right member is zero by the first equality in (1.5.2) and, after decomposing $\frac{\partial N_h}{\partial X_0}$ in tangent and normal part, the tangent term is zero by induction and the normal term is zero for (1.5.2) again. This shows the second equality in (1.5.2).

So we suppose $1<s<k+1$, by the inductive hypothesis,

$$A^k(X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})=p^{k,s-1}_{j_1\dots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}$$

thus, differentiating along X_0, which is a tangent field, we obtain

$$\begin{aligned}
A^{k+1}&(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=\frac{\partial}{\partial X_0}\Big(p^{k,s-1}_{j_1\dots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\Big)\\
&\quad-\sum_{h=1}^{s-1}A^k\Big(X_1,\dots,\Big(\frac{\partial X_h}{\partial X_0}\Big)^M,\dots,X_{s-1},N_1,\dots,N_{k-s+1}\Big)\\
&\quad-\sum_{h=1}^{s-1}A^k\Big(X_1,\dots,\Big(\frac{\partial X_h}{\partial X_0}\Big)^\perp,\dots,X_{s-1},N_1,\dots,N_{k-s+1}\Big)\\
&\quad-\sum_{h=1}^{k-s+1}A^k\Big(X_1,\dots,X_{s-1},N_1,\dots,\Big(\frac{\partial N_h}{\partial X_0}\Big)^M,\dots,N_{k-s+1}\Big)\\
&\quad-\sum_{h=1}^{k-s+1}A^k\Big(X_1,\dots,X_{s-1},N_1,\dots,\Big(\frac{\partial N_h}{\partial X_0}\Big)^\perp,\dots,N_{k-s+1}\Big).
\end{aligned}$$

We use now the symmetry of A^k and we substitute recursively $p^{k,s}$, $p^{k,s-1}$ and $p^{k,s-2}$ to A^k, according to the number of tangent vectors inside A^k,

$$\begin{aligned}
A^{k+1}&(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=\frac{\partial}{\partial X_0}\Big(p^{k,s-1}_{j_1\dots j_{k-s+1}}(X_1,\dots,X_{s-1})\Big)N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad+\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\dots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots\frac{\partial N_h^{j_h}}{\partial X_0}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-1}_{j_1\dots j_{k-s+1}}(X_1,\dots,\nabla_{X_0}X_h,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-2}_{rj_1\dots j_{k-s+1}}(X_1,\dots,X_{h-1},X_{h+1},\dots,X_{s-1})\Big[\Big(\frac{\partial X_h}{\partial X_0}\Big)^\perp\Big]^r N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s}_{j_1\dots j_{h-1}j_{h+1}\dots j_{k-s+1}}\Big(X_1,\dots,X_{s-1},\Big(\frac{\partial N_h}{\partial X_0}\Big)^M\Big)N_1^{j_1}\dots N_{h-1}^{j_{h-1}}N_{h+1}^{j_{h+1}}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\dots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots\Big[\Big(\frac{\partial N_h}{\partial X_0}\Big)^\perp\Big]^{j_h}\dots N_{k-s+1}^{j_{k-s+1}}.
\end{aligned}$$

Adding the first and the third line on the right hand side we get the covariant derivative of the tensor $p^{k,s-1}_{j_1\dots j_{k-s+1}}$ times $N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}$, adding the second and the last line we get

$$\begin{aligned}
A^{k+1}&(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=\nabla p^{k,s-1}_{j_1\dots j_{k-s+1}}(X_0,X_1,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad+\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\dots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots\Big[\Big(\frac{\partial N_h}{\partial X_0}\Big)^M\Big]^{j_h}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-2}_{rj_1\dots j_{k-s+1}}(X_1,\dots,X_{h-1},X_{h+1},\dots,X_{s-1})\Big[\Big(\frac{\partial X_h}{\partial X_0}\Big)^\perp\Big]^r N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s}_{j_1\dots j_{h-1}j_{h+1}\dots j_{k-s+1}}\Big(X_1,\dots,X_{s-1},\Big(\frac{\partial N_h}{\partial X_0}\Big)^M\Big)N_1^{j_1}\dots N_{h-1}^{j_{h-1}}N_{h+1}^{j_{h+1}}\dots N_{k-s+1}^{j_{k-s+1}}.
\end{aligned}$$

Taking now into account that

$$\Big[\Big(\frac{\partial N_h}{\partial X_0}\Big)^M\Big]^r = \Big[\Big\langle\frac{\partial N_h}{\partial X_0}, \frac{\partial}{\partial x_i}\Big\rangle\frac{\partial}{\partial x_i}\Big]^r = -\Big\langle N_h, \frac{\partial}{\partial X_0}\frac{\partial}{\partial x_i}\Big\rangle\Big\langle\frac{\partial}{\partial x_i}, e_r\Big\rangle = -\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}N_h^{j_h}\,,$$

where $\Big\{\frac{\partial}{\partial x_i}\Big\}_{i=1,\dots,n}$ is a basis of the tangent space of M, and

$$\Big[\Big(\frac{\partial X_h}{\partial X_0}\Big)^\perp\Big]^r = \mathrm{B}^r_{i_0i_h}X_0^{i_0}X_h^{i_h}\,,$$

substituting, we get

$$\begin{aligned}
A^{k+1}&(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=\nabla p^{k,s-1}_{j_1\dots j_{k-s+1}}(X_0,X_1,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\dots j_{h-1}rj_{h+1}\dots j_{k-s+1}}(X_1,\dots,X_{s-1})\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-2}_{rj_1\dots j_{k-s+1}}(X_1,\dots,X_{h-1},X_{h+1},\dots,X_{s-1})\mathrm{B}^r_{i_0i_h}X_0^{i_0}X_h^{i_h}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad+\sum_{h=1}^{k-s+1}p^{k,s}_{j_1\dots j_{h-1}j_{h+1}\dots j_{k-s+1}}(X_1,\dots,X_{s-1},\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}e_r)N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\,.
\end{aligned}$$

Then, expressing the tensors in coordinates, we have

$$\begin{aligned}
A^{k+1}&(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=(\nabla p^{k,s-1}_{j_1\dots j_{k-s+1}})_{i_0i_1\dots i_{s-1}}X_0^{i_0}X_1^{i_1}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\dots j_{h-1}rj_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}}\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-2}_{rj_1\dots j_{k-s+1},i_1\dots i_{h-1}i_{h+1}\dots i_{s-1}}\mathrm{B}^r_{i_0i_h}X_0^{i_0}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad+\sum_{h=1}^{k-s+1}p^{k,s}_{j_1\dots j_{h-1}j_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}r}\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\,,
\end{aligned}$$

which is formula (1.5.3).
In the special case $s = k+1$, to get formula (1.5.4), we just have to repeat the computations dropping all the lines containing sums like $\sum_{h=1}^{k-s+1}$ ..., which are not present.
Finally, assuming inductively that the polynomial tensors $p^{k,s}$, $p^{k,s-1}$ and $p^{k,s-2}$ are symmetric in the j–indices and contain covariant derivatives of B only up to the order $k-3$ (when $k \geq 3$), also the claims about the symmetry and the order of the derivatives of B follow.

□

EXAMPLE 1.5.2. We compute some $p^{k,s}$ as a consequence of this proposition.

(1) When $k=2$ we saw that

$$p^{2,0}_{j_1j_2}=0,\qquad p^{2,1}_{j_1}=0,\qquad p^{2,2}=g\,.$$

(2) When $k=3$ we have, by means of formulas (1.5.2) and (1.5.3),

$$\begin{aligned}
&p^{3,0}_{j_1j_2j_3}=0,\qquad p^{3,1}_{j_1j_2}=0\\
&p^{3,2}_{j_1,i_1i_2}=p^{2,2}_{i_2r}\mathrm{B}^{j_1}_{ri_1}=\mathrm{B}^{j_1}_{i_1i_2}\\
&p^{3,3}_{i_1i_2i_3}=(\nabla p^{2,2})_{i_1i_2i_3}+p^{2,1}_{r,i_2}\mathrm{B}^r_{i_1i_3}+p^{2,1}_{r,i_3}\mathrm{B}^r_{i_1i_2}=0
\end{aligned}$$

that is,

$$p^{3,2}_{j_1}=\mathrm{B}^{j_1}\text{ and }p^{3,3}=0\,.$$

(3) When $k = 4$ we have,

$$
\begin{aligned}
p^{4,0}_{j_1j_2j_3j_4} &= 0, \qquad p^{4,1}_{j_1j_2j_3} = 0\\
p^{4,2}_{j_1j_2,i_1i_2} &= p^{3,2}_{j_1,i_1r}\mathrm{B}^{j_2}_{ri_2} + p^{3,2}_{j_2,i_1r}\mathrm{B}^{j_1}_{ri_1} = \mathrm{B}^{j_1}_{i_1r}\mathrm{B}^{j_2}_{ri_2} + \mathrm{B}^{j_2}_{i_2r}\mathrm{B}^{j_1}_{ri_1}\\
p^{4,3}_{j_1,i_1i_2i_3} &= (\nabla p^{3,2}_{j_1})_{i_1i_2i_3} + p^{3,2}_{r,i_2i_3}\mathrm{B}^{j_1}_{ri_1} = (\nabla p^{3,2}_{j_1})_{i_1i_2i_3} + \mathrm{B}^{r}_{i_2i_3}\mathrm{B}^{j_1}_{ri_1} = (\nabla \mathrm{B}^{j_1})_{i_1i_2i_3}\\
&\quad \text{since we contracted a normal vector with a tangent one,}\\
p^{4,4}_{i_1i_2i_3i_4} &= -\,p^{3,2}_{r}i_3i_4\mathrm{B}^{r}_{i_1i_2} - p^{3,2}_{r}i_2i_4\mathrm{B}^{r}_{i_1i_3} - p^{3,2}_{r}i_2i_3\mathrm{B}^{r}_{i_1i_4}\\
&= -\,\mathrm{B}^{r}_{i_3i_4}\mathrm{B}^{r}_{i_1i_2} - \mathrm{B}^{r}_{i_2i_4}\mathrm{B}^{r}_{i_1i_3} - \mathrm{B}^{r}_{i_2i_3}\mathrm{B}^{r}_{i_1i_4}\,.
\end{aligned}
$$

Proposition 1.5.1 allows us to write A^k in terms of the tensors $p^{k,s}$ and the projections on the tangent and normal spaces (hence contracting with the scalar product of $\mathbb{R}^{n+m}$), so we get the following corollary.

COROLLARY 1.5.3. *For every $k \geq 3$ the symmetric tensor A^k can be expressed as a polynomial tensor in B and its covariant derivatives, contracted with the scalar product of $\mathbb{R}^{n+m}$.*
The maximum order of differentiation of B which appears in A^k is $k-3$. More precisely, the only tensors among the $p^{k,s}$ containing such highest derivative are $p^{k,k-1}_{j_1}$, given by

$$p^{k,k-1}_{j_1} = \nabla^{k-3}\mathrm{B}^{j_1} + \mathrm{LOT}\,.$$

*where we denoted with LOT (*lower order terms*) a polynomial term containing only derivatives of B up to the order k-4.*

PROOF. Looking at the tensors with the derivative of B of maximum order among the $p^{k,s}_{j_1\dots j_{k-s}}$, by formula (1.5.3) and the fact that the only non zero polynomials $p^{3,s}_{j_1\dots j_{3-s},i_1\dots i_s}$ are $p^{3,2}_{j_1,i_1i_2} = \mathrm{B}^{j_1}_{i_1i_2}$ (see Example 1.5.2), it is clear that they come from the derivative $\nabla p^{k-1,k-2}_{j_1}$. Iterating the argument, the leading term in $p^{k,k-1}_{j_1}$ is given by $\nabla^{k-3}p^{3,2}_{j_1} = \nabla^{k-3}\mathrm{B}^{j_1}$. □

REMARK 1.5.4. We can see in Example 1.5.2 that when $k = 3$ and 4, the lower order term which appears above is zero. Actually, by a tedious computation, one can see that for $k \geq 5$ this is no more true.

COROLLARY 1.5.5. *For every $k \geq 3$ we have the following estimates at every point $x \in M$,*

$$C_1|\nabla^{k-3}\mathrm{B}|^2 + \mathrm{LOT}_1 \leq |A^k|^2 \leq C_2|\nabla^{k-3}\mathrm{B}|^2 + \mathrm{LOT}_2$$

where the two constants C_1 and C_2 depends only on k, n and m, and LOT_1 and LOT_2 are polynomial terms containing only derivatives of B up to the order k-4.
Moreover, for a couple of "universal" functions F_1 and F_2 depending only on k, n and m, we have

$$
\begin{aligned}
\sum_{i=3}^{k}|A^i|^2 &\leq F_1\Big(\sum_{i=0}^{k-3}|\nabla^i\mathrm{B}|^2\Big)\\
\sum_{i=0}^{k-3}|\nabla^i\mathrm{B}|^2 &\leq F_2\Big(\sum_{i=3}^{k}|A^i|^2\Big)\,.
\end{aligned}
$$

PROOF. The first estimates follow by Corollary 1.5.3 and the structure of A^k obtained in Proposition 1.5.1. The second statement is obtained by such estimates, by iteration. □

The decomposition of A^k in its tangent and normal components is very useful in studying in even more detail the norm of A^k.

Fixing at a point $x \in M$ an orthonormal basis $\{e_1, \dots, e_{n+m}\}$ of $\mathbb{R}^{n+m}$ such that $\{e_1, \dots, e_n\}$ is a basis of T_xM, we have obviously

$$\begin{aligned}|A^k|^2 &= \sum_{1\leq i_1,\dots,i_k\leq n+m} [A^k(e_{i_1}, \dots, e_{i_k})]^2\\ &\geq \sum_{\substack{1\leq i_1,i_2\leq n\\ n<i_3,\dots,i_k\leq n+m}} [A^k(e_{i_1}, e_{i_2}, e_{i_3}, \dots, e_{i_k})]^2\\ &\geq \sum_{n<j\leq n+m}\sum_{1\leq i_1,i_2\leq n} [A^k(e_{i_1}, e_{i_2}, e_j, \dots, e_j)]^2\\ &= \sum_{n<j\leq n+m}\sum_{1\leq i_1,i_2\leq n} [p^{k,2}_{j\dots j,i_1i_2}]^2\,,\end{aligned}$$

that is,

$$|A^k|^2 \geq \sum_{n<j\leq n+m} |p^{k,2}_{j\dots j}|^2\,.$$

We analyze this last term by means of formula (1.5.3). We have $p^{2,2} = g$ and for every $k \geq 2$,

$$p^{k+1,2}_{j\dots j,i_0i_1} = \sum_{h=1}^{k-1} p^{k,2}_{j\dots j,i_1r}\mathrm{B}^j_{ri_0} = (k-1)\,p^{k,2}_{j\dots j,i_1r}\mathrm{B}^j_{ri_0}\,.$$

Then, by induction, it is easy to see that

$$p^{k,2}_{j\dots j,i_0i_1} = (k-2)!\,\mathrm{B}^j_{i_0r_1}\mathrm{B}^j_{r_1r_2}\dots\mathrm{B}^j_{r_{k-3}i_1}$$

hence, as the bilinear form B^j is symmetric, denoting with λ^j_s its eigenvalues at the point $x \in M$, we conclude

$$|p^{k,2}_{j\dots j}|^2 = [(k-2)!]^2\sum_{s=1}^n(\lambda^j_s)^{2(k-2)} \geq \widetilde{C}|\mathrm{B}^j|^{2k-4}\,.$$

Coming back to our estimate,

$$|A^k|^2 \geq \widetilde{C}\sum_{n<j\leq n+m}|\mathrm{B}^j|^{2k-4} \geq C\Big(\sum_{n<j\leq n+m}|\mathrm{B}^j|^2\Big)^{k-2} = C|\mathrm{B}|^{2k-4}\,.$$

PROPOSITION 1.5.6. *The following estimate holds,*

$$|A^k|^2 \geq C|\mathrm{B}|^{2k-4}$$

where C is a universal constant depending only on k, n and m.

1.6. The Distance Function on Riemannian Manifolds

In this section we discuss more in detail some analytic properties of the distance function that we stated without proof in Section 1.4.
We consider in full generality the distance function d^K from a closed set K of a Riemannian manifold (M, g) and we analyze the connection with the theory of *viscosity solutions* of Hamilton–Jacobi equations. Indeed, we will see that the distance function is a viscosity solution of the following Hamilton–Jacobi problem

$$\begin{cases}|\nabla u| = 1 & \text{in } M\setminus K\,,\\ u = 0 & \text{on } \partial K\end{cases}$$

and we will use the property of *semiconcavity* shared by such solutions to analyze the properties of d^K.

1.6.1. Stationary Hamilton–Jacobi Equations on Manifolds. Let M be a smooth and connected, n–dimensional, differentiable manifold.

We consider the following Hamilton–Jacobi problem in $\Omega \subset M$,

$$\begin{cases} \mathrm{H}(x, du(x), u(x)) = 0 & \text{in } \Omega\,, \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

where $\mathrm{H} : T^*\Omega \times \mathbb{R} \to \mathbb{R}$ and T^* denotes the cotangent bundle.

DEFINITION 1.6.1. Given a continuous function $u : \Omega \to \mathbb{R}$ and a point $x \in M$, the *superdifferential* of u at x is the subset of T_x^*M defined by

$$\partial^+ u(x) = \left\{ d\varphi(x) \,|\, \varphi \in C^1(M), \varphi(x) - u(x) = \min_M \varphi - u \right\}.$$

Similarly, the set

$$\partial^- u(x) = \left\{ d\psi(x) \,|\, \psi \in C^1(M), \psi(x) - u(x) = \max_M \psi - u \right\}$$

is called the *subdifferential* of u at x.
Notice that it is equivalent to replace the $\max$ ($\min$) on all M with the maximum (minimum) in an open neighborhood of x in M.

It is easy to see that $\partial^+ u(x)$ and $\partial^- u(x)$ are both nonempty if and only if u is differentiable at $x \in M$. In this case we have

$$\partial^+ u(x) = \partial^- u(x) = \{du(x)\}\,.$$

We list here without proof some of the standard properties of the sub and superdifferential which will be needed later.

PROPOSITION 1.6.2. *If $\psi : N \to M$ is a map between the smooth manifolds N and M which is C^1 around $x \in N$, then*

$$\partial^+(u \circ \psi)(x) \supset \partial^+ u(\psi(x)) \circ d\psi(x) \;=\; \{v \circ d\psi(x) \,|\, v \in \partial^+ u(\psi(x))\}\,.$$

If ψ is a local diffeomorphism near x, the inclusion becomes an equality. An analogous statement holds for ∂^-.

PROPOSITION 1.6.3. *If $\theta : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $\theta'(u(x)) \geq 0$, then*

$$\partial^+(\theta \circ u)(x) \supset d\theta(u(x)) \circ \partial^+ u(x) \;=\; \{d\theta(u(x)) \circ v \,|\, v \in \partial^+ u(x)\}\,,$$

similarly for ∂^-. If $\theta'(u(x)) > 0$ then the inclusion is an equality.

For a locally Lipschitz function u on a Riemannian manifold (M, g), $\partial^+ u(x)$ and $\partial^- u(x)$ are compact convex sets, almost everywhere coinciding with the differential of the function u, by Rademacher's theorem.
For a generic continuous function u we prove in the next proposition that $\partial^+ u(x)$ and $\partial^- u(x)$ are not empty in a dense subset.

PROPOSITION 1.6.4. *Let $u : \Omega \to \mathbb{R}$ be a continuous function on an open subset Ω of M. Then the subdifferential $\partial^- u(x)$ (the superdifferential $\partial^+ u(x)$) is not empty for every x in a dense subset of Ω.*

PROOF. It is always possible to endow M with a Riemannian structure giving a metric $d(\cdot\,,\cdot)$ on M which generates the same topology.
Consider a generic point $y \in \Omega$ and a geodesic ball B contained in Ω with center y. If the ball B is small enough, the function $x \mapsto d^2(x, y)$ is smooth in $\overline{B}$. Taking a large positive constant A, the function $F_A(x) = u(x) + Ad^2(x, y)$ has a local minimum at a point x_A in the interior of B. At x_A the subdifferential of the function F_A must contain the origin of $T^*_{x_A}M$, hence, being $d^2(x, y)$ differentiable in the ball B, the differential of $-d^2(x, y)$ at x_A belongs to $\partial^- u(x_A)$. As the point y and the ball B were arbitrarily chosen, the set of points where the subdifferential of u is not empty is dense in Ω.
The same argument holds for the superdifferential of u, considering the function $-u$. □

Now we introduce the notion of semiconcavity which will play a central role.

DEFINITION 1.6.5. Given an open set $\Omega \subset \mathbb{R}^n$, a continuous function $u : \Omega \to \mathbb{R}$ is called *locally semiconcave* if, for any open convex set $\Omega' \subset \Omega$ with compact closure in Ω, there exists a constant C such that one of the following three equivalent conditions is satisfied,

(1) $\forall x, h$ with $x, x+h, x-h \in \Omega'$,

$$u(x+h) + u(x-h) - 2u(x) \leq 2C|h|^2\,,$$

(2) $u(x) - C|x|^2$ is a concave function in Ω',
(3) $D^2u \leq 2C\,\mathrm{Id}$ in Ω', as distributions (Id is the $n \times n$ identity matrix).

In order to give a meaning to the concept of semiconcavity when the ambient space is a differentiable manifold M, we analyze the stability of this property under composition with C^2 maps.

PROPOSITION 1.6.6. *Let Ω and Ω' two open subsets of $\mathbb{R}^n$. If $u : \Omega \to \mathbb{R}$ is a Lipschitz function such that $u(x) - C\,|x|^2$ is concave and $\psi : \Omega' \to \Omega$ is a C^2 function with bounded first and second derivatives, then $u \circ \psi : \Omega' \to \mathbb{R}$ is a Lipschitz function and $u \circ \psi(y) - C'|y|^2$ is concave, for a suitable constant C'.*

The proof is straightforward. Then, the following definition is well–posed.

DEFINITION 1.6.7. A continuous function $u : M \to \mathbb{R}$ is called *locally semiconcave* if, for any local chart $\psi : \mathbb{R}^n \to \Omega \subset M$, the function $u \circ \psi$ is locally semiconcave in $\mathbb{R}^n$.

The importance of semiconcave functions in connection with the generalized differentials is expressed by the following proposition (see [**21**]).

PROPOSITION 1.6.8. *Let the function $u : M \to \mathbb{R}$ be locally semiconcave, then the superdifferential $\partial^+ u$ is not empty at each point, moreover, $\partial^+ v$ is* upper semicontinuous*, namely*

$$x_k \to x, \quad v_k \to v, \quad v_k \in \partial^+ u(x_k) \quad \Longrightarrow \quad v \in \partial^+ u(x)\,.$$

In particular, if the differential du exists at every *point of $\Omega \in M$, then $u \in C^1(\Omega)$.*

Now we introduce the definition of viscosity solution.
Let Ω be an open subset of M and H, called *Hamiltonian* function, a continuous real function on $T^*\Omega \times \mathbb{R}$. We are interested in the following Hamilton–Jacobi problem

$$\mathrm{H}(x, du(x), u(x)) = 0 \qquad \text{in } \Omega\,. \tag{1.6.1}$$

DEFINITION 1.6.9. We say that a continuous function u is a *viscosity solution* of equation (1.6.1) if for every $x \in \Omega$,

$$\begin{cases} \mathrm{H}(x, v, u(x)) \leq 0 & \forall v \in \partial^+ u(x)\,, \\ \mathrm{H}(x, v, u(x)) \geq 0 & \forall v \in \partial^- u(x)\,. \end{cases} \tag{1.6.2}$$

If only the first condition is satisfied (respectively, the second) u is called a *viscosity subsolution* (respectively, a *viscosity supersolution*).

If Ω' is an open subset of another smooth differentiable manifold N and $\psi : \Omega' \to \Omega$ is a C^1 local diffeomorphism, we define the *pull–back* of the Hamiltonian function $\psi^*\mathrm{H} : T^*\Omega' \times \mathbb{R} \to \mathbb{R}$ by

$$\psi^*\mathrm{H}(y, v, r) = \mathrm{H}(\psi(y), v \circ d\psi(y)^{-1}, r)\,.$$

Taking into account Proposition 1.6.2, the following statement is obvious.

PROPOSITION 1.6.10. *If u is a viscosity solution of $\mathrm{H} = 0$ in $\Omega \subset M$ and $\psi : \Omega' \to \Omega$ is a C^1 local diffeomorphism, then $u \circ \psi$ is a viscosity solution of $\psi^*\mathrm{H} = 0$ in $\Omega' \subset N$.*

1.6.2. The Distance Function from a Closed Subset of a Manifold. From now on, (M, g) will be a smooth, connected and complete, Riemannian manifold without boundary, of dimension n.

We consider a closed and not empty subset K and the distance function $d^K : M \to \mathbb{R}$ from K, which is defined as the infimum of the lengths of the C^1 curves starting at x and ending at K. As M is complete, by the Theorem of Hopf–Rinow, such infimum is reached by at least one curve which will be a smooth geodesic. We will also consider the function $\eta^K = [d^K]^2/2$ as in the previous sections.

In the following we will denote the distance between two points $x, y \in M$ with $d(x, y)$ and the *exponential map* of (M, g) with $\mathrm{Exp} : TM \times \mathbb{R} \to M$. For simplicity, we will write $|v|$ for the modulus of a vector $v \in TM$, defined as $\sqrt{g(v, v)}$.

PROPOSITION 1.6.11. *The distance function d^K is the* unique *viscosity solution of the following Hamilton–Jacobi problem*

$$\begin{cases} |\nabla u|^2 - 1 = 0 & \textit{in } M \setminus K\,, \\ u = 0 & \textit{on } K \end{cases} \tag{1.6.3}$$

in the class of continuous functions bounded from below.
The function η^K is the unique *viscosity solution of*

$$\begin{cases} |\nabla u|^2 - 2u = 0 & \textit{in } M\,, \\ u = 0 & \textit{on } K \end{cases} \tag{1.6.4}$$

in the class of continuous functions on M such that their zero set is K.

REMARK 1.6.12. The restriction to lower bounded functions is necessary, $\|x\|$ and $-\|x\|$ are both viscosity solutions of Problem (1.6.3) with $M = \mathbb{R}^n$ and $K = \{0\}$. Moreover, the completeness of M plays an important role here, if M is the open unit ball of $\mathbb{R}^n$ the same example shows that the uniqueness does not hold.
Notice also that every function $[d^H]^2/2$ where H is a closed subset of M with $H \supset K$, is a viscosity solution of Problem (1.6.4), equal to zero on K.

PROOF. The quantity $d^K(x)$ is the minimum time $t \geq 0$ for any curve γ to reach a point $\gamma(t) \in K$, subject to the conditions $\gamma(0) = 0$ and $|\gamma'| \leq 1$; the function d^K is then the value function of a "minimum time problem"; this proves that d^K is also a viscosity solution of Problem (1.6.3), by well known results (see for example [**10**, Chap. 4, Prop. 2.3]). Then we show that the function η^K is a solution of Problem (1.6.4).
First of all, notice that the distance function from K is a 1–Lipschitz function, hence η^K is locally Lipschitz.
As d^K is 1–Lipschitz, at every point of K the function η^K is differentiable and its differential is zero. Hence, the definition of viscosity solution holds also for points belonging to K. In order to prove the thesis, it is then sufficient to test conditions (1.6.2) on the generalized differentials at the points of the open set $M \setminus K$.
Since η^K is positive in $M \setminus K$, applying Proposition 1.6.3 with the function $\theta(t) = \sqrt{2t}$, we see that the function η^K is a viscosity solution of

$$g\left(\frac{\nabla u}{\sqrt{2u}}, \frac{\nabla u}{\sqrt{2u}}\right) - 1 = 0$$

in $M \setminus K$. Being there positive, it also solves

$$g(\nabla u, \nabla u) - 2u = 0$$

in $M \setminus K$. This fact together with the previous remark about the behavior of η^K at the points of K gives the claim.

Suppose now that u is a viscosity solution of Problem (1.6.3) then, u is also a solution of

$$\begin{cases} |\nabla u| - 1 = 0 & \text{in } M \setminus K\,, \\ u = 0 & \text{on } K\,. \end{cases}$$

As in the work of Kružhkov **[65]**, we consider the function $v = -e^{-u}$ which, by Proposition 1.6.3, turns out to be a viscosity solution of

$$\begin{cases} |\nabla v| + v = 0 & \text{in } M \setminus K\,, \\ v = -1 & \text{on } K \end{cases} \tag{1.6.5}$$

moreover, $|v| \leq e^{-\inf u}$.
We establish an uniqueness result for this last problem in the class of bounded functions v, which clearly implies the first uniqueness result. We remark that the proof is based on similar ones in **[28, 29, 49]**.
We argue by contradiction, suppose that u and v are two bounded solutions of (1.6.5), $|u|, |v| \leq C$, and that at a point $\overline{x}$ we have $u(\overline{x}) \geq 2\varepsilon + v(\overline{x})$ with $\varepsilon > 0$.
Let $b(x, y) : M \times M \to \mathbb{R}$ be a smooth function satisfying

- $b \geq 0$
- $|\nabla_x b(x, y)|, |\nabla_y b(x, y)| \leq 2$
- $\sup_{M\times M} |d(x, y) - b(x, y)| < \infty$

such a function can be obtained *smoothing* the distance function in $M \times M$.
We fix a point x_0 in K and we define the smooth function $B(x) = b(x, x_0)^2$. By the properties of b and the boundedness of u and v, the following function $\Psi : M \times M \to \mathbb{R}$

$$\Psi(x, y) = u(x) - v(y) - \lambda d(x, y)^2 - \delta B(x) - \delta B(y)$$

has a maximum at a point $(\widehat{x}, \widehat{y})$ (dependent on the positive parameters δ and λ) and such maximum $\Psi(\widehat{x}, \widehat{y})$ is less than $2C$. Hence, the function

$$x \mapsto [v(\widehat{y}) + \lambda d(x, \widehat{y})^2 + \delta B(x) + \delta B(\widehat{y})] - u(x) \tag{1.6.6}$$

has a minimum at $\widehat{x}$ while

$$y \mapsto [u(\widehat{x}) - \lambda d(\widehat{x}, y)^2 - \delta B(\widehat{x}) - \delta B(y)] - v(y) \tag{1.6.7}$$

has a maximum at $\widehat{y}$.
If $2\delta \leq \varepsilon/B(\overline{x})$ then

$$\Psi(\widehat{x}, \widehat{y}) \geq \Psi(\overline{x}, \overline{x}) \geq 2\varepsilon - 2\delta B(\overline{x}) \geq \varepsilon$$

hence, we get

$$\delta B(\widehat{x}) + \delta B(\widehat{y}) + \lambda d(\widehat{x}, \widehat{y})^2 + \varepsilon \leq u(\widehat{x}) - v(\widehat{y}) \leq 2C\,. \tag{1.6.8}$$

This shows that, for a fixed δ, the minimizing pairs $(\widehat{x}, \widehat{y})$, for λ varying, are all contained in a bounded set and, if λ goes to $+\infty$ the distance between $\widehat{x}$ and $\widehat{y}$ goes to zero. Possibly passing to a subsequence for λ going to infinity, $\widehat{x}$ and $\widehat{y}$ converge to a common limit point z which cannot belong to K, otherwise we would get $\varepsilon \leq u(z) - v(z) = 0$, thus, for some λ large enough also $\widehat{x}$ and $\widehat{y}$ do not belong to K.
As the function $d^2(x, y)$ is smooth in $B_z \times B_z \subset M \times M$, where B_z is a small geodesic ball around z, choosing a suitable λ large enough we can differentiate the functions inside the square brackets in equations (1.6.6) and (1.6.7) obtaining

$$\widehat{v} = \delta\nabla B(\widehat{x}) + \lambda\nabla_x d^2(\widehat{x}, \widehat{y}) \;\in \partial^+ u(\widehat{x})\,,$$
$$\widehat{w} = -\delta\nabla B(\widehat{y}) - \lambda\nabla_y d^2(\widehat{x}, \widehat{y}) \;\in \partial^- v(\widehat{y})\,.$$

By Definition 1.6.9 we have that $|\widehat{v}| + u(\widehat{x}) \leq 0$ and $|\widehat{w}| + v(\widehat{y}) \geq 0$, hence

$$u(\widehat{x}) - v(\widehat{y}) + |\widehat{v}| - |\widehat{w}| \leq 0\,.$$

Moreover,

$$\begin{aligned} |\widehat{v}| - |\widehat{w}| &= \left|\delta\,\nabla B(\widehat{v}) + \lambda\nabla_x d^2(\widehat{x}, \widehat{y})\right| - \left|\delta\,\nabla B(\widehat{y}) + \lambda\nabla_y d^2(\widehat{x}, \widehat{y})\right| \\ &\geq \left|\lambda\nabla_x d^2(\widehat{x}, \widehat{y})\right| - \left|\lambda\nabla_y d^2(\widehat{x}, \widehat{y})\right| - |\delta\,\nabla B(\widehat{y})| - |\delta\,\nabla B(\widehat{x})| \\ &= 2\lambda d(\widehat{x}, \widehat{y})\,|\nabla_x d(\widehat{x}, \widehat{y})| - 2\lambda d(\widehat{x}, \widehat{y})\,|\nabla_y d(\widehat{x}, \widehat{y})| - |\delta\,\nabla B(\widehat{y})| - |\delta\,\nabla B(\widehat{x})| \\ &= 2\lambda d(\widehat{x}, \widehat{y}) - 2\lambda d(\widehat{x}, \widehat{y}) - |\delta\,\nabla B(\widehat{y})| - |\delta\,\nabla B(\widehat{x})| \\ &= -\,|\delta\,\nabla B(\widehat{y})| - |\delta\,\nabla B(\widehat{x})| \end{aligned}$$

which implies,

$$u(\widehat{x}) - v(\widehat{y}) - \delta|\nabla B(\widehat{y})| - \delta|\nabla B(\widehat{x})| \leq 0\,.$$

Finally, we have that

$$\delta|\nabla B(\widehat{x})| = 2\delta|b(\widehat{x},x_0)\nabla b(\widehat{x},x_0)| \leq 4\delta\sqrt{B(\widehat{x})}$$

and using the estimate $\delta B(\widehat{x}) \leq 2C$ which follows from equation (1.6.8),

$$\delta|\nabla B(\widehat{x})| \leq 8\sqrt{2\delta C} \leq \varepsilon/4$$

if δ was chosen small enough. Holding the same for $\widehat{y}$, we conclude that

$$u(\widehat{x}) - v(\widehat{y}) - \varepsilon/2 \leq 0$$

which is in contradiction with the fact that $u(\widehat{x}) - v(\widehat{y}) \geq \varepsilon$.

About the second uniqueness claim, if u is a continuous viscosity solution of Problem (1.6.4) then, by Proposition 1.6.4 the superdifferential of u is not empty in a dense subset of $M \setminus K$, hence, directly by the equation and by continuity, u is non negative. By the hypothesis on its zero set we conclude that u is positive in all $M \setminus K$. Composing u with the function $t \mapsto \sqrt{2t}$, we see that $\sqrt{2u}$ is a positive, continuous viscosity solution of Problem (1.6.3), then it must coincide with d^K, by the previous result. It follows that $u = \eta^K$. □

We now study the *singular set* of d^K,

$$Sing = \left\{x \in M \mid \eta^K \text{ is not differentiable at } x\right\}\,.$$

REMARK 1.6.13. In this definition we used the squared distance function instead of the distance in order to avoid to consider also the points of the boundary of K, which are singular for d^K but not for η^K. It is trivial to see that outside K the distance and its square have the same regularity.

PROPOSITION 1.6.14. *The function d^K is locally semiconcave in $M \setminus K$.*

PROOF. The distance function d^K is a viscosity solution of $\mathrm{H} = 0$ in $M \setminus K$, where the Hamiltonian function is given by $\mathrm{H}(x, v, t) = |v|^2 - 1$. We choose a smooth local chart $\psi : \mathbb{R}^n \to \Omega \subset M$ and we define $v = d^K \circ \psi$, which is a locally Lipschitz function and, by Proposition 1.6.10, it is a viscosity solution of $\psi^*\mathrm{H} = 0$.
The pull–back of the Hamiltonian function on $\mathbb{R}^n$ takes the form

$$\psi^*\mathrm{H}(y, w, s) = g_{\psi(y)}(d\psi(w), d\psi(w)) - 1 = g_{ij}(y)w_iw_j - 1$$

for $(y, w, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and where $g_{ij}(y)$ are the components of the metric tensor of M in local coordinates.
Since the matrix $g_{ij}(y)$ is positive definite $\psi^*\mathrm{H}(y, w, s)$ is locally uniformly convex in w, hence, by [**71**, Thm. 5.3], it follows that $v = d^K \circ \psi$ is locally semiconcave in $\mathbb{R}^n$. Recalling Definition 1.6.7, this means that d^K is locally semiconcave in $M \setminus K$. □

The semiconcavity of d^K allows us to work with the superdifferential when the gradient does not exist. Indeed, it follows that the points of $Sing$ are precisely those where the superdifferential is not a singleton and the following result is a straightforward consequence of Proposition 1.6.8.

PROPOSITION 1.6.15. *The function η^K is of class C^1 in the open set $M \setminus \overline{Sing}$ and d^K is C^1 in $M \setminus \big(K \cup \overline{Sing}\big)$.*

The semiconcavity property also gives information about the relations between the structure of the superdifferential at a point x and the set of minimal geodesics from x to K (see [**3, 5**]).
The set $Ext(\partial^+\eta^K(x))$ of extremal points of the (convex) superdifferential set of η^K at x is in one–to–one correspondence with the family $\mathcal{G}(x)$ of minimal geodesics from x to K. Precisely $\mathcal{G}(x)$ is described by

$$\mathcal{G}(x) = \left\{\mathrm{Exp}(x, -v, \cdot) \,|\, [0, 1] \to M \,|\, \text{for } v \in Ext(\partial^+\eta^K(x))\right\}\,. \tag{1.6.9}$$

Hence, the set of points of K at minimum distance from x are given by $\mathrm{Exp}(x, -v, 1)$ for v in the set of extremal points of the superdifferential set of η^K at x. As a particular case we have that if the function η^K is differentiable at x if and only if the point of K closest to x is uniquely determined and given by $\mathrm{Exp}(x, -\nabla\eta^K(x), 1)$.

We consider now a set K which is a k–dimensional, embedded C^r submanifold of M without boundary, with $0 \leq k \leq n-1$ (the case $k = n$ is trivial) and $r \geq 2$.

For every $p \in K$ we consider the following three subsets of T_pM,

- T_pK, the vector subspace of tangent vectors to K at p,
- $N_pK = \{w \in T_pM \mid g_p(w, T_pK) = 0\}$, the vector subspace of normal vectors to K at p,
- $U_pK = \{w \in N_pK \mid g_p(w, w) = 1\}$, the subset of unit normal vectors to K at p,

then the bundles $NK = \{(p, v) \mid v \in N_pK\}$ and $UK = \{(p, v) \mid v \in U_pK\}$ inherit the structure of TM. Being K a C^r submanifold of M, the bundles NK and UK are respectively n–dimensional and $(n-1)$–dimensional C^{r-1} submanifolds of TM.

Notice that in the special case $K = \{p\}$, we have that $NK = T_pM$ and $UK = \mathbb{S}^{n-1} \subset T_pM$.

We define the map $F : UK \times \mathbb{R}^+ \to M$ as the restriction of the exponential map of M to UK,

$$F(p, v, t) = \mathrm{Exp}(p, v, t) \qquad \forall (p, v) \in UK \text{ and } t \in \mathbb{R}^+.$$

Since UK is a C^{r-1} manifold and the exponential map of M is smooth, F and all its derivatives with respect to the variable t are of class C^{r-1}.

REMARK 1.6.16. If a minimal geodesic, parametrized by arc length, starts at a point $p \in M$ and arrives at a point $q \in K$, its velocity vector v at q has to belong to U_qK, otherwise the condition of minimality is easily contradicted.

Since the geodesics, parametrized by arc length, ending on K are given by the family of maps $t \mapsto F(q, v, t)$ with $(q, v) \in UK$, the distance from K of a point p is given by the formula

$$d^K(p) = \inf\{t \in \mathbb{R}^+ \mid (q, v, t) \in F^{-1}(p)\}, \tag{1.6.10}$$

which obviously becomes $d^K(p) = \pi_{\mathbb{R}^+}\left(F^{-1}(p)\right)$ when the counterimage is a singleton (the map $\pi_{\mathbb{R}^+}$ is the projection on the second factor of the product $UK \times \mathbb{R}^+$).

The study of the singularities of the squared distance function then reduces to the analysis of the (possibly set valued) map F^{-1}.

This problem, from the topological point of view, is naturally connected with the study of the singularities of continuous maps between Euclidean spaces. For instance, when K coincides with a single point of M the singular sets were shown to be related to the classes of singularities considered by the *Theory of Catastrophes*, see **[20]**.

Let us define the C^{r-1} map $\exp : NK \to M$ by

$$\exp(p, v) = \mathrm{Exp}(p, v, 1) \qquad \forall (p, v) \in NK.$$

At the points $(p, 0) \in NK$ the map exp is differentiable and $d\exp(p, 0)$ is invertible between $T_{(p,0)}NK$ and T_pM, indeed $T_{(p,0)}NK$ can be identified with T_pM and under such identification $d\exp(p, 0)$ is the identity. Since, by hypothesis, the map exp is at least C^1, it follows that in a neighborhood of $(p, 0)$ in NK the differential of exp is invertible, hence the map exp is a C^{r-1} local diffeomorphism. Holding the relation $F(p, v, t) = \exp(p, vt)$, we conclude that for small $t > 0$, the map F is a local diffeomorphism.

Being K at least C^2, by a standard result in differential geometry, there exists an open *tubular neighborhood* Ω' of K in M formed by non intersecting, minimal geodesics starting normally from K. Hence, by the previous discussion and possibly choosing a smaller tubular neighborhood Ω of K, the map F^{-1} is well defined and C^{r-1} in $\Omega \setminus K$ (see for instance, **[7]**).

Then, the gradient of η^K exists in Ω and we have, by relations (1.6.9) and (1.6.10),

$$\nabla\eta^K(p) = d^K(p)\frac{\partial F}{\partial t}(F^{-1}(p)).$$

Since $d^K = \pi_{\mathbb{R}^+}\left(F^{-1}(p)\right) \in C^{r-1}$ in Ω and the functions F, $\frac{\partial F}{\partial t}$ are of class C^{r-1}, it follows that $\nabla\eta^K$ is C^{r-1} and η^K is C^r in $\Omega \setminus K$. The same C^r regularity in $\Omega \setminus K$ follows immediately also for the distance function d^K.

Moreover, the function η^K is C^r regular also on the set K, hence in the whole neighborhood Ω, as the square regularizes the jump of the gradient in the direction normal to K, see [**6, 7**].

We summarize these results in the following proposition which has as a particular case Proposition 1.4.1.

PROPOSITION 1.6.17. *If K is a regular submanifold of class C^r, with $r \geq 2$, then there exists an open subset Λ of $UK \times \mathbb{R}^+$ with the property that if $(q, v, t) \in \Lambda$ then also $(q, v, s) \in \Lambda$ for every $0 < s < t$, and an open neighborhood Ω of K in M, such that the map $F|_\Lambda : \Lambda \to \Omega \setminus K$ is a diffeomorphism. Moreover,*

- *for every point in Ω there is an unique point of minimum distance in K (*unique projection property in Ω*),*
- *the distance function d^K is C^r in $\Omega \setminus K$,*
- *the squared distance function η^K is C^r in Ω.*

REMARK 1.6.18. It can be proved that $C^{1,1}$ is the minimal regularity of K to have the unique projection property in a neighborhood, in this case also the squared distance function turns out to be of class $C^{1,1}$ (see [**42, 44**] and also [**33, 34**] for a detailed discussion of the relation between the regularity of K and of d^K.

CHAPTER 2

Functionals on Submanifolds of the Euclidean Space

In this chapter we are going to discuss the Euler equations of geometric functionals defined on submanifolds of the Euclidean space. We will show an algorithm to compute the first variation in general and we will analyze its "structural" properties for some selected functionals that will be the main subject of the next chapters.

2.1. Geometric Functionals

We are interested in studying functionals defined on compact, n–dimensional, smooth submanifolds of $\mathbb{R}^{n+m}$ depending on their geometric properties. More precisely, we consider *integral* functionals as follows

$$\mathcal{F}(\varphi) = \int_M f\left(\varphi, P, \mathrm{B}, \nabla\mathrm{B}, \dots, \nabla^s\mathrm{B}\right)\, d\mu\,, \tag{2.1.1}$$

defined on the smooth immersions $\varphi : M \to \mathbb{R}^{n+m}$ of n–dimensional differentiable manifolds, where f is a real smooth function and μ is the canonical measure on M associated to the induced metric via the immersion.

In codimension one, when $m = 1$, if ν is the inner normal vector field to the hypersurface, we will also consider the functionals

$$\mathcal{F}(\varphi) = \int_M f\left(\varphi, P, \mathrm{B}, \nabla\mathrm{B}, \dots, \nabla^s\mathrm{B}, \nu, \nabla\nu, \dots, \nabla^r\nu\right)\, d\mu\,,$$

which are anyway expressible as the ones above, by the Gauss–Weingarten relations (1.1.7) relating $\nabla\nu$ and B (clearly also P and ν have a one–to–one relation).

By simplicity and in order to have invariance under translation and rotation of the submanifold, in the sequel we will assume that the function f does not depend on the "position" φ and the tangent space (hence, not on P and ν), that is, the functional is "autonomous" and "anisotropic", but with minor variations all the conclusions follow also without such hypotheses.

Interesting examples of these functionals are the following:

- Taking $f = 1$ we have the *Area functional* which is the volume of the submanifold. This is clearly the simplest geometric functional on submanifolds.
- The functionals $\mathcal{W}_p(\varphi) = \int_M |\mathrm{B}|^p\, d\mu$ and $\mathcal{H}_p(\varphi) = \int_M |\mathrm{H}|^p\, d\mu$.
  In the special case of surfaces in $\mathbb{R}^3$ and if $p = 2$, the two functionals coincide up to a constant (by Gauss–Bonnet Theorem) and $\int_M |\mathrm{H}|^2\, d\mu$ takes the name of *Willmore functional* (see **[93]**).
- The functionals $\mathcal{B}_s(\varphi) = \int_M |\nabla^s\mathrm{B}|^2\, d\mu$ and $\mathcal{C}_r(\varphi) = \int_M |\nabla^r\nu|^2\, d\mu$, in codimension one, which are connected by the Gauss–Weingarten relations (1.1.7).

Seeing the submanifold M at least locally as a smooth subset of $\mathbb{R}^{n+m}$, by the relations between the second fundamental form and the distance function established in the previous chapter, it follows that actually all these functionals can be expressed in terms of functional in the function $A^M(x) = \frac{|x|^2 - [d^M(x)]^2}{2}$ and its derivatives as follows

$$\mathcal{F}(M) = \int_M f\left(A^M_{ij\dots k}(x)\right)\, d\mathcal{H}^n(x)\,, \tag{2.1.2}$$

with the meaning that where M is not embedded, we consider the distance from a local embedded piece and where $\mathcal{H}^n$ is the n–dimensional Hausdorff measure in $\mathbb{R}^{n+m}$, counting multiplicities. Notice that, by the hypotheses above, the function f does not depend on $\nabla A^M(x) = x$.

Viceversa, every functional in the distance function and its derivatives can be expressed as a geometric functional in the second fundamental form (and its derivatives).

It is then natural to consider among the interesting functionals also the following family that we call *De Giorgi functionals*, see [**31**, Sect. 5] and [**32**, Sect. 5] for an English translation,

$$\mathcal{G}_\gamma(M) = \int_M \sum_{|\alpha|=\gamma} |A^M_\alpha|^2 \, d\mathcal{H}^n = \int_M |A^\gamma|^2 \, d\mathcal{H}^n \,,$$

with the notation of Section 1.4.

2.2. First Variation

In this section we show how to compute the first variation of the general functional (2.1.2). As it is possible to express in this form the functionals (2.1.1) also, our procedure provides a method to compute the first variation for these latter too.

We consider an n–dimensional smooth submanifold $M \hookrightarrow \mathbb{R}^{n+m}$ and a smooth one–parameter family of diffeomorphisms Φ_t of $\mathbb{R}^{n+m}$ in itself such that there exists a bounded open subset U of $\mathbb{R}^{n+m}$ with $\Phi_t = \mathrm{Id}$ in $\complement U$, for every $t \in \mathbb{R}$, and $\Phi_0 = \mathrm{Id}$. The smooth field with compact support $X(x) = \frac{\partial \Phi_t(x)}{\partial t}\Big|_{t=0}$ is called the *infinitesimal generator* of the family Φ_t.
By the above properties of $\Phi_t : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, for $|t|$ small enough, this map gives a compactly supported *deformation* of M that we denote with $M_t = \Phi_t(M)$, which is again a smooth n–dimensional submanifold. We want now to compute the derivative of $\mathcal{F}(M_t)$ at $t = 0$,

$$\frac{d}{dt} \int_{M_t} f\left(A^{M_t}_{i_1 i_2}, \dots, A^{M_t}_{j_1 \dots j_\gamma}\right) d\mathcal{H}^n \Big|_{t=0} .$$

The main result of this section is the following.

PROPOSITION 2.2.1. *There exists a unique vector field $E_\mathcal{F}(A^M)$ such that*

$$\frac{d}{dt}\mathcal{F}(M_t)\Big|_{t=0} = \int_M \langle E_\mathcal{F}\left(A^M\right) \mid X \rangle \, d\mathcal{H}^n$$

for any family Φ_t whose infinitesimal generator is X. Moreover $E_\mathcal{F}(A^M)$ is normal and

(1) *if f depends on the derivatives of A^M up to the order γ, then $E_\mathcal{F}(A^M)$ depends on the derivatives of A^M up to the order $(2\gamma - 1)$;*
(2) *if the function f in the functional* (2.1.2) *is a polynomial, then $E_\mathcal{F}(A^M)$ is a polynomial in the derivatives of A^M.*

By the same argument leading to Proposition 1.4.1, we can find an open neighborhood Ω of $M = M_0$ such that for any $t \in (-\varepsilon, \varepsilon)$ all the manifolds M_t are contained in Ω and the function $A^t(x) = A^{M_t}(x)$ is a globally smooth function for $t \in (-\varepsilon, \varepsilon)$ and $x \in \Omega$. Notice that, by construction, all the M_t are coincident outside some compact subset of $\mathbb{R}^{n+m}$.

Applying the Area Formula (1.3.2) to the map $\Phi_t : M \to M_t$ we can rewrite the derivative as

$$\frac{d}{dt} \int_M f\left(A^t_{i_1 i_2}(\Phi_t(x)), \dots, A^t_{j_1 \dots j_\gamma}(\Phi_t(x))\right) J^M \Phi_t(x) \, d\mathcal{H}^n(x) \Big|_{t=0}$$

where $J^M \Phi_t(x)$ denotes the tangential Jacobian on M of the map Φ_t.
Hence, carrying the derivative under the integral sign, we find out

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(M_t)\Big|_{t=0} &= \int_M \sum_\alpha \frac{\partial f}{\partial A^M_\alpha} \, \frac{\partial}{\partial t}\left[A^t_\alpha(\Phi_t(x))\right]\Big|_{t=0} d\mathcal{H}^n(x) \\ &\quad + \int_M f\left(A^{M_t}_{i_1 i_2}, \dots, A^{M_t}_{j_1 \dots j_\gamma}\right) \frac{\partial}{\partial t} J^M \Phi_t(x) \Big|_{t=0} d\mathcal{H}^n(x) \,. \end{aligned}$$

where γ is a multiindex such that $|\alpha| \leq \gamma$.
Now, the derivative of the Jacobian is simply the tangential divergence of the infinitesimal generator field X and the derivative of the function $[A^t_\alpha(\Phi_t(x))]$ can be expressed by

$$\frac{\partial}{\partial t}\left[A^t_\alpha(\Phi_t(x))\right]\Big|_{t=0} = \frac{\partial A^t_\alpha}{\partial t}(x)\Big|_{t=0} + \langle \nabla A^M_\alpha(x)\,|\,X(x)\rangle\,.$$

Using the fact that the function $A^t(x)$ is smooth, we can exchange the order of differentiation in the middle term of this equation to get

$$\frac{\partial A^t_\alpha}{\partial t}(x)\Big|_{t=0} = D^\alpha\left\{\frac{\partial}{\partial t}A^t(x)\Big|_{t=0}\right\}\,.$$

To go on, we need to compute the t–derivative of the function $A^t(x)$ at $t = 0$.

LEMMA 2.2.2. *Under the above smoothness assumptions, we have*

$$\frac{\partial}{\partial t}A^t(x)\Big|_{t=0} = -\langle \nabla A^M(x) - x\,|\,X(\nabla A^M(x))\rangle$$

in Ω, where X is the infinitesimal generator field of Φ_t.

PROOF. We consider a point $x \in \Omega$ and we define $y = \pi^M(x) \in M$ and $z = \Phi_t(y) \in M_t$. We have $d^2(x, M) = \|x - y\|^2$ and $d^2(x, M_t) \leq \|x - z\|^2$, hence

$$\begin{aligned}\frac{A^{M_t}(x) - A^M(x)}{t} &= -\frac{d^2(x, M_t) - d^2(x, M)}{2t}\\ &\geq \frac{\|x - y\|^2 - \|x - z\|^2}{2t} = \frac{\langle z - y\,|\,2x - y - z\rangle}{2t}\,.\end{aligned}$$

Now $z - y = \Phi_t(y) - y$ is infinitesimal as $t \to 0$, moreover

$$\Phi_t(y) = y + tX(y) + o(t).$$

Then the last term of the equation above tends to

$$\langle X(\pi^M(x))\,|\,x - \pi^M(x)\rangle = -\langle \nabla A^M(x) - x\,|\,X(\nabla A^M(x))\rangle\,.$$

This proves that

$$\liminf_{t\to 0}\frac{A^{M_t}(x) - A^M(x)}{t} \geq -\langle \nabla A^M(x) - x\,|\,X(\nabla A^M(x))\rangle\,.$$

Now, using a similar reasoning with $y = \pi^{M_t}(x)$ and $z = \Phi_t^{-1}(y)$, we obtain the opposite estimate

$$\limsup_{t\to 0}\frac{A^{M_t}(x) - A^M(x)}{t} \leq -\langle \nabla A^M(x) - x\,|\,X(\nabla A^M(x))\rangle$$

and this proves the lemma. $\square$

We can now write the following general formula

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(M_t)\Big|_{t=0} &= \int_M f\left(A^{M_t}_{i_1 i_2}, \dots, A^M_{j_1\dots j_\gamma}\right)\mathrm{div}^M X(x)\,d\mathcal{H}^n(x)\\ &+ \int_M \sum_\alpha \frac{\partial f}{\partial A^M_\alpha}\langle \nabla A^M_\alpha(x)\,|\,X(x)\rangle\,d\mathcal{H}^n(x)\\ &- \int_M \sum_\alpha \frac{\partial f}{\partial A^M_\alpha} D^\alpha\left[\langle \nabla A^M(x) - x\,|\,X(\nabla A^M(x))\rangle\right]\,d\mathcal{H}^n(x)\,.\end{aligned}$$

Applying the tangential divergence theorem 1.3.4 to the first term and adding together gradient and tangential gradient of the functions A^M we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(M_t)\Big|_{t=0} = &-\int_M f\left(A^M_{i_1 i_2}, \ldots, A^M_{j_1\ldots j_\gamma}\right)\langle \mathrm{H}\,|\,X\rangle\, d\mathcal{H}^n \\ &+\int_M \sum_\alpha \frac{\partial f}{\partial A^M_\alpha}\langle \nabla^\perp A^M_\alpha\,|\,X\rangle\, d\mathcal{H}^n \\ &-\int_M \sum_\alpha \frac{\partial f}{\partial A^M_\alpha} D^\alpha\left[\langle \nabla A^M(x) - x\,|\,X(\nabla A^M(x))\rangle\right]\, d\mathcal{H}^n(x)\end{aligned} \tag{2.2.1}$$

recalling that H is the mean curvature vector and that the sign "$^\perp$" denotes the projection on the normal space to M.

It is now clear that the last step in getting to the Euler equations for $\mathcal{F}$ relies on the computation of the last term, and in particular on the study of the derivatives

$$D^\alpha\left[\langle \nabla A^M(x) - x\,|\,X(\nabla A^M(x))\rangle\right]. \tag{2.2.2}$$

Before proceeding to this computation, we want to make some remarks on the first variation.

PROPOSITION 2.2.3. *The first variation of the functional* (2.1.2) *depends only on the values on M of the infinitesimal generator X. Moreover if the vector field X is tangent to M, the first variation is zero.*

PROOF. Since $\nabla A^M(x) \in M$ for any $x \in \Omega$, formula (2.2.1) clearly implies that the first variation depends only on $X|_M$. If X is tangential, the first term is zero because $\mathrm{H}(x) \in N_x M$, the second one is clearly zero and the last one vanishes because $\nabla A^M(x) - x$ is normal to M at $\nabla A^M(x)$ for any $x \in \Omega$. □

Since (2.2.1) is linear in X, splitting $X(x)$ in $P(x)X(x)$ and $(I - P(x))X(x)$, we can assume in the following that $X(x)$ is normal to M at $\nabla A^M(x)$ for any $x \in \Omega$.

Now we go on with the study of the equation (2.2.2) assuming that the multiindex α is described by $(i_1 \ldots i_r)$ with $\gamma \geq r \geq 2$. We can distribute the derivatives on the two terms inside the scalar product. If all the derivatives act on the right term in the scalar product the result is zero, because the quantity $\nabla A^M(x) - x$ is zero on the manifold M. If all the derivatives go on the left term, it is simple to see that we obtain exactly the second term, with the opposite sign, in equation (2.2.1) which simplifies. So we study the terms with at least one derivative on $X(\nabla A^M(x))$ and at least one on $\nabla A^M(x) - x$.
Forgetting the term on the left in the scalar product, which will produce functions of kind $A^M_{j_1\ldots j_t}$, we reduce ourselves to study the derivatives of functions like $\varphi(\nabla A^M(x))$, where $\varphi : M \to \mathbb{R}$.

PROPOSITION 2.2.4. *For every multiindex β the derivative $D^\beta[\varphi(\nabla A^M(x))]$ can be expressed on M by a sum of terms*

$$g(A^M)\nabla^M_{j_1} \circ \nabla^M_{j_2} \circ \ldots \circ \nabla^M_{j_l}\varphi(x)$$

with $l \leq |\beta|$ and with the functions g being polynomials in the derivatives of A^M up to the order $|\beta| + 1$.

PROOF. We extend our notion of tangential gradient, denoting by $\nabla^M f(x)$ the projection of the gradient of the function f on the tangent space of M at the point $\pi^M(x)$ even if $x \notin M$. This vector clearly coincides with the tangential gradient if $x \in M$.
We argue by induction on $s = |\beta|$, that every derivative can be written as a sum of terms of the following kind

$$g\left(A^M\right)\nabla^M_{j_1} \circ \nabla^M_{j_2} \circ \ldots \circ \nabla^M_{j_l}\varphi(\nabla A^M(x)) \tag{2.2.3}$$

for $x \in \Omega$, $l \leq s$ and where $g\left(A^M\right)$ denotes a function of the derivatives of A^M up to the order $(s+1)$ (here we tangentially differentiate l–times the function $\varphi(y)$ and we evaluate the derivatives at $\nabla A^M(x)$).
If $s = 1$ we have only one derivative, hence

$$\frac{\partial}{\partial x_i}[\varphi(\nabla A^M(x))] = \nabla_k\varphi(\nabla A^M(x))\, A^M_{ki}(x) = \nabla^M_k\varphi(\nabla A^M(x))\,.$$

as $\nabla^2 A^M$ is the projection on the tangent space.
Now, assuming that the proposition is true for $(s-1)$, to get the induction step we have to differentiate with respect to x_i a formula like (2.2.3). If the additional derivative ∇_i acts on $g\left(A^M\right)$ it does not matter, while when it acts on the other factor we apply the same reasoning of the case $s=1$ to get a term of the form

$$g\left(A^M\right) A^M_{ik}(x)\nabla^M_k \circ \nabla^M_{j_1} \circ \nabla^M_{j_2} \circ \ldots \circ \nabla^M_{j_l}\varphi(\nabla A^M(x)).$$

Finally, if x belongs to M we have $\nabla A^M(x) = x$ and the statement follows. □

PROOF OF PROPOSITION 2.2.1. The uniqueness of $E_{\mathcal{F}}(A^M)$ easily follows by the possibility to choose $\Phi_t(x) = x + tX(x)$ where X is any vector field.
The existence of $E_{\mathcal{F}}(A^M)$ and its computing algorithm are described by the following steps:

- **Step 1.** Distribute the derivatives on the two terms in the scalar product in the last line of (2.2.1), avoiding to have all the derivatives acting on a single term.
- **Step 2.** Write the derivative operator on the field X in terms of tangential gradients, following Proposition 2.2.4.
- **Step 3.** Bring derivatives away from the field X, using the identity $f\nabla^M_i X^s = \nabla^M_i(fX^s) - X^s\nabla^M_i f$, and then the tangential divergence theorem 1.3.4 to exchange the integral of $\nabla^M_i(fX^s)$ with the integral of $-\mathrm{H}^i f X^s$. Iterating this procedure we get to a final expression $h^s(A^M)X^s$, which we are interested in.

In particular, we obtain that $E_{\mathcal{F}}(A^M)$ has a polynomial dependence on the derivatives of A^M if the same is true for f. Applying Proposition 2.2.4 to expressions like

$$\frac{\partial f}{\partial A^M_\alpha}\left[\langle D^\beta\left(\nabla A^M(x) - x\right) \mid D^\tau X(\nabla A^M(x))\rangle\right]$$

with $\beta + \tau = \alpha$ and $\beta,\, \tau \neq 0$, one finds terms of the following form

$$g_\sigma(A^M)\nabla^M_{\sigma_1} \circ \ldots \circ \nabla^M_{\sigma_l} X(x)$$

with g_σ depending on the derivatives of A^M up to the order $|\alpha|$ and $l \leq |\alpha| - 1$. Integrating by parts we obtain terms depending on the derivatives of A^M up to the order $l + |\alpha|$. Since $l \leq |\alpha| - 1$ and $|\alpha| \leq \gamma$, we get terms with derivatives of order at most $(2\gamma - 1)$. □

2.3. Euler Equations of some Special Functionals

We study now and compute explicitly the Euler equations in some interesting cases.
We will first consider the following functionals,

$$\mathcal{H}_p(M) = \int_M |\mathrm{H}|^p \, d\mathcal{H}^n \qquad \text{and} \qquad \mathcal{G}_\gamma(M) = \int_M \sum_{|\alpha|=\gamma} |A^M_\alpha|^2 \, d\mathcal{H}^n\,, \tag{2.3.1}$$

defined on compact, smooth n–dimensional submanifolds M of $\mathbb{R}^{n+m}$ with $\partial M = \emptyset$ and we will compute their first variations by means of the procedure of the previous section,

Then, we will discuss the Euler equations, for any $m \geq 1$, of the functionals

$$\mathcal{C}_m(\varphi) = \int_M |\nabla^m \nu|^2 \, d\mu\,,$$

defined on the immersions $\varphi : M \to \mathbb{R}^{n+1}$ of a smooth closed hypersurface in $\mathbb{R}^{n+1}$, where ν is a local unit normal vector field to M and $|\nabla^m\nu|^2$ means $\sum_{\alpha=1}^{n+1} |\nabla^m \nu^\alpha|^2$. The norm $|\cdot|$, the connection ∇ and the measure μ are all relative to the Riemannian metric g which is induced on M by $\mathbb{R}^{n+1}$ via the immersion φ. Notice that these functionals are well defined also without a global unit normal vector field, when M is not orientable, because of the modulus.
Even if these functionals can be expressed in terms of the function A^M, we will compute their first variation by means of a more "intrinsic", direct computation, differentiating the geometric objects associated to the Riemannian manifold (M, g), with the metric g induced by the immersion φ.

REMARK 2.3.1.

- The mean curvature vector H of M, appearing inside the first integral in (2.3.1), can be expressed as $\Delta\left(\nabla A^M\right)$ (see Proposition 1.4.11).
- The Willmore functional corresponds to the case of surfaces in $\mathbb{R}^3$ with $p=2$, for further references on this topic see **[93]**.
- For $\gamma=2$ the functional $\mathcal{G}_\gamma$ reduces to $n\mathcal{H}^n(M)$, whose first variation is $-n\mathrm{H}$.
  If $\gamma=3$, by Corollary 1.4.12, the functional $\mathcal{G}_\gamma$ is equal to 3 times the integral of the square of the quadratic norm of B.
- By the Gauss–Bonnet theorem, in the case $n=2$, $m=1$ the functionals $\mathcal{H}_2$ and $\mathcal{G}_3$ are proportional, if we consider a fixed genus family, as $|\mathrm{B}|^2$ is equal to $|\mathrm{H}|^2-2\lambda_1\lambda_2$, where $\lambda_1,\ \lambda_2$ are the principal curvatures. In particular, in this case we have $\mathrm{E}_{\mathcal{G}_3}=3\,\mathrm{E}_{\mathcal{H}_2}$ (see also Remark 2.3.3).

In the computations of this section we will need the following lemma expressing the Codazzi equations (1.1.5) in extrinsic terms.

LEMMA 2.3.2. *At every point of the manifold M, the following relation holds,*

$$\begin{aligned}\nabla_i^M \mathrm{B}_{jk}^l-\nabla_j^M \mathrm{B}_{ik}^l \;=\; \sum_{s=1}^{n+m}\big\{\,&\mathrm{B}_{ks}^l\nabla_i^M P_{js}-\,\mathrm{B}_{ks}^l\nabla_j^M P_{is}\\ &+\mathrm{B}_{js}^l\nabla_i^M P_{ks}-\,\mathrm{B}_{is}^l\nabla_j^M P_{ks}\\ &+\mathrm{B}_{ik}^s\nabla_j^M P_{ls}\,-\,\mathrm{B}_{jk}^s\nabla_i^M P_{ls}\,\big\}\,,\end{aligned}$$

where ∇^M denotes the tangential gradient.

The proof is a straightforward computation starting by formula (1.1.5) and using the relations of Section 1.4.

In codimension one this relation becomes very simple, denoting with ν a locally smooth, unit normal vector field and with B^ν the symmetric bilinear form $\langle \mathrm{B}|\nu\rangle$, by means of relations in Proposition 1.4.7, we have

$$\nabla_i^M \mathrm{B}_{jk}^\nu-\nabla_j^M \mathrm{B}_{ik}^\nu \;=\; \nu_j\,[\mathrm{B}^\nu]_{ik}^2-\nu_i\,[\mathrm{B}^\nu]_{jk}^2\;.$$

Moreover, setting in this formula $j=k$ and summing over the index k, we get the equation

$$\sum_{k=1}^{n+1}\nabla_k^M \mathrm{B}_{ik}^\nu \;=\; \nabla_i^M H+\nu_i|\mathrm{B}|^2 \tag{2.3.2}$$

where $H=\langle \mathrm{H}|\nu\rangle$.

2.3.1. The Euler Equation of $\mathcal{H}_p$. We have seen in formula (2.2.1) that the first variation is expressed by

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_p(M_t)\bigg|_{t=0} =&-\int_M |\mathrm{H}|^p\,\langle \mathrm{H}\,|\,X\rangle\,d\mathcal{H}^n\\ &+\int_M\langle\nabla^\perp|\mathrm{H}|^p\,|\,X\rangle\,d\mathcal{H}^n\\ &-p\sum_{ijl}\int_M |\mathrm{H}|^{p-2}\,A_{ill}^M D^{ijj}\left[\langle\nabla A^M(x)-x\,|\,X(\nabla A^M(x))\rangle\right]\,d\mathcal{H}^n(x)\,.\end{aligned} \tag{2.3.3}$$

By Proposition 2.2.3, we can assume that $X(x)$ is normal to M at $\nabla A^M(x)$ for any $x\in\Omega$. Studying the last term and distributing the derivatives in the scalar product we obtain the following:

- With 3 derivatives on the left term we get

  $$-\int_M\langle\nabla^\perp|\mathrm{H}|^p\,|\,X\rangle\,d\mathcal{H}^n$$

  that simplifies with the second term in equation (2.3.3).

- 3 derivatives on the right term give zero, because the function $\nabla A^M(x) - x$ is zero on M.
- 2 derivatives on the left term,

$$-p\int_M |\mathrm{H}|^{p-2}\, A^M_{ill}\langle D^{jj}\left(\nabla A^M(x)-x\right) \mid \nabla^M_i X(x)\rangle\, d\mathcal{H}^n(x)$$
$$-2p\int_M |\mathrm{H}|^{p-2}\, A^M_{ill}\langle D^{ij}\left(\nabla A^M(x)-x\right) \mid \nabla^M_j X(x)\rangle\, d\mathcal{H}^n(x)\,.$$

The first term is zero because A^M_{ill} is a normal vector and $\nabla^M_i X$ is a tangential gradient. The second one, using the tangential divergence theorem can be expressed as

$$2p\int_M |\mathrm{H}|^{p-2}\,\mathrm{H}^j A^M_{ill}\langle \nabla A^M_{ij}(x)\,|\,X(x)\rangle\, d\mathcal{H}^n(x)$$
$$+2p\int_M \nabla^M_j\left\{|\mathrm{H}|^{p-2}\,A^M_{ill}A^M_{ijs}\right\}X^s\,d\mathcal{H}^n\,.$$

Finally, by the fact that the 3–tensor A^M_{ijk} gives zero when applied to the two normal vectors A^M_{ill} and H^j (see Proposition 1.4.11(b)), we finally get

$$2p\int_M \nabla^M_j\left\{|\mathrm{H}|^{p-2}\,\mathrm{H}^i A^M_{ijs}\right\}X^s\,d\mathcal{H}^n\,.$$

- 2 derivatives on the right term,

$$-p\int_M |\mathrm{H}|^{p-2}\, A^M_{ill}D^i\left(\nabla_s A^M(x)-x_s\right)\,D^{jj}\left[X^s(\nabla A^M(x))\right]\,d\mathcal{H}^n(x)$$
$$-2p\int_M |\mathrm{H}|^{p-2}\, A^M_{ill}D^j\left(\nabla_s A^M(x)-x_s\right)\,D^{ij}\left[X^s(\nabla A^M(x))\right]\,d\mathcal{H}^n(x)\,.$$

Using an orthogonality argument like above, we see that the second of these two terms vanishes, while the first one gives

$$\begin{aligned}
&p\int_M |\mathrm{H}|^{p-2}\,A^M_{sll}\frac{\partial}{\partial x_j}\left\{A^M_{jr}(x)A^M_{rt}(\nabla A^M(x))X^s_t(\nabla A^M(x))\right\}\,d\mathcal{H}^n(x)\\
=\,&p\int_M |\mathrm{H}|^{p-2}\,A^M_{sll}A^M_{jr}(x)\frac{\partial}{\partial x_j}\left\{A^M_{rt}(\nabla A^M(x))X^s_t(\nabla A^M(x))\right\}\,d\mathcal{H}^n(x)\\
+\,&p\int_M |\mathrm{H}|^{p-2}\,A^M_{sll}A^M_{jjr}(x)A^M_{rt}(\nabla A^M(x))X^s_t(\nabla A^M(x))\,d\mathcal{H}^n(x)\\
=\,&p\int_M |\mathrm{H}|^{p-2}\,A^M_{sll}\nabla^M_r\circ\nabla^M_r X^s(x)\,d\mathcal{H}^n(x) = p\int_M |\mathrm{H}|^{p-2}\,A^M_{sll}\Delta^M X^s\,d\mathcal{H}^n(x)
\end{aligned}$$

where we used extensively Proposition 1.4.11(a) and in particular the identity

$$A^M_{jr}(x)A^M_{rt}(\nabla A^M(x)) = A^M_{jt}(x)\,.$$

Substituting A^M_{sll} with H^s and using the properties of the tangential Laplacian, this final term is equal to

$$p\int_M \Delta^M\left(|\mathrm{H}|^{p-2}\,\mathrm{H}^s\right)X^s\,d\mathcal{H}^n.$$

Finally, adding all these results together, we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{H}_p(M_t)\Big|_{t=0} &= -\int_M |\mathrm{H}|^p\,\langle \mathrm{H}\,|\,X\rangle\,d\mathcal{H}^n + p\int_M \Delta^M\left(|\mathrm{H}|^{p-2}\,\mathrm{H}^i\right)X^i\,d\mathcal{H}^n\\
&\quad +2p\int_M \nabla^M_j\left\{|\mathrm{H}|^{p-2}\,\mathrm{H}^s A^M_{ijs}\right\}X^i\,d\mathcal{H}^n\,.
\end{aligned}$$

Using the orthogonality of X and Proposition 1.4.11(b) we can simplify again the last term to get

$$2p\,e^\perp_i\nabla^M_j\left\{|\mathrm{H}|^{p-2}\mathrm{H}^s A^M_{ijs}\right\} = 2p\,e^\perp_i|\mathrm{H}|^{p-2}\nabla^M_j\left\{\mathrm{H}^s A^M_{ijs}\right\}.$$

Now we have,

$$e_i^\perp \nabla_j^M \left\{ \mathrm{H}^s A_{ijs}^M \right\} = e_i^\perp \nabla_j^M \left\{ \mathrm{H}^s \mathrm{B}_{ij}^s \right\} = e_i^\perp \mathrm{H}^s \nabla_j^M \mathrm{B}_{ij}^s$$

and using the relations of Lemma 2.3.2,

$$e_i^\perp \nabla_j^M \mathrm{B}_{ij}^s = \mathrm{B}_{jt}^s \nabla_j^M P_{it}$$

hence, substituting this quantity in the equation above, the term we are dealing with becomes

$$\mathrm{H}^s \mathrm{B}_{jt}^s \nabla_j^M P_{it} = \mathrm{H}^s \mathrm{B}_{jt}^s A_{jti}^M = \mathrm{H}^s \mathrm{B}_{jt}^s \mathrm{B}_{jt}^i .$$

Then we get the Euler equation of $\mathcal{H}_p$,

$$\mathrm{E}_{\mathcal{H}_p} = -|\mathrm{H}|^p\,\mathrm{H} + 2p\,|\mathrm{H}|^{p-2}\mathrm{H}^s\mathrm{B}_{jt}^s\mathrm{B}_{jt}^i e_i + p\,\Delta^M\left(|\mathrm{H}|^{p-2}\,\mathrm{H}^i\right) e_i^\perp\,, \tag{2.3.4}$$

where we denoted by $e_i^\perp = (I - \nabla^2 A^M) e_i$ the normal projections of the vectors of the canonical basis of $\mathbb{R}^{n+m}$.

In the codimension one case $m = 1$, we have a scalar form of the Euler equations. Indeed, considering the *scalar* second fundamental form $\mathrm{B}^\nu = \langle \mathrm{B} | \nu \rangle$ and the scalar mean curvature $H = \langle \mathrm{H} | \nu \rangle$, locally we can write $X(x) = \varphi(x)\nu(x)$ and $\mathrm{B}_{jt}^l = \nu_l \mathrm{B}_{jt}^\nu$, hence $\mathrm{H}(x) = H(x)\nu(x)$, with ν smooth unit normal vector field and φ in $C^\infty(M)$.
Equation (2.3.4) then becomes

$$\begin{aligned}\langle \mathrm{E}_{\mathcal{H}_p} \,|\, X \rangle = & -|H|^p H\varphi + 2p\varphi|H|^{p-2}H\mathrm{tr}\,[\mathrm{B}^\nu]^2 + p\varphi\Delta^M\left(|H|^{p-2}H\right) \\ & + p\varphi\,|H|^{p-2}\,H\nu_i\Delta^M\nu_i\end{aligned}$$

where we used the fact that $\nu_i \nabla^M \nu_i$ is equal to zero because ν is a unit vector field.
By the same reason $\nu_i \Delta^M \nu_i = -\langle \nabla^M \nu_i | \nabla^M \nu_i \rangle = -\sum_i \left|\nabla^M \nu_i\right|^2$, which is the square of the quadratic norm of the bilinear form B^ν, indeed, by relation (1.3.1) we have

$$\mathrm{B}_{ij}^\nu = -\nabla_i^M \nu_j = -\nabla_j^M \nu_i\,.$$

The term $\mathrm{tr}\,[\mathrm{B}^\nu]^2$ is clearly also equal to the norm of the second fundamental form $|\mathrm{B}|^2$, hence

$$\mathrm{E}_{\mathcal{H}_p} = \left[-|H|^p H + p\,|H|^{p-2}\,H|\mathrm{B}|^2 + p\,\Delta^M\left(|H|^{p-2}H\right)\right]\nu\,.$$

In particular, for the Willmore functional we have the nice equation (see **[93]**)

$$\mathrm{E}_{\mathcal{H}_2} = \left[2\,\Delta^M H + 2H|\mathrm{B}|^2 - H^3\right]\nu\,.$$

2.3.2. The Euler Equation of $\mathcal{G}_\gamma$. By Remark 2.3.1, we assume $\gamma > 2$ and we perform the full computation of $\mathrm{E}_{\mathcal{G}_\gamma}$ only for the case $\gamma = 3$ in codimension one. In the general case we only study the part of $\mathrm{E}_{\mathcal{G}_\gamma}$ containing the greatest number of derivatives of the function A^M.

Reasoning like in the previous example, the first variation of $\mathcal{G}_3$ in codimension one is given by

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_3(M_t)\Big|_{t=0} = & -\int_M |A_{ijk}^M|^2 \langle \mathrm{H} | X \rangle\, d\mathcal{H}^n \\ & - 2\int_M A_{ijk}^M\, A_{ijs}^M \nabla_k^M X^s\, d\mathcal{H}^n \\ & - 2\int_M A_{ijk}^M\left(A_{is}^M - \delta_{is}\right)\nabla_j^M \nabla_k^M X^s\, d\mathcal{H}^n\,,\end{aligned}$$

permuting cyclically the indices i, j and k in the last two integrals. Integrating by parts,

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_3(M_t)\Big|_{t=0} = & -3\int_M |\mathrm{B}|^2 \langle \mathrm{H} | X \rangle\, d\mathcal{H}^n \\ & + 6\int_M \nabla_k^M\left(\nabla_k^M A_{ij}^M\, A_{ijs}^M\right) X^s\, d\mathcal{H}^n \\ & + 3\int_M A_{ijk}^M\left(\delta_{is} - A_{is}^M\right)\left(\nabla_j^M \circ \nabla_k^M + \nabla_k^M \circ \nabla_j^M\right) X^s\, d\mathcal{H}^n\,.\end{aligned}$$

Now we use the fact that $A^M_{is} = \delta_{is} - \nu_i\nu_s$, moreover we set $\mathrm{H} = H\nu$ and $X = \varphi\nu$. Substituting these quantities in the formula above and simplifying the terms equal to zero by orthogonality, we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{G}_3(M_t)\Big|_{t=0} = & -3\int_M \varphi H|\mathrm{B}|^2\, d\mathcal{H}^n \\
& + 6\int_M \varphi\Delta^M\left(\nu_i\nu_j\right)\nabla^M_i \nu_j\, d\mathcal{H}^n \\
& - 6\int_M \varphi\nabla^M_k\left(\nu_i\nu_j\right)\nabla^M_k A^M_{ijs}\nu_s\, d\mathcal{H}^n \\
& - 3\int_M \nabla^M_j \nu_k\left(\nabla^M_j\nabla^M_k\varphi + \nabla^M_k\nabla^M_j\varphi\right)\, d\mathcal{H}^n \\
& - 3\int_M \varphi\nabla^M_j \nu_k\nu_s\left(\nabla^M_j\nabla^M_k\nu_s + \nabla^M_k\nabla^M_j\nu_s\right)\, d\mathcal{H}^n\,.
\end{aligned}
$$

Indeed, using the properties stated in Proposition 1.4.11, we can compute

$$
\begin{aligned}
\varphi\nabla^M_k\left(\nabla^M_k A^M_{ij}A^M_{ijs}\right)\nu_s &= -\varphi\nabla^M_k\left(\nabla^M_k\nu_i\nu_j\nabla^M_i A^M_{js} + \nabla^M_k\nu_j\nu_i\nabla^M_j A^M_{is}\right)\nu_s \\
&= \varphi\nabla^M_k\left\{\left(\nabla^M_k\nu_i\right)\left(\nabla^M_i\nu_s\right) + \left(\nabla^M_k\nu_j\right)\left(\nabla^M_j\nu_s\right)\right\}\nu_s \\
&= -2\left(\nabla^M_k\nu_i\right)\left(\nabla^M_i\nu_s\right)\left(\nabla^M_k\nu_s\right)\,.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{G}_3(M_t)\Big|_{t=0} = & -3\int_M \varphi H|\mathrm{B}|^2\, d\mathcal{H}^n - 12\int_M \varphi\left(\nabla^M_k\nu_i\right)\left(\nabla^M_k\nu_j\right)\left(\nabla^M_i\nu_j\right)\, d\mathcal{H}^n \\
& - 3\int_M \nabla^M_j\nu_k\left(\nabla^M_j\nabla^M_k\varphi + \nabla^M_k\nabla^M_j\varphi\right)\, d\mathcal{H}^n \\
& + 6\int_M \varphi\left(\nabla^M_j\nu_k\right)\left(\nabla^M_s\nu_j\right)\left(\nabla^M_k\nu_s\right)\, d\mathcal{H}^n \\
= & -3\int_M \varphi H|\mathrm{B}|^2\, d\mathcal{H}^n \\
& - 3\int_M \nabla^M_j\nu_k\left(\nabla^M_j\nabla^M_k\varphi + \nabla^M_k\nabla^M_j\varphi\right)\, d\mathcal{H}^n \\
& - 6\int_M \varphi\left(\nabla^M_k\nu_i\right)\left(\nabla^M_j\nu_k\right)\left(\nabla^M_i\nu_j\right)\, d\mathcal{H}^n\,.
\end{aligned}
$$

We introduce now the following elementary symmetric functions of the eigenvalues λ_i of $\mathrm{B}^\nu = \langle \mathrm{B}|\nu\rangle$,

$$
\mathcal{S}_s = \sum_{i_1<i_2\,\dots\,<i_s} \lambda_{i_1}\lambda_{i_2}\,\dots\,\lambda_{i_s}, \qquad \text{for } s\leq n
$$

and we define $\mathcal{S}_s = 0$ for $s > n$. The last term in the equation above can be written as

$$
\left(\nabla^M_k\nu_i\right)\left(\nabla^M_k\nu_j\right)\left(\nabla^M_i\nu_j\right) = -\mathrm{tr}\left[\mathrm{B}^\nu\right]^3 = -\sum_i \lambda_i^3\,.
$$

Using the formula

$$
\mathcal{S}_1^3 = -2(\lambda_1^3 + \lambda_2^3 + \,\dots\, + \lambda_n^3) + 3\mathcal{S}_1[\mathcal{S}_1^2 - 2\mathcal{S}_2] + 6\mathcal{S}_3
$$

and recalling that $H = \mathcal{S}_1$ and $\mathcal{S}_1^2 - 2\mathcal{S}_2 = |\mathrm{B}|^2$, we have

$$
2(\lambda_1^3 + \lambda_2^3 + \,\dots\, + \lambda_n^3) = H^3 - 3H|\mathrm{B}|^2 + 6\mathcal{S}_3\,.
$$

Substituting this term in the equation above, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_3(M_t)\Big|_{t=0} =& -3\int_M \varphi H|\mathrm{B}|^2\,d\mathcal{H}^n + 3\int_M \mathrm{B}^\nu_{jk}\left(\nabla^M_j\nabla^M_k\varphi + \nabla^M_k\nabla^M_j\varphi\right)\,d\mathcal{H}^n\\ &+ 18\int_M \varphi\mathcal{S}_3\,d\mathcal{H}^n - 3\int_M \varphi H^3\,d\mathcal{H}^n + 9\int_M \varphi H|\mathrm{B}|^2\,d\mathcal{H}^n\,,\end{aligned}$$

and finally

$$\begin{aligned}\frac{d}{dt}\int_M |\mathrm{B}|^2\,d\mathcal{H}^n\Big|_{t=0} =& \, 2\int_M \varphi H|\mathrm{B}|^2\,d\mathcal{H}^n + 6\int_M \varphi\mathcal{S}_3\,d\mathcal{H}^n - \int_M \varphi H^3\,d\mathcal{H}^n\\ &+ \int_M \mathrm{B}^\nu_{jk}\left(\nabla^M_j\nabla^M_k\varphi + \nabla^M_k\nabla^M_j\varphi\right)\,d\mathcal{H}^n\,.\end{aligned}$$

Now, to conclude we show that the last term of the formula above is equal to

$$2\int_M \varphi\,\Delta^M H\,d\mathcal{H}^n.$$

This can be done with the help of Codazzi equations, in particular using the relation (2.3.2),

$$\begin{aligned}\int_M \mathrm{B}^\nu_{jk}\,\nabla^M_j\nabla^M_k\varphi\,d\mathcal{H}^n &= -\int_M \nabla^M_j\mathrm{B}^\nu_{jk}\nabla^M_k\varphi\,d\mathcal{H}^n\\ &= -\int_M \left(\nabla^M_k H + \nu_k|\mathrm{B}|^2\right)\nabla^M_k\varphi\,d\mathcal{H}^n = -\int_M \nabla^M_k H\nabla^M_k\varphi\,d\mathcal{H}^n\\ &= \int_M \varphi\nabla^M_k\nabla^M_k H\,d\mathcal{H}^n = \int_M \varphi\Delta^M H\,d\mathcal{H}^n\,.\end{aligned}$$

Hence, the Euler equation of the functional $\mathcal{G}_3$ (which is three times the integral of $|\mathrm{B}|^2$) is given by

$$\mathrm{E}_{\mathcal{G}_3} = 3\,\left[2\Delta^M H + 2H|\mathrm{B}|^2 - H^3 + 6\mathcal{S}_3\right]\nu.$$

REMARK 2.3.3. For a complete discussion of Euler equations of functionals depending on the elementary symmetric functions of the eigenvalues of the second fundamental form, see **[90]**.

Now we deal with the general functional $\mathcal{G}_\gamma$.

PROPOSITION 2.3.4. *For any $\gamma > 2$ the Euler equation of the functional $\mathcal{G}_\gamma$ is given by*

$$\begin{aligned}\mathrm{E}_{\mathcal{G}_\gamma} &= 2\gamma(-1)^{\gamma-1}\sum_{j,i_2,k_2\,\dots\,i_\gamma,k_\gamma}\left(A^M_{i_2k_2}\,\dots\,A^M_{i_\gamma k_\gamma}A^M_{ji_2k_2\,\dots\,i_\gamma k_\gamma}\right)e_j^\perp + g(A^M)\\ &= 2\gamma(-1)^{\gamma-1}\sum_{j=1}^{n+m}\Big(\overbrace{\Delta^M\Delta^M\dots\Delta^M}^{(\gamma-2)\text{–}times}\,\mathrm{H}^j\Big)\,e_j^\perp + h(A^M)\end{aligned}\tag{2.3.5}$$

where the vector fields $g(A^M)$, $h(A^M)$ are polynomials in the derivatives of A^M up to the order $(2\gamma-2)$.

PROOF. By Proposition 2.2.3, we can assume that the infinitesimal generator X is a normal vector field. We can see, following the proof of Proposition (2.2.1), that the term with the highest number of derivatives arises from the integral

$$-2\int_M A^M_{i_1\,\dots\,i_\gamma}D^{i_1\,\dots\,i_\gamma}\langle\nabla A^M(x) - x\,|\,X(\nabla A^M(x))\rangle\,d\mathcal{H}^n(x)$$

when all but one of the derivatives D^{i_j} act on the field X. We suppose that the only derivative going on the left is D^{i_1}. Hence, we have to study

$$-2\int_M A^M_{i_1\,\dots\,i_\gamma}\left(A^M_{i_1k}(x) - \delta_{i_1k}\right)\,D^{i_2\,\dots\,i_\gamma}\left[X^k(\nabla A^M(x))\right]\,d\mathcal{H}^n(x)\,.$$

After doing the first derivative on $X(\nabla A^M(x))$ we get $A^M_{i_\gamma j}(x)(\nabla^M_j X^k)(\nabla A^M(x))$. It is clear that if we are only interested in the term containing the highest derivative, we can avoid to distribute derivatives on $A^M_{i_\gamma j}(x)$ and then consider only the term containing the derivatives of the field. Iterating this argument we get

$$-2\int_M A^M_{i_1\dots i_\gamma}\left(A^M_{i_1 k}(x)-\delta_{i_1 k}\right)A^M_{i_\gamma j_\gamma}\,\dots\,A^M_{i_2 j_2}\nabla^M_{j_2}\circ\,\dots\,\circ\nabla^M_{j_\gamma}X^k(x)\,d\mathcal{H}^n(x)\,.$$

Now we have to apply the tangential divergence theorem 1.3.4, noticing again that if we are interested only in the highest derivative term, we can limit ourselves to differentiate the term $A^M_{i_1\dots i_\gamma}$. Moreover, since we apply the theorem with tangential fields, no term containing H appears. After doing this we obtain

$$(-1)^\gamma 2\int_M\left[\nabla^M_{j_\gamma}\circ\,\dots\,\circ\nabla^M_{j_2}A^M_{i_1\dots i_\gamma}\right]\left(A^M_{i_1 k}-\delta_{i_1 k}\right)A^M_{i_\gamma j_\gamma}\,\dots\,A^M_{i_2 j_2}X^k(x)\,d\mathcal{H}^n(x)\,.$$

Using the orthogonality of X we get

$$-2(-1)^\gamma\int_M\left[\nabla^M_{j_\gamma}\circ\,\dots\,\circ\nabla^M_{j_2}\left(\nabla_{i_1}A^M\right)_{i_2\dots i_\gamma}\right]A^M_{i_\gamma j_\gamma}\,\dots\,A^M_{i_2 j_2}X^{i_1}(x)\,d\mathcal{H}^n(x)\,.$$

Hence, performing the tangential derivatives and adding on all indices we get the first equality in formula (2.3.5).
To obtain the second equality, we apply in the inverse direction the derivative of a product formula to *carry inside* the components of the projection $A^M_{i_t k_t}$, in order to obtain the tangential Laplacians. Notice that, with a reasoning similar to the one above, in doing this we only introduce terms with an order of differentiation at most $(2\gamma-2)$. In this way we obtain

$$\mathrm{E}_{\mathcal{G}_\gamma}=2\gamma(-1)^{\gamma-1}\sum_{j=1}^{n+m}\Big(\overbrace{\Delta^M\Delta^M\dots\Delta^M}^{(\gamma-1)\text{–times}}\left(\nabla_j A^M\right)\Big)e_j^\perp+h(A^M)\,.$$

The last step in proving this representation formula for $\mathcal{G}_\gamma$ is to recall that $\mathrm{H}=\Delta^M\left(\nabla A^M\right)$, by Proposition 1.4.11). $\square$

In codimension one we have another more standard form of the Euler equation $\mathrm{E}_{\mathcal{G}_\gamma}$.

COROLLARY 2.3.5. *If $m=1$, for any $\gamma>2$ the Euler equation of the functional $\mathcal{G}_\gamma$ is given by*

$$\mathrm{E}_{\mathcal{G}_\gamma}=2\gamma(-1)^{\gamma-1}\Big(\overbrace{\Delta\Delta\dots\Delta}^{(\gamma-2)\text{–times}}H\Big)\,\nu+h(A^M)$$

where $h(A^M)$ is a normal vector field which is a polynomial in the derivatives of A^M up to the order $(2\gamma-2)$ and Δ is the intrinsic Laplacian of the Riemannian manifold (M,g), with g the metric induced by the immersion of M in $\mathbb{R}^{n+1}$.

PROOF. If the codimension is one, we have $\mathrm{H}=H\nu$ and the second formula in Proposition 2.3.4 becomes

$$\mathrm{E}_{\mathcal{G}_\gamma}=2\gamma(-1)^{\gamma-1}\sum_{j=1}^{n+1}\Big(\overbrace{\Delta^M\Delta^M\dots\Delta^M}^{(\gamma-2)\text{–times}}(H\nu^j)\Big)\,e_j^\perp+h(A^M)\,,$$

hence, after distributing all the Laplacians on the product $H\nu^j$, the leading term is obtained when all the Laplacians go on the mean curvature factor H. It is straightforward to check that all the other lower order terms we obtain, expressing them with the function A^M and its derivatives by means of the relations of the previous section (after using the Gauss–Weingarten relations to write any $\nabla^M\nu$ in terms of B), are polynomials in the derivatives of A^M up to the order $(2\gamma-2)$, hence we can absorb them into the term $h(A^M)$. Finally, the normality of this latter follows by Proposition 2.2.3.
We conclude the proof noticing that, since H is a function, the tangential Laplacian Δ^M and the intrinsic one Δ on (M,g) coincide. $\square$

REMARK 2.3.6. We remark that in the two expressions in Proposition 2.3.4for the leading term, we cannot substitute $e_i^\perp$ with e_i, because of the fact that neither the first nor the second are in general normal vectors. This can be seen considering a torus T in $\mathbb{R}^3$ with the biggest radius equal to 2 and the smallest one equal to 1, for instance the one defined by

$$T \equiv \left\{(2-\cos\alpha)\cos\beta, (2-\cos\alpha)\sin\beta, \sin\alpha) \in \mathbb{R}^3 \mid (\alpha,\beta) \in \mathbb{R}^2\right\}$$

and computing such two vectors in the first meaningful case $\gamma = 3$ at the point $(2,0,1)$ on the top of T.

The function η^T for the torus is given by

$$\eta^T(x,y,z) = \frac{1}{2}\left[\sqrt{\left(\sqrt{x^2+y^2}-2\right)^2+z^2}-1\right]^2 .$$

Considering

$$A^T(x,y,z) = \frac{\|(x,y,z)\|^2}{2} - \eta^T(x,y,z)$$

and using the *Mathematica*[1] package we computed

$$\sum_{i_2,k_2,i_3,k_3} A^T_{i_2k_2}A^T_{i_3k_3}A^T_{ji_2k_2i_3k_3} = A^T_{j1111} + 2A^T_{j2211} + A^T_{j2222}$$

with $j = 1$ at $(2,0,1)$ and we found the value -3, hence there is a tangential component in the leading term of the first representation in expression (2.3.5).

For the second term we show the computation explicitly. We have that $\Delta^T\mathrm{H}^i = \Delta^T(H\nu_i)$, hence

$$\begin{aligned}\Delta^T\mathrm{H}^i &= \nu_i\Delta^T H + 2\langle\nabla^T\nu_i|\nabla^T H\rangle + H\Delta\nu_i\\ &= \nu_i\Delta^T H + 2\nabla^T_k\nu_i\nabla^T_k H + H\nabla^T_k\nabla^T_k\nu_i\\ &= \nu_i\Delta^T H - 2\mathrm{B}^\nu_{ik}\nabla^T_k H - H\nabla^T_k\mathrm{B}^\nu_{ik} .\end{aligned}$$

Now we apply the relation (2.3.2) to the last term in the equation above to get

$$\begin{aligned}\Delta^T\mathrm{H}^i &= \nu_i\Delta^T H - 2\mathrm{B}^\nu_{ik}\nabla^T_k H - H\nabla^T_i H - \nu_i H|\mathrm{B}|^2\\ &= \nu_i\left(\Delta^T H - H|\mathrm{B}|^2\right) - (H\delta_{ik} + 2\mathrm{B}^\nu_{ik})\nabla^T_k H .\end{aligned}$$

Since at the point $(2,0,1)$ of the torus T we have $\mathrm{B}^\nu_{11} = 1$ and $\mathrm{B}^\nu_{2j} = 0$, hence $H = 1$, the vector $e_i\Delta^T\mathrm{H}^i$ has a tangential part given by

$$-3(\nabla^T_1 H)e_1 - H(\nabla^T_2 H)e_2 . \tag{2.3.6}$$

The quantity $H = \langle\mathrm{H}|\nu\rangle$ in a neighborhood of the point $(2,0,1)$ is

$$H(x,y,z) = 2 - \frac{2}{\sqrt{x^2+y^2}} ,$$

then

$$\nabla H(x,y,z) = 2\frac{xe_1+ye_2}{(x^2+y^2)^{3/2}} .$$

At the point $(2,0,1)$ we have

$$\nabla H(2,0,1) = \frac{1}{2}e_1 = \nabla^M H(2,0,1)$$

because the gradient is a tangent vector.
This, with the computation (2.3.6) shows that $e_i\Delta\mathrm{H}^i$ can have a non zero tangential component.

[1] *Mathematica* is a registered trademark of Wolfram Research.

2.3.3. The Euler Equation of $\mathcal{C}_m$. As in the previous case of $\mathcal{G}_\gamma$, we are going to analyze the main properties and the structure of the first variation of the functional $\mathcal{C}_m$, with particular attention to the leading term, since computing the exact form can be quite complicated.

Instead of expressing the functional in terms of the function A^M and applying the procedure of Section 2.2, we compute directly its first variation, differentiating the geometric objects associated to the Riemannian manifold (M, g), with the metric g induced by the immersion φ.

Suppose that we have a one–parameter family $\mathcal{I}$ of immersions $\varphi_t : M \to \mathbb{R}^{n+1}$, with $\varphi_0 = \varphi$, we compute

$$\frac{d}{dt}\mathcal{C}_m(\varphi_t)\Big|_{t=0} = \frac{d}{dt}\int_M |\nabla^m \nu|^2\, d\mu_t\Big|_{t=0} \tag{2.3.7}$$

where clearly the metric g, the covariant derivative ∇ and the normal ν depend on t.
Setting $X(p) = \frac{\partial}{\partial t}\varphi_t(p)\big|_{t=0}$ we obtain a vector field along M as a submanifold of $\mathbb{R}^{n+1}$ via φ. It is well known that

$$\frac{\partial}{\partial t}\mu_t\Big|_{t=0} = -\langle \mathrm{H}\,|\,X\rangle\,\mu$$

so it follows,

$$\begin{aligned}
\frac{d}{dt}\mathcal{C}_m(\varphi_t)\Big|_{t=0} &= -\int_M |\nabla^m\nu|^2 \langle \mathrm{H}\,|\,X\rangle\, d\mu + \int_M \frac{\partial}{\partial t}|\nabla^m\nu|^2\Big|_{t=0}\, d\mu\\
&= -\int_M |\nabla^m\nu|^2 H\langle \nu\,|\,X\rangle\, d\mu + \int_M \frac{\partial}{\partial t}\left(g^{i_1 j_1}\dots g^{i_m j_m}\nabla_{i_1\dots i_m}\nu\nabla_{j_1\dots j_m}\nu\right)\Big|_{t=0}\, d\mu\,.
\end{aligned}$$

Then, we need to compute the derivative in the last term.
For the metric tensor g_{ij} we have

$$\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= \frac{\partial}{\partial t}\left\langle \frac{\partial\varphi}{\partial x_i}\,\middle|\,\frac{\partial\varphi}{\partial x_j}\right\rangle\\
&= \left\langle \frac{\partial X}{\partial x_i}\,\middle|\,\frac{\partial\varphi}{\partial x_j}\right\rangle + \left\langle \frac{\partial X}{\partial x_j}\,\middle|\,\frac{\partial\varphi}{\partial x_i}\right\rangle\\
&= \frac{\partial}{\partial x_i}\left\langle X\,\middle|\,\frac{\partial\varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X\,\middle|\,\frac{\partial\varphi}{\partial x_i}\right\rangle - 2\left\langle X\,\middle|\,\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right\rangle\\
&= a_{ij}(X)\,.
\end{aligned}$$

Differentiating the formula $g_{is}g^{sj} = \delta_i^j$ we get

$$\frac{\partial}{\partial t}g^{ij} = -g^{is}\frac{\partial}{\partial t}g_{sl}g^{lj} = -g^{is}a_{sl}(X)g^{lj}\,.$$

The derivative of the normal ν is given by

$$\begin{aligned}
\frac{\partial}{\partial t}\nu &= \left\langle \frac{\partial\nu}{\partial t}\,\middle|\,\frac{\partial\varphi}{\partial x_i}\right\rangle \frac{\partial\varphi}{\partial x_j}g^{ij} = -\left\langle \nu\,\middle|\,\frac{\partial^2\varphi}{\partial t\partial x_i}\right\rangle\frac{\partial\varphi}{\partial x_j}g^{ij}\\
&= -\left\langle \nu\,\middle|\,\frac{\partial X}{\partial x_i}\right\rangle\frac{\partial\varphi}{\partial x_j}g^{ij} = -\nabla\langle\nu\,|\,X\rangle + \left\langle\frac{\partial\nu}{\partial x_i}\,\middle|\,X\right\rangle\frac{\partial\varphi}{\partial x_j}g^{ij}\\
&= -\nabla\langle\nu\,|\,X\rangle + \nabla\nu^\alpha X^\alpha = b(X)\,.
\end{aligned}$$

Finally the derivative of the Christoffel symbols is

$$
\begin{aligned}
\frac{\partial}{\partial t}\Gamma^i_{jk} =& \frac{1}{2}g^{il}\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial t}g_{kl}\right)+\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial t}g_{jl}\right)-\frac{\partial}{\partial x_l}\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}\\
&+\frac{1}{2}\frac{\partial}{\partial t}g^{il}\left\{\frac{\partial}{\partial x_j}g_{kl}+\frac{\partial}{\partial x_k}g_{jl}-\frac{\partial}{\partial x_l}g_{jk}\right\}\\
=& \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right)+\nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right)-\nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}\\
&+\frac{1}{2}g^{il}\left\{\frac{\partial}{\partial t}g_{kz}\Gamma^z_{jl}+\frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk}+\frac{\partial}{\partial t}g_{jz}\Gamma^z_{kl}+\frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk}-\frac{\partial}{\partial t}g_{jz}\Gamma^z_{kl}-\frac{\partial}{\partial t}g_{kz}\Gamma^z_{jl}\right\}\\
&-\frac{1}{2}g^{is}\frac{\partial}{\partial t}g_{sz}g^{zl}\left\{\frac{\partial}{\partial x_j}g_{kl}+\frac{\partial}{\partial x_k}g_{jl}-\frac{\partial}{\partial x_l}g_{jk}\right\}\\
=& \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right)+\nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right)-\nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}\\
&+g^{il}\frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk}-g^{is}\frac{\partial}{\partial t}g_{sz}\Gamma^z_{jk}\\
=& \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right)+\nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right)-\nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}\\
=& \frac{1}{2}g^{il}\left\{\nabla_j a_{kl}(X)+\nabla_k a_{jl}(X)-\nabla_l a_{jk}(X)\right\}\,.
\end{aligned}
$$

Notice that all these derivatives are linear in the field X, since the $a_{ij}(X)$ and $b(X)$ are such.

Recalling the conventions we set in Section 1.2, we have the following lemma.

LEMMA 2.3.7. *If $a(X)=\frac{\partial}{\partial t}g$ is the tensor defined before, for every covariant tensor $T=T_{i_1\dots i_l}$ we have*

$$\frac{\partial}{\partial t}\nabla^s T=\nabla^s\frac{\partial T}{\partial t}+\mathfrak{p}_{s-1}(T,\nabla a(X))$$

where the constants in the polynomials $\mathfrak{p}_{s-1}(T,\nabla a(X))$ are universal.
Moreover, if the tensor T is a function $f: M\to\mathbb{R}^k$ the last term $\mathfrak{p}_{s-1}(f,\nabla a(X))$ can be substituted with another polynomial $\widetilde{\mathfrak{p}}_{s-2}(\nabla f,\nabla a(X))$.

PROOF. We prove the lemma by induction on $s\geq 1$.
If $s=1$ then

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla_j T_{i_1\dots i_l} &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x_j}T_{i_1\dots i_l}-\Gamma^r_{ji_z}T_{i_1\dots i_{z-1}ri_{z+1}\dots i_l}\right)\\
&=\frac{\partial}{\partial x_j}\frac{\partial}{\partial t}T_{i_1\dots i_l}-\Gamma^r_{ji_z}\frac{\partial}{\partial t}T_{i_1\dots i_{z-1}ri_{z+1}\dots i_l}\\
&\quad-\frac{\partial}{\partial t}\Gamma^r_{ji_z}T_{i_1\dots i_{z-1}ri_{z+1}\dots i_l}\\
&=\nabla\frac{\partial T}{\partial t}+T*\nabla a(X)
\end{aligned}
$$

by the previous computation, hence

$$\frac{\partial}{\partial t}\nabla T=\nabla\frac{\partial T}{\partial t}+\mathfrak{p}_0(T,\nabla a(X))$$

and the initial case is proved.
Supposing the lemma holds for $s-1$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^s T &= \frac{\partial}{\partial t}\nabla(\nabla^{s-1}T)\\ &= \nabla\left(\frac{\partial}{\partial t}\nabla^{s-1}T\right) + \mathfrak{p}_0(\nabla^{s-1}T, \nabla a(X))\\ &= \nabla\left(\nabla^{s-1}\frac{\partial T}{\partial t} + \mathfrak{p}_{s-2}(T, \nabla a(X))\right)\\ &\quad + \mathfrak{p}_0(\nabla^{s-1}T, \nabla a(X))\\ &= \nabla^s\frac{\partial T}{\partial t} + \nabla\mathfrak{p}_{s-2}(T, \nabla a(X))\\ &\quad + \mathfrak{p}_0(\nabla^{s-1}T, \nabla a(X))\\ &= \nabla^s\frac{\partial T}{\partial t} + \mathfrak{p}_{s-1}(T, \nabla a(X))\end{aligned}$$

where we set

$$\mathfrak{p}_{s-1}(T, \nabla a(X)) = \nabla\mathfrak{p}_{s-2}(T, \nabla a(X)) + \mathfrak{p}_0(\nabla^{s-1}T, \nabla a(X))\,.$$

By this last formula, it is clear that the constants involved are universal. Moreover, if T is a function $f : M \to \mathbb{R}^k$ then the term $\mathfrak{p}_0(f, \nabla a(X))$ vanishes and the same formula says that $\mathfrak{p}_{s-1}(f, \nabla a(X))$ does not contain f without being differentiated. □

REMARK 2.3.8. In the following we will omit to underline that all the coefficients of the polynomials $\mathfrak{p}_s$ and $\mathfrak{q}^s$ which will appear are *algebraic*, that is, they are the result of formal manipulations. In particular, such coefficients are independent of the manifold (M, g) where the tensors are defined. This is crucial in view of the geometry–independent estimates we need in the analysis of the following chapters.

PROPOSITION 2.3.9. *The derivative*

$$\frac{\partial}{\partial t}\left(g^{i_1 j_1}\dots g^{i_m j_m}\nabla_{i_1\dots i_m}\nu\nabla_{j_1\dots j_m}\nu\right)\Big|_{t=0}$$

depends only on the vector field $X = \frac{\partial}{\partial t}\varphi_t\,\big|_{t=0}$ and such dependence is linear. Hence, the first variation of $\mathcal{C}_m$ is a linear function of the field X.

PROOF. Distributing the derivative in t on the terms of the product, we have seen that the derivatives of the metric coefficients depends linearly on X, it lasts to check the derivative of $\nabla_{i_1\dots i_m}\nu$.
By the last assertion of Lemma 2.3.7, we have

$$\frac{\partial}{\partial t}\nabla^m\nu = \nabla^m\frac{\partial\nu}{\partial t} + \mathfrak{p}_{m-2}(\nabla\nu, \nabla a(X))$$

and since $\frac{\partial\nu}{\partial t} = b(X)$ we get

$$\frac{\partial}{\partial t}\nabla^m\nu = \nabla^m b(X) + \mathfrak{p}_{m-2}(\nabla\nu, \nabla a(X))$$

which proves the first part of the lemma as $a(X)$ and $b(X)$ are linear in X.
The second statement clearly follows by the previous computations and the first part of the lemma. □

Now we want to prove that actually the first variation depends only on the *normal component* of the field X, that is, $\langle\nu \mid X\rangle$, by linearity, it is clearly sufficient to show that $\delta\mathcal{C}_m(\varphi)(X) = 0$ for every tangent vector field X. By the previous proposition, in order to compute the derivative (2.3.7) we can choose any family φ_t of immersions with $X = \frac{\partial}{\partial t}\varphi_t\,\big|_{t=0}$.

Given a vector field X along M as a submanifold of $\mathbb{R}^{n+1}$ which is tangent, there exists a tangent vector field Y on M such that $d\varphi_p(Y(p)) = X(p)$ for every $p \in M$.
Then we consider the smooth flow $L(p,t) : M \times (-\varepsilon, \varepsilon) \to M$ generated by Y on M as the solution of the ODE's system

$$\begin{cases} \frac{\partial}{\partial t} L(p,t) \,=\, Y(L(p,t))\,, \\ L(p,0) \,=\, p \end{cases}$$

for every $p \in M$ and $t \in (-\varepsilon, \varepsilon)$, and we define $\varphi_t(p) = \varphi(L(p,t))$.
Clearly $\varphi_0 = \varphi$ and

$$\frac{\partial}{\partial t}\varphi_t(p)\bigg|_{t=0} = d\varphi_{L(p,t)}\left(\frac{\partial}{\partial t}L(p,t)\right)\bigg|_{t=0} = d\varphi_p(Y(p)) = X(p)\,.$$

If now g_t is the metric tensor on M induced by $\mathbb{R}^{n+1}$ via the immersion φ_t, then the Riemannian manifolds (M, g_t) and (M, g) are isometric for every $t \in (-\varepsilon, \varepsilon)$, being $I(\cdot\,,t) = \varphi^{-1} \circ \varphi_t : (M, g_t) \to (M, g)$ an isometry between them. Since the functional $\mathcal{C}_m$ is invariant by isometry, $\mathcal{C}_m(\varphi_t)$ does not depend on t and its derivative is zero, hence, the first variation of $\mathcal{C}_m$ in the tangent vector field X is zero.
By the previous discussion we have then the following proposition.

PROPOSITION 2.3.10. *The first variation of $\mathcal{C}_m$ depends only on the normal component of the field X.*

This means that we can suppose that X is a normal field in studying the first variation of $\mathcal{C}_m$. Hence, we can strengthen the previous computations as follows,

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= a_{ij}(X) \,=\, -2\left\langle X \,\middle|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle \,=\, -2\,\langle \nu \,|\, X\rangle \mathrm{B}_{ij} \\
\frac{\partial}{\partial t} g^{ij} &= -\, g^{is}\frac{\partial}{\partial t} g_{sl} g^{lj} \,=\, 2\,\langle \nu \,|\, X\rangle \mathrm{B}^{ij} \\
\frac{\partial}{\partial t}\nu &= -\,\nabla\,\langle \nu \,|\, X\rangle \\
\frac{\partial}{\partial t}\Gamma^i_{jk} &= -\, g^{il}\,\{\nabla_j(\langle \nu \,|\, X\rangle \mathrm{B}_{kl}) + \nabla_k(\langle \nu \,|\, X\rangle \mathrm{B}_{jl}) - \nabla_l(\langle \nu \,|\, X\rangle \mathrm{B}_{jk})\} \\
&= \nabla \mathrm{B} * \langle \nu \,|\, X\rangle + \mathrm{B} * \nabla\langle \nu \,|\, X\rangle\,.
\end{aligned}$$

Supposing X normal, we have immediately the following modification of Lemma 2.3.7 substituting the tensor $a_{ij}(X)$ with $-2\,\langle \nu \,|\, X\rangle \mathrm{B}_{ij}$.

LEMMA 2.3.11. *For every covariant tensor $T = T_{i_1 \dots i_l}$, we have*

$$\frac{\partial}{\partial t}\nabla^s T = \nabla^s \frac{\partial T}{\partial t} + \mathfrak{p}_s(T, \mathrm{B}, \langle \nu \,|\, X\rangle)$$

where in $\mathfrak{p}_s(T, \mathrm{B}, \langle \nu \,|\, X\rangle)$ the derivative $\nabla^s T$ does not appear. If T is a function $f : M \to \mathbb{R}^k$

$$\frac{\partial}{\partial t}\nabla^s f = \nabla^s \frac{\partial f}{\partial t} + \mathfrak{p}_{s-1}(\nabla f, \mathrm{B}, \langle \nu \,|\, X\rangle)$$

and $\mathfrak{p}_{s-1}(\nabla f, \mathrm{B}, \langle \nu \,|\, X\rangle)$ does not contain $\nabla^s f$.

This lemma and the fact that $\frac{\partial \nu}{\partial t} = -\nabla\langle \nu \,|\, X\rangle$ lead to the following proposition.

PROPOSITION 2.3.12. *Letting $\{e_\alpha\}$ the canonical basis of $\mathbb{R}^{n+1}$ and setting $\nu = \nu^\alpha e_\alpha \in \mathbb{R}^{n+1}$, we have*

$$\frac{\partial}{\partial t}\nabla_{i_1 \dots i_m}\nu^\alpha = -\nabla_{i_1 \dots i_m}\nabla^\alpha\langle \nu \,|\, X\rangle + \mathfrak{p}_{m-1}(\nabla\nu, \mathrm{B}, \langle \nu \,|\, X\rangle)$$

where we denoted with $\nabla^\alpha\langle \nu \,|\, X\rangle$ the α component of the gradient $\nabla\langle \nu \,|\, X\rangle$ in the canonical basis of $\mathbb{R}^{n+1}$. Moreover, the derivative $\nabla^m \nu$ is not present in $\mathfrak{p}_{m-1}(\nabla\nu, \mathrm{B}, \langle \nu \,|\, X\rangle)$.

We are finally ready to compute

$$\begin{aligned}
\frac{d}{dt}\int_M |\nabla^m\nu|^2\,d\mu_t\Big|_{t=0} &= -\int_M |\nabla^m\nu|^2 H\langle\nu\,|\,X\rangle\,d\mu\\
&\quad+\int_M g^{i_1j_1}\dots\frac{\partial}{\partial t}g^{i_kj_k}\dots g^{i_mj_m}\nabla_{i_1\dots i_m}\nu\nabla_{j_1\dots j_m}\nu\,d\mu\\
&\quad-2\int_M g^{i_1j_1}\dots g^{i_mj_m}\,\nabla_{i_1\dots i_m}\nabla^\alpha\langle\nu\,|\,X\rangle\,\nabla_{j_1\dots j_m}\nu^\alpha\,d\mu\\
&\quad+2\int_M \nabla^m\nu*\mathfrak{p}_{m-1}(\nabla\nu,\mathrm{B},\langle\nu\,|\,X\rangle)\,d\mu\\
&= -\int_M |\nabla^m\nu|^2 H\langle\nu\,|\,X\rangle\,d\mu\\
&\quad+2m\int_M \nabla^m\nu*\nabla^m\nu*\mathrm{B}\langle\nu\,|\,X\rangle\,d\mu\\
&\quad-2\int_M g^{i_1j_1}\dots g^{i_mj_m}\,\nabla_{i_1\dots i_m}\nabla^\alpha\langle\nu\,|\,X\rangle\,\nabla_{j_1\dots j_m}\nu^\alpha\,d\mu\\
&\quad+\int_M \mathfrak{p}_{m-1}(\nabla^m\nu,\nabla\nu,\mathrm{B},\langle\nu\,|\,X\rangle)\,d\mu\,.
\end{aligned}$$

Now, in order to "carry away" derivatives from $\langle\nu\,|\,X\rangle$ in the last integral, we integrate by parts with the divergence theorem, "moving" all the derivatives on the other terms of the products. Hence, we can rewrite it as

$$\int_M \mathfrak{p}_{2m-2}(\nabla\nu,\nabla\nu,\mathrm{B})\langle\nu\,|\,X\rangle\,d\mu\,,$$

which is equal to

$$\int_M \mathfrak{q}^{2m+1}(\nabla\nu,\mathrm{B})\langle\nu\,|\,X\rangle\,d\mu$$

with the conventions of Section 1.

Since also the second integral has this form, collecting them together, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_M |\nabla^m\nu|^2\,d\mu_t\Big|_{t=0} &= \int_M \mathfrak{q}^{2m+1}(\nabla\nu,\mathrm{B})\langle\nu\,|\,X\rangle\,d\mu\\
&\quad-2\int_M g^{i_1j_1}\dots g^{i_mj_m}\,\nabla_{i_1\dots i_m}\nabla^\alpha\langle\nu\,|\,X\rangle\,\nabla_{j_1\dots j_m}\nu^\alpha\,d\mu\,.
\end{aligned}$$

Finally, we deal with this last term. First, by the divergence theorem it can be transformed in

$$-2(-1)^m\int_M \nabla^\alpha\langle\nu\,|\,X\rangle\,\nabla^{j_m\dots j_1}\nabla_{j_1\dots j_m}\nu^\alpha\,d\mu\,,$$

second, using the tangential divergence formula (1.3.3), it is equal to

$$2(-1)^m\int_M \langle\nu\,|\,X\rangle\,\nabla^\alpha\nabla^{j_m\dots j_1}\nabla_{j_1\dots j_m}\nu^\alpha\,d\mu+\int_M \mathfrak{q}^{2m+1}(\nabla\nu,\mathrm{B})\langle\nu\,|\,X\rangle\,d\mu\,,$$

where the extra term $\mathfrak{q}^{2m+1}(\nabla\nu,\mathrm{B})\langle\nu\,|\,X\rangle$, which has a differentiation order lower than the first term, comes from the product with the mean curvature in the tangential divergence formula.

Notice now that the permutation of derivatives introduces additional lower order terms of the form

$$\int_M \mathfrak{q}^{2m+1}(\nabla\nu,\mathrm{B})\langle\nu\,|\,X\rangle\,d\mu$$

by formulas (1.1.4), hence we get

$$2(-1)^m\int_M \langle\nu\,|\,X\rangle\,\nabla^{j_1}\nabla_{j_1}\dots\nabla^{j_m}\nabla_{j_m}\nabla^\alpha\nu^\alpha\,d\mu+\int_M \mathfrak{q}^{2m+1}(\nabla\nu,\mathrm{B})\langle\nu\,|\,X\rangle\,d\mu$$

that is,

$$2(-1)^m \int_M \langle \nu \,|\, X\rangle \overbrace{\Delta\Delta\ldots\Delta}^{m\text{–times}} \nabla^\alpha \nu^\alpha \, d\mu + \int_M \mathfrak{q}^{2m+1}(\nabla\nu, \mathrm{B})\langle \nu \,|\, X\rangle \, d\mu \,.$$

By Gauss–Weingarten relations (1.1.7), we have

$$\nabla^\alpha \nu^\alpha = -\frac{\partial \varphi^\alpha}{\partial x_i} g^{ij} \mathrm{B}_{jl} g^{ls} \frac{\partial \varphi^\alpha}{\partial x_s} = -^{ij}\mathrm{B}_{jl} g^{ls} g_{si} = -g^{ij}\mathrm{B}_{ji} = -H \,,$$

so we conclude

$$\frac{d}{dt} \int_M |\nabla^m \nu|^2 \, d\mu_t \Big|_{t=0} = \int_M \mathfrak{q}^{2m+1}(\nabla\nu, \mathrm{B})\langle \nu \,|\, X\rangle \, d\mu - 2(-1)^m \int_M \overbrace{\Delta\Delta\ldots\Delta}^{m\text{–times}} H \langle \nu \,|\, X\rangle \, d\mu \,.$$

By the previous discussion this formula holds in general for every vector field X along M. We summarize all these facts in the following proposition.

PROPOSITION 2.3.13. *For any $m \geq 1$ the Euler equation of the functional $\mathcal{C}_m$ is given by*

$$\mathrm{E}_{\mathcal{C}_m} = \Big(2(-1)^{m+1} \overbrace{\Delta\Delta\ldots\Delta}^{m\text{–times}} H + \mathfrak{q}^{2m+1}(\nabla\nu, \mathrm{B}) \Big)\, \nu \,.$$

CHAPTER 3

The Evolution Problems and Short Time Existence of the Flows

In this chapter and in the next ones we will study the evolution of hypersurfaces by (minus) the gradient flow associated to the functionals

$$\mathcal{DG}_\gamma = \int_M 1 + |A^\gamma|^2 \, d\mathcal{H}^n \,, \qquad \text{and} \qquad \mathcal{F}_m = \int_M 1 + |\nabla^m \nu|^2 \, d\mu \,,$$

which are simply, the functionals $\mathcal{G}_\gamma$ and $\mathcal{C}_m$ with the addition of an Area term.

We have seen in the previous chapter in Corollary 2.3.5 and Proposition 2.3.13 that in codimension one the Euler equations of the two functionals $\mathcal{G}_\gamma$ and $\mathcal{C}_m$, when $\gamma \geq 3$ and $m \geq 1$ respectively, have analogous leading terms

$$2\gamma(-1)^{\gamma-1}\Big(\overbrace{\Delta\Delta\ldots\Delta}^{(\gamma-2)\text{–times}} H\Big)\,\nu \qquad \text{and} \qquad 2(-1)^{m+1}\Big(\overbrace{\Delta\Delta\ldots\Delta}^{m\text{–times}} H\Big)\,\nu$$

which actually coincide, up to the constant $m+2$, when $\gamma = m+2$.
Notice moreover, that in this case also the lower order terms have the same form, as $\mathfrak{q}^{2m+1}(\nabla\nu, \mathrm{B})\nu$ can be expressed as a polynomial in the function A^M and its derivatives up to the order $2m+2$ (by the very definition of the polynomials $\mathfrak{q}^{2m+1}(\nabla\nu, \mathrm{B})$ in Section 1.2), which is the maximal order of derivatives of A^M that the lower order term in the Euler equation of the functional $\mathcal{G}_{m+2}$ can contain.
Conversely, the term $h(A^M)$ which appears in the expression of $\mathrm{E}_{\mathcal{G}_{m+2}}$ given by Corollary 2.3.5, by means of Corollary 1.5.3, can be written as $\mathfrak{q}^{2m+1}(\nabla\nu, \mathrm{B})\nu$.

Hence, as that Euler equation of the Area functional is simply minus the mean curvature H, we have that

$$\mathrm{E}_{\mathcal{DG}_{m+2}} = \Big(-H + 2(m+2)(-1)^{m+1}\, \overbrace{\Delta\Delta\ldots\Delta}^{m\text{–times}} H + \mathfrak{q}^{2m+1}(\nabla\nu, \mathrm{B}) \Big)\,\nu$$

and

$$\mathrm{E}_{\mathcal{F}_m} = \Big(-H + 2(-1)^{m+1}\, \overbrace{\Delta\Delta\ldots\Delta}^{m\text{–times}} H + \mathfrak{q}^{2m+1}(\nabla\nu, \mathrm{B}) \Big)\,\nu \,,$$

for $m \geq 1$.

All this should not come as a surprise, since the two functionals are strictly related, indeed, roughly speaking, the derivative of the normal is "more or less" the curvature of M.

3.1. The Evolution Problems

DEFINITION 3.1.1. The gradient flows of an initial, smooth, compact, n–dimensional, immersed hypersurface $\varphi_0 : M \to \mathbb{R}^{n+1}$, associated to the two functionals $\mathcal{DG}_{m+2}$ and $\mathcal{F}_m$ are given by some smooth functions $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ such that

(1) the map $\varphi_t = \varphi(\cdot, t) : M \to \mathbb{R}^{n+1}$ is an immersion for every $t \in [0,T)$;
(2) $\varphi(p, 0) = \varphi_0(p)$ for every $p \in M$;

(3) the following PDE's systems are satisfied, respectively,

$$\frac{\partial\varphi}{\partial t} = -\,\mathrm{E}_{\mathcal{DG}_{m+2}} = \mathrm{H} + \Big(2(m+2)(-1)^m\,\overbrace{\Delta^{M_t}\Delta^{M_t}\dots\Delta^{M_t}}^{m\text{–times}}H + \mathfrak{q}^{2m+1}(\nabla\nu,\mathrm{B})\Big)\,\nu\,,$$

$$\frac{\partial\varphi}{\partial t} = -\,\mathrm{E}_{\mathcal{F}_m} = \mathrm{H} + \Big(2(-1)^m\,\overbrace{\Delta^{M_t}\Delta^{M_t}\dots\Delta^{M_t}}^{m\text{–times}}H + \mathfrak{q}^{2m+1}(\nabla\nu,\mathrm{B})\Big)\,\nu\,.$$

We denoted with Δ^{M_t} the Laplacian of the Riemannian manifolds $M_t = (M, g_t)$, where g_t is the metric induced on M by the evolving immersion φ_t.

Notice that, despite the appealing form of the leading terms, the PDE's systems are *quasilinear parabolic systems* of partial differential equations or order $2m+2$ on the manifold M. Indeed, once expressing everything in local coordinates, the coefficients of any Laplacian operator Δ^{M_t} depend on the derivatives of the immersion φ_t up to the order $2m+1$.

In the study of the *mean curvature flow* of a hypersurface $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$,

$$\frac{\partial\varphi}{\partial t} = H\nu = \Delta^{M_t}\varphi$$

(which is a second order parabolic flow), via techniques such as varifolds, level sets, viscosity solutions (see [**4, 7, 18, 40, 62**]), the maximum principle is the key tool to get comparison results and estimates on solutions (the quasilinear parabolic system is of second order).
In our case, even if $m = 1$, the first variations and hence the corresponding quasilinear parabolic problems turn out to be of order higher than two, precisely of order $2m+2$, so we have to deal with equations of fourth order at least. This fact has the relevant consequence that we cannot employ the maximum principle to get pointwise estimates and to compare two solutions, thus losing a whole bunch of geometric results holding for the mean curvature flow. In particular, we cannot expect that an initially embedded hypersurface remains embedded during the flow, since self–intersections can develop (an example is given by Giga and Ito in [**50**]). By these reasons, techniques based on the description of the hypersurfaces as level sets of functions seems of difficult application in this case and therefore we adopt a parametric approach as in the work of Huisken [**54**].

In the simplest one–dimensional case, with $m = 1$, the two functionals (not only their Euler equations) coincide, up to a constant in front of κ^2 and we are concerned with curves in the plane evolving by the gradient flow of the functional

$$\int_{\mathbb{S}^1} 1 + \kappa^2\,ds\,, \tag{3.1.1}$$

since the curvature κ of a curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ satisfies $\kappa^2 = |\nabla\nu|^2$ and we have seen in Corollary 1.4.12 that $|\nabla^3 A^M|^2 = 3|\mathrm{B}|^2 = 3\kappa^2$.
In this special case, the flow was shown to exist globally smooth for every positive time by Polden in the papers [**79, 80**], which have been a starting point for our work. We also mention that Wen in [**92**] found results similar to Polden's ones, in considering the flow for $\int_{\mathbb{S}^1} \kappa^2\,ds$ of curves with a fixed length.

In the paper [**31**, Sect. 5] (see [**32**, Sect. 5] for an English translation) De Giorgi stated the following conjecture (Conj. 2, Pag. 267).

CONJECTURE 3.1.2 (De Giorgi). Any compact, n–dimensional, smooth submanifold M of $\mathbb{R}^{n+m}$ without boundary, moving by the gradient of the functional

$$\mathcal{DG}_k(M) = \int_M 1 + |\nabla^k\eta^M|^2\,d\mathcal{H}^n\,,$$

where η^M is the square of the distance function from M and $\mathcal{H}^n$ is the n–dimensional Hausdorff measure in $\mathbb{R}^{n+m}$, does not develop singularities, if $k > n+1$.

An analogous conjecture (only in codimension one) can be stated for the flow by the gradient of the functional $\mathcal{F}_m$ and, rereading the conjecture of De Giorgi with our notation in codimension one, they actually state that the flows of any initial, smooth, compact, immersed hypersurface associated to the two functionals $\mathcal{DG}_{m+2}$ and $\mathcal{F}_m$, have global existence and regularity, for a suitably large $m \geq 1$ (we will be more precise later on).

In the next section we will see that for both flows there exists for some positive time a unique smooth evolution φ_t of any initial, smooth, compact, immersed submanifold. Then, to analyze the long time existence and regularity of these flows we will need suitable a priori estimates, that will be the goal of the next chapters.

In order to show regularity, a good substitute of the pointwise estimates coming from the maximum principle, are suitable estimates on the second fundamental form in Sobolev spaces, using Gagliardo–Nirenberg interpolation type inequalities for tensors. Since the constants involved in these inequalities depends on the Sobolev constants and these latter on the geometry of the hypersurface where the tensors are defined, before doing estimates we absolutely need some uniform control independent of time on these constants. In **[79]** these controls are obvious as the constants depend only on the length, on the contrary, much more work is needed here because of the richer geometry of higher dimensional hypersurfaces.

The final result of our analysis will be the following theorem, which in particular, answers affirmatively De Giorgi's Conjecture 3.1.2 above.

THEOREM 3.1.3. *If the differentiation order m is strictly larger than $[n/2]$, then the flows by the gradient of $\mathcal{DG}_{m+2}$ and $\mathcal{F}_m$ of any initial, smooth, compact, n–dimensional, immersed hypersurface of $\mathbb{R}^{n+1}$ exist, are unique and smooth for every positive time ($[n/2]$ means the integer part of $n/2$).*
Moreover, as $t \to +\infty$, the evolving hypersurface φ_t sub–converges (up to reparametrization and translation) to a smooth critical point of the respective functional.

REMARK 3.1.4. Notice that the hypothesis $m > [n/2]$ in general is weaker than the original one in De Giorgi's conjecture, the two conditions actually coincide in dimension one and two.

3.2. Short Time Existence

By means of a slight extension (see **[38]** for details) of the following theorem of Polden in **[57]** (and [**80**, Sect. 2, Thm. 2.5.2]), there exists for some positive time a unique smooth evolution φ_t of any initial, compact, smooth submanifold M for any of the two flows above.

THEOREM 3.2.1. *If $m \geq 1$, for any smooth immersion $\varphi_0 : M \to N$ of an n–dimensional, compact, hypersurface M in a smooth $(n+1)$–dimensional Riemannian manifold (N, h), there exists a unique smooth solution to the flow problem*

$$\frac{\partial \varphi}{\partial t} = \Big((-1)^m \overbrace{\Delta^{M_t}\Delta^{M_t}\dots\Delta^{M_t}}^{m\text{–times}} H + \Phi(\varphi, \nu, \mathrm{B}, \nabla\mathrm{B}, \dots, \nabla^{2m-1}\mathrm{B})\Big)\nu\,,$$

defined on some positive time interval $0 \leq t < T$ and taking φ_0 as its initial value.

Looking at Polden's proof, it is possible to allow the function Φ to depend also on the metric g_t induced by the immersion, moreover the higher covariant derivatives of the normal ν, by and induction argument using the Gauss–Weingarten relations (1.1.7), can be expressed in terms of the covariant derivatives of the second fundamental form B, hence both our evolution problems fit into the hypotheses of Polden's theorem, as the constants multiplying the leading terms can be eliminated by a time–only rescaling.

REMARK 3.2.2.

- Notice that if we find a smooth solution of the evolution problem for some interval of time, by the fact that the initial submanifold is immersed, the solution is still an immersion (we assumed that M is compact) at least for short time, so such condition is automatically satisfied.
- Once choosing a good parametrization for the evolving hypersurfaces, Polden is able to reduce the evolution problem to solving a quasilinear parabolic *equation* on the compact manifold M. He develops an existence/uniqueness/regularity theory for linear equations in parabolic Sobolev spaces and pass from the linear case to the quasilinear one by

means of the inverse function theorem. Unfortunately, as pointed out by Sharples **[85]**, such procedure has a gap in the convergence of the solutions of the "frozen" linear problems to a solution of the quasilinear one.
In Appendix A, we attached the paper **[74]** with Luca Martinazzi where we filled the gap in Polden's proof. We assume his linear result and we show that his linearization procedure actually works if one linearizes at a suitably chosen function and discusses in details the above mentioned convergence.

- We have seen that from the point of view of short time existence the two functionals $\mathcal{DG}_{m+2}$ and $\mathcal{F}_m$ behave the same and no restrictions on $m \geq 1$ are needed. When instead in the next chapters we will study the global existence and smoothness of the two flows, we will need to put some hypotheses on $m \geq 1$ (depending on the dimension) and we will analyze more in detail their properties in order to get a priori estimates on the geometric quantities leading to the regularity of the flows.

CHAPTER 4

A Priori Estimates

To prove long time existence we need a priori estimates on the second fundamental form and its derivatives which will be obtained via Sobolev and Gagliardo–Nirenberg interpolation inequalities for functions defined on M_t.
Since the hypersurfaces are moving, also the constants appearing in such inequalities change during the flow, hence, before proceeding with the estimates, we need some uniform control on them.

In this chapter we will see that if the integer m is larger than $[n/2]$ then we have a uniform control, independent of time, on the L^{n+1} norm of the second fundamental form during the flows by the gradient of the functionals $\mathcal{DG}_{m+2}$ and $\mathcal{F}_m$. This is the crucial point where such hypothesis on m is necessary.

This fact will allow us to show in the next section that also the above constants are uniformly bounded during the flow.
Moreover, using an inequality of Michael and Simon, we will also prove also an a priori lower bound on the volume of the evolving hypersurfaces.

4.1. A Priori Estimates on the Sobolev Constants and on the Volume of the Evolving Hypersurfaces

We start dealing with the gradient flow associated to the functional $\mathcal{F}_m$.
By the very definition of the flow, the value of the functional $\mathcal{F}_m$ decreases in time, as in general if E is the Euler equation of any functional $\mathcal{F}$, we have

$$\frac{d}{dt}\mathcal{F}(\varphi_t) = -\int_M [\mathrm{E}(\varphi_t)]^2\, d\mu_t \leq 0\,,$$

hence, as long as the flow associated to $\mathcal{F}_m$ remains smooth, we have the uniform estimate

$$\int_M 1 + |\nabla^m \nu|^2\, d\mu_t = \mathcal{F}_m(\varphi_t) \leq \mathcal{F}_m(\varphi_0) \tag{4.1.1}$$

for every $t \geq 0$.

Now we want to prove that if $m > [n/2]$, this estimate implies that the $L^{n+1}(\mu_t)$ norms of the second fundamental form B of M_t are uniformly bounded independently of time.

Our starting point is the following universal interpolation type inequalities for tensors.

PROPOSITION 4.1.1. *Suppose that (M, g) is a smooth and compact n–dimensional Riemannian manifold without boundary and μ the measure associated to g.*
Then for every covariant tensor T and exponents $q \in [1, +\infty)$ and $r \in [1, +\infty]$, we have

$$\|\nabla^j T\|_{L^p(\mu)} \leq C\|\nabla^s T\|^{\frac{j}{s}}_{L^q(\mu)} \|T\|^{\frac{s-j}{s}}_{L^r(\mu)} \qquad \forall j \in [0, s]\,, \tag{4.1.2}$$

with

$$\frac{1}{p} = \frac{j}{sq} + \frac{s-j}{sr}\,,$$

where the constant C depends only on n, s, j, p, q, r and not on the metric or the geometry of M.

The proof of the case $r = +\infty$ can be found in [**53**, Sect. 12], along the same lines also the case $r < +\infty$ follows (see also [**9**, Chap. 3, Sect. 7.6]).

Suppose that M is orientable and that g is the metric induced by the immersion $\varphi : M \to \mathbb{R}^{n+1}$, let ν be a global unit normal vector field on M.

If in (4.1.2) we consider $T = \nu$, $s = m$, $j = 1$, $q = 2$ and $r = +\infty$, then we have $|T| = 1$ and $p = 2m$, hence

$$\|\nabla\nu\|_{L^{2m}(\mu)} \leq C\|\nabla^m\nu\|^{\frac{1}{m}}_{L^2(\mu)}\,,$$

for a constant $C = C(n, m)$.
Since by (1.1.7) $|\nabla\nu| = |\mathrm{B}|$, we conclude

$$\int_M |\mathrm{B}|^{2m}\,d\mu \leq C\int_M |\nabla^m\nu|^2\,d\mu \leq C\mathcal{F}_m(\varphi)\,.$$

If M is not orientable, then there exists a two–fold Riemannian covering $\widetilde{M}$ of M, with a locally isometric projection map $\pi : \widetilde{M} \to M$ which is orientable and immersed in $\mathbb{R}^{n+1}$ via the map $\varphi\circ\pi$. Repeating the previous argument for $\widetilde{M}$ we get

$$\int_{\widetilde{M}} |\mathrm{B}|^{2m}\,d\widetilde{\mu} \leq C\int_{\widetilde{M}} |\nabla^m\nu|^2\,d\widetilde{\mu}\,.$$

Since π is a local isometry and noticing that the global unit normal field on $\widetilde{M}$ gives locally a unit normal field on M, all the quantities which appear inside the integrals above do not change passing from $\widetilde{M}$ to M, only when we integrate we need to take into account the two–fold structure of the covering. This means that for every smooth function $u : M \to \mathbb{R}$ we have

$$\int_{\widetilde{M}} u\circ\pi\,d\widetilde{\mu} = 2\int_M u\,d\mu\,.$$

Hence, we deduce

$$2\int_M |\mathrm{B}|^{2m}\,d\mu \leq 2\,C\int_M |\nabla^m\nu|^2\,d\mu \leq 2\,C\mathcal{F}_m(\varphi)$$

which clearly gives the same estimate as in the orientable case.

As $2m > 2[n/2]$, then $2m \geq n+1$, we have

$$\int_M |\mathrm{B}|^{n+1}\,d\mu \leq \left(\int_M |\mathrm{B}|^{2m}\,d\mu\right)^{\frac{n+1}{2m}} (\mathrm{Vol}\ M)^{\frac{2m-n-1}{2m}} \leq C\mathcal{F}_m(\varphi) \tag{4.1.3}$$

with a constant $C = C(n, m)$.

Now we show that also the volume of (M, g) is well controlled by the value of $\mathcal{F}_m(\varphi)$ under the hypothesis $m > [n/2]$.
The bound from above is obvious, the bound from below in dimension $n > 1$ can be obtained via the following universal Sobolev inequality due to Michael and Simon (see **[76, 86]**).

PROPOSITION 4.1.2. *Let $\varphi : M \to \mathbb{R}^{n+1}$ be an immersion of an n–dimensional, compact hypersurface without boundary. On M we consider the Riemannian metric induced by $\mathbb{R}^{n+1}$ and the corresponding measure μ.*
Then, there exists a constant $C = C(n, p)$ depending only on the dimension n and the exponent p such that, for every smooth function $u : M \to \mathbb{R}$

$$\left(\int_M |u|^{p^*}\,d\mu\right)^{1/p^*} \leq C(n,p)\left(\int_M |\nabla u|^p\,d\mu + \int_M |\mathrm{H}u|^p\,d\mu\right)^{1/p}\,, \tag{4.1.4}$$

where $p \in [1, n)$, $n > 1$ and $p^* = \frac{np}{n-p}$.

Considering the function $u : M \to \mathbb{R}$ constantly equal to 1 in the inequality for $p = 1$, and taking into account inequality (4.1.3), we get

$$\begin{aligned}(\mathrm{Vol}\ M)^{\frac{n-1}{n}} \leq& C\int_M |\mathrm{H}|\,d\mu\\ \leq& C\|\mathrm{B}\|_{L^{n+1}(\mu)}\,(\mathrm{Vol}\ M)^{\frac{n}{n+1}}\\ \leq& C\mathcal{F}_m(\varphi)^{\frac{1}{n+1}}\,(\mathrm{Vol}\ M)^{\frac{n}{n+1}}\,.\end{aligned}$$

Dividing both members by $(\mathrm{Vol}\, M)^{\frac{n-1}{n}}$, as $\frac{n}{n+1} > \frac{n-1}{n}$ we conclude

$$1 \leq C\mathcal{F}_m(\varphi)^{\frac{1}{n+1}}\,(\mathrm{Vol}\, M)^{\frac{1}{n(n+1)}}$$

that is,

$$\frac{C}{\mathcal{F}_m(\varphi)^n} \leq \mathrm{Vol}\, M \leq \mathcal{F}_m(\varphi)$$

for a constant $C = C(n, m)$.

REMARK 4.1.3. With the same argument, it follows that also $\|\mathrm{B}\|_{L^{n+1}(\mu)}$ can be controlled above and below with $\mathcal{F}_m(\varphi)$ and that the functional $\mathcal{F}_m$ is uniformly bounded from below by a constant greater than zero.

In the special case $n = 1$, we recall that for every closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ in the plane the integral of the modulus of its curvature κ is at least 2π, then

$$2\pi \leq \int_{\mathbb{S}^1} |\kappa|\, ds \leq \left(\int_{\mathbb{S}^1} \kappa^2\, ds\right)^{1/2} \sqrt{\mathrm{Length}\,\gamma} \leq C\sqrt{\mathcal{F}_m(\gamma)}\sqrt{\mathrm{Length}\,\gamma}\,.$$

Hence,

$$\frac{C}{\mathcal{F}_m(\gamma)} \leq \mathrm{Length}\,\gamma \leq \mathcal{F}_m(\gamma)$$

with $C = C(m)$.

Putting together all these inequalities and the uniform estimate (4.1.1) we obtain the following result.

PROPOSITION 4.1.4. *As long as the flow by the gradient of $\mathcal{F}_m$, with $m > [n/2]$, of a hypersurface in $\mathbb{R}^{n+1}$ exists, we have the estimates*

$$\|\mathrm{B}\|_{L^{n+1}(\mu_t)} \leq C_1 < +\infty$$

$$0 < C_2 \leq \mathrm{Vol}\, M_t \leq C_3 < +\infty$$

where the three constants C_1, C_2 and C_3 are independent of time.
They depend only on n, m and the value of $\mathcal{F}_m$ for the initial hypersurface.

Now we turn our attention to the functional $\mathcal{DG}_{m+2}$. Again, as the flow φ_t is variational, the value of $\mathcal{DG}_{m+2}(M_t)$ is monotone non increasing in time, hence it is bounded by its value on the initial submanifold. This implies that, for all the evolving submanifolds,

$$\mathrm{Vol}(M_t) + \int_M |A^{m+2}|^2\, d\mu_t \leq C\,.$$

Hence, by means of Proposition 1.5.6 we get

$$\mathrm{Vol}(M_t) + \int_M |\mathrm{B}|^{2m}\, d\mu_t \leq C$$

for a constant C independent of time.
Since when $m > [n/2]$ we have $2m \geq n+1$ we conclude, by the same argument used for the functional $\mathcal{F}_m$, that

$$\mathrm{Vol}(M_t) + \|\mathrm{H}\|_{L^{n+1}(\mu_t)} \leq C_1 \tag{4.1.5}$$

uniformly in time, for a constant C_1 depending only on the initial submanifold. Analogously, we also get the following uniform lower bound on the *Volume* of M_t,

$$0 < C_2 \leq \mathrm{Vol}(M_t) \leq C_3 < +\infty$$

with a couple of constants C_2 and C_3 independent of time (moreover, notice that Remark 4.1.3 applies too).

REMARK 4.1.5. We underline the two key points where the properties of the distance function play a role here. First, when the order m is larger than $[n/2]$, the estimate $|A^{m+2}|^2 \geq C_m|\mathrm{B}|^{2m}$ implies the a priori estimate (4.1.5) leading to the geometry–independent interpolation inequalities that we will see in Proposition 4.2.7. Second, the "nice" structure of the leading term of the Euler equation of the functional $\mathcal{DG}_{m+2}$.

4.2. Interpolation Inequalities for Tensors

We show now that the uniform bound on the L^{n+1} norm of the second fundamental form during the evolution, that we got in the previous section, implies that the constants involved in some Sobolev and Gagliardo–Nirenberg interpolation type inequalities are also uniformly bounded in time.

Recalling inequality (4.1.4), we have

$$\|u\|_{L^{p^*}(\mu)} \leq C(n,p)\left(\|\nabla u\|_{L^p(\mu)} + \|\mathrm{H}u\|_{L^p(\mu)}\right) \tag{4.2.1}$$

for every $u \in C^1(M)$, where $p^* = \frac{np}{n-p}$ and $p \in [1,n)$.

PROPOSITION 4.2.1. *If the manifold (M,g) satisfies* Vol $M + \|\mathrm{H}\|_{L^{n+\delta}(\mu)} \leq D$ *for some $\delta > 0$ then for every $p \in [1,n)$,*

$$\|u\|_{L^{p^*}(\mu)} \leq C\left(\|\nabla u\|_{L^p(\mu)} + \|u\|_{L^p(\mu)}\right) \qquad \forall u \in C^1(M)\,,$$

where the constant C depends only on n, p, δ and D.

PROOF. Applying Hölder inequality to the last term of inequality (4.2.1), we get

$$\|u\|_{L^{p^*}(\mu)} \leq C(n,p)\|\nabla u\|_{L^p(\mu)} + C(n,p,\delta,D)\|u\|_{L^{\widetilde{p}}(\mu)}$$

where $\widetilde{p}$ is given by

$$\widetilde{p} = \frac{p(n+\delta)}{n+\delta-p} = p^*\frac{n(n+\delta)}{n(n+\delta)+p^*\delta}\,,$$

then $p < \widetilde{p} < p^*$.
Hence, we can interpolate $\|u\|_{L^{\widetilde{p}}(\mu)}$ between a small fraction of $\|u\|_{L^{p^*}(\mu)}$ and a possibly large multiple of $\|u\|_{L^p(\mu)}$,

$$\|u\|_{L^{p^*}(\mu)} \leq C(n,p)\|\nabla u\|_{L^p(\mu)} + C(n,p,\delta,D)\left(\varepsilon\|u\|_{L^{p^*}(\mu)} + C(\varepsilon,p)\|u\|_{L^p(\mu)}\right)\,.$$

Choosing $\varepsilon > 0$ such that $\varepsilon C(n,p,\delta,D) \leq 1/2$ and collecting terms we obtain

$$\|u\|_{L^{p^*}(\mu)} \leq C(n,p,\delta,D)\left(\|\nabla u\|_{L^p(\mu)} + \|u\|_{L^p(\mu)}\right)\,.$$

□

When $p > n$ we prove the following L^∞ result (see also [**67**, Thm. 5.6]).

PROPOSITION 4.2.2. *If the manifold (M,g) satisfies* Vol $M + \|\mathrm{H}\|_{L^{n+\delta}(\mu)} \leq D$ *for some $\delta > 0$ then for every $p > n$, we have*

$$\max_M |u| \leq C\left(\|\nabla u\|_{L^p(\mu)} + \|u\|_{L^p(\mu)}\right) \qquad \forall u \in C^1(M)\,,$$

where the constant C depends only on n, p, δ and D.

PROOF. Suppose first that M is embedded and $n+\delta \geq p > n$, clearly $\|\mathrm{H}\|_{L^p(\mu)}$ is bounded by a value depending on the constant D.
We consider M as a subset of $\mathbb{R}^{n+1}$ via the embedding φ and μ as a measure on $\mathbb{R}^{n+1}$ which is supported on M. Then the following result holds ([**86**, Thm. 17.7]), let $B_\rho(x)$ be the ball of radius ρ centered at x in $\mathbb{R}^{n+1}$, for every $0 < \sigma < \rho < +\infty$ we have

$$\left(\frac{\mu(B_\sigma(x))}{\sigma^n}\right)^{1/p} \leq \left(\frac{\mu(B_\rho(x))}{\rho^n}\right)^{1/p} + C(n,p,\delta,D)\left(\rho^{1-n/p} - \sigma^{1-n/p}\right)\,.$$

Hence,

$$\left(\frac{\mu(B_\sigma(x))}{\sigma^n}\right)^{1/p} \leq \frac{C_1}{\rho^{n/p}} + C_2\rho^{1-n/p}\,,$$

and choosing $\rho = 1$, for every $0 < \sigma < 1$ we get the inequality

$$\mu(B_\sigma(x)) \leq C(n,p,\delta,D)\sigma^n \,.$$

Then we need the following formula which is proved in [**86**, Sect. 18], as a consequence of the tangential divergence formula (1.3.3).
For every $0 < \sigma < \rho < +\infty$ we have

$$\frac{\int_{B_\sigma(x)} u\,d\mu}{\sigma^n} \leq \frac{\int_{B_\rho(x)} u\,d\mu}{\rho^n} + \int_\sigma^\rho \tau^{-n-1} \int_{B_\tau(\varepsilon)} r(|\nabla u| + |u\mathrm{H}|)\,d\mu(y)\,d\tau$$

where $r = |x-y|$ and u is any smooth non negative function.

Noticing that $r \leq \tau$ and using Hölder inequality we estimate

$$\begin{aligned}
\frac{\int_{B_\sigma(x)} u\,d\mu}{\sigma^n} &\leq \frac{\int_{B_\rho(x)} u\,d\mu}{\rho^n} + \left(\int_M |\nabla u|^p + |u\mathrm{H}|^p\,d\mu\right)^{1/p} \int_\sigma^\rho \tau^{-n}\mu(B_\tau(x))^{1-1/p}\,d\tau \\
&\leq \int_{B_1(x)} u\,d\mu + C\left(\|\nabla u\|_{L^p(\mu)} + \|u\mathrm{H}\|_{L^p(\mu)}\right) \int_\sigma^1 \tau^{-n}\tau^{n-n/p}\,d\tau
\end{aligned}$$

where in the last passage we set $\rho = 1$ used the previous estimate on $\mu(B_\tau(x))$. The function $\tau^{-n/p}$ is integrable since $p > n$ and we get

$$\frac{\int_{B_\sigma(x)} u\,d\mu}{\sigma^n} \leq \int_{B_1(x)} u\,d\mu + C\left(\|\nabla u\|_{L^p(\mu)} + \|u\mathrm{H}\|_{L^p(\mu)}\right) \frac{1-\sigma^{1-n/p}}{1-n/p}\,,$$

now sending σ to zero, on the left side we obtain the value of $u(x)$ times ω_n which is the volume of the unit ball of $\mathbb{R}^n$, hence

$$\begin{aligned}
\omega_n u(x) &\leq \int_{B_1(x)} u\,d\mu + C\left(\|\nabla u\|_{L^p(\mu)} + \|u\mathrm{H}\|_{L^p(\mu)}\right) \\
&\leq C(n,p,\delta,D)\left(\|u\|_{L^1(\mu)} + \|\nabla u\|_{L^p(\mu)} + \|u\mathrm{H}\|_{L^p(\mu)}\right)\,.
\end{aligned}$$

For a general u we apply this inequality to the function u^2, thus

$$\begin{aligned}
u^2(x) &\leq C\left(\int_M |u|^2\,d\mu + \left(\int_M |u\nabla u|^p\,d\mu\right)^{1/p} + \left(\int_M |u^2\mathrm{H}|^p\,d\mu\right)^{1/p}\right) \\
&\leq C\max_M |u| \left(\int_M |u|\,d\mu + \left(\int_M |\nabla u|^p\,d\mu\right)^{1/p} + \left(\int_M |u\mathrm{H}|^p\,d\mu\right)^{1/p}\right)\,.
\end{aligned}$$

Since $x \in \mathbb{R}^{n+1}$ was arbitrary we conclude that

$$\max_M |u| \leq C(n,p,\delta,D)\left(\|u\|_{L^1(\mu)} + \|\nabla u\|_{L^p(\mu)} + \|u\mathrm{H}\|_{L^p(\mu)}\right)\,.$$

for a constant C depending on n, p, δ and D.
If M is only immersed, we consider the embeddings of M in $\mathbb{R}^{n+1}\times\mathbb{R}^k$ given by the map $\varphi\times\varepsilon\psi : M \to \mathbb{R}^{n+1}\times\mathbb{R}^k$, where $\psi : M \to \mathbb{R}^k$ is an embedding of M in some Euclidean space. Then, repeating the previous argument (it is possible since the starting inequalities from [**86**] hold for embeddings in any $\mathbb{R}^l$) we will get the same conclusion with a constant C_ε. Finally, as C_ε depends only on Vol M and H, and all the geometric quantities converge uniformly when ε goes to zero, we conclude that the inequality holds also in the immersed case.

Now, given any $p > n$, we choose $\widetilde{p} = \frac{1}{2}\min\{n+p, 2n+\delta\}$, then clearly $n < \widetilde{p} < \min\{p, n+\delta/2\}$. By the inequality above we have

$$\max_M |u| \leq C(n,\widetilde{p},\delta,D)\left(\|u\|_{L^1(\mu)} + \|\nabla u\|_{L^{\widetilde{p}}(\mu)} + \|u\mathrm{H}\|_{L^{\widetilde{p}}(\mu)}\right)\,,$$

then using Hölder inequality and an interpolation argument as in the proof of Proposition 4.2.1 we get

$$\max_M |u| \leq C(n,\widetilde{p},\delta,D)\left(\|u\|_{L^1(\mu)} + \|\nabla u\|_{L^{\widetilde{p}}(\mu)} + \|u\|_{L^p(\mu)}\right)\,.$$

Applying again Hölder inequality, as $\widetilde{p} < p$, we conclude that

$$\max_M |u| \leq C(n, \widetilde{p}, \delta, D) \left(\|\nabla u\|_{L^p(\mu)} + \|u\|_{L^p(\mu)} \right) ,$$

which gives the thesis since $\widetilde{p}$ depends only on n, p and δ. $\square$

We now extend these propositions to tensors (see [**9**, Prop. 2.11] and also [**22, 23**]). Since $|T|$ is not necessarily smooth we apply the previous inequalities first to the smooth functions $\sqrt{|T|^2 + \varepsilon^2}$, converging to $|T|$ when $\varepsilon \to 0$. As

$$\left| \nabla \sqrt{|T|^2 + \varepsilon^2} \right| = \left| \frac{\langle \nabla T, T \rangle}{\sqrt{|T|^2 + \varepsilon^2}} \right| \leq \frac{|T|}{\sqrt{|T|^2 + \varepsilon^2}} |\nabla T| \leq |\nabla T|$$

we get then easily the following result.

PROPOSITION 4.2.3. *If the manifold (M, g) satisfies* $\operatorname{Vol} M + \|\mathrm{H}\|_{L^{n+\delta}(\mu)} \leq D$ *for some $\delta > 0$ then for every covariant tensor $T = T_{i_1 \dots i_l}$ we have,*

$$\|T\|_{L^{p^*}(\mu)} \leq C \left(\|\nabla T\|_{L^p(\mu)} + \|T\|_{L^p(\mu)} \right) \qquad \text{if } 1 \leq p < n, \tag{4.2.2}$$

$$\max_M |T| \leq C \left(\|\nabla T\|_{L^p(\mu)} + \|T\|_{L^p(\mu)} \right) \qquad \text{if } p > n, \tag{4.2.3}$$

where the constants depend only on n, l, p, δ and D.

We define the Sobolev norm of a tensor T on (M, g) as

$$\|T\|_{W^{s,q}(\mu)} = \sum_{i=0}^{s} \|\nabla^i T\|_{L^q(\mu)} .$$

COROLLARY 4.2.4. *In the same hypothesis on (M, g) we have*

$$\|\nabla^j T\|_{L^p(\mu)} \leq C \|T\|_{W^{s,q}(\mu)} \qquad \text{with} \qquad \frac{1}{p} = \frac{1}{q} - \frac{s-j}{n} > 0 , \tag{4.2.4}$$

$$\max_M |\nabla^j T| \leq C \|T\|_{W^{s,q}(\mu)} \qquad \text{when} \qquad \frac{1}{q} - \frac{s-j}{n} < 0 . \tag{4.2.5}$$

The constants depend only on n, l, s, j, p, q, δ and D.

PROOF. By inequality (4.2.2) applied to the tensor $\nabla^j T$ we get

$$\begin{aligned}
\|\nabla^j T\|_{L^p(\mu)} &\leq C \left(\|\nabla^{j+1} T\|_{L^{p_1}(\mu)} + \|\nabla^j T\|_{L^{p_1}(\mu)} \right) \\
&\leq C \left(\|\nabla^{j+2} T\|_{L^{p_2}(\mu)} + 2 \, \|\nabla^{j+1} T\|_{L^{p_2}(\mu)} + \|\nabla^j T\|_{L^{p_2}(\mu)} \right) \\
&\leq \quad \dots \\
&\leq C \left(\|\nabla^s T\|_{L^{p_{s-j}}(\mu)} + \dots + \|\nabla^j T\|_{L^{p_{s-j}}(\mu)} \right) \\
&\leq C \|T\|_{W^{s,p_{s-j}}(\mu)} .
\end{aligned}$$

Since the p_i are related by

$$\frac{1}{p_i} = \frac{1}{p_{i+1}} - \frac{1}{n} ,$$

$p_0 = p$ and $p_{s-j} = q$, we have

$$\frac{1}{p} = \frac{1}{p_{s-j}} - \frac{s-j}{n} = \frac{1}{q} - \frac{s-j}{n} ,$$

and the first part of the corollary is proved.
The second part follows analogously using also inequality (4.2.3). $\square$

Now we put together this result and the universal inequalities

$$\|\nabla^j T\|_{L^p(\mu)} \leq C \, \|T\|_{W^{s,q}(\mu)}^{\frac{j}{s}} \|T\|_{L^r(\mu)}^{\frac{s-j}{s}} , \tag{4.2.6}$$

which are obviously implied by Proposition 4.1.1, to get the following interpolation type inequalities.

PROPOSITION 4.2.5. *In the same hypothesis on (M, g) as before, there exist a constant C depending only on n, l, s, j, p, q, r, δ and D, such that for every covariant tensor $T = T_{i_1 \dots i_l}$, the following inequality hold*

$$\|\nabla^j T\|_{L^p(\mu)} \leq C \, \|T\|^a_{W^{s,q}(\mu)} \|T\|^{1-a}_{L^r(\mu)} \,, \tag{4.2.7}$$

for all $j \in [0, s]$, $p, q, r \in [1, +\infty)$ and $a \in [j/s, 1]$ with the compatibility condition

$$\frac{1}{p} = \frac{j}{n} + a \left(\frac{1}{q} - \frac{s}{n} \right) + \frac{1-a}{r} \,.$$

If such condition gives a negative value for p, the inequality holds for every $p \in [1, +\infty)$ on the left side.

PROOF. The cases $a = j/s$ and $a = 1$ are inequalities (4.2.6) and (4.2.4), respectively, the intermediate cases, when $j/s < a < 1$, are obtained immediately by the log–convexity of $\| \cdot \|_{L^p(\mu)}$ in $1/p$, which is a linear function of a, and the fact that the right side is exponential in a.
If p is negative then $\frac{1}{q} - \frac{s}{n} < 0$ and

$$\frac{1}{q} - \frac{s-j}{n} \leq \frac{j}{n} + a \left(\frac{1}{q} - \frac{s}{n} \right) + \frac{1-a}{r} \,,$$

hence, the L^∞ estimate of inequality (4.2.5) together with (4.2.6) gives the inequality for every $p \in [1, +\infty)$. □

REMARK 4.2.6. By simplicity, we avoided to discuss in all the section the critical cases of the inequalities, for instance $p = n$ in Proposition 4.2.3. Actually, for our purposes, we just need to say that in a critical case we can allow any value of $p \in [1, +\infty)$ in the left side of inequalities like (4.2.7). This can be seen easily, by considering a suitable inequality with a lower integrability exponent on the right side and then applying Hölder inequality.

Putting together the estimates of this section with Proposition 4.1.4 we obtain the following result.

PROPOSITION 4.2.7. *As long as the flow of a hypersurface in $\mathbb{R}^{n+1}$ by the gradient of $\mathcal{F}_m$ or $\mathcal{DG}_{m+2}$ exists, with $m > [n/2]$, for every smooth covariant tensor $T = T_{i_1 \dots i_l}$ the following inequalities hold*

$$\|\nabla^j T\|_{L^p(\mu)} \leq C \, \|T\|^a_{W^{s,q}(\mu)} \|T\|^{1-a}_{L^r(\mu)} \,, \tag{4.2.8}$$

for all $j \in [0, s]$, $p, q, r \in [1, +\infty)$ and $a \in [j/s, 1]$ with the compatibility condition

$$\frac{1}{p} = \frac{j}{n} + a \left(\frac{1}{q} - \frac{s}{n} \right) + \frac{1-a}{r} \,.$$

If such condition gives a negative value for p, the inequality holds for every $p \in [1, +\infty)$ on the left side. The constant C depends only on m, n, l, s, j, p, q, r and the value of the relative functional for the initial hypersurface.

CHAPTER 5

Long Time Existence of the Flow and Convergence

Suppose that at a certain time $T > 0$ the evolving hypersurface by the gradient flow of the functional $\mathcal{F}_m$ or $\mathcal{DG}_{m+2}$, with $m > [n/2]$, develops a singularity. Then, considering the family $\{M_t\}_{t\in[0,T)}$, we are going to use the time–independent inequalities (4.2.8) to show that we have uniform estimates

$$\max_{M_t} |\nabla^k \mathrm{B}| \leq C_k < +\infty \qquad \forall t \in [0,T)$$

for all $k \in \mathbb{N}$. We will see that such estimates are in contradiction with the development of a singularity at time $t = T$, hence the flow must be smooth for every positive time.
To this aim we are going to study the evolution of the following integrals,

$$\int_M |\nabla^k \mathrm{B}|^2 \, d\mu_t \, .$$

REMARK 5.0.8. As in the previous sections, we will omit to say in the computations that all the polynomials $\mathfrak{p}_s$ and $\mathfrak{q}^s$ which will appear are *algebraic*, that is, they are the result of formal manipulations. In particular, such coefficients are independent of the manifold (M, g) where the tensors are defined.

The subsequent analysis is in common between the functionals $\mathcal{F}_m$ and $\mathcal{DG}_{m+2}$, being the discussion of the a priori estimates of the previous chapter the only step needing a separate treatment, hence, we will denote with E_m the first variations of both functional, that we know, by Chapter 3 have the common structure

$$\mathrm{E}_m = \Big(-H + 2(-1)^{m+1} \overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}} H + \mathfrak{q}^{2m+1}(\nabla\nu, \mathrm{B}) \Big)\, \nu \, .$$

5.1. Estimates on the Geometric Quantities

First we derive the evolution equations for g, ν, Γ^i_{jk} and B.
Essentially repeating the computations of Section 2.3.3, we get

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -\,2\mathrm{E}_m \mathrm{B}_{ij} \\
\frac{\partial}{\partial t} g^{ij} &= 2\mathrm{E}_m \mathrm{B}^{ij} \\
\frac{\partial}{\partial t} \nu &= \nabla \mathrm{E}_m \\
\frac{\partial}{\partial t} \Gamma^i_{jk} &= \nabla \mathrm{E}_m * \mathrm{B} + \mathrm{E}_m * \nabla \mathrm{B} \, .
\end{aligned}$$

LEMMA 5.1.1. *The second fundamental form of M_t satisfies the evolution equation*

$$\frac{\partial}{\partial t} \mathrm{B}_{ij} = 2(-1)^m \overbrace{\Delta\dots\Delta}^{(m+1)\text{–times}} \mathrm{B}_{ij} + \mathfrak{q}^{2m+3}(\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2m+3}(\nabla\nu, \mathrm{B}) + \mathfrak{q}^3(\mathrm{B}) \, .$$

PROOF. Keeping in mind the Gauss–Weingarten relations (1.1.7) and the equations above, we compute

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{B}_{ij} &= -\frac{\partial}{\partial t}\left\langle \nu \,\Big|\, \frac{\partial^2\varphi}{\partial x_i\partial x_j}\right\rangle \\
&= \left\langle \nu \,\Big|\, \frac{\partial^2(\mathrm{E}_m\nu)}{\partial x_i\partial x_j}\right\rangle - \left\langle \nabla\mathrm{E}_m \,\Big|\, \frac{\partial^2\varphi}{\partial x_i\partial x_j}\right\rangle \\
&= \frac{\partial^2\mathrm{E}_m}{\partial x_i\partial x_j} + \mathrm{E}_m\left\langle \nu \,\Big|\, \frac{\partial}{\partial x_i}\left(\mathrm{B}_{jl}g^{ls}\frac{\partial\varphi}{\partial x_s}\right)\right\rangle \\
&\quad - \left\langle \frac{\partial\mathrm{E}_m}{\partial x_l}\cdot\frac{\partial\varphi}{\partial x_s}g^{ls} \,\Big|\, \Gamma_{ij}^k\frac{\partial\varphi}{\partial x_k} - \mathrm{B}_{ij}\nu\right\rangle \\
&= \frac{\partial^2\mathrm{E}_m}{\partial x_i\partial x_j} - \Gamma_{ij}^k\frac{\partial\mathrm{E}_m}{\partial x_k} + \mathrm{E}_m\mathrm{B}_{jl}g^{ls}\left\langle \nu \,\Big|\, \Gamma_{is}^z\frac{\partial\varphi}{\partial x_z} - \mathrm{B}_{is}\nu\right\rangle \\
&= \nabla_i\nabla_j\mathrm{E}_m - \mathrm{E}_m\mathrm{B}_{is}g^{sl}\mathrm{B}_{lj}\,.
\end{aligned}$$

Expanding E_m we continue,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{B}_{ij} &= \nabla_i\nabla_j\Big(2(-1)^m\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}\mathrm{H} + \mathfrak{q}^{2m+1}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^1(\mathrm{B})\Big) \\
&\quad - \Big(2(-1)^m\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}\mathrm{H} + \mathfrak{q}^{2m+1}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^1(\mathrm{B})\Big)\mathrm{B}_{is}g^{sl}\mathrm{B}_{lj} \\
&= 2(-1)^m\,\nabla_i\nabla_j\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}\mathrm{H} + \mathfrak{q}^{2m+3}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^3(\mathrm{B})\,.
\end{aligned}$$

Interchanging repeatedly derivatives in the first term we introduce some extra terms of the form $\mathfrak{q}^{2m+3}(\mathrm{B},\mathrm{B})$ and we get

$$\frac{\partial}{\partial t}\mathrm{B}_{ij} = 2(-1)^m\,\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}\nabla_i\nabla_j\mathrm{H} + \mathfrak{q}^{2m+3}(\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2m+3}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^3(\mathrm{B})\,,$$

then using equation (1.1.8) we conclude

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{B}_{ij} &= 2(-1)^m\,\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}(\Delta\mathrm{B}_{ij} - \mathrm{H}\mathrm{B}_{il}g^{ls}\mathrm{B}_{sj} - |\mathrm{B}|^2\mathrm{B}_{ij}) \\
&\quad + \mathfrak{q}^{2m+3}(\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2m+3}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^3(\mathrm{B}) \\
&= 2(-1)^m\,\overbrace{\Delta\Delta\dots\Delta}^{(m+1)\text{–times}}\,\mathrm{B}_{ij} + \mathfrak{q}^{2m+3}(\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2m+3}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^3(\mathrm{B})\,.
\end{aligned}$$

□

Now we deal with the covariant derivatives of B.

LEMMA 5.1.2. *We have*

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^k\mathrm{B}_{ij} &= 2(-1)^m\,\overbrace{\Delta\Delta\dots\Delta}^{(m+1)\text{–times}}\,\nabla^k\mathrm{B}_{ij} \\
&\quad + \mathfrak{q}^{k+2m+3}(\mathrm{B},\mathrm{B}) + \mathfrak{q}^{k+2m+3}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^{k+3}(\mathrm{B})\,.
\end{aligned}$$

PROOF. With a reasoning analogous to the one of Lemma 2.3.11 applied to the tensor B and by the previous lemma, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^k \mathrm{B}_{ij} &= \nabla^k \frac{\partial}{\partial t}\mathrm{B}_{ij} + \mathfrak{p}_k(\mathrm{B},\mathrm{B},\mathrm{E}_m)\\
&= \nabla^k \frac{\partial}{\partial t}\mathrm{B}_{ij} + \mathfrak{q}^{k+2m+3}(\mathrm{B},\mathrm{B}) + \mathfrak{q}^{k+2m+3}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^{k+3}(\mathrm{B},\mathrm{B})\\
&= 2(-1)^m\nabla^k \overbrace{\Delta\Delta\ldots\Delta}^{(m+1)\text{–times}} \mathrm{B}_{ij}\\
&\quad + \nabla^k\mathfrak{q}^{2m+3}(\mathrm{B},\mathrm{B}) + \nabla^k\mathfrak{q}^{2m+3}(\nabla\nu,\mathrm{B}) + \nabla^k\mathfrak{q}^{3}(\mathrm{B})\\
&\quad + \mathfrak{q}^{k+2m+3}(\mathrm{B},\mathrm{B}) + \mathfrak{q}^{k+2m+3}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^{k+3}(\mathrm{B},\mathrm{B})\\
&= 2(-1)^m\nabla^k \overbrace{\Delta\Delta\ldots\Delta}^{(m+1)\text{–times}} \mathrm{B}_{ij}\\
&\quad + \mathfrak{q}^{k+2m+3}(\mathrm{B},\mathrm{B}) + \mathfrak{q}^{k+2m+3}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^{k+3}(\mathrm{B})\,.
\end{aligned}$$

Interchanging the operator ∇^k with the Laplacians in the first term and including the extra terms in $\mathfrak{q}^{k+2m+3}(\mathrm{B},\mathrm{B})$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^k \mathrm{B}_{ij} &= 2(-1)^m \overbrace{\Delta\Delta\ldots\Delta}^{(m+1)\text{–times}} \nabla^k\mathrm{B}_{ij}\\
&\quad + \mathfrak{q}^{k+2m+3}(\mathrm{B},\mathrm{B}) + \mathfrak{q}^{k+2m+3}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^{k+3}(\mathrm{B})\,.
\end{aligned}$$

□

PROPOSITION 5.1.3. *The following formula holds,*

$$\begin{aligned}
\frac{\partial}{\partial t}\int_M |\nabla^k\mathrm{B}|^2\,d\mu_t &= -4\int_M |\nabla^{k+m+1}\mathrm{B}|^2\,d\mu_t\\
&\quad + \int_M \mathfrak{q}^{2(k+m+2)}(\mathrm{B},\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+m+2)}(\nabla\nu,\mathrm{B},\mathrm{B})\,d\mu_t\\
&\quad + \int_M \mathfrak{q}^{2(k+2)}(\mathrm{B},\mathrm{B})\,d\mu_t\,.
\end{aligned}$$

PROOF. By the previous results we have

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k\mathrm{B}|^2 &= 2g^{i_1j_1}\ldots g^{i_kj_k}g^{is}g^{jz}\frac{\partial}{\partial t}\nabla_{i_1\ldots i_k}\mathrm{B}_{ij}\nabla_{j_1\ldots j_k}\mathrm{B}_{sz}\\
&\quad + g^{i_1j_1}\ldots\frac{\partial}{\partial t}g^{i_lj_l}\ldots g^{i_kj_k}g^{is}g^{jz}\nabla_{i_1\ldots i_k}\mathrm{B}_{ij}\nabla_{j_1\ldots j_k}\mathrm{B}_{sz}\\
&= 4(-1)^m g^{i_1j_1}\ldots g^{i_kj_k}g^{is}g^{jz}\overbrace{\Delta\Delta\ldots\Delta}^{(m+1)\text{–times}}\nabla_{i_1\ldots i_k}\mathrm{B}_{ij}\nabla_{j_1\ldots j_k}\mathrm{B}_{sz}\\
&\quad + \Big(\mathfrak{q}^{k+2m+3}(\mathrm{B},\mathrm{B}) + \mathfrak{q}^{k+2m+3}(\nabla\nu,\mathrm{B}) + \mathfrak{q}^{k+3}(\mathrm{B})\Big) * \nabla^k\mathrm{B}\\
&\quad + 2\mathrm{E}_m g^{i_1j_1}\ldots\mathrm{B}^{i_lj_l}\ldots g^{i_kj_k}g^{is}g^{jz}\nabla_{i_1\ldots i_k}\mathrm{B}_{ij}\nabla_{j_1\ldots j_k}\mathrm{B}_{sz}\\
&= 4(-1)^m g^{i_1j_1}\ldots g^{i_kj_k}g^{is}g^{jz}\overbrace{\Delta\Delta\ldots\Delta}^{(m+1)\text{–times}}\nabla_{i_1\ldots i_k}\mathrm{B}_{ij}\nabla_{j_1\ldots j_k}\mathrm{B}_{sz}\\
&\quad + \mathfrak{q}^{2(k+m+2)}(\mathrm{B},\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+m+2)}(\nabla\nu,\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+2)}(\mathrm{B},\mathrm{B})\\
&= 4(-1)^m g^{is}g^{jz}\nabla_{i_{k+1}}\nabla^{i_{k+1}}\ldots\nabla_{i_{k+m+1}}\nabla^{i_{k+m+1}}\nabla_{i_1\ldots i_k}\mathrm{B}_{ij}\nabla^{i_1\ldots i_k}\mathrm{B}_{sz}\\
&\quad + \mathfrak{q}^{2(k+m+2)}(\mathrm{B},\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+m+2)}(\nabla\nu,\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+2)}(\mathrm{B},\mathrm{B})\,.
\end{aligned}$$

Interchanging the covariant derivatives in the first term we introduce some extra terms of the form $\mathfrak{q}^{2(k+m+2)}(\mathrm{B},\mathrm{B},\mathrm{B})$, hence we get

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_M |\nabla^k \mathrm{B}|^2\,d\mu_t = \\
&\quad 4(-1)^m \int_M g^{is}g^{jz}\nabla^{i_{k+1}}\dots\nabla^{i_{k+m+1}}\nabla_{i_{k+m+1}}\dots\nabla_{i_{k+1}}\nabla_{i_1\dots i_k}\mathrm{B}_{ij}\nabla^{i_1\dots i_k}\mathrm{B}_{sz}\,d\mu_t \\
&\quad + \int_M \mathfrak{q}^{2(k+m+2)}(\mathrm{B},\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+m+2)}(\nabla\nu,\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+2)}(\mathrm{B},\mathrm{B})\,d\mu_t \\
&\quad + \int_M \mathfrak{q}^{2(k+2)}(\mathrm{B},\mathrm{B})\,d\mu_t\,,
\end{aligned}
$$

where the last integral comes from the time derivative of μ_t.
Then, carrying the $m+1$ derivatives $\nabla^{i_{k+1}}\dots\nabla^{i_{k+m+1}}$ on $\nabla^{i_1\dots i_k}\mathrm{B}_{sz}$ by means of the divergence theorem, we finally obtain the claimed result,

$$
\begin{aligned}
&= -4\int_M g^{is}g^{jz}\nabla_{i_{k+m+1}}\dots\nabla_{i_{k+1}}\nabla_{i_1\dots i_k}\mathrm{B}_{ij}\nabla^{i_{k+m+1}}\dots\nabla^{i_{k+1}}\nabla^{i_1\dots i_k}\mathrm{B}_{sz}\,d\mu_t \\
&\quad + \int_M \mathfrak{q}^{2(k+m+2)}(\mathrm{B},\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+m+2)}(\nabla\nu,\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+2)}(\mathrm{B},\mathrm{B})\,d\mu_t \\
&= -4\int_M |\nabla^{k+m+1}\mathrm{B}|^2\,d\mu_t \\
&\quad + \int_M \mathfrak{q}^{2(k+m+2)}(\mathrm{B},\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+m+2)}(\nabla\nu,\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+2)}(\mathrm{B},\mathrm{B})\,d\mu_t\,.
\end{aligned}
$$

The leading coefficient became -4 since we multiplied $4(-1)^m$ for $(-1)^{m+1}$ while doing the $m+1$ integrations by parts. ☐

Now we analyze the terms

$$
\int_M \mathfrak{q}^{2(k+m+2)}(\mathrm{B},\mathrm{B},\mathrm{B})\,d\mu_t \quad \text{and} \quad \int_M \mathfrak{q}^{2(k+m+2)}(\nabla\nu,\mathrm{B},\mathrm{B})\,d\mu_t\,.
$$

If one of the two polynomials contains a derivative $\nabla^i\mathrm{B}$ or $\nabla^i(\nabla\nu)$ of order $i > k+m+1$, then all the other derivatives must be of order lower than $k+m$, since the rescaling order of the polynomials is $2(k+m+2)$ and the fact that there are at least three factors in every additive term. In this case, using repeatedly the divergence theorem as before, to lower such highest derivative, we get the integral of a new polynomial which does not contain derivatives of order higher than $k+m+1$. Moreover, if there is a derivative of order $k+m+1$ then the order of all the other derivatives in $\mathfrak{q}^{2(k+m+2)}$ must be lower or equal than $k+m$, by the same argument.
With the same reasoning, the term

$$
\int_M \mathfrak{q}^{2(k+2)}(\mathrm{B},\mathrm{B})\,d\mu_t\,,
$$

can be transformed it in a term without derivatives of order higher or equal than $k+m+1$.

Hence, we can suppose that the last three terms in

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_M |\nabla^k\mathrm{B}|^2\,d\mu_t = &-4\int_M |\nabla^{k+m+1}\mathrm{B}|^2\,d\mu_t \\
&+ \int_M \mathfrak{q}^{2(k+m+2)}(\mathrm{B},\mathrm{B},\mathrm{B}) + \mathfrak{q}^{2(k+m+2)}(\nabla\nu,\mathrm{B},\mathrm{B})\,d\mu_t \\
&+ \int_M \mathfrak{q}^{2(k+2)}(\mathrm{B},\mathrm{B})\,d\mu_t
\end{aligned}
\tag{5.1.1}
$$

do not contain derivatives of B or of $\nabla\nu$ of order higher than $k+m+1$, possibly, only one derivative of order $k+m+1$ can appear.

LEMMA 5.1.4. *The following inequality holds*

$$|\nabla^s \nu| \leq |\nabla^{s-1}\mathrm{B}| + |\mathfrak{q}^s(\mathrm{B})|\,,$$

where $\mathfrak{q}^s(\mathrm{B})$ *does not contain derivatives of* B *of order higher than* $s-2$.

PROOF. By equations (1.1.7) it follows that $\nabla\nu = \mathrm{B} * \nabla\varphi$, hence

$$\nabla^s \nu = \nabla^{s-1}\mathrm{B} * \nabla\varphi + \sum_{i+j=s-2} \nabla^i \mathrm{B} * \nabla^j \nabla^2 \varphi$$

and since $\nabla^2_{ij}\varphi = -\mathrm{B}_{ij}\nu$, we get

$$\begin{aligned}\nabla^s \nu &= \nabla^{s-1}\mathrm{B} * \nabla\varphi + \sum_{i+j=s-2} \nabla^i \mathrm{B} * \nabla^j (\mathrm{B}\nu)\\ &= \nabla^{s-1}\mathrm{B} * \nabla\varphi + \sum_{i+j+k=s-2} \nabla^i \mathrm{B} * \nabla^j \mathrm{B} * \nabla^k \nu\,.\end{aligned}$$

Then, by an induction argument we can express $\nabla^s\nu$ as

$$\nabla^s \nu = \nabla^{s-1}\mathrm{B} * \nabla\varphi + \mathfrak{q}^s(\mathrm{B})$$

where $\mathfrak{q}^s(\mathrm{B})$ does not contain derivatives of order higher than $s-2$.
Taking the norm of both sides we get

$$|\nabla^s \nu| \leq |\nabla^{s-1}\mathrm{B} * \nabla\varphi| + |\mathfrak{q}^s(\mathrm{B})|$$

and we conclude the proof computing

$$\begin{aligned}|\nabla^{s-1}\mathrm{B} * \nabla\varphi| &= \left|\nabla_{i_1\dots i_{s-1}}\mathrm{B}_{il} g^{lk}\frac{\partial\varphi}{\partial x_k}\right|\\ &= \left(\nabla_{i_1\dots i_{s-1}}\mathrm{B}_{il} g^{lk}\frac{\partial\varphi}{\partial x_k} g^{i_1j_1}\dots g^{i_{s-1}j_{s-1}} g^{ij} \nabla_{j_1\dots j_{s-1}}\mathrm{B}_{jw} g^{wz}\frac{\partial\varphi}{\partial x_z}\right)^{1/2}\\ &= \left(\nabla_{i_1\dots i_{s-1}}\mathrm{B}_{il} g^{lk} g_{kz} g^{wz} g^{i_1j_1}\dots g^{i_{s-1}j_{s-1}} g^{ij} \nabla_{j_1\dots j_{s-1}}\mathrm{B}_{jw}\right)^{1/2}\\ &= \left(\nabla_{i_1\dots i_{s-1}}\mathrm{B}_{il} g^{lw} g^{i_1j_1}\dots g^{i_{s-1}j_{s-1}} g^{ij} \nabla_{j_1\dots j_{s-1}}\mathrm{B}_{jw}\right)^{1/2}\\ &= |\nabla^{s-1}\mathrm{B}|\,.\end{aligned}$$

□

Taking the absolute values inside the integrals and using this lemma to substitute every derivative of ν in (5.1.1), we obtain

$$\frac{\partial}{\partial t}\int_M |\nabla^k \mathrm{B}|^2\,d\mu_t \leq -4\int_M |\nabla^{k+m+1}\mathrm{B}|^2\,d\mu_t + \int_M |\mathfrak{q}^{2(k+m+2)}(\mathrm{B})| + |\mathfrak{q}^{2(k+2)}(\mathrm{B})|\,d\mu_t$$

where, as before, the two polynomials do not contain derivatives of B of order higher than $k+m+1$, possibly, only one derivative of order $k+m+1$ can appear in every multiplicative term of $\mathfrak{q}^{2(k+m+2)}(\mathrm{B})$.

Before going on, we remark that the $*$ product of tensors satisfies the following metric property,

$$|T * S| \leq |T|\cdot|S|\,. \tag{5.1.2}$$

This can be easily seen choosing an orthonormal basis at a point of M, in such coordinates we have

$$\begin{aligned}
|T * S|^2 &= \sum_{\substack{\text{free}\\\text{indices}}} \Bigl(\sum_{\substack{\text{contracted}\\\text{indices}}} T_{i_1 \dots i_k} S_{j_1 \dots j_l} \Bigr)^2 \\
&\leq \sum_{\substack{\text{free}\\\text{indices}}} \Bigl(\sum_{\substack{\text{contracted}\\\text{indices}}} T^2_{i_1 \dots i_k} \Bigr) \Bigl(\sum_{\substack{\text{contracted}\\\text{indices}}} S^2_{j_1 \dots j_l} \Bigr) \\
&\leq \Bigl(\sum_{\substack{\text{free}\\\text{indices}}} \sum_{\substack{\text{contracted}\\\text{indices}}} T^2_{i_1 \dots i_k} \Bigr) \Bigl(\sum_{\substack{\text{free}\\\text{indices}}} \sum_{\substack{\text{contracted}\\\text{indices}}} S^2_{j_1 \dots j_l} \Bigr) \\
&= |T|^2 \cdot |S|^2 .
\end{aligned}$$

Now by definition we have

$$\mathfrak{q}^{2(k+m+2)}(\mathrm{B}) = \sum_j \mathop{\circledast}_{l=1}^{N_j} \nabla^{c_{jl}} \mathrm{B}$$

with

$$\sum_{l=1}^{N_j} (c_{jl} + 1) = 2(k + m + 2)$$

for every j, hence

$$|\mathfrak{q}^{2(k+m+2)}(\mathrm{B})| \leq \sum_j \prod_{l=1}^{N_j} |\nabla^{c_{jl}} \mathrm{B}|$$

by (5.1.2). Setting

$$Q_j = \prod_{l=1}^{N_j} |\nabla^{c_{jl}} \mathrm{B}|$$

we clearly obtain

$$\int_M |\mathfrak{q}^{2(k+m+2)}(\mathrm{B})| \, d\mu_t \leq \sum_j \int_M Q_j \, d\mu_t .$$

If Q_j contains a derivative of B of order $k + m + 1$, we have seen that all the others have order lower or equal than $k + m$, then collecting derivatives of the same order, Q_j can be estimated as follows

$$Q_j \leq |\nabla^{k+m+1} \mathrm{B}| \cdot \prod_{i=0}^{k+m} |\nabla^i \mathrm{B}|^{\alpha_{ji}}$$

for some α_{ji} satisfying the rescaling condition

$$(k + m + 2) + \sum_{i=0}^{k+m} (i + 1)\alpha_{ji} = 2(k + m + 2) .$$

Hence, using Young inequality, for every $\varepsilon_j > 0$ we have

$$\begin{aligned}
\int_M Q_j \, d\mu_t &\leq \varepsilon_j \int_M |\nabla^{k+m+1} \mathrm{B}|^2 \, d\mu_t + \frac{1}{4\varepsilon_j} \int_M \prod_{i=0}^{k+m} |\nabla^i \mathrm{B}|^{2\alpha_{ji}} \, d\mu_t \\
&= \varepsilon_j \int_M |\nabla^{k+m+1} \mathrm{B}|^2 \, d\mu_t + \int_M |\mathfrak{q}^{2(k+m+2)}(\mathrm{B})| \, d\mu_t ,
\end{aligned}$$

where we put in evidence the fact that the last term satisfies again the rescaling condition and no more contains the derivative $\nabla^{k+m+1}\mathrm{B}$.

Collecting all together such "bad" terms, and choosing suitable $\varepsilon_j > 0$ such that their total sum is less than one, we obtain

$$\frac{\partial}{\partial t}\int_M |\nabla^k \mathrm{B}|^2\,d\mu_t \leq -3\int_M |\nabla^{k+m+1}\mathrm{B}|^2\,d\mu_t + \int_M |\mathfrak{q}^{2(k+m+2)}(\mathrm{B})| + \int_M |\mathfrak{q}^{2(k+2)}(\mathrm{B})|\,d\mu_t$$

where now in the last two terms all the derivatives of B have order lower than $k+m+1$. We are then ready to estimate them via interpolation inequalities.

As before,

$$|\mathfrak{q}^{2(k+m+2)}(\mathrm{B})| \leq \sum_j Q_j$$

and after collecting derivatives of the same order in Q_j,

$$Q_j = \prod_{i=0}^{k+m} |\nabla^i \mathrm{B}|^{\alpha_{ji}} \qquad \text{with} \qquad \sum_{i+1}^{k+m} \alpha_{ji}(i+1) = 2(k+m+2)\,.$$

Then,

$$\begin{aligned}\int_M Q_j\,d\mu_t &= \int_M \prod_{i=0}^{k+m} |\nabla^i \mathrm{B}|^{\alpha_{ji}}\,d\mu_t \\ &\leq \prod_{i=0}^{k+m}\left(\int_M |\nabla^i \mathrm{B}|^{\alpha_{ji}\gamma_i}\,d\mu_t\right)^{\frac{1}{\gamma_i}} \\ &= \prod_{i=0}^{k+m} \|\nabla^i \mathrm{B}\|^{\alpha_{ij}}_{L^{\alpha_{ji}\gamma_i}(\mu_t)}\end{aligned}$$

where the γ_i are arbitrary positive values such that $\sum 1/\gamma_i = 1$.

We apply interpolation inequalities. If in (4.2.7) we take $q=2$, $r=n+1$, $s=k+m+1$, $j=i$ and $T=\mathrm{B}$ we get

$$\|\nabla^i \mathrm{B}\|_{L^{p_i}(\mu_t)} \leq C\|\mathrm{B}\|^a_{W^{2,k+m+1}(\mu_t)}\|\mathrm{B}\|^{1-a}_{L^{n+1}(\mu_t)}$$

with

$$a = \frac{\frac{1}{p_i} - \frac{i}{n} - \frac{1}{n+1}}{\frac{1}{2} - \frac{k+m+1}{n} - \frac{1}{n+1}} \in \left[\frac{i}{k+m+1}, 1\right] \tag{5.1.3}$$

and $p_i > 1$.

Now, since the volumes of M_t and $\|\mathrm{B}\|_{L^{n+1}(\mu_t)}$ are uniformly bounded in time, also $\|\mathrm{B}\|_{L^2(\mu_t)}$ is uniformly bounded and using the universal inequalities (4.2.6) with $p=q=r=2$ we have

$$\begin{aligned}\|\mathrm{B}\|_{W^{2,k+m+1}(\mu_t)} &\leq \sum_{s=0}^{k+m+1} C\|\nabla^{k+m+1}\mathrm{B}\|^{\frac{s}{k+m+1}}_{L^2(\mu_t)} \\ &\leq \sum_{s=0}^{k+m+1} \|\nabla^{k+m+1}\mathrm{B}\|_{L^2(\mu_t)} + C \\ &\leq D\|\nabla^{k+m+1}\mathrm{B}\|_{L^2(\mu_t)} + C\,,\end{aligned}$$

where we applied Young inequality.

Hence, we conclude that we have constants C, D independent of t such that

$$\|\nabla^i \mathrm{B}\|_{L^{p_i}(\mu_t)} \leq \left(D\|\nabla^{k+m+1}\mathrm{B}\|_{L^2(\mu_t)} + C\right)^a \tag{5.1.4}$$

for a as in (5.1.3) and $p_i > 1$.

Choosing $\gamma_i = 0$ if $\alpha_{ji} = 0$ and $\gamma_i = \frac{2(k+m+2)}{\alpha_{ji}(i+1)}$ otherwise, we have clearly

$$\sum_{i=0}^{k+m} \frac{1}{\gamma_i} = \sum_{i=0}^{k+m} \frac{\alpha_{ji}(i+1)}{2(k+m+2)} = 1$$

by the rescaling condition on the α_{ji}.
We claim that for every $i \in \{0, \dots, k+m\}$, the product $p_i = \alpha_{ji}\gamma_i$ satisfies the condition (5.1.3). By definition, $p_i = \frac{2(k+m+2)}{i+1}$, hence we must check that the following inequality holds

$$\frac{i}{k+m+1} \le \frac{\frac{i+1}{2(k+m+2)} - \frac{i}{n} - \frac{1}{n+1}}{\frac{1}{2} - \frac{k+m+1}{n} - \frac{1}{n+1}} \le 1$$

for every $i \in \{0, \dots, k+m\}$. Since every term is an affine function of i, the claim follows if we show that the inequality holds for $i = 0$ and $i = k+m+1$.
If $i = 0$ we have to prove that

$$0 \le \frac{\frac{1}{2(k+m+2)} - \frac{1}{n+1}}{\frac{1}{2} - \frac{k+m+1}{n} - \frac{1}{n+1}} \le 1\,,$$

that is, since the denominator of the fraction is negative (as $2m \ge n+1$),

$$\frac{1}{2} - \frac{k+m+1}{n} - \frac{1}{n+1} \le \frac{1}{2(k+m+2)} - \frac{1}{n+1} \le 0\,.$$

The right inequality is clearly true, again since $2m \ge n+1$, the left one becomes

$$\frac{k+m+1}{2(k+m+2)} = \frac{1}{2} - \frac{1}{2(k+m+2)} \le \frac{k+m+1}{n}$$

which is true as $2(k+m+2) \ge n$.
When $i = k+m+1$ the fraction is equal to 1, hence the inequality obviously holds.

Then, the exponents $p_i = \alpha_{ji}\gamma_i$ are allowed in inequality (5.1.4) and we get

$$\|\nabla^i \mathrm{B}\|_{L^{\alpha_{ji}\gamma_i}(\mu_t)} \le \big(D\|\nabla^{k+m+1}\mathrm{B}\|_{L^2(\mu_t)} + C\big)^{a_{ji}}$$

where a_{ji} is the relative value we obtain from (5.1.3).

Hence,

$$\begin{aligned}\int_M Q_j\, d\mu_t \le& \prod_{i=0}^{k+m} \|\nabla^i \mathrm{B}\|^{\alpha_{ij}}_{L^{\alpha_{ji}\gamma_i}(\mu_t)}\\ \le& \prod_{i=0}^{k+m} \big(D\|\nabla^{k+m+1}\mathrm{B}\|_{L^2(\mu_t)} + C\big)^{a_{ji}\alpha_{ji}}\\ \le& \big(D\|\nabla^{k+m+1}\mathrm{B}\|_{L^2(\mu_t)} + C\big)^{\sum_{i=0}^{k+m} a_{ji}\alpha_{ji}}\end{aligned}$$

where the constants C and D are independent of t and

$$a_{ji} = \frac{\frac{1}{\alpha_{ji}\gamma_i} - \frac{i}{n} - \frac{1}{n+1}}{\frac{1}{2} - \frac{k+m+1}{n} - \frac{1}{n+1}}\,.$$

Multiplying this relation by α_{ji} and summing on i from 0 to $k+m$ we get

$$\begin{aligned}\sum_{i=0}^{k+m} \alpha_{ji}a_{ji} =& \sum_{i=0}^{k+m} \frac{\frac{1}{\gamma_i} - \frac{i\alpha_{ji}}{n} - \frac{\alpha_{ji}}{n+1}}{\frac{1}{2} - \frac{k+m+1}{n} - \frac{1}{n+1}}\\ =& \frac{1 - \sum_{i=0}^{k+m}\left(\frac{i\alpha_{ji}}{n} + \frac{\alpha_{ji}}{n+1}\right)}{\frac{1}{2} - \frac{k+m+1}{n} - \frac{1}{n+1}}\\ =& \frac{1 - \sum_{i=0}^{k+m}\frac{\alpha_{ji}(i+1)}{n} - \sum_{i=0}^{k+m}\alpha_{ji}\left(\frac{1}{n+1} - \frac{1}{n}\right)}{\frac{1}{2} - \frac{k+m+1}{n} - \frac{1}{n+1}}\end{aligned}$$

recalling that $\sum_{i=0}^{k+m}\alpha_{ji}(i+1)=2(k+m+2)$ we continue,

$$
\begin{aligned}
&= \frac{1-2\frac{k+m+2}{n}+\sum_{i=0}^{k+m}\frac{\alpha_{ji}}{n(n+1)}}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}} \\
&= \frac{1-2\frac{k+m+1}{n}-\frac{2}{n}+\sum_{i=0}^{k+m}\frac{\alpha_{ji}}{n(n+1)}}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}}\,.
\end{aligned}
$$

Now, the denominator is negative and clearly

$$
\sum_{i=0}^{k+m}\alpha_{ji}\geq\sum_{i=0}^{k+m}\frac{\alpha_{ji}(i+1)}{k+m+1}=2\frac{k+m+2}{k+m+1}\,,
$$

so we obtain

$$
\begin{aligned}
\sum_{i=0}^{k+m}\alpha_{ji}a_{ji}\ &\leq \frac{1-2\frac{k+m+1}{n}-\frac{2}{n}+2\frac{k+m+2}{k+m+1}\frac{1}{n(n+1)}}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}} \\
&= \frac{1-2\frac{k+m+1}{n}-\frac{2}{n}+\frac{2}{n(n+1)}+\frac{2}{k+m+1}\frac{1}{n(n+1)}}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}} \\
&= \frac{1-2\frac{k+m+1}{n}-\frac{2}{n+1}+\frac{2}{k+m+1}\frac{1}{n(n+1)}}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}} \\
&= 2-\frac{\frac{2}{k+m+1}\frac{1}{n(n+1)}}{\frac{k+m+1}{n}+\frac{1}{n+1}-\frac{1}{2}} \\
&= 2-\frac{4}{(k+m+1)[2(k+m+1)(n+1)-n(n-1)]}\ <\ 2\,.
\end{aligned}
$$

Hence, we finally get

$$
\int_M Q_j\,d\mu_t\leq\left(D\int_M|\nabla^{k+m+1}\mathrm{B}|^2\,d\mu_t+C\right)^{1-\delta}
$$

for a positive δ and using again Young inequality, we have

$$
\int_M Q_j\,d\mu_t\leq\varepsilon_j\int_M|\nabla^{k+m+1}\mathrm{B}|^2\,d\mu_t+C
$$

for arbitrarily small ε_j. Repeating this argument for all the Q_j and choosing suitable ε_j whose sum is less than one, we conclude that

$$
\frac{d}{dt}\int_M|\nabla^k\mathrm{B}|^2\,\mu_t\ \leq\ -2\int_M|\nabla^{k+m+1}\mathrm{B}|^2\,\mu_t+C+\int_M|\mathfrak{q}^{2(k+2)}(\mathrm{B})|\,d\mu_t
$$

with a constant C independent of time.

The last term can be treated in the same way. It can be estimated by the sum of the multiplicative terms Q_j and collecting derivatives of the same order as before, we have

$$
Q_j\leq\prod_{i=0}^{k+m}|\nabla^i\mathrm{B}|^{\beta_{ji}}\quad\text{with}\quad\sum_{i=0}^{k+m}\beta_{ji}(i+1)=2k+4\,.
$$

In this case the coefficients γ_i, when $\beta_{ji}\neq 0$, are given by $\gamma_i=\frac{2(k+2)}{\alpha_{ji}(i+1)}$, hence

$$
\sum_{i=0}^{k+m}\frac{1}{\gamma_i}=\sum_{i=0}^{k+m}\frac{\alpha_{ji}(i+1)}{2(k+2)}=1
$$

by the rescaling condition.

With an analogous control, one can see that the conditions on the exponent p_i are satisfied. It lasts to compute

$$\begin{aligned}\sum_{i=0}^{k+m}\beta_{ji}a_{ji} &= \sum_{i=0}^{k+m}\frac{\frac{1}{\gamma_i}-\frac{i\beta_{ji}}{n}-\frac{\beta_{ji}}{n+1}}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}}\\ &= \frac{1-\sum_{i=0}^{k+m}\left(\frac{i\beta_{ji}}{n}+\frac{\beta_{ji}}{n+1}\right)}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}}\\ &= \frac{1-\sum_{i=0}^{k+m}\frac{\beta_{ji}(i+1)}{n}+\sum_{i=0}^{k+m}\frac{\beta_{ji}}{n(n+1)}}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}}\\ &= \frac{1-\frac{2k+4}{n}+\sum_{i=0}^{k+m}\frac{\beta_{ji}}{n(n+1)}}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}}\,.\end{aligned}$$

As the denominator is negative and

$$\sum_{i=0}^{k+m}\beta_{ji}\geq\sum_{i=0}^{k+m}\frac{\beta_{ji}(i+1)}{k+m+1}=\frac{2k+4}{k+m+1}\,,$$

we obtain

$$\begin{aligned}\sum_{i=0}^{k+m}\beta_{ji}a_{ji} &\leq \frac{1-\frac{2k+4}{n}+\sum_{i=0}^{k+m}\frac{\beta_{ji}(i+1)}{k+m+1}\frac{1}{n(n+1)}}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}}\\ &= \frac{1-\frac{2k+4}{n}+\frac{2k+4}{k+m+1}\frac{1}{n(n+1)}}{\frac{1}{2}-\frac{k+m+1}{n}-\frac{1}{n+1}}<2\,,\end{aligned}$$

since this last inequality is equivalent to

$$1-\frac{2k+4}{n}+\frac{2k+4}{k+m+1}\frac{1}{n(n+1)}>1-\frac{2(k+m+1)}{n}-\frac{2}{n+1}$$

and simplifying, to

$$\frac{2k+4}{k+m+1}\frac{1}{n(n+1)}>-\frac{2(m-1)}{n}-\frac{2}{n+1}$$

which is obviously true.

Concluding as before we finally get

$$\frac{d}{dt}\int_M|\nabla^k\mathrm{B}|^2\,\mu_t\;\leq\;-\int_M|\nabla^{k+m+1}\mathrm{B}|^2\,\mu_t+C \tag{5.1.5}$$

for a constant C independent of time.
By (4.1.2) and Young inequality, we have

$$\begin{aligned}\int_M|\nabla^k\mathrm{B}|^2\,\mu_t+C &\leq D\|\nabla^{k+m+1}\mathrm{B}\|_{L^2(\mu_t)}^{\frac{k}{k+m+1}}\|\mathrm{B}\|_{L^2(\mu_t)}^{\frac{m+1}{k+m+1}}+C\\ &\leq D\|\nabla^{k+m+1}\mathrm{B}\|_{L^2(\mu_t)}^{\frac{k}{k+m+1}}+C\\ &\leq \frac{1}{2}\int_M|\nabla^{k+m+1}\mathrm{B}|^2\,\mu_t+C\end{aligned}$$

again with a uniform constant. Combining this inequality with (5.1.5), we obtain

$$\frac{d}{dt}\int_M|\nabla^k\mathrm{B}|^2\,\mu_t\;\leq\;-\frac{1}{2}\int_M|\nabla^k\mathrm{B}|^2\,\mu_t+C$$

and a simple ODE's argument proves that there exists constants C_k independent of time such that

$$\int_M|\nabla^k\mathrm{B}|^2\,d\mu_t\leq C_k\,.$$

To pass from $W^{2,p}(\mu_t)$ to pointwise estimates, first we notice that being all the derivatives of B bounded in $L^2(\mu_t)$, by inequalities (4.2.2), for every $p \geq 1$ and $k \in \mathbb{N}$ we have constants $C_{k,p}$ such that

$$\int_M |\nabla^k \mathrm{B}|^p \, d\mu_t \leq C_{k,p} \,.$$

Then choosing a $p > n$, we apply inequalities (4.2.3) to every $\nabla^k \mathrm{B}$ to conclude that for every $k \in \mathbb{N}$ we have constants C_k, independent of t, such that

$$\max_{M_t} |\nabla^k \mathrm{B}| \leq C_k \,. \tag{5.1.6}$$

Looking back at the way we obtained them, we can see that the constants C_k depend only on the dimension n, the differentiation order k and the initial hypersurface φ_0.

5.2. Long Time Existence and Regularity

Following Huisken [**54**, Sect. 8] and Kuwert and Schätzle [**67**, Sect. 8], these estimates imply the smoothness of the map $\varphi(p,t)$.
Since $\nabla^k \mathrm{B}$ are uniformly bounded in time, supposing that $[0,T)$ is the maximal interval of existence of the flow, we have

$$|\varphi(p,t) - \varphi(p,s)| \leq \int_s^t |\mathrm{E}_m(\varphi_\xi)(p)| \, d\xi \leq C(t-s)$$

for every $0 \leq s \leq t < T$, then φ_t uniformly converge to a continuous limit φ_T as $t \to T$.
We recall Lemma 8.2 in [**54**] (Lemma 14.2 in [**53**]).

LEMMA 5.2.1. *Let g_{ij} a time–dependent metric on a compact manifold M for $0 \leq t < T \leq +\infty$. Suppose that*

$$\int_0^T \max_{M_t} \left| \frac{\partial}{\partial t} g_{ij} \right| \, dt \leq C \,.$$

Then the metrics $g_{ij}(t)$ are all equivalent, and they converge as $t \to T$ uniformly to a positive definite metric tensor $g_{ij}(T)$ which is continuous and also equivalent.

In our situation, if $T < +\infty$, the hypotheses of this lemma are clearly satisfied, hence $\varphi(\cdot, T)$ represents a hypersurface. Moreover, it also follows that there exists a positive constant C depending only on n and φ_0 such that for every $0 \leq t < T$ we have

$$\frac{1}{C} \leq g_{ij}(t) \leq C \,.$$

Since

$$\frac{\partial}{\partial t} g_{ij} = -2\mathrm{E}_m \mathrm{B}_{ij}$$

by (5.1.6), for every $k \in \mathbb{N}$ we have

$$\left\| \nabla^k \frac{\partial}{\partial t} g_{ij} \right\|_{L^\infty(\mu)} \leq C_k \,,$$

analogously, as the time derivative of the Christoffel symbols is given by

$$\frac{\partial}{\partial t} \Gamma^i_{jk} = \nabla \mathrm{E}_m * \mathrm{B} + \mathrm{E}_m * \nabla \mathrm{B}$$

it follows that

$$\left\| \nabla^k \frac{\partial}{\partial t} \Gamma^i_{jk} \right\|_{L^\infty(\mu)} \leq C_k \,.$$

for every $k \in \mathbb{N}$.
With an induction argument, we can prove the following formula (where we avoid to indicate the indices) relating the iterated covariant and coordinate derivatives of a tensor T,

$$\nabla^m T = \partial^m T + \sum_{i=1}^m \sum_{j_1 + \cdots + j_i + k \leq m-1} \partial^{j_1} \Gamma \ldots \partial^{j_i} \Gamma \partial^k T \,. \tag{5.2.1}$$

By this formula and induction, it follows that

$$\|\partial^k \Gamma^i_{jl}\|_{L^\infty(\mu)}\,, \qquad \left\|\partial^k \frac{\partial}{\partial t}\Gamma^i_{jl}\right\|_{L^\infty(\mu)} \leq C_k\,,$$

for every $t \in [0,T)$.
Applying again formula (5.2.1) to $T = \nabla^s \mathrm{B}$ we see that

$$\partial^k \nabla^s \mathrm{B} - \nabla^{k+s}\mathrm{B} = \sum_{i=1}^{k} \sum_{j_1+\dots+j_i+l\leq k-1} \partial^{j_1}\Gamma \dots \partial^{j_i}\Gamma\, \partial^l \nabla^s \mathrm{B}\,,$$

and by induction and estimates (5.1.6) we obtain

$$\|\partial^k \nabla^s \mathrm{B}\|_{L^\infty(\mu)} \leq C_{k,s}$$

for every $k, s \in \mathbb{N}$.
Since we already know that $|\varphi|$ is bounded and $|\partial\varphi| = 1$, by the Gauss–Weingarten relations (1.1.7)

$$\partial^2\varphi = \Gamma\partial\varphi + \mathrm{B}\nu\,, \qquad \partial\nu = \mathrm{B} * \partial\varphi$$

and the previous estimates, we can conclude that

$$\|\partial^k \varphi\|_{L^\infty(\mu)} \leq C_k$$

for every $k \in \mathbb{N}$ and $t \in [0,T)$.
The regularity of the time derivatives also follows by these estimates and the evolution equation.

Hence, the convergence $\varphi_t \to \varphi_T$, when $t \to T$, is in the C^∞ topology and M_T is smooth. Then, using Theorem 3.2.1 to restart the flow with φ_T as initial hypersurface, we get a contradiction with the fact that $[0,T)$ is the maximal interval of existence.

REMARK 5.2.2. Though this argument shows that the solution is classical, we cannot conclude that the estimates on the parametrization hold uniformly for every $t \in [0,+\infty)$ which is instead the case for the estimates (5.1.6) on the curvature.

We have then shown Theorem 3.1.3, with the extra estimate of the following proposition.

PROPOSITION 5.2.3. *If $m > [n/2]$, the unique smooth solution of the evolution problem*

$$\frac{\partial\varphi}{\partial t}(p,t) = -\mathrm{E}_m(\varphi_t)(p)\nu(p,t)\,,$$

with an initial smooth, compact, immersed hypersurface $\varphi_0 : M \to \mathbb{R}^{n+1}$ that is, the gradient flow associated to the functional $\mathcal{F}_m$ or $\mathcal{DG}_{m+2}$, satisfies

$$\max_{M_t} |\nabla^k \mathrm{B}| \leq C_k\,.$$

for constants C_k depending only on n, k and φ_0.

REMARK 5.2.4. A natural extension would be to consider ambient spaces different by $\mathbb{R}^{n+1}$ and a codimension s greater than one, that is, a general Riemannian manifold (N,h) of dimension $n+s$ (notice that Polden's Theorem 3.2.1 about short time existence of the flow already consider hypersurfaces in a general target manifold), in particular to deal with the original conjecture of De Giorgi 3.1.2 which was stated in arbitrary codimension. In this context the "analogous" $\mathcal{F}_m$–functional which can be considered is

$$\mathcal{F}_m(\varphi) = \int_M 1 + |\nabla^m \omega|^2\, d\mu$$

where $\omega = \nu_1 \wedge \dots \wedge \nu_s$ is an s–vector obtained by a local orthonormal basis of the normal space to the n–dimensional immersed submanifold $\varphi : M \to N^{n+s}$.
This extension can actually be obtained by some technical and sometimes heavy but straightforward modifications of the arguments and computations of the previous chapters.

We remark that Kuwert, Schätzle and Dziuk in **[37]** extended Polden's results to space curves.

5.3. Convergence

Let us consider the function $\sigma : [0, +\infty) \to \mathbb{R}$,

$$\sigma(t) = \int_M [\mathrm{E}_m(\varphi_t)]^2 \, d\mu_t \geq 0 \,.$$

Clearly we have

$$\frac{d}{dt}\mathcal{F}_m(\varphi_t) = -\int_M [\mathrm{E}_m(\varphi_t)]^2 \, d\mu_t = -\sigma(t)\,,$$

and integrating both sides in t on $[0, +\infty)$ we get

$$\int_0^{+\infty} \sigma(t)\, dt = \mathcal{F}_m(\varphi_0) - \mathcal{F}_m(\varphi_t) \leq \mathcal{F}_m(\varphi_0)\,.$$

Moreover,

$$\left|\frac{d}{dt}\sigma(t)\right| = \int_M \left| 2\,\frac{\partial \mathrm{E}_m(\varphi_t)}{\partial t}\,\mathrm{E}_m(\varphi_t) - \mathrm{H}\,[\mathrm{E}_m(\varphi_t)]^3 \right| d\mu_t \leq C$$

by the bounds (5.1.6). Then the function σ, being Lipschitz and integrable on $[0, +\infty)$, converges to zero at $+\infty$. This means that every C^∞ limit hypersurface of the flow $\psi : M \to \mathbb{R}^{n+1}$ satisfies $\mathrm{E}_m(\psi) = 0$, that is, it is a critical point of $\mathcal{F}_m$.

To find limit hypersurfaces, we need the following compactness result of Langer and Delladio [**35, 69**].

THEOREM 5.3.1. *Let be given a family (M, g_i) of closed, oriented, n–dimensional hypersurfaces, isometrically immersed in $\mathbb{R}^{n+1}$ via the maps $\varphi_i : M \to \mathbb{R}^{n+1}$, let μ_i the associated measures on M and Bar_i the* center of gravity *of φ_i, that is,*

$$\mathrm{Bar}_i = \int_M \varphi_i \, d\mu_i \,.$$

Let h be any metric tensor on M, if for some exponent $p > n$ and $C > 0$ we have

$$\int_M 1 + |\mathrm{B}|^p \, d\mu_i + |\mathrm{Bar}_i| \;\leq\; C < +\infty\,,$$

then there exist a subsequence of $\{\varphi_i\}$ (not relabeled) and diffeomorphisms $\sigma_i : M \to M$ such that, $\{\varphi_i \circ \sigma_i\}$ converges in the $W^{2,p}$ weak topology of maps from $(M, h) \to \mathbb{R}^{n+1}$ to an immersion $\varphi : M \to \mathbb{R}^{n+1}$.

Translating the hypersurfaces $\varphi_t : M \to \mathbb{R}$ in order to have $\mathrm{Bar}_t = 0 \in \mathbb{R}^{n+1}$, we are in the above hypotheses. Hence, we can extract a subsequence of smooth hypersurfaces $\varphi_i = \varphi_{t_i}$ and diffeomorphisms $\sigma_i : M \to M$ such that, for a fixed metric h on M, the sequence $\{\varphi_i \circ \sigma_i\}$ converges in the $W^{2,p}$ weak topology to an immersion $\psi : M \to \mathbb{R}^{n+1}$.
With the arguments of the proof of Theorem 5.3.1 in [**35, 69**] and keeping into account that in our case we have also the estimates (5.1.6), it is possible to conclude that actually the convergence is in the C^∞ topology and the limit hypersurface is smooth (see also [**55**, Prop. 3.4]).

As the analysis for the functional $\mathcal{DG}_{m+2}$ is analogous, we resume this discussion in the following theorem.

THEOREM 5.3.2. *The family of smooth hypersurfaces $\varphi_0 : M \to \mathbb{R}^{n+1}$, immersed in $\mathbb{R}^{n+1}$, evolving by the gradient flow for the functional $\mathcal{F}_m$ or $\mathcal{DG}_{m+2}$, with $m > \left[\frac{n}{2}\right]$, up to reparametrizations and translations, is compact in the C^∞ topology of maps. Moreover, every limit point for $t \to +\infty$ is a C^∞ critical hypersurface of the functional $\mathcal{F}_m$ or $\mathcal{DG}_{m+2}$, respectively.*

REMARK 5.3.3. A natural open problem arising from the discussion of this section is the question of the uniqueness of the limit hypersurface. It is also unknown to the author if actually it can happen that the evolving hypersurface goes to the infinity when $t \to +\infty$.
To conclude, we mention the problem of classification of the possible limit points of these flows, or equivalently of the critical hypersurfaces of $\mathcal{F}_m$ and $\mathcal{DG}_{m+2}$. In his work [**79**] Polden completely classifies the limit curves of the flow of the functional (3.1.1), the analogous n–dimensional result seems to be a much more difficult task.

5.4. Other Functionals

It would be very interesting to study the flows in the "critical" case $2m = n$, where our proof fails since we are no more able to bound the constants in the inequalities independently of time, as we did in Chapter 4.
Notice that the well known Willmore functional (see **[67, 87, 93]**)

$$\mathcal{W}(\varphi) = \int_M |\mathrm{B}|^2 \, d\mu$$

falls exactly in this case if we add an area term, since $|\mathrm{B}|^2$ is equal to $|\nabla\nu|^2$.
To the author's knowledge, up to now nor there is a proof of regularity of the flow, neither an example showing the development of a singularity. Some important steps in this direction come from the works of Kuwert and Schätzle **[67, 66]**.

When $2m < n$ we do not expect regularity of the flow by the gradient of $\mathcal{F}_m$ or $\mathcal{DG}_{m+2}$, since the curvature term should not be sufficient to give regularity and dumbbell–like separation phenomena should appear during the flow of certain hypersurfaces. It should also be noticed that in this and in the critical case, the n–dimensional unit sphere in $\mathbb{R}^{n+1}$ "collapses" in finite time.

Moreover, one can consider also "non–quadratic" functionals, for instance,

$$\mathcal{F}_{m,p}(\varphi) = \int_M 1 + |\nabla^m \nu|^p \, d\mu \qquad \text{when } mp > n$$

(following the analogy with the Sobolev spaces), in particular,

$$\mathcal{F}_{1,p}(\varphi) = \int_M 1 + |\mathrm{B}|^p \, d\mu \qquad \text{for } p > n$$

which would give rise to a flow of order lower than the one of $\mathcal{F}_m$ when $n > 1$.
In the same spirit another interesting functional is

$$\mathcal{H}_p(\varphi) = \int_M 1 + |\mathrm{H}|^p \, d\mu \qquad \text{for } p > n\,.$$

In all these cases the smoothness of the associated flows is an open problem.

CHAPTER 6

Singular Approximation of the Mean Curvature Flow

Slightly modifying our analysis in the previous chapters, it easily follows that if $m > [n/2]$, for every pair of positive constants α and β also the gradient flows of the functionals

$$\mathcal{F}_m^{\alpha\beta}(\varphi) = \int_M \alpha + \beta|\nabla^m \nu|^2 \, d\mu$$

and

$$\mathcal{DG}_{m+2}^{\alpha\beta}(M) = \int_M \alpha + \beta|A^{m+2}|^2 \, d\mathcal{H}^n$$

exists and are smooth for every positive time.
Moreover, if we consider a general, positive, geometric functional

$$\mathcal{G}(\varphi) = \int_M f(\varphi, g, \mathrm{B}, \nu, \ldots, \nabla^s \mathrm{B}, \nabla^l \nu) \, d\mathcal{H}^n \,,$$

such that the function f is smooth and has polynomial growth, choosing an integer m large enough, the gradient flow of the "perturbed" functional, for any $\varepsilon > 0$,

$$\mathcal{G}_m^\varepsilon(\varphi) = \mathcal{G}(\varphi) + \varepsilon \mathcal{F}_m(\varphi)$$

does not develop singularities (the same if we perturb the functional $\mathcal{G}$ with $\varepsilon\mathcal{DG}_{m+2}$).
This can be shown by first noticing that, as $\mathcal{G}$ is positive, the estimates on the constants in the inequalities of Chapter 4 hold for the flow by the gradient of $\mathcal{G}_m^\varepsilon$, then by choosing the order m large enough in order that the term $|\nabla^m \nu|^2$ (or $|A^{m+2}|^2$) "dominates" all the others in $f(\varphi, g, \mathrm{B}, \nu, \ldots, \nabla^s \mathrm{B}, \nabla^l \nu)$. This leads to the short time existence of the gradient flow and its global regularity.
To be more precise, assuming for instance that $f(\varphi, g, \mathrm{B}, \nu, \ldots, \nabla^s \mathrm{B}, \nabla^l \nu)$ is bounded by $C + \mathfrak{q}^s(\nabla\nu, \mathrm{B})$ for some polynomial $\mathfrak{q}^s(\nabla\nu, \mathrm{B})$ in the covariant derivatives of $\nabla\nu$ and B (see Section 1.2), as in the hypotheses, we consider an integer $m \geq 1$ such that

- m is larger than the maximal order of differentiation of ν present in $\mathfrak{q}^s(\nabla\nu, \mathrm{B})$,
- $m - 1$ is larger than the maximal order of differentiation of B present in $\mathfrak{q}^s(\nabla\nu, \mathrm{B})$,
- $2m$ is larger than s, the *rescaling order* of $\mathfrak{q}^s(\nabla\nu, \mathrm{B})$.
  We recall that this latter is defined as

  $$s = \sum_{k=1}^{N}(i_k + 1) \, + \, \sum_{l=1}^{M}(j_l + 1)$$

  for a polynomial $\mathfrak{q}^s$ in $\nabla\nu$, B and their derivatives, of the form

  $$\sum (\nabla^{i_1}\nabla\nu) \ldots (\nabla^{i_k}\nabla\nu) \ldots (\nabla^{i_N}\nabla\nu) \, \nabla^{j_1}\mathrm{B} \ldots \nabla^{j_l}\mathrm{B} \ldots \nabla^{j_M}\mathrm{B} \, g^{w_1 z_1} \ldots g^{w_t z_t}$$

  (see Remark 1.2.1 and the discussion therein).

By the first two conditions on m, the terms coming from the first variation of $\mathcal{G}$ are of lower order than the leading term of the first variation of $\varepsilon\mathcal{F}_m$ (or $\varepsilon\mathcal{DG}_{m+2}$, see the beginning of Chapter 3), hence, the leading term of the Euler equation of $\mathcal{G}_m^\varepsilon$ is still given by

$$2\varepsilon(m+2)(-1)^{m+1}\Big(\overbrace{\Delta\Delta\ldots\Delta}^{m\text{–times}} H\Big)\,\nu$$

and we can apply again Polden's Theorem 3.2.1 in order to have short time existence and uniqueness of the gradient flow of the functional $\mathcal{G}^\varepsilon_m$, for every initial, smooth, compact, immersed hypersurface $\varphi_0 : M \to \mathbb{R}^{n+1}$.
Getting the global regularity of the flow is a little bit more involved. Actually, the third condition above on m is exactly what is needed in order that, after a careful inspection of all the arguments, the estimates of Chapter 5 still hold for the flow by the gradient of $\mathcal{G}^\varepsilon_m$.

REMARK 6.0.1. We underline that we could have also assumed that the integrand function f in the functional $\mathcal{G}$ above, depends also on A^M and its derivatives, by the relations between the second fundamental form and the distance function established in the Chapter 1 (see also the discussion at the beginning of Chapter 2).

We then say that $\mathcal{F}_m$ and $\mathcal{DG}_{m+2}$ are *smoothing terms* for the functional $\mathcal{G}$, that possibly does not admit a gradient flow even for short time starting from a generic initial, smooth, compact, immersed hypersurface.
It this then natural to investigate what happens when the constant $\varepsilon > 0$ in front of these smoothing terms goes to zero.

This program, suggested by De Giorgi in [**31, 32**, Sect. 5], can be described as follows: given a geometric functional $\mathcal{G}$ defined on submanifolds of the Euclidean space (or a more general ambient space),

- find a functional $\mathcal{F}$ such that the perturbed functionals $\mathcal{G}^\varepsilon = \mathcal{G} + \varepsilon\mathcal{F}$ give rise to smooth flows for every $\varepsilon > 0$;
- study what happens when $\varepsilon \to 0$, in particular, the existence of a limit flow and in this case its relation with the gradient flow of $\mathcal{G}$ (if it exists, smooth or singular).

If proved successful, this scheme would give a *singular approximation* procedure of the gradient flow of $\mathcal{G}$ with a family of globally smooth flows.

Our work shows that the functionals $\mathcal{F}_m$ and $\mathcal{DG}_{m+2}$ satisfy the first point for any geometric functional $\mathcal{G}$ defined on hypersurfaces in $\mathbb{R}^{n+1}$ with polynomial growth, provided we choose an order m large enough (depending on $\mathcal{G}$).

About the second point, the very first case is concerned with the possible limits when $\varepsilon \to 0$ of the gradient flows of $\int_M 1 + \varepsilon|\nabla^m\nu|^2\,d\mu$ when $m > [n/2]$ and their relation with the mean curvature flow, which is the gradient flow of the Area functional, obtained by letting $\varepsilon = 0$.

De Giorgi, in the same paper [**31**, Sect. 5, Cong. 3 and Oss. 2/3] cited above (see also [**32**, Sect. 5, Conj. 3 and Rem. 2/3]), *essentially* stated the following conjecture.

CONJECTURE 6.0.2 (De Giorgi). Let $m > [n/2]$, if the parameter $\varepsilon > 0$ goes to zero, the flows φ^ε_t associated to the functionals

$$\mathcal{DG}^\varepsilon_m(M) = \int_M 1 + \varepsilon|A^{m+2}|^2\,d\mathcal{H}^n$$

and starting from a common initial, smooth, compact, immersed hypersurface $\varphi_0 : M \to \mathbb{R}^{n+1}$, *converge in some sense* to the mean curvature flow of φ_0,

$$\frac{\partial\varphi}{\partial t} = H\nu$$

(which is the gradient flow associated to the limit Area functional, as $\varepsilon \to 0$).

REMARK 6.0.3. De Giorgi proposed this conjecture in general codimension, in the following we will discuss only the case of evolving hypersurfaces, see anyway Remark 5.2.4 and Remark 6.3.3.

REMARK 6.0.4. Clearly, an analogous conjecture can be stated for the ε–parametrized family of functionals

$$\mathcal{F}^\varepsilon_m(M) = \int_M 1 + \varepsilon|\nabla^m\nu|^2\,d\mu\,.$$

The goal of this chapter will be to show the following theorem, related to such conjecture.

THEOREM 6.0.5. *Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth, compact, n–dimensional, immersed submanifold of $\mathbb{R}^{n+1}$. Let $T_{\mathrm{sing}} > 0$ be the first singularity time of the mean curvature flow $\varphi : M \times [0, T_{\mathrm{sing}}) \to \mathbb{R}^{n+1}$ of M. For any $\varepsilon > 0$ let $\varphi^\varepsilon : M \times [0, +\infty) \to \mathbb{R}^{n+1}$ be the flow associated to the functional $\mathcal{DG}^\varepsilon_m$ (or $\mathcal{F}^\varepsilon_m$) with $m > [n/2]$, that is,*

$$\begin{aligned}\frac{\partial \varphi^\varepsilon}{\partial t} &= \mathrm{H} + \varepsilon\Big(2(m+2)(-1)^m \overbrace{\Delta^{M_t}\Delta^{M_t}\dots\Delta^{M_t}}^{m\text{–times}} H + \mathfrak{q}^{2m+1}(\nabla\nu, \mathrm{B})\Big)\,\nu \qquad (6.0.1)\\ &= \mathrm{H} + 2\varepsilon(m+2)(-1)^m \Big(\overbrace{\Delta^{M_t}\Delta^{M_t}\dots\Delta^{M_t}}^{m\text{–times}} H\Big)\,\nu + \varepsilon \mathrm{LOT}\,\nu\end{aligned}$$

(LOT denotes terms of lower order in the curvature and its derivatives), all starting from the same initial immersion φ_0.
Then the maps φ^ε converge locally in $C^\infty(M \times [0, T_{\mathrm{sing}}))$ to the map φ, as $\varepsilon \to 0$.

Since the proofs of this theorem for the two functionals $\mathcal{DG}^\varepsilon_m$ or $\mathcal{F}^\varepsilon_m$ are exactly the same, in the sequel we will discuss only the case of $\mathcal{DG}^\varepsilon_m$.

EXAMPLE 6.0.6. In case of immersed plane curves $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, that is $n = 1$, the simplest choice is $m = 1$. Since $|A^3|^2 = 3\kappa^2$, where κ is the curvature of γ, in this simple case the approximating functionals read as

$$\int_\gamma \left(1 + 3\varepsilon\kappa^2\right)\, ds$$

where s is the arclength parameter and we. The regularized system which should approximate the curve shortening flow is then

$$\frac{\partial \gamma}{\partial t} = \left(\kappa - 6\varepsilon\partial_s^2\kappa - 3\varepsilon\kappa^3\right)\nu,$$

where ν is a suitable choice of the normal unit vector to the curve.

The crucial point in order to prove Theorem 6.0.5 is to obtain ε–independent estimates of the curvature and its derivatives in order to gain sufficient compactness properties. We will get these latter by computing the evolution equations satisfied by the L^2 norms of the derivatives of the second fundamental form of the flowing manifolds and by estimating via Gagliardo–Nirenberg interpolation inequalities.
At the moment we are not able to characterize the possible limits of the approximating flows *after the first singularity time*, as the proof of Theorem 6.0.5 relies heavily on the smoothness of the mean curvature flow in the time interval of existence. Our future goal would be to get *some* limit flow defined *for all times*, thus providing a new weak definition of solution to the mean curvature flow.
We mention the simplest open problem in defining a limit flow after the first singularity.
It is well known (Gage–Hamilton [**45, 46**] and Huisken [**54**]) that a convex curve in the plane (or hypersurface in $\mathbb{R}^{n+1}$) moving by mean curvature shrinks to a point in finite time, becoming exponentially round. In this case we expect that the approximating flows converge (in a way to be made precise) to such point for every time after the extinction.

The plan of this chapter is the following. In the next section, in order to make the line of the proof clearer, we work out in detail the ε–independent estimates in the simplest case of plane immersed curves; also in this special case, the result appears to be nontrivial. In Section 6.2 we consider the general case of a n–dimensional submanifold of $\mathbb{R}^{n+1}$. Section 6.3 is devoted to show Theorem 6.0.5.

6.1. Curves in the Plane

Let $\gamma \in C^\infty(\mathbb{S}^1; \mathbb{R}^2)$ be a regular immersed closed curve in the plane $\mathbb{R}^2$. Let $\tau = \gamma_x/|\gamma_x| = \gamma_s$ and $\nu = \mathrm{R}\tau$ be respectively the tangent and the normal to the curve γ, where R is the counterclockwise rotation of $\pi/2$ in the plane, and $\gamma_x = \partial_x\gamma$.
We recall that $\partial_s = \partial_x/|\gamma_x|$ and

$$\partial_s\tau = \kappa\nu, \qquad \partial_s\nu = -\kappa\tau \qquad (6.1.1)$$

where κ is the curvature of γ. In the sequel we let $\mathrm{L} = \mathrm{L}(\gamma) = \int_\gamma 1\,ds$ be the length of the curve γ.

By simplicity, we will consider the functional

$$\mathcal{DG}_3^{\varepsilon/3}(\gamma) = \int_\gamma \left(1 + \varepsilon\kappa^2\right)\,ds\,,$$

instead of $\mathcal{DG}_3^{\varepsilon}$ (with $n = m = 1$), all the conclusions will holds clearly for this latter. Set

$$\mathrm{E}^\varepsilon = -\kappa + 2\varepsilon\partial_s^2\kappa + \varepsilon\kappa^3\,,$$

then the gradient flow by $\mathcal{DG}_3^{\varepsilon/3}$ is given by a smooth map $\gamma : \mathbb{S}^1 \times [0,+\infty) \to \mathbb{R}^2$ which is an immersion for any $t \in [0,+\infty)$, equals a given immersion γ_0 at time $t = 0$, and satisfies

$$\partial_t\gamma = -\mathrm{E}^\varepsilon\,\nu\,, \tag{6.1.2}$$

where $\partial_t = \frac{\partial}{\partial t}$. For notational simplicity, we omit the dependence of γ on ε.

LEMMA 6.1.1. *We have*

$$\partial_s\partial_t\gamma = -(\partial_s\mathrm{E}^\varepsilon)\nu + \kappa\mathrm{E}^\varepsilon\tau\,,$$

in particular

$$\langle\partial_s\partial_t\gamma, \tau\rangle = \kappa\mathrm{E}^\varepsilon\,. \tag{6.1.3}$$

PROOF. It follows from equations (6.1.1) and the evolution equation (6.1.2). □

LEMMA 6.1.2. *Let γ be a smooth closed curve, then*

$$\frac{1}{\mathrm{L}} \leq \frac{1}{4\pi^2}\int_\gamma \kappa^2\,ds\,. \tag{6.1.4}$$

PROOF. By Borsuk and Schwartz–Hölder inequalities we have

$$2\pi \leq \int_\gamma |\kappa|\,ds \leq \left(\int_\gamma \kappa^2\,ds\right)^{1/2}\mathrm{L}^{1/2}\,.$$

□

LEMMA 6.1.3. *The following commutation rule holds:*

$$\partial_t\partial_s = \partial_s\partial_t - \kappa\mathrm{E}^\varepsilon\partial_s\,. \tag{6.1.5}$$

PROOF. Observing that $\frac{\partial_t\partial_x}{|\gamma_x|} = \frac{\partial_x}{|\gamma_x|}\partial_t = \partial_s\partial_t$, we have

$$\begin{aligned}\partial_t\partial_s &= \partial_t\left(\frac{\partial_x}{|\gamma_x|}\right) = \frac{\partial_t\partial_x}{|\gamma_x|} - \frac{\langle\gamma_x, \partial_t\gamma_x\rangle\partial_x}{|\gamma_x|^3} = \frac{\partial_x}{|\gamma_x|}\partial_t - \left\langle\frac{\gamma_x}{|\gamma_x|}, \frac{\partial_t\gamma_x}{|\gamma_x|}\right\rangle\frac{\partial_x}{|\gamma_x|}\\ &= \partial_s\partial_t - \langle\tau, \partial_s\partial_t\gamma\rangle\partial_s\,.\end{aligned}$$

Then the commutation rule (6.1.5) follows from equation (6.1.3). □

LEMMA 6.1.4. *We have*

$$\partial_t\kappa = -\partial_s^2\mathrm{E}^\varepsilon - \kappa^2\mathrm{E}^\varepsilon = \partial_s^2\kappa + \kappa^3 - 2\varepsilon\partial_s^4\kappa - 6\varepsilon\kappa(\partial_s\kappa)^2 - 5\varepsilon\kappa^2\partial_s^2\kappa - \varepsilon\kappa^5\,. \tag{6.1.6}$$

PROOF. We have

$$\partial_t\kappa = \partial_t\langle\partial_s\tau, \nu\rangle = \langle\partial_t\partial_s\tau, \nu\rangle\,.$$

Therefore, using formula (6.1.5) we have

$$\partial_t\kappa = \langle\partial_s\partial_t\partial_s\gamma, \nu\rangle - \kappa\mathrm{E}^\varepsilon\langle\partial_s\tau, \nu\rangle = \langle\partial_s^2\partial_t\gamma, \nu\rangle - \langle\partial_s[\kappa\mathrm{E}^\varepsilon\partial_s\gamma], \nu\rangle - \kappa^2\mathrm{E}^\varepsilon\,.$$

Using the evolution law (6.1.2) we get

$$\langle\partial_s^2\partial_t\gamma, \nu\rangle = -\langle\partial_s^2(\mathrm{E}^\varepsilon\nu), \nu\rangle = -\partial_s^2\mathrm{E}^\varepsilon + \mathrm{E}^\varepsilon\langle\partial_s(\kappa\tau), \nu\rangle = -\partial_s^2\mathrm{E}^\varepsilon + \kappa^2\mathrm{E}^\varepsilon\,.$$

In addition,

$$\langle\partial_s[\kappa\mathrm{E}^\varepsilon\partial_s\gamma], \nu\rangle = \kappa\mathrm{E}^\varepsilon\langle\partial_s\tau, \nu\rangle = \kappa^2\mathrm{E}^\varepsilon\,.$$

Hence $\partial_t\kappa = -\partial_s^2\mathrm{E}^\varepsilon - \kappa^2\mathrm{E}^\varepsilon$ and the last equality in (6.1.6) follows by expanding E^ε. □

REMARK 6.1.5. For $\varepsilon = 0$, formula (6.1.6) gives the well known evolution equation $\kappa_t = \partial_s^2\kappa + \kappa^3$, valid for motion by curvature, see [**46**, Lemma 3.1.6].

By pushing a little the analysis in [**9**, Chap. 3, Sect. 7.6] and [**9**, Chap. 4] in the case of closed curves, we can get the following special form of interpolation inequalities. We underline that the "special" here refers to the fact that the influence of the geometry on the constants is explicit and it is given only by the length of the curve.

PROPOSITION 6.1.6. *Let γ be a regular closed curve in $\mathbb{R}^2$ with finite length* L. *Let u be a smooth function defined on γ, $m \geq 1$ and $p \in [2, +\infty]$. If $n \in \{0, \dots, m-1\}$ we have the estimates*

$$\|\partial_s^n u\|_{L^p} \leq C_{n,m,p} \|\partial_s^m u\|_{L^2}^{\sigma} \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m,p}}{\mathrm{L}^{m\sigma}} \|u\|_{L^2}\,, \tag{6.1.7}$$

where

$$\sigma = \frac{n + 1/2 - 1/p}{m} \in [0,1)$$

and the constants $C_{n,m,p}$ and $B_{n,m,p}$ are independent of γ.

Clearly inequalities (6.1.7) hold with uniform constants if applied to a family of curves having lengths uniformly bounded below by some positive value.
We underline that the "special" here refers

REMARK 6.1.7. In the special case $p = +\infty$, we have $\sigma = \frac{n+1/2}{m}$, and

$$\|\partial_s^n u\|_{L^\infty} \leq C_{n,m} \|\partial_s^m u\|_{L^2}^{\sigma} \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m}}{\mathrm{L}^{m\sigma}} \|u\|_{L^2}\,.$$

REMARK 6.1.8. In the particular case $n = 0$, $m = 2$, $p = 6$ we get $\sigma = 1/6$ and

$$\|u\|_{L^6} \leq C \|\partial_s^2 u\|_{L^2}^{\frac{1}{6}} \|u\|_{L^2}^{\frac{5}{6}} + \frac{C}{\mathrm{L}^{\frac{1}{3}}} \|u\|_{L^2}\,,$$

for some $C > 0$, hence, by means of Young inequality $|xy| \leq \frac{1}{a}|x|^a + \frac{1}{b}|y|^b$, $1/a + 1/b = 1$, choosing $a = b = 2$, $x = \sqrt{2}\|\partial_s^2 u\|_{L^2}$ and $y = \frac{C^6}{\sqrt{2}}\|u\|_{L^2}^5$, we obtain

$$\int_\gamma u^6\, ds \leq \int_\gamma (\partial_s^2 u)^2\, ds + C\left(\int_\gamma u^2\, ds\right)^5 + \frac{C}{\mathrm{L}^2}\left(\int_\gamma u^2\, ds\right)^3. \tag{6.1.8}$$

In the particular case $n = 0$, $m = 1$, $p = 4$ we get $\sigma = 1/4$ and

$$\|u\|_{L^4} \leq C \|\partial_s u\|_{L^2}^{\frac{1}{4}} \|u\|_{L^2}^{\frac{3}{4}} + \frac{C}{\mathrm{L}^{\frac{1}{4}}} \|u\|_{L^2}\,,$$

hence, reasoning as before,

$$\int_\gamma u^4\, ds \leq \int_\gamma (\partial_s u)^2\, ds + C\left(\int_\gamma u^2\, ds\right)^3 + \frac{C}{\mathrm{L}}\left(\int_\gamma u^2\, ds\right)^2. \tag{6.1.9}$$

We are now ready for the estimates. We recall that

$$\partial_t\, ds = \kappa \mathrm{E}^\varepsilon\, ds = (-\kappa^2 + 2\varepsilon\kappa\partial_s^2\kappa + \varepsilon\kappa^4)\, ds. \tag{6.1.10}$$

LEMMA 6.1.9. *We have*

$$\partial_t \int_\gamma \kappa^2\, ds = \int_\gamma \big(-2(\partial_s\kappa)^2 + \kappa^4 - 4\varepsilon(\partial_s^2\kappa)^2 - \varepsilon\kappa^6 - 4\varepsilon\kappa^3\partial_s^2\kappa\big)\, ds\,. \tag{6.1.11}$$

PROOF. From (6.1.10) and Lemma 6.1.4 we get

$$\begin{aligned}
\partial_t \int_\gamma \kappa^2\, ds =& \,2\int_\gamma \kappa\partial_t\kappa\, ds + \int_\gamma (-\kappa^4 + 2\varepsilon\kappa^3\partial_s^2\kappa + \varepsilon\kappa^6)\, ds \\
=& \,2\int_\gamma \kappa\big(\partial_s^2\kappa + \kappa^3 - 2\varepsilon\partial_s^4\kappa - 6\varepsilon\kappa(\partial_s\kappa)^2 - 5\varepsilon\kappa^2\partial_s^2\kappa - \varepsilon\kappa^5\big)\, ds \\
&+ \int_\gamma (-\kappa^4 + 2\varepsilon\kappa^3\partial_s^2\kappa + \varepsilon\kappa^6)\, ds \\
=& \int_\gamma \big(2\kappa\partial_s^2\kappa + \kappa^4 - 4\varepsilon\kappa\partial_s^4\kappa - 12\varepsilon\kappa^2(\partial_s\kappa)^2 - 8\varepsilon\kappa^3\partial_s^2\kappa - \varepsilon\kappa^6\big)\, ds.
\end{aligned}$$

Therefore, integrating by parts, we obtain

$$\begin{aligned}\partial_t \int_\gamma \kappa^2\,ds &= \int_\gamma \big(-2(\partial_s\kappa)^2+\kappa^4-4\varepsilon(\partial_s^2\kappa)^2-\varepsilon\kappa^6-12\varepsilon\kappa^2(\partial_s\kappa)^2-8\varepsilon\kappa^3\partial_s^2\kappa\big)\,ds\\ &= \int_\gamma \big(-2(\partial_s\kappa)^2+\kappa^4-4\varepsilon(\partial_s^2\kappa)^2-\varepsilon\kappa^6-4\varepsilon\kappa^3\partial_s^2\kappa\big)\,ds,\end{aligned}$$

where in the last equality we used the fact that $-3\int_\gamma \kappa^2(\partial_s\kappa)^2\,ds=\int_\gamma \kappa^3\partial_s^2\kappa\,ds$. ☐

PROPOSITION 6.1.10. *The following estimate holds*

$$\partial_t \int_\gamma \kappa^2\,ds \leq C\left(\int_\gamma \kappa^2\,ds\right)^3 + C\left(\int_\gamma \kappa^2\,ds\right)^5, \tag{6.1.12}$$

where C is a constant independent of ε.

PROOF. Adding to the right hand side of equation (6.1.11) the positive quantity $2\varepsilon(\partial_s^2\kappa+\kappa^3)^2$ we get

$$\partial_t \int_\gamma \kappa^2\,ds \leq \int_\gamma \big(-2(\partial_s\kappa)^2+\kappa^4-2\varepsilon(\partial_s^2\kappa)^2+\varepsilon\kappa^6\big)\,ds\,.$$

Using now inequalities (6.1.8) and (6.1.9) we obtain

$$\begin{aligned}\partial_t \int_\gamma \kappa^2 \leq& \int_\gamma \big(-(\partial_s\kappa)^2-\varepsilon(\partial_s^2\kappa)^2\big)\,ds + C\varepsilon\left(\int_\gamma \kappa^2\,ds\right)^5+\frac{C\varepsilon}{\mathrm{L}^2}\left(\int_\gamma \kappa^2\,ds\right)^3\\ &+C\left(\int_\gamma \kappa^2\,ds\right)^3+\frac{C}{\mathrm{L}}\left(\int_\gamma \kappa^2\,ds\right)^2\\ \leq& C\left(\int_\gamma \kappa^2\,ds\right)^5+C\left(\int_\gamma \kappa^2\,ds\right)^3+\frac{C}{\mathrm{L}}\left(\int_\gamma \kappa^2\,ds\right)^2+\frac{C}{\mathrm{L}^2}\left(\int_\gamma \kappa^2\,ds\right)^3\\ \leq& C\left(\int_\gamma \kappa^2\,ds\right)^5+C\left(\int_\gamma \kappa^2\,ds\right)^3,\end{aligned}$$

where we supposed $\varepsilon<1$ and in the last inequality we used the geometric estimate (6.1.4). ☐

We deal now with the higher derivatives of the curvature.
Since here we are working in dimension and codimension one, all polynomials in the curvature κ and its derivatives are completely contracted, that is, they belong to the "family" $\mathfrak{q}^r(\partial_s^l\kappa)$ (see Section 1.2); moreover, every of their monomials is of the form

$$\prod_{i=1}^N \partial_s^{j_i}\kappa \qquad \text{with } 0\leq j_i\leq l \text{ and } N\geq 1$$

with

$$r=\sum_{i=1}^N (j_i+1)\,,$$

as the $*$–product in this case is simply the usual product.

LEMMA 6.1.11. *For any $j\in\mathbb{N}$ the following formula holds:*

$$\partial_t\partial_s^j\kappa=\partial_s^{j+2}\kappa+\mathfrak{q}^{j+3}(\partial_s^j\kappa)-2\varepsilon\partial_s^{j+4}\kappa-5\varepsilon\kappa^2\partial_s^{j+2}\kappa+\varepsilon\mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa)\,. \tag{6.1.13}$$

PROOF. We argue by induction on j.
The case $j=0$ in (6.1.13) is equation (6.1.6), where $\mathfrak{q}^5(\partial_s\kappa)=-6\kappa(\partial_s\kappa)^2-\kappa^5$.
Suppose that (6.1.13) holds for $(j-1)$; using the commutation rule (6.1.5) we get

$$\begin{aligned}\partial_t\partial_s^j\kappa =& \partial_s\partial_t\partial_s^{j-1}\kappa+\kappa(\kappa-2\varepsilon\partial_s^2\kappa-\varepsilon\kappa^3)\partial_s^j\kappa\\ =& \partial_s\left[\partial_s^{j+1}\kappa+\mathfrak{q}^{j+2}(\partial_s^{j-1}\kappa)-2\varepsilon\partial_s^{j+3}\kappa-5\varepsilon\kappa^2\partial_s^{j+1}\kappa+\varepsilon\mathfrak{q}^{j+4}(\partial_s^j\kappa)\right]\\ &+\mathfrak{q}^{j+3}(\partial_s^j\kappa)+\varepsilon\mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa)\,,\end{aligned}$$

where we expressed $\mathfrak{q}^{j+3}(\partial_s^j\kappa)=\kappa^2\partial_s^j\kappa$ and $\mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa)=-(2\kappa\partial_s^2\kappa+\kappa^4)\partial_s^j\kappa$. Hence, we deduce

$$\partial_t\partial_s^j\kappa=\partial_s^{j+2}\kappa+\mathfrak{q}^{j+3}(\partial_s^j\kappa)-2\varepsilon\partial_s^{j+4}\kappa-5\varepsilon\kappa^2\partial_s^{j+2}\kappa+\varepsilon\mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa)\,,$$

which gives the inductive step. □

LEMMA 6.1.12. *For any $j\in\mathbb{N}$ we have*

$$\begin{aligned}\partial_t\int_\gamma|\partial_s^j\kappa|^2\,ds=&-2\int_\gamma|\partial_s^{j+1}\kappa|^2\,ds-4\varepsilon\int_\gamma|\partial_s^{j+2}\kappa|^2\,ds\\&+\int_\gamma\mathfrak{q}^{2j+4}(\partial_s^j\kappa)\,ds+\varepsilon\int_\gamma\mathfrak{q}^{2j+6}(\partial_s^{j+1}\kappa)\,ds\,.\end{aligned}$$

PROOF. Using (6.1.10), (6.1.13) and integrating by parts we deduce

$$\begin{aligned}\partial_t\int_\gamma|\partial_s^j\kappa|^2\,ds=&\,2\int_\gamma\partial_s^j\kappa\,\partial_t\partial_s^j\kappa\,ds+\int_\gamma|\partial_s^j\kappa|^2\kappa\mathrm{E}^\varepsilon\,ds\\=&\,2\int_\gamma\partial_s^j\kappa\left(\partial_s^{j+2}\kappa+\mathfrak{q}^{j+3}(\partial_s^j\kappa)\right)ds\\&+\varepsilon\int_\gamma2\partial_s^j\kappa\left(-2\partial_s^{j+4}\kappa-5\kappa^2\partial_s^{j+2}\kappa+\mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa)\right)ds\\&-\int_\gamma|\partial_s^j\kappa|^2\kappa(\kappa-2\varepsilon\partial_s^2\kappa-\varepsilon\kappa^3)\,ds\\=&-2\int_\gamma\left(|\partial_s^{j+1}\kappa|^2+\mathfrak{q}^{2j+4}(\partial_s^j\kappa)\right)\,ds\\&-4\varepsilon\int_\gamma\left(|\partial_s^{j+2}\kappa|^2+\mathfrak{q}^{2j+6}(\partial_s^{j+1}\kappa)\right)\,ds\,.\end{aligned}\tag{6.1.14}$$

□

PROPOSITION 6.1.13. *For any $j\in\mathbb{N}$ we have the ε–independent estimate, for $\varepsilon<1$,*

$$\partial_t\int_\gamma|\partial_s^j\kappa|^2\,ds\le C\left(\int_\gamma\kappa^2\,ds\right)^{2j+3}+C\left(\int_\gamma\kappa^2\,ds\right)^{2j+5}+C\tag{6.1.15}$$

where the constant C depends only on $1/\mathrm{L}$.

PROOF. We estimate the term $\int_\gamma\mathfrak{q}^{2j+4}(\partial_s^j\kappa)\,ds$ as in Section 5.1. By definition, we have

$$\mathfrak{q}^{2j+4}(\partial_s^j\kappa)=\sum_m\prod_{l=1}^{N_m}\partial_s^{c_{ml}}\kappa$$

with all the c_{ml} less than or equal to j and

$$\sum_{l=1}^{N_m}(c_{ml}+1)=2j+4$$

for every m. Hence,

$$|\mathfrak{q}^{2j+4}(\partial_s^j\kappa)|\le\sum_m\prod_{l=1}^{N_m}|\partial_s^{c_{ml}}\kappa|$$

and setting

$$Q_m=\prod_{l=1}^{N_m}|\partial_s^{c_{ml}}\kappa|\,,$$

we clearly obtain

$$\int_\gamma|\mathfrak{q}^{2j+4}(\partial_s^j\kappa)|\,ds\le\sum_m\int_\gamma Q_m\,ds\,.$$

We now estimate any term Q_m via interpolation inequalities. After collecting derivatives of the same order in Q_m we can write

$$Q_m = \prod_{i=0}^{j} |\partial_s^i \kappa|^{\alpha_{mi}} \qquad \text{with} \qquad \sum_{i=0}^{j} \alpha_{mi}(i+1) = 2j+4\,. \tag{6.1.16}$$

Then

$$\int_\gamma Q_m\,ds = \int_\gamma \prod_{i=0}^{j} |\partial_s^i \kappa|^{\alpha_{mi}}\,ds \leq \prod_{i=0}^{j} \left(\int_\gamma |\partial_s^i \kappa|^{\alpha_{mi}\lambda_i}\,ds \right)^{\frac{1}{\lambda_i}} = \prod_{i=0}^{j} \|\partial_s^i \kappa\|_{L^{\alpha_{mi}\lambda_i}}^{\alpha_{mi}}$$

where the values λ_i are chosen as follows: $\lambda_i = 0$ if $\alpha_{mi} = 0$ (in this case the corresponding term is not present in the product) and $\lambda_i = \frac{2j+4}{\alpha_{mi}(i+1)}$ if $\alpha_{mi} \neq 0$. Clearly, $\alpha_{mi}\lambda_i = \frac{2j+4}{i+1} \geq \frac{2j+4}{j+1} > 2$ and by the condition in (6.1.16), $\sum_{\substack{i=0\\ \lambda_i \neq 0}}^{j} \frac{1}{\lambda_i} = \sum_{\substack{i=0\\ \lambda_i \neq 0}}^{j} \frac{\alpha_{mi}(i+1)}{2j+4} = 1$.
As $\alpha_{mi}\lambda_i > 2$ these values are allowed as exponents p in inequality (6.1.7) and taking $m = j+1$, $n = i$, $u = \kappa$, we get

$$\|\partial_s^i \kappa\|_{L^{\alpha_{mi}\lambda_i}} \leq C \|\partial_s^{j+1}\kappa\|_{L^2}^{\sigma_{mi}} \|\kappa\|_{L^2}^{1-\sigma_{mi}} + \frac{C}{L^{(j+1)\sigma_{mi}}} \|\kappa\|_{L^2} \leq C \left(\|\partial_s^{j+1}\kappa\|_{L^2} + \|\kappa\|_{L^2} \right)^{\sigma_{mi}} \|\kappa\|_{L^2}^{1-\sigma_{mi}}$$

with

$$\sigma_{mi} = \frac{i + 1/2 - 1/(\alpha_{mi}\lambda_i)}{j+1}$$

and the constant C depends only on $1/\mathrm{L}$.

Multiplying together all the estimates,

$$\begin{aligned} \int_\gamma Q_m\,ds &\leq C \prod_{i=0}^{j} \left(\|\partial_s^{j+1}\kappa\|_{L^2} + \|\kappa\|_{L^2} \right)^{\alpha_{mi}\sigma_{mi}} \|\kappa\|_{L^2}^{\alpha_{mi}(1-\sigma_{mi})} \\ &\leq C \left(\|\partial_s^{j+1}\kappa\|_{L^2} + \|\kappa\|_{L^2} \right)^{\sum_{i=0}^{j} \alpha_{mi}\sigma_{mi}} \|\kappa\|_{L^2}^{\sum_{i=0}^{j} \alpha_{mi}(1-\sigma_{mi})}\,. \end{aligned} \tag{6.1.17}$$

Then we compute

$$\sum_{i=0}^{j} \alpha_{mi}\sigma_{mi} = \sum_{i=0}^{j} \frac{\alpha_{mi}(i+1/2) - 1/\lambda_i}{j+1} = \frac{\sum_{i=0}^{j} \alpha_{mi}(i+1/2) - 1}{j+1}$$

and using again the rescaling condition in (6.1.16),

$$\sum_{i=0}^{j} \alpha_{mi}\sigma_{mi} = \frac{4j+6-\sum_{i=0}^{j}\alpha_{mi}}{2(j+1)}\,.$$

Since

$$\sum_{i=0}^{j} \alpha_{mi} \geq \sum_{i=0}^{j} \alpha_{mi} \frac{i+1}{j+1} = \frac{2j+4}{j+1}$$

we get

$$\sum_{i=0}^{j} \alpha_{mi}\sigma_{mi} \leq \frac{2j^2+4j+1}{(j+1)^2} = 2 - \frac{1}{(j+1)^2} < 2\,.$$

Hence, we can apply Young inequality to the product in the last term of inequality (6.1.17), in order to get the exponent 2 on the first quantity, that is,

$$\int_\gamma Q_m\,ds \leq \frac{\delta_m}{2} \left(\|\partial_s^{j+1}\kappa\|_{L^2} + \|\kappa\|_{L^2} \right)^2 + C_m \|\kappa\|_{L^2}^{\beta} \leq \delta_m \int_\gamma |\partial_s^{j+1}\kappa|^2\,ds + \delta_m \int_\gamma \kappa^2\,ds + C_m \|\kappa\|_{L^2}^{\beta}\,,$$

for arbitrarily small $\delta_m > 0$ and some constant $C_m > 0$. The exponent β is given by

$$\begin{aligned}
\beta =& \sum_{i=0}^{j} \alpha_{mi}(1-\sigma_{mi}) \frac{1}{1 - \frac{\sum_{i=0}^{j} \alpha_{mi}\sigma_{mi}}{2}} \\
=& \frac{2\sum_{i=0}^{j} \alpha_{mi}(1-\sigma_{mi})}{2 - \sum_{i=0}^{j} \alpha_{mi}\sigma_{mi}} \\
=& \frac{2\sum_{i=0}^{j} \alpha_{mi} - \frac{4j+6-\sum_{i=0}^{j} \alpha_{mi}}{j+1}}{2 - \frac{4j+6-\sum_{i=0}^{j} \alpha_{mi}}{2(j+1)}} \\
=& 2\frac{2(j+1)\sum_{i=0}^{j} \alpha_{mi} - 4j - 6 + \sum_{i=0}^{j} \alpha_{mi}}{4j+4-4j-6+\sum_{i=0}^{j} \alpha_{mi}} \\
=& 2\frac{(2j+3)\sum_{i=0}^{j} \alpha_{mi} - 2(2j+3)}{\sum_{i=0}^{j} \alpha_{mi} - 2} \\
=& 2(2j+3)\,.
\end{aligned}$$

Therefore, we conclude

$$\int_\gamma Q_m\,ds \leq \delta_m \int_\gamma |\partial_s^{j+1}\kappa|^2\,ds + \delta_m \int_\gamma \kappa^2\,ds + C_m\left(\int_\gamma \kappa^2\,ds\right)^{2j+3}.$$

Repeating this argument for all the Q_m and choosing suitable δ_m whose sum over m is less than one, we conclude that there exists a constant C depending only on $1/\mathrm{L}$ and $j \in \mathbb{N}$ such that

$$\int_\gamma \mathfrak{q}^{2j+4}(\partial_s^j\kappa)\,ds \ \leq \int_\gamma |\partial_s^{j+1}\kappa|^2\,ds + C\left(\int_\gamma \kappa^2\right)^{2j+3} + C\,.$$

Reasoning similarly for the term $\mathfrak{q}^{2j+6}(\partial_s^{j+1}\kappa)$, we obtain

$$\int_\gamma \mathfrak{q}^{2j+6}(\partial_s^j\kappa)\,ds \leq \int_\gamma |\partial_s^{j+2}\kappa|^2\,ds + C\left(\int_\gamma \kappa^2\,ds\right)^{2j+5} + C\,.$$

Hence, from (6.1.14) we get

$$\begin{aligned}
\partial_t \int_\gamma |\partial_s^j\kappa|^2\,ds \leq& -\int_\gamma |\partial_s^{j+1}\kappa|^2\,ds - \varepsilon\int_\gamma |\partial_s^{j+2}\kappa|^2\,ds \\
&+ C\left(\int_\gamma \kappa^2\,ds\right)^{2j+3} + C\varepsilon\left(\int_\gamma \kappa^2\,ds\right)^{2j+5} + C \\
\leq& C\left(\int_\gamma \kappa^2\,ds\right)^{2j+3} + C\left(\int_\gamma \kappa^2\,ds\right)^{2j+5} + C
\end{aligned}$$

when $\varepsilon < 1$ and the constant C depends only on $1/\mathrm{L}$. □

By means of Propositions 6.1.10 and 6.1.13 we have then the following result.

THEOREM 6.1.14. *For any $j \in \mathbb{N}$ there exists a smooth function $\mathrm{Z}^j : \mathbb{R} \to (0, +\infty)$ such that*

$$\partial_t \int_\gamma |\partial_s^j\kappa|^2\,ds \leq \mathrm{Z}^j\left(\int_\gamma \kappa^2\,ds\right)$$

for every $\varepsilon < 1$ and curve γ evolving by the gradient of the functional $\mathcal{DG}_3^{\varepsilon/3}$.

PROOF. The statement clearly follows by Propositions 6.1.10 and 6.1.13, since by Lemma 6.1.2 the quantity $1/\mathrm{L}$ is controlled by $\int_\gamma \kappa^2\,ds$.
The smoothness of the functions Z^j is obtained choosing possibly slightly larger constants in inequalities (6.1.15) and (6.1.12). □

This proposition, like the analogous one for the general case, Theorem 6.2.3, is the key tool in order to get ε–independent compactness estimates. Indeed, for example, one can see that, by an ODE's argument, since all the flows (letting $0 < \varepsilon < 1$ vary) start from a common initial, closed, smooth curve, fixing any $j \in \mathbb{N}$, there exists a common positive interval of time such that all the quantities $\|\partial_s^i \kappa\|_{L^2}$, for $i \in \{0, \dots, j\}$ are equibounded. This will allow us to get compactness and C^∞ convergence to the mean curvature flow as $\varepsilon \to 0$.

6.2. The General Case

By the computations in Chapter 2 (in particular Corollary 2.3.5) and the discussion at the beginning of Chapter 3, we can write the evolution problem (6.0.1) as follows,

$$\frac{\partial \varphi^\varepsilon}{\partial t} = -\mathrm{E}^\varepsilon_m = \mathrm{H} + 2\varepsilon(m+2)(-1)^m \Big(\overbrace{\Delta^{M_t}\Delta^{M_t}\dots\Delta^{M_t}}^{m\text{–times}} H\Big)\,\nu + \varepsilon\mathfrak{q}^{2m+1}(\nabla^{2m-1}\mathrm{B})\nu\,,$$

where E^ε_m is the Euler equation of the functional $\mathcal{DG}^\varepsilon_m$.
Essentially repeating the computations of Section 5.1, we have the following evolution equations for the second fundamental form and its derivatives under the flow by the gradient of $\mathcal{DG}^\varepsilon_m$,

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^s\mathrm{B} = {}& 2\varepsilon(m+2)(-1)^m\nabla^{s+2}\,\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}\, H + \nabla^{s+2}H\\ &+ \varepsilon\mathfrak{p}^{2m+s+3}(\nabla^{2m+s+1}\mathrm{B}) + \mathfrak{p}^{s+3}(\nabla^{s+1}\mathrm{B})\,.\end{aligned}$$

We notice that every monomial of the terms $\mathfrak{p}^{2m+s+3}(\nabla^{2m+s+1}\mathrm{B})$ and $\mathfrak{p}^{s+3}(\nabla^{s+1}\mathrm{B})$ contains at least *two* factors, since for both, the difference between the *rescaling order* and the highest possible order of differentiation of B is two.

LEMMA 6.2.1. *Flowing by the gradient of the functional $\mathcal{DG}^\varepsilon_m$, for any $s \in \mathbb{N}$ we have*

$$\begin{aligned}\frac{\partial}{\partial t}\int_M |\nabla^s\mathrm{B}|^2\,d\mu = {}& -2\int_M |\nabla^{s+1}\mathrm{B}|^2\,d\mu + \int_M \mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})\,d\mu\\ &-4\varepsilon(m+2)\int_M |\nabla^{m+s+1}\mathrm{B}|^2\,d\mu + \varepsilon\int_M \mathfrak{q}^{2m+4+2s}(\nabla^{2m+s+1}\mathrm{B})\,d\mu\end{aligned}\tag{6.2.1}$$

and

$$\frac{\partial}{\partial t}\int_M |\mathrm{B}|^{2s+2}\,d\mu = \int_M \mathfrak{q}^{2s+4}(\nabla^2\mathrm{B})\,d\mu + \varepsilon\int_M \mathfrak{q}^{2m+4+2s}(\nabla^{2m+1}\mathrm{B})\,d\mu\,.\tag{6.2.2}$$

Moreover, every monomial of the $\mathfrak{q}$–terms in the two formulas contains at least three factors, for the $*$–product.

PROOF. We compute

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^s\mathrm{B}|^2 = {}& 4\varepsilon(m+2)(-1)^m\nabla^{s+2}_{i_1\dots i_s lw}\,\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}\,H\,\nabla^s_{j_1\dots j_s}\mathrm{B}_{pz}\,g^{i_1j_1}\dots g^{i_sj_s}g^{lp}g^{wz}\\ &+ 2\nabla^{s+2}_{i_1\dots i_s lw}H\,\nabla^s_{j_1\dots j_s}\mathrm{B}_{pz}\,g^{i_1j_1}\dots g^{i_sj_s}g^{lp}g^{wz}\\ &+ [\varepsilon\mathfrak{p}^{2m+s+3}(\nabla^{2m+s+1}\mathrm{B}) + \mathfrak{p}^{s+3}(\nabla^{s+1}\mathrm{B})] * \nabla^s\mathrm{B}\\ = {}& 4\varepsilon(m+2)(-1)^m\nabla^{s+2}_{i_1\dots i_s lw}\,\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}\,H\,\nabla^s_{j_1\dots j_s}\mathrm{B}_{pz}\,g^{i_1j_1}\dots g^{i_sj_s}g^{lp}g^{wz}\\ &+ 2\nabla^{s+2}_{i_1\dots i_s lw}H\,\nabla^s_{j_1\dots j_s}\mathrm{B}_{pz}\,g^{i_1j_1}\dots g^{i_sj_s}g^{lp}g^{wz}\\ &+ \varepsilon\mathfrak{q}^{2m+2s+4}(\nabla^{2m+s+1}\mathrm{B}) + \mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})\end{aligned}$$

and every monomial of the terms $\mathfrak{q}^{2m+2s+4}(\nabla^{2m+s+1}\mathrm{B})$ and $\mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})$ contains at least *three* factors, by the previous remark.

Thus, we have that the time derivative of the quantity $\int_M |\nabla^s \mathrm{B}|^2\,d\mu$ is given by

$$
\begin{aligned}
&4\varepsilon(m+2)(-1)^m\int_M \nabla^{s+2}_{i_1\dots i_s lw}\,\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}H\,\nabla^s_{j_1\dots j_s}\mathrm{B}_{pz}\,g^{i_1j_1}\dots g^{i_sj_s}g^{lp}g^{wz}\,d\mu\\
&+2\int_M \nabla^{s+2}_{i_1\dots i_s lw}H\,\nabla^s_{j_1\dots j_s}\mathrm{B}_{pz}\,g^{i_1j_1}\dots g^{i_sj_s}g^{lp}g^{wz}\,d\mu\\
&+\varepsilon\int_M \mathfrak{q}^{2m+2s+4}(\nabla^{2m+s+1}\mathrm{B})\,d\mu+\int_M \mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})\,d\mu\,,
\end{aligned}
$$

where we used that

$$\frac{\partial}{\partial t}d\mu=\langle \mathrm{H},\mathrm{E}^\varepsilon_m\rangle\,d\mu,$$

hence its contribution can be absorbed in the last two terms (notice that, in doing this, the condition of at least three factors in the monomials of $\mathfrak{q}^{2m+2s+4}(\nabla^{2m+s+1}\mathrm{B})$ and $\mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})$ is preserved).

Reasoning then as in Section 5.1 (integrating by parts and interchanging derivatives), we eventually obtain equation (6.2.1).
The final polynomials $\mathfrak{q}^{2m+2s+4}(\nabla^{2m+s+1}\mathrm{B})$ and $\mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})$ still have the three factors property, as every interchange of covariant derivatives in the previous process always introduces an extra lower order term, with one more factor of kind $\nabla^l\mathrm{B}$ (since the formula of interchange of covariant derivatives involves the Riemann tensor, that we express in terms of B), which is absorbed in $\mathfrak{q}^{2m+2s+4}(\nabla^{2m+s+1}\mathrm{B})$ and $\mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})$.
Equation (6.2.2) follows analogously. □

We set now, for any integer $s>n/2$ and $\varepsilon>0$,

$$Q^s_\varepsilon(t)=\int_M\big(1+|\nabla^s\mathrm{B}|^2+|\mathrm{B}|^{2s+2}\big)\,d\mu,\qquad t\in[0,+\infty).\tag{6.2.3}$$

Letting $\varepsilon>0$ vary, we want to study the evolution of Q^s_ε under the flows φ^ε associated with the functionals $\mathcal{DG}^\varepsilon_m$ (we recall that $m>[n/2]$).

By Lemma 6.2.1, we have

$$
\begin{aligned}
\frac{\partial Q^s_\varepsilon}{\partial t}=&-2\int_M|\nabla^{s+1}\mathrm{B}|^2\,d\mu-4\varepsilon(m+2)\int_M|\nabla^{m+s+1}\mathrm{B}|^2\,d\mu\\
&+\int_M\mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})\,d\mu+\varepsilon\int_M\mathfrak{q}^{2m+2s+4}(\nabla^{2m+s+1}\mathrm{B})\,d\mu\,.
\end{aligned}\tag{6.2.4}
$$

In order to deal with the polynomial terms we need the following easy consequence of Proposition 4.2.5.

LEMMA 6.2.2. *There exists a constant C depending only on n, l, z, j, p, q, r and $Q^{[n/2]+1}_\varepsilon$, such that for every compact, n–dimensional manifold (M,g), isometrically immersed in $\mathbb{R}^{n+1}$, and covariant tensor $T=T_{i_1\dots i_l}$, the following inequality holds*

$$\|\nabla^jT\|_{L^p(\mu)}\le C\,\|T\|^\sigma_{W^{z,q}(\mu)}\|T\|^{1-\sigma}_{L^r(\mu)}\,,\tag{6.2.5}$$

for all $z\in\mathbb{N}$, $j\in\{0,\dots,z\}$, $p,q,r\in[1,+\infty)$ and $\sigma\in[j/z,1]$ with the compatibility condition

$$\frac1p=\frac jn+\sigma\left(\frac1q-\frac zn\right)+\frac{1-\sigma}{r}\,.$$

If such a condition gives a negative value for p, the inequality holds in (6.2.5) for every $p\in[1,+\infty)$ on the left hand side.

PROOF. By Proposition 4.2.5, choosing any $\delta>0$ and setting $D=\mathrm{Vol}(M)+\|\mathrm{H}\|_{L^{n+\delta}(\mu)}$, the inequality holds for a constant C depending on n, l, z, j, p, q, r, δ and D. Hence, since by its definition

$$Q_\varepsilon^{[n/2]+1} \geq \int_M \left(1+|\mathrm{B}|^{n+4}\right)\,d\mu\,, \qquad \text{for } n \text{ even and}$$

$$Q_\varepsilon^{[n/2]+1} \geq \int_M \left(1+|\mathrm{B}|^{n+3}\right)\,d\mu\,, \qquad \text{for } n \text{ odd,}$$

the thesis follows. □

Working now as in Section 5.1, with $s > n/2$ fixed, we can interpolate the polynomial terms as follows,

$$\int_M \mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})\,d\mu \leq \int_M |\nabla^{s+1}\mathrm{B}|^2\,d\mu + C_1(Q_\varepsilon^{[n/2]+1})$$

$$\int_M \mathfrak{q}^{2m+2s+4}(\nabla^{2m+s+1}\mathrm{B})\,d\mu \leq 3(m+2)\int_M |\nabla^{m+s+1}\mathrm{B}|^2\,d\mu + C_2(Q_\varepsilon^{[n/2]+1})$$

where $C_1(Q_\varepsilon^{[n/2]+1})$ and $C_2(Q_\varepsilon^{[n/2]+1})$ are some constants depending only on n, m, s and $Q_\varepsilon^{[n/2]+1}$.

We discuss briefly a key point of such estimates.

By Lemma 6.2.1, we know that the every monomial of $\mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})$ and $\mathfrak{q}^{2m+2s+4}(\nabla^{2m+s+1}\mathrm{B})$ contains at least three factors for the $*$–product.

Then, if a monomial of $\mathfrak{q}^{2s+4}(\nabla^{s+1}\mathrm{B})$ contains a factor $\nabla^{s+1}\mathrm{B}$, all the other factors $\nabla^l\mathrm{B}$ must have $0 \leq l < s+1$, because every other factor (which are at least two) contributes with $l+1 \geq 1$ to the total sum $2s+4$, giving the rescaling order. Hence, since $\nabla^{s+1}\mathrm{B}$ can eventually occur only one time, we can "eliminate" it by means of Young inequality and interpolate.

Regarding the term $\mathfrak{q}^{2m+2s+4}(\nabla^{2m+s+1}\mathrm{B})$, by the same argument, if some monomial contains at least one occurrence of a derivative $\nabla^{m+s+1+j}\mathrm{B}$ for some integer $j \geq 0$, then, all the other factors $\nabla^l\mathrm{B}$ (which are at least two) must have $0 \leq l < m+s+1-j$. Then, integrating repeatedly by parts, we can "move" j derivatives from the factor $\nabla^{m+s+1+j}\mathrm{B}$ to the other factors, obtaining a polynomial $\mathfrak{q}^{2m+2s+4}(\nabla^{m+s+1}\mathrm{B})$ whose monomials can contain at most only one occurrence of the derivative $\nabla^{m+s+1}\mathrm{B}$. At this point, we conclude like for the other polynomial, with Young inequality and interpolation.

Hence, for every $s > n/2$, by (6.2.4) we have the estimate

$$\frac{\partial Q_\varepsilon^s}{\partial t} \leq -\int_M |\nabla^{s+1}\mathrm{B}|^2\,d\mu - \varepsilon(m+2)\int_M |\nabla^{m+s+1}\mathrm{B}|^2\,d\mu + C(Q_\varepsilon^{[n/2]+1})\,,$$

where $C = C_1 + \varepsilon C_2$ and the constants C_1, C_2 depend on ε only through $Q_\varepsilon^{[n/2]+1}$.

THEOREM 6.2.3. *For any integer $s > n/2$ there exists a smooth function $\mathrm{Z}^s : \mathbb{R} \to (0,+\infty)$ such that*

$$\partial_t \int_M \left(1+|\nabla^s\mathrm{B}|^2+|\mathrm{B}|^{2s+2}\right)\,d\mu \leq \mathrm{Z}^s\left(\int_M \left(1+|\nabla^{[n/2]+1}\mathrm{B}|^2+|\mathrm{B}|^{2[n/2]+4}\right)\,d\mu\right) \tag{6.2.6}$$

for every $\varepsilon \in (0,1)$ and any smooth evolution by the gradient of the functional $\mathcal{DG}_m^\varepsilon$.

PROOF. The functions Z^s can be clearly chosen to be smooth, possibly slightly enlarging the constants in the last inequality above. □

As a consequence we get the following proposition.

PROPOSITION 6.2.4. *In the same hypotheses of Theorem 6.2.3, there exists a continuous nonincreasing function $\Theta : (0,+\infty) \to (0,+\infty)$, independent of $\varepsilon \in (0,1)$, such that for every $T \in \mathbb{R}$ and $t \in [T, T+\Theta(Q_\varepsilon^{[n/2]+1}(T))]$ we have $Q_\varepsilon^{[n/2]+1}(t) \leq 2Q_\varepsilon^{[n/2]+1}(T)$.*

PROOF. The statement follows by a standard ODE's argument applied to the differential inequality

$$\partial_t \int_M \left(1+|\nabla^{[n/2]+1}\mathrm{B}|^2+|\mathrm{B}|^{2[n/2]+4}\right)\,d\mu \leq \mathrm{Z}^{[n/2]+1}\left(\int_M \left(1+|\nabla^{[n/2]+1}\mathrm{B}|^2+|\mathrm{B}|^{2[n/2]+4}\right)\,d\mu\right)$$

which is the first case of Theorem 6.2.3. □

In other words, this proposition says that (for ε small) we have a uniform control $Q_\varepsilon^{[n/2]+1}(t) \leq C$ in some time interval $[T, T+\Theta]$ (hence also a control the constants in Lemma 6.2.2 and on the right hand side of inequalities (6.2.6) for every $s > n/2$), with C and $\Theta > 0$ depending (smoothly) only on the value of $Q_\varepsilon^{[n/2]+1}$ at the starting time T.

6.3. Convergence to the Mean Curvature Flow

In this section we prove the convergence of solutions $\varphi^\varepsilon : M \times [0, +\infty) \to \mathbb{R}^{n+1}$ to (6.0.1) (all starting from a common immersion φ_0) to the mean curvature flow $\varphi : M \times [0, T_{\text{sing}}) \to \mathbb{R}^{n+1}$ before its first singularity time. Since we are considering $\varepsilon \to 0$, we can assume in all this section that $\varepsilon \in (0, 1)$.

Let $Q^s(t)$ denote, for each nonnegative time t before the first singularity, the right hand side of equation (6.2.3) for the mean curvature flow φ at time t.

LEMMA 6.3.1. *Let ε belong to some interval $(0, \varepsilon_0)$ with $\varepsilon_0 < 1$.*
If the family of immersions $\varphi^\varepsilon(\cdot, T) : M \to \mathbb{R}^{n+1}$ are bounded in the C^∞ topology, for any $s \in \mathbb{N}$ all the quantities $|\nabla^s \mathrm{B}|$ are uniformly bounded by ε–independent constants $C_s < +\infty$, in the time interval $[T, T+\theta]$, where $\theta = \Theta(\sup_{\varepsilon \in (0,\varepsilon_0)} Q_\varepsilon^{[n/2]+1}(T)) > 0$ and Θ is the function in Proposition 6.2.4.

PROOF. By the C^∞ boundedness of the family $\varphi^\varepsilon(\cdot, T) : M \to \mathbb{R}^{n+1}$, all the quantities $Q_\varepsilon^{[n/2]+1}(T)$ are equibounded, as $\varepsilon \in (0, \varepsilon_0)$. Since the function Θ is continuous and nonincreasing, setting $\theta = \Theta(\sup_{\varepsilon \in (0,\varepsilon_0)} Q_\varepsilon^{[n/2]+1}(T)) > 0$, by Proposition 6.2.4 there exists a constant $C > 0$ such that $Q_\varepsilon^{[n/2]+1}(t) \leq C$ for every $\varepsilon \in (0, \varepsilon_0)$ and $t \in [T, T+\theta]$.
Then, again by the boundedness of the family $\varphi^\varepsilon(\cdot, T)$ and Theorem 6.2.3, in the same time interval $[T, T+\theta]$ all the quantities

$$\int_M \left(1 + |\nabla^s \mathrm{B}|^2 + |\mathrm{B}|^{2s+2}\right) d\mu\,,$$

for every $s > n/2$, are bounded by ε–independent constants $C_s < +\infty$. Moreover, all the constants in the interpolation inequalities of Lemma 6.2.2 and Proposition 4.2.3 are also bounded.
Now, as a first step we see that, by means of Lemma 6.2.2, we get the following estimates, for every $p \in [2, +\infty)$ and $s > n/2$,

$$\int_M |\nabla^s \mathrm{B}|^p \, d\mu \leq C_{s,p}$$

in the same time interval $[T, T+\theta]$. Here again the constants $C_{s,p} < +\infty$ are ε–independent.
Then, we conclude the proof by means of Proposition 4.2.3. $\square$

LEMMA 6.3.2. *Assume that at time $t = T$ the family of maps $\varphi^\varepsilon(\cdot, T) : M \to \mathbb{R}^{n+1}$ converges as $\varepsilon \to 0$ in the C^∞ topology to the immersion $\varphi_T : M \to \mathbb{R}^{n+1}$. Then the maps φ^ε smoothly converge in the time interval $[T, T + \Theta(Q^{[n/2]+1}(T)))$ to the solution of the mean curvature flow starting from φ_T (here, $Q^{[n/2]+1}(T)$ is the quantity relative to the immersion φ_T).*

PROOF. Chosen any $\varepsilon_0 < 1$, by the previous lemma, we have uniform bounds on B and its derivatives in the time interval $[T, T+\theta]$ with $\theta = \Theta(\sup_{\varepsilon \in (0,\varepsilon_0)} Q_\varepsilon^{[n/2]+1}(T))$. Then, there exists $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$\left|\frac{\partial \varphi^\varepsilon(p,t)}{\partial t}\right| = |\mathrm{E}_m^\varepsilon(p,t)| < C \qquad \forall (p,t) \in M \times [T, T+\theta],\ \varepsilon \in (0, \varepsilon_0)\,.$$

Now we consider the metric tensors $g_{ij}^\varepsilon(p,t) = \left\langle \frac{\partial \varphi^\varepsilon(p,t)}{\partial x_i}, \frac{\partial \varphi^\varepsilon(p,t)}{\partial x_j} \right\rangle$, and fix a vector $V = \{v^i\} \in T_pM$. Then we have

$$\left|\frac{\partial}{\partial t} |V|^2_{g^\varepsilon(p,t)}\right| = \left|\partial_t g_{ij}^\varepsilon(p,t) v^i v^j\right| = 2\left|\langle \mathrm{E}_m^\varepsilon, \mathrm{B}_{ij}\rangle v^i v^j\right| \leq 2|\mathrm{E}_m^\varepsilon|\,|\mathrm{B}|_{g^\varepsilon(p,t)} |V|^2_{g^\varepsilon(p,t)} \leq C|V|^2_{g^\varepsilon(p,t)}$$

where C does not depend on $\varepsilon \in (0, \varepsilon_0)$.
Then a simple ODE's argument shows that the metrics g^ε_{ij} are all equivalent; more precisely, there exists a positive constant C depending only on φ_T such that

$$\frac{\mathrm{Id}}{C} \leq g^\varepsilon_{ij}(p,t) \leq C\mathrm{Id}\,, \tag{6.3.1}$$

as matrices.
Moreover, as functions, all the $g^\varepsilon_{ii} = \left|\frac{\partial\varphi^\varepsilon}{\partial x_i}\right|^2$ are equibounded above by a common constant.
Hence, by Ascoli–Arzelà's Theorem, up to a subsequence, the immersions φ^ε uniformly converge, as $\varepsilon \to 0$ to some Lipschitz map $\widehat{\varphi} : M \times [T, T+\theta] \to \mathbb{R}^{n+1}$, which clearly satisfies $\widehat{\varphi}(p,T) = \varphi_T(p)$ for every $p \in M$.

Similarly, as the time derivative of the Christoffel symbols is given by

$$\frac{\partial}{\partial t}\Gamma^l_{ij} = \nabla\mathrm{E}^\varepsilon_m * \mathrm{B} + \mathrm{E}^\varepsilon_m * \nabla\mathrm{B} \tag{6.3.2}$$

(see the beginning of Section 6.2) and all the metrics are equivalent, it follows that all the Christoffel symbols are equibounded. This means that estimating the covariant derivatives is equivalent to estimate the standard derivatives in coordinates, hence, we have immediately $|\partial^s\nabla^l\mathrm{B}| \leq C_{s,l}$ for every $s, l \in \mathbb{N}$.
Since

$$\frac{\partial}{\partial t}g^\varepsilon_{ij} = 2\langle \mathrm{E}^\varepsilon_m, \mathrm{B}_{ij}\rangle$$

we get

$$\left|\nabla^s\frac{\partial}{\partial t}g_{ij}\right| \leq C_s\,,$$

and, by formula (6.3.2),

$$\left|\nabla^s\frac{\partial}{\partial t}\Gamma^l_{ij}\right| \leq C_s\,.$$

for every $s \in \mathbb{N}$.
Hence, we get $\left|\partial^s\frac{\partial}{\partial t}\Gamma^l_{ij}\right| \leq C_s$ which implies, as the family of maps φ^ε_T is bounded in the C^∞–topology, that $|\partial^s\Gamma^l_{ij}| \leq C_s$.
Since we already know that $|\varphi^\varepsilon|$ are equibounded, $|\partial\varphi^\varepsilon| \leq C$ and $\partial^2\varphi^\varepsilon = \Gamma\partial\varphi^\varepsilon + \mathrm{B}$, by the estimates $|\partial^s\nabla^l\mathrm{B}| \leq C_{s,l}$, we can conclude that the derivatives $|\partial^s\varphi^\varepsilon|$ are all bounded by ε–independent constants C_s, for every $s \in \mathbb{N}$.
Finally, the uniform control on the time and mixed derivatives of φ^ε follows using the evolution equation.

Hence, the sub–convergence $\varphi^\varepsilon \to \widehat{\varphi}$, as $\varepsilon \to 0$, is in the C^∞ topology and $\widehat{\varphi}$ is smooth, moreover, the limit metric is positive definite by (6.3.1).
Passing to the limit in the evolution equation $\partial_t\varphi^\varepsilon = \mathrm{E}^\varepsilon_m$, by the bounds on B and its derivatives, shows that $\widehat{\varphi} : M \times [T, T+\theta] \to \mathbb{R}^{n+1}$ is the flow by mean curvature of the starting smooth datum φ_T. Since this flow is unique, *all* the sequence of maps φ^ε converges to $\widehat{\varphi}$ in the time interval $[T, T+\theta]$.

Chosen now any $\delta > 0$, we take $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon\in(0,\varepsilon_0)} Q^{[n/2]+1}_\varepsilon(T) - Q^{[n/2]+1}(T) < \delta\,,$$

this is clearly possible as $\varphi^\varepsilon(\cdot, T)$ converges smoothly to φ_T.
Since the function Θ is nonincreasing (see Lemma 6.3.1), then we conclude that for any $\delta > 0$ the sequence φ^ε converges to the mean curvature flow of φ_T in the time interval $[T, T + \Theta(Q^{[n/2]+1}(T) + \delta)]$.
Letting δ to zero, as the function Θ is continuous, we get the thesis. □

We are now in the position to prove Theorem 6.0.5.

PROOF OF THEOREM 6.0.5. Let $T_{\max}$ be the maximal time such that φ^ε converge to the solution of the mean curvature flow equation φ in $C^\infty(M \times [0, T_{\max}))$ starting at time $t = 0$ from the common immersion φ_0. Observe that $T_{\max}$ is positive by Lemma 6.3.2. We want to show that $T_{\max}$ coincides with the first singularity time T_{sing} for φ.

Assume by contradiction that $T_{\max} < T_{\text{sing}}$. Then $\varphi(\cdot, t) \to \varphi(\cdot, T_{\max})$ in $C^\infty(M)$ as $t \to T_{\max}$. As the function Θ is continuous, there exists

$$\lim_{t \to T_{\max}} \Theta(Q^{[n/2]+1}(t)) = \Theta(Q^{[n/2]+1}(T_{\max})) = \tau > 0\,.$$

Choosing now a time $T \in [T_{\max} - \tau/4, T_{\max})$ such that $\Theta(Q^{[n/2]+1}(T)) > \tau/2$, and applying Lemma 6.3.2, we see that $\varphi^\varepsilon(\cdot, t)$ converges to the mean curvature flow also for t in the interval $[T, T + \tau/2]$. As $T + \tau/2 \geq T_{\max} - \tau/4 + \tau/2 > T_{\max}$, we have a contradiction. $\square$

REMARK 6.3.3. As we discussed in Remark 5.2.4 the extension to higher codimension of the results of the previous chapters, all the material of this chapter also can be generalized (considering a suitable functional) leading to a full proof of the original conjecture of De Giorgi (see the remark immediately after Conjecture 6.0.2).

REMARK 6.3.4. We remark here that this method works in general for any geometric evolution of submanifolds in a Riemannian manifold till the first singularity time, even when the equations are of high order (like, for instance, in the Willmore flow, see **[66, 67, 93]**), choosing an appropriate regularizing term (of *higher order*).

APPENDIX A

Quasilinear Parabolic Equations on Manifolds

(In Collaboration with Luca Martinazzi)

Let (M, g) be a compact, smooth Riemannian manifold without boundary of dimension n and let $d\mu$ be the canonical measure associated to the metric tensor g.

We consider the parabolic problem with a smooth initial datum $u_0 : M \to \mathbb{R}$,

$$\begin{cases} u_t = Q[u] & \text{in } M \times [0,T] \\ u(\,\cdot\,,0) = u_0 & \text{on } M\,, \end{cases} \tag{A.3}$$

where Q is a smooth, quasilinear, locally elliptic operator of order $2p$, defined in $M \times [0, \mathcal{T})$ for some $\mathcal{T} > 0$ which, adopting (as in all the sequel) the Einstein convention of summing over repeated indices, can be expressed in local coordinates as

$$Q[u](x,t) = A^{i_1\dots i_{2p}}(x,t,u,\nabla u,\dots,\nabla^{2p-1}u)\nabla^{2p}_{i_1\dots i_{2p}}u(x,t) + b(x,t,u,\nabla u,\dots,\nabla^{2p-1}u)\,,$$

where A is a locally elliptic smooth $(2p, 0)$–tensor of the form

$$A^{i_1 j_1\dots i_p j_p} = (-1)^{p-1}E_1^{i_1 j_1}\cdots E_p^{i_p j_p} \tag{A.4}$$

for some $(2, 0)$–tensors $E_1, \dots, E_p$ and a function b smoothly depending on their arguments. Local ellipticity here means that for every $L > 0$ there exists a positive constant $\lambda \in \mathbb{R}$ such that each tensor E_ℓ satisfies

$$E_\ell^{ij}(x,t,u,\psi_1,\dots,\psi_{2p-1})\xi_i\xi_j \geq \lambda|\xi|^2_{g(x)}, \quad \text{for every } \xi \in T^*_x M\,, \tag{A.5}$$

when $x \in M$, $t \in [0,T]$ with $T < \mathcal{T}$, $u \in \mathbb{R}$ with $|u| \leq L$, $\psi_k \in \otimes^k T^*_x M$ with $|\psi_k|_{g(x)} \leq L$. In other words we require that condition (A.5) holds for some positive λ whenever the arguments of E_ℓ^{ij} lie in a compact set K of their natural domain of definition and assume that λ depends only on K. If $\lambda > 0$ can be chosen independent of K (i.e. of L), then we shall say that A is *uniformly* elliptic.

Clearly, this is not the most general notion of quasilinear parabolic problems, due to the special "product" structure of the operator, anyway it covers several important situations. For instance, our definition includes the case of standard locally parabolic equations of order two in non–divergence form. Notice that we make no growth assumptions on the tensor A and the function b.

Interchanging covariant derivatives, integrating by parts and using interpolation inequalities (see **[80]** for details), the following Gårding's inequality holds for this class of operators. For every smooth u and $t \in [0, \mathcal{T})$, we have

$$-\int_M \psi A^{i_1\dots i_{2p}}(u)\nabla^{2p}_{i_1\dots i_{2p}}\psi\, d\mu \geq \sigma\|\psi\|^2_{W^{p,2}(M)} - C\|\psi\|^2_{L^2(M)} \qquad \forall \psi \in C^\infty(M)\,, \tag{A.6}$$

where the constants $\sigma > 0$ and $C > 0$ depend continuously only on the C^p–norm of the tensor A and on the C^{3p-1}–norm of the function u at time t (and on the curvature tensor of (M, g) and its covariant derivatives). In particular, if u depends smoothly on time, $\sigma = \sigma(t)$ and $C = C(t)$ are continuous functions of time.

The aim of this note is to prove the following short time existence result.

THEOREM A.1. *For every $u_0 \in C^\infty(M)$ there exists a positive time $T > 0$ such that problem* (A.3) *has a smooth solution. Moreover, the solution is unique and depends continuously on u_0 in the C^∞–topology.*

Our interest in having a handy proof of this result is related to geometric evolution problems, like for instance the Ricci flow, the mean curvature flow, the Willmore flow [**67**], the Q–curvature flow [**73**], the Yamabe flow [**19, 84, 94**], etc. In all these problems, the very first step is to have a short time existence theorem showing that for an initial geometric structure (hypersurface, metric) the flow actually starts. Usually, after some manipulations in order to eliminate the degeneracies due to the geometric invariances, one has to face a quasilinear parabolic equation with smooth coefficients and smooth initial data.

If we replace the compact manifold M with a bounded domain $\Omega \subset \mathbb{R}^n$, short time existence for quasilinear systems of order two, with prescribed boundary conditions and initial data, was proven by Giaquinta and Modica [**48**] in the setting of Hölder spaces.

A different approach to Theorem A.1 was developed by Polden in his PhD Thesis [**80**] (see also [**57**]), by means of an existence result for linear equations in parabolic Sobolev spaces and the inverse function theorem. Unfortunately, as pointed out by Sharples [**85**], such procedure has a gap in the convergence of the solutions of the "frozen" linear problems to a solution of the quasilinear one.

In the same paper [**85**] Sharples, pushing further the estimates of Polden and allowing non-smooth coefficients, was able by means of an iteration scheme to show the existence of a short time solution of the quasilinear problem on a two–dimensional manifold, when the operator is of order two and *in divergence form*.

Our goal here is instead to simply fill the gap in Polden's proof. We start with his linear result and we show that his linearization procedure actually works if one linearizes at a suitably chosen function and discusses in details the above mentioned convergence.
As we do not assume any condition on the operator (only its product structure) and on the dimension of the manifold, we have a complete proof of the short time existence of a smooth solution to these quasilinear locally parabolic equations of arbitrary order on compact manifolds and of its uniqueness and smooth dependence on the initial data. We refer the interested reader to the nice and detailed introduction in [**85**] for the different approaches to the problem.

In the next section we present the linearization procedure, assuming Polden's linear result (Proposition A.3 below) and we prove Theorem A.1 by means of Lemma A.6 which is the core of our argument. Roughly speaking, when a candidate solution u stays in some parabolic Sobolev space of order high enough, the functions $u, \nabla u, \dots, \nabla^{2p-1}u$ are continuous (or even more regular), hence the same holds for the tensor A and the function b. This implies that the map $u \mapsto (u_t - Q[u])$ is of class C^1 between some suitable spaces, as it closely resembles a linear map with regular coefficients. This allows the application of the inverse function theorem which, in conjunction with an approximation argument, yields the existence of a solution. The last two sections are devoted to the proof of Lemma A.6 and to the discussion of the parabolic Sobolev embeddings on which such proof relies.

We mention that the results can be extended to quasilinear parabolic *systems* as the linearization procedure remains the same and Polden's linear estimates (Proposition A.3) can be actually easily generalized, assuming a suitable definition of ellipticity. In fact one easily sees that our result applies to all quasilinear systems whose linearization is invertible in the sense of Proposition A.4 below. For more general definition of elliptic or parabolic operators of higher–order see [**2**].

In the following the letter C will denote a constant which can change from a line to another and even within the same formula.

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Proof of the Main Theorem

We recall Polden's result for linear parabolic equations. Let us consider the problem

$$\begin{cases} u_t - A^{i_1\dots i_{2p}}\nabla^{2p}_{i_1\dots i_{2p}}u - \sum_{k=0}^{2p-1} R_k^{j_1\dots j_k}\nabla^k_{j_1\dots j_k}u = b \\ u(\,\cdot\,,0) = u_0\,, \end{cases} \tag{A.7}$$

where all the tensors A and R_k depend only on $(x,t) \in M \times [0,+\infty)$, are smooth and uniformly bounded with all their derivatives. Moreover, we assume that the tensor A has the product structure (A.4), where each E_ℓ is uniformly elliptic.

The Gårding's inequality for the linear operator

$$L(u) = A^{i_1\dots i_{2p}}\nabla^{2p}_{i_1\dots i_{2p}}u - \sum_{k=0}^{2p-1} R_k^{j_1\dots j_k}\nabla^k_{j_1\dots j_k}u$$

reads (see again **[80]** for details)

$$-\int_M \psi L(\psi)\,d\mu \geq \frac{\lambda}{2}\|\psi\|^2_{W^{p,2}(M)} - C\|\psi\|^2_{L^2(M)} \qquad \forall \psi \in C^\infty(M)\,, \tag{A.8}$$

where the constant $C > 0$ depends only on the C^p–norm of the tensors A and R_k. Clearly, by approximation this inequality holds also for every $\psi \in W^{2p,2}(M)$.

DEFINITION A.2. For any $m \in \mathbb{N}$ and $a \in \mathbb{R}^+$ we define $P^m_a(M)$ to be the completion of $C^\infty_c(M \times [0,+\infty))$ under the *parabolic norm*

$$\|f\|^2_{P^m_a(M)} = \sum_{j,\,k\,\in\,\mathbb{N} \text{ and } 2pj+k\,\leq\,2pm} \int_{M\times[0,+\infty)} e^{-2at}|\partial_t^j\nabla^k f|^2\,d\mu\,dt$$

and analogously $P^m(M,T)$ as the completion of $C^\infty(M \times [0,T])$ under the norm

$$\|f\|^2_{P^m(M,T)} = \sum_{j,\,k\,\in\,\mathbb{N} \text{ and } 2pj+k\,\leq\,2pm} \int_{M\times[0,T]} |\partial_t^j\nabla^k f|^2\,d\mu\,dt\,,$$

for every $T \in \mathbb{R}^+$.

Clearly for every $T \in \mathbb{R}^+$ there is a natural continuous embedding $P^m_a(M) \hookrightarrow P^m(M,T)$.

We have then the following global existence result for problem (A.7), by Polden **[80**, Thm. 2.3.5].

PROPOSITION A.3. *For every $m \in \mathbb{N}$ there exists $a \in \mathbb{R}^+$ large enough such that the linear map*

$$\Phi(u) = \Big(u_0, u_t - A^{i_1\dots i_{2p}}\nabla^{2p}_{i_1\dots i_{2p}}u - \sum_{k=0}^{2p-1} R_k^{j_1\dots j_k}\nabla^k_{j_1\dots j_k}u\Big) = (u_0, L(u))\,, \tag{A.9}$$

where $u_0 = u(\,\cdot\,,0)$, is an isomorphism of $P^m_a(M)$ onto $W^{p(2m-1),2}(M) \times P^{m-1}_a(M)$.

In the following it will be easier (though conceptually equivalent) to use the spaces $P^m(M,T)$ instead of the weighted spaces $P^m_a(M)$. For this reason we translate Proposition A.3 into the setting of $P^m(M,T)$ spaces.

PROPOSITION A.4. *For every $T > 0$ and $m \in \mathbb{N}$ the map Φ given by formula* (A.9) *is an isomorphism of $P^m(M,T)$ onto $W^{p(2m-1),2}(M) \times P^{m-1}(M,T)$.*

PROOF. The continuity of the second component of Φ is obvious while the continuity of the first component follows as in the Polden's proof of Proposition A.3 in **[80]**. Hence, the map Φ is continuous, now we show that it is an isomorphism.
Given any $b \in P^{m-1}(M,T)$ we consider an extension $\widetilde{b} \in P^{m-1}_a(M)$ of the function b and we let $\widetilde{u} \in P^m_a(M)$ be the solution of problem (A.7) for $\widetilde{b}$. Clearly, $u = \widetilde{u}|_{M\times[0,T]}$ belongs to $P^m(M,T)$

and satisfies $\Phi(u) = (u_0, b)$ in $M \times [0, T]$. Suppose that $v \in P^m(M,T)$ is another function such that $\Phi(v) = (u_0, b)$ in $M \times [0,T]$, then setting $w = u - v \in P^m(M,T)$ we have that

$$\begin{cases} w_t - A^{i_1\dots i_{2p}}\nabla^{2p}_{i_1\dots i_{2p}} w - \displaystyle\sum_{k=0}^{2p-1} R_k^{j_1\dots j_k}\nabla^k_{j_1\dots j_k} w = w_t - L(w) = 0 \\ w(\,\cdot\,, 0) = 0\,. \end{cases}$$

By the very definition of solution in $P^m(M,T)$ (see **[80]**) and Gårding's inequality (A.8), we get

$$\begin{aligned} \int_M w^2(x,t)\, d\mu(x) = & \int_0^t \int_M 2ww_t\, d\mu\, ds \\ =& \,2\int_0^t \int_M wL(w)\, d\mu\, ds \\ \le & -\frac{\lambda}{2}\int_0^t \int_M |\nabla^p w|^2\, d\mu\, ds + C\int_0^t \int_M w^2\, d\mu\, ds \\ \le & \,C\int_0^t \int_M w^2(x,s)\, d\mu(x)\, ds\,, \end{aligned}$$

as $w(\,\cdot\,, t) \in W^{2p,2}(M)$ for almost every $t \in [0,T]$ and where the constant $C > 0$ depends only on T as the coefficients of the operator L are smooth. Then, by Gronwall's lemma (in its integral version) it follows that $\int_M w^2(\,\cdot\,,t)\, d\mu$ is zero for every $t \in [0,T]$, as it is zero at time $t = 0$. It follows that w is zero on all $M \times [0,T]$, hence the two functions u and v must coincide.

Since the map $\Phi : P^m(M,T) \to W^{p(2m-1),2}(M) \times P^{m-1}(M,T)$ is continuous, one–to–one and onto, it is an isomorphism by the open mapping theorem. □

REMARK A.5. When u_0 and b are smooth the unique solution u of problem (A.7) belongs to all the spaces $P^m(M,T)$ for every $m \in \mathbb{N}$. As by Sobolev embeddings for any $k \in \mathbb{N}$ we can find a large $m \in \mathbb{N}$ so that $P^m(M,T)$ continuously embeds into $C^k(M \times [0,T])$, we can conclude that u actually belongs to $C^\infty(M \times [0,T])$.

Now we are ready to prove Theorem A.1. The tensor A and the function b from now on will depend on $x, t, u, \nabla u, \dots, \nabla^{2p-1}u$ as in the introduction. Since M is compact there exists a constant $C > 0$ such that the initial datum satisfies $|u_0| + |\nabla u_0|_g + \dots + |\nabla^{2p-1}u_0|_g \le C$. Then, since we are interested in existence for short time, possibly modifying the tensor A and the function b outside a compact set with some "cut–off" functions, we can assume that if $|u| + |\nabla u|_g + \dots + |\nabla^{2p-1}u|_g + t \ge 2C$, then

$$E^{ij}_\ell(x,t,u,\nabla u,\dots,\nabla^{2p-1}u) = g^{ij}(x), \quad \text{and} \quad b(x,t,u,\nabla u,\dots,\nabla^{2p-1}u) = 0\,.$$

In particular we can assume that the tensors E_ℓ are uniformly elliptic.

For a fixed $m \in \mathbb{N}$, we consider the map defined on $P^m(M,T)$ given by

$$\mathcal{F}(u) = (u_0, u_t - Q[u]) = \Big(u(\,\cdot\,,0), u_t - A(u)\cdot\nabla^{2p}u - b(u)\Big)\,,$$

where in order to simplify we used the notation

$$A(u)\cdot\nabla^{2p}v(x,t) = A^{i_1\dots i_{2p}}(x,t,u(x,t),\dots,\nabla^{2p-1}u(x,t))\nabla^{2p}_{i_1\dots i_{2p}}v(x,t)\,,$$

and

$$b(u)(x,t) = b(x,t,u(x,t),\dots,\nabla^{2p-1}u(x,t))$$

for $u, v \in P^m(M,T)$.

We have seen in Proposition A.4 that if $A(u)$ and $b(u)$ only depend on $x \in M$ and $t \in [0,T]$ (and not on u and its space derivatives), then $\mathcal{F}$ is a continuous map from $P^m(M)$ onto $W^{p(2m-1),2}(M)\times P^{m-1}(M)$. This is not the case in general when A and b depend on u and its derivatives, but it is true if $m \in \mathbb{N}$ is large enough and in this case $\mathcal{F}$ is actually C^1.

LEMMA A.6. *Assume that*

$$m > \frac{\dim M + 6p - 2}{4p} = \frac{n + 6p - 2}{4p}, \tag{A.10}$$

and $u \in P^m(M,T)$. *Then* $\mathcal{F}(u) \in W^{p(2m-1),2}(M) \times P^{m-1}(M,T)$ *and the map*

$$\mathcal{F} : P^m(M,T) \to W^{p(2m-1),2}(M) \times P^{m-1}(M,T)$$

is of class C^1.

We postpone the proof of this lemma to next section.

We fix $m \in \mathbb{N}$ such that the hypothesis of Lemma A.6 holds and we set

$$\widetilde{u}_0(x,t) = \sum_{\ell=0}^{m-1} \frac{a_\ell(x)\, t^\ell}{\ell!}$$

for some functions $a_0, \dots, a_{m-1} \in C^\infty(M)$ to be determined later. Let $w \in P^m(M,T)$ be the unique solution of the linear problem

$$\begin{cases} w_t = A(\widetilde{u}_0) \cdot \nabla^{2p} w + b(\widetilde{u}_0) \\ w(\,\cdot\,,0) = u_0\,. \end{cases}$$

Such solution exists by Proposition A.4 and it is smooth by Remark A.5, as u_0 and $\widetilde{u}_0$ are smooth (thus also $A(\widetilde{u}_0)$ and $b(\widetilde{u}_0)$).

Hence, we have

$$\mathcal{F}(w) = (u_0, w_t - Q[w]) = \Big(u_0, (A(\widetilde{u}_0) - A(w)) \cdot \nabla^{2p} w + b(\widetilde{u}_0) - b(w)\Big) =: (u_0, f)\,,$$

where we set $f = (A(\widetilde{u}_0) - A(w)) \cdot \nabla^{2p} w + b(\widetilde{u}_0) - b(w)$.

If we compute the differential $d\mathcal{F}_w$ of the map $\mathcal{F}$ at the "point" $w \in C^\infty(M \times [0,T])$, acting on $v \in P^m(M,T)$, we obtain

$$\begin{aligned} d\mathcal{F}_w(v) = \Big(v_0, v_t &- A^{i_1\dots i_{2p}}(w) \nabla^{2p}_{i_1\dots i_{2p}} v - D_w A^{i_1\dots i_{2p}}(w) v \nabla^{2p}_{i_1\dots i_{2p}} w \dots \\ &\dots - D_{w_{j_1\dots j_{2p-1}}} A^{i_1\dots i_{2p}}(w) \nabla^{2p-1}_{j_1\dots j_{2p-1}} v \nabla^{2p}_{i_1\dots i_{2p}} w \\ &- D_w b(w) v \dots - D_{w_{j_1\dots j_{2p-1}}} b(w) \nabla^{2p-1}_{j_1\dots j_{2p-1}} v\Big)\,, \end{aligned} \tag{A.11}$$

where $v_0 = v(\,\cdot\,,0)$ and we denoted by $D_{w_{j_1\dots j_k}} A^{i_1\dots i_{2p}}(w)$, $D_{w_{j_1\dots j_k}} b(w)$ the derivatives of the tensor A and of the function b with respect to their variables $\nabla^k_{j_1\dots j_k} w$, respectively.

Then, we can see that $d\mathcal{F}_w(v) = (z,h) \in W^{p(2m-1),2}(M) \times P^{m-1}(M,T)$ implies that v is a solution of the linear problem

$$\begin{cases} v_t - \widetilde{A}^{i_1\dots i_{2p}} \nabla^{2p}_{i_1\dots i_{2p}} v - \displaystyle\sum_{k=0}^{2p-1} \widetilde{R}_k^{j_1\dots j_k} \nabla^k_{j_1\dots j_k} v = h \\ v(\,\cdot\,,0) = z\,, \end{cases}$$

where

$$\begin{aligned} \widetilde{A}^{i_1\dots i_{2p}} &= A^{i_1\dots i_{2p}}(w)\,, \\ \widetilde{R}_k^{j_1\dots j_k} &= D_{w_{j_1\dots j_k}} A^{i_1\dots i_{2p}}(w) \nabla^{2p}_{i_1\dots i_{2p}} w + D_{w_{j_1\dots j_k}} b(w) \end{aligned}$$

are smooth tensors independent of v.

By Proposition A.4 for every $(z,h) \in W^{p(2m-1),2}(M) \times P^{m-1}(M,T)$ there exists a unique solution v of this problem, hence $d\mathcal{F}_w$ is a Hilbert space isomorphism and the inverse function theorem can be applied, as the map $\mathcal{F}$ is C^1 by Lemma A.6. Hence, the map $\mathcal{F}$ is a diffeomorphism of a neighborhood $U \subset P^m(M,T)$ of w onto a neighborhood $V \subset W^{p(2m-1),2}(M) \times P^{m-1}(M,T)$ of (u_0, f).

Getting back to the functions a_ℓ, we claim that we can choose them such that $a_\ell = \partial_t^\ell w|_{t=0} \in C^\infty(M)$ for every $\ell = 0, \dots, m-1$.

We apply the following recurrence procedure. We set $a_0 = u_0 \in C^\infty(M)$ and, assuming to have defined $a_0, \dots, a_\ell$, we consider the derivative

$$\partial_t^{\ell+1} w|_{t=0} = \partial_t^\ell [A^{i_1\dots i_{2p}}(x,t,\widetilde{u}_0, \nabla\widetilde{u}_0, \dots, \nabla^{2p-1}\widetilde{u}_0)\nabla^{2p}_{i_1\dots i_{2p}} w + b(x,t,\widetilde{u}_0, \nabla\widetilde{u}_0, \dots, \nabla^{2p-1}\widetilde{u}_0)]\Big|_{t=0}$$

and we see that the right–hand side contains time-derivatives at time $t = 0$ of $\widetilde{u}_0, \dots, \nabla^{2p-1}\widetilde{u}_0$ and $\nabla^{2p}_{i_1\dots i_{2p}} w$ only up to the order ℓ, hence it only depends on the functions $a_0, \dots, a_\ell$. Then, we define $a_{\ell+1}$ to be equal to such expression. Iterating up to $m-1$, the set of functions $a_0, \dots, a_{m-1}$ satisfies the claim.

Then, $a_\ell = \partial_t^\ell \widetilde{u}_0|_{t=0} = \partial_t^\ell w|_{t=0}$ and it easily follows by the "structure" of the function $f \in C^\infty(M \times [0,T])$, that we have $\partial_t^\ell f|_{t=0} = 0$ and $\nabla^j \partial_t^\ell f|_{t=0} = 0$ for any $0 \leq \ell \leq m-1$ and $j \in \mathbb{N}$.

We consider now for any $k \in \mathbb{N}$ the "translated" functions $f_k : M \times [0,T] \to \mathbb{R}$ given by

$$f_k(x,t) = \begin{cases} 0 & \text{if } t < 1/k \\ f(x, t-1/k) & \text{if } 1/k \leq t \leq T\,. \end{cases}$$

Since $f \in C^\infty(M \times [0,T])$ and $\nabla^j \partial_t^\ell f|_{t=0} = 0$ for every $0 \leq \ell \leq m-1$ and every $j \in \mathbb{N}$, all the functions $\nabla^j \partial_t^\ell f_k \in C^0(M \times [0,T])$ for every $0 \leq \ell \leq m-1$ and $j \geq 0$, it follows easily that

$$\nabla^j \partial_t^\ell f_k \to \nabla^j \partial_t^\ell f \quad \text{in } L^2(M \times [0,T]) \text{ for } 0 \leq \ell \leq m-1, j \geq 0\,,$$

hence $f_k \to f$ in $P^m(M,T)$.

Hence, there exists a function $\widetilde{f} \in P^{m-1}(M,T)$ such that $(u_0, \widetilde{f})$ belongs to the neighborhood V of $\mathcal{F}(w)$ and $\widetilde{f} = 0$ in $M \times [0,T']$ for some $T' \in (0,T]$. Since $\mathcal{F}|_U$ is a diffeomorphism between U and V, we can find a function $u \in U$ such that $\mathcal{F}(u) = (u_0, \widetilde{f})$. Clearly such $u \in P^m(M,T')$ is a solution of problem (A.3) in $M \times [0,T']$. Since $u \in P^m(M,T')$ implies that $\nabla^{2p-1}u \in C^0(M \times [0,T'])$, parabolic regularity implies that actually $u \in C^\infty(M \times [0,T'])$.

We now prove uniqueness by a standard energy estimate, which we include for completeness. In the sequel for simplicity we relabel T the time T' found above.

Suppose that we have two smooth solutions $u, v : M \times [0,T] \to \mathbb{R}$ of Problem (A.3). Setting $w := u - v$, we compute in an orthonormal frame

$$\begin{aligned}
\frac{d}{dt}\int_M |\nabla^p w|^2\,d\mu = &\int_M 2\nabla^p_{i_1\dots i_p} w \nabla^p_{i_1\dots i_p}\Big(A(u)\cdot\nabla^{2p}u - A(v)\cdot\nabla^{2p}v\Big)\,d\mu \\
&+ \int_M 2\nabla^p_{i_1\dots i_p} w \nabla^p_{i_1\dots i_p}\Big(b(u) - b(v)\Big)\,d\mu \\
= &\,2\int_M \nabla^p_{i_1\dots i_p} w \nabla^p_{i_1\dots i_p}\Big(A(u)\cdot\nabla^{2p}w\Big)\,d\mu \\
&+ 2\int_M \nabla^p_{i_1\dots i_p} w \nabla^p_{i_1\dots i_p}\Big((A(u)-A(v))\cdot\nabla^{2p}v\Big)\,d\mu \\
&+ 2(-1)^p\int_M \nabla^p_{i_1\cdots i_p}\nabla^p_{i_1\cdots i_p} w\Big(b(u)-b(v)\Big)\,d\mu \\
\leq &\,2\int_M \nabla^p_{i_1\dots i_p} w \nabla^p_{i_1\dots i_p}\Big(A(u)\cdot\nabla^{2p}w\Big)\,d\mu \\
&+ 2\int_M |\nabla^{2p}w|\Big(|A(u)-A(v)|\,|\nabla^{2p}v| + |b(u)-b(v)|\Big)\,d\mu\,,
\end{aligned}$$

where the integrals over M are intended at time $t \in [0,T]$.

Now we consider the integral $\int_M \nabla^p_{i_1\dots i_p} w \nabla^p_{i_1\dots i_p}(A^{j_1\cdots j_{2p}}(u)\nabla^{2p}_{j_1\dots j_{2p}} w)\,d\mu$. Expanding the derivative $\nabla^p_{i_1\dots i_p}(A^{j_1\cdots j_{2p}}(u)\nabla^{2p}_{j_1\dots j_{2p}} w)$ we will get one special term $A^{j_1\cdots j_{2p}}(u)\nabla^{3p}_{i_1\dots i_p j_1\dots j_{2p}} w$ and several other terms of the form $B(x,t,u,\dots,\nabla^{3p-1}u)\#\nabla^q w$ with $2p \leq q < 3p$, for some tensor B smoothly depending on its arguments, where the symbol $\#$ means metric contraction on some indices. For each of these terms, integrating repeatedly by parts, we can write

$$\int_M \nabla^p w \# B(x,t,u,\dots,\nabla^{3p-1}u)\#\nabla^q w\,d\mu = \sum_{\ell=p}^{2p}\int_M \nabla^\ell w\# D_\ell(x,t,u,\dots,\nabla^{4p-1}u)\#\nabla^{q-p}w\,d\mu$$

where the tensors D_ℓ are smoothly depending on their arguments.
Since $u\in C^\infty(M\times[0,T])$, all the tensors D_ℓ are bounded, hence we can estimate

$$\begin{aligned}\int_M \nabla^p_{i_1\dots i_p} w\nabla^p_{i_1\dots i_p}(A(u)\cdot\nabla^{2p}w)\,d\mu \leq{}& \int_M \nabla^p_{i_1\dots i_p} w A^{j_1\dots j_{2p}}(u)\nabla^{3p}_{i_1\dots i_p j_1\dots j_{2p}} w\,d\mu\\ &+ C\sum_{r=p}^{2p-1}\sum_{\ell=p}^{2p}\int_M |\nabla^\ell w|\,|\nabla^r w|\,d\mu\,.\end{aligned}$$

where C is a constant independent of time (actually C depends only on the structure of A). Interchanging the covariant derivatives we have

$$\nabla^{3p}_{i_1\dots i_p j_1\dots j_{2p}} w = \nabla^{3p}_{j_1\dots i_{2p} i_1\dots i_p} w + \sum_{q=0}^{3p-1} R_q\#\nabla^q w$$

where the tensors R_q are functions of the Riemann tensor and its covariant derivatives, hence they are smooth and bounded. We can clearly deal with this sum of terms as above, by means of integrations by parts, obtaining the same result. Then we conclude, also using Gårding's inequality (A.6)

$$\begin{aligned}\int_M \nabla^p_{i_1\dots i_p} w\nabla^p_{i_1\dots i_p}(A(u)\cdot\nabla^{2p}w)\,d\mu \leq{}& \int_M \nabla^p_{i_1\dots i_p} w A^{j_1\dots j_{2p}}(u)\nabla^{2p}_{j_1\dots j_{2p}}\nabla^p_{i_1\dots i_p} w\,d\mu\\ &+ C\sum_{r=p}^{2p-1}\sum_{\ell=p}^{2p}\int_M |\nabla^\ell w|\,|\nabla^r w|\,d\mu\\ \leq{}& -\alpha\int_M|\nabla^{2p}w|^2\,d\mu + C\sum_{r=p}^{2p-1}\sum_{\ell=p}^{2p}\int_M |\nabla^\ell w|\,|\nabla^r w|\,d\mu\,,\end{aligned}$$

for some positive constant α. Getting back to the initial computation and using Peter–Paul inequality we get

$$\begin{aligned}\frac{d}{dt}\int_M|\nabla^p w|^2\,d\mu \leq{}& -2\alpha\int_M|\nabla^{2p}w|^2\,d\mu + C\sum_{r=p}^{2p-1}\sum_{\ell=p}^{2p}\int_M |\nabla^\ell w|\,|\nabla^r w|\,d\mu\\ &+ C\int_M|\nabla^{2p}w|\Big(|A(u)-A(v)|\,|\nabla^{2p}v| + |b(u)-b(v)|\Big)\,d\mu\\ \leq{}& -2\alpha\int_M|\nabla^{2p}w|^2\,d\mu + C\sum_{r=p}^{2p-1}\sum_{\ell=p}^{2p-1}\int_M |\nabla^\ell w||\nabla^r w|\,d\mu\\ &+\sum_{r=0}^{2p-1}\Big(\varepsilon_r\int_M|\nabla^{2p}w|^2\,d\mu + C_{\varepsilon_r}\int_M|\nabla^r w|^2 d\mu\Big)\\ &+\delta\int_M|\nabla^{2p}w|^2\,d\mu + C_\delta\int_M\Big(|A(u)-A(v)|^2+|b(u)-b(v)|^2\Big)\,d\mu\\ \leq{}& -\alpha\int_M|\nabla^{2p}w|^2\,d\mu + C\sum_{r=0}^{2p-1}\int_M|\nabla^r w|^2\,d\mu\\ &+ C_\delta\int_M\Big(|A(u)-A(v)|^2+|b(u)-b(v)|^2\Big)\,d\mu\,,\end{aligned}$$

where we chose $\delta + \sum_{r=0}^{2p-1} \varepsilon_r = \alpha$ and we used the fact that $|\nabla^{2p} v|$ is bounded.
As the tensor A and the function b are smooth, we can easily bound

$$|A(u) - A(v)|^2 + |b(u) - b(v)^2| \leq C \sum_{r=0}^{2p-1} |\nabla^r u - \nabla^r v|^2 = C \sum_{r=0}^{2p-1} |\nabla^r w|^2 \,,$$

so finally

$$\frac{d}{dt} \int_M |\nabla^p w|^2 \, d\mu \leq -\alpha \int_M |\nabla^{2p} w|^2 \, d\mu + C \sum_{r=0}^{2p-1} \int_M |\nabla^r w|^2 \, d\mu \,.$$

Now we have, using again Gårding's and Peter–Paul inequalities,

$$\begin{aligned}
\frac{d}{dt} \int_M w^2 \, d\mu =& \, 2 \int_M w \Big(A(u) \cdot \nabla^{2p} u - A(v) \cdot \nabla^{2p} v \Big) \, d\mu + 2 \int_M w \Big(b(u) - b(v)) \Big) \, d\mu \\
=& \, 2 \int_M w A(u) \cdot \nabla^{2p} w \, d\mu + 2 \int_M w \Big((A(u) - A(v)) \cdot \nabla^{2p} v + b(u) - b(v) \Big) \, d\mu \\
\leq& - \beta \int_M |\nabla^p w|^2 \, d\mu + C \int_M w^2 \, d\mu + C \int_M w (A(u) - A(v) + b(u) - b(v)) \, d\mu \\
\leq& - \beta \int_M |\nabla^p w|^2 \, d\mu + C \int_M w^2 \, d\mu + C \int_M \Big(|A(u) - A(v)|^2 + |b(u) - b(v)|^2 \Big) \, d\mu \,.
\end{aligned}$$

Estimating the last integral as before and putting the two computation together we obtain

$$\frac{d}{dt} \int_M \Big(|\nabla^p w|^2 + w^2 \Big) \, d\mu \leq - \frac{\alpha}{2} \int_M |\nabla^{2p} w|^2 \, d\mu + C \sum_{r=0}^{2p-1} \int_M |\nabla^r w|^2 \, d\mu \,.$$

In order to deal with the last term, we apply the following Gagliardo–Nirenberg interpolation inequalities (see [**9**, Prop. 2.11] and [**1**, Thm. 4.14]): for every $0 \leq r < 2p$ and $\varepsilon > 0$ there exists a constant C_ε such that

$$\|\nabla^r f\|^2_{L^2(M)} \leq \varepsilon \|\nabla^{2p} f\|^2_{L^2(M)} + C_\varepsilon \|f\|^2_{L^2(M)}$$

for every function $f \in W^{2p,2}(M)$.
Hence, for some $\varepsilon > 0$ small enough we get,

$$\begin{aligned}
\frac{d}{dt} \int_M \Big(|\nabla^p w|^2 + w^2 \Big) \, d\mu \leq& - \frac{\alpha}{4} \int_M |\nabla^{2p} w|^2 \, d\mu + C \sum_{r=0}^{2p-1} \varepsilon \int_M |\nabla^{2p} w|^2 \, d\mu + C \sum_{r=0}^{2p-1} C_\varepsilon \int_M w^2 \, d\mu \\
\leq& \, C \int_M w^2 \, d\mu \,.
\end{aligned}$$

From this ordinary differential inequality and Gronwall's lemma, it follows that if the quantity $\int_M (|\nabla^p w|^2 + w^2) \, d\mu$ is zero at some time t_0, then it must be zero for every time $t \in [t_0, T]$. Since at $t = 0$ we have $w(\,\cdot\,, 0) = u_0 - v_0 = 0$, we are done.

We now prove the continuous dependence of a solution $u \in C^\infty(M \times [0,T])$ on its initial datum $u_0 = u(\,\cdot\,, 0) \in C^\infty(M)$. Fix any $m \in \mathbb{N}$ satisfying condition (A.10), so that by the Sobolev embeddings $u \in P^m(M,T)$ implies $\nabla^{2p-1} u \in C^0(M \times [0,T])$. By the above argument, $u = (\mathcal{F}|_U)^{-1}(u_0, 0) \in P^m(M,T)$ where $\mathcal{F}|_U$ is a diffeomorphism of an open set $U \subset P^m(M,T)$ onto $V \subset W^{p(2m-1),2}(M) \times P^{m-1}(M,T)$, with $(u_0, 0) \in V$. Then, assuming that $u_{k,0} \to u_0$ in $C^\infty(M)$ as $k \to \infty$, we also have $u_{k,0} \to u_0$ in $W^{p(2m-1),2}(M)$, hence for k large enough $(u_{k,0}, 0) \in V$ and there exists $u_k \in U$ such that $\mathcal{F}(u_k) = (u_{k,0}, 0)$. This is the unique solution in $P^m(M,T)$ (hence in $C^\infty(M \times [0,T])$ by parabolic bootstrap) with initial datum $u_{k,0}$. Moreover, since $\mathcal{F}|_U$ is a diffeomorphism, we have $u_k \to u$ in $P^m(M,T)$.
By uniqueness, we can repeat the same procedure for any $m \in \mathbb{N}$ satisfying condition (A.10) concluding that $u_k \to u$ in $P^m(M,T)$ for every such $m \in \mathbb{N}$, hence in $C^\infty(M \times [0,T])$.

Proof of Lemma A.6

We shall write $P^m = P^m(M,T)$, $L^q = L^q(M\times[0,T])$, $C^0 = C^0(M\times[0,T])$ etc..., so that for instance $C^0(P^m;C^1)$ will denote the space of continuous maps from $P^m(M,T)$ to $C^1(M\times[0,T])$. The first component of $\mathcal{F}$, i.e. the map $u\mapsto u(\,\cdot\,,0)$ is linear and bounded from P^m to $W^{p(2m-1),2}(M)$, by Proposition A.4, therefore it is C^1. Obviously the map $u\mapsto\partial_t u$ is linear and bounded from P^m to P^{m-1}, hence also C^1. Thus, it remains to show that the two maps

$$\mathcal{F}_A(u) := A(u)\cdot\nabla^{2p}u\,,\quad \mathcal{F}_b(u) := b(u)$$

belong to $C^1(P^m;P^{m-1})$.

We first prove that $\mathcal{F}_A,\mathcal{F}_b\in C^0(P^m;P^{m-1})$. By an induction argument, it is easy to see that for every $k\in\mathbb{N}$

$$\nabla^k\big(A(u)\cdot\nabla^{2p}u\big) = \sum_{j=0}^{k}\sum_{\substack{i_1,i_2,\dots,i_{j+1}\geq 1\\ i_1+\dots+i_{j+1}\leq k+2p+(2p-1)j}}\partial^j A(u)\#\nabla^{i_1}u\#\dots\#\nabla^{i_{j+1}}u\,,\tag{A.12}$$

where $\partial^j A(u)$ denotes the j–th derivative of A with respect to any of its arguments and $D\#E$ denotes an arbitrary contraction with the metric of two tensors D and E.

Taking into account formula (A.12) with $k\leq 2p(m-1)$, in order to prove that the map $u\mapsto\nabla^{2p(m-1)}(A(u)\cdot\nabla^{2p}u)$ belongs to $C^0(P^m;L^2)$ we have to show that any map of the form

$$u\mapsto\partial^j A(u)\#\nabla^{i_1}u\#\cdots\#\nabla^{i_{j+1}}u\tag{A.13}$$

belongs to $C^0(P^m;L^2)$ whenever

$$i_1+\dots+i_{j+1}\leq 2pm+(2p-1)j\qquad\text{and}\qquad i_1,\dots,i_{j+1}\geq 1\,.\tag{A.14}$$

The case $r=0$ and $\ell=2p-1$ of the Sobolev embeddings (A.24) below and condition (A.10) imply that if $u\in P^m$ then $\nabla^{2p-1}u\in C^0$ (and the immersion is bounded), hence all the maps $u\mapsto\partial^j A(u)$ belong to $C^0(P^m;C^0)$.
We can assume from now on that $j\geq 1$, since in the case $j=0$, we get the term $A(u)\#\nabla^{2p+k}u$ which is continuous from P^m to L^2 as a function of u for $k\leq 2p(m-1)$.
As for the factors $\nabla^{i_\ell}u$ appearing in formula (A.13), first we assume that each i_ℓ is such that we are in case (A.22) of Sobolev embeddings, i.e.

$$\frac{1}{q_\ell} := \frac{1}{2}-\frac{2pm-i_\ell}{n+2p}>0,\tag{A.15}$$

so that the map $u\mapsto\nabla^{i_\ell}u$ lies in $C^0(P^m;L^{q_\ell})$. By Hölder's inequality, the condition

$$\frac{1}{q} := \sum_{\ell=1}^{j+1}\frac{1}{q_\ell} = \sum_{\ell=1}^{j+1}\left(\frac{1}{2}-\frac{2pm-i_\ell}{n+2p}\right)\leq\frac{1}{2}\,,\tag{A.16}$$

implies that the map $u\mapsto\nabla^{i_1}u\#\cdots\#\nabla^{i_{j+1}}u$ belongs to $C^0(P^m;L^q)$, hence also to $C^0(P^m;L^2)$, as L^q embeds continuously into L^2 for $q\geq 2$. Then, if we show inequality (A.16), the map defined by formula (A.13) belongs to $C^0(P^m;L^2)$. From inequalities (A.10), (A.14) and $j\geq 1$ it follows,

$$\sum_{\ell=1}^{j+1}\frac{1}{q_\ell}\leq\frac{j+1}{2}-\frac{2pm(j+1)-2pm-(2p-1)j}{n+2p}=\frac{1}{2}+\frac{j}{2}-\frac{(2pm-2p+1)j}{n+2p}<\frac{1}{2}\,.\tag{A.17}$$

Now, if for some i_ℓ, say $i_1,\dots,i_s$, we have $\frac{2pm-i_\ell}{n+2p}>\frac{1}{2}$, then we are in case (A.24) of Sobolev embeddings and the corresponding maps $u\mapsto\nabla^{i_\ell}u$ belong to $C^0(P^m;C^0)$, hence we can avoid to estimate such factors, as for $A(u)$. Then, since (A.15) holds for $\ell\in\{s+1,\dots,j+1\}$, arguing again by induction, in this case we have to deal with functions $u\mapsto\nabla^{i_{s+1}}u\#\cdots\#\nabla^{i_{j+1}}u$ under the conditions

$$i_{s+1}+\dots+i_{j+1}\leq 2pm+(2p-1)(j-s)\qquad\text{and}\qquad i_{s+1},\dots i_{j+1}\geq 1\,.$$

Then, computing as in inequality (A.17) one shows

$$\begin{aligned}\sum_{\ell=s+1}^{j+1}\frac{1}{q_\ell} &\leq \frac{j+1-s}{2}-\frac{2pm(j+1-s)-2pm-(2p-1)(j-s)}{n+2p} \\ &= \frac{1}{2}+\frac{j-s}{2}-\frac{(2pm-2p+1)(j-s)}{n+2p} \\ &\leq \frac{1}{2}\,,\end{aligned} \tag{A.18}$$

where we intend that if $s=j+1$ there is nothing to sum. Notice that the last inequality is strict if $s\neq j$, and in the case $s=j$ the map $u\mapsto\nabla^{i_{j+1}}u$ is continuous from P^m to L^2 as $i_{j+1}\leq 2pm$.

If in addition for some i_ℓ, say $i_{s+1},\dots,i_r$, we have $\frac{2pm-i_\ell}{n+2p}=\frac{1}{2}$ (i.e. we are in the critical case (A.23) of the Sobolev embeddings), we know that for such indices the maps $u\mapsto\nabla^{i_\ell}u$ belong to $C^0(P^m;L^q)$ for every $1\leq q<\infty$. Then inequality (A.18) still holds true if we choose $q_{s+1},\dots,q_r$ large enough, since, unless $s=r=j$, the last inequality in (A.18) is strict.
Hence, we conclude as before that the map $u\mapsto\nabla^{2p(m-1)}(A(u)\cdot\nabla^{2p}u)$ lies in $C^0(P^m;L^2)$.

The time or mixed space-time derivatives $\partial_t^r\nabla^k(A(u)\cdot\nabla^{2p}u)$ with $2pr+k\leq 2p(m-1)$ can be treated in a similar way, observing that the functions $\partial_t^r\nabla^\ell u$ have the same integrability of $\nabla^{2pr+\ell}u$ from the point of view of the embeddings (A.22)–(A.24).
Starting from formula (A.12) and differentiating in time, again by an induction argument, one gets

$$\partial_t^r\nabla^k\big(A(u)\cdot\nabla^{2p}u\big)=\sum_{j=0}^{r+k}\sum_{\substack{i_1,\dots,i_{j+1},\iota_1,\dots,\iota_{j+1}\geq 0\\ i_1+\dots+i_{j+1}\leq k+2p+(2p-1)j\\ \iota_1+\dots+\iota_{j+1}\leq r}}\partial^jA(u)\#\partial_t^{\iota_1}\nabla^{i_1}u\#\cdots\#\partial_t^{\iota_{j+1}}\nabla^{i_{j+1}}u\,. \tag{A.19}$$

Then, with the same proof as before one shows that a map of the form

$$u\mapsto\partial^jA(u)\#\partial_t^{\iota_1}\nabla^{i_1}u\#\cdots\#\partial_t^{\iota_{j+1}}\nabla^{i_{j+1}}u$$

belongs to $C^0(P^{m+1};L^2)$ whenever $i_1,\dots,i_{j+1},\iota_1,\dots,\iota_{j+1}\geq 0$ and

$$i_1+\dots+i_{j+1}+2p(\iota_1+\dots+\iota_{j+1})\leq 2pm+(2p-1)j\,.$$

Hence the map $u\mapsto\partial_t^r\nabla^k\big(A(u)\cdot\nabla^{2p}u\big)$ belongs to $C^0(P^m;L^2)$ for $2pr+k\leq 2p(m-1)$, which means that $\mathcal{F}_A\in C^0(P^m;P^{m-1})$ as wished.

The map $\mathcal{F}_b$ can be treated in a similar way, so also $\mathcal{F}_b\in C^0(P^m;P^{m-1})$.

It remains to prove that $d\mathcal{F}_A,d\mathcal{F}_b\in C^0(P^m;L(P^m;P^{m-1}))$, where $L(P^m;P^{m-1})$ denotes the Banach space of bounded linear maps from P^m into P^{m-1}. We first claim that the Gateaux derivative

$$(u,v)\mapsto d\mathcal{F}_A(u)(v):=\frac{d}{dt}\mathcal{F}_A(u+tv)\Big|_{t=0}$$

belongs to $C^0(P^m\times P^m;P^{m-1})$. Indeed, $d\mathcal{F}_A(u)(v)$ can be written as

$$B(u,v)\#\nabla^{2p}u+A(u)\cdot\nabla^{2p}v\,,$$

where B is a tensor depending smoothly on $x,t,u,\dots,\nabla^{2p-1}u$ and linearly on some derivative of v up to the order $2p-1$, that is, $B(u,v)=\sum_{\ell=0}^{2p-1}B_\ell(u)\cdot\nabla^\ell v$, compare with formula (A.11). The estimates proven for $\mathcal{F}_A$ can be applied to any term of the form $\partial_t^r\nabla^k(B(u,v)\#\nabla^{2p}u)$, since they can be expressed as a sum similar to the right–hand side of identity (A.19). The only difference is that now in every term of such sum one linear occurrence of u is replaced by v. Precisely, writing $u_1:=u$, $u_2:=v$ every term $\partial^jA(u)\#\partial_t^{\iota_1}\nabla^{i_1}u\#\cdots\#\partial_t^{\iota_{j+1}}\nabla^{i_{j+1}}u$ has to be replaced by some

$$D(u)\#\partial_t^{\iota_1}\nabla^{i_1}u_{\tau_1}\#\cdots\#\partial_t^{\iota_{j+1}}\nabla^{i_{j+1}}u_{\tau_{j+1}} \tag{A.20}$$

where exactly one of the indices $\tau_1,\dots,\tau_{j+1}$ is equal to 2, and the others are equal to 1.
An analogous reasoning applies to the term $A(u)\cdot\nabla^{2p}v$. It is then easy to see, since $v\in P^m$ like u, that we can repeat the same estimates used to show the continuity of $u\mapsto\mathcal{F}_A(u)$. This proves

in particular that $d\mathcal{F}_A(u) \in L(P^m; P^{m-1})$.
In order now to prove that $d\mathcal{F}_A \in C^0(P^m; L(P^m; P^{m-1}))$ we need to show that

$$\sup_{\|v\|_{P^m} \le 1} \|d\mathcal{F}_A(\widetilde{u})(v) - d\mathcal{F}_A(u)(v)\|_{P^{m-1}} \to 0 \quad \text{as } \widetilde{u} \to u \text{ in } P^m .$$

Again, this estimate is similar to what we have already done. Indeed, supposing that τ_{j+1} is the only index equal to 2 in (A.20) and assuming that there are no time derivatives for the sake of simplicity, we want to see that, as $\widetilde{u} \to u$ in P^m,

$$\sup_{\|v\|_{P^m} \le 1} \|D(\widetilde{u})\#\nabla^{i_1}\widetilde{u}\# \cdots \#\nabla^{i_j}\widetilde{u}\,\nabla^{i_{j+1}}v - D(u)\#\nabla^{i_1}u\# \cdots \#\nabla^{i_j}u\,\nabla^{i_{j+1}}v\|_{L^2} \to 0\,, \tag{A.21}$$

where $i_1 + \cdots + i_{j+1} \le 2pm + (2p-1)j$ (see formula (A.12) and condition (A.14)).
Adding and subtracting terms, one gets

$$\begin{aligned} \Big| D(\widetilde{u})\#\nabla^{i_1}\widetilde{u}\# \cdots \#\nabla^{i_j}\widetilde{u}\,\nabla^{i_{j+1}}v - D(u)\#\nabla^{i_1}u\# \cdots \#\nabla^{i_j}u\,\nabla^{i_{j+1}}v \Big| & \\ \le \Big\{ |D(\widetilde{u}) - D(u)|\,|\nabla^{i_1}\widetilde{u}| \cdots |\nabla^{i_j}\widetilde{u}| & \\ + |D(u)|\,|\nabla^{i_1}(\widetilde{u}-u)|\,|\nabla^{i_2}\widetilde{u}| \cdots |\nabla^{i_j}\widetilde{u}| & \\ + \cdots + |D(u)|\,|\nabla^{i_1}u| \cdots |\nabla^{i_j}(\widetilde{u}-u)| \Big\}\, |\nabla^{i_{j+1}}v|\,. & \end{aligned}$$

Studying now the L^2 norm of this sum, the first term can be bounded as before and it goes to zero as $D(u)$ is continuous from P^m to L^∞. The L^2 norm of all the other terms, repeating step by step the previous estimates, using Hölder's inequality and embeddings (A.22)–(A.24), will be estimated by some product

$$C\|u\|^\alpha_{P^m}\|\widetilde{u}\|^\beta_{P^m}\|v\|^\gamma_{P^m}\|\widetilde{u}-u\|^\sigma_{P^m} \le C\|u\|^\alpha_{P^m}\|\widetilde{u}\|^\beta_{P^m}\|\widetilde{u}-u\|^\sigma_{P^m}$$

for a constant C and some nonnegative exponents $\alpha, \beta, \gamma, \sigma$ satisfying $\alpha + \beta + \gamma + \sigma \le 1$ and $\sigma > 0$. Here we we used the fact that $\|v\|_{P^m} \le 1$.
As $\widetilde{u} - u \to 0$ in P^m, this last product goes to zero in L^2, hence uniformly for $\|v\|_{P^m} \le 1$ and inequality (A.21) follows, as claimed. The analysis of the estimates with mixed time/space derivatives is similar and all this argument works analogously for the term $A(u) \cdot \nabla^{2p}v$.
Then, the Gateaux derivative $d\mathcal{F}_A$ is continuous which implies that it coincides with the Frechét derivative, hence $\mathcal{F}_A \in C^1(P^m; P^{m-1})$.

The map $\mathcal{F}_b$ can be dealt with in the same way and we are done.

Parabolic Sobolev Embeddings

PROPOSITION A.7. *Let $u \in P^m(M,T)$. Then for $r, \ell \in \mathbb{N}$ with $2pr + \ell \le 2mp$, we have*

$$\|\partial_t^r \nabla^\ell u\|_{L^q(M\times[0,T])} \le C\|u\|_{P^m(M,T)} \quad \textit{if} \quad \frac{1}{q} = \frac{1}{2} - \frac{2pm - \ell - 2pr}{n+2p} > 0\,; \tag{A.22}$$

$$\|\partial_t^r \nabla^\ell u\|_{L^q(M\times[0,T])} \le C\|u\|_{P^m(M,T)} \quad \textit{if} \quad \frac{1}{2} - \frac{2pm - \ell - 2pr}{n+2p} = 0 \ \textit{and}\ 1 \le q < \infty\,; \tag{A.23}$$

the function $\partial_t^r \nabla^\ell u$ is continuous and

$$\|\partial_t^r \nabla^\ell u\|_{C^0(M\times[0,T])} \le C\|u\|_{P^m(M,T)} \quad \textit{if} \quad \frac{1}{2} - \frac{2pm - \ell - 2pr}{n+2p} < 0\,, \tag{A.24}$$

where the constant C does not depend on u.

PROOF. Of course we can write

$$P^m(M,T) = L^2([0,T]; H^{2mp}(M)) \cap H^1([0,T]; H^{2p(m-1)}(M)) \cap \cdots \cap H^m([0,T]; L^2(M))\,.$$

By standard interpolation theory, see e.g. [**70**, Thm. 2.3], we have the continuous immersion

$$P^m(M,T) \hookrightarrow H^s([0,T]; H^{2p(m-s)}(M)), \quad \text{for all } s \in [0,m]\,.$$

We shall now assume that $\frac{1}{2} - \frac{2pm-\ell-2pr}{n+2p} > 0$ and prove inequality (A.22). For $0 \leq \sigma < \frac{1}{2}$ and for any Hilbert space X we have the Sobolev embedding

$$H^\sigma([0,T];X) \hookrightarrow L^q([0,T];X) \quad \text{for} \quad \frac{1}{q} = \frac{1}{2} - \sigma\,.$$

Then, for $\ell, r \in \mathbb{N}$ with $2pr + \ell \leq 2pm$ and for any $s \in \big(m - \frac{\ell}{2p} - \frac{n}{4p}, m - \frac{\ell}{2p}\big] \cap \big[r, r+\frac{1}{2}\big)$, also using the standard Sobolev embeddings on M, for every $u \in P^m(M,T)$ one gets

$$\begin{aligned}\partial_t^r \nabla^\ell u \in H^{s-r}([0,T]; H^{2p(m-s)-\ell}(M)) &\hookrightarrow L^q([0,T]; H^{2p(m-s)-\ell}(M))\\ &\hookrightarrow L^q([0,T]; L^{\widetilde{q}}(M))\,,\end{aligned}$$

with

$$\frac{1}{q} = \frac{1}{2} - s + r \quad \text{and} \quad \frac{1}{\widetilde{q}} = \frac{1}{2} - \frac{2p(m-s)-\ell}{n}\,.$$

We now choose $s = \frac{rn+2pm-\ell}{n+2p}$ and claim that $s \in \big(m - \frac{\ell}{2p} - \frac{n}{4p}, m - \frac{\ell}{2p}\big] \cap \big[r, r+\frac{1}{2}\big)$. Then

$$\frac{1}{q} = \frac{1}{\widetilde{q}} = \frac{1}{2} - \frac{2pm-\ell-2pr}{n+2p}\,,$$

hence for such $q \in \mathbb{R}$ we have

$$u \in L^q([0,T]; L^q(M)) \simeq L^q(M \times [0,T])\,,$$

and embedding (A.22) is proven. As for the claim, the inequalities $s \geq r$ and $s \leq m - \frac{\ell}{2p}$ easily follow from the inequality $2pr + \ell \leq 2pm$, while inequality $s < r + \frac{1}{2}$ is equivalent to $\frac{1}{2} - \frac{2pm-\ell-2pr}{n+2p} > 0$. This means $\frac{1}{q} > 0$ which implies $s > m - \frac{\ell}{2p} - \frac{n}{4p}$.

The proof of inequality (A.23) is analogous.

Finally, if $\frac{1}{2} - \frac{2pm-\ell-2pr}{n+2p} < 0$, using that for $\sigma > \frac{1}{2}$ one has $H^\sigma([0,T];X) \hookrightarrow C^0([0,T];X)$ and that for $\sigma > \frac{n}{2}$ one has $H^\sigma(M) \hookrightarrow C^0(M)$, for every $u \in P^m(M,T)$ we infer

$$\partial_t^r \nabla^\ell u \in H^{s-r}([0,T]; H^{2p(m-s)-\ell}(M)) \hookrightarrow C^0([0,T]; C^0(M)) \simeq C^0(M \times [0,T])\,,$$

for $s = \frac{rn+2pm-\ell}{n+2p} \in \big(r + \frac{1}{2}, m - \frac{\ell}{2p} - \frac{n}{4p}\big)$. This proves embedding (A.24). $\square$

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