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Abstract

This thesis is devoted to study delay/path-dependent stochastic differential equations and their connection with partial differential equations in infinite dimensional spaces, possibly path-dependent. We address mathematical problems arising in hedging a derivative product for which the volatility of the underlying assets as well as the claim may depend on the past history of the assets themselves.

The starting point is to provide a robust framework for working with mild solutions to path-dependent SDEs: well-posedness, continuity with respect to the data, regularity with respect to the initial condition. This is done in Chapter 1. In Chapter 2, under Lipschitz conditions on the data, we prove the directional regularity needed in order to write the hedging strategy. In Chapter 3 we introduce a new notion of viscosity solution to semilinear path-dependent PDEs in Hilbert spaces (PPDEs), we prove well-posedness and show that the solution is given by the Fyenman-Kac formula. In Chapter 4 we extend to Hilbert spaces the functional Itō calculus and, under smooth assumptions on the data, we prove a path-dependent Itō's formula, show existence of classical solutions to PPDEs, and obtain a Clark-Ocone type formula. In Chapter 5 we introduce a new notion of C_0-semigroup suitable to be applied to Markov transition semigroups, hence to mild solutions to Kolmogorov PDEs, and we prove all the basic results analogous to those available for C_0-semigroups in Banach spaces. Additional theoretical results for stochastic analysis in Hilbert spaces, regarding stochastic convolutions, are given in Appendix A.

Our methodology varies among different chapters. Path-dependent models can be studied in their original path-dependent form or by representing them as non-path-dependent models in infinite dimension. We exploit both approaches. We treat path-dependent Kolmogorov equations in infinite dimension with two notions of solution: classical and viscosity solutions. Each approach leads to original results in each chapter.

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Contents

Introduction

The present thesis is mainly devoted to study delay/path-dependent stochastic differential equations and their connection with partial differential equations in infinite dimensional spaces, possibily path-dependent (PPDEs), and with problems in mathematical finance.

Motivation

The motivation for studying path-dependent equations comes from the classical problem in finance of hedging a derivative product for which the volatility of the underlying assets as well as the claim may depend on the past history of the assets themselves.

We briefly recall this problem in the case without path-dependence and outline a standard procedure used to solve it, following e.g. [5, Ch. 8].

Consider a financial market composed of two assets: a risk free asset B (a bond price), and a risky asset R (a stock price). We assume that B follows the deterministic dynamics $dB_s = rB_s ds$, where r is the (constant) spot interest rate, and that R follows the dynamics

$$\begin{cases} dR_s = rR_s ds + \nu(s, R_s)dW_s & s \in (t, T], \\ R_t = x, \end{cases} \tag{1}$$

where $T > 0$ is the maturity, $x \in \mathbb{R}$ is the initial datum at time $t \in [0, T)$, and W is a Brownian motion on a filtered probability space $(\Omega, \mathbb{F} = \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$. Assume that ν satisfies the usual Lipschitz conditions and denote by $R^{t,x}$ the unique strong solution to the stochastic differential equation (1).

Given a function $\varphi\colon \mathbb{R} \to \mathbb{R}$, the problem of hedging the derivative $\varphi(R_T^{0,x})$ consists in finding a replicating self-financing portfolio strategy for $\varphi(R_T^{0,x})$, i.e. a couple of real-valued processes $\{(h_s^B, h_s^R)\}_{s\in[0,T]}$ such that the value process

$$V_s = h_s^B B_s + h_s^R R_s^{0,x} \qquad \forall s \in [0, T],$$

of the portfolio composed by h_s^B shares of B and h_s^R shares of $R^{0,x}$, satisfies

$$\begin{cases} dV_s = h_s^B dB_s + h_s^R dR_s^{0,x} & s \in [0, T) \\ V_T = \varphi(R_T^{0,x}). \end{cases} \tag{2}$$

The first equation in (2) is the self-financing conditions. It means that, at every instant t, the variation dV_t of the value of the portfolio is due only to the variation of the value of the two assets B,R. The second equation in (2) is the replicating condition, which states that the value V_T of the portfolio at maturity is exactly the value of the derivative $\varphi(R_T^{0,x})$. Let us introduce the function

$$u(t,x) := e^{-r(T-t)}\mathbb{E}\left[\varphi(R_T^{t,x})\right] \qquad \forall (t,x) \in [0,T]\times\mathbb{R}. \tag{3}$$

Notice that, by Markov property of R, we have

$$u(t,x) := e^{-rh}\mathbb{E}\left[u(t+h,R_{t+h}^{t,x})\right] \qquad \forall 0 \le t < t+h \le T,\ x\in\mathbb{R}. \tag{4}$$

It is well-known that, if the data ν,φ are sufficiently smooth, e.g. C^2 with respect to the variable x with bounded differentials, then $u(t,x)$ is Fréchet differentiable up to order 2 with respect to x, with derivatives $D_x u$, $D_{xx}^2 u$ which are bounded and continuous, jointly in (t,x) (see e.g. [23, Ch. 7]). In such a case, Itō's formula and (4) permit to show that u is $C^{1,2}$ and solves the following backward Kolmogorov equation ([23, Ch. 7]):

$$\begin{cases} u_t + rxD_x u + \frac{1}{2}\nu^2 D_{xx}^2 u - ru = 0 & \text{on } (0,T]\times\mathbb{R}, \\ u(T,x) = \varphi(x) & x\in\mathbb{R}. \end{cases} \tag{5}$$

By using (5) and applying Itō's formula to $u(s,X_s^{0,x})$, we end up with the following representation formula

$$u(s,R_s^{0,x}) = u(0,x) + \int_0^s ru(\zeta,R_\zeta^{0,x})d\zeta + \int_0^s D_x u(\zeta,R_\zeta^{0,x})\nu(\zeta,R_\zeta^{0,x})dW_\zeta. \tag{6}$$

By recalling the definition of u and considering (6), we see that the portfolio strategy defined by

$$h_s^B = \frac{u(s,R_s^{0,x}) - D_x u(s,R_s^{0,x})R_s^{0,x}}{B_s} \quad \text{and} \quad h_s^R = D_x u(s,R_s^{0,x}) \quad \forall s\in[0,T) \tag{7}$$

solves the hedging problem. Indeed, we have

$$V_s = h_s^B B_s + h_s^R R_s^{0,x} = u(s,R_s^{0,x}) \qquad \forall s\in[0,T), \tag{8}$$

hence, by continuity, $V_T = u(T,R_T^{0,x}) = \varphi(R_T^{0,x})$. Moreover, by (6) and (8), we verify the self-financing condition

$$dV_s = h_s^B dB_s + h_s^R dR_s^{0,x} \qquad \forall s\in[0,T).$$

There are three essential features that allow to implement the program above:

(F1) The Markov property of R, which makes (4) possible.

(F2) The existence of $D_x u$, which lets the portfolio strategy be defined by (7).

(F3) The availability of Itō's formula and the fact that u solves (5), in order to derive (6), hence to see that (7) is a hedging strategy.

It is relevant to notice that, if v is non-degenerate, the regularity of u can be obtained under weaker assumptions (see e.g. [69]).

Let us now consider a more general risky asset R, in which the volatility depends not only on the present value R_s of R at time s, but also on the past values of R. The dynamics of R has the following *path-dependent* form:

$$\begin{cases} dR_s = rR_s ds + v(s,\{R_{s'}\}_{s'\in[0,s]})dW_s & s\in(t,T] \\ R_{t'} = \mathbf{x}(t') & t'\in[0,t], \end{cases} \tag{9}$$

where $\mathbf{x}\colon [0,t]\to\mathbb{R}$ is a given deterministic funtion, belonging to some functional space $\mathbb{S}$, to be chosen, and expressing the past history of the stock price R up to time t. We also would like to face the case in which the claim depends itself on the history of R, i.e. it has the form $\varphi(\{R^{0,\mathbf{x}}_{t'}\}_{t'\in[0,T]})$, where, for each couple $(t,\mathbf{x})$, we denote by $R^{t,\mathbf{x}}$ the solution to (9).

A natural question is if we can solve the hedging problem for the path-dependent case by implementing the arguments outlined above for the case in which R was given by (1). More precisely, if we consider the function u, now defined by

$$u(t,\mathbf{x}) := e^{-r(T-t)}\mathbb{E}\left[\varphi(\{R^{t,\mathbf{x}}_{t'}\}_{t'\in[0,T]})\right] \qquad \forall (t,\mathbf{x})\in[0,T]\times\mathbb{S}, \tag{10}$$

we would like to find a stochastic representation for $u(s,\{R^{0,\mathbf{x}}_{t'}\}_{t'\in[0,s]})$ of the form (6), by using the fact that u is a solution to a PDE analogous to (5), but now path-dependent, and after showing that u is sufficiently regular in order to write the hedging strategy. Briefly, we are concerned with the features (F1)–(F3) in the path-dependent case.

We immediately see that the extension of (F1)–(F3) to the present case is not trivial. The problems that we have to face are the following.

(P1) Firstly, Markov property does not hold for $R^{t,\mathbf{x}}$, due to the path-dependence of its dynamics.

(P2) Secondly, even under smooth assumptions on the data, a detailed study of path-dependent stochastic systems is required in order to show that the function u is sufficiently regular to apply stochastic calculus and to define the hedging strategy. More precisely, regularity of solutions to path-dependent SDEs with respect to the starting path should be considered, together with a path-dependent (or functional) stochastic calculus. If we wish to go beyond smooth data, it is reasonable to ask for a set of stability results regarding continuity with respect to the data of path-dependent dynamics and their derivatives.

(P3) Thirdly, even if u is regular and the use of Markov property can be somehow avoided, the derivation of the PDE cannot be done as in the non-path-dependent case. This is related to the previous issue and it is due to the fact that a fully-developed path-dependent stochastic calculus is not available.

(P4) Finally, since the PDE is now path-dependent, not only well-posedness, but even the very definition of solution to such type of PDEs becomes subject of discussion.

The framework sketched above to deal with models whose dynamics depends on the past is characterized by keeping (9) in the form as it is and facing (P1)–(P4) as they are stated, meaning without changing the state space ($\mathbb{R}$) in which the system $R^{t,\mathbf{x}}$ evolves. To distinguish it from the framework that we describe below, we call it the *path-dependent* setting.

There is another framework to deal with models like (9) and with the associated mathematical tools and objects (as stochastic calculus, Kolmogorov equation, representation formula for $u(s,\{R^{0,\mathbf{x}}_{t'}\}_{t'\in[0,s]})$, in our case) that we call the *delay* setting. It consists (roughly speaking) in describing the dynamics of R trough the dynamics of its paths in a space $\mathbb{S}_0$ of functions $[-T,0]\to\mathbb{R}$ and reading (9) as a non-path-dependent SDE evolving in $\mathbb{S}_0$. The idea is then to associate (9) with the following (informally written) SDE in $\mathbb{S}_0$

$$\begin{cases} dX_s = \tilde{r}(s,X_s)ds + \tilde{v}(s,X_s)dW_s & s\in(t,T],\\ X_t(t') = \mathbf{1}_{[-T,-t)}(t')\mathbf{x}(0) + \mathbf{1}_{[-t,0]}(t')\mathbf{x}(t+t'), & t'\in[-T,0], \end{cases} \tag{11}$$

where X now is an $\mathbb{S}_0$-valued process and $\tilde{r},\tilde{v}$ are associated to r,v in such a way that $R^{t,\mathbf{x}}$ can be obtained once having the solution $X^{t,\mathbf{x}}$ to (11) by the pointwise evaluation $R^{t,\mathbf{x}}_s = X^{t,\mathbf{x}}_s(0)$, $s\in[0,T]$.

Within the delay approach, Markov property is restored and stochastic calculus can be used, even though it is in infinite dimensional spaces, like Hilbert or more generally Banach spaces, and it is not the standard stochastic calculus in finite dimension. Nevertheless, there are some drawbacks coming from the reformulation $R\rightsquigarrow X$:

(P1)′ The rephrasing of the evolution equation in the new state space $\mathbb{S}_0$ entails the appearance of a term of the form AX_s in the drift $\tilde{r}$ of (11), where A is a linear unbounded operator.

(P2)′ Even if the Kolmogorov equation associated to (11) is no more path-dependent, the domain of the PDE is now an infinite dimensional space and a first-order linear term with an unbounded linear coefficient appears.

(P3)′ If the chosen functional space $\mathbb{S}_0$ is not Hilbert, then no well-developed PDE theory is available on such domain, particularly if we want to deal with non-smooth data by making use of viscosity solutions.

(P4)′ Even if $\mathbb{S}_0$ is Hilbert, the second order term in the PDE is highly degenerate and the study of the regularity needed to define the hedging strategy is difficult to obtain if the data are non-smooth, e.g. Lipschitz.

With various theoretical approaches, some of the problems listed above, arising in the path-dependent or in the delay setting, are the subject of the present thesis.

In Chapter 1 we build a general framework for Hilbert space-valued path-dependent SDEs, improving the theory available in the literature, particularly regarding regularity with respect to the initial condition and continuity with respect to the data of the system. In Chapter 2 we solve one of the problems arising in the precedure outlined above in order to find the hedging strategy: we prove that the value function has the needed regularity when the data are only Lipschitz continuous. Chapter 3 develops the recent theory of path-dependent PDEs in the infinite dimensional case. Chapter 4 develops the functional Itō calculus in infinite dimension. Chapter 5 is also indirectly related to PDEs in Hilbert spaces associated to delay problems, since it introduces a notion of C_0-semigroup suited to be applied to Markov transition semigroups, for which the standard theory of C_0-semigroups in Banach spaces cannot be used. Finally, some contributions are given in Appendix A, where a stochastic Fubini's theorem is proved for a generic stochastic integration considered merely as a linear and continuous operator, and applications to stochastic convolution are considered.

Contents of the thesis

We now describe the contents of the single chapters.

Chapter 1. We begin in Chapter 1 by studying path-dependent SDEs in Hilbert spaces. This is the first step in order to develop any functional Itō calculus in Hilbert spaces. We prove existence and uniqueness of mild solution (Theorem 1.2.6), Gâteaux differentiability of order n of mild solutions with respect to the starting point (Theorem 1.2.9), continuity with respect to all the data of the system of the mild solution and of its Gâteaux derivatives (Theorem 1.2.14 and Theorem 1.2.16).

Existence and uniqueness of mild solution are well-known and obtained by standard contraction arguments. Differently, there are no available results regarding differentiability of generic order with respect to the initial condition of mild solutions to path-dependent SDEs in Hilbert spaces. Also for the non-path-dependent case, Gâteaux and Fréchet differentiability with respect to the initial datum are always studied at most up to order 2. (see e.g. [23, 24, 48, 62]). As far as we know, no result providing Gâteaux differentiability of order $n > 2$ is available in the literature for path-dependent SDEs in finite or infinite dimension, or for non-path-dependent SDEs in infinite dimension. Theorem 1.2.9 fills this gap. Also regarding the dependence of the mild solution on the given data, we prove a general result providing joint continuity with respect to all the data of the system, i.e. the coefficients (including the unbounded term), the starting time, the starting point: in Theorem 1.2.14, we generalize to the infinite dimensional path-dependent case the previous related results in [23, 24, 48, 76]. We study also continuity with respect to the data of Gâteaux derivatives of order n with respect to the initial datum of mild solutions. The stability result provided by Theorem 1.2.16 generalizes to an arbitrary order of differentiability and to the path-dependent case the related results in [8, 23].

Our method to study path-dependent SDEs relies entirely on the properties of fixed-point maps associated to parametric contractions in Banach spaces. Because of that, we begin in Section 1.1 by recalling some notions regarding strongly continuous Gâteaux differentiability and basic results for contractions in Banach space. Then, we provide a result for Gâteaux differentiability of order n of fixed-point maps associated to contractions which are Gâteaux differentiable only with respect to some subspaces (Theorem 1.1.13) and we give continuity results for the fixed-point maps and their derivatives of any order (Proposition 1.1.15). These results, needed for applications to SDEs, were previously available only up to order 2 ([8, 23]).

In Section 1.2 we apply the theory recalled or developed in the first section to mild solutions to path-dependent SDEs. As said above, we focus on existence and uniqueness of mild solutions (Theorem 1.2.6), continuity and differentiability of order n with respect to the initial datum (Theorem 1.2.9), stability of mild solutions and their derivatives under perturbations of all the data of the system (Theorem 1.2.14 and Theorem 1.2.16). We develop the theory in a slightly more general setting than the one usually adopted in the literature when addressing path-dependent SDEs with the Wiener process as stochastic integrator. Firstly, the assumptions on the diffusion coefficient is close to the minimal required in order to obtain a continuous version of the stochastic convolution (for more details on this point, see Appendix A, Theorems A.2.10 and A.2.13) and to construct the contraction that provides the solution. Secondly, the path-dependence is considered with respect to a generic closed subspace $\mathbb{S}$ of the space $B_b([0,T],H)$ of H-valued bounded Borel functions (for example $\mathbb{S}$ may be the space of continuous functions, or the space of càdlàg functions) and the initial datum can be any process taking values in $\mathbb{S}$. This choice turns out to be useful when dealing with derivatives of mild solutions with respect to step functions as initial datum, since these derivatives naturally arise in the context of functional Itō calculus.

The contents presented in the chapter appear in the manuscript [89], submitted to *Stochastic Processes and their Applications*.

Chapter 2. In Chapter 2 we adopt the delay approach and address the regularity needed for the hedging strategy. If X is solves (11), with $\mathbf{x}(0) = x_0$ and $\mathbf{x}_{|[-T,0)} = x_1$, we prove the partial regularity of

$$u(t,(x_0,x_1)) := \mathbb{E}\left[\varphi(\{X_{t'}^{t,(x_0,x_1)}\}_{t'\in[0,T]})\right] \tag{12}$$

with respect to the component x_0, under Lipschitz assumptions on the coefficients of the state dynamics and on φ with respect to a suitably chosen norm. The derivative $D_{x_0}u$ is interesting because it turns out to be the only one relevant to define the hedging strategy.

Such a regularity is difficult to obtain in infinite dimension. For fixed x_1, we rely on approximations of $v(t,x_0) := u(t,(x_0,x_1))$ by a sequence of functions $v_n(t,x_0)$ that are regular with respect to x_0 and are viscosity solutions to finite dimensional parabolic PDEs. Then, through parabolic regularity estimates, we obtain the $C^{1+\alpha}$ partial regularity of the limit $v(t,x_0) = u(t,(x_0,x_1))$, in the component x_0, for all $\alpha \in (0,1)$. Partial regularity results for first order unbounded HJB equations in Hilbert spaces associated to certain deterministic optimal control problems with delays have been obtained in [39]. The technique of [39] is different from ours, relies on arguments using concavity of the data and strict convexity of the Hamiltonian, and provides C^1 regularity on one-dimensional sections corresponding to the so-called "present" variable.

In Section 2.1, we consider the mild solution X to an SDE in a Hilbert space that, when thinking to the financial application, represents the process R in (1) together with its past. Then we outline the relationship between the state dynamics X and the dynamics of a process $\overline{X}$ which evolves in a larger Hilbert space H_B, built accordingly to the continuity assumptions on the coefficients of the SDE driving X. In this larger state space H_B, we approximate $\overline{X}$ with a sequence $\overline{X}_n$ of processes which are differentiable with respect to the initial datum. In Section 2.2, we show that the section in the direction x_0 of the semigroup associated to $\overline{X}_n$ provides the viscosity solution v_n to a "frozen" finite dimensional PDE, and, since v_n is also an L^p-viscosity solutions ([18]), this let us to obtain a parabolic estimate for v_n. Since v_n converges to $u(\cdot,(\cdot,x_1))$, we derive in this way the partial regularity of u (Theorem 2.2.9).

The regularity result is original and has been obtained in a joint work with Andrzej Święch. The paper ([90]) has been accepted for publication by *Journal of Differential Equations*.

Chapter 3. Chapter 3 contains the first work in the literature studying viscosity solutions to path-dependent Kolmogorov equations in Hilbert spaces. We adopt a new notion of viscosity solution, recently introduced, in the finite dimensional case, in [33], and further developed in [35, 34, 85], whose main idea consists in replacing the usual tangency condition asked for test functions in classical viscosity solution theory with a tangency in expectation, local in probability, and we extend the definition to our infinite dimensional framework.

In Section 3.1 we introduce the process X, whose distribution determines the support on which viscosity solutions are tested. Section 3.2 contains the core of the chapter. Firstly, smooth and test functions are defined in terms of X and the new notion of viscosity solution is introduced (Definition 3.2.2). Secondly, we prove the martingale characterisation theorem (Theorem 3.2.7), that shows how a viscosity solution is associated to a certain martingale process. This characterisation is the key step to obtain the comparison result (Corollary 3.2.14).

Our definition of viscosity solution leads to use mainly probabilistic tools and to avoid the Crandall-Ishii lemma (see [17]), which is needed to prove uniqueness of viscosity solutions in the second order case and which is not available in infinite dimension. In this way, even in the Markovian case, we can treat a larger class of problems than those treatable with the available viscosity theory in Hilbert spaces ([37, Ch. 3]).

The chapter results from a joint work with Andrea Cosso, Salvatore Federico, Fausto Gozzi and Nizar Touzi. The manuscript ([15]) is under second review by *The Annals of Probability*.

Chapter 4. Chapter 4 is devoted to study functional Itō calculus in Hilbert spaces and application to path-dependent PDEs.

In [31] the main ideas for a functional Itō calculus are presented for one-dimensional continuous semimartingales. In [12, 13, 14] these ideas are developed in a more rigorous setting and generalized. In [12] a functional Itō's formula is proved for a large class of finite-dimensional càdlàg processes, including semimartingales and Dirichlet processes, and for functionals which can depend on the quadratic variation. In [14] the notion of vertical derivative is extended to square integrable continuous martingales and showed to coincide with the integrand appearing in the martingale representation theorem. Differently than in [12, 13, 14, 31], functional Itō calculus in finite dimension can be seen as application to the space of continuous/càdlàg functions of stochastic calculus in Banach spaces ([26, 27, 28, 29, 43]). In [29] the notion of χ-quadratic variation is introduced for Banach space-valued processes (not necessarily semimartingales), the related Itō's formula is discussed and then applied to "windows" processes in $C([-T,0],\mathbb{R}^n)$, letting a Clark-Ocone type representation formula be derived by recurring to solutions to a path-dependent Kolmogorov equation.

Our approach to functional Itō calculus does not rely on the methods used in [12, 13, 14, 31] nor on the stochastic calculus in Banach spaces developed in [29]. Our results extend the functional Itō calculus to Hilbert spaces, for which no literature was previously available.

In Section 4.1 we introduce and study the notion of weak continuity for the differentials of the smooth functionals that we consider for our functional Itō calculus. Then, in Section 4.2, we consider an Hilbert space valued Itō process X and we prove that Itō's formula holds for processes of the form $u(t,X)$, where u is a suitably smooth non-anticipative functional depending on the path of X (Theorem 4.2.8). In Section 4.3 we apply the path-dependent Itō's formula to obtain a representation for classical solutions to path-dependent Kolmogorov equations through strong solutions to path-dependent SDEs (Theorem 4.3.2) and derive a Clark-Ocone type formula (Corollary 4.3.3). Finally, in Section 4.4, we apply the developed theory to the case in which the path-dependent SDE driving X has a constant diffusion coefficient and the drift is the composition of a nonlinear non-path-dependent function with the convolution of the path of X with a Radon measure. The chosen test model is truly path-dependent, in the sense that it cannot be reduced to a non-path-dependent model by adding a finite number of SDEs evolving in the same state state space of X (or in a space with "comparable" dimension), as it can be done e.g. for the Hobson-Rogers model, introduced in [57] for reasons other than making a path-dependent analysis, but that is often assumed as a test case of this sort.

The contents presented in the chapter appear in the manuscript [87], submitted to *The Annals of Probability*.

Chapter 5. The aim of Chapter 5 is to propose and study a new notion of semigroup in locally convex spaces, based on sequential continuity, for which all the standard basic results available for C_0-semigroups in Banach spaces can be obtained. Our purpose for future research is to exploit the theory here developed as a starting point for studying semilinear elliptic PDEs in infinite dimensional spaces and their application to optimal control problems.

The main motivation that led us to consider sequential continuity is that it allows a convenient treatment of Markov transition semigroups. The employment of Markov transition semigroups to the study of linear and semilinear PDEs, both in finite and infinite dimension, is the subject of a wide mathematical literature (see e.g. [8, 24]). The regularizing properties of such semigroups are the core of a regularity theory for second order PDEs ([74] and [37, Ch. 4,5]). Unfortunately, the framework of C_0-semigroups in Banach spaces is not suited to treat Markov transition semigroups, when considering Banach spaces of functions not vanishing at infinity. In this case, the C_0-property fails already at the very basic level, such as for the one-dimensional Ornstein-Uhlenbeck semigroup acting on the space of bounded uniformly continuous functions on the real line, if the drift of the associated stochastic differential equation is not zero (see [7, Ex. 6.1] and [21, Lemma 3.2])

A strategy to overcome this difficulty consists in finding locally convex topologies on spaces of continuous functions in order to frame Markov transition semigroups within the theory of C_0-locally equicontinuous semigroups. To find the right topology is not an easy task and the chosen topology is in general difficult to handle. In the case of the Ornstein-Uhlenbeck semigroup, this approach is adopted in [49]. A different approach is to introduce other notions of semigroup, specified according to various continuity assumptions with respect to sequences, as for the weakly continuous semigroups in [7], the π-continuous semigroups in [84], the bi-continuous semigroups in [65]). Basically, we follow this second approach, but we provide a more general notion, in our opinion also cleaner at the level of the definition, by which we can gather in a single framework the aforementioned approaches.

In Section 5.2 we introduce the notion of C_0-sequentially equicontinuous semigroup (Definition 5.2.12), and we show that the usual relations between generator and resolvent still hold within this setting, when considering sequential continuity in place of continuity. In particular, in Section 5.2.6 we provide a generation theorem (Theorem 5.2.37) characterizing the linear operators generating C_0-sequentially equicontinuous semigroups, in the same spirit of the Hille-Yosida theorem.

In Section 5.3 we apply the theory of the previous sections in spaces of Borel functions, continuous functions, or uniformly continuous functions, with prescribed growth at infinity. In these function spaces, we find and study appropriate locally convex topologies allowing a comparison between our notion of C_0-sequentially equicontinuous semigroup and other notions of semigroup studied in [7, 49, 65, 84].

Finally, in Section 5.4, we apply the results of Section 5.3 to transition semigroups, in particular to semigroups associated to SDEs in Hilbert spaces.

The chapter comes from a joint work with Salvatore Federico, and the manuscript ([41]) has been submitted to *Proceedings of the London Mathematical Society*.

Appendix A. In this thesis, the role of the appendix is to provide a precise presentation of those properties of the stochastic convolution operator that are fundamental to Chapter 1. Beyond that, Appendix A presents results which are interesting in themselves and extend previous results appearing in [24, 48, 83]. A stochastic Fubini's theorem is presented regardless of the particular stochastic integrator considered, as long as the stochastic integral can be considered as a continuous linear map from a space of L^p integrable processes to another L^p space, containing the integrated processes (Theorem A.1.3). Then, in this setting, we consider stochastic convolutions with strongly continuous functions, not necessary semigroups, and we show existence of predictable versions (Theorem A.2.5). A supplementary result is the characterisation of the measurability required for processes by the stochastic convolution (Theorem A.2.10). Finally, in case the convolution is made with a C_0-semigroup and the stochastic integral provides continuous paths, we show that the standard factorization argument can be adapted to our framework (Theorem A.2.13).

The contents of the appendix appear in the manuscript [88].

Chapters 2, 3, 4, and 5, are independent from each other. Chapters 2, 3, and 4 depend on Chapter 1. Some tools used in Chapter 1 have been detailed in Appendix A.

Notation

We collect here the basic notation that we will use. Further notation will be introduced in the single chapters.

If $\mathcal{T}$ is a topological space, $\mathcal{B}_{\mathcal{T}}$ denotes the Borel σ-algebra of $\mathcal{T}$. If $\mathcal{T}'$ is another topological space, $C(\mathcal{T},\mathcal{T}')$ denotes the space of continuous functions from $\mathcal{T}$ to $\mathcal{T}'$ and $B(\mathcal{T},\mathcal{T}')$ denotes the space of Borel measurable functions from $\mathcal{T}$ to $\mathcal{T}'$.

If E and F are topological vector spaces, $L(E,F)$ denotes the space of linear continuous functions from E to F. The topological dual of E is denoted by E^*.

If E is a normed the space, then $|\cdot|_E$ denotes its norm. If F is another normed space, $L(E,F)$ is endowed with its operator norm, unless otherwise specified.

Unless otherwise specified, every normed space E is considered endowed with its Borel σ-algebra $\mathcal{B}_E$.

Let $(E,|\cdot|_E)$ be a normed space. If $\mathcal{T}$ is a topological space, $C_b(\mathcal{T},E)$ and $B_b(\mathcal{T},E)$ denote the subspaces of $C(\mathcal{T},E)$ and $B(\mathcal{T},E)$ containing continuous and bounded functions, respectively. If $\mathcal{T}$ is a uniform space, then $UC(\mathcal{T},E)$ (resp. $UC_b(\mathcal{T},E)$) denotes the subspace of $C(\mathcal{T},E)$ (resp. $C_b(\mathcal{T},E)$) containing the uniformly continuous functions. Unless otherwise specified, $C_b(\mathcal{T},E)$, $B_b(\mathcal{T},E)$, $UC_b(\mathcal{T},E)$, are always considered as normed spaces endowed with the supremum norm $|f|_\infty := \sup_{x\in\mathcal{T}} |f(x)|_E$, $f \in B_b(\mathcal{T},E)$. We will use the same notation $|\cdot|_\infty$ for functions taking values in different normed spaces.

If $f\colon E \to F$ is Gâteaux differentiable in $x \in E$, we denote by $\partial f(x)$ the Gâteaux differential. We will need to consider Gâteaux differentiability of f with respect subspaces $V \subset E$. In such a case, the notation $\partial_V f$ is used (see Section 1.1.1 for the definition).

If U,H are separable Hilbert spaces, $L_2(U,H)$ denotes the space of Hilbert-Schmidt operators, endowed with the scalar product $\langle\cdot,\cdot\rangle_{L_2(U,H)}$ and the norm $|\cdot|_{L_2(U,H)}$ defined by

$$\langle T,Q\rangle_{L_2(U,H)} := \sum_{n=1}^{\infty} \langle Te_n,Qe_n\rangle_H \qquad |T|_{L_2(U,H)} := \sum_{n=1}^{\infty} |Te_n|_H^2$$

for $T,Q \in L_2(U,H)$ and for any orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of U. The numbers $\langle T,Q\rangle_{L_2(U,H)}$ and $|T|_{L_2(U,H)}$ are independent from the choice of the basis $\{e_n\}_{n\in\mathbb{N}}$. The space $L_2(U,H)$, endowed with $\langle\cdot,\cdot\rangle_{L_2(U,H)}$, is a separable Hilbert space continuously embedded in the subspace of $L(U,H)$ of compact operators.

For $T > 0$, $M([0,T])$ denotes the space of Radon measures on the interval $[0,T]$. The Lebesgue measure on $[0,T]$ is denoted by m. The space $M([0,T])$ endowed with the total-variation norm is a Banach space, and is canonically identified with the topological dual of $C([0,T],\mathbb{R})$.

Let $(G,\mathscr{G},\mu)$ be a measure space and let E be a Banach space. We use the notation $L^0\left((G,\mathscr{G},\mu),E\right)$ for the space of classes of equivalence of E-valued measurable functions $f\colon (G,\mathscr{G},\mu)\to E$, where $f = g$ if and only if $f(x) = g(x)$ μ-a.e. $x\in G$. For $p\in[1,\infty)$, we denote by $L^p\left((G,\mathscr{G},\mu),E\right)$ the space of $f\in L^0\left((G,\mathscr{G},\mu),E\right)$ for which there exists a μ-null set N such that $f_{|G\setminus N}$ has separable range, and

$$|f|_{L^p((G,\mathscr{G},\mu),E)} := \left(\int_G |f|_E^p d\mu\right)^{1/p} < \infty.$$

If $f\in L^p\left((G,\mathscr{G},\mu),E\right)$, then the Bochner integral $\int_G f d\mu\in E$ is well-defined, and

$$\left(L^p\left((G,\mathscr{G},\mu),E\right), |\cdot|_{L^p((G,\mathscr{G},\mu),E)}\right)$$

is a Banach space.

Let $T>0$. We use the notation $\Omega_T := \Omega\times[0,T]$. Let $\mathbb{F} = \{\mathscr{F}_t\}_{t\in[0,T]}$ be a complete right-continuous filtration on $(\Omega,\mathscr{F},\mathbb{P}_T)$. On Ω_T, we will often consider the σ-algebra $\mathscr{P}_T$ of predictable sets associated to the filtration $\mathbb{F}$ (see [24] for the definition). If $(E,|\cdot|_E)$ is a Banach space, an E-valued predictable process is any measurable function from $(\Omega_T,\mathscr{P}_T)$ to E.

We denote by $L^0_{\mathscr{P}_T}(E)$ the space $L^0((\Omega_T,\mathscr{P}_T,\mathbb{P}\otimes m),E)$ and, for $p\in[1,\infty)$, we denote by $L^p_{\mathscr{P}_T}(E)$ the space $L^0((\Omega_T,\mathscr{P}_T,\mathbb{P}\otimes m),E)$.

Chapter 1

Path-dependent SDEs in Hilbert spaces

In this chapter we deal with mild solutions to path-dependent SDEs, evolving in a separable Hilbert space H, of the form

$$\begin{cases} dX_t = (AX_t + b((\cdot,t),X))dt + \sigma((\cdot,s),X)dW_s & \forall s \in (t,T] \\ X_s = Y_s & s \in [0,t], \end{cases} \tag{1.0.1}$$

where $t \in [0,T)$, Y is a H-valued process, W is a cylindrical Wiener process taking values in a separable Hilbert space U, $b((\omega,s),X)$ is a H-valued random variable depending on $\omega \in \Omega$, on the time s, and on the path X, $\sigma((\omega,s),X)$ is a $L_2(U,H)$-valued random variable depending on $\omega \in \Omega$, on the time s, and on the path X, and A is the generator of a C_0-semigroup S on H. By using methods based on implicit functions associated to contractions in Banach spaces, we study continuity of the mild solution $X^{t,Y}$ to (1.0.1) with respect to t,Y,A,b,σ under standard Lipschitz conditions on b,σ, Gâteaux differentiability of generic order $n \geq 1$ of $X^{t,Y}$ with respect to Y under Gâteaux differentiability assumptions on b,σ, and continuity with respect to t,Y,A,b,σ of the Gâteaux differentials $\partial_Y^n X^{t,Y}$.

Path-dependent SDEs in finite dimensional spaces are studied in [76]. The standard reference for SDEs in Hilbert spaces is [24]. More generally, in addition to SDEs in Hilbert spaces, also the case of path-dependent SDEs in Hilbert spaces is considered in [48, Ch. 3], but for the path-dependent case the study is there limited mainly to existence and uniqueness of mild solutions. Our framework generalize the latter one by weakening the Lipschitz conditions on the coefficients, by letting the starting process Y belong to a generic space of paths contained in $B_b([0,T],H)$ ([1]) obeying few conditions, but not necessarily assumed to be $C([0,T],H)$, and by providing results on differentiability with respect to the initial datum and on continuity with respect to all the data.

[1] $B_b([0,T],H)$ denotes the space of bounded Borel functions $[0,T] \to H$.

In the literature on mild solutions to SDEs in Hilbert spaces, differentiability with respect to the initial datum is always proved only up to order $n = 2$, in the sense of Gâteaux ([23, 24]) or Frećhet ([48, 62]). In [23, Theorem 7.3.6] the case $n > 2$ is stated but not proved. There are no available results regarding differentiability with respect to the initial condition of mild solutions to SDEs of the type (1.0.1). One of the contributions of the present chapter is to fill this gap in the literature, by extending to a generic order n, in the Gâteaux sense, and to the path-dependent case the results so far available.

In case (1.0.1) is not path-dependent, the continuity of $X^{t,Y}$, $\partial_Y X^{t,Y}$, and $\partial^2_Y X^{t,Y}$, separately with respect to t, Y and A, b, σ, is considered and used in [23, Ch. 7]. We extend these previous results to the path-dependent case and to Gâteaux derivatives $\partial^n_Y X^{t,Y}$ of generic order n, proving joint continuity with respect to all the data t, Y, A, b, σ.

Similarly as in the cited literature, we obtain our results for mild solutions (differentiability and continuity with respect to the data) starting from analogous results for implicit functions associated to Banach space-valued contracting maps. Because of that, the first part of the chapter is entirely devoted to study parametric contractions in Banach spaces and regularity of the associated implicit functions. In this respect, regarding Gâteaux differentiability of implicit functions associated to parametric contractions and continuity of the derivatives under perturbation of the data, we prove a general result, for a generic order n of differentiability, extending the results in [8, 23, 24], that were limited to the case $n = 2$.

In a unified framework, this chapter provides a collection of results for mild solutions to path-dependent SDEs which are very general, within the standard case of Lipschitz-type assumptions on the coefficients, a useful toolbox for starting dealing with path-dependent stochastic analysis in Hilbert spaces. For example, the so called "vertical derivative" in the finite dimensional functional Itō calculus ([14, 31]) of functionals like $F(t, \mathbf{x}) = \mathbb{E}[\varphi(X^{t,\mathbf{x}})]$, where φ is a functional on the space $\mathbf{D}$ of càdlàg functions and $\mathbf{x} \in \mathbf{D}$, is easily obtained starting from the partial derivative of $X^{t,\mathbf{x}}$ with respect to a step function, which can be treated in our setting by choosing $\mathbf{D}$ as state space for SDE (1.0.1) (see Remark 1.2.11). Another field in which the tools here provided can be employed is the study of stochastic representations of classical solutions to path dependent Kolmogorov equations, where second order derivatives are required. Furthermore, the continuity of the mild solution and of its derivatives with respect to all the data, including the coefficients, reveals to be useful e.g. when merely continuous Lipschitz coefficients need to be approximated by smoothed out coefficients, which is in general helpful when dealing with Kolmogorov equations in Hilbert spaces (path- or non-path-dependent) for which notions other than classical solutions are considered, as strong-viscosity solutions ([16]) or strong solutions ([8]).

The contents of the chapter are organized as follows. First, in Section 1.1, we recall some notions regarding strongly continuous Gâteaux differentiability and some basic results for contractions in Banach spaces. Then we provide the first main result (Theorem 1.1.13): the strongly continuous Gâteaux differentiability up to a generic order n of fixed-point maps associated to parametric contractions which are differentiable only with respect to some subspaces. We conclude the section with a result regarding the continuity of the Gâteaux differentials of the implicit function with respect to the data (Proposition 1.1.15).

In Section 1.2 we consider path-dependent SDEs. After a standard existence and uniqueness result (Theorem 1.2.6), we move to study Gâteaux differentiability with respect to the initial datum up to order n of mild solutions, in Theorem 1.2.9, which is the other main result and justifies the study made in Section 1.1. We conclude with Theorem 1.2.16, which concerns the continuity of the Gâteaux differentials with respect to all the data of the system (coefficients, initial time, initial condition).

1.1 Preliminaries

In this section we recall the notions and develop the tools that we will apply to study path-dependent SDEs in Section 1.2. We focus on strongly continuous Gâteaux differentiability of fixed-point maps associated to parametric contractions in Banach spaces.

1.1.1 Strongly continuous Gâteaux differentials

We begin by recalling the basic definitions regarding Gâteaux differentials, mainly following [44]. Then we will define the space of strongly continuously Gâteaux differentiable functions, that will be the reference spaces in the following sections.

If X, Y are topological vector spaces, $U \subset X$ is a set, $f : U \to Y$ is a function, $u \in U$, $x \in X$ is such that $[u - \varepsilon x, u + \varepsilon x] \subset U$ ([2]) for some $\varepsilon > 0$, the directional derivative of f at u for the increment x is the limit

$$\partial_x f(u) := \lim_{t \to 0} \frac{f(u + tx) - f(u)}{t}$$

whenever it exists. Also in the case in which the directional derivative $\partial_x f(u)$ is defined for all $x \in X$, it need not be linear.

Higher order directional derivatives are defined recursively. For $n \geq 1$, $u \in U$, the nth-order directional derivative $\partial^n_{x_1 \ldots x_n} f(u)$ at u for the increments $x_1, \ldots, x_n \in X$ is the directional derivative of $\partial^{n-1}_{x_1 \ldots x_{n-1}} f$ at u for the increment x_n (notice that this implies, by definition, the existence of $\partial^n_{x_1 \ldots x_{n-1}} f(u')$ for u' in some neighborhood of u' in $U \bigcap (u + \mathbb{R} x_n)$)

[2] If $x, x' \in X$, the segment $[x, x']$ is the set $\{\zeta x + (1 - \zeta) x' | \zeta \in [0, 1]\}$.

If Y is locally convex, we denote by $L_s(X,Y)$ the space $L(X,Y)$ endowed with the coarsest topology wich makes continuous the linear functions of the form

$$L(X,Y) \to Y,\ \Lambda \mapsto \Lambda(x),$$

for all $x \in X$. Then $L_s(X,Y)$ is a locally convex space.

Let X_0 be a topological vector space continuously embedded into X. If $u \in U$, if $\partial_x f(u)$ exists for all $x \in X_0$ and $X_0 \to Y,\ x \mapsto \partial_x f(u)$, belongs to $L(X_0,Y)$, then f is said to be *Gâteaux differentiable at u with respect to X_0* and the map $X_0 \to Y,\ x \mapsto \partial_x f(u)$, is the *Gâteaux differential of f at u with respect to X_0*. In this case, we denote the Gâteaux differential of f at u by $\partial_{X_0} f(u)$ and its evaluation $\partial_x f(u)$ by $\partial_{X_0} f(u).x$. If $\partial_{X_0} f(u)$ exists for all $u \in U$, then we say that f is *Gâteaux differentiable with respect to X_0*, or, in case $X_0 = X$, we just say that f is *Gâteaux differentiable* and we use the notation $\partial f(u)$ in place of $\partial_X f(u)$.

A function $f\colon U \to Y$ is said to be *strongly continuously Gâteaux differentiable with respect to X_0* if it is Gâteaux differentiable with respect to X_0 and

$$U \to L_s(X_0,Y),\ u \mapsto \partial_{X_0} f(u)$$

is continuous. If $n > 1$, we say that f is *strongly continuously Gâteaux differentiable up to order n with respect to X_0* if it is strongly continuously Gâteaux differentiable up to order $n-1$ with respect to X_0 and

$$\partial^{n-1}_{X_0} f\colon U \to \overbrace{L_s(X_0,L_s(X_0,\cdots L_s(X_0,Y)}^{n-1 \text{ times } L_s}\cdots))$$

exists and is strongly continuously Gâteaux differentiable with respect to X_0. In this case, we denote $\partial^n_{X_0} f := \partial_{X_0}\partial^{n-1}_{X_0} f$ and $\partial^n f := \partial\partial^{n-1} f$.

The following proposition shows that, if f is strongly continuously Gâteaux differentiable with respect to X_0 and Y is a normed space, than f is continuous when restricted to the affine subspaces of the form $u + X_0$, with $u \in U$.

Proposition 1.1.1. *Let X_0 be a topological vector space continuously embedded into a topological vector space X, let $U \subset X$ be a non-empty open set, let Y be a normed space. Suppose that $f\colon U \to Y$ is Gâteaux differentiable with respect to X_0 and that $U \to L_s(X_0,Y)$, $u \mapsto \partial_{X_0} f(u)$, is continuous. Then, for all $u \in U$, the restriction of f to $U \bigcap (u + X_0)$ is continuous, when $U \bigcap (u + X_0)$ is endowed with the topology induced by $u + \tau_{X_0}$, where τ_{X_0} denotes the vector topology of X_0.*

Proof. Let $u \in U$ and let $\{u_\iota\}_{\iota \in \mathscr{I}} \subset U \bigcap (u + X_0)$ be a net converging to u in the topology $U \bigcap (u + \tau_{X_0})$. By the mean value theorem [44, Corollary 1.6.3], for all $\iota \in \mathscr{I}$ there exists $u'_\iota \in [u,u_\iota]$ such that $|f(u_\iota) - f(u)|_Y \le |\partial_{X_0} f(u'_\iota).(u_\iota - u)|_Y$. In particular, $u'_\iota \to u$ and $\partial_{X_0} f(u'_\iota).(u_\iota - u) \to 0$, by strong continuity of $\partial_{X_0} f$. Hence $f(u_\iota) \to f(u)$. ■

Let X, X_0 be topological vector spaces, with X_0 continuously embedded into X, let U be an open subset of X, and let Y be a locally convex space. We denote by $\mathscr{G}^n(U,Y;X_0)$ the space of functions $f\colon U \to Y$ which are continuous and strongly continuously Gâteaux differentiable up to order n with respect to X_0. In case $X_0 = X$, we use the notation $\mathscr{G}^n(U,Y)$ instead of $\mathscr{G}^n(U,Y;X)$.

Let $L_s^{(n)}(X_0^n,Y)$ be the vector space of n-linear functions from X_0^n into Y which are continuous with respect to each variable separately, endowed with the coarsest vector topology making continuous all the linear functions of the form

$$L_s^{(n)}(X_0^n,Y) \to Y,\ \Lambda \to \Lambda(x_1,\ldots,x_n)$$

for $x_1,\ldots,x_n \in X_0$. Then $L_s^n(X_0^n,Y)$ is a locally convex space. Trough the canonical identification (as topological vector spaces)

$$\overbrace{L_s(X_0,L_s(X_0,\cdots L_s(X_0,Y)}^{n \text{ times } L_s}\cdots)) \cong L_s^{(n)}(X_0^n,Y),$$

we can consider $\partial^n_{X_0} f$ as taking values in $L_s^{(n)}(X_0^n,Y)$, whenever $f \in \mathscr{G}^n(U,Y;X_0)$.

If X_0, X, Y are normed spaces, U is an open subset of X, $\partial_x f(u)$ exists for all $u \in U$, $x \in X_0$, $\partial_x f(u)$ is continuous with respect to u, for all $x \in X_0$, then $\partial_x f(u)$ is linear in x (see [44, Lemma 4.1.5]).

The following proposition is a characterisation for the continuity conditions on the directional derivatives of a function $f \in \mathscr{G}^n(U,Y;X_0)$, when X_0, X, Y are normed spaces.

Proposition 1.1.2. *Let $n \geq 1$, let X_0, X, Y be normed spaces, with X_0 continuously embedded into X, and let U be an open subset of X. Then $f \in \mathscr{G}^n(U,Y;X_0)$ if and only if f is continuous, the directional derivatives $\partial^j_{x_1\ldots x_j} f(u)$ exist for all $u \in U$, $x_1,\ldots,x_j \in X_0$, $j = 1,\ldots,n$, and the functions*

$$U \times X_0^j \to Y,\ (u,x_1,\ldots,x_j) \mapsto \partial^j_{x_1\ldots x_j} f(u) \tag{1.1.1}$$

are separately continuous in each variable. In this case,

$$\partial^j_{X_0} f(u).(x_1,\ldots,x_j) = \partial^j_{x_1\ldots x_j} f(u) \qquad \forall u \in U,\ \forall x_1,\ldots,x_j \in X_0,\ j = 1,\ldots,n. \tag{1.1.2}$$

Proof. Suppose that the derivatives $\partial^j_{x_1\ldots x_j} f(u)$ exists for all $u \in U$, $x_1,\ldots,x_j \in X_0$, $j = 1,\ldots,n$, separately continuous in $u, x_1,\ldots,x_j$. We want to show that $f \in \mathscr{G}^n(U,Y;X_0)$.

We proceed by induction on n. Let $n = 1$. Since $\partial_x f(u)$ is continuous in u, for all $x \in X_0$, we have that $X_0 \to Y$, $x \mapsto \partial_x f(u)$, is linear ([44, Lemma 4.1.5]). By assumption, it is also continuous. Hence $x \mapsto \partial_x f(u) \in L(X_0,Y)$ for all $u \in U$. This shows the existence of $\partial_{X_0} f$. The continuity of $U \to L_s(X_0,Y)$, $u \mapsto \partial_{X_0} f(u)$, comes from the separate continuity of

(1.1.1) and from the definition of the locally convex topology on $L_s(X_0,Y)$. This shows that $f \in \mathscr{G}^1(U,Y;X_0)$.

Let now $n>1$. By inductive hypothesis, we may assume that $f \in \mathscr{G}^{n-1}(U,Y;X_0)$ and

$$\partial^j_{X_0} f(u).(x_1,\dots,x_j) = \partial^j_{x_1\dots x_j} f(u) \qquad \forall u \in U,\ \forall j=1,\dots,n-1,\ \forall (x_1,\dots,x_j) \in X_0^j.$$

Let $x_n \in X_0$. The limit

$$\lim_{t\to 0} \frac{\partial^{n-1}_{X_0} f(u+tx_n) - \partial^{n-1}_{X_0} f(u)}{t} = \Lambda \tag{1.1.3}$$

exists in $L_s^{(n-1)}(X_0^{n-1},Y)$ if and only if $\Lambda \in L_s^{(n-1)}(X_0^{n-1},Y)$ and, for all $x_1,\dots,x_{n-1} \in X_0$, the limit

$$\lim_{t\to 0} \frac{\partial^{n-1}_{x_1\dots x_{n-1}} f(u+tx_n) - \partial^{n-1}_{x_1\dots x_{n-1}} f(u)}{t} = \Lambda(x_1,\dots,x_{n-1}) \tag{1.1.4}$$

holds in Y. By assumption, the limit (1.1.4) is equal to $\partial^n_{x_1\dots x_{n-1}x_n} f(u)$, for all $x_1,\dots,x_{n-1}$. Since, by assumption, $\partial^n_{x_1\dots x_{n-1}x_n} f(u)$ is separately continuous in $u,x_1,\dots,x_{n-1},x_n$, we have that the limit (1.1.3) exists in $L_s^{(n-1)}(X_0^{n-1},Y)$ and is given by

$$\partial_{x_n}\partial^{n-1}_{X_0} f(u).(x_1,\dots,x_{n-1}) = \Lambda(x_1,\dots,x_{n-1}) = \partial^n_{x_1\dots x_{n-1}x_n} f(u) \qquad \forall x_1,\dots,x_{n-1} \in X_0.$$

Since u and x_n were arbitrary, we have proved that $\partial_{x_n}\partial^{n-1}_{X_0} f(u)$ exists for all u, x_n. Moreover, for all $x_1,\dots,x_n \in X_0$, the function

$$U \to Y,\ u \mapsto \partial_{x_n}\partial^{n-1}_{X_0} f(u).(x_1,\dots,x_{n-1}) = \partial_{x_n}\partial^n_{x_1\dots x_{n-1}} f(u)$$

is continuous, by separate continuity of (1.1.1). Then $\partial^n_{x_1\dots x_{n-1}x_n} f(u)$ is linear in x_n. The continuity of

$$X_0 \to L_s^{(n-1)}(X_0^{n-1},Y),\ x \mapsto \partial_x\partial^{n-1}_{X_0} f(u) \tag{1.1.5}$$

comes from the continuity of $\partial^n_{x_1\dots x_{n-1}x} f(u)$ in each variable, separately. Hence (1.1.5) belongs to $L_s(X_0, L_s^{n-1}(X_0^{n-1},Y))$ for all $u \in U$. This shows that $\partial^{n-1}_{X_0} f$ is Gâteaux differentiable with respect to X_0 and that

$$\partial^n_{X_0} f(u).(x_1,\dots,x_n) = \partial^n_{x_1\dots x_n} f(u) \qquad \forall u \in U,\ \forall x_1,\dots,x_n \in X_0,$$

and shows also the continuity of

$$U \to L_s^{(n)}(X_0^n,Y),\ u \mapsto \partial^n_{X_0} f(u),$$

due to the continuity of the derivatives of f, separately in each direction. Then we have proved that $f \in \mathscr{G}^n(U,Y;X_0)$ and that (1.1.2) holds.

Now suppose that $f \in \mathscr{G}^n(U,Y;X_0)$. By the very definition of $\partial_{X_0} f$, $\partial_x f(u)$ exists for all $x \in X_0$ and $u \in U$, it is separately continuous in u,x, and coincides with $\partial_{X_0} f(u).x$. By induction, assume that $\partial^{n-1}_{x_1\dots x_{n-1}} f(u)$ exists and that

$$\partial^{n-1}_{X_0} f(u).(x_1,\dots,x_{n-1}) = \partial^{n-1}_{x_1\dots x_{n-1}} f(u) \qquad \forall u \in U,\ \forall x_1,\dots,x_{n-1} \in X_0. \tag{1.1.6}$$

Since $\partial^{n-1}_{X_0} f(u)$ is Gâteaux differentiable, the directional derivative $\partial_{x_n}\partial^{n-1}_{X_0} f(u)$ exists. Hence, by (1.1.6), the derivative $\partial^n_{x_1\ldots x_{n-1}x_n} f(u)$ exists for all $x_1,\ldots,x_{n-1},x_n \in X_0$. The continuity of $\partial^n_{x_1\ldots x_{n-1}x_n} f(u)$ with respect to u comes from the continuity of $\partial^n_{X_0} f$. The continuity of $\partial^n_{x_1\ldots x_j\ldots x_n} f(u)$ with respect to x_j comes from the fact that, for all $u \in U$, $x_{j+1},\ldots,x_n \in X_0$,

$$X_0^j \to Y,\ (x'_1,\ldots,x'_j) \mapsto \partial^n_{X_0} f(u).(x'_1,\ldots,x'_j,x_{j+1},\ldots,x_n)$$

belongs to $L_s^{(j)}(X_0^j,Y)$. ∎

Remark 1.1.3. If X_0 is Banach, X is normed, Y is locally convex, and $f \in \mathscr{G}^n(X,Y;X_0)$, then, by Proposition 1.1.2 and the Banach-Steinhaus theorem, if follows that the map

$$U \times X_0^n \to Y,\ (u,x_1,\ldots,x_n) \mapsto \partial^n_{X_0} f(u).(x_1,\ldots,x_n)$$

is continuous, jointly in $u,x_1,\ldots,x_n$.

Remark 1.1.4. Under the assumption of Proposition 1.1.2, by Schwarz' theorem,

$$y^*(\partial^2_{zw} f(u)) = \partial^2_{zw}(y^* f)(u) = \partial^2_{wz}(y^* f)(u) = y^*(\partial^2_{wz} f(u)),\ \forall u \in U,\ \forall w,x \in X_0,\ \forall y^* \in Y^*.$$

Hence $\partial^2_{wz} f = \partial^2_{wz} f$ for all $w,z \in X_0$.

Chain rule

In this subsection, we show the classical Faà di Bruno's formula, together with a corresponding stability result, for derivatives of order $n \geq 1$ of compositions of strongly continuously Gâteaux differentiable functions. We will use this formula in order to prove the main results of Section 1.1.3 (Theorem 1.1.13 and Proposition 1.1.15).

In [11], a version of Proposition 1.1.7 is provided for the case of "chain differentials". We could prove that the strongly continuously Gâteaux differentiable functions that we consider satisfy the assumptions of [11, Theorem 2]. This would provide Proposition 1.1.7 as a corollary of [11, Theorem 2]. Since the proof of Proposition 1.1.7 is quite concise, we prefer to report it, and avoid introducing other notions of differential. Besides, we give the related stability results.

Lemma 1.1.5. *Let $k \geq 0$, let X_1, X_2, X_3 be Banach spaces, let U be an open subset of X_1, and let X_0 be a subspace of X_1. Let $f,f_1,\ldots,f_k\colon U \to X_2$ be functions having directional derivatives $\partial_x f,\partial_x f_1,\ldots,\partial_x f_k$ with respect to all $x \in X_0$ and let $g \in \mathscr{G}^{k+1}(X_2,X_3)$. Then*

$$\gamma\colon U \to X_3,\ u \mapsto \partial^k_{f_1(u)\ldots f_k(u)} g(f(u)) \tag{1.1.7}$$

has directional derivatives $\partial_x\gamma$ with respect to all $x \in X_0$ and

$$\partial_x\gamma(u) = \partial^{k+1}_{\partial_x f(u) f_1(u)\ldots f_k(u)} g(f(u)) + \sum_{i=1}^{k} \partial^k_{f_1(u)\ldots\partial_x f_i(u)\ldots f_k(u)} g(f(u))\ \ \forall u \in U,\ \forall x \in X_0. \tag{1.1.8}$$

If X_0 is a Banach space continuously embedded in X_1 and if $f, f_1, \ldots, f_k \in \mathcal{G}^1(U, X_2; X_0)$, then $\gamma \in \mathcal{G}^1(U, X_3; X_0)$.

Proof. Let $u \in U$, $x \in X_0$, and let $[u - \varepsilon x, u + \varepsilon x] \subset U$, for some $\varepsilon > 0$. Let $h \in [-\varepsilon, \varepsilon] \setminus \{0\}$. By strong continuity of $\partial^{k+1} g$ and by k-linearity of $\partial^k_{x_1 \ldots x_k} g$ with respect to $x_1, \ldots, x_k$, we can write

$$\begin{aligned}
\frac{\gamma(u+hx) - \gamma(u)}{h} &= \\
&= \frac{1}{h}\Big(\partial^k_{f_1(u+hx) f_2(u+hx) \ldots f_k(u+hx)} g(f(u+hx)) - \partial^k_{f_1(u+hx) f_2(u+hx) \ldots f_k(u+hx)} g(f(u))\Big) \\
&\quad + \frac{1}{h} \sum_{i=1}^k \Big(\partial^k_{f_1(u) f_2(u) \ldots f_{i-1}(u) f_i(u+hx) f_{i+1}(u+hx) \ldots f_k(u+hx)} g(f(u)) \\
&\qquad\qquad - \partial^k_{f_1(u) f_2(u) \ldots f_{i-1}(u) f_i(u) f_{i+1}(u+hx) \ldots f_k(u+hx)} g(f(u))\Big) \\
&= \int_0^1 \partial_{\frac{f(u+hx) - f(u)}{h}} \partial^k_{f_1(u+hx) f_2(u+hx) \ldots f_k(u+hx)} g(f(u) + \theta(f(u+hx) - f ux))) d\theta \\
&\quad + \sum_{i=1}^k \partial^k_{f_1(u) f_2(u) \ldots f_{i-1}(u) \frac{f_i(u+hx) - f_i(u)}{h} f_{i+1}(u+hx) \ldots f_k(u+hx)} g(f(u)).
\end{aligned}$$

By continuity of $f, f_1, \ldots, f_k$ on the set $(u + \mathbb{R}x) \bigcap U$ and by joint continuity of $\partial^{k+1} g$, the integrand function is uniformly continuous in $(h, \theta) \in ([-\varepsilon, \varepsilon] \setminus \{0\}) \times [0,1]$. Then we can pass to the limit $h \to 0$ and obtain (1.1.8).

If $f, f_1, \ldots, f_k \in \mathcal{G}^1(U, X_2; X_0)$, then the strong continuity of $\partial_{X_0} \gamma$ comes from Proposition 1.1.2 and formula (1.1.8), by recalling also Remark 1.1.3. ■

Lemma 1.1.6. *Let $n \in \mathbb{N}$. Let X_0, X_1, X_2, X_3 be Banach spaces, with X_0 continuously embedded in X_1, and let $U \subset X_1$ be an open set. Let*

$$\begin{cases}
f_0, \ldots, f_n \in \mathcal{G}^1(U, X_2; X_0) \\
f_0^{(k)}, \ldots, f_n^{(k)} \in \mathcal{G}^1(U, X_2; X_0) & \forall k \in \mathbb{N} \\
g \in \mathcal{G}^{n+1}(X_2, X_3) \\
g^{(k)} \in \mathcal{G}^{n+1}(X_2, X_3) & \forall n \in \mathbb{N}.
\end{cases}$$

Suppose that, for $i = 0, \ldots, n$,

$$\begin{cases}
\lim_{k \to \infty} f_i^{(k)}(u) = f_i(u) \\
\lim_{k \to \infty} \partial_x f_i^{(k)}(u) = \partial_x f_i(u),
\end{cases}$$

uniformly for u on compact subsets of U and x on compact subsets of X_0, and that

$$\lim_{k \to \infty} \partial^j_{x_1 \ldots x_j} g^{(k)}(x_0) = \partial^j_{x_1 \ldots x_j} g(x_0) \qquad j = n, n+1,$$

uniformly for $x_0, x_1, \dots, x_j$ on compact subsets of X_2. Define

$$\begin{cases} \gamma & : U \to X_3,\ u \mapsto \partial^n_{f_1(u)\dots f_n(u)} g(f_0(u)) \\ \gamma^{(k)} & : U \to X_3,\ u \mapsto \partial^n_{f_1^{(k)}(u)\dots f_n^{(k)}(u)} g^{(k)}(f_0^{(k)}(u)), \qquad \forall k \in \mathbb{N}. \end{cases} \tag{1.1.9}$$

Then

$$\lim_{k\to\infty} \partial_x \gamma^{(k)}(u) = \partial_x \gamma(u) \tag{1.1.10}$$

uniformly for u on compact subsets of U and x on compact subsets of X_0.

Proof. Since the composition of sequences of continuous functions uniformly convergent on compact sets is convergent to the composition of the limits, uniformly on compact sets, it is sufficient to recall Remark 1.1.3, apply Lemma 1.1.5, and consider (1.1.8). ■

Let X_0, X_1 be Banach spaces, with X_0 continuously embedded in X_1, and let U be an open subset of X_1. Let $n \in \mathbb{N}$, $n \geq 1$, $\mathbf{x}_n := \{x_1, \dots, x_n\} \subset X_0^n$, $j \in \{1, \dots, n\}$. Then

- $P^j(\mathbf{x}_n)$ denotes the set of partitions of $\mathbf{x}_n$ in j non-empty subsets.
- If $f \in \mathscr{G}^n(U, X_1; X_2)$ and $\mathbf{q} := \{y_1, \dots, y_j\} \subset \mathbf{x}_n$, then $\partial^j_{\mathbf{q}} f(u)$ denotes the derivative $\partial^j_{y_1\dots y_j} f(u)$ ([3]).
- $|\mathbf{q}|$ denotes the cardinality of $\mathbf{q}$.

Proposition 1.1.7 (Faà di Bruno's formula). *Let $n \geq 1$. Let X_0, X_1, X_2, X_3 be Banach spaces, with X_0 continuously embedded in X_1, and let U be an open subset of X_1. If $f \in \mathscr{G}^n(U, X_2; X_0)$ and $g \in \mathscr{G}^n(X_2, X_3)$, then $g \circ f \in \mathscr{G}^n(U, X_3; X_0)$. Moreover*

$$\partial^j_{\mathbf{x}_j} g \circ f(u) = \sum_{i=1}^{j} \sum_{\{\mathbf{p}_1^i, \dots, \mathbf{p}_i^i\} \in P^i(\mathbf{x}_j)} \partial^j_{\partial^{|\mathbf{p}_1^i|}_{\mathbf{p}_1^i} f(u) \dots \partial^{|\mathbf{p}_i^i|}_{\mathbf{p}_i^i} f(u)} g(f(u)). \tag{1.1.11}$$

for all $u \in U$, $j = 1, \dots, n$, $\mathbf{x}_j = \{x_1, \dots, x_j\} \subset X_0^j$.

Proof. The proof is standard and is obtained by induction on n and by making use of Lemma 1.1.5 at each step of the inductive argument. The case $n = 1$ is obtained by applying Lemma 1.1.5 with $k = 0$. Now consider the case $n \geq 2$. By inductive hypothesis, formula (1.1.11) holds true for $j = 1, \dots, n-1$, and we need to prove that it holds for $j = n$. Let $u \in U$, $x_1, \dots, x_n \in X_0$, $\mathbf{x}_{n-1} := \{x_1, \dots, x_{n-1}\}$. Then, by (1.1.11),

$$\partial^{n-1}_{x_1\dots x_{n-1}} g \circ f(u) = \sum_{i=1}^{n-1} \sum_{\{\mathbf{p}_1^i, \dots, \mathbf{p}_i^i\} \in P^i(\mathbf{x}_{n-1})} \partial^{n-1}_{\partial^{|\mathbf{p}_1^i|}_{\mathbf{p}_1^i} f(u) \dots \partial^{|\mathbf{p}_i^i|}_{\mathbf{p}_i^i} f(u)} g(f(u)).$$

[3] By Remark 1.1.4, there is no ambiguity due to the fact that $\mathbf{q}$ is not ordered.

By applying Lemma 1.1.5, with $k = i$ and $f_j = \partial^{|\mathbf{p}^i_j|}_{\mathbf{p}^i_j} f$, for $j = 1,\dots,i$, to each member of the sum over $P^i(\mathbf{x}_{n-1})$, we obtain, for all $x_n \in X_0$,

$$\begin{aligned}
\partial^n_{x_1\dots x_{n-1}x_n} g\circ f(u) &= \sum_{i=1}^{n-1} \sum_{\{\mathbf{p}^i_1,\dots,\mathbf{p}^i_i\}\in P^i(\mathbf{x}_{n-1})} \Big(\partial^n_{\partial_{x_n} f(u)\partial^{|\mathbf{p}^i_1|}_{\mathbf{p}^i_1} f(u)\dots\partial^{|\mathbf{p}^i_i|}_{\mathbf{p}^i_i} f(u)} g(f(u)) \\
&\qquad + \sum_{l=1}^{i} \partial^n_{\partial^{|\mathbf{p}^i_1|}_{\mathbf{p}^i_1} f(u)\dots\partial_{x_n}\partial^{|\mathbf{p}^i_l|}_{\mathbf{p}^i_l} f(u)\dots\partial^{|\mathbf{p}^i_i|}_{\mathbf{p}^i_i} f(u)} g(f(u))\Big) \\
&= \sum_{i=1}^{n-1}\left(\sum_{\substack{\{\mathbf{p}^{i+1}_1,\dots,\mathbf{p}^{i+1}_{i+1}\}\in P^{i+1}(\mathbf{x}_n):\\ \{x_n\}\in\{\mathbf{p}^{i+1}_1,\dots,\mathbf{p}^{i+1}_{i+1}\}}} \partial^n_{\partial^{|\mathbf{p}^{i+1}_1|}_{\mathbf{p}^{i+1}_1} f(u)\dots\partial^{|\mathbf{p}^{i+1}_{i+1}|}_{\mathbf{p}^{i+1}_{i+1}} f(u)} g(f(u)) \right. \\
&\qquad \left. + \sum_{\substack{\{\mathbf{p}^i_1,\dots,\mathbf{p}^i_i\}\in P^i(\mathbf{x}_n):\\ \{x_n\}\notin\{\mathbf{p}^i_1,\dots,\mathbf{p}^i_i\}}} \partial^n_{\partial^{|\mathbf{p}^i_1|}_{\mathbf{p}^i_1} f(u)\dots\partial^{|\mathbf{p}^i_i|}_{\mathbf{p}^i_i} f(u)} g(f(u)) \right) \\
&= \sum_{i=1}^{n} \sum_{\{\mathbf{p}^i_1,\dots,\mathbf{p}^i_i\}\in P^i(\mathbf{x}_n)} \partial^n_{\partial^{|\mathbf{p}^i_1|}_{\mathbf{p}^i_1} f(u)\dots\partial^{|\mathbf{p}^i_i|}_{\mathbf{p}^i_i} f(u)} g(f(u)).
\end{aligned}$$

This concludes the proof of (1.1.11). ■

Proposition 1.1.8. *Let $n \geq 1$. Let X_0, X_1, X_2, X_3 be Banach spaces, with X_0 continuously embedded in X_1, and let U be an open subset of X_1. Let*

$$\begin{cases} f \in \mathcal{G}^n(U, X_2; X_0) \\ f^{(k)} \in \mathcal{G}^n(U, X_2; X_0) & \forall k \in \mathbb{N} \\ g \in \mathcal{G}^n(X_2, X_3) \\ g^{(k)} \in \mathcal{G}^n(X_2, X_3) & \forall k \in \mathbb{N}. \end{cases}$$

Suppose that

$$\begin{cases} \lim_{k\to\infty} f^{(k)}(u) = f(u) \\ \lim_{k\to\infty} \partial^j_{x_1\dots x_j} f^{(k)}(u) = \partial^j_{x_1\dots x_j} f(u) & \text{for } j = 1,\dots,n, \end{cases}$$

uniformly for u on compact subsets of U and $x_1,\dots,x_j$ on compact subsets of X_0, and that

$$\begin{cases} \lim_{k\to\infty} g^{(k)}(x) = g(x) \\ \lim_{k\to\infty} \partial^j_{x_1\dots x_j} g^{(k)}(x) = \partial^j_{x_1\dots x_j} f(x) & \text{for } j = 1,\dots,n, \end{cases}$$

uniformly for $x, x_1,\dots,x_j$ on compact subsets of X_2. Then

$$\begin{cases} \lim_{k\to\infty} g^{(k)}\circ f^{(k)}(u) = g\circ f(u) \\ \lim_{k\to\infty} \partial^j_{x_1\dots x_j} g^{(k)}\circ f^{(k)}(u) = \partial^j_{x_1\dots x_j} g\circ f(u) & \text{for } j = 1,\dots,n, \end{cases}$$

uniformly for u on compact subsets of U and $x_1,\dots,x_j$ on compact subsets of X_0.

Proof. Use recursively formula (1.1.11) and Lemma 1.1.6. ■

1.1.2 Contractions in Banach spaces: survey of basic results

In this section, we assume that X and Y are Banach spaces, and that U is an open subset of X. We recall that, if $\alpha \in [0,1)$ and $h\colon U \times Y \to Y$, then h is said a *parametric α-contraction* if

$$|h(u,y)-h(u,y')|_Y \leq \alpha|y-y'| \qquad \forall u \in U,\ \forall y,y' \in Y.$$

By the Banach contraction principle, to any such h we can associate a uniquely defined map $\varphi\colon U \to Y$ such that $h(u,\varphi(u)) = \varphi(u)$ for all $u \in U$. We refer to φ as to *the fixed-point map associated to h*. For future reference, we summurize some basic continuity properties that φ inherites from h.

The following lemma can be found in [55, p. 13].

Lemma 1.1.9. *Let $\alpha \in [0,1)$ and let $h(u,\cdot)\colon U \times Y \to Y$, $h_n(u,\cdot) : U \times Y \to Y$, for $n \in \mathbb{N}$, be parametric α-contractions. Denote by φ (resp. φ_n) the fixed-point map associated to h (resp. h_n).*

(i) If $h_n \to h$ pointwise on $U \times Y$, then $\varphi_n \to \varphi$ pointwise on U.

(ii) If $A \subset U$ is a set and if there exists an increasing concave function w on $\mathbb{R}^+$ such that $w(0) = 0$ and

$$|h(u,y)-h(u',y)|_Y \leq w(|u-u'|_X) \qquad \forall u,u' \in A,\ \forall y \in Y, \tag{1.1.12}$$

then

$$|\varphi(u)-\varphi(u')|_Y \leq \frac{1}{1-\alpha} w(|u-u'|_X) \qquad \forall u,u' \in A.$$

(iii) If h is continuous, then φ is continuous.

Proof. Since h and h_n are α-contractions, we have

$$|\varphi_n(u)-\varphi(u')| \leq \frac{|h_n(u,\varphi(u)))-h(u',\varphi(u'))|}{1-\alpha}, \tag{1.1.13}$$

$$|\varphi(u)-\varphi(u')| \leq \frac{|h(u,\varphi(u'))-h(u',\varphi(u'))|}{1-\alpha}, \tag{1.1.14}$$

for all $u,u' \in U$. Then (1.1.13) yields *(i)* by taking $u = u'$ and letting $n \to \infty$, and (1.1.14) yields *(ii)* by using (1.1.12).

Regarding *(iii)*, let $u' \in U$, $u_n \to u'$ in U, let $V \subset U$ be an open set containing u', and let $\bar{n} \in \mathbb{N}$ such that $u_n - u' + V \subset U$ for all $n \geq \bar{n}$. Define $h_n\colon V \times Y \to Y$ by $h_n(u,y) :=$

$h(u+u_n-u',y)$ for all $(u,y)\in V\times Y$. Then h_n is a parametric α-contraction. Denote by φ_n its associated fixed-point map. Then, by continuity of h and by *(i)*, $\varphi_n(u)\to\varphi(u)$ for all $u\in V$. In particular, $\varphi(u_n)=\varphi_n(u')\to\varphi(u')$, hence φ is continuous. ■

Remark 1.1.10. If $h\colon U\times Y\to Y$ is a parametric α-contraction ($\alpha\in[0,1)$) belonging to $\mathscr{G}^1(U\times Y,Y;\{0\}\times Y)$, then

$$|\partial_Y h(u,y)|_{L(Y)}\leq\alpha \qquad \forall u\in U,\ y\in Y, \tag{1.1.15}$$

where $|\cdot|_{L(Y)}$ denotes the operator norm on $L(Y)$. Hence $\partial_Y h(u,y)$ is invertible and the family $\{(I-\partial_Y h(u,y))^{-1}\}_{(u,y)\in U\times Y}$ is uniformly bounded in $L(Y)$. For what follows, it is important to notice that, for all $y\in Y$,

$$U\times Y\to Y,\ (u,y')\mapsto(I-\partial_Y h(u,y'))^{-1}y \tag{1.1.16}$$

is continuous, hence, because of the formula

$$(I-\partial_Y h(u,y'))^{-1}y=\sum_{n\in\mathbb{N}}\left(\partial_Y h(u,y')\right)^n y$$

and of Lebesgue's dominated convergence theorem (for series), $(I-\partial_Y h(u,y'))^{-1}y$ is jointly continuous in u,y',y.

The following proposition shows that the fixed-point map φ associated to a parametric α-contraction h inherits from h the strongly continuous Gâteaux differentiability.

Proposition 1.1.11. *If $h\in\mathscr{G}^1(U\times Y,Y)$ is a parametric α-contraction and if φ is the fixed-point map associated to h, then $\varphi\in\mathscr{G}^1(U,Y)$ and*

$$\partial_x\varphi(u)=\left(I-\partial_Y h\left(u,\varphi(u)\right)\right)^{-1}\left(\partial_x h\left(u,\varphi(u)\right)\right) \qquad \forall u\in U,\forall x\in X. \tag{1.1.17}$$

Proof. For the proof, see [24, Lemma 2.9], or [8, Proposition C.0.3], taking into account also [8, Remark C.0.4], Lemma 1.1.9*(iii)*, Remark 1.1.10. ■

1.1.3 Gâteaux differentiability of order n of fixed-point maps

In this section we provide a result for the Gâteux differentiability up to a generic order n of a fixed-point map φ associated to a parametric α-contraction h, under the assumption that h is Gâteaux differentiable only with respect to some invariant subspaces of the domain.

The main result of this section is Theorem 1.1.13, which is suitable to be applied to mild solutions to SDEs (Section 1.2.2). When $n=1$, Theorem 1.1.13 reduces to Proposition 1.1.11. In the case $n=2$, Theorem 1.1.13 is also well-known, and a proof can be found in [8, Proposition C.0.5]. On the other hand, when the order of differentiability

n is generic, the fact that the parametric α-contraction is assumed to be differentiable only with respect to certain subspaces makes non-trivial the proof of the theorem. To our knowledge, a reference for the case $n \geq 3$ is not available in the literature. The main issue consists in providing a precise formulation of the statement, with its assumptions, that can be proved by induction.

For the sake of readability, we collect the assumptions of Theorem 1.1.13 in the following

Assumption 1.1.12.

(1) $n \geq 1$ *and* $\alpha \in [0,1)$*;*

(2) X *is a Banach space and* U *is an open subset of* X.

(3) $Y_1 \supset Y_2 \supset \ldots \supset Y_n$ *is a decreasing sequence of Banach spaces, with norms* $|\cdot|_1, \ldots, |\cdot|_n$*, respectively.*

(4) For $k = 1,\ldots,n$ *and* $j = 1,2,\ldots,k$*, the canonical embedding of* Y_k *into* Y_j*, denoted by* $i_{k,j}\colon Y_k \to Y_j$*, is continuous.*

(5) $h_1\colon U \times Y_1 \to Y_1$ *is a function such that* $h_1(U \times Y_k) \subset Y_k$ *for* $k = 2,\ldots,n$*. For* $k = 2,\ldots,n$*, we denote by* h_k *the induced function*

$$h_k\colon U \times Y_k \to Y_k,\ (u,y) \mapsto h_1(u,y). \tag{1.1.18}$$

(6) For $k = 1,\ldots,n$*,* h_k *is continuous and satisfies*

$$\left|h_k(u,y) - h_k(u,y')\right|_k \leq \alpha |y - y'|_k \qquad \forall u \in U,\ \forall y,y' \in Y_k. \tag{1.1.19}$$

(7) For $k = 1,\ldots,n$*,* $h_k \in \mathscr{G}^n(U \times Y_k, Y_k; X \times \{0\})$*.*

(8) For $k = 1,\ldots,n-1$*,* $h_k \in \mathscr{G}^n(U \times Y_k, Y_k; X \times Y_{k+1})$

(9) For $k = 1,\ldots,n$*,* $j = 1,\ldots,n-1$*, for all* $u \in U$*,* $z_1,\ldots,z_j \in X$*,* $y, z_{j+1} \in Y_k$*, and for all permutations* σ *of* $\{1,\ldots,j+1\}$*, the directional derivative* $\partial^{j+1}_{z_{\sigma(1)}\ldots z_{\sigma(j+1)}} h_k(u,y)$ *exists, and*

$$U \times Y_k \times X^j \times Y_k \to Y_k,\ (u,y,z_1,\ldots,z_j,z_{j+1}) \mapsto \partial^{j+1}_{z_{\sigma(1)}\ldots z_{\sigma(j)} z_{\sigma(j+1)}} h_k(u,y) \tag{1.1.20}$$

is continuous.

Theorem 1.1.13. *Let Assumption 1.1.12 be satisfied and let* $\varphi\colon U \to Y_1$ *denote the fixed-point function associated to the parametric* α*-contraction* h_1*. Then, for* $j = 1,\ldots,n$*, we* *have* $\varphi \in \mathscr{G}^j(U, Y_{n-j+1})$ *and, for all* $u \in U$, $x_1, \ldots, x_j \in X$, $\partial^j_{x_1 \ldots x_j}\varphi(u)$ *is given by the formula*

$$\begin{aligned}\partial^j_{x_1\ldots x_j}\varphi(u) &= \left(I - \partial_{Y_1} h_1(u,\varphi(u))\right)^{-1} \partial^j_{x_1\ldots x_j} h_1(u,\varphi(u))\\ &+ \sum_{\substack{\mathbf{x}\in 2^{\{x_1,\ldots,x_j\}}\\ \mathbf{x}\neq\emptyset}} \sum_{i=\max\{1,2-j+|\mathbf{x}|\}}^{|\mathbf{x}|} \sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\ldots,\mathbf{p}_i)}} \left(I - \partial_{Y_1} h_1(u,\varphi(u))\right)^{-1} \partial^j[\mathbf{x}^c,\mathbf{p}] h_1(u,\varphi(u))\end{aligned} \tag{1.1.21}$$

where $2^{\{x_1,\ldots,x_i\}}$ *is the power set of* $\{x_1,\ldots,x_i\}$, $P^i(\mathbf{x})$ *is the set of partitions of* $\mathbf{x}$ *in* i *non-empty parts,* $\mathbf{x}^c := \{x_1,\ldots,x_j\}\setminus\mathbf{x}$, *and* $\partial^j[\mathbf{x}^c,\mathbf{p}] := \partial^{j-|\mathbf{x}|}_{\mathbf{x}^c}\partial^{|\mathbf{x}|}_{\partial^{|\mathbf{p}_1|}_{\mathbf{p}_1}\varphi(u),\ldots,\partial^{|\mathbf{p}_i|}_{\mathbf{p}_i}\varphi(u)}$ *(*[4]*).*

Proof. The proof is by induction on n. The case $n = 1$ is provided by Proposition 1.1.11.

Let $n \geq 2$. Clearly, it is sufficient to prove that $\varphi \in \mathscr{G}^n(U, Y_n)$ and that (1.1.21) holds true for $j = n$. Since we are assuming that the theorem holds true for $n-1$, we can apply it with the data

$$\widetilde{h}_1 : U \times \widetilde{Y}_2 \to \widetilde{Y}_2,\ \ldots, \widetilde{h}_{n-1} : U \times \widetilde{Y}_n \to \widetilde{Y}_n,$$

where $\widetilde{h}_k := h_{k+1}$, $\widetilde{Y}_k := Y_{k+1}$, for $k = 1,\ldots,n-1$. According to the claim, the fixed-point function $\widetilde{\varphi}$ of $\widetilde{h}_1$ belongs to $\mathscr{G}^j(U, \widetilde{Y}_{(n-1)-j+1})$, for $j = 1,\ldots,n-1$, and formula (1.1.21) holds true for $\widetilde{\varphi}$ and $j = 1,\ldots,n-1$. Since $\varphi(u) = (i_{2,1}\circ\widetilde{\varphi})(u)$, for $u \in U$, we have $\varphi \in \mathscr{G}^j(U, \widetilde{Y}_{n-j}) = \mathscr{G}^j(U, Y_{n-j+1})$, for $j = 1,\ldots,n-1$, and

$$\partial^j_{x_1\ldots x_j}\varphi(u) = \partial^j_{x_1\ldots x_j}\widetilde{\varphi}(u) \in \widetilde{Y}_{n-j} = Y_{n-j+1}, \qquad \forall u \in U,\ \forall x_1,\ldots,x_j \in X.$$

Then (1.1.21) holds true for φ up to order $j = n-1$. In particular $\varphi \in \mathscr{G}^{n-1}(U, Y_2)$, hence, for $x_1,\ldots,x_n \in X$, $\varepsilon > 0$, we can write

$$\begin{aligned}&\partial^{n-1}_{x_1\ldots x_{n-1}}\varphi(u+\varepsilon x_n) - \partial^{n-1}_{x_1\ldots x_{n-1}}\varphi(u)\\ &= \left(\partial_{Y_1} h_1(u+\varepsilon x_n,\varphi(u+\varepsilon x_n)).\partial^{n-1}_{x_1\ldots x_{n-1}}\varphi(u+\varepsilon x_n) - \partial_{Y_1} h_1(u,\varphi(u)).\partial^{n-1}_{x_1\ldots x_{n-1}}\varphi(u)\right)\\ &\quad + (\mathscr{S}(u+\varepsilon x_n) - \mathscr{S}(u))\\ &=: \mathbf{I} + \mathbf{II},\end{aligned} \tag{1.1.22}$$

where $\mathscr{S}(\cdot)$ denotes the sum

$$\mathscr{S}(v) := \partial^{n-1}_{x_1\ldots x_{n-1}} h_1(v,\varphi(v)) + \sum_{\substack{\mathbf{x}\in 2^{\{x_1,\ldots,x_{n-1}\}}\\ \mathbf{x}\neq\emptyset}} \sum_{i=\max\{1,2-(n-1)+|\mathbf{x}|\}}^{|\mathbf{x}|} \sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\ldots,\mathbf{p}_i)}} \partial^{n-1}[\mathbf{x}^c,\mathbf{p}] h_1(v,\varphi(v)),$$

for $v \in U$. By recalling that $\varphi \in \mathscr{G}^j(U, Y_{n-j+1})$, $j = 1,\ldots,n-1$, hence by taking into account

[4]Recall notation at p. 22.

with respect to which space the derivatives of φ are continuous, we write

$$\begin{aligned}\mathbf{I} =&\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u+\varepsilon x_n)}h_1(u+\varepsilon x_n,\varphi(u+\varepsilon x_n)) - \partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}h_1(u,\varphi(u))\\ =&\int_0^1 \partial_{x_n}\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u+\varepsilon x_n)}h_1(u+\theta\varepsilon x_n,\varphi(u+\varepsilon x_n))\varepsilon d\theta\\ &+\int_0^1 \partial_{\frac{\varphi(u+\varepsilon x_n)-\varphi(u)}{\varepsilon}}\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u+\varepsilon x_n)}h_1(u,\varphi(u)+\theta(\varphi(u+\varepsilon x_n)-\varphi(u)))\varepsilon d\theta\\ &+\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u+\varepsilon x_n)-\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}h_1(u,\varphi(u))\\ =&\mathbf{I_1}+\mathbf{I_2}+\partial_{Y_1}h_1(u,\varphi(u)).\left(\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u+\varepsilon x_n)-\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)\right),\end{aligned} \tag{1.1.23}$$

with ([5])

$$\lim_{\varepsilon\to 0}\frac{\mathbf{I_1}}{\varepsilon} = \partial_{x_n}\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}h_1(u,\varphi(u)) \qquad \text{and} \qquad \lim_{\varepsilon\to 0}\frac{\mathbf{I_2}}{\varepsilon} = \partial_{\partial_{x_n}\varphi(u)}\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}h_1(u,\varphi(u)).$$

In a similar way,

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\mathbf{II}}{\varepsilon} =& \partial_{x_n}\partial^{n-1}_{x_1\dots x_{n-1}}h_1(u,\varphi(u))+\partial_{\partial_{x_n}\varphi(u)}\partial^{n-1}_{x_1\dots x_{n-1}}h_1(u,\varphi(u))\\ &+\sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_{n-1}\}}\\ \mathbf{x}\neq\emptyset}}\sum_{i=\max\{1,2-(n-1)+|\mathbf{x}|\}}^{|\mathbf{x}|}\sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)}}\partial_{x_n}\partial^{n-1}[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u))\\ &+\sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_{n-1}\}}\\ \mathbf{x}\neq\emptyset}}\sum_{i=\max\{1,2-(n-1)+|\mathbf{x}|\}}^{|\mathbf{x}|}\sum_{\substack{\mathbf{p}\in P^i(\mathbf{p})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)}}\left(\partial_{\partial_{x_n}\varphi(u)}\partial^{n-1}[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u))\right.\\ &\left.+\sum_{j=1}^{i}\partial^{|\mathbf{x}^c|}_{\mathbf{x}^c}\partial_{\partial^{|\mathbf{p}_1|}_{\mathbf{p}_1}\varphi(u)}\dots\partial_{\partial^{|\mathbf{p}_{j-1}|}_{\mathbf{p}_{j-1}}\varphi(u)}\partial_{\partial_{x_n}\partial^{|\mathbf{p}_j|}_{\mathbf{p}_j}\varphi(u)}\partial_{\partial^{|\mathbf{p}_{j+1}|}_{\mathbf{p}_{j+1}}\varphi(u)}\dots\partial_{\partial^{|\mathbf{p}_i|}_{\mathbf{p}_i}\varphi(u)}h_1(u,\varphi(u))\right).\end{aligned} \tag{1.1.24}$$

Notice that

$$\begin{aligned}&\sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_{n-1}\}}\\ \mathbf{x}\neq\emptyset}}\sum_{i=\max\{1,2-(n-1)+|\mathbf{x}|\}}^{|\mathbf{x}|}\sum_{\substack{p_\pi\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)}}\partial_{x_n}\partial^{n-1}[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u))\\ =&\sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_n\}}\\ \mathbf{x}\neq\emptyset\\ x_n\notin\mathbf{x}}}\sum_{i=\max\{1,2-n+|\mathbf{x}|\}}^{|\mathbf{x}|}\sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)}}\partial^n[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u))-\partial_{x_n}\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}h_1(u,\varphi(u))\end{aligned} \tag{1.1.25}$$

[5] The limits should be understood in the suitable spaces Y_k. For instance, when computing $\lim_{\varepsilon\to 0}\frac{\mathbf{I_1}}{\varepsilon}$, the object $\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u+\varepsilon x_n)$ should be considered in the space Y_2, which can be done thanks to the inductive hypothesis.

and

$$\sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_{n-1}\}}\\ \mathbf{x}\neq\emptyset}} \sum_{i=\max\{1,2-(n-1)+|\mathbf{x}|\}}^{|\mathbf{x}|} \sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)}} \partial_{\partial_{x_n}\varphi(u)}\partial^{n-1}[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u))$$

$$= \sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_n\}}\\ x_n\in\mathbf{x}\\ \mathbf{x}\neq\{x_n\}}} \sum_{i=\max\{1,2-n+|\mathbf{x}|\}}^{|\mathbf{x}|} \sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)\\ \{x_n\}\in\mathbf{p}}} \partial^n[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u)) \qquad (1.1.26)$$

$$-\partial_{\partial_{x_n}\varphi(u)}\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}h_1(u,\varphi(u))$$

and

$$\sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_{n-1}\}}\\ \mathbf{x}\neq\emptyset}} \sum_{i=\max\{1,2-(n-1)+|\mathbf{x}|\}}^{|\mathbf{x}|} \sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)}} \sum_{j=1}^{i} L(\mathbf{p},j;u)$$

$$= \sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_n\}}\\ x_n\in\mathbf{x}\\ \mathbf{x}\neq\{x_n\}}} \sum_{i=\max\{1,2-n+|\mathbf{x}|\}}^{|\mathbf{x}|} \sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)\\ \{x_n\}\notin\mathbf{p}}} \partial^n[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u)) \qquad (1.1.27)$$

where

$$L(\mathbf{p},j;u) := \partial^{|\mathbf{x}^c|}_{\mathbf{x}^c}\partial^{|\mathbf{x}|}_{\partial^{|\mathbf{p}_1|}_{\mathbf{p}_1}\varphi(u)\dots\partial^{|\mathbf{p}_{j-1}|}_{\mathbf{p}_{j-1}\varphi(u)}\partial_{x_n}\partial^{|\mathbf{p}_j|}_{\mathbf{p}_j}\varphi(u)\partial^{|\mathbf{p}_{j+1}|}_{\mathbf{p}_{j+1}}\varphi(u)\dots\partial^{|\mathbf{p}_i|}_{\mathbf{p}_i}\varphi(u)} h_1(u,\varphi(u)).$$

By collecting (1.1.24), (1.1.25), (1.1.26), (1.1.27), we obtain

$$\lim_{\varepsilon\to 0}\frac{\mathbf{II}}{\varepsilon} = \partial_{\partial_{x_n}\varphi(u)}\partial^{n-1}_{x_1\dots x_{n-1}}h_1(u,\varphi(u)) + \partial^n_{x_1\dots x_n}h_1(u,\varphi(u)) - \partial_{\partial_{x_n}\varphi(u)}\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}h_1(u,\varphi(u))$$

$$+ \sum_{\substack{\mathbf{x}\in 2^{\{x_1\dots x_n\}}\\ \mathbf{x}\neq\emptyset\\ \mathbf{x}\neq\{x_n\}}} \sum_{i=\max\{1,2-n+|\mathbf{x}|\}}^{|\mathbf{x}|} \sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)}} \partial^n[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u)) - \partial_{x_n}\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}h_1(u,\varphi(u))$$

$$= \partial^n_{x_1\dots x_n}h_1(u,\varphi(u)) - \partial_{\partial_{x_n}\varphi(u)}\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}h_1(u,\varphi(u))$$

$$+ \sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_n\}}\\ \mathbf{x}\neq\emptyset}} \sum_{i=\max\{1,2-n+|\mathbf{x}|\}}^{|\mathbf{x}|} \sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)}} \partial^n[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u)) - \partial_{x_n}\partial_{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}h_1(u,\varphi(u)).$$

Hence

$$\lim_{\varepsilon\to 0}\left(\frac{\mathbf{I_1}}{\varepsilon}+\frac{\mathbf{I_2}}{\varepsilon}+\frac{\mathbf{II}}{\varepsilon}\right) = \sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_n\}}\\ \mathbf{x}\neq\emptyset}} \sum_{i=\max\{1,2-n+|\mathbf{x}|\}}^{|\mathbf{x}|} \sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)}} \partial^n[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u)) + \partial^n_{x_1\dots x_n}h_1(u,\varphi(u)),$$

and, by recalling (1.1.22), (1.1.23), we obtain

$$\lim_{\varepsilon\to 0}\left(I-\partial_{Y_1}h_1(u,\varphi(u))\right).\frac{\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u+\varepsilon x_n)-\partial^{n-1}_{x_1\dots x_{n-1}}\varphi(u)}{\varepsilon}$$

$$= \sum_{\substack{\mathbf{x}\in 2^{\{x_1,\dots,x_n\}}\\ \mathbf{x}\neq\emptyset}} \sum_{i=\max\{1,2-n+|\mathbf{x}|\}}^{|\mathbf{x}|} \sum_{\substack{\mathbf{p}\in P^i(\mathbf{x})\\ \mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_i)}} \partial^n[\mathbf{x}^c,\mathbf{p}]h_1(u,\varphi(u)) + \partial^n_{x_1\dots x_n}h_1(u,\varphi(u)).$$

Finally, we can conclude the proof by recalling that $I - \partial_{Y_1} h_1(u,\varphi(u))$ is invertible with strongly continuous inverse. ■

Theorem 1.1.13 says that φ is Y_n-valued, continuous as a map from U into Y_n, and, for $j = 1,\dots,n$, for all $u \in U$, $x_1,\dots,x_j \in X$, the directional derivative $\partial^j_{x_1\dots x_j}\varphi(u)$ exists, it belongs to Y_{n-j+1}, the map

$$U \times X^j \to Y_{n-j+1},\ (u,x_1,\dots,x_j) \mapsto \partial^j_{x_1\dots x_j}\varphi(u)$$

is continuous, and (1.1.21) holds true.

Formula (1.1.21) can be useful e.g. when considering the boundedness of the derivatives of φ, or when studying convergences of derivatives under perturbations of h, as Corollary 1.1.14 and Proposition 1.1.15 show.

Corollary 1.1.14. *Let Assumption 1.1.12 be satisfied. Suppose that there exists $M > 0$ such that*

$$\begin{cases} |\partial_y h_k(u,y')|_k \le M|y|_k & \begin{cases}\forall u \in U,\\ \forall y,y' \in Y_k,\ k = 1,\dots,n\end{cases}\\ |\partial^j_{x_1\dots x_j} h_k(u,y)|_k \le M\prod_{l=1}^{j}|x_l|_X & \begin{cases}\forall u \in U,\ \forall x_1,\dots,x_j \in X,\\ \forall y \in Y_k,\ j,k = 1,\dots,n\end{cases}\\ |\partial^{j+i}_{x_1\dots x_j y_1\dots y_i} h_k(u,y)|_k \le M\prod_{l=1}^{j}|x_l|_X \cdot \prod_{l=1}^{i}|y_l|_{k+1} & \begin{cases}\forall u \in U,\ \forall x_1,\dots,x_j \in X,\\ \forall y \in Y_k,\ \forall y_1,\dots,y_i \in Y_{k+1},\\ k = 1,\dots,n-1,\\ j,i = 1,\dots,n-1,\ 1 \le j+i \le n-1.\end{cases}\end{cases} \tag{1.1.28}$$

Then, for $k = 1,\dots,n$,

$$\sup_{\substack{u\in U\\ x_1,\dots,x_k\in X\\ |x_1|_X=\dots=|x_k|_X=1}} |\partial^k_{x_1\dots x_k}\varphi(u)|_{n-k+1} \le C(\alpha,M),$$

where $C(\alpha,M) \in \mathbb{R}$ depends only on α, M.

Proof. Reason by induction taking into account (1.1.21) and (1.1.15). ■

Proposition 1.1.15. *Suppose that Assumption 1.1.12 holds true for a given h_1 and that $h_1^{(1)}, h_1^{(2)}, h_1^{(3)}\dots$ is a sequence of functions, each of which satisfies Assumption 1.1.12, uniformly with respect to the same n, α. Let $h_k^{(m)}$ denote the map associated to $h_1^{(m)}$ according to* (1.1.18) *and let $\varphi^{(m)}$ denote the fixed-point map associated to the parametric α-contraction $h_1^{(m)}$. Suppose that the following convergences occurs.*

(i) For $k = 1,\dots,n$, $y \in Y_k$,

$$\lim_{m\to\infty} h_k^{(m)}(u,y) = h_k(u,y) \text{ in } Y_k \tag{1.1.29}$$

uniformly for u on compact subsets of U;

(ii) for $k = 1,\dots,n$,

$$\begin{cases} \lim_{m\to\infty} \partial_x h_k^{(m)}(u,y) = \partial_x h_k(u,y) & \text{in } Y_k \\ \lim_{m\to\infty} \partial_y h_k^{(m)}(u,y') = \partial_y h_k(u,y') & \text{in } Y_k \end{cases} \tag{1.1.30}$$

uniformly for u on compact subsets of U, x on compact subsets of X, and y, y' *on compact subsets of* Y_k*;*

(iii) for all $k = 1,\dots,n-1$, $u \in U$, $j, i = 0,\dots,n$, $1 \le j+i \le n$,

$$\lim_{m\to\infty} \partial^{j+i}_{x_1\dots x_j y_1\dots y_i} h_k^{(m)}(u,y) = \partial^{j+i}_{x_1\dots x_j y_1\dots y_i} h_k(u,y) \text{ in } Y_k \tag{1.1.31}$$

uniformly for u on compact subsets of U, $x_1,\dots,x_j$ *on compact subsets of X, y on compact subsets of* Y_k, $y_1,\dots,y_i$ *on compact subsets of* Y_{k+1}.

Then $\varphi_m \to \varphi$ *uniformly on compact subsets of* Y_n *and, for all* $j = 1,\dots,n$

$$\lim_{m\to\infty} \partial^j_{x_1\dots x_j} \varphi^{(m)}(u) = \partial^j_{x_1\dots x_j} \varphi(u) \text{ in } Y_{n-j+1} \tag{1.1.32}$$

uniformly for u on compact subsets of U and $x_1,\dots,x_j$ *on compact subsets of X.*

Proof. Notice that (1.1.29) and the fact that each $h_k^{(m)}$ is a parametric α-contraction (with the same α) imply the uniform convergence $h_k^{(m)} \to h_k$ on compact subsets of Y_k. In particular, the sequence $h_k^{(1)}, h_k^{(2)}, h_k^{(3)}, \dots$ is uniformly equicontinuous on compact sets. Then, by Lemma 1.1.9*(i),(ii)*, $\varphi^{(m)} \to \varphi$ in Y_k uniformly on compact subsets of Y_k, for $k = 1,\dots,n$. Moreover, by (1.1.15), that holds for all $h_1^{(m)}$ uniformly in m, we have the boundedness of $(I - \partial_{Y_1} h_1^{(m)})^{-1}$, uniformly in m. Convergence (1.1.32) is then obtained by reasoning by induction on (1.1.21), taking into account the strong continuity of $(I - \partial_{Y_1} h_1)^{-1}$. ■

1.2 Path-dependent SDEs in Hilbert spaces

In this section we study mild solutions to path-dependent SDEs in Hilbert spaces. In particular, by applying the results of the previous section, we address differentiability with respect to the initial datum and stability of the derivatives. By emulating the arguments of [23, Ch. 7] for the Markovian case and for differentiability up to order 2, we extend the results there provided to the following path-dependent setting and to differentiability of generic order n.

Let H and U be real separable Hilbert spaces, with scalar product denoted by $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_U$, respectively. Let $\mathfrak{e} := \{e_n\}_{n\in\mathcal{N}}$ be an orthonormal basis of H, where $\mathcal{N} = \{1,\ldots,N\}$ if H has dimension $N \in \mathbb{N}\setminus\{0\}$, or $\mathcal{N} = \mathbb{N}$ if H has infinite dimension, and let $\mathfrak{e}' := \{e'_m\}_{m\in\mathcal{M}}$ be an orthonormal basis of U, where $\mathcal{M} = \{1,\ldots,M\}$ if U has dimension $M \in \mathbb{N}\setminus\{0\}$, or $\mathcal{M} = \mathbb{N}$ if U has infinite dimension. If $\mathbf{x}\colon [0,T] \to \mathcal{S}$ is a function taking values in any set $\mathcal{S}$ and if $t \in [0,T]$, we denote by $\mathbf{x}_{t\wedge\cdot}$ the function defined by

$$\begin{cases} \mathbf{x}_{t\wedge\cdot}(s) := \mathbf{x}(s) & s \in [0,t] \\ \mathbf{x}_{t\wedge\cdot}(s) := \mathbf{x}(t) & s \in (t,T]. \end{cases}$$

For elements of stochastic analysis in infinite dimension used hereafter, we refer to [24, 48].

We begin by considering the SDE

$$\begin{cases} dX_s = (AX_s + b((\cdot,s),X))dt + \sigma((\cdot,s),X)dW_s & s \in (t,T] \\ X_s = Y_s & s \in [0,t], \end{cases} \tag{1.2.1}$$

where $t \in [0,T]$, Y is a H-valued process, W is a U-valued cylindrical Wiener process, $b((\omega,s),X)$ is a H-valued random variable depending on $\omega \in \Omega$, on the time s, and on the path X, $\sigma((\omega,s),X)$ is a $L_2(U,H)$-valued random variable depending on $\omega \in \Omega$, on the time s, and on the path X, and A is the generator of a C_0-semigroup S on H.

We introduce the following notation:

- $\mathbb{S}$ denotes a closed subspace of $B_b([0,T],H)$ (6) such that

$$\begin{cases} (a)\ C([0,T],H) \subset \mathbb{S} \\ (b)\ \mathbf{x}_{t\wedge\cdot} \in \mathbb{S},\ \forall \mathbf{x} \in \mathbb{S},\ \forall t \in [0,T] \\ (c)\ \text{for all } T \in L(H) \text{ and } \mathbf{x} \in \mathbb{S}, \text{ the map } [0,T] \to H,\ t \mapsto T\mathbf{x}_t, \text{ belongs to } \mathbb{S}. \end{cases} \tag{1.2.2}$$

  Hereafter, unless otherwise specified, $\mathbb{S}$ will be always considered as Banach space endowed with the norm $|\cdot|_\infty$.

- $\mathcal{L}^0_{\mathcal{P}_T}(\mathbb{S})$ denotes the space of functions $X\colon \Omega_T \to H$ such that

$$\begin{cases} (a)\ \forall \omega \in \Omega, \text{ the map } [0,T] \to H,\ t \mapsto X_t(\omega), \text{ belongs to } \mathbb{S} \\ (b)\ (\Omega_T,\mathcal{P}_T) \to \mathbb{S},\ (\omega,t) \mapsto X_{t\wedge\cdot}(\omega) \text{ is measurable.} \end{cases} \tag{1.2.3}$$

  Two processes $X, X' \in \mathcal{L}^0_{\mathcal{P}_T}(\mathbb{S})$ are equal if and only if $\mathbb{P}(|X - X'|_\infty = 0) = 1$.

- For $p \in [1,\infty)$, $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$ denotes the space of (classes of) functions $X \in \mathcal{L}^0_{\mathcal{P}_T}(\mathbb{S})$ such that $\Omega_T \to \mathbb{S}$, $(\omega,t) \mapsto X_{t\wedge\cdot}(\omega)$ has separable range and

$$|X|_{\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})} := \left(\mathbb{E}\left[|X|^p_\infty\right]\right)^{1/p} < \infty. \tag{1.2.4}$$

[6] We recall that $B_b([0,T],H)$ is endowed with the norm $|\cdot|_\infty$.

- For $p,q \in [1,\infty)$ and $\beta \in [0,1)$, $\Lambda^{p,q,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$ denotes the space of functions $\Phi\colon \Omega_T \to L(U,H)$ such that

$$\begin{cases} \Phi u\colon (\Omega_T,\mathscr{P}_T) \to H,\ (\omega,t) \mapsto \Phi_t(\omega)u, \text{ is measurable}, \forall u \in U \\ |\Phi|_{p,q,S,\beta} := \left(\int_0^T \left(\int_0^t (t-s)^{-\beta q} \left(\mathbb{E}\left[|S_{t-s}\Phi_s|^p_{L_2(U,H)} \right] \right)^{q/p} ds \right)^{p/q} dt \right)^{1/p} < \infty. \end{cases}$$

  The space $\Lambda^{p,q,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$ is normed by $|\cdot|_{p,q,S,\beta}$ (see Remark 1.2.1 below and Appendix A.2.2).
- $\overline{\Lambda}^{p,q,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$ denotes the completion of $\Lambda^{p,q,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$. We keep the notation $|\cdot|_{p,q,S,\beta}$ for the extended norm.

It can be seen that $(\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}), |\cdot|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})})$ is a Banach space ($\mathbb{F}$ is supposed to be complete). For example, $\mathbb{S}$ could be $C([0,T],H)$, the space of càdlàg functions $[0,T] \to H$, $\mathbb{B}^1(H)$ (see Chapter 4 for the definition), or $B_b([0,T],H)$ itself.

Remark 1.2.1. To see that $|\cdot|_{p,q,S,\beta}$ is a norm and not just a seminorm, suppose that $|\Phi|_{p,q,S,\beta} = 0$. In particular, for $u \in U$,

$$\int_{[0,T]^2} \mathbf{1}_{(0,T]}(t-s)(t-s)^{-\beta} \mathbb{E}[|S_{t-s}\Phi_s u|_H]\, ds \otimes dt = 0,$$

which entails, for $\mathbb{P} \otimes m$-a.e. $(\omega,s) \in \Omega_T$,

$$S_{t-s}\Phi_s(\omega)u = 0 \qquad m\text{-a.e. } t \in (s,T]. \tag{1.2.5}$$

Since S is strongly continuous, (1.2.5) gives

$$\Phi_s(\omega)u = 0 \qquad \mathbb{P} \otimes m\text{-a.e. } (\omega,s) \in \Omega_T,$$

which provides $\Phi = 0$ $\mathbb{P} \otimes m$-a.e., since U is supposed to be separable and $\Phi_s(\omega) \in L(U,H)$ for all ω, s.

Remark 1.2.2. The space $\overline{\Lambda}^{p,q,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$ can be naturally identified with a closed subspace of the space of all those measurable functions

$$\zeta\colon (\Omega_T \times [0,T], \mathscr{P}_T \otimes \mathscr{B}_T) \to L_2(U,H)$$

such that

$$\begin{cases} \zeta((\omega,s),t) = 0,\ \forall ((\omega,s),t) \in \Omega_T \times [0,T],\ s > t, \\ |\zeta|_{p,q,p} := \left(\int_0^T \left(\int_0^t \left(\mathbb{E}\left[|\zeta((\cdot,s),t)|^p_{L_2(U,H)} \right] \right)^{q/p} ds \right)^{p/q} dt \right)^{1/p} < \infty. \end{cases}$$

Indeed, if we denote by $L^{p,q,p}_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$ such a space, then $L^{p,q,p}_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$ endowed with $|\cdot|_{p,q,p}$ is a Banach space and the map

$$\iota\colon \Lambda^{p,q,p}_{\mathscr{P}_T,S,\beta}(L(U,H))\to L^{p,q,p}_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$$

defined by

$$\iota(\Phi)(\omega,s,t):=\begin{cases}(t-s)^{-\beta}S_{t-s}\Phi_s(\omega) & \forall((\omega,s),t)\in\Omega_T\times[0,T],\ s\leq t,\\ 0 & \text{otherwise.}\end{cases}$$

is an isometry.

The reason to introduce the space $\overline{\Lambda}^{p,q,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$ is related to the existence of a continuous version of the stochastic convolution and to the factorization method used to construct such a version. Let $p>\max\{2,1/\beta\}$, $t\in[0,T]$, and $\Phi\in\Lambda^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$. If we consider the two stochastic convolutions

$$Y_{t'}:=\mathbf{1}_{[t,T]}(t')\int_t^{t'}S_{t'-s}\Phi_s dW_s,\quad Z_{t'}:=\mathbf{1}_{[t,T]}(t')\int_t^{t'}(t'-s)^{-\beta}S_{t'-s}\Phi_s dW_s, \tag{1.2.6}$$

then $Y_{t'}$ is well-defined for all $t'\in[0,T]$, $Z_{t'}$ is well-defined for m-a.e. $t\in[0,T]$, and $Y_{t'},Z_{t'}$ belong to $L^p((\Omega,\mathscr{F}_{t'},\mathbb{P}),H)$ (for details, see the discussion in Appendix A.2.2). By using the stochastic Fubini's theorem and the factorization method (see [24] or Theorem A.2.13 and Example A.2.14 in Appendix A.2.2), we can find a predictable process $\widetilde{Z}$ such that:

(a) for m-a.e. $t\in[0,T]$, $\widetilde{Z}_t=Z_t$ $\mathbb{P}$-a.e.;

(b) for all $t'\in[0,T]$, the following formula holds

$$Y_{t'}=c_\beta\mathbf{1}_{[t,T]}(t')\int_t^{t'}(t'-s)^{\beta-1}\widetilde{Z}_s ds\qquad \mathbb{P}\text{-a.e.}, \tag{1.2.7}$$

where c_β is a constant depending only on β.

By (1.2.6), (a), [22, Lemma 7.7], it follows that $\widetilde{Z}(\omega)\in L^p((0,T),H)$ for $\mathbb{P}$-a.e. $w\in\Omega$, hence, by [48, Lemma 3.2], the right-hand side of (1.2.7) is continuous in t'.

This classical argument shows that there exists a pathwise continuous process $S\overset{dW}{*_t}\Phi$ such that, for all $t'\in[0,T]$, $(S\overset{dW}{*_t}\Phi)_{t'}=Y_{t'}$ $\mathbb{P}$-a.e.. In particular, $S\overset{dW}{*_t}\Phi\in\mathscr{L}^0_{\mathscr{P}_T}(C([0,T],H))$. By (1.2.6), (1.2.7), Hölder's inequality, and [22, Lemma 7.7], we also have

$$\mathbb{E}\left[|S\overset{dW}{*_t}\Phi|^p_\infty\right]\leq c^p_\beta\left(\int_0^T v^{\frac{(\beta-1)p}{p-1}}dv\right)^{p-1}\mathbb{E}\left[\int_0^T|\widetilde{Z}_s|^p_H ds\right]\leq c'_{\beta,T,p}|\Phi|^p_{p,2,S,\beta}, \tag{1.2.8}$$

where $c'_{\beta,T,p}$ is a constant depending only on β,T,p. This shows that the linear map

$$\Lambda^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H))\to\mathscr{L}^p_{\mathscr{P}_T}(C([0,T],H)),\ \Phi\mapsto S\overset{dW}{*_t}\Phi \tag{1.2.9}$$

is well-defined and continuous. Then, we can uniquely extend (1.2.9) to a continuous linear map on $\overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$, that we can see as $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$-valued, since, by assumption, $C([0,T],H)\subset\mathbb{S}$. We end up with a continuous linear map, again denoted by $S \overset{dW}{*}_t \#$,

$$S \overset{dW}{*}_t \#\colon \overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H)) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}). \tag{1.2.10}$$

Summarizing,

(1) the map $S \overset{dW}{*}_t \#$ is linear, continuous, $\mathscr{L}^p_{\mathscr{P}_T}(C([0,T],H))$-valued;

(2) the operator norm of $S \overset{dW}{*}_t \#$ depends only on β, T, p;

(3) if $\Phi\in\Lambda^{p,2,p}_{\mathscr{P}_T,S,\beta}(L_2(U,H))$, $S \overset{dW}{*}_t \Phi$ is a continuous version of the process Y in (1.2.6).

Within the approach using the factorization method, the space $\overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$ is then naturally introduced if we want to see the stochastic convolution as a continuous linear operator acting on a Banach space and providing pathwise continuous processes, and this perspective is useful when applying to SDEs the results based on parametric α-contractions obtained in the first part of the chapter.

We make some observations that will be useful later. Let $\hat{S}$ be another C_0-semigroup on H, and let $\Phi\in\Lambda^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$, $\hat{\Phi}\in\Lambda^{p,2,p}_{\mathscr{P}_T,\hat{S},\beta}(L(U,H))$. Then, by using the factorization formula (1.2.7) both with respect to the couples (S,Φ) and $(\hat{S},\hat{\Phi})$, and by an estimate analogous to (1.2.8), we obtain

$$\begin{aligned}\mathbb{E}\left[|S \overset{dW}{*}_t \Phi - \hat{S} \overset{dW}{*}_t \hat{\Phi}|^p_\infty\right] &\leq \\ &\leq c'_{\beta,T,p}\int_0^T\left(\int_0^t (t-s)^{-2\beta}\left(\mathbb{E}\left[|S_{t-s}\Phi_s-\hat{S}_{t-s}\hat{\Phi}_s|^p_{L_2(U,H)}\right]\right)^{2/p} ds\right)^{p/2} dt.\end{aligned} \tag{1.2.11}$$

For $0\leq t_1\leq t_2\leq T$ and $\Phi\in\Lambda^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$, we also have

$$(S \overset{dW}{*}_{t_1} \Phi - S \overset{dW}{*}_{t_2} \Phi)_s = \mathbf{1}_{[t_1,t_2]}(s)(S \overset{dW}{*}_{t_1} \Phi)_s + \mathbf{1}_{(t_2,T]}(s) S_{s-t_2}(S \overset{dW}{*}_{t_1} \Phi)_{t_2} \qquad \forall s\in[0,T]. \tag{1.2.12}$$

Since

$$\sup_{s\in[t_1,t_2]} |(S \overset{dW}{*}_{t_1} \Phi)_s|_H \leq |S \overset{dW}{*}_{t_1} (\mathbf{1}_{[t_1,t_2]}(\cdot)\Phi)|_\infty \qquad \mathbb{P}\text{-a.e.},$$

we obtain, by (1.2.8),

$$\lim_{t_2-t_1\to0^+}\mathbb{E}\left[\sup_{s\in[t_1,t_2]}|(S \overset{dW}{*}_{t_1} \Phi)_s|^p_H\right] \leq \lim_{t_2-t_1\to0^+} c'_{\beta,T,p}|\mathbf{1}_{[t_1,t_2]}(\cdot)\Phi|^p_{p,2,S,\beta}=0, \tag{1.2.13}$$

where the latter limit can be seen by applying Lebesgue's dominated convergence theorem three times, to the three integrals defining $|\cdot|_{p,2,S,\beta}$. Actually, since the linear map

$$\overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H)) \to \overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H)),\ \Phi\to\mathbf{1}_{[t_1,t_2]}(\cdot)\Phi$$

is bounded, uniformly in t_1, t_2, the limit (1.2.13) is uniform for Φ in compact subsets of $\overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$ and $t_1,t_2 \in [0,T]$, $t_2 - t_1 \to 0^+$. Then, by (1.2.12) and (1.2.13), we finally obtain

$$\lim_{|t_2-t_1|\to 0} |S \overset{dW}{*}_{t_1} \Phi - S \overset{dW}{*}_{t_2} \Phi|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})} = 0 \tag{1.2.14}$$

uniformly for Φ in compact subsets of $\overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$. In particular, thanks to the uniform boundedness of $\{S \overset{dW}{*}_t \#\}_{t\in[0,T]}$ (see (1.2.8)), the map

$$[0,T] \times \overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H)) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\ (t,\Phi) \mapsto S \overset{dW}{*}_t \Phi \tag{1.2.15}$$

is continuous.

1.2.1 Existence and uniqueness of mild solution

The following assumption will be standing for the remaining part of this chapter. We recall that, if E is a Banach space, then $\mathscr{B}_E$ denotes its Borel σ-algebra.

Assumption 1.2.3.

(i) $b\colon (\Omega_T \times \mathbb{S}, \mathscr{P}_T \otimes \mathscr{B}_{\mathbb{S}}) \to (H, \mathscr{B}_H)$ *is measurable;*

(ii) $\sigma\colon (\Omega_T \times \mathbb{S}, \mathscr{P}_T \otimes \mathscr{B}_{\mathbb{S}}) \to L(U,H)$ *is strongly measurable, i.e., for all* $u \in U$*, the map* $(\Omega_T \times \mathbb{S}, \mathscr{P}_T \otimes \mathscr{B}_{\mathbb{S}}) \to H$, $((\omega,t),\mathbf{x}) \mapsto \sigma((\omega,t),\mathbf{x})u$ *is measurable;*

(iii) (non-anticipativity condition) for all $((\omega,t),\mathbf{x}) \in \Omega_T \times \mathbb{S}$, $b((\omega,t),\mathbf{x}) = b((\omega,t),\mathbf{x}_{t\wedge\cdot})$ *and* $\sigma((\omega,t),\mathbf{x}) = \sigma((\omega,t),\mathbf{x}_{t\wedge\cdot})$*;*

(iv) there exists $g \in L^1((0,T),\mathbb{R})$ *such that*

$$\begin{cases} |b((\omega,t),\mathbf{x})|_H \le g(t)(1+|\mathbf{x}|_\infty) & \forall ((\omega,t),\mathbf{x}) \in \Omega_T \times \mathbb{S}, \\ |b((\omega,t),\mathbf{x}) - b((\omega,t),\mathbf{x}')|_H \le g(t)|\mathbf{x}-\mathbf{x}'|_\infty & \forall (\omega,t) \in \Omega_T,\ \forall \mathbf{x},\mathbf{x}' \in \mathbb{S}; \end{cases}$$

(v) there exist $M > 0$, $\gamma \in [0,1/2)$ *such that*

$$\begin{cases} |S_t\sigma((\omega,s),\mathbf{x})|_{L_2(U,H)} \le Mt^{-\gamma}(1+|\mathbf{x}|_\infty) & \forall ((\omega,s),\mathbf{x}) \in \Omega_T \times \mathbb{S},\ \forall t \in (0,T], \\ |S_t((\omega,s),\mathbf{x}) - S_t\sigma((\omega,s),\mathbf{x}')|_{L_2(U,H)} \le Mt^{-\gamma}|\mathbf{x}-\mathbf{x}'|_\infty & \forall (\omega,s) \in \Omega_T,\ \forall t \in (0,T],\ \forall \mathbf{x},\mathbf{x}' \in \mathbb{S}. \end{cases}$$

Remark 1.2.4. Assumption 1.2.3*(iv)* can be generalized to the form

$$\begin{cases} |S_tb((\omega,s),\mathbf{x})|_H \le t^{-\gamma}g(s)(1+|\mathbf{x}|_\infty) & \forall ((\omega,s),\mathbf{x}) \in \Omega_T \times \mathbb{S},\ \forall t \in (0,T] \\ |S_t(b((\omega,s),\mathbf{x}) - b((\omega,s),\mathbf{x}'))|_H \le t^{-\gamma}g(s)|\mathbf{x}-\mathbf{x}'|_\infty & \forall (\omega,s) \in \Omega_T,\ \forall t \in (0,T],\ \forall \mathbf{x},\mathbf{x}' \in \mathbb{S}, \end{cases}$$

with g suitably integrable, and similarly for Assumption 1.2.3*(v)*. The results obtained and the methods used hereafter in this chapter could be adapted to cover these more general assumptions.

Definition 1.2.5 (Mild solution)**.** *Let* $Y \in \mathscr{L}^0_{\mathscr{P}_T}(\mathbb{S})$ *and* $t \in [0,T)$*. A function* $X \in \mathscr{L}^0_{\mathscr{P}_T}(\mathbb{S})$ *is a* mild solution *to* (1.2.1) *if, for all* $t' \in [t,T]$*,*

$$\mathbb{P}\left(\int_t^{t'} |S_{t-s}b(\cdot,s,X)|_H ds + \int_t^{t'} |S_{t-s}\sigma(\cdot,s,X)|^2_{L_2(U,H)} ds\right) < \infty,$$

and

$$\begin{cases} \forall t' \in [0,t], & X_{t'} = Y_{t'} \ \mathbb{P}\text{-a.e.}, \\ \forall t' \in (t,T], & X_{t'} = S_{t'-t}Y_t + \int_t^{t'} S_{t'-s}b((\cdot,s),X)ds + \int_t^{t'} S_{t'-s}\sigma((\cdot,s),X)dW_s \ \mathbb{P}\text{-a.e.}. \end{cases}$$

Using a classical contraction argument, we are going to prove existence and uniqueness of mild solution in the space $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$, when the initial datum Y belongs to $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$, for p large enough. This will let us apply the theory developed in Section 1.1.

For $t \in [0,T]$ and

$$p > p^* := \frac{2}{1-2\gamma}, \ \beta \in (1/p, 1/2-\gamma),$$

we define the following maps:

$$\mathrm{id}^S_t : \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\ Y \mapsto \mathbf{1}_{[0,t]}(\cdot)Y + \mathbf{1}_{(t,T]}(\cdot)S_{\cdot-t}Y_t$$

$$F_b : \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}) \to L^{p,1}_{\mathscr{P}_T}(H),\ X \mapsto b((\cdot,\cdot),X)$$

$$F_\sigma : \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}) \to \overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H)),\ X \mapsto \sigma((\cdot,\cdot),X)$$

$$S *_t \# : L^{p,1}_{\mathscr{P}_T}(H) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\ X \mapsto \mathbf{1}_{[t,T]}(\cdot)\int_t^\cdot S_{\cdot-s}X_s ds,$$

and we recall the map

$$S \overset{dW}{*}_t \# : \overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H)) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\ \Phi \mapsto S \overset{dW}{*}_t \Phi.$$

Then id^S_t is well-defined, due to (a) and (b) in (1.2.2), because we can write

$$\mathrm{id}^S_t(Y) = Y_{t\wedge\cdot} + \mathbf{1}_{(t,T]}(\cdot)(S_{\cdot-t} - I)Y_t. \tag{1.2.17}$$

As regarding F_b, by Assumption 1.2.3*(i)*,*(iii)*, and by (b) in (1.2.2), the map

$$\Omega_T \to H,\ (\omega,t) \mapsto b((\omega,t),X(\omega)) = b((\omega,t),X_{t\wedge\cdot}(\omega))$$

is predictable. Moreover, by Assumption 1.2.3*(iv)*, we have

$$\int_0^T \left(\mathbb{E}\left[|b(\cdot,t,X_{t\wedge\cdot})|^p\right]\right)^{1/p} dt \le \int_0^T g(t)\left(\mathbb{E}\left[(1+|X|_\infty)^p\right]\right)^{1/p} dt \le |g|_{L^1((0,T),\mathbb{R})}(1+|X|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})}),$$

which shows that $F_b(X) \in L^{p,1}_{\mathscr{P}_T}(H)$. By Assumption 1.2.3*(iv)*, we also have that F_b is Lipschitz, with Lipschitz constant dominated by $|g|_{L^1((0,1),\mathbb{R})}$. Similarly as done for F_b, by using Assumption 1.2.3*(ii)*, one can see that, for $X \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$, the map

$$(\Omega_T, \mathscr{P}_T) \to L(U,H),\ (\omega,t) \mapsto \sigma((\omega,t), X_{t\wedge\cdot}(\omega))$$

is strongly measurable. Moreover, by Assumption 1.2.3*(v)*, we have

$$\begin{aligned}|F_\sigma(X)|_{p,2,S,\beta} &= \left(\int_0^T \left(\int_0^t (t-s)^{-\beta 2}\left(\mathbb{E}\left[|S_{t-s}\sigma((\cdot,s),X_{s\wedge\cdot})|^p_{L_2(U,H)}\right]\right)^{2/p} ds\right)^{p/2} dt\right)^{1/p}\\ &\leq M\left(\int_0^T \left(\int_0^t v^{-(\beta+\gamma)2} dv\right)^{p/2} dt\right)^{1/p} (1+|X|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})})\end{aligned}$$

and the latter term is finite because $\beta < 1/2-\gamma$ and $X \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$. Then F_σ is well-defined. With similar computations, we have that F_σ is Lipschitz, with Lipschitz constant depending only on M, β, γ, p. Regarding $S *_t \#$, if $X \in L^{p,1}_{\mathscr{P}_T}(H)$, then $X(\omega) \in L^1((0,T),H)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, hence it is easily checked that

$$[0,T] \to H,\ t' \mapsto \mathbf{1}_{[0,t]}(t')\int_t^{t'} S_{t'-s}X_s(\omega)ds$$

is continuous, and then it belongs to $\mathbb{S}$. Since $\mathbb{F}$ is complete, we can assume that $S *_t X(\omega)$ is continuous for all ω, hence it is predictable, because it is $\mathbb{F}$-adapted. Since the trajectories are continuous, we also have the measurability of

$$(\Omega_T, \mathscr{P}_T) \to C([0,T],H) \subset \mathbb{S},\ (\omega,t') \mapsto (S *_t X)_{t'\wedge\cdot}(\omega).$$

Then, to show that $S *_t X \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$, it remains to verify the integrability condition. We have

$$|S *_t X|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})} \leq M'\left(\mathbb{E}\left[\left(\int_0^T |X_s|_H ds\right)^p\right]\right)^{1/p} \leq M'\int_0^T \left(\mathbb{E}\left[|X_s|^p_H\right]\right)^{1/p} ds = M'|X|_{p,1},$$

where

$$M' \text{ is any upper bound for } \sup_{t\in[0,T]} |S_t|_{L(H)}.$$

The good definition of $S \overset{dW}{*_t} \#$ was discussed above (observe that $p > \max\{2, 1/\beta\}$ (we refer the reader to Appendix A.2.2 for further details).

We can then build the map

$$\psi\colon \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}) \times \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\ (Y,X) \mapsto \mathrm{id}^S_t(Y) + S *_t F_b(X) + S \overset{dW}{*_t} F_\sigma(X). \tag{1.2.18}$$

In what follows, whenever we need to make explicit the dependence of $\psi(Y,X)$ on the data t,S,b,σ, we write $\psi(Y,X;t,S,b,\sigma)$.

We first show that, for ech $Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, $\psi(Y,\cdot)$ has a unique fixed point X. Such a fixed point is a mild solution to (1.2.1).

The advantage of introducing the setting above is that it permits to see ψ as a composition of maps that have different regularity and that can be considered individually when studying the regularity of the mild solution $X^{t,Y}$ with respect to Y or the dependence of of $X^{t,Y}$ with respect to a perturbation of the data Y,t,S,b,σ.

For $\lambda > 0$, we consider the following norm on $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$

$$|X|_{\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}),\lambda} := \left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\lambda pt}|X_t|^p\right]\right)^{1/p} \qquad \forall X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}).$$

Then $|\cdot|_{\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}),\lambda}$ is equivalent to $|\cdot|_{\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})}$.

We proceed to show that there exists $\lambda > 0$ with respect to which ψ is a parametric contraction.

For $X,X' \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, $\lambda > 0$, and $t' \in [0,T]$, we have

$$\begin{aligned}
e^{-\lambda pt'}|(S *_t F_b(X))_{t'} - (S *_t F_b(X'))_{t'}|^p_H &\leq (M')^p \left(\int_0^{t'} e^{-\lambda t'}|b((\cdot,s),X) - b((\cdot,s),X')|_H ds\right)^p \\
&\leq (M')^p \left(\int_0^{t'} e^{-\lambda(t'-s)} g(s) e^{-\lambda s}|X_{s\wedge\cdot} - X'_{s\wedge\cdot}|_\infty ds\right)^p \\
&\leq C^p_{\lambda,g,M'} \sup_{s\in[0,T]} \left\{e^{-\lambda ps}|X_s - X'_s|^p_H\right\},
\end{aligned}$$

where $C_{\lambda,g,M'} := M' \sup_{t'\in[0,T]} \int_0^{t'} e^{-\lambda v} g(t'-v)dv$. We then obtain

$$|S *_t F_b(X) - S *_t F_b(X')|_{\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}),\lambda} \leq C_{\lambda,g,M'}|X - X'|_{\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}),\lambda}. \tag{1.2.19}$$

It is not difficult to see that $C_{\lambda,g,M'} \to 0$ as $\lambda \to \infty$.

Now, if $\Phi \in \Lambda^{p,2,p}_{\mathcal{P}_T,S,\beta}(L(U,H))$, then $e^{-\lambda\cdot}\Phi \in \Lambda^{p,2,p}_{\mathcal{P}_T,e^{-\lambda\cdot}S,\beta}(L(U,H))$ for all $\lambda \geq 0$ and, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$e^{-\lambda t'}(S \overset{dW}{*_t} \Phi)_{t'} = ((e^{-\lambda\cdot}S) \overset{dW}{*_t} (e^{-\lambda\cdot}\Phi))_{t'} \qquad \forall t' \in [0,T]. \tag{1.2.20}$$

For $X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, we have

$$\int_t^{t'} \mathbb{E}\left[|e^{-\lambda(t'-s)}S_{t'-s}(e^{-\lambda\cdot}F_\sigma(X))_s|^2_{L_2(U,H)}\right] ds < \infty \qquad \forall t' \in [t,T].$$

Then, for $X,X' \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, $\lambda \geq 0$, and for all $t' \in [t,T]$, formula (1.2.7) provides

$$((e^{-\lambda\cdot}S) \overset{dW}{*_t} (e^{-\lambda\cdot}F_\sigma(X)))_{t'} - ((e^{-\lambda\cdot}S) \overset{dW}{*_t} (e^{-\lambda\cdot}F_\sigma(X')))_{t'} = c_\beta \int_t^{t'} (t'-s)^{\beta-1}\hat{Z}_s ds \qquad \mathbb{P}\text{-a.e.},$$

where $\hat{Z}$ is an H-valued predictable process such that, for a.e. $t' \in [t,T]$,

$$\hat{Z}_{t'} = \int_t^{t'} (t'-s)^{-\beta} e^{-\lambda(t'-s)} S_{t'-s}(e^{-\lambda\cdot} F_\sigma(X) - e^{-\lambda\cdot} F_\sigma(X'))_s dW_s \qquad \mathbb{P}\text{-a.e.}.$$

By collecting the observations above, we can write, for $\lambda \geq 0$ and for all $t' \in [t,T]$,

$$e^{-\lambda p t'} |(S \overset{dW}{*}_t F_\sigma(X))_{t'} - (S \overset{dW}{*}_t F_\sigma(X'))_{t'}|_H^p \leq c_\beta^p \left(\int_0^T v^{\frac{(\beta-1)p}{p-1}} dv \right)^{p-1} \int_t^T |\hat{Z}_s|_H^p ds,$$

then, by applying [22, Lemma 7.7],

$$|S \overset{dW}{*}_t F_\sigma(X) - S \overset{dW}{*}_t F_\sigma(X')|^p_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda} \leq c'_{\beta,T,p} |e^{-\lambda\cdot} F_\sigma(X) - e^{-\lambda\cdot} F_\sigma(X')|^p_{p,2,e^{-\lambda\cdot}S,\beta}$$

where $c'_{\beta,T,p}$ is a constant depending only on β, T, p. Now, by using Assumption 1.2.3*(v)*, we have

$$|e^{-\lambda\cdot} F_\sigma(X) - e^{-\lambda\cdot} F_\sigma(X')|^p_{p,2,e^{-\lambda\cdot}S,\beta} \leq M^p \left(\int_0^T \left(\int_0^t v^{-(\beta+\gamma)2} e^{-\lambda v} dv \right)^{p/2} dt \right) |X - X'|^p_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda}.$$

We finally obtain

$$|S \overset{dW}{*}_t F_\sigma(X) - S \overset{dW}{*}_t F_\sigma(X')|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda} \leq c''_{\beta,\gamma,T,p,M,\lambda} |X - X'|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda}, \tag{1.2.21}$$

where $c''_{\beta,\gamma,T,p,M,\lambda}$ is a constant depending only on $\beta, \gamma, T, p, M, \lambda$, and is such that

$$\lim_{\lambda\to\infty} c''_{\beta,\gamma,T,p,M,\lambda} = 0.$$

By (1.2.19) and (1.2.21), we have, for all Y, X, Y', X',

$$\begin{aligned} &|\psi(Y,X) - \psi(Y',X')|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda} \leq \\ &\qquad \leq M'|Y - Y'|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda} + C'_{\lambda,g,\gamma,M',\beta,T,p,M} |X - X'|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda}, \end{aligned} \tag{1.2.22}$$

where $C'_{\lambda,g,\gamma,M',\beta,T,p,M}$ is a constant depending only on $\lambda, g, \gamma, M', \beta, T, p, M$, such that

$$\lim_{\lambda\to\infty} C'_{\lambda,g,\gamma,M',\beta,T,p,M} = 0. \tag{1.2.23}$$

Theorem 1.2.6. *Let Assumption 1.2.3 hold and let $t \in [0,T]$, $p > p^*$. Then there exists a unique mild solution $X^{t,Y} \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$ to SDE* (1.2.1)*. Moreover, there exists a constant C, depending only on g, γ, M, M', T, p, such that,*

$$|X^{t,Y} - X^{t,Y'}|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})} \leq C|Y - Y'|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})} \qquad \forall Y, Y' \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}).$$

Proof. Let us fix any $\beta \in (1/p, 1/2 - \gamma)$ and let ψ be defined by (1.2.18). It is clear that any fixed point of $\psi(Y,\cdot)$ is a mild solution to SDE (1.2.1). Then, it is sufficient to apply Lemma 1.1.9 to ψ, taking into account (1.2.22) and (1.2.23), and recalling the equivalence of the norms $|\cdot|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})}, |\cdot|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda}$. ■

Remark 1.2.1. Since, for $p^* < p < q$, we have $\mathscr{L}^q_{\mathscr{P}_T}(\mathbb{S}) \subset \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$, then, if $Z \in \mathscr{L}^q_{\mathscr{P}_T}(\mathbb{S})$, the associated mild solution $X^{t,Z} \in \mathscr{L}^q_{\mathscr{P}_T}(\mathbb{S})$ is also a solution in $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$ and, by uniqueness, it is *the* solution in that space. Hence, the solution does not depend on the specific $p > p^*$ chosen.

1.2.2 Gâteaux differentiability with respect to the initial datum

We now study the differentiability of the mild solution $X^{t,Y}$ with respect to the initial datum Y.

Assumption 1.2.7. *Let b,σ,g,γ be as in Assumption 1.2.3. Let $n \in \mathbb{N}$, $n \geq 1$.*

(i) For all $(\omega,t) \in \Omega_T$ and $u \in U$, $b((\omega,t),\cdot) \in \mathcal{G}^n(\mathbb{S},H)$, $\sigma((\omega,t),\cdot)u \in \mathcal{G}^n(\mathbb{S},H)$.

(ii) There exists M'' and $c := \{c_m\}_{m\in\mathcal{M}} \in \ell^2(\mathcal{M})$ such that

$$\sup_{j=1,\dots,n} \sup_{\substack{\omega\in\Omega \\ \mathbf{x},\mathbf{y}_1,\dots,\mathbf{y}_j\in\mathbb{S} \\ |\mathbf{y}_1|_\infty=\dots=|\mathbf{y}_j|_\infty=1}} |\partial^j_{\mathbf{y}_1\dots\mathbf{y}_j} b((\omega,s),\mathbf{x})|_H \leq M'' g(s), \tag{1.2.24}$$

$$\sup_{j=1,\dots,n} \sup_{\substack{\omega\in\Omega \\ \mathbf{x},\mathbf{y}_1,\dots,\mathbf{y}_j\in\mathbb{S} \\ |\mathbf{y}_1|_\infty=\dots=|\mathbf{y}_j|_\infty=1}} |S_t\partial^j_{\mathbf{y}_1\dots\mathbf{y}_j}(\sigma((\omega,s),\mathbf{x})e'_m))|_H \leq M'' t^{-\gamma} c_m, \tag{1.2.25}$$

for all $s \in [0,T]$, $t \in (0,T]$, $m \in \mathcal{M}$.

In accordance with Assumption 1.2.7*(i)*, by writing $\partial^j_{\mathbf{y_1}\dots\mathbf{y_j}}(\sigma((\omega,s),\mathbf{x})u)$, we mean the Gâteaux derivative of the map $\mathbf{x} \mapsto \sigma((\omega,s),\mathbf{x}).u$, for fixed $u \in U$.

Lemma 1.2.8. *Suppose that Assumption 1.2.3 and Assumption 1.2.7 are satisfied. Let $p > p^*$, $\beta \in (1/p, 1/2-\gamma)$. Then, for $j = 1,\dots,n$,*

$$F_b \in \mathcal{G}^j(\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}), L^{p,1}_{\mathcal{P}_T}(H); \mathcal{L}^{jp}_{\mathcal{P}_T}(\mathbb{S})),$$

$$F_\sigma \in \mathcal{G}^j(\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}), \overline{\Lambda}^{p,2,p}_{\mathcal{P}_T,S,\beta}(L(U,H)); \mathcal{L}^{jp}_{\mathcal{P}_T}(\mathbb{S})).$$

and, for $X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, $Y_1,\dots,Y_j \in \mathcal{L}^{jp}_{\mathcal{P}_T}(\mathbb{S})$, $u \in U$, $\mathbb{P}\otimes m$-a.e. $(\omega,t) \in \Omega_T$,

$$\begin{cases} \partial^j_{Y_1\dots Y_j} F_b(X)(\omega,t) = \partial^j_{Y_1(\omega)\dots Y_j(\omega)} b((\omega,t),X(\omega)) \\ \partial^j_{Y_1\dots Y_j} F_\sigma(X)(\omega,t)u = \partial^j_{Y_1(\omega)\dots Y_j(\omega)}(\sigma((\omega,t),X(\omega))u). \end{cases} \tag{1.2.26}$$

Moreover,

$$\sup_{j=1,\dots,n} \sup_{\substack{X\in\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}) \\ Y_1,\dots,Y_j\in\mathcal{L}^{jp}_{\mathcal{P}_T}(\mathbb{S}) \\ |Y_1|_{\mathcal{L}^{jp}_{\mathcal{P}_T}(\mathbb{S})}=\dots=|Y_j|_{\mathcal{L}^{jp}_{\mathcal{P}_T}(\mathbb{S})}=1}} \left(|\partial^j_{Y_1\dots Y_j} F_b(X)|_{L^{p,1}_{\mathcal{P}_T}(H)} + |\partial^j_{Y_1\dots Y_j} F_\sigma(X)|_{p,2,S,\beta} \right) \leq M''',$$

where M''' depends only on $T,p,\beta,\gamma,|g|_{L^1((0,T),\mathbb{R})},M'',|c|_{\ell^2(\mathcal{M})}$.

Proof. We prove the lemma by induction on n.

Case $n=1$. Let $X,Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$. First notice that the function

$$(\Omega_T,\mathcal{P}_T)\to H,\ (\omega,t)\mapsto \partial_{Y(\omega)}b((\omega,t),X(\omega))$$

is measurable. Let $\varepsilon\in\mathbb{R}\setminus\{0\}$. Since $b((\omega,t),\cdot)\in\mathcal{G}^1(\mathbb{S},H)$ for all $(\omega,t)\in\Omega_T$, we can write

$$\begin{aligned}\Delta_{\varepsilon Y}F_b(X)(\omega,t) &:= \varepsilon^{-1}(F_b(X+\varepsilon Y)(\omega,t)-F_b(X)(\omega,t))\\ &= \varepsilon^{-1}(b((\omega,t),X(\omega)+\varepsilon Y(\omega))-b((\omega,t),X(\omega)))\\ &= \int_0^1 \partial_{Y(\omega)}b((\omega,t),X(\omega)+\varepsilon\theta Y(\omega))d\theta \qquad \mathbb{P}\otimes m\text{-a.e. }(\omega,t)\in\Omega_T.\end{aligned} \tag{1.2.27}$$

By (1.2.24), we also have

$$|\partial_{Y(\omega)}b((\omega,t),X(\omega)+\varepsilon Y(\omega))|_H \le M''g(t)|Y(\omega)|_\infty \qquad \forall(\omega,t)\in\Omega_T,\ \forall\varepsilon\in\mathbb{R}. \tag{1.2.28}$$

By (1.2.27) and (1.2.28), we can apply Lebesgue's dominated convergence theorem and obtain

$$\lim_{\varepsilon\to 0}\int_0^T \left(\mathbb{E}\left[|\Delta_{\varepsilon Y}F_b(X)(\cdot,t)-\partial_Y b((\cdot,t),X)|_H^p\right]\right)^{1/p}dt=0.$$

This proves that F_b has directional derivative at X for the increment Y and that

$$\partial_Y F_b(X)(\omega,t)=\partial_{Y(\omega)}b((\omega,t),X(\omega)) \qquad \mathbb{P}\otimes m\text{-a.e. }(\omega,t)\in\Omega_T. \tag{1.2.29}$$

We now show that $\partial_Y F_b(X)$ is continuous in $(X,Y)\in\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$. Notice that, by (1.2.24), the linear map $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})\to L^{p,1}_{\mathcal{P}_T}(H)$, $Y\mapsto\partial_Y F_b(X)$, is bounded, uniformly in X. Then it is sufficient to verify the continuity of $\partial_Y F_b(X)$ in X, for fixed Y. Let $X_k\to X$ in $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$. By (1.2.24), (1.2.29), and Lebesgue's dominated convergence theorem, we have

$$\lim_{k\to\infty}\partial_Y F_b(X_k)=\partial_Y F_b(X)\text{ in } L^{p,1}_{\mathcal{P}_T}(H).$$

This concludes the proof that $F_b\in\mathcal{G}^1(\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}),L^{p,1}_{\mathcal{P}_T(H)})$ and that the differential is uniformly bounded.

Similarly, as regarding F_σ, we have that, for all $u\in U$, the function

$$(\Omega_T,\mathcal{P}_T)\to H,\ (\omega,t)\mapsto\partial_{Y(\omega)}(\sigma(t,X(\omega))u)$$

is measurable, and

$$\begin{aligned}\Delta_{\varepsilon Y}(F_\sigma(X)u)(\omega,t) &:= \varepsilon^{-1}((F_\sigma(X+\varepsilon Y)u)(\omega,t)-(F_\sigma(X)u)(\omega,t))\\ &= \varepsilon^{-1}(\sigma((\omega,t),X(\omega)+\varepsilon Y(\omega))u-\sigma((\omega,t),X(\omega))u)\\ &= \int_0^1 \partial_{Y(\omega)}(\sigma((\omega,t),X(\omega)+\varepsilon\theta Y(\omega))u)d\theta \qquad \mathbb{P}\otimes m\text{-a.e. }(\omega,t)\in\Omega_T.\end{aligned} \tag{1.2.30}$$

By (1.2.25), for all $0 \le s < t \le T$, $\omega \in \Omega$, $\varepsilon \in \mathbb{R}$, $m \in \mathcal{M}$,

$$|S_{t-s}\partial_{Y(\omega)}(\sigma((\omega,s),X(\omega)+\varepsilon Y(\omega))e'_m)|_H \le M''(t-s)^{-\gamma}c_m|Y(\omega)|_\infty. \tag{1.2.31}$$

By repeatedly applying Lebesgue's dominated convergence theorem, we have that

$$\int_0^T \left(\int_0^t (t-s)^{-2\beta}\left(\mathbb{E}\left[\left(\sum_{m\in\mathcal{M}} \left|S_{t-s}\left(\Delta_{\varepsilon Y}F_\sigma(X)(\cdot,s).e'_m - \partial_Y(\sigma((\cdot,s),X).e'_m)\right)\right|_H^2\right)^{p/2}\right]\right)^{2/p} ds\right)^{p/2} dt$$

goes to 0 as $\varepsilon \to 0$. This proves that F_σ has directional derivative at X for the increment Y and, taking into account the separability of U, that

$$\partial_Y F_\sigma(X)(\omega,t) = \partial_{Y(\omega)}(\sigma((\omega,t),X(\omega))\#) \qquad \mathbb{P}\otimes m\text{-a.e. } (\omega,t)\in\Omega_T. \tag{1.2.32}$$

By (1.2.31) and arguing similarly as done for $\partial_Y F_b(X)$, in order to show the continuity of $\partial_Y F_\sigma(X)$ in $(X,Y) \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, it is sufficient to verify the continuity of $\partial_Y F_\sigma(X)$ in X, for fixed Y. Let $X_k \to X$ in $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$. By (1.2.25), (1.2.32), and Lebesgue's dominated convergence theorem, we have

$$\lim_{k\to\infty} \partial_Y F_\sigma(X_k) = \partial_Y F_\sigma(X) \text{ in } \overline{\Lambda}^{p,2,p}_{\mathcal{P}_T,S,\beta}(L(U,H)).$$

This shosws that $F_\sigma \in \mathcal{G}^1(\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}), \overline{\Lambda}^{p,2,p}_{\mathcal{P}_T,S,\beta}(L(U,H)))$ and that the differential is uniformly bounded.

<u>*Case $n > 1$.*</u> Let $X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$ and $Y_1,\dots,Y_n \in \mathcal{L}^{np}_{\mathcal{P}_T}(\mathbb{S})$. By inductive hypothesis, we can assume that $\partial^{n-1}_{Y_1\dots Y_{n-1}}F_b(X) \in L^{p,1}_{\mathcal{P}_T}(H)$ exists, jointly continuous in $X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$ and $Y_1,\dots,Y_{n-1} \in \mathcal{L}^{(n-1)p}_{\mathcal{P}_T}(H)$, and that

$$\partial^{n-1}_{Y_1\dots Y_{n-1}}F_b(X)(\omega,t) = \partial^{n-1}_{Y_1(\omega)\dots Y_{n-1}(\omega)}b((\omega,t),X(\omega)) \qquad \mathbb{P}\otimes m\text{-a.e. } (\omega,t)\in\Omega_T.$$

The argument goes like the case $n = 1$. Let $\varepsilon \in \mathbb{R}\setminus\{0\}$. Since $b((\omega,t),\cdot) \in \mathcal{G}^n(\mathbb{S},H)$ for $(\omega,t)\in\Omega_T$, we can write, for $\mathbb{P}\otimes m$-a.e. $(\omega,t)\in\Omega_T$,

$$\begin{aligned}
\Delta_{\varepsilon Y_n}\partial^{n-1}_{Y_1\dots Y_{n-1}}F_b(X)(\omega,t) &:= \varepsilon^{-1}\left(\partial^{n-1}_{Y_1\dots Y_{n-1}}F_b(X+\varepsilon Y_n)(\omega,t) - \partial^{n-1}_{Y_1\dots Y_{n-1}}F_b(X)(\omega,t)\right)\\
&= \varepsilon^{-1}\left(\partial^{n-1}_{Y_1(\omega)\dots Y_{n-1}(\omega)}b((\omega,t),X(\omega)+\varepsilon Y_n(\omega)) - \partial^{n-1}_{Y_1(\omega)\dots Y_{n-1}(\omega)}b((\omega,t),X(\omega))\right)\\
&= \int_0^1 \partial^n_{Y_1(\omega)\dots Y_{n-1}(\omega)Y_n(\omega)}b((\omega,t),X(\omega)+\varepsilon\theta Y_n(\omega))d\theta.
\end{aligned}$$

By (1.2.24) we have

$$|\partial^n_{Y_1(\omega)\dots Y_n(\omega)}b((\omega,t),X(\omega)+\varepsilon Y_n(\omega))|_H \le M''g(t)\prod_{j=1}^n |Y_j(\omega)|_\infty \qquad \forall(\omega,t)\in\Omega_T,\ \forall\varepsilon\in\mathbb{R}.$$

Since $Y_j \in \mathscr{L}^{np}_{\mathscr{P}_T}(H)$, by the generalized Hölder's inequality $\prod_{j=1}^n |Y_j|_\infty \in L^p((\Omega,\mathscr{F}_T,\mathbb{P}),\mathbb{R})$. Then we can apply Lebesgue's dominated convergence theorem twice to obtain

$$\lim_{\varepsilon\to 0}\int_0^T \left(\mathbb{E}\left[|\Delta_{\varepsilon Y_n}\partial^{n-1}_{Y_1\dots Y_{n-1}}F_b(X)(\cdot,t)-\partial^n_{Y_1\dots Y_n}b((\cdot,t),X)|^p_H\right]\right)^{1/p}dt=0.$$

This proves that $\partial^{n-1}_{Y_1\dots Y_{n-1}}F_b$ has directional derivative at X for the increment Y_n and that

$$\partial^n_{Y_1\dots Y_{n-1}Y_n}F_b(X)(\omega,t)=\partial^n_{Y_1(\omega)\dots Y_n(\omega)}b((\omega,t),X(\omega))\qquad \mathbb{P}\otimes m\text{-a.e. }(\omega,t)\in\Omega_T. \tag{1.2.33}$$

The continuity of $\partial^n_{Y_1\dots Y_{n-1}Y_n}F_b(X)$ in $X\in\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$, $Y_1,\dots,Y_n\in\mathscr{L}^{np}_{\mathscr{P}_T}(H)$, is proved similarly as for the case $n=1$, again by invoking the generalized Hölder's inequality. This concludes the proof that $F_b\in\mathscr{G}^n(\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),L^{p,1}_{\mathscr{P}_T(H)};\mathscr{L}^{np}_{\mathscr{P}_T}(H))$. The uniform boundedness of the differentials is obtained by (1.2.24), (1.2.33), and the generalized Hölder's inequality.

Finally, as regarding F_σ, let again $X\in\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$ and $Y_1,\dots,Y_n\in\mathscr{L}^{np}_{\mathscr{P}_T}(\mathbb{S})$. By inductive hypothesis, we can assume that $\partial^{n-1}_{Y_1\dots Y_{n-1}}F_\sigma(X)\in\overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$ exists, that it is continuous in $X\in\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$, $Y_1,\dots,Y_{n-1}\in\mathscr{L}^{(n-1)p}_{\mathscr{P}_T}(\mathbb{S})$, and that, for all $u\in U$,

$$\partial^{n-1}_{Y_1\dots Y_{n-1}}F_\sigma(X)(\omega,t)u=\partial^{n-1}_{Y_1(\omega)\dots Y_{n-1}(\omega)}(\sigma((\omega,t),X(\omega))u)\qquad \mathbb{P}\otimes m\text{-a.e. }(\omega,t)\in\Omega_T.$$

For $\varepsilon\in\mathbb{R}\setminus\{0\}$, by strongly continuous Gâteaux differentiability of

$$x\mapsto\partial^{n-1}_{Y_1(\omega)\dots Y_{n-1}(\omega)}(\sigma(t,x)u),$$

we can write,

$$\begin{aligned}\Delta_{\varepsilon Y_n}\partial^{n-1}_{Y_1\dots Y_{n-1}}F_\sigma(X)(\omega,t)u &:= \varepsilon^{-1}\left(\partial^{n-1}_{Y_1\dots Y_{n-1}}F_\sigma(X+\varepsilon Y_n)(\omega,t)u-\partial^{n-1}_{Y_1\dots Y_{n-1}}F_\sigma(X)(\omega,t)u\right)\\ &=\varepsilon^{-1}\left(\partial^{n-1}_{Y_1(\omega)\dots Y_{n-1}(\omega)}(\sigma((\omega,t),X(\omega)+\varepsilon Y_n(\omega))u)-\partial^{n-1}_{Y_1(\omega)\dots Y_{n-1}(\omega)}(\sigma((\omega,t),X(\omega))u)\right)\\ &=\int_0^1\partial^n_{Y_1(\omega)\dots Y_n(\omega)}(\sigma((\omega,t),X(\omega)+\varepsilon\theta Y_n(\omega))u)d\theta.\end{aligned}$$

By (1.2.25) we have, for all $\omega\in\Omega$, $\varepsilon\in\mathbb{R}$, $0\le s<t\le T$, $m\in\mathscr{M}$,

$$|S_{t-s}\partial^n_{Y_1(\omega)\dots Y_n(\omega)}(\sigma((\omega,s),X(\omega)+\varepsilon Y_n(\omega))e'_m)|_H\le M''(t-s)^{-\gamma}c_m\prod_{j=1}^n|Y_j(\omega)|_\infty.$$

By the generalized Hölder's inequality and by Lebesgue's dominated convergence theorem, we conclude

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^T\Bigg(\int_0^t(t-s)^{-2\beta}\Bigg(\mathbb{E}\Bigg[\Bigg(\sum_{m\in\mathscr{M}}\Big|S_{t-s}\Big(\Delta_{\varepsilon Y_n}\partial^{n-1}_{Y_1\dots Y_{n-1}}F_\sigma(X)(\omega,s)e'_m\\ -\partial^n_{Y_1(\omega)\dots Y_n(\omega)}(\sigma((\cdot,s),X)e'_m)\Big|^2_H\Bigg)^{p/2}\Bigg]\Bigg)^{2/p}ds\Bigg)^{p/2}dt=0.\end{aligned} \tag{1.2.34}$$

Then $\partial^{n-1}_{Y_1\dots Y_{n-1}}F_\sigma$ has directional derivative at X for the increment Y_n, given by, for all $u\in U$,

$$\partial_{Y_n}\partial^{n-1}_{Y_1\dots Y_{n-1}}F_\sigma(X)(\omega,t)u=\partial^n_{Y_1(\omega)\dots Y_n(\omega)}(\sigma((\omega,t),X(\omega))u)\qquad \mathbb{P}\otimes m\text{-a.e. }(\omega,t)\in\Omega_T.$$

The continuity of $\partial_{Y_n}\partial^{n-1}_{Y_1\dots Y_{n-1}}F_\sigma(X)$ with respect to $X\in\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, $Y_1,\dots,Y_n\in\mathcal{L}^{np}_{\mathcal{P}_T}(H)$, is proved as for the case $n=1$. Then $F_\sigma\in\mathcal{G}^n(\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}),\overline{\Lambda}^{p,2,p}_{\mathcal{P}_T,S,\beta}(L(U,H));\mathcal{L}^{np}_{\mathcal{P}_T}(H))$. The uniform boundedness of the differentials is obtained by (1.2.25), (1.2.34), and the generalized Hölder's inequality. ■

Due to the fact that $X^{t,Y}$ is the fixed point of $\psi(Y,\cdot)$ and due to the structure of ψ, the previous lemma permits to easily obtain the following

Theorem 1.2.9. *Suppose that Assumption 1.2.7 is satisfied. Let $t\in[0,T]$, $p>p^*$, $p\geq n$. Then the map*

$$\mathcal{L}^{p^n}_{\mathcal{P}_T}(\mathbb{S})\to\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}),\ Y\mapsto X^{t,Y}\tag{1.2.35}$$

belongs to $\mathcal{G}^n(\mathcal{L}^{p^n}_{\mathcal{P}_T}(\mathbb{S}),\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}))$ and the Gâteaux differentials up to order n are uniformly bounded by a constant depending only on $T,p,\gamma,g,M,M',M'',|c|_{\ell^2(\mathcal{M})}$.

Proof. Let $\beta\in(1/p,1/2-\gamma)$. We have $p^k>p^*$ and $\beta\in(1/p^k,1/2-\gamma)$ for all $k=1,\dots,n$. Then, for $k=1,\dots,n$, the map

$$\psi_k:\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S})\times\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S})\to\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S}),\ (Y,X)\mapsto \mathrm{id}^S_t(Y)+S*_tF_b(X)+S\overset{dW}{*_t}F_\sigma(X)$$

is well-defined, where we have implicitly chosen the space $L^{p^k,1}_{\mathcal{P}_T}(H)$ as codomain of F_b and $\overline{\Lambda}^{p^k,2,p^k}_{\mathcal{P}_T,S,\beta}(L(U,H))$ as codomain of F_σ. Since the functions

$$\begin{gathered}\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S})\to\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S})\\ S*_t\#:L^{p^k,1}_{\mathcal{P}_T}(H)\to\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S})\\ S\overset{dW}{*_t}\#:\overline{\Lambda}^{p^k,2,p^k}_{\mathcal{P}_T,S,\beta}(L(U,H))\to\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S})\end{gathered}$$

are linear and continuous, with an upper bound for the operator norms depending only on β,M',T,p, we have, by applying Lemma 1.2.8, for $k,j=1,\dots,n$,

$$\psi_k\in\mathcal{G}^j(\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S})\times\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S}),\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S});\mathcal{L}^{p^k}_{\mathcal{P}_T}(\mathbb{S})\times\mathcal{L}^{jp^k}_{\mathcal{P}_T}(\mathbb{S})),$$

with differentials bounded by a constant depending only on $g,\gamma,M,M',M'',|c|_{\ell^2(\mathcal{M})},T$, on p^k (hence on p), and on β, which depends on p,γ. In particular, since $np^k\leq p^{k+1}$, we have, for the rescritions $\psi_{k|\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\times\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})}$ of ψ_k to $\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\times\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})$,

$$\begin{cases}\psi_{k|\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\times\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})}\in\mathscr{G}^1(\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\times\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}),\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}))\\ \psi_{k|\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\times\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})}\in\mathscr{G}^n(\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\times\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}),\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S});\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\times\mathscr{L}^{p^{k+1}}_{\mathscr{P}_T}(\mathbb{S}))\end{cases}$$

for $k=1,\dots,n$, with the Gâteaux differentials that are unformly bounded by a constant depending only on g,γ, M,M',M'',$|c|_{\ell^2(\mathscr{M})}$,$T$, on β (hence on p,γ), and on p^n,p^k,p^{k+1} (hence on p).

By (1.2.22) and (1.2.23) (where p should be replaced by p^k), there exists $\lambda>0$, depending only on g,γ,M,M',β,T, and on p^k (hence on p), such that ψ_k is a parametric 1/2-contraction with respect to the second variable, uniformly in the first one, when the space $\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})$ is endowed with the equivalent norm $|\cdot|_{\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}),\lambda}$. Then we can assume that the uniform bound of the Gâteaux differentials of ψ_k, for $k=1,\dots,n$, holds with respect to the equivalent norms $|\cdot|_{\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}),\lambda}$, and is again depending only on g, γ, M,M',M'',$|c|_{\ell^2(\mathscr{M})}$, T, p.

Now consider Assumption 1.1.12, after setting:

- $\alpha:=1/2$;
- $U:=X:=(\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}),|\cdot|_{\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}),\lambda})$;
- $Y_1:=(\mathscr{L}^{p}_{\mathscr{P}_T}(\mathbb{S}),|\cdot|_{\mathscr{L}^{p}_{\mathscr{P}_T}(\mathbb{S}),\lambda})$, $\dots$, $Y_k:=(\mathscr{L}^{p^k}_{\mathscr{P}_T},|\cdot|_{\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}),\lambda})$, $\dots$, $Y_n:=(\mathscr{L}^{p^n}_{\mathscr{P}_T},|\cdot|_{\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}),\lambda})$;
- $h_1:=\psi_{1|\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\times\mathscr{L}^{p}_{\mathscr{P}_T}(\mathbb{S})}$, $\dots$, $h_k:=\psi_{k|\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\times\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})}$, $\dots$, $h_n:=\psi_{n|\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\times\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})}$.

The discussion above, together with the smooth dependence of h_k on the first variable, shows that Assumption 1.1.12 is verified. We can then apply Theorem 1.1.13, which provides

$$((1.2.35)=)\ \mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\to\mathscr{L}^{p}_{\mathscr{P}_T}(\mathbb{S}),\ Y\mapsto X^{t,Y},\in\mathscr{G}^n(\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}),\mathscr{L}^{p}_{\mathscr{P}_T}(\mathbb{S})).$$

Finally, by applying Corollary 1.1.14, we obtain the uniform boundedness of the Gâteaux differentials up to order n of (1.2.35), with a bound that depends only on T,γ,g,M,M',M'', $|c|_{\ell^2(\mathscr{M})}$,$p$. ■

Remark 1.2.10. As said in the introduction to the chapter, we obtain the Gâteaux differentiability of $x\mapsto X^{t,x}$ by studying the parametric contraction providing $X^{t,x}$ as its unique fixed point, similarly as done in [23] for the non-path-dependent case. A different approach consists in studying directly the variations $\lim_{h\to0}\frac{X^{t,x+hv}-X^{t,x}}{h}$, showing that the limit exists (under suitable smooth assumptions on the coefficients) and is continuous with respect to v, for fixed t,x. This would provide the existence of the Gâteaux differential $\partial X^{t,x}$. Usually, in this way one shows also that $\partial X^{t,x}.v$ solves to an SDE. By using this SDE, one could go further and prove that the second order derivative $\partial^2 X^{t,x}.(v,w)$ exists, and that it is continuous in v,w, for fixed t,x. This would provide the second order Gâteaux differentiability of $x \mapsto X^{t,x}$. In this way, it is possible also to study the continuity of the Gâteaux differentials, by considering the SDEs solved by the directional derivatives, and to obtain Fréchet differentiability (under suitable assumptions on the coefficients, e.g. uniformly continuous Fréchet differentiability). By doing so, first- and second-order Fréchet differentiability are proved in [62]. But if one wants to use these methods to obtain derivatives of a generic order $n \geq 3$, then a recursive formula providing the SDE solved by the $(n-1)$th-order derivatives is needed, hence we fall back to a statement like Theorem 1.1.13. One could also try to prove the Fréchet differentiability of $x \mapsto X^{t,x}$ by studying directly the Fréchet differentiability of the parametric contractions providing the mild solution $X^{t,x}$. This is the approach followed in [48, Theorem 3.9], for orders $n = 1,2$. Nevertheless, we notice that the proof of [48, Theorem 3.8], on which [48, Theorem 3.9] relies, contains some inaccuracy: it is not clear why the term $|\eta(s)|_H/|\eta|_{\tilde{\mathcal{H}}_2}$ is bounded by 1, uniformly in (ω,s), when η is only supposed to be a process such that $|\eta|^2_{\tilde{\mathcal{H}}_2} := \sup_{s\in[0,T]} \mathbb{E}[|\eta(s)|^2_H] < \infty$.

Let $n = 2$ and let h_1 as in the proof of Theorem 1.2.9. By continuity and linearity of id^S_t, $S *_t \#$, $S \overset{dW}{*_t} \#$, and by recalling Lemma 1.2.8, we have, for $Y, Y_1, Y_2 \in \mathcal{L}^{p^2}_{\mathcal{P}_T}(\mathbb{S})$ (the space of the first variable of h_1), $X, X_1, X_2 \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$ (the space of the second variable of h_1),

$$\begin{cases} \partial_{Y_1} h_1(Y,X) = \mathrm{id}^S_t(Y_1) \\ \partial_{X_1} h_1(Y,X) = S *_t \partial_{X_1} F_b(X) + S \overset{dW}{*_t} \partial_{X_1} F_\sigma(X) \\ \partial^2_{Y_1 Y_2} h_1(Y,X) = \partial^2_{Y_1 X_1} h_1(Y,X) = 0 \\ \partial^2_{X_1 X_2} h_1(Y,X) = S *_t \partial^2_{X_1 X_2} F_b(X) + S \overset{dW}{*_t} \partial^2_{X_1 X_2} F_\sigma(X). \end{cases}$$

Then, by Theorem 1.1.13, we have

$$\partial_{Y_1} X^{t,Y} = \mathrm{id}^S_t(Y_1) + S *_t \partial_{\partial_{Y_1} X^{t,Y}} F_b(X^{t,Y}) + S \overset{dW}{*_t} \partial_{\partial_{Y_1} X^{t,Y}} F_\sigma(X^{t,Y}) \tag{1.2.36a}$$

$$\begin{aligned} \partial^2_{Y_1 Y_2} X^{t,Y} =& S *_t \partial_{\partial^2_{Y_1 Y_2} X^{t,Y}} F_b(X) + S \overset{dW}{*_t} \partial_{\partial^2_{Y_1 Y_2} X^{t,Y}} F_\sigma(X) \\ &+ S *_t \partial^2_{\partial_{Y_1} X^{t,Y} \partial_{Y_2} X^{t,Y}} F_b(X) + S \overset{dW}{*_t} \partial^2_{\partial_{Y_1} X^{t,Y} \partial_{Y_2} X^{t,Y}} F_\sigma(X) \end{aligned} \tag{1.2.36b}$$

where the equality (1.2.36a) holds in the space $\mathcal{L}^{p^2}_{\mathcal{P}_T}(\mathbb{S})$ and the equality (1.2.36b) holds in the space $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$. Formulae (1.2.36a) and (1.2.36b) generalize to the present setting the well-known SDEs for the first- and second-order derivatives with respect to the initial datum of mild solutions to non-path-dependent SDEs ([24, Theorem 9.8 and Theorem 9.9]).

Remark 1.2.11. Suppose that $\mathbb{S} = \mathbf{D}$, where $\mathbf{D}$ is the space of right-continuous left-limited functions $[0,T] \to H$. Notice that $\mathbf{D}$ satisfies all the properties required at p. 32. Then our setting applies and (1.2.36b)-(1.2.36b) provide equations for the first- and second-order directional derivatives of $X^{t,Y}$ with respect to vectors belonging to $\mathscr{L}^{p^2}_{\mathscr{P}_T}(\mathbf{D})$. In particular, if $\varphi\colon \mathbf{D} \to \mathbb{R}$ is a suitably regular functional, then the so-called "vertical derivatives" in the sense of Dupire of $F(t,\mathbf{x}) := \mathbb{E}[\varphi(X^{t,\mathbf{x}})]$, used in the finite dimensional Itō calculus developed by [12, 13, 14, 31] to show that F solves a path-dependent Kolmogorov equation associated to X, can be classically obtained by the chain rule starting from the Gâteaux derivatives $\partial_{Y_1} X^{t,Y}$, $\partial^2_{Y_1 Y_2} X^{t,Y}$, where $y_1, y_1 \in H$ and $Y_1 := \mathbf{1}_{[t,T]}(\cdot) y_1, Y_2 := \mathbf{1}_{[t,T]}(\cdot) y_2$.

1.2.3 Perturbation of path-dependent SDEs

In this section we study the stability of the mild solution $X^{t,Y}$ and of its Gâteaux derivatives with respect to perturbations of the data t, Y, S, b, σ.

Let us fix sequences $\mathbf{t} := \{t_j\}_{j\in\mathbb{N}} \subset [0,T]$, $\{S_j\}_{j\in\mathbb{N}} \subset L(H)$, $\{b_j\}_{j\in\mathbb{N}}$, $\{\sigma_j\}_{j\in\mathbb{N}}$, satisfying the following assumption.

Assumption 1.2.12. *Let b, σ, g, γ, M, be as in Assumption 1.2.3. Assume that*

(i) $\{t_j\}_{j\in\mathbb{N}}$ *is a sequence converging to* $\hat{t}$ *in* $[0,T]$*;*

(ii) *for all* $j \in \mathbb{N}$, $b_j\colon (\Omega_T \times \mathbb{S}, \mathscr{P}_T \otimes \mathscr{B}_{\mathbb{S}}) \to (H, \mathscr{B}_H)$ *is measurable;*

(iii) *for all* $j \in \mathbb{N}$, $\sigma_j\colon (\Omega_T \times \mathbb{S}, \mathscr{P}_T \otimes \mathscr{B}_{\mathbb{S}}) \to L(U,H)$ *is strongly measurable;*

(iv) *for all* $j \in \mathbb{N}$ *and all* $((\omega,t),\mathbf{x}) \in \Omega_T \times \mathbb{S}$, $b_j((\omega,t),\mathbf{x}) = b_j((\omega,t),\mathbf{x}_{t\wedge\cdot})$ *and* $\sigma_j((\omega,t),\mathbf{x}) = \sigma_j((\omega,t),\mathbf{x}_{t\wedge\cdot})$*;*

(v) *for all* $j \in \mathbb{N}$,

$$\begin{cases} |b_j((\omega,t),\mathbf{x})|_H \le g(t)(1+|\mathbf{x}|_\infty) & \forall ((\omega,t),\mathbf{x}) \in \Omega_T \times \mathbb{S}, \\ |b_j((\omega,t),\mathbf{x}) - b_j((\omega,t),\mathbf{x}')|_H \le g(t)|\mathbf{x}-\mathbf{x}'|_\infty & \forall (\omega,t) \in \Omega_T,\ \forall \mathbf{x},\mathbf{x}' \in \mathbb{S}; \end{cases}$$

(vi) *for all* $j \in \mathbb{N}$,

$$\begin{cases} |(S_j)_t \sigma_j((\omega,s),\mathbf{x})|_{L_2(U,H)} \le M t^{-\gamma}(1+|\mathbf{x}|_\infty) & \forall ((\omega,s),\mathbf{x}) \in \Omega_T \times \mathbb{S},\ \forall t \in (0,T], \\ |(S_j)_t \sigma_j((\omega,s),\mathbf{x}) - (S_j)_t \sigma_j((\omega,s),\mathbf{x}')|_{L_2(U,H)} \le M t^{-\gamma}|\mathbf{x}-\mathbf{x}'|_\infty & \forall (\omega,s) \in \Omega_T, \forall \mathbf{x},\mathbf{x}' \in \mathbb{S}, \forall t \in (0,T]; \end{cases}$$

(vii) for all $t \in [0,T]$, $\{(S_j)_t\}_{j\in\mathbb{N}}$ converges strongly to S_t, i.e.

$$\lim_{j\to\infty}(S_j)_t x = S_t x \qquad \forall x \in H;$$

(viii) the following convergences hold true:

$$\begin{cases} \lim_{j\to\infty} |b((\omega,t),\mathbf{x}) - b_j((\omega,t),\mathbf{x})|_H = 0 & \forall(\omega,t)\in\Omega_T,\ \forall \mathbf{x}\in\mathbb{S} \\ \lim_{j\to\infty} |S_t\sigma((\omega,s),\mathbf{x}) - (S_j)_t\sigma_j((\omega,s),\mathbf{x})|_{L_2(U,H)} = 0 & \forall(\omega,s)\in\Omega_T,\ \forall t\in(0,T],\ \forall\mathbf{x}\in\mathbb{S}. \end{cases}$$

Under Assumption 1.2.12, for $p > p^*$ and $\beta \in (1/p, 1/2-\gamma)$, we define $\mathrm{id}_{t_j}^{S_j}$, F_{b_j}, F_{σ_j}, $S_j *_{t_j} \#$, $S_j \overset{dW}{*}_{t_j} \#$, ψ_j, similarly as done for id_t^S, F_b, F_σ, $S *_t \#$, $S \overset{dW}{*}_t \#$, ψ, i.e.

$$\mathrm{id}_{t_j}^{S_j} \colon \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\ Y \mapsto \mathbf{1}_{[0,t_j]}(\cdot)Y + \mathbf{1}_{(t_j,T]}(\cdot)(S_j)_{\cdot-t_j}Y_{t_j}$$

$$F_{b_j} \colon \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}) \to L^{p,1}_{\mathscr{P}_T}(H),\ X \mapsto b_j((\cdot,\cdot),X)$$

$$F_{\sigma_j} \colon \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}) \to \overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S_j,\beta}(L(U,H)),\ X \mapsto \sigma_j((\cdot,\cdot),X)$$

$$S_j *_{t_j} \# \colon L^{p,1}_{\mathscr{P}_T}(H) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\ X \mapsto \mathbf{1}_{[t_j,T]}(\cdot)\int_{t_j}^{\cdot}(S_j)_{\cdot-s}X_s ds$$

$$S_j \overset{dW}{*}_{t_j} \# \colon \overline{\Lambda}^{p,2,p}_{\mathscr{P}_T,S_j,\beta}(L(U,H)) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\ \Phi \mapsto (S_j) \overset{dW}{*}_{t_j} \Phi.$$

$$\psi^{(j)} \colon \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}) \times \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\ (Y,X) \mapsto \mathrm{id}_{t_j}^{S_j}(Y) + S_j *_{t_j} F_{b_j}(X) + S_j \overset{dW}{*}_{t_j} F_{\sigma_j}(X).$$

In a similar way as done for ψ, we can obtain (1.2.22) for each $\psi^{(j)}$, with a constant $C'_{\lambda,g,\gamma,M',\beta,T,p,M}$ independent of j. In particular, there exists λ_0 large enough such that, for all $\lambda > \lambda_0$ and all $Y, X \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$,

$$\begin{aligned} &|\psi^{(j)}(Y,X) - \psi^{(j)}(Y',X')|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda} \leq \\ &\qquad \leq M'|Y-Y'|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda} + \frac{1}{2}|X-X'|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}),\lambda}, \qquad \forall j \in \mathbb{N}, \end{aligned} \tag{1.2.38}$$

where

$$M' \text{ is any upper bound for } \sup_{\substack{t\in[0,T]\\ j\in\mathbb{N}}} |(S_j)_t|_{L(H)}.$$

Let A_j denotes the infinitesimal generator of S_j. By arguing as done in the proof of Theorem 1.2.6, we have that, for each $j \in \mathbb{N}$, there exists a unique mild solution $X_j^{t,Y}$ in $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$ to

$$\begin{cases} d(X_j)_s = \left(A_j(X_j)_s + b_j\left((\cdot,s),X_j\right)\right)dt + \sigma_j\left((\cdot,s),X_j\right)dW_s & s\in(t_j,T] \\ (X_j)_s = Y_s & s\in[0,t_j], \end{cases} \tag{1.2.39}$$

and that, due to the equivalence of the norms $|\cdot|_{\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}),\lambda}$, the map $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}) \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, $Y \mapsto X^{t_j,Y}_j$ is Lipschitz, with Lipschitz constant bounded by some $C''_{g,\gamma,M,M',T,p}$ depending only on g,γ,M,M',T,p and independent of j.

For a given set $B \subset [0,T]$, let us denote

$$\mathbb{S}_B := \{\mathbf{x} \in \mathbb{S} \colon \forall t \in B,\ \mathbf{x} \text{ is continuous in } t\}.$$

Then $\mathbb{S}_B$ is a closed subspace of $\mathbb{S}$ and it satisfies all the three conditions required for $\mathbb{S}$ at p. 32. Moreover, if $t \in [0,T]$ and $Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_B)$, then $X^{t,Y} \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_B)$, because $X^{t,Y}$ is continuous on $[t,T]$ (recall that $S *_t \#$ and $S \overset{dW}{*_t} \#$ are $\mathcal{L}^p_{\mathcal{P}_T}(C([0,T],H))$-valued) and coincides with Y on $[0,t]$.

Proposition 1.2.13. *Suppose that Assumption 1.2.3 and Assumption 1.2.12 are satisfied and let $p > p^*$. Then*

$$\lim_{j\to\infty} X^{t_j,Y}_j = X^{\hat{t},Y} \tag{1.2.40}$$

in $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})$, uniformly for Y on compact subsets of $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})$.

Proof. Let $\psi^{(j)}$ be defined as above (p. 49). It is clear that, if $Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})$ and $X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, then $\psi(Y,X) \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})$, because it is continuous on $[\hat{t},T]$ and coincides with Y on $[0,\hat{t}]$. Similarly, $\psi^{(j)}(Y,X)$ is continuous on $[t_j,T]$ and coincides with Y on $[0,t_j]$, than also $\psi^{(j)}(Y,X) \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})$. Then, if the claimed convergence occurs, it does in $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})$.

In order to prove the convergence, we consider the restrictions

$$\begin{cases} \hat{\psi}^{(j)} := \psi^{(j)}_{|\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})\times\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})} & \forall j \in \mathbb{N} \\ \hat{\psi} := \psi_{|\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})\times\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})}, \end{cases}$$

which are $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})$-valued, as noticed above. Clearly (1.2.38) still holds true with $\hat{\psi}^{(j)}$, $\hat{\psi}$ in place of $\psi^{(j)}$, ψ, respectively, and then

$$\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}}) \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}}),\ Y \mapsto X^{t_j,Y}_j$$

is Lipschitz in Y, uniformly in j. We then need only to prove the convergence

$$X^{t_j,Y}_j \to X^{\hat{t},Y} \text{ in } \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}}), \forall Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}}).$$

Thanks to Lemma 1.1.9*(i)*, the latter convergence reduces to the pointwise convergence

$$\hat{\psi}^{(j)} \to \hat{\psi}.$$

Let $Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}(\{\hat{t}\}))$. Due to the continuity of $Y(\omega)$ in $\hat{t}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, the strong continuity of S_j and S, and the strong convergence $S_j \to S$, we have $\mathrm{id}^{S_j}_{t_j}(Y) \to \mathrm{id}^{S}_{\hat{t}}(Y)$ in $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})$ for all $Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$ (this can be seen by (1.2.17)).

We show that $S_j \overset{dW}{*}_{t_j} F_{\sigma_j}(X) \to S \overset{dW}{*}_{\hat{t}} F_\sigma(X)$, for all $X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$. Write

$$S_j \overset{dW}{*}_{t_j} F_{\sigma_j} - S \overset{dW}{*}_{\hat{t}} F_\sigma = (S_j \overset{dW}{*}_{t_j} F_{\sigma_j} - S \overset{dW}{*}_{t_j} F_\sigma) + (S \overset{dW}{*}_{t_j} F_\sigma - S \overset{dW}{*}_{\hat{t}} F_\sigma).$$

By Lebesgue's dominated convergence theorem and by Assumption 1.2.12, we have, for $\beta \in (1/p, 1/2-\gamma)$,

$$\lim_{j\to\infty} \int_0^T \left(\int_0^t (t-s)^{-2\beta} \left(\mathbb{E}\left[|(S_j)_{t-s}\sigma_j((\cdot,s),X)) - S_{t-s}\sigma((\cdot,s),X))|^p_{L_2(U,H)} \right] \right)^{2/p} ds \right)^{p/2} dt = 0$$

Then, by (1.2.11) (which holds uniformly in t),

$$S_j \overset{dW}{*}_{t_j} F_{\sigma_j}(X) - S \overset{dW}{*}_{t_j} F_\sigma(X) \to 0 \text{ in } \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}).$$

By (1.2.14), we also have

$$S \overset{dW}{*}_{t_j} F_\sigma(X) - S \overset{dW}{*}_{\hat{t}} F_\sigma(X) \to 0 \text{ in } \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}).$$

Then, we conclude

$$S_j \overset{dW}{*}_{t_j} F_{\sigma_j} - S \overset{dW}{*}_{\hat{t}} F_\sigma \to 0 \text{ in } \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}).$$

By arguing in a very similar way as done for $S_j \overset{dW}{*}_{t_j} F_{\sigma_j} - S \overset{dW}{*}_{\hat{t}} F_\sigma$, one can prove that

$$\forall X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}),\ S_j *_{t_j} F_{b_j}(X) - S *_{\hat{t}} F_b(X) \to 0 \text{ in } \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}).$$

Then $\hat{\psi}^{(j)} \to \hat{\psi}$ pointwise and the proof is complete. ■

The following result provides continuity of the mild solution with respect to perturbations of all the data of the system.

Theorem 1.2.14. *Suppose that Assumption 1.2.3 and Assumption 1.2.12 are satisfied, let* $p > p^*$, $Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}_{\{\hat{t}\}})$, *and let* $\{Y_j\}_{j\in\mathbb{N}} \subset \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$ *be a sequence converging to* Y *in* $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$. *Then*

$$\lim_{j\to\infty} X_j^{t_j,Y_j} = X^{\hat{t},Y} \text{ in } \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}).$$

Proof. Write

$$X^{\hat{t},Y} - X_j^{t_j,Y_j} = (X^{\hat{t},Y} - X_j^{t_j,Y}) + (X_j^{t_j,Y} - X_j^{t_j,Y_j}), \tag{1.2.41}$$

The term $X^{\hat{t},Y} - X_j^{t_j,Y}$ tends to 0 by Proposition 1.2.13, whereas the term $X_j^{t_j,Y} - X_j^{t_j,Y_j}$ tends to 0 by uniform equicontinuity of the family

$$\left\{ \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}) \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}),\ Y \mapsto X_j^{t_j,Y} \right\}_{j\in\mathbb{N}}.$$ ■

We end this chapter with a result regarding stability of Gâteaux differentials of mild solutions.

Assumption 1.2.15. *Let $b,\sigma,g,\gamma,n,c,M''$ be as in Assumption 1.2.7, and let $\{b_j\}_{j\in\mathbb{N}}$, $\{\sigma\}_{j\in\mathbb{N}}$, $\{S_j\}_{j\in\mathbb{N}}$, be as in Assumption 1.2.12. Assume that*

(i) for all $j\in\mathbb{N}$, $(\omega,t)\in\Omega_T$, and $u\in U$, $b_j((\omega,t),\cdot)\in\mathscr{G}^n(\mathbb{S},H)$ and $\sigma_j((\omega,t),\cdot)u\in\mathscr{G}^n(\mathbb{S},H)$;

(ii) for all $s\in[0,T]$,

$$\sup_{\substack{i=1,\dots,n\\ j\in\mathbb{N}}}\ \sup_{\substack{\omega\in\Omega\\ \mathbf{x},\mathbf{y}_1,\dots,\mathbf{y}_j\in\mathbb{S}\\ |\mathbf{y}_1|_\infty=\dots=|\mathbf{y}_i|_\infty=1}} |\partial^i_{\mathbf{y}_1\dots\mathbf{y}_i} b_j((\omega,s),\mathbf{x})|_H \le M''g(s), \tag{1.2.42}$$

and, for all $s\in[0,T]$, $t\in(0,T]$, and all $m\in\mathscr{M}$,

$$\sup_{\substack{i=1,\dots,n\\ j\in\mathbb{N}}}\ \sup_{\substack{\omega\in\Omega\\ \mathbf{x},\mathbf{y}_1,\dots,\mathbf{y}_i\in\mathbb{S}\\ |\mathbf{y}_1|_\infty=\dots=|\mathbf{y}_i|_\infty=1}} |(S_j)_t\partial^i_{\mathbf{y}_1\dots\mathbf{y}_i}(\sigma_j((\omega,s),\mathbf{x})e'_m))|_H \le M''t^{-\gamma}c_m; \tag{1.2.43}$$

(iii) for all $X\in\mathbb{S}$,

$$\begin{cases} \lim_{j\to\infty} |\partial^i_{\mathbf{y}_1\dots\mathbf{y}_i} b((\omega,t),\mathbf{x}) - \partial^i_{\mathbf{y}_1\dots\mathbf{y}_i} b_j((\omega,t),\mathbf{x})|_H = 0 & \forall(\omega,t)\in\Omega_T \\ \lim_{j\to\infty} |S_t\partial^i_{\mathbf{y}_1\dots\mathbf{y}_i}(\sigma((\omega,s),\mathbf{x})e'_m) - (S_j)_t\partial^i_{\mathbf{y}_1\dots\mathbf{y}_i}(\sigma_j((\omega,s),\mathbf{x})e'_m)|_H = 0 & \begin{cases}\forall\omega\in\Omega,\\ \forall s\in[0,T],\forall t\in(0,T],\\ \forall m\in\mathscr{M}.\end{cases} \end{cases}$$

Theorem 1.2.16. *Suppose that Assumption 1.2.3 and Assumption 1.2.12 are satisfied, and that, for some $n\in\mathbb{N}$, $n\ge 1$, Assumption 1.2.7 and Assumption 1.2.15 are satisfied. Let $p>p^*$, $p\ge n$. Then, for $i=1,\dots,n$,*

$$\partial^i_{Y_1\dots Y_i} X_j^{t_j,Y} \to \partial^i_{Y_1\dots Y_i} X^{\hat{t},Y} \ \text{in}\ \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}}), \tag{1.2.44}$$

uniformly for $Y,Y_1,\dots,Y_i$ in compact subsets of $\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$.

Proof. By Theorem 1.2.9, $\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S})\to\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S})$, $Y\mapsto X_j^{t_j,Y}$ belongs to $\mathscr{G}^n(\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}),\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}))$. Then, since $X_j^{t_j,Y}\in\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$ if $Y\in\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$, the map $\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})\to\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$, $Y\mapsto X_j^{t_j,Y}$ belongs to $\mathscr{G}^n(\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}}),\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}}))$.

To prove (1.2.44), we wish to apply Proposition 1.1.15. In the proof of Theorem 1.2.9, we associated the map ψ and the spaces $\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})$ to Assumption 1.1.12. In the same way, here, we associate the restrictions

$$\psi^{(1)}_{|\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})\times\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})},\ \psi^{(2)}_{|\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})\times\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})},\ \psi^{(3)}_{|\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})\times\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})},\dots,$$

respectively to the functions $h_1^{(1)},h_1^{(2)},h_1^{(3)},\dots$ appearing in the assumption of Proposition 1.1.15, and, to each $h_1^{(m)}$, we associate the functions $h_k^{(m)}$, for $k=1,\dots,n$, defined by $h_k^{(m)} := \psi_{k|\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})\times\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})}$ and considered as $\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})$-valued functions.

As argued several times above, we can choose $\lambda > 0$ such that, for $m = 1,2,\dots$ and $k = 1,\dots,n$, each function $h_k^{(m)}$ is a parametric 1/2-contractions with respect to the norm $|\cdot|_{\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}),\lambda}$. With respect to this equivalent norm, for each $h_1^{(m)}$, Assumption 1.1.12 can be verified in exactly the same way as it was verified for the function h_1 appearing in the proof of Theorem 1.2.9. Then, in order to apply Proposition 1.1.15, it remains to verify hypotheses *(i)*,*(ii)*,*(iii)*, appearing in the statement of that proposition. Since the norms $|\cdot|_{\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}),\lambda}$, $\lambda \geq 0$, are equivalent, the three hypotheses reduce to the following convergences:

(i) for all $k = 1,\dots,n$, $X \in \mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$,

$$\psi^{(j)}(Y,X) \to \psi(Y,X) \text{ in } (\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}}), |\cdot|_{\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})}) \tag{1.2.45}$$

uniformly for Y on compact subsets of $\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$;

(ii) for $k = 1,\dots,n$

$$\begin{cases} \lim_{j\to\infty} \partial_{Y'}\psi^{(j)}(Y,X) = \partial_{Y'}\psi(Y,X) & \text{in } (\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}}), |\cdot|_{\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})}) \\ \lim_{j\to\infty} \partial_{X'}\psi^{(j)}(Y,X) = \partial_{X'}\psi(Y,X) & \text{in } (\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}}), |\cdot|_{\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})}) \end{cases} \tag{1.2.46}$$

uniformly for Y,Y' on compact subsets of $\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$ and X,X' on compact subsets of $\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$;

(iii) for all $k = 1,\dots,n-1$, $Y \in \mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$, $l,i = 0,\dots,n$, $1 \leq l+i \leq n$,

$$\lim_{j\to\infty} \partial^{l+i}_{Y_1\dots Y_l X_1\dots X_i}\psi^{(j)}(Y,X) = \partial^{l+i}_{Y_1\dots Y_l X_1\dots X_i}\psi(Y,X) \text{ in } (\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}}), |\cdot|_{\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})}) \tag{1.2.47}$$

uniformly for $Y, Y_1,\dots,Y_l$ on compact subsets of $\mathscr{L}^{p^n}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$, X on compact subsets of $\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$, $X_1,\dots,X_i$ on compact subsets of $\mathscr{L}^{p^{k+1}}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$.

Taking into account the equicontinuity of the family $\{\psi^{(j)}\}_{j\in\mathbb{N}}$ with respect to the second variable, (i) is contained in the proof Proposition 1.2.13. As regarding (ii) and (iii), since the linear term $\mathrm{id}^{S_j}_{t_j}$ is easily treated in $\mathscr{L}^{p}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$ (as shown in the proof of Proposition 1.2.13), the only comments to make are about the convergences of the derivatives

$$\begin{cases} \partial_{Y'}(S_j *_{t_j} F_{b_j})(X) \\ \partial_{X'}(S_j *_{t_j} F_{b_j})(X) \\ \partial_{Y'}(S_j \overset{dW}{*}_{t_j} F_{\sigma_j})(X) \\ \partial_{X'}(S_j \overset{dW}{*}_{t_j} F_{\sigma_j})(X) \end{cases} \quad \text{and} \quad \begin{cases} \partial^{l+i}_{Y_1\dots Y_l X_1\dots X_i}(S_j *_{t_j} F_{b_j})(X) \\ \partial^{l+i}_{Y_1\dots Y_l X_1\dots X_i}(S_j *_{t_j} F_{\sigma_j})(X). \end{cases}$$

Due to linearity and continuity of the convolution operators, to the independence of the first variable of F_b and F_σ, and to Lemma 1.2.8, the above derivatives are respectively equal to

$$\begin{cases} 0 \\ S_j *_{t_j} (\partial_{X'} F_{b_j})(X) \\ 0 \\ S_j \overset{dW}{*}_{t_j} (\partial_{X'} F_{\sigma_j})(X) \end{cases} \quad \text{and} \quad \begin{cases} \begin{cases} S_j *_{t_j} (\partial^i_{X_1 \dots X_i} F_{b_j})(X) & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases} \\ \begin{cases} S_j *_{t_j} (\partial^i_{X_1 \dots X_i} F_{\sigma_j})(X) & \text{if } l = 0 \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{1.2.48}$$

Let us consider, for example, the difference

$$S_j *_{t_j} (\partial^i_{X_1 \dots X_i} F_{\sigma_j})(X_j) - S *_{\hat{t}} (\partial^i_{X_1 \dots X_i} F_\sigma)(X) \tag{1.2.49}$$

for some sequence $\{X_j\}_{j\in\mathbb{N}}$ converging to X in $\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S})$. We can decompose the above difference as done in (1.2.41), and then use the same arguments, together with expressions (1.2.26), the bounds (1.2.42) and (1.2.42), the generalized Hölder's inequality, the pointwise convergences in Assumption 1.2.15*(iii)*, and Lebesgue's dominated convergence theorem, to conclude

$$S_j *_{t_j} (\partial^i_{X_1 \dots X_i} F_{\sigma_j})(X_j) - S *_{\hat{t}} (\partial^i_{X_1 \dots X_i} F_\sigma)(X) \to 0$$

in $\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$, for all $X_1, \dots, X_i \in \mathscr{L}^{p^{k+1}}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$. By recalling the continuity of $X \mapsto \partial^i_{X_1 \dots X_i} F_\sigma(X)$ (Lemma 1.2.8), this shows the convergence

$$S_j *_{t_j} (\partial^i_{X_1 \dots X_i} F_{\sigma_j})(X) - S *_{\hat{t}} (\partial^i_{X_1 \dots X_i} F_\sigma)(X) \to 0, \tag{1.2.50}$$

uniformly for X on compact sets of $\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$, for fixed $X_1, \dots, X_i \in \mathscr{L}^{p^{k+1}}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$. But, since by Lemma 1.2.8 the derivatives (1.2.48) are jointly continuous in $X, X', X_1, \dots, X_i$, and uniformly bounded, the convergence (1.2.50) occurs uniformly for X on compact sets of $\mathscr{L}^{p^k}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$ and $X_1, \dots, X_i$ on compact sets of $\mathscr{L}^{p^{k+1}}_{\mathscr{P}_T}(\mathbb{S}_{\{\hat{t}\}})$. The arguments for the other derivatives are similar. This shows that we can apply Proposition 1.1.15, which provides (1.2.44). ■

Chapter 2

Partial regularity of viscosity solutions for a class of Kolmogorov equations arising from mathematical finance

In this chapter we study partial regularity of viscosity solutions for a class of Kolmogorov equations associated to stochastic delay problems. They are linear second order partial differential equations in an infinite dimensional Hilbert space with a drift term which contains an unbounded operator and a second order term which only depends on a finite dimensional component of the Hilbert space. Such equations are typically investigated using the notion of the so-called B-continuous viscosity solutions (see [37, 60, 95]). We impose conditions under which our Kolmogorov equations have unique B-continuous viscosity solutions. However general Hamilton-Jacobi-Bellman equations associated to stochastic delay optimal control problems which are rewritten as optimal control problems for stochastic differential equations (SDE) in an infinite dimensional Hilbert space are difficult, not well studied yet, and few results are available in the literature.

We work directly with the value function here since its partial regularity is of interest in the hedging problem outlined in the Introduction and it is well known that under our assumptions the value function is the unique B-continuous viscosity solution to the Kolmogorov equation (see e.g. [37, 60]). We thus never use the theory of B-continuous viscosity solutions. Instead our strategy for proving partial regularity of the value function is the following. We consider SDEs with smoothed out coefficients and the unbounded operator replaced by its Yosida approximations and study the corresponding value functions with smoothed out payoff function. The new value functions are Gâteaux differentiable and converge on compact sets to the original value function. They also satisfy their associated Kolmogorov equations. We then prove that their finite dimensional sections are viscosity solutions to certain linear finite dimensional parabolic equations for which we establish $C^{1+\alpha}$ estimates. Passing to the limit with the approximations, these estimates are preserved, giving $C^{1+\alpha}$ partial regularity for finite dimensional sections of the original value function.

Partial regularity results for first order unbounded HJB equations in Hilbert spaces associated to certain deterministic optimal control problems with delays have been obtained in [39]. The technique of [39] relied on arguments using concavity of the data and strict convexity of the Hamiltonian and provided C^1 regularity on one-dimensional sections corresponding to the so-called "present" variable. Here the equations are of second order, we rely on approximations and parabolic regularity estimates, and we obtain regularity on m-dimensional sections. The reader can also consult [71] for various global and partial regularity results for bounded HJB equations in Hilbert spaces (see also [94]).

We refer the reader to [37, 71, 73] for the theory of viscosity solutions for bounded second order HJB equations in Hilbert spaces and to [37, 60, 95] for the theory of the so-called B-continuous viscosity solutions for unbounded second order HJB equations in Hilbert spaces. A fully nonlinear equation with a similar separated structure to our Kolmogorov equation (2.2.14) but with a nonlinear unbounded operator A was studied in [63]. For classical results about Kolmogorov equation in Hilbert spaces we refer the reader to [23].

Let us recall the financial problem outlined in the Introduction. In a financial market composed by a risk free asset B and a risky asset R with path-dependent dynamics, we are concerned with the hedging of a path-dependent derivative. Let us assume that the path-dependent dynamics of R (see (9) at page 1) separates the dependence of ν on the present from the dependence on the past and that the dependence on the path involves the whole past history from $-\infty$ up to now, as follows:

$$\begin{cases} dR_s = rR_s ds + \nu(t, R_s, \{R_{s'+s}\}_{s'\in(-\infty,0)})dW_s & s \in (t,T] \\ R_t = x_0 & \\ R_{t+t'} = x_1(t') & t' \in (-\infty, 0), \end{cases} \tag{2.0.1}$$

where $x_0 \in \mathbb{R}$ and $x_1 : (-\infty, 0) \to \mathbb{R}$ is a given deterministic funtion belonging to $L^2(\mathbb{R}^-, \mathbb{R})$, expressing the past of the stock price R. The claim that we aim to hedge has the form $\varphi(R_T, \{R_{T+t'}\}_{t'\in(-\infty,0)})$, where $\varphi : \mathbb{R} \times L^2(\mathbb{R}^-, \mathbb{R}) \to \mathbb{R}$. In $\nu = \nu(t, x_0, x_1)$ and $\varphi = \varphi(x_0, x_1)$, the "present" is then represented by x_0 and the "past" by x_1.

We point out that model (2.0.1) can also include the case in which the path-dependence is only relative to a finite past window $[-d, 0]$. To fit this case into (2.0.1), it is sufficient to replace the coefficient ν in (2.0.1) by a ν' defined by

$$\nu'(s, R_s, \{R_{s+s'}\}_{s'\in(-\infty,0)}) := \nu(s, R_s, \mathbf{1}_{[-d,0)}(\cdot)\{R_{s+s'}\}_{s'\in(-\infty,0)}).$$

In such a case, it is easily seen that R does not depend on the tail $\mathbf{1}_{(-\infty,-d)}(\cdot)x_1$ of the initial datum. Hence a delay model with a finite delay window can be rewritten in the form (2.0.1).

If $R^{t,(x_0,x_1)}$ solves (2.0.1), then in general it is not Markovian. Moreover, since both the claim φ and the function u, corresponding to (10) at p. 3 but now defined by

$$u(t,x_0,x_1) := e^{-r(T-t)}\mathbb{E}\left[\varphi\big(R_T^{t,(x_0,x_1)},\{R_{t'}^{t,(x_0,x_1)}\}_{t'\in(-\infty,T)}\big)\right] \quad \forall (t,x_0,x_1)\in[0,T]\times\mathbb{R}\times L^2(\mathbb{R}^-,\mathbb{R}),$$

are path-dependent, the PDE analogous to (5) at p. 2 would now be path-dependent, and it would be necessary to employ a stochastic calculus for path-dependent Itō processes in order to relate u with the PDE, as in the non-path-dependent case.

A classical workaround tool to regain Markovianity and avoid the complications of a path-dependent stochastic calculus consists in rephrasing the model in a functional space setting. What we lose by doing so is that the dynamics will evolve in an infinite dimensional space. We briefly recall how the rephrasing works. We refer the reader to [10] for the case with finite delay. The argument extends without difficulty to the case with infinite delay.

We first introduce the Hilbert space $H := \mathbb{R}\times L^2(\mathbb{R}^-,\mathbb{R})$, the functions

$$\begin{aligned} &F\colon [0,T]\times H\to H,\ (x_0,x_1)\mapsto (rx_0,0)\\ &\Sigma\colon [0,T]\times H\to H,\ (x_0,x_1)\mapsto (\nu(t,x_0,x_1),0), \end{aligned} \tag{2.0.2}$$

and the strongly continuous semigroup of translations on H, i.e. the family $\hat{S} := \{\hat{S}_t\}_{t\in\mathbb{R}^+}$ of linear continuous operators defined by

$$\hat{S}_t\colon H\to H,\ (x_0,x_1)\mapsto (x_0, x_1(t+\cdot)\mathbf{1}_{(-\infty,-t)} + x_0\mathbf{1}_{[-t,0]}).$$

The infinitesimal generator $\hat{A}$ of $\hat{S}$ is given by

$$\hat{A}\colon D(\hat{A})\to H,\ (x_0,x_1)\mapsto (0,x_1'),$$

where

$$D(\hat{A}) = \left\{(x_0,x_1)\in H\colon x_1\in W^{1,2}(\mathbb{R}^-) \text{ and } x_0 = x_1(0)\right\}.$$

Then we consider the H-valued dynamics

$$\begin{cases} d\hat{X}_s = \big(\hat{A}\hat{X}_s + F\big(t,\hat{X}_s\big)\big)\,ds + \Sigma\big(t,\hat{X}_s\big)\,dW_s & s\in(t,T],\\ \hat{X}_t = (x_0,x_1), \end{cases} \tag{2.0.3}$$

where $(x_0,x_1)\in H$. Under usual Lipschitz assumptions on ν, it can be shown that (2.0.3) has a unique mild solution $\hat{X}^{t,(x_0,x_1)}$ (we refer to [24] for stochastic differential equations in Hilbert spaces). The link between (2.0.1) and (2.0.3) is given by the following equation:

$$\text{for all } s\in[0,T],\ \hat{X}_s^{t,(x_0,x_1)} = \big(R_s^{t,(x_0,x_1)},\{R_{s'+s}^{t,(x_0,x_1)}\}_{s'\in(-\infty,0)}\big)\ \mathbb{P}\text{-a.s.,} \tag{2.0.4}$$

where $R^{t,(x_0,x_1)}$ denotes the unique strong solution to (2.0.1). Observe that $\hat{X}$ is Markovian and no path-dependence appears in the coefficients F, Σ. This is the natural rephrasing of the dynamics of R to get a Markovian setting for which the basic tools of stochastic calculus in Hilbert spaces (such as Itō's formula) are available.

We need an additional step to let the model studied in this chapter apply to the financial problem we are considering. We rephrase (2.0.3) as an SDE in the same Hilbert space H, but with a maximal dissipative unbounded operator. To this goal, we observe that $A := \hat{A} - \frac{1}{2}$ is a maximal dissipative operator generating the semigroup of contractions $S := \{S_t := e^{-t/2}\hat{S}_t\}_{t\in\mathbb{R}^+}$. Let us define $G(t,x) := F(t,x) + \frac{x}{2}$, $(t,x) \in [0,T]\times H$. Denote by $X^{t,(x_0,x_1)}$ the unique mild solution to the SDE

$$\begin{cases} dX_s = (AX_s + G(t,X_s))ds + \Sigma(t,X_s)dW_s & s \in (t,T] \\ X_t = (x_0,x_1), \end{cases} \tag{2.0.5}$$

where $(x_0,x_1) \in H$, $t \in [0,T)$. It is not difficult to see that $\hat{X}^{t,(x_0,x_1)} = X^{t,(x_0,x_1)}$. Indeed, if $\{\hat{A}_\lambda\}_{\lambda>1/2}$ denote the Yosida approximations of $\hat{A}$, then the strong solution to

$$\begin{cases} dX_{\lambda,s} = \big(\hat{A}_\lambda X_{\lambda,s} + F\big(t,X_{\lambda,s}\big)\big)ds + \Sigma\big(t,X_{\lambda,s}\big)dW_s & s \in (t,T], \\ X_{\lambda,t} = (x_0,x_1), \end{cases} \tag{2.0.6}$$

coincides with the strong solution $X_\lambda^{t,(x_0,x_1)}$ to

$$\begin{cases} X_{\lambda,s} = \left(\left(\hat{A}_\lambda - \dfrac{1}{2}\right)X_{\lambda,s} + G\left(t,X_{\lambda,s}\right)\right)ds + \Sigma\left(t,X_{\lambda,s}\right)dW_s & s \in (t,T], \\ X_{\lambda,t} = (x_0,x_1), \end{cases} \tag{2.0.7}$$

by the very definition and by uniqueness of strong solutions. Recalling that strong and mild solutions coincide when the linear operator appearing in the drift is bounded[1], $X_\lambda^{t,(x_0,x_1)}$ solves (2.0.7) in the mild sense. Now observe that $\hat{A}_\lambda - \frac{1}{2}$ generates the semigroup $\hat{S}_\lambda := \{\hat{S}_{\lambda,t} := e^{-t/2}e^{\hat{A}_\lambda t}\}_{t\in\mathbb{R}^+}$. Since $e^{\hat{A}_\lambda t} \to \hat{S}_t$ strongly as $\lambda \to +\infty$, we have also $\hat{S}_{\lambda,t} \to S_t$ strongly. Then the mild solution $X_\lambda^{t,(x_0,x_1)}$ converges to the mild solution $X^{t,(x_0,x_1)}$ as $\lambda \to +\infty$ (see e.g. the argument used to show Proposition 2.1.10*(ii)*). Similarly, $X_\lambda^{t,(x_0,x_1)}$ solves (2.0.6) in the mild sense and then $X_\lambda^{t,(x_0,x_1)} \to \hat{X}^{t,(x_0,x_1)}$ as $\lambda \to +\infty$. We thus conclude that $\hat{X}^{t,(x_0,x_1)} = X^{t,(x_0,x_1)}$ in a suitable space of processes where the well-posedness of the SDEs and the convergences above are considered.

It follows that equation (2.0.4) can be rewritten as:

$$\text{for all } s \in [0,T],\ X_s^{t,(x_0,x_1)} = \left(R_s^{t,(x_0,x_1)}, \{R_{s'+s}^{t,(x_0,x_1)}\}_{s'\in(-\infty,0)}\right)\ \mathbb{P}\text{-a.s.}. \tag{2.0.8}$$

Having (2.0.8), the function u can be written as

$$u(t,x_0,x_1) = e^{-r(T-t)}\mathbb{E}\left[\varphi\big(X_T^{t,(x_0,x_1)}\big)\right] \qquad \forall (t,(x_0,x_1)) \in [0,T]\times H. \tag{2.0.9}$$

[1] This can be seen by an easy application of Ito's formula, together with uniqueness of mild solutions.

Thanks to the special structure of Σ in SDE (2.0.5), if u has sufficient regularity to perform the computations, it turns out that, for $s \in [0,T]$,

$$\begin{aligned} u\big(s, X_s^{0,(x_0,x_1)}\big) = u\big(0,(x_0,x_1)\big) + \int_0^s r u\big(w, X_w^{0,(x_0,x_1)}\big) dw \\ + \int_0^s D_{x_0} u\big(w, X_w^{0,(x_0,x_1)}\big) \nu\big(w, X_w^{0,(x_0,x_1)}\big) dW_w, \end{aligned} \tag{2.0.10}$$

and the only derivative of u appearing in the above formula is the directional derivative $D_{x_0} u$ with respect to the variable x_0, representing the "present", according to the rephrasing $R \rightsquigarrow X$. Once that (2.0.10) is available, one can verify, as it was done for the case without delay (p. 2), that

$$h_s^B = \frac{u\big(s, X_s^{0,(x_0,x_1)}\big) - D_{x_0} u\big(s, X_s^{0,(x_0,x_1)}\big) X_{0,s}^{0,(x_0,x_1)}}{B_s} \text{ and } h_s^R = D_{x_0} u\big(s, X_s^{0,(x_0,x_1)}\big) \quad \forall s \in [0,T),$$

solve the hedging problem in the delay case.

The goal of this chapter is to show the regularity of the function u, defined by (2.0.9), with respect to the component x_0, when all the data are assumed to be Lipschitz with respect to a particular norm associated to the operator A.

The contents of the chapter are organized as follows. Section 2.1 contains some notation and various results about the mild solution X to (2.0.5), its extension to a bigger space with a weaker topology related to the original unbounded operator A, and various approximation results. In Section 2.2 we study viscosity solutions to the approximating equations, investigate finite dimensional sections of viscosity solutions, prove their regularity, and obtain the partial regularity for the original value function (Theorem 2.2.9).

2.1 Preliminaries

In the present chapter, differently than in Chapter 1, we will not need to distinguish between Gâteaux differentials with respect to various subspaces, and we will deal only with Gâteaux or Fréchet differentials up to order 2. Because of that, for a function $f = f(t,x)$, we adopt the more standard notation f_t, $\nabla_x f$, $D_x f$, $D_x^2 f$, for the partial derivative of f in t, the Gâteaux differential, the Fréchet differential, the second order Fréchet differential of f with respect to the second variable x, respectively. For E, F real Banach spaces:

- $\mathscr{G}_b^1(E,F)$ denotes the space of continuous functions $f: E \to F$ such that the Gâteaux derivative $\nabla f(x)$ exists for every $x \in E$, the function

$$\nabla f : E \to L(E,F)$$

is strongly continuous, and

$$\sup_{x\in E} |\nabla f(x)|_{L(E,F)} < +\infty.$$

When E is a Hilbert space and $F = \mathbb{R}$, we identify ∇f with an element of E through the Riesz representation $E^* = E$.

- $\mathscr{G}_b^{0,1}([0,T]\times E, F)$ denotes the space of continuous functions $f\colon [0,T]\times E \to F$ such that the Gâteaux derivative in the x variable $\nabla_x f(t,x)$ exists for every $x \in E$, the function

$$\nabla_x f\colon [0,T]\times E \to L(E,F)$$

is strongly continuous, and

$$\sup_{(t,x)\in[0,T]\times E} |\nabla_x f(t,x)|_{L(E,F)} < +\infty.$$

- $C_b^1(E,F)$ denotes the space of continuous functions $f\colon E \to F$, continuously Fréchet differentiable, and such that

$$\sup_{x\in E} |Df(x)|_{L(E,F)} < +\infty,$$

where Df denotes the Fréchet derivative of f.

- $C^{0,1}([0,T]\times E,F)$ denotes the space of continuous functions $f\colon [0,T]\times E \to F$, continuously Fréchet differentiable with respect to the second variable.
- $C_b^{0,1}([0,T]\times E,F)$ denotes the space of functions $f \in C^{0,1}([0,T]\times E,F)$ such that

$$\sup_{(t,x)\in[0,T]\times E} |D_x f(t,x)|_{L(E,F)} < +\infty,$$

where $D_x f$ denotes the Fréchet derivative of f with respect to x.

When $F = \mathbb{R}$, we drop $\mathbb{R}$ and simply write $\mathscr{G}_b^1(E)$, $\mathscr{G}_b^{0,1}(E)$, $C_b^{0,1}([0,T]\times E)$, and $C_b^{0,1}([0,T]\times E)$.

Though the notation could appear to be misleading, observe that if $f \in C_b^{0,1}([0,T]\times E,F)$ or $f \in C_b^1(E,F)$, then f is not supposed to be bounded.

Let $m > 0$ be a positive integer, and let U be an open subset of $\mathbb{R}^m$. Let a, b be real numbers such that $a < b$. Define $Q := [a,b)\times U$ and $\partial_P Q := \partial U \times [a,b] \cup \{b\}\times U$.

(i) For $\alpha \in (0,1)$, $C^{1+\alpha}(Q)$ denotes the space of continuous functions $f\colon Q \to \mathbb{R}$ such that $D_x f(t,x)$ exists classically for every $(t,x)\in Q$, and such that

$$|f|_{C^{1+\alpha}(Q)} := |f|_\infty + |D_x f|_\infty + \sup_{\substack{(t,x),(s,y)\in Q\\(t,x)\neq(s,y)}} \frac{|u(s,y)-u(t,x)-\langle D_x f(t,x), y-x\rangle_m|}{\left(|t-s|+|x-y|_m^2\right)^{(1+\alpha)/2}} < +\infty,$$

where $|\cdot|_\infty$ is the supremum norm, and $|\cdot|_m$ and $\langle\cdot,\cdot\rangle_m$ are the Euclidean norm and scalar product in $\mathbb{R}^m$ respectively.

(ii) For $\alpha \in (0,1)$, $C^{1+\alpha}_{\text{loc}}((0,T)\times\mathbb{R}^m)$ denotes the space of continuous functions $f\colon (0,T)\times\mathbb{R}^m \to \mathbb{R}$ such that, for every point $(t,x)\in(0,T)\times\mathbb{R}^m$, there exists $\varepsilon>0$ and $a,b\in(0,T)$, with $a<b$, such that $f\in C^{1+\alpha}([a,b)\times B(x,\varepsilon))$ ([2]).

(iii) For $p\geq 1$, $W^{1,2,p}(Q)$ denotes the usual Sobolev space of functions $f\in L^p(Q)$, whose weak partial derivatives u_t, f_{x_i} and $f_{x_ix_j}$ belong to $L^p(Q)$. $W^{1,2,p}(Q)$ is equipped with the norm

$$|f|_{W^{1,2,p}(Q)} := \left(|f|^p_{L^p(Q)} + |f_t|^p_{L^p(Q)} + |D_xf|^p_{L^p(Q)} + |D^2_xf|^p_{L^p(Q)}\right)^{1/p}.$$

Let $(\Omega,\mathbb{F}:=\{\mathscr{F}_t\}_{t\in[0,T]},\mathscr{F},\mathbb{P})$ be a complete filtered probability space. We will make use of the notation $\mathscr{L}^p_{\mathscr{P}_T}(C([0,T],E))$ introduced in Chapter 1, Section 1.2.

2.1.1 H_B-extension of SDEs

Let $m\geq 1$, and let H_1 be a real separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{H_1}$. Define $H:=\mathbb{R}^m\times H_1$. Whenever x is a point of H, we will denote by x_0 the component of x in $\mathbb{R}^m$ and by x_1 the component of x in H_1. We endow H with the natural scalar product

$$\langle (x_0,x_1),(y_0,y_1)\rangle := \langle x_0,y_0\rangle_m + \langle x_1,y_1\rangle_{H_1} \qquad \forall (x_0,x_1),(y_0,y_1)\in H.$$

We denote by $\mathbb{W}$ the space $C([0,T],H)$ of continuous functions $[0,T]\to H$.

Let $G\colon [0,T]\times H\to H$ and $\sigma\colon [0,T]\times H\to L(\mathbb{R}^m)$. We will consider the following assumptions on them.

Assumption 2.1.1. *The functions G and σ are continuous, and there exists $M>0$ such that*

$$|G(t,x)-G(t,y)|_H + |\sigma(t,x)-\sigma(t,y)|_{L(\mathbb{R}^m)} \leq M|x-y|_H \qquad \forall (t,x),(t,y)\in[0,T]\times H.$$

We associate to σ the following function:

$$\Sigma\colon [0,T]\times H\to L(\mathbb{R}^m,H),$$

defined by

$$\Sigma(t,x)y = (\sigma(t,x)y, 0_1) \tag{2.1.1}$$

for $(t,x)\in[0,T]\times H$, $y\in\mathbb{R}^m$, and where 0_1 denotes the origin in H_1.

The following assumption will be standing for the remaining part of the chapter.

Assumption 2.1.2. *S is a strongly continuous semigroup of contractions, with A as its infinitesimal generator.*

[2] $B(x,\varepsilon)$ denotes the open ball centered at x of radius ε.

We remark that Assumption 2.1.2 implies that A is a linear densely defined maximal dissipative operator on H. In the rest of the chapter A is an abstract operator which may be different from the operator $A = \hat{A} - \frac{1}{2}$ introduced at p. 58.

Let W be a standard m-dimensional Brownian motion with respect to the filtration $\mathbb{F}$. For $t \in [0,T)$ and $x \in H$, consider the SDE

$$\begin{cases} dX_s = (AX_s + G(s,X_s))ds + \Sigma(s,X_s)dW_s & s \in (t,T] \\ X_t = x. \end{cases} \tag{2.1.2}$$

It is well known (see [24, Ch. 7], or Chapter 1 in this thesis) that, under Assumption 2.1.1, for $p > 2$, there exists a unique mild solution in $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W})$ to (2.1.2), i.e. a unique process $X^{t,x} \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W})$ such that

$$X^{t,x}_s = \begin{cases} x & s \in [0,t] \\ S_{s-t}x + \int_t^s S_{s-w}G(w,X^{t,x}_w)dw + \int_t^s S_{s-w}\Sigma(w,X^{t,x}_w)dW_w & s \in (t,T]. \end{cases}$$

Moreover, for every $t \in [0,T]$, the map

$$H \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}), \quad x \mapsto X^{t,x} \tag{2.1.3}$$

is continuous and Lipschitz.

For future reference, we state existence and uniqueness of mild solution in the following proposition, where we also show continuity in t, and we introduce tools useful for later proofs.

Proposition 2.1.3. *For any $p > 2$, under Assumption 2.1.1, there exists a unique mild solution $X^{t,x} \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W})$ to SDE* (2.1.2)*, and the map*

$$[0,T] \times (H, |\cdot|_H) \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}),\ (t,x) \mapsto X^{t,x} \tag{2.1.4}$$

is continuous in (t,x), and Lipschitz in x, uniformly in t.

Proof. The proposition can be seen as a particular case of Theorem 1.2.6, which provides existence, uniqueness, and uniform Lipschitzianity, and of Proposition 1.2.13, which provides also continuity in time. ∎

We are going to endow H with a weaker norm, and give conditions such that the above continuity in (t,x) of $X^{t,x}$ extends to the new norm. We will also make assumptions which will guarantee the Gâteaux differentiability of the mild solution with respect to the initial datum x in the space with the weaker norm and the strong continuity of the the Gâteaux derivative.

Let $R\colon D(R)\to H$ be a densely defined linear operator such that $R\colon D(R)\to H$ has inverse $R^{-1}\in L(H)$. Then $B=(R^*)^{-1}R^{-1}\in L(H)$ is selfadjoint and positive. For $x\in H$, define

$$|x|_B^2=\langle Bx,x\rangle=\left|R^{-1}x\right|_H^2 \tag{2.1.5}$$

Such norms have been introduced in the context of the so-called B-continuous viscosity solutions to HJB equations in [19, 20] and used in many later works on HJB equations in infinite dimensional spaces (see [37, Ch. 3] for more on this). The space H endowed with the norm $|\cdot|_B$ is pre-Hilbert, since $|\cdot|_B$ is inherited by the scalar product $\langle x,y\rangle_B=\langle B^{1/2}x,B^{1/2}y\rangle$, where $B^{1/2}$ is the unique positive self-adjoint continuous linear operator such that $B=B^{1/2}B^{1/2}$. Denote by H_B the completion of the pre-Hilbert space $(H,|\cdot|_B)$. With some abuse of notation, we also denote by $|\cdot|_B$ the extension of $|\cdot|_B$ to H_B. We denote by $\mathbb{W}_B$ the space $C([0,T],H_B)$ of continuous functions $[0,T]\to H_B$.

By definition of $|\cdot|_B$, $R\colon(D(R),|\cdot|_H)\to(H,|\cdot|_B)$ is a full-range isometry. This implies the following facts:

(1) there exists a unique extension $\widetilde{R}\colon H\to H_B$;

(2) $\widetilde{R}$ and $\widetilde{R}^{-1}$ are isometries;

(3) $\widetilde{R}^{-1}=\widetilde{R^{-1}}$, where $\widetilde{R^{-1}}\colon H_B\to H$ is the unique continuous extension of R^{-1}.

Denote by $\overline{R}$ the operator $\widetilde{R}$ considered as an operator $H_B\supset H=D(\overline{R})\to H_B$. The above facts imply that $\overline{R}$ is a densely-defined full-range closed linear operator in H_B, and that $D(R)$ is a core for $\overline{R}$.

We will need the following proposition.

Proposition 2.1.4. *Let $R\colon D(R)\subset H\to H$ be a densely defined linear operator such that $R^{-1}\in L(H)$. Let H_B be the Hilbert space defined above as the completion of H with respect to the norm $|\cdot|_B$ given by* (2.1.5)*.*

(i) Suppose that

$$S_tR\subset RS_t \qquad \forall t\in\mathbb{R}^+. \tag{2.1.6}$$

Then, for every $t\in\mathbb{R}^+$, there exists a unique continuous extension $\overline{S}_t$ of S_t to H_B, the family $\overline{S}:=\{\overline{S}_t\}_{t\in\mathbb{R}^+}$ is a strongly continuous semigroup of contractions on H_B, and

$$\overline{S}_t\overline{R}\subset\overline{R}\overline{S}_t \qquad \forall t\in\mathbb{R}^+, \tag{2.1.7}$$

$$\overline{A}=\overline{R}A\overline{R}^{-1}, \tag{2.1.8}$$

where $\overline{A}$ is the infinitesimal generator of $\overline{S}$.

(ii) Suppose that

$$AR = RA, \tag{2.1.9}$$

$$D(A) \subset D(R). \tag{2.1.10}$$

Then (2.1.6) *is satisfied and the Yosida approximations* $\{\overline{A}_n\}_{n\geq 1}$ *of the infinitesimal generator* $\overline{A}$ *of* $\overline{S}$ *are given by the unique continuous extensions to* H_B *of the Yosida approximations* $\{A_n\}_{n\geq 1}$ *of* A*, i.e.*

$$\overline{A}_n = \overline{A_n} \qquad \forall n \geq 1.$$

Proof. (i) Suppose that (2.1.6) holds true. Observe that (2.1.6) implies

$$AR \subset RA. \tag{2.1.11}$$

Since $R^{-1}S_t = S_t R^{-1}$, we have

$$|S_t x|_B = |R^{-1}S_t x|_H = |S_t R^{-1}x|_H \leq |R^{-1}x|_H = |x|_B \qquad \forall t \in \mathbb{R}^+, x \in H.$$

We can then extend each S_t to an operator $\overline{S}_t \in L(H_B)$ with the operator norm less than or equal to 1. By density of H in H_B, it is clear that the family $\{\overline{S}_t\}_{t\in\mathbb{R}^+}$ is a semigroup of contractions. Moreover, for $x \in H$,

$$\lim_{t\to 0^+} |\overline{S}_t x - x|_B = \lim_{t\to 0^+} |R^{-1}(S_t x - x)|_H = \lim_{t\to 0^+} |S_t R^{-1}x - R^{-1}x|_H = 0.$$

The above observations imply that the family $\{\overline{S}_t\}_{t\in\mathbb{R}^+}$ is uniformly bounded and strongly continuous on a dense subspace of H_B. Thus, by [36, Proposition 5.3], $\overline{S}$ is a strongly continuous semigroup on H_B.

We now prove (2.1.7). Let $(x,\overline{R}x) \in \Gamma(\overline{R})$, where $\Gamma(\overline{R})$ is the graph of $\overline{R}$. We noticed that $D(R)$ is a core for $\overline{R}$. Then we can choose a sequence $\{(x_n, Rx_n)\}_{n\in\mathbb{N}} \in \Gamma(R)$ such that $(x_n, Rx_n) \to (x, \overline{R}x)$ in $H_B \times H_B$. Hence, using (2.1.6), we can write

$$\overline{S}_t \overline{R}x = \lim_{n\to+\infty} \overline{S}_t R x_n = \lim_{n\to+\infty} S_t R x_n = \lim_{n\to+\infty} R S_t x_n = \lim_{n\to+\infty} \overline{R}\,\overline{S}_t x_n,$$

where all the limits are considered in H_B. This means that $\{\overline{R}\,\overline{S}_t x_n\}_{n\in\mathbb{N}}$ is convergent in H_B. We recall that $\overline{R}$ is closed in H_B and we observe that $\overline{S}_t x_n \to \overline{S}_t x$ in H_B by continuity. Thus we conclude that $\overline{R}\,\overline{S}_t x_n \to \overline{R}\,\overline{S}_t x$ in H_B. This proves (2.1.7).

Now let $\overline{A}$ be the generator of the semigroup $\{\overline{S}_t : H_B \to H_B\}_{t\in\mathbb{R}^+}$. Obviously $\overline{A}$ is an extension of A, i.e. $\overline{A}x = Ax$ for $x \in D(A)$. We will show that $D(\overline{A}) = \overline{R}(D(A))$. Using (2.1.7) we have for $x \in H_B$,

$$\lim_{t\to 0^+} \frac{\overline{S}_t - I}{t} x = \lim_{t\to 0^+} \frac{\overline{S}_t - I}{t} \overline{R}\,\overline{R}^{-1} x = \text{(by (2.1.7))} = \lim_{t\to 0^+} \overline{R} \frac{\overline{S}_t - I}{t} \overline{R}^{-1} x = \lim_{t\to 0^+} \overline{R} \frac{S_t - I}{t} \overline{R}^{-1} x.$$

The last limit exists in H_B if and only if the limit

$$\lim_{t\to 0^+} \frac{S_t - I}{t}\overline{R}^{-1}x$$

exists in H. Therefore we conclude that

$$D(\overline{A}) = \overline{R}(D(A)) \qquad \text{and} \qquad \overline{A}x = \overline{R}A\overline{R}^{-1}x \quad \forall x \in D(\overline{A}), \tag{2.1.12}$$

which can be written as (2.1.8).

(ii) Let $\{A_n\}_{n\ge1}$ be the Yosida approximations of A. We begin by showing that

$$(n-A)^{-1}R \subset R(n-A)^{-1} \quad \forall n \ge 1. \tag{2.1.13}$$

By (2.1.10), it follows that

$$D((n-A)^{-1}R) = D(R) \subset H = D(R(n-A)^{-1}).$$

By (2.1.10), we have, for $x \in D(R)$,

$$A(n-A)^{-1}x = n(n-A)^{-1}x - x \subset D(A) + D(R) \subset D(R), \tag{2.1.14}$$

hence $(n-A)^{-1}x \in D(RA)$. Then, by using (2.1.9), we can write, for $x \in D(R)$,

$$(n-A)^{-1}Rx = (n-A)^{-1}R(n-A)(n-A)^{-1}x = (n-A)^{-1}(n-A)R(n-A)^{-1}x = R(n-A)^{-1}x.$$

This shows (2.1.13).

We now claim that

$$e^{tA_n}R \subset Re^{tA_n}, \tag{2.1.15}$$

where e^{tA_n} is the semigroup generated by A_n. By (2.1.13), we have

$$A_nRx = n^2(n-A)^{-1}Rx - nRx = n^2R(n-A)^{-1}x - nRx = RA_nx \qquad \forall x \in D(R),$$

hence

$$A_nR \subset RA_n. \tag{2.1.16}$$

Let $x \in D(R)$. By (2.1.14) and (2.1.16),

$$A_n^kRx = RA_n^kx \qquad \forall k \in \mathbb{N}. \tag{2.1.17}$$

For $t \in \mathbb{R}^+$ and $m \in \mathbb{N}$, define

$$y_m := \sum_{k=0}^{m} \frac{t^k}{k!}A_n^kx.$$

By (2.1.14), $y_m \in D(R)$. Moreover, $\lim_{m\to+\infty} y_m = e^{tA_n}x$ and, by (2.1.17),

$$\lim_{m\to+\infty} Ry_m = \lim_{m\to+\infty} \sum_{k=0}^{m} \frac{t^k}{k!}A_n^kRx = e^{tA_n}Rx.$$

Since R is closed, it follows that $e^{tA_n}x \in D(R)$, and $Re^{tA_n}x = e^{tA_n}Rx$. Since this holds for every $x \in D(R)$, we conclude $e^{tA_n}R \subset Re^{tA_n}$.

We can now prove that (2.1.6) is satisfied. Let $x \in D(R)$. By (2.1.15),

$$\lim_{n\to\infty} Re^{tA_n}x = \lim_{n\to\infty} e^{tA_n}Rx = S_tRx.$$

Since R is closed, we have $\lim_{n\to\infty} e^{tA_n}x = S_tx \in D(R)$ and $RS_tx = S_tRx$. Then (2.1.6) is verified.

We can now conclude the proof. By (2.1.15), arguing as it was done for S, we obtain that every S_n can be uniquely extended to the semigroup $e^{t\overline{A_n}}$ on H_B generated by $\overline{A_n}$. Similarly to (2.1.12), we have

$$D(\overline{A_n}) = \overline{R}(D(A_n)) \qquad \text{and} \qquad \overline{A_n}x = \overline{R}A_n\overline{R}^{-1}x \quad \forall x \in D(\overline{A_n}). \tag{2.1.18}$$

We observe that $\overline{R}(D(A_n)) = \overline{R}(H) = H_B$. If $x \in H$, by (2.1.10), (2.1.12), (2.1.13), and (2.1.18), we have

$$\begin{aligned}\overline{A_n}x = \overline{R}A_n\overline{R}^{-1}x = \overline{R}nA(n-A)^{-1}\overline{R}^{-1}x = RnA(n-A)^{-1}R^{-1}x \\ = n(RAR^{-1})(R(n-A)^{-1}R^{-1})x = n(RAR^{-1})(n-A)^{-1}x = nA(n-A)^{-1}x,\end{aligned}$$

which can be written as

$$\overline{A_n}x = n\overline{A}(n-\overline{A})^{-1}x = \overline{A}_nx \qquad \forall x \in H,$$

where $\overline{A}_n$ is the Yosida approximation of $\overline{A}$. Finally, since both $\overline{A_n}$ and $\overline{A}_n$ are continuous on H_B, and since H is dense in H_B, we obtain

$$\overline{A_n} = \overline{A}_n,$$

and then $e^{t\overline{A}_n} = e^{t\overline{A_n}}$, where $e^{t\overline{A}_n}$ is the semigroup generated by $\overline{A}_n$. ■

In the remaining of this section we will assume that (2.1.9) and (2.1.10) hold true.

Assumption 2.1.5. *The functions G and Σ are Lipschitz with respect to the norm $|\cdot|_B$, with respect to the second variable and uniformly in the first one, i.e. there exists $M > 0$ such that*

$$|G(t,x) - G(t,y)|_B + |\Sigma(t,x) - \Sigma(t,y)|_{L(\mathbb{R}^m,H_B)} \le M|x-y|_B \tag{2.1.19}$$

for all $t \in [0,T]$, $x,y \in H$. Denote by $\overline{G}$ (resp. $\overline{\Sigma}$) the unique extension of G (resp. Σ) to a function from $[0,T]\times H_B$ into H_B (resp. from $[0,T]\times H_B$ into $L(\mathbb{R}^m,H_B)$).

Remark 2.1.6. It is obvious that Assumptions 2.1.1 and 2.1.5 are satisfied if

$$|G(t,x)-G(t,y)|_H+|\sigma(t,x)-\sigma(t,y)|_{L(\mathbb{R}^m)}\le M|x-y|_B. \tag{2.1.20}$$

It is then easy to see that the functions $G_0(t,x)=G(t,Rx)$ and $\sigma_0(t,x)=\sigma(t,Rx)$ defined on $[0,T]\times D(R)$ satisfy

$$|G_0(t,x)-G_0(t,y)|_H+|\sigma_0(t,x)-\sigma_0(t,y)|_{L(\mathbb{R}^m)}\le M|x-y|_H \tag{2.1.21}$$

for $t\in[0,T]$ and $x,y\in D(R)$, and hence they uniquely extend to functions defined on $[0,T]\times H$ satisfying (2.1.21) for all $t\in[0,T]$ and $x,y\in H$. The converse is also true, i.e. (2.1.21) implies (2.1.20). Thus (2.1.20) is satisfied if and only if $G(t,x)=G_0(t,R^{-1}x)$, $\sigma(t,x)=\sigma_0(t,R^{-1}x)$, for $(t,x)\in[0,T]\times H$, for some G_0, σ_0 which satisfy (2.1.21) for all $t\in[0,T]$ and $x,y\in H$. We notice that for σ, (2.1.20) is also necessary for Assumptions 2.1.1 and 2.1.5.

For instance, focusing on σ (which corresponds to ν in the financial problem considered in the introduction to the present chapter, pp. 56–59), this condition is easily seen to be satisfied if

$$\sigma(t,x)=f(t,\langle x,\bar y^1\rangle,\dots,\langle x,\bar y^n\rangle)$$

for some $f\colon[0,T]\times\mathbb{R}^n\to L(\mathbb{R}^m)$ Lipschitz continuous in the last n variables (uniformly for $t\in[0,T]$) and $\bar y^1,\dots,\bar y^n\in D(R^*)$. Indeed, in such a case we can write

$$\sigma(t,x)=f(t,\langle x,\bar y^1\rangle,\dots,\langle x,\bar y^n\rangle)=\sigma_0(t,R^{-1}x), \tag{2.1.22}$$

where $\sigma_0(t,x)=f(t,\langle x,R^*\bar y^1\rangle,\dots,\langle x,R^*\bar y^n\rangle)$. Since later in (2.1.37) we take $R=A-I$, in applications to our financial problem (pp. 56–59) this would mean that

$$\bar y^i=(\bar y^i_0,\bar y^i_1)\in\mathbb{R}\times W^{1,2}(\mathbb{R}^-)\qquad i=1,\dots,n.$$

Thus a function of the form

$$\sigma(t,x)=f\left(t,x_0^1\bar y_0^1,\int_{-\infty}^0 x_1^1(s)\bar y_1^1(s)ds,\dots,x_0^n\bar y_0^n,\int_{-\infty}^0 x_1^n(s)\bar y_1^n(s)ds\right),$$

where $f\colon[0,T]\times\mathbb{R}^{2n}\to L(\mathbb{R}^m)$ is continuous in the $2n+1$ variables and Lipschitz continuous in the last $2n$ variables, uniformly for $t\in[0,T]$, satisfies Assumptions 2.1.1 and 2.1.5.

One can also give an equivalent condition which may be easier to check. We can only require that $G(t,x)=G_0(t,Kx)$, $\sigma(t,x)=\sigma_0(t,Kx)$, for some G_0,σ_0 satisfying (2.1.21) for all $t\in[0,T]$ and $x,y\in H$, and a bounded operator K on H such that $|Kx|_H\le C|R^{-1}x|_H$ for all $x\in H$. The last requirement (see e.g. [24, p. 429, Proposition B.1]) is equivalent to $K^*(H)\subset(R^{-1})^*(H)=D(R^*)$. In particular, if K is the orthogonal projection onto a finite dimensional subspace H_0 of H, then we need $H_0\subset D(R^*)$. By assuming without loss of generality that $\bar y^1,\dots,\bar y^n$ in (2.1.22) are orthonormal, then the previous example is readily reduced to the present if K is the orthogonal projection onto $\operatorname{span}\{\bar y^1,\dots,\bar y^n\}$.

Under Assumption 2.1.5 we can consider the following SDE on H_B

$$\begin{cases} d\overline{X}_s = \left(\overline{A}\overline{X}_s + \overline{G}\left(s,\overline{X}_s\right)\right)ds + \overline{\Sigma}\left(s,\overline{X}_s\right)dW_s, & s\in(t,T],\\ \overline{X}_t = x, \end{cases} \tag{2.1.23}$$

where $x\in H_B$, $t\in[0,T)$. By changing the reference Hilbert space from H to H_B, we can apply Proposition 2.1.3 and say that SDE (2.1.23) has a unique mild solution $\overline{X}^{t,x}$ in $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}_B)$, and $[0,T]\times H_B\to\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}_B)$, $(t,x)\mapsto\overline{X}^{t,x}$, is continuous and $|\cdot|_B$-Lipschitz with respect to x, uniformly in t.

Proposition 2.1.7. *For any $p>2$, under Assumptions 2.1.1 and 2.1.5, there exists a unique mild solution $\overline{X}^{t,x}\in\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}_B)$ to SDE* (2.1.23)*, and the map*

$$[0,T]\times H_B\to\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}_B),(t,x)\mapsto\overline{X}^{t,x} \tag{2.1.24}$$

is continuous in (t,x), and Lipschitz in x, uniformly in t. If $x\in H$, $\overline{X}^{t,x}\in\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$ and $\overline{X}^{t,x}=X^{t,x}$, where $X^{t,x}\in\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$ is the unique mild solution to SDE (2.1.2)*.*

Proof. The first part follows from Proposition 2.1.3. It remains to comment on the fact that $X^{t,x}=\overline{X}^{t,x}$ if $x\in H$. The space $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$ is continuously embedded in $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}_B)$. Thus, if G and Σ satisfy Assumptions 2.1.1 and 2.1.5, and if the initial value x belongs to H, the mild solution $X^{t,x}$ to (2.1.2) is also a mild solution to (2.1.23), and then, by uniqueness of mild solutions, $X^{t,\bar{x}}=\overline{X}^{t,x}$ in $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}_B)$. ■

In order to obtain an a-priori estimate giving the regularity in which we are interested, we will need to approximate mild solutions with other mild solutions to SDEs with smoother coefficients.

Proposition 2.1.8. *Let G and σ satisfy Assumptions 2.1.1 and 2.1.5. There exist sequences $\{G_n\}_{n\in\mathbb{N}}\subset C^{0,1}_b([0,T]\times H,H)$, $\{\Sigma_n\}_{n\in\mathbb{N}}\subset C^{0,1}_b([0,T]\times H,L(\mathbb{R}^m,H))$, with $\Sigma_n(t,x)y=(\sigma_n(t,x)y,0_1)$ for some $\sigma_n\in C^{0,1}_b([0,T]\times H,L(\mathbb{R}^m))$, satisfying:*

(i) For every $n\in\mathbb{N}$, G_n and Σ_n have extensions $\overline{G}_n\in C^{0,1}_b([0,T]\times H_B,H_B)$ and $\overline{\Sigma}_n\in C^{0,1}_b([0,T]\times H_B,L(\mathbb{R}^m,H_B))$.

(ii) For all $(t,x),(t,y)\in[0,T]\times H_B$,

$$\sup_{n\in\mathbb{N}}|\overline{G}_n(t,x)-\overline{G}_n(t,y)|_B\le M|x-y|_B \tag{2.1.25}$$

$$\sup_{n\in\mathbb{N}}|\overline{\Sigma}_n(t,x)-\overline{\Sigma}_n(t,y)|_{L(\mathbb{R}^m,H_B)}\le M|x-y|_B. \tag{2.1.26}$$

(iii) For every compact set $K\subset H_B$,

$$\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times K}|\overline{G}(t,x)-\overline{G}_n(t,x)|_B=0 \tag{2.1.27}$$

$$\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times K}|\overline{\Sigma}(t,x)-\overline{\Sigma}_n(t,x)|_{L(\mathbb{R}^m,H_B)}=0. \tag{2.1.28}$$

Remark 2.1.9. We remark that, due to the fact that the range of Σ is finite-dimensional (see (2.1.1)), once the above continuity/differentiability/approximation conditions for $\overline{\Sigma}_n$ are satisfied with respect to H_B, they automatically hold for Σ_n with respect to H.

Proof of Proposition 2.1.8. The proof uses approximations similar to those in [82]. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of H_B contained in H. For $n \in \mathbb{N}$, let us define the functions

$$I_n\colon \mathbb{R}^n \to H_B,\ y \mapsto \sum_{k=1}^{n} y_k e_k$$

and

$$P_n\colon H_B \to \mathbb{R}^n,\ x \mapsto (\langle x, e_1\rangle_B, \ldots, \langle x, e_n\rangle_B).$$

It is clear that $|I_n|_{L(\mathbb{R}^n,H_B)} = 1$ and $|I_nP_n|_{L(H_B)} = 1$. We observe also that, for every $n \in \mathbb{N}$, the linear operator

$$H_B \to H,\ x \mapsto I_nP_n x = \sum_{k=1}^{n} \langle x, e_k\rangle_B e_k$$

is well defined and continuous. Denote $c_n := |I_nP_n|_{L(H_B,H)}$.

Let

$$\varphi(r) := \begin{cases} e^{-\frac{1}{1-r^2}} & \text{if } r \in (-1,1) \\ 0 & \text{otherwise,} \end{cases}$$

and, for every $n \in \mathbb{N}$,

$$C_n := \left(\int_{\mathbb{R}^n} \varphi(n|y|_n)dy\right)^{-1},$$

where $|\cdot|_n$ denotes the Euclidean norm in $\mathbb{R}^n$. Define

$$g_n\colon [0,T]\times\mathbb{R}^n \to H$$

by standard mollification

$$g_n(t,y) := C_n\left(G(t, I_n\cdot) * \varphi(n|\cdot|_n)\right)(y) = C_n \int_{\mathbb{R}^n} G\left(t, \sum_{k=0}^{n-1} z_k e_k\right)\varphi(n|y-z|_n)dz,$$

for all $(t,y) \in [0,T]\times\mathbb{R}^n$. We observe that g_n is well-defined, because G is H-valued and continuous, and φ has compact support. By Lebesgue's dominated convergence theorem, g_n is continuous.

Since the map $\mathbb{R}^n \to \mathbb{R},\ z \mapsto \varphi(n|z|)$, is continuously differentiable and has compact support and since G is continuous, by a standard argument we can differentiate under the integral sign to obtain g_n is differentiable with respect to y and

$$D_y g_n(t,y)v = nC_n \int_{\mathbb{R}^n} G(t, I_n z)\varphi'(n|y-z|_n)\frac{\langle y-z, v\rangle_n}{|y-z|_n}dz.$$

By Lebesgue's dominated convergence theorem, the map

$$[0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to H,\ (t,y,v) \mapsto D_y g_n(t,y)v$$

is continuous. Thus $g_n \in C^{0,1}([0,T]\times\mathbb{R}^n, H)$. Define

$$\overline{G}_n\colon [0,T]\times H_B \to H_B$$

by

$$\overline{G}_n(t,x) := g_n(t,P_nx) = C_n\int_{\mathbb{R}^n} G(t, I_nP_nx - I_nz)\varphi(n|z|_n)dz \qquad \forall (t,x)\in[0,T]\times H_B.$$

Since $\overline{G}_n([0,T]\times H_B)\subset H$, we can also define $G_n\colon [0,T]\times H\to H$ by $G_n(t,x) := \overline{G}_n(t,x)$ for every $(t,x)\in[0,T]\times H$. Then $G_n\in C^{0,1}([0,T]\times H,H)$ and $\overline{G}_n\in C^{0,1}([0,T]\times H_B, H_B)$. Moreover, by Assumption 2.1.1,

$$\begin{aligned}
|G_n(t,x) - G_n(t,x')|_H &= |g_n(t,P_nx) - g_n(t,P_nx')|_H \\
&\le C_n\int_{\mathbb{R}^n}\left|G(t,I_nP_nx - I_nz) - G\left(t,I_nP_nx' - I_nz\right)\right|_H \varphi(n|z|_n)dz \qquad (2.1.29)\\
&\le M\left|I_nP_nx - I_nP_nx'\right|_H \le Mc_n|x-x'|_B \le Mc_n|R^{-1}|_{L(H)}|x-x'|_H,
\end{aligned}$$

for every $t\in[0,T]$ and $x,x'\in H$. Similarly, by Assumption 2.1.5,

$$\begin{aligned}
|\overline{G}_n(t,x) - \overline{G}_n(t,x')|_B &= |g_n(t,P_nx) - g_n(t,P_nx')|_B \\
&\le M\left|I_nP_nx - I_nP_nx'\right|_B \le M\left|x-x'\right|_B,
\end{aligned} \qquad (2.1.30)$$

for every $t\in[0,T]$ and $x,x'\in H_B$. Thus $G_n\in C_b^{0,1}([0,T]\times H,H)$ and $\overline{G}_n\in C_b^{0,1}([0,T]\times H_B,H_B)$.

To prove (2.1.27) for every compact $K\subset H_B$, we first notice that

$$\sup_{x\in K}|I_nP_nx - x|_B = \varepsilon_n \to 0 \text{ as } n\to+\infty.$$

Thus by (2.1.19),

$$\lim_{n\to+\infty}\sup_{(t,x)\in[0,T]\times K}|\overline{G}(t,I_nP_nx) - \overline{G}(t,x)|_B \le \lim_{n\to+\infty} M\varepsilon_n = 0. \qquad (2.1.31)$$

Moreover, for $(t,x)\in[0,T]\times H_B$,

$$\begin{aligned}
|\overline{G}(t,I_nP_nx) - \overline{G}_n(t,x)|_B &\le C_n\int_{\mathbb{R}^n}|G(t,I_nP_nx - I_nz) - G(t,I_nP_nx)|_B\,\varphi(n|z|_n)dz \\
&\le MC_n\int_{\mathbb{R}^n}|I_nz|_B\varphi(n|z|_n)dz \le MC_n\int_{\mathbb{R}^n}|z|_n\varphi(n|z|_n)dz \le \frac{M}{n}.
\end{aligned}$$

This, together with (2.1.31), gives (2.1.27).

We have thus proved that $\{G_n\}_{n\in\mathbb{N}}\subset C_b^{0,1}([0,T]\times H,H)$, that $\{\overline{G}_n\}_{n\in\mathbb{N}}\subset C_b^{0,1}([0,T]\times H_B,H_B)$, and that (2.1.25) and (2.1.27) hold true.

The other half of the proof, regarding Σ, is similar. We only make a few comments. For $n\in\mathbb{N}$, define

$$\zeta_n\colon \mathbb{R}^n \to L(\mathbb{R}^m)$$

by

$$\zeta_n(t,y) := C_n\big(\sigma(t,I_n\cdot) * \varphi(n|\cdot|_n)\big)(y) = C_n \int_{\mathbb{R}^n} \sigma\left(t, \sum_{k=1}^n z_k e_k\right) \varphi(n|y-z|_n)dz,$$

for all $(t,y) \in [0,T] \times \mathbb{R}^n$, and $\overline{\sigma}_n : [0,T] \times H_B \to L(\mathbb{R}^m)$ by $\overline{\sigma}_n(t,x) := \zeta_n(t, I_n P_n x)$ for all $(t,y) \in [0,T] \times \mathbb{R}^n$, and $n \in \mathbb{N}$. Arguing as it was done done for g_n, we have that $\zeta_n \in C^{0,1}([0,T] \times \mathbb{R}^n, L(\mathbb{R}^m))$, and then $\overline{\sigma}_n \in C^{0,1}([0,T] \times H_B, L(\mathbb{R}^m))$. Moreover,

$$|\overline{\sigma}_n(t,x) - \overline{\sigma}_n(t,x')|_B \le M\left|x - x'\right|_B, \tag{2.1.32}$$

and hence $\overline{\sigma}_n \in C_b^{0,1}([0,T] \times H_B, L(\mathbb{R}^m))$. The proof of (2.1.28) is done in the same way as for $\overline{G}_n$. Finally we define

$$\overline{\Sigma}_n(t,x)y := (\overline{\sigma}_n(t,x)y, 0_1) \quad \forall (t,x) \in [0,T] \times H_B,\ \forall y \in \mathbb{R}^n,\ \forall n \in \mathbb{N}$$

and

$$\Sigma_n(t,x)y := \overline{\Sigma}_n(t,x)y \quad \forall (t,x) \in [0,T] \times H,\ \forall y \in \mathbb{R}^n,\ \forall n \in \mathbb{N}.$$

This concludes the proof. ■

Unless otherwise specified, Assumptions 2.1.1 and 2.1.5 will be standing for the remaining part of the chapter, and $\{G_n\}_{n\in\mathbb{N}}$, $\{\Sigma_n\}_{n\in\mathbb{N}}$, $\{\overline{G}_n\}_{n\in\mathbb{N}}$, $\{\overline{\Sigma}_n\}_{n\mathbb{N}}$ will denote the sequences introduced in Proposition 2.1.8.

Let $\{A_n\}_{n\ge 1}$ be the Yosida approximation of A. We recall that for every $n \ge 1$, by Proposition 2.1.4, A_n has a unique continuous extension $\overline{A_n}$ to H_B, and $\overline{A_n} = \overline{A}_n$, where $\{\overline{A}_n\}_{n\in\mathbb{N}}$ is the Yosida approximation of the infinitesimal generator $\overline{A}$ of $\overline{S}$. We remind that we denote by $e^{t\overline{A}_n}$ the semigroup generated by $\overline{A}_n$. For $t \in [0,T)$ and $n \ge 1$, we denote by $X_n^{t,x}$, $\overline{X}_n^{t,x}$, respectively the unique mild solutions to

$$\begin{cases} dX_{n,s} = \left(A_n X_{n,s} + G_n\left(s, X_{n,s}\right)\right)ds + \Sigma_n\left(s, X_{n,s}\right)dW_s & s \in (t,T] \\ X_{n,t} = x \in H, \end{cases} \tag{2.1.33}$$

$$\begin{cases} d\overline{X}_{n,s} = \left(\overline{A}_n \overline{X}_{n,s} + \overline{G}_n\left(s, \overline{X}_{n,s}\right)\right)ds + \overline{\Sigma}_n\left(s, \overline{X}_{n,s}\right)dW_s & s \in (t,T] \\ \overline{X}_{n,t} = x \in H_B. \end{cases} \tag{2.1.34}$$

For any $p > 2$, existence and uniqueness of mild solution are provided by Propositions 2.1.3 and 2.1.7, together with the continuity of the maps

$$[0,T] \times H \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}), (t,x) \mapsto X_n^{t,x} \qquad [0,T] \times H_B \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B), (t,x) \mapsto \overline{X}_n^{t,x}. \tag{2.1.35}$$

Proposition 2.1.10. *Let Assumptions 2.1.1 and 2.1.5 hold and let $p > 2$. Then:*

(i) For every $n \in \mathbb{N}$ and $x \in H$, $X_n^{t,x} = \overline{X}_n^{t,x}$ (in $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B)$).

(ii) $\lim_{n\to+\infty}\overline{X}_n^{t,x} = \overline{X}^{t,x}$ *in* $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B)$ *uniformly for* (t,x) *on compact sets of* $[0,T]\times H_B$.

(iii) *For every* $n\in\mathbb{N}$ *the map*

$$[0,T]\times H_B \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B), (t,x)\mapsto \overline{X}_n^{t,x} \tag{2.1.36}$$

belongs to $\mathcal{G}_b^{0,1}([0,T]\times H_B, \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B))$.

(iv) *The set* $\{\nabla_x \overline{X}_n^{t,x}\}_{n\in\mathbb{N}}$ *is bounded in* $L(H_B, \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B))$*, uniformly for* $(t,x)\in[0,T]\times H_B$.

Proof. *(i)* Let $(t,x)\in[0,T]\times H$. Since $A_n = \overline{A}_n$ on H, we have $e^{sA_n} = e^{s\overline{A}_n}$ on H for all $s\in\mathbb{R}^+$. Recalling that $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W})$ is continuously embedded in $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B)$, we then have that the mild solution $X_n^{t,x}$ is also a mild solution to (2.1.34) in $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B)$. By uniqueness we conclude that $X_n^{t,x} = \overline{X}_n^{t,x}$ in $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B)$.

(ii) This is a consequence of Theorem 1.2.14 and of the continuity of

$$[0,T]\times H_B \mapsto \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B),\ (t,x)\mapsto \overline{X}^{t,x},$$

provided by Proposition 2.1.7.

(iii)+ (iv) Theorem 1.2.9 provides the Gâteaux differentiability of $\overline{X}^{t,\cdot}$, for fixed t, and the boundedness of $\nabla_x\overline{X}^{t,x}$, uniformly in (t,x). Theorem 1.2.16 implies that

$$\lim_{h\to 0}\nabla_x\overline{X}^{t+h,x}y = \nabla_x\overline{X}^{t,x}y$$

uniformly for (x,y) on compact subsets of H_B. The two facts implies the strong continuity of

$$[0,T]\times H_B \mapsto \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{W}_B),\ (t,x)\mapsto \nabla_x\overline{X}^{t,x} \qquad \blacksquare$$

We will make a particular choice of R and thus B. Recall that $(0,+\infty)$ is contained in the resolvent set of A (and hence of A^*). For $\lambda>0$, let $A_\lambda := A-\lambda$, $A^*_\lambda := A^*-\lambda = (A-\lambda)^*$. If $R = A_\lambda$, then (2.1.6), (2.1.9), and (2.1.10), are satisfied. We can then apply all of the above arguments with

$$B = B_{A,\lambda} := (A^*_\lambda)^{-1}A_\lambda^{-1}.$$

Notice that

$$|x|_{B_{A,\lambda}} \le \big(1+|\lambda-\lambda'||A_\lambda^{-1}|_{L(H)}\big)|x|_{B_{A,\lambda'}} \qquad \forall \lambda,\lambda'\in(0,+\infty), x\in H,$$

hence the norms $|\cdot|_{B_{A,\lambda}}$ and $|\cdot|_{B_{A,\lambda'}}$ are equivalent. We will thus pick $\lambda=1$ and from now on we set

$$B := B_{A,1} = (A_1^*)^{-1}A_1^{-1}. \tag{2.1.37}$$

We observe that with this choice of B we have

$$|x|_B = |(\overline{A}-I)^{-1}x|_H \quad \forall x\in H_B,$$

and

$$\langle x,y\rangle_B = \langle(\overline{A}-I)^{-1}x,(\overline{A}-I)^{-1}y\rangle \quad \forall\, x,y\in H_B.$$

In particular

$$\langle x,y\rangle_B = \langle (A^*-I)^{-1}(\overline{A}-I)^{-1}x,y\rangle \quad \forall x\in H_B,\ \forall y\in H.$$

2.2 Viscosity solutions to Kolmogorov PDEs in Hilbert spaces with finite-dimensional second-order term

We remind that throughout the rest of the chapter B is defined by (2.1.37). For this B, Assumptions 2.1.1 and 2.1.5 will be standing for the remaining part of the chapter, $\{G_n\}_{n\in\mathbb{N}}$, $\{\Sigma_n\}_{n\in\mathbb{N}}$, $\{\overline{G}_n\}_{n\in\mathbb{N}}$, $\{\overline{\Sigma}_n\}_{n\mathbb{N}}$ denote the sequences introduced in Proposition 2.1.8, the operators A_n, $n\geq 1$, are the Yosida approximations of A, and $X_n^{t,x}$, $\overline{X}_n^{t,x}$ are respectively the mild solutions to (2.1.33), (2.1.34), with $B=B_{A,1}$, $n\geq 1$. We recall that, by Proposition 2.1.10, $X^{t,x}=\overline{X}^{t,x}$ and $X_n^{t,x}=\overline{X}_n^{t,x}$ for every $(t,x)\in[0,T]\times H$, $n\geq 1$.

2.2.1 Existence and uniqueness of solution

The following assumption will be standing for the remaining part of the chapter.

Assumption 2.2.1. *The function $h\colon H_B\to\mathbb{R}$ is such that there is a constant $M\geq 0$ such that*

$$|h(x)-h(y)|\leq M|x-y|_B \quad \forall x,y\in H. \tag{2.2.1}$$

The function h extends uniquely to $\overline{h}: H_B\to\mathbb{R}$ which also satisfies (2.2.1). Taking the inf-sup convolutions of $\overline{h}$ in H_B (see [37, 68]) we can obtain a sequence of functions $\{\overline{h}_n\}_{n\in\mathbb{N}}\subset C_b^1(H_B)$ (and even more regular) such that

$$\sup_{\substack{n\in\mathbb{N}\\ x\in H_B}} |D\overline{h}_n(x)|_B<+\infty \qquad \text{and} \qquad \lim_{n\to+\infty}\sup_{x\in H_B}\left|\overline{h}(x)-\overline{h}_n(x)\right|=0. \tag{2.2.2}$$

The restriction of $\overline{h}_n$ to H will be denoted by h_n.

We define the functions

$$u\colon[0,T]\times H\to\mathbb{R},\ (t,x)\mapsto\mathbb{E}\left[h(X_T^{t,x})\right], \tag{2.2.3}$$

$$u_n\colon[0,T]\times H\to\mathbb{R},\ (t,x)\mapsto\mathbb{E}\left[h_n(X_{n,T}^{t,x})\right],\quad n\geq 1. \tag{2.2.4}$$

By sublinear growth of h and h_n, u and u_n are well defined. Each of the above functions has an associated Kolmogorov equation in $(0,T]\times H$. However we will only need to consider the equation satisfied by u_n. We also define

$$\overline{u}_n\colon[0,T]\times H_B\to\mathbb{R},\ (t,x)\mapsto\mathbb{E}\left[\overline{h}_n(\overline{X}_{n,T}^{t,x})\right],\quad n\geq 1.$$

We observe that $u_n=\overline{u}_{n|[0,T]\times H}$.

Proposition 2.2.2. *Let $p > 2$. Then:*

(i) *u_n is uniformly continuous on bounded sets of $[0,T]\times(H,|\cdot|_B)$ and, for every $t \in [0,T]$, $u_n(t,\cdot)$ is $|\cdot|_B$-Lipschitz continuous, with a Lipschitz constant uniform in $t \in [0,T]$ and $n \geq 1$.*

(ii) *For every $n \geq 1$, the sequence $\{u_n\}_{n\geq 1}$ converges to u uniformly on compact sets of $[0,T]\times H$.*

(iii) *For every $n \geq 1$, $u_n \in \mathcal{G}^{0,1}_b([0,T]\times H)$, and*

$$\sup_{\substack{(t,x)\in[0,T]\times H\\ n\geq 1}} |\nabla_x u_n(t,x)|_H < +\infty, \tag{2.2.5}$$

$$\sup_{\substack{(t,x)\in[0,T]\times H\\ n\geq 1}} \left|A_n^* \nabla_x u_n(t,x)\right|_H < +\infty. \tag{2.2.6}$$

Proof. *(i)* From (2.2.2) and Proposition 2.1.10*(i),(iii),(iv)*, it follows that u_n is continuous and $|\cdot|_B$-Lipschitz continuous in x with a Lipschitz constant uniform in $t \in [0,T]$ and $n \geq 1$.

The uniform continuity of u_n on bounded sets is standard since we are dealing with bounded evolution and can be deduced from a more general result, see e.g. [24, Theorem 9.1], however we present a short argument. We first notice that it follows from Proposition 2.1.10*(iii),(iv)* that, for any $r > 0$ and $n \geq 1$, there exists $K > 0$ such that

$$|\overline{X}_n^{t,x}|_{\mathcal{L}^2_{\mathcal{P}_T}(\mathbb{W}_B)} \leq K \qquad \forall t \in [0,T],\ \forall x \in H_B,\ |x|_B \leq r.$$

Secondly, we recall that, for $t \in [0,T]$ and $x \in H_B$, $\overline{X}_n^{t,x}$ is a strong solution to (2.1.34), because $\overline{A}_n$ is bounded (see footnote 1 on p. 58). Then if $0 \leq t \leq t' \leq T$ and $x \in H_B$, $|x|_B \leq r$, for some constants C_1, C_2 depending only on T, K, $|\overline{A}_n|_{L(H_B)}$, and on the Lipschitz and the linear-growth constants of $\overline{G}_n$ and $\overline{\Sigma}_n$, by standard estimates we have

$$\mathbb{E}\left[\left|\overline{X}_{n,s}^{t,x} - \overline{X}_{n,s}^{t',x}\right|_B^2\right] \leq C_1(t'-t) + C_2\int_{t'}^{s} \mathbb{E}\left[\left|\overline{X}_{n,w}^{t,x} - \overline{X}_{n,w}^{t',x}\right|_B^2\right] dw \qquad \forall s \in [t',T].$$

By Gronwall's lemma, the inequality above provides

$$\mathbb{E}\left[\left|\overline{X}_{n,T}^{t,x} - \overline{X}_{n,T}^{t',x}\right|_B^2\right] \leq C_1 e^{C_2 T}(t'-t). \tag{2.2.7}$$

The uniform continuity of u_n on $[0,T]\times\{x \in H : |x|_B \leq r\}$ is then obtained by (2.2.2), (2.2.7), and by the $|\cdot|_B$-Lipschitz continuity of $\overline{X}_n^{t,x}$ in x with a Lipschitz constant uniform in $t \in [0,T]$.

(ii) Part *(ii)* is a consequence of Proposition 2.1.10*(i),(ii)* and (2.2.2).

(iii) Let $n \geq 1$. By [23, Ch. 7, Proposition 7.3.3], the map

$$\Xi_n \colon \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}_B) \to L^p((\Omega,\mathscr{F}_T,\mathbb{P}),H_B),\ Z \mapsto \overline{h}_n(Z_T)$$

belongs to $\mathscr{G}^1_b(\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}_B), L^p((\Omega,\mathscr{F}_T,\mathbb{P}),H_B))$, and

$$(\nabla_Z \Xi_n(Z)Y)_T = D\overline{h}_n(Z_T)Y_T \qquad \forall Z,Y \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}_B). \tag{2.2.8}$$

By Proposition 2.1.10*(iii)*, linearity and continuity of $\mathbb{E}$ on $L^p((\Omega,\mathscr{F}_T,\mathbb{P}),H_B)$, formula (2.2.8), composition of strongly continuously Gâteaux differentiable functions with bounded differentials, we obtain $\overline{u}_n \in \mathscr{G}^{0,1}_b([0,T]\times H_B)$ and

$$\langle \nabla_x \overline{u}_n(t,x), y\rangle_B = \mathbb{E}\left[D\overline{h}_n(\overline{X}^{t,x}_{n,T})\left(\nabla_x \overline{X}^{t,x}_n y\right)_T\right] \qquad \forall (t,x,y) \in [0,T]\times H_B \times H_B. \tag{2.2.9}$$

By Proposition 2.1.10*(iv)*, (2.2.2), (2.2.9),

$$\sup_{\substack{(t,x)\in[0,T]\times H_B\\ n\geq 1}} |\nabla_x \overline{u}_n(t,x)|_B < +\infty. \tag{2.2.10}$$

By continuous embedding $H \to H_B$ and by (2.2.10) we have also

$$u_n \in \mathscr{G}^{0,1}_b([0,T]\times H), \qquad \sup_{\substack{(t,x)\in[0,T]\times H\\ n\geq 1}} |\nabla_x u_n(t,x)|_H < +\infty, \tag{2.2.11}$$

which shows (2.2.5). Moreover, since

$$\nabla_x u_n(t,x) = (A^*-1)^{-1}(\overline{A}-1)^{-1}\nabla_x \overline{u}_n(t,x),$$

we obtain from (2.2.10) that

$$\sup_{\substack{(t,x)\in[0,T]\times H\\ n\geq 1}} |A^*\nabla_x u_n(t,x)|_H < +\infty. \tag{2.2.12}$$

Therefore, recalling that S is a semigroup of contractions, we have

$$|A^*_n \nabla_x u_n(t,x)|_H \leq |n(n-A)^{-1}|_{L(H)} |A^*\nabla_x u_n(t,x)|_H \leq |A^*\nabla_x u_n(t,x)|_H$$

for all $(t,x)\in[0,T]\times H$ which, together with (2.2.12), shows (2.2.6). ■

We now define for $n \geq 1$

$$L_n \colon [0,T]\times H \times H \times \mathbf{S}_m \to \mathbb{R},\ (t,x,p,P) \mapsto \langle p, G_n(t,x)\rangle + \frac{1}{2}\mathrm{Tr}(\sigma_n(t,x)\sigma^*_n(t,x)P)$$

where $\mathbf{S}_m$ is the set of $m\times m$ symmetric matrices.

We consider the following terminal value problems

$$\begin{cases} -v_t - \langle A_n x, D_x v\rangle - L_n(t,x,D_x v, D^2_{x_0x_0}v) = 0 & (t,x)\in(0,T)\times H \\ v(T,x) = h_n(x) & x \in H. \end{cases} \tag{2.2.13}$$

Since the operator A_n is bounded we will use the definition of viscosity solution from [73].

Definition 2.2.3. *A locally bounded* ([3]) *upper semi-continuous function* v *on* $(0,T]\times H$ *is a viscosity subsolution to* (2.2.13) *if* $v(T,x)\le h_n(x)$ *for all* $x\in H$*, and whenever* $v-\varphi$ *has a local maximum at a point* $(\hat t,\hat x)\in(0,T)\times H$*, for some* $\varphi\in C^{1,2}((0,T)\times H)$*, then*

$$-\varphi_t(\hat t,\hat x)-\langle A_n\hat x, D_x\varphi(\hat t,\hat x)\rangle - L_n(\hat t,\hat x,D_x\varphi(\hat t,\hat x),D^2_{x_0}\varphi(\hat t,\hat x))\le 0.$$

A locally bounded lower semi-continuous function v *on* $(0,T]\times H$ *is a viscosity supersolution to* (2.2.13) *if* $v(T,x)\ge h_n(x)$ *for all* $x\in H$*, and whenever* $v-\varphi$ *has a local minimum at a point* $(\hat t,\hat x)\in(0,T)\times H$*, for some* $\varphi\in C^{1,2}((0,T)\times H)$*, then*

$$-\varphi_t(\hat t,\hat x)-\langle A_n\hat x, D_x\varphi(\hat t,\hat x)\rangle - L_n(\hat t,\hat x,D_x\varphi(\hat t,\hat x),D^2_{x_0}\varphi(\hat t,\hat x))\ge 0.$$

A viscosity solution to (2.2.13) *is a function which is both a viscosity subsolution and a viscosity supersolution to* (2.2.13).

Theorem 2.2.4. *For* $n\ge 1$*, the function* u_n *is the unique (within the class of, say locally uniformly continuous functions with at most polynomial growth) viscosity solution to* (2.2.13).

Proof. Since A_n is a bounded operator this is a standard result, see e.g. [37, 60, 73]. Notice that Proposition 2.2.2*(i)* guarantees that the function u_n is locally uniformly continuous on $[0,T]\times H$ and is Lipschitz continuous in x. ∎

Remark 2.2.5. This is not needed here however it is worth noticing that the function u is the unique so called $B_{A,1}$-continuous viscosity solution (unique within the class of $B_{A,1}$-continuous functions with at most polynomial growth which attain the terminal condition locally uniformly), of the equation

$$\begin{cases} -u_t-\langle Ax,D_xu\rangle - L(t,x,D_xu,D^2_{x_0x_0}u)=0 & (t,x)\in(0,T)\times H\\ u(T,x)=h(x) & x\in H,\end{cases}\tag{2.2.14}$$

where

$$L\colon[0,T]\times H\times H\times \mathbf{S}_m\to\mathbb{R},(t,x,p,P)\mapsto\langle p,G(t,x)\rangle+\frac{1}{2}\mathrm{Tr}(\sigma(t,x)\sigma^*(t,x)P).$$

For the proof of this we refer the reader to [37, Theorem 3.64].

[3] By "locally bounded" we mean "bounded on bounded subsets of the domain", and by "locally uniformly continuous" we mean "uniformly continuous on bounded subsets of the domain".

2.2.2 Space sections of viscosity solutions

We skip the proof of the following basic lemma (for a very similar version, see [17, Proposition 3.7]).

Lemma 2.2.6. *Let D be a set, and $f,g\colon D\to\mathbb{R}$ be functions, with $g\geq 0$. Let*

$$Z=\{y\in D\colon g(y)=0\}$$

be the set of zeros of g. Suppose that $Z\neq\emptyset$. Let $\{h_i\colon D\to\mathbb{R}\}_{i\in\mathbb{N}}$ be a sequence of functions converging uniformly to 0 in D as $i\to+\infty$. Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a sequence of positive numbers decreasing to 0. Define

$$\psi_i(y):=f(y)-\frac{g(y)}{\varepsilon_i}+h_i(y)\qquad\forall i\in\mathbb{N},\ \forall y\in D.$$

Suppose that $\{y_i\}_{i\in\mathbb{N}}\subset D$ is a sequence such that

$$\lim_{i\to\infty}\left[\sup_{y\in D}\psi_i(y)-\psi_i(y_i)\right]=0.$$

Then $\lim_{i\to\infty}\dfrac{g(y_i)}{\varepsilon_i}=0$.

Fix $\overline{x}_1\in H_1$. Let $\varphi\in C^{1,2}((0,T)\times\mathbb{R}^m)$ and let $(\hat{t},\hat{x}_0)\in(0,T)\times\mathbb{R}^m$ be a maximum point of $u_n(\cdot,(\cdot,\overline{x}_1))-\varphi(\cdot,\cdot)$ over $[0,T]\times\mathbb{R}^m$. Without loss of generality we can assume that $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^m)$ and that the maximum is strict and global.

For $\varepsilon>0$, define the function

$$\Phi_\varepsilon(t,x_0,x_1)=\varphi(t,x_0)+\frac{1}{\varepsilon}|(0,x_1-\overline{x}_1)|_H^2,\tag{2.2.15}$$

where $t\in(0,T)$, $(x_0,x_1)\in H$. Observe that $\Phi_\varepsilon\in C^{1,2}([0,T]\times H)$, and

$$\begin{aligned}
&D_t\Phi_\varepsilon(t,x)=\varphi_t(t,x_0)\\
&D_x\Phi_\varepsilon(t,x)=\big(D_{x_0}\varphi(t,x_0),0\big)+\frac{2}{\varepsilon}(0,x_1-\overline{x}_1)\\
&D^2_{x_0}\Phi_\varepsilon(t,x)=D^2_{x_0}\varphi(t,x_0).
\end{aligned}\tag{2.2.16}$$

Lemma 2.2.7. *For each $n\geq 1$, there exist real sequences $\{a_i\}_{i\in\mathbb{N}}$, $\{\varepsilon_i\}_{i\in\mathbb{N}}$ converging to 0, and a sequence $\{p_i\}_{i\in\mathbb{N}}$ converging to the origin in H, such that the function*

$$(0,T)\times H\to\mathbb{R},\ (t,x)\mapsto u_n(t,x)-\Phi_{\varepsilon_i}(t,x)-\langle p_i,x\rangle-a_it\tag{2.2.17}$$

has a strict global maximum at (t_i,x_i) and the sequence $\{(t_i,x_i)\}_{i\in\mathbb{N}}$ converges to $(\hat{t},(\hat{x}_0,\overline{x}_1))$.

Proof. Let $R > |(\hat{x}_0, \overline{x}_1)|_H$ and $B_H(0,R) := \{x \in H : |x|_H \le R\}$. Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a sequence converging to 0. Applying the classical result of Ekeland and Lebourg [32, 93], there exist sequences $\{a_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ and $\{p_i\}_{i\in\mathbb{N}} \subset H$ such that $|a_i| \le 1/i$, $|p_i|_H \le 1/i$, and such that the function

$$[0,T] \times B_H(0,R) \to \mathbb{R},\; u_n(t,x) - \Phi_{\varepsilon_i}(t,x) - \langle p_i, x\rangle - a_i t$$

has a strict global maximum at some point $(t_i, x_i) \in [0,T] \times B_H(0,R)$. By Lemma 2.2.6, with $D = [0,T] \times B_H(0,R)$, $f(t,x) = u_n(t,x) - \varphi(t,x_0)$, $g(t,x) = |(0, x_1 - \overline{x}_1)|_H^2$, $h_i(t,x) = -\langle p_i, x\rangle - a_i t$, $y_i = (t_i, x_i)$, we obtain

$$\lim_{i\to\infty} |(0, x_{i,1} - \overline{x}_1)|_H = 0. \tag{2.2.18}$$

To conclude the proof it is then sufficient to show that $(t_i, x_{i,0}) \to (\hat{t}, \hat{x}_0)$. Indeed, suppose that this does not hold. Up to a subsequence, we can suppose that $(t_i, x_{i,0}) \to (\tilde{t}, \tilde{x}_0) \ne (\hat{t}, \hat{x}_0)$. Since, by assumption, $(\hat{t}, \hat{x}_0)$ is a strict global maximum point of $u_n(\cdot, (\cdot, \overline{x}_1)) - \varphi(\cdot,\cdot)$, there exists $\eta > 0$ such that, for i sufficiently large, we have

$$\begin{aligned}
u_n(\hat{t}, (\hat{x}_0, \overline{x}_1)) - \varphi(\hat{t}, \hat{x}_0) \ge& \eta + u_n(t_i, (x_{i,0}, \overline{x}_1)) - \varphi(t_i, x_{i,0}) \\
\ge& \eta + u_n(t_i, (x_{i,0}, \overline{x}_1)) - \Phi_{\varepsilon_i}(t_i, x_i) \\
=& \eta + \big(u_n(t_i, (x_{i,0}, \overline{x}_1)) - u_n(t_i, x_i)\big) + u_n(t_i, x_i) - \Phi_{\varepsilon_i}(t_i, x_i) \\
\ge& \eta + \big(u_n(t_i, (x_{i,0}, \overline{x}_1)) - u_n(t_i, x_i)\big) + u_n(\hat{t}, (\hat{x}_0, \overline{x}_1)) - \varphi(\hat{t}, \hat{x}_0) \\
&+ \langle p_i, x_i - (\hat{x}_0, \overline{x}_1)\rangle + a_i(t_i - \hat{t}).
\end{aligned} \tag{2.2.19}$$

By (2.2.18), $\lim_{i\to\infty} (x_{i,0}, x_{i,1}) = (\tilde{x}_0, \overline{x}_1)$. Thus by continuity of u_n, for i sufficiently large, we have

$$|u_n(t_i, (x_{i,0}, \overline{x}_1)) - u_n(t_i, x_i)| \le \frac{\eta}{2}$$

and then if follows from 2.2.19 that

$$u_n(\hat{t}, (\hat{x}_0, \overline{x}_1)) - \varphi(\hat{t}, \hat{x}_0) \ge \frac{\eta}{2} + u_n(\hat{t}, (\hat{x}_0, \overline{x}_1)) - \varphi(\hat{t}, \hat{x}_0) + \langle p_i, x_i - (\hat{x}_0, \overline{x}_1)\rangle + a_i(t_i - \hat{t}).$$

This produces a contradiction by letting $i \to +\infty$, recalling that $p_i \to 0$ and $a_i \to 0$. Thus we must have $\lim_{i\to\infty} (t_i, x_{i,0}) = (\hat{t}, \hat{x}_0)$. ■

For any $\overline{x}_1 \in H_1$ and $n \in \mathbb{N}$, we define the following functions

$$v_{n,\overline{x}_1} : [0,T] \times \mathbb{R}^m \to \mathbb{R},\, (t, x_0) \mapsto u_n(t, (x_0, \overline{x}_1)), \tag{2.2.20}$$

$$a_{n,\overline{x}_1} : [0,T] \times \mathbb{R}^m \to \mathbf{S}_m,\, (t, x_0) \mapsto \sigma_n(t, (x_0, \overline{x}_1))\sigma_n^*(t, (x_0, \overline{x}_1)) \tag{2.2.21}$$

and

$$\beta_{n,\overline{x}_1} : [0,T] \times \mathbb{R}^m \to \mathbb{R},\, (t, x_0) \mapsto \langle A_n(x_0, \overline{x}_1) + G_n(t, (x_0, \overline{x}_1)), \nabla_x u_n(t, (x_0, \overline{x}_1))\rangle. \tag{2.2.22}$$

We associate to (2.2.13) the following terminal value problem

$$\begin{cases} -v_t(t,x_0) - \dfrac{1}{2}\mathrm{Tr}(a_{n,\overline{x}_1}(t,x_0)D^2_{x_0}v(t,x_0)) - \beta_{n,\overline{x}_1}(t,x_0) = 0 & (t,x_0) \in (0,T)\times\mathbb{R}^m \\ v(T,x_0) = h_n(x_0,\overline{x}_1) & x_0 \in \mathbb{R}^m. \end{cases} \tag{2.2.23}$$

We recall that it follows from Proposition 2.2.2*(iii)* that for every $\overline{x}_1 \in H_1$ the function $\beta_{n,\overline{x}_1}$ is continuous and for every compact set $K \subset \mathbb{R}^m$,

$$\sup_{\substack{n\geq 1\\ (t,x_0)\in[0,T]\times K}} |\beta_{n,\overline{x}_1}(t,x_0)| < +\infty. \tag{2.2.24}$$

In the following proposition we show that the section functions $v_{n,\overline{x}_1}$ are the viscosity solutions to (2.2.23). For the definition of viscosity solution in finite dimensions, we refer to [17].

Proposition 2.2.8. *For every $\overline{x}_1 \in H_1$ and $n \geq 1$, $v_{n,\overline{x}_1}$ is a viscosity solution to* (2.2.23)*.*

Proof. We prove that $v_{n,\overline{x}_1}$ is a subsolution. The supersolution case is similar. The continuity of u_n implies the continuity of $v_{n,\overline{x}_1}$. Let $\varphi \in C^{1,2}((0,T)\times\mathbb{R}^m)$ be such that $v_{n,\overline{x}_1} - \varphi$ has a local maximum at $(\hat{t},\hat{x}_0) \in (0,T)\times\mathbb{R}^m$. Without loss of generality, we can assume that the maximum is strict and global and that $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^m)$. By Lemma 2.2.7, there exist real sequences $\{\varepsilon_i\}_{i\in\mathbb{N}}, \{a_i\}_{i\in\mathbb{N}}$ converging to 0, and a sequence $\{p_i\}_{i\in\mathbb{N}}$ in H converging to 0, such that the functions

$$[0,T]\times\mathbb{R}^m \to \mathbb{R}, (t,x) \mapsto u_n(t,x) - \Phi_{\varepsilon_i}(t,x) - \langle p_i, x\rangle - a_i t$$

have local maxima at (t_i,x_i) and the sequence $\{(t_i,x_i)\}_{i\in\mathbb{N}}$ converges to $(\hat{t},(\hat{x}_0,\overline{x}_1))$. Since u_n is a viscosity solution to (2.2.13), we have

$$-D_t\Phi_{\varepsilon_i}(t_i,x_i) - a_i - \langle A_n x_i, D_x\Phi_{\varepsilon_i}(t_i,x_i) + p_i\rangle - L_n\left(t_i,x_i,D_x\Phi_{\varepsilon_i}(t_i,x_i) + p, D^2_{x_0}\Phi_{\varepsilon_i}(t_i,x_i)\right) \leq 0. \tag{2.2.25}$$

Since $u_n \in \mathscr{G}^{0,1}_b([0,T]\times H,\mathbb{R})$, we must have

$$\nabla_x u_n(t_i,x_i) = D_x\Phi_{\varepsilon_i}(t_i,x_i) + p_i. \tag{2.2.26}$$

Thus, by recalling (2.2.16), we have

$$-D_t\Phi_{\varepsilon_i}(t_i,x_i) - a_i - \langle A_n x_i, \nabla_x u_n(t_i,x_i)\rangle - L_n\left(t_i,x_i,\nabla_x u_n(t_i,x_i), D^2_{x_0}\varphi(t_i,x_{i,0})\right) \leq 0. \tag{2.2.27}$$

We now pass to the limit $i \to +\infty$ and, by (2.2.16) and the strong continuity of $\nabla_x u_n$, we obtain

$$-\varphi_t\left(\hat{t},\hat{x}_0\right) - \langle A_n(\hat{x}_0,\overline{x}_1), \nabla_x u_n(\hat{t},(\hat{x}_0,\overline{x}_1))\rangle - L_n\left(\hat{t},(\hat{x}_0,\overline{x}_1), \nabla_x u_n(\hat{t},(\hat{x}_0,\overline{x}_1)), D^2_{x_0}\varphi(\hat{t},\hat{x}_0)\right) \leq 0,$$

which can be written, by using the definition of $\beta_{n,\overline{x}_1}$,

$$-\varphi_t(\hat{t},\hat{x}_0) - \frac{1}{2}\mathrm{Tr}\left(\left(D^2_{x_0}\varphi(\hat{t},\hat{x}_0)\right)\sigma_n(\hat{t},(\hat{x}_0,\overline{x}_1))\sigma_n^*(\hat{t},(\hat{x}_0,\overline{x}_1))\right) - \beta_{n,\overline{x}_1}(\hat{t},\hat{x}_0) \leq 0.$$

Thus $v_{n,\overline{x}_1}$ is a viscosity subsolution to (2.2.23). ■

2.2.3 Regularity with respect to the finite dimensional component

In this last section we show that, if σ is non-degenerate, then the function u defined by (2.2.3) is differentiable with respect to x_0 and $D_{x_0}u$ enjoys some Hölder continuity.

Theorem 2.2.9. *Suppose that, for every $(t,x)\in[0,T]\times H$ and $y\in\mathbb{R}^m$, $\sigma(t,x)y\neq 0$. Then, for every $\overline{x}_1\in H_1$, the function $v_{\overline{x}_1}$ defined by $v_{\overline{x}_1}(t,x_0):=u(t,(x_0,\overline{x}_1))$ belongs to $C^{1+\alpha}_{loc}((0,T)\times\mathbb{R}^m)$, for every $\alpha\in(0,1)$.*

Proof. Let $(t,x_0)\in(0,T)\times\mathbb{R}^m$. Let $Q:=[c,d)\times B(x_0,\varepsilon)$ be a neighborhood of (t,x_0) in $(0,T)\times\mathbb{R}^m$ such that, for some $M>0$ and $\delta>0$,

$$\delta < a_{\overline{x}_1}(s,y) := \sigma(s,(y,\overline{x}_1))\sigma^*(s,(y,\overline{x}_1)) < M \qquad \forall (s,y)\in Q.$$

Since $\Sigma_n(s,(y,\overline{x}_1))z=(\sigma_n(s,(y,\overline{x}_1))z,0_1)$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ converges to σ uniformly on compact sets (Remark 2.1.9), we can suppose that $\delta<a_{n,\overline{x}_1}(s,y)<M$ for all $n\in\mathbb{N}$ and $(s,y)\in Q$ and that the family $\{a_{n,\overline{x}_1}\}_{n\in\mathbb{N}}$ is equi-uniformly continuous.

By Proposition 2.2.8, for $n\geq 1$, $v_{n,\overline{x}_1}$ is a viscosity solution to (2.2.23), in particular it is a viscosity solution to the terminal boundary value problem

$$\begin{cases} -v_t(s,y)-\dfrac{1}{2}\mathrm{Tr}(a_{n,\overline{x}_1}(s,y)D^2_y v(s,y))-\beta_{n,\overline{x}_1}(s,y)=0 & (s,y)\in Q\\ v(s,y)=u_n(s,(y,\overline{x}_1)) & (s,y)\in\partial_P Q \end{cases} \tag{2.2.28}$$

Thus, for instance by [18, Lemma 2.9, Proposition 2.10, and Theorem 9.1], $v_{n,\overline{x}_1}$ is the unique viscosity solution (in particular also a unique L^p-viscosity solution ([4])) of (2.2.28), and

$$|v_{n,\overline{x}_1}|_{W^{1,2,p}(Q')}\leq C\left(\sup_{(s,y)\in Q}|u_n(s,(y,\overline{x}_1))|+\sup_{(s,y)\in Q}|\beta_n(s,(y,\overline{x}_1))|\right) \tag{2.2.29}$$

for all $m+1\leq p<+\infty$ and for all $Q'=[c',d')\times B(x,\varepsilon')$, with $c<c'<d'<d$ and $0<\varepsilon'<\varepsilon$, and where C depends only on m, p, δ, M, Q, Q', and the uniform modulus of continuity of the functions $a_{n,\overline{x}_1}$. Thus, by Proposition 2.2.2 and (2.2.24), the set $\{v_{n,\overline{x}_1}\}_{n\geq 1}$ is uniformly bounded in $W^{1,2,p}(Q')$. Therefore applying an embedding theorem, see e.g. [67, Lemma 3.3, p. 80], we obtain that for every $\alpha\in(0,1)$

$$|v_{n,\overline{x}_1}|_{C^{1+\alpha}(Q')}\leq C_\alpha$$

for some constant C_α independent of n. Since the sequence $\{v_{n,\overline{x}_1}\}_{n\geq 1}$ converges uniformly on compact sets to the function $v_{\overline{x}_1}$ as $n\to+\infty$, it follows that the function $v_{\overline{x}_1}$ satisfies the above estimate too. This completes the proof. ■

[4]See [18] for the definition of L^p-viscosity solution.

Chapter 3

Viscosity solutions to semilinear path-dependent PDEs in Hilbert spaces

Given $T > 0$ and a real separable Hilbert space H, we recall that $\mathbb{W}$ denotes the Banach space $C([0,T],H)$ of continuous functions from $[0,T]$ to H, endowed with the supremum norm $|\mathbf{x}|_\infty := \sup_{t\in[0,T]} |\mathbf{x}(t)|_H$, for all $\mathbf{x} \in \mathbb{W}$. Let

$$\Lambda := [0,T] \times \mathbb{W}$$

and consider the following pseudometric on Λ:

$$\mathbf{d}_\infty\big((t,\mathbf{x}),(t',\mathbf{x}')\big) := |t-t'| + |\mathbf{x}_{\cdot\wedge t} - \mathbf{x}'_{\cdot\wedge t'}|_\infty, \qquad (t,\mathbf{x}),\ (t',\mathbf{x}') \in \Lambda.$$

This pseudo-metric allows to account for the *non-anticipativity* condition: each function $v\colon (\Lambda,\mathbf{d}_\infty) \to E$, where E is a Banach space, which is measurable with respect to the Borel σ-algebra induced by $\mathbf{d}_\infty$, is such that $v(t,\mathbf{x}) = v(t,\mathbf{x}_{\cdot\wedge t})$ for all $(t,\mathbf{x}) \in \Lambda$. Let A be the generator of a strongly continuous semigroup S on H, and let $b\colon \Lambda \to H$, $\sigma\colon \Lambda \to L(U,H)$, where U is another real separable Hilbert space (the noise space, as we will see in Section 3.1.2). In this chapter, we study the wellposedness of the following infinite dimensional semilinear path-dependent partial differential equation (PPDE):

$$-\partial_t u - \langle A\mathbf{x}_t, \partial_\mathbf{x} u\rangle - \langle b(t,\mathbf{x}), \partial_\mathbf{x} u\rangle - \frac{1}{2}\mathrm{Tr}\left[\sigma(t,\mathbf{x})\sigma^*(t,\mathbf{x})\partial^2_{\mathbf{xx}} u\right] - F(t,\mathbf{x},u) = 0, \qquad (3.0.1)$$

for all $t \in [0,T)$ and $\mathbf{x} \in \mathbb{W}$, where $F\colon \Lambda \times \mathbb{R} \to \mathbb{R}$ and $\partial_t u$, $\partial_\mathbf{x} u$, $\partial^2_{\mathbf{xx}} u$ denote the so-called pathwise (or functional or Dupire, see [12, 14, 31]) derivatives. The unknown is a non-anticipative functional $u\colon \Lambda \to \mathbb{R}$. We are deliberately restricting the nonlinearity F to depend only on u, and not on $\partial_\mathbf{x} u$, in order to focus on our main wellposedness objective. More general nonlinearities are left for future research.

In addition to the infinite dimensional feature of equation (3.0.1), we emphasize that its coefficients b, σ, and F are path-dependent. Such a path-dependence may be addressed through the standard PDE approach, by introducing a "second level" of infinite-dimensionality, embedding the state space H in a larger infinite-dimensional space like e.g. $L^2((-T,0),H)$, and converting equation (3.0.1) into a PDE on this larger space (see e.g., in the context of delay equations and when the original space H is finite-dimensional, [10, 23, 37], and the introduction to Chapter 2). The latter methodology turns out to be problematic when the data, as in our case, are required to have continuity properties with respect to the supremum norm, as the PDE should be considered basically in spaces of continuous functions, which are not reflexive. However, we should mention that some attempts have been achieved along this direction, we refer to [29, 28, 38, 43, 46].

When the space H is finite-dimensional, PPDEs with a structure more general than (3.0.1) have been investigated by means of a new concept of viscosity solution recently introduced in [33], and further developed in [34, 35, 85]. This new notion enlarges the class of test functions, by defining the smoothness only "with respect to the dynamics" of the underlying stochastic system and requiring the usual "tangency condition" — required locally pointwise in the standard viscosity definition — only in mean. These two weakenings, on one hand, keep safe the existence of solutions; on the other hand, simplify a lot the proof of uniqueness — as it does not require anymore the passage through the Crandall-Ishii Lemma.

The main objective of this chapter is to extend to our infinite-dimensional path-dependent context such new notion of viscosity solution. Before illustrating our results, we recall that, for equation like (3.0.1), when all coefficient are Markovian, results on existence and uniqueness of classical solutions (that can be found e.g. in [23, Chapter 7]) are much weaker than in the finite dimensional case, due to the lack of local compactness and to the absence of a reference measure like the Lebesgue one. This makes quite relevant the notion of viscosity solution, introduced in the infinite-dimensional case by [70, 72, 73], see also [95] and, for a survey, [37, Chapter 3]. The infinite dimensional extension of the usual notion of viscosity solution to these PDEs is not trivial, as the comparison results are established only under strong continuity assumptions on the coefficients (needed to generate maxima and minima) and under a nuclearity condition on the diffusion coefficient σ. The latter purely technical condition is a methodological bound of this notion of viscosity solutions, as it is only needed in order to adapt the Crandall-Ishii Lemma to the infinite-dimensional context.

The core results of the present chapter (contained in Section 3.2) are as follows. First, on the line of [85], we show that the infinite-dimensional definition has an equivalent version with semijets (Proposition 3.2.5). Then, under natural assumptions on the operator A and the coefficients b, σ, F, we prove sub/supermartingale characterisation of sub/supersolutions which extends the corresponding result in [85] (Theorem 3.2.7). As a corollary of this characterisation we get that the PPDE satisfies the desired stability property of viscosity solutions (Proposition 3.2.12). Furthermore, still applying Theorem 3.2.7, we prove that equation (3.0.1) satisfies the comparison principle in the class of continuous functions with polynomial growth on Λ (Corollary 3.2.14). In particular, since the Crandall-Ishii Lemma is not needed to establish comparison, we emphasize that the nuclearity condition on σ is completely by-passed in our framework. Similarly, this happens for the strong continuity properties mentioned above. Finally, given a continuous terminal condition $u(T, \mathbf{x}) = \xi(\mathbf{x})$ with sublinear growth, we establish existence of a unique solution (Theorem 3.2.16). We observe that our unique viscosity solution is closely related to the solution of the infinite dimensional backward stochastic differential equation (BSDE) of [47], which can be viewed as a Sobolev solution to equation (3.0.1) (see e.g. [4]).

From what we have said, it follows that the passage from finite to infinite dimension makes meaningful considering the new notion of viscosity solution *even* in the Markovian (non-path-dependent case). Indeed, while in the finite dimensional case the theory based on the usual definition of viscosity solutions is so well-developed to cover basically a huge class of PDEs, in the infinite dimensional case the known theory of viscosity solutions collides with the structural constraints described above, which can be by-passed with the new notion.

Finally, we point out that our results may be extended to suitable nonlinearities depending on the gradient $\partial_{\mathbf{x}} u$. In our formalism, a way to do it could be by introducing a control process in the drift of the underlying stochastic system, which basically corresponds, in the formalism [33], to replace the expectation in the tangency condition on test function by a nonlinear expectation operator defined as sup/inf of expectations under a convenient family of probability measures. We deliberately avoid this additional complication in order to focus on the infinite-dimensional feature of the equations, and we leave the study of more general nonlinearities to future work.

The present chapter is organized as follows. In Section 3.1, we give the additional notation required for our framework, then we introduce the process X from which the definition of smooth and test functions will depend.

In Section 3.2, we introduce the notion of viscosity solution for path-dependent PDEs in Hilbert spaces, in terms of both test functions and semijets (Section 3.2.1); we prove a martingale characterisation of viscosity sub/supersolutions and a stability result (Section 3.2.2); finally, we prove the comparison principle (Section 3.2.3) and we provide an existence and uniqueness result for the path-dependent PDE (Section 3.2.4).

In the last section, Section 3.3, we consider the Markovian case, i.e. when all data depend only on the present, and we compare the notion of viscosity solution studied in Section 3.2 to the usual notions of viscosity solutions adopted in the literature for partial differential equations in Hilbert spaces.

Finally, the Appendix of the chapter is devoted to a clarification on the definition of test functions given in Section 3.2.1.

3.1 Preliminaries

3.1.1 Notation

Consider a real separable Hilbert space H. Denote by $\langle\cdot,\cdot\rangle_H$ and $|\cdot|_H$ the scalar product and norm on H, respectively. For $T>0$, we recall that $\mathbb{W}$ is a short notation for $C([0,T],H)$, that the generic element of $\mathbb{W}$ is denoted by $\mathbf{x}$, and that $\mathbb{W}$ is endowed with the supremum norm $|\cdot|_\infty$. We introduce the space

$$\Lambda := [0,T]\times\mathbb{W}$$

and the map $\mathbf{d}_\infty\colon \Lambda\times\Lambda\to\mathbb{R}^+$ defined by

$$\mathbf{d}_\infty\big((t,\mathbf{x}),(t',\mathbf{x}')\big) := |t-t'| + |\mathbf{x}_{\cdot\wedge t} - \mathbf{x}'_{\cdot\wedge t'}|_\infty.$$

Then $\mathbf{d}_\infty$ is a pseudometric on Λ. In particular, $(\Lambda,\mathbf{d}_\infty)$ is a topological space with the topology induced by the pseudometric $\mathbf{d}_\infty$. The quotient space $(\Lambda/\sim)$, where $\sim$ is the equivalence relation defined by

$$(t,\mathbf{x})\sim(t',\mathbf{x}') \qquad \text{whenever} \qquad t=t',\ \mathbf{x}_s=\mathbf{x}'_s,\ \forall s\in[0,t],$$

is a complete separable metric space when endowed with the quotient metric. Λ becomes a measurable space when endowed with the Borel σ-algebra induced by $\mathbf{d}_\infty$. Throughout the present chapter, the topology and σ-algebra on Λ are those induced by $\mathbf{d}_\infty$.

Let E be a Banach space. We recall the that an E-valued non-anticipative function on Λ is a map $v\colon\Lambda\to E$ such that

$$v(t,\mathbf{x}) = v(t,\mathbf{x}_{\cdot\wedge t}), \qquad \forall (t,\mathbf{x})\in\Lambda.$$

We denote by $C(\Lambda,E)$ the space of continuous functions $v\colon\Lambda\to E$. For $p\geq 1$, $C_p(\Lambda,E)$ denotes the space of continuous functions $v\colon\Lambda\to E$ satisfying the following polynomial growth condition:

$$|v(t,\mathbf{x})|_E \leq M(1+|\mathbf{x}|_\infty^p), \qquad \forall (t,\mathbf{x})\in\Lambda,$$

for some constant $M>0$. $C_p(\Lambda,E)$ is a Banach space when endowed with the norm

$$|v|_{C_p(\Lambda,E)} := \sup_{(t,\mathbf{x})\in\Lambda} \frac{|v(t,\mathbf{x})|_E}{(1+|\mathbf{x}|_\infty)^p}.$$

We denote by $UC(\Lambda, E)$ the space of uniformly continuous functions $v\colon \Lambda \to E$. When $E = \mathbb{R}$, we drop $\mathbb{R}$ and simply write $C(\Lambda)$, $C_p(\Lambda)$, and $UC(\Lambda)$.

Clearly, for all $1 \le p \le q$, we have the inclusions

$$UC(\Lambda, E) \subset C_p(\Lambda, E) \subset C_q(\Lambda, E) \subset C(\Lambda, E).$$

We notice that a measurable map $v\colon \Lambda \to E$ is automatically non-anticipative. For this reason, we will drop the term non-anticipative when v is measurable.

Remark 3.1.1. In Chapter 4, in place of $C(\Lambda, E)$, we will use the space $CNA([0,T] \times \mathbb{W}, E)$, which contains the functions of $C([0,T] \times \mathbb{W}, E)$ which are non-anticipative, and in Remark 4.1.1 we briefly show that our analysis on Λ can be reduced to analysis on subspaces of more customary spaces. In the present chapter we prefer to keep the functional setting introduced first in [31], for the finite dimensional case, where non-anticipative functions are defined on Λ.

3.1.2 The reference process X

In this section we introduce the path-dependent SDE in H that determines our reference process for the definition of viscosity solution in the following section.

Let $(\Omega, \mathscr{F}, \mathbb{F} = \{\mathscr{F}_t\}_{t \in [0,T]}, \mathbb{P})$ denote a filtered probability space satisfying the usual conditions. We recall that $\mathscr{P}_T$ denotes the σ-algebra of predictable sets with respect to $\mathbb{F}$ on $\Omega \times [0,T]$. From Chapter 1, we recall that $\mathscr{L}^0_{\mathscr{P}_T}(\mathbb{W})$ denotes the space of continuous H-valued predictable processes, and that for $p \ge 1$, $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$ denotes the subspace of $X \in \mathscr{L}^0_{\mathscr{P}_T}(\mathbb{W})$ such that

$$|X|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})} := \left(\mathbb{E}\left[|X|^p_\infty\right]\right)^{1/p} < \infty,$$

where two processes X, Y are identified if $|X - Y|_\infty = 0$ $\mathbb{P}$-a.e..

We observe that, for $X \in \mathscr{L}^0_{\mathscr{P}_T}(\mathbb{W})$, since the $\mathbb{W}$-valued process $\{X_{t\wedge\cdot}\}_{t\in[0,T]}$ is $\mathbb{F}$-adapted and continuous, hence predictable, the map

$$g\colon \Omega \times [0,T] \to \Lambda,\ (\omega, t) \mapsto (t, X_{t\wedge\cdot})$$

is predictable. Then, for $v \in C(\Lambda, H)$, the process $v(\cdot, X)$ is predictable, and, since it is clearly continuous, it belongs to $\mathscr{L}^0_{\mathscr{P}_T}(\mathbb{W})$. In particular, if $v \in C_q(\Lambda, H)$ and $X \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$, then $v(\cdot, X) \in \mathscr{L}^{q/p}_{\mathscr{P}_T}(\mathbb{W})$.

Remark 3.1.1. In the present chapter, as it is usually done in the literature on infinite dimensional second order PDEs (see, e.g., [22, 95]), we distinguish between the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, whose generic element is ω, and the path space $\mathbb{W}$, whose generic element is $\mathbf{x}$. Instead, in [33], the authors identify these two spaces (up to the translation of the initial point), taking as probability space the canonical space $\{\mathbf{x} \in \mathbb{W} : \mathbf{x}_0 = 0\}$ and calling ω its generic element. Clearly everything done here can be rephrased in the setting of [33] (again up to a translation of the initial point), by taking as probability space $(\mathbb{W}, \mathscr{B}_{\mathbb{W}}, \mathbb{P}^X)$, where $\mathscr{B}_{\mathbb{W}}$ is the σ-algebra of Borel subsets of $\mathbb{W}$ and $\mathbb{P}^X$ is the law of the process X that we will define in the next section as mild solution to a path-dependent SDE. □

Let U be a real separable Hilbert space and let W be a U-valued cylindrical Wiener process on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$. For $t \in [0,T]$ and $Z \in \mathscr{L}^0_{\mathscr{P}_T}(\mathbb{W})$, we consider the following path-dependent SDE:

$$\begin{cases} dX_s = AX_s ds + b(s,X)ds + \sigma(s,X)dW_s, & s \in [t,T], \\ X_{t\wedge\cdot} = Z_{t\wedge\cdot}, \end{cases} \tag{3.1.1}$$

under the following

Assumption 3.1.2.

(i) *The operator* $A : D(A) \subset H \to H$ *is the generator of a strongly continuous semigroup* S *in the Hilbert space* H.

(ii) $b : \Lambda \to H$ *is measurable and such that, for some constant* $M > 0$,

$$|b(t,\mathbf{x}) - b(t,\mathbf{x}')| \le M|\mathbf{x} - \mathbf{x}'|_\infty, \qquad |b(t,\mathbf{x})| \le M(1 + |\mathbf{x}|_\infty),$$

for all $\mathbf{x}, \mathbf{x}' \in \mathbb{W}$, $t \in [0,T]$.

(iii) $\sigma : \Lambda \to L(U,H)$ *is such that* $\sigma(\cdot,\cdot)v : \Lambda \to H$ *is measurable for each* $v \in U$ *and*

$$S_s\sigma(t,\mathbf{x}) \in L_2(U,H) \qquad \forall s \in (0,T],\ (t,\mathbf{x}) \in \Lambda.$$

Moreover, there exist $\hat{M} > 0$ *and* $\gamma \in [0,1/2)$ *such that, for all* $\mathbf{x}, \mathbf{x}' \in \mathbb{W}$, $t \in [0,T]$, $s \in (0,T]$,

$$|e^{sA}\sigma(t,\mathbf{x})|_{L_2(U,H)} \le \hat{M}s^{-\gamma}(1 + |\mathbf{x}|_\infty), \tag{3.1.2}$$

$$|e^{sA}\sigma(t,\mathbf{x}) - e^{sA}\sigma(t,\mathbf{x}')|_{L_2(U,H)} \le \hat{M}s^{-\gamma}|\mathbf{x} - \mathbf{x}'|_\infty. \tag{3.1.3}$$

Regarding Assumption 3.1.2*(iii)*, we observe that one could do the more demanding assumption of sublinear growth and Lipschitz continuity of $\sigma(t,\cdot)$ as function valued in the space $L_2(K,H)$ (see Chapter 1 in this thesis, and [48]). The assumptions we give, which are the closely related to the minimal ones used in literature to give sense to the stochastic integral and to ensure the continuity of the stochastic convolution (see Appendix A for further clarifications), are taken from [22, Hypothesis 7.2] and [47].

Regarding Assumption 3.1.2*(ii)*, we observe that it could be relaxed giving assumptions on the composition of the map b with the semigroup, as done for σ in part *(iii)* of the same Assumption. Here, we follow [22, 47] and we do not perform it.

We recall from Chapter 1 the definition of mild solution.

Definition 3.1.3. *Let $Z \in \mathscr{L}^0_{\mathscr{P}_T}(\mathbb{W})$. We call mild solution to* (3.1.1) *a process $X \in \mathscr{L}^0_{\mathscr{P}_T}(\mathbb{W})$ such that $X_{\cdot\wedge t} = Z_{\cdot\wedge t}$ and*

$$X_s = S_{s-t}Z_t + \int_t^s S_{s-r}b(r,X)dr + \int_t^s S_{s-r}\sigma(r,X)dW_r, \qquad \forall s \in [t,T].$$

Existence and uniqueness of mild solution to (3.1.1) in $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$ come from Chapter 1. We state the result for future reference. Let

$$p^* := \frac{2}{1-2\gamma}.$$

Theorem 3.1.4. *Let Assumption 3.1.2 hold. Then, for every $p > p^*$, $t \in [0,T]$ and $Z \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$, there exists a unique mild solution $X^{t,Z}$ to* (3.1.1). *Moreover, $X^{t,Z} \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$, and the map*

$$[0,T] \times \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}),\ (t,Z) \mapsto X^{t,Z} \tag{3.1.4}$$

is Lipschitz continuous with respect to Z, uniformly in $t \in [0,T]$, and jointly continuous. In particular, there exists K_0 such that

$$|X^{t,Z}|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})} \le K_0(1+|Z|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})}), \qquad \forall (t,Z) \in [0,T] \times \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}). \tag{3.1.5}$$

Proof. Apply Theorem 1.2.6 and Theorem 1.2.14. ■

We notice that uniqueness of mild solutions yields the flow property for the solution with initial data $(t,\mathbf{x}) \in \Lambda$:

$$X^{t,\mathbf{x}} = X^{s,X^{t,\mathbf{x}}}, \qquad \forall (t,\mathbf{x}) \in \Lambda,\ \forall s \in [t,T]. \tag{3.1.6}$$

In the sequel, we will use the following generalized dominated convergence result.

Lemma 3.1.5. *Let $(\Sigma,\mathscr{S},\mu)$ be a measure space. Assume that $f_n, g_n, f, g \in L^1((\Sigma,\mathscr{S},\mu),\mathbb{R})$, $f_n \to f$ and $g_n \to g$ μ-a.e., $|f_n| \le g_n$ and $\int_\Sigma g_n d\mu \to \int_\Sigma g d\mu$. Then $\int_\Sigma f_n d\mu \to \int_\Sigma f d\mu$.*

We have the following corollary of Theorem 3.1.4, that will be used in Section 3.2.

Corollary 3.1.6. *Let $\kappa \in L^\infty((\Omega,\mathscr{F},\mathbb{P}),C_p(\Lambda))$ and $p > p^*$. Then the map*

$$[0,T] \times [0,T] \times \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}) \to \mathbb{R},\ (s,t,Z) \mapsto \mathbb{E}\left[\kappa(\cdot)(s,X^{t,Z})\right] \tag{3.1.7}$$

is well-defined and continuous.

Proof. We first comment the measurability of

$$\Omega \to \mathbb{R},\ \omega \mapsto \kappa(\omega)(s,Y), \tag{3.1.8}$$

for $Y \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$. Since $|Y|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})} < \infty$, there exists e separable subset of $\mathbb{W}$ such that $Y(\omega) \in \mathbb{W}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then we can suppose that κ takes values on a separable subspace V of $C_p(\Lambda)$. In particular, κ is the pointwise limit, everywhere, of a sequence of V valued simple functions [24, Lemma 1.3]. We then only need to prove the measurability of (3.1.8) for κ simple. As previously observed, if $v \in C_p(\Lambda)$, then $v(\cdot,Y)$ is predictable. Hence, for $B \in \mathscr{F}$ and $s \in [0,T]$, the simple function

$$\Omega \to C_p(\Lambda),\ \omega \mapsto v(\cdot,\cdot)\mathbf{1}_B(\omega)$$

gives rise to a measurable map

$$\Omega \to \mathbb{R},\ \omega \mapsto v(s,Y(\omega))\mathbf{1}_B(\omega).$$

This concludes the proof that (3.1.8) is measurable.

Regarding the integrability required in the definition of (3.1.7), it comes from Theorem 3.1.4 and from the polynomial growth for $\kappa(\omega)(\cdot)$.

Concerning continuity, again in view of Theorem 3.1.4, it suffices to show that the map

$$[0,T] \times \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}) \to \mathbb{R},\ (s,Y) \mapsto \mathbb{E}[\kappa(\cdot)(s,Y)]$$

is continuous. Let $\{Y^{(n)}\}_{n\in\mathbb{N}}$ be a sequence converging to Y in $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$, and let $s_n \to s$ in $[0,T]$. Let $\{Y^{(n_k)}\}_{k\in\mathbb{N}}$ be a subsequence such that $|Y - Y^{(n_k)}|_\infty \to 0$ $\mathbb{P}$-a.s.. Then, using the continuity of $\kappa(\omega)(\cdot,\cdot)$ we get, by applying Lemma 3.1.5, the convergence

$$\mathbb{E}[\kappa(\cdot)(s_{n_k}, Y^{(n_k)})] \to \mathbb{E}[\kappa(\cdot)(s,Y)].$$

Since the original converging sequence $\{(s_n, Y^{(n)})\}_{n\in\mathbb{N}}$ was arbitrary, we get the claim. ■

The following stability result for SDE (3.1.1) will be used to prove the stability of viscosity solutions in the next section. It is a particular case of Theorem 1.2.14, to which we refer for the proof.

Proposition 3.1.7. *Let Assumption 3.1.2 hold and assume that it holds also, for each $n \in \mathbb{N}$, for analogous objects A_n, b_n and σ_n, such that the estimates of parts (ii)-(iii) in Assumption 3.1.2 hold with the constants $M, \hat{M}, \gamma$. Assume that the following convergences hold for every $(t,\mathbf{x}) \in \Lambda$ and every $s \in [0,T]$:*

(i) $(S_n)_s\mathbf{x}_s \to S_s\mathbf{x}_s$ *in* H*;*

(ii) $(S_n)_s b_n(t,\mathbf{x}) \to S_s b(t,\mathbf{x})$ *in* H*;*

(iii) $(S_n)_s \sigma_n(t,\mathbf{x}) \to S_s \sigma(t,\mathbf{x})$ *in* $L_2(K,H)$*;*

where S_n *denotes the* C_0*-semigroup generated by* A_n*. Let* $t \in [0,T]$ *and* $Z \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$*, with* $p > p^*$*. Then, calling* $X_n^{t,Z}$ *the mild solution to* (3.1.1)*, where* A, b, σ *are replaced by* A_n, b_n, σ_n*, one has the convergence* $X_n^{t,Z} \overset{n\to\infty}{\longrightarrow} X^{t,Z}$ *in* $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$ *and, for fixed t, there exists* K_0 *such that*

$$|X_n^{t,Z}|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})} \leq K_0 \left(1 + |Z|_{\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})}\right), \qquad \forall Z \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}),\ \forall n \in \mathbb{N}. \tag{3.1.9}$$

3.2 Path-dependent PDEs and viscosity solutions in Hilbert spaces

In the present section, we introduce a path-dependent PDE in the space H and study it through the concept of viscosity solutions in the spirit of the definition given in [33, 34, 85]. As in [85], we also provide an equivalent definition in terms of jets. The key result is a martingale characterisation for viscosity sub/supersolution, from which a stability result and the comparison principle follow. We finally prove the existence of a viscosity solution through a fixed point argument.

3.2.1 Definition: test functions and semijets

We begin by introducing the set $C_X^{1,2}(\Lambda)$ of smooth functions, which will be used to define test functions. We note that the definition of the last set will depend on the process $X^{t,\mathbf{x}}$ solution to (3.1.1), hence on the coefficients A, b, σ. The subscript X in the notation $C_X^{1,2}(\Lambda)$ stays there to recall that.

Definition 3.2.1. *We say that* $u \in C_X^{1,2}(\Lambda)$ *if there exists* $p \geq 1$ *such that* $u \in C_p(\Lambda)$ *and there exist* $\alpha \in C_p(\Lambda)$*,* $\beta \in C_p(\Lambda, U)$ *such that*

$$du(s, X^{t,\mathbf{x}}) = \alpha(s, X^{t,\mathbf{x}})ds + \langle \beta(s, X^{t,\mathbf{x}}), dW_s\rangle, \qquad \forall (t,\mathbf{x}) \in \Lambda,\ \forall s \in [t,T]. \tag{3.2.1}$$

Notice that α and β in Definition 3.2.1 are uniquely determined, as it can be easily shown by identifying the finite variation part and the Brownian part in (3.2.1). Given $u \in C_X^{1,2}(\Lambda)$, we denote

$$\mathscr{L}u := \alpha.$$

We refer to the Appendix of the chapter for an insight on the above notation for α and for a link with the pathwise derivatives introduced in [31].

One of the key ingredients of the notion of viscosity solution we are going to define is the concept of test function introduced in Definition 3.2.1. Notice that, the larger the class of test functions, the easier should be the proof of the comparison principle and the harder the proof of the existence. In order to make easier the proof of uniqueness, we weaken the concept of test functions as much as possible — but, clearly, still keeping "safe" the existence part. The space $C_X^{1,2}(\Lambda)$ is the result of this trade-off. It is a quite large class of test functions: for example, as it will be shown in Lemma 3.2.11 below, if $f \in C_p(\Lambda)$, $p \geq 1$, then $\varphi(t,\mathbf{x}) := \int_0^t f(s,\mathbf{x})ds$ is in $C_X^{1,2}(\Lambda)$, whereas, even if $H = \mathbb{R}^n$ and f is Markovian (which means $f(s,\mathbf{x}) = f(s,\mathbf{x}_s)$), it does not belong, in general, to the usual class $C^{1,2}([0,T]\times\mathbb{R}^n,\mathbb{R})$ of smooth functions.

We are concerned with the study the following *path-dependent* PDE (from now on, PPDE):

$$\mathscr{L}u(t,\mathbf{x}) + F(t,\mathbf{x},u(t,\mathbf{x})) = 0, \qquad (t,\mathbf{x}) \in \Lambda,\ t < T, \tag{3.2.2}$$

with terminal condition

$$u(T,\mathbf{x}) = \xi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{W}, \tag{3.2.3}$$

where $F\colon \Lambda \times \mathbb{R} \to \mathbb{R}$ and $\xi\colon \mathbb{W} \to \mathbb{R}$.

We introduce the concept of viscosity solution for the path-dependent PDE (3.2.2), following [33, 34, 85]. To this end, we denote

$$\mathscr{T} := \big\{\tau\colon \tau \text{ is an } [0,T]\text{-valued } \mathbb{F}\text{-stopping time}\big\}.$$

Given $u \in C_p(\Lambda)$ for some $p \geq 1$, we define the following two classes of test functions:

$$\underline{\mathscr{A}}u(t,\mathbf{x}) := \Big\{\varphi \in C_X^{1,2}(\Lambda)\colon \text{there exists } \mathrm{H} \in \mathscr{T},\ \mathrm{H} > t, \text{ such that} \\ (\varphi - u)(t,\mathbf{x}) = \min_{\tau\in\mathscr{T},\tau\geq t} \mathbb{E}\big[(\varphi-u)(\tau\wedge\mathrm{H}, X^{t,\mathbf{x}})\big]\Big\},$$

$$\overline{\mathscr{A}}u(t,\mathbf{x}) := \Big\{\varphi \in C_X^{1,2}(\Lambda)\colon \text{there exists } \mathrm{H} \in \mathscr{T},\ \mathrm{H} > t, \text{ such that} \\ (\varphi - u)(t,\mathbf{x}) = \max_{\tau\in\mathscr{T},\tau\geq t} \mathbb{E}\big[(\varphi-u)(\tau\wedge\mathrm{H}, X^{t,\mathbf{x}})\big]\Big\}.$$

Definition 3.2.2. *Let $p \geq 1$, $u \in C_p(\Lambda)$.*

(i) We say that u is a viscosity subsolution *(resp.* supersolution*) to the path-dependent PDE* (3.2.2) *if*

$$-\mathscr{L}\varphi(t,\mathbf{x}) - F(t,\mathbf{x},u(t,\mathbf{x})) \leq 0, \qquad (\textit{resp.}\ \leq 0)$$

for all $(t,\mathbf{x}) \in \Lambda$, $t < T$, and all $\varphi \in \underline{\mathscr{A}}u(t,\mathbf{x})$ (resp. $\varphi \in \overline{\mathscr{A}}u(t,\mathbf{x})$).

(ii) We say that u is a viscosity solution *to the path-dependent PDE* (3.2.2) *if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 3.2.3. As usual, in Definition 3.2.2, without loss of generality, one can consider only the test functions $\varphi \in \underline{\mathscr{A}}u(t,\mathbf{x})$ (resp. $\overline{\mathscr{A}}u(t,\mathbf{x})$) such that $(\varphi - u)(t,\mathbf{x}) = 0$.

Remark 3.2.4. The notion of viscosity solution that we introduced is designed for our path-dependent PDE and it should be modified in a suitable way if we want to consider more general nonlinearities. For example, if we take F depending also on $\partial_{\mathbf{x}} u$ as in [33], this would entail a substantial change in our definition of viscosity solution. In [33] this corresponds to take an optimal stopping problem under nonlinear expectation, that is under a family of probability measures; in our formalism, which separates the (fixed) probability space from the state space (see Remark 3.1.1), this would correspond to take a mixed control/stopping problem, with the control acting on the drift of the SDE. In our infinite-dimensional framework, the case under study already presents some specific difficulties and interesting features (for instance, already in the comparison with the literature on viscosity solutions in infinite dimension in the Markovian case, see Section 3.3).

Following [85], we now provide an equivalent definition of viscosity solution in terms of semijets. Given $u \in C_p(\Lambda)$, for some $p \geq 1$, define the *subjet* and *superjet* of u at $(t,\mathbf{x}) \in \Lambda$ as

$$\underline{\mathscr{J}}u(t,\mathbf{x}) := \{\alpha \in \mathbb{R} \colon \exists \varphi \in \underline{\mathscr{A}}u(t,\mathbf{x}) \text{ such that } \varphi(s,\mathbf{y}) = \alpha s,\ \forall (s,\mathbf{y}) \in \Lambda\},$$

$$\overline{\mathscr{J}}u(t,\mathbf{x}) := \{\alpha \in \mathbb{R} \colon \exists \varphi \in \overline{\mathscr{A}}u(t,\mathbf{x}) \text{ such that } \varphi(s,\mathbf{y}) = \alpha s,\ \forall (s,\mathbf{y}) \in \Lambda\}.$$

We have the following equivalence result.

Proposition 3.2.5. *Suppose that Assumption 3.1.2 holds. Then $u \in C_p(\Lambda)$, $p \geq 1$, is a viscosity subsolution (resp. supersolution) to the path-dependent PDE* (3.2.2) *if and only if*

$$-\alpha - F(t,\mathbf{x},u(t,\mathbf{x})) \leq 0, \qquad (\textit{resp. } \geq 0),$$

for all $\alpha \in \underline{\mathscr{J}}u(t,\mathbf{x})$ (resp. $\alpha \in \overline{\mathscr{J}}u(t,\mathbf{x})$).

Proof. We focus on the *if* part, since the other implication is clear. Fix $(t,\mathbf{x}) \in \Lambda$ and $\varphi \in \underline{\mathscr{A}}u(t,\mathbf{x})$ (the supersolution part has a similar proof). From Definition 3.2.1 we know that there exists $\mathscr{L}\varphi := \alpha \in C_p(\Lambda)$ and $\beta \in C_p(\Lambda, H)$ such that (3.2.1) holds, with φ in place of u. Set

$$\alpha_0 := \mathscr{L}\varphi(t,\mathbf{x}) = \alpha(t,\mathbf{x})$$

and, for every $\varepsilon > 0$, consider $\varphi_\varepsilon(s,\mathbf{y}) := (\alpha_0 + \varepsilon)s$, for all $(s,\mathbf{y}) \in \Lambda$. Then $\varphi_\varepsilon \in C_X^{1,2}(\Lambda)$. Since $\mathscr{L}\varphi$ is continuous, we can find $\delta > 0$ such that

$$\left|\mathscr{L}\varphi(t',\mathbf{x}') - \alpha_0\right| = \left|\mathscr{L}\varphi(t',\mathbf{x}') - \mathscr{L}\varphi(t,\mathbf{x})\right| \leq \varepsilon, \qquad \text{if } \mathbf{d}_\infty\big((t',\mathbf{x}'),(t,\mathbf{x})\big) \leq \delta.$$

Let H be the stopping time associated to φ appearing in the definition of $\underline{\mathscr{A}}u(t,\mathbf{x})$ and define

$$\mathrm{H}_\varepsilon := \mathrm{H} \wedge \{s \geq t : \mathbf{d}_\infty((s, X^{t,\mathbf{x}}), (t,\mathbf{x})) > \delta\}.$$

Note that $\mathrm{H}_\varepsilon > 0$. Then, for any $\tau \in \mathscr{T}$ with $\tau \geq t$, we have

$$\begin{aligned}(u-\varphi_\varepsilon)(t,\mathbf{x}) &- \mathbb{E}[(u-\varphi_\varepsilon)(\tau \wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})] \\ &= (u-\varphi)(t,\mathbf{x}) - \mathbb{E}[(u-\varphi)(\tau \wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})] + \mathbb{E}[(\varphi_\varepsilon - \varphi)(\tau \wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})] - (\varphi_\varepsilon - \varphi)(t,\mathbf{x}) \quad (3.2.4)\\ &\geq \mathbb{E}[(\varphi_\varepsilon - \varphi)(\tau \wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})] - (\varphi_\varepsilon - \varphi)(t,\mathbf{x}),\end{aligned}$$

where the last inequality follows from the fact that $\varphi \in \underline{\mathscr{A}}u(t,\mathbf{x})$. Since φ and φ_ε belong to $C_X^{1,2}(\Lambda)$, we can write

$$\mathbb{E}\left[\varphi(\tau \wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})\right] = \varphi(t,\mathbf{x}) + \mathbb{E}\left[\int_t^{\tau\wedge \mathrm{H}_\varepsilon} \mathscr{L}\varphi(s, X^{t,\mathbf{x}}) ds\right] \tag{3.2.5}$$

and, clearly, we also have

$$\mathbb{E}\left[\varphi_\varepsilon(\tau \wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})\right] = \varphi_\varepsilon(t,\mathbf{x}) + \mathbb{E}\left[\int_t^{\tau\wedge \mathrm{H}_\varepsilon} (\alpha_0 + \varepsilon) ds\right]. \tag{3.2.6}$$

Plugging (3.2.5) and (3.2.6) into (3.2.4), we obtain

$$(\varphi_\varepsilon - u)(t,\mathbf{x}) - \mathbb{E}[(\varphi_\varepsilon - u)(\tau \wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})] \leq \mathbb{E}\left[\int_t^{\tau\wedge \mathrm{H}_\varepsilon} (\mathscr{L}\varphi(s, X^{t,\mathbf{x}}) - (\alpha_0 + \varepsilon)) ds\right] \leq 0,$$

where the last inequality follows by definition of H_ε. It follows that $\varphi_\varepsilon \in \underline{\mathscr{A}}(t,\mathbf{x})$, hence that $\alpha_0 + \varepsilon \in \underline{\mathscr{J}}u(t,\mathbf{x})$, therefore

$$-(\mathscr{L}\varphi(t,\mathbf{x}) + \varepsilon) - F(t,\mathbf{x},u(t,\mathbf{x})) = -(\alpha_0 + \varepsilon) - F(t,\mathbf{x},u(t,\mathbf{x})) \leq 0.$$

By arbitrariness of ε we conclude. ■

3.2.2 Martingale characterisation and stability

In the sequel, we will consider the following conditions on F.

Assumption 3.2.6.

(i) $F : \Lambda \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies the growth condition: there exists* $L > 0$ *such that*

$$|F(t,\mathbf{x},y)| \leq L(1 + \|\mathbf{x}\|_\infty^p + |y|), \qquad \forall (t,\mathbf{x}) \in \Lambda,\ \forall y \in \mathbb{R}. \tag{3.2.7}$$

(ii) F *is Lipschitz with respect to the third variable, uniformly in the other ones: there exists* $\hat{L} > 0$ *such that*

$$|F(t,\mathbf{x},y) - F(t,\mathbf{x},y')| \leq \hat{L}|y - y'|, \qquad \forall (t,\mathbf{x}) \in \Lambda,\ \forall y, y' \in \mathbb{R}. \tag{3.2.8}$$

We now state the main result of this section, the sub(super)martingale characterisation for viscosity sub(super)solutions to PPDE (3.2.2).

Theorem 3.2.7. *Let $u \in C_p(\Lambda)$, $p \geq 1$, and let Assumptions 3.1.2 and 3.2.6(i) hold. The following facts are equivalent.*

(i) For every $(t,\mathbf{x}) \in \Lambda$

$$u(t,\mathbf{x}) \leq \mathbb{E}\Big[u(s,X^{t,\mathbf{x}}) + \int_t^s F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr\Big], \qquad \forall s \in [t,T], \tag{3.2.9}$$

(resp. $\geq$).

(ii) For every $(t,\mathbf{x}) \in \Lambda$ with $t < T$ the process

$$\left(u(s,X^{t,\mathbf{x}}) + \int_t^s F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr\right)_{s\in[t,T]} \tag{3.2.10}$$

is a $(\mathscr{F}_s)_{s\in[t,T]}$-submartingale (resp. supermartingale).

(iii) u is a viscosity subsolution (resp. supersolution) to PPDE (3.2.2).

To prove Theorem 3.2.7 we need some technical results from the optimal stopping theory. For this reason, we first look at them. Let $u,f \in C_p(\Lambda)$ for some $p \geq 1$. Given $s \in [0,T]$, define $\Lambda_s := \{(t,\mathbf{x}) \in \Lambda \colon t \in [0,s]\}$ and consider the optimal stopping problems

$$\Psi_s(t,\mathbf{x}) := \sup_{\tau\in\mathscr{T},\tau\geq t} \mathbb{E}\left[u(\tau\wedge s,X^{t,\mathbf{x}}) + \int_t^{\tau\wedge s} f(r,X^{t,\mathbf{x}})dr\right], \qquad (t,\mathbf{x}) \in \Lambda_s. \tag{3.2.11}$$

Remark 3.2.8. We observe that we need only to consider optimal stopping problems (3.2.11) with deterministic finite horizon $s \in [0,T]$, rather than random finite horizon as in [85].

Lemma 3.2.9. *Let Assumption 3.1.2 hold and let $u,f \in C_p(\Lambda)$ for some $p \geq 1$. Then Ψ_s is lower semicontinuous on Λ_s.*

Proof. Using the fact that $u,f \in C_p(\Lambda)$ for some $p \geq 1$, we see, by Corollary 3.1.6, that the functional

$$\Lambda_s \to \mathbb{R},\ (t,\mathbf{x}) \mapsto \mathbb{E}\left[u((\tau\wedge s)\vee t,X^{t,\mathbf{x}}) + \int_t^{(\tau\wedge s)\vee t} f(r,X^{t,\mathbf{x}})dr\right]$$

is well-defined and continuous for every $\tau \in \mathscr{T}$. We deduce that

$$\begin{aligned}\Psi_s(t,\mathbf{x}) &= \sup_{\tau\in\mathscr{T},\tau\geq t} \mathbb{E}\left[u(\tau\wedge s,X^{t,\mathbf{x}}) + \int_t^{\tau\wedge s} f(r,X^{t,\mathbf{x}})dr\right] \\ &= \sup_{\tau\in\mathscr{T}} \mathbb{E}\left[u((\tau\wedge s)\vee t,X^{t,\mathbf{x}}) + \int_t^{(\tau\wedge s)\vee t} f(r,X^{t,\mathbf{x}})dr\right], \qquad (t,\mathbf{x}) \in \Lambda_s,\end{aligned} \tag{3.2.12}$$

is lower semicontinuous, as it is supremum of continuous functions. ∎

Define the continuation region

$$\mathscr{C}_s := \{(t,\mathbf{x}) \in \Lambda_s : \Psi_s(t,\mathbf{x}) > u(t,\mathbf{x})\}.$$

Due to the continuity of u and the lower semicontinuity of Ψ_s, it follows that $\mathscr{C}_s$ is an open subset of Λ_s. From the general theory of optimal stopping we have the following result.

Theorem 3.2.10. *Let Assumption 3.1.2 hold. Let $s \in [0,T]$, $(t,\mathbf{x}) \in \Lambda_s$ and define the random time $\tau^*_{t,\mathbf{x}} := \inf\{r \in [t,s] : (r, X^{t,\mathbf{x}}) \notin \mathscr{C}_s\}$, with the convention $\inf\emptyset = s$. Then $\tau^*_{t,\mathbf{x}}$ is the first optimal stopping time for problem* (3.2.11).

Proof. First of all, we notice that, since $u, f \in C_p(\Lambda)$ for some $p \geq 1$, we have, for every $(t,\mathbf{x}) \in \Lambda$,

$$\mathbb{E}\left[\sup_{r\in[t,T]} |u(r,X^{t,\mathbf{x}})|\right] < +\infty, \qquad \mathbb{E}\left[\int_t^T |f(r,X^{t,\mathbf{x}})|dr\right] < +\infty. \tag{3.2.13}$$

Now, given $(t,\mathbf{x}) \in \Lambda$, consider the window process

$$[0,T]\times\Omega \to \mathbb{W},\ (r,\omega) \mapsto \mathbb{X}^{t,\mathbf{x}}_r(\omega),$$

where

$$\mathbb{X}^{t,\mathbf{x}}_r(\omega)(\alpha) := \begin{cases} \mathbf{x}_0, & \alpha + r < T, \\ X^{t,\mathbf{x}}_{\alpha+r-T}(\omega), & \alpha + r \geq T, \end{cases} \qquad r, \alpha \in [0,T].$$

Clearly this process is Markovian and we can write the optimal stopping problem in terms of it. Then, the standard theory of optimal stopping of Markovian processes allows to conclude. More precisely, taking into account (3.2.13), we can use [81, Corollary 2.9, Ch. I.1] when $f = 0$; when $f \neq 0$, the integral part of the functional can be reduced to u by adding one dimension to the problem in a standard way (see, e.g., [81, Ch. III.6]). ■

Lemma 3.2.11. *Let Assumption 3.1.2 hold. Let $u, f \in C_p(\Lambda)$ and assume that there exist $s \in [0,T]$ and $(t,\mathbf{x}) \in \Lambda_s$, with $t < s$, such that*

$$u(t,\mathbf{x}) > \mathbb{E}\left[u(s,X^{t,\mathbf{x}}) + \int_t^s f(r,X^{t,\mathbf{x}})dr\right] \qquad (resp.\ <). \tag{3.2.14}$$

Then there exists $(a,\mathbf{y}) \in \Lambda_s$ such that the function φ defined as $\varphi(s,\mathbf{z}) := -\int_0^s f(r,\mathbf{z})dr$ belongs to $\underline{\mathscr{A}}u(a,\mathbf{y})$ (resp. belongs to $\overline{\mathscr{A}}u(a,\mathbf{y})$).

Proof. We prove the claim for the "sub-part". The proof of the "super-part" is completely symmetric.

First, we notice that $\varphi \in C^{1,2}_X(\Lambda)$, as it satisfies (3.2.1) with $\alpha = -f$ and $\beta \equiv 0$. Let us now focus on the maximum property. Consider the optimal stopping problem (3.2.11) and let $\tau^*_{t,\mathbf{x}}$ be the stopping time of Theorem 3.2.10. Due to (3.2.14) we have $\mathbb{P}\{\tau^*_{t,\mathbf{x}} < s\} > 0$. This implies that there exists $(a,\mathbf{y}) \in \Lambda_s \backslash \mathscr{C}_s$. Hence

$$-u(a,\mathbf{y}) = -\Psi_s(a,\mathbf{y}) = \min_{\tau \in \mathscr{T}, \tau \geq a} \mathbb{E}\left[-\int_a^{\tau\wedge s} f(r, X^{a,\mathbf{y}})dr - u(\tau \wedge s, X^{a,\mathbf{y}}) \right].$$

By adding $-\int_0^a f(r,\mathbf{y})dr$ to the above equality, we get the claim ([1]). ■

Proof of Theorem 3.2.7. We prove the claim for the case of the subsolution/submartingale. The other claim can be proved in a completely symmetric way.

(i)⇒(ii) We need to prove that, for every pair of times (s_1, s_2) with $t \leq s_1 \leq s_2 \leq T$,

$$u(s_1, X^{t,\mathbf{x}}) \leq \mathbb{E}\left[u(s_2, X^{t,\mathbf{x}}) + \int_{s_1}^{s_2} F(r, X^{t,\mathbf{x}}, u(r, X^{t,\mathbf{x}}))dr \middle| \mathscr{F}_{s_1} \right]. \tag{3.2.15}$$

Using (3.1.6) and the equality $X^{s_1, X^{t,\mathbf{x}}} = X^{s_1, X^{t,\mathbf{x}}_{\cdot\wedge s_1}}$, we have ([2])

$$\begin{aligned}\mathbb{E}&\left[u(s_2, X^{t,\mathbf{x}}) + \int_{s_1}^{s_2} F(r, X^{t,\mathbf{x}}, u(r, X^{t,\mathbf{x}}))dr \middle| \mathscr{F}_{s_1} \right] \\ &= \mathbb{E}\left[u(s_2, X^{s_1, X^{t,\mathbf{x}}_{\cdot\wedge s_1}}) + \int_{s_1}^{s_2} F(r, X^{s_1, X^{t,\mathbf{x}}_{\cdot\wedge s_1}}, u(r, X^{s_1, X^{t,\mathbf{x}}_{\cdot\wedge s_1}}))dr \middle| \mathscr{F}_{s_1} \right].\end{aligned}$$

Note that $X^{s_1,\mathbf{x}'}$ is independent of $\mathscr{F}_{s_1}$ for each $\mathbf{x}'$ and $X^{t,\mathbf{x}}_{\cdot\wedge s_1}$ is $\mathscr{F}_{s_1}$-measurable. Hence, using [3, Lemma 3.9, p. 55],

$$\begin{aligned}\mathbb{E}&\left[u(s_2, X^{s_1, X^{t,\mathbf{x}}_{\cdot\wedge s_1}}) + \int_{s_1}^{s_2} F(r, X^{s_1, X^{t,\mathbf{x}}_{\cdot\wedge s_1}}, u(r, X^{s_1, X^{t,\mathbf{x}}_{\cdot\wedge s_1}}))dr \middle| \mathscr{F}_{s_1} \right] \\ &= \mathbb{E}\left[u(s_2, X^{s_1,\mathbf{x}'}) + \int_{s_1}^{s_2} F(r, X^{s_1,\mathbf{x}'}, u(r, X^{s_1,\mathbf{x}'}))dr \right]_{|\mathbf{x}' = X^{t,\mathbf{x}}}.\end{aligned}$$

Now we conclude, as *(i)* holds.

(ii)⇒(iii) Let $\varphi \in \underline{\mathscr{A}}(t,\mathbf{x})$. Then, by definition of test function, there exists $\mathrm{H} \in \mathscr{T}$, with $\mathrm{H} > t$, such that

$$(\varphi - u)(t,\mathbf{x}) \geq \mathbb{E}\left[(\varphi - u)\left(\tau \wedge \mathrm{H}, X^{t,\mathbf{x}}\right) \right], \qquad \forall \tau \in \mathscr{T},\ t \leq \tau. \tag{3.2.16}$$

As $\varphi \in C^{1,2}_X(\Lambda)$, we can write

$$\mathbb{E}\left[\varphi(\tau \wedge \mathrm{H}, X^{t,\mathbf{x}}) \right] = \varphi(t,\mathbf{x}) + \mathbb{E}\left[\int_t^{\tau\wedge \mathrm{H}} \mathscr{L}\varphi(s, X^{t,\mathbf{x}})ds \right]. \tag{3.2.17}$$

[1]The role of the localizing stopping time H in the definition of test functions is here played by s.

[2]The flow property of $X^{t,\mathbf{x}}$ used here plays the role of the method based on regular conditional probability used in [33, 34, 35].

By combining (3.2.16)-(3.2.17), we get

$$-\mathbb{E}\left[\int_t^{\tau\wedge \mathrm{H}} \mathscr{L}\varphi(s,X^{t,\mathbf{x}})ds\right] \le u(t,\mathbf{x}) - \mathbb{E}\left[u(\tau\wedge \mathrm{H}, X^{t,\mathbf{x}})\right]$$

or, equivalently,

$$\begin{aligned}&-\mathbb{E}\left[\int_t^{\tau\wedge \mathrm{H}} \left(\mathscr{L}\varphi(s,X^{t,\mathbf{x}}) + F(s,X^{t,\mathbf{x}},u(s,X^{t,\mathbf{x}}))\right)ds\right] \\ &\qquad \le u(t,\mathbf{x}) - \mathbb{E}\left[u(\tau\wedge \mathrm{H}, X^{t,\mathbf{x}}) + \int_t^{\tau\wedge \mathrm{H}} F(s,X^{t,\mathbf{x}},u(s,X^{t,\mathbf{x}}))ds\right].\end{aligned} \tag{3.2.18}$$

Now observe that the submartingale assumption (3.2.10) implies that the right-hand side of (3.2.18) is smaller than 0. Hence, we can conclude by considering in (3.2.18) stopping times of the form $\tau = t+\varepsilon$, with $\varepsilon > 0$, dividing by ε and letting $\varepsilon \to 0^+$.

(iii)⇒(i) Let $\varepsilon > 0$ and consider the function $u_\varepsilon(r,\mathbf{z}) := u(r,\mathbf{z}) + \varepsilon r$. Assume that there exist $\varepsilon > 0$, $(t,\mathbf{x}) \in \Lambda$ and $t < s \le T$ such that

$$u_\varepsilon(t,\mathbf{x}) > \mathbb{E}\left[u_\varepsilon(s,X^{t,\mathbf{x}}) + \int_t^s F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr\right]. \tag{3.2.19}$$

By applying Lemma 3.2.11, we get that φ^ε defined as $\varphi^\varepsilon(r,\mathbf{z}) := \varphi(r,\mathbf{z}) - \varepsilon r$, where φ is defined as in Lemma 3.2.11 taking $f(r,\cdot) := F(r,\cdot,u(r,\cdot))$, belongs to $\underline{\mathscr{A}}u(a,\mathbf{y})$ for some $(a,\mathbf{y})$. By the viscosity subsolution property of u, we then obtain the contradiction $\varepsilon \le 0$. Hence we deduce that

$$u_\varepsilon(t,\mathbf{x}) \le \mathbb{E}\left[u_\varepsilon(s,X^{t,\mathbf{x}})) + \int_t^s F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr\right]. \tag{3.2.20}$$

As ε is arbitrary in the argument above, we can take $\varepsilon \downarrow 0$ in (3.2.20), getting (3.2.9). ■

As a direct consequence of the martingale characterisation in Theorem 3.2.7, we have the following stability result.

Proposition 3.2.12. *Let the assumptions of Proposition 3.1.7 hold. Let Assumption 3.2.6(i) hold and assume that it also holds, for each $n \in \mathbb{N}$, for analogous objects F_n with the same constant L. Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded subset of $C_p(\Lambda)$, for some $p \ge 1$, and let $u \in C_p(\Lambda)$. Assume that the following convergences hold:*

(i) $F_n(s,\cdot,y) \to F(s,\cdot,y)$ uniformly on compact subsets of $\mathbb{W}$ for each $(s,y) \in [0,T]\times\mathbb{R}$.

(ii) $u_n(s,\cdot) \to u(s,\cdot)$ uniformly on compact subsets of $\mathbb{W}$ for each $s \in [0,T]$.

Finally, assume that, for each $n \in \mathbb{N}$, the function u_n is a viscosity subsolution (resp. supersolution) to PPDE (3.2.2) *associated to the coefficients A_n, b_n, σ_n, F_n. Then u is a viscosity subsolution (resp. supersolution) to* (3.2.2) *associated to the coefficients A, b, σ, F.*

Proof. For any $n > 0$ and $(t, \mathbf{x}) \in \Lambda$, it follows from Proposition 3.1.4 that there exists a unique mild solution $X_n^{t,\mathbf{x}}$ to SDE (3.1.1) with coefficients A_n, b_n, σ_n. By Proposition 3.1.7

$$X_n^{t,\mathbf{x}} \overset{n\to\infty}{\longrightarrow} X^{t,\mathbf{x}}, \text{ in } \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}), \ \forall (t,\mathbf{x}) \in \Lambda. \tag{3.2.21}$$

Since u_n is a viscosity subsolution (the supersolution case can be proved in a similar way) to PPDE (3.2.2), from Theorem 3.2.7 we have, for every $(t, \mathbf{x}) \in \Lambda$ with $t < T$,

$$u_n(t,\mathbf{x}) \le \mathbb{E}\left[u_n(s, X_n^{t,\mathbf{x}}) + \int_t^s F_n(r, X_n^{t,\mathbf{x}}, u_n(r, X_n^{t,\mathbf{x}}))dr \right], \qquad \forall s \in [t,T]. \tag{3.2.22}$$

In view of the same theorem, to conclude the proof we just need to prove that, letting $n \to \infty$, the same inequality holds true when u_n, F_n and $X^{(n),t,\mathbf{x}}$ are replaced by u, F and $X^{t,\mathbf{x}}$, respectively.

Clearly the left-hand side of the above inequality tends to $u(t,\mathbf{x})$ as $n \to \infty$. Let us consider the right-hand side. From (3.2.21), up to extracting a subsequence, we have for $\mathbb{P}$-a.e. ω, the convergence $X_n^{t,\mathbf{x}}(\omega) \to X^{t,\mathbf{x}}(\omega)$ in $\mathbb{W}$. Fix such an ω. Then

$$\mathscr{S}(\omega) := \left\{ X_n^{t,\mathbf{x}}(\omega) \right\}_{n\in\mathbb{N}} \bigcup \{ X^{t,\mathbf{x}}(\omega) \}$$

is a compact subset of $\mathbb{W}$. Then, for each $s \in [t,T]$,

$$\begin{aligned} &|u_n(s, X_n^{t,\mathbf{x}}(\omega)) - u(s, X^{t,\mathbf{x}}(\omega))| \\ &\qquad \le \sup_{\mathbf{z}\in\mathscr{S}(\omega)} |u_n(s,\mathbf{z}) - u(s,\mathbf{z})| + |u(s, X_n^{t,\mathbf{x}}(\omega)) - u(s, X^{t,\mathbf{x}}(\omega))| \overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

because $u_n(s,\cdot) \to u(s,\cdot)$ on compact subsets of $\mathbb{W}$, u is continuous, and $X_n^{t,\mathbf{x}}(\omega) \to X^{t,\mathbf{x}}(\omega)$ in $\mathbb{W}$. This shows that $u_n(s, X_n^{t,\mathbf{x}}(\omega)) \to u(s, X^{t,\mathbf{x}}(\omega))$ for every $s \in [t,T]$. Arguing analogously, we have for each $s \in [t,T]$

$$F_n(s, X_n^{t,\mathbf{x}}(\omega), u_n(s, X_n^{t,\mathbf{x}}(\omega))) \overset{n\to\infty}{\longrightarrow} F(s, X^{t,\mathbf{x}}(\omega), u(s, X^{t,\mathbf{x}}(\omega))).$$

Now we can conclude by applying Lemma 3.1.5. Indeed, assuming without loss of generality $t < s$, the hypotheses are verified for $(\Sigma, \mu) = (\Omega \times [t,s], \mathbb{P} \otimes m)$, and

$$\begin{aligned} f_n(\omega, r) &= \frac{1}{s-t} u_n(s, X_n^{t,\mathbf{x}}(\omega)) + F_n(r, X_n^{t,\mathbf{x}}(\omega), u_n(r, X_n^{t,\mathbf{x}}(\omega))), \\ f(\omega, r) &= \frac{1}{s-t} u(s, X^{t,\mathbf{x}}(\omega)) + F(r, X^{t,\mathbf{x}}(\omega), u(r, X^{t,\mathbf{x}}(\omega))), \\ g_n(\omega, r) &= g_n(\omega) = M'(1 + |X_n^{t,\mathbf{x}}(\omega)|_\infty^p), \\ g(\omega, r) &= g(\omega) = M'(1 + |X^{t,\mathbf{x}}(\omega)|_\infty^p), \end{aligned}$$

for a sufficiently large $M' > 0$, since $\int_\Sigma g_n d\mu \to \int_\Sigma g d\mu$ by (3.2.21). ∎

3.2.3 Comparison principle

In this section we provide a comparison result for viscosity sub and supersolutions to (3.2.2), which, through the use of a technical lemma provided here, turns out to be a corollary of the characterisation of Theorem 3.2.7.

Lemma 3.2.13. *Let $Z \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$ and $g\colon [0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$ be such that $g(\cdot,\cdot,z)\in L^1_{\mathscr{P}_T}(\mathbb{R})$, for all $z\in\mathbb{R}$, and, for some constant $C_g>0$,*

$$g(\cdot,\cdot,z)\le C_g|z|, \qquad \forall z\in\mathbb{R}. \tag{3.2.23}$$

Assume that the process

$$\left(Z_s+\int_t^s g(r,\cdot,Z_r)dr\right)_{s\in[t,T]} \tag{3.2.24}$$

is an $\{\mathscr{F}_s\}_{s\in[t,T]}$-submartingale. Then $Z_T\le 0$, $\mathbb{P}$-a.s., implies $Z_t\le 0$, $\mathbb{P}$-a.s..

Proof. Let $Z_T\le 0$ and define

$$\tau^* := \inf\{s\ge t\colon Z_s\le 0\}.$$

Clearly $t\le\tau^*\le T$ and, since Z has continuous trajectories,

$$Z_{\tau^*}\le 0. \tag{3.2.25}$$

Using the submartingale property, we obtain

$$Z_s\le\mathbb{E}\left[Z_{\tau^*\vee s}+\int_s^{\tau^*\vee s} g(r,\cdot,Z_r)dr\,\middle|\,\mathscr{F}_s\right], \qquad \forall s\in[t,T]. \tag{3.2.26}$$

Multiplying (3.2.26) by the $\mathscr{F}_s$ -measurable random variable $\mathbf{1}_{\{s\le\tau^*\}}$, and recalling (3.2.25), we find

$$\begin{aligned}\mathbf{1}_{\{s\le\tau^*\}}Z_s &\le \mathbb{E}\left[\mathbf{1}_{\{s\le\tau^*\}}\left(Z_{\tau^*}+\int_s^{\tau^*} g(r,\cdot,Z_r)dr\right)|\mathscr{F}_s\right]\\ &\le \mathbb{E}\left[\mathbf{1}_{\{s\le\tau^*\}}\int_s^{\tau^*} g(r,\cdot,Z_r)dr\,\middle|\,\mathscr{F}_s\right]\\ &= \mathbb{E}\left[\int_s^T \mathbf{1}_{\{r\le\tau^*\}}g(r,\cdot,Z_r)dr\,|\mathscr{F}_s\right], \qquad \forall s\in[t,T].\end{aligned} \tag{3.2.27}$$

Now from (3.2.23) and the definition of τ^*, we have

$$\mathbf{1}_{\{r\le\tau^*\}}g(r,\cdot,Z_r)\le\mathbf{1}_{\{r\le\tau^*\}}C_g|Z_r| = \mathbf{1}_{\{r\le\tau^*\}}C_gZ_r, \qquad \forall r\in[t,T].$$

Plugging the latter inequality into (3.2.27) and taking the conditional expectations with respect to $\mathscr{F}_t$, we obtain

$$\mathbb{E}\left[\mathbf{1}_{\{s\le\tau^*\}}Z_s|\mathscr{F}_t\right]\le C_g\int_s^T\mathbb{E}[\mathbf{1}_{\{r\le\tau^*\}}Z_r|\mathscr{F}_t]dr, \qquad \forall s\in[t,T]. \tag{3.2.28}$$

Now, setting $h(s) := \mathbb{E}[\mathbf{1}_{\{s\leq\tau^*\}}Z_s|\mathscr{F}_t]$, (3.2.28) becomes

$$h(s) \leq C_g \int_s^T h(r)dr, \qquad \forall s \in [t,T]. \tag{3.2.29}$$

Gronwall's lemma yields $h(s) \leq 0$, for all $s \in [t,T]$. In particular, for $s = T$, we obtain, $\mathbb{P}$-a.s., $Z_t = \mathbb{E}[Z_t|\mathscr{F}_t] = \mathbb{E}[\mathbf{1}_{\{t\leq\tau^*\}}Z_t|\mathscr{F}_t] = h(t) \leq 0$. ■

Corollary 3.2.14 (Comparison principle)**.** *Let Assumptions 3.1.2 and 3.2.6 hold. Let $p \geq 1$ and let $u^{(1)} \in C_p(\Lambda)$ (resp. $u^{(2)} \in C_p(\Lambda)$) be a viscosity subsolution (resp. supersolution) to PPDE* (3.2.2)*. If $u^{(1)}(T,\cdot) \leq u^{(2)}(T,\cdot)$ on $\mathbb{W}$, then $u^{(1)} \leq u^{(2)}$ on Λ.*

Proof. Let $(t,\mathbf{x}) \in \Lambda$. Set

$$g(r,\omega,z) := F(r,X^{t,\mathbf{x}}(\omega), z + u^{(2)}(r,X^{t,\mathbf{x}}(\omega))) - F(r,X^{t,\mathbf{x}}(\omega), u^{(2)}(r,X^{t,\mathbf{x}}(\omega)))$$

and

$$Z_r(\omega) := u^{(1)}(r,X^{t,\mathbf{x}}(\omega)) - u^{(2)}(r,X^{t,\mathbf{x}}(\omega)).$$

Due to Assumption 3.2.6, the map g satisfies the assumptions of Lemma 3.2.13. Moreover, by using the inequality $u^1(T,\cdot) - u^2(T,\cdot) \leq 0$ and the implication *(iii)⇒(ii)* of Theorem 3.2.7, we see that Z satisfies the assumption of Lemma 3.2.13. Then the claim follows as, $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$u^{(1)}(t,X^{t,\mathbf{x}}(\omega)) - u^{(2)}(t,X^{t,\mathbf{x}}(\omega)) = u^{(1)}(t,\mathbf{x}) - u^{(2)}(t,\mathbf{x}).$$ ■

3.2.4 Existence and uniqueness

In this section we provide our main result. We will consider the following assumption on the terminal condition ξ.

Assumption 3.2.15. *$\xi \in C(\mathbb{W},\mathbb{R})$ and, for some $C_\xi > 0$, $p \geq 1$,*

$$|\xi(\mathbf{x})| \leq C_\xi(1 + |\mathbf{x}|_\infty^p), \qquad \forall \mathbf{x} \in \mathbb{W}. \tag{3.2.30}$$

Theorem 3.2.16. *Let Assumption 3.1.2 hold and let Assumptions 3.2.6, 3.2.15, hold with the same growth rate $p \geq 1$. Then PPDE* (3.2.2) *has a unique viscosity solution in the space $C_p(\Lambda)$ satisfying the terminal condition* (3.2.3)*.*

Remark 3.2.17. Uniqueness of viscosity solutions to PPDE (3.2.2) is already implied by the comparison principle in Corollary 3.2.14. However, it will be also a by-product of the fixed-point argument used to prove the existence (Proposition 3.2.18).

Due to Theorem 3.2.7, the proof of the result above reduces to the study of the functional equation

$$u(t,\mathbf{x}) = \mathbb{E}\left[u(s,X^{t,\mathbf{x}}) + \int_t^s F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr\right], \qquad \forall (t,\mathbf{x}) \in \Lambda,\ s \in [t,T], \tag{3.2.31}$$

with terminal condition

$$u(T,\cdot) = \xi(\cdot). \tag{3.2.32}$$

Existence and uniqueness of solutions to the functional equation (3.2.31)-(3.2.32) could be deduced from the theory of backward stochastic differential equations in Hilbert spaces (see Remark 3.2.20 below). However, for the reader's convenience, we provide here a direct proof that does not rely on the theory of BSDEs.

Proposition 3.2.18. *Let Assumption 3.1.2 hold and let Assumptions 3.2.6, 3.2.15 hold with the same growth rate $p \geq 1$. There exists a unique $\hat{u} \in C_p(\Lambda)$ solution to* (3.2.31) *with terminal condition* (3.2.32)*.*

Proof. Step 1. Fix a function $\zeta \in C_p(\Lambda)$, and let $0 \leq a \leq b \leq T$. Consider the nonlinear operator $\Gamma\colon C_p(\Lambda) \to C_p(\Lambda)$, $u \mapsto \Gamma(u)$, defined by

$$\Gamma(u)(t,\mathbf{x}) := \mathbb{E}\left[\zeta(X^{t,\mathbf{x}}) + \mathbf{1}_{[a,b]}(t)\int_t^b F(s,X^{t,\mathbf{x}},u(s,X^{t,\mathbf{x}}))ds\right], \qquad \forall (t,\mathbf{x}) \in \Lambda. \tag{3.2.33}$$

First we note that actually Γ is well defined and maps $C_p(\Lambda)$ into itself: it follows from Assumption 3.2.6 and Corollary 3.1.6.

We now show that there exists $\varepsilon > 0$ such that, if $b - a < \varepsilon$, then Γ is a contraction on $C_p(\Lambda)$, hence admits a unique fixed point. Let $u, v \in C_p(\Lambda)$. Using Assumption 3.2.6*(ii)*,

$$\begin{aligned}
|\Gamma(u)(t,\mathbf{x}) - \Gamma(v)(t,\mathbf{x})| &\leq \mathbb{E}\left[\mathbf{1}_{[a,b]}(t)\int_t^b \left|F(s,X^{t,\mathbf{x}},u(s,X^{t,\mathbf{x}})) - F(s,X^{t,\mathbf{x}},v(s,X^{t,\mathbf{x}})\right| ds\right] \\
&\leq \hat{L}\mathbb{E}\left[\mathbf{1}_{[a,b]}(t)\int_t^b |u(s,X^{t,\mathbf{x}}) - v(s,X^{t,\mathbf{x}})|ds\right] \\
&\leq \hat{L}|u-v|_{C_p(\Lambda)}\mathbb{E}\left[\mathbf{1}_{[a,b]}(t)\int_t^b \left(1 + |X^{t,\mathbf{x}}|_\infty^p\right) ds\right] \\
&\leq \hat{L}|u-v|_{C_p(\Lambda)}\mathbf{1}_{[a,b]}(t)\int_t^b \left(1 + M(1+|\mathbf{x}|_\infty^p)\right) ds \\
&\leq \varepsilon\hat{L}\left(1+M)(1+|\mathbf{x}|_\infty^p\right)|u-v|_{C_p(\Lambda)}
\end{aligned}$$

which yields

$$|\Gamma(u) - \Gamma(v)|_{C_p(\Lambda)} \leq \varepsilon\hat{L}(1+M)|u-v|_{C_p(\Lambda)}. \tag{3.2.34}$$

Thus, Γ is a contraction whenever $\varepsilon < (\hat{L}(1+M))^{-1}$. For such ε, it admits a unique fixed point $\hat{u}$:

$$\hat{u}(t,\mathbf{x}) = \mathbb{E}\left[\zeta(X^{t,\mathbf{x}}) + \mathbf{1}_{[a,b]}(t)\int_t^b F(s,X^{t,\mathbf{x}},\hat{u}(s,X^{t,\mathbf{x}}))ds\right], \qquad \forall (t,\mathbf{x}) \in \Lambda. \tag{3.2.35}$$

Step 2. We prove that, if a function $\hat{u}$ satisfies (3.2.35) for $(t,\mathbf{x}) \in \Lambda$, $a \le t \le b$, then it also satisfies, for every $(t,\mathbf{x}) \in \Lambda$ and every $(s,\mathbf{x}) \in \Lambda$ with $a \le t \le s \le b$, the equality

$$\hat{u}(t,\mathbf{x}) = \mathbb{E}\left[\hat{u}(s,X^{t,\mathbf{x}}) + \int_t^s F(r,X^{t,\mathbf{x}},\hat{u}(r,X^{t,\mathbf{x}}))dr\right], \tag{3.2.36}$$

Indeed, using (3.1.6) and [3, Lemma 3.9, p. 55]

$$\begin{aligned}
\hat{u}(s,X^{t,\mathbf{x}}) &= \mathbb{E}\left[\zeta(X^{s,\mathbf{y}}) + \int_s^b F(r,X^{s,\mathbf{y}},\hat{u}(r,X^{s,\mathbf{y}}))dr\right]_{|\mathbf{y}=X^{t,\mathbf{x}}} \\
&= \mathbb{E}\left[\zeta(X^{s,X^{t,\mathbf{x}}}) + \int_s^b F(r,X^{s,X^{t,\mathbf{x}}},\hat{u}(r,X^{s,X^{t,\mathbf{x}}}))dr \middle| \mathscr{F}_s\right] \\
&= \mathbb{E}\left[\zeta(X^{t,\mathbf{x}}) + \int_s^b F(r,X^{t,\mathbf{x}},\hat{u}(r,X^{t,\mathbf{x}}))dr \middle| \mathscr{F}_s\right].
\end{aligned}$$

Hence

$$\mathbb{E}\left[\hat{u}(s,X^{t,\mathbf{x}})\right] = \mathbb{E}\left[\zeta(X^{t,\mathbf{x}}) + \int_s^b F(r,X^{t,\mathbf{x}},\hat{u}(r,X^{t,\mathbf{x}}))dr\right]$$

and we conclude by (3.2.35).

Step 3. In this step we conclude the proof. Let a,b as in *Step 1* and let us assume, without loss of generality, that $T/(b-a) = n \in \mathbb{N}$. By *Step 1*, there exists a unique $\hat{u}_n \in C_p(\Lambda)$ satisfying

$$\hat{u}_n(t,\mathbf{x}) := \mathbb{E}\left[\xi(X^{t,\mathbf{x}}) + \mathbf{1}_{[T-(b-a),T]}(t)\int_t^T F(s,X^{t,\mathbf{x}},\hat{u}_n(s,X^{t,\mathbf{x}}))ds\right], \qquad \forall (t,\mathbf{x}) \in \Lambda.$$

With a backward recursion argument, using *Step 1*, we can find (uniquely determined) functions $\hat{u}_i \in C_p(\Lambda)$, $i = 1,\dots,n$, such that

$$\hat{u}_{i-1}(t,\mathbf{x}) := \mathbb{E}\left[\hat{u}_i(i(b-a),X^{t,\mathbf{x}}) + \mathbf{1}_{[(i-1)(b-a),i(b-a)]}(t)\int_t^{i(b-a)} F(s,X^{t,\mathbf{x}},\hat{u}_i(s,X^{t,\mathbf{x}}))ds\right],$$

for all $(t,\mathbf{x}) \in \Lambda$. Now define $\hat{u}(t,\cdot) = \sum_{1\le i\le n} \mathbf{1}_{[(i-1)(b-a),i(b-a))}(t)\hat{u}_i(t,\cdot) + \mathbf{1}_{\{T\}}(t)\xi(\cdot)$. To conclude the existence, we use recursively *Step 2* to prove that $\hat{u}$ satisfies (3.2.31) with terminal condition (3.2.32).

Uniqueness follows from local uniqueness. Indeed, let $\hat{u},\hat{v}$ be two solutions in $C_p(\Lambda)$ to (3.2.31)-(3.2.32) and define

$$T^* := \sup\left\{t \in [0,T] \colon \sup_{\mathbf{x}\in\mathbb{W}} |\hat{u}(t,\mathbf{x}) - \hat{v}(t,\mathbf{x})| > 0\right\},$$

with the convention $\sup\emptyset = 0$. By continuity of $\hat{u},\hat{v}$, and since $\hat{u}(T,\cdot) = \hat{v}(T,\cdot)$, we have $\hat{u}(t,\cdot) \equiv \hat{v}(t,\cdot)$ for every $t \in [T^*,T]$. If $T^* = 0$, we have done. Assume, by contradiction, that $T^* > 0$. As done in *Step 2*, one can prove that both $\hat{u}$ and $\hat{v}$ satisfy (3.2.36). In particular, if we consider the definition (3.2.33) with $\zeta(\cdot) = \hat{u}(T^*,\cdot) = \hat{v}(T^*,\cdot)$, $a = 0 \vee (T^* - \varepsilon)$, $b = T^*$, where $\varepsilon < (\hat{L}(1+M))^{-1}$, we have

$$\Gamma(\hat{u})(t,\mathbf{x}) = \hat{u}(t,\mathbf{x}) \quad \text{and} \quad \Gamma(\hat{v})(t,\mathbf{x}) = \hat{v}(t,\mathbf{x}), \qquad \forall (t,\mathbf{x}) \in \Lambda,\ \forall t \in [T^* - \varepsilon T^*].$$

Then, recalling (3.2.34), we get a contradiction and conclude. ■

Remark 3.2.19. If there exists a modulus of continuity w_F such that

$$|F(t,\mathbf{x},y)-F(t',\mathbf{x}',y')| \le w_F(\mathbf{d}_\infty((t,\mathbf{x}),(t',\mathbf{x}'))) + \hat{L}|y-y'|,$$

then Γ defined in (3.2.33) maps $UC(\Lambda)$ into itself. Hence, if ξ is uniformly continuous and the condition above on F holds, then the solution $\hat{u}$ belongs to $UC(\Lambda)$.

Remark 3.2.20. Another way to solve the functional equation (3.2.31) is to consider the following backward stochastic differential equation

$$Y_s = \xi(X^{t,\mathbf{x}}) + \int_s^T F(r,X^{t,\mathbf{x}},Y_r)dr - \int_s^T Z_r dW_r, \qquad s\in[t,T]. \tag{3.2.37}$$

Then, it follows from Proposition 4.3 in [47] that, under Assumptions 3.1.2, 3.2.6, and 3.2.15 (with the same growth rate $p\ge 1$), for any $(t,\mathbf{x})\in\Lambda$ there exists a unique solution $(Y_s^{t,\mathbf{x}},Z_s^{t,\mathbf{x}})_{s\in[0,T]} \in \mathscr{L}^2_{\mathscr{P}_T}(C([0,T],\mathbb{R}))\times L^2_{\mathscr{P}_T}(H^*)$ to equation (3.2.37), which can be viewed as a Sobolev solution to PPDE (3.2.2) (see e.g. [4]). We also know that $Y_t^{t,\mathbf{x}}$ is constant, then we may define

$$\hat{u}(t,\mathbf{x}) := Y_t^{t,\mathbf{x}} = \mathbb{E}\left[\xi(X^{t,\mathbf{x}}) + \int_t^T F(s,X^{t,\mathbf{x}},Y_s^{t,\mathbf{x}})ds\right], \tag{3.2.38}$$

for all $(t,\mathbf{x})\in\Lambda$. It can be shown, using the flow property of $X^{t,\mathbf{x}}$ and the uniqueness of the backward equation (3.2.37), that $Y_s^{t,\mathbf{x}} = \hat{u}(s,X^{t,\mathbf{x}})$ for all $s\in[t,T]$, $\mathbb{P}$-a.e.. Moreover, using the backward equation (3.2.37), the regularity of ξ and F, and the flow property of $X^{t,\mathbf{x}}$ with respect to $(t,\mathbf{x})$, we can prove that $\hat{u}\in C_p(\Lambda)$. This implies that $\hat{u}$ solves the functional equation (3.2.31) with terminal condition (3.2.32), and it is the same function of Proposition 3.2.18. Viceversa, we can also prove an existence and uniqueness result for the backward equation (3.2.37) if we know that there exists a unique solution $\hat{u}\in C_p(\Lambda)$ to the functional equation (3.2.31) with terminal condition (3.2.32). In conclusion, $\hat{u}$ admits a nonlinear Feynman-Kac representation formula through a non-Markovian forward-backward stochastic differential equation given by:

$$\begin{cases} X_s = S_{s-t}\mathbf{x}_t + \int_t^s S_{s-r}b(r,X)dr + \int_t^s S_{s-r}\sigma(r,X)dW_r, & s\in[t,T], \\ X_s = \mathbf{x}_s, & s\in[0,t), \\ Y_s = \xi(X) + \int_s^T F(r,X,Y_r)dr - \int_s^T Z_r dW_r, & s\in[0,T]. \end{cases}$$

3.3 The Markovian case

In the Markovian case, i.e., when all data depend only on the present, infinite-dimensional PDEs of type (3.2.2)-(3.2.3) have been studied from the point of view of viscosity solutions starting from [70, 72, 73]. In this section we compare the results of the literature with the statement of our main Theorem 3.2.7 in this Markovian framework.

Hence, let us assume that the data b, σ, F, ξ satisfy all the assumptions used in the previous sections and, moreover, that they depend only on $x = \mathbf{x}_t$, instead of the whole path $\mathbf{x}$. SDE (3.1.1) is no more path-dependent and takes the following form:

$$\begin{cases} dX_s = AX_s ds + b(s,X_s)ds + \sigma(s,X_s)dW_s, & s \in [t,T], \\ X_t = x \in H. \end{cases} \tag{3.3.1}$$

Accordingly, (3.0.1) becomes a non path-dependent ([3]) second order parabolic PDE in the Hilbert space H, which is formally written for $(t,x) \in [0,T) \times D(A)$ as ([4])

$$\begin{aligned} -\partial_t u(t,x) - \frac{1}{2}\operatorname{Tr}\left[\sigma(t,x)\sigma^*(t,x)D^2u(t,x)\right] - \langle Ax, Du(t,x)\rangle - \\ - \langle b(t,x), Du(t,x)\rangle - F(t,x,u(t,x)) = 0. \end{aligned} \tag{3.3.2}$$

In such Markovian framework, the results of the previous sections still hold. Indeed, defining viscosity solutions to (3.3.2) as in Definition 3.2.2, with x in place of $\mathbf{x}$, we know from Theorem 3.2.16 that there exists a unique viscosity solution $\hat{u}$ to (3.3.2) and that it admits the probabilistic representation formula (3.2.38) of Remark 3.2.20, with x in place of $\mathbf{x}$.

On the other hand, equations like (3.3.2) have been studied in the literature, by means of what we call here the "standard" viscosity solution approach. This is performed, in the spirit of the finite-dimensional case, by computing the terms of (3.3.2) on smooth test functions suitably defined and using the method of doubling variables to prove the comparison. Such "standard" approach in infinite dimension has been first introduced in [70, 72, 73] and then developed in various papers (see e.g. [52, 54, 53, 61, 95]).

To compare our results with the ones obtained in the literature quoted above, we first introduce a concept of classical solution to (3.3.2).

First of all, observe that (3.3.2) is well defined only in $[0,T) \times D(A)$. In order to formally extend this set of definition we can consider the operator A^*, adjoint of A, defined on $D(A^*) \subset H$, and express the term containing Ax in (3.3.2) by writing

$$\langle Ax, Du(t,x)\rangle = \langle x, A^* Du(t,x)\rangle,$$

which is well defined in $[0,T) \times H$ provided that $Du \in D(A^*)$. Hence, to define classical solutions to such equation, we define the operator $\mathscr{L}_1$ as follows: the domain of definition

[3]In this section we drop the final condition ξ. But it is important to notice that the PDE must be considered path-dependent even if only ξ depends on the past, while b, σ, F do not.

[4]Notice that the time derivative $\partial_t u(t,x)$ here appearing can denote equivalently the Dupire time-derivative defined in the Appendix of the chapter or the standard partial right time-derivative, as in this Markovian case they coincide each other on $[0,T)$.

of the solution is ($UC^{1,2}([0,T]\times H)$ denotes the space of maps $\psi\colon [0,T]\times H \to \mathbb{R}$ which are uniformly continuous together with their first time Fréchet derivative and their first and second spatial Fréchet derivatives)

$$D(\mathscr{L}_1) = \Big\{ \psi \in UC^{1,2}([0,T]\times H)\colon \text{ the maps } (t,x) \mapsto \langle x, A^* D\psi(t,x)\rangle, A^* D\psi(t,x), \frac{1}{2}\mathrm{Tr}\left[\sigma(t,x)\sigma^*(t,x)D^2\psi(t,x)\right], \text{ belong to } UC([0,T]\times H) \Big\},$$

and

$$\mathscr{L}_1\psi(t,x) = \partial_t\psi(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma(t,x)\sigma^*(t,x)D^2\psi(t,x)\right] + \langle x, A^*D\psi(t,x)\rangle + \langle b(t,x), D\psi(t,x)\rangle.$$

Then we say that u is a classical solution to (3.3.2) if $u \in D(\mathscr{L}_1)$ and satisfies

$$-\mathscr{L}_1 u(t,x) - F(t,x,u(t,x)) = 0, \qquad \forall (t,x) \in [0,T)\times H. \tag{3.3.3}$$

The standard definition of viscosity subsolution (supersolution) for (3.3.2) says roughly that, at any given $(t,x)\in[0,T)\times H$, the equation must be satisfied with $\le$ ($\ge$), when we substitute to the derivatives of $u(t,x)$ the derivatives of $\varphi(t,x)$, where φ is a suitably chosen test function.

Clearly, in this context test functions should be chosen in such a way that all terms of (3.3.2) have classical sense. Hence, their regularity must be substantially the one required for classical solutions, i.e., roughly, $\varphi \in D(\mathscr{L}_1)$. This regularity is very demanding, much more than the one required in the finite dimensional case: requiring that $D\varphi \in D(A^*)$ and the finite trace condition in the second order term strongly restricts the set of test functions. In this way the proof of the existence has not a greater structural difficulty with respect to the finite-dimensional case, but the uniqueness, which is based on a delicate construction of suitable test functions, becomes much harder.

To be more explicit, let us first give a definition of "naive" viscosity solution to (3.3.2).

Definition 3.3.1.

(i) An upper semicontinuous function $u\colon [0,T]\times H \to \mathbb{R}$ is called a naive viscosity subsolution *to* (3.3.2) *if*

$$-\mathscr{L}_1\varphi(t,x) - F(t,x,u(t,x)) \le 0,$$

for any $(t,x)\in[0,T)\times H$ and any function $\varphi \in D(\mathscr{L}_1)$ such that $\varphi - u$ has a local minimum at (t,x).

(ii) A lower semicontinuous function $u\colon [0,T]\times H \to \mathbb{R}$ is called a naive viscosity supersolution *to* (3.3.2) *if*

$$-\mathscr{L}_1\varphi(t,x) - F(t,x,u(t,x)) \ge 0,$$

for any $(t,x)\in[0,T)\times H$ and any function $\varphi \in D(\mathscr{L}_1)$ such that $\varphi - u$ has a local maximum at (t,x).

(iii) A continuous function $u:[0,T]\times H\to\mathbb{R}$ *is called a* naive viscosity solution *to* (3.3.2) *if it is both a viscosity subsolution and a viscosity supersolution.*

If we adopt this definition, it is clear that the set of test functions used is strictly included in the one used in our Definition 3.2.2. Hence, if a function is a viscosity solution according to Definition 3.2.2, it must also be a viscosity solution according to Definition 3.3.1, while the opposite is, a priori, not true. Hence, if one were able to prove a uniqueness result for viscosity solution according to Definition 3.3.1, such a result would be more powerful than our existence and uniqueness Theorem 3.2.16. However, the technique used to prove uniqueness in finite dimension does not work with such a definition and there are no general uniqueness results with this definition.

In the literature concerning "standard" viscosity solutions in infinite dimension this problem has been overcome by introducing suitable restrictions on the family of equations and adding an ad hoc radial term g to each test function φ. We explain more in detail what is needed to apply such techniques to our equation (3.3.2); then we give a result obtained with such technique and compare it with our previous results.

To start, it is useful to rewrite equation (3.3.2) as follows:

$$-\partial_t u(t,x)-\langle x,A^*Du(t,x)\rangle-Lu(t,x)-F(t,x,u(t,x))=0,\quad \text{on } [0,T)\times H, \tag{3.3.4}$$

with, for any $u\in C^{1,2}([0,T]\times H)$ in the sense of Fréchet,

$$Lu(t,x)=\langle b(t,x),Du(t,x)\rangle+\frac{1}{2}\operatorname{Tr}\left[\sigma(t,x)\sigma^*(t,x)D^2u(t,x)\right].$$

To account for the "difficult" term $\langle x,A^*Du(t,x)\rangle$ we impose the following assumption.

Assumption 3.3.2. *The operator A is a maximal dissipative operator in H.*

Under Assumptions 3.1.2 and 3.3.2, it follows from [86] that there exists a symmetric, strictly positive, and bounded operator B on H such that A^*B is a bounded operator on H and

$$-A^*B+c_0B\geq 0,$$

for some $c_0>0$.

Definition 3.3.3 (B-convergence, B-upper/-lower semicontinuity, B-continuity)**.** *Let $x\in H$ and let $\{x_n\}_{n\in\mathbb{N}}\subset H$ be a sequence. We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is B-convergent to x, if $x_n\rightharpoonup x$ and $Bx_n\to Bx$ in H.*

A function $u:[0,T]\times H\to\mathbb{R}$ is said to be B-upper semicontinuous (resp. B-lower semicontinuous) if for any $\{t_n\}_{n\in\mathbb{N}}\subset[0,T]$ convergent to $t\in[0,T]$, and for any $\{x_n\}_{n\in\mathbb{N}}\subset H$ B-convergent to $x\in H$, we have

$$\limsup_{n\to\infty}u(t_n,x_n)\leq u(t,x)\qquad \textit{(resp. } \liminf_{n\to\infty}u(t_n,x_n)\geq u(t,x)\textit{)}.$$

Finally, u is B-continuous if it is B-upper and B-lower semicontinuous.

We consider two classes of smooth (test) functions:

(C1) (the "smooth" part) $\varphi \in C^{1,2}([0,T]\times H)$, $D\varphi$ is $D(A^*)$-valued, $\partial_t\varphi$, $A^*D\varphi$, and $D^2\varphi$ are uniformly continuous on $[0,T]\times H$, and φ is B-lower semicontinuous.

(C2) (the "radial" part) $g \in C^{1,2}([0,T]\times \mathbb{R})$ and, for every $t\in[0,T]$, the function $g(t,\cdot)$ is even on $\mathbb{R}$ and nondecreasing on $[0,\infty)$.

Definition 3.3.4.

(i) A B-upper semicontinuous function $u\colon[0,T]\times H\to\mathbb{R}$, which is bounded on bounded sets, is called a viscosity subsolution *to* (3.3.4) *if*

$$-\partial_t(\varphi+g)(t,x)-\langle x,A^*D\varphi(t,x)\rangle-L(\varphi+g)(t,x)-F(t,x,u(t,x))\le 0,$$

for any $(t,x)\in[0,T)\times H$ and any pair of functions (φ,g) belonging, respectively, to the classes (C1)-(C2) *above and such that $\varphi+g-u$ has a local minimum at (t,x).*

(ii) A B-lower semicontinuous function $u\colon[0,T]\times H\to\mathbb{R}$, which is bounded on bounded sets, is called a viscosity supersolution *to* (3.3.4) *if*

$$-\partial_t(\varphi-g)(t,x)-\langle x,A^*D\varphi(t,x)\rangle-L(\varphi-g)(t,x)-F(t,x,u(t,x))\ge 0,$$

for any $(t,x)\in[0,T)\times H$ and any pair of functions (φ,g) belonging, respectively, to the classes (C1)-(C2) *above and such that $\varphi-g-u$ has a local maximum at (t,x).*

(iii) A function $u\colon[0,T]\times H\to\mathbb{R}$ is called a viscosity solution *to* (3.3.4) *if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 3.3.5. The radial function g belonging to the class (C2) introduced in Definition 3.3.4 plays the role of cut-off function and is needed to produce, together with the B-continuity property, local/global minima and maxima of $\varphi+g-u$ and $\varphi-g-u$, respectively. However, the introduction of the radial function forces to impose Assumption 3.3.2 to get rid of the term $\langle Ax, Dg(t,x)\rangle$ which would come out from the gradient of g.

Radial test functions could also be included in our Definition 3.2.2 when A is a maximal monotone operator without compromising the existence result (but note that it would be redundant including them in our definition, as they are not needed to prove uniqueness in Theorem 3.2.16). In this case, our Definition 3.2.2 would be stronger than Definition 3.3.4 in the sense that a viscosity subsolution (supersolution) in the sense of Definition 3.2.2 must be necessarily also a viscosity subsolution (supersolution) according to Definition 3.3.4. Indeed, a test function in the sense of Definition 3.3.4 would be also a test function in the sense of Definition 3.2.2. □

We can now state a comparison theorem and an existence result for equation (3.3.4). First, we need to introduce some notations. Let H_{-1} be the completion of H with respect to the norm

$$|x|_{-1}^2 := \langle Bx, x\rangle.$$

Notice that H_{-1} is a Hilbert space with the inner product

$$\langle x, x'\rangle_{-1} := \big\langle B^{1/2}x, B^{1/2}x'\big\rangle.$$

Let now $\{e_1, e_2, \ldots\}$ be an orthonormal basis in H_{-1} made of elements of H. For $N > 2$ we denote $H_N = \text{span}\{e_1, \ldots, e_N\}$. Let $P_N \colon H_{-1} \to H_{-1}$ be the orthogonal projection onto H_N and denote $P_N^\perp = I - P_N$.

Theorem 3.3.6. *Let Assumptions 3.1.2, 3.2.30, 3.2.6, and 3.3.2 hold. In addition, let us impose the following assumptions.*

(i) *The map $y \mapsto F(t,x,y)$ is nonincreasing on $\mathbb{R}$, for any $(t,x) \in [0,T] \times H$.*

(ii) *There exists a positive constant $L_{b,\sigma}$ and a modulus of continuity $\omega_{\xi,F}$ such that*

$$|b(t,x) - b(t,x')| + |\sigma(t,x) - \sigma(t,x')|_2 \le L_{b,\sigma}|x - x'|_{-1},$$
$$|\xi(x) - \xi(x')| + |F(t,x,y) - F(t,x',y)| \le \omega_{\xi,F}(|x - x'|_{-1}),$$

for all $t \in [0,T]$, $x, x' \in H$, and $y \in \mathbb{R}$.

(iii) *$\sigma(t,x) \in L_2(H)$ for every $(t,x) \in \Lambda$ and the following limit holds*

$$\lim_{N\to\infty} \text{Tr}\big[\sigma(t,x)\sigma^*(t,x)BP_N^\perp\big] = 0, \qquad \forall (t,x) \in [0,T] \times H.$$

Then, the following statements hold true.

(a) *Let u (resp. v) be a viscosity subsolution (resp. supersolution) to* (3.3.4) *satisfying a polynomial growth condition. If $u(T,\cdot) \le v(T,\cdot)$, then $u \le v$ on* $[0,T] \times H$.

(b) *Assume that $F = F(t,x)$ does not depend on y. Then, there exists a unique viscosity solution $\hat{u}$ to equation* (3.3.4) *satisfying the terminal condition $\hat{u}(T,\cdot) = \xi(\cdot)$ and it admits the probabilistic representation* ([5])

$$\hat{u}(t,x) = \mathbb{E}\left[\xi(X_T^{t,x}) + \int_t^T F(s, X_s^{t,x})ds\right], \qquad \forall (t,x) \in [0,T] \times H.$$

Proof. See [95, Th. 3.2] ([6]). ■

[5] When H is finite dimensional, the probabilistic representation formula (3.2.38) provides the unique "standard" viscosity solution to (3.3.4) also when F depends on y, see [78].

[6] Actually, under the assumption that u, v are bounded in part *(a)*, but this assumption can be relaxed to the polynomial growth case.

Notice that Assumption *(i)* of Theorem 3.3.6 is actually redundant in the framework of Assumption 3.2.6, due to the uniform Lipschitz property of F with respect to the last argument required therein. Indeed, let u (resp. v) be a viscosity subsolution (resp. supersolution) to (3.3.4) satisfying $u(T,\cdot) \leq \xi(\cdot)$ (resp. $v(T,\cdot) \geq \xi(\cdot)$). Our aim is to prove point *(a)* of Theorem 3.3.6, that is that $u \leq v$ on $[0,T]\times H$, without imposing Assumption *(i)* of the same theorem. To this end, set $\tilde{u}(t,x) := e^{\hat{L}t}u(t,x)$ and $\tilde{v}(t,x) := e^{\hat{L}t}v(t,x)$, for all $(t,x) \in [0,T]\times H$, where $\hat{L}$ is the constant in Assumption 3.2.6*(ii)*. Then, by standard arguments (see, e.g., point (i) of Remark 3.9 in [33]), we can prove that $\tilde{u}$ (resp. $\tilde{v}$) is a viscosity subsolution (resp. supersolution) to (3.3.4) with $\tilde{F}(t,x,y) = -\hat{L}y + e^{\hat{L}t}F(t,x,e^{-\hat{L}t}y)$ in place of F. The Lipschitz property of F implies that the map $y \mapsto \tilde{F}(t,x,y)$ is nonincreasing, therefore we can apply point *(a)* of Theorem 3.3.6 to $\tilde{u}$ and $\tilde{v}$, which yields $\tilde{u} \leq \tilde{v}$ on $[0,T]\times H$. Then $u \leq v$ on $[0,T]\times H$ follows.

Assumption *(ii)* in Theorem 3.3.6 is needed to exploit the B-continuity. Indeed the requirement of B-continuity on the sub(super)solutions is needed to generate maxima and minima in the proof of comparison. In this way one is obliged to assume these stronger conditions on the coefficients to ensure the existence of solutions (see [95]).

Assumption *(iii)* in Theorem 3.3.6 is needed since, to prove uniqueness, one has to use the so-called Ishii's lemma which allows to perform the procedure of doubling variables. Up to now Ishii's lemma is known to hold only in finite dimension, so the proof is performed through finite dimensional approximations: the condition *(iii)* ensures the convergence of such approximations.

We can conclude that, under the assumptions of Theorem 3.3.6*(b)*, the two definitions of viscosity solution select the same solution. However, adopting our Definition 3.2.2 requires weaker assumptions to prove that the function $\hat{u}$ in (3.2.38) is the unique viscosity solution. First, the map σ does not need to satisfy assumptions *(iii)* (which, in the constant σ case, would imply that $\sigma\sigma^*$ is a nuclear operator, hence reducing the applicability of the theory) as the proof of uniqueness does not require the use of Ishii's lemma on the corresponding finite-dimensional approximations. Secondly, the coefficients b, σ, F, and ξ do not need to be B-continuous with respect to x, as no local compactness is needed to produce local max/min in our sense. Finally, the operator A does not need to be maximal monotone, as radial test functions are not needed to produce local max/min in our sense.

Roughly speaking, we can say that the definition here adopted allows to cover more general cases since the relation with the PDE is different in the following sense: the PDE is tested in analytical sense, but over test functions which satisfy the min/max condition only in a probabilistic sense and only when composed with the process $X^{t,x}$; indeed minimum (maximum) of $\varphi - u$ is not pointwise in a neighborhood of (t,x), but only in mean when composed with the process $X^{t,x}$.

Appendix

Pathwise derivatives

The class of test functions used to define viscosity solutions for path-dependent PDEs has evolved from [33] and [34] to the recent work [85]. In Definition 3.2.1, which is inspired by [85], there is no more reference to the so-called pathwise (or functional, or Dupire) derivatives (for which we refer to [31] and also to [13, 12, 14, 16]), which are instead adopted in [33] and [34] (actually in [34] only the pathwise time derivative is used). This allows to go directly to the definition of viscosity solution, without pausing on the definition of pathwise derivatives, and, more generally, on recalling tools from functional Itō calculus. However, the class of test functions used in [33] or [34] has the advantage to be defined in a similar way to $C^{1,2}$, the standard class of smooth real-valued functions. In this case the object $\mathscr{L}u$ of (3.2.1), which for us is only abstract, can be expressed in terms of the pathwise derivatives, as in the non path-dependent case, where $\mathscr{L}$ corresponds to a parabolic operator and can be written by means of time and spatial derivatives.

For this reason, in order to better understand Definition 3.2.1 and the notation $\mathscr{L}u$, we now define a subset of test functions $\mathscr{C}^{1,2}_X(\Lambda) \subset C^{1,2}_X(\Lambda)$ which admit the pathwise derivatives we are going to define. Here we follow [34], generalizing it to the present infinite dimensional setting.

Definition 3.A.7. *Given $u \in C_p(\Lambda)$, for some $p \geq 1$, we define the* pathwise time derivative *of u at $(t,\mathbf{x}) \in \Lambda$ as follows:*

$$\begin{cases} \partial_t u(s,\mathbf{x}) := \lim\limits_{h\to 0^+} \dfrac{u(s+h,\mathbf{x}_{\cdot\wedge s}) - u(s,\mathbf{x})}{h}, & s \in [0,T), \\ \partial_t u(T,\mathbf{x}) := \lim\limits_{s\to T^-} \partial_t u(s,\mathbf{x}), & s = T, \end{cases}$$

when these limits exist.

In the following definition A^* is the adjoint operator of A, defined on $D(A^*) \subset H$.

Definition 3.A.8. *Denote by $S(H)$ the Banach space of bounded and self-adjoint operators in the Hilbert space H endowed with the operator norm, and let $D(A^*)$ be endowed with the graph norm, which renders it a Hilbert space. We say that $u \in C_p(\Lambda)$, for some $p \geq 1$, belongs to $\mathscr{C}^{1,2}_X(\Lambda)$ if:*

(i) there exists $\partial_t u$ in Λ in the sense of Definition 3.A.7 and it belongs to $C_p(\Lambda)$;

(ii) there exist two maps $\partial_{\mathbf{x}} u \in C_p(\Lambda, D(A^*))$ and $\partial^2_{\mathbf{xx}} u \in C_p(\Lambda, S(H))$ such that

$$\mathrm{Tr}\left[\sigma\sigma^* \partial^2_{\mathbf{xx}} u\right] < \infty$$

in Λ *and the following* functional Itō's formula *holds for all* $(t,\mathbf{x}) \in \Lambda$ *and* $s \in [t,T]$:

$$du(s,X^{t,\mathbf{x}}) = \mathscr{L}u(s,X^{t,\mathbf{x}})ds + \langle \sigma^*(s,X^{t,\mathbf{x}})\partial_{\mathbf{x}}u(s,X^{t,\mathbf{x}}), dW_s\rangle, \tag{3.A.5}$$

where, for $(s,\mathbf{y}) \in \Lambda$,

$$\begin{aligned}\mathscr{L}u(s,\mathbf{y}) := \partial_t u(s,\mathbf{y}) + \langle \mathbf{y}_t, A^*\partial_{\mathbf{x}}u(s,\mathbf{y})\rangle + \langle b(s,\mathbf{y}), \partial_{\mathbf{x}}u(s,\mathbf{y})\rangle \\ + \frac{1}{2}\mathrm{Tr}\big[\sigma(s,\mathbf{y})\sigma^*(s,\mathbf{y})\partial^2_{\mathbf{xx}}u(s,\mathbf{y})\big].\end{aligned} \tag{3.A.6}$$

Given (i)-(ii) above, we call $\partial_{\mathbf{x}}u$ *a* pathwise first order spatial derivative *of* u *with respect to* X *and* $\partial^2_{\mathbf{xx}}u$ *a* pathwise second order spatial derivative *of* u *with respect to* X *and denote*

$$\partial^2_X u := \left\{(\partial_{\mathbf{x}}u, \partial^2_{\mathbf{xx}}u) \in C_p(\Lambda, D(A^*)) \times C_p(\Lambda, S(H)) \colon \partial_{\mathbf{x}}u \textit{ and } \partial^2_{\mathbf{xx}}u \textit{ as in (ii)} \right\}.$$

Notice that, given $u \in \mathscr{C}^{1,2}_X(\Lambda)$ and $(t,\mathbf{x}) \in \Lambda$, the objects $\partial_{\mathbf{x}}u$ and $\partial^2_{\mathbf{xx}}u$ are not necessarily uniquely determined, while $\mathscr{L}u$ defined as in (3.A.6) and $\sigma^*\partial_{\mathbf{x}}u$ are uniquely determined. Indeed, this can be shown by identifying the finite variation part and the Brownian part in the functional Itō's formula (3.A.5). Moreover, (3.2.1) is satisfied with

$$\alpha(t,\mathbf{x}) = \partial_t u(t,\mathbf{x}) + \langle \mathbf{x}_t, A^*\partial_{\mathbf{x}}u(t,\mathbf{x})\rangle + \langle b(t,\mathbf{x}), \partial_{\mathbf{x}}u(t,\mathbf{x})\rangle + \frac{1}{2}\mathrm{Tr}\big[\sigma(t,\mathbf{x})\sigma^*(t,\mathbf{x})\partial^2_{\mathbf{xx}}u(t,\mathbf{x})\big],$$

$$\beta(t,\mathbf{x}) = \sigma^*(t,\mathbf{x})\partial_{\mathbf{x}}u(t,\mathbf{x}).$$

In particular, $\mathscr{C}^{1,2}_X(\Lambda) \subset C^{1,2}_X(\Lambda)$ and the notation $\mathscr{L}u := \alpha$ introduced in Subsection 3.2.1 becomes clear.

Chapter 4

Functional Itō calculus in Hilbert spaces and application to PPDEs

The present chapter extends to infinite dimensional spaces the so called functional Itō calculus, so far developed in finite-dimensional spaces, and some of its applications.

In [31] the first ideas for a functional Itō calculus were presented for one-dimensional continuous semimartingales, by introducing suitable notions of time/space derivatives which reveal to be adequate for dealing with non-anticipative functionals. In that paper, a functional Itō's formula is provided and then employed to represent solutions to backward Kolmogorov equations with path-dependent terminal value. This allows to obtain an explicit representation of the stochastic integrand in the martingale representation theorem, when the martingale is closed by a functional of the process solving the SDE associated to the Kolmogorov equation. In [12, 13, 14] these ideas are furtherly developed and generalized. In [12] the functional Itō's formula is proved for a large class of finite-dimensional càdlàg processes, including semimartingales and Dirichlet processes, and for functionals which can depend on the quadratic variation. In [14] the notion of vertical derivative is extended to square integrable continuous martingales and it is proved that it coincides with the stochastic integrand in the martingale representation theorem.

Functional Itō calculus in finite dimension can be also viewed as an application to the spaces of continuous/càdlàg functions of stochastic calculus in Banach spaces ([26, 27, 28, 29, 43]). In [29] the notion of χ-quadratic variation is introduced for Banach space-valued processes (not necessarily semimartingales) and the related Itō's formula is discussed. This general framework finds application to "window" processes in $C([-T,0],\mathbb{R}^n)$, whose values, at each time $t \in [0,T]$, is essentially the path up to time t of an $\mathbb{R}^n$-valued continuous process. When applied to window processes, such Itō's formula allows to derive a Clark-Ocone type representation formula by recurring to the solution to a path-dependent Kolmogorov equation. In [43] finite dimensional Itō processes X

with constant diffusion coefficient and path-dependent drift are considered. By embedding the dynamics of X into a Banach space of functions $[-T,0] \to \mathbb{R}^n$, it is proved that the Feynman-Kac formula provides a solution to the path-dependent backward Kolmogorov equation associated to X, with a non-path-dependent terminal value.

Another approach to path-dependent functionals and path dependent stochastic systems is represented by the embedding in infinite dimensional Hilbert spaces. Indeed, when the dependence on the history is sufficiently regular — precisely regular with respect to a L^2 norm — a representation in the Hilbert space of the form $\mathbb{R} \times L^2$ is possible. This approach goes back to [10] and was further developed in other papers ([40, 42, 51]). With this approach, the very well-developed theory of stochastic calculus in Hilbert space ([24]) can be applied. On the other hand, it leaves out some important classes of problems, in particular all those where the dependence on the history involves pointwise evaluations at past times.

Up to our knowledge, so far the functional Itō calculus has been developed only in finite dimensional spaces. We generalize it to infinite dimension as follows. Consider two real separable Hilbert spaces U,H and a U-valued cylindrical Wiener process W. Given $T>0$, denote by $\mathbb{W}$ the space $C([0,T],H)$ of continuous functions $[0,T] \to H$. Given $t \in [0,T]$ and $\mathbf{x} \in \mathbb{W}$, consider the process

$$X_s^{t,\mathbf{x}} = \mathbf{x}_{t\wedge\cdot} + \int_t^{t\vee s} b_r dr + \int_t^{t\vee s} \Phi_r dW_r \qquad s \in [0,T],$$

where

$$\mathbf{x}_{t\wedge\cdot}(s) := \begin{cases} \mathbf{x}(s) & s \in [0,t] \\ \mathbf{x}(t) & s \in (t,T], \end{cases}$$

b is a square-integrable H-valued process, and Φ is a square-integrable process valued in the space of Hilbert-Schmidt operators $L_2(U,H)$. We develop a functional Itō calculus for processes of the form

$$u(\cdot, X^{t,\mathbf{x}}) := \left\{u(s, X_s^{t,\mathbf{x}})\right\}_{s\in[0,T]}$$

where $u\colon [0,T] \times \mathbb{W} \to \mathbb{R}$ is a *non-anticipative* functional, meaning that $u(s,\mathbf{y}) = u(s,\mathbf{y}')$ whenever $\mathbf{y} = \mathbf{y}'$ on $[0,s]$ for a given $s \in [0,T]$. Under suitable regularity assumptions on u, we prove an Itō formula for $u(\cdot, X^{t,\mathbf{x}})$. Then, assuming that $X^{t,\mathbf{x}}$ is driven by an SDE of the form

$$\begin{cases} dX_s = b(s,X)ds + \Phi(s,X)dW_s & \forall s \in [t,T] \\ X_{t\wedge\cdot} = \mathbf{x}_{t\wedge\cdot}, \end{cases} \tag{4.0.1}$$

where $b\colon [0,T] \times \mathbb{W} \to H$, $\Phi\colon [0,T] \times \mathbb{W} \to L_2(U,H)$ are non-anticipative coefficients satisfying usual Lipschitz conditions, and letting $f\colon \mathbb{W} \to \mathbb{R}$ be a function, we show that, if the non-anticipative function φ defined by

$$\varphi(t,\mathbf{x}) := \mathbb{E}\left[f(X^{t,\mathbf{x}})\right] \qquad (t,\mathbf{x}) \in [0,T] \times \mathbb{W}$$

is suitably regular, then φ solves the path-dependent backward Kolmogorov equation associated to (4.0.1) with terminal value f at time T. As a corollary, we obtain a Clark-Ocone type formula for the process $\varphi(\cdot, X^{t,\mathbf{x}})$. Finally, we accomplish a complete study of the regularity of the solution $X^{t,\mathbf{x}}$ to SDE (4.0.1) with respect to $t, \mathbf{x}$, when Φ is constant and b contains a convolution of the path of X with a Radon measure. In particular, the case of pointwise delay in the coefficient b will be covered. For the latter class of dynamics, by a pathwise analysis, we show in detail that the assumptions required by the general results previously obtained (Itō's formula, representation of solution to the path-dependent Kolmogorov equation, Clark-Ocone type formula) are satisfied, hence the theory can be applied.

Our methods deviate from the ones used in the aforementioned liteature. In [12, 13, 14, 31] non-anticipative functionals are considered on the metric space Λ of couples "(time t,càdlàg path on $[0,t]$)". Due to the lack of a linear structure for Λ, this choice leads to introduce non-standard notions of derivatives (vertical/horizontal) and to deal with ad-hoc continuity assumptions. On the contrary, we do not use the space Λ and, in a more standard perspective, we look at the set of continuous non-anticipative functionals as a subvector space of the space of continuous functions on $[0,T] \times \mathbb{W}$. Our choice is equivalent to take the restriction of Λ to couples with continuous path in the second component as working space, but shows the advantage to allow to deal with classical Gâteaux derivatives in space. The choice of Gâteaux derivatives in space reveals to be particularly adequate when proving regularity of solutions to path-dependent SDEs with respect to the intial value by using contraction methods in Banach spaces, as in Section 4.4: if one wishes to apply the theoretical results in practice, this is a key step in order to show that the assumptions of the theory are satisfied. Nevertheless, also in our setting, the introduction of an ad-hoc time derivative for non-anticipative functionals cannot be avoided. It is remarkable that it is convenient for us to use a left-sided time derivative, instead of the right-sided derivative introduced in [31] and then adopted also in [12, 13, 14]. Our choice turns out to be very natural when studying the link between the path-dependent SDE and the associated Kolmogorov equation. Moreover, unlike [26, 27, 28, 29, 43], we do not rephrase our path-dependent problem in a Banach space. This allows to avoid to work with stochastic calculus in Banach spaces.

The present chapter is organized as follows. In Section 4.1, after introducing some notation, we define the locally convex space with respect to which the regularity of non-anticipative functionals will be considered. In Section 4.2 we prove the path-dependent Itō's formula (Theorem 4.2.8). In Section 4.3 we show that the Feynman-Kac formula for the strong solution to a path-dependent SDE in Hilbert spaces, if it is sufficiently regular, provides a solution to the associated Kolmogorov equation (Theorem 4.3.2). We then use this fact to derive a Clark-Ocone type formula (Corollary 4.3.3). Finally, in Section 4.4, we explicitly show that the previously developed theory can be applied to a class of SDEs with path-dependent drift and constant diffusion coefficient (Theorem 4.4.9).

4.1 Preliminaries

4.1.1 Notation

Let $T > 0$, let $(\Omega, \mathbb{F} := \{\mathscr{F}_t\}_{t\in[0,T]}, \mathscr{F}, \mathbb{P})$ be a complete filtered probability space, and let $(E, |\cdot|_E)$ be a Banach space. Unless otherwise specified, every Banach space E is considered endowed with its Borel σ-algebra $\mathscr{B}_E$. We will make use of the notation $\mathscr{L}^p_{\mathscr{P}_T}(C([0,T],E))$ introduced in Section 1.2. $B_b([0,T],E)$ denotes the space of bounded Borel measurable functions $\mathbf{x}\colon [0,T] \to E$. If $\mathbf{x} \in B_b([0,T],E)$, then $\mathbf{x}_t$ and $\mathbf{x}(t)$ denote the evaluation at time $t \in [0,T]$ of the function $\mathbf{x}$, whereas $\mathbf{x}_{t\wedge\cdot}$ denotes the function defined by $(\mathbf{x}_{t\wedge\cdot})_s := \mathbf{x}_{t\wedge s}$ for $s \in [0,T]$. We denote by $B_{b,0}([0,T],E)$ the subspace of $B_b([0,T],E)$ of bounded Borel functions $\mathbf{x}\colon [0,T] \to E$ with separable range. Unless otherwise specified, $B_{b,0}([0,T],E)$ is considered with the topology of the uniform convergence. Then $B_{b,0}([0,T],E)$ is a Banach space and $C([0,T],E) \subset B_{b,0}([0,T],E)$.

By $M([0,T])$ we denote the space of Radon measures on the interval $[0,T]$. For $\nu \in M([0,T])$, $|\nu|_1$ denotes the total variation of ν.

Let F be another Banach space. From Section 1.1.1, we recall that $\mathscr{G}^n(E,F)$ denotes the space of functions $f\colon E \to F$ which are strongly Gâteaux differentiable on E up to order n (see Section 1.1.1 for details). If $f \in [0,T] \times E \to F$ is such that $f(t,\cdot) \in \mathscr{G}^n(E,F)$ for all $t \in [0,T]$, then we denote by $\partial^j_E f$, $j = 1,\dots,n$, the Gâteaux differentials of f with respect E. Similarly, if $f(t,\cdot) \in C^n(E,F)$, i.e. $f(t,\cdot)$ is continuously Fréchet differentiable up to order n, we denote by $D^j_E f$, $j = 1,\dots,n$, the Fréchet differentials of f with respect to E.

Let $NA([0,T] \times C([0,T],E),F)$ denote the subspace of $F^{[0,T]\times C([0,T],E)}$ whose members are non-anticipative functions, i.e.

$$NA([0,T] \times C([0,T],E),F) := \Big\{ f \in F^{[0,T]\times C([0,T],E)} : \\ f(t,\mathbf{x}) = f(t,\mathbf{x}_{t\wedge\cdot})\ \forall (t,\mathbf{x}) \in [0,T] \times C([0,T],E) \Big\}.$$

By $CNA([0,T] \times C([0,T],E),F)$ we denote the subspace of $C([0,T] \times C([0,T],E),F)$ whose members are non-anticipative functions, i.e.

$$CNA([0,T] \times C([0,T],E),F) := C([0,T] \times C([0,T],E),F) \bigcap NA([0,T] \times C([0,T],E),F).$$

H and U denote two real separable Hilbert spaces, with scalar product denoted by $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_U$, respectively. Let $\mathfrak{e} := \{e_n\}_{n\in\mathscr{N}}$ be an orthonormal basis of H, where $\mathscr{N} = \{1,\dots,N\}$ if H has dimension $N \in \mathbb{N}\setminus\{0\}$, or $\mathscr{N} = \mathbb{N}$ if H has infinite dimension.

Similarly, $\mathfrak{e}' := \{e'_m\}_{m\in\mathscr{M}}$ denotes an orthonormal basis of U, where $\mathscr{M} = \{1,\dots,M\}$ if U has dimension $M \in \mathbb{N}\setminus\{0\}$, or $\mathscr{M} = \mathbb{N}$ if U has infinite dimension. We use the short notation $\mathbb{W}$ for the space $C([0,T],H)$ of continuous functions $[0,T]\to H$.

Remark 4.1.1. It can be easily seen that the space $C(\Lambda,E)$ used in Chapter 3 coincides with $CNA([0,T]\times\mathbb{W},E)$. Let $\mathbf{d}_\infty$ be the pseudo-metric on $\Lambda = [0,T]\times\mathbb{W}$ defined in Section 3.1.1 (p. 84). We have

$$\mathbf{d}_\infty\big((t,\mathbf{x}),(t',\mathbf{x}')\big) \le |t-t'| + (w_{\mathbf{x}}\wedge w_{\mathbf{x}'})(|t-t'|) + |\mathbf{x}-\mathbf{x}'|_\infty \qquad \forall (t,\mathbf{x}),\ (t',\mathbf{x}')\in[0,T]\times\mathbb{W},$$

where $w_{\mathbf{x}}$ and $w_{\mathbf{x}'}$ denote the modulus of continuity of $\mathbf{x}$ and of $\mathbf{x}'$, respectively. Then $C(\Lambda,E)\subset CNA([0,T]\times\mathbb{W},E)$. Conversely, let $f\in CNA([0,T]\times\mathbb{W},E)$ and let $\{(t_n,\mathbf{x}^{(n)})\}_{n\in\mathbb{N}}$ be a sequence converging to $(t,\mathbf{x})$ with respect to the pseudometric $\mathbf{d}_\infty$. By definition of $\mathbf{d}_\infty$, the sequence $\{(t_n,\mathbf{x}^{(n)}_{t_n\wedge\cdot})\}_{n\in\mathbb{N}}$ converges to $(t,\mathbf{x}_{t\wedge\cdot})$ in $([0,T]\times\mathbb{W},|\cdot|+|\cdot|_\infty)$. Then, since f is non-anticipative and continuous on $([0,T]\times\mathbb{W},|\cdot|+|\cdot|_\infty)$, we can write

$$\lim_{n\to\infty} f(t_n,\mathbf{x}^{(n)}) = \lim_{n\to\infty} f(t_n,\mathbf{x}^{(n)}_{t_n\wedge\cdot}) = f(t,\mathbf{x}_{t\wedge\cdot}) = f(t,\mathbf{x})$$

hence $f\in C(\Lambda,E)$. We conclude $CNA([0,T]\times\mathbb{W},E) = C(\Lambda,E)$.

Let $(\tilde\Lambda,\tilde{\mathbf{d}}_\infty)$ denote the quotient metric space associated to the psedudometric space $(\Lambda,\mathbf{d}_\infty)$, and let us consider the "stopping map"

$$\mathfrak{s}\colon ([0,T]\times\mathbb{W},|\cdot|+|\cdot|_\infty)\to(\tilde\Lambda,\tilde{\mathbf{d}}_\infty),\ (t,\mathbf{x})\mapsto[(t,\mathbf{x}_{t\wedge\cdot})],$$

where $[(t,\mathbf{x}_{t\wedge\cdot})]$ denotes the class of $(t,\mathbf{x}_{t\wedge\cdot})$ in $\tilde\Lambda$. Then $\mathfrak{s}$ is onto. With arguments similar as above, one shows that $\mathfrak{s}$ is continuous and open. It is also clear that a function $f\colon[0,T]\times\mathbb{W}\to\mathbb{R}$ is non-anticipative if and only if it can be written as a composition $\tilde f\circ\mathfrak{s}$, with $\tilde f\colon\tilde\Lambda\to\mathbb{R}$. Then there is a bijection through $\mathfrak{s}$ of the (whatever valued) non-anticipative Borel functions on $[0,T]\times\mathbb{W}$ and the Borel functions on $\tilde\Lambda$. Finally, since $(\Lambda,\mathbf{d}_\infty)$ and $(\tilde\Lambda,\tilde{\mathbf{d}}_\infty)$ are Borel isomorphic through the quotien map, we conclude that the non-anticipative Borel functions on $[0,T]\times\mathbb{W}$ can be identified with the Borel functions on Λ.

4.1.2 The space $\mathbb{B}^1_{\sigma^s}(E)$

In this section we introduce a topology with respect to which we will often consider the regularity of the differentials of path-dependent functions in the remaining of the chapter.

We begin by introducing on $B_{b,0}([0,T],E)$ the family of seminorms $\mathbf{p}^s := \{p^s_\nu\}_{\nu\in M([0,T])}$ defined by

$$p^s_\nu(\mathbf{x}) := \left|\int_{[0,T]}\mathbf{x}(s)\nu(ds)\right|_E \qquad \forall\mathbf{x}\in B_{b,0}([0,T],E),\ \forall\nu\in M([0,T]).$$

Since we are considering only bounded Borel functions $\mathbf{x}$ with separable range, the integral $\int_{[0,T]} \mathbf{x} d\mu$ is well defined.

We denote by σ^s the locally convex vector topology induced on $B_{b,0}([0,T],E)$ by $\mathbf{p}^s$. If τ_∞ denotes the topology of the uniform convergence $B_{b,0}([0,T],E)$, it is easily seen that

$$\sigma^s \subsetneq \tau_\infty. \tag{4.1.1}$$

The inclusion $\sigma^s \subset \tau_\infty$ is immediate, whereas the strict inclusion is due to the fact that σ^s is contained in the weak topology of $(B_{b,0}([0,T],E), |\cdot|_\infty)$, and, unless E is trivial, the weak topology is strictly weaker than the topology induced by the norm, because $B_{b,0}([0,T],E)$ is infinite dimensional. The same holds for the restrictions to $C([0,T],E)$, i.e. $\sigma^s_{|C([0,T],E)} \subsetneq \tau_{\infty|C([0,T],E)}$.

Proposition 4.1.2. *Convergent and Cauchy sequences in σ^s are characterized as follows.*

(i) A sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ converges to $\mathbf{x}$ in $(B_{b,0}([0,T],E),\sigma^s)$ if and only if

$$\begin{cases} (a) & \sup_{n\in\mathbb{N}} |\mathbf{x}_n|_\infty < \infty \\ (b) & \lim_{n\to\infty} \mathbf{x}_n(s) = \mathbf{x}(s) \quad \forall s \in [0,T]. \end{cases} \tag{4.1.2}$$

(ii) A sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is Cauchy in $(B_{b,0}([0,T],E),\sigma^s)$ if and only if (4.1.2)(a) *holds and the sequence $\{\mathbf{x}_n(s)\}_{n\in\mathbb{N}}$ is Cauchy for every $s \in [0,T]$.*

Proof. We prove only *(i)*. The proof of *(ii)* is similar. Suppose that $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ converges to $\mathbf{x}$ in $(B_{b,0}([0,T],E),\sigma^s)$. For $s \in [0,T]$, if δ_s is the Dirac measure in s, we have

$$\lim_{n\to\infty} |\mathbf{x}_n(s) - \mathbf{x}(s)|_H = \lim_{n\to\infty} p_{\delta_s}(\mathbf{x}_n - \mathbf{x}) = 0 \qquad ,$$

which shows (4.1.2)*(b)*.

To show (4.1.2)*(a)*, consider the family of continuous linear operators

$$\Phi_n : M([0,T]) \to E,\ \nu \mapsto \int_{[0,T]} \mathbf{x}_n(s)\nu(ds),$$

for $n \in \mathbb{N}$. Since $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is convergent, the orbit $\{\Phi_n(\nu)\}_{n\in\mathbb{N}}$ is bounded, for all $\nu \in M([0,T])$, then, by Banach-Steinhaus theorem, we have

$$\sup_{n\in\mathbb{N}} |\mathbf{x}_n|_\infty = \sup_{n\in\mathbb{N}} \sup_{\substack{\nu\in M([0,T]) \\ |\nu|_1\le 1}} \left| \int_{[0,T]} \mathbf{x}_n(s)\nu(ds) \right|_E = \sup_{n\in\mathbb{N}} |\Phi_n|_{L(M([0,T]),E)} < \infty,$$

where $|\Phi_n|_{L(M([0,T]),E)}$ denotes the operator norm of Φ_n. This shows (4.1.2)*(a)* and concludes the proof for one direction of the claim.

Conversely, if (4.1.2) holds, then $p_\nu(\mathbf{x}_n - \mathbf{x}) \to 0$ by Lebesgue's dominated convergence theorem, for all $\nu \in M([0,T])$, hence $\mathbf{x}_n \to \mathbf{x}$ in σ^s. ■

By (4.1.1), it follows that bounded sets in τ_∞ are bounded in σ^s. By using Banach-Steinhaus theorem similarly as done in the proof of Proposition 4.1.2, one can see that bounded sets in σ^s are bounded in τ_∞. Then the bounded sets in σ^s and τ_∞ are the same.

Definition 4.1.3. *We define $\mathbb{B}^1(E)$ as the vector space of all functions $\mathbf{x}\colon [0,T] \to E$ which are pointwise limit of a uniformly bounded sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \subset C([0,T],E)$, i.e.*

$$\mathbb{B}^1(E) := \left\{ \mathbf{x} \in E^{[0,T]} \colon \exists\, \{\mathbf{x}_n\}_{n\in\mathbb{N}} \subset C([0,T],E) \text{ s.t. } \begin{cases} \lim_{n\to\infty} \mathbf{x}_n(s) = \mathbf{x}(s) & \forall s \in [0,T] \\ \sup_{n\in\mathbb{N}} |\mathbf{x}_n|_\infty < \infty \end{cases} \right\}.$$

We denote by $\mathbb{B}^1_{\sigma^s}(E)$ the space $\mathbb{B}^1(E)$ endowed with the locally convex topology induced by σ^s. Then a net $\{\mathbf{x}_\iota\}_{\iota\in\mathscr{I}}$ converges to 0 in $\mathbb{B}^1_{\sigma^s}(E)$ if and only if $\lim_\iota p_\nu(\mathbf{x}_\iota) = 0$ for all $\nu \in M([0,T])$.

Remark 4.1.4. By Proposition 4.1.2*(i)*, it follows that $\mathbb{B}^1(E)$ is the sequential closure $[(C([0,T],E),\sigma^s)]_{\mathrm{seq}}$ of $C([0,T],E)$ in $(B_{b,0}([0,T],E),\sigma^s)$. In particular, for any T_2-space $\mathscr{T}$ and any function $C([0,T],E) \to \mathscr{T}$, there exists at most one sequentially continuous extension $(\mathbb{B}^1(E),\sigma^s) \to \mathscr{T}$.

Remark 4.1.5. In Definition 4.1.3, by multiplying $\mathbf{x}_n$ by $|\mathbf{x}|_\infty/|\mathbf{x}_n|_\infty$ if necessary, we can assume without loss of generality that $\sup_{n\in\mathbb{N}} |\mathbf{x}_n|_\infty \leq |\mathbf{x}|_\infty$. By Proposition 4.1.2*(i)*, we then see that the unit ball of $(C([0,T],E),|\cdot|_\infty)$ is σ^s-sequentially dense in the unit ball of $(\mathbb{B}^1(E),|\cdot|_\infty)$.

Since we have the inclusion $\mathbb{B}^1(\mathbb{R}) \subsetneq B_b([0,T],\mathbb{R})$ (see [96, Theorem 11.4]), through the identification $B_b([0,T],\mathbb{R}) = B_b([0,T],\mathbb{R}e)$ in $B_{b,0}([0,T],E)$, for some $e \in E$, $|e|_E = 1$ ($E \neq \{0\}$), we also have the strict inclusion $\mathbb{B}^1(E) \subsetneq B_{b,0}([0,T],E)$.

The space $\mathbb{B}^1(E)$ is closed in $B_{b,0}([0,T],E)$ (hence in $B_b([0,T],E)$) with respect to the uniform norm. The proof of [96, Theorem 11.7], that is made for the case $E = \mathbb{R}$ and for a space of Borel functions larger than our $\mathbb{B}^1(\mathbb{R})$, can be adapted to cover our case. Since the completeness of $\mathbb{B}^1(E)$ is essential to the present chapter, we prove it.

Proposition 4.1.6. *$(\mathbb{B}^1(E),|\cdot|_\infty)$ is a Banach space.*

Proof. We show that every absolutely convergent sum is convergent in $\mathbb{B}^1(E)$. To this end, let $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \subset \mathbb{B}^1(E)$ be sequence such that $\sum_{n\in\mathbb{N}} |\mathbf{x}_n|_\infty < \infty$. By completeness of $B_b([0,T],E)$, $\sum_{n\in\mathbb{N}} \mathbf{x}_n$ is convergent in $B_b([0,T],E)$, say to $\mathbf{z}$. We are done if we show that $\mathbf{z} \in \mathbb{B}^1(E)$. By definition of $\mathbb{B}^1(E)$, for each $n \in \mathbb{N}$, there exists a sequence $\{\mathbf{y}_n^{(k)}\}_{k\in\mathbb{N}} \subset C([0,T],E)$ such that

$$M_n := \sup_{k\in\mathbb{N}} |\mathbf{y}_n^{(k)}|_\infty < \infty \qquad \text{and} \qquad \lim_{k\to\infty} \mathbf{y}_n^{(k)}(s) = \mathbf{x}_n(s)\ \forall s \in [0,T].$$

By multiplying $\mathbf{y}_n^{(k)}$ by $|\mathbf{x}_n|_\infty/|\mathbf{y}_n^{(k)}|_\infty$ if necessary, without loss of generality we can assume that $M_n \leq |\mathbf{x}_n|_\infty$. Define $\mathbf{z}_k := \sum_{n=1}^k \mathbf{y}_n^{(k)}$, $k \in \mathbb{N}$. Then $\mathbf{z}_k \in C([0,T],E)$ and

$$\sup_{k\in\mathbb{N}} |\mathbf{z}_k|_\infty \leq \sup_{k\in\mathbb{N}} \sum_{n=1}^{\infty} |\mathbf{y}_n^{(k)}|_\infty \leq \sum_{n=1}^{\infty} |\mathbf{x}_n|_\infty < \infty. \tag{4.1.3}$$

Moreover, for $s \in [0,T]$, $0 \leq \bar{k} \leq k$,

$$\begin{aligned}|\mathbf{z}(s) - \mathbf{z}_k(s)|_E &= |\sum_{n=1}^{\infty} \mathbf{x}_n(s) - \sum_{n=1}^{k} \mathbf{y}_n^{(k)}(s)|_E \leq \sum_{n=\bar{k}}^{\infty} (|\mathbf{x}_n|_\infty + |\mathbf{y}_n^{(k)}|_\infty) + \sum_{n=1}^{\bar{k}} |\mathbf{x}_n(s) - \mathbf{y}_n^{(k)}(s)|_E \\ &\leq 2\sum_{n=\bar{k}}^{\infty} |\mathbf{x}_n|_\infty + \sum_{n=1}^{\bar{k}} |\mathbf{x}_n(s) - \mathbf{y}_n^{(k)}(s)|_E.\end{aligned}$$

By taking first the $\limsup_{k\to\infty}$, recalling the pointwise convergence $\mathbf{y}_n^{(k)}(s) \to \mathbf{x}_n(s)$ as $k \to \infty$, and then taking the $\lim_{\bar{k}\to\infty}$, we obtain $\mathbf{z}_k(s) \to \mathbf{z}(s)$ as $k \to \infty$. Since $s \in [0,T]$ was arbitrary, this, together with (4.1.3), proves that $\mathbf{z} \in \mathbb{B}^1(E)$. ■

4.1.3 $\mathbb{V}_{\sigma^s}(E)$-sequentially continuous derivatives

We introduce the following subspace of $\mathbb{B}^1(E)$:

$$\mathbb{V}(E) := \operatorname{span}\left\{\mathbf{x} + v\mathbf{1}_{[t,T]} : \mathbf{x} \in C([0,T],E),\ v \in E,\ t \in [0,T]\right\}. \tag{4.1.4}$$

A member of $\mathbb{V}(E)$ is the sum of a continuous function and a right-continuous step function (with finite number of jumps). We denote by $\mathbb{V}_{\sigma^s}(E)$ the space $\mathbb{V}(E)$ endowed with the locally convex topology induced by $\mathbb{B}^1_{\sigma^s}(E)$ and by $\mathbb{V}_\infty(E)$ the space $\mathbb{V}(E)$ endowed with the topology induced by the supremum norm $|\cdot|_\infty$.

Definition 4.1.7. *We say that a function* $f \in \mathcal{G}^2(C([0,T],E),F)$ *has* derivatives with $\mathbb{V}_{\sigma^s}(E)$-sequentially continuous extensions *if*

$$\partial f : C([0,T],E) \times C([0,T],E) \to F,\ (\mathbf{x},\mathbf{v}) \mapsto \partial_{\mathbf{v}} f(\mathbf{x})$$

and

$$\partial^2 f : C([0,T],E) \times C([0,T],E) \times C([0,T],E) \to F,\ (\mathbf{x},\mathbf{v},\mathbf{w}) \mapsto \partial^2_{\mathbf{vw}} f(\mathbf{x})$$

admit sequentially continuous extensions, respectively,

$$\overline{\partial f} : C([0,T],E) \times \mathbb{V}_{\sigma^s}(E) \to F,\ (\mathbf{x},\mathbf{v}) \mapsto \overline{\partial f}(\mathbf{x}).\mathbf{v}$$

and

$$\overline{\partial^2 f} : C([0,T],E) \times \mathbb{V}_{\sigma^s}(E) \times \mathbb{V}_{\sigma^s}(E) \to F,\ (\mathbf{x},\mathbf{v},\mathbf{w}) \mapsto \overline{\partial^2 f}(\mathbf{x}).(\mathbf{v},\mathbf{w}).$$

We denote by $\mathcal{G}^2_{\sigma^s}(C([0,T],E),F)$ *the subspace of* $\mathcal{G}^2(C([0,T],E),F)$ *containing the functions having derivatives with* $\mathbb{V}_{\sigma^s}(E)$*-sequentially continuous extensions.*

If $u \in NA([0,T] \times C([0,T],E),F)$, $t \in [0,T]$, and $u(t,\cdot) \in \mathscr{G}^2_{\sigma^s}(C([0,T],E),F)$, then the notation $\overline{\partial_E u}(t,\mathbf{x}).\mathbf{v}$, for $\mathbf{x} \in C([0,T],E)$ and $\mathbf{v} \in \mathbb{V}(E)$, stands for $\overline{\partial_E u(t,\cdot)}(\mathbf{x}),\mathbf{v}$. Similarly, $\overline{\partial_E u}(t,\cdot)$ stands for $\overline{\partial_E u(t,\cdot)}$.

Remark 4.1.8. If $u \in NA([0,T] \times C([0,T],E),F)$ is such that, for some $t \in [0,T]$, $u(t,\cdot) \in \mathscr{G}^2(E,F)$, then, by non-anticipativity,

$$\partial_E u(t,\mathbf{x}).\mathbf{v} = \partial_E u(t,\mathbf{x}).\mathbf{v}' \qquad \forall \mathbf{x},\mathbf{v},\mathbf{v}' \in C([0,T],E) \text{ s.t. } \mathbf{v}(s) = \mathbf{v}'(s) \text{ for } s \in [0,t].$$

If $u(t,\cdot) \in \mathscr{G}^2_{\sigma^s}(C([0,T],E),F)$, then it also holds

$$\overline{\partial_E u}(t,\mathbf{x}).\mathbf{v} = \overline{\partial_E u}(t,\mathbf{x}).\mathbf{v}' \qquad \forall \mathbf{x} \in C([0,T],E),\ \forall \mathbf{v},\mathbf{v}' \in \mathbb{V}(E) \text{ s.t. } \mathbf{v}(s) = \mathbf{v}'(s) \text{ for } s \in [0,t].$$

In particular,

$$\overline{\partial_E u}(t,\mathbf{x}).(\mathbf{1}_{[t,T]}v) = \overline{\partial_E u}(t,\mathbf{x}).(\mathbf{1}_{[t,T')}v) \qquad \forall \mathbf{x} \in C([0,T],E),\ \forall v \in E,\ \forall T' \in (t,T).$$

A similar remark holds for the second-order differential. Because of that, the directional derivatives $\overline{\partial_E u}(t,\mathbf{x}).(\mathbf{1}_{[t,T]}v), \overline{\partial^2_E u}(t,\mathbf{x}).(\mathbf{1}_{[t,T]}v, \mathbf{1}_{[t,T]}w)$, $\mathbf{x} \in C([0,T],E)$, $v,w \in E$, express in our framework the so-called vertical derivatives of [12, 13, 14].

Example 4.1.9. Let $\mu \in M([0,T])$ and $g \in C([0,T] \times E,F)$ such that $g(t,\cdot) \in \mathscr{G}^2(E,F)$ for all $t \in [0,T]$, and let us assume that $\partial_E g$ and $\partial^2_E g$ are bounded on bounded sets of $[0,T] \times E$. Define

$$f(\mathbf{x}) := \int_{[0,T]} g(s,\mathbf{x}(s))\mu(ds) \qquad \forall \mathbf{x} \in C([0,T],E).$$

Then $f \in \mathscr{G}^2(C([0,T],E),F)$, with

$$\partial f(\mathbf{x}).\mathbf{v} = \int_{[0,T]} \partial_E g(s,\mathbf{x}(s)).\mathbf{v}(s)\mu(ds) \qquad \forall \mathbf{x},\mathbf{v} \in C([0,T],H)$$

$$\partial^2 f(\mathbf{x}).(\mathbf{v},\mathbf{w}) = \int_{[0,T]} \partial^2_E g(s,\mathbf{x}(s)).(\mathbf{v}(s),\mathbf{w}(s))\mu(ds) \qquad \forall \mathbf{x},\mathbf{v},\mathbf{w} \in C([0,T],H).$$

It is clear that $\partial f(\mathbf{x}).\mathbf{v}$ and $\partial^2 f(\mathbf{x}).(\mathbf{v},\mathbf{w})$ can be computed with the same expressions when $\mathbf{v},\mathbf{w} \in \mathbb{V}(E)$. Moreover, by Proposition 4.1.2*(i)*, by Lebesgue's dominated convergence theorem, and by strong continuity of the Gâteaux differentials of g, we have that $\partial f(\mathbf{x}).\mathbf{v}$ and $\partial f(\mathbf{x}).(\mathbf{v},\mathbf{w})$ are sequentially continuous with respect to $(\mathbf{x},\mathbf{v}) \in C([0,T],E) \times \mathbb{V}_{\sigma^s}(E)$ and $(\mathbf{x},\mathbf{v},\mathbf{w}) \in C([0,T],E) \times \mathbb{V}_{\sigma^s}(E) \times \mathbb{V}_{\sigma^s}(E)$, respectively. Then $f \in \mathscr{G}^2_{\sigma^s}(C([0,T],E),F)$.

4.2 A path-dependent Itō's formula

In this section we prove an Itō's formula for processes of the form $\{u(t,X)\}_{t\in[0,T]}$, where X is a diffusion with values in H and u is a non-anticipative function with regular time-space derivatives, in a sense specified below by Assumption 4.2.3.

For a non-anticipative function u, we introduce the following left-sided time derivative.

Definition 4.2.1. *For $u \in NA([0,T]\times C([0,T],E),F)$ and $(t,\mathbf{x}) \in (0,T)\times C([0,T],E)$, we define the following left-sided derivative, if it exists:*

$$\mathscr{D}_t^- u(t,\mathbf{x}) := \lim_{h\to 0^+} \frac{u(t,\mathbf{x}_{(t-h)\wedge\cdot}) - u(t-h,\mathbf{x})}{h}. \tag{4.2.1}$$

Remark 4.2.2. Notice that, by the very definition, for $t,t' \in (0,T)$, $t<t'$, and $\mathbf{x}\in C([0,T],E)$, the derivative $\mathscr{D}_t^- u(t',\mathbf{x}_{t\wedge\cdot})$ concides with the left-sided derivative of the map

$$(t,T)\to F,\ s\mapsto u(s,\mathbf{x}_{t\wedge\cdot})$$

computed in t'.

We will prove the path-dependent Itō's formula under the following assumption.

Assumption 4.2.3. *The function u belongs to $CNA([0,T]\times\mathbb{W},\mathbb{R})$ and has the following properties.*

(i) For all $t\in(0,T)$, $\mathscr{D}_t^- u(t,\mathbf{x})$ exists for all $\mathbf{x}\in\mathbb{W}$. For a.e. $t\in(0,T)$, the map

$$\mathbb{W}\to\mathbb{R},\ \mathbf{x}\mapsto \mathscr{D}_t^- u(t,\mathbf{x})$$

is continuous. For all compact set $K\subset\mathbb{W}$ there exists $M_K>0$ such that

$$\sup_{\mathbf{x}\in K} |\mathscr{D}_t^- u(t,\mathbf{x})| \le M_K \qquad \text{for a.e. } t\in(0,T). \tag{4.2.2}$$

(ii) For all $t\in[0,T]$, $u(t,\cdot)\in\mathscr{G}^2_{\sigma^s}(\mathbb{W},\mathbb{R})$ and the differentials $\partial_{\mathbb{W}}u$ and $\partial^2_{\mathbb{W}}u$ are bounded:

$$\sup_{t\in[0,T]} \sup_{\substack{\mathbf{x},\mathbf{v}\in\mathbb{W}\\ |\mathbf{v}|_\infty\le 1}} |\partial_{\mathbb{W}}u(t,\mathbf{x}).\mathbf{v}| < \infty \tag{4.2.3}$$

$$\sup_{t\in[0,T]} \sup_{\substack{\mathbf{x},\mathbf{v},\mathbf{w}\in\mathbb{W}\\ |\mathbf{w}|\vee|\mathbf{v}|_\infty\le 1}} \left|\partial^2_{\mathbb{W}}u(t,\mathbf{x}).(\mathbf{v},\mathbf{w})\right| < \infty. \tag{4.2.4}$$

(iii) For a.e. $t\in(0,T)$,

$$\lim_{h\to 0^+} \overline{\partial_{\mathbb{W}}u}(t+h,\mathbf{x}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}(\cdot)v) = \overline{\partial_{\mathbb{W}}u}(t,\mathbf{x}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}(\cdot)v), \tag{4.2.5}$$

$$\lim_{h\to 0^+} \overline{\partial^2_{\mathbb{W}}u}(t+h,\mathbf{x}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}(\cdot)v,\mathbf{1}_{[t,T]}(\cdot)v) = \overline{\partial^2_{\mathbb{W}}u}(t,\mathbf{x}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}(\cdot)v,\mathbf{1}_{[t,T]}(\cdot)v), \tag{4.2.6}$$

for all $\mathbf{x}\in\mathbb{W}$ and all $v\in H$.

We give some simple examples for which Assumption 4.2.3 is verified.

Example 4.2.4. Let $\hat{u} \in C_b^{1,2}([0,T] \times H, \mathbb{R})$ and $u(t,\mathbf{x}) := \hat{u}(t,\mathbf{x}(t))$, $(t,\mathbf{x}) \in [0,T] \times \mathbb{W}$. Then Assumption 4.2.3 is verified, with $\mathscr{D}_t^- u(t,\mathbf{x}) = \partial_t \hat{u}(t,\mathbf{x}(t))$, for $t \in (0,T)$, $\mathbf{x} \in \mathbb{W}$, and with $\overline{\partial_{\mathbb{W}} u}(t,\mathbf{x}).\mathbf{v} = D_H \hat{u}(t,\mathbf{x}(t)).\mathbf{v}(t)$, $\overline{\partial^2_{\mathbb{W}} u}(t,\mathbf{x}).(\mathbf{v}.\mathbf{w}) = D_H \hat{u}(t,\mathbf{x}(t)).(\mathbf{v}(t),\mathbf{w}(t))$, for $t \in [0,T]$, $\mathbf{x} \in \mathbb{W}$, $\mathbf{v},\mathbf{w} \in \mathbb{V}(H)$.

Example 4.2.5. Let $\gamma \in C^1([0,T],\mathbb{R})$, $h \in C_b^{0,2}([0,T] \times H, \mathbb{R})$. For $(t,\mathbf{x}) \in [0,T] \times \mathbb{W}$, define

$$u(t,\mathbf{x}) := \int_0^t h(s,\mathbf{x}(s))\gamma(t-s)ds.$$

A direct computation gives, for $(t,\mathbf{x}) \in [0,T] \times \mathbb{W}$,

$$\begin{aligned}
\mathscr{D}_t^- u(t,\mathbf{x}) &= h(t,\mathbf{x}(t))\gamma(0) + \int_0^t h(s,\mathbf{x}(s))\gamma'(t-s)ds \\
\overline{\partial_{\mathbb{W}} u}(t,\mathbf{x}).\mathbf{v} &= \int_0^t D_H h(s,\mathbf{x}(s)).\mathbf{v}(s)\gamma(t-s)ds \\
\overline{\partial^2_{\mathbb{W}} u}(t,\mathbf{x}).(\mathbf{v},\mathbf{w}) &= \int_0^t D_H^2 h(s,\mathbf{x}(s)).(\mathbf{v}(s),\mathbf{w}(s))\gamma(t-s)ds
\end{aligned}$$

and one can easiliy see that Assumption 4.2.3 is verified by u.

Example 4.2.6. Let u be a function verifying Assumption 4.2.3 and let $h \in C_b^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$. For $(t,\mathbf{x}) \in [0,T] \times \mathbb{W}$, define $\hat{u}(t,\mathbf{x}) := h(t,u(t,\mathbf{x}))$. We have

$$\mathscr{D}_t^- \hat{u}(t,\mathbf{x}) = \partial_t h(t,u(t,\mathbf{x})) + D_H u(t,u(t,\mathbf{x})).\mathscr{D}_t^- u(t,\mathbf{x})$$

and $\overline{\partial_{\mathbb{W}} \hat{u}}, \overline{\partial^2_{\mathbb{W}} \hat{u}}$ are given by the chain rule. Assumption 4.2.3 are verified.

Let $B \colon \mathbb{V}_\infty(H) \times \mathbb{V}_\infty(H) \to \mathbb{R}$ be a continuous bilinear functional and let $C > 0$ such that $|B(\mathbf{x},\mathbf{y})| \leq C|\mathbf{x}|_\infty |\mathbf{y}|_\infty$, for all $\mathbf{x},\mathbf{y} \in \mathbb{V}_\infty(H)$. Let $\mathbf{a} \in \mathbb{V}(\mathbb{R})$, $|\mathbf{a}|_\infty \leq 1$, and $T \in L_2(U,H)$. Then $\mathbf{a}Tu \in \mathbb{V}(H)$, for all $u \in U$, and $\mathbf{a}v \in \mathbb{V}(H)$, for all $v \in H$. Clearly

$$U \times H \to \mathbb{R},\ (u,v) \mapsto B(\mathbf{a}Tu,\mathbf{a}v)$$

is bilinear and continuous. Let $Q \in L(U,H)$ be the unique linear and continuous operator such that

$$\langle Qu, v\rangle_H = B(\mathbf{a}Tu,\mathbf{a}v) \qquad \forall u \in U,\ \forall v \in H. \tag{4.2.7}$$

We claim that $Q \in L_2(U,H)$. Indeed,

$$\sum_{m \in \mathscr{M}} |Qe'_m|_H^2 = \sum_{m \in \mathscr{M}} \sup_{\substack{v \in H \\ |v|_H \leq 1}} \big(B(\mathbf{a}Te'_m,\mathbf{a}v)\big)^2 \leq \sum_{m \in \mathscr{M}} C^2 |\mathbf{a}Te'_m|_\infty^2 \leq C^2 |T|^2_{L_2(U,H)} < \infty.$$

Then $Q^* \in L_2(H,U)$ and, by [24, Proposition C.4], $Q^*T \in L(U)$ is a nuclear operator. In particular, the number

$$\sum_{m \in \mathscr{M}} B(\mathbf{a}Te'_m,\mathbf{a}Te'_m) = \sum_{m \in \mathscr{M}} \langle Qe'_m, Te'_m\rangle_H = \sum_{m \in \mathscr{M}} \langle e'_m, Q^*Te'_m\rangle_U = \operatorname{Tr}(Q^*T)$$

is well-defined, finite, and does not depend on the chosen orthonormal basis $\{e'_m\}_{m \in \mathscr{M}}$. This observation leads to introduce the following well-defined notion.

Definition 4.2.7. *Let $B\colon \mathbb{V}_\infty(H)\times\mathbb{V}_\infty(H)\to\mathbb{R}$ be a continuous bilinear functional, $\mathbf{a}\in\mathbb{V}(\mathbb{R})$, $T\in L_2(U,H)$. We define*

$$\mathbf{T}[B,\mathbf{a}T] := \sum_{m\in\mathcal{M}} B(\mathbf{a}Te'_m,\mathbf{a}Te'_m). \tag{4.2.8}$$

Let $b\in\mathcal{L}^1_{\mathcal{P}_T}(\mathbb{W})$, $\Phi\in\mathcal{L}^2_{\mathcal{P}_T}(C([0,T],L_2(U,H)))$, and let W be a U-valued cylindrical Wiener process. For $(\hat{t},\hat{Y})\in[0,T]\times\mathcal{L}^1_{\mathcal{P}_T}(\mathbb{W})$, let $X^{\hat{t},\hat{Y}}\in\mathcal{L}^1_{\mathcal{P}_T}(\mathbb{W})$ be the process defined by

$$X_t = \hat{Y}_{\hat{t}\wedge t} + \int_{\hat{t}}^{\hat{t}\vee t} b_s ds + \int_{\hat{t}}^{\hat{t}\vee t}\Phi_s dW_s \qquad \forall t\in[0,T]. \tag{4.2.9}$$

The first main result of this chapter is the following path-dependent Itō's formula.

Theorem 4.2.8. *Suppose that u satisfies Assumption 4.2.3. For $\hat{Y}\in\mathcal{L}^1_{\mathcal{P}_T}(\mathbb{W})$ and $\hat{t}\in[0,T]$, let $X^{\hat{t},\hat{Y}}$ be the process defined by* (4.2.9)*. Then*

(i) for all $\omega\in\Omega$, $\mathscr{D}_t^- u(\cdot,X^{\hat{t},\hat{Y}}(\omega))\in L^1((0,T),\mathbb{R})$;

(ii) $\left\{\overline{\partial_{\mathbb{W}}u}(t,X^{\hat{t},\hat{Y}}).(\mathbf{1}_{[t,T]}b_t)\right\}_{t\in[0,T]}\in L^1_{\mathcal{P}_T}(\mathbb{R})$;

(iii) $\left\{\overline{\partial_{\mathbb{W}}u}(t,X^{\hat{t},\hat{Y}}).(\mathbf{1}_{[t,T]}\Phi_t)\right\}_{t\in[0,T]}\in L^2_{\mathcal{P}_T}(U^*)$;

(iv) $\left\{\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}u}(t,X^{\hat{t},\hat{Y}}),\mathbf{1}_{[t,T]}\Phi_t\right]\right\}_{t\in[0,T]}\in L^1_{\mathcal{P}_T}(\mathbb{R})$.

For $t\in[\hat{t},T]$,

$$\begin{aligned} u(t,X^{\hat{t},\hat{Y}}) &= u(\hat{t},\hat{Y}) + \int_{\hat{t}}^t \Big(\mathscr{D}_t^- u(s,X^{\hat{t},\hat{Y}})ds + \overline{\partial_{\mathbb{W}}u}(s,X^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}b_s)\Big)ds \\ &\quad + \frac{1}{2}\int_{\hat{t}}^t \mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}u}(s,X^{\hat{t},\hat{Y}}),\mathbf{1}_{[s,T]}\Phi_s\right]ds + \int_{\hat{t}}^t \overline{\partial_{\mathbb{W}}u}(s,X^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}\Phi_s)dW_s, \qquad \mathbb{P}\text{-a.e.}. \end{aligned} \tag{4.2.10}$$

Remark 4.2.9. Notice that, by Example 4.2.4, (4.2.10) is a generalization of the standard Itō's formula in the non-path-dependent case.

The proof of Theorem 4.2.8 is obtained through several partial results. We begin by preparing a setting useful to approximate path-dependent functionals by non-path-dependent ones, for which we can use the standard (non-path-dependent) stochastic analysis on Hilbert spaces, as presented e.g. in [24].

For $n\geq 1$, we consider the product Hilbert space H^n endowed with the scalar product $\langle\cdot,\cdot\rangle_{H^n}$ defined by

$$\langle x,x'\rangle_{H^n} := \sum_{k=1}^n \langle x_k,x'_k\rangle_H \qquad \forall x=(x_1,\ldots,x_n),\ x'=(x'_1,\ldots,x'_n)\in H^n.$$

Let $\pi := \{0 = t_1 < t_2 < \ldots < t_n = T\}$ be a partition of the interval $[0,T]$ and let

$$\delta(\pi) := \sup_{i=1,\ldots,n-1} |t_{i+1} - t_i|.$$

Define the operator

$$\ell_\pi : H^n \to \mathbb{W}$$

as the linear interpolation on the partition π, i.e.

$$\ell_\pi(x_1,\ldots,x_n)(t) := x_1 + \sum_{i=1}^{n-1} \frac{t \wedge t_{k+1} - t \wedge t_k}{t_{k+1} - t_k}(x_{k+1} - x_k) \qquad \forall t \in [0,T]. \tag{4.2.11}$$

The operator ℓ_π is linear and continuous, with operator norm 1. If $\mathbf{x} \in \mathbb{W}$ and if $w_{\mathbf{x}}$ denotes a modulus of continuity for $\mathbf{x}$, then

$$|\ell_\pi\left(\mathbf{x}_{t_2\wedge\cdot}(t), \mathbf{x}_{t_3\wedge\cdot}(t), \ldots \mathbf{x}_{t_{n-1}\wedge\cdot}(t), \mathbf{x}_{t_n\wedge\cdot}(t), \mathbf{x}_{t_n\wedge\cdot}(t)\right) - \mathbf{x}_{t\wedge\cdot}|_\infty \le 2w_{\mathbf{x}}(\delta(\pi)). \tag{4.2.12}$$

Let X be given by (4.2.9). We introduce the following H-valued processes, obtained by stopping X at certain fixed times. For $i = 1,\ldots,n-1$ and $t \in [0,T]$, let $X_t^{(\pi,i)}$ be the continuous process defined by

$$X_t^{(\pi,i)} := X_{t_{i+1}\wedge t}^{\hat{t},\hat{Y}} = \hat{Y}_{\hat{t}\wedge t\wedge t_{i+1}} + \int_{\hat{t}}^{\hat{t}\vee t} \mathbf{1}_{[0,t_{i+1})}(s) b_s ds + \int_{\hat{t}}^{\hat{t}\vee t} \mathbf{1}_{[0,t_{i+1})}(s)\Phi_s dW_s \tag{4.2.13}$$

and let $X_t^{(\pi,n)} := X_t^{\hat{t},\hat{Y}}$, $t \in [0,T]$. We define the H^n-valued process $X^{(\pi)}$ by

$$X_t^{(\pi)} := (X_t^{(\pi,1)},\ldots,X_t^{(\pi,n)}) \qquad \forall t \in [0,T].$$

Notice that $X^{(\pi)} \in \mathscr{L}^1_{\mathscr{P}_T}(C([0,T],H^n))$. The dynamics of $X^{(\pi)}$ is given by

$$X_t^{(\pi)} = X_{\hat{t}}^{(\pi)} + \int_{\hat{t}}^t b_s^{(\pi)} ds + \int_{\hat{t}}^t \Phi_s^{(\pi)} dW_s \qquad \forall t \in [\hat{t},T],$$

where

$$X_{\hat{t}}^{(\pi)} = (\hat{Y}_{\hat{t}\wedge t_2}, \hat{Y}_{\hat{t}\wedge t_3}, \ldots, \hat{Y}_{\hat{t}}, \hat{Y}_{\hat{t}}) \in H^n$$

and where the coefficients $b^{(\pi)}$ and $\Phi^{(\pi)}$ are the following

$$\begin{cases} b_s^{(\pi)} := (\mathbf{1}_{[0,t_2)}(s)b_s, \mathbf{1}_{[0,t_3)}(s)b_s, \ldots, \\ \qquad \ldots, \mathbf{1}_{[0,t_{n-1})}(s)b_s, \mathbf{1}_{[0,t_n)}(s)b_s, \mathbf{1}_{[0,t_n]}(s)b_s) \qquad \forall s \in [0,T] \\ \Phi_s^{(\pi)}u := (\mathbf{1}_{[0,t_2)}(s)\Phi_s u, \mathbf{1}_{[0,t_3)}(s)\Phi_s u, \ldots, \\ \qquad \ldots, \mathbf{1}_{[0,t_{n-1})}(s)\Phi_s u, \mathbf{1}_{[0,t_n)}(s)\Phi_s u, \mathbf{1}_{[0,t_n]}(s)\Phi_s u) \quad \forall s \in [0,T],\ \forall u \in U. \end{cases} \tag{4.2.14}$$

We can verify that $b^{(\pi)} \in L^1_{\mathscr{P}_T}(H^n)$ by

$$\mathbb{E}\left[\int_0^T |b_s^{(\pi)}|_{H^n}\right] = \mathbb{E}\left[\int_0^T |b_s|_H \left(1 + \sum_{j=2}^n \mathbf{1}_{[0,t_i)}(s)\right)^{1/2} ds\right] \le n^{1/2}\mathbb{E}\left[\int_0^T |b_s|_H\right]$$

and that $\Phi^{(\pi)} \in L^2_{\mathscr{P}_T}(L_2(U,H^n))$ by

$$\mathbb{E}\left[\int_0^T |\Phi_s^{(\pi)}|^2_{L_2(U,H^n)}\right] = \mathbb{E}\left[\int_0^T |\Phi_s|^2_{L_2(U,H)}\left(1+\sum_{j=2}^n \mathbf{1}_{[0,t_i)}(s)\right)ds\right] \le n\mathbb{E}\left[\int_0^T |\Phi_s|^2_{L_2(U,H)}\right].$$

We notice that, by (4.2.12) and (4.2.13),

$$\lim_{\delta\to 0^+} \sup_{\pi:\, \delta(\pi)\le\delta} \sup_{t\in[0,T]} \left|\ell_\pi(X_t^{(\pi)}(\omega)) - X_{t\wedge\cdot}^{\hat{t},\hat{Y}}(\omega)\right|_\infty = 0 \qquad \forall\omega\in\Omega. \tag{4.2.15}$$

Remark 4.2.10. The importance of the choice of $b^{(\pi)}$ as in (4.2.14) can be understood when we consider the composition $\ell_\pi(b_s^{(\pi)}(\omega))$. If $\delta(\pi)\to 0$, then $\ell_\pi(b_s^{(\pi)}(\omega))$ converges pointwise to $\mathbf{1}_{[s,T]}(\cdot)b_s(\omega)$ everywhere on $[0,T]$. On the contrary, if we consider

$$\tilde{b}_s^{(\pi)} := (\mathbf{1}_{[0,t_1)}(s)b_s, \mathbf{1}_{[0,t_2)}(s)b_s, \dots, \mathbf{1}_{[0,t_{n-1})}(s)b_s, \mathbf{1}_{[0,t_n]}(s)b_s)$$

then the pointwise limit as $\delta(\pi)\to 0$ of $\ell_\pi(\tilde{b}_s^{(\pi)}(\omega))$ is 0 on $[0,s)$ and b_s on $(s,T]$, but it is not guaranteed that the limit in s exists. In our approximation framework, we deal with sequential continuity with respect to the topology σ^s in $\mathbb{V}(H)$, wich implies pointwise convergence, as clarified by Proposition 4.1.2*(i)*. Because of that, the choice of $b^{(\pi)}$ as in (4.2.14) will be relevant. The same comment holds for $\Phi^{(\pi)}$.

We will need the following measurability lemma.

Lemma 4.2.11. *Let V,Y,Z be H-valued continuous $\mathbb{F}$-adapted processes. Let E be a Banach space and let*

$$\bar{f}: \mathbb{W}\times\mathbb{V}_{\sigma^s}(H)\times\mathbb{V}_{\sigma^s}(H)\to E$$

be a sequentially continuous function. Then the process

$$\Psi := \left\{\bar{f}(V_{t\wedge\cdot}, \mathbf{1}_{[t,T]}Y_t, \mathbf{1}_{[t,T]}Z_t)\right\}_{t\in[0,T]}$$

is $\mathbb{F}$-adapted and left-continuous.

Proof. For all $\mathbf{x}\in\mathbb{W}$, the map

$$[0,T]\to\mathbb{W},\ t\mapsto \mathbf{x}_{t\wedge\cdot}$$

is continuous. Then $\{V_{t\wedge\cdot}\}_{t\in[0,T]}$ is a $\mathbb{W}$-valued continuous process. We now show that $\{V_{t\wedge\cdot}\}_{t\in[0,T]}$ is $\mathbb{F}$-adapted. Let $t\in[0,T]$. Let $\pi=\{0=t_1<\ldots<t_n=T\}$ be a partition of $[0,T]$. It is clear that $(V_{t_1\wedge t},\dots,V_{t_n\wedge t})$ is an H^n-valued $\mathscr{F}_t$-measurable random variable. Then $\ell_\pi(V_{t_1\wedge t},\dots,V_{t_n\wedge t})$ is a $\mathbb{W}$-valued $\mathscr{F}_t$-adapted random variable. For all $\mathbf{x}\in\mathbb{W}$,

$$|\ell_\pi\left(\mathbf{x}_{t_1\wedge t},\dots,\mathbf{x}_{t_n\wedge t})\right) - \mathbf{x}_{t\wedge\cdot}|_\infty \le w_{\mathbf{x}}(\delta(\pi)),$$

where $w_{\mathbf{x}}$ is a modulus of continuity for $\mathbf{x}$, hence, for all $\omega \in \Omega$,

$$\lim_{\delta(\pi)\to 0} \ell_\pi(V_{t_1\wedge t}(\omega),\dots,V_{t_n\wedge t}(\omega)) = V_{t\wedge\cdot}(\omega) \text{ in } \mathbb{W}, \text{ uniformly for } t \in [0,T].$$

This shows that $\{V_{t\wedge\cdot}\}_{t\in[0,T]}$ is a $\mathbb{W}$-valued $\mathbb{F}$-adapted process. The same considerations hold for $\{Y_{t\wedge\cdot}\}_{t\in[0,T]}$ and for $\{Z_{t\wedge\cdot}\}_{t\in[0,T]}$.

Now let $t \in [0,T]$ and let $\{\varphi_n\}_{n\in\mathbb{N}} \subset C([0,T],\mathbb{R})$ be a sequence such that

$$\begin{cases} 0 \le \varphi_n \le 1 & \forall n \in \mathbb{N} \\ \lim_{n\to\infty} \varphi_n(s) = \mathbf{1}_{[t,T]}(s) & \forall s \in [0,T]. \end{cases} \tag{4.2.16}$$

Since, for every $n \in \mathbb{N}$, the map $H \to \mathbb{W}$, $h \mapsto \varphi_n h$ is linear and continuous, we have that $\varphi_n Y_t$ and $\varphi_n Z_t$ are $\mathbb{W}$-valued, $\mathscr{F}_t$-measurable random variables. It follows that $(V_{t\wedge\cdot}, \varphi_n Y_t, \varphi_n Z_t)$ is a $\mathbb{W}\times\mathbb{W}\times\mathbb{W}$-valued $\mathscr{F}_t$-measurable random variable. The sequential continuity of $\bar{f}$ implies the continuity of the restriction of $\bar{f}$ to $\mathbb{W}\times\mathbb{W}\times\mathbb{W}$, then $\bar{f}(V_{t\wedge\cdot}, \varphi_n Y_t, \varphi_n Z_t)$ is an E-valued $\mathscr{F}_t$-measurable random variable. Now, by (4.2.16) and Proposition 4.1.2*(i)*, we have

$$\begin{cases} \lim_{n\to\infty} \varphi_n Y_t(\omega) = \mathbf{1}_{[t,T]} Y_t(\omega) & \text{in } \mathbb{V}_{\sigma^s}(H),\ \forall \omega \in \Omega, \\ \lim_{n\to\infty} \varphi_n Z_t(\omega) = \mathbf{1}_{[t,T]} Z_t(\omega) & \text{in } \mathbb{V}_{\sigma^s}(H),\ \forall \omega \in \Omega. \end{cases}$$

By sequential continuity of $\bar{f}$, we conclude

$$\lim_{n\to\infty} \bar{f}(V_{t\wedge\cdot}, \varphi_n Y_t, \varphi_n Z_t) = \bar{f}(V_{t\wedge\cdot}, \mathbf{1}_{[t,T]} Y_t, \mathbf{1}_{[t,T]} Z_t) \text{ pointwise.}$$

This shows that Ψ_t is an E-valued $\mathscr{F}_t$-measurable random variable, hence Ψ is $\mathbb{F}$-adapted.

Let $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ be a sequence converging to t in $(0,T]$ from the left. Then the sequence $\{V_{t_n\wedge\cdot}(\omega)\}_{n\in\mathbb{N}}$ converges to $V_{t\wedge\cdot}(\omega)$ in $\mathbb{W}$, for all $\omega \in \Omega$. Moreover, by Proposition 4.1.2*(i)* and continuity of Y, Z,

$$\forall \omega \in \Omega,\ \begin{cases} \lim_{n\to\infty} \mathbf{1}_{[t_n,T]}(\cdot) Y_{t_n}(\omega) = \mathbf{1}_{[t,T]}(\cdot) Y_t & \text{in } \mathbb{V}_{\sigma^s}(H) \\ \lim_{n\to\infty} \mathbf{1}_{[t_n,T]}(\cdot) Z_{t_n} = \mathbf{1}_{[t,T]}(\cdot) Z_t & \text{in } \mathbb{V}_{\sigma^s}(H). \end{cases}$$

Then, by sequential continuity of $\bar{f}$, we conclude $\Psi_{t_n}(\omega) \to \Psi_t(\omega)$. This proves the left continuity of Ψ. ■

The following proposition provides a version of Itō's formula for Gâteaux differentiable functions that will be used later.

Proposition 4.2.12. *Let $\tilde{b} \in \mathscr{L}^1_{\mathscr{P}_T}(\mathbb{W})$, $\tilde{\Phi} \in \mathscr{L}^2_{\mathscr{P}_T}(C([0,T],L_2(U,H)))$, and let W be a U-valued cylindrical Wiener process. Let $t_0 \in [0,T]$ and $Y \in \mathscr{L}^1_{\mathscr{P}_T}(\mathbb{W})$. Let $\tilde{X} \in \mathscr{L}^1_{\mathscr{P}_T}(\mathbb{W})$ be the Itō process defined by*

$$\tilde{X}_t = Y_{t\wedge t_0} + \int_{t_0}^{t_0\vee t} \tilde{b}_s ds + \int_{t_0}^{t_0\vee t} \tilde{\Phi}_s dW_s \qquad \forall t \in [0,T]. \tag{4.2.17}$$

Let $f\colon [0,T]\times H \to \mathbb{R}$ be such that the derivatives $\partial_t f(t,x)$, $\partial_v f(t,x)$, $\partial^2_{vw} f(t,x)$ exist for all $t\in[0,T]$, $x,v,w\in H$, and are jointly continuous with respect to t,x,v,w. Suppose that

$$\begin{cases} \sup\limits_{(t,x)\in[0,T]\times H} \dfrac{|\partial_t f(t,x)|}{1+|x|_H} < \infty \\ \sup\limits_{\substack{(t,x)\in[0,T]\times H\\ v\in H,\ |v|_H\le 1}} |\partial_v f(t,x)| < \infty \\ \sup\limits_{\substack{(t,x)\in[0,T]\times H\\ v,w\in H,\ |v|_H\vee|w|_H\le 1}} |\partial^2_{vw} f(t,x)| < \infty. \end{cases} \tag{4.2.18}$$

Then

(i) $\{\partial_t f(t,\tilde{X}_t)\}_{t\in[0,T]} \in \mathscr{L}^1_{\mathscr{P}_T}(C([0,T],\mathbb{R}))$*;*

(ii) $\{\partial_H f(t,\tilde{X}_t).\tilde{b}_t\}_{t\in[0,T]} \in L^1_{\mathscr{P}_T}(\mathbb{R})$*;*

(iii) $\{\partial_H f(t,\tilde{X}_t).\tilde{\Phi}_t\}_{t\in[0,T]} \in L^2_{\mathscr{P}_T}(U^*)$*;*

(iv) $\{\mathrm{Tr}[\tilde{\Phi}^*_t \partial^2_H(t,\tilde{X}_t)\tilde{\Phi}_t]\}_{t\in[0,T]} \in L^1_{\mathscr{P}_T}(\mathbb{R})$*;*

and, for $t\in[t_0,T]$,

$$\begin{aligned} f(t,\tilde{X}_t) =& f(t_0,Y_{t_0}) + \int_{t_0}^t \left(\partial_t f(s,\tilde{X}_s) + \partial_H f(s,\tilde{X}_s).\tilde{b}_s + \frac{1}{2}\mathrm{Tr}[\tilde{\Phi}^*_s \partial^2_H(s,\tilde{X}_s)\tilde{\Phi}_s]\right) ds \\ &+ \int_{t_0}^t \partial_H f(s,\tilde{X}_s).\tilde{\Phi}_s dW_s \qquad \mathbb{P}\text{-a.e..} \end{aligned} \tag{4.2.19}$$

Proof. *(i)*, *(ii)*, *(iii)*, and *(iv)* are easily obtained by the assumptions on continuity and boundedness of the differentials of f.

We show how to obtain (4.2.19). Let $\{H_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite dimensional subspaces of H such that $\bigcup_{n\in\mathbb{N}} H_n$ is dense in H. Let $P_n\colon H\to H_n$ be the orthogonal projection of H onto H_n. Define $f_n(t,x) := f(t,P_n x)$ for $(t,x)\in[0,T]\times H$, $n\in\mathbb{N}$. Due to the continuity assumptions on $\partial_t f$, $\partial_H f$, $\partial^2_H f$, the restriction $f_{|[0,T]\times H_n}$ of f to $[0,T]\times H_n$ belongs to $C^{1,2}([0,T]\times H_n,\mathbb{R})$, hence $f_n\in C^{1,2}([0,T]\times H,\mathbb{R})$. Moreover, (4.2.18) holds also for f_n, with bounds uniform in n. Then, by [48, p. 69, Theorem 2.10]), formula (4.2.19) holds for all f_n. To conclude the proof it is enough to prove the following limits

$$f_n(t,\tilde{X}_t) \to f(t,\tilde{X}_t) \qquad \mathbb{P}\text{-a.s., } \forall t\in[0,T] \tag{4.2.20}$$

$$\partial_t f_n(\cdot,\tilde{X}_\cdot) \to \partial_t f(\cdot,\tilde{X}_\cdot) \qquad \text{in } L^1_{\mathscr{P}_T}(\mathbb{R}) \tag{4.2.21}$$

$$\partial_H f_n(\cdot,\tilde{X}_\cdot).\tilde{b}_\cdot \to \partial_H f(\cdot,\tilde{X}_\cdot).\tilde{b}_\cdot \qquad \text{in } L^1_{\mathscr{P}_T}(\mathbb{R}) \tag{4.2.22}$$

$$\mathrm{Tr}[\tilde{\Phi}^*_\cdot \partial^2_H f_n(\cdot,\tilde{X}_\cdot)\tilde{\Phi}_\cdot] \to \mathrm{Tr}[\tilde{\Phi}^*_\cdot \partial^2_H f(\cdot,\tilde{X}_\cdot)\tilde{\Phi}_\cdot] \qquad \text{in } L^1_{\mathscr{P}_T}(\mathbb{R}) \tag{4.2.23}$$

$$\partial_H f_n(\cdot,\tilde{X}_\cdot).\tilde{\Phi}_\cdot \to \partial_H f(\cdot,\tilde{X}_\cdot).\tilde{\Phi}_\cdot \qquad \text{in } L^2_{\mathscr{P}_T}(U^*). \tag{4.2.24}$$

Convergence (4.2.20) is clear. Since (4.2.18) holds with f_n in place of f, with bounds uniform in n, in order to prove (4.2.21), (4.2.22), (4.2.23), (4.2.24), it is sufficient to show that those convergences hold pointwise. Let $\varphi \in L_2(U,H)$ and $(t,x) \in [0,T] \times H$. Let $\{u_n\}_{n\in\mathbb{N}} \subset U$ be a sequence such that $|u_n|_U \leq 1$ for all n and $u_n \rightharpoonup u$. Since φ is compact, $\varphi u_n \to \varphi u$ in H, hence $P_n \varphi u_n \to \varphi u$. By continuity of $\partial_v f(t,x)$ in t,x,v, we then have

$$\partial_H f_n(t,x).(\varphi u_n) = \partial_H f(t,P_n x).(P_n \varphi u_n) \to \partial_H f(t,x).(\varphi u).$$

Since we also have $\partial_H f(t,x).(\varphi u_n) \to \partial_H f(t,x).(\varphi u)$, we conclude $\partial_H f_n(t,x).\varphi \to \partial_H f(t,x).\varphi$ in U^*. This provides (4.2.24). The other pointwise convergences can be proved with similar arguments. ■

Under the following assumption, we prove in Proposition 4.2.14 a less general version of Theorem 4.2.8, in which the functional u is of the form $u(t,\mathbf{x}) = f(\mathbf{x}_{t\wedge\cdot})$.

Assumption 4.2.13. *The function f belongs to $\mathscr{G}^2_{\sigma^s}(\mathbb{W},\mathbb{R})$ and its differentials ∂f and $\partial^2 f$ are bounded, i.e.*

$$M_1 := \sup_{\substack{\mathbf{x},\mathbf{v}\in\mathbb{W} \\ |\mathbf{v}|_\infty \leq 1}} |\partial f(\mathbf{x}).\mathbf{v}| < \infty \tag{4.2.25}$$

$$M_2 := \sup_{\substack{\mathbf{x},\mathbf{v},\mathbf{w}\in\mathbb{W} \\ |\mathbf{w}|\vee|\mathbf{v}|_\infty \leq 1}} \left|\partial^2 f(\mathbf{x}).(\mathbf{v},\mathbf{w})\right| < \infty. \tag{4.2.26}$$

By Remark 4.1.5, due to the sequential continuity of the differentials, (4.2.25) and (4.2.26) are equivalent to

$$M_1 = \sup_{\substack{\mathbf{x}\in\mathbb{W} \\ \mathbf{v}\in\mathbb{V}(H),\ |\mathbf{v}|_\infty \leq 1}} \left|\overline{\partial f}(\mathbf{x}).\mathbf{v}\right| < \infty, \tag{4.2.27}$$

$$M_2 = \sup_{\substack{\mathbf{x}\in\mathbb{W} \\ \mathbf{v},\mathbf{w}\in\mathbb{V}(H),\ |\mathbf{w}|\vee|\mathbf{v}|_\infty \leq 1}} \left|\overline{\partial^2 f}(\mathbf{x}).(\mathbf{v},\mathbf{w})\right| < \infty. \tag{4.2.28}$$

Proposition 4.2.14. *Suppose that f satisfies Assumption 4.2.13. For $\hat{Y} \in \mathscr{L}^1_{\mathscr{P}_T}(\mathbb{W})$ and $\hat{t} \in [0,T]$, let $X^{\hat{t},\hat{Y}}$ be the process defined by* (4.2.9). *Then*

(i) $\left\{\overline{\partial f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]} b_t)\right\}_{t\in[0,T]} \in L^1_{\mathscr{P}_T}(\mathbb{R})$;

(ii) $\left\{\overline{\partial f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]} \Phi_t)\right\}_{t\in[0,T]} \in L^2_{\mathscr{P}_T}(U^*)$;

(iii) $\left\{\mathbf{T}\left[\overline{\partial^2 f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}), \mathbf{1}_{[t,T]} \Phi_t\right]\right\}_{t\in[0,T]} \in L^1_{\mathscr{P}_T}(\mathbb{R})$.

Moreover, for $t \in [\hat{t}, T]$,

$$f(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}) = f(\hat{Y}_{\hat{t}\wedge\cdot}) + \int_{\hat{t}}^{t} \left(\overline{\partial f}(X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}b_s) + \frac{1}{2}\mathbf{T}\left[\overline{\partial^2 f}(X^{\hat{t},\hat{Y}}_{s\wedge\cdot}), \mathbf{1}_{[s,T]}\Phi_s\right]\right) ds \\ + \int_{\hat{t}}^{t} \overline{\partial f}(X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}\Phi_s) dW_s, \qquad \mathbb{P}\text{-}a.e.. \tag{4.2.29}$$

Proof. By Lemma 4.2.11, the process

$$\left\{\overline{\partial f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}b_t)\right\}_{t\in[0,T]}$$

is left-continuous and adapted, hence predictable. Similarly, the process

$$\left\{\overline{\partial f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}\Phi_t u)\right\}_{t\in[0,T]} \tag{4.2.30}$$

is left-continuous and adapted, hence predictable, for all $u \in U$.

If $(\omega, t) \in \Omega_T$ and $\{u_n\}_{n\in\mathbb{N}}$ is a sequence converging to 0 in U, then

$$\{\mathbf{1}_{[t,T]}\Phi_t(\omega)u_n\}_{n\in\mathbb{N}} \tag{4.2.31}$$

is a uniformly bounded sequence in $\mathbb{V}(H)$, converging pointwise to 0. Then, by Proposition 4.1.2*(i)*, the sequence (4.2.31) converges to 0 in $\mathbb{V}_{\sigma^s}(H)$. By $\mathbb{V}_{\sigma^s}(H)$-sequential continuity of $\overline{\partial f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}(\omega))$, we conclude

$$\lim_{n\to\infty} \overline{\partial f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}(\omega)).(\mathbf{1}_{[t,T]}\Phi_t(\omega)u_n) = 0.$$

This shows that, for all $(\omega, t) \in \Omega_T$, $\overline{\partial f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}(\omega)).(\mathbf{1}_{[t,T}\Phi_t(\omega)) \in U^*$. Then, by separability of U and by Pettis's measurability theorem, we have that

$$\left\{\overline{\partial f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}\Phi_t)\right\}_{t\in[0,T]}$$

is a U^*-valued predictable process.

We now show the integrability properties in *(i)* and *(ii)*. By (4.2.27), we have

$$\mathbb{E}\left[\int_0^T \left|\overline{\partial f}(X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}b_s)\right| ds\right] \leq M_1 T |b|_{\mathscr{L}^1_{\mathscr{P}_T}(C([0,T],H))},$$

which concludes the proof of *(i)*. Similarly, by (4.2.28),

$$\mathbb{E}\left[\int_0^T \sup_{\substack{u\in U\\|u|_U\leq 1}} \left|\overline{\partial f}(X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}\Phi_s u)\right|^2 ds\right] \leq M_1^2 \mathbb{E}\left[\int_0^T |\Phi_s|^2_{L(U,H)} ds\right] \\ \leq M_1^2 T |\Phi|^2_{\mathscr{L}^2_{\mathscr{P}_T}(C([0,T],L_2(U,H)))}.$$

This concludes the proof of *(ii)*.

To show *(iii)*, we first prove that the sum defining $\mathbf{T}\left[\overline{\partial^2 f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}), \mathbf{1}_{[t,T]}\Phi_t\right]$ is convergent. By (4.2.28), we have,

$$\begin{aligned}\sum_{m\in\mathcal{M}}\left|\overline{\partial^2 f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}\Phi_t e'_m, \mathbf{1}_{[t,T]}\Phi_t e'_m)\right| &\le M_2\sum_{m\in\mathcal{M}}|\mathbf{1}_{[t,T]}\Phi_t e'_m|^2_\infty \\ &= M_2\sum_{m\in\mathcal{M}}|\Phi_t e'_m|^2_H \\ &= M_2|\Phi_t|^2_{L_2(U,H)}.\end{aligned} \tag{4.2.32}$$

Then $\mathbf{T}\left[\overline{\partial^2 f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}), \mathbf{1}_{[t,T]}\Phi_t\right]$ is well defined, for all $t\in[0,T]$. By Lemma 4.2.11, for every $m\in\mathcal{M}$, the process

$$\left\{\overline{\partial^2 f}(t, X^{\hat{t},\hat{Y}}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}\Phi_t e'_m, \mathbf{1}_{[t,T]}\Phi_t e'_m)\right\}_{t\in[0,T]} \tag{4.2.33}$$

is adapted and left-continuous, hence predictable. Then $\left\{\mathbf{T}\left[\overline{\partial^2 f}(X^{\hat{t},\hat{Y}}_{t\wedge\cdot}), \mathbf{1}_{[t,T]}\Phi_t\right]\right\}_{t\in[0,T]}$ is predictable. It is also integrable, by (4.2.32).

We finally address formula (4.2.29). We will derive it from the standard Itō's formula in Hilbert spaces, by using the approximation framework introduced at pp. 122–124.

Since, by Assumption 4.2.13, $f\in\mathcal{G}^2(\mathbb{W},\mathbb{R})$, by linearity of ℓ_π we have that

$$f_\pi\colon H^n\to\mathbb{R},\ x\mapsto f(\ell_\pi(x))$$

f_π is strongly continuously Gâteaux differentiable up to order 2 on H^n, with

$$\partial f_\pi(x).v = \partial f(\ell_\pi(x)).\ell_\pi(v), \tag{4.2.34}$$

for all $(x,v)\in H^n\times H^n$,

$$\partial^2 f_\pi(x).(v,w) = \partial^2 f(\ell_\pi(x)).(\ell_\pi(v),\ell_\pi(w)), \tag{4.2.35}$$

for all $(x,v,w)\in H^n\times H^n\times H^n$. Then we can apply the standard Itō's formula, in the version provided by Proposition 4.2.12, to the predictable pathwise continuous process

$$\left\{f_\pi(X^{(\pi)}_t)\right\}_{t\in[0,T]} = \left\{f(\ell_\pi(X^{(\pi)}_t))\right\}_{t\in[0,T]}.$$

For $t\in[\hat{t},T]$, we have

$$\begin{aligned}f_\pi(X^{(\pi)}_t) = f_\pi(X^{(\pi)}_{\hat{t}}) + \int_{\hat{t}}^t\left(\partial f_\pi(X^{(\pi)}_s).b^{(\pi)}_s\frac{1}{2}\mathrm{Tr}\left((\Phi^{(\pi)}_s)^*\partial^2 f_\pi(X^{(\pi)}_s)\Phi^{(\pi)}_s\right)\right)ds \\ + \int_{\hat{t}}^t\partial f_\pi(X^{(\pi)}_s).\Phi^{(\pi)}_s dW_s \quad \mathbb{P}\text{-a.e..}\end{aligned} \tag{4.2.36}$$

Through several steps, we are going to prove that the terms appearing in (4.2.36) converge to the corresponding terms in (4.2.29), as $\delta(\pi)\to 0$.

Let $\{\pi_n\}_{n\in\mathbb{N}}$ be a sequence of partition of $[0,T]$ such that $\lim_{n\to\infty}\delta(\pi_n)=0$.

Step 1. By (4.2.15) and by continuity of f, we immediately have that, for $t\in[0,T]$, $f_\pi(X_t^{(\pi_n)})\to f(X_{t\wedge\cdot}^{\hat{t},\hat{Y}})$ $\mathbb{P}$-a.e..

Step 2. We show that

$$\lim_{n\to\infty}\partial f_{\pi_n}(X_\#^{(\pi_n)}).(b_\#^{(\pi_n)})=\overline{\partial f}(X_{\#\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[\#,T]}b_\#) \text{ in } L^1_{\mathcal{P}_T}(\mathbb{R}). \tag{4.2.37}$$

We notice that, by the very definition of $b_s^{(\pi_n)}$ in (4.2.14) and of ℓ_π (see also Remark 4.2.10), we have, for all $\omega\in\Omega$ and $s\in[0,T]$,

$$\ell_{\pi_n}(b_s^{(\pi_n)}(\omega))=\begin{cases} b_s(\omega) & \text{on } [s,T] \\ 0 & \text{on } [0,s-2\delta(\pi_n)] \end{cases}$$

and $\sup_{n\in\mathbb{N}}|\ell_{\pi_n}(b_s^{(\pi_n)}(\omega))|_\infty\le|b_s(\omega)|_H$. By Proposition 4.1.2*(i)*, it follows

$$\lim_{n\to\infty}\ell_{\pi_n}(b_s^{(\pi_n)}(\omega))=\mathbf{1}_{[s,T]}b_s(\omega) \text{ in } \mathbb{V}_{\sigma^s}(H),\ \forall(\omega,s)\in\Omega_T. \tag{4.2.38}$$

By (4.2.27) and (4.2.34),

$$\begin{aligned}\sup_{s\in[0,T]}|\partial f_{\pi_n}(X_s^{(\pi_n)}).b_s^{(\pi_n)}|+\sup_{s\in[0,T]}|\overline{\partial f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}b_s)|\\ \le M_1\left(\sup_{s\in[0,T]}|\ell_{\pi_n}(b_s^{(\pi_n)})|_\infty+\sup_{s\in[0,T]}|\mathbf{1}_{[s,T]}b_s|_\infty\right)=2M_1|b|_\infty.\end{aligned} \tag{4.2.39}$$

By (4.2.15), (4.2.38), (4.2.39), sequential continuity of $\overline{\partial f}$, and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^T\left|\partial f_{\pi_n}(X_s^{(\pi_n)}).b_s^{(\pi_n)}-\overline{\partial f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}b_s)\right|ds\right]=0,$$

which provides (4.2.37).

Step 3. We show that

$$\lim_{n\to\infty}\partial f_{\pi_n}(X_\#^{(\pi_n)}).\Phi_\#^{(\pi_n)}=\overline{\partial f}(X_{\#\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[\#,T]}\Phi_\#) \text{ in } L^2_{\mathcal{P}_T}(U^*). \tag{4.2.40}$$

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence weakly convergent to u in the unit ball of U. Since $\Phi_s(\omega)$ is compact, $\Phi_s(\omega)u_n\to\Phi_s(\omega)u$ strongly in H for all $(\omega,s)\in\Omega_T$. We also have, for $n\in\mathbb{N}$,

$$\ell_{\pi_n}(\Phi_s^{(\pi_n)}(\omega)u_n)=\begin{cases}\Phi_s(\omega)u_n & \text{on } [s,T] \\ 0 & \text{on } [0,s-2\delta(\pi_n)).\end{cases}$$

and

$$\sup_{n\in\mathbb{N}}\left|\ell_{\pi_n}(\Phi_s^{(\pi_n)}(\omega)u_n)\right|_\infty\le\sup_{n\in\mathbb{N}}|\Phi_s(\omega)u_n|_H\le|\Phi_s(\omega)|_{L(U,H)}.$$

Then, by Proposition 4.1.2*(i)*,

$$\lim_{n\to\infty} \ell_{\pi_n}(\Phi_s^{(\pi_n)}(\omega)u_n) = \mathbf{1}_{[s,T]}\Phi_s(\omega)u \text{ in } \mathbb{V}_{\sigma^s}(H),\ \forall(\omega,s)\in\Omega_T. \tag{4.2.41}$$

By (4.2.15), (4.2.34), (4.2.41), we obtain

$$\lim_{n\to\infty}\left|\partial f_{\pi_n}(X_s^{(\pi_n)}).(\Phi_s^{(\pi_n)}u_n) - \overline{\partial f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}\Phi_s u)\right| = 0 \qquad \forall(\omega,s)\in\Omega_T.$$

By (4.2.41) and sequential continuity of $\overline{\partial f}$, we have

$$\lim_{n\to\infty}\left|\overline{\partial f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}\Phi_s(u_n-u))\right| = 0 \qquad \forall(\omega,s)\in\Omega_T.$$

Since the weakly convergent sequence $\{u_n\}_{n\in\mathbb{N}}$ is arbitrary, the two limits above let us to conclude

$$\lim_{n\to\infty}\left|\partial f_{\pi_n}(X_s^{(\pi_n)}).\Phi_s^{(\pi_n)} - \overline{\partial f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}\Phi_s)\right|_{U^*} = 0 \qquad \forall(\omega,s)\in\Omega_T. \tag{4.2.42}$$

Moreover, by (4.2.27) and (4.2.34), for $u\in U$, $|u|_U=1$,

$$\begin{aligned}\sup_{s\in[0,T]}&|\partial f_{\pi_n}(X_s^{(\pi_n)}).\Phi_s^{(\pi_n)}u| + \sup_{s\in[0,T]}|\overline{\partial f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}\Phi_s u)|\\ &\le M_1\left(\sup_{s\in[0,T]}\left|\ell_{\pi_n}(\Phi_s^{(\pi_n)}u)\right|_\infty + \sup_{s\in[0,T]}\left|\mathbf{1}_{[s,T]}\Phi_s u\right|_\infty\right)\\ &\le 2M_1\sup_{s\in[0,T]}|\Phi_s|_{L(U,H)} \le 2M_1\sup_{s\in[0,T]}|\Phi_s|_{L_2(U,H)}.\end{aligned} \tag{4.2.43}$$

By (4.2.42), (4.2.43), and by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^T \left|\partial f_{\pi_n}(X_s^{(\pi_n)})(\Phi_s^{(\pi_n)}u) - \overline{\partial f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}})(\mathbf{1}_{[s,T]}(\Phi_s u)\right|_{U^*}^2 ds\right] = 0.$$

This provides (4.2.40).

Step 4. We show that

$$\lim_{n\to\infty}\operatorname{Tr}\left((\Phi_\#^{(\pi_n)})^*\partial^2 f_{\pi_n}(X_\#^{(\pi_n)})\Phi_\#^{(\pi_n)}\right) = \mathbf{T}\left[\overline{\partial^2 f}(X_{\#\wedge\cdot}^{\hat{t},\hat{Y}}),\mathbf{1}_{[\#,T]}\Phi_\#\right] \text{ in } L^1_{\mathscr{P}_T}(\mathbb{R}). \tag{4.2.44}$$

By

$$\begin{aligned}&\left|\partial^2 f_{\pi_n}(X_s^{(\pi_n)}).(\Phi_s^{(\pi_n)}e'_m,\Phi_s^{(\pi_n)}e'_m)\right| + \left|\overline{\partial^2 f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}\Phi_s e'_m,\mathbf{1}_{[s,T]}\Phi_s e'_m)\right|\\ &\qquad\le M_2\left(|\ell_{\pi_n}(\Phi_s^{(\pi_n)}e'_m)|_\infty^2 + |\mathbf{1}_{[s,T]}\Phi_s e'_m|_\infty^2\right) = 2M_2|\Phi_s e'_m|_H^2,\end{aligned}$$

and

$$\sum_{m\in\mathcal{M}}\mathbb{E}\left[\int_0^T|\Phi_s e'_m|_H^2 ds\right] = \mathbb{E}\left[\int_0^T|\Phi_s|^2_{L_2(U,H)}ds\right] < \infty,$$

we can apply Lebesgue's dominated convergence theorem and obtain

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}\left[\int_0^T\left|\mathrm{Tr}\left((\Phi_s^{(\pi_n)})^*\partial^2 f_{\pi_n}(X_s^{(\pi_n)})\Phi_s^{(\pi_n)}\right)-\mathbf{T}\left[\overline{\partial^2 f}(X_{s\wedge\cdot}),\mathbf{1}_{[s,T]}\Phi_s\right]\right|ds\right]\\
&\quad\le \lim_{n\to\infty}\sum_{m\in\mathcal{M}}\mathbb{E}\left[\int_0^T\left|\partial^2 f_{\pi_n}(X_s^{(\pi_n)}).(\Phi_s^{(\pi_n)}e'_m,\Phi_s^{(\pi_n)}e'_m)\right.\right.\\
&\qquad\qquad\left.\left.-\overline{\partial^2 f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}\Phi_s e'_m,\mathbf{1}_{[s,T]}\Phi_s e'_m)\right|ds\right]\\
&\quad=\sum_{m\in\mathcal{M}}\mathbb{E}\left[\int_0^T\lim_{n\to\infty}\left|\partial^2 f_{\pi_n}(X_s^{(\pi_n)}).(\Phi_s^{(\pi_n)}e'_m,\Phi_s^{(\pi_n)}e'_m)\right.\right.\\
&\qquad\qquad\left.\left.-\overline{\partial^2 f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}\Phi_s e'_m,\mathbf{1}_{[s,T]}\Phi_s e'_m)\right|ds\right]\\
&\quad=0
\end{aligned}$$

where the pointwise convergence of the latter integrand comes from the sequential continuity of $\overline{\partial^2 f}$, from (4.2.15), and from

$$\lim_{n\to\infty}\ell_{\pi_n}(\Phi_s^{(\pi_n)}(\omega)e'_m)=\mathbf{1}_{[s,T]}\Phi_s(\omega)e'_m \text{ in } \mathbb{V}_{\sigma^s}(H),\ \forall(\omega,s)\in\Omega_T,\ \forall m\in\mathcal{M}$$

(that comes from (4.2.41) with $u_n=u=e'_m$ for all n).

<u>*Step 5.*</u> We can now conclude the proof of the theorem, because (4.2.29) is obtained by passing to the limit $n\to\infty$ in (4.2.36) (with π replaced by π_n), and by considering the partial results of Step 1, Step 2, Step 3, Step 4. ■

We can now prove Theorem 4.2.8.

Proof of Theorem 4.2.8. *(i)* By continuity of u, for $h\in(0,T)$, both $\{u(t,X_{(t-h)\wedge\cdot}^{\hat{t},\hat{Y}})\}_{t\in[h,T]}$ and $\{u(t-h,X_{(t-h)\wedge\cdot}^{\hat{t},\hat{Y}})\}_{t\in[h,T]}$ are pathwise continuous and $\mathbb{F}$-adapted, hence predictable. In particular, $\mathscr{D}_t^- u(\cdot,X^{\hat{t},\hat{Y}})$ is predictable on $(0,T)$ and then $\mathscr{D}_t^- u(\cdot,X^{\hat{t},\hat{Y}}(\omega))$ is measurable for all $\omega\in\Omega$. Moreover, for $\omega\in\Omega$, the map $[0,T]\to\mathbb{W}$, $t\mapsto X_{t\wedge\cdot}^{\hat{t},\hat{Y}}(\omega)$ is continuous, hence $\{X_{t\wedge\cdot}^{\hat{t},\hat{Y}}(\omega)\}_{t\in[0,T]}$ is compact in $\mathbb{W}$ and (4.2.2) implies $\mathscr{D}_t^- u(\cdot,X^{\hat{t},\hat{Y}}(\omega))\in L^1((0,T),\mathbb{R})$.

(ii)+ *(iii)*+ *(iv)* For $n\ge 1$, let $t_k^n:=kT/n$, for $k=0,\dots,n$. By applying Lemma 4.2.11 to $\overline{\partial_{\mathbb{W}}u}(t_k^n,\cdot)$, for $k=1,\dots,n$, we obtain the predictability of the process

$$\left\{\overline{\partial_{\mathbb{W}}u}(t_k^n,X_{t\wedge\cdot}^{\hat{t},\hat{Y}}).(\mathbf{1}_{[t,T]}b_t)\right\}_{t\in[0,T]}\in L^1_{\mathscr{P}_T}(\mathbb{R})\qquad\forall k=1,\dots,n.$$

By Assumption 4.2.3*(iii)*, for all $t\in(0,T]$ and all $\omega\in\Omega$,

$$\overline{\partial_{\mathbb{W}}u}(t,X^{\hat{t},\hat{Y}}(\omega)).(\mathbf{1}_{[t,T]}b_t(\omega))=\lim_{n\to\infty}\sum_{k=1}^n\mathbf{1}_{(t_{k-1}^n,t_k^n]}(t)\overline{\partial_{\mathbb{W}}u}(t_k^n,X_{t\wedge\cdot}^{\hat{t},\hat{Y}}(\omega)).(\mathbf{1}_{[t,T]}b_t(\omega)),$$

which shows that $\left\{\overline{\partial_{\mathbb{W}}u}(t,X^{\hat{t},\hat{Y}}).(\mathbf{1}_{[t,T]}b_t)\right\}_{t\in[0,T]}$ is predictable.

In the same way, by applying Lemma 4.2.11 and Pettis's measurability theorem, we see that the U^*-valued process $\left\{\overline{\partial_{\mathbb{W}}u}(t,X^{\hat{t},\hat{Y}}).(\mathbf{1}_{[t,T]}\Phi_t)\right\}_{t\in[0,T]}$ is predictable.

We now address $\left\{\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}} u}(t,X^{\hat{t},\hat{Y}}),\mathbf{1}_{[t,T]}\Phi_t\right]\right\}_{t\in[0,T]}$. Again by Lemma 4.2.11, the process

$$\left\{\overline{\partial^2_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}\Phi_t e'_m,\mathbf{1}_{[t,T]}\Phi_t e'_m)\right\}_{t\in[0,T]}$$

is predictable, for all $m\in\mathscr{M}$. Thanks to Assumption 4.2.3*(iii)*, we have, for all $t\in(0,T]$ and $\omega\in\Omega$,

$$\begin{aligned}&\overline{\partial^2_{\mathbb{W}} u}(t,X^{\hat{t},\hat{Y}}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}\Phi_t e'_m,\mathbf{1}_{[t,T]}\Phi_t e'_m)\\&\qquad=\lim_{n\to\infty}\sum_{k=1}^n\mathbf{1}_{(t^n_{k-1},t^n_k]}(t)\overline{\partial^2_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{t\wedge\cdot}).(\mathbf{1}_{[t,T]}\Phi_t e'_m,\mathbf{1}_{[t,T]}\Phi_t e'_m).\end{aligned}$$

Then $\left\{\overline{\partial^2_{\mathbb{W}} u}(t,X^{\hat{t},\hat{Y}}).(\mathbf{1}_{[t,T]}\Phi_t e'_m,\mathbf{1}_{[t,T]}\Phi_t e'_m)\right\}_{t\in[0,T]}$ is predictable, hence

$$\left\{\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}} u}(t,X^{\hat{t},\hat{Y}}),\mathbf{1}_{[t,T]}\Phi_t\right]\right\}_{t\in[0,T]}$$

is predictable too.

Finally, the integrability properties claimed in *(ii)*,*(iii)*,*(iv)* are proved exactly as for Proposition 4.2.14*(i)*,*(ii)*,*(iii)* by using Assumption 4.2.3*(ii)*.

We now prove formula (4.2.10). Considering Remark 4.1.8, without loss of generality we can assume $t=T$. Let $n\geq 1$ and let $\hat{t}=t^n_0<\ldots<t^n_n=T$ be a partition of $[\hat{t},T]$, with $t^n_k-t^n_{k-1}=(T-\hat{t})/n$, for $k=1,\ldots,n$. We first write

$$\begin{aligned}u(T,X^{\hat{t},\hat{Y}})-u(\hat{t},\hat{Y})&=\sum_{k=1}^n\left(u(t^n_k,X^{\hat{t},\hat{Y}})-u(t^n_{k-1},X^{\hat{t},\hat{Y}})\right)\\&=\sum_{k=1}^n\left(u(t^n_k,X^{\hat{t},\hat{Y}})-u(t^n_k,X^{\hat{t},\hat{Y}}_{t^n_{k-1}\wedge\cdot})\right)+\sum_{k=1}^n\left(u(t^n_k,X^{\hat{t},\hat{Y}}_{t^n_{k-1}\wedge\cdot})-u(t^n_{k-1},X^{\hat{t},\hat{Y}})\right)\\&=:\mathbf{I}_n+\mathbf{II}_n.\end{aligned}\tag{4.2.45}$$

For $k=1,\ldots,n$, due to our assumptions on u, we can apply Proposition 4.2.14 to $u(t^n_k,\cdot)$, then (4.2.29) gives

$$\begin{aligned}u(t^n_k,X^{\hat{t},\hat{Y}})=&u(t^n_k,\hat{Y}_{\hat{t}\wedge\cdot})+\int_{\hat{t}}^{t^n_k}\left(\overline{\partial_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}b_s)+\frac{1}{2}\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}),\mathbf{1}_{[s,T]}\Phi_s\right]\right)ds\\&+\int_{\hat{t}}^{t^n_k}\overline{\partial_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}\Phi_s)dW_s\\=&u(t^n_k,X^{\hat{t},\hat{Y}}_{t^n_{k-1}\wedge\cdot})+\int_{t^n_{k-1}}^{t^n_k}\left(\overline{\partial_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}b_s)+\frac{1}{2}\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}),\mathbf{1}_{[s,T]}\Phi_s\right]\right)ds\\&+\int_{t^n_{k-1}}^{t^n_k}\overline{\partial_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}\Phi_s)dW_s,\qquad\mathbb{P}\text{-a.e.}.\end{aligned}$$

Then

$$\begin{aligned}\mathbf{I}_n=&\int_{\hat{t}}^T\sum_{k=1}^n\mathbf{1}_{(t^n_{k-1},t^n_k]}(s)\left(\overline{\partial_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}b_s)+\frac{1}{2}\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}),\mathbf{1}_{[s,T]}\Phi_s\right]\right)ds\\&+\int_{\hat{t}}^T\sum_{k=1}^n\mathbf{1}_{(t^n_{k-1},t^n_k]}(s)\overline{\partial_{\mathbb{W}} u}(t^n_k,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}\Phi_s)dW_s,\qquad\mathbb{P}\text{-a.e.}.\end{aligned}$$

By Assumption 4.2.3*(ii)*,*(iii)*, we can apply Lebesgue's dominated convergence theorem (the integrands are estimated similarly as done in Steps 2–4 of the proof of Proposition 4.2.14) and obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbf{1}_{(t^n_{k-1},t^n_k]}(\#)\left(\overline{\partial_{\mathbb{W}}u}(t^n_k,X^{\hat{t},\hat{Y}}_{\#\wedge\cdot}).(\mathbf{1}_{[\#,T]}b_\#)+\frac{1}{2}\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}u}(t^n_k,X^{\hat{t},\hat{Y}}_{\#\wedge\cdot}),\mathbf{1}_{[\#,T]}\Phi_\#\right]\right)$$
$$=\overline{\partial_{\mathbb{W}}u}(\#,X^{\hat{t},\hat{Y}}_{\#\wedge\cdot}).(\mathbf{1}_{[\#,T]}b_\#)+\frac{1}{2}\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}u}(\#,X^{\hat{t},\hat{Y}}_{\#\wedge\cdot}),\mathbf{1}_{[\#,T]}\Phi_\#\right]\ \text{in } L^1_{\mathscr{P}_T}(\mathbb{R})$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbf{1}_{(t^n_{k-1},t^n_k]}(\#)\overline{\partial_{\mathbb{W}}u}(t^n_k,X^{\hat{t},\hat{Y}}_{\#\wedge\cdot}).(\mathbf{1}_{[\#,T]}\Phi_\#)=\overline{\partial_{\mathbb{W}}u}(\#,X^{\hat{t},\hat{Y}}_{\#\wedge\cdot}).(\mathbf{1}_{[\#,T]}\Phi_\#)\ \text{in } L^2_{\mathscr{P}_T}(U^*).$$

The two limits above permit to obtain the following limit in $L^1(\Omega,\mathbb{R})$:

$$\begin{aligned}\lim_{n\to\infty}\mathbf{I}_n=&\int_{\hat{t}}^{T}\left(\overline{\partial_{\mathbb{W}}u}(s,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}b_s)+\frac{1}{2}\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}u}(s,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}),\mathbf{1}_{[s,T]}\Phi_s\right]\right)ds\\&+\int_{\hat{t}}^{T}\overline{\partial_{\mathbb{W}}u}(s,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}\Phi_s)dW_s.\end{aligned}\tag{4.2.46}$$

We now address the term $\mathbf{II}_n$. By Assumption 4.2.3*(i)*, continuity of u, and recalling Remark 4.2.2, we can apply [44, (1.4.4), p. 23] and conclude that $(t,T)\to\mathbb{R}$, $t\mapsto u(s,\mathbf{x}_{t\wedge\cdot})$ is Lipschitz. We can then write

$$\begin{aligned}\mathbf{II}_n&=\sum_{k=1}^{n}\int_{t^n_{k-1}}^{t^n_k}\frac{d}{ds}u(s,X^{\hat{t},\hat{Y}}_{t^n_{k-1}\wedge\cdot})ds=\sum_{k=1}^{n}\int_{t^n_{k-1}}^{t^n_k}\mathscr{D}^-_t u(s,X^{\hat{t},\hat{Y}}_{t^n_{k-1}\wedge\cdot})ds\\&=\int_{\hat{t}}^{T}\left(\sum_{k=1}^{n}\mathbf{1}_{(t^n_{k-1},t^n_k]}(s)\mathscr{D}^-_t u(s,X^{\hat{t},\hat{Y}}_{t^n_{k-1}\wedge\cdot})\right)ds.\end{aligned}\tag{4.2.47}$$

Fix $\omega\in\Omega$. As noticed at the beginning of the proof, the set $K:=\{X^{\hat{t},\hat{Y}}_{t\wedge\cdot}(\omega)\}_{t\in[0,T]}$ is compact in $\mathbb{W}$. Then, by Assumption 4.2.3*(i)*, there exists $M_K>0$ (depending on ω, since our compact set K depends on ω too) such that

$$\left|\sum_{k=1}^{n}\mathbf{1}_{(t^n_{k-1},t^n_k]}(s)\mathscr{D}^-_t u(s,X^{\hat{t},\hat{Y}}_{t^n_{k-1}\wedge\cdot}(\omega))\right|_H\le M_K\ \text{for a.e. } s\in(0,T).\tag{4.2.48}$$

For fixed $s\in(0,T)$, let $\{k_n\}_{n\in\mathbb{N}}$ be the sequence such that $s\in(t^n_{k_n-1},t^n_{k_n}]$ for all $n\in\mathbb{N}$, $n\ge1$. Then $X^{\hat{t},\hat{Y}}_{t^n_{k_n-1}\wedge\cdot}(\omega)\to X^{\hat{t},\hat{Y}}_{s\wedge\cdot}(\omega)$ in $\mathbb{W}$ as $n\to\infty$. Since this holds for all $s\in(0,T)$ and since $\mathbb{W}\to\mathbb{R}$, $\mathbf{x}\mapsto\mathscr{D}^-_t u(s,\mathbf{x})$, is continuous for a.e. $s\in(0,T)$ because of Assumption 4.2.3*(i)*, we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbf{1}_{(t^n_{k-1},t^n_k]}(s)\mathscr{D}^-_t u(s,X^{\hat{t},\hat{Y}}_{t^n_{k-1}\wedge\cdot}(\omega))=\mathscr{D}^-_t u(s,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}(\omega))\ \text{for a.e. } s\in(0,T).\tag{4.2.49}$$

By (4.2.48) and (4.2.49), we can apply Lebesgue's dominated convergence theorem to (4.2.47) evaluated in ω and obtain

$$\lim_{n\to\infty}\mathbf{II}_n(\omega)=\int_{\hat{t}}^{T}\mathscr{D}^-_t u(s,X^{\hat{t},\hat{Y}}_{s\wedge\cdot}(\omega))ds.$$

Since $\omega \in \Omega$ was arbitrary, we have

$$\lim_{n\to\infty} \mathbf{II}_n = \int_{\hat{t}}^{T} \mathscr{D}_t^- u(s, X_{s\wedge\cdot}^{\hat{t},\hat{Y}})ds \text{ pointwise on } \Omega. \tag{4.2.50}$$

This concludes the proof, because, by passing to the limit $n \to \infty$ in (4.2.45) and considering (4.2.46) and (4.2.50), we obtain (4.2.10) with $t = T$. ■

4.3 Application to path-dependent PDEs

In this section we use the path-dependent Itō's formula to relate the solution to an H-valued path-dependent SDE with a path-dependent Kolmogorov equation, similarly as in the classical non-path-dependent case (see e.g. [23, Ch. 7]). As a corollary, we will derive a Clark-Ocone type formula.

The following assumption on b, Φ will be standing for the remaining of the present section.

Assumption 4.3.1. *$b \in CNA([0,T]\times\mathbb{W}, H)$, $\Phi \in CNA([0,T]\times\mathbb{W}, L_2(U,H))$, and there exists $M > 0$ such that*

$$\begin{cases} |b(t,\mathbf{x}) - b(t,\mathbf{x}')|_H \leq M|\mathbf{x}-\mathbf{x}'|_\infty \\ |b(t,\mathbf{x})|_H \leq M(1+|\mathbf{x}|_\infty) \end{cases} \qquad \begin{cases} |\Phi(t,\mathbf{x}) - \Phi(t,\mathbf{x}')|_{L_2(U,H)} \leq M|\mathbf{x}-\mathbf{x}'|_\infty \\ |\Phi(t,\mathbf{x})|_{L_2(U,H)} \leq M(1+|\mathbf{x}|_\infty) \end{cases}$$

for all $t \in [0,T]$, $\mathbf{x}, \mathbf{x}' \in \mathbb{W}$.

For $p > 2$, $\hat{Y} \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$, $\hat{t} \in [0,T]$, we consider the following path-dependent SDE

$$\begin{cases} dX_s = b(s,X)ds + \Phi(s,X)dW_s & \forall s \in [\hat{t}, T] \\ X_{\hat{t}\wedge\cdot} = \hat{Y}_{\hat{t}\wedge\cdot}. \end{cases} \tag{4.3.1}$$

By Theorem 1.2.6, there exists a unique strong solution $X^{\hat{t},\hat{Y}}$ to (4.3.1) in $\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$, i.e. a unique process $X^{\hat{t},\hat{Y}} \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$ such that, for all $t \in [0,T]$,

$$X_t^{\hat{t},\hat{Y}} = \hat{Y}_{\hat{t}\wedge t} + \int_{\hat{t}}^{\hat{t}\vee t} b(r, X^{\hat{t},\hat{Y}})dr + \int_{\hat{t}}^{\hat{t}\vee t} \Phi(r, X^{\hat{t},\hat{Y}})dW_r \qquad \mathbb{P}\text{-a.e.}.$$

Moreover, the map

$$[0,T]\times\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}) \to \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}),\ (t,Y) \mapsto X^{t,Y} \tag{4.3.2}$$

is Lipschitz continuous with respect to Y, uniformly for $t \in [0,T]$, and jointly continuous in (t,Y). Uniqueness of solution yields the flow property

$$X^{t,\mathbf{x}} = X^{s,X^{t,\mathbf{x}}} \text{ in } \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W}),\ \forall (t,\mathbf{x}) \in [0,T]\times\mathbb{W},\ \forall s \in [t,T]. \tag{4.3.3}$$

Let $f\colon \mathbb{W} \to \mathbb{R}$ be a Lipschitz function. Hereafter in this section, we denote by φ the function

$$\varphi\colon [0,T]\times\mathbb{W}\to\mathbb{R}$$

defined by

$$\varphi(t,\mathbf{x}) := \mathbb{E}\left[f(X^{t,\mathbf{x}})\right] \qquad \forall (t,\mathbf{x})\in[0,T]\times\mathbb{W}. \tag{4.3.4}$$

Due to the continuity properties of the map (4.3.2), $\varphi(t,\mathbf{x})$ is Lipschitz continuous with respect to $\mathbf{x}$, uniformly for $t\in[0,T]$, and jointly continuous in $(t,\mathbf{x})$. It is clear that $\varphi(t,\mathbf{x}) = \varphi(t,\mathbf{x}_{t\wedge\cdot})$. Then $\varphi\in CNA([0,T]\times\mathbb{W},\mathbb{R})$. Since $X^{t,\mathbf{x}}$ is independent of $\mathscr{F}_t$, we can write, by (4.3.3) and [3, Lemma 3.9, p. 55],

$$\begin{aligned}\varphi(t',\mathbf{x}) =& \mathbb{E}\left[f(X^{t',\mathbf{x}})\right] = \mathbb{E}\left[f(X^{t,X^{t',\mathbf{x}}_{t\wedge\cdot}})\right]\\ =& \mathbb{E}\left[\mathbb{E}\left[f(X^{t,X^{t',\mathbf{x}}_{t\wedge\cdot}})|\mathscr{F}_t\right]\right] = \mathbb{E}\left[\varphi(t,X^{t',\mathbf{x}}_{t\wedge\cdot})\right] = \mathbb{E}\left[\varphi(t,X^{t',\mathbf{x}})\right] \qquad \forall t\in[t',T].\end{aligned} \tag{4.3.5}$$

In what follows, we will show that, in case $\varphi(t,\mathbf{x})$ is sufficiently regular with respect to the variable $\mathbf{x}$, then Proposition 4.2.14 can be used to conclude that $\mathscr{D}_t^-\varphi$ exists everywhere and that φ solves a path-dependent backward Kolmogorov equation associated to SDE (4.3.1). We argue similarly as in [23, Ch. 7], where, differently than in our case, the setting is non-path-dependent. The two main tools of the argument are (4.3.5) and formula (4.2.29).

In order to use formula (4.2.29), we need to make some assumptions regarding existence and regularity of the spatial derivatives of φ. In this section, we make such assumptions without any further investigation under which conditions they can be obtained. We only guess that, at least in the Markovian case, i.e. when b and Φ are not path-dependent, and the only path-dependence is due to f, the regularity assumptions on $\varphi(t,\cdot)$ should come from continuity assumption on ∂f and $\partial^2 f$ with respect to σ^s, and from regularity assumptions on the coeffiecients b and Φ, thanks to the results in [23, Ch. 7]. In the following section, we will prove that the regularity assumptions on the spatial derivatives of φ are satisfied for a particular class of dynamics X.

For a function $v(t,\mathbf{x})$, defined for $(t,\mathbf{x})\in[0,T]\times\mathbb{B}^1(H)$, the more concise notation $\partial_{\mathbb{B}^1}v$ stands for $\partial_{\mathbb{B}^1(H)}v$, and $\partial^2_{\mathbb{B}^1}v$ stands for $\partial^2_{\mathbb{B}^1(H)}v$. For a function v such that, for all $t\in(0,T)$, $v(t,\cdot)$ satisfies Assumption 4.2.13, we define $\mathscr{L}v$ by

$$\mathscr{L}v(t,\mathbf{x}) := \overline{\partial_{\mathbb{W}}v}(t,\mathbf{x}).(\mathbf{1}_{[t,T]}b(t,\mathbf{x})) + \frac{1}{2}\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}v}(t,\mathbf{x}),\mathbf{1}_{[t,T]}\Phi(t,\mathbf{x})\right] \qquad \forall (t,\mathbf{x})\in(0,T)\times\mathbb{W}.$$

Theorem 4.3.2. *Let φ be defined by* (4.3.4)*. If φ satisfies Assumption 4.2.3(ii), then φ satisfies also Assumption 4.2.3(i) and*

$$\mathscr{D}_t^-\varphi(t,\mathbf{x}) + \mathscr{L}\varphi(t,\mathbf{x}) = 0 \qquad \forall (t,\mathbf{x})\in(0,T)\times\mathbb{W}. \tag{4.3.6}$$

Proof. Let $t', t \in (0,T)$, $t' < t$, $\mathbf{x} \in \mathbb{W}$. By assumption on the spatial derivatives of $\varphi(t,\cdot)$, we can apply Proposition 4.2.14 to $\varphi(t, X^{t',\mathbf{x}}_{t\wedge\cdot})$, and obtain

$$\begin{aligned}\mathbb{E}\left[\varphi(t,X^{t',\mathbf{x}}_{t\wedge\cdot})\right] = \varphi(t,\mathbf{x}_{t'\wedge\cdot}) &+ \int_{t'}^{t}\mathbb{E}\left[\overline{\partial_{\mathbb{W}}\varphi}(t,X^{t',\mathbf{x}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}b(s,X^{t',\mathbf{x}}))\right]ds \\ &+ \frac{1}{2}\int_{t'}^{t}\mathbb{E}\left[\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}\varphi}(t,X^{t',\mathbf{x}}_{s\wedge\cdot}),\mathbf{1}_{[s,T]}\Phi(s,X^{t',\mathbf{x}})\right]\right]ds.\end{aligned} \tag{4.3.7}$$

By non-anticipativity, $\varphi(t,X^{t',\mathbf{x}}) = \varphi(t,X^{t',\mathbf{x}}_{t\wedge\cdot})$. Then, by (4.3.5) and (4.3.7), we have

$$\begin{aligned}\varphi(t,\mathbf{x}_{t'\wedge\cdot}) - \varphi(t',\mathbf{x}) = &-\int_{t'}^{t}\mathbb{E}\left[\overline{\partial_{\mathbb{W}}\varphi}(t,X^{t',\mathbf{x}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}b(s,X^{t',\mathbf{x}}))\right]ds \\ &- \frac{1}{2}\int_{t'}^{t}\mathbb{E}\left[\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}\varphi}(t,X^{t',\mathbf{x}}_{s\wedge\cdot}),\mathbf{1}_{[s,T]}\Phi(s,X^{t',\mathbf{x}})\right]\right]ds.\end{aligned}$$

By continuity of (4.3.2),

$$\lim_{t'\to t^-}\sup_{s\in[t',t]}|X^{t',\mathbf{x}}_{s\wedge\cdot} - \mathbf{x}_{t\wedge\cdot}|_H = 0 \text{ on } \Omega. \tag{4.3.8}$$

By non-anticipativity and continuity of b and Φ, we then obtain, on Ω,

$$\lim_{t'\to t^-}\sup_{s\in[t',t]}|b(s,X^{t',\mathbf{x}}) - b(t,\mathbf{x})|_H = 0$$
$$\lim_{t'\to t^-}\sup_{s\in[t',t]}|\Phi(s,X^{t',\mathbf{x}}) - \Phi(t,\mathbf{x})|_{L_2(U,H)} = 0.$$

Then, by Proposition 4.1.2*(i)*, for any sequence $\{(t'_n,s_n)\}_{n\in\mathbb{N}}$ with $t'_n \le s_n \le t$ and $t'_n \to t$, we have

$$\begin{cases}\lim_{n\to\infty}\mathbf{1}_{[s_n,T]}b(s_n,X^{t'_n,\mathbf{x}}) = \mathbf{1}_{[t,T]}b(t,\mathbf{x}) \text{ in } \mathbb{V}_{\sigma^s}(H)\\ \lim_{n\to\infty}\mathbf{1}_{[s_n,T]}\Phi(s_n,X^{t'_n,\mathbf{x}}) = \mathbf{1}_{[t,T]}\Phi(t,\mathbf{x}) \text{ in } \mathbb{V}_{\sigma^s}(L_2(U,H)).\end{cases} \tag{4.3.9}$$

By assumption, $\overline{\partial_{\mathbb{W}}\varphi}(t,\mathbf{x}).\mathbf{v}$ and $\overline{\partial^2_{\mathbb{W}}\varphi}(t,\mathbf{x}).(\mathbf{v},\mathbf{v})$ are uniformly bounded for $\mathbf{x} \in \mathbb{W}$ and $\mathbf{v} \in \mathbb{V}(H)$, $|\mathbf{v}|_\infty \le 1$, and sequentially continuous in $(\mathbf{x},\mathbf{v}) \in \mathbb{W} \times \mathbb{V}_{\sigma^s}(H)$. Then, by (4.3.8), (4.3.9), and Lebesgue's dominated convergence theorem, we have

$$\lim_{t'\to t^-}\sup_{s\in[t',t]}\mathbb{E}\left[\left|\overline{\partial_{\mathbb{W}}\varphi}(t,X^{t',\mathbf{x}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}b(s,X^{t',\mathbf{x}})) - \overline{\partial_{\mathbb{W}}\varphi}(t,\mathbf{x}).(\mathbf{1}_{[t,T]}b(t,\mathbf{x}))\right|\right] = 0 \tag{4.3.10}$$

and

$$\lim_{t'\to t^-}\sup_{s\in[t',t]}\mathbb{E}\left[\left|\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}\varphi}(t,X^{t',\mathbf{x}}_{s\wedge\cdot}),\mathbf{1}_{[s,T]}\Phi(s,X^{t',\mathbf{x}})\right] - \mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}\varphi}(t,\mathbf{x}),\mathbf{1}_{[t,T]}\Phi(t,\mathbf{x})\right]\right|\right] = 0 \tag{4.3.11}$$

Thanks to (4.3.10) and (4.3.11), we can finally write

$$\lim_{t'\to t^-}\frac{1}{t-t'}\int_{t'}^{t}\mathbb{E}\left[\overline{\partial_{\mathbb{W}}\varphi}(t,X^{t',\mathbf{x}}_{s\wedge\cdot}).(\mathbf{1}_{[s,T]}b(s,X^{t',\mathbf{x}}))\right]ds = \overline{\partial_{\mathbb{W}}\varphi}(t,\mathbf{x}).(\mathbf{1}_{[t,T]}b(t,\mathbf{x}))$$

and

$$\lim_{t'\to t^-}\frac{1}{t-t'}\int_{t'}^{t}\mathbb{E}\left[\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}\varphi}(t,X^{t',\mathbf{x}}_{s\wedge\cdot}),\mathbf{1}_{[s,T]}\Phi(s,X^{t',\mathbf{x}})\right]\right]ds=\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}\varphi}(t,\mathbf{x}),\mathbf{1}_{[t,T]}\Phi(t,\mathbf{x})\right]$$

This proves that $\mathscr{D}^-_t\varphi(t,\mathbf{x})$ exists and that (4.3.6) holds true.

We now show that $\mathscr{D}^-_t\varphi(t,\mathbf{x})$ is continuous in $\mathbf{x}$ and that

$$\sup_{\substack{t\in(0,T)\\ \mathbf{x}\in K}}|\mathscr{D}^-_t\varphi(t,\mathbf{x})|<\infty,$$

for all compact sets $K\subset\mathbb{W}$. By (4.3.6), it is sufficient to show that

$$\mathbb{W}\to\mathbb{R},\ \mathbf{x}\mapsto\mathscr{L}\varphi(t,\mathbf{x})$$

is continuous, for all $t\in(0,T)$, and that

$$\sup_{\substack{t\in(0,T)\\ \mathbf{x}\in K}}|\mathscr{L}\varphi(t,\mathbf{x})|<\infty.$$

But this is straightforward from the sublinear growth and continuity assumptions in $\mathbf{x}$ of b,Φ and from the boundedness and continuity assumption on $\overline{\partial_{\mathbb{W}}\varphi},\overline{\partial^2_{\mathbb{W}}\varphi}$. ■

Corollary 4.3.3. *If φ satisfies Assumption 4.2.3(ii),(iii), then, for all $t\in[\hat{t},T]$, we have the following representation:*

$$\varphi(t,X^{\hat{t},\hat{Y}})=\varphi(\hat{t},\hat{Y})+\int_{\hat{t}}^{t}\overline{\partial_{\mathbb{W}}\varphi}(s,X^{\hat{t},\hat{Y}}).(\mathbf{1}_{[s,T]}\Phi_s)dW_s\qquad\mathbb{P}\text{-a.e.}.\tag{4.3.12}$$

Proof. By Theorem 4.3.2, the assumptions of Theorem 4.2.8 are satisfied for φ. By applying formula (4.2.10) to $\varphi(t,X^{\hat{t},\hat{Y}})$ and recalling (4.3.6), we obtain (4.3.12). ■

4.4 The case $b(t,\mathbf{x})=b(t,\int_{[0,T]}\tilde{\mathbf{x}}(t-s)\mu(ds))$ and additive noise

In this section, in a case of interest, we show that Theorem 4.3.2 and Corollary 4.3.3 can be applied.

The following assumption will be standing for the remaining of this section.

Assumption 4.4.1.

(i) $\mu\in M([0,T])$;

(ii) $b\colon [0,T]\times H \to H$ *is continuous and there exists* $N>0$ *such that*

$$\sup_{t\in[0,T]} |b(t,y)|_H \le N(1+|y|_H) \qquad \forall y\in H, \tag{4.4.1}$$

$$\sup_{t\in[0,T]} |b(t,y)-b(t,y')|_H \le N|y-y'|_H \qquad \forall y,y'\in H. \tag{4.4.2}$$

(iii) for all $t\in[0,T]$*,* $b(t,\cdot)\in\mathscr{G}^2(H,H)$*,*

$$N_1 := \sup_{\substack{(t,y)\in[0,T]\times H\\ v\in H,\ |v|_H\le 1}} |\partial_H b(t,y).v|_H < \infty, \tag{4.4.3}$$

$$N_2 := \sup_{\substack{(t,y)\in[0,T]\times H\\ v,w\in H,\ |v|_H\vee|w|_H\le 1}} |\partial_H^2 b(t,y).(v,w)|_H < \infty, \tag{4.4.4}$$

and $\partial_H b(t,y).v$*,* $\partial_H^2 b(t,y).(v,w)$ *are jointly continuous in* t,y,v,w*.*

We define

$$\hat b(t,\mathbf{y}) := b\left(t, \int_{[0,T]} \tilde{\mathbf{y}}(t-s)\mu(ds)\right) \qquad \forall (t,\mathbf{y})\in[0,T]\times\mathbb{B}^1(H).$$

where

$$\tilde{\mathbf{y}}(r) := \mathbf{1}_{[-T,0)}(r)\mathbf{y}(0) + \mathbf{1}_{[0,T]}(r)\mathbf{y}(r) \qquad \forall r\in[-T,T]. \tag{4.4.5}$$

Then $\hat b(t,\mathbf{y})$ is a function of t and the convolution between μ and $\mathbf{y}$ computed taking into account the past history of $\mathbf{y}$ on the time window $[t-T,t]$.

Remark 4.4.2. The fact that $b(t,\cdot)\in\mathscr{G}^2(H,H)$, with differentials uniformly bounded, implies that $b(t,\cdot)\in C^1_b(H,H)$, i.e. $b(t,\cdot)$ is Fréchet differentiable and the Fréchet differential $Db(t,\cdot)$ is continuous and bounded (with bound uniform in t, due to our assumptions on b). For the proof, see [23, Proposition 7.4.1].

Let again W denote a U-valued cylindrical Wiener process and let $B\in L_2(U,H)$. Consider the following SDE:

$$\begin{cases} dX_s = \hat b(s,X)ds + BdW_s & \forall s\in[\hat t,T]\\ X_{\hat t\wedge\cdot} = \hat Y_{\hat t\wedge\cdot}, \end{cases} \tag{4.4.6}$$

for $\hat Y\in\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$, $p>2$. Notice that Assumption 4.3.1 is verified with the present coefficients b and $\Phi\equiv B$. Our aim is to prove a certain regularity of the solution $X^{\hat t,\hat Y}$ to (4.4.6) with respect to the initial datum $\hat Y\in\mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$, $p>2$, suitable to apply Theorem 4.3.2 and Corollary 4.3.3.

Remark 4.4.3. The choice $\mu=\delta_0$, Dirac measure in 0, corresponds to the Markovian case $\hat b(s,\mathbf{y}) = b(t,\mathbf{y}(t))$. By choosing $\mu=\delta_a$, Dirac measure centered in $a\in(0,T]$, we obtain a drift $\hat b(t,\mathbf{y}) = b(t,\mathbf{y}(t-a))$ with a pointwise delay.

By Assumption 4.4.1, we have that $\hat{b}(\cdot,\mathbf{y})$ is continuous for all $\mathbf{y} \in C([0,T],H)$. Moreover,

$$\left|\hat{b}(t,\mathbf{y}_1) - \hat{b}(t,\mathbf{y}_2)\right|_H \leq N \int_{[0,T]} |\tilde{\mathbf{y}}_1(t-r) - \tilde{\mathbf{y}}_2(t-r)|_H \, \mu(dr) \qquad \forall \mathbf{y}_1,\mathbf{y}_2 \in \mathbb{B}^1(H).$$

Then, if $\{\mathbf{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{W}$ and $\mathbf{y}_n \to \mathbf{y}$ in $\mathbb{B}^1_{\sigma^s}(H)$, we have $\hat{b}(t,\mathbf{y}_n) \to \hat{b}(t,\mathbf{y})$ for all $t \in [0,T]$. Hence $\hat{b}(\cdot,\mathbf{y}) \in \mathbb{B}^1(H)$, for all $\mathbf{y} \in \mathbb{B}^1(H)$. In particular, for all $\mathbf{y} \in \mathbb{B}^1(H)$, the indefinite integral

$$[0,T] \to H, \; \xi \mapsto \int_t^{t\vee\xi} \hat{b}(s,\mathbf{y})ds$$

is continuous.

These considerations entails the well-posedness, for any fixed $\omega \in \Omega$, of the map

$$\psi\colon [0,T] \times \mathbb{B}^1(H) \times \mathbb{B}^1(H) \to \mathbb{B}^1(H) \tag{4.4.7}$$

defined by

$$\psi(t,\mathbf{x},\mathbf{y}) := \mathbf{x}_{t\wedge\cdot} + \int_t^{t\vee\cdot} \hat{b}(s,\mathbf{y})ds + (W^B_{t\vee\cdot}(\omega) - W^B_t(\omega)) \quad \forall (t,\mathbf{x},\mathbf{y}) \in [0,T] \times \mathbb{B}^1(H) \times \mathbb{B}^1(H),$$

where W^B is a short notation for a fixed representant of $\int_0^{\cdot} B dW_s$.

In the following propositions, we prove existence and uniqueness of a fixed point for $\psi(t,\mathbf{x},\cdot)$ and study how the fixed point depends on $t,\mathbf{x}$. The arguments are very close to those used in the more general setting developed in Chapter 1, with the difference that the SDE is here considered pathwise, in order to have better insight about the regularity of the paths $X^{t,\mathbf{x}}(\omega)$ with respect to $\mathbf{x}$.

Remark 4.4.4. In the notation ψ, the dependence on ω is not explicit. Nevertheless, we stress the very important fact that all the bounds for the Lipschitz constants and the differentials, which appear in the following propositions, are independent of ω. More precisely, the terms λ,α appearing in Proposition 4.4.5*(i)*, the bounds for (4.4.9) and (4.4.10), the bounds for $\partial_{\mathbb{B}^1}\Lambda^{t,\cdot}$ and $\partial^2_{\mathbb{B}^1}\Lambda^{t,\cdot}$ in Proposition 4.4.6, can be — and we assume that they are — chosen independently of ω.

For $\lambda > 0$, we introduce on $\mathbb{B}^1(H)$ the norm

$$|\mathbf{x}|_\lambda := \sup_{t\in[0,T]} e^{-\lambda t}|\mathbf{x}(t)|_H, \qquad \forall \mathbf{x} \in \mathbb{B}^1(H).$$

Then $|\cdot|_\lambda$ is equivalent to $|\cdot|_\infty$.

Hereafter, we denote by $\mathbb{B}^1_\infty(H)$ the Banach space $(\mathbb{B}^1(H),|\cdot|_\infty)$ and by $\mathbb{B}^1_\lambda(H)$ the equivalent Banach space $(\mathbb{B}^1(H),|\cdot|_\lambda)$.

Proposition 4.4.5.

(i) There exists $\lambda > 0$ and $\alpha \in (0,1)$ such that

$$\sup_{(t,\mathbf{x})\in[0,T]\times\mathbb{B}^1(H)} |\psi(t,\mathbf{x},\mathbf{y}) - \psi(t,\mathbf{x},\mathbf{y}')|_\lambda \leq \alpha|\mathbf{y}-\mathbf{y}'|_\lambda \qquad \forall \mathbf{y},\mathbf{y} \in \mathbb{B}^1(H). \tag{4.4.8}$$

(ii) The restriction of ψ to $[0,T]\times\mathbb{W}\times\mathbb{B}^1_\infty(H)$ is $\mathbb{W}$-valued and continuous.

(iii) For all $t \in [0,T]$, the section

$$\psi(t,\cdot,\cdot)\colon \mathbb{B}^1_\infty(H)\times\mathbb{B}^1_\infty(H) \to \mathbb{B}^1_\infty(H),\ (\mathbf{x},\mathbf{y}) \mapsto \psi(t,\mathbf{x},\mathbf{y})$$

is strongly continuously Gâteaux differentiable up to order 2, i.e.

$$\psi(t,\cdot,\cdot) \in \mathscr{G}^2(\mathbb{B}^1_\infty(H)\times\mathbb{B}^1_\infty(H),\mathbb{B}^1_\infty(H)).$$

Moreover,

$$\sup_{\substack{t\in[0,T],\ \mathbf{x},\mathbf{y}\in\mathbb{B}^1_\infty(H)\\ \mathbf{v}\in\mathbb{B}^1_\infty(H),\ |\mathbf{v}|_\infty\leq 1}} |\partial_2\psi(t,\mathbf{x},\mathbf{y}).\mathbf{v}|_\infty + \sup_{\substack{t\in[0,T],\ \mathbf{x},\mathbf{y}\in\mathbb{B}^1_\infty(H)\\ \mathbf{v}\in\mathbb{B}^1_\infty(H),\ |\mathbf{v}|_\infty\leq 1}} |\partial_3\psi(t,\mathbf{x},\mathbf{y}).\mathbf{v}|_\infty < \infty \tag{4.4.9}$$

$$\sup_{\substack{t\in[0,T],\ \mathbf{x},\mathbf{y}\in\mathbb{B}^1_\infty(H)\\ \mathbf{v},\mathbf{w}\in\mathbb{B}^1_\infty(H),\ |\mathbf{v}|_\infty\vee|\mathbf{w}|_\infty\leq 1}} |\partial_3^2\psi(t,\mathbf{x},\mathbf{y}).(\mathbf{v},\mathbf{w})|_\infty < \infty, \tag{4.4.10}$$

where $\partial_i\psi$ and $\partial_i^2\psi$ denote the first- and second-order Gâteaux differential of ψ with respect to the i-th variable.

(iv) If $t_n \to t$ in $[0,T]$, $\mathbf{x}_n \to \mathbf{x}$ in $\mathbb{B}^1_\infty(H)$, $\mathbf{y}_n \to \mathbf{y}$ in $\mathbb{B}^1_{\sigma^s}(H)$, $\mathbf{v}_n \to \mathbf{v}$ in $\mathbb{B}^1_{\sigma^s}(H)$, $\mathbf{w}_n \to \mathbf{w}$ in $\mathbb{B}^1_{\sigma^s}(H)$, then

$$\partial_3\psi(t_n,\mathbf{x}_n,\mathbf{y}_n).\mathbf{v}_n \to \partial_3\psi(t,\mathbf{x},\mathbf{y}).\mathbf{v} \text{ in } \mathbb{B}^1_\infty(H) \tag{4.4.11}$$

$$\partial_3^2\psi(t_n,\mathbf{x}_n,\mathbf{y}_n).(\mathbf{v}_n,\mathbf{w}_n) \to \partial_3^2\psi(t,\mathbf{x},\mathbf{y}).(\mathbf{v},\mathbf{w}) \text{ in } \mathbb{B}^1_\infty(H). \tag{4.4.12}$$

Proof. (i) For $t \in [0,T]$ and $\mathbf{x} \in \mathbb{B}^1(H)$, by standard computations, we have

$$\begin{aligned}
e^{-\lambda s}|\psi(t,\mathbf{x},\mathbf{y})(s) - \psi(t,\mathbf{x},\mathbf{y}')(s)|_H &\leq e^{-\lambda s}\int_0^s |\hat{b}(r,\mathbf{y}) - \hat{b}(r,\mathbf{y}')|_H dr \\
&\leq \int_0^s e^{-\lambda(s-r)}e^{-\lambda r}|\hat{b}(r,\mathbf{y}) - \hat{b}(r,\mathbf{y}')|_H dr \\
&\leq \frac{1-e^{-\lambda T}}{\lambda}|\hat{b}(\cdot,\mathbf{y}) - \hat{b}(\cdot,\mathbf{y}')|_\lambda \\
&\leq \frac{1-e^{-\lambda T}}{\lambda}N|\mu|_1|\mathbf{y}-\mathbf{y}'|_\lambda,
\end{aligned}$$

for all $\mathbf{y},\mathbf{y}' \in \mathbb{B}^1(H)$ and all $s \in [0,T]$. Then, for all $t,\mathbf{x},\mathbf{y},\mathbf{y}$,

$$|\psi(t,\mathbf{x},\mathbf{y}) - \psi(t,\mathbf{x},\mathbf{y}')|_\lambda \leq \frac{1-e^{-\lambda T}}{\lambda}N|\mu|_1|\mathbf{y}-\mathbf{y}'|_\lambda. \tag{4.4.13}$$

By defining $\alpha := \frac{1-e^{-\lambda T}}{\lambda} N|\mu|_1$, for λ sufficiently large we obtain *(i)*.

(ii) Due to *(i)*, it is sufficient to prove that $\psi(\cdot,\cdot,\mathbf{y})$ is $\mathbb{W}$-valued and continuous on $[0,T]\times\mathbb{W}$, for all $\mathbf{y}\in\mathbb{B}^1(H)$. But this comes from the continuity of the maps

$$[0,T]\times\mathbb{W}\to\mathbb{W},\ (t,\mathbf{x})\mapsto\mathbf{x}_{t\wedge\cdot}\qquad [0,T]\to H,\ s\mapsto\int_0^s \hat{b}(r,\mathbf{y})dr + W^B_s(\omega).$$

(iii)+(iv) We begin by showing that, for all $t\in[0,T]$,

$$\Psi_t\colon \mathbb{B}^1_\infty(H)\to\mathbb{B}^1_\infty(H),\ \mathbf{y}\mapsto\int_t^{t\vee\cdot}\hat{b}(r,\mathbf{y})dr \tag{4.4.14}$$

is strongly continuously Gâteaux differentiable up to order 2, with bounded differentials (bound uniform in t). By standard computations, due to Assumption 4.4.1*(iii)*, we have

$$h^{-1}(\Psi_t(\mathbf{y}+h\mathbf{v})-\Psi_t(\mathbf{y}))$$
$$=\int_t^{t\vee\cdot}\left(\int_0^1\langle\nabla_H b\left(r,\int_{[0,T]}\tilde{\mathbf{y}}(r-s)\mu(ds)+\theta h\int_{[0,T]}\tilde{\mathbf{v}}(r-s)\mu(ds)\right),\int_{[0,T]}\tilde{\mathbf{v}}(r-s)\mu(ds)\rangle_H d\theta\right)dr,$$

where $\nabla_H b$ represents $\partial_H b$ in H. Due to the assumptions on $\partial_H b$, we can pass to the limit $h^{-1}(\Psi_t(\mathbf{y}+h\mathbf{v})-\Psi_t(\mathbf{y}))$ in $\mathbb{B}^1_\infty(H)$ as $h\to 0$ and obtain

$$\partial\Psi_t(\mathbf{y}).\mathbf{v}=\int_t^{t\vee\cdot}\langle\nabla_H b\left(r,\int_{[0,T]}\tilde{\mathbf{y}}(r-s)\mu(ds)\right),\int_{[0,T]}\tilde{\mathbf{v}}(r-s)\mu(ds)\rangle_H dr. \tag{4.4.15}$$

Notice that, if $\mathbf{y}_n\to\mathbf{y}$ and $\mathbf{v}_n\to\mathbf{v}$ in $\mathbb{B}^1_{\sigma^s}(H)$, then , by Proposition 4.1.2*(i)*,

$$\int_{[0,T]}\tilde{\mathbf{y}}_n(r-s)\mu(ds)\to\int_{[0,T]}\tilde{\mathbf{y}}(r-u)\mu(du)\qquad\forall r\in[0,T]$$

and the family

$$\left\{\int_{[0,T]}\tilde{\mathbf{y}}_n(r-u)\mu(du)\right\}_{\substack{r\in[0,T]\\ n\in\mathbb{N}}}$$

is bounded in H. The same holds with respect to $\tilde{\mathbf{v}}_n$ and $\tilde{\mathbf{v}}$. If $t_n\to t$ in $[0,T]$, by strong continuity of $\partial_H b$ and using (4.4.15), we conclude that

$$|\partial\Psi_t(\mathbf{y}).\mathbf{v}-\partial\Psi_{t_n}(\mathbf{y}_n).\mathbf{v}_n|_\infty\to 0. \tag{4.4.16}$$

This proves (4.4.11), because $\partial_3\psi(t,\mathbf{x},\mathbf{y})=\partial\Psi_t(\mathbf{y})$ for all $(t,\mathbf{x},\mathbf{y})\in[0,T]\times\mathbb{B}^1(H)\times\mathbb{B}^1(H)$. In particular, the limit (4.4.16) holds when $t_n=t$, for all $n\in\mathbb{N}$, and the convergences $\mathbf{y}_n\to\mathbf{y}$ and $\mathbf{v}_n\to\mathbf{v}$ take place in $\mathbb{B}^1_\infty(H)$. This shows that $\Psi_t\in\mathcal{G}^1(\mathbb{B}^1_\infty(H),\mathbb{B}^1_\infty(H))$, and, by (4.4.3) and (4.4.15), that the first order differentials are bounded, with bound uniform in t. By observing that $\partial_2\psi(t,\mathbf{x},\mathbf{y}).\mathbf{v}=\mathbf{v}_{t\wedge\cdot}$ for all $t\in[0,T]$, $\mathbf{x},\mathbf{y},\mathbf{v}\in\mathbb{B}^1(H)$, we have then proved that $\psi(t,\cdot,\cdot)\in\mathcal{G}^1(\mathbb{B}^1_\infty(H)\times\mathbb{B}^1_\infty(H),\mathbb{B}^1_\infty(H))$ and that (4.4.9) holds true.

Regarding the second order derivative, by using similar arguments as above, we obtain

$$\partial^2\Psi_t(\mathbf{x}).(\mathbf{v},\mathbf{w}) = \\ = \int_t^{t\vee\cdot} \partial_H^2 b\left(r, \int_{[0,T]} \tilde{\mathbf{y}}(r-s)\mu(ds)\right).\left(\int_{[0,T]} \tilde{\mathbf{v}}(r-s)\mu(ds), \int_{[0,T]} \tilde{\mathbf{w}}(r-s)\mu(ds)\right) dr \tag{4.4.17}$$

and the continuity of

$$[0,T]\times \mathbb{B}^1_{\sigma^s}(H)\times \mathbb{B}^1_{\sigma^s}(H)\times \mathbb{B}^1_{\sigma^s}(H)\to \mathbb{B}^1_\infty(H),\ (t,\mathbf{y},\mathbf{v},\mathbf{w})\mapsto \partial^2\Psi_t(\mathbf{y}).(\mathbf{v},\mathbf{w}).$$

Then, since $\partial_2^2\psi(t,\mathbf{x},\mathbf{y})=0$ and $\partial_3^2\psi(t,\mathbf{x},\mathbf{y})=\partial^2\Psi_t(\mathbf{y})$, $\psi(t,\cdot,\cdot)\in\mathscr{G}^2(\mathbb{B}^1_\infty(H)\times\mathbb{B}^1_\infty(H),\mathbb{B}^1_\infty(H))$. By (4.4.4), also the second order differentials $\partial_2^2\psi,\partial_3^2\psi$ are bounded, with bound uniform in t. ■

In the following proposition we see how the regularity properties of ψ are inherited by the associated fixed-point map.

Proposition 4.4.6.

(i) *For all $(t,\mathbf{x})\in[0,T]\times\mathbb{B}^1(H)$, there exists a unique $\Lambda^{t,\mathbf{x}}\in\mathbb{B}^1(H)$ such that*

$$\Lambda^{t,\mathbf{x}}=\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}).$$

(ii) *The map*

$$\Lambda\colon [0,T]\times\mathbb{B}^1_\infty(H)\to\mathbb{B}^1_\infty(H),\ (t,\mathbf{x})\mapsto\Lambda^{t,\mathbf{x}}$$

is Lipschitz in $\mathbf{x}$, with a bound for the Lipschitz constant independent of t.

(iii) *The restriction of Λ to $[0,T]\times\mathbb{W}$ is continuous and $\mathbb{W}$-valued.*

(iv) *For all $t\in[0,T]$, $\Lambda^{t,\cdot}\in\mathscr{G}^2(\mathbb{B}^1_\infty(H),\mathbb{B}^1_\infty(H))$ and $\partial_{\mathbb{B}^1}\Lambda^{t,\cdot}$, $\partial^2_{\mathbb{B}^1}\Lambda^{t,\cdot}$ are uniformly bounded, uniformly in t.*

(v) *For all $t\in[0,T]$ and $\mathbf{x}\in\mathbb{B}^1(H)$, $I-\partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}})\in L(\mathbb{B}^1_\infty(H))$ is invertible and*

$$\mathbb{B}^1_\infty(H)\to L(\mathbb{B}^1_\infty(H)),\ \mathbf{x}\mapsto(I-\partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}))^{-1} \tag{4.4.18}$$

is strongly continuous.

(vi) *For all $t\in[0,T]$, $\mathbf{x},\mathbf{v},\mathbf{w}\in\mathbb{B}^1(H)$, we have*

$$\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{v}=\left(I-\partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}})\right)^{-1}\left(\partial_2\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}).\mathbf{v}\right)$$

$$\partial^2_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.(\mathbf{v},\mathbf{w})=\left(I-\partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}})\right)^{-1}\left(\partial_3^2\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}).\left((\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{v}),(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{w})\right)\right)$$

Proof. By Proposition 4.4.5*(i)*, we can choose $\lambda > 0$ such that $\psi(t,\mathbf{x},\cdot)$ is an α-contraction on $\mathbb{B}^1_\lambda(H)$, with $\alpha \in (0,1)$, uniformly in $(t,\mathbf{x}) \in [0,T] \times \mathbb{B}^1(H)$.

(i) Apply Banach's contraction principle to $\psi(t,\mathbf{x},\cdot)$ on $\mathbb{B}^1_\lambda(H)$.

(ii) For every $t \in [0,T]$, we have

$$|\psi(t,\mathbf{x},\mathbf{y}) - \psi(t,\mathbf{x}',\mathbf{y})|_\lambda \leq |\mathbf{x} - \mathbf{x}'|_\lambda \qquad \forall \mathbf{x},\mathbf{x}' \in \mathbb{B}^1(H).$$

The conclusion follows by Lemma 1.1.9*(ii)*.

(iii) Since ψ maps $[0,T] \times \mathbb{W} \times \mathbb{W}$ into $\mathbb{W}$ by Proposition 4.4.5*(ii)*, we also have that Λ maps $[0,T] \times \mathbb{W}$ into $\mathbb{W}$. Let us denote by $\Lambda_{\mathbb{W}}$ the map

$$\Lambda_{\mathbb{W}} \colon [0,T] \times \mathbb{W} \to \mathbb{W},\ (t,\mathbf{x}) \mapsto \Lambda^{t,\mathbf{x}}.$$

By *(ii)*, to prove the continuity of $\Lambda_{\mathbb{W}}$, it is sufficient to show the continuity of $\Lambda^{\cdot,\mathbf{x}}$, for fixed $\mathbf{x} \in \mathbb{W}$. Let $t_n \to t$ in $[0,T]$. We have

$$\psi(t_n,\mathbf{x},\mathbf{y}) \to \psi(t,\mathbf{x},\mathbf{y}) \text{ in } \mathbb{W},\ \forall \mathbf{x},\mathbf{y} \in \mathbb{W}.$$

Then the conclusion follows by Lemma 1.1.9*(i)*.

(iv)+(v)+(vi) Thanks to Proposition 4.4.5*(iii)*, we can apply [23, Theorems 7.1.2 and 7.1.3] to all maps $\psi(t,\cdot,\cdot)$, for all $t \in [0,T]$. This shows *(iv)* and *(vi)*.

It remains only to comment the strong continuity of (4.4.18) (which is indeed contained in the proof of [23, Theorems 7.1.2 and 7.1.3]). This comes from the fact that, for all $t \in [0,T]$, by *(ii)* and Proposition 4.4.5*(iii)*, the map

$$\mathbb{B}^1_\lambda(H) \to L(\mathbb{B}^1_\lambda(H)),\ \mathbf{x} \mapsto \partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}})$$

is strongly continuous and $|\partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}})|_{L(\mathbb{B}^1_\lambda(H))} \leq \alpha$ for all $\mathbf{x} \in \mathbb{B}^1(H)$. By writing

$$(I - \partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}))^{-1}\mathbf{v} = \sum_{n\in\mathbb{N}} (\partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}))^n\mathbf{v} \qquad \forall \mathbf{v} \in \mathbb{B}^1(H) \tag{4.4.19}$$

and by Lebesgue's dominated convergence theorem (for series), we see the strong continuity of

$$\mathbb{B}^1_\lambda(H) \to L(\mathbb{B}^1_\lambda(H)),\ \mathbf{x} \mapsto (I - \partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}))^{-1}. \qquad \blacksquare$$

The following proposition provides the good continuity of the differentials of Λ with respect to $\mathbf{x}$, that we will later need in order to apply Theorem 4.3.2 and Corollary 4.3.3 when the process X has the dynamics (4.4.6).

Proposition 4.4.7. *Let* $t \in [0,T]$.

(i) If $\mathbf{x}_n \to \mathbf{x}$ *in* $\mathbb{B}^1_\infty(H)$, $\mathbf{v}_n \to \mathbf{v}$ *in* $\mathbb{B}^1_{\sigma^s}(H)$, *and* $\mathbf{w}_n \to \mathbf{w}$ *in* $\mathbb{B}^1_{\sigma^s}(H)$, *then*

$$\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_n}.\mathbf{v}_n \to \partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{v} \text{ in } \mathbb{B}^1_{\sigma^s}(H) \tag{4.4.20}$$

$$\partial^2_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_n}.(\mathbf{v}_n,\mathbf{w}_n) \to \partial^2_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.(\mathbf{v},\mathbf{w}) \text{ in } \mathbb{B}^1_\infty(H). \tag{4.4.21}$$

(ii) If $t_n \to t^+$ *in* $[0,T]$, $\mathbf{x} \in \mathbb{W}$, $\mathbf{v},\mathbf{w} \in \mathbb{B}^1(H)$, *then*

$$\partial_{\mathbb{B}^1}\Lambda^{t_n,\mathbf{x}}.\mathbf{v} \to \partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{v} \text{ in } \mathbb{B}^1_{\sigma^s}(H) \tag{4.4.22}$$

$$\partial^2_{\mathbb{B}^1}\Lambda^{t_n,\mathbf{x}}.(\mathbf{v},\mathbf{w}) \to \partial^2_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.(\mathbf{v},\mathbf{w}) \text{ in } \mathbb{W}. \tag{4.4.23}$$

Proof. (i) Let $t \in [0,T]$, $\mathbf{x}_n \to \mathbf{x}$ in $\mathbb{B}^1_\infty(H)$, $\mathbf{v}_n \to \mathbf{v}$ in $\mathbb{B}^1_{\sigma^s}(H)$. By Proposition 4.1.2*(i)*, $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{B}^1_\infty(H)$. By Proposition 4.4.6*(iv)*, $\{\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_n}.\mathbf{v}_n\}_{n\in\mathbb{N},t\in[0,T]}$ is bounded in $\mathbb{B}^1_\infty(H)$. In particular, $\{\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_n}.\mathbf{v}_n\}_{n\in\mathbb{N}}$ is bounded in the Hilbert space $L^2([0,T],H)$, which is separable. Then we can find subsequences $\{\mathbf{x}_{n_k}\}_{k\in\mathbb{N}}$ and $\{\mathbf{v}_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k} = Z \text{ weakly in } L^2([0,T],H),$$

for some $Z \in L^2([0,T],H)$. We recall that $\partial_3\psi(t,\mathbf{x}',\mathbf{y}') = \partial\Psi_t(\mathbf{y}')$ for all $\mathbf{x}',\mathbf{y}' \in \mathbb{B}^1(H)$, where Ψ_t was defined in the proof of Proposition 4.4.5 by (4.4.14). By (4.4.15), for $\mathbf{y}',\mathbf{v}' \in \mathbb{B}^1(H)$, we have

$$\begin{aligned}
(\partial\Psi_t(\mathbf{y}').\mathbf{v}')(\xi) &= \int_t^{t\vee\xi} \left\langle \nabla_y b\left(r, \int_{[0,T]} \tilde{\mathbf{y}}'(r-u)\mu(du)\right), \int_{[0,T]} \tilde{\mathbf{v}}'(r-u)\mu(du)\right\rangle_H dr \\
&= \int_t^{t\vee\xi} \left(\int_{[0,T]} \left\langle \nabla_y b\left(r, \int_{[0,T]} \tilde{\mathbf{y}}'(r-u)\mu(du)\right), \tilde{\mathbf{v}}'(r-s)\right\rangle_H \mu(ds)\right) dr \\
&= \int_{[0,T]} \left(\int_t^{t\vee\xi} \left\langle \nabla_y b\left(r, \int_{[0,T]} \tilde{\mathbf{y}}'(r-u)\mu(du)\right), \tilde{\mathbf{v}}'(r-s)\right\rangle_H dr\right) \mu(ds).
\end{aligned}$$

By replacing $\mathbf{x}'$ by $\mathbf{x}_{n_k}$, $\mathbf{y}'$ by $\Lambda^{t,\mathbf{x}_{n_k}}$, and $\mathbf{v}'$ by $\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}$, we obtain ([1])

$$\begin{aligned}
&\left(\partial_3\psi(t,\mathbf{x}_{n_k},\Lambda^{t,\mathbf{x}_{n_k}}).\left(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}\right)\right)(\xi) \\
&\quad = \int_{[0,T]} \left(\int_t^{t\vee\xi} \left\langle \nabla_y b\left(r, \int_{[0,T]} (\Lambda^{t,\mathbf{x}_{n_k}})^\sim(r-u)\mu(du)\right), (\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k})^\sim(r-s)\right\rangle_H dr\right) \mu(ds).
\end{aligned} \tag{4.4.24}$$

Due to the fact that $\{\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}\}_{k\in\mathbb{N}}$ is uniformly bounded in $\mathbb{B}^1_\infty(H)$, passing to another subsequence if necessary, we can assume that $(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k})(0)$ is weakly convergent in H to some $z_0 \in H$. Then

$$\lim_{k\to\infty} (\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k})^\sim = \mathbf{1}_{[-T,0)}(\cdot)z_0 + \mathbf{1}_{[0,T]}Z \text{ weakly in } L^2([-T,T],H). \tag{4.4.25}$$

By Proposition 4.4.6*(ii)*, we have

$$\Lambda^{t,\mathbf{x}_{n_k}} \to \Lambda^{t,\mathbf{x}} \text{ in } \mathbb{B}^1_\infty(H),$$

[1] If the argument $\mathbf{y}$ of the notation $\tilde{\mathbf{y}}$ is long, we write $(\mathbf{y})^\sim$.

then, since $b(r,\cdot)\in C^1_b(H,H)$ (see Remark 4.4.2),

$$\lim_{k\to\infty}\left|\nabla_y b\left(r,\int_{[0,T]}(\Lambda^{t,\mathbf{x}_{n_k}})^{\sim}(r-u)\mu(du)\right)-\nabla_y b\left(r,\int_{[0,T]}(\Lambda^{t,\mathbf{x}})^{\sim}(r-u)\mu(du)\right)\right|_H=0,$$

for all $r\in[0,T]$. In particular, by Lebesgue's dominated convergence theorem,

$$\nabla_y b\left(\cdot,\int_{[0,T]}(\Lambda^{t,\mathbf{x}_{n_k}})^{\sim}(\cdot-u)\mu(du)\right)\to\nabla_y b\left(\cdot,\int_{[0,T]}(\Lambda^{t,\mathbf{x}})^{\sim}(\cdot-u)\mu(du)\right)\tag{4.4.26}$$

strongly in $L^2([0,T],H)$. By (4.4.25) and (4.4.26), we have, for all $s,\xi\in[0,T]$,

$$\begin{aligned}\lim_{k\to\infty}\int_t^{t\vee\xi}&\left\langle\nabla_y b\left(r,\int_{[0,T]}(\Lambda^{t,\mathbf{x}_{n_k}})^{\sim}(r-u)\mu(du)\right),(\partial_{\mathbb{B}^1}\Lambda(t,\mathbf{x}_{n_k}).\mathbf{v}_{n_k})^{\sim}(r-s)\right\rangle_H dr=\\&=\int_t^{t\vee\xi}\left\langle\nabla_y b\left(r,\int_{[0,T]}(\Lambda^{t,\mathbf{x}})^{\sim}(r-u)\mu(du)\right),\mathbf{1}_{[-T,0)}(r-s)z_0+\mathbf{1}_{[0,T]}(r-s)Z\right\rangle_H dr.\end{aligned}$$

Since the latter limit holds for all ξ and s, by (4.4.24) we have that the limit

$$\lim_{k\to\infty}\left(\partial_3\psi(t,\mathbf{x}_{n_k},\Lambda^{t,\mathbf{x}_{n_k}}).\left(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}\right)\right)(\xi)$$

exists for all $\xi\in[0,T]$. By Proposition 4.1.2*(i)*, since the sequence is uniformly bounded, we can finally conclude that

$$\left\{\partial_3\psi(t,\mathbf{x}_{n_k},\Lambda^{t,\mathbf{x}_{n_k}}).\left(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}\right)\right\}_{k\in\mathbb{N}}$$

converges in $\mathbb{B}^1_{\sigma^s}(H)$. Now we are almost done. By Proposition 4.4.6*(vi)*, we have

$$\begin{aligned}\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}&=\partial_3\psi(t,\mathbf{x}_{n_k},\Lambda^{t,\mathbf{x}_{n_k}}).\left(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}\right)+\partial_2\psi(t,\mathbf{x}_{n_k},\Lambda^{t,\mathbf{x}_{n_k}}).\mathbf{v}_{n_k}\\&=\partial_3\psi(t,\mathbf{x}_{n_k},\Lambda^{t,\mathbf{x}_{n_k}}).\left(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}\right)+(\mathbf{v}_{n_k})_{t\wedge\cdot}.\end{aligned}\tag{4.4.27}$$

By considering what proved above and the assumptions on $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$, there exists $\gamma\in\mathbb{B}^1(H)$ such that

$$\partial_3\psi(t,\mathbf{x}_{n_k},\Lambda^{t,\mathbf{x}_{n_k}}).\left(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}\right)+(\mathbf{v}_{n_k})_{t\wedge\cdot}\to\gamma\ \text{in}\ \mathbb{B}^1_{\sigma^s}(H).\tag{4.4.28}$$

Then, (4.4.27), (4.4.28), and Proposition 4.4.5*(iv)*, we have

$$\partial_3\psi(t,\mathbf{x}_{n_k},\Lambda^{t,\mathbf{x}_{n_k}}).\left(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}\right)\to\partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}).\gamma\ \text{in}\ \mathbb{B}^1_\infty(H),\ \text{hence in}\ \mathbb{B}^1_{\sigma^s}(H).$$

By taking the limit in $\mathbb{B}^1_{\sigma^s}(H)$ in (4.4.27), we have

$$\gamma=\partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}).\gamma+\mathbf{v}_{t\wedge\cdot},$$

which entails $\gamma=\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{v}$, by Proposition 4.4.6*(vi)*. This shows that

$$\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_{n_k}}.\mathbf{v}_{n_k}\to\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{v}\ \text{ in }\ \mathbb{B}^1_{\sigma^s}(H).$$

Since the original sequences $\{\mathbf{x}\}_{n\in\mathbb{N}},\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ were arbitrary, (4.4.20) is proved.

To prove (4.4.21), we use Proposition 4.4.6*(vi)*. But now most of the work is done. By (4.4.20), we have

$$\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_n}.\mathbf{v}_n \to \partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{v} \text{ in } \mathbb{B}^1_{\sigma^s}(H)$$
$$\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_n}.\mathbf{w}_n \to \partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{w} \text{ in } \mathbb{B}^1_{\sigma^s}(H).$$

By Proposition 4.4.5*(iv)*, we have

$$\lim_{k\to\infty}\partial_3^2\psi(t,\mathbf{x}_n,\Lambda^{t,\mathbf{x}_n}).\left((\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_n}.\mathbf{v}_n),(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}_n}.\mathbf{w}_n)\right) =$$
$$= \partial_3^2\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}).\left((\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{v}),(\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{w})\right)$$

where the limit is taken in $\mathbb{B}^1_\infty(H)$. We can now conclude by using the strong continuity claimed in Proposition 4.4.6*(v)* and the formula for the second order derivative provided by Proposition 4.4.6*(vi)*.

(ii) Let $t_n \to t^+$ in $[0,T]$, $\mathbf{x}\in\mathbb{W}$, $\mathbf{v}\in\mathbb{B}^1(H)$. By Proposition 4.4.6*(vi)* and by taking into account formula (4.4.19), we can write

$$\begin{aligned}\partial_{\mathbb{B}^1}\Lambda^{t_n,\mathbf{x}}.\mathbf{v} &= (I-\partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}))^{-1}\left(\partial_2\psi(t_n,\mathbf{x},\Lambda^{t_n,\mathbf{x}}).\mathbf{v}\right)\\ &= \sum_{k\in\mathbb{N}}(\partial_3\psi(t_n,\mathbf{x},\Lambda^{t_n,\mathbf{x}}))^k\mathbf{v}_{t_n\wedge\cdot} = \mathbf{v}_{t_n\wedge\cdot} + \sum_{k\geq 1}(\partial_3\psi(t_n,\mathbf{x},\Lambda^{t_n,\mathbf{x}}))^k\mathbf{v}_{t_n\wedge\cdot}.\end{aligned}$$

The fact that $t_n \to t$ from the right assures that $\mathbf{v}_{t_n\wedge\cdot}\to\mathbf{v}_{t\wedge\cdot}$ in $\mathbb{B}^1_{\sigma^s}(H)$. Moreover, by Proposition 4.4.6*(iii)*, $\Lambda^{t_n,\mathbf{x}}\to\Lambda^{t,\mathbf{x}}$ in $\mathbb{W}$. Then, by Proposition 4.4.5*(iii)*,*(iv)*, and Lebesgue's dominated convergence theorem for series, we have

$$\sum_{k\geq 1}(\partial_3\psi(t_n,\mathbf{x},\Lambda^{t_n,\mathbf{x}}))^k\mathbf{v}_{t_n\wedge\cdot} \to \sum_{k\geq 1}(\partial_3\psi(t,\mathbf{x},\Lambda^{t,\mathbf{x}}))^k\mathbf{v}_{t\wedge\cdot} \text{ in } \mathbb{B}^1_\infty(H).$$

Then $\partial_{\mathbb{B}^1}\Lambda^{t_n,\mathbf{x}}.\mathbf{v}\to\partial_{\mathbb{B}^1}\Lambda^{t,\mathbf{x}}.\mathbf{v}$ in $\mathbb{B}^1_{\sigma^s}(H)$ and (4.4.22) is proved.

Regarding (4.4.23), the argument is similar, by using the expression for $\partial^2_{\mathbb{B}^1}\Lambda^{t_n,\mathbf{x}}.(\mathbf{v},\mathbf{w})$ provided by Proposition 4.4.6*(vi)*, the convergence (4.4.22) just proved, and (4.4.12) in Proposition 4.4.5*(iv)* ■

We defined ψ for a given, fixed, $\omega\in\Omega$ (p. 140). For every such ψ, Propositions 4.4.5, 4.4.6, 4.4.7 apply. We can then define the map

$$\Omega\times[0,T]\times\mathbb{B}^1(H)\to\mathbb{B}^1(H),\ (\omega,t,\mathbf{x})\mapsto X^{t,\mathbf{x}}(\omega) \tag{4.4.29}$$

where $X^{t,\mathbf{x}}(\omega)$ is the function $\Lambda^{t,\mathbf{x}}$ provided by Proposition 4.4.6, when ψ is associated to ω. It should be clear that $X^{t,\mathbf{x}}$ is the unique strong solution to SDE (4.4.6) in $\mathscr{L}^0_{\mathscr{P}_T}(\mathbb{W})$.

Let $f:\mathbb{B}^1(H)\to\mathbb{R}$ be a function. Hereafter, we assume that f satisfies the following assumption.

Assumption 4.4.8.

(i) $f \in \mathscr{G}^2(\mathbb{B}^1_\infty(H), \mathbb{R})$*;*

(ii) the differentials ∂f *and* $\partial^2 f$ *are bounded;*

(iii) $\mathbb{B}^1_\infty(H) \times \mathbb{B}^1_{\sigma^s}(H) \to \mathbb{R}$, $(\mathbf{x}, \mathbf{v}) \mapsto \partial f(\mathbf{x}).\mathbf{v}$ *is sequentially continuous;*

(iv) $\mathbb{B}^1_\infty(H) \times \mathbb{B}^1_{\sigma^s}(H) \times \mathbb{B}^1_{\sigma^s}(H) \to \mathbb{R}$, $(\mathbf{x}, \mathbf{v}, \mathbf{w}) \mapsto \partial^2 f(\mathbf{x}).(\mathbf{v}, \mathbf{w})$ *is sequentially continuous.*

The following theorem shows that the main results of Section 4.3 can be applied in the present framework.

Theorem 4.4.9. *Let X be the unique strong solution to* (4.4.6) *and let*

$$\varphi \colon [0,T] \times \mathbb{W} \to \mathbb{R},\ (t, \mathbf{x}) \mapsto \mathbb{E}\left[f(X^{t,\mathbf{x}})\right].$$

Then φ verifies Assumption 4.2.3. Moreover, for all $t \in (0,T)$ and all $\mathbf{x} \in \mathbb{W}$,

$$\mathscr{D}_t^- \varphi(t, \mathbf{x}_{t\wedge\cdot}) + \overline{\partial_{\mathbb{W}}\varphi}(t,\mathbf{x}).(\mathbf{1}_{[t,T]} b(t,\mathbf{x})) + \frac{1}{2}\mathbf{T}\left[\overline{\partial^2_{\mathbb{W}}\varphi}(t,\mathbf{x}), B\right] = 0 \tag{4.4.30}$$

and for all $t \in [0,T]$, $t' \in [t,T]$, $Y \in \mathscr{L}^p_{\mathscr{P}_T}(\mathbb{W})$, $p > 2$,

$$\varphi(t', X^{t,Y}) = \varphi(t, Y) + \int_t^{t'} \overline{\partial_{\mathbb{W}}\varphi}(s, X^{t,Y}).(\mathbf{1}_{[s,T]} b(s, X^{t,Y}))dW_s \qquad \mathbb{P}\text{-a.e.}. \tag{4.4.31}$$

Proof. It is sufficient to show that φ verifies Assumption 4.2.3*(ii),(iii)*, since the remaining part of the theorem comes from Theorem 4.3.2 and Corollary 4.3.3.

We begin by verifying Assumption 4.2.3*(ii)*. By Propositin 4.4.6*(iv)*, for all $(\omega, t) \in \Omega \times [0,T]$, the map $\mathbf{x} \mapsto X^{t,\mathbf{x}}(\omega)$ belongs to $\mathscr{G}^2(\mathbb{B}^1_\infty(H), \mathbb{B}^1_\infty(H))$ and has differentials $\partial_{\mathbb{B}^1} X^{t,\cdot}(\omega)$ and $\partial^2_{\mathbb{B}^1} X^{t,\cdot}(\omega)$ bounded, with bound uniform in ω, t (recall Remark 4.4.4). Then, since $f \in \mathscr{G}^2(\mathbb{B}^1_\infty(H), \mathbb{R})$ and ∂f and $\partial^2 f$ are uniformly bounded, the composition $\mathbf{x} \mapsto f(X^{t,\mathbf{x}}(\omega))$ belongs to $\mathscr{G}^2(\mathbb{B}^1_\infty(H), \mathbb{R})$ and has differentials $\partial_{\mathbb{B}^1} f(X^{t,\cdot}(\omega))$ and $\partial^2_{\mathbb{B}^1} f(X^{t,\cdot}(\omega))$ bounded, with bound uniform in ω, t. We have

$$\partial_{\mathbb{B}^1} f(X^{t,\mathbf{x}}(\omega)).\mathbf{v} = \partial f(X^{t,\mathbf{x}}(\omega)).(\partial_{\mathbb{B}^1} X^{t,\mathbf{x}}(\omega).\mathbf{v}) \tag{4.4.32}$$

for all $t \in [0,T]$, $\omega \in \Omega$, $\mathbf{x}, \mathbf{v} \in \mathbb{B}^1(H)$, and

$$\begin{aligned} &\partial^2_{\mathbb{B}^1} f(X^{t,\mathbf{x}}(\omega)).(\mathbf{v}, \mathbf{w}) = \\ &\quad = \partial^2 f(X^{t,\mathbf{x}}(\omega)).((\partial_{\mathbb{B}^1} X^{t,\mathbf{x}}(\omega).\mathbf{v}).(\partial_{\mathbb{B}^1} X^{t,\mathbf{x}}(\omega).\mathbf{w})) + \partial f(X^{t,\mathbf{x}}(\omega)).(\partial^2_{\mathbb{B}^1} X^{t,\mathbf{x}}(\omega).(\mathbf{v}, \mathbf{w})) \end{aligned} \tag{4.4.33}$$

for all $t \in [0,T]$, $\omega \in \Omega$, $\mathbf{x}, \mathbf{v}, \mathbf{w} \in \mathbb{B}^1(H)$. Since $\partial_{\mathbb{B}^1} f(X^{t,\mathbf{x}}(\omega))$ and $\partial^2_{\mathbb{B}^1} f(X^{t,\mathbf{x}}(\omega))$ are bounded, with bound uniform in $\omega, t, \mathbf{x}$, we can easily see that

$$\partial_{\mathbb{B}^1} \varphi(t, \mathbf{x}).\mathbf{v} = \mathbb{E}\left[\partial_{\mathbb{B}^1} f(X^{t,\mathbf{x}}).\mathbf{v}\right] \tag{4.4.34}$$

$$\partial^2_{\mathbb{B}^1} \varphi(t, \mathbf{x}).(\mathbf{v}, \mathbf{w}) = \mathbb{E}\left[\partial^2_{\mathbb{B}^1} f(X^{t,\mathbf{x}}).(\mathbf{v}, \mathbf{w})\right], \tag{4.4.35}$$

for all $t \in [0,T]$, $\mathbf{x},\mathbf{v},\mathbf{w} \in \mathbb{B}^1(H)$. Finally, by (4.4.34), (4.4.35), boundedness of $\partial_{\mathbb{B}^1} f(X^{t,\cdot})(\omega)$ and $\partial^2_{\mathbb{B}^1} f(X^{t,\cdot}(\omega))$, strong continuity of $\partial_{\mathbb{B}^1} f(X^{t,\cdot})(\omega)$ and $\partial^2_{\mathbb{B}^1} f(X^{t,\cdot}(\omega))$, we obtain that $\varphi(t,\cdot)$ belongs to $\mathscr{G}^2(\mathbb{B}^1_\infty(H),\mathbb{R})$ and has bounded first and second order differentials. To conclude the verification of Assumption 4.2.3*(ii)*, it is sufficient to show that, for all $t \in [0,T]$, the maps

$$\mathbb{W} \times \mathbb{B}^1_{\sigma^s}(H) \to \mathbb{R},\ (\mathbf{x},\mathbf{v}) \mapsto \partial_{\mathbb{B}^1}\varphi(t,\mathbf{x}).\mathbf{v}$$
$$\mathbb{W} \times \mathbb{B}^1_{\sigma^s}(H) \times \mathbb{B}^1_{\sigma^s}(H) \to \mathbb{R},\ (\mathbf{x},\mathbf{v},\mathbf{w}) \mapsto \partial^2_{\mathbb{B}^1}\varphi(t,\mathbf{x}).(\mathbf{v},\mathbf{w})$$

are sequentially continuous. This comes immediately by combining (4.4.32), (4.4.33), (4.4.34), (4.4.35), Proposition 4.4.6*(ii)*, Proposition 4.4.7*(i)*, Assumption 4.4.8*(iii)*,*(iv)*, the uniform boundedness of the differentials involved (we recall again Remark 4.4.4) and of the convergent sequences in $\mathbb{B}^1_{\sigma^s}(H)$.

Similarly, we can see that Assumption 4.2.3*(iii)* is verified by taking into account (4.4.32), (4.4.33), (4.4.34), (4.4.35), Proposition 4.4.6*(iii)*, Proposition 4.4.7*(ii)*, Assumption 4.4.8*(iii)*,*(iv)*, the uniform boundedness of the differentials involved and of the convergent sequences in $\mathbb{B}^1_{\sigma^s}(H)$. ■

Remark 4.4.10. Let $g\colon [0,T]\times H \to H$ be a continuous function, with $g(t,\cdot) \in C^2_b(H,H)$ and with differentials $D_H g, D^2_H g$ uniformly continuous. Let us introduce the function $\hat{b}^g$ defined by

$$\hat{b}^g(t,\mathbf{y}) := b\left(t, \int_{[0,T]} \tilde{g}(t-s,\tilde{\mathbf{y}}(t-s))\mu(ds)\right) \qquad \forall (t,\mathbf{y}) \in [0,T]\times\mathbb{B}^1(H),$$

where $\tilde{g}(r,x) := g(0,x)$ if $r<0$. Consider the function

$$G\colon \mathbb{B}^1(H) \to \mathbb{B}^1(H),\ \mathbf{y} \mapsto \{g(t,\mathbf{y}(t))\}_{t\in[0,T]}.$$

Then G is well-defined, G belongs to $C^2_b(\mathbb{B}^1_\infty(H),\mathbb{B}^1_\infty(H))$, and $\hat{b}^g(t,\mathbf{y}) = \hat{b}(t,G(\mathbf{y}))$. By using these observations and the explicit expressions of DG, D^2G in terms of $D_H g$, $D^2_H g$, it is not difficult to show that the results proved in this section can be extended to the case in which the drift $\hat{b}$ in SDE (4.4.6) is replaced by the more general drift $\hat{b}^g$.

Chapter 5

C_0-sequentially equicontinuous semigroups

The aim of this chapter is to present and apply a notion of one parameter strongly continuous (C_0) semigroups of linear operators in locally convex spaces based on the notion of sequential equicontinuity and following the spirit and the methods of the classical theory in Banach spaces.

The theory of C_0-semigroups was first stated in Banach spaces (a widespread presentation can be found in several monographs, e.g. [36, 56, 80]). The theory was extended to locally convex spaces by introducing the notions of C_0-equicontinuous semigroup ([98, Ch. IX]), C_0-quasi-equicontinuous semigroup ([9]), C_0-locally equicontinuous semigroup ([25, 64]), weakly integrable semigroup ([58, 59]). A mixed approach is the one followed by [65], which introduces the notion of bi-continuous semigroup: in a framework of Banach spaces, semigroups that are strongly continuous with respect to a weaker locally convex topology are considered.

Here, we deal with semigroups of linear operators in locally convex spaces that are only *sequentially* continuous. The idea is due to the following key observation: the theory of C_0-(locally) equicontinuous semigroups can be developed, with appropriate adjustments, to semigroups of operators which are only C_0-(locally) *sequentially* equicontinuous (in the sense specified by Definition 5.2.1). On the other hand, as we will show by examples, the passage from equicontinuity to sequential equicontinuity is motivated and fruitful: as discussed in Remark 5.2.13 and shown by Example 5.4.5, in concrete applications, replacing equicontinuity with sequential equicontinuity is convenient or even, in some cases, necessary.

The main motivation that led us to consider sequential continuity is that it allows a convenient treatment of Markov transition semigroups. The employment of Markov transition semigroups to the study of partial differential equations through the use of stochastic representation formulas is the subject of a wide mathematical literature (here we only refer to [8] in finite and infinite dimension and to [24] in infinite dimension). Also, the regularizing properties of such semigroups is the core of a regularity theory for second order PDEs (see, e.g., [74]). Unfortunately, the framework of C_0-semigroup in Banach spaces is not always appropriate to treat such semigroups. Indeed, on Banach spaces of functions not vanishing at infinity, the C_0-property fails already in basic cases, such as the Ornstein-Uhlenbeck semigroup, when considering it in the space of bounded uniformly continuous real-valued functions $(UC_b(\mathbb{R}), |\cdot|_\infty)$ (see, e.g., [7, Ex. 6.1] for a counterexample, or [21, Lemma 3.2], which implies this semigroup is strongly continuous in $(UC_b(\mathbb{R}), |\cdot|_\infty)$ if and only if the drift of the associated stochastic differential equation vanishes). On the other hand, finding a locally convex topology on these spaces to frame Markov transition semigroups within the theory of C_0-locally equicontinuous semigroups is not an easy task (see also the considerations of Remark 5.2.13). In the case of the Ornstein-Uhlenbeck semigroup, such approach is adopted by [49]. Some authors have bypassed these difficulties by introducing some (more or less *ad hoc*) notions, relying on some sequential continuity properties, to treat such semigroups (weakly continuous semigroups [7], π-continuous semigroups [84], bi-continuous semigroups [65]). The theory developed in this chapter allows to gather all the aforementioned notions under a unified framework.

We end this introductory part by describing in detail the contents of the forthcoming sections.

Section 5.1 contains some notation that will hold throughout the chapter, in addition to the notation given at p. 12.

In Section 5.2 we first provide and study the notions of sequential continuity of linear operators and sequential equicontinuity of families of linear operators on locally convex spaces. Then, we give the definition of C_0-sequentially (locally) equicontinuous semigroup in locally convex spaces. Next, we define the generator of the semigroup and the resolvent of the generator. In order to guarantee the existence of the resolvent, the theory is developed under Assumption 5.2.16, requiring the existence of the Laplace transform (5.2.10) as Riemann integral (see Remark 5.2.17). This assumption is immediately verified if the underlying space X is sequentially complete. Otherwise, the Laplace transform always exists in the (sequential) completion of X and then one should check that it lies in X, as we do in Proposition 5.3.19. The properties of generator and resolvent are stated through a series of results: their synthesis is represented by Theorem 5.2.25, stating that the semigroup is uniquely identified by its generator, and by Theorem 5.2.27, stating that the resolvent coincides with the Laplace transform. Then we provide a generation theorem (Theorem 5.2.37), characterizing, in the same spirit of the Hille-Yosida theorem, the linear operators generating C_0-sequentially equicontinuous semigroups. Afterwards, we show that the notion of bi-continuous semigroups can be seen as a specification of ours

(Proposition 5.2.42). Finally, we provide some examples which illustrate our notion in relation to the others.

Section 5.3 implements the theory of Section 5.2 in spaces of bounded Borel functions, continuous and bounded functions, or uniformly continuous and bounded functions defined on a metric space. The main aim of this section is to find and study appropriate locally convex topologies in these functional spaces allowing a comparison between our notion with the aforementioned other ones. We identify them in two topologies belonging to a class of locally convex topologies defined through the family of seminorms (5.3.1). We study the relation between them and the topology induced by the uniform norm (Proposition 5.3.6). Then, we study these topological spaces through a series of results ending with Proposition 5.3.15 and we characterize their topological dual in Proposition 5.3.16. We end the section with the desired comparison: in Subsections 5.3.2, 5.3.3, and 5.3.4, we show that the notions developed in [7], [84], and [49] to treat Markov transition semigroups can be reintepreted in our framework.

Section 5.4 applies the results of Section 5.3 to transition semigroups. This is done, in Subsection 5.4.1, in the space of bounded continuous functions endowed with the topology $\tau_{\mathscr{K}}$ defined in (5.3.7). Then, in Subsection 5.4.3, we provide an extension to weighted spaces of continuous functions, not necessarily bounded. Finally, in Subsection 5.4.3, we treat the case of Markov transition semigroups associated to stochastic differential equations in Hilbert spaces. Our purpose for future research is to exploit these latter results as a starting point for studying semilinear elliptic partial differential equations in infinite dimensional spaces and their application to optimal control problems.

5.1 Notation

In this chapter, we adopt the following notation.

- X,Y denote Hausdorff topological vector spaces. Starting from Subsection 5.2.2, Assumption 5.2.3 will hold and X,Y will be Hausdorff locally convex topological vector spaces.
- The topological dual of a topological vector space X is denoted by X^*.
- If X is a vector space and Γ is a vector space of linear functionals on X separating points in X, we denote by $\sigma(X,\Gamma)$ the weakest locally convex topology on X making continuous the elements of Γ.
- The weak topology on the topological vector space X is denoted by τ_w, that is $\tau_w := \sigma(X,X^*)$.
- If X and Y are topological vector spaces, the space of continuous operators from X into Y is denoted by $L(X,Y)$, and the space of sequentially continuous operators

from X into Y (see Definition 5.2.1) is denoted by $\mathscr{L}_0(X,Y)$. We also denote $L(X) := L(X,X)$ and $\mathscr{L}_0(X) := \mathscr{L}_0(X,X)$.

- Given a locally convex topological vector space X, the symbol $\mathscr{P}_X$ denotes a family of seminorm on X inducing the locally convex topology.
- E denotes a metric space; $\mathscr{E} := \mathscr{B}(E)$ denotes the Borel σ-algebra of subsets of E .
- Given the metric space E, $\mathbf{ba}(E)$ denotes the space of finitely additive signed measures with bounded total variation on $\mathscr{E}$, $\mathbf{ca}(E)$ denotes the subspace of $\mathbf{ba}(E)$ of countably additive finite measure, and $\mathbf{ca}^+(E)$ denotes the subspace of $\mathbf{ca}(E)$ of positive countably additive finite measures.
- Given the metric space E, we denote by $B(x,r)$ the open ball centered at $x \in E$ and with radius r and by $B(x,r]$ the closed ball centered at x and with radius r.
- The common symbol $\mathscr{S}(E)$ denotes indifferently one of the spaces $B_b(E)$, $C_b(E)$, $UC_b(E)$, that is, respectively, the space of real-valued *bounded Borel / continuous and bounded / uniformly continuous and bounded* functions defined on E.
- On $\mathscr{S}(E)$, we consider the sup-norm $|f|_\infty := \sup_{x\in E} |f(x)|$, which makes it a Banach space. The topology on $\mathscr{S}(E)$ induced by such norm is denoted by τ_∞.
- On $\mathscr{S}(E)$, the symbol $\tau_{\mathscr{C}}$ denotes the topology of the uniform convergence on compact sets.
- By $\mathscr{S}(E)^*_\infty$ we denote the topological dual of $(\mathscr{S}(E), |\cdot|_\infty)$ and by $|\cdot|_{\mathscr{S}(E)^*_\infty}$ the operator norm in $\mathscr{S}(E)^*_\infty$.

5.2 C_0-sequentially equicontinuous semigroups

In this section, we introduce and investigate the notion of C_0-sequentially equicontinuous semigroups on locally convex topological vector spaces.

5.2.1 Sequential continuity and equicontinuity

We recall the notion of sequential continuity for functions and define the notion of sequential equicontinuity for families of functions on topological spaces.

Definition 5.2.1. *Let X, Y be Hausdorff topological spaces.*

(i) A function $f : X \to Y$ is said to be sequentially continuous *if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x in X, we have $f(x_n) \to f(x)$ in Y.*

(ii) If Y is a vector space, a family of functions $\mathscr{F} = \{f_\iota : X \to Y\}_{\iota\in\mathscr{I}}$ is said to be sequentially equicontinuous *if for every $x \in X$, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x in X and for every neighborhood U of 0 in Y, there exists $\overline{n} \in \mathbb{N}$ such that $f_\iota(x_n) \in f_\iota(x) + U$ for every $\iota \in \mathscr{I}$ and $n \geq \overline{n}$.*

Remark 5.2.2. Let E be a metric space. If $g: X \to Y$ is sequentially continuous and $f: E \to X$ is continuous, then $g \circ f: E \to Y$ is continuous. It is sufficient to recall that continuity for a function defined on a metric space is equivalent to sequential continuity.

If Y is a locally convex topological vector space, then Definition 5.2.1*(ii)* is equivalent to

$$\{x_n\}_{n\in\mathbb{N}} \subset X,\ x_n \to x \text{ in } X \implies \lim_{n\to+\infty} \sup_{\iota\in\mathcal{I}} q(f_\iota(x_n) - f_\iota(x)) = 0, \qquad \forall q \in \mathcal{P}_Y, \tag{5.2.1}$$

where $\mathcal{P}_Y$ is a set of seminorms inducing the topology on Y. The characterisation of sequential continuity (5.2.1) will be very often used hereafter.

5.2.2 The space of sequentially continuous linear operators

Starting from this subsection, we make the following

Assumption 5.2.3. *X and Y are Hausdorff locally convex topological vector spaces, and $\mathcal{P}_X$, $\mathcal{P}_Y$ denote families of seminorms inducing the topology on X, Y, respectively.*

Remark 5.2.4. We recall that a subset $B \subset X$ is bounded if and only if $\sup_{x\in B} p(x) < +\infty$ for every $p \in \mathcal{P}_X$ and that Cauchy (and, therefore, also convergent) sequences are bounded in X.

We define the vector space

$$\mathcal{L}_0(X,Y) := \{F: X \to Y \text{ s.t. } F \text{ is linear and sequentially continuous}\}.$$

We will use $\mathcal{L}_0(X)$ to denote the space $\mathcal{L}_0(X,X)$. Clearly, we have the inclusion

$$L(X,Y) \subset \mathcal{L}_0(X,Y). \tag{5.2.2}$$

We recall that a linear operator $F: X \to Y$ is a called *bounded* if $F(B)$ is bounded in Y for each bounded subset $B \subset X$. As well known (see [91, Th. 1.32, p. 24])

$$F \in L(X,Y) \implies F \text{ is bounded}. \tag{5.2.3}$$

On the other hand, if X is bornological (see [77, p. 95, Definition 4.1]), then, by [77, Ch. 4, Prop. 4.12], also the converse holds true:

$$X \text{ bornological},\ F: X \to Y \text{ linear and bounded} \implies F \in L(X,Y). \tag{5.2.4}$$

Proposition 5.2.5. *Let $F \in \mathcal{L}_0(X,Y)$. Then*

(i) F is a bounded operator;

(ii) F maps Cauchy sequences into Cauchy sequences.

Proof. *(i)* See [77, Ch. 4, Prop. 4.12].

(ii) Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in X. In order to prove that $\{Fx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, we need to prove that, for every $q \in \mathscr{P}_Y$ and $\varepsilon > 0$, there exists $\overline{n}$ such that $n,m \geq \overline{n}$ implies $q(F(x_m - x_n)) \leq \varepsilon$. Fix $q \in \mathscr{P}_Y$ and $\varepsilon > 0$. As, by Remark 5.2.4, $\{x_n\}_{n\in\mathbb{N}}$ is bounded in X, by *(i)* the sequence $\{Fx_n\}_{n\in\mathbb{N}}$ is bounded in Y. Then, for every $n \in \mathbb{N}$, we can choose $k_n \in \mathbb{N}$, with $k_n \geq n$, such that

$$q(F(x_{k_n} - x_n)) + 2^{-n} \geq \sup_{k\geq n} q(F(x_k - x_n)). \tag{5.2.5}$$

Define $z_n := x_{k_n} - x_n$, for $n \in \mathbb{N}$. As $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X, we have $z_n \to 0$ as $n \to +\infty$. By sequential continuity of F, also $Fz_n \to 0$. Then (5.2.5) entails, for every $\overline{n} \in \mathbb{N}$ and every $n,m \geq \overline{n}$,

$$q(F(x_m - x_n)) \leq q(F(x_m - x_{\overline{n}})) + q(F(x_n - x_{\overline{n}})) \leq 2\sup_{k\geq\overline{n}} q(F(x_k - x_{\overline{n}})) \leq 2^{1-\overline{n}} + 2q(Fz_{\overline{n}}).$$

Passing to the limit $\overline{n} \to +\infty$, we conclude that $\{Fx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. ■

Remark 5.2.6. We notice that the fact that F is a bounded linear operator from X into Y does not guarantee, in general, that it belongs to the space $\mathscr{L}_0(X,Y)$. Indeed, the bounded sets in the weak topology τ_w of any Banach space X are exactly the originally bounded sets (see Lemma 5.2.40; actually this is true for locally convex spaces: see [91, p. 70, Theorem 3.18]). Then, if τ denotes the norm-topology in X, the identity $\mathbf{id}\colon (X,\tau_w) \to (X,\tau)$ is bounded. Nevertheless, this identity is in general not sequentially continuous (any infinite dimensional Hilbert space provides an immediate counterexample).

Corollary 5.2.7. *If X is bornological, then*

$$\mathscr{L}_0(X,Y) = \mathscr{L}_0(X_w,Y_w) = L(X,Y) = L(X_w,Y_w),$$

where X_w, Y_w denote, respectively, the spaces X,Y endowed with their weak topologies.

Proof. Since X (resp. Y) is locally convex, by [91, p. 70, Theorem 3.18], the weakly bounded sets of X (resp. Y) are exactly the originally bounded sets in X (resp. in Y). Hence, (5.2.3) and (5.2.4) yield $L(X_w,Y_w) \subset L(X,Y)$. On the other hand, the opposite inclusion holds true for every X,Y vector topological spaces. So, we have proved that $L(X_w,Y_w) = L(X,Y)$.

Now, by Proposition 5.2.5*(i)* and by (5.2.4), we have $\mathscr{L}_0(X,Y) \subset L(X,Y)$. The opposite inclusion is obvious. So, $\mathscr{L}_0(X,Y) = L(X,Y)$.

Finally, considering that $\mathscr{L}_0(X_w,Y_w) \supset L(X_w,Y_w)$, in order to conclude we need to show that $\mathscr{L}_0(X_w,Y_w) \subset L(X,Y)$. Recalling that the weakly bounded sets of X (resp. Y) are exactly the originally bounded sets in X (resp. in Y), the latter follows from (5.2.4) and Proposition 5.2.5*(i)*, as X is bornological. ■

Let $\mathbf{B}$ be the set of all bounded subsets of X. We introduce on $\mathscr{L}_0(X,Y)$ a locally convex topology as follows. By Proposition 5.2.5*(i)*

$$\rho_{q,D}(F) := \sup_{x\in D} q(Fx) \tag{5.2.6}$$

is finite for all $F \in \mathscr{L}_0(X,Y)$, $D \in \mathbf{B}$, and $q \in \mathscr{P}_Y$. Given $D \in \mathbf{B}$ and $q \in \mathscr{P}_Y$, (5.2.6) defines a seminorm in the space $\mathscr{L}_0(X,Y)$. We denote by $\mathscr{L}_{0,b}(X,Y)$ the space $\mathscr{L}_0(X,Y)$ endowed with the locally convex vector topology τ_b induced by the family of seminorms $\{\rho_{q,D}\}_{q\in\mathscr{P}_Y,\, D\in\mathbf{B}}$. We notice that τ_b does not depend on the choice of family $\mathscr{P}_Y$ inducing the topology of Y. Since $\mathbf{B}$ contains all singletons $\{x\}_{x\in X}$ and Y is Hausdorff, also $\mathscr{L}_{0,b}(X,Y)$ is Hausdorff.

Proposition 5.2.8. *The map*

$$\mathscr{L}_{0,b}(X) \times \mathscr{L}_{0,b}(X) \to \mathscr{L}_{0,b}(X), \quad (F,G) \mapsto FG,$$

is sequentially continuous.

Proof. Let $(F,G) \in \mathscr{L}_0(X) \times \mathscr{L}_0(X)$, and let $D \subset X$ be bounded. Let $\{(F_n,G_n)\}_{n\in\mathbb{N}}$ be a sequence converging to (F,G) in $\mathscr{L}_{0,b}(X) \times \mathscr{L}_{0,b}(X)$. Consider the set $D' := \bigcup_{n\in\mathbb{N}} G_n D$. We have

$$\sup_{n\in\mathbb{N}} \sup_{x\in D} q(G_n x) \le \sup_{n\in\mathbb{N}} \sup_{x\in D} q((G_n - G)x) + \sup_{x\in D} q(Gx) \qquad \forall q \in \mathscr{P}_X.$$

On the other hand, $G_n \to G$ yields

$$\sup_{n\in\mathbb{N}} \sup_{x\in D} q((G_n - G)x) = \sup_{n\in\mathbb{N}} \rho_{q,D}(G_n - G) < +\infty, \quad \forall q \in \mathscr{P}_X.$$

Then, combining with Proposition 5.2.5*(i)*, we conclude that D' is bounded.

Now fix $q \in \mathscr{P}_X$. For every $n \in \mathbb{N}$, we can write

$$\rho_{q,D}((FG - F_nG_n)) \le \rho_{q,D}(F(G-G_n)) + \rho_{q,D}((F-F_n)G_n) \le \rho_{q,D}(F(G-G_n)) + \rho_{q,D'}(F-F_n).$$

Now $\lim_{n\to+\infty} \rho_{q,D'}(F-F_n) = 0$, because $D' \in \mathbf{B}$ and $F_n \to F$ in $\mathscr{L}_{0,b}(X)$. Hence we conclude if we show $\lim_{n\to+\infty} \rho_{q,D}(F(G-G_n)) = 0$. Assume, by contradiction, that there exist $\varepsilon > 0$, $\{x_k\}_{k\in\mathbb{N}} \subset D$, and a subsequence $\{G_{n_k}\}_{k\in\mathbb{N}}$, such that

$$q(F(G - G_{n_k})x_k) \ge \varepsilon \qquad \forall k \in \mathbb{N}. \tag{5.2.7}$$

Since

$$\lim_{n\to+\infty} q'((G-G_{n_k})x_k) \le \lim_{n\to+\infty} \rho_{q',D}(G-G_{n_k}) = 0 \qquad \forall q' \in \mathscr{P}_X,$$

then $\{z_k := (G - G_{n_k} x_k)\}_{k\in\mathbb{N}}$ is a sequence converging to 0 in X. By sequential continuity of F, we have $\lim_{k\to+\infty} q(Fz_k) = 0$, contradicting (5.2.7) and concluding the proof. ■

Proposition 5.2.9. *(i) If Y is complete, then $\mathscr{L}_{0,b}(X,Y)$ is complete.*

(ii) If Y is sequentially complete, then $\mathscr{L}_{0,b}(X,Y)$ is sequentially complete.

Proof. *(i)* Let $\{F_\iota\}_{\iota\in\mathscr{I}}$ be a Cauchy net in $\mathscr{L}_{0,b}(X,Y)$. Then , by definition of τ_b, the net $\{F_\iota(x)\}_{\iota\in\mathscr{I}}$ is Cauchy in Y, for every $x\in X$. Since Y is complete, for every $x\in X$, the limit $F(x):=\lim_\iota F_\iota(x)$ exists in Y. Clearly, F is linear. Now we show that it is sequentially continuous. Let $q\in\mathscr{P}_Y$ and denote by D the bounded set $D:=\{x_n\}_{n\in\mathbb{N}}\subset X$, where $x_n\to 0$ in X. Then, for all $\bar\iota\in\mathscr{I}$, $n\in\mathbb{N}$,

$$\begin{aligned} q(Fx_n) &= \lim_{\substack{\iota\\ \iota\succeq\bar\iota}} q(F_\iota x_n) \leq \lim_{\substack{\iota\\ \iota\succeq\bar\iota}} q((F_\iota-F_{\bar\iota})x_n)+q(F_{\bar\iota}x_n)\\ &\leq \sup_{\iota\succeq\bar\iota}\rho_{q,D}(F_\iota-F_{\bar\iota})+q(F_{\bar\iota}x_n). \end{aligned}$$

Taking the $\limsup_{n\to+\infty}$ in the inequality above and taking into account that $\{F_\iota\}_{\iota\in\mathscr{I}}$ is a Cauchy net in $\mathscr{L}_{0,b}(X,Y)$ yield the sequential continuity of F.

We now show that $\lim_\iota F_\iota = F$ in $\mathscr{L}_{0,b}(X,Y)$. Let $D\in\mathbf{B}$ and let $q\in\mathscr{P}_Y$. We have

$$q((F-F_{\bar\iota})x)=\lim_{\substack{\iota\\ \iota\succeq\bar\iota}} q((F_\iota-F_{\bar\iota})x)\leq \sup_{\iota\succeq\bar\iota}\rho_{q,D}(F_\iota-F_{\bar\iota}) \qquad \forall\bar\iota\in\mathscr{I},\ \ \forall x\in D,$$

and the conclusion follows as $\{F_\iota\}_{\iota\in\mathscr{I}}$ is a Cauchy net in $\mathscr{L}_{0,b}(X,Y)$.

(ii) It follows by similar arguments as those above, taking now Y sequentially complete and replacing $\mathscr{I}$ by $\mathbb{N}$. ■

5.2.3 Families of sequentially equicontinuous functions

Proposition 5.2.10. *For $n\in\mathbb{N}$ and $i=1,\dots,n$, let $\mathscr{F}^{(i)}=\{F^{(i)}_\iota : X\to X\}_{\iota\in\mathscr{I}_i}$ be families of sequentially equicontinuous linear operators. Then the following hold.*

(i) The family $\mathscr{F}=\{F^{(1)}_{\iota_1}F^{(2)}_{\iota_2}\dots F^{(n)}_{\iota_n} : X\to X\}_{\iota_1\in\mathscr{I}_1,\dots,\iota_n\in\mathscr{I}_n}$ is sequentially equicontinuous.

(ii) The family $\mathscr{F}=\{F^{(1)}_{\iota_1}+F^{(2)}_{\iota_2}+\dots+F^{(n)}_{\iota_n} : X\to Y\}_{\iota_1\in\mathscr{I}_1,\dots,\iota_n\in\mathscr{I}_n}$ is sequentially equicontinuous.

(iii) The family $\mathscr{F}$ is equibounded, i.e., if D is a bounded subset of X, then $\left\{F^{(i)}_{\iota_i}x\right\}_{\substack{\iota_i\in\mathscr{I}_i,\\ i=1,\dots,n\\ x\in D}}$ is bounded in X.

Proof. *(i)* It suffices to prove the statement for $n=2$. By contradiction, assume that there exist a sequence $\{x_k\}_{k\in\mathbb{N}}$ converging to 0 in X, sequences $\{\iota_1^{(k)}\}_{k\in\mathbb{N}}$ in $\mathscr{I}_1$ and $\{\iota_2^{(k)}\}_{k\in\mathbb{N}}$ in $\mathscr{I}_2$, $p\in\mathscr{P}_X$, and $\varepsilon>0$, such that

$$p\left(\left(F^{(1)}_{\iota_1^{(k)}}F^{(2)}_{\iota_2^{(k)}}\right)x_k\right)\geq\varepsilon \qquad \forall k\in\mathbb{N}.$$

Since $\mathscr{F}^{(2)}$ is sequentially equicontinuous, we have

$$\limsup_{k\to+\infty} q\left(F^{(2)}_{\iota_2^{(k)}} x_k\right) \le \lim_{k\to+\infty} \sup_{\iota_2\in\mathscr{I}_2} q\left(F^{(2)}_{\iota_2} x_k\right) = 0, \qquad \forall q \in \mathscr{P}_X.$$

This means that the sequence $\{y_k := F^{(2)}_{\iota_2^{(k)}} x_k\}_{k\in\mathbb{N}}$ converges to 0 in X. Then, in the same way, since $\mathscr{F}^{(1)}$ is sequentially equicontinuous,

$$\limsup_{k\to+\infty} p\left(\left(F^{(1)}_{\iota_1^{(k)}} F^{(2)}_{\iota_2^{(k)}}\right) x_k\right) = \limsup_{k\to+\infty} p\left(F^{(1)}_{\iota_1^{(k)}} y_k\right) \le \limsup_{k\to+\infty} \sup_{\iota_1\in\mathscr{I}_1} p\left(F^{(1)}_{\iota_1} y_k\right) = 0,$$

and the contradiction arises.

(ii) The proof follows by the triangular inequality.

(iii) Assume, by contradiction, that there exist a bounded set D and $p \in \mathscr{P}_X$ such that

$$\sup_{\substack{\iota_i\in\mathscr{I}_i\\ i=1,\dots,n,\\ x\in D}} p\left(F^{(i)}_{\iota_i} x\right) = +\infty.$$

Then there exist $\bar{\iota} \in \{1,\dots,n\}$ and sequences $\{x_k\}_{k\in\mathbb{N}} \subset D$, $\{\iota_k\}_{k\in\mathbb{N}} \subset \mathscr{I}_{\bar{\iota}}$, such that

$$p\left(F^{(\bar{\iota})}_{\iota_k} x_k\right) \ge k, \qquad \forall k \in \mathbb{N}. \tag{5.2.8}$$

On the other hand, since D is bounded, the sequence $\left\{\frac{x_k}{k}\right\}_{k\in\mathbb{N}\setminus\{0\}}$ converges to 0, and then, since the family $\{F^{(\bar{\iota})}_{\iota}\}_{\iota\in\mathscr{I}_{\bar{\iota}}}$ is sequentially equicontinuous, we have

$$\lim_{k\to+\infty} p\left(F^{(\bar{\iota})}_{\iota_k} \frac{x_k}{k}\right) = 0,$$

which contradicts (5.2.8), concluding the proof. ∎

The following proposition clarifies when the notion of sequential equicontinuity for a family of linear operators is equivalent to the notion of equicontinuity.

Proposition 5.2.11. *Let $\mathscr{F} := \{F_\iota : X \to X\}_{\iota\in\mathscr{I}}$ be a family of linear operators. If $\mathscr{F} \subset L(X)$ is equicontinuous, then $\mathscr{F} \subset \mathscr{L}_0(X)$ and $\mathscr{F}$ is sequentially equicontinuous.*

Conversely, if X is metrizable and $\mathscr{F} \subset \mathscr{L}_0(X)$ is sequentially equicontinuous, then $\mathscr{F} \subset L(X)$ and $\mathscr{F}$ is equicontinuous.

Proof. The first statement being obvious, we will only show the second one.

Assume that $\mathscr{F}$ is sequentially equicontinuous and that X is metrizable. Since X is metrizable, we have $\mathscr{L}_0(X) = L(X)$. Assume, by contradiction, that $\mathscr{F}$ is not equicontinuous. Since the topology of X is induced by a countable family of seminorms $\{p_n\}_{n\in\mathbb{N}}$ (see [77, Th. 3.35, p. 77]), it then follows that there exist a continuous seminorm q on X and sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$, $\{\iota_n\}_{n\in\mathbb{N}} \subset \mathscr{I}$ such that

$$\sup_{k=1,\dots,n} p_k(x_n) < \frac{1}{n}, \qquad q(F_{\iota_n} x_n) > 1, \qquad \forall n \in \mathbb{N}.$$

But then

$$\lim_{n\to+\infty} x_n = 0 \qquad \text{and} \qquad \liminf_{n\to+\infty}\left(\sup_{\iota\in\mathscr{I}} q(F_\iota x_n)\right) \geq \liminf_{n\to+\infty} q(F_{\iota_n} x_n) \geq 1,$$

which implies that $\mathscr{F}$ is not sequentially equicontinuous, getting a contradiction and concluding the proof. ■

5.2.4 C_0-sequentially equicontinuous semigroups

We now introduce the notion of C_0-sequentially (locally) equicontinuous semigroups.

Definition 5.2.12 (C_0-sequentially (locally) equicontinuous semigroup)**.** *A family of linear operators (not necessarily continuous)*

$$T := \{T_t : X \to X\}_{t\in\mathbb{R}^+}$$

is called a C_0-sequentially equicontinuous semigroup *on X if the following properties hold.*

(i) (Semigroup property) $T_0 = I$ and $T_{t+s} = T_t T_s$ for all $t, s \geq 0$.

(ii) (C_0- or strong continuity property) $\lim_{t\to 0^+} T_t x = x$, for every $x \in X$.

(iii) (Sequential equicontinuity) T is a sequentially equicontinuous family.

The family T is said to be a C_0-sequentially locally equicontinuous semigroup if (iii) is replaced by

(iii)′ (Sequential local equicontinuity) $\{T_t\}_{t\in[0,R]}$ is sequentially locally equicontinuous for every $R > 0$.

Remark 5.2.13. The notion of C_0-sequentially (locally) equicontinuous semigroup that we introduced is clearly a generalization of the notion of C_0-(locally) equicontinuous semigroup considered, e.g., in [98, Ch. IX], [64]. By Proposition 5.2.11 the two notions coincide if X is metrizable. In order to motivate the introduction of C_0-sequentially equicontinuous semigroups, we stress two facts.

(1) Even if a semigroup on a sequentially complete space is C_0-(locally) equicontinuous, proving this property might be harder than proving that it is only C_0-sequentially equicontinuous. For instance, in locally convex functional spaces with topologies defined by seminorms involving integrals, one can use integral convergence theorems for sequence of functions which do not hold for nets of functions.

(2) The notion of C_0-sequentially equicontinuous semigroup is a genuine generalization of the notion of C_0-equicontinuous semigroup of [98], as shown by Example 5.2.47.

As for C_0-semigroups in Banach spaces, given a C_0-sequentially locally equicontinuous semigroup T, we define

$$D(A) := \left\{ x \in X : \exists \lim_{h\to 0^+} \frac{T_h x - x}{h} \in X \right\}.$$

Clearly, $D(A)$ is a linear subspace of X. Then, we define the linear operator $A : D(A) \to X$ as

$$Ax := \lim_{h\to 0^+} \frac{T_h x - x}{h}, \qquad x \in D(A),$$

and call it the *infinitesimal generator* of T.

Proposition 5.2.14. *Let $T := \{T_t : X \to X\}_{t\in\mathbb{R}^+}$ be a C_0-sequentially locally equicontinuous semigroup.*

(i) For every $x \in X$, the function $Tx : \mathbb{R}^+ \to X$, $t \mapsto T_t x$, is continuous.

(ii) If T is sequentially equicontinuous, then, for every $x \in X$, the function $Tx : \mathbb{R}^+ \to X$, $t \mapsto T_t x$, is bounded.

Proof. (i) Let $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be a sequence converging from the right (resp., from the left) to $t \in \mathbb{R}$. By Definition 5.2.12*(i)*, we have, for every $p \in \mathscr{P}_X$ and $x \in X$,

$$p(T_{t_n}x - T_t x) \ = \ p(T_t(T_{t_n - t}x - x)) \quad (\text{resp.}, \ p(T_{t_n}x - T_t x) \ = p(T_{t_n}(T_{t-t_n}x - x))).$$

By Definition 5.2.12*(ii)*, $\{T_{t_n-t}x - x\}_{n\in\mathbb{N}}$ (resp. $\{T_{t-t_n}x - x\}_{n\in\mathbb{N}}$) converges to 0. Now conclude by using local sequential equicontinuity and (5.2.1).

(ii) This is provided by Proposition 5.2.10*(iii)*. ■

As well known, unlike the Banach space case, in locally convex spaces the passage from C_0-locally *equicontinuous* semigroups to C_0-*equicontinuous* semigroups through a renormalization with an exponential function is not obtainable in general (see Examples 5.2.44 and 5.2.45 in Subsection 5.2.9). Nevertheless, we have the following partial result.

Proposition 5.2.15. *Let τ denote the locally convex topology on X and let $|\cdot|_X$ be a norm on X. Assume that a set is τ-bounded if and only if it is $|\cdot|_X$-bounded. Let T be a C_0-sequentially locally equicontinuous semigroup on (X,τ).*

(i) If there exist $\alpha \in \mathbb{R}$ and $M \geq 1$ such that

$$|T_t|_{L((X,|\cdot|_X))} \leq M e^{\alpha t}, \quad \forall t \in \mathbb{R}^+, \tag{5.2.9}$$

then, for every $\lambda > \alpha$, the family $\{e^{-\lambda t}T_t : (X,\tau) \to (X,\tau)\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially equicontinuous semigroup.

(ii) If $(X,|\cdot|_X)$ is Banach, then there exist $\alpha \in \mathbb{R}$ and $M \geq 1$ such that (5.2.9) *holds.*

Proof. *(i)* Let $\lambda > \alpha$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to 0 in (X,τ). Then $\{x_n\}_{n\in\mathbb{N}}$ is bounded in (X,τ), thus, by assumption, also in $(X,|\cdot|_X)$. Set $N := \sup_{n\in\mathbb{N}}|x_n|_X$ and let $p \in \mathscr{P}_{(X,\tau)}$. Then

$$\begin{aligned}\sup_{t\in\mathbb{R}^+} p(e^{-\lambda t}T_t x_n) &\leq \sup_{0\leq t\leq s} p(e^{-\lambda t}T_t x_n) + \sup_{t>s} p(e^{-\lambda t}T_t x_n)\\ &\leq \sup_{0\leq t\leq s} p(e^{-\lambda t}T_t x_n) + L_p e^{(\alpha-\lambda)s} MN,\end{aligned}$$

where $L_p := \sup_{x\in X\setminus\{0\}} p(x)/|x|_X$ is finite, because $|\cdot|_X$-bounded sets are τ-bounded. Now we can conclude by applying to the right hand side of the inequality above first the $\limsup_{n\to+\infty}$ and considering that T is a C_0-sequentially locally equicontinuous semigroup on (X,τ), then the $\lim_{s\to+\infty}$ and taking into account that $\lambda > \alpha$.

(ii) By assumption, the bounded sets of $(X,|\cdot|_X)$ coincide with the bounded sets of (X,τ). By Proposition 5.2.5*(i)*, we then have $\mathscr{L}_0((X,\tau)) \subset L((X,|\cdot|_X))$. In particular $T_t \in L((X,|\cdot|_X))$, for all $t \in \mathbb{R}^+$. Now, by Proposition 5.2.14*(i)*, the set $\{T_t x\}_{t\in[0,t_0]}$ is compact in (X,τ) for every $x \in X$ and $t_0 > 0$, hence bounded. We can then apply the Banach-Steinhaus Theorem in $(X,|\cdot|_X)$ and conclude that there exists $M \geq 0$ such that $|T_t|_{L((X,|\cdot|_X))} \leq M$ for all $t \in [0,t_0]$. The conclusion now follows in a standard way from the semigroup property. ■

From here on in this subsection and in Subsections 5.2.5-5.2.6, unless differently specified, we will deal with C_0-sequentially equicontinuous semigroups and, to simplify the exposition, we will adopt a standing notation for them and their generator, i.e.

- $T = \{T_t\}_{t\in\mathbb{R}^+}$ denotes a C_0-sequentially equicontinuous semigroup;
- A denotes the infinitesimal generator of T.

Also, unless differently specified, from here on in this subsection and in Subsections 5.2.5-5.2.6, we will assume the following

Assumption 5.2.16. *For every $x \in X$ and $\lambda > 0$, there exists the generalized Riemann integral in X ([1])*

$$R(\lambda)x := \int_0^{+\infty} e^{-\lambda t} T_t x dt. \tag{5.2.10}$$

Remark 5.2.17. By Proposition 5.2.14, the generalized Riemann integral (5.2.10) always exists in the sequential completion of X. In particular, Assumption 5.2.16 is satisfied if X is sequentially complete.

[1]For every $a \geq 0$, the Riemann integral $\int_0^a e^{-\lambda t}T_t x dt$ exists in X, together with the limit $\int_0^{+\infty} e^{-\lambda t}T_t x dt := \lim_{a\to+\infty}\int_0^a e^{-\lambda t}T_t x dt$.

For every $p \in \mathscr{P}_X$, and every $\lambda, \hat{\lambda} \in (0,+\infty)$, we have the following inequalities, whose proof is straightforward, by triangular inequality and definition of Riemann integral, and by recalling Proposition 5.2.14:

$$p(R(\lambda)x - y) \leq \int_0^{+\infty} e^{-\lambda t} p(T_t x - \lambda y) dt, \qquad \forall x, y \in X \tag{5.2.11}$$

$$p(R(\lambda)x - R(\hat{\lambda})x) \leq \int_0^{+\infty} |e^{-\lambda t} - e^{-\hat{\lambda} t}| p(T_t x) dt, \qquad \forall x \in X. \tag{5.2.12}$$

Proposition 5.2.18. *If $L \in \mathscr{L}_0(X,Y)$, then $\mathbb{R}^+ \to Y, x \mapsto LT_t x$ is continuous and bounded. Moreover, for every $x \in X$, every $a \geq 0$, and every $\lambda > 0$,*

$$L\int_0^a e^{-\lambda t} T_t x dt = \int_0^a e^{-\lambda t} L T_t x dt \quad \textit{and} \quad L\int_0^{+\infty} e^{-\lambda t} T_t x dt = \int_0^{+\infty} e^{-\lambda t} L T_t dt, \tag{5.2.13}$$

where the Riemann integrals on the right-hand side of the equalities exist in Y.

Proof. Continuity of the map $\mathbb{R}^+ \to X,\ t \mapsto LT_t x$, follows from Remark 5.2.2, from sequential continuity of L and from Proposition 5.2.14*(i)*. By Proposition 5.2.14*(ii)*, we have that $\{T_t x\}_{t \in \mathbb{R}^+}$ is bounded, for all $x \in X$. From Proposition 5.2.5*(i)*, it then follows that $\{LT_t x\}_{t \in \mathbb{R}^+}$ is bounded.

Let $\{\pi^k\}_{k \in \mathbb{N}}$ be a sequence of partitions of $[0,a] \subset \mathbb{R}^+$ of the form $\pi^k := \{0 = t_0^k < t_1^k < \ldots < t_{n_k}^k = a\}$, with $|\pi^k| \to 0$ as $k \to +\infty$, where $|\pi^k| := \sup\{|t_{i+1} - t_i| : i = 0, \ldots, n_k - 1\}$. Then, by recalling Assumption 5.2.16 and by continuity of $\mathbb{R}^+ \to X,\ t \mapsto T_t x$, we have in Y

$$\int_0^a e^{-\lambda t} T_t x dt = \lim_{k \to +\infty} \sum_{i=0}^{n_k - 1} e^{-\lambda t_i^k} T_{t_i^k} x (t_{i+1}^k - t_i^k).$$

By sequential continuity of L we then have

$$L\int_0^a e^{-\lambda t} T_t x dt = \lim_{k \to +\infty} \sum_{i=0}^{n_k - 1} e^{-\lambda t_i^k} L T_{t_i^k} x (t_{i+1}^k - t_i^k). \tag{5.2.14}$$

Since $\mathbb{R}^+ \to X,\ t \mapsto LT_t x$ is continuous, equality (5.2.14) entails that $\mathbb{R}^+ \to X,\ t \mapsto e^{-\lambda t} LT_t x$ is Riemann integrable and that the first equality of (5.2.13) holds true.

The second equality of (5.2.13) follows from the first one and from sequential continuity of L, by letting $a \to +\infty$. ■

Proposition 5.2.19. *(i) For every $\lambda > 0$, the operator $R(\lambda) : X \to X$ is linear and sequentially continuous.*

(ii) For every $x \in X$, the function $(0,+\infty) \to X,\ \lambda \mapsto R(\lambda)x$, is continuous.

Proof. (i) The linearity of $R(\lambda)$ is clear. It remains to show its sequential continuity. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence convergent to 0. Then, for all $p \in \mathscr{P}_X$,

$$\lim_{n\to+\infty} p(R(\lambda)x_n) \leq \lim_{n\to+\infty} \int_0^{+\infty} e^{-\lambda t} p(T_t x_n) dt = \lambda^{-1} \lim_{n\to+\infty} \sup_{t\in\mathbb{R}^+} p(T_t x_n) = 0$$

where the last limit is obtained by sequential equicontinuity and by recalling (5.2.1).

(ii) For $p \in \mathscr{P}_X$, $x \in X$, $\lambda, \hat{\lambda} \in (0,+\infty)$, by (5.2.12),

$$p\left(R(\lambda)x - R(\hat{\lambda})x\right) \leq \int_0^{+\infty} |e^{-\lambda t} - e^{-\hat{\lambda}t}| p(T_t x) dt \leq \sup_{r\in\mathbb{R}^+} p(T_r x) \int_0^{+\infty} |e^{-\lambda t} - e^{-\hat{\lambda}t}| dt.$$

The last integral converges to 0 as $\lambda \to \hat{\lambda}$, and we conclude as $\sup_{r\in\mathbb{R}^+} p(T_r x) < +\infty$ by Proposition 5.2.14*(ii)*. ∎

The following proposition will be used in Subsection 5.3.2 to fit the theory of weakly continuous semigroups of [7, 8].

Proposition 5.2.20. *Let $C \subset X$ be sequentially closed, convex, and containing the origin, let $\hat{t} > 0$, and let $x \in X$. If $T_t x \in C$ for all $t \in [0, \hat{t}]$, then*

$$\int_0^{\hat{t}} e^{-\lambda t} T_t x dt \in \frac{1}{\lambda} C, \quad \forall \lambda > 0. \tag{5.2.15}$$

If $T_t x \in C$ for all $t \in \mathbb{R}^+$ then,

$$R(\lambda)x \in \frac{1}{\lambda} C, \quad \forall \lambda > 0. \tag{5.2.16}$$

Proof. We prove the first claim, as the second one is a straightforward consequence of it because of the sequential completeness of C.

Let $\hat{t} > 0$. The Riemann integral in (5.2.15) is the limit of a sequence of Riemann sums $\{\sigma(\pi^k)\}_{k\in\mathbb{N}}$ of the form

$$\sigma(\pi^k) = \sum_{i=1}^{m_k} e^{-\lambda t_i^k} (t_i^k - t_{i-1}^k) T_{t_i^k} x,$$

with $\pi^k := \{0 = t_0^k < t_1^k < \ldots < t_{m_k}^k = \hat{t}\}$ and $|\pi^k| \to 0$ as $k \to +\infty$, where $|\pi^k| := \sup\{|t_i - t_{i-1}| : i = 1, \ldots, m_k\}$. Then, by sequential closedness of C, we are reduced to show that $\sigma(\pi^k) \in \frac{1}{\lambda} C$ for every $k \in \mathbb{N}$. Denote

$$\alpha_k := \sum_{i=1}^{m_k} e^{-\lambda t_i^k} (t_i^k - t_{i-1}^k), \qquad \forall k \in \mathbb{N}.$$

Then

$$0 < \alpha_k < \int_0^{+\infty} e^{-\lambda t} dt = \lambda^{-1}, \quad \forall k \in \mathbb{N}.$$

As $\sigma(\pi^k)/\alpha_k$ is a convex combination of the elements $\{T_{t_i^k} x\}_{i=1,\ldots,m_k}$, which belong to C by assumption, recalling that C is convex and contains the origin, we conclude $\sigma(\pi^k) \in \alpha_k C \subset \frac{1}{\lambda} C$, for every $k \in \mathbb{N}$, and the proof is complete. ∎

5.2.5 Generators of C_0-sequentially equicontinuous semigroups

In this subsection we study the generator A of the C_0-sequentially equicontinuous semigroup T.

Recall that a subset U of a topological space Z is said to be sequentially dense in Z if, for every $z \in Z$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset U$ converging to z in Z. In such a case, it is clear that U is also dense in Z.

Proposition 5.2.21. *$D(A)$ is sequentially dense in X.*

Proof. Let $\lambda > 0$ and set $\psi_\lambda := \lambda R(\lambda) \in X$. By (5.2.13),

$$T_h R(\lambda)x = \int_0^{+\infty} e^{-\lambda t} T_{h+t} x dt \ \in \ X, \qquad \forall x \in X.$$

Then, following the proof of [98, p. 237, Theorem 1]([2]), we have

$$\frac{T_h\psi_\lambda x - \psi_\lambda x}{h} = \frac{e^{\lambda h}-1}{h}\left(\psi_\lambda x - \lambda\int_0^h e^{-\lambda t}T_t x dt\right) - \frac{\lambda}{h}\int_0^h e^{-\lambda t}T_t x dt \ \in \ X, \qquad \forall x \in X.$$

Passing to the limit for $h \to 0^+$, we obtain

$$\lim_{h\to 0^+}\frac{T_h\psi_\lambda x - \psi_\lambda x}{h} = \lambda(\psi_\lambda x - x) \ \in \ X, \qquad \forall x \in X.$$

Then $\psi_\lambda x \in D(A)$ and

$$A\psi_\lambda x = \lambda(\psi_\lambda - I)x \ \in \ X, \qquad \forall x \in X. \tag{5.2.17}$$

For future reference, we notice that this shows, in particular, that

$$\operatorname{Im}(R(\lambda)) \subset D(A). \tag{5.2.18}$$

Now we prove that

$$\lim_{\lambda\to+\infty}\psi_\lambda x = x \qquad \forall x \in X, \tag{5.2.19}$$

which concludes the proof. By (5.2.11), we have

$$p(\psi_\lambda x - x) = \lambda p(R(\lambda)x - \lambda^{-1}x) \le \int_0^{+\infty}\lambda e^{-\lambda t}p(T_t x - x)dt \quad \forall x \in X, \ \forall p \in \mathscr{P}_X.$$

By Proposition 5.2.14*(ii)*, we can apply the dominated convergence theorem to the last integral above when $\lambda \to +\infty$. Then we have

$$p(\psi_\lambda x - x) \to 0, \quad \forall x \in X, \ \forall p \in \mathscr{P}_X,$$

and we obtain (5.2.19) by arbitrariness of $p \in \mathscr{P}_X$. ∎

[2] In the cited result, X is assumed sequentially complete. However, the completeness of X is used in the proof only to define the integrals. In our case, existence for the integrals involved in the proof holds by assumption.

Remark 5.2.22. We notice that, if X is sequentially complete, then Proposition 5.2.21 can be refined. Indeed, as for C_0-semigroups in Banach spaces, we can define $D_\infty := \bigcap_{n=1}^{+\infty} D(A^n)$. If X is sequentially complete, then, for every $\varphi \in C_c^\infty((0,+\infty))$ and every $x \in X$, we can define the integral

$$\varphi_T x := \int_0^{+\infty} \varphi(t) T_t x dt.$$

Then one can show that $\varphi_T x \in D_\infty$, $A^n \varphi_T x = (-1)^n (\varphi^{(n)})_T x$, for all $n \geq 1$, and the set $\{\varphi_T x \colon \varphi \in C_c^\infty((0,+\infty)),\ x \in X\}$ is sequentially dense in X.

Proposition 5.2.23. *Let $x \in D(A)$. Then*

(i) $T_t x \in D(A)$ for all $t \in \mathbb{R}^+$;

(ii) the map $Tx \colon \mathbb{R}^+ \to X$, $t \mapsto T_t x$ is differentiable;

(iii) the following identity holds

$$\frac{d}{dt} T_t x = A T_t x = T_t A x, \qquad \forall t \in \mathbb{R}^+. \tag{5.2.20}$$

Proof. Let $x \in D(A)$. Consider the function $\Delta \colon \mathbb{R}^+ \to X$ defined by

$$\Delta(h) := \begin{cases} \dfrac{T_h - I}{h} x, & \text{if } h \neq 0 \\ \Delta(0) = Ax. \end{cases}$$

This function is continuous by definition of A. Then, by Remark 5.2.2,

$$T_t A x = T_t \lim_{h \to 0^+} \Delta(h) = \lim_{h \to 0^+} T_t \Delta(h) = \lim_{h \to 0^+} \frac{T_h T_t x - T_t x}{h}, \qquad \forall t \in \mathbb{R}^+,$$

which shows that *(i)* holds and that

$$T_t A x = A T_t x, \qquad \forall t \in \mathbb{R}^+.$$

The rest of the proof follows exactly as in [98, p. 239, Theorem 2]. ■

We are going to show that the infinitesimal generator identifies uniquely the semigroup T. For that, we need the following lemma, which will be also used afterwards.

Lemma 5.2.24. *Let $0 \leq a < b$, $f, g \colon (a,b) \to \mathscr{L}_0(X)$, $t_0 \in (a,b)$, and $x \in X$. Assume that*

(i) the family $\{f(t)\}_{t \in [a',b']}$ is sequentially equicontinuous, for every $a < a' < b' < b$;

(ii) $g(\cdot)x \colon (a,b) \to X$ is differentiable at t_0;

(iii) $f(\cdot) g(t_0) x \colon (a,b) \to X$ is differentiable at t_0.

Then there exists the derivative of $f(\cdot)g(\cdot)x\colon (a,b)\to X$ *at* $t=t_0$ *and*

$$\frac{d}{dt}[f(t)g(t)x)]|_{t=t_0} = \frac{d}{dt}[f(t)g(t_0)x]|_{t=t_0} + f(t_0)\frac{d}{dt}[g(t)x]|_{t=t_0}.$$

Proof. For $h\in\mathbb{R}\setminus\{0\}$ such that $[t_0-|h|,t_0+|h|]\subset(a,b)$, write

$$\begin{aligned} f(t_0+h)g(t_0+h)x-f(t_0)g(t_0)x =& f(t_0+h)\left(g(t_0+h)-g(t_0)-h\frac{d}{dt}[g(t)x]|_{t=t_0}\right)\\ &+hf(t_0+h)\frac{d}{dt}[g(t)x]|_{t=t_0}+(f(t_0+h)-f(t_0))g(t_0)x\\ =:& I_1(h)+I_2(h)+I_3(h). \end{aligned}$$

Letting $h\to 0$, we have $h^{-1}I_2(h)\to f(t_0)\frac{d}{dt}[g(t)x]|_{t=t_0}x$ and $h^{-1}I_3(h)\to \frac{d}{dt}[f(t)g(t_0)x]|_{t=t_0}$. Moreover,

$$p\left(h^{-1}I_1(h)\right)\le \sup_{s\in[t_0-|h|,t_0+|h|]} p\left(f(s)\left(\frac{g(t_0+h)-g(t_0)}{h}-\frac{d}{dt}[g(t)x]|_{t=t_0}\right)x\right),\qquad \forall p\in\mathscr{P}_X,$$

and the member at the right-hand side of the inequality above tends to 0 as $h\to 0$, because of sequential local equicontinuity of the family $\{f(s)\}_{s\in(a,b)}$ (part *(i)* of the assumptions) and because of differentiability of $g(\cdot)x$ in t_0. ■

Theorem 5.2.25. *Let S be a C_0-sequentially equicontinuous semigroup on X with infinitesimal generator $A_S=A$. Then $S=T$.*

Proof. For $t>0$ and $x\in D(A)$, consider the function $f\colon[0,t]\to X$, $s\mapsto T_{t-s}S_sx$. By Proposition 5.2.23 and Lemma 5.2.24, $f'(s)=0$ for all $s\in[0,t]$, and then $T_tx=f(0)=f(t)=S_tx$. Since $D(A)$ is sequentially dense in X and the operators T_t, S_t are sequentially continuous, we have $T_tx=S_tx$ for all $x\in X$, and we conclude by arbitrariness of $t>0$. ■

Definition 5.2.26. *Let $D(C)\subset X$ be a linear subspace. For a linear operator $C\colon D(C)\to X$, we define the* spectrum *$\sigma_0(C)$ as the set of $\lambda\in\mathbb{R}$ such that one of the following holds:*

(i) *$\lambda-C$ is not one-to-one;*

(ii) $\mathrm{Im}(\lambda-C)\neq X$*;*

(iii) *there exists $(\lambda-C)^{-1}$, but it is not sequentially continuous.*

We denote $\rho_0(C):=\mathbb{R}\setminus\sigma_0(C)$, and call it resolvent set *of C. If $\lambda\in\rho_0(C)$, we denote by $R(\lambda,C)$ the sequentially continuous inverse $(\lambda-C)^{-1}$ of $\lambda-C$.*

Theorem 5.2.27. *If $\lambda>0$, then $\lambda\in\rho_0(A)$ and $R(\lambda,A)=R(\lambda)$.*

Proof. Step 1. Here we show that $\lambda - A$ is one-to-one for every $\lambda > 0$. Let $x \in D(A)$. By Proposition 5.2.23, for any $f \in X^*$, the function $F\colon \mathbb{R}^+ \to \mathbb{R}$, $t \mapsto f(e^{-\lambda t}T_t x)$ is differentiable, and $F'(t) = f(e^{-\lambda t}T_t(A-\lambda)x)$. If $(A-\lambda)x = 0$, then F is constant. By Proposition 5.2.14*(ii)*, $F(t) \to 0$ as $t \to +\infty$, hence it must be $F \equiv 0$. Then $f(x) = F(0) = 0$. As f is arbitrary, we conclude that $x = 0$ and, therefore, that $\lambda - A$ is one-to-one.

Step 2. Here we show that $\lambda - A$ is invertible and $R(\lambda, A) = R(\lambda)$, for every $\lambda > 0$. By (5.2.18) and (5.2.17),

$$(\lambda - A)R(\lambda) = I \tag{5.2.21}$$

which shows that $\lambda - A$ is onto, and then invertible (by recalling also Step 1), and that $(\lambda - A)^{-1} = R(\lambda)$.

Step 3. The fact $(\lambda - A)^{-1} \in \mathscr{L}_0(X)$ follows from Step 2 and Proposition 5.2.19*(i)*. ■

Corollary 5.2.28. *The operator A is sequentially closed, i.e., its graph* $\mathrm{Gr}(A)$ *is sequentially closed in $X \times X$.*

Proof. Observe that $(x,y) \in \mathrm{Gr}(A)$ if and only if $(x, x-y) \in \mathrm{Gr}(I - A)$, and hence if and only if $(x-y, x) \in \mathrm{Gr}(R(1,A))$. As $R(1,A) \in \mathscr{L}_0(X)$, then its graph is sequentially closed in $X \times X$, and we conclude. ■

Corollary 5.2.29. *We have the following.*

(i) $AR(\lambda,A)x = \lambda R(\lambda,A)x - x$, *for all* $\lambda > 0$ *and* $x \in X$.

(ii) $R(\lambda,A)Ax = AR(\lambda,A)x$, *for all* $\lambda > 0$ *and* $x \in D(A)$.

(iii) (Resolvent equation) For every $\lambda > 0$ *and* $\mu > 0$,

$$R(\lambda,A) - R(\mu,A) = (\mu - \lambda)R(\lambda,A)R(\mu,A). \tag{5.2.22}$$

(iv) For every $x \in X$, $\lim_{\lambda\to\infty} \lambda R(\lambda,A)x = x$.

Proof. (i) It follows from (5.2.21).

(ii) By *(i)* and considering that $x \in D(A)$, we can write

$$AR(\lambda,A)x = \lambda R(\lambda,A)x - x = \lambda R(\lambda,A)x - R(\lambda,A)(\lambda - A)x = R(\lambda,A)Ax.$$

(iii) It follows from *(i)* by standard algebraic computations.

(iv) This follows from (5.2.19) and from Theorem 5.2.27. ■

Remark 5.2.30. The computations involved in the proof of Corollary 5.2.29(iii) require only that $A\colon D(A) \subset X \to X$ is a linear operator and $\lambda, \mu \in \rho_0(A)$.

Proposition 5.2.31. *The family of operators* $\{\lambda^n R(\lambda,A)^n : X \to X\}_{\lambda>0, n\in\mathbb{N}}$ *is sequentially equicontinuous.*

Proof. Arguing as in the proof of [98, p. 241, Theorem 2] ([3]), we obtain the inequality

$$\sup_{\substack{n\in\mathbb{N}\\ \lambda>0}} p\left(\lambda^{n+1}R(\lambda,A)^{n+1}x\right) \le \sup_{t\in\mathbb{R}^+} p(T_t x), \qquad \forall p\in\mathscr{P}_X,$$

which provides the sequential equicontinuity due to sequential equicontinuity of T. ■

Proposition 5.2.32. *Let $\lambda_1,\dots,\lambda_j$ be strictly positive real numbers. Then*

$$p\left(\left(\prod_{i=1}^{j}\lambda_i R(\lambda_i,A)\right)x\right) \le \sup_{t\in\mathbb{R}^+} p(T_t x), \qquad \forall p\in\mathscr{P}_X,\ \ \forall x\in X.$$

Proof. By Theorem 5.2.27 and by Proposition 5.2.18, for every $x\in X$ we have

$$\left(\prod_{i=1}^{j} R(\lambda_i,A)\right)x = \int_0^{+\infty} e^{-\lambda_1 t_1}\int_0^{+\infty} e^{-\lambda_2 t_2}\dots\int_0^{+\infty} e^{-\lambda_j t_j} T_{\sum_{i=1}^{j} t_i} x\, dt_j \dots dt_2 dt_1$$

and then

$$\begin{aligned} p\left(\left(\prod_{i=1}^{j} R(\lambda_i,A)\right)x\right) &\le \left(\int_0^{+\infty} e^{-\lambda_1 t_1}\int_0^{+\infty} e^{-\lambda_2 t_2}\dots\int_0^{+\infty} e^{-\lambda_j t_j} dt_j \dots dt_2 dt_1\right)\sup_{t\in\mathbb{R}^+} p(T_t x)\\ &= \left(\prod_{i=1}^{j}\lambda_i^{-1}\right)\sup_{t\in\mathbb{R}^+} p(T_t x). \end{aligned}$$

This concludes the proof, because T is sequentially equicontinuous. ■

5.2.6 Generation of C_0-sequentially equicontinuous semigroups

The aim of this subsection is to state a generation theorem for C_0-sequentially equicontinuous semigroups in the spirit of the Hille-Yosida theorem stated for C_0-semigroups in Banach spaces. In order to implement the classical arguments (with slight variations due to our "sequential continuity" setting), and, more precisely, in order to define the Yosida approximation, we need the sequential completeness of the space X.

Proposition 5.2.33. *Let X be sequentially complete and let $B\in\mathscr{L}_0(X)$. Assume that the family $\{B^n : X\to X\}_{n\in\mathbb{N}}$ is sequentially equicontinuous. Let $f:\mathbb{R}\to\mathbb{R}$ be an analytic function of the form $f(t)=\sum_{n=0}^{+\infty}a_n t^n$, with $t\in\mathbb{R}$. Then the following hold.*

(i) The series

$$f_B(t) := \sum_{n=0}^{+\infty} a_n t^n B^n \tag{5.2.23}$$

converges in $\mathscr{L}_{0,b}(X)$ uniformly for t on compact sets of $\mathbb{R}$.

[3] Also here, we remark that the sequential completeness of the space is not necessary, once that Assumption 5.2.16 is standing.

(ii) The function $f_B : \mathbb{R} \to \mathscr{L}_{0,b}(X)$, $t \mapsto f_B(t)$ *is continuous.*

(iii) The family $\{f_B(t)\}_{t\in[-r,r]}$ *is sequentially equicontinuous for every* $r > 0$.

Proof. (i) For $0 \le n \le m$, $p \in \mathscr{P}_X$, $D \subset X$ bounded, $r > 0$, $x \in D$, $t \in [-r,r]$, we write

$$\begin{aligned} p\left(\sum_{k=n}^{m} a_k t^k B^k x\right) &\le \sum_{k=n}^{m} |a_k||t|^k p\left(B^k x\right) \le \left(\sum_{k=n}^{+\infty} |a_k| r^k\right) \sup_{i\in\mathbb{N}} p\left(B^i x\right) \\ &\le \left(\sum_{k=n}^{+\infty} |a_k| r^k\right) \sup_{y\in\bigcup_{i\in\mathbb{N}} B^i D} p(y). \end{aligned} \tag{5.2.24}$$

Observe that, by Proposition 5.2.10*(iii)*, the supremum appearing in the last term of (5.2.24) is finite. Then

$$\sup_{t\in[-r,r]} \rho_{p,D}\left(\sum_{k=n}^{m} a_k t^k B^k\right) \le \left(\sum_{k=n}^{+\infty} |a_k| r^k\right) \sup_{y\in\bigcup_{i\ge 0} B^i D} p(y) \qquad \forall n \in \mathbb{N} \tag{5.2.25}$$

shows that the sequence of the partials sums of (5.2.23) is Cauchy in $\mathscr{L}_{0,b}(X)$, uniformly for $t \in [-r,r]$, and then, by Proposition 5.2.9*(ii)*, the sum is convergent, uniformly for $t \in [-r,r]$.

(ii) This follows from convergence of the partial sums in the space $C([-r,r], \mathscr{L}_{0,b}(X))$ endowed with the compact-open topology, as shown above.

(iii) By continuity of p, estimate (5.2.24) shows that

$$\sup_{t\in[-r,r]} p\left(f_B(t)x\right) = \sup_{t\in[-r,r]} \lim_{n\to+\infty} p\left(\sum_{k=0}^{n} a_k t^k B^k x\right) \le \left(\sum_{k=0}^{+\infty} |a_k| r^k\right) \sup_{i\in\mathbb{N}} p\left(B^i x\right) \qquad \forall x \in X,$$

which provides the sequential equicontinuity of $\{f_B(t)\}_{t\in[-r,r]}$. ■

Lemma 5.2.34. *Let* X *be sequentially complete. Let* $B, C \in \mathscr{L}_0(X)$ *be such that* $\{B^n\}_{n\in\mathbb{N}}$ *and* $\{C^n\}_{n\in\mathbb{N}}$ *are sequentially equicontinous. Let* $f(t) = \sum_{n=0}^{+\infty} a_n t^n$, $g(t) = \sum_{n=0}^{+\infty} b_n t^n$ *be analytic functions defined on* $\mathbb{R}$. *Then, for all* $p \in \mathscr{P}_X$, $x \in X$, $t, s \in \mathbb{R}$,

$$p\left(f_B(t) g_C(s) x\right) \le \left(\sum_{n=0}^{+\infty} |a_n||t|^n\right)\left(\sum_{n=0}^{+\infty} |b_n||s|^n\right) \sup_{i,j\in\mathbb{N}} p\left(B^i C^j x\right), \tag{5.2.26}$$

and the family $\{f_B(t) g_C(s)\}_{t,s\in[-r,r]}$ *is sequentially equicontinuous for every* $r > 0$.

Proof. By Proposition 5.2.33 and by recalling that every partial sum $\sum_{i=0}^{n} a_i t^i B^i$ is sequentially continuous, we can write

$$p\left(f_B(t) g_C(s) x\right) = \lim_{n\to+\infty} \lim_{m\to+\infty} p\left(\left(\sum_{i=0}^{n} a_i t^i B^i\right)\left(\sum_{j=0}^{m} b_j s^j C^j\right) x\right) \qquad \forall p \in \mathscr{P}_X,\ \forall x \in X,\ \forall t, s \in \mathbb{R}.$$

Then, we obtain (5.2.26) by the properties of the seminorms. The sequential equicontinuity of the family $\{f_B(t) g_C(s)\}_{t,s\in[-r,r]}$ comes from (5.2.26) and Proposition 5.2.10*(i)*. ■

Proposition 5.2.35. *Let X be sequentially complete. Let B, C, f, g, as in Lemma 5.2.34. We have the following:*

(i) $(f+g)_B = f_B + g_B$ *and* $(fg)_B = f_B g_B$;

(ii) *if* $BC = CB$, *then* $f_B(t)g_C(s) = g_C(s)f_B(t)$, *for every* $t,s \in \mathbb{R}$, *and* $\{f_B(t)g_C(s)\}_{t,s\in[-r,r]}$ *is sequentially equicontinuous for every* $r > 0$.

Proof. The proof follows by algebraic computations on the partial sums and then passing to the limit. ■

Notation. We denote $e^{tB} := f_B(t)$ when $f(t) = e^t$.

Proposition 5.2.36. *Let X be sequentially complete.*

(i) *Let* $B,C \in \mathscr{L}_0(X)$ *be such that* $BC = CB$, *and assume that the families* $\{B^n\}_{n\in\mathbb{N}}$ *and* $\{C^n\}_{n\in\mathbb{N}}$ *are sequentially equicontinous. Then, for every* $t,s \in \mathbb{R}$,

(a) *the sum* $e^{tB+sC} := \sum_{n=0}^{+\infty} \frac{(tB+sC)^n}{n!}$ *converges in* $\mathscr{L}_{0,b}(X)$;

(b) $e^{tB+sC} = e^{tB}e^{sC} = e^{sC}e^{tB}$;

(c) *the family* $\left\{e^{tB+sC}\right\}_{t,s\in[-r,r]}$ *is sequentially equicontinuous for every* $r > 0$.

(ii) *Let* $B \in \mathscr{L}_0(X)$ *be such that the family* $\{B^n\}_{n\in\mathbb{N}}$ *is sequentially equicontinous. Then* $\{e^{tB}\}_{t\in\mathbb{R}^+}$ *is a C_0-sequentially locally equicontinuous semigroup on X with infinitesimal generator B.*

Proof. *(i)* Let $r > 0$ and $t \in [-r,r]$. By standard computations, we have

$$\sum_{i=0}^{n} \frac{(B+C)^i}{i!} t^i = \left(\sum_{i=0}^{n} \frac{B^i}{i!} t^i\right)\left(\sum_{i=0}^{n} \frac{C^i}{i!} t^i\right) - \sum_{i=0}^{n} \frac{B^i}{i!} t^i \left(\sum_{k=n-i+1}^{n} \frac{C^k}{k!} t^k\right). \tag{5.2.27}$$

Let $D \subset X$ be a bounded set. For $x \in D$ and $p \in \mathscr{P}_X$, we have

$$\begin{aligned} p\left(\sum_{i=0}^{n} \frac{B^i}{i!} t^i \left(\sum_{k=n-i+1}^{n} \frac{C^k x}{k!} t^k\right)\right) &\le \sum_{i=0}^{n} \sum_{k=n-i+1}^{n} \frac{1}{i!k!} r^{i+k} p\left(B^i C^k x\right) \\ &\le \left(\sum_{i=0}^{n} \sum_{k=n-i+1}^{n} \frac{1}{i!k!} r^{i+k}\right) \sup_{i,k\in\mathbb{N}} \rho_{p,D}\left(B^i C^k\right). \end{aligned}$$

By Proposition 5.2.10*(i)*, the family $\{B^i C^k\}_{i,k\in\mathbb{N}}$ is sequentially equicontinuous. Hence, by Proposition 5.2.10*(iii)*, we have $\sup_{i,k\in\mathbb{N}} \rho_{p,D}\left(B^i C^k\right) < +\infty$. Moreover, Lebesgue's dominated convergence theorem applied in discrete spaces yields

$$\lim_{n\to+\infty} \sum_{i=0}^{n} \sum_{k=n-i+1}^{n} \frac{1}{i!k!} r^{i+k} = 0.$$

So, we conclude

$$\lim_{n\to+\infty}\rho_{p,D}\left(\sum_{i=0}^{n}\frac{B^i}{i!}t^i\left(\sum_{k=n-i+1}^{n}\frac{C^k}{k!}t^k\right)\right)=0. \tag{5.2.28}$$

On the other hand, by Proposition 5.2.8,

$$\lim_{n\to+\infty}\left(\sum_{i=0}^{n}\frac{B^i}{i!}t^i\right)\left(\sum_{i=0}^{n}\frac{C^i}{i!}t^i\right)=\lim_{n\to+\infty}\left(\sum_{i=0}^{n}\frac{B^i}{i!}t^i\right)\lim_{n\to+\infty}\left(\sum_{i=0}^{n}\frac{C^i}{i!}t^i\right)=e^{tB}e^{tC}, \tag{5.2.29}$$

where the limits are taken in the space $\mathscr{L}_{0,b}(X)$. By (5.2.27), (5.2.28) and (5.2.29), we obtain

$$\lim_{n\to+\infty}\sum_{i=0}^{n}\frac{(B+C)^i}{i!}t^i=e^{tB}e^{tC}, \tag{5.2.30}$$

with the limit taken in $\mathscr{L}_{0,b}(X)$.

Now, let $t\neq 0$ and $|s|\leq|t|$[4]. Then $\left\{\left(\frac{s}{t}C\right)^n\right\}_{n\in\mathbb{N}}$ is sequentially equicontinuous. By replacing C by $\frac{s}{t}C$ in (5.2.30), we have

$$\lim_{n\to+\infty}\sum_{i=0}^{n}\frac{(tB+sC)^i}{i!}=e^{tB}e^{\left(\frac{s}{t}C\right)t}=e^{tB}e^{sC}, \tag{5.2.31}$$

where the limits are in $\mathscr{L}_{0,b}(X)$. So we have proved *(a)*. Properties *(b)* and *(c)* now follow from (5.2.31) and from Proposition 5.2.35*(ii)*.

(ii) First we notice that $e^{0B}=I$ by definition. The semigroup property for $\{e^{tB}\}_{t\in\mathbb{R}^+}$ is given by *(i)*, which also provides the sequential local equicontinuity. Proposition 5.2.33 provides the continuity of the map $\mathbb{R}^+\to X,\ t\mapsto e^{tB}x$, for every $x\in X$. Hence, we have proved that $\{e^{tB}\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially locally equicontinuous semigroup. It remains to show that the infinitesimal generator is B. For $h>0$, define $f(t;h):=e^{ht}-1-ht$. By applying (5.2.26) to the map $\mathbb{R}\to\mathbb{R},\ t\mapsto f(t;h)$, with B in place of B, and with $C=I$ and $g\equiv 1$, we obtain

$$p\left(\frac{e^{hB}-I}{h}x-Bx\right)=h^{-1}p\left(f_B(1;h)\right)\leq h^{-1}f(1;h)\sup_{n\in\mathbb{N}}p\left(B^nx\right)$$

and the last term converges to 0 as $h\to 0^+$, because of sequential equicontinuity of $\{B^n\}_{n\in\mathbb{N}}$. This shows that the domain of the generator is the whole space X and that the generator is B. ■

We can now state the equivalent of the Hille-Yosida generation theorem in our framework of C_0-sequentially equicontinuous semigroups.

Theorem 5.2.37. *Let $\hat{A}: D(\hat{A})\subset X\to X$ be a linear operator. Consider the following two statements.*

[4] If $|t|<|s|$, we can exchange the role of B and C, by simmetry of the sums appearing in (5.2.31).

(i) $\hat{A}$ *is the infinitesimal generator of a* C_0*-sequentially equicontinuous semigroup* $\hat{T}$ *on* X.

(ii) $\hat{A}$ *is a sequentially closed linear operator,* $D(\hat{A})$ *is sequentially dense in* X*, and there exists a sequence* $\{\lambda_n\}_{n\in\mathbb{N}}\subset\rho_0(\hat{A})$*, with* $\lambda_n\to+\infty$*, such that the family*

$$\left\{\left(\lambda_n R(\lambda_n,\hat{A})\right)^m\right\}_{n,m\in\mathbb{N}}$$

is sequentially equicontinuous.

Then (i)⇒(ii). If X *is sequentially complete, then (ii)⇒(i).*

Proof. (i)⇒(ii). The fact that $\hat{A}$ is a sequentially closed linear operator was proved in Corollary 5.2.28. The fact that $D(\hat{A})$ is sequentially dense in X was proved in Proposition 5.2.21. The remaining facts follow by Proposition 5.2.32 and Theorem 5.2.27.

(ii)⇒(i). We split this part of the proof in several steps.

Step 1. Let $\{\lambda_n\}_{n\in\mathbb{N}}\subset\rho_0(\hat{A})$ be a sequence as in *(ii)*. For $n\in\mathbb{N}$, define $J_{\lambda_n}:=\lambda_n R(\lambda_n,\hat{A})$. Observe that, for all $x\in D(\hat{A})$, it is $(J_{\lambda_n}-I)x=R(\lambda_n,\hat{A})\hat{A}x$. By assumption, the family $\{J_{\lambda_n}\}_{n\in\mathbb{N}}$ is sequentially equicontinuous, and then, for every $x\in D(\hat{A})$ and $p\in\mathscr{P}_X$,

$$\lim_{n\to+\infty}p\left(J_{\lambda_n}x-x\right)=\lim_{n\to+\infty}p\left(R(\lambda_n,\hat{A})\hat{A}x\right)\leq\lim_{n\to+\infty}\lambda_n^{-1}\left(\sup_{k\in\mathbb{N}}p\left(J_k\hat{A}x\right)\right)=0. \tag{5.2.32}$$

Now let $x\in X$. By assumption, there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $D(\hat{A})$ converging to x in X. We have

$$p\left(J_{\lambda_n}x-x\right)\leq p(x-x_k)+p\left(J_{\lambda_n}x_k-x_k\right)+p\left(J_{\lambda_n}(x-x_k)\right),\qquad\forall k\in\mathbb{N},\ \forall n\in\mathbb{N},\ \forall p\in\mathscr{P}_X.$$

By taking first the lim sup in n and then the limit as $k\to+\infty$ in the inequality above, and recalling (5.2.32) and the sequential equicontinuity of $\{J_{\lambda_n}\}_{n\in\mathbb{N}}$, we conclude

$$\lim_{n\to+\infty}J_{\lambda_n}x=x,\qquad\forall x\in X. \tag{5.2.33}$$

Step 2. Here we show that, for $t\in\mathbb{R}^+$ and $n\in\mathbb{N}$, $T_t^{(n)}:=e^{t\hat{A}J_{\lambda_n}}$ is well-defined as a convergent series in $\mathscr{L}_{0,b}(X)$, and that $\{T_t^{(n)}\}_{t\in\mathbb{R}^+,n\in\mathbb{N}}$ is sequentially equicontinuous. Taking into account that $\hat{A}J_{\lambda_n}=\lambda_n(J_{\lambda_n}-I)$, we have (as formal sums) $T_t^{(n)}=e^{t\hat{A}J_{\lambda_n}}=e^{t\lambda_n(J_{\lambda_n}-I)}$. Since $\{J_{\lambda_n}^k\}_{k\in\mathbb{N}}$ is assumed to be sequentially equicontinuous, by Proposition 5.2.36*(i)*, $T_t^{(n)}$ is well-defined as a convergent series in $\mathscr{L}_{0,b}(X)$, and

$$T_t^{(n)}=e^{-t\lambda_n I}e^{t\lambda_n J_{\lambda_n}}. \tag{5.2.34}$$

Hence, using Proposition 5.2.36*(ii)*, the family $\{T_t^{(n)}\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially locally equicontinuous semigroup for each fixed $n\in\mathbb{N}$. On the other hand, by (5.2.34) and by Lemma 5.2.34, we have

$$\sup_{n\in\mathbb{N}}p\left(T_t^{(n)}x\right)=\sup_{n\in\mathbb{N}}\left(e^{-t\lambda_n}p\left(e^{t\lambda_n J_{\lambda_n}}x\right)\right)\leq\sup_{n,k\in\mathbb{N}}p(J_{\lambda_n}^k x),\qquad\forall p\in\mathscr{P}_X,\ \forall x\in X.$$

As, by assumption, $\{J^k_{\lambda_n}\}_{n,k\in\mathbb{N}}$ is sequentially equicontinuous, this shows that $\{T^{(n)}_t\}_{t\in\mathbb{R}^+,n\in\mathbb{N}}$ is sequentially equicontinuous.

Step 3. Here we show that the sequence $\{T^{(n)}_t x\}_{n\in\mathbb{N}}$ is Cauchy for every $t\in\mathbb{R}^+$ and $x\in D(\hat{A})$. First note that, since the family $\{R(\lambda_n,\hat{A})\}_{n\in\mathbb{N}}$ is a commutative set (see (5.2.22) and Remark 5.2.30), also the family $\{J_{\lambda_n}\}_{n\in\mathbb{N}}$ is a commutative set. Then $\lambda_m(J_{\lambda_m}-I)$ commutes with every J_{λ_n}. Since the sum defining $T^{(m)}_t$ is convergent in $\mathscr{L}_{0,b}(X)$, we have $T^{(m)}_t J_{\lambda_n}=J_{\lambda_n}T^{(m)}_t$ and $T^{(m)}_t T^{(n)}_s=T^{(n)}_s T^{(m)}_t$ for every $m,n\in\mathbb{N}$, $t,s\in\mathbb{R}^+$. By Lemma 5.2.24 and by the commutativity just noticed, if $x\in X$ and $t\in\mathbb{R}^+$, the map $F\colon[0,t]\to X$, $s\mapsto T^{(n)}_{t-s}T^{(m)}_s x$, is differentiable and

$$T^{(m)}_t x-T^{(n)}_t x=\int_0^t F'(s)ds=\int_0^t T^{(n)}_{t-s}T^{(m)}_s\hat{A}\left(J_{\lambda_m}-J_{\lambda_n}\right)x\,ds,$$

where the integral is well-defined by sequential completeness of X. We notice that $J_{\lambda_n}\hat{A}=\hat{A}J_{\lambda_n}$ on $D(\hat{A})$. Then, from the equality above we deduce

$$p\left(T^{(m)}_t x-T^{(n)}_t x\right)\le\int_0^t p\left(T^{(n)}_{t-s}T^{(m)}_s\left(J_{\lambda_m}-J_{\lambda_n}\right)\hat{A}x\right)ds,\qquad\forall x\in D(\hat{A}),\ \forall p\in\mathscr{P}_X,$$

and then, for all $\hat{t}>0$, $x\in D(\hat{A})$, $p\in\mathscr{P}_X$,

$$\sup_{t\in[0,\hat{t}]}p\left(T^{(m)}_t x-T^{(n)}_t x\right)\le\hat{t}\sup_{t,s\in[0,\hat{t}]}p\left(T^{(n)}_t T^{(m)}_s\left(J_{\lambda_m}-J_{\lambda_n}\right)\hat{A}x\right).\tag{5.2.35}$$

Now observe that, by Proposition 5.2.10*(i)* and Step 2, the family $\{T^{(m)}_t T^{(n)}_s\}_{\substack{t,s\in\mathbb{R}^+\\ m,n\in\mathbb{N}}}$ is sequentially equicontinuous, and then the term on the right-hand side of (5.2.35) goes to 0 as $n,m\to+\infty$, because of (5.2.33). Hence, the sequence $\{T^{(n)}_t x\}_{n\in\mathbb{N}}$ is Cauchy for every $t\in\mathbb{R}$ and $x\in D(\hat{A})$.

Step 4. By Step 3 and by sequential completeness of X, we conclude that there exists in X

$$\hat{T}_t x:=\lim_{n\to+\infty}T^{(n)}_t x,\qquad\forall t\in\mathbb{R}^+,\ \forall x\in D(\hat{A}).\tag{5.2.36}$$

Moreover, by (5.2.35), the limit (5.2.36) is uniform in $t\in[0,\hat{t}]$, for every $\hat{t}>0$.

Step 5. We extend the result of Step 4, stated for $x\in D(\hat{A})$, to all $x\in X$. Let $\hat{t}>0$ and let $\{x_k\}_{k\in\mathbb{N}}\subset D(\hat{A})$ be a sequence converging to x in X. We can write

$$T^{(m)}_t x-T^{(n)}_t x=\left(T^{(m)}_t-T^{(n)}_t\right)(x-x_k)+\left(T^{(m)}_t-T^{(n)}_t\right)x_k,\qquad\forall t\in[0,\hat{t}],\ \forall m,n,k\in\mathbb{N}.$$

Then, using Step 4, we have, uniformly for $t\in[0,\hat{t}]$,

$$\begin{aligned}\limsup_{n,m\to+\infty}\sup_{t\in[0,\hat{t}]}p\left(T^{(m)}_t x-T^{(n)}_t x\right)&\le\limsup_{n,m\to+\infty}\sup_{t\in[0,\hat{t}]}p\left(\left(T^{(m)}_t-T^{(n)}_t\right)(x-x_k)\right)\\&\le\sup_{\substack{n,m\in\mathbb{N}\\ t\in[0,\hat{t}]}}p\left(\left(T^{(m)}_t-T^{(n)}_t\right)(x-x_k)\right),\qquad\forall k\in\mathbb{N},\ \forall p\in\mathscr{P}_X.\end{aligned}$$

The last term goes to 0 as $k \to +\infty$, because of sequential equicontinuity of the family $\{T_t^{(n)}\}_{n\in\mathbb{N},t\in\mathbb{R}^+}$ (Step 2).

Hence, recalling that $D(\hat{A})$ is sequentially dense in X, we have proved that that there exists in X, uniformly for $t \in [0,\hat{t}]$,

$$\hat{T}_t x := \lim_{n\to+\infty} T_t^{(n)} x, \quad \forall x \in X. \tag{5.2.37}$$

Step 6. We show that the family $\hat{T} = \{\hat{T}_t\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially equicontinuous semigroup on X. First we notice that, as by Step 5 the limit in (5.2.37) defining $\hat{T}_t x$ is uniform for $t \in [0,\hat{t}]$, for every $\hat{t} > 0$, then the function $\mathbb{R}^+ \to X$, $t \mapsto \hat{T}_t x$, is continuous. In particular, $\hat{T}_t x \to \hat{T}_0 x$ as $t \to 0^+$ for every $x \in X$. Moreover, $\hat{T}_0 = I$ as $T_0^{(n)} = I$ for each $n \in \mathbb{N}$. The linearity of $\hat{T}_t$ and the semigroup property come from the same properties holding for every $T_t^{(n)}$. It remains to show that the family $\hat{T}$ is sequentially equicontinuous. This comes from sequential equicontinuity of the family $\{T_t^{(n)}\}_{n\in\mathbb{N},t\in\mathbb{R}^+}$ (Step 2), and from the estimate

$$p\left(\hat{T}_t x\right) \le p\left(\hat{T}_t x - T_t^{(n)} x\right) + p\left(T_t^{(n)} x\right) \le p\left(\hat{T}_t x - T_t^{(n)} x\right) + \sup_{\substack{t\in\mathbb{R}^+\\ n\in\mathbb{N}}} p\left(T_t^{(n)} x\right) \qquad \forall t \in \mathbb{R}^+,\ \forall n \in \mathbb{N},$$

by taking first the limit as $n \to +\infty$ and then the supremum over t.

Step 7. To conclude the proof, we only need to show that the infinitesimal generator of $\hat{T}$ is $\hat{A}$. Let $p \in \mathscr{P}_X$ and $x \in D(\hat{A})$. By applying Proposition 5.2.23 to $T^{(n)}$, we can write

$$\hat{T}_t x - x = \lim_{n\to+\infty} (T_t^{(n)} x - x) = \lim_{n\to+\infty} \int_0^t T_s^{(n)} \hat{A} J_{\lambda_n} x\, ds,$$

where the integral on the right-hand side exists because of sequential completeness of X and of continuity of the integrand function, and where the latter equality is obtained, as usual, by pairing the two members of the equality with funtionals $\Lambda \in X^*$ and by using (5.2.13).

Now we wish to exchange the limit with the integral. This is possible, as, by Step 2, Step 5, and and (5.2.33), we have

$$\lim_{n\to+\infty} T_t^{(n)} J_{\lambda_n} \hat{A} x = \hat{T}_t \hat{A} x \quad \text{uniformly for } t \text{ over compact sets.}$$

Then

$$\hat{T}_t x - x = \int_0^t \lim_{n\to+\infty} T_s^{(n)} \hat{A} J_{\lambda_n} x\, ds = \int_0^t \hat{T}_s \hat{A} x.$$

Dividing by t and letting $t \to 0^+$, we conclude that $x \in D(\tilde{A})$, where $\tilde{A}$ is the infinitesimal generator of $\hat{T}$, and that $\tilde{A} = \hat{A}$ on $D(\hat{A})$. But, by assumption, for some $\lambda_n > 0$, the operator $\lambda_n - \hat{A}$ is one-to-one and full-range. By Theorem 5.2.27, the same thing holds true for $\lambda_n - \tilde{A}$. Then we conclude $D(\tilde{A}) = D(\hat{A})$ and $\tilde{A} = \hat{A}$. ■

Remark 5.2.38. Let X be a Banach space with norm $|\cdot|_X$ and let τ be a sequentially complete locally convex topology on X such that the τ-bounded sets are exactly the $|\cdot|_X$-bounded sets. Then, by Proposition 5.2.5(i), we have $\mathscr{L}_0((X,\tau)) \subset L((X,|\cdot|_X))$. Let $\hat{T}$ be a C_0-sequentially equicontinuous semigroup on (X,τ) with infinitesimal generator $\hat{A}$. By referring to the notation of the proof of Theorem 5.2.37, we make the following observations.

(1) Since $R(\lambda_n,\hat{A}) \in \mathscr{L}_0((X,\tau)) \subset L((X,|\cdot|_X))$, then the Yosida approximations $\{T^{(n)}\}_{n\in\mathbb{N}}$, approximating $\hat{T}$ according to (5.2.37), are uniformly continuous semigroups on the Banach space $(X,|\cdot|_X)$.

(2) The fact that $\left\{(\lambda_n R(\lambda_n,\hat{A}))^m\right\}_{n,m\in\mathbb{N}}$ is sequentially equicontinuous implies that such a family is uniformly bounded in $L((X,|\cdot|_X))$. Indeed, as the unit ball B in $(X,|\cdot|_X)$ is bounded in (X,τ), by Proposition 5.2.10(iii) the set $\left\{(\lambda_n R(\lambda_n,\hat{A}))^m x\right\}_{n,m\in\mathbb{N},x\in B}$ is bounded in (X,τ). Hence, it is also bounded in $(X,|\cdot|_X)$, as we are assuming that the bounded sets are the same in both the topologies. As a consequence, by recalling the Hille-Yosida theorem for C_0-semigroups in Banach spaces, we have that $\hat{T}$ is also a C_0-semigroup in the Banach space $(X,|\cdot|_X)$ if and only if $D(\hat{A})$ is norm dense in X.

5.2.7 Relationship with *bi-continuous* semigroups

In this subsection we establish a comparison of our notion of C_0-sequentially equicontinuous semigroup with the notion of *bi-continuous* semigroup developed in [65, 66]. The latter requires to deal with Banach spaces as underlying spaces.

Definition 5.2.39. *Let $(X,|\cdot|_X)$ be a Banach space and let X^* be its topological dual. A linear subspace $\Gamma \subset X^*$ is called* norming *for $(X,|\cdot|_X)$ if $|x|_X = \sup_{\gamma\in\Gamma,\ |\gamma|_{X^*}\leq 1} |\gamma(x)|$, for every $x \in X$.*

Lemma 5.2.40. *Let $(X,|\cdot|_X)$ be a Banach space and let $\Gamma \subset X^*$ be norming for $(X,|\cdot|_X)$ and closed with respect to the operator norm $|\cdot|_{X^*}$. Then $B \subset X$ is $\sigma(X,\Gamma)$-bounded if and only if it is $|\cdot|_X$-bounded.*

Proof. As $\sigma(X,\Gamma)$ is weaker than the $|\cdot|_X$-topology, clearly $|\cdot|_X$-bounded sets are also $\sigma(X,\Gamma)$-bounded. Conversely, let $B \subset X$ be $\sigma(X,\Gamma)$-bounded and consider the family of continuous functionals

$$\{\Lambda_b : \Gamma \to \mathbb{R},\ \gamma \mapsto \gamma(b)\}_{b\in B},$$

By assumption, $\sup_{b\in B} |\gamma(b)| < +\infty$ for every $\gamma \in \Gamma$. The Banach-Steinhaus theorem applied in the Banach space $(\Gamma,|\cdot|_{X^*})$ yields

$$M := \sup_{b\in B} \sup_{\gamma\in\Gamma,\ |\gamma|_{X^*}\leq 1} |\gamma(b)| < +\infty.$$

Then, since Γ is norming for $(X, |\cdot|_X)$, we have

$$|b|_X = \sup_{\gamma\in\Gamma,\ |\gamma|_{X^*}\leq 1} |\gamma(b)| \leq M < +\infty \qquad \forall b \in B,$$

and then B is $|\cdot|_X$-bounded. ■

We recall the definition of bi-continuous semigroup as given in [66, Def. 3] and [65, Def. 1.3].

Definition 5.2.41. *Let $(X, |\cdot|_X)$ be a Banach space with topological dual X^*. Let τ be a Hausdorff locally convex topology on X with the following properties.*

(i) The space (X, τ) is sequentially complete on $|\cdot|_X$-bounded sets.

(ii) τ is weaker than the topology induced by the norm $|\cdot|_X$.

(iii) The topological dual of (X, τ) is norming for $(X, |\cdot|_X)$.

A family of linear operators $T = \{T_t : X \to X\}_{t\in\mathbb{R}^+} \subset L((X, |\cdot|_X))$ is called a bi-continuous semigroup with respect to τ and of type $\alpha \in \mathbb{R}$ *if the following conditions hold:*

(iv) $T_0 = I$ and $T_t T_s = T_{t+s}$ for every $t, s \in \mathbb{R}^+$;

(v) for some $M \geq 0$, $|T_t|_{L((X,|\cdot|))} \leq Me^{\alpha t}$, for every $t \in \mathbb{R}^+$;

(vi) T is strongly τ-continuous, i.e. the map $\mathbb{R}^+ \to (X, \tau)$, $t \mapsto T_t x$ is continuous for every $x \in X$;

(vii) T is locally bi-continuous, i.e., for every $|\cdot|_X$-bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ τ-convergent to $x \in X$ and every $\hat{t} > 0$, we have

$$\lim_{n\to+\infty} T_t x_n = T_t x \qquad \text{in } (X, \tau), \text{ uniformly in } t \in [0, \hat{t}].$$

The following proposition shows that the notion of bi-continuous semigroup is a specification of our notion of C_0-sequentially locally equicontinuous semigroup in sequentially complete spaces. Indeed, given a bi-continuous semigroup on a Banach space $(X, |\cdot|_X)$ with respect to a topology τ, one can define a locally convex sequentially complete topology $\tau' \supset \tau$ and see the bi-continuous semigroup as a C_0-sequentially locally equicontinuous semigroup on (X, τ').

Proposition 5.2.42. *Let $\{T_t\}_{t\in\mathbb{R}^+}$ be a bi-continuous semigroup on X with respect to τ and of type α. Then there exists a locally convex topology τ' with the following properties:*

(i) $\tau \subset \tau'$ and τ' is weaker than the $|\cdot|_X$-topology;

(ii) *a sequence converges in τ' if and only if it is $|\cdot|_X$-bounded and convergent in τ;*

(iii) *(X,τ') is sequentially complete;*

(iv) *T is a C_0-sequentially locally equicontinuous semigroup in (X,τ'); moreover, for every $\lambda > \alpha$, $\{e^{-\lambda t}T_t\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially equicontinuous semigroup on (X,τ') satisfying Assumption 5.2.16.*

Proof. Denote by X^* the topological dual of $(X,|\cdot|_X)$, and let $\mathscr{P}_X$ be a set of seminorms on X inducing τ. Denote by Γ the dual of (X,τ). On X, define the seminorms

$$q_{p,\gamma}(x) = p(x) + |\gamma(x)|, \qquad p \in \mathscr{P}_X,\ \gamma \in \overline{\Gamma},$$

where $\overline{\Gamma}$ is the closure of Γ with respect to the norm $|\cdot|_{X^*}$. Let τ' be the locally convex topology induced by the family of seminorms $\{q_{p,\gamma}\}_{p\in\mathscr{P}_X,\gamma\in\overline{\Gamma}}$.

(i) Clearly $\tau\subset\tau'$ and τ' is weaker than the $|\cdot|_X$-topology.

(ii) As $\tau\subset\tau'$, the τ'-convergent sequences are τ-convergent. Moreover, as Γ is norming, $\overline{\Gamma}$ is norming too. Then, by Lemma 5.2.40, every $\sigma(X,\overline{\Gamma})$-bounded set is $|\cdot|_X$-bounded. In particular, every convergent sequence in τ' is $|\cdot|_X$-bounded.

Conversely, consider a sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ which is τ-convergent to 0 in X and $|\cdot|_X$-bounded by a constant $M>0$. To show that $x_n \xrightarrow{\tau'} 0$, we only need to show that $\gamma(x_n)\to 0$ for every $\gamma\in\overline{\Gamma}$. For that, notice first that the convergence to 0 with respect to τ implies the convergence $\gamma(x_n)\to 0$ for every $\gamma\in\Gamma$. Take now $\gamma\in\overline{\Gamma}$ and a sequence $\{\gamma_k\}_{k\in\mathbb{N}}\subset\Gamma$ converging to γ with respect to $|\cdot|_{X^*}$. Then the estimate

$$|\gamma(x_n - x)| \le M|\gamma-\gamma_k|_{X^*} + |\gamma_k(x_n)| \qquad \forall n,k\in\mathbb{N},$$

yields

$$\limsup_{n\to+\infty}|\gamma(x_n)| \le M|\gamma-\gamma_k|_{X^*} \qquad \forall k\in\mathbb{N}.$$

Since $\gamma_k\to\gamma$ with respect to $|\cdot|_{X^*}$ when $k\to+\infty$, we now conclude that sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to 0 also with respect to τ'.

(iii) A Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,τ') is τ'-bounded. By Lemma 5.2.40, it is $|\cdot|_X$-bounded. Clearly, $\{x_n\}_{n\in\mathbb{N}}$ is also τ-Cauchy. Then, by Definition 5.2.41*(i)*, $\{x_n\}_{n\in\mathbb{N}}$ converges to some x in (X,τ). Since the sequence is $|\cdot|_X$-bounded, by *(ii)* the convergence takes place also in τ'. This proves that (X,τ') is sequentially complete.

(iv) We begin by proving that $\{T_t\}_{t\in\mathbb{R}^+}$ is a sequentially locally equicontinuous family of operators in the space (X,τ'). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence τ'-convergent to 0. By *(ii)*, $\{x_n\}_{n\in\mathbb{N}}$ is $|\cdot|_X$-bounded and τ-convergent to 0. By Definition 5.2.41*(vii)*

$$\lim_{n\to+\infty}\sup_{t\in[0,\hat{t}]} p(T_t x_n) = 0, \qquad \forall p\in\mathscr{P}_X,\ \forall \hat{t}>0. \tag{5.2.38}$$

Assume now, by contradiction, that there exist $R > 0$, $p \in \mathscr{P}_X$, $\gamma \in \overline{\Gamma}$, and $\varepsilon > 0$, such that

$$\limsup_{n\to+\infty} \sup_{t\in[0,R]} q_{p,\gamma}(T_t x_n) \geq \varepsilon.$$

Then, due to (5.2.38), there exist a sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0,R]$ convergent to some $t \in [0,R]$ and a subsequence of $\{x_n\}_{n\in\mathbb{N}}$, still denoted by $\{x_n\}_{n\in\mathbb{N}}$, such that

$$|\gamma(T_{t_n} x_n)| \geq \varepsilon \qquad \forall n \in \mathbb{N}. \tag{5.2.39}$$

By Definition 5.2.41*(v)*, the family $\{T_t\}_{t\in[0,R]}$ is uniformly bounded in the operator norm. Then, by recalling that $\{x_n\}_{n\in\mathbb{N}}$ is $|\cdot|_X$-bounded, we have

$$\hat{M} := \sup_{n\in\mathbb{N}} |T_{t_n} x_n|_X < +\infty.$$

Let $\hat{\gamma} \in \Gamma$ be such that $|\hat{\gamma} - \gamma|_{X^*} \leq \varepsilon/(2\hat{M})$. Then

$$\limsup_{n\to+\infty} |\gamma(T_{t_n} x_n)| \leq \frac{\varepsilon}{2} + \limsup_{n\to+\infty} |\hat{\gamma}(T_{t_n} x_n)| = \frac{\varepsilon}{2}, \tag{5.2.40}$$

where the last equality is due (5.2.38) and to the fact that $\hat{\gamma} \in \Gamma = (X,\tau)^*$. But (5.2.40) contradicts (5.2.39). The fact that T is strongly continuous with respect to τ' follows from *(ii)* and from Definition 5.2.41*(v)-(vi)*.

Finally, by Definition 5.2.41*(v)* we can apply Proposition 5.2.15*(i)* and conclude that $\{e^{-\lambda t} T_t\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially equicontinuous semigroup on (X,τ') for every $\lambda > \alpha$. Due to part *(iii)*, such a semigroup satisfies Assumption 5.2.16 (recall Remark 5.2.17). ■

5.2.8 A note on a weaker definition

In this subsection we point out how, under weaker requirements in Definition 5.2.12, some of the results appearing in the previous sections still hold. The definition that we are going to introduce below will not be used in the sequel, except in Subsection 5.3.3, where we briefly clarify the relationship between the notion of π-semigroup, introduced in [84], and our notion of C_0-sequentially locally equicontinuos semigroup.

Definition 5.2.43. *Let X be a Hausdorff locally convex space. Let $T := \{T_t\}_{t\in\mathbb{R}^+} \subset \mathscr{L}_0(X)$ be a family of sequentially continuous linear operators. We say that T is a* bounded C_0-sequentially continuous semigroup *if*

(i) $T_0 = I$ *and* $T_{t+s} = T_t T_s$ *for all* $t,s \in \mathbb{R}^+$*;*

(ii) for each $x \in X$, the map $\mathbb{R}^+ \to X$, $t \mapsto T_t x$, is continuous and bounded.

By recalling Proposition 5.2.14, we see that Definition 5.2.12 is stronger than Definition 5.2.43.

Let T be a bounded C_0-sequentially continuous semigroup on X and let us assume that, for every $x \in X$, the Riemann integral

$$R(\lambda)x := \int_0^{+\infty} e^{-\lambda t} T_t x dt, \tag{5.2.41}$$

(which exists in the completion of X, by Definition 5.2.43*(ii)*) belongs to X (this happens, for example, if X is sequentially complete).

Then, a straightforward inspection of the proofs shows that the following results still hold: Proposition 5.2.15*(ii)*; Proposition 5.2.18; Proposition 5.2.19*(ii)*; Proposition 5.2.21; Proposition 5.2.23; Theorem 5.2.27, except for the conclusion $(\lambda - A)^{-1} \in \mathcal{L}_0(X)$; Corollary 5.2.29.

To summarize, if the Laplace transform (5.2.41) of a bounded C_0-sequentially continuous semigroup is well-defined, then the domain $D(A)$ of the generator A is sequentially dense in X and $\lambda - A$ is one-one and onto for every $\lambda > 0$.

We outline that, without the sequential local equicontinuity of T, the proof of Lemma 5.2.24 does not work, and consequently the proof of Theorem 5.2.25 does not work.

5.2.9 Examples and counterexamples

In this subsection we provide some examples to clarify some features of the notion of C_0-sequentially (locally) equicontinuous semigroup.

First, with respect to the case of C_0-semigroups on Banach spaces, we notice two relevant basic implications that we loose when dealing with strong continuity and (sequential) local equicontinuity in locally convex spaces. The first one is related to the growth rate of the orbits of the semigroup, and consequently to the possibility to define the Laplace transform. The fact that T is a C_0-locally (sequentially) equicontinuous semigroup does not imply, in general, the existence of $\alpha > 0$ such that $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$ is a C_0-(sequentially) locally equicontinuous semigroup. We give two examples.

Example 5.2.44. Consider the vector space $X := C(\mathbb{R})$, endowed with the topology of the uniform convergence on compact sets, which makes X a Fréchet space. Define $T_t : X \to X$ by

$$T_t \varphi(s) := e^{st} \varphi(s) \qquad \forall s \in \mathbb{R},\ \forall t \in \mathbb{R}^+,\ \forall \varphi \in X.$$

One verifies that $T = \{T_t\}_{t \in \mathbb{R}^+}$ is a C_0-sequentially locally equicontinuous semigroup on X (actually, locally equicontinuous, by Proposition 5.2.11). On the other hand, for whatever $\alpha > 0$, the family $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$ is not sequentially equicontinuous. Indeed, one has that $\{e^{-\alpha t} T_t f\}_{t \in \mathbb{R}^+}$ is unbounded in X for every f not identically zero on $(\alpha, +\infty)$.

Example 5.2.45. Another classical example is given in [64]. Let X be as in Example 5.2.44, with the same topology. For $t \in \mathbb{R}^+$, we define $T := \{T_t\}_{t\in\mathbb{R}^+}$ by

$$T_t \colon X \to X,\ \varphi \mapsto \varphi(t+\cdot).$$

Then T is a C_0-sequentially locally equicontinuous semigroup on X (equivalently, T is a C_0-locally equicontinuous semigroup, by Proposition 5.2.11), but there does not exist any $\alpha > 0$ such that $\{e^{-\alpha t}T_t\}_{t\in\mathbb{R}^+}$ is equicontinuous.

The second relevant difference with respect to C_0-semigroups in Banach spaces is that the strong continuity does not imply, in general, the sequential local equicontinuity. The following example shows that Definition 5.2.12*(iii′)* in general cannot be derived by Definition 5.2.12*(i)-(ii)*, even if Definition 5.2.12*(ii)* is strengthened by requiring the continuity of $\mathbb{R}^+ \to X,\ t \mapsto T_t x,\ x \in X$.

Example 5.2.46. Let $X := C(\mathbb{R})$ be endowed with the topology of the pointwise convergence. Define the semigroup $T := \{T_t\}_{t\in\mathbb{R}^+}$ by

$$T_t \colon X \to X,\ \varphi \mapsto \varphi(t+\cdot).$$

Then $T_t \in \mathscr{L}_0(X)$ for all $t \in \mathbb{R}^+$. It is clear that, for every $\varphi \in C(\mathbb{R})$, the map $\mathbb{R}^+ \to X,\ t \mapsto T_t\varphi$, is continuous. Nevertheless, for each $\hat{t} > 0$ we can find a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C(\mathbb{R})$ of functions converging pointwise to 0 and such that

$$\liminf_{n\to+\infty} \sup_{t\in[0,\hat{t}]} |(T_t\varphi_n)(0)| = \liminf_{n\to+\infty} \sup_{t\in[0,\hat{t}]} |\varphi_n(t)| > 0.$$

Hence, T is not a C_0-sequentially locally equicontinuous semigroup. We observe that the same conclusion holds true if we restrict the action of T to the space $C_b(\mathbb{R})$.

Referring to Remark 5.2.13*(2)*, we provide the following example([5]).

Example 5.2.47. Consider the Banach space ℓ^1, with its usual norm $|\mathbf{x}|_1 = \sum_{k=0}^{+\infty} |x_k|$, where $\mathbf{x} := \{x_k\}_{k\in\mathbb{N}} \in \ell^1$, and denote by τ_1 and τ_w the $|\cdot|_1$-topology and the weak topology respectively. Define $Z := \ell^1 \times \ell^1$ and endow it with the product topology $\tau_w \otimes \tau_1$. Let

$$B \colon Z \to Z,\ (x_1,x_2) \mapsto (x_1,x_1).$$

We recall that ℓ^1 enjoys Schur's property (weak convergent sequences are strong convergent; see [30, p. 85]). As a consequence, we have that Z is sequentially complete and $B \in \mathscr{L}_0(Z)$. On the other hand, as τ_w is strictly weaker than τ_1, we have $B \notin L(Z)$. By induction, we see that $(I-B)^n = (I-B)$ for each $n \geq 1$, and then $\{(I-B)^n\}_{n\in\mathbb{N}}$ is a family of sequentially equicontinuous operators. By Proposition 5.2.36, if we define $T_t := e^{t(B-I)}$ for $t \in \mathbb{R}^+$, then $T := \{T_t\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially equicontinuous semigroup on Z. Actually, we have $e^{t(B-I)} = e^{-t}(I-B)+B$. However, if $t > 0$, the operators $e^{t(B-I)} = e^{-t}I + (1-e^{-t})B$ are not continuous on Z.

[5]Example 5.2.47 could seem a bit artificious and *ad hoc*. In the next section we will provide another more meaningful example by a very simple Markov transition semigroup (Example 5.4.5).

5.3 Developments in functional spaces

The aim of this section is to develop the theory of the previous section in some specific functional spaces. Throughout the rest of the chapter, E will denote a metric space, $\mathcal{E}$ will denote the associated Borel σ-algebra, and $\mathcal{S}(E)$ will denote one of the spaces $UC_b(E)$, $C_b(E)$, $B_b(E)$. We recall that $(\mathcal{S}(E),|\cdot|_\infty)$, where $|\cdot|_\infty$ is the usual sup-norm, is a Banach space. For simplicity of notation, we denote by $\mathcal{S}(E)^*_\infty$ the dual of $(\mathcal{S}(E),|\cdot|_\infty)$ and by $|\cdot|_{\mathcal{S}(E)^*_\infty}$ the operator norm in $\mathcal{S}(E)^*_\infty$.

We are going to define on $\mathcal{S}(E)$ two particular locally convex topologies. The motivation for introducing such topologies is that they allow to frame under a general unified viewpoint some of the approaches used in the literature of Markov transition semigroups. In particular, we are able to cover the following types of semigroups.

1. Weakly continuous semigroups, introduced in [7] for the space $UC_b(E)$ with E separable Hilbert space (an overview can also be found in [8, Appendix B], with E separable Banach space).

2. π-semigroups, introduced in [84] for the space $UC_b(E)$, with E separable metric space.

3. C_0-locally equicontinuous semigroups with respect to the so called mixed topology in the space $C_b(E)$, considered by [49], with E separable Hilbert space.

5.3.1 A family of locally convex topologies on $\mathcal{S}(E)$

Let $\mathbf{P}$ be a set of non-empty parts of E such that $E=\bigcup_{P\in\mathbf{P}}P$. For every $P\in\mathbf{P}$ and every $\mu\in\mathbf{ca}(E)$, let us introduce the seminorm

$$p_{P,\mu}(f):=[f]_P+\left|\int_E f\,d\mu\right|,\quad \forall f\in\mathcal{S}(E), \tag{5.3.1}$$

where

$$[f]_P:=\sup_{x\in P}|f(x)|.$$

Denote by $\tau_{\mathbf{P}}$ the locally convex topology on $\mathcal{S}(E)$ induced by the family of seminorms

$$\{p_{P,\mu}\colon P\in\mathbf{P},\ \mu\in\mathbf{ca}(E)\}.$$

Since $E=\bigcup_{P\in\mathbf{P}}P$, $\tau_{\mathbf{P}}$ is Hausdorff.

In the following, by $\mathbf{ba}(E)$ we denote the space of finitely additive signed measures on $(E,\mathcal{E})$ with bounded total variation. The space $\mathbf{ba}(E)$ is Banach when endowed with the norm $|\cdot|_1$ given by the total variation and is canonically identified with $(B_b(E)^*_\infty,|\cdot|_{B_b(E)^*_\infty})$ (see [1, Theorem 14.4]) through the isometry

$$\Phi\colon(\mathbf{ba}(E),|\cdot|_1)\to(B_b(E)^*_\infty,|\cdot|_{B_b(E)^*_\infty}),\ \mu\mapsto\Phi_\mu, \tag{5.3.2}$$

where

$$\Phi_\mu(f) := \int_E f\, d\mu \qquad \forall f \in B_b(E), \tag{5.3.3}$$

with $\int_E \# d\mu$ interpreted in the Darboux sense (see [1, Sec. 11.2]).

We denote by $\mathbf{ca}(E)$ the space of elements of $\mathbf{ba}(E)$ that are countably additive. The space $(\mathbf{ca}(E), |\cdot|_1)$ is Banach as well. If $\mu \in \mathbf{ca}(E)$, then the Darboux integral in (5.3.3) coincides with the Lebesgue integral.

For future reference, we recall the following result (see [79, Th. 5.9, p. 39]).

Lemma 5.3.1. *Let $\nu \in \mathbf{ca}(E)$ be such that $\int_E f d\nu = 0$ for all $f \in UC_b(E)$. Then $\nu = 0$.*

Proposition 5.3.2. *The space $(\mathbf{ca}(E), |\cdot|_1)$ is isometrically embedded into $(\mathscr{S}(E)^*_\infty, |\cdot|_{\mathscr{S}(E)^*_\infty})$ by*

$$\Phi\colon \mathbf{ca}(E) \to \mathscr{S}(E)^*_\infty,\ \mu \mapsto \Phi_\mu, \tag{5.3.4}$$

where

$$\Phi_\mu(f) := \int_E f\, d\mu, \qquad \forall f \in \mathscr{S}(E). \tag{5.3.5}$$

Proof. It is clear that Φ is linear.

Let $\mu \in \mathbf{ca}(E)$. As $|\Phi_\mu(f)| \le |f|_\infty |\mu|_1$ for every $f \in \mathscr{S}(E)$, then $\Phi_\mu \in \mathscr{S}(E)^*$ and $|\Phi_\mu|_{\mathscr{S}(E)^*} \le |\mu|_1$. To show that Φ is an isometry it remains to show that $|\Phi_\mu|_{\mathscr{S}(E)^*} \ge |\mu|_1$. Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ, and let $C^+ := \mathrm{supp}(\mu^+)$, $C^- := \mathrm{supp}(\mu^-)$. Let $\varepsilon > 0$. Then we can find a closed set $C^+_\varepsilon \subset C^+$ such that $\mu^+(C^+ \setminus C^+_\varepsilon) < \varepsilon$, and $d(C^+_\varepsilon, C^-) > 0$. Let f be defined by

$$f(x) := \frac{d(x, C^-) - d(x, C^+_\varepsilon)}{d(x, C^-) + d(x, C^+_\varepsilon)} \qquad \forall x \in E.$$

Then $f \in UC_b(E)$, $f \equiv 1$ on C^+_ε, $f \equiv -1$ on C^-, and $|f|_\infty = 1$. Therefore,

$$\int_E f\, d\mu = \int_{C^+_\varepsilon} f\, d\mu^+ + \int_{C^+ \setminus C^+_\varepsilon} f\, d\mu^+ - \int_{C^-} f\, d\mu^- \ge \mu^+(C^+_\varepsilon) - \varepsilon + \mu^-(C^-) \ge |\mu|_1 - 2\varepsilon.$$

Then $|\Phi_\mu|_{\mathscr{S}(E)^*_\infty} \ge |\mu|_1 - 2\varepsilon$. We conclude by arbitrariness of ε. ∎

Let us denote by τ_∞ the topology induced by the norm $|\cdot|_\infty$ on $\mathscr{S}(E)$. Since the functional Φ_μ defined in (5.3.5) is $\tau_{\mathbf{P}}$-continuous for every $\mu \in \mathbf{ca}(E)$, and since $p_{P,\mu}$ is τ_∞-continuous for every $P \in \mathbf{P}$ and every $\mu \in \mathbf{ca}(E)$, we have the inclusions

$$\sigma(\mathscr{S}(E), \mathbf{ca}(E)) \subset \tau_{\mathbf{P}} \subset \tau_\infty. \tag{5.3.6}$$

Observe that, when $\mathbf{P}$ contains only finite parts of E, then $\tau_{\mathbf{P}} = \sigma(\mathscr{S}(E), \mathbf{ca}(E))$, because $\mathbf{ca}(E)$ contains all Dirac measures. The opposite case is when $E \in \mathbf{P}$, and then $\tau_{\mathbf{P}} = \tau_\infty$.

Proposition 5.3.3. *Let $B \subset \mathscr{S}(E)$. The following are equivalent.*

(i) B is $\sigma(\mathscr{S}(E), \mathbf{ca}(E))$-bounded.

(ii) B *is* $\tau_{\mathbf{P}}$*-bounded.*

(iii) B *is* τ_∞*-bounded.*

Proof. By (5.3.6), it is sufficient to prove that *(i)⇒(iii)*. Let B be $\sigma(\mathscr{S}(E),\mathbf{ca}(E))$-bounded. By Proposition 5.3.2, $\mathbf{ca}(E)$ is closed in $\mathscr{S}(E)^*_\infty$. Moreover, since $\mathbf{ca}(E)$ contains the Dirac measures, it is norming. Then we conclude by applying Lemma 5.2.40. ■

Corollary 5.3.4. $\mathscr{L}_0((\mathscr{S}(E),\tau_{\mathbf{P}}))\subset L((\mathscr{S}(E),|\cdot|_\infty))$.

Proof. By Proposition 5.3.3, the bounded sets of $\tau_{\mathbf{P}}$ are exactly the bounded sets of τ_∞. Then, we conclude by applying Proposition 5.2.5*(i)*. ■

Corollary 5.3.5. *Let T be a C_0-sequentially locally equicontinuous semigroup on $(\mathscr{S}(E),\tau_{\mathbf{P}})$. Then there exists $M\geq 1$ and $\alpha>0$ such that $|T_t|_{L((\mathscr{S}(E),|\cdot|_\infty))}\leq Me^{\alpha t}$ for all $t\in\mathbb{R}^+$.*

Proof. Due to Proposition 5.3.3, we can conclude by applying Proposition 5.2.15*(ii)*. ■

We now focus on the following two cases:

(a) $\mathbf{P}$ is the set of all finite subsets of E, and then $\tau_{\mathbf{P}}=\sigma(\mathscr{S}(E),\mathbf{ca}(E))$;

(b) $\mathbf{P}$ is the set of all non-empty compact subsets of E; in this case, we denote $\tau_{\mathbf{P}}$ by $\tau_{\mathscr{K}}$, i.e.

$$\tau_{\mathscr{K}}:=\text{l.c. topology on } \mathscr{S}(E) \text{ generated by } \{p_{K,\mu}: K\subset E \text{ compact}, \ \mu\in\mathbf{ca}(E)\}. \quad (5.3.7)$$

Proposition 5.3.6. *We have the following characterisations.*

(i) $\tau_{\mathscr{K}}=\tau_\infty$ *if and only if E is compact.*

(ii) $\sigma(\mathscr{S}(E),\mathbf{ca}(E))=\tau_\infty$ *if and only if E is finite.*

Proof. First, note that the inclusions $\sigma(\mathscr{S}(E),\mathbf{ca}(E))\subset\tau_\infty$ and $\tau_{\mathscr{K}}\subset\tau_\infty$ have been already observed in (5.3.6).

(i) If E is compact, we have $|\cdot|_\infty=p_{E,0}$, hence $\tau_{\mathscr{K}}=\tau_\infty$. Conversely, assume that $\tau_{\mathscr{K}}=\tau_\infty$ on $\mathscr{S}(E)$. Then there exist a non-empty compact set $K\subset E$, measures $\mu_1,\ldots,\mu_n\in\mathbf{ca}(E)$, and $L>0$, such that

$$|f|_\infty\leq L\left([f]_K+\sum_{i=1}^n\left|\int_E f\,d\mu_i\right|\right),\qquad \forall f\in\mathscr{S}(E). \quad (5.3.8)$$

For $\varepsilon>0$, define $A_\varepsilon:=\{x\in E: B(x,\varepsilon)\subset K^c\}$, and define, with the convention $d(\cdot,\emptyset)=+\infty$, the function $r_\varepsilon(x):=\frac{d(x,K)}{d(x,A_\varepsilon)+d(x,K)}$. Then $0\leq r_\varepsilon\leq 1$, $r_\varepsilon=0$ on K, $r_\varepsilon=1$ on A_ε, $r_\varepsilon\uparrow\mathbf{1}_{K^c}$ pointwise as $\varepsilon\downarrow 0$, and r_ε is uniformly continuous (the latter is due to the fact that

$d(A_\varepsilon,K)\geq\varepsilon$). Hence, for every $f\in\mathscr{S}(E)$, the function fr_ε belongs to $\mathscr{S}(E)$ and $|fr_\varepsilon|\uparrow |f\mathbf{1}_{K^c}|$ pointwise as $\varepsilon\downarrow 0$, which entails $|fr_\varepsilon|_\infty\uparrow|f\mathbf{1}_{K^c}|_\infty$ as $\varepsilon\downarrow 0$. We can then apply (5.3.8) to every fr_ε and pass to the limit for $\varepsilon\downarrow 0$ to obtain

$$|f\mathbf{1}_{K^c}|_\infty\leq L\sum_{i=1}^n\left|\int_E f\,d(\mu_i\lfloor K^c)\right|,\qquad \forall f\in\mathscr{S}(E),$$

where $\mu_i\lfloor K^c$ denotes the restriction of μ_i to K^c. Let $\nu\in\mathbf{ca}(E)$ be such that $|\nu|(K)=0$. Then

$$\left|\int_E f\,d\nu\right|=\left|\int_E f\mathbf{1}_{K^c}d\nu\right|\leq\ |\nu|_1|f\mathbf{1}_{K^c}|_\infty\leq|\nu|_1L\sum_{i=1}^n\left|\int_E f\,d(\mu_i\lfloor K^c)\right|,\qquad\forall f\in\mathscr{S}(E).$$

Then, by [91, Lemma 3.9, p. 63] and by Proposition 5.3.2, there exist $\alpha_1,\dots,\alpha_n\in\mathbb{R}$ such that $\nu=\sum_{i=1}^n\alpha_i(\mu_i\lfloor K^c)$. By arbitrariness of ν this implies that $E\setminus K$ is finite, and then E is compact.

(ii) If E is finite, clearly $\sigma(\mathscr{S}(E),\mathbf{ca}(E))=\tau_\infty$. Conversely, assume that $\sigma(\mathscr{S}(E),\mathbf{ca}(E))=\tau_\infty$. Then there exist $K\subset E$ compact, $\mu_1,\dots,\mu_n\in\mathbf{ca}(E)$, and $L>0$ such that

$$|f|_\infty\leq L\sum_{i=1}^n\left|\int_E f\,d\mu_i\right|,\qquad\forall f\in\mathscr{S}(E).$$

By arguing as for concluding the proof of *(i)*, we obtain

$$\mathbf{ca}(E)=\operatorname{span}\{\mu_1,\dots,\mu_n\},$$

and then E must be finite. ■

We recall the following definition.

Definition 5.3.7. *A locally convex topological vector space is said to be* infrabarreled *if every closed, convex, balanced set, absorbing every bounded set, is a neighborhood of* 0.

Corollary 5.3.8. *We have the following characterisations.*

(i) $(\mathscr{S}(E),\sigma(\mathscr{S}(E),\mathbf{ca}(E)))$ *is infrabarrelled if and only if E is finite.*

(ii) $(\mathscr{S}(E),\tau_{\mathscr{K}})$ *is infrabarrelled if and only if E is compact.*

Proof. If E is finite (resp. E is compact), then, by Proposition 5.3.6, $\sigma(\mathscr{S}(E),\mathbf{ca}(E))$ (resp. $\tau_{\mathscr{K}}$) coincides with the topology τ_∞ of the Banach space $(\mathscr{S}(E),|\cdot|_\infty)$, and then it is infrabarreled, because every Banach space is so (see [77, Theorem 4.5, p. 97]).

Conversely, let E be not finite (resp. not compact) and consider the $|\cdot|_\infty$-closed ball

$$B_\infty(0,1]:=\{f\in\mathscr{S}(E):\ |f|_\infty\leq 1\}.$$

The set $B_\infty(0,1]$ is convex, balanced, absorbent. Moreover,

$$B_\infty(0,1] = \bigcap_{x\in E}\left\{f\in\mathscr{S}(E)\colon \left|\int_E f\,d\delta_x\right|\le 1\right\},$$

where $\delta_x\in\mathbf{ca}(E)$ is the Dirac measure centered in x. Hence $B_\infty(0,1]$ is $\sigma(\mathscr{S}(E),\mathbf{ca}(E))$-closed (and then $\tau_{\mathscr{K}}$-closed). So $B_\infty(0,1]$ is a barrel for the topology $\sigma(\mathscr{S}(E),\mathbf{ca}(E))$ (resp. $\tau_{\mathscr{K}}$). Moreover, by Proposition 5.3.3, it absorbs every $\sigma(\mathscr{S}(E),\mathbf{ca}(E))$- (resp. $\tau_{\mathscr{K}}$-) bounded set. Assuming now, by contradiction, that $(\mathscr{S}(E),\sigma(\mathscr{S}(E),\mathbf{ca}(E)))$ (resp. $(\mathscr{S}(E),\tau_{\mathscr{K}})$) is infrabarreled, we would have that $B_\infty(0,1]$ is a $\sigma(\mathscr{S}(E),\mathbf{ca}(E))$-neighborhood (resp. $\tau_{\mathscr{K}}$-neighborhood) of the origin. This would contradict Proposition 5.3.6. ∎

Remark 5.3.9. Corollary 5.3.8 has an important consequence. If E is not finite (resp. not compact), then $\sigma(\mathscr{S}(E),\mathbf{ca}(E))$ (resp. $(\mathscr{S}(E),\tau_{\mathscr{K}})$) is not infrabarreled, so the Banach-Steinhaus theorem cannot be invoked to deduce that strongly continuous semigroups in $(\mathscr{S}(E),\sigma(\mathscr{S}(E),\mathbf{ca}(E)))$ (resp. $(\mathscr{S}(E),\tau_{\mathscr{K}})$) are necessarily locally equicontinuous — as it is usually done for C_0-semigroups in Banach spaces (cf. also Example 5.2.46).

We now investigate the relationship between $\tau_{\mathscr{K}}$ and $\tau_{\mathscr{C}}$, where $\tau_{\mathscr{C}}$ denotes the topology on $\mathscr{S}(E)$ defined by the uniform convergence on compact sets of E, induced by the family of seminorms

$$\{p_K = [\cdot]_K\colon\ K \text{ non-empty compact subset of } E\}.$$

Clearly $\tau_{\mathscr{C}}\subset\tau_{\mathscr{K}}$. In order to understand when the equality $\tau_{\mathscr{C}}=\tau_{\mathscr{K}}$ is possible, we proceed with two preparatory lemmas.

Lemma 5.3.10. *$UC_b(E)\neq C_b(E)$ if and only if there exists a sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}}\subset E\times E$ having the following properties.*

- *(i) $\{d(x_n,y_n)\}_{n\in\mathbb{N}}$ is a strictly positive sequence, converging to 0;*
- *(ii) the sequence $\{d_n\}_{n\in\mathbb{N}}$ defined by $d_n := d(\{x_n,y_n\},\bigcup_{k>n}\{x_k,y_k\})$, for $n\in\mathbb{N}$, is strictly positive;*
- *(iii) the sequence $\{x_n\}_{n\in\mathbb{N}}$ does not have any convergent subsequence.*

Proof. We first prove that, if $UC_b(E)\neq C_b(E)$, then there exists a sequence satisfying *(i)*,*(ii)*,*(iii)*. Let $f\in C_b(E)\setminus UC_b(E)$. Then there exist $\varepsilon>0$ and a sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}}\subset E\times E$ such that $\lim_{n\to+\infty}d(x_n,y_n)=0$ and $\inf_{n\in\mathbb{N}}|f(x_n)-f(y_n)|\ge\varepsilon$. Then *(i)* is satisfied by $\{(x_n,y_n)\}_{n\in\mathbb{N}}$. Now we show that *(ii)* holds. Assume, by contradiction, that $d_{\hat{n}}=0$ for some $\hat{n}\in\mathbb{N}$. Then $d(z,\bigcup_{k>\hat{n}}\{x_k,y_k\})=0$ for $z=x_{\hat{n}}$ or $z=y_{\hat{n}}$. Therefore z is an accumulation point for $\bigcup_{k>\hat{n}}\{x_k,y_k\}$. Hence, as $d(x_n,y_n)\to 0$, there exists a subsequence $\{(x_{n_k},y_{n_k})\}_{k\in\mathbb{N}}$

such that $x_{n_k} \to z$ and $y_{n_k} \to z$ as $k \to +\infty$. Now, as f is continuous, we have the contradiction $f(z) - f(z) = \lim_{k\to+\infty} |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$. Finally, property *(iii)* can be proved by using the same argument as for proving *(ii)*.

Conversely, take a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset E \times E$ satisfying *(i)*,*(ii)*,*(iii)*. Consider the balls

$$B_n := \{x \colon d(x_n, x) < \varepsilon_n\}, \quad n \in \mathbb{N}, \tag{5.3.9}$$

where $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is recursively defined by

$$\begin{cases} \varepsilon_0 := \dfrac{d_0 \wedge d(x_0, y_0)}{2} \\ \varepsilon_n := \dfrac{d_n \wedge d(x_n, y_n) \wedge \varepsilon_{n-1}}{2} & n \geq 1. \end{cases}$$

By the properties *(i)*,*(ii)*, the balls $\{B_n\}_{n\in\mathbb{N}}$ are pairwise disjoint and $\lim_{n\to+\infty} \varepsilon_n = 0$. It is also clear that $y_n \notin B_n$, for $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we can construct a uniformly continuous function ρ_n such that $0 \leq \rho_n \leq 1$, $\rho_n(x_n) = 1$, and $\rho_n = 0$ on B_n^c. For $n \in \mathbb{N}$, the function $f_n := \sum_{i=0}^{n} \rho_i$ is uniformly continuous. Let $f := \sum_{i=0}^{+\infty} \rho_i$. By *(iii)* and since $\varepsilon_n \to 0$, one can show that every converging sequence in E can intersect only a finite number of the pairwise disjoint balls $\{B_n\}_{n\in\mathbb{N}}$. Hence, any compact set $K \subset E$ intersects only a finite number of balls $\{B_n\}_{n\in\mathbb{N}}$. Then f restricted to any compact set $K \subset E$ is actually a finite sum of the form $\sum_{i=1}^{n_K} \rho_i$, hence it coincides with f_{n_K}, for some $n_K \in \mathbb{N}$ depending on K. In particular, $f \in C_b(E)$. On the other hand, $f(x_n) - f(y_n) = 1$ and $d(x_n, y_n) \to 0$ as $n \to +\infty$, so $f \notin UC_b(E)$. ■

Lemma 5.3.11. *If E is not complete, then $UC_b(E) \neq C_b(E)$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a non-convergent Cauchy sequence in E and define $y_n := x_{2n}$, for $n \in \mathbb{N}$. We now show that, up to extract a subsequence, the sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ satisfies *(i)*,*(ii)*,*(iii)* of Lemma 5.3.10.

We prove property *(i)*. As $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and non-convergent, up to extract a subsequence, we can assume that $x_n \neq x_k$, if $n \neq k$, hence $d(x_n, y_n) > 0$. On the other hand, since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, we have $\lim_{n\to+\infty} d(x_n, y_n) = 0$.

We prove property *(ii)*. Let $\{d_n\}_{n\in\mathbb{N}}$ be defined as in Lemma 5.3.10*(ii)*. Assume, by contradiction, that $d_{\overline{n}} = 0$ for some $\overline{n} \in \mathbb{N}$. Then $z = x_{\overline{n}}$ or $z = y_{\overline{n}}$ should be an accumulation point for the sequence $\{x_n\}_{n\in\mathbb{N}}$ or for the sequence $\{y_n = x_{2n}\}_{n\in\mathbb{N}}$, which cannot be true by assumption on $\{x_n\}_{n\in\mathbb{N}}$.

Finally, property *(iii)* is clear from the fact that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy but is not convergent. ■

Proposition 5.3.12. *$\tau_{\mathscr{K}} = \tau_{\mathscr{C}}$ on $\mathscr{S}(E)$ if and only if E is compact.*

Proof. If E is compact, it is clear that $\tau_{\mathscr{K}} = \tau_{\mathscr{C}}$. Suppose now that E is not compact. We recall that E is not compact if and only if E is not complete or E is not totally bounded. In both cases, we will show that there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset UC_b(E)$ convergent to 0 in $\tau_{\mathscr{C}}$, but unbounded in $\tau_{\mathscr{K}}$.

Case E non-complete. By Lemma 5.3.11, there exists a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset E \times E$ satisfying *(i)*,*(ii)*,*(iii)* of Lemma 5.3.10. Let $\{B_n\}_{n\in\mathbb{N}}$ and $\{\rho_n\}_{n\in\mathbb{N}}$ be as in the second part of the proof of Lemma 5.3.10. Define $\varphi_n := 2^{2n}\rho_n$ for every $n \in \mathbb{N}$. As proved in that lemma, any compact set $K \subset E$ intersects only a finite numbers of balls $\{B_n\}_{n\in\mathbb{N}}$, therefore $\lim_{n\to+\infty}\varphi_n = 0$ in $(UC_b(E), \tau_{\mathscr{C}})$.

Now, let $\mu \in \mathbf{ca}(E)$ be defined by $\mu := \sum_{n\in\mathbb{N}} 2^{-n}\delta_{x_n}$. We have

$$\sup_{n\in\mathbb{N}} \left| \int_E \varphi_n d\mu \right| = \sup_{n\in\mathbb{N}} 2^{-n}\varphi_n(x_n) = \sup_{n\in\mathbb{N}} 2^n = +\infty,$$

which shows that $\{\varphi_n\}_{n\in\mathbb{N}}$ is $\tau_{\mathscr{K}}$-unbounded.

Case E not totally bounded. Let $\varepsilon > 0$ be such that E cannot be covered by a finite number of balls of radius ε. By induction, we can construct a sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ such that, for every $n \in \mathbb{N}$, $x_{n+1} \notin \bigcup_{j=0}^n B(x_j, \varepsilon)$. For every $n \in \mathbb{N}$, let $\varphi_n \in UC_b(E)$ be such that $\varphi_n(x_n) = 2^{2n}$, $\varphi_n(x) = 0$ if $d(x, x_n) \geq \varepsilon/2$, $|\varphi_n|_\infty = 2^{2n}$ ([6]). Then we conclude as in the previous case. ■

Propositions 5.3.6 and 5.3.12 yield the following inclusions of topologies in the space $\mathscr{S}(E)$

$$\tau_{\mathscr{C}} \subset \tau_{\mathscr{K}} \subset \tau_\infty$$

and state that such inclusions are equalities if and only if E is compact. The following proposition makes clearer the connection between $\tau_{\mathscr{K}}$ and $\tau_{\mathscr{C}}$ when E is not compact.

Proposition 5.3.13. *The following statements hold.*

(i) If a net $\{f_\iota\}_{\iota\in\mathscr{I}}$ is bounded and convergent to f in $(\mathscr{S}(E), \tau_{\mathscr{K}})$, then

$$\sup_{\iota\in\mathscr{I}} |f_\iota|_\infty < +\infty \quad \textit{and} \quad \lim_\iota f_\iota = f \textit{ in } (\mathscr{S}(E), \tau_{\mathscr{C}}).$$

If either $\mathscr{I} = \mathbb{N}$ or E is homeomorphic to a Borel subset of a Polish space, then also the converse holds true.

(ii) If a net $\{f_\iota\}_{\iota\in\mathscr{I}}$ is bounded and Cauchy in $(\mathscr{S}(E), \tau_{\mathscr{K}})$, then

$$\sup_{\iota\in\mathscr{I}} |f_\iota|_\infty < +\infty \quad \textit{and} \quad \{f_\iota\}_\iota \textit{ is Cauchy in } (\mathscr{S}(E), \tau_{\mathscr{C}}).$$

If either $\mathscr{I} = \mathbb{N}$ or E is homeomorphic to a Borel subset of a Polish space, then also the converse holds true.

[6] For instance, $\varphi_n(x) := 2^{2n}\frac{d(x, B(x_n, \varepsilon/2)^c)}{d(x, x_n) + d(x, B(x_n, \varepsilon/2)^c)}$.

Proof. (i) Let $\{f_\iota\}_{\iota\in\mathscr{I}}$ be a $\tau_{\mathscr{K}}$-bounded net converging to f in $(\mathscr{S}(E),\tau_{\mathscr{K}})$. By Proposition 5.3.3 we have $\sup_{\iota\in\mathscr{I}}|f_\iota|_\infty<+\infty$, and, since $\tau_{\mathscr{C}}\subset\tau_{\mathscr{K}}$, the net converges to f also with respect to $\tau_{\mathscr{C}}$.

Conversely, let $\{f_\iota\}_{\iota\in\mathscr{I}}\subset\mathscr{S}(E)$ be such that $\sup_\iota|f_\iota|_\infty=M<+\infty$ and $\lim_\iota f_\iota=f$ in $(\mathscr{S}(K),\tau_{\mathscr{C}})$. Then $\{f_\iota\}_{\iota\in\mathscr{I}}$ is $\tau_{\mathscr{K}}$-bounded, because $\tau_{\mathscr{K}}\subset\tau_\infty$. We want to prove that $\{f_\iota\}_{\iota\in\mathscr{I}}$ is $\tau_{\mathscr{K}}$-convergent to f if $\mathscr{I}=\mathbb{N}$ or if E homeomorphic to a Borel subset of a Polish space. Assume without loss of generality $f=0$. We already know that $[f_\iota]_K$ converges to 0 for every compact set $K\subset E$, then it remains to show that $\int_E f_\iota d\mu$ converges to 0 for every $\mu\in\mathbf{ca}(E)$. If $\mathscr{I}=\mathbb{N}$, this follows by dominated convergence theorem, because $\sup_\iota|f_\iota|_\infty<+\infty$. If E is homeomorphic to a Borel subset of a Polish space, then $|\mu|$ is tight (see [79, p. 29, Theorem 3.2]), so, given $\varepsilon>0$, there exists $K_\varepsilon\subset E$ compact such that $|\mu|(K_\varepsilon^c)<\varepsilon$. Let $\bar\iota\in\mathscr{I}$ be such that $\iota\geq\bar\iota$ implies $\sup_{\iota\geq\bar\iota}[f_\iota]_{K_\varepsilon}<\varepsilon$ (this is possible by uniform convergence of $\{f_\iota\}_{\iota\in\mathscr{I}}$ to 0 on compact sets). Then

$$\begin{aligned}\left|\int_E f_\iota d\mu\right| &\leq \int_E |f_\iota| d|\mu| \leq [f_\iota]_{K_\varepsilon}|\mu|_1+\int_{K_\varepsilon^c}|f_\iota|d|\mu|\\ &\leq |\mu|_1\sup_{\iota\geq\bar\iota}[f_\iota]_{K_\varepsilon}+|f_\iota|_\infty|\mu|(K_\varepsilon^c)\leq(|\mu|_1+M)\varepsilon,\quad\forall\iota\geq\bar\iota,\end{aligned}$$

and we conclude by arbitrariness of ε.

(ii) The proof is analogous to that of *(i)*. ∎

We have a similar proposition relating $\sigma(\mathscr{S}(E),\mathbf{ca}(E))$ and the pointwise convergence in $\mathscr{S}(E)$. Actually, a part of this proposition is implicitly provided by [84, Theorem 2.2], where the separability of E and the choice $\mathscr{S}(E)=UC_b(E)$ play no role.

Proposition 5.3.14. *The following statements hold.*

(i) If a net $\{f_\iota\}_{\iota\in\mathscr{I}}$ is bounded and convergent to f in $(\mathscr{S}(E),\sigma(\mathscr{S}(E),\mathbf{ca}(E)))$, then

$$\sup_{\iota\in\mathscr{I}}|f_\iota|_\infty<+\infty \quad \textit{and} \quad \lim_\iota f_\iota=f \textit{ pointwise}.$$

If $\mathscr{I}=\mathbb{N}$ then also the converse holds true.

(ii) If a net $\{f_\iota\}_{\iota\in\mathscr{I}}$ is bounded and Cauchy in $(\mathscr{S}(E),\sigma(\mathscr{S}(E),\mathbf{ca}(E)))$, then

$$\sup_{\iota\in\mathscr{I}}|f_\iota|_\infty<+\infty \quad \textit{and} \quad \{f_\iota(x)\}_\iota \textit{ is Cauchy for every } x\in E.$$

If $\mathscr{I}=\mathbb{N}$ then also the converse holds true.

Proof. (i) Let $\{f_\iota\}_{\iota\in\mathscr{I}}$ be a bounded net in $(\mathscr{S}(E),\sigma(\mathscr{S}(E),\mathbf{ca}(E)))$, converging to f in this space. By Proposition 5.3.3 we have $\sup_{\iota\in\mathscr{I}}|f_\iota|_\infty<+\infty$, and, since $\mathbf{ca}(E)$ contains the Dirac measures, the net converges to f also pointwise. Conversely, let $\{f_n\}_{n\in\mathbb{N}}\subset\mathscr{S}(E)$ be such that $\sup_{n\in\mathbb{N}}|f_n|_\infty=M<+\infty$ and $\lim_{n\to+\infty}f_n=f$ pointwise. Then an application of Lebesgue's dominated convergence theorem provides $\lim_{n\to+\infty} f_n = f$ in $(\mathscr{S}(E), \sigma(\mathscr{S}(E), \mathbf{ca}(E)))$.

(ii) The proof is analogous to that of *(i)*. ■

Proposition 5.3.15. *The following statements hold.*

(i) $(B_b(E), \sigma(B_b(E), \mathbf{ca}(E)))$ *and* $(B_b(E), \tau_{\mathscr{K}})$ *are sequentially complete.*

(ii) $C_b(E)$ *is* $\tau_{\mathscr{K}}$*-closed in* $B_b(E)$ *(hence, by (i),* $(C_b(E), \tau_{\mathscr{K}})$ *is sequentially complete).*

(iii) If E *is homeomorphic to a Borel subset of a Polish space, then* $UC_b(E)$ *is dense in* $(C_b(E), \tau_{\mathscr{K}})$*.*

(iv) $(UC_b(E), \tau_{\mathscr{K}})$ *is sequentially complete if and only if* $UC_b(E) = C_b(E)$*.*

(v) $(\mathscr{S}(E), \tau_{\mathscr{K}})$ *is metrizable if and only if* E *is compact.*

Proof. (i) Let $\{f_n\}_{n\in\mathbb{N}}$ be $\tau_{\mathscr{K}}$-Cauchy in $B_b(E)$. Then, as every Cauchy sequence is bounded, by Proposition 5.3.13*(ii)*, the sequence is τ_∞-bounded. Then its pointwise limit f (that clearly exists) belongs to $B_b(E)$. By Proposition 5.3.13*(ii)*, the convergence is uniform on every compact subset of E. Then Proposition 5.3.13*(i)* implies that $\{f_n\}_{n\in\mathbb{N}}$ is $\tau_{\mathscr{K}}$-convergent to f. This shows that $(B_b(E), \tau_{\mathscr{K}})$ is sequentially complete.

By using Proposition 5.3.14, a similar argument shows that also $(B_b(E), \sigma(B_b(E), \mathbf{ca}(E)))$ is sequentially complete.

(ii) Let $\{f_\iota\}_{\iota\in\mathscr{I}} \subset C_b(E)$ be a net $\tau_{\mathscr{K}}$-converging to f in $B_b(E)$. In particular, the convergence is uniform on compact sets, hence $f \in C_b(E)$.

(iii) Let $f \in C_b(E)$, let K be a compact subset of E, let $\mu_1, \ldots, \mu_n \in \mathbf{ca}(E)$, and let $\varepsilon > 0$. We show that there exists $g \in UC_b(E)$ such that $\max_{i=1,\ldots,n} p_{K,\mu_i}(f-g) \leq \varepsilon$. This will prove the density of $UC_b(E)$ in $C_b(E)$ with respect to $\tau_{\mathscr{K}}$. Since E is homeomorphic to a Borel subset of a Polish space, the finite family $|\mu_1|, \ldots, |\mu_n|$ is tight (see [79, Theorem 3.2, p. 29]). Hence, there exists a compact set K_ε such that $\max_{i=1,\ldots,n} |\mu_i|(K_\varepsilon^c) < \frac{\varepsilon}{2(1+|f|_\infty)}$. Let $g \in UC_b(E)$ be a uniformly continuous extension of $f_{|K\cup K_\varepsilon}$ such that $|g|_\infty \leq |f|_\infty$. Then

$$\max_{i=1,\ldots,n} p_{K,\mu_i}(f-g) \leq [f-g]_K + \max_{i=1,\ldots,n} \int_E |f-g| d|\mu_i| \leq 2|f|_\infty \max_{i=1,\ldots,n} |\mu_i|(K_\varepsilon^c) \leq \varepsilon.$$

(iv) If $UC_b(E) = C_b(E)$, then the sequential completeness of $(UC_b(E), \tau_{\mathscr{K}})$ follows from *(ii)* of the present proposition.

Suppose that $UC_b(E) \neq C_b(E)$. Let $\{B_n\}_{n\in\mathbb{N}}$, $\{f_n\}_{n\in\mathbb{N}} \subset UC_b(E)$, and $f \in C_b(E) \setminus UC_b(E)$ be as in the second part of the proof of Lemma 5.3.10. To show that $UC_b(E)$ is not sequentially complete, we will show that $\lim_{n\to+\infty} f_n = f$ in $(C_b(E), \tau_{\mathscr{K}})$. Let $K \subset E$ be compact and $\mu \in \mathbf{ca}(E)$. As observed in the proof of Lemma 5.3.10, $f = \sum_{i=1}^{n_K} \rho_i$ on K, for some $n_K \in \mathbb{N}$ depending on K, and then $[f - f_n]_K = 0$ for every $n \geq n_K$. Then

$$\begin{aligned}
\limsup_{n\to+\infty} p_{K,\mu}(f-f_n) &= \limsup_{n\to+\infty} p_{K,\mu}\left(\sum_{i=n+1}^{+\infty}\rho_i\right) = \limsup_{n\to+\infty}\left|\int_E\left(\sum_{i=n+1}^{+\infty}\rho_i\right)d\mu\right| \\
&\leq \lim_{n\to+\infty}\sum_{i=n+1}^{+\infty}\int_E \rho_i d|\mu| \leq \lim_{n\to+\infty}\sum_{i=n+1}^{+\infty}|\mu|(B_i) \\
&= \lim_{n\to+\infty}|\mu|\left(\bigcup_{i\geq n+1} B_i\right) = |\mu|\left(\bigcap_{n\geq 1}\bigcup_{i\geq n+1} B_i\right).
\end{aligned}$$

As the balls $\{B_n\}_{n\in\mathbb{N}}$ are pairwise disjoint, we have $\bigcap_{n\geq 1}\bigcup_{i\geq n} B_i = \emptyset$. Hence, the last term in the inequality above is 0 and we conclude.

(v) If E is compact, then Proposition 5.3.6 yields $\tau_{\mathscr{K}} = \tau_\infty$, hence $(\mathscr{S}(E), \tau_{\mathscr{K}})$ is metrizable.

If E is not compact, in order to prove that $(\mathscr{S}(E), \tau_{\mathscr{K}})$ is not metrizable, it will be sufficient to prove that every $\tau_{\mathscr{K}}$-neighborhood of 0 contains a non-degenerate vector space. Indeed, in such a case, if $\hat{d}$ was a metric inducing $\tau_{\mathscr{K}}$, there would exist a sequence $\{x_n\}_{\in\mathbb{N}}$, such that $\lim_{n\to+\infty}\hat{d}(x_n,0) = 0$ and $\lim_{n\to\infty}|x_n|_\infty = +\infty$. But then $\{x_n\}_{n\in\mathbb{N}}$ would converge to 0 in $\tau_{\mathscr{K}}$, and then the sequence would be $|\cdot|_\infty$-bounded, by Proposition 5.3.13*(i)*, providing the contradiction.

To show that every neighborhood of 0 in $\tau_{\mathscr{K}}$ contains a non-degenerate vector space, let $K \subset E$ be compact, $\mu_1, \ldots, \mu_m \in \mathbf{ca}(E)$, $\varepsilon > 0$, and consider the neighborhood

$$\mathscr{I} := \{f \in \mathscr{S}(E) : \ p_{K,\mu_i}(f) < \varepsilon, \ \forall i = 1, \ldots, m\}.$$

Since E is not compact, by Lemma 5.3.11, $UC_b(E) \neq C_b(E)$. Hence, we can construct the sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset UC_b(E) \subset \mathscr{S}(E)$ as in the second part of the proof of Lemma 5.3.10. This is a sequence of linearly independent functions. Setting

$$Z_K := \{f \in UC_b(E) : f(x) = 0, \ \forall x \in K\},$$

we have $\rho_n \in Z_K$ for every $n \geq n_K$ (where n_K is as in the proof of Lemma 5.3.10). This shows that the subspace $Z_K \subset \mathscr{S}(E)$ is infinite dimensional. For $i = 1, \ldots, m$, define the functionals

$$\Lambda_i : Z_K \to \mathbb{R}, \ \ \varphi \mapsto \int_E \varphi d\mu_i.$$

Since Z_K is infinite dimensional, $\mathscr{N} := \bigcap_{i=1}^m \ker \Lambda_i$ is infinite dimensional too. On the other hand, by construction, $\mathscr{N} \subset \mathscr{I}$. This concludes the proof. ■

Characterisation of $(\mathscr{S}(E), \tau_{\mathscr{K}})^*$

The aim of this subsection is to provide a characterizion of $(\mathscr{S}(E), \tau_{\mathscr{K}})^*$, for the cases $\mathscr{S}(E) = B_b(E)$ and $\mathscr{S}(E) = C_b(E)$. Denote by $\mathbf{ba}_{\mathscr{C}}(E)$ the subspace of $\mathbf{ba}(E)$ defined by

$$\mathbf{ba}_{\mathscr{C}}(E) := \{\mu \in \mathbf{ba}(E) : \exists\, K \subset E \text{ compact} : |\mu|(K^c) = 0\}.$$

If E is compact, we clearly have $\mathbf{ba}_{\mathscr{C}}(E) = \mathbf{ba}(E)$. Conversely, if E is not compact, then $\mathbf{ba}_{\mathscr{C}}(E)$ is a non-closed subspace of $\mathbf{ba}(E)$. Indeed, if the sequence $\{x_n\}_{n\in\mathbb{N}}$ in E does not admit any convergent subsequence, then

$$\mu_n = \sum_{k=1}^{n} 2^{-k}\delta_{x_k} \in \mathbf{ba}_{\mathscr{C}}(E),\ \forall n \in \mathbb{N}, \text{ and } \lim_{n\to+\infty} \mu_n = \sum_{k=1}^{+\infty} 2^{-k}\delta_{x_k} \in \mathbf{ca}(E) \setminus \mathbf{ba}_{\mathscr{C}}(E).$$

Denote by $C_b(E)^{\perp}$ the annihilator of $C_b(E)$ in $(B_b(E), |\cdot|_\infty)^* \cong (\mathbf{ba}(E), |\cdot|_1)$ (see (5.3.2),(5.3.3)), i.e.

$$C_b(E)^{\perp} := \left\{\mu \in \mathbf{ba}(E)\colon\ \int_E f\,d\mu = 0,\ \ \forall f \in C_b(E)\right\}.$$

By Lemma 5.3.1, we have $C_b(E)^{\perp}\setminus\{0\} \subset \mathbf{ba}(E) \setminus \mathbf{ca}(E)$.

Proposition 5.3.16. *The following statements hold.*

(i) $(B_b(E), \tau_{\mathscr{K}})^* = \left(\mathbf{ba}_{\mathscr{C}}(E) \bigcap C_b(E)^{\perp}\right) \oplus \mathbf{ca}(E)$*. More explicitly, for each* $\Lambda \in (B_b(E), \tau_{\mathscr{K}})^*$ *there exist unique* $\mu \in \mathbf{ba}_{\mathscr{C}}(E) \bigcap C_b(E)^{\perp}$ *and* $\nu \in \mathbf{ca}(E)$ *such that*

$$\Lambda(f) = \int_E f\,d(\mu+\nu) \qquad \forall f \in B_b(E),$$

where the integral is in the Darboux sense.

(ii) $(C_b(E), \tau_{\mathscr{K}})^* = \mathbf{ca}(E)$*. More explicitly, for each* $\Lambda \in (C_b(E), \tau_{\mathscr{K}})^*$ *there exists a unique* $\nu \in \mathbf{ca}(E)$ *such that*

$$\Lambda(f) = \int_E f\,d\nu \qquad \forall f \in C_b(E).$$

Proof. *(i)* Let $\Lambda \in (B_b(E), \tau_{\mathscr{K}})^*$. Then there exist $L > 0$, a compact set $K \subset E$, a natural number N, and measures $\mu_1, \ldots, \mu_N \in \mathbf{ca}(E)$, such that

$$|\Lambda(f)| \leq L\left([f]_K + \sum_{n=1}^{N}\left|\int_E f\,d\mu_n\right|\right) \qquad \forall f \in B_b(E).$$

Define $\Lambda_{K^c}(f) = \Lambda(f\mathbf{1}_{K^c})$, for $f \in B_b(E)$. Then

$$|\Lambda_{K^c}(f)| \leq L\sum_{n=1}^{N}\left|\int_E f\,d(\mu_n\lfloor K^c)\right|, \qquad \forall f \in B_b(E), \tag{5.3.10}$$

where $\mu_n\lfloor K^c$ denotes the restriction of μ_n to K^c. Hence $\Lambda_{K^c} \in (B_b(E), \tau_{\mathscr{K}})^*$. Moreover, by [91, Lemma 3.9, p. 63], (5.3.10) implies that there exists $\nu \in \operatorname{span}\{\mu_i\lfloor K^c\colon i = 1, \ldots, N\} \subset \mathbf{ca}(E)$ such that

$$\Lambda_{K^c}(f) = \int_E f\,d\nu \qquad \forall f \in B_b(E).$$

Define $\Lambda_K(f) := \Lambda(f\mathbf{1}_K)$, for $f \in B_b(E)$. Since $\Lambda_K = \Lambda - \Lambda_{K^c}$, $\Lambda_K \in (B_b(E), \tau_{\mathscr{K}})^*$. By the identification $(B_b(E), |\cdot|_\infty)^* \cong (\mathbf{ba}(E), |\cdot|_1)$ (see (5.3.2)–(5.3.3)), there exists a unique $\mu \in \mathbf{ba}(E)$ such that

$$\Lambda_K(f) = \int_E f\,d\mu \qquad \forall f \in B_b(E),$$

where the integral above is defined in the Darboux sense. We notice that $\mu(A) = 0$ for every Borel set $A \subset K^c$. Hence $\mu \in \mathbf{ba}_{\mathscr{C}}(E)$, and the existence part of the claim is proved.

As regarding uniqueness, let $\mu_1 + \nu_1$ and $\mu_2 + \nu_2$ be two decompositions as in the statement. Then $\nu_1 - \nu_2 \in \mathbf{ca}(E) \bigcap C_b(E)^{\perp}$. Therefore, by Lemma 5.3.1, $\nu_1 - \nu_2 = 0$, and then $\mu_1 = \mu_2$.

(ii) Let $\Lambda \in (C_b(E), \tau_{\mathscr{K}})^*$. Since $\tau_{\mathscr{K}}$ is locally convex, by the Hahn-Banach Theorem we can extend Λ to some $\overline{\Lambda} \in (B_b(E), \tau_{\mathscr{K}})^*$. Let $\mu + \nu$ be the decomposition of $\overline{\Lambda}$ provided by *(i)*, with $\mu \in \mathbf{ba}_{\mathscr{C}}(E) \bigcap C_b(E)^{\perp}$ and $\nu \in \mathbf{ca}(E)$. Then

$$\Lambda(f) = \overline{\Lambda}(f) = \int_E f\, d(\mu + \nu) = \int_E f\, d\nu \qquad \forall f \in C_b(E).$$

Uniqueness is provided by Lemma 5.3.1. ∎

Remark 5.3.17. In general, the dual of $(C_b(E), \tau_\infty)$ cannot be identified with $\mathbf{ca}(E)$ through the integral, that is the isometric embedding (5.3.4) is not onto([7]). An example where $(C_b(E), \tau_\infty)^* \neq \mathbf{ca}(E)$ is provided by the case $E = \mathbb{N}$. Then $C_b(\mathbb{N}) = \ell^\infty$ and $(C_b(\mathbb{N}), \tau_\infty)^* = (\ell^\infty)^* \supsetneq \ell^1 \cong \mathbf{ca}(\mathbb{N})$ (where the symbol "$\cong$" is consistent with the action of ℓ^1 and of $\mathbf{ca}(\mathbb{N})$ on ℓ^∞). In view of this observation, Proposition 5.3.16(ii) cannot be seen, in its generality, as a straightforward consequence of the inclusions $\sigma(C_b(E), \mathbf{ca}(E)) \subset \tau_{\mathscr{K}} \subset \tau_\infty$.

5.3.2 Relationship with weakly continuous semigroups

In this subsection we first recall the notions of $\mathscr{K}$-convergence and of weakly continuous semigroup in the space $UC_b(E)$, introduced and studied first in [7, 8] in the case of E separable Banach space ([8]). So, throughout this subsection E is assumed to be a Banach space. We will show that every weakly continuous semigroup is a C_0-sequentially locally equicontinuous semigroup and, up to a renormalization, a C_0-sequentially equicontinuous semigroup on $(UC_b(E), \tau_{\mathscr{K}})$ (Proposition 5.3.19).

The notion of $\mathscr{K}$-convergence was introduced in [7, 8] for sequences. We recall it in its natural extension to nets. A net of functions $\{f_\iota\}_{\iota \in \mathscr{I}} \subset UC_b(E)$ is said $\mathscr{K}$-convergent to $f \in UC_b(E)$ if it is $|\cdot|_\infty$-bounded and if $\{f_\iota\}_{\iota \in \mathscr{I}}$ converges to f uniformly on compact sets of E, i.e.

$$\begin{cases} \sup_{\iota \in \mathscr{I}} |f_\iota|_\infty < +\infty \\ \lim_\iota [f_\iota - f]_K = 0 \quad \text{for every non-empty compact } K \subset E. \end{cases} \tag{5.3.11}$$

[7] For a characterisation of $(C_b(E), \tau_\infty)^*$, see [1, Sec. 14.2].

[8] In order to avoid misunderstanding, we stress that [8] uses the notation $C_b(E)$ to denote the space of *uniformly* continuous bounded functions on E, i.e. our space $UC_b(E)$. Also we notice that the separability of E is not needed here for our discussion.

In such a case, we write $f_\iota \xrightarrow{\mathcal{K}} f$. If E is separable, in view of Proposition 5.3.13*(i)*, the convergence (5.3.11) is equivalent to the convergence with respect to the locally convex topology $\tau_{\mathcal{K}}$. In this sense, $\tau_{\mathcal{K}}$ is the natural vector topology to treat weakly continuous semigroups (whose definition is recalled below) within the framework of C_0-sequentially locally equicontinuous semigroups.

Definition 5.3.18. *A* weakly continuous semigroup *on $UC_b(E)$ is a family $T = \{T_t\}_{t\in\mathbb{R}^+}$ of bounded linear operators on $(UC_b(E), |\cdot|_\infty)$ satisfying the following conditions.*

(P1) $T_0 = I$ and $T_tT_s = T_{t+s}$ for $t,s \in \mathbb{R}^+$.

(P2) There exist $M \geq 1$ and $\alpha \in \mathbb{R}$ such that $|T_tf|_\infty \leq Me^{\alpha t}|f|_\infty$ for every $t \in \mathbb{R}^+$, $f \in UC_b(E)$.

(P3) For every $f \in UC_b(E)$ and every $\hat{t} > 0$, the family of functions $\{T_tf : E \to \mathbb{R}\}_{t\in[0,\hat{t}]}$ is equi-uniformly continuous, i.e., there exists a modulus of continuity w (depending on $\hat{t}$) such that

$$\sup_{t\in[0,\hat{t}]} |T_tf(\xi) - T_tf(\xi')| \leq w(|\xi - \xi'|_E), \qquad \forall \xi, \xi' \in E. \tag{5.3.12}$$

(P4) For every $f \in UC_b(E)$, we have $T_tf \xrightarrow{\mathcal{K}} f$ as $t \to 0^+$; in view of (P2) the latter convergence is equivalent to

$$\lim_{t\to 0^+} [T_tf - f]_K = 0 \text{ for every non-empty compact } K \subset E. \tag{5.3.13}$$

(P5) If $f_n \xrightarrow{\mathcal{K}} f$, then $T_tf_n \xrightarrow{\mathcal{K}} T_tf$ uniformly in $t \in [0,\hat{t}]$ for every $\hat{t} > 0$; in view of (P2), the latter convergence is equivalent to

$$\lim_{n\to+\infty} \sup_{t\in[0,\hat{t}]} [T_tf_n - T_tf]_K = 0 \text{ for every non-empty compact } K \subset E,\ \forall \hat{t} \in \mathbb{R}^+. \tag{5.3.14}$$

Proposition 5.3.19. *Let $T := \{T_t\}_{t\in\mathbb{R}^+}$ be a weakly continuous semigroup on $UC_b(E)$. Then T is a C_0-sequentially locally equicontinuous semigroup on $(UC_b(E), \tau_{\mathcal{K}})$ and, for every $\lambda > \alpha$ (where α is as in (P2)), $\{e^{-\lambda t}T_t\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially equicontinuous semigroup on $(UC_b(E), \tau_{\mathcal{K}})$ satisfying Assumption 5.2.16.*

Conversely, if T is a C_0-sequentially locally equicontinuous semigroup on $(UC_b(E), \tau_{\mathcal{K}})$ satisfying (P3), then T is a weakly continuous semigroup on $UC_b(E)$.

Proof. Let $f \in UC_b(E)$. By *(P4)* and by Proposition 5.3.13*(i)*, $T_tf \to f$ *in* $(UC_b(E), \tau_{\mathcal{K}})$ *when* $t \to 0^+$. This shows the strong continuity of T in $(UC_b(E), \tau_{\mathcal{K}})$.

Now let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence converging to 0 in $(UC_b(E), \tau_{\mathcal{K}})$ and let $\hat{t} \in \mathbb{R}^+$. By Proposition 5.3.13*(i)*, it follows that $f_n \xrightarrow{\mathcal{K}} 0$. By (P5) we then have $T_tf_n \xrightarrow{\mathcal{K}} 0$ uniformly in $t \in [0, \hat{t}]$. Using again Proposition 5.3.13*(i)*, we conclude that T is locally sequentially equicontinuous in $(UC_b(E), \tau_{\mathscr{K}})$.

By (P2) and by Proposition 5.3.3, we can apply Proposition 5.2.15*(i)* to T conclude that $\{e^{-\lambda t}T_t\}_{t\mathbb{R}^+}$ is a C_0-sequentially equicontinuous semigroup on $(UC_b(E), \tau_{\mathscr{K}})$.

We finally show that, for $\lambda > \alpha$, $\{e^{-\lambda t}T_t\}_{t \in \mathbb{R}^+}$ satisfies Assumption 5.2.16. Let $\alpha < \lambda' < \lambda$ and $f \in UC_b(E)$. By Proposition 5.3.15, $(C_b(E), \tau_{\mathscr{K}})$ is sequentially complete. By Proposition 5.2.14, the map

$$\mathbb{R}^+ \to (UC_b(E), \tau_{\mathscr{K}}),\ t \mapsto e^{-\lambda' t}T_t f,$$

is continuous and bounded. It then follows that the Riemann integral $R(\lambda)f$ exists in $C_b(E)$. We show that $R(\lambda)f \in UC_b(E)$. Since the Dirac measures are contained in $(C_b(E), \tau_{\mathscr{K}})^*$, by Proposition 5.2.18 we have

$$R(\lambda)f(\xi) = \int_0^{+\infty} e^{-\lambda t}T_t f(\xi)dt \qquad \forall \xi \in E.$$

On the other hand, by (P2), for every $\varepsilon > 0$ there exists $\hat{t} \in \mathbb{R}^+$ such that

$$\sup_{\xi \in E} \int_{\hat{t}}^{+\infty} e^{-\lambda t}T_t f(\xi)dt < \varepsilon.$$

Hence, to prove that $R(\lambda)f \in UC_b(E)$, it suffices to show that, for every $\hat{t} \in \mathbb{R}^+$,

$$\int_0^{\hat{t}} e^{-\lambda t}T_t f dt \in UC_b(E). \tag{5.3.15}$$

Let us define the set

$$C := \left\{ g \in C_b(E) \colon \sup_{\xi, \xi' \in E} |g(\xi) - g(\xi')| \le w(|\xi - \xi'|_E) \right\},$$

where w is as in (5.3.12). Clearly C is a subset of $UC_b(E)$, it is convex, it contains the origin, and is closed in $(C_b(E), \tau_{\mathscr{K}})$. By (5.3.12), $\{e^{-\lambda' t}T_t f\}_{t \in [0,\hat{t}]} \subset C$. Hence, we conclude by Proposition 5.2.20 that

$$\int_0^{t_0} e^{-\lambda t}T_t f dt \in \frac{1}{\lambda - \lambda'} C \quad \forall \lambda > \lambda',$$

which shows (5.3.15), concluding the proof of the first part of the proposition.

Now let T be a C_0-sequentially locally equicontinuous on $(UC_b(E), \tau_{\mathscr{K}})$ satisfying (P3). We only need to show that T verifies (P2), (P4), and (P5). Now, (P2) follows from Proposition 5.3.3 and Proposition 5.2.15, whereas (P4) comes once again by Proposition 5.3.13*(i)*. Finally, (P5) is due to Proposition 5.3.13*(i)* and to sequential local equicontinuity of T. ■

5.3.3 Relationship with π-semigroups

In this subsection we provide a connection between the notion of π-semigroups in $UC_b(E)$ introduced in [84] and bounded C_0-sequentially continuous semigroups (see Definition 5.2.43) in the space $(UC_b(E), \sigma(UC_b(E), \mathbf{ca}(E)))$ ([9]). We recall that the assumption E Banach space was standing only in the latter subsection, and that in the present subsection we restore the assumption that E is a generic metric space. We begin by recalling the definition of π-semigroup in $UC_b(E)$.

Definition 5.3.20. *A* π-semigroup on $UC_b(E)$ *is a family* $T = \{T_t\}_{t\in\mathbb{R}^+}$ *of bounded linear operators on* $(UC_b(E), |\cdot|_\infty)$ *satisfying the following conditions.*

(P1) $T_0 = I$ *and* $T_tT_s = T_{t+s}$ *for* $t, s \in \mathbb{R}^+$.

(P2) There exist $M \geq 1$, $\alpha \in \mathbb{R}$ *such that* $|T_tf|_\infty \leq Me^{\alpha t}|f|_\infty$ *for every* $t \in \mathbb{R}^+$, $f \in UC_b(E)$.

(P3) For each $\xi \in E$ *and* $f \in UC_b(E)$*, the map* $\mathbb{R}^+ \to \mathbb{R}$, $t \mapsto T_tf(\xi)$ *is continuous.*

(P4) If a sequence $\{f_n\}_{n\in\mathbb{N}} \subset UC_b(E)$ *is such that*

$$\sup_{n\in\mathbb{N}} |f_n|_\infty < +\infty \qquad \text{and} \qquad \lim_{n\to+\infty} f_n = f \qquad \text{pointwise,}$$

then, for every $t \in \mathbb{R}^+$,

$$\lim_{n\to+\infty} T_tf_n = T_tf \quad \text{pointwise.}$$

Proposition 5.3.21. *T is a π-semigroup in $UC_b(E)$ if and only if $\{e^{-\alpha t}T_t\}_{t\in\mathbb{R}^+}$ is a bounded C_0-sequentially continuous semigroup in $(UC_b(E), \sigma(UC_b(E), \mathbf{ca}(E)))$ (see Definition 5.2.43).*

Proof. Let us denote $\sigma := \sigma(UC_b(E), \mathbf{ca}(E))$. Let T be a π-semigroup in $UC_b(E)$. By Definition 5.3.20*(P2)*,*(P4)* and Proposition 5.3.14*(i)*, we have $\{e^{-\alpha t}T_t\}_{t\in\mathbb{R}^+} \subset \mathscr{L}_0((UC_b(E), \sigma))$. By Definition 5.3.20*(P2)*,*(P3)* and by Proposition 5.3.14*(i)*, the map $\mathbb{R}^+ \to (UC_b(E), \sigma)$, $t \mapsto e^{-\alpha t}T_tf$ is continuous for every $f \in UC_b(E)$. Moreover, by Definition 5.3.20*(P2)* and by Proposition 5.3.3, it is also bounded. This shows that $\{e^{-\alpha t}T_t\}_{t\in\mathbb{R}^+}$ is a bounded C_0-sequentially continuous semigroup in $(UC_b(E), \sigma)$.

Conversely, let $\{e^{-\alpha t}T_t\}_{t\in\mathbb{R}^+}$ be a bounded C_0-sequentially continuous semigroup in $(UC_b(E), \sigma)$. By Proposition 5.3.3, for every $f \in UC_b(E)$ the family $\{e^{-\alpha t}T_tf\}_{t\in\mathbb{R}^+}$ is bounded in $(UC_b(E), |\cdot|_\infty)$. By the Banach-Steinhaus theorem we conclude that there exists $M > 0$ such that

$$|e^{-\alpha t}T_t|_{L((UC_b(E),|\cdot|_\infty))} \leq M \qquad \forall t \in \mathbb{R}^+,$$

[9] Also in this case, in order to avoid misunderstanding, we stress that [84] uses the notation $C_b(E)$ to denote the space of *uniformly* continuous bounded functions on E, i.e. our space $UC_b(E)$. We also notice that in [84] the metric space E is assumed to be separable, but, for our discussion, this is not needed.

which provides $T \subset L((UC_b(E), |\cdot|_\infty))$ and *(P2)*. Then, *(P3)* is implied by the fact that the map $\mathbb{R}^+ \to (UC_b(E), \sigma)$, $t \mapsto e^{-\alpha t} T_t f$, is continuous and that Dirac measures are contained in σ. Finally, *(P4)* is due to the assumption $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+} \subset \mathscr{L}_0((UC_b(E), \sigma))$ and to Proposition 5.3.14*(i)*. ■

As observed in Subsection 5.2.8, if the Laplace transform (5.2.41) of a bounded C_0-sequentially continuous semigroup in $(UC_b(E), \sigma(UC_b(E), \mathbf{ca}(E)))$ is well-defined, several results that we stated for C_0-sequentially equicontinuous semigroups still hold. Nevertheless, some other important results, as the generation theorem, or the fact that two semigroups with the same generator are equal, cannot be proved for bounded C_0-sequentially continuous semigroups within the approach of the previous sections. Due to Proposition 5.3.21, this is reflected in the fact that, as far as we know, such results are not available in the literature for π-semigroups.

5.3.4 Relationship with locally equicontinuous semigroups with respect to the *mixed* topology

When E is a separable Hilbert space, in [49] the so called *mixed topology* (introduced in [97]) is employed in the space $C_b(E)$ to frame a class of Markov transition semigroups within the theory of C_0-locally equicontinuous semigroups. The same topology, but in the more general case of E separable Banach space, is used in [50] to deal with Markov transition semigroups associated to the Ornstein-Uhlenbeck processe in Banach spaces.

In this subsection, we assume that E is a separable Banach space and we briefly precise what is the relation between the mixed topology and $\tau_{\mathscr{K}}$ in the space $C_b(E)$, and between C_0-locally equicontinuous semigroups with respect to the mixed topology and C_0-sequentially locally equicontinuous semigroups with respect to $\tau_{\mathscr{K}}$.

The mixed topology on $C_b(E)$, denoted by $\tau_{\mathscr{M}}$, can be defined by seminorms as follows. Let $\mathbf{K} := \{K_n\}_{n \in \mathbb{N}}$ be a sequence of compact subsets of E, and let $\mathbf{a} := \{a_n\}_{n \in \mathbb{N}}$ be a sequence of strictly positive real numbers such that $a_n \to 0$. Define

$$p_{\mathbf{K},\mathbf{a}}(f) = \sup_{n \in \mathbb{N}} \{a_n [f]_{K_n}\} \qquad \forall f \in C_b(E). \tag{5.3.16}$$

Then $p_{\mathbf{K},\mathbf{a}}$ is a seminorm, and $\tau_{\mathscr{M}}$ is the locally convex topology induced by the family of seminorms $p_{\mathbf{K},\mathbf{a}}$, when $\mathbf{K}$ ranges on the set of countable families of compact subsets of E, and $\mathbf{a}$ ranges on the set of sequences of strictly positive real numbers converging to 0.

It can be proved (see [92, Theorem 2.4]), that $\tau_{\mathscr{M}}$ is the finest locally convex topology on $C_b(E)$ such that a net $\{f_\iota\}_{\iota \in \mathscr{I}}$ is bounded in the uniform norm and converges to f in $\tau_{\mathscr{M}}$ if and only if it is $\mathscr{K}$-convergent, hence if and only if (5.3.11) is verified.

To establish the relation between $\tau_{\mathscr{M}}$ and $\tau_{\mathscr{K}}$, we begin with a lemma.

Lemma 5.3.22. *Let $S \subset E$ be a Borel set and assume that S is a retract of E, i.e., there exists a continuous map $r\colon E \to S$ such that $r(s) = s$ for every $s \in S$. We denote by $\tau^S_{\mathcal{K}}$ the topology $\tau_{\mathcal{K}}$ when considered in the spaces $C_b(S)$. Then*

$$\Psi\colon C_b(E) \to C_b(S),\ \ f \mapsto f_{|S} \tag{5.3.17}$$

is continuous and open as a map from $(C_b(E), \tau_{\mathcal{K}})$ onto $(C_b(S), \tau^S_{\mathcal{K}})$.

Proof. First we show that Ψ is continuous. Let $\{f_\iota\}_{\iota\in\mathcal{I}} \subset C_b(E)$ be a net converging to 0 in $\tau_{\mathcal{K}}$, let $K \subset S$ be compact, and let $\mu \in \mathbf{ca}(S)$. Since K is also compact in E, we immediately have $[f_{\iota|S}]_K \to 0$. Moreover, since S is Borel, the set function μ^S defined by $\mu^S(A) := \mu(A \cap S)$, $A \in \mathcal{E}$, belongs to $\mathbf{ca}(E)$. Then we also have $\int_S f_{\iota|S} d\mu = \int_E f_\iota d\mu^S \to 0$. So Ψ is continuous.

Let us prove that Ψ is open. Let $K \subset E$ be compact, $\mu_1, \dots, \mu_n \in \mathbf{ca}(E)$, $\varepsilon > 0$. Define the neighborhood of 0 in $(C_b(E), \tau^E_{\mathcal{K}})$

$$U := \left\{ f \in C_b(E) \colon [f]_K < \varepsilon,\ \left| \int_E f d\mu_i \right| < \varepsilon,\ i = 1, \dots, n \right\}$$

and define the the neighborhood of 0 in $(C_b(S), \tau^S_{\mathcal{K}})$

$$V := \left\{ g \in C_b(S) \colon [g]_{r(K)} < \varepsilon,\ \left| \int_E g d(r_\# \mu_i) \right| < \varepsilon,\ i = 1, \dots, n \right\}$$

where $r_\#\mu_i$ is the pushforward measure of μ_i through r. Then $g \in V$ if and only if $f := g \circ r \in U$. As $g = (g \circ r)_{|S}$ for every $g \in C_b(S)$, we see that $V \subset \Psi(U)$. Hence, we conclude that Ψ is open. ■

Proposition 5.3.23. *If* $\dim E \geq 1$, *then* $\tau_{\mathcal{K}} \subsetneq \tau_{\mathcal{M}}$ *on* $C_b(E)$.

Proof. We already observed that $\tau_{\mathcal{M}}$ is the finest locally convex topology $\tau_{\mathcal{M}}$ such that $\{f_\iota\}_{\iota\in\mathcal{I}}$ is bounded in the uniform norm and converges to f in $\tau_{\mathcal{M}}$ if and only if it is $\mathcal{K}$-convergent. Then, by Proposition 5.3.13*(i)*, we have $\tau_{\mathcal{K}} \subset \tau_{\mathcal{M}}$.

Now we show that $\tau_{\mathcal{M}} \not\subset \tau_{\mathcal{K}}$ if $\dim(E) \geq 1$. Let S be a one dimensional subspace of E and let

$$\Psi\colon C_b(E) \to C_b(S),\ f \mapsto f_{|S}.$$

By using the seminorms defined in (5.3.16), one checks that Ψ, defined in (5.3.17), is continuous from $(C_b(E), \tau_{\mathcal{M}})$ onto $(C_b(S), \tau^S_{\mathcal{M}})$, where $\tau^S_{\mathcal{M}}$ denotes the topology $\tau_{\mathcal{M}}$ in the space $C_b(S)$. Clearly S is a retract of E. Then, by Lemma 5.3.22, to show that $\tau_{\mathcal{M}} \not\subset \tau_{\mathcal{K}}$ on $C_b(E)$, it is sufficient to show that $\tau^S_{\mathcal{M}} \not\subset \tau^S_{\mathcal{K}}$ on $C_b(S)$. Let us identify S with $\mathbb{R}$. Let W be a Wiener process in $\mathbb{R}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By [49, Theorem 4.1], the transition semigroup $T := \{T_t\}_{t\in\mathbb{R}^+}$ defined by

$$T_t\colon C_b(\mathbb{R}) \to C_b(\mathbb{R}),\ f \mapsto \mathbb{E}[f(\cdot + W_t)],$$

is a C_0-locally equicontinuous semigroup in $(C_b(\mathbb{R}),\tau^{\mathbb{R}}_{\mathcal{M}})$. But Example 5.4.5 below shows that T is not locally equicontinuous in $(C_b(\mathbb{R}),\tau^{\mathbb{R}}_{\mathcal{K}})$. Then $\tau^{\mathbb{R}}_{\mathcal{M}} \not\subset \tau^{\mathbb{R}}_{\mathcal{K}}$. Since we already know that $\tau^{\mathbb{R}}_{\mathcal{K}} \subset \tau^{\mathbb{R}}_{\mathcal{M}}$, we deduce that $\tau^{\mathbb{R}}_{\mathcal{M}} \not\subset \tau^{\mathbb{R}}_{\mathcal{K}}$ and conclude. ■

By Proposition 5.3.13*(i)*, every sequence convergent in $\tau_{\mathcal{K}}$ is bounded and convergent uniformly on compact sets, and then it is convergent in $\tau_{\mathcal{M}}$. Since we also know $\tau_{\mathcal{K}} \subset \tau_{\mathcal{M}}$, we immediately obtain the following

Proposition 5.3.24. *A semigroup T is C_0-sequentially (locally) equicontinuous in $(C_b(E),\tau_{\mathcal{M}})$ if and only if it is C_0-sequentially (locally) equicontinuous in $(C_b(E),\tau_{\mathcal{K}})$.*

5.4 Application to transition semigroups

In this section we apply the results of Section 5.3 to transition semigroups in spaces of (not necessarily bounded) continuous functions.

5.4.1 Transition semigroups in $(C_b(E),\tau_{\mathcal{K}})$

Let $\boldsymbol{\mu} := \{\mu_t(\xi,\cdot)\}_{\substack{t\in\mathbb{R}^+\\ \xi\in E}}$ be a subset of $\mathbf{ca}^+(E)$ and consider the following assumptions.

Assumption 5.4.1. *The family $\boldsymbol{\mu} := \{\mu_t(\xi,\cdot)\}_{\substack{t\in\mathbb{R}^+\\ \xi\in E}} \subset \mathbf{ca}^+(E)$ has the following properties.*

(i) *The family $\boldsymbol{\mu}$ is bounded in $\mathbf{ca}^+(E)$ and $p_0(\xi,\Gamma) = \mathbf{1}_\Gamma(\xi)$ for every $\xi \in E$ and every $\Gamma \in \mathcal{E}$.*

(ii) *For every $f \in C_b(E)$ and $t \in \mathbb{R}^+$, the map*

$$E \to \mathbb{R},\ \xi \mapsto \int_E f(\xi')\mu_t(\xi,d\xi') \tag{5.4.1}$$

is continuous.

(iii) *For every $f \in C_b(E)$, every $t,s \in \mathbb{R}^+$, and every $\xi \in E$,*

$$\int_E f(\xi')\mu_{t+s}(\xi,d\xi') = \int_E \left(\int_E f(\xi'')\mu_t(\xi',d\xi'')\right)\mu_s(\xi,d\xi').$$

(iv) *For every $\hat{t} > 0$ and every compact $K \subset E$, the family $\{\mu_t(\xi,\cdot)\colon t \in [0,\hat{t}],\ \xi \in K\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K_0 \subset E$ such that*

$$\mu_t(\xi,K_0) > \mu_t(\xi,E) - \varepsilon \qquad \forall t \in [0,\hat{t}],\ \forall \xi \in K.$$

(v) *For every $r > 0$ and every non-empty compact $K \subset E$,*

$$\lim_{t\to 0^+} \sup_{\xi\in K} |\mu_t(\xi,B(\xi,r)) - 1| = 0, \tag{5.4.2}$$

where $B(\xi,r)$ denotes the open ball $B(\xi,r) := \{\xi' \in E : d(\xi,\xi') < r\}$.

We observe that in Assumption 5.4.1 it is not required that $p_t(\xi,E) = 1$ for every $t \in \mathbb{R}^+$, $\xi \in E$, hence the family $\boldsymbol{\mu}$ is not necessarily a probability kernel in $(E,\mathscr{E})$. Assumptions 5.4.1*(ii)*,*(iii)* can be rephrased by saying that

$$T_t : C_b(E) \to C_b(E),\ f \mapsto \int_E f(\xi)\mu_t(\cdot, d\xi)$$

is well defined for all $t \in \mathbb{R}^+$ and $T := \{T_t\}_{t\in\mathbb{R}^+}$ is a transition semigroup in $C_b(E)$. If $\boldsymbol{\mu}$ is a probability kernel, then T is a Markov transition semigroup.

Proposition 5.4.2. *Let Assumption 5.4.1 holds and let $T := \{T_t\}_{t\in\mathbb{R}^+}$ be defined as in* (5.4.1). *Then T is a C_0-sequentially locally equicontinuous semigroup on $(C_b(E),\tau_{\mathscr{K}})$. Moreover, for every $\alpha > 0$, the normalized semigroup $\{e^{-\alpha t}T_t\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially equicontinuous semigroup on $(C_b(E),\tau_{\mathscr{K}})$ satisfying Assumption 5.2.16.*

Proof. Assumptions 5.4.1*(i)*,*(ii)*,*(iii)* imply that T maps $C_b(E)$ into itself and that it is a semigroup. We show that the C_0-property holds, i.e. $\lim_{t\to 0^+} T_t f = f$ in $(C_b(E),\tau_{\mathscr{K}})$ for every $f \in C_b(E)$. Let $M := \sup_{\substack{t\in\mathbb{R}^+\\ \xi\in E}} |\mu_t(\xi,E)|$. By Assumption 5.4.1*(i)*, $M < +\infty$ and

$$|T_t f|_\infty \le M|f|_\infty \qquad \forall f \in C_b(E),\ \ \forall t \in \mathbb{R}^+. \tag{5.4.3}$$

Let $f \in C_b(E)$. By (5.4.3) and by Proposition 5.3.13*(i)*, in order to show that $\lim_{t\to 0^+} T_t f = f$ in $(C_b(E),\tau_{\mathscr{K}})$, it is sufficient to show that $\lim_{t\to 0^+}[T_t f - f]_K = 0$, for every $K \subset E$ nonempty compact. Let $K \subset E$ be such a set. We claim that

$$\lim_{t\to 0^+} \sup_{\xi\in K} |\mu_t(\xi,E) - 1| = 0. \tag{5.4.4}$$

Indeed, let ε and K_0 as in Assumption 5.4.1*(iv)*, when $\hat{t} = 1$, and let $r := \sup_{(\xi,\xi')\in K\times K_0} d(\xi,\xi') + 1$. Then $K_0 \subset B(\xi,r)$ for every $\xi \in K$. For $t \in [0,1]$ and $\xi \in K$, we have

$$\begin{aligned} |\mu_t(\xi,E) - 1| &\le |\mu_t(\xi,E \setminus B(\xi,r))| + |\mu_t(\xi,B(\xi,r)) - 1| \\ &\le |\mu_t(\xi,E \setminus K_0)| + |\mu_t(\xi,B(\xi,r)) - 1| \\ &\le \varepsilon + |\mu_t(\xi,B(\xi,r)) - 1|. \end{aligned}$$

By taking the supremum over $x \in K$, by passing to the limit as $t \to 0^+$, by using (5.4.2), and by arbitrariness of ε, we obtain (5.4.4). In particular, (5.4.4) implies

$$\lim_{t\to 0^+} \sup_{\xi\in K} |f(\xi) - \mu_t(\xi,E)f(\xi)| = 0, \tag{5.4.5}$$

and then $T_t f \to f$ in $\tau_{\mathscr{K}}$ as $t \to 0^+$ if and only if

$$\lim_{t\to 0^+} \sup_{\xi\in K} |T_t f(\xi) - \mu_t(\xi,E)f(\xi)| = 0. \tag{5.4.6}$$

Again, let $\varepsilon > 0$ and K_0 be as in Assumption 5.4.1*(iv)*, when $\hat{t} = 1$. Let w be a modulus of continuity for $f_{|K_0}$. For $\delta > 0$, $t \in [0,1]$, and $\xi \in K$, we write

$$\begin{aligned}|T_t f(\xi) - \mu_t(\xi,E)f(\xi)| \leq \int_E |f(\xi') - f(\xi)|\mu_t(\xi,d\xi') = \int_{K_0 \cap B(\xi,\delta)} |f(\xi') - f(\xi)|\mu_t(\xi,d\xi') \\ + \int_{K_0 \cap B(\xi,\delta)^c} |f(\xi') - f(\xi)|\mu_t(\xi,d\xi') + \int_{K_0^c} |f(\xi') - f(\xi)|\mu_t(\xi,d\xi') \\ \leq w(\delta) + 2|f|_\infty \left(\mu_t(\xi,B(\xi,\delta)^c) + \varepsilon\right).\end{aligned}$$

We then obtain

$$\sup_{\xi \in K} |T_t f(\xi) - \mu_t(\xi,E)f(\xi)| \leq \omega(\delta) + 2|f|_\infty \left(\sup_{\xi \in K} \mu_t(\xi,B(\xi,\delta)^c) + \varepsilon\right) \qquad \forall \delta > 0,\ \forall t \in [0,1],\ \forall \xi \in K.$$

By passing to the limit as $t \to 0^+$, by (5.4.2), by (5.4.4), and by arbitrariness of δ and ε, we obtain (5.4.6).

We now show that $\{T_t\}_{t \in [0,\hat{t}]}$ is sequentially equicontinuous for every $\hat{t} > 0$. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence converging to 0 in $(C_b(E), \tau_{\mathcal{K}})$ and let $\hat{t} > 0$. By Proposition 5.3.13*(i)*, $\{|f_n|_\infty\}_{n \in \mathbb{N}}$ is bounded by some $b > 0$. Then, by (5.4.3), $\{T_t f_n\}_{t \in \mathbb{R}^+, n \in \mathbb{N}}$ is bounded. To show that $T_t f_n \to 0$ in $(C_b(E), \tau_{\mathcal{K}})$, uniformly for $t \in [0,\hat{t}]$, it is then sufficient to show that

$$\lim_{n \to +\infty} \sup_{t \in [0,\hat{t}]} [T_t f_n]_K = 0 \qquad \forall K \subset E \text{ non-empty compact.}$$

Let $\varepsilon > 0$ and K_0 be as in Assumption 5.4.1*(iv)*, when $\hat{t} = 1$. Then, for $t \in [0,\hat{t}]$, $\xi \in K$, $n \in \mathbb{N}$, we have

$$|T_t f_n(\xi)| \leq \int_{K_0} |f_n(\xi')|\mu_t(\xi,d\xi') + \int_{K_0^c} |f_n(\xi')|\mu_t(\xi,d\xi') \leq M[f_n]_{K_0} + b\varepsilon.$$

Since $[f_n]_{K_0} \to 0$ as $n \to +\infty$, by arbitrariness of ε we conclude $\sup_{t \in [0,\hat{t}]} [T_t f_n]_K \to 0$ as $n \to +\infty$. This concludes the proof that T is a C_0-sequentially locally equicontinuous semigroup on $(C_b(E), \tau_{\mathcal{K}})$.

Next, by Proposition 5.3.3 and by (5.4.3), we can apply Proposition 5.2.15 and obtain that, for every $\alpha > 0$, $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$ is C_0-sequentially locally equicontinuous semigroup on $(C_b(E), \tau_{\mathcal{K}})$. Finally, by Remark 5.2.17 and Proposition 5.3.15*(ii)*, we conclude that Assumption 5.2.16 holds true for $\{e^{-\alpha t} T_t\}_{t \in \mathbb{R}^+}$. ■

5.4.2 Extension to weighted spaces of continuous functions

In this subsection, we briefly discuss how to deal with transition semigroups in weighted spaces of continuous functions. Let $\gamma \in C(E)$ such that $\gamma > 0$. We introduce the following γ-weighted space of continuous functions

$$C_\gamma(E) := \left\{ f \in C(E) : f\gamma \in C_b(E) \right\}.$$

A typical case is when E is an unbounded subset of a Banach space and $\gamma(x) = (1+|x|_E^p)^{-1}$, for some $p \in \mathbb{N}$. Then $C_\gamma(E)$ is the space of continuous functions on E having at most polynomial growth of order p. By the very definition of $C_\gamma(E)$, the multiplication by γ

$$\varphi_\gamma : C_\gamma(E) \to C_b(E),\ f \mapsto f\gamma,$$

defines an algebraic isomorphism. Hence, by endowing $C_\gamma(E)$ with the topology $\tau_{\mathscr{K}}^\gamma := \gamma^{-1}(\tau_{\mathscr{K}})$, this space becomes a locally convex Hausdorff topological vector space. A family of seminorms inducing $\tau_{\mathscr{K}}^\gamma$ is given by

$$p_{K,\mu}^\gamma := [f\gamma]_K + \left|\int_E f\gamma d\mu\right| \qquad \forall f \in C_\gamma(E),$$

when K ranges on the set of non-empty compact subsets of E and μ ranges on $\mathbf{ca}(E)$. Clearly, $\left(C_\gamma(E), \tau_{\mathscr{K}}^\gamma\right)$ and $(C_b(E), \tau_{\mathscr{K}})$ enjoy the same topological properties and γ is an isomorphism of topological vector spaces. This basic observation will be used now to frame C_0-sequentially locally equicontinuous semigroups on $(C_\gamma(E), \tau_{\mathscr{K}}^\gamma)$ induced by transition functions.

Let $\boldsymbol{\mu} := \{\mu_t(\xi,\cdot)\}_{t\in\mathbb{R}^+,\xi\in E} \subset \mathbf{ca}^+(E)$ and let $t \in \mathbb{R}^+$. Define the family $\boldsymbol{\mu}^\gamma := \{\mu_t^\gamma(\xi,\cdot)\}_{t\in\mathbb{R}^+,\xi\in E}$ by

$$\mu_t^\gamma(\xi,\Gamma) := \gamma(\xi)\int_\Gamma \gamma^{-1}(\xi')\mu_t(\xi,d\xi') \qquad \forall \Gamma \in \mathscr{E},\ \forall \xi \in E, \tag{5.4.7}$$

and

$$T_t f(\xi) := \int_E f(\xi')\mu_t(\xi,d\xi') \qquad \forall \xi \in E,\ \forall f \in C_\gamma(E). \tag{5.4.8}$$

Given $f \in C_\gamma(E)$, $\xi \in E$, the latter is well defined and finite if and only if and only if

$$\int_E \varphi_\gamma(f)(\xi')\mu_t^\gamma(\xi,d\xi') = \gamma(\xi)\int_E f(\xi')\mu_t(\xi,d\xi')$$

is well defined and finite. Then, $T_t f(\xi)$ is well defined and finite if and only if, setting $g := f\gamma$,

$$T_t^\gamma g(\xi) := \int_E g(\xi')\mu_t^\gamma(\xi,d\xi')$$

is well defined and finite. At the end, we get that $T_t f(\xi)$ is well defined and finite for every $\xi \in E$ and every $f \in C_\gamma(E)$ if and only if $T_t^\gamma g(f)$ is well defined and finite for every $\xi \in E$ and every $g \in C_b(E)$. In such a case

$$(\varphi_\gamma^{-1} \circ T_t^\gamma \circ \varphi_\gamma)f = T_t f \qquad \forall f \in C_\gamma(E), \tag{5.4.9}$$

hence the diagram

$$\begin{array}{ccc} C_\gamma(E) & \xrightarrow{\ \varphi_\gamma\ } & C_b(E) \\ \big\downarrow{\scriptstyle T_t} & & \big\downarrow{\scriptstyle T_t^\gamma} \\ C_\gamma(E) & \xleftarrow[\ \varphi_\gamma^{-1}\ ]{} & C_b(E) \end{array}$$

is commutative. Due to this fact, Proposition 5.4.2 can be immediately stated in the following equivalent form.

Proposition 5.4.3. *Let* $\boldsymbol{\mu} := \{\mu_t(\xi,\cdot)\}_{t\in\mathbb{R}^+,\xi\in E} \subset \mathbf{ca}^+(E)$ *and let* $\boldsymbol{\mu}^\gamma := \{\mu_t^\gamma(\xi,\cdot)\}_{\substack{t\in\mathbb{R}^+\\ \xi\in E}}$ *be defined starting from* $\boldsymbol{\mu}$ *through* (5.4.7). *Assume that* $\boldsymbol{\mu}^\gamma$ *satisfies Assumption 5.4.1 (when* $\boldsymbol{\mu}$ *is replaced by* $\boldsymbol{\mu}^\gamma$*). Then* $T := \{T_t\}_{t\in\mathbb{R}^+}$ *defined in* (5.4.8) *is a* C_0*-sequentially locally equicontinuous semigroup on* $(C_\gamma(E), \tau_{\mathcal{K}}^\gamma)$*. Moreover, for every* $\alpha > 0$*, the normalized semigroup* $\{e^{-\alpha t}T_t\}_{t\in\mathbb{R}^+}$ *is a* C_0*-sequentially equicontinuous semigroup on* $(C_\gamma(E), \tau_{\mathcal{K}}^\gamma)$ *satisfying Assumption 5.2.16.*

5.4.3 Markov transition semigroups associated to stochastic differential equations

Propositions 5.4.2 and 5.4.3 have a straightforward application to transition functions associated to mild solutions to stochastic differential equations in Hilbert spaces. Let $(U,|\cdot|_U)$, $(H,|\cdot|_H)$ be separable Hilbert spaces, let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}^+},\mathbb{P})$ be a complete filtered probability space, let Q be a positive self-adjoint operator, and let W^Q be a U-valued Q-Wiener process defined on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}^+},\mathbb{P})$ (see [24, Ch. 4]). Denote by $L_2(U_0,H)$ the space of Hilbert-Schmidt operators from $U_0 := Q^{1/2}(U)$ ([10]) into H, let A be the generator of a strongly continuous semigroup $\{S_A(t)\}_{t\in\mathbb{R}^+}$ in $(H,|\cdot|_H)$, and let $F\colon H\to H$, $B\colon H\to L_2(U_0,H)$. Then, under suitable assumptions on the coefficients F and B (e.g., [24, p. 187, Hypotehsis 7.1]), for every $\xi\in H$, the SDE in the space H

$$\begin{cases} dX(t) = AX(t) + F(X(t))dt + B(X(t))dW^Q(t) & t\in(0,T] \\ X(0) = \xi, \end{cases} \tag{5.4.10}$$

admits a unique (up to undistinguishability) mild solution $X(\cdot,\xi)$ with continuous trajectories (see [24, p. 188, Theorem 7.2]), i.e., there exists a unique H-valued process $X(\cdot,\xi)$ with continuous trajectories satisfying the integral equation

$$X(t,\xi) = S_A(t)\xi + \int_0^t S_A(t-s)F(X(s,\xi))ds + \int_0^t S_A(t-s)B(X(s,\xi))dW^Q(s) \qquad \forall t\in\mathbb{R}^+.$$

By standard estimates (see, e.g., [24, p. 188, Theorem 7.2] ([11])), for every $p\geq 2$ we have, for some $K_p>0$ and $\hat{\alpha}_p\in\mathbb{R}$,

$$\mathbb{E}\left[|X(t,\xi)|_H^p\right] \leq K_p e^{\hat{\alpha}_p t}(1+|\xi|_H^p) \qquad \forall (t,\xi)\in\mathbb{R}^+\times H. \tag{5.4.11}$$

Moreover, by [24, p. 235, Theorem 9.1],

$$(t,\xi) \mapsto X(t,\xi) \text{ is stochastically continuous.} \tag{5.4.12}$$

[10]The scalar product on U_0 is defined by $\langle u,v\rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v\rangle_H$.

[11]The constant in that estimate can be taken exponential in time, because the SDE is autonomous.

Proposition 5.4.4. *Let [24, Hypothesis 7.1] hold and let $X(\cdot,\xi)$ be the mild solution to* (5.4.10).

(i) Define

$$T_t f(\xi) := \mathbb{E}[f(X(t,\xi))] \qquad \forall f \in C_b(H)\ \forall \xi \in H,\ \forall t \in \mathbb{R}^+. \tag{5.4.13}$$

Then $T := \{T_t\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially locally equicontinuous semigroup in $(C_b(H),\tau_{\mathscr{K}})$. Moreover, $\{e^{-\alpha t}T_t\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially equicontinuous semigroup in $(C_b(H),\tau_{\mathscr{K}})$ for every $\alpha > 0$.

(ii) Let $p \geq 2$ and set $\gamma(\xi) := (1+|\xi|_H^p)^{-1}$ for $\xi \in H$. Define

$$T_t f(\xi) := \mathbb{E}[f(X(t,\xi))] \qquad \forall f \in C_\gamma(H),\ \forall \xi \in H,\ \forall t \in \mathbb{R}^+. \tag{5.4.14}$$

Then $T := \{T_t\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially locally equicontinuous semigroup in $(C_\gamma(H),\tau^\gamma_{\mathscr{K}})$. Moreover, $\{e^{-\alpha t}T_t\}_{t\in\mathbb{R}^+}$ is a C_0-sequentially equicontinuous semigroup in $(C_\gamma(H),\tau^\gamma_{\mathscr{K}})$ for every $\alpha > \hat{\alpha}_p$, where $\hat{\alpha}_p$ is the constant appearing in (5.4.11).

Proof. (i) Define

$$\mu_t(\xi,\Gamma) := \mathbb{P}(X(t,\xi) \in \Gamma) \qquad \forall t \in \mathbb{R}^+,\ \forall \xi \in H,\ \forall \Gamma \in \mathscr{B}(H). \tag{5.4.15}$$

We show that we can apply Proposition 5.4.2 with the family $\boldsymbol{\mu} := \{\mu_t(\xi,\cdot)\}_{\substack{t\in\mathbb{R}^+\\ \xi\in H}}$ given by (5.4.15). The condition of Assumption 5.4.1*(i)* is clearly verified. The condition of Assumption 5.4.1*(ii)* is consequence of (5.4.12). The condition of Assumption 5.4.1*(iii)* is verified by [24, p. 249, Corollaries 9.15 and 9.16].

Now we verify the condition of Assumption 5.4.1*(iv)*. Let $\hat{t} > 0$ and let $K \subset E$ compact. By (5.4.12) the map

$$\mathbb{R}^+ \times H \to (\mathbf{ca}(H), \sigma(\mathbf{ca}(H), C_b(H))),\ \ (t,\xi) \mapsto \mu(\xi,\cdot)$$

is continuous. Then the family of probability measures $\{\mu_t(\xi,\cdot)\}_{(t,\xi)\in[0,\hat{t}]\times H}$ is $\sigma(\mathbf{ca}(H),C_b(H))$-compact. Hence, by [1, p. 519, Theorem 15.22], it is tight.

We finally verify the condition of Assumption 5.4.1*(v)*. Let $r > 0$, let $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be a sequence converging to 0, and let $\{\xi_n\}_{n\in\mathbb{N}}$ be sequence converging to ξ in H. By (5.4.12) and recalling that $X(0,\xi) = \xi$, we get

$$\lim_{n\to+\infty} \mu_{t_n}(\xi_n, B(\xi_n,r)) = \lim_{n\to+\infty} \mathbb{P}(|\xi_n - X(t_n,\xi_n)|_H < r) = 0.$$

By arbitrariness of the sequences $\{t_n\}_{n\in\mathbb{N}}$, $\{\xi_n\}_{n\in\mathbb{N}}$ and of ξ, this implies the condition of Assumption 5.4.1*(v)*.

(ii) First, we notice that $T_t f$ in (5.4.14) is well defined due to (5.4.11). Consider now the family $\boldsymbol{\mu} := \{\mu_t(\xi,\cdot)\}_{\substack{t\in\mathbb{R}^+\\ \xi\in H}}$ defined in (5.4.15) and the renormalized family $\boldsymbol{\nu} := \{\nu_t(\xi,\cdot)\}_{\substack{t\in\mathbb{R}^+\\ \xi\in H}}$ defined by $\nu_t(\xi,\cdot) := e^{-\hat{\alpha}_p t}\mu_t$. Then, consider the weighted family

$$\boldsymbol{\nu}^\gamma := \{\nu_t^\gamma(\xi,\cdot)\}_{\substack{t\in\mathbb{R}^+\\ \xi\in H}}$$

defined by

$$\nu_t^\gamma(\xi,\Gamma) := \frac{1}{1+|\xi|^p}\int_\Gamma (1+|\xi'|^p)\nu_t(\xi,d\xi') \qquad \forall\Gamma\in\mathscr{B}(H),\ \forall\xi\in H.$$

We have

$$T_t f(\xi) = e^{\hat{\alpha}_p t}\int_H f(\xi')\nu_t(\xi,d\xi') \qquad \forall f\in C_\gamma(H),\ \forall\xi\in H,\ \forall t\in\mathbb{R}^+.$$

Hence, by Proposition 5.4.3, the proof reduces to show that Assumption 5.4.1 is verified by $\boldsymbol{\nu}^\gamma$. The latter follows straightly from its definition by taking into account the properties already proved for $\boldsymbol{\mu}$ in part *(i)* of the proof and (5.4.11). ■

Example 5.4.5. Let H be a non-trivial separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $Q\in L(H)$ be a positive self-adjoint trace-class operator and let W^Q be a Q-Wiener process in H on some filtered probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}^+},\mathbb{P})$ (see [24, Ch. 4]). Let $T=\{T_t\}_{t\in\mathbb{R}^+}$ be defined by

$$T_t f(\xi) := \mathbb{E}[f(\xi+W_t^Q)] = \int_H f(\xi')\mu_t(\xi,d\xi') \qquad \forall f\in C_b(H),\ \forall\xi\in H,\ \forall t\in\mathbb{R}^+,$$

where $\mu_t(\xi,\cdot)$ denotes the law of $\xi+W_t^Q$. Then, by Proposition 5.4.4, T is a C_0-sequentially locally equicontinuous semigroup in $(C_b(H),\tau_{\mathscr{K}})$. We claim that T is not locally equicontinuous. Indeed, if T was locally equicontinuous, for any fixed $\hat{t}>0$, there should exist $L>0$, a compact set $K\subset H$, and $\eta_1,\dots,\eta_n\in\mathbf{ca}(H)$ such that

$$\sup_{t\in[0,\hat{t}]}|T_t f(0)| \le L\left([f]_K + \sum_{i=1}^n\left|\int_H f\,d\eta_i\right|\right) \qquad \forall f\in C_b(H). \tag{5.4.16}$$

Let $v\in H\setminus\{0\}$ and let $a:=\max_{h\in K}|\langle v,h\rangle|$. Then, denoting by λ_t the pushforward measure of $\mu_t(0,\cdot)$ through the application $\langle v,\cdot\rangle$ (i.e. the law of the real-valued random variable $\langle v,W_t^Q\rangle$), and by ν_i, $i=1,\dots,n$, the pushforward measure of η_i through the same application, inequality (5.4.16) provides, in particular,

$$\sup_{t\in[0,\hat{t}]}\left|\int_a^{+\infty} g\,d\lambda_t\right| \le L\sum_{i=1}^n\left|\int_a^{+\infty} g\,d\nu_i\right|, \qquad \forall g\in C_{0,b}([a,+\infty)), \tag{5.4.17}$$

where $C_{0,b}([a,+\infty))$ is the space of bounded continuous functions f on $[a,+\infty)$ such that $f(a)=0$. Then, by [91, p. 63, Lemma 3.9], every λ_t restricted to $(a,+\infty)$ must be a linear combination of the measures $\nu_1 \dots, \nu_n$ restricted to $(a, +\infty)$. In particular, choosing any sequence $0 < t_1 < \dots < t_n < t_{n+1} \leq \hat{t}$, the family $\{\lambda_{t_i} \lfloor (a, +\infty)\}_{i=1,\dots,n+1}$ is linearly dependent. This is not possible, as they are restrictions of nondegenerate Gaussian laws having all different variances.

Remark 5.4.6. In this subsection we have considered a Hilbert space setting, as the theory of SDEs in Hilbert spaces is very well developed and the properties of their solutions allow to state our results for a large class of SDEs. Nevertheless, the same kind of results hold for suitable classes of SDEs in Banach spaces (see e.g. [50]).

Appendix A

Stochastic Fubini's theorem and stochastic convolution

In this appendix we prove a stochastic Fubini's theorem and apply it to obtain existence of predictable/continuous versions of stochastic convolutions.

We do not choose any particular stochastic integrator. We look at the stochastic integration simply as a linear and continuous operator $\mathfrak{L}$ from an L^p space of Banach space-valued processes, the stochastically integrable processes, to another L^p space, containing functions whose values are the paths of the stochastic integrals. The paths do not need to be continuous. Within this setting, the continuity assumption on $\mathfrak{L}$ plays the role of Itō's isometry or of the Burkholder-Davis-Gundy inequality in the standard construction of stochastic integrals with respect to square integrable continuous martingales.

For such an operator $\mathfrak{L}$, we prove the stochastic Fubini's theorem (Theorem A.1.3). The result can be applied e.g. to stochastic integration in infinite dimensional spaces with respect to L^p-integrable martingales ([83, Ch. 8]) or more general martingale-valued measures (for the finite dimensional case, see e.g. [2, Ch. 4]), generalizing standard results as [24, Theorem 4.33], [48, Theorem 2.8], [83, Theorem 8.14].

Secondly, we particularize the study to the case in which $\mathfrak{L}$ is defined on a space of $L_2(U,H)$-valued processes, where U,H are separable Hilbert spaces and $L_2(U,H)$ is the vector space of Hilbert-Schmidt operators from U into H. Denote $\mathfrak{L}$ by $\mathfrak{I}$, in this particular case. For a strongly continuous map $R\colon (0,T] \to L(H)$ and for a process $\Phi\colon \Omega \times [0,T] \to L(U,H)$ such that the composition $\mathbf{1}_{(0,t]}(\cdot)R(t-\cdot)\Phi$ belongs to the domain of $\mathfrak{I}$, we consider the convolution process

$$(\mathfrak{I}(\mathbf{1}_{[0,t)}(\cdot)R(t-\cdot)\Phi))_t \qquad t \in [0,T]. \tag{A.0.1}$$

By using the stochastic Fubini's theorem, we show that (A.0.1) admits a jointly measurable version (Theorem A.2.5). The joint measurability of the stochastic convolution is of interest e.g. when its paths must be integrated, as it happens in the factorization formula

([24, Theorem 5.10]). We also provide a characterisation of the measurability needed by functions $\Phi\colon \Omega\times[0,T]\to L(U,H)$ in order that $\mathbf{1}_{(0,t]}(\cdot)R(t-\cdot)\Phi$ has the necessary measurability required by the operator $\mathfrak{J}$ (Theorem A.2.10). This measurability result turns out to be useful e.g. in order to understand what are the most general measurability conditions for coefficients of stochastic differential equations in Hilbert spaces for which mild solutions are considered.

Finally, in case $\mathfrak{J}$ takes values in a space of processes with continuous paths and $R=S$ is a C_0-semigroups, by adapting the factorization method to the present setting, we show that (A.0.1) admits a continuous version (Theorem A.2.13).

A.1 Stochastic Fubini's theorem

Throughout this section, $(G,\mathscr{G},\mu)$ and $(D_2,\mathscr{D}_2,\nu_2)$ are positive finite measure spaces, $(D_1,\mathscr{D}_1)$ is a measurable space, and ν_1 is a kernel from D_2 to D_1, i.e.

$$\nu_1\colon \mathscr{D}_1\times D_2\to\mathbb{R}^+$$

is such that

(i) $\nu_1(A,\cdot)$ is $\mathscr{D}_2$-measurable, for all $A\in\mathscr{D}_1$;

(ii) $\nu_1(\cdot,x)$ is a positive measure, for all $x\in D_2$.

We assume that

$$C:=\int_{D_2}\nu_1(D_1,x)\nu_2(dx)<\infty.$$

Let $D:=D_1\times D_2$. On $(D,\mathscr{D}_1\otimes\mathscr{D}_2)$, we define the meaure ν by

$$\nu(A):=\int_{D_2}\left(\int_{D_1}\mathbf{1}_A(x_1,x_2)\nu_1(dx_1,x_2)\right)\nu_2(dx_2),\qquad \forall A\in\mathscr{D}_1\otimes\mathscr{D}_2.$$

Notice that $\nu(D)=C$ is finite.

Let $\mathscr{D}$ be a given sub-σ-algebra of $\mathscr{D}_1\otimes\mathscr{D}_2$. When we consider measurability or integrability with respect to G (resp. D_1, D_2, $D_1\times D_2$, D), we always mean it with respect to the space $(G,\mathscr{G},\mu)$ (resp. $(D_1,\mathscr{D}_1,\nu_1)$, $(D_2,\mathscr{D}_2,\nu_2)$, $(D,\mathscr{D},\nu)$). According to that, if we write, for example $L^1(D,V)$, for some Banach space V, we mean $L^1((D,\mathscr{D},\nu),V)$, and similarly for other spaces of integrable functions on G, D_1, D_2, D.

Let E be a given Banach space. For $p,q\in[1,\infty)$, we denote by $L^{p,q}_{\mathscr{D}}(E)$ the space of measurable functions $f\colon(D,\mathscr{D})\to E$ such that

(i) there exists $N\in\mathscr{D}$ such that $\nu(N)=0$ and $f(G\setminus N)$ is separable;

(ii) the following integrability condition holds:

$$|f|_{p,q}:=\left(\int_{D_2}\left(\int_{D_1}|f(x,y)|_E^p d\nu_1(dx,y)\right)^{q/p}\nu_2(dy)\right)^{1/q}<\infty.$$

It is not difficult to see that $(L^{p,q}_{\mathscr{D}}(E), |\cdot|_{L^{p,q}_{\mathscr{D}}(E)})$ is a Banach space, with the usual identification $f = g$ if and only if $f = g$ ν-a.e.. Indeed, if $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^{p,q}_{\mathscr{D}}(E)$, then it is Cauchy also in $L^1((D,\mathscr{D},\nu),E)$. Passing to a subsequence if necessary, we may assume that $f_n \to f$ ν-a.e., for some $f \in L^1((D,\mathscr{D},\nu),E)$. Now Fatou's lemma gives $f \in L^{p,q}_{\mathscr{D}}(E)$ and $f_n \to f$ in $L^{p,q}_{\mathscr{D}}(E)$.

Finally, we use the short notation $L^1(D\times G,E)$ for the space

$$L^1((D\times G,\mathscr{D}\otimes\mathscr{G},\nu\otimes\mu),E).$$

We will prove the stochastic Fubini's theorem first for simple functions and then for the general case through approximation. We need the following preparatory lemma.

Lemma A.1.1. *Let $p,q \in [1,\infty)$ and $f \in L^1(G,L^{p,q}_{\mathscr{D}}(E))$. If $q>1$, assume that*

$$C(p,q) := \left(\int_{D_2} (\nu_1(D_1,x))^{\frac{q(p-1)}{p(q-1)}}\,\nu_2(dx)\right)^{\frac{q-1}{q}} < \infty. \tag{A.1.1}$$

If $q=1$ and $p>1$, assume that

$$C(p,1) := \sup_{x\in D_2} (\nu_1(D_1,x))^{\frac{p-1}{p}} < \infty. \tag{A.1.2}$$

Define $C(1,1) := 1$. Then there exist measurable functions

$$\tilde{f}\colon (D\times G,\mathscr{D}\otimes\mathscr{G}) \to E \tag{A.1.3}$$

$$\tilde{f}_n\colon (D\times G,\mathscr{D}\otimes\mathscr{G}) \to E,\ n\in\mathbb{N} \tag{A.1.4}$$

such that

$$\tilde{f}(\cdot,y) \in L^{p,q}_{\mathscr{D}}(E),\ \forall y\in G, \tag{A.1.5}$$

$$G \to L^{p,q}_{\mathscr{D}}(E),\ y\mapsto \tilde{f}(\cdot,y) \text{ is measurable} \tag{A.1.6}$$

$$\tilde{f}(\cdot,y) = f(y) \text{ in } L^{p,q}_{\mathscr{D}}(E)\ \mu\text{-a.e. } y\in G, \tag{A.1.7}$$

$$\tilde{f}_n(\cdot,y) \in L^{p,q}_{\mathscr{D}}(E),\ \forall y\in G,\ \forall n\in\mathbb{N}, \tag{A.1.8}$$

$$G \to L^{p,q}_{\mathscr{D}}(E),\ y\mapsto \tilde{f}_n(\cdot,y) \text{ is a simple function, } \forall n\in\mathbb{N}, \tag{A.1.9}$$

$$\lim_{n\to\infty}\int_G \left(\int_{D_2}\left(\int_{D_1} |\tilde{f}_n((x_1,x_2),y) - \tilde{f}((x_1,x_2),y)|_E^p\,\nu_1(dx_1,x_2)\right)^{q/p}\nu_2(dx_2)\right)^{1/q}\mu(dy) = 0 \tag{A.1.10}$$

$$\tilde{f}(x,\cdot) \in L^1(G,E),\ \forall x\in D, \tag{A.1.11}$$

$$D \to L^1(G,E),\ x\mapsto \tilde{f}(x,\cdot),\ \text{belongs to } L^{p,q}(D,L^1(G,E)) \tag{A.1.12}$$

$$\tilde{f}_n(x,\cdot) \in L^1(G,E),\ \forall x \in D,\ \forall n \in \mathbb{N}, \tag{A.1.13}$$

$$D \to L^1(G,E),\ x \mapsto \tilde{f}_n(x,\cdot),\ \textit{belongs to}\ L^{p,q}\left(D,L^1(G,E)\right) \tag{A.1.14}$$

$$\lim_{n\to\infty}\int_{D_2}\left(\int_{D_1}\left|\tilde{f}_n((x_1,x_2),\cdot)-\tilde{f}((x_1,x_2),\cdot)\right|^p_{L^1(G,E)}\nu_1(dx_1,x_2)\right)^{q/p}\nu_2(dx_2)=0. \tag{A.1.15}$$

Proof. Since f is Bochner integrable, without loss of generality we can assume that $f(G)$ is separable. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of $L^{p,q}_{\mathscr{D}}(E)$-valued simple functions such that

$$\lim_{n\to\infty} f_n(y) = f(y) \text{ in } L^{p,q}_{\mathscr{D}}(E),\ \forall y \in G, \tag{A.1.16}$$

$$\lim_{n\to\infty} |f_n - f|_{L^1\left(G,L^{p,q}_{\mathscr{D}}(E)\right)} = 0. \tag{A.1.17}$$

Each f_n can be written in the form

$$f_n(y) = \sum_{i=1}^{M(n)} \mathbf{1}_{A_i^n}(y)\varphi_{n,i} \qquad \forall y \in G, \tag{A.1.18}$$

where $M(n) \in \mathbb{N}$, $A_i^n \in \mathscr{G}$, and $\varphi_{n,i}$ is a fixed representant of its equivalence class in $L^{p,q}_{\mathscr{D}}(E)$. For $n \in \mathbb{N}$, define

$$\tilde{f}_n\colon (D\times G, \mathscr{D}\otimes\mathscr{G}) \to E,\ (x,y) \mapsto f_n(y)(x).$$

By using (A.1.18), we have the measurability of (A.1.4), and (A.1.8), (A.1.9), (A.1.13), (A.1.14), are immediately verified.

We claim that the sequence $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^1(D\times G,E)$. Indeed, since $\varphi_{n,i} \in L^{p,q}_{\mathscr{D}}(E)$, we have $\tilde{f}_n \in L^1(D\times G,E)$, for every $n \in \mathbb{N}$. Moreover, by Hölder's inequality,

$$\begin{aligned}
&\int_{D\times G} |\tilde{f}_n - \tilde{f}_m|_E\, d(\nu\otimes\mu) = \\
&\quad = \int_G \left(\int_{D_2}\left(\int_{D_1} |\tilde{f}_n((x_1,x_2),y) - \tilde{f}_m((x_1,x_2),y)|_E \nu_1(dx_1,x_2)\right)\nu_2(dx_2)\right)\mu(dy) \\
&\quad \le C(p,q)\int_G \left(\int_{D_2}\left(\int_{D_1} |\tilde{f}_n((x_1,x_2),y) - \tilde{f}_m((x_1,x_2),y)|^p_E \nu_1(dx_1,x_2)\right)^{q/p}\nu_2(dx_2)\right)^{1/q}\mu(dy) \\
&\quad = C(p,q)|f_n - f_m|_{L^1\left(G,L^{p,q}_{\mathscr{D}}(E)\right)},
\end{aligned}$$

and the last member tends to 0 as n and m tend to ∞, by (A.1.17). Then there exists $\tilde{f} \in L^1(D\times G,E)$ such that, after replacing $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ by a subsequence if necessary,

$$\lim_{n\to\infty} \tilde{f}_n(x,y) = \tilde{f}(x,y) \qquad \forall (x,y) \in (D\times G)\setminus N \tag{A.1.19}$$

$$\lim_{n\to\infty} \tilde{f}_n = \tilde{f} \qquad \text{in } L^1(D\times G,E), \tag{A.1.20}$$

where N is a $\nu\otimes\mu$-null set. We redefine $\tilde{f}$ on N by $\tilde{f}(x,y) := 0$ for $(x,y) \in N$. After such a redefinition, the partial results of the theorem till now proved still hold true.

By (A.1.19), since we can assume that each $\varphi_{n,i}$ has separable range, we see that the range of $\tilde{f}$ is separable. By measurability of sections of real-valued measurable functions and by Pettis's measurability theorem (use the fact that the range of $\tilde{f}$ is separable and then use Hahn-Banach theorem to extend continuous linear functionals on the space generated by the range of $\tilde{f}$ to the whole space E), we have that

$$(D,\mathscr{D}) \to E,\ x \mapsto \tilde{f}(x,y') \quad \text{and} \quad (G,\mathscr{G}) \to E,\ y \mapsto \tilde{f}(x',y)$$

are measurable, for all $y' \in G$ and all $x' \in D$. Since

$$\begin{aligned}
&\int_G \liminf_{m\to\infty} \left|\tilde{f}(\cdot,y) - \tilde{f}_m(\cdot,y)\right|_{p,q} \mu(dy) \\
&\quad \le \lim_{m\to\infty} \int_G \left(\int_{D_2} \left(\int_{D_1} \left|\tilde{f}((x_1,x_2),y) - \tilde{f}_m((x_1,x_2),y)\right|_E^p \nu_1(dx_1,x_2) \right)^{q/p} \nu_2(dx_2) \right)^{1/q} \mu(dy) \\
&\qquad \le \lim_{m\to\infty} \liminf_{n\to\infty} |f_n - f_m|_{L^1(G,L^{p,q}_{\mathscr{D}}(E))} = 0, \qquad \text{(A.1.21)}
\end{aligned}$$

we have

$$\liminf_{m\to\infty} \left|\tilde{f}(\cdot,y) - \tilde{f}_m(\cdot,y)\right|_{p,q} = 0 \qquad \mu\text{-a.e. } y \in G. \tag{A.1.22}$$

By recalling that $\tilde{f}_n(\cdot,y) \in L^{p,q}_{\mathscr{D}}(E)$ for all $y \in G$, (A.1.22) shows that the map

$$D \to E,\ x \mapsto \tilde{f}(x,y)$$

belongs to $L^{p,q}_{\mathscr{D}}(E)$ for all $y \in G \setminus N'$, where N' is a μ-null set. We redefine $\tilde{f}$ on N' by $\tilde{f}(x,y) := 0$ for $(x,y) \in D \times N'$. Again, we notice that the partial results of the theorem till now proved still hold true after the redefinition on $D \times N_1$. In addition,

$$\forall y \in G, \text{ the map } D \to E,\ x \mapsto \tilde{f}(x,y), \text{ belongs to } L^{p,q}_{\mathscr{D}}(E).$$

This provides (A.1.5). Moreover, since N' can be chosen such that (A.1.22) holds for all $y \in G \setminus N'$ and since $G \to L^{p,q}_{\mathscr{D}}(E)$, $y \mapsto \tilde{f}_n(\cdot,y) = f_n(y)$, is measurable, for all $n \in \mathbb{N}$, also (A.1.6) is proved. From the last inequality of (A.1.21), (A.1.10) follows. From (A.1.16) and (A.1.22), (A.1.7) follows as well.

By Hölder's inequality, we have $|\tilde{f}|_{L^1(D\times G,E)} \le C(p,q)|\tilde{f}|_{L^1(G,L^{p,q}_{\mathscr{D}}(E))} < \infty$. Then, after redefining $\tilde{f}$ on a set $N'' \times G$, where N'' is a ν-null set, by $\tilde{f}(x,y) := 0$ for $(x,y) \in N'' \times G$, we have

$$\forall x \in D, \text{ the map } G \to E,\ y \mapsto \tilde{f}(x,y), \text{ belongs to } L^1(G,E).$$

This provides (A.1.11). By applying Minkowski's inequality for integrals twice (see [45,

p. 194, 6.19]), we have

$$\lim_{n\to\infty}\left(\int_{D_2}\left(\int_{D_1}\left(\int_G |\tilde f((x_1,x_2),y)-\tilde f_n((x_1,x_2),y)|_E\,\mu(dy)\right)^p \nu_1(dx_1,x_2)\right)^{q/p}\nu_2(dx_2)\right)^{1/q}$$

$$\le \lim_{n\to\infty}\left(\int_{D_2}\left(\int_G\left(\int_{D_1} |\tilde f((x_1,x_2),y)-\tilde f_n((x_1,x_2),y)|_E^p\,\nu_1(dx_1,x_2)\right)^{1/p}\mu(dy)\right)^{q}\nu_2(dx_2)\right)^{1/q}$$

$$\le \lim_{n\to\infty}\int_G\left(\int_{D_2}\left(\int_{D_1} |\tilde f((x_1,x_2),y)-\tilde f_n((x_1,x_2),y)|_E^p\,\nu_1(dx_1,x_2)\right)^{q/p}\nu_2(dx_2)\right)^{1/q}\mu(dy).$$

Since the latter member tends to 0 because of the second inequality in (A.1.21), the estimate above provides (A.1.12) and (A.1.15), after redefining $\tilde f$ on a set $N''' \times G$, where N''' is a suitably chosen ν-null set, by $\tilde f(x,y) := 0$ for $(x,y)\in N'''\times G$. ■

Let $T>0$ and let $\mathscr{B}_T$ be a short notation for the Borel σ-algebra $\mathscr{B}_{[0,T]}$ on $[0,T]$. We recall that, if $\mathscr{T}$ is a topological space, then $\mathscr{B}_{\mathscr{T}}$ denotes the Borel σ-algebra of $\mathscr{T}$ ([1]). Let $\left(\Omega,\mathscr{F},\mathbb{F}:=\{\mathscr{F}_t\}_{t\in[0,T]},\mathbb{P}\right)$ be a complete filtered probability space. We endow the product space $\Omega_T:=\Omega\times[0,T]$ with the σ-algebra $\mathscr{P}_T$ of predictables sets associated to the filtration $\mathbb{F}$ and the measurable space $(\Omega_T,\mathscr{P}_T)$ with the product measure $\mathbb{P}\otimes m$, where m denotes the Lebesgue's measure. We need to introduce some further notation.

- F is a Banach space;
- $\mathbb{T}\subset B_b([0,T],F)$ is a closed subspace (with respect to the norm $|\cdot|_\infty$) such that

$$\mathbb{T}\times[0,T]\to F,\ (\mathbf{x},t)\mapsto \mathbf{x}(t); \tag{A.1.23}$$

  is Borel measurable, when $\mathbb{T}\times[0,T]$ is endowed with the product σ-algebra $\mathscr{B}_{\mathbb{T}}\otimes\mathscr{B}_{[0,T]}$ (and not just with the Borel σ-algebra of the product topology!).
- $\mathscr{P}'$ is a given sub-σ-algebra of $\mathscr{F}_T\otimes\mathscr{B}_T$ such that, for all $A\in\mathscr{F}_T$ with $\mathbb{P}(A)=0$, $A\times[0,T]\in\mathscr{P}'$.
- $\mathscr{L}^0_{\mathscr{P}'}(\mathbb{T})$ is the vector space of measurable functions

$$X:(\Omega_T,\mathscr{P}')\to F$$

  such that, for $\mathbb{P}$-a.e. $\omega\in\Omega$, the path

$$X(\omega)\colon[0,T]\to F,\ t\mapsto X_t(\omega)$$

  belongs to $\mathbb{T}$, and the $\mathbb{P}$-a.e. defined map

$$(\Omega,\mathscr{F}_T)\to\mathbb{T},\ \omega\mapsto X(\omega) \tag{A.1.24}$$

  is measurable, when $\mathbb{T}$ is endowed with the Borel σ-algebra induced by the norm $|\cdot|_\infty$.

[1]No topological space will be denoted by T, hence there will not be any confusion with $\mathscr{B}_T$.

- For $r \in [1,\infty)$, $\mathscr{L}^r_{\mathscr{P}'}(\mathbb{T})$ denotes the space of (equivalence classes of) $X \in \mathscr{L}^0_{\mathscr{P}'}(\mathbb{T})$ such that (A.1.24) has separable range and

$$|X|_{\mathscr{L}^r_{\mathscr{P}'}(\mathbb{T})} := \left(\mathbb{E}\left[|X|^r_\infty\right]\right)^{1/r} < \infty.$$

Then $(\mathscr{L}^r_{\mathscr{P}'}(\mathbb{T}), |\cdot|_{\mathscr{L}^r_{\mathscr{P}'}(\mathbb{T})})$ is a Banach space.

Remark A.1.2. The space $\mathbb{T}$ can be e.g. $C_b([0,T],F)$, because in such a case (A.1.23) is continuous, hence measurable. This permits also to consider $\mathbb{T}$ as the space of left-limited right-continuous functions, because, if φ is real valued and continuous with support $[0,1]$ and if $\varphi_\varepsilon(t) = \varepsilon^{-1}\varphi(\varepsilon^{-1}t)$, then $\varphi_\varepsilon * \mathbf{x}$ converges pointwise to $\mathbf{x}$ everywhere on $[0,T]$ as $\varepsilon \to 0^+$, after extending $\mathbf{x}$ by continuity beyond T. We observe also that (A.1.23) is measurable whenever $\mathbb{T}$ is separable. To see it, let $\mathscr{U} \subset F$ be an open set, and let $\{\mathbf{x}_i\}_{i\in\mathbb{N}}$ be a dense subset of $\mathbb{T}$. If we define

$$\mathscr{U}_n := \{x \in F : B_F(x, 2^{-n}) \subset F\},$$

where $B_F(x,2^{-n})$ is the open ball in F centered in x with radius 2^{-n}, then

$$\begin{aligned}\{(\mathbf{x},t) \in \mathbb{T} \times [0,T] : \mathbf{x}(t) \in \mathscr{U}\} &= \bigcup_{n\in\mathbb{N}}\bigcup_{i\in\mathbb{N}} \left\{(\mathbf{x},t) \in \mathbb{T}\times[0,T] : |\mathbf{x}-\mathbf{x}_i|_\infty < 2^{-n},\ \mathbf{x}_i(t) \in \mathscr{U}_n\right\}\\ &= \bigcup_{n\in\mathbb{N}}\bigcup_{i\in\mathbb{N}} B_{\mathbb{T}}(\mathbf{x}_i, 2^{-n}) \times \left(\mathbf{x}_i^{-1}(\mathscr{U}_n)\right) \in \mathscr{B}_{\mathbb{T}} \otimes \mathscr{B}_T,\end{aligned}$$

where $B_{\mathbb{T}}(\mathbf{x}_i, 2^{-n})$ is the open ball in $\mathbb{T}$ centered in $\mathbf{x}_i$ with radius 2^{-n}.

We now provide the main result of this section.

Theorem A.1.3 (Stochastic Fubini's theorem)**.** *Let $p,q,r \in [1,\infty)$, $g \in L^1(G, L^{p,q}_{\mathscr{D}}(E))$. Let*

$$\mathfrak{L} \colon L^{p,q}_{\mathscr{D}}(E) \to \mathscr{L}^r_{\mathscr{P}'}(\mathbb{T})$$

be a linear and continuous operator. Then there exist measurable functions

$$X_1 \colon (D \times G, \mathscr{D} \otimes \mathscr{G}) \to E$$
$$X_2 \colon (\Omega_T \times G, (\mathscr{F}_T \otimes \mathscr{B}_T) \otimes \mathscr{G}) \to F$$

such that

$$X_1(x,\cdot) \in L^1(G,E),\ \forall x \in D, \text{ and } X_2((\omega,t),\cdot) \in L^1(G,F),\ \forall (\omega,t) \in \Omega_T$$
$$D \to L^1(G,E),\ x \mapsto X_1(x,\cdot),\ \in L^{p,q}_{\mathscr{D}}(L^1(G,E))$$
$$(\Omega_T, \mathscr{P}_T) \to L^1(G,F),\ (\omega,t) \mapsto X_2((\omega,t),\cdot),\ \text{is measurable}$$
$$X_1(\cdot,y) \in L^{p,q}_{\mathscr{D}}(E),\ \forall y \in G$$
$$G \to L^{p,q}_{\mathscr{D}}(E),\ y \mapsto X_1(\cdot,y),\ \in L^1(G, L^{p,q}_{\mathscr{D}}(E))$$

$$X_1(\cdot, y) = g(y) \text{ in } L^{p,q}_{\mathcal{D}}(E) \text{ for } \mu\text{-a.e. } y \in G$$
$$X_2(\cdot, y) \in \mathscr{L}^r_{\mathscr{P}'}(\mathbb{T}), \ \forall y \in G$$
$$X_2(\cdot, y) = \mathfrak{L}g(y) \text{ in } \mathscr{L}^r_{\mathscr{P}'}(\mathbb{T}), \ \mu\text{-a.e. } y \in G \tag{A.1.25a}$$

and such that

$$\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \ (\mathfrak{L}Y)(\omega, t) = \int_G X_2((\omega, t), y)\mu(dy), \ \forall t \in [0, T], \tag{A.1.26}$$

where

$$Y(x) := \int_G X_1(x, y)\mu(dy), \ \forall x \in D.$$

Proof. By Lemma A.1.1 , there exist measurable functions

$$\tilde{f}: (D \times G, \mathscr{D} \otimes \mathscr{G}) \to E,$$
$$\tilde{f}_n: (D \times G, \mathscr{D} \otimes \mathscr{G}) \to E, \qquad n \in \mathbb{N}$$

satisfying (A.1.11)–(A.1.15). For $n \in \mathbb{N}$, $\tilde{f}_n$ has the form

$$\tilde{f}_n(x, y) = \sum_{i=1}^{M(n)} \mathbf{1}_{A_i^n}(y)\varphi_{n,i}(x) \qquad \forall x \in D, \ \forall y \in G,$$

where $\varphi_{n,i}$ is a fixed representant of its class in $L^{p,q}_{\mathscr{D}}(E)$. For all $n \in \mathbb{N}$, the function $\tilde{f}_n^{(\mu)}$ defined by

$$\tilde{f}_n^{(\mu)}: D \to E \ x \mapsto \int_G \tilde{f}_n(x, y)\mu(dy) = \sum_{i=1}^{M(n)} \varphi_{n,i}(x)\mu(A_i^n)$$

belongs to $L^{p,q}_{\mathscr{D}}(E)$. Then, if we define

$$\tilde{f}^{(\mu)}: D \to E, \ x \mapsto \int_G \tilde{f}(x, y)\mu(dy),$$

due to (A.1.15), we have

$$\lim_{n\to\infty} \tilde{f}_n^{(\mu)} = \tilde{f}^{(\mu)} \text{ in } L^{p,q}_{\mathscr{D}}(E). \tag{A.1.27}$$

By linearity of $\mathfrak{L}$, we have

$$\mathfrak{L}\tilde{f}_n^{(\mu)} = \sum_{i=1}^{M(n)} \mu(A_i^n)\mathfrak{L}\varphi_{n,i} \text{ in } \mathscr{L}^r_{\mathscr{P}'}(\mathbb{T}). \tag{A.1.28}$$

By continuity of $\mathfrak{L}$, (A.1.27) and (A.1.28) give

$$\lim_{n\to\infty} \sum_{i=1}^{M(n)} \mu(A_i^n)\mathfrak{L}\varphi_{n,i} = \mathfrak{L}\tilde{f}^{(\mu)} \text{ in } \mathscr{L}^r_{\mathscr{P}'}(\mathbb{T}). \tag{A.1.29}$$

For $n \in \mathbb{N}$, we now consider the measurable function

$$\tilde{f}_n^{(\mathfrak{L})}: \left(\Omega_T \times G, \mathscr{P}' \otimes \mathscr{G}\right) \to F, \ ((\omega, t), y) \mapsto \sum_{i=1}^{M(n)} \mathbf{1}_{A_i^n}(y)\left(\mathfrak{L}\varphi_{n,i}\right)(\omega, t)$$

where here $\mathfrak{L}\varphi_{n,i}$ is a fixed representant of its class in $\mathscr{L}^r_{\mathscr{P}'}(\mathbb{T})$. For all $y \in G$, $\tilde{f}_n^{(\mathfrak{L})}(\cdot, y)$ is a representant of the class of $\mathfrak{L}\left(\tilde{f}_n(\cdot, y)\right)$ in $\mathscr{L}^r_{\mathscr{P}'}(\mathbb{T})$. Moreover,

$$\int_G \tilde{f}_n^{(\mathfrak{L})}((\omega,t),y)\mu(dy) = \sum_{i=1}^{M(n)} \mu(A_i^n)\left(\mathfrak{L}\varphi_{n,i}\right)(\omega,t) \qquad \forall (\omega,t) \in \Omega_T,\ \forall n \in \mathbb{N}.$$

By (A.1.28), we obtain

$$\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega,\ \int_G \tilde{f}_n^{(\mathfrak{L})}((\omega,t),y)\mu(dy) = (\mathfrak{L}\tilde{f}_n^{(\mu)})(\omega,t)\ \forall t \in [0,T]. \tag{A.1.30}$$

We now show that we can pass to the limit in (A.1.30). By (A.1.10),

$$\lim_{n\to\infty} \int_G \left|\tilde{f}_n(\cdot,y) - \tilde{f}(\cdot,y)\right|_{L^{p,q}_{\mathscr{D}}(E)} \mu(dy) = 0,$$

hence, by continuity of $\mathfrak{L}$,

$$\lim_{n\to\infty} \int_G \left|\mathfrak{L}\left(\tilde{f}_n(\cdot,y)\right) - \mathfrak{L}\left(\tilde{f}(\cdot,y)\right)\right|_{\mathscr{L}^r_{\mathscr{P}'}(\mathbb{T})} \mu(dy) = 0. \tag{A.1.31}$$

Since $\mathscr{L}^r_{\mathscr{P}'}(\mathbb{T})$ is a closed subspace of

$$L^r(\Omega,\mathbb{T}) := L^r\left((\Omega,\mathscr{F}_T,\mathbb{P}),(\mathbb{T},|\cdot|_\infty)\right),$$

the map

$$(G,\mathscr{G}) \to L^r(\Omega,\mathbb{T}),\ y \mapsto \mathfrak{L}\left(\tilde{f}(\cdot,y)\right) \tag{A.1.32}$$

is measurable and integrable (the range of (A.1.32) is separable). By applying Lemma A.1.1 again, now to (A.1.32), we have that there exists a measurable function

$$g\colon (\Omega \times G, \mathscr{F}_T \otimes \mathscr{G}) \to \mathbb{T} \tag{A.1.33}$$

such that, for some $A \in \mathscr{G}$ with $\mu(A^c) = 0$,

$$g(\cdot,y) = \mathfrak{L}\left(\tilde{f}(\cdot,y)\right) \text{ in } L^r(\Omega,\mathbb{T}),\ \forall y \in A. \tag{A.1.34}$$

Define

$$X_2((\omega,t),y) := \begin{cases} g(\omega,y)(t) & \forall((\omega,t),y) \in \Omega_T \times A \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, since $\mathfrak{L}\left(\tilde{f}(\cdot,y)\right)$ is $\mathscr{P}'$-measurable for all $y \in G$ (by definition of $\mathfrak{L}$) and since $\mathscr{P}'$ contains the sets $N \times [0,T]$ when $N \in \mathscr{F}_T$ and $\mathbb{P}(N) = 0$, we have, by (A.1.34), that $X_2(\cdot,y)$ is $\mathscr{P}'$-measurable for all $y \in G$. Moreover, since the evaluation map (A.1.23) is assumed to be measurable, by measurability of (A.1.33) and by definition of X_2 we have that

$$X_2\colon (\Omega_T \times G, (\mathscr{F}_T \otimes \mathscr{B}_T) \otimes \mathscr{G}) \to F$$

is measurable. By (A.1.31), we can write

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega\left(\int_G\sup_{t\in[0,T]}\left|\tilde f_n^{(\mathfrak{L})}((\omega,t),y)-X_2((\omega,t),y)\right|_F\mu(dy)\right)\mathbb{P}(d\omega)\\ \leq\lim_{n\to\infty}\int_G\left(\int_\Omega\sup_{t\in[0,T]}\left|\tilde f_n^{(\mathfrak{L})}((\omega,t),y)-X_2((\omega,t),y)\right|_F^r\mathbb{P}(d\omega)\right)^{1/r}\mu(dy)\\ =\lim_{n\to\infty}\int_G\left|\mathfrak{L}\left(\tilde f_n(\cdot,y)\right)-\mathfrak{L}\left(\tilde f(\cdot,y)\right)\right|_{\mathscr{L}^r_{\mathscr{P}'}(\mathbb{T})}\mu(dy)=0,\end{aligned}\tag{A.1.35}$$

where the measurability of $|\tilde f_n^{(\mathfrak{L})}((\omega,\cdot),y)-X_2((\omega,\cdot),y)|_\infty$, jointly in (ω,y), is due to the measurability of (A.1.33), to the definition of X_2, and to the definition of $\tilde f_n^{(\mathfrak{L})}$. By (A.1.35), by considering a subsequence if necessary, it follows that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left|\int_G\tilde f_n^{(\mathfrak{L})}((\omega,t),y)\mu(dy)-\int_G X_2((\omega,t),y)\mu(dy)\right|_F=0\qquad\mathbb{P}\text{-a.e. }\omega\in\Omega.\tag{A.1.36}$$

By (A.1.28), (A.1.29), (A.1.30), and (A.1.36), we conclude that, for $\mathbb{P}$-a.e. $\omega\in\Omega$,

$$(\mathfrak{L}\tilde f^{(\mu)})(\omega,t)=\int_G X_2((\omega,t),y)\mu(dy),\ \forall t\in[0,T],$$

which provides (A.1.26), after defining $X_1:=\tilde f$. ■

A.2 Stochastic convolution

One of the contents of Theorem A.1.3 is the existence of the jointly measurable function X_2, whose sections $X_2(\cdot,y)$ coincide with the "stochastic integral" $\mathfrak{L}g(y)$, for a.e. y. This fact permits to obtain a jointly measurable version of a stochastic convolution, as we will explain in the present section.

Let us recall/introduce the following notation. We consider separable Hilbert spaces H and U, with scalar product $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_U$, respectively.

- $L_2(U,H)$ denotes the space of Hilbert-Schmidt linear operators from U into H.

Let E be a Banach space.

- If E is a Banach space, $L^0_{\mathscr{P}_T}(E)$ denotes the space of E-valued $\mathscr{P}_T/\mathscr{B}_E$-measurable processes $\Phi\colon\Omega_T\to E$, for which there exists $N\in\mathscr{P}_T$ with $\mathbb{P}\otimes m(N)=0$ such that $X(\Omega_T\setminus N)$ is separable. Two processes are equal in $L^0_{\mathscr{P}_T}(E)$ if they coincides $\mathbb{P}\otimes m$-a.e.. The space $L^0_{\mathscr{P}_T}(E)$ is a complete metrizable space when endowed with the topology induced by the convergence in measure (see [75, Sec. 5.2]).

- $L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(E)$ denotes the space of (equivalence classes of) E-valued $\mathscr{P}_T\otimes\mathscr{B}_T/\mathscr{B}_E$-measurable processes $\zeta\colon\Omega_T\times[0,T]\to E$, with separable range, up to a modification on a $(\mathbb{P}\otimes m)\otimes m$-null set if necessary. Two processes are equal in $L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(E)$ if they coincides $(\mathbb{P}\otimes m)\otimes m$-a.e.. $L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(E)$ is endowed with the metrizable complete vector topology induced by the convergence in measure.
- For $p,q\in[1,\infty)$, $L^{p,q}_{\mathscr{P}_T}(E)$ denotes the subspace of $L^0_{\mathscr{P}_T}(E)$ whose members X satisfy

$$|X|_{p,q}=\left(\int_0^T\left(\mathbb{E}\left[|X_t|^p_E\right]\right)^{q/p}dt\right)^{1/q}<\infty.$$

$(L^{p,q}_{\mathscr{P}_T}(E),|\cdot|_{p,q})$ is a Banach space. The space $L^{p,q}_{\mathscr{F}_T\otimes\mathscr{B}_T}(E)$ is defined similarly to $L^{p,q}_{\mathscr{P}_T}(E)$, after replacing $\mathscr{P}_T$ by $\mathscr{F}_T\otimes\mathscr{B}_T$. We use the notation $L^p_{\mathscr{P}_T}(E)$, $L^p_{\mathscr{F}_T\otimes\mathscr{B}_T}(E)$, for $L^{p,p}_{\mathscr{P}_T}(E)$, $L^{p,p}_{\mathscr{F}_T\otimes B_T}(E)$, respectively.
- For $p,q,r\in[1,\infty)$. $L^{p,q,r}_{\mathscr{P}_T\otimes\mathscr{B}_T}(E)$ denotes the space containing those $\zeta\in L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(E)$ such that

$$|\zeta|_{p,q,r}:=\left(\int_0^T\left(\int_0^T\left(\mathbb{E}\left[|\zeta((\omega,s),t)|^p_E\right]\right)^{q/p}ds\right)^{r/q}dt\right)^{1/r}<\infty. \tag{A.2.1}$$

$(L^{p,q,r}_{\mathscr{P}_T\otimes\mathscr{B}_T}(E),|\cdot|_{p,q,r})$ is a Banach space.

A.2.1 Jointly measurable version

In this section we employ Theorem A.1.3 to obtain jointly measurable versions of stochastic integrals (represented, as in the previous section, by a generic continuous linear operator $\mathfrak{I}$) depending on parameter.

We will often need to consider sections of measurable functions and their measurability with respect to some codomains. We begin with the following lemma.

Lemma A.2.1. *Let $\zeta\in L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$. Then*

$$f_\zeta\colon[0,T]\to L^0_{\mathscr{P}_T}(L_2(U,H)),\ t\mapsto\zeta(\cdot,t) \tag{A.2.2}$$

is measurable.

Proof. Let us first suppose that $U=H=\mathbb{R}$, hence $L_2(U,H)=\mathbb{R}$. Define

$$\mathscr{C}:=\left\{A\in\mathscr{P}_T\otimes\mathscr{B}_T\text{ s.t. }f_{\mathbf{1}_A}\text{ is measurable}\right\}.$$

It is clear that the rectangles of the form $B\times C$, with $B\in\mathscr{P}_T$ and $C\in\mathscr{B}_T$, belong to $\mathscr{C}$, because $f_{\mathbf{1}_{B\times C}}$ assumes only the two values 0 ans $\mathbf{1}_B$ on $\Omega_T\setminus C$ and on B, respectively. If $A\in\mathscr{C}$, $B\in\mathscr{C}$, $B\subset A$, then $f_{\mathbf{1}_{A\setminus B}}=f_{\mathbf{1}_A}-f_{\mathbf{1}_B}$ is measurable, and then $A\setminus B\in\mathscr{C}$. If $\{A_n\}_{n\in\mathbb{N}}\subset\mathscr{C}$ is an increasing sequence, then $f_{\mathbf{1}_{\cup_{n\in\mathbb{N}}A_n}}(t)=\lim_{n\to\infty}f_{\mathbf{1}_{A_n}}(t)$ in $L^0_{\mathscr{P}_T}(\mathbb{R})$ for all

$t \in [0,T]$, hence $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{C}$. This shows that $\mathscr{C}$ is a λ-class containing the rectangles $B \times C$, with $B \in \mathscr{P}_T$ and $C \in \mathscr{B}_T$, hence $\mathscr{P}_T \otimes \mathscr{B}_T \subset \mathscr{C}$. By linearity and by monotone convergence, we have that f_ζ is measurable for all $\zeta \in L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(\mathbb{R})$.

Now let U, H, be generic separable Hilbert spaces and let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L_2(U,H)$ (we consider the case $\dim L_2(U,H) = \infty$; the case $< \infty$ is similar). If $\zeta \in L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$, then, for all $t \in [0,T]$,

$$f_\zeta(t)(\omega,s) = \sum_{n\in\mathbb{N}} \langle \varphi_n, \zeta((\omega,s),t)\rangle_{L_2(U,H)} \varphi_n = \sum_{n\in\mathbb{N}} f_{\langle \varphi_n,\zeta\rangle_{L_2(U,H)}\varphi_n}(t)(\omega,s) \qquad \forall(\omega,s) \in \Omega_T.$$

From the first part of the proof, $f_{\langle \varphi_n,\zeta\rangle_{L_2(U,H)}\varphi_n}$ is measurable, after the identification $L^0_{\mathscr{P}_T}(\mathbb{R}) = L^0_{\mathscr{P}_T}(\mathbb{R}\varphi_n)$ and the continuous, hence measurable, embedding

$$L^0_{\mathscr{P}_T}(\mathbb{R}\varphi_n) \subset L^0_{\mathscr{P}_T}(L_2(U,H)).$$

We conclude that f_ζ is measurable, because it is the pointwise limit of the sequence

$$\left\{ \sum_{n=0}^{N} f_{\langle \varphi_n,\zeta\rangle_{L_2(U,H)}\varphi_n} : [0,T] \to L^0_{\mathscr{P}_T}(L_2(U,H)) \right\}_{N\in\mathbb{N}}. \qquad \blacksquare$$

Remark A.2.2. If $p,q \in [1,\infty)$, the map

$$|\cdot|_{p,q} : L^0_{\mathscr{P}_T}(L_2(U,H)) \to [0,\infty],\ \xi \mapsto |\xi|_{p,q}$$

is lower-semicontinuous (Fatou's Lemma). By Lemma A.2.1, if $\zeta \in L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$, then

$$f_\zeta : [0,T] \to L^0_{\mathscr{P}_T}(L_2(U,H)),\ t \mapsto \zeta(\cdot,t) \tag{A.2.3}$$

is measurable. By combining f_ζ with $|\cdot|_{p,q}$, we have that the set

$$B_\zeta := \left\{ t \in [0,T] \colon \zeta(\cdot,t) \in L^{p,q}_{\mathscr{P}_T}(L_2(U,H)) \right\} \tag{A.2.4}$$

is a Borel set.

Clearly the set B_ζ defined in Remark A.2.2 depends on the representant of ζ chosen in $L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H)$. Hereafter, whenever a notion associated to some function f belonging to some quotient space of mesurable functions is pointwise dependent, we mean that the notion is actually associated to a chosen representant f.

Notation. In what follows, we will always use the notation B_ζ for the set defined by (A.2.4), when $\zeta \in L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$. In the notation, we omit the dependence of B_ζ on p,q, as it will be always clear from the context.

The next result is a variant of Lemma A.2.1 for $L^{p,q,r}_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$. It will be used to derive jointly measurable versions of stochastic convolutions.

Lemma A.2.3. *Let $p,q,r \in [1,\infty)$ and let $\zeta \in L^{p,q,r}_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$. Let B_ζ be the Borel set defined by* (A.2.4). *Then $m([0,T]\setminus B_\zeta)=0$ and*

$$f_\zeta\colon B_\zeta \to L^{p,q}_{\mathcal{P}_T}(L_2(U,H)),\ t\mapsto \zeta(\cdot,t)$$

is Borel measurable.

Proof. It is clear that $m([0,T]\setminus B_\zeta)=0$, because $\zeta \in L^{p,q,r}_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$ and then $\zeta(\cdot,t)\in L^{p,q}_{\mathcal{P}_T}(L_2(U,H))$ for m-a.e. $t\in[0,T]$. By redefining $\zeta((\omega,s),t):=0$ for $((\omega,s),t)\in\Omega_T\times[0,T]$, $t\in[0,T]\setminus B_\zeta$, we can assume that $B_\zeta=[0,T]$. In such a case, to show that f_ζ is Borel measurable, we argue as in the proof of Lemma A.2.1, after replacing $L^0_{\mathcal{P}_T\otimes\mathcal{B}_T}$ by $L^{p,q,r}_{\mathcal{P}_T\otimes\mathcal{B}_T}$ and $L^0_{\mathcal{P}_T}$ by $L^{p,q}_{\mathcal{P}_T}$. ■

For $p,q,r\in[1,\infty)$, let

$$\mathfrak{I}\colon L^{p,q}_{\mathcal{P}_T}(L_2(U,H))\to \mathcal{L}^r_{\mathcal{F}_T\otimes\mathcal{B}_T}(\mathbb{T}) \tag{A.2.5}$$

be a linear and continuous operator, where $\mathcal{L}^r_{\mathcal{F}_T\otimes\mathcal{B}_T}(\mathbb{T})$ is defined as in Section A.1 (p. 212), with $\mathcal{P}'=\mathcal{F}_T\otimes\mathcal{B}_T$.

Let $\zeta\in L^0_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$ be a given representant of its class. Our aim is to show that there exists a (ω,t)-jointly measurable version of the family of random variables

$$\mathfrak{I}^\zeta_t := (\mathfrak{I}(\zeta(\cdot,t)))_{t\in B_\zeta}, \tag{A.2.6}$$

where B_ζ is defined by (A.2.4).

Remark A.2.4. Definition A.2.6 depends on the chosen representant ζ. If $\zeta=\zeta'$ in $L^0_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$, then $m(B_\zeta\triangle B_{\zeta'})=0$, and, due to the fact that $\mathfrak{I}$ has values in $\mathcal{L}^r_{\mathcal{F}_T\otimes\mathcal{B}_T}(\mathbb{T})$, we have $\mathfrak{I}^\zeta_t=\mathfrak{I}^{\zeta'}_t$ $\mathbb{P}$-a.e., for all $t\in B_\zeta\bigcap B_{\zeta'}$.

Theorem A.2.5. *Let $p,q,r\in[1,\infty)$, let $\zeta\in L^{p,q,r}_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$, and let B_ζ be the set defined by* (A.2.4). *Then there exists a process*

$$\Sigma^\zeta\in L^r_{\mathcal{F}_T\otimes\mathcal{B}_T}(F) \tag{A.2.7}$$

such that

$$\text{for } m\text{-a.e. } t\in B_\zeta,\ \Sigma^\zeta_t(\omega)=(\mathfrak{I}(\zeta(\cdot,t)))_t(\omega)\ \mathbb{P}\text{-a.e. } \omega\in\Omega. \tag{A.2.8}$$

Moreover, the map

$$\mathbf{J}\colon L^{p,q,r}_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))\to L^r_{\mathcal{F}_T\otimes\mathcal{B}_T}(F),\ \zeta\mapsto\Sigma^\zeta \tag{A.2.9}$$

is linear, continuous, uniquely determined by (A.2.7), (A.2.8). *The operator norm of* $\mathbf{J}$ *is bounded by the operator norm of* $\mathfrak{I}$.

Proof. We apply Theorem A.1.3, with the following data:

- $G = [0,T]$, $\mathcal{G} = \mathcal{B}_T$, $\mu = m$;
- $D_1 = \Omega$, $D_2 = [0,T]$, $D = \Omega_T$, $\mathcal{D} = \mathcal{P}_T$, $\nu_1 = \mathbb{P}$, $\nu_2 = m$;
- $E = L_2(U,H)$;
- $\mathfrak{L} = \mathfrak{I}$;
- $g \colon [0,T] \to L^{p,q}_{\mathcal{P}_T}(L_2(U,H))$ defined by

$$g(t) := \begin{cases} \zeta(\cdot,t) & \text{if } t \in B_\zeta \\ 0 & \text{if } t \in [0,T] \setminus B_\zeta. \end{cases}$$

  By Lemma A.2.3, g is well-defined and measurable. Moreover, $\zeta \in L^{p,q,r}_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$ implies $g \in L^1([0,T], L^{p,q}_{\mathcal{P}_T}(L_2(U,H))$.

Let X_2 be the process provided by application of the theorem. Then

$$X_2(\cdot,t) = \mathfrak{I}(\zeta(\cdot,t)) \text{ in } \mathcal{L}^r_{\mathcal{F}_T\otimes\mathcal{B}_T}(\mathbb{T}),\ \mathbb{P}\text{-a.e. } t \in B_\zeta. \tag{A.2.10}$$

Define

$$\Sigma^\zeta_t(\omega) := X_2((\omega,t,),t) \qquad \forall(\omega,t) \in \Omega_T,\ t \in [0,T].$$

Then Σ^ζ is jointly measurable in (ω,t), and, by (A.2.10), for m-a.e. $t \in B_\zeta$,

$$\Sigma^\zeta_t(\omega) = X_2((\omega,t),t) = (\mathfrak{I}(g(t)))_t(\omega) = (\mathfrak{I}(\zeta(\cdot,t)))_t(\omega) \qquad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Moreover,

$$\begin{aligned}
\int_0^T \mathbb{E}\left[|\Sigma^\zeta_t|^r_F\right] dt &= \int_0^T \mathbb{E}\left[|X_2((\cdot,t),t)|^r_F\right] dt \\
&\le \int_0^T \mathbb{E}\left[\sup_{s\in[0,T]} |X_2((\cdot,s),t)|^r_F\right] dt \\
&= \int_0^T |X_2(\cdot,t)|^r_{\mathcal{L}^r_{\mathcal{F}_T\otimes\mathcal{B}_T}(\mathbb{T})} dt \\
&= (\text{by (A.2.10)}) = \int_0^T |\mathfrak{I}(\zeta(\cdot,t))|^r_{\mathcal{L}^r_{\mathcal{F}_T\otimes\mathcal{B}_T}(\mathbb{T})} dt \\
&\le C\int_0^T |\zeta(\cdot,t)|^r_{L^{p,q}_{\mathcal{P}_T(L_2(U,H))}} dt = C|\zeta|^r_{L^{p,q,r}_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))}.
\end{aligned} \tag{A.2.11}$$

This shows (A.2.7).

Now, if Σ_1 and Σ_2 satisfy (A.2.7) and (A.2.8), with respect to the same ζ, then they belong to the same class in $L^r_{\mathcal{F}_T\otimes\mathcal{B}_T}(F)$, because $m([0,T]\setminus B_\zeta) = 0$. Similarly, if $\zeta_1 = \zeta_2$ in $L^{p,q,r}_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$, then, as noticed in Remark A.2.4, for m-a.e. $t \in [0,T]$, $\mathfrak{I}^{\zeta_1}_t(\omega) = \mathfrak{I}^{\zeta_2}_t(\omega)$ $\mathbb{P}$-a.e. $\omega \in \Omega$. Then (A.2.8) entails $\Sigma^{\zeta_1} = \Sigma^{\zeta_2}$ for $\mathbb{P}\otimes m$-a.e. $(\omega,t) \in \Omega_T$. This shows that (A.2.9) is well-defined. Linearity is clear. Continuity comes from (A.2.11). ■

In general, we cannot hope to have versions of $\mathfrak{I}^{\zeta}$ with a better measurability than the one provided by Theorem A.2.5, without further assumptions on $\mathfrak{I}$ (observe that our assumptions on $\mathfrak{I}$ do not take in consideration any progressive measurability of the values of $\mathfrak{I}$).

We now address the case when $\zeta \in L^{p,q,r}_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$ has the form

$$\zeta((\omega,s),t) = R(t-s)\Phi_s(\omega) =: \Phi_R((\omega,s),t) \qquad \forall (\omega,s)\in\Omega_T,\ t\in(s,T],$$

where $R\colon(0,T]\to L(H)$ is strongly continuous and $\Phi\in L(U,H)^{\Omega_T}$ is a function.

Under a technical assumption on R, we characterize those functions $\Phi\in L(U,H)^{\Omega_T}$ for which Φ_R belongs to $L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$. This fact is of interest because it is the minimal requirement in order to define the family $\mathfrak{I}^{\Phi_R} = \{\mathfrak{I}^{\Phi_R}_t\}_{t\in B_{\Phi_R}}$ by (A.2.6) (with $\zeta = \Phi_R$), and to obtain the joint measurability of $\mathfrak{I}^{\Phi_R}$ through Theorem A.2.5.

Assumption A.2.6. *The function $R\colon(0,T]\to L(H)$ is strongly continuous and there exists a sequence $\{t_n\}_{n\in\mathbb{N}}\subset(0,T]$ converging to 0 such that, if $C\subset H$ is closed, convex, and bounded, then $u\in C$ if and only if $\exists m\in\mathbb{N}\colon R(t_n)u\in R(t_n)C\ \forall n\geq m$.*

Remark A.2.7. Due to the fact that the closed convex sets in H are the same in the weak and in the strong topology, then, if the following implication holds for some $\{t_n\}_{n\in\mathbb{N}}\subset(0,T]$ converging to 0:

$$\{x_n\}_{n\in\mathbb{N}}\subset H \text{ bounded such that } \{R(t_n)x_n\}_{n\in\mathbb{N}} \text{ is definitely null} \quad\Longrightarrow\quad x_n \rightharpoonup 0, \tag{A.2.12}$$

Assumption A.2.6 holds true. To see it, let ut suppose that there exists $m\in\mathbb{N}$ such that $R(t_n)u\in R(t_n)C$ for $n\geq m$. This means that $R(t_n)(u-c_n)=0$ for $n\geq m$. By (A.2.12), $c_n\rightharpoonup u$, hence u belongs to C.

In particular, we notice that (A.2.12) is satisfied whenever $R\colon\mathbb{R}^+\to L(H)$ is a C_0-semigroup on H. In such a case, R^* is a C_0-semigroup (see [36, pp. 43–44, Section 5.14], and then we can write, if $\{t_n\}_{n\in\mathbb{N}}$ is any bounded sequence converging to 0 and if $\{x_n\}_{n\in\mathbb{N}}$ is such that $\{R(t_n)x_n\}_{n\in\mathbb{N}}$ is definitely null,

$$\lim_{n\to\infty}\langle x_n,y\rangle = \lim_{n\to\infty}\langle x_n,R^*(t_n)y\rangle = \lim_{n\to\infty}\langle R(t_n)x_n,y\rangle = 0 \qquad \forall y\in H.$$

In what follows, we denote by $\overline{\mathscr{P}_T}$ the completion of $\mathscr{P}_T$ with respect to $\mathbb{P}\otimes m$. If $\Phi\in L(U,H)^{\Omega_T}$, we denote by Φ_R the map defined by

$$\Phi_R\colon\Omega_T\times[0,T]\to L(U,H),\ ((\omega,s),t)\mapsto \mathbf{1}_{[0,t)}(s)R(t-s)\Phi_s(\omega). \tag{A.2.13}$$

By saying that $\Phi\in L(U,H)^{\Omega_T}$ *is strongly measurable*, we mean that

$$(\Omega_T,\mathscr{P}_T)\to H,\ (\omega,t)\mapsto\Phi_t(\omega)u$$

is measurable, for all $u\in U$. Similarly, if $\Phi\in L(U,H)^{\Omega_T}$, then Φ_R *is strongly measurable* if $\Phi_R(\cdot)u$ is $\mathscr{P}_T\otimes\mathscr{B}_T/\mathscr{B}_H$-measurable, for all $u\in U$.

Proposition A.2.8. *Let $R\colon (0,T] \to L(H)$ be strongly continuous and let $\Phi \in L(U,H)^{\Omega_T}$.*

(i) If Φ is strongly measurable, then Φ_R is strongly measurable.

(ii) Suppose that R satisfies Assumption A.2.6. If Φ_R is strongly measurable, then there exists $\hat{\Phi} \in L(U,H)^{\Omega_T}$ and a $\mathbb{P}\otimes m$-null set $A \in \mathscr{P}_T$ such that $\Phi = \hat{\Phi}$ on $\Omega_T \setminus A$ and $\hat{\Phi}$ is strongly measurable.

Proof. *(i)* Let $\Phi \in L(U,H)^{\Omega_T}$ be strongly measurable. Let

$$\rho := \{0 = t_0 < \ldots < t_k = T\} \subset [0,T].$$

Denote $\delta(\rho) := \sup_{i=0,\ldots,k-1}\{|t_{i+1} - t_i|\}$. Define the function

$$\Phi_{R,\rho}\colon (\Omega_T \times [0,T], \mathscr{P}_T \otimes \mathscr{B}_T) \to L(U,H)$$

by

$$\Phi_{R,\rho}((\omega,s),t) := \sum_{i=0}^{k-1} \mathbf{1}_{[t_i,t_{i+1})}(t)\mathbf{1}_{[0,t_i)}(s)R(t_i - s)\Phi_s(\omega) + \mathbf{1}_{\{T\}}(t)\mathbf{1}_{[0,T)}(s)R(T-s)\Phi_s(\omega).$$

For all $t \in [0,T]$ and $h \in H$, the map

$$(\Omega_T, \mathscr{P}_T) \to H,\ (\omega,s) \mapsto \mathbf{1}_{[0,t)}(s)R^*(t-s)h$$

is measurable, by strong continuity of R and Pettis's measurability theorem. Moreover, for $u \in U$,

$$(\Omega_T, \mathscr{P}_T) \to H,\ (\omega,s) \mapsto \Phi_s(\omega)u$$

is measurable by assumption, we conclude that, for $u \in U$ and $t \in [0,T]$,

$$(\Omega_T, \mathscr{P}_T) \to \mathbb{R},\ (\omega,s) \mapsto \langle \mathbf{1}_{[0,t)}(s)R(t-s)\Phi_s(\omega)u, h\rangle_H$$

is measurable. Then, again by Pettis's measurablity theorem,

$$(\Omega_T, \mathscr{P}_T) \to H,\ (\omega,s) \mapsto \mathbf{1}_{[0,t)}(s)R(t-s)\Phi_s(\omega)u$$

is measurable, for every $u \in U$ and $t \in [0,T]$. Hence $\Phi_{R,\rho}$ is strongly measurable. By strong continuity of R, we have

$$\lim_{\delta(\rho)\to 0} \Phi_{R,\rho}((\omega,s),t)u = \Phi_R((\omega,s),t)u \qquad \forall ((\omega,s),t) \in \Omega_T \times [0,T],$$

for every $u \in U$. This shows that Φ_R is strongly measurable.

(ii) Suppose that Φ_R is strongly measurable. Let $u \in U$ and let $C \subset H$ be closed, convex, and bounded. Let $\{t_n\}_{n\in\mathbb{N}}$ be as in Assumption A.2.6. For $n \in \mathbb{N}$, define

$$\Delta_n := \{((\omega,s),t) \in \Omega_T \times [0,T]\colon t - s = t_n\}$$

$$B_n := \{((\omega,s),t) \in \Omega_T \times [0,T]\colon \Phi_R((\omega,s),t)u \in R(t_n)C\}$$

$$F_n := \{(\omega,s) \in \Omega_T\colon R(t_n)\Phi_s(\omega)u \in R(t_n)C\}.$$

It is clear that $\Delta_n \in \mathscr{P}_T \otimes \mathscr{B}_T$. By weak compactness of C, $R(t_n)C$ is closed. Then, by strong measurability of Φ_R, $B_n \in \mathscr{P}_T \otimes \mathscr{B}_T$, hence $B_n \bigcap \Delta_n \in \mathscr{P}_T \otimes \mathscr{B}_T$.

Let $\pi_{\Omega_T} : \Omega_T \times [0,T] \to \Omega_T$ be the projection defined by

$$\pi_{\Omega_T}((\omega,s),t) := (\omega,s).$$

By the projection theorem (see [6, p. 75, Theorem III-23]), $\pi_{\Omega_T}(B_n \bigcap \Delta_n) \in \overline{\mathscr{P}_T}$. Notice that

$$\begin{aligned}\pi_{\Omega_T}(B_n \bigcap \Delta_n) &= \{(\omega,s) \in \Omega_T : s + t_n \le T \text{ and } R(t_n)\Phi_s(\omega)u \in R(t_n)C\} \\ &= F_n \bigcap (\Omega \times [0,T-t_n]).\end{aligned} \tag{A.2.14}$$

By Assumption A.2.6 and by recalling that $\{t_n\}_{n\in\mathbb{N}} \subset (0,T]$ converges to 0, we have

$$\{(\omega,s) \in \Omega_T : \Phi_s(\omega)u \in C,\ s < T\} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} (F_n \bigcap (\Omega \times [0,T-t_n])). \tag{A.2.15}$$

By (A.2.14) and (A.2.15), we conclude $\{(\omega,s) \in \Omega_T : \Phi_s(\omega)u \in C,\ s < T\} \in \overline{\mathscr{P}_T}$. The slice $\{(\omega,T) \in \Omega_T : \Phi_T(\omega)u \in C\}$ is a $\mathbb{P} \otimes m$-null set. Then

$$\{(\omega,s) \in \Omega_T : \Phi_s(\omega)u \in C\} \in \overline{\mathscr{P}_T}.$$

Since this holds for every closed, convex, bounded set C, hence for balls, and since H is separable, we have that Φu is $\overline{\mathscr{P}_T}/\mathscr{B}_H$-measurable, for every $u \in U$.

Now let $\{u_n\}_{n\in\mathbb{N}}$ be a dense subset of U. Since $\overline{\mathscr{P}_T}$ is the completion of $\mathscr{P}_T$ with respect to $\mathbb{P} \otimes m$, and since H is separable, for every $n \in \mathbb{N}$ there exists $A_n \in \mathscr{P}_T$ such that $\mathbb{P} \otimes m(A_n) = 0$ and $\mathbf{1}_{A_n}\Phi u_n$ is $\mathscr{P}_T/\mathscr{B}_H$-measurable. Let $A := \bigcup_{n\in\mathbb{N}} A_n$. Then $A \in \mathscr{P}_T$, $\mathbb{P} \otimes m(A) = 0$, and $\mathbf{1}_A \Phi u_n$ is $\mathscr{P}_T/\mathscr{B}_H$-measurable for every $n \in \mathbb{N}$. Since $\Phi_s(\omega) \in L(U,H)$ for every $(\omega,s) \in \Omega_T$, by density of $\{u_n\}_{n\in\mathbb{N}}$ we conclude that $\mathbf{1}_A \Phi u$ is $\mathscr{P}_T/\mathscr{B}_H$-measurable for every $u \in U$. This concludes the proof of *(ii)* and of the proposition. ■

We will make use of the following lemma, whose proof can be found in [37, Ch. 1].

Lemma A.2.9. *Let $(G,\mathscr{G})$ be a measurable space. Let $f : (G,\mathscr{G}) \to L_2(U,H)$. Then $f(\cdot)u$ is $\mathscr{G}/\mathscr{B}_H$-measurable, for all $u \in U$, if and only if f is $\mathscr{G}/\mathscr{B}_{L_2(U,H)}$-measurable.*

Under Assumption A.2.6, the following theorem characterizes those functions $\Phi \in L(U,H)^{\Omega_T}$ for which Φ_R belongs to $L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$.

Theorem A.2.10. *Let $R : (0,T] \to L(H)$ be strongly continuous and let $\Phi \in L(U,H)^{\Omega_T}$.*

(i) If Φ is strongly measurable and if $\mathbf{1}_{[0,t)}(s)R(t-s)\Phi_s(\omega) \in L_2(U,H)$ for all $((\omega,s),t) \in \Omega_T \times [0,T]$, then Φ_R is measurable as an $L_2(U,H)$-valued map (that is when $L_2(U,H)$ is endowed with its Borel σ-algebra).

(ii) Suppose that R satisfies Assumption A.2.6. If Φ_R has values in $L_2(U,H)$ and if it is measurable as an $L_2(U,H)$-valued map, then there exists $\hat{\Phi} \in L(U,H)^{\Omega_T}$ and a $\mathbb{P} \otimes m$-null set $A \in \mathscr{P}_T$ such that $\Phi = \hat{\Phi}$ on $\Omega_T \setminus A$, $\hat{\Phi}$ is strongly measurable, and $\mathbf{1}_{[0,t)}(s)R(t-s)\hat{\Phi}_s(\omega) \in L_2(U,H)$ for all $((\omega,s),t) \in \Omega_T \times [0,T]$.

Proof. Apply Proposition A.2.8 and Lemma A.2.9. ∎

Example A.2.11. Let Q be a positive self-adjoint operator of trace class in H and let W be a U-valued Q-Wiener process with respect to $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$. Let $U_0 := Q^{1/2}(U)$ be the Hilbert space isometric to U through $Q^{-1/2}\colon U_0 \to U$. By [24, p. 114, Theorem 4.37], for $p \geq 2$, the stochastic integral is a linear and continuous map

$$\mathfrak{I}_W \colon L^{p,2}_{\mathscr{P}_T}(L_2(U_0,H)) \to \mathscr{L}^p_{\mathscr{P}_T}(C([0,T],H)),\ \Psi \mapsto \Psi \cdot W := \int_0^{\cdot} \Psi_s dW_s.$$

Let R be as in Assumption A.2.6. Let $\Phi \in L(U_0,H)^{\Omega_T}$ be strongly measurable and such that $R(t-s)\Phi_s(\omega) \in L_2(U_0,H)$ for $(\omega,s) \in \Omega_T$, $t \in (s,T]$. Then, by Theorem A.2.10*(i)*, $\Phi_R \in L^0_{\mathscr{P}_T \otimes \mathscr{B}_T}(L_2(U_0,H))$. If $|\Phi_R|_{p,2,p} < \infty$, then we can apply Theorem A.2.5, according to which the process

$$\left\{ \int_0^t R(t-s)\Phi_s dW_s \right\}_t,$$

which is well-defined for a.e. $t \in [0,T]$, has an $\mathscr{F}_T \otimes \mathscr{B}_T$-jointly measurable version.

A.2.2 Continuous version

In this section we review the factorization method used to show existence of continuous version of stochastic convolutions made with respect to a C_0-semigroup.

Notation. Throughout this section

- S denotes a strongly continuous semigroup on H and $M := \sup_{t \in [0,T]} |S_t|_{L(H)}$;
- $\mathbb{W} := C([0,T],H)$;
- for $\beta \in (0,1)$, c_β denotes the number $c_\beta := \left(\int_0^1 (1-w)^{\beta-1} w^{-\beta} dw \right)^{-1}$.

As noticed in Remark A.2.7, S verifies Assumption A.2.6.

The factorization method relies on the semigroup property of S and on the fact that continuous linear operator commutes with stochastic integral. We rephrase this commutativity assumption in our setting through the following

Assumption A.2.12. *Let $p,q,r \in [1,\infty)$, and let*

$$\mathfrak{I} \colon L^{p,q}_{\mathscr{P}_T}(L_2(U,H)) \mapsto \mathscr{L}^r_{\mathscr{F}_T \otimes \mathscr{B}_T}(\mathbb{W})$$

be a linear and continuous operator such that

$$Q(\Im\Phi) = \Im(Q\Phi) \text{ in } \mathscr{L}^r_{\mathscr{F}_T\otimes\mathscr{B}_T}(\mathbb{W})\ (^2) \qquad \forall Q\in L(H). \tag{A.2.16}$$

For $p,q,r\in[1,\infty)$ and $\beta\in[0,1)$, $\Lambda^{p,q,r}_{\mathscr{P}_T,S,\beta}(L(U,H))$ denotes the vector space of equivalence classes of strongly measurable functions $\Phi\in L(U,H)^{\Omega_T}$ such that

$$\left(\int_0^T\left(\int_0^t (t-s)^{-\beta q}\left(\mathbb{E}\left[|S(t-s)\Phi_s|^p_{L_2(U,H)}\right]\right)^{q/p} ds\right)^{r/q} dt\right)^{1/r} < \infty. \tag{A.2.17}$$

Two functions Φ_1, Φ_2, are in the same class if the quantity (A.2.17) is 0 for $\Phi=\Phi_1-\Phi_2$. This implies, for all $u\in U$, for $\mathbb{P}\otimes m$-a.e. $(\omega,s)\in\Omega_T$,

$$\mathbf{1}_{[0,t)}(s)S(t-s)(\Phi_1)_s u = \mathbf{1}_{[0,t)}(s)S(t-s)(\Phi_2)_s u \qquad m\text{-a.e. } t\in[0,T],$$

hence, by strong continuity of S, for all $u\in U$,

$$(\Phi_1)_s u = (\Phi_2)_s u \qquad \mathbb{P}\otimes m\text{-a.e. } (\omega,s)\in\Omega_T.$$

By separability of U we conclude that $\Phi_1=\Phi_2$ in $\Lambda^{p,q,r}_{\mathscr{P}_T,S,\beta}(L(U,H))$ if and only if $(\Phi_1)_s(\omega) = (\Phi_2)_s(\omega)$ in $L(U,H)$ $\mathbb{P}\otimes m$-a.e. $(\omega,s)\in\Omega_T$.

For $\Phi\in\Lambda^{p,q,r}_{\mathscr{P}_T,S,\beta}(L(U,H))$, we define, for all $(\omega,s)\in\Omega_T$ and $t\in[0,T]$,

$$\Phi_{S,\beta}((\omega,s),t) := \mathbf{1}_{[0,t)}(s)(t-s)^{-\beta}S(t-s)\Phi_s(\omega)$$

$$\Phi_S((\omega,s),t) := \mathbf{1}_{[0,t)}(s)S(t-s)\Phi_s(\omega),$$

By Theorem A.2.10*(i)*, $\Phi_{S,\beta}\in L^0_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$, and (A.2.17) can be written as

$$|\Phi_{S,\beta}|_{p,q,r} < \infty. \tag{A.2.18}$$

Then, through the well-defined map

$$\Lambda^{p,q,r}_{\mathscr{P}_T,S,\beta}(L(U,H)) \to L^{p,q,r}_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H)),\ \Phi\mapsto\Phi_{S,\beta},$$

$\Lambda^{p,q,r}_{\mathscr{P}_T,S,\beta}(L(U,H))$ is identified with a subspace of $L^{p,q,r}_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$. In particular, the map

$$\Lambda^{p,q}_{\mathscr{P}_T,S,\beta}(L(U,H)) \to \mathbb{R}^+,\ \Phi\mapsto|\Phi_{S,\beta}|_{p,q,r}$$

is a norm. In what follows we always consider $\Lambda^{p,q}_{\mathscr{P}_T,S,\beta}(L(U,H))$ endowed with the norm $|\#_{S,\beta}|_{p,q,r}$.

[2] Q applied to a process Φ means the pointwise composition $Q(\Phi_t(\omega))$, for $(\omega,t)\in\Omega_T$.

Again by Theorem A.2.10*(i)*, $\Phi_S \in L^0_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$. Moreover, for all $t' \in [0,T]$, we have, by applying Minkowski's inequality for integrals (see [45, p. 194, 6.19]),

$$|\Phi_S(\cdot,t)|_{p,q} = c_\beta \left(\int_0^{t'}\left(\int_s^{t'}(t'-t)^{\beta-1}(t-s)^{-\beta}\left(\mathbb{E}\left[|S(t'-s)\Phi_s|^p_{L_2(U,H)}\right]\right)^{1/p}dt\right)^q ds\right)^{1/q}$$
$$\le c_\beta \int_0^{t'}(t'-t)^{\beta-1}\left(\int_0^t (t-s)^{-\beta q}\left(\mathbb{E}\left[|S(t'-s)\Phi_s|^p_{L_2(U,H)}\right]\right)^{q/p}ds\right)^{1/q}dt.$$

Now, if we take $r>1$ and $\beta\in(1/r,1)$, by applying Hölder's inequality to the last term and writing $S(t'-s)=S(t'-t)S(t-s)$,

$$|\Phi_S(\cdot,t')|_{p,q} \le c_\beta M\left(\int_0^T w^{\frac{(\beta-1)r}{r-1}}dw\right)^{(r-1)/r}|\Phi_{S,\beta}|_{p,q,r}<\infty. \tag{A.2.19}$$

This shows that

$$\Phi_S(\cdot,t')\in L^{p,q}_{\mathcal{P}_T}(L_2(U,H)),\ \forall t'\in[0,T]. \tag{A.2.20}$$

Theorem A.2.13. *Let $p,q\in[1,\infty)$, $r\in(1,\infty)$, $\beta\in(1/r,1)$. Let $\mathfrak{I}$ be as in Assumption A.2.12. Then there exists a unique linear and continuous function*

$$\mathbf{C}\colon \Lambda^{p,q,r}_{\mathcal{P}_T,S,\beta}(L(U,H))\to \mathcal{L}^r_{\mathcal{F}_T\otimes\mathcal{B}_T}(\mathbb{W}) \tag{A.2.21}$$

such that, for all $\Phi\in\Lambda^{p,q,r}_{\mathcal{P}_T,S,\beta}(L(U,H))$, for all $t\in[0,T]$,

$$\left(\mathfrak{I}\left(\mathbf{1}_{[0,t)}(\cdot)S(t-\cdot)\Phi\right)\right)_t = (\mathbf{C}(\Phi))_t \qquad \mathbb{P}\text{-a.e.}. \tag{A.2.22}$$

The operator norm of $\mathbf{C}$ is bounded by a constant depending only on β, r, T, M, and on the operator norm of $\mathfrak{I}$.

Proof. Let $\Phi\in\Lambda^{p,q,r}_{\mathcal{P}_T,S,\beta}(L(U,H))$. First notice that the left-hand side of (A.2.22) is meaningful because of (A.2.20). We now construct $\mathbf{C}(\Phi)$. Fix $t'\in[0,T]$, and define

$$\Phi^{(t')}_{S,\beta}((\omega,s),t) := c_\beta \mathbf{1}_{[0,t')}(t)(t'-t)^{\beta-1}\mathbf{1}_{[0,t)}(s)(t-s)^{-\beta}S(t'-s)\Phi_s(\omega) \qquad (\omega,s)\in\Omega_T,\ t\in[0,T].$$

By Theorem A.2.10*(i)*, $\Phi^{(t')}_{S,\beta}\in L^0_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$. Moreover,

$$|\Phi^{(t')}_{S,\beta}|_{p,q,1} = c_\beta\int_0^{t'}(t'-t)^{\beta-1}\left(\int_0^t(t-s)^{-q\beta}\left(\mathbb{E}\left[|S(t'-s)\Phi_s|^p_{L_2(U,H)}\right]\right)^{q/p}ds\right)^{1/q}dt$$
$$\le c_\beta M\left(\int_0^T w^{\frac{(\beta-1)r}{r-1}}dw\right)^{(r-1)/r}|\Phi_{S,\beta}|_{p,q,r}<\infty.$$

Then $\Phi^{(t')}_{S,\beta}\in L^{p,q,1}_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$. By Lemma A.2.3, the map

$$g\colon B_0\to L^{p,q}_{\mathcal{P}_T}(L_2(U,H)),\ t\mapsto \Phi^{(t')}_{S,\beta}(\cdot,t), \tag{A.2.23}$$

where B_0 is the set of t such that $|\Phi^{(t')}_{S,\beta}(\cdot,t)|_{p,q}<\infty$, is Borel measurable. Let us define $g=0$ on $[0,T]\setminus B_0$. By $\Phi^{(t')}_{S,\beta}\in L^{p,q,1}_{\mathcal{P}_T\otimes\mathcal{B}_T}(L_2(U,H))$ and by measurability of (A.2.23), we have $g\in L^1([0,T],L^{p,q}_{\mathcal{P}_T}(L_2(U,H))$. We can then apply Theorem A.1.3, with the following data:

- $G = [0,T]$, $\mathscr{G} = \mathscr{B}_T$, $\mu = m$;
- $D_1 = \Omega$, $D_2 = [0,T]$, $D = \Omega_T$, $\mathscr{D} = \mathscr{P}_T$, $\nu_1 = \mathbb{P}$, $\nu_2 = m$;
- $E = L_2(U,H)$;
- $F = H$;
- $\mathfrak{L} = \mathfrak{I}$;
- g as above.

The theorem provides measurable functions

$$X_1 \colon (\Omega_T \times [0,T], \mathscr{P}_T \otimes \mathscr{B}_T) \to L_2(U,H) \qquad X_2 \colon (\Omega_T \times [0,T], (\mathscr{F}_T \otimes \mathscr{B}_T) \otimes \mathscr{B}_T) \to H$$

such that, for m-a.e. $t \in [0,T]$,

$$\begin{cases} X_1(\cdot,t) = g(t) \text{ in } L^{p,q}_{\mathscr{P}_T}(L_2(U,H)) \\ X_2(\cdot,t) = \mathfrak{I}(g(t)) \text{ in } \mathscr{L}^r_{\mathscr{F}_T \otimes \mathscr{B}_T}(\mathbb{W}), \end{cases} \tag{A.2.24}$$

and

$$\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega,\ (\mathfrak{I}Y)_t(\omega) = \int_0^T X_2((\omega,t),s)ds \ \forall t \in [0,T], \tag{A.2.25}$$

where

$$Y_t(\omega) = \int_0^T X_1((\omega,t),s)ds,\ \forall (\omega,t) \in \Omega_T. \tag{A.2.26}$$

By (A.2.24), by definition of g, and by joint measurability of X_1 and $\Phi^{(t')}_{S,\beta}$, we have

$$X_1((\omega,s),t) = \Phi^{(t')}_{S,\beta}((\omega,s),t) \qquad (\mathbb{P} \otimes m) \otimes m\text{-a.e. } ((\omega,s),t) \in \Omega_T \times [0,T]. \tag{A.2.27}$$

Then (A.2.26) becomes

$$Y_t(\omega) = \int_0^T \Phi^{(t')}_{S,\beta}((\omega,t),s)ds = \mathbf{1}_{[0,t')}(t) S(t'-t)\Phi_t(\omega) \qquad \mathbb{P} \otimes m\text{-a.e. } (\omega,t) \in \Omega_T, \tag{A.2.28}$$

hence, in particular,

$$(\mathfrak{I}Y)_{t'}(\omega) = \mathfrak{I}\left(\mathbf{1}_{[0,t')} S(t'-\cdot)\Phi\right)_{t'}(\omega) \qquad \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{A.2.29}$$

On the other hand, for m-a.e. $t \in [0,T]$,

$$g(t) = c_\beta \mathbf{1}_{[0,t')}(t)(t'-t)^{\beta-1} S(t'-t)\Phi_{S,\beta}(\cdot,t) \text{ in } L^{p,q}_{\mathscr{P}_T}(L_2(U,H)).$$

Then, by assumption on $\mathfrak{I}$, we have, for m-a.e. $t \in [0,T]$,

$$\mathfrak{I}(g(t)) = c_\beta \mathbf{1}_{[0,t')}(t)(t'-t)^{\beta-1} S(t'-t)\mathfrak{I}\left(\Phi_{S,\beta}(\cdot,t)\right) \text{ in } \mathscr{L}^r_{\mathscr{F}_t \otimes \mathscr{B}_T}(\mathbb{W}), \tag{A.2.30}$$

hence, in particular, for m-a.e. $t \in [0,T]$,

$$(\mathfrak{I}(g(t)))_t(\omega) = c_\beta \mathbf{1}_{[0,t')}(t)(t'-t)^{\beta-1} S(t'-t)\left(\mathfrak{I}\left(\Phi_{S,\beta}(\cdot,t)\right)\right)_t(\omega) \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{A.2.31}$$

Now our aim is to replace the last factor in (A.2.31) with a process jointly measurable in (ω,t). We noticed in (A.2.18) that $\Phi_{S,\beta} \in L^{p,q,r}_{\mathscr{P}_T\otimes\mathscr{B}_T}(L_2(U,H))$. We can then apply Theorem A.2.5. Let

$$\Sigma^{\Phi_{S,\beta}} := \mathbf{J}(\Phi_{S,\beta}) \in L^r_{\mathscr{F}_T\otimes\mathscr{B}_T}(H) \tag{A.2.32}$$

be the process obtained by applying the map (A.2.9) to $\Phi_{S,\beta}$. We know that $\Sigma^{\Phi_{S,\beta}}_t(\omega)$ is $\mathscr{F}_T \otimes \mathscr{B}_T$-measurable in $(\omega,t) \in \Omega_T$ and that, for m-a.e. $t \in [0,T]$,

$$\left(\mathfrak{I}(\Phi_{S,\beta}(\cdot,t))\right)_t(\omega) = \Sigma^{\Phi_{S,\beta}}_t(\omega) \qquad \mathbb{P}\text{-a.e. } \omega\in\Omega. \tag{A.2.33}$$

By (A.2.31) and (A.2.33), we can write, for m-a.e. $t \in [0,T]$,

$$(\mathfrak{I}(g(t)))_t(\omega) = c_\beta \mathbf{1}_{[0,t')}(t)(t'-t)^{\beta-1}S(t'-t)\Sigma^{\Phi_{S,\beta}}_t(\omega) \qquad \mathbb{P}\text{-a.e. } \omega\in\Omega. \tag{A.2.34}$$

Then, by (A.2.24) and taking into account the joint measurability of X_2 and $\Sigma^{\Phi_{S,\beta}}$,

$$X_2((\omega,t'),t) = c_\beta \mathbf{1}_{[0,t')}(t)(t'-t)^{\beta-1}S(t'-t)\Sigma^{\Phi_{S,\beta}}_t(\omega) \qquad \mathbb{P}\otimes m\text{-a.e. } (\omega,t)\in\Omega_T. \tag{A.2.35}$$

By (A.2.25), (A.2.35), (A.2.29), we finally obtain, for $\mathbb{P}$-a.e. $\omega\in\Omega$,

$$\begin{aligned}\mathfrak{I}\left(\mathbf{1}_{[0,t')}S(t'-\cdot)\Phi\right)_{t'}(\omega) = (\mathfrak{I}Y)_{t'}(\omega) &= \int_0^T X_2((\omega,t'),s)ds \\ &= c_\beta \int_0^{t'} (t'-t)^{\beta-1}S(t'-t)\Sigma^{\Phi_{S,\beta}}_t(\omega)dt.\end{aligned} \tag{A.2.36}$$

Now define the process $\mathbf{C}(\Phi)$ by

$$(\mathbf{C}(\Phi))_t(\omega) := \begin{cases} c_\beta \displaystyle\int_0^t (t-s)^{\beta-1}S(t-s)\Sigma^{\Phi_{S,\beta}}_s(\omega)ds & \text{if } \Sigma^{\Phi_{S,\beta}}(\omega) \in L^r([0,T],H) \\ 0 & \text{otherwise}\end{cases} \tag{A.2.37}$$

for all $(\omega,t)\in\Omega_T$. By [24, p. 129, Proposition 5.9], $\mathbf{C}(\Phi)$ is well-defined and pathwise continuous. By Hölder's inequality,

$$|\mathbf{C}(\Phi)(\omega)|_\infty \le C_{\beta,r,T,M}|\Sigma^{\Phi_{S,\beta}}(\omega)|_{L^r([0,T],H)} \qquad \forall \omega\in\Omega, \tag{A.2.38}$$

where $C_{\beta,r,T,M}$ depends only on β,r,T,M. Hence, by recalling (A.2.32),

$$|\mathbf{C}(\Phi)|_{\mathscr{L}^r_{\mathscr{F}_T\otimes\mathscr{B}_T}(\mathbb{W})} \le C_{\beta,r,T,M}|\Sigma^{\Phi_{S,\beta}}|_{L^r_{\mathscr{F}_T\otimes\mathscr{B}_T}(H)} \le C_{\beta,r,T,M,|\mathfrak{I}|}|\Phi_{S,\beta}|_{p,q,r}. \tag{A.2.39}$$

where $C_{\beta,r,T,M,|\mathfrak{I}|}$ depends only on β,r,T,M, and on the operator norm $|\mathfrak{I}|$ of $\mathfrak{I}$. Moreover, since $t'\in[0,T]$ was arbitrary chosen, and since the choice of $\Sigma^{\Phi_{S,\beta}}$ does not depend on t', (A.2.36) gives, for all $t'\in[0,T]$,

$$\mathfrak{I}\left(\mathbf{1}_{[0,t')}S(t'-\cdot)\Phi\right)_{t'} = (\mathbf{C}(\Phi))_{t'} \qquad \mathbb{P}\text{-a.e.}. \tag{A.2.40}$$

It is clear that the process $\mathbf{C}(\Phi)$ is uniquely identified by (A.2.40) in $\mathscr{L}^r_{\mathscr{F}_T\otimes\mathscr{B}_T}(\mathbb{W})$, because it is continuous. Moreover, if $\Phi=\Phi'$ in $\Lambda^{p,q,r}_{\mathscr{P}_T,S,\beta}(L(U,H))$, then $\mathbf{C}(\Phi)=\mathbf{C}(\Phi')$ in $\mathscr{L}^r_{\mathscr{F}_T\otimes\mathscr{B}_T}(\mathbb{W})$. Linearity of $\mathbf{C}$ is clear as well. This concludes the proof that the map (A.2.21) is well-defined on $\Lambda^{p,q,r}_{\mathscr{P}_T,S,\beta}(L(U,H))$, linear, and that (A.2.22) is satisfied. Finally, continuity, with operator norm bounded by a constant depending only on β, r, T, M, $|\mathfrak{I}|$, is due to (A.2.39). ■

We remark that the joint measurability of X_1, X_2, $\Sigma^{\Phi_{S,\beta}}$, provided by Theorem A.1.3 and Theorem A.2.5, play a central role in order to obtain the factorization formula (A.2.37).

Example A.2.14. We can apply Theorem A.2.13 within the framework of Example A.2.11, when $R=S$ is a C_0-semigroup. If $p=r\geq q=2$, $\beta\in(1/p,1)$, $\Phi\in\Lambda^{p,2,p}_{\mathscr{P}_T,S,\beta}(L(U,H))$, then there exists $\mathbf{C}(\Phi)\in\mathscr{L}^p_{\mathscr{F}_T\otimes\mathscr{B}_T}(\mathbb{W})$ such that, for all $t\in[0,T]$,

$$(\mathbf{C}(\Phi))_t=\int_0^t S(t-s)\Phi_s dW_s \qquad \mathbb{P}\text{-a.e..} \tag{A.2.41}$$

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