

Geometric and regularity properties of solutions of some problems related to the average distance functional

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Introduction

Optimal transport theory, as first formulated by Monge in [43], involves finding the “optimal way” to move an initial configuration of material (“déblais”) to build a castle (“remblais”), minimizing the total “effort”. In the Monge formulation, there are given Polish spaces (X, μ) and (Y, ν) endowed with Borel probability measures μ, ν , and a cost function $c : X \times Y \rightarrow [0, \infty]$. The goal is to minimize

$$\int_X c(x, T(x)) d\mu(x)$$

among all Borel maps $T : X \rightarrow Y$ satisfying $T_{\#}\mu = \nu$.

As this formulation has several undesirable properties, Kantorovich proposed in [28] a relaxed problem: given Polish spaces (X, μ) and (Y, ν) endowed with Borel probability measures μ, ν , and a cost function $c : X \times Y \rightarrow [0, \infty]$. The goal is to minimize

$$\int_{X \times Y} c(x, y) d\gamma(x, y)$$

among measures γ on $X \times Y$ satisfying $\pi_{X\#}\gamma = \mu$, $\pi_{Y\#}\gamma = \nu$ where $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ denote the projections on X and Y respectively.

In this formulation both measures are probability measures, which can be relaxed to require $\mu(X) = \nu(Y) < \infty$. Over the there has been great progress in understanding the Monge-Kantorovich problem, (see for instance [21], [18] and [56] and references therein).

Average distance problem

A related problem is the transport in presence of “Dirichlet regions”, i.e. subsets on which the transport is essentially “free”. Dirichlet regions will be assumed pathwise connected. Given a domain $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$), a Dirichlet region $\Sigma \subseteq \Omega$ is such that for any points $x, y \in \Sigma$, then the “cost” to transport x in y is 0. Thus instead of the path distance of Ω , which will be denoted with $\text{dist}_\Omega(\cdot, \cdot)$, one considers $\text{dist}_{\Omega, \Sigma}(\cdot, \cdot)$ defined as

$$\text{dist}_{\Omega, \Sigma}(x, y) := \min\{\text{dist}_\Omega(x, y), \min_{z_1 \in \Sigma} \text{dist}_\Omega(x, z_1) + \min_{z_2 \in \Sigma} \text{dist}_\Omega(z_2, y)\}.$$

Indeed, if $x, y \in \Sigma$, then the cost to transport x to y is 0, independently of $\text{dist}_\Omega(x, y)$.

Dirichlet regions considered here will always be compact, pathwise connected, Hausdorff one dimensional sets with finite $\mathcal{H}^1$ measure (the notation $\mathcal{H}^1$ denotes the Hausdorff-1 measure). The $\mathcal{H}^1$ measure of a set will be often referred as “length”.

A natural problem here is to determine the “optimal” Dirichlet region, which in some sense best “serves” the domain Ω , leading to the so-called average distance problem:

- Given a domain Ω , a measure μ , a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$ and a parameter $L \geq 0$ solve

$$\min F_{\mu,A}$$

where

$$F_{\mu,A}(\Sigma) := \int_{\Omega} A(\text{dist}_{\Omega}(x, \Sigma)) d\mu(x),$$

among all admissible Dirichlet regions Σ satisfying $\mathcal{H}^1(\Sigma) \leq L$.

This problem was first introduced in [14], and later studied in several articles (e.g. [16], [17]). A variant (see for instance [13]) is to solve

$$\min \int_{\Omega} A(\text{dist}_{\Omega}(x, \Sigma)) d\mu(x) + \lambda \mathcal{H}^1(\Sigma)$$

among all admissible Σ , with $\lambda > 0$ a given parameter. These formulations are often referred as “constrained” and “penalized” problem respectively. Such constraints are essential to the well-posedness of this problem.

The average distance problem (in both formulations) can be used to model problems arising from urban planning, or data cloud approximation.

In the constrained problem, if the length constraint is $L = 0$, connectedness imposes that all admissible Dirichlet regions are single points. However, a related problem, referred in literature as “location” problem, involves minimizing

$$\min \int_{\Omega} A(\text{dist}_{\Omega}(x, K)) d\mu(x)$$

among $K = \{P_i\}_{i=1}^N$ set of single points, where N is a given parameter.

Both problems have a quite simple formulation, yet even with simple geometry and stringent conditions on Ω, μ, A , no explicit solution can be determined: indeed the average distance problem, in both constrained and penalized formulation, cannot be solved for general L (or λ for the penalized problem) even with $\Omega = \overline{B}((0,0), 1) \subseteq \mathbb{R}^2$, $\mu = \mathcal{L}^2|_{\Omega}$ and $A = id$. The location problem is easier to solve, especially in highly regular and symmetric domains, but for general domains it cannot be solved either.

Some link with the classic optimal transport problem will be explained in Chapter 3.

The average distance problem is also related to the so-called q -compliance problem: given a domain Ω , consider

$$\begin{cases} -\Delta_q u = 1 & \text{in } \Omega \setminus \Sigma \\ u = 0 & \text{on } \Sigma \cup \partial\Omega \end{cases} \quad (0.0.1)$$

where Σ verifies the same properties of Dirichlet regions, and Δ_q denotes the q -Laplacian. The energy associated is

$$\mathcal{C}_q(\Sigma) := \left(1 - \frac{1}{q}\right) \int_{\Omega} u_{\Sigma}(x) dx,$$

with u_Σ denoting a solution of (0.0.1), and given a parameter $l > 0$, the associated q -compliance problem is solving

$$\min_{\mathcal{H}^1(\Sigma) \leq l} \mathcal{C}_q(\Sigma)$$

among Σ satisfying the same properties of Dirichlet regions. As proven in [15], passing to the limit $q \rightarrow \infty$, the energy $\mathcal{C}_q(\Sigma)$ Γ -converges to

$$\int_{\Omega} \text{dist}(x, \partial\Omega \cup \Sigma) dx.$$

Another related problem, which can be considered the “dual” problem, is the so called “maximal distance problem”: given a domain Ω , and a parameter $L \geq 0$, solve

$$\min_{\Sigma} F^*(\Sigma),$$

where

$$F^*(\Sigma) := \max_{y \in \Omega} \min_{z \in \Sigma} \text{dist}_{\Omega}(y, z).$$

Similarly to the average distance problem, the maximal distance problem is not possible to solve explicitly in general.

Due to the impossibility to solve the average distance problem (in both formulations), qualitative properties of minimizers have been studied. In particular, it has been proven that under mild assumptions on Ω, μ, A (more details will be given in Chapter 3), such minimizers must verify:

- Absence of loops: if the Radon-Nykodim density of μ , $\frac{d\mu}{d\mathcal{L}^n}$ belongs to L^p with $p \geq 1$, then any minimizer Σ_{opt} cannot contain subsets homeomorphic to S^1 ,
- Absence of crosses: if Ω is a two dimensional domain, and $\frac{d\mu}{d\mathcal{L}^2}$ belongs to L^p with $p > 4/3$ then any minimizer does not contain points with order greater than 3, and there are only a finite number of order 3. Moreover, if a point has order 3, then all the three angles have value $2\pi/3$. The term “cross” is used as using Menger n -Beinsatz (see [30]), a point P with order 4 will have at least 4 disjoint arcs $\{\gamma_i\}_{i=1}^4$ with endpoint (in all the thesis “endpoint” will refer to a point with order 1, i.e. removing such point would preserve connectedness) in P and disjoint outside P ,
- Ahlfors regularity: if $\frac{d\mu}{d\mathcal{L}^n}$ belongs to L^p with $p > n/(n-1)$ ($p > 4/3$ in two dimensional domains), then any minimizer Σ_{opt} is Ahlfors regular, i.e. there exist constants $c, C > 0$ such that

$$c \leq \frac{\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(P, \rho))}{\rho} \leq C$$

for any $P \in \Sigma_{\text{opt}}, \rho > 0$,

- in two dimensional domains, if $\frac{d\mu}{d\mathcal{L}^2}$ belongs to L^p with $p > 4/3$ then any minimizer is finite union of Lipschitz curves.

These results were first proven in two dimension cases by Buttazzo, Oudet and Stepanov (see for instance [16], [17]); then in [44], Paolini and Stepanov extended some of these results to higher dimensional domains. The thesis proves that condition $\frac{d\mu}{d\mathcal{L}^n} \in L^p$ with $p \geq 1$ is optimal; moreover, it is proven that if $\frac{d\mu}{d\mathcal{L}^2} \notin L^1$ then the absence of crosses is false too.

An interesting problem concerning solutions of the average distance problem is the regularity of minimizers: indeed Ahlfors regularity is a very weak property (nonetheless, it implies uniform rectifiability).

Two results have given a partial answer:

- in [55], Tilli has proven that any $C^{1,1}$ curve is minimizer under suitable choice of the domain (indeed the bulk of the argument involves determining such domain),
- in [52], Slepčev has shown that C^1 regularity can be false, by exhibiting an example of minimizer which is not C^1 regular.

These two results can be considered together with results by Santambrogio and Tilli, in [50], which prove $C^{1,1}$ regularity on certain points, and limit corners to points verifying specific conditions, i.e. those on which positive mass is projected.

In Chapter 6, a new weak second order regularity has been proven for minimizers of the penalized problem: in collaboration with Slepčev, in [38] it has been proven that for the penalized problem, any minimizer is finite union of curves $\{\gamma_k\}_{k=1}^h$, with $\sum_{k=1}^h \|\gamma'_k\|_{BV}$ uniformly bounded from above.

Evolution

An interesting problem is to extend results proven for minimizers of the average distance problem to solutions of evolution schemes related to the average distance functional. Two important evolution schemes will be analyzed.

Given $\Omega \subseteq \mathbb{R}^N$, μ, A as in the average distance problem, and an initial datum S_0 , consider the recursive sequence

$$\begin{cases} w(0) := S_0 \\ w(n) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \eta(\mathcal{H}^1(\cdot \Delta w(n-1))), \end{cases} \quad (0.0.2)$$

where Δ denotes the symmetric difference, and η is a given function. Some of the natural conditions on η (which will be assumed) include $\eta(0) = 0$, and η non decreasing. However it will follow from the arguments in Chapter 4 that the analysis when the penalization term has form $\tilde{\eta}(t) = at^b$, $a > 0, b \geq 1$, can be easily reduced to the case $\eta(t) = kt$, with $k > 0$. The presence of symmetric difference in the penalization term forces the evolution to be monotone w.r.t. set inclusion, i.e. $w(k_1) \subseteq w(k_2)$ whenever $k_1 \leq k_2$. Here, given $l \geq 0$, $\mathcal{A}_l(\Omega)$ denotes the collection of subsets $\Sigma \subset \Omega$

which are compact, path-wise connected, $\dim_{\mathcal{H}} \Sigma = 1$ and $\mathcal{H}^1(\Sigma) \leq l$. The union $\cup_{j \geq 0} \mathcal{A}_j(\Omega)$ will be denoted with $\mathcal{A}(\Omega)$. When Ω is the whole Euclidean space, $\mathcal{A}$ will be used instead of $\mathcal{A}(\Omega)$. A variant is

$$\begin{cases} w(0) := S_0 \\ w(n) \in \operatorname{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0) + n\varepsilon}(\Omega)} F_{\mu, A}, \end{cases} \quad (0.0.3)$$

where $\varepsilon > 0$ is a time step. The first evolution scheme is related to the penalized problem, the latter to the constrained problem. These schemes are often referred as “quasi static” evolution.

Another important class of evolutions are gradient flows related to the average distance functional. The framework of gradient flows has been first developed in Hilbert spaces. Given a Hilbert space $(H, \langle \cdot, \cdot \rangle)$, a functional $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying some weak properties (mainly λ -convexity, lower semicontinuity, coercivity and compactness of sublevels), a “gradient flow” x is essentially a curve which at each instant descends along the steepest descent direction, i.e.

$$\begin{cases} x_0 := \bar{x} \\ x'_t \in -\partial^- \phi(x_t). \end{cases}$$

This construction is not reproducible in a purely metric space (X, d) due to the explicit involvement of the scalar product. Moreover, the purely metric setting requires a definition for “gradient” and “speed”: these will be replaced by the “slope” and “metric derivative”. In [3], Ambrosio, Gigli and Savaré used an approach (based on approximation via time discretized evolutions), which is the one we will use. A variant (actually, three different variants) of the “gradient flow” can be obtained using the discrete approximation method: consider a sequence $\{\tau_j\}_{j \in \mathbb{N}} \downarrow 0$ and consider (here we omit all the details about existence and passage to the limit, as in the metric setting such properties are not guaranteed - a more detailed discussion can be found in Chapter 2)

$$\begin{cases} w_0 := \bar{x} \\ w_{n+1} \in \operatorname{argmin} \phi(\cdot) + \frac{1}{2\tau_j} d(\cdot, w_n)^2 \end{cases}.$$

Here, in the generic framework ϕ and d can be quite general functional and distance. More details about the choice of ϕ and d will be presented in Chapter 5. Then define the function

$$x_j : I \rightarrow X, \quad x_j(t) := w([t/\tau_j])$$

where I is a time interval, and $[\cdot]$ denotes the integer part mapping. Then under suitable assumptions for a subsequence (which will not be relabeled) there exists a limit function

$$x : I \rightarrow X, \quad x(t) = \lim_{j \rightarrow \infty} x_j(t)$$

for any $t \in I$. This function x (often called “minimizing movement” in literature, see for instance [3]) can be considered an analogous of the gradient flow from Hilbert context. In the metric setting, discussed more in detail in Chapter 2, there are three formulations; the function x is:

1. Gradient flow in the Energy Dissipation Inequality sense if:

$$\begin{aligned} E(x(s)) + \frac{1}{2} \int_t^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_t^s |\nabla E|^2(x(r)) dr &\leq E(x(t)) \quad a.e. \ t > 0, \ \forall s \geq t \\ E(x(s)) + \frac{1}{2} \int_0^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_0^s |\nabla E|^2(x(r)) dr &\leq E(\bar{x}) \quad \forall s \geq 0, \end{aligned}$$

2. Gradient flow in the Energy Dissipation Equality sense if:

$$\begin{aligned} E(x(s)) + \frac{1}{2} \int_t^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_t^s |\nabla E|^2(x(r)) dr &= E(x(t)) \quad a.e. \ t > 0, \ \forall s \geq t \\ E(x(s)) + \frac{1}{2} \int_0^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_0^s |\nabla E|^2(x(r)) dr &= E(\bar{x}) \quad \forall s \geq 0, \end{aligned}$$

3. Gradient flow in the Energy Variation Inequality sense if:

$$E(x(t)) + \frac{1}{2} \frac{d}{dt} d(x(t), y)^2 + \frac{\lambda}{2} d(x(t), y)^2 \leq E(y), \quad \forall y \in X, \ a.e. \ t > 0.$$

The main goal of this thesis is to analyze whether solutions of these evolutions schemes (including solutions of discrete evolution schemes) related to the average distance functional satisfy properties similar to those proven for minimizers of the average distance problem.

In particular it will be proven:

- Absence of loops: if the Radon-Nykodim density of μ , $\frac{d\mu}{d\mathcal{L}^n}$ belongs to L^p with $p \geq 1$, and the initial datum does not contain loops, then solutions of both quasi static and gradient flow evolution schemes, in the discrete case, do not contain loops
- Absence of crosses: if Ω is a two dimensional domain, and $\frac{d\mu}{d\mathcal{L}^2}$ belongs to L^p with $p > 4/3$, and the initial datum does not contain crosses, then solutions of gradient flow evolution schemes, in the discrete case, do not contain crosses
- Ahlfors regularity: if $\frac{d\mu}{d\mathcal{L}^n}$ belongs to L^p with $p > n/(n-1)$ ($p > 4/3$ in two dimensional domains), and the initial datum is Ahlfors regular, then solutions of both quasi static and gradient flow evolution schemes, in the discrete case, are Ahlfors regular.

These results are proven for the discrete evolutions, by adapting techniques used to prove similar results for solutions of the classic average distance problem.

An important subclass of these evolution schemes is the irreversible evolution, i.e. when the additional condition of monotonicity w.r.t. set inclusion is imposed. This comes free for some type of quasi static evolution. This condition is useful to model physical processes involving some kind of irreversibility, e.g. fracture propagation and membrane debonding, or in urban planning where removing the old network it is not advantageous, e.g. when planning to extend an existing subway network. The irreversibility condition can alter qualitative properties of solutions, as the absence of points with order at least 4 is not true anymore even if the initial datum does not contain points with order at least 4. In other words, one can say that solutions exhibit a “branching behavior” at some positive time, and one result of this thesis is to construct an example where an upper bound for such time can be determined.

Average distance problem with density penalization

The average distance problem, especially in the penalized formulation

$$\min \int \text{dist}(x, \cdot) d\mu + \lambda \mathcal{H}^1(\cdot) \quad (0.0.4)$$

can be used to approximate data distributions, which in this case would be represented by μ . Differently from the classic formulation in [14], it is not required that μ is absolutely continuous w.r.t. Lebesgue measure.

In data approximation it is often more convenient to work with parameterized curves instead of elements of $\mathcal{A}$, due to computational costs. The main result of Chapter 7 will be proving that the average distance problem is still well posed if restricted to parameterized curves (an *ad hoc* notion of convergence on the space of parameterized curves will be introduced), and injectivity is true for minimizers.

However, as proven in [52], even under strong assumptions on μ (indeed in [52] the counterexample was with $\mu \in \mathcal{L}^\infty$), minimizers of (0.0.4) can fail to be C^1 regular simple curves, with the second (distributional) derivative having an atom of positive mass.

In data approximation, this is equivalent to a strong loss of injectivity, as much data (a positive fraction of the data) is projected onto one single point. In order to overcome this issue, a term penalizing the density on Σ is introduced, and to avoid excessive geometric rigidity caused by projecting each point onto one of the closest points on Σ , a relaxed version of (0.0.5) is introduced:

Problem 0.0.1. *Given probability measure μ on $\mathbb{R}^d$ with compact support, and parameters $\lambda, \eta > 0$, $\alpha, q > 1$, solve*

$$\min \int_{\mathbb{R}^d \times \Sigma} |x - y|^\alpha d\Pi(x, y) + \lambda \mathcal{H}^1(\Sigma) + \varepsilon \int_{\Sigma} \nu^q d\mathcal{L}^1, \quad (0.0.5)$$

among triples (Σ, ν, Π) , where Σ varies among parameterized curves, ν among probability measure on Σ , and Π among measures on $\mathbb{R}^d \times \Sigma$ having first marginal μ and second marginal ν .

The term $\int_{\Sigma} \nu^q d\mathcal{L}^1$ is to be interpreted as $+\infty$ if $\nu \perp \mathcal{L}^1 \neq 0$, and $\int_{\Sigma} \left(\frac{d\nu}{d\mathcal{L}^1}\right)^q d\mathcal{L}^1$ if $\nu \ll \mathcal{L}^1$. This choice stems from Proposition 8.1.3. Injectivity is not guaranteed anymore, as the average distance functional has been replaced by a different functional, but still desirable. Thus a term penalized non injectivity will be introduced. The main result of Chapter 8 (Theorem 8.2.1) deals with regularity properties of ν , when (Σ, ν, Π) is a minimizer.

Outline

This thesis will be structured as follows:

- Chapter 1 will present a general overview of the optimal transport problem, in the classic Kantorovich formulation,
- Chapter 2 will recall notions from the theory of gradient flows in a purely metric spaces,
- Chapter 3 will recall results about solutions of the average distance problem,

- Chapter 4 (based on [33, 35]) will analyze solutions of the quasi static evolution,
- Chapter 5 (based on [34, 35]) will analyze solutions of gradient flow evolutions,
- Chapter 6 (based on [38]) will prove a weak second order regularity for minimizers of the average distance problem in the penalized formulation, i.e. BV regularity of the derivative, and a sort of “topological lower semicontinuity”,
- Chapter 7 (based on [37]) will analyze the average distance problem restricted among parameterized curves. The main result is to prove injectivity of minimizers,
- Chapter 8 (based on [36]) will analyze some regularity of ν when (Σ, ν, Π) is a minimizer.

Chapters 1 and 2, along with most of Chapter 3 are not original results, but intended to recall preliminary notions; Chapters 4, 5, 6, 7 and 8 contain new results.

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Chapter 1

Optimal Transport Theory

1.1 Introduction

The optimal transport theory was introduced in 1781 by Monge in [43], who proposed the following optimization problem: given an initial deposit of rock (“déblais”), one wants to build a castle (“remblais”) from it, with the minimum “effort”.

This chapter is dedicated to recall previous results concerning optimal transport theory. Sections 1.1, 1.2 and 1.3 are based on the work of several authors (see for instance [4], [18], [21], [24], [29], [56] among others) during the last decade, while Section 1.4 (based on a work by Brenier) deals with the quadratic cost penalization case.

The mathematical formulation, referred as *Monge formulation*, can be given in the following way:

Problem 1.1.1. *Given Polish spaces (X, μ) , (Y, ν) , with μ, ν probability measures, and a cost function $c : X \times Y \rightarrow [0, \infty]$ define*

$$\mathcal{T}(\mu, \nu) := \{f : X \rightarrow Y : f_{\#}\mu = \nu\}$$

and consider the minimization problem

$$\min_{T \in \mathcal{T}(\mu, \nu)} \int_X c(x, T(x)) d\mu(x).$$

For the original formulation proposed by Monge in [43], data were $X = Y = \mathbb{R}^d$, $c(x, y) := |x - y|$, μ and ν denoted the “déblais” and the “remblais” respectively.

Elements of $\mathcal{T}$ are often referred as *transport maps*, between μ and ν . This formulation presents several undesirable problems:

1. $\mathcal{T}(\mu, \nu) \neq \emptyset$ it is not guaranteed: a very easy example is $X = Y := \mathbb{R}$, $c(x, y) := |x - y|$, $\mu := \delta_0$, $\nu := \frac{\delta_{-1} + \delta_1}{2}$;
2. existence may not occur, i.e. (1.1.1) can admit no minima: an easy counterexample is $X = Y := B((0, 0), 1) \setminus \{(0, 0)\} \subseteq \mathbb{R}^2$, $\mu = \delta_{(1/2, 0)}$, $\nu := \delta_{(-1/2, 0)}$;

3. condition $f_{\#}\mu = \nu$ is not weakly sequentially closed: a counterexample is $T_n : \mathbb{R} \rightarrow \mathbb{R}$, $T_n(x) := T(nx)$ with $T : \mathbb{R} \rightarrow \mathbb{R}$ a 1-periodic function equal to 1 on $[0, 1/2)$ and -1 on $[1/2, 1)$, $\mu := \mathcal{L}_{|[0,1]}$, $\nu := \frac{1}{2}(\delta_{-1} + \delta_1)$. For every n equality $T_{n\#}\mu = \nu$ is true, but passing to the limit this becomes $0_{\#}\mu = \nu$ (0 denoting the null function on $\mathbb{R}$), clearly false.

A way to overcome these difficulties is provided by the Kantorovich formulation, first proposed in [28]:

Problem 1.1.2. Given Polish spaces (X, μ) , (Y, ν) , with μ, ν probability measures, and a cost function $c : X \times Y \rightarrow [0, \infty]$, define $\text{Adm}(\mu, \nu) := \{\xi \in \mathcal{M}(X \times Y) : \pi_{X\#}\xi = \mu, \pi_{Y\#}\xi = \nu\}$ where $\mathcal{M}(X \times Y)$ denotes the set of probability measures on $X \times Y$, $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ the natural projections, and consider the minimization problem

$$\min_{\xi \in \text{Adm}(\mu, \nu)} \int_{X \times Y} c(x, y) d\xi(x, y).$$

Elements of $\text{Adm}(\mu, \nu)$ are often referred as “transport plans”. This formulation provides several advantages over formulation 1.1.1:

- $\text{Adm}(\mu, \nu) \ni \mu \times \nu$, while $\mathcal{T}(\mu, \nu)$ can be empty,
- there exists a natural injection

$$i : \mathcal{T}(\mu, \nu) \rightarrow \text{Adm}(\mu, \nu), \quad i(T) := (id \times T)_{\#}\mu,$$

- $\text{Adm}(\mu, \nu)$ is convex and compact with respect to the narrow convergence, and

$$\xi \mapsto \int_{X \times Y} c(x, y) d\xi(x, y)$$

is linear,

- importantly, as proven in [24], [4] and [46], under some additional conditions the infimum of Monge problem is equal to the minimum of Kantorovich problem, which effectively renders the latter a relaxation of the former.

Existence is not guaranteed in general, but requires very mild conditions:

Theorem 1.1.3. Problem 1.1.2 admits a solution if the cost function $c : X \times Y \rightarrow \mathbb{R}$ is lower semicontinuous and bounded from below.

Proof. From inequality

$$\xi((X \times Y) \setminus (K_1 \times K_2)) \leq \mu(X \setminus K_1) + \nu(Y \setminus K_2)$$

for any $\xi \in \text{Adm}(\mu, \nu)$ one gets that if $K_1 \subseteq \mathcal{M}(X)$, $K_2 \subseteq \mathcal{M}(Y)$ are tight, then $\{\eta : \pi_{X\#}\eta \in K_1, \pi_{Y\#}\eta \in K_2\}$ is tight too. Combined with Ulam theorem, this gives $\text{Adm}(\mu, \nu)$ tight in $\mathcal{M}(X \times Y)$, and by Prokhorov theorem, relatively compact.

Given a sequence $\{\xi_n\}_{n=0}^\infty$ converging to ξ narrowly, and $\phi \in C_b(X)$, equality

$$\begin{aligned} \int_X \phi d\pi_{X\#}\xi &= \int_{X \times Y} \phi(x) d\xi(x, y) \\ &= \lim_{n \rightarrow \infty} \int_{X \times Y} \phi(x) d\xi_n(x, y) \\ &= \lim_{n \rightarrow \infty} \int_X \phi d\pi_{X\#}\xi_n \\ &= \int_X \phi d\mu \end{aligned}$$

holds, which gives compactness with respect to the narrow topology.

Assumptions on cost function c give the existence of a non decreasing sequence $\{c_n\} : X \times Y \rightarrow \mathbb{R}$ of continuous bounded functions verifying $c(x, y) = \sup_n c_n(x, y)$, and by monotone convergence theorem

$$\int_{X \times Y} c d\xi = \sup_n \int_{X \times Y} c_n d\xi$$

holds, which concludes the proof. $\square$

The set of measures belonging to $\operatorname{argmin}_{\xi \in \operatorname{Adm}(\mu, \nu)} \int_{X \times Y} c(x, y) d\xi(x, y)$ will be denoted by $\operatorname{Opt}(\mu, \nu)$, and referred as “optimal plans” from μ to ν with respect to the cost function c (this dependence will be omitted if there is no risk of confusion).

In the following, unless explicitly specified, the cost function c will always be considered lower semicontinuous and bounded from below. A natural connection between optimal maps and plans is provided by the following result:

Lemma 1.1.4. *Given Polish measure spaces (X, μ) , (Y, ν) , a transport plan $\xi \in \operatorname{Adm}(\mu, \nu)$ is induced by a map if and only if $\operatorname{supp}(\xi) \subseteq X \times Y$ is the graph of a function T . In this case $\xi = (id \times T)_\# \mu$.*

Proof. If a plan $\xi \in \operatorname{Adm}(\mu, \nu)$ is induced by a map $T \in \mathcal{T}(\mu, \nu)$, then obviously $\xi = (id \times T)_\# \mu$.

For the converse implication, define $\Gamma := \operatorname{supp}(\xi)$, and upon ξ -negligible sets, assume Γ is the graph of a function T . Due to inner regularity of measures it is possible to assume $\Gamma = \bigcup_{n=0}^\infty \Gamma_n$ σ -compact, thus $\pi_X(\Gamma)$ (the domain of T) is σ -compact, and $T|_{\pi_X(\Gamma_n)}$ is continuous, yielding T Borel map. Since $(x, y) \in \Gamma$ implies $y = T(x)$, given a test function $\phi : X \times Y \rightarrow \mathbb{R}$:

$$\begin{aligned} \int_{X \times Y} \phi(x, y) d\xi(x, y) &= \int_{X \times Y} \phi(x, T(x)) d\xi(x, y) \\ &= \int_X \phi(x, T(x)) d\mu(x) \end{aligned}$$

and the proof is complete. $\square$

1.2 Optimality conditions

This section is aimed to discuss optimality of transport plans; several preliminary notions are required.

Definition 1.2.1. Given Polish spaces X, Y , a cost function $c : X \times Y \rightarrow \mathbb{R}$, a set $\Gamma \in X \times Y$ is c -cyclically monotone if for any N -ple $\{(x_i, y_i)\}_{i=1}^N \subseteq \Gamma$ inequality

$$\sum_{i=1}^N c(x_i, y_i) \leq \sum_{i=1}^N c(x_i, y_{\sigma(i)})$$

holds for any permutation σ .

Definition 1.2.2. Given Polish spaces X, Y , a cost function $c : X \times Y \rightarrow \mathbb{R}$, a function $\psi : Y \rightarrow \mathbb{R}$:

- its c_+ -transform is

$$\psi^{c+} : X \rightarrow \mathbb{R}, \quad \psi^{c+}(x) = \inf_{y \in Y} c(x, y) - \psi(y);$$

- its c_- -transform is

$$\psi^{c-} : X \rightarrow \mathbb{R}, \quad \psi^{c-}(x) = \sup_{y \in Y} -c(x, y) - \psi(y).$$

The c_+ and c_- -transform for functions on X are defined in a similar way.

Definition 1.2.3. Given Polish spaces X, Y , a cost function $c : X \times Y \rightarrow \mathbb{R}$, a function $\psi : Y \rightarrow \mathbb{R}$:

- is c -concave if there exists $\varphi : X \rightarrow \mathbb{R}$ such that $\psi = \varphi^{c+}$;
- is c -convex if there exists $\varphi : X \rightarrow \mathbb{R}$ such that $\psi = \varphi^{c-}$.

For functions on X notions of c -concavity and c -convexity are defined in a similar way.

Definition 1.2.4. Given Polish spaces X, Y , a cost function $c : X \times Y \rightarrow \mathbb{R}$, and a c -concave function $\varphi : X \rightarrow \mathbb{R}$, the c -superdifferential is

$$\partial^{c+} \varphi := \{(x, y) \in X \times Y : \varphi(x) + \varphi^{c+}(y) = c(x, y)\}.$$

Given a c -convex function $\psi : Y \rightarrow \mathbb{R}$, the c -subdifferential is

$$\partial^{c-} \psi := \{(x, y) \in X \times Y : \varphi(x) + \varphi^{c-}(y) = -c(x, y)\}.$$

The next result, first studied in [47], is crucial in characterizing optimal transport plans:

Theorem 1.2.5. Given Polish spaces X, Y , a cost function $c : X \times Y \rightarrow \mathbb{R}$, continuous and bounded from below, assume there exist $\mu \in \mathcal{M}(X)$, $\nu \in \mathcal{M}(Y)$ such that

$$c(x, y) \leq f(x) + g(y)$$

for some $f \in L^1(X, \mu)$, $g \in L^1(Y, \nu)$. Then given any measure $\xi \in \text{Adm}(\mu, \nu)$ the following statements are equivalent:

1. $\xi \in \text{Opt}(\mu, \nu)$;
2. $\text{supp}(\xi) \subseteq X \times Y$ is cyclically monotone;
3. there exists $\varphi : X \rightarrow \mathbb{R}$ c -concave such that $\varphi \vee 0 \in L^1(X, \mu)$ and $\text{supp}(\xi) \in \partial^{c+} \varphi$.

Proof. Integrating $c(x, y) \leq f(x) + g(y)$ yields

$$\begin{aligned} \int_{X \times Y} c(x, y) d\eta(x, y) &\leq \int_{X \times Y} f(x) + g(y) d\eta(x, y) \\ &= \int_X f(x) d\mu(x) + \int_Y g(y) d\nu(y) < \infty \end{aligned}$$

for any $\eta \in \text{Adm}(\mu, \nu)$; using c bounded from below, this yields $c \in L^1(X \times Y, \eta)$ for any $\eta \in \text{Adm}(\mu, \nu)$.

(1) $\implies$ (2).

Assume there exists $\xi \in \text{Opt}(\mu, \nu)$ not cyclically monotone, thus there exists a N -ple $\{(x_i, y_i)\}_{i=1}^N \subseteq X \times Y$ and a permutation σ on $\{1, \dots, N\}$ such that

$$\sum_{i=1}^N c(x_i, y_i) > \sum_{i=1}^N c(x_i, y_{\sigma(i)}).$$

Arguing by continuity, for any $i = 1, \dots, N$ there exists neighborhoods $U_i \ni x_i, V_i \ni y_i$ such that

$$\sum_{i=1}^N c(u_i, v_i) > \sum_{i=1}^N c(u_i, v_{\sigma(i)})$$

for any $u_i \in U_i, v_i \in V_i$. Define

$$\Omega := \prod_{i=1}^N (U_i \times V_i)$$

and

$$\eta := \prod_{i=1}^N \frac{1}{\xi(U_i \times V_i) \xi|_{U_i \times V_i}}.$$

Denoting π_{U_i} and π_{V_i} projections of Ω on U_i and V_i , define

$$\zeta := \frac{1}{N} \min_{1 \leq i \leq N} \xi(U_i \times V_i) \sum_{i=1}^N ((\pi_{U_i} - \pi_{V_{\sigma(i)}})_{\#} \eta - (\pi_{U_i} - \pi_{V_i})_{\#} \eta)$$

and consider the competitor $\xi' := \xi + \zeta$: ζ verifies

- $\zeta^- \leq \xi$ and has null first and second marginal;
- $\int_{X \times Y} c(x, y) d\zeta(z, y) < 0$.

Thus the optimality of ξ is contradicted.

(2) $\implies$ (3).

Fix $(\bar{x}, \bar{y}) \in X \times Y$, as the goal is to determine a c -concave function φ such that $\Gamma \subseteq \partial^{c+} \varphi$, for any N -ple $\{(x_i, y_i)\}_{i=1}^N$

$$\begin{aligned} \varphi(x) &\leq c(x, y_1) - \varphi^{c+}(y_1) = c(x_1, y_1) + \varphi(x_1) \\ &\leq (c(x, y_1) - c(x_1, y_1)) + c(x_1, y_1) - \varphi^{c+}(y_1) \\ &\leq (c(x, y_1) - c(x_1, y_1)) + (c(x_1, y_2) - c(x_2, y_2)) + \varphi(x_2) \\ &\vdots \\ &\leq (c(x, y_1) - c(x_1, y_1)) + (c(x_1, y_2) - c(x_2, y_2)) + \cdots + \varphi(\bar{x}) \end{aligned}$$

Define

$$\varphi(x) := \inf_{N \geq 1, (x_i, y_i) \in \Gamma} (c(x, y_1) - c(x_1, y_1)) + (c(x_1, y_2) - c(x_2, y_2)) + \cdots + (c(x_N, \bar{y}) - c(\bar{x}, \bar{y})).$$

This function is c -concave. As Γ is c -cyclically monotone, $\varphi(\bar{x}) \geq 0$ follows, and choosing $N = 1$, i.e. $(x_1, y_1) = (\bar{x}, \bar{y})$, $\varphi(\bar{x}) = 0$ follows.

Apart from the definition, another (equivalent) characterization of c -superdifferential is

$$y \in \partial^{c+} \phi(x) \iff \varphi(x) - c(x, y) \geq \varphi(z) - c(z, x) \quad \forall z \in X.$$

Choosing again $N = 1$, $(x_1, y_1) = (\bar{x}, \bar{y})$, inequality

$$\varphi(x) \leq c(x, \bar{y}) - c(\bar{x}, \bar{y}) < f(x) + g(\bar{y}) - c(\bar{x}, \bar{y})$$

follows, yielding $\varphi \vee 0 \in L^1(X, \mu)$.

To prove $\Gamma \subseteq \partial^{c+} \varphi$, choose an arbitrary $(\bar{x}, \bar{y}) \in \Gamma$, impose $(x_1, y_1) = (\bar{x}, \bar{y})$ and

$$\begin{aligned} \varphi(x) &\leq c(x, \bar{y}) - c(\bar{x}, \bar{y}) + \inf (c(\bar{x}, y_2) - c(x_2, y_2)) + \cdots + (c(x_N, \bar{y}) - c(\bar{x}, \bar{y})) \\ &= c(x, \bar{y}) - c(\bar{x}, \bar{y}) + \varphi(\bar{x}). \end{aligned}$$

(3) $\implies$ (1).

Given an arbitrary $\eta \in \text{Adm}(\mu, \nu)$, as φ is c -concave, $\varphi(x) + \varphi^{c+}(y) \leq c(x, y)$ for any $x \in X, y \in Y$, with equality holding if and only if $(x, y) \in \text{supp}(\xi)$. Integrating on $X \times Y$ we have

$$\begin{aligned} \int_{X \times Y} c(x, y) d\xi(x, y) &= \int_{X \times Y} \varphi(x) + \varphi^{c+}(y) d\xi(x, y) \\ &= \int_X \varphi(x) d\mu(x) + \int_Y \varphi^{c+}(y) d\nu(y) \\ &= \int_{X \times Y} \varphi(x) + \varphi^{c+}(y) d\eta(x, y) \\ &\leq \int_{X \times Y} c(x, y) d\eta(x, y), \end{aligned}$$

and the proof is complete. □

A similar argument holds for transport maps: if for a map $T : X \rightarrow Y$ there exists a c -concave function φ such that $T(x) \in \partial^{c+}\varphi$ for any x , then for any measure $\mu \in \mathcal{T}(X)$ such that there exists $f \in L^1(X, \mu)$, $g \in L^1(Y, T_{\#}\mu)$ such that

$$c(x, y) \leq f(x) + g(y) \quad (x, y) \in X \times Y,$$

we have $T \in \text{Opt}(\mu, T_{\#}\mu)$.

This consequence allows to somewhat break the dependence between transport maps and reference measures.

1.3 Duality

The Kantorovich formulation of optimal transport problems involves minimizing a linear map with affine constraints. This kind of problem has often an associated dual problem; for Problem 1.1.2 this is:

Problem 1.3.1. *Given Polish measure spaces (X, μ) , (Y, ν) , and a cost function $C : X \times Y \rightarrow \mathbb{R}$, maximize*

$$\int_X \varphi(x) d\mu(x) + \int_Y \psi(y) d\nu(y)$$

among $\varphi \in L^1(X, \mu)$, $\psi \in L^1(Y, \nu)$, $\varphi(x) + \psi(y) \leq c(x, y)$ for any $(x, y) \in X \times Y$.

Under some additional hypothesis, Problems 1.1.2 and 1.3.1 are related by

$$\inf_{\xi \in \text{Adm}(\mu, \nu)} \int_{X \times Y} c(x, y) d\xi(x, y) = \sup_{\varphi \in L^1(X, \mu), \psi \in L^1(Y, \nu), \varphi + \psi \leq c} \int_X \varphi(x) d\mu(x) + \int_Y \psi(y) d\nu(y).$$

Before proving the main result, observe that Problem 1.3.1 admits an equivalent formulation: indeed

$$\inf_{\xi \in \text{Adm}(\mu, \nu)} \int c d\xi = \inf_{\eta} \int c d\eta + \chi(\eta),$$

where η varies among non negative probability measures on $X \times Y$, and χ is defined as

$$\chi(\eta) := \begin{cases} 0 & \text{if } \eta \in \text{Adm}(\mu, \nu) \\ \infty & \text{if } \eta \notin \text{Adm}(\mu, \nu) \end{cases}.$$

We claim

- the function χ can be written as

$$\chi(\xi) := \sup_{\varphi, \psi} \left\{ \int_X \varphi d\mu + \int_Y \psi d\nu - \int_{X \times Y} (\varphi(x) + \psi(y)) d\xi(x, y) \right\}$$

where $\pi_{X\#}\xi = \mu$, $\pi_{Y\#}\xi = \nu$ and (φ, ψ) varies in $C_b(X) \times C_b(Y)$.

Indeed if $\xi \in \text{Adm}(\mu, \nu)$ then $\chi(\xi) = 0$, while if $\xi \notin \text{Adm}(\mu, \nu)$ then $\chi(\xi) = \infty$, that is the argument of the supremum is greater than 0, thus multiplying (φ, ψ) by a suitable real number the supremum goes to ∞ .

Thus it holds:

$$\inf_{\xi \in \text{Adm}(\mu, \nu)} \int_{X \times Y} c d\xi = \inf_{\xi \in \mathcal{M}_+(X \times Y)} \sup_{\varphi, \psi} \left\{ \int_{X \times Y} c d\xi + \int_X \varphi d\mu + \int_Y \psi d\nu - \int_{X \times Y} \varphi(x) + \psi(y) d\xi(x, y) \right\}$$

where $\mathcal{M}_+(X \times Y)$ denotes the set of non negative probability measures on $X \times Y$, and φ, ψ vary in $C_b(X) \times C_b(Y)$ ($C_b(\cdot)$ denotes the set of continuous bounded functions). Define

$$\Phi(\xi, \varphi, \psi) := \int_{X \times Y} c d\xi + \int_X \varphi d\mu + \int_Y \psi d\nu - \int_{X \times Y} \varphi(x) + \psi(y) d\xi(x, y),$$

and notice that $\xi \mapsto \Phi(\xi, \varphi, \psi)$ and $(\varphi, \psi) \mapsto \Phi(\xi, \varphi, \psi)$ are convex and concave respectively, then from the min-max principle it holds

$$\inf_{\xi \in \text{Adm}(\mu, \nu)} \sup_{\varphi, \psi} \Phi(\xi, \varphi, \psi) = \sup_{\varphi, \psi} \inf_{\xi \in \mathcal{M}_+(X \times Y)} \Phi(\xi, \varphi, \psi)$$

which yields

$$\begin{aligned} \inf_{\xi \in \text{Adm}(\mu, \nu)} \int_{X \times Y} c d\xi &= \sup_{\varphi, \psi} \inf_{\xi \in \mathcal{M}_+(X \times Y)} \left\{ \int_{X \times Y} c d\xi + \int_X \varphi d\mu + \int_Y \psi d\nu - \int_{X \times Y} \varphi(x) + \psi(y) d\xi(x, y) \right\} \\ &= \sup_{\varphi, \psi} \left\{ \int_X \varphi d\mu + \int_Y \psi d\nu + \inf_{\xi \in \mathcal{M}_+(X \times Y)} \left\{ \int_{X \times Y} c(x, y) - \varphi(x) - \psi(y) d\xi(x, y) \right\} \right\}. \end{aligned}$$

The integrand in the quantity

$$\inf_{\xi \in \mathcal{M}_+(X \times Y)} \left\{ \int_{X \times Y} c(x, y) - \varphi(x) - \psi(y) d\xi(x, y) \right\}$$

is non negative and has infimum 0 if $c(x, y) \geq \varphi(x) + \psi(y)$ for any $(x, y) \in X \times Y$, while conversely if there exists $(x, y) \in X \times Y$ such that $c(x, y) < \varphi(x) + \psi(y)$ then define $\xi_n := n\delta_{(x, y)}$, and for $n \rightarrow \infty$ the infimum is $-\infty$. Thus it holds

$$\inf_{\xi \in \text{Adm}(\mu, \nu)} \int_{X \times Y} c d\xi = \sup_{\varphi, \psi} \int_X \varphi d\mu + \int_Y \psi d\nu$$

with (φ, ψ) varying in $C_b(X) \times C_b(Y)$, $\varphi(x) + \psi(y) \leq c(x, y)$ for any $(x, y) \in X \times Y$. Thus in Problem 1.3.1 it is possible to impose the additional condition $(\varphi, \psi) \in C_b(X) \times C_b(Y)$. The following result is crucial for the dual problem:

Theorem 1.3.2. *Given Polish measure spaces (X, μ) , (Y, ν) , a cost function $c : X \times Y \rightarrow \mathbb{R}$ continuous and bounded from below, and assume there exist $\mu \in \mathcal{M}(X)$, $\nu \in \mathcal{M}(Y)$, and $f \in L^1(X, \mu)$, $g \in L^1(Y, \nu)$ such that*

$$c(x, y) \leq f(x) + g(y) \quad \forall (x, y) \in X \times Y.$$

Then the following results hold:

- *the supremum of problem 1.3.1 is attained for some (φ, φ^{c+}) with $\varphi : X \rightarrow \mathbb{R}$ c -concave;*
- *the supremum of problem 1.3.1 and infimum of problem 1.1.2 are equal.*

Proof. Choose an arbitrary $\xi \in \text{Adm}(\mu, \nu)$: by hypothesis there exist $f \in L^1(X, \mu)$, $g \in L^1(Y, \nu)$ such that

$$c(x, y) \geq f(x) + g(y) \quad \forall (x, y) \in X \times Y,$$

and integrating on $X \times Y$ yields

$$\begin{aligned} \int_{X \times Y} c(x, y) d\xi(x, y) &\geq \int_{X \times Y} f(x) + g(y) d\xi(x, y) \\ &= \int_X f(x) d\mu(x) + \int_Y g(y) d\nu(y). \end{aligned}$$

This gives

$$\inf_{\eta \in \text{Adm}(\mu, \nu)} \int_{X \times Y} c(x, y) d\eta(x, y) \geq \sup_{\varphi \in L^1(X, \mu), \psi \in L^1(Y, \nu)} \int_X \varphi(x) d\mu(x) + \int_Y \psi(y) d\nu(y).$$

For the converse inequality, choose an arbitrary $\xi \in \text{Opt}(\mu, \nu)$ and by Theorem 1.2.5 there exists $\varphi : X \rightarrow \mathbb{R}$ c -concave such that $\text{supp}(\xi) \subseteq \partial^{c+} \varphi$. This yields, using the argument found in the proof of Theorem 1.2.5,

$$\begin{aligned} \int_{X \times Y} c(x, y) d\xi(x, y) &= \int_{X \times Y} \varphi(x) + \varphi^{c+}(y) d\xi(x, y) \\ &= \int_X \varphi(x) d\mu(x) + \int_Y \varphi^{c+}(y) d\nu(y) \end{aligned}$$

and $c \in L^1(X \times Y, \xi)$, implying $\varphi \in L^1(X, \mu)$, $\varphi^{c+} \in L^1(Y, \nu)$, thus (φ, φ^{c+}) is admissible solution for Problem 1.3.1, and the proof is complete. $\square$

This result shows that c -concave functions $\varphi : X \rightarrow \mathbb{R}$ belonging to $\mathcal{L}^1(X, \mu)$, such that (φ, φ^{c+}) is a point where

$$\sup_{\varphi \in L^1(X, \mu), \psi \in L^1(Y, \nu)} \int_X \varphi(x) d\mu(x) + \int_Y \psi(y) d\nu(y)$$

is attained, have a “special” role:

Definition 1.3.3. Given Polish measure spaces (X, μ) , (Y, ν) , a cost function $c : X \times Y \rightarrow \mathbb{R}$, a “ c -concave Kantorovich potential” is a function $\varphi : X \rightarrow \mathbb{R}$ belonging to $\mathcal{L}^1(X, \mu)$ such that (φ, φ^{c+}) maximizes

$$\int_X \varphi(x) d\mu(x) + \int_Y \psi(y) d\nu(y).$$

1.4 Quadratic cost function case

In this Section our goal is to analyze some properties of optimal transport maps when the cost function is quadratic. As seen previously even existence can be not true, as in the counterexample $X = Y := [0, 1]$, $\mu := \delta_0$, $\nu := \frac{1}{2}(\delta_{-1/2} + \delta_{1/2})$ and $c(x, y) := |x - y|$; similarly we cannot expect neither uniqueness nor continuity in the general case. We restrict the discussion to the special case:

- $X = Y = \mathbb{R}^d$, $d \geq 1$ with cost function $c(x, y) := |x - y|^2/2$.

In this case the following result provides a simple characterization of c -concavity and c -superdifferential:

Proposition 1.4.1. Given an arbitrary function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$, $d \geq 1$, φ is c -concave if and only if $x \mapsto \varphi^*(x) := \frac{|x|^2}{2} - \varphi(x)$ is convex and lower semicontinuous. In this case $y \in \partial^{c+}\varphi(x)$ if and only if $y \in \partial^-\varphi^*(x)$.

Proof. For the first part observe that

$$\begin{aligned} \varphi(x) = \inf_y \frac{|x - y|^2}{2} - \psi(y) &\iff \varphi(x) = \inf_y \frac{|x|^2}{2} + \langle x, -y \rangle + \frac{|y|^2}{2} - \psi(y) \\ &\iff \varphi(x) - \frac{|x|^2}{2} = \inf_y \langle x, -y \rangle + \left(\frac{|y|^2}{2} - \psi(y)\right) \\ &\iff \varphi^*(x) = \sup_y \langle x, y \rangle - \left(\frac{|y|^2}{2} - \psi(y)\right). \end{aligned}$$

For the second part observe that

$$\begin{aligned} y \in \partial^{c+}\varphi(x) &\iff \varphi(x) = \frac{|x - y|^2}{2} - \varphi^{c+}(y), \quad \varphi(z) \leq \frac{|z - y|^2}{2} - \varphi^{c+}(y) \quad \forall z \in \mathbb{R}^d \\ &\iff \varphi(x) - \frac{|x|^2}{2} = \langle x, -y \rangle + \frac{|y|^2}{2} - \varphi^{c+}(y), \\ \varphi(z) - \frac{|z|^2}{2} &\leq \langle z, -y \rangle + \frac{|y|^2}{2} - \varphi^{c+}(y), \quad \forall z \in \mathbb{R}^d \\ &\iff \varphi(z) - \frac{|z|^2}{2} \leq \varphi(x) - \frac{|x|^2}{2} + \langle z - x, -y \rangle, \quad \forall z \in \mathbb{R}^d \\ &\iff -y \in \partial^+(\varphi - \frac{|\cdot|^2}{2})(x) \\ &\iff y \in \partial^-\varphi^*(x), \end{aligned}$$

and the proof is complete. □

This result essentially transforms the problem of existence of optimal transport maps to the (better understood) one of understanding how the set of non differentiability points of a convex map is made. Some preliminary discussion is required:

Definition 1.4.2. A set $E \subseteq \mathbb{R}^d$ is a $c - c$ (“convex minus convex”) hypersurface if in a suitable coordinate system there exist convex functions $f, g : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ such that

$$E = \{(y, t) \in \mathbb{R}^d : y \in \mathbb{R}^{d-1}, t = f(y) - g(y)\}.$$

The following convex analysis result (whose proof will be skipped, see [3] for more details) holds:

Theorem 1.4.3. Given an arbitrary set $A \subseteq \mathbb{R}^d$, there exists a convex function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that A is contained in the set of non differentiability points of φ if and only if A can be covered by countably many $c - c$ hypersurfaces.

Definition 1.4.4. A probability measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is “regular” if any $c - c$ hypersurface is μ -negligible.

The next result is an important one concerning existence and uniqueness of optimal transport maps:

Theorem 1.4.5. Given an arbitrary $\mu \in \mathcal{M}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |x|^2 d\mu(x) < \infty$, the following statements are equivalent:

1. for any $\nu \in \mathcal{M}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |x|^2 d\nu(x) < \infty$, there exists a unique optimal transport plan from μ to ν , and this plan is induced by a transport map T ,
2. μ is regular.

In this case the map T is the gradient of a convex function.

Proof. (1) $\implies$ (2).

It is obvious that

$$c(x, y) := \frac{|x - y|^2}{2} \leq \frac{|x|^2}{2} + \frac{|y|^2}{2};$$

define $a(x) := |x|^2/2$, and by hypothesis $a \in L^1(\mathbb{R}^d, \mu)$. Thus we are under hypothesis of Theorems 1.2.5 and 1.3.2, and for any c -concave Kantorovich potential φ and any optimal plan ξ it holds $\text{supp}(\xi) \subseteq \partial^{c+}\varphi$. From Proposition 1.4.1 the map $\varphi^* := |\cdot|^2 - \varphi$ is convex and $\partial^{c+}\varphi = \partial^-\varphi^*$. Since φ^* is convex, $\nabla\varphi^*$ is well defined μ -a.e., as the set of its non differentiability points must be a $c - c$ hypersurface, and every optimal plan must be concentrated on its graph. Hence the optimal plan is unique and induced by $\nabla\varphi^*$.

(2) $\implies$ (1).

Assume there exists a convex function $\varphi^* : \mathbb{R}^d \rightarrow \mathbb{R}$ such that the set E of non differentiability points is not μ -negligible. Upon modifying φ^* outside a compact set, assume it has linear growth at infinity, and define $T(x)$ and $S(x)$ the element with smallest and biggest norm in $\partial^-\varphi^*(x)$ respectively, and the plan

$$\xi := \frac{1}{2}((id, T)_\# \mu + (id, S)_\# \mu).$$

The linear growth at infinity implies that $\nu := \pi_{Y\#} \xi$ has compact support, hence $\int_Y |x|^2 d\nu(x) < \infty$. Then $\xi \in \text{Adm}(\mu, \nu)$ is c -cyclically monotone, thus optimal, but it is not induced by a map, contradiction. $\square$

An interesting consequence is the following result about factorization of vector fields in $\mathbb{R}^d$: given a compact domain $\Omega \subseteq \mathbb{R}^d$, define $\mu_d := \frac{1}{\mathcal{L}^d(\Omega)} \mathcal{L}^d|_\Omega$, and

$$S(\Omega) := \{s : \Omega \longrightarrow \Omega : s \text{ Borel}, s_\# \mu_\Omega = \mu_\Omega\}.$$

The following result holds:

Proposition 1.4.6. *Given an arbitrary $S \in L^2(\mu_\Omega, \mathbb{R}^n)$ such that $\nu := S_\# \mu_\Omega$ is regular, then there exists unique $s \in S(\Omega)$ and $\nabla \varphi$ with φ convex such that $S = (\nabla \varphi) \circ s$. Moreover, s is the unique minimizer of*

$$\int_\Omega |S - f|^2 d\mu$$

among $f \in S(\Omega)$.

Proof. By hypothesis both μ_Ω and ν are regular with finite second moment. The claim

$$\inf_{f \in S(\Omega)} \int_\Omega |S - f| d\mu = \min_{\xi \in \text{Adm}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\xi(x, y) \quad (1.4.1)$$

would conclude the proof except for uniqueness.

Associate to each $f \in S(\Omega)$ a plan $\xi_f := (f, S)_\# \mu \in \text{Adm}(\mu, \nu)$, yielding

$$\inf_{f \in S(\Omega)} \int_\Omega |S - f| d\mu \geq \min_{\xi \in \text{Adm}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\xi(x, y).$$

Denote with ξ^* the unique optimal plan, and applying Theorem 1.4.5 twice yields

$$\xi^* = (id, \nabla \varphi)_\# \mu_\Omega = (\nabla \varphi^*, id)_\# \nu$$

for suitable convex functions φ, φ^* , which therefore satisfy $\nabla \varphi \circ \nabla \varphi^* = id$ μ -a.e.. Define $s := \nabla \varphi^* \circ S$, and $s_\# \mu_\Omega = \mu_\Omega$. Also $S = \nabla \varphi \circ s$, which proves the existence part. Identity

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\xi_f(x, y) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |f - S|^2 d\mu_\Omega &&= \int_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla \varphi^* \circ S - S|^2 d\mu_\Omega \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla \varphi^* - id|^2 d\nu &&= \min_{\xi \in \text{Adm}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\xi(x, y) \end{aligned}$$

proves the converse inequality in (1.4.1), and uniqueness of optimal plan ensures uniqueness of such minimizer.

To prove the uniqueness of such factorization, assume $S = (\nabla\varphi') \circ s'$ is another polar factorization, and notice that $\nabla\varphi'_\# \mu_\Omega = \nu$. Thus $\nabla\varphi'$ is a transport map from μ_Ω to ν , and gradient of a convex function, thus optimal map and $\nabla\varphi' = \nabla\varphi$ follows. $\square$

Chapter 2

Gradient flows

In this chapter we present the gradient flow theory, first in the Hilbertian context, and then in a metric setting. Recall that given a Riemannian manifold M , a point $\bar{x} \in M$ and a smooth function $F : M \rightarrow \mathbb{R}$, the gradient flow starting from $\bar{x}$ is a differentiable curve $x : \mathbb{R}_0^+ \rightarrow M$ verifying

$$\begin{cases} x(0) := \bar{x} \\ x'(t) = -\nabla F(x(t)) \end{cases}$$

An interpretation of this formulation is that the curve x is forced at each time to descend along the steepest descent direction, i.e. along the opposite of the direction of the gradient.

Section 2.1 will briefly recall the Hilbertian case (based on the work [8] by Brézis), while the remaining sections mainly deal with the purely metric setting (mainly based on [3] by Ambrosio, Gigli and Savaré, but including ideas of other authors).

2.1 Hilbertian theory

Let us quickly recall some notions about gradient flows in Hilbert spaces. The following extension of convexity will be useful:

Definition 2.1.1. *Given a Hilbert space H , a parameter $\lambda > 0$, a functional $F : H \rightarrow \mathbb{R} \cup \{\infty\}$ is λ -convex if for any $x, y \in H$, $t \in [0, 1]$ inequality*

$$F((1-t)x + ty) \leq (1-t)F(x) + tF(y) - \frac{\lambda}{2}t(1-t)|x - y|^2$$

holds.

Obviously a λ -convex function is convex too. As the functional F can take value the ∞ , we will denote with $D(F)$ the set $F^{-1}(\mathbb{R})$.

The subdifferential of a λ -convex function F at a point $x \in D(F)$ is defined as:

$$\partial^- F(x) := \{v \in H : F(x) + \langle v, y - x \rangle + \frac{\lambda}{2}|x - y|^2 \leq F(y) \text{ for any } y \in H\}.$$

The set $\partial^- F(x)$ is closed and convex, independently of the point x . Thus if $\partial^- F(x) \neq \emptyset$, then $\partial F(x)$ has an element of minimal norm, which we will denote with $\nabla F(x)$.

Moreover, given arbitrary points $x, y \in D(F)$, the “monotonicity inequality”

$$\langle v - w, x - y \rangle \geq \lambda |x - y|^2 \quad \forall v \in \partial F(x), w \in \partial^- F(y)$$

holds. A natural generalization of gradient flow in this context is:

Definition 2.1.2. Given a Hilbert space H , an element $\bar{x} \in H$, a λ -convex functional $F : H \rightarrow \mathbb{R}$, a gradient flow starting from $\bar{x}$ is a curve $x : \mathbb{R}_0^+ \rightarrow H$, locally absolutely continuous in $\mathbb{R}^+$, verifying

$$\begin{cases} x(0) := \bar{x} \\ x'(t) \in -\partial^- F(x(t)) \end{cases} \quad (2.1.1)$$

The following result (see for instance [8] and [9]) is crucial in dealing with existence and uniqueness in the Hilbertian case:

Theorem 2.1.3. Given a Hilbert space H , a λ -convex, lower semicontinuous functional $F : H \rightarrow \mathbb{R} \cup \{\infty\}$, the following results hold:

1. for any $\bar{x} \in \overline{D(F)}$ the curve defined in (2.1.1) exists and is unique;
2. for every time $t > 0$ the right derivative, i.e. the derivative computed considering only times $s \rightarrow t^+$, which will be denoted with $\frac{d_+}{dt}$, exists, and it hold
 - $\frac{d_+}{dt} x(t) = -\nabla F(x(t))$;
 - $\frac{d_+}{dt} F(x(t)) = -|\nabla F|^2(x(t))$, with $|\nabla F|$ denoting the slope;
 - $F(x(t)) \leq \inf_{v \in D(F)} \left\{ F(v) + \frac{1}{2t} |v - \bar{x}|^2 \right\}$;
 - $|\nabla F|^2(x(t)) \leq \inf_{v \in D(\partial F)} \left\{ |\nabla F|^2(v) + \frac{1}{t^2} |v - \bar{x}|^2 \right\}$.
3. $|x'(t)|, |\nabla F|(x(t))$ are in $L_{loc}^2(\mathbb{R}^+)$, $F(x(t)) \in AC(\mathbb{R}^+)$ ($AC(\cdot)$ denotes the set of absolutely continuous functions), and the “Energy Dissipation Equality”

$$F(x(t)) - F(x(s)) = \frac{1}{2} \int_t^s |\nabla F|^2(x(r)) dr + \frac{1}{2} \int_t^s |x'(r)|^2 dr$$

holds for any $0 < t \leq s < \infty$;

4. $x : \mathbb{R}_0^+ \rightarrow H$ is the unique solution of the “Evolution Variational Inequality”

$$\frac{1}{2} \frac{d}{dt} |\bar{x}(t) - y|^2 + F(x(t)) + \frac{\lambda}{2} |\bar{x}(t) - y|^2 \leq F(y) \quad y \in H, \text{ a.e. } t$$

with $\bar{x}(t)$ varying in $AC(\mathbb{R}^+)$ and converging to $\bar{x}$ for $t \rightarrow 0$. Moreover, given another element $\bar{y} \in \overline{D(F)}$, denoting by $y : \mathbb{R}_0^+ \rightarrow H$ the unique solution of (2.1.1) starting from $\bar{y}$, inequality

$$|x(t) - y(t)| \leq e^{-\lambda t} |\bar{x} - \bar{y}|$$

holds for any t ;

5. there exists a unique minimum x_{min} of F and inequality

$$F(x(t)) - F(x_{min}) \leq (F(\bar{x}) - F(x_{min}))e^{-2\lambda t}$$

holds. Then this gives

$$F(x) \geq F(x_{min}) + \frac{\lambda}{2} |x - x_{min}|^2 \quad \forall x \in H$$

and

$$|x(t) - x_{min}| \leq \sqrt{\frac{2(F(x(t)) - F(x_{min}))}{\lambda}} e^{-\lambda t}.$$

2.2 Metric space setting

In the previous Section we have given an introduction, with some results, about gradient flow theory in Hilbert spaces. As derivatives can be defined in a purely metric space, without requiring neither norms nor scalar products, the gradient flow theory can be extended to this context too.

Notice that in (2.1.1) the subdifferential is explicitly involved, i.e. a scalar product is required, thus cannot be extended to the metric context in this form. As we will see in the following, there are several analogues of (2.1.1) in the metric setting. Firstly, we need to define the notions of “gradient” and “speed” in the metric context:

Definition 2.2.1. Given a metric space (X, d) , a functional $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$, a point $x \in X$ such that $E(x) < \infty$, the “slope” of E in x is

$$|\nabla E|(x) := \limsup_{y \rightarrow x} \frac{(E(x) - E(y))^+}{d(x, y)}.$$

Definition 2.2.2. Given a metric space (X, d) , a curve $x : [0, 1] \rightarrow X$, the “speed” of x at a time $t \in [0, 1]$ is

$$|x'(t)| := \lim_{s \rightarrow t} \frac{d(x(s), x(t))}{|s - t|}.$$

This gives three different formulations of gradient flow:

Definition 2.2.3. Given a metric space (X, d) , a functional $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$, a point $\bar{x} \in X$ such that $E(\bar{x}) < \infty$. Then the curve $x : [0, \infty) \rightarrow X$ is gradient flow in the Energy Dissipation Inequality (EDI) sense starting in $\bar{x}$ if x is absolutely continuous, $x(0) = \bar{x}$ and

$$\begin{aligned} E(x(s)) + \frac{1}{2} \int_t^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_t^s |\nabla E|^2(x(r)) dr &\leq E(x(t)) \quad \text{a.e. } t > 0, \forall s \geq t \\ E(x(s)) + \frac{1}{2} \int_0^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_0^s |\nabla E|^2(x(r)) dr &\leq E(\bar{x}) \quad \forall s \geq 0 \end{aligned}$$

Definition 2.2.4. Given a metric space (X, d) , a functional $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$, a point $\bar{x} \in X$ such that $E(\bar{x}) < \infty$. Then the curve $x : [0, \infty) \rightarrow X$ is gradient flow in the Energy Dissipation Equality (EDE) sense starting in $\bar{x}$ if x is absolutely continuous, $x(0) = \bar{x}$ and

$$\begin{aligned} E(x(s)) + \frac{1}{2} \int_t^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_t^s |\nabla E|^2(x(r)) dr &= E(x(t)) \quad \text{a.e. } t > 0, \forall s \geq t \\ E(x(s)) + \frac{1}{2} \int_0^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_0^s |\nabla E|^2(x(r)) dr &= E(\bar{x}) \quad \forall s \geq 0 \end{aligned}$$

Definition 2.2.5. Given a metric space (X, d) , a parameter $\lambda > 0$, a functional $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$, a point $\bar{x} \in X$ such that $E(\bar{x}) < \infty$. Then the curve $x : [0, \infty) \rightarrow X$ is gradient flow with respect to λ in the Evolution Variation Inequality (EVI) sense starting in $\bar{x}$ if x is absolutely continuous, $x(0) = \bar{x}$ and

$$E(x(t)) + \frac{1}{2} \frac{d}{dt} d(x(t), y)^2 + \frac{\lambda}{2} d(x(t), y)^2 \leq E(y), \quad \forall y \in X, \text{ a.e. } t > 0. \quad (2.2.1)$$

In the Hilbert context all these formulations are equivalent, while in the metric setting

$$EVI \implies EDE \implies EDI$$

holds, with no converse implication holding true.

Proposition 2.2.6. Given a metric space (X, d) , a parameter $\lambda > 0$, a lower semicontinuous functional $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$, a point $\bar{x} \in X$, and assume $x : [0, \infty) \rightarrow X$ is gradient flow in EVI sense with respect to λ . Then $x : [0, \infty) \rightarrow X$ is gradient flow in EDE sense too.

Proof. We will first assume that $x : [0, \infty) \rightarrow X$ is locally Lipschitz. From triangular inequality

$$\frac{1}{2} \frac{d}{dt} d(x(t), y)^2 \geq -|\dot{x}(t)| d(x(t), y), \quad \text{a.e. } t > 0, \forall y \in X$$

holds, and combining with inequality (2.2.1) we get

$$-|\dot{x}(t)| d(x(t), y) + \frac{\lambda}{2} d(x(t), y)^2 + E(x(t)) \leq E(y), \quad \text{a.e. } t > 0, \forall y \in X$$

and then

$$|\nabla E|(x(t)) = \limsup_{y \rightarrow x(t)} \frac{(E(x(t)) - E(y))^+}{d(x(t), y)} \leq |\dot{x}(t)|, \quad \text{a.e. } t > 0.$$

Fix an interval $[a, b] \subseteq (0, \infty)$, let L be the Lipschitz constant of x in $[a, b]$, and for any $y \in X$

$$\frac{d}{dt}d(x(t), y)^2 \geq -|\dot{x}(t)|d(x(t), y) \geq -Ld(x(t), y)$$

holds for a.e. $t \in [a, b]$. Combining with inequality (2.2.1)

$$-Ld(x(t), y) + \frac{\lambda}{2}d(x(t), y)^2 + E(x(t)) \leq E(y), \quad a.e. t \in [a, b], \forall y \in X$$

follows, and by lower semicontinuity of $t \mapsto E(x(t))$ this holds for every $t \in [a, b]$. Choosing $y = x(s)$ yields

$$|E(x(s)) - E(x(t))| \leq Ld(x(t), x(s)) - \frac{\lambda}{2}d(x(t), x(s))^2 \leq L|t - s| \left(L + \frac{|\lambda|}{2}L|t - s| \right), \quad \forall t, s \in [a, b]$$

implying $t \mapsto E(x(t))$ locally Lipschitz. Moreover

$$\begin{aligned} -\frac{d}{dt}E(x(t)) &= \lim_{h \rightarrow 0} \frac{E(x(t)) - E(x(t+h))}{h} \\ &= \lim_{h \rightarrow 0} \frac{E(x(t)) - E(x(t+h))}{d(x(t+h), x(t))} \frac{d(x(t+h), x(t))}{h} \\ &\leq |\nabla E|(x(t))|\dot{x}(t)| \\ &\leq \frac{1}{2}|\nabla E|^2(x(t)) + \frac{1}{2}|\dot{x}(t)|^2, \quad a.e. t > 0. \end{aligned}$$

The opposite inequality remains: integrating inequality (2.2.1) from t to $t+h$ we get

$$\frac{d(x(t+h), y)^2 - d(x(t), y)^2}{2} + \int_t^{t+h} E(x(s))ds + \frac{\lambda}{2} \int_t^{t+h} d(x(s), y)^2 ds \leq hE(y).$$

Putting $y = x(t)$ this reads

$$\frac{d(x(t+h), x(t))^2}{2} + \int_t^{t+h} E(x(s))ds + \frac{\lambda}{2} \int_t^{t+h} d(x(s), x(t))^2 ds \leq hE(x(t))$$

thus

$$\frac{d(x(t+h), x(t))^2}{2} \leq \int_t^{t+h} E(x(t)) - E(x(s))ds + \frac{|\lambda|}{6}L^2h^3 \quad (2.2.2)$$

$$= h \int_0^1 E(x(t)) - E(x(t+hr))dr + \frac{|\lambda|}{6}L^2h^3. \quad (2.2.3)$$

Let A be the set of differentiability points of $t \mapsto E(x(t))$ and where $|\dot{x}(t)|$ exists, choose $t \in A \cap (0, \infty)$, inequality (2.2.2) yields

$$\frac{d(x(t+h), x(t))^2}{2h^2} \leq \frac{1}{h} \int_0^1 E(x(t)) - E(x(t+hr))dr + \frac{|\lambda|}{6}L^2h,$$

taking the limit $h \rightarrow 0$ this reads

$$\frac{1}{2}|\dot{x}(t)|^2 \leq \lim_{h \rightarrow 0} \int_0^1 \frac{E(x(t))E(x(t+hr))}{r} dr = -\frac{d}{dt}E(x(t)) \int_0^1 r dr = -\frac{1}{2} \frac{d}{dt}E(x(t)),$$

and combining with

$$|\nabla E|(x(t)) \leq |\dot{x}(t)|,$$

we finally get

$$-\frac{d}{dt}E(x(t)) \geq |\dot{x}(t)|^2 + \frac{1}{2}|\nabla E|^2(x(t)), \quad a.e. \ t > 0.$$

Lastly, we have to prove the local Lipschitz continuity of x . It is immediate to verify that $t \mapsto x(t+h)$ is gradient flow in EVI sense starting in $x(h)$ for any $h > 0$. The last point of Theorem 2.1.3 gives that distance between two such curves is contractive up to an exponential factor, thus we have

$$d(x(s), x(s+h)) \leq \exp(-\lambda(s-t))d(x(t), x(t+h)), \quad \forall s > t.$$

Let B the set where the metric derivative of x exists, choosing $t \in B \cap (0, \infty)$ we get

$$\frac{1}{h}d(x(s), x(s+h)) \leq \exp(-\lambda(s-t))\frac{1}{h}d(x(t), x(t+h)), \quad \forall s > t$$

and taking the limit $h \rightarrow 0$

$$|\dot{x}(s)| = \lim_{h \rightarrow 0} \frac{d(x(s), x(s+h))}{|h|} \leq \exp(-\lambda(s-t))|\dot{x}(t)|$$

for any $B \ni s > t$, thus the curve x is locally Lipschitz in $(0, \infty)$. □

2.2.1 Discrete evolution

In this subsection we will present a “discretized” version of gradient flow evolutions: let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and F a convex and lower semicontinuous function. Fix $\bar{x} \in \overline{D(F)}$ and $\tau > 0$, we can define recursively the sequence

$$\mathbb{N} \ni n \mapsto w_\tau(n)$$

as

$$w_\tau(0) := \bar{x}$$

and $w_\tau(n+1)$ chosen among the minimizers of

$$X \ni x \mapsto F(x) + \frac{|x - w_\tau(n)|^2}{2\tau}.$$

Existence and uniqueness of such minimizer is easy in the Hilbertian case, thus the sequence $\{w_\tau(n)\}_{n \in \mathbb{N}}$ is well defined; the Euler-Lagrange equation of $w_\tau(n+1)$ is

$$\frac{w_\tau(n+1) - w_\tau(n)}{\tau} \in -\partial^- F(w_\tau(n+1)),$$

and this is a time discretization of (2.1.1). Here it is natural to consider the curve

$$x : [0, \infty) \longrightarrow X, \quad x(t) := w_\tau \left(\left\lceil \frac{t}{\tau} \right\rceil \right)$$

with $\lceil \cdot \rceil$ denoting the integer part mapping.

The same construction can be done in a purely metric setting: given a metric space (X, d) , a lower semicontinuous functional F , $\tau > 0$, and an element $\bar{x} \in \overline{D(F)}$, consider the sequence $\{w_\tau(n)\}_{n \in \mathbb{N}}$ defined recursively as

$$\begin{cases} w_\tau(0) := \bar{x}' \\ w_\tau(n+1) \in \operatorname{argmin}_{y \in Y} F(y) + \frac{d(y, w_\tau(n))^2}{2\tau} \end{cases}, \quad (2.2.4)$$

and similarly the curve

$$y : [0, \infty) \longrightarrow Y, \quad y(t) := w_\tau \left(\left\lceil \frac{t}{\tau} \right\rceil \right)$$

can be associated. This construction is often referred as “implicit Euler scheme”.

Definition 2.2.7. Let (X, d) be a metric space, $E : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous functional, $\bar{x} \in \overline{D(E)}$ a given point, and $\tau > 0$ a given parameter. A “discrete solution” is a map $x : [0, \infty) \longrightarrow X$ defined by

$$x(t) := w \left(\left\lceil \frac{t}{\tau} \right\rceil \right)$$

where $w(\cdot)$ is defined as in (2.2.4).

Differently from the Hilbertian case, in the purely metric context neither existence nor uniqueness is guaranteed, and without further assumptions, neither of them is true; two sets of assumptions, one ensuring existence and the other allowing the passage to the limit as $\tau \rightarrow 0$, are required. We will first assume existence:

Assumption 2.2.8. Let (X, d) be a metric space, $E : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function, bounded from below. We will assume that there exists $\bar{\tau}$ such that for every $\tau \in [0, \bar{\tau}]$ and $\bar{x} \in \overline{D(E)}$ the map

$$x \mapsto E(x) + \frac{d(x, \bar{x})^2}{2\tau}$$

has at least a minimum.

This assumption ensures that for any point $\bar{x}$, discrete solutions exist for an uniform interval of time steps, with length not depending on $\bar{x}$. The key problem here is to prove that these solutions verify a discrete variant of Energy Dissipation Inequality, which passes to the limit $\tau \rightarrow 0$. The next result is crucial:

Theorem 2.2.9. *Given a metric space (X, d) , and a lower semicontinuous function $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$ bounded from below verifying Assumption 2.2.8, fix a point $\bar{x} \in \overline{D(E)}$ and consider the function*

$$x : [0, \bar{\tau}] \rightarrow X$$

such that for any $\tau \in [0, \bar{\tau}]$, $x(\tau)$ is a minimizer of

$$x \mapsto E(x) + \frac{d(x, \bar{x})^2}{2\tau}.$$

Then the map

$$[0, \bar{\tau}] \ni \tau \mapsto E(x(\tau)) + \frac{d(x(\tau), \bar{x})^2}{2\tau}$$

is locally Lipschitz in $(0, \bar{\tau})$, and

$$\frac{d}{d\tau}(E(x(\tau)) + \frac{d(x(\tau), \bar{x})^2}{2\tau}) = -\frac{d(x(\tau), \bar{x})^2}{2\tau^2} \quad (2.2.5)$$

holds for a.e. $\tau \in (0, \bar{\tau})$.

Proof. Fix $\tau_0 \in (0, \bar{\tau})$, due to minimality properties of $x(\tau_0)$ inequality

$$E(x(\tau_0)) + \frac{d(x(\tau_0), \bar{x})^2}{2\tau} \leq E(x(\tau_1)) + \frac{d(x(\tau_1), \bar{x})^2}{2\tau}$$

holds for any $\tau_1 \in (0, \bar{\tau})$, thus

$$E(x(\tau_0)) + \frac{d(x(\tau_0), \bar{x})^2}{2\tau} - E(x(\tau_1)) + \frac{d(x(\tau_1), \bar{x})^2}{2\tau} \leq \left(\frac{1}{2\tau_0} - \frac{1}{2\tau_1} \right) d(x(\tau_1), \bar{x})^2 = \frac{\tau_1 - \tau_0}{2\tau_0\tau_1} d(x(\tau_1), \bar{x})^2.$$

With a symmetrical argument

$$E(x(\tau_0)) + \frac{d(x(\tau_0), \bar{x})^2}{2\tau} - E(x(\tau_1)) + \frac{d(x(\tau_1), \bar{x})^2}{2\tau} \geq \frac{\tau_1 - \tau_0}{2\tau_0\tau_1} d(x(\tau_1), \bar{x})^2,$$

thus $\tau \mapsto E(x(\tau)) + \frac{d(x(\tau), \bar{x})^2}{2\tau}$ is locally Lipschitz, and the proof is complete by taking the limit $\tau_1 \rightarrow \tau_0$, which gives immediately (2.2.5). $\square$

Lemma 2.2.10. *In the context of Theorem 2.2.9, using the same notations, the following properties hold:*

1. $\tau \mapsto \frac{d(x(\tau), \bar{x})^2}{2\tau}$ is non decreasing,
2. $\tau \mapsto E(x(\tau))$ is non increasing,
3. $|\nabla E|(x(\tau)) \leq \frac{d(x(\tau), \bar{x})}{\tau}.$

Proof. Let $\tau_0, \tau_1 \in (0, \bar{\tau})$, $\tau_0 < \tau_1$, and from minimality properties of $x : [0, \bar{\tau}] \rightarrow X \cup \{+\infty\}$ the following inequalities hold:

$$\begin{aligned} E(x(\tau_0)) + \frac{d(x(\tau_0), \bar{x})^2}{2\tau_0} &\leq E(x(\tau_1)) + \frac{d(x(\tau_1), \bar{x})^2}{2\tau_0} \\ E(x(\tau_1)) + \frac{d(x(\tau_1), \bar{x})^2}{2\tau_1} &\leq E(x(\tau_0)) + \frac{d(x(\tau_0), \bar{x})^2}{2\tau_1} \end{aligned}$$

and using $\tau_0 < \tau_1$, summing side by side yields $d(x(\tau_0), \bar{x}) \leq d(x(\tau_1), \bar{x})$, i.e. $\tau \mapsto d(x(\tau), \bar{x})$ non decreasing. This leads to

$$E(x(\tau_1)) + \frac{d(x(\tau_0), \bar{x})^2}{2\tau_1} \leq E(x(\tau_1)) + \frac{d(x(\tau_1), \bar{x})^2}{2\tau_1}$$

and combined with the minimality of $x(\tau_1)$ we have

$$E(x(\tau_1)) + \frac{d(x(\tau_0), \bar{x})^2}{2\tau_1} \leq E(x(\tau_1)) + \frac{d(x(\tau_1), \bar{x})^2}{2\tau_1} \leq E(x(\tau_0)) + \frac{d(x(\tau_0), \bar{x})^2}{2\tau_1}$$

which implies $\tau \mapsto E(x(\tau))$ non increasing.

For the last point, fix $\tau \in (0, \bar{\tau})$, and from minimality properties of $x(\tau)$ we have

$$E(x(\tau)) + \frac{d(x(\tau), \bar{x})^2}{2\tau} \leq E(y) + \frac{d(y, \bar{x})^2}{2\tau}, \quad \forall y \in X$$

which leads to

$$\frac{E(x(\tau)) - E(y)}{d(x(\tau), y)} \leq \frac{d(y, \bar{x})^2 - d(x(\tau), \bar{x})^2}{2\tau d(x(\tau), y)} \leq \frac{d(x(\tau), \bar{x}) + d(y, \bar{x})}{2\tau}$$

and

$$\begin{aligned} |\nabla E|(x(\tau)) &= \limsup_{y \rightarrow x(\tau)} \frac{(E(x(\tau)) - E(y))^+}{d(x(\tau), y)} \\ &\leq \limsup_{y \rightarrow x(\tau)} \frac{d(x(\tau), \bar{x}) + d(y, \bar{x})}{2\tau} = \frac{d(x(\tau), \bar{x})}{\tau}, \end{aligned}$$

completing the proof. □

Another definition is useful:

Definition 2.2.11. In a metric space (X, d) , given an initial datum $\bar{x}$, a sequence $\{\tau_j\}_{j \in \mathbb{N}} \downarrow 0$, and implicit Euler schemes as defined in (2.2.4)

$$\begin{cases} w_j(0) := \bar{x} \\ w_j(k+1) \in \operatorname{argmin} E(\cdot) + \frac{1}{2\tau_j} d(\cdot, w_j(k))^2 \end{cases}$$

with associated functions (where I is a given interval)

$$x_j : I \rightarrow X, \quad x_j(t) := w_j(\lfloor t/\tau_j \rfloor),$$

a function $x : I \rightarrow X$ is a minimizing movement with initial datum $\bar{x}$ if there exists subsequence $\{\tau_{j_h}\}_{h \in \mathbb{N}}$ such that

$$x(t) = \lim_{h \rightarrow \infty} x_{j_h}(t)$$

for any t .

In view of these results it is natural to introduce another time-discretized variant in the minimizing movement scheme, the “variational interpolation”:

Definition 2.2.12. Given a metric space (X, d) and a lower semicontinuous functional $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$ bounded from below, satisfying Assumption 2.2.8, fix a point $\bar{x} \in \overline{D(E)}$. The map

$$[0, \infty) \ni t \mapsto x_\tau(t)$$

defined by

- $x_\tau(0) := \bar{x}$,
- $x_\tau((n+1)\tau)$ chosen among minimizers of (2.2.4) with $\bar{x}$ replaced by $x_\tau(n\tau)$,
- $x_\tau(t)$, $t \in (n\tau, (n+1)\tau)$ chosen among minimizers of (2.2.4) with $\bar{x}$ and τ replaced by $x_\tau(n\tau)$ and $t - n\tau$ respectively.

In the context of variational interpolation, we are able to extend notions of speed and slope here. Using notations from Definition 2.2.12:

1. the “discrete speed” is the map

$$Dsp_\tau : [0, \infty) \rightarrow [0, \infty), \quad Dsp_\tau(t) := \frac{d(x_\tau(n\tau), x_\tau((n+1)\tau))}{\tau}, \quad t \in (n\tau, (n+1)\tau),$$

2. the “discrete slope” is the map

$$Dsl_\tau : [0, \infty) \rightarrow [0, \infty), \quad Dsl_\tau(t) := \frac{d(x_\tau(n\tau), x_\tau(t))}{t - n\tau}, \quad t \in (n\tau, (n+1)\tau).$$

Despite this definition of discrete slope seems unrelated to Definition 2.2.1, from Lemma 2.2.10 it holds $|\nabla E|(x_\tau(t)) \leq Dsl_\tau(t)$, and passing to the limit as $\tau \downarrow 0$ will produce the slope from Definition 2.2.1 (see the proof of Theorem 2.2.14).

With these definitions, and notations from Theorem 2.2.9, equation (2.2.5) can be rewritten as

$$E(x_\tau(s)) + \frac{1}{2} \int_t^s |Dsp_\tau(r)|^2 dr + \frac{1}{2} \int_t^s |Dsl_\tau(r)|^2 dr = E(x_\tau(t)), \quad \forall t = n\tau, s = m\tau, n < m \in \mathbb{N}. \quad (2.2.6)$$

Assumption 2.2.8 is quite general, and guarantees existence of discrete solutions. Our next goal will be to pass to the limit for $\tau \rightarrow 0$, and stronger assumptions are required:

Assumption 2.2.13. Let (X, d) be a metric space, $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$ a functional, assume the following conditions hold:

1. E is bounded from below, and its sublevels are boundedly compact, i.e. $\{E \leq c\} \cap \overline{B(x, r)}$ is compact for any $c \in \mathbb{R}, r > 0$ and $x \in X$,
2. the slope $|\nabla E| : D(E) \rightarrow [0, \infty]$ is lower semicontinuous,
3. for any sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to x , implication

$$\sup_{n \in \mathbb{N}} \{|\nabla E|(x_n), E(x_n)\} < \infty \implies E(x_n) \rightarrow E(x)$$

is true.

Under these assumptions the following result holds:

Theorem 2.2.14. Let (X, d) be a metric space and $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$ a functional satisfying Assumption 2.2.13 and equation (2.2.5). Fix $\bar{x} \in \overline{D(E)}$, $\tau \in (0, \bar{\tau})$, and consider a discrete solution $x : [0, \bar{\tau}] \rightarrow X$ defined via variational interpolation. Then the following results hold:

- the set $\{x_\tau(t)\}_\tau$ is relatively compact in the set of curves in X with respect to the uniform local convergence,
- any limit curve is a gradient flow in the EDI sense.

Proof. The proof is divided in two parts: the first deals with compactness, while the second concerns the passage to the limit $\tau \rightarrow 0$.

Compactness: from inequality (2.2.6) we have

$$d(x_\tau(t), \bar{x}) \leq \left(\int_0^T |Dsp_\tau(r)| dr \right)^2 \leq T \int_0^T |Dsp_\tau(r)|^2 dr \leq 2T(E(\bar{x}) - \inf E)$$

for any $t \leq T$, $T = n\tau$ with $n \in \mathbb{N}$. Therefore for any $T > 0$ the set $\{x_\tau(t)\}_{t \leq T}$ is uniformly bounded in τ ; as it is also contained in $\{E \leq E(\bar{x})\}$, it is relatively compact. Using an Ascoli-Arzelà like argument on inequality

$$d(x_\tau(t), x_\tau(s)) \leq \left(\int_s^t |Dsp_\tau(r)| dr \right)^2 \leq 2(s-t)(E(\bar{x}) - \inf E), \quad \forall t = n\tau, s = m\tau, n < m \in \mathbb{N}$$

relative compactness with respect to local uniform convergence follows.

Limit $\tau \rightarrow 0$: consider a sequence $\{\tau_n\}_{n \in \mathbb{N}} \rightarrow 0$ such that $\{x_{\tau_n}(t)\}$ converges (locally uniformly) to a limit curve $x : [0, \infty) \rightarrow X$. It is not difficult to check that $t \mapsto x(t)$ is absolutely continuous and satisfies

$$\int_t^s |\dot{x}(r)|^2 dr \leq \liminf_{n \rightarrow \infty} \int_t^s |Dsp_{\tau_n}(r)|^2 dr \quad \forall 0 \leq t < s.$$

By lower semicontinuity of $|\nabla E|$ and Lemma 2.2.10 inequality

$$|\nabla E|(x(t)) \leq \liminf_{n \rightarrow \infty} |\nabla E|(x_{\tau_n}(t)) \leq \liminf_{n \rightarrow \infty} Dsl_{\tau_n}(t)$$

holds, and Fatou's lemma gives that for any $t < s$

$$\int_t^s |\nabla E|^2(x(r))dr \leq \int_t^s \liminf_{n \rightarrow \infty} |\nabla E|^2(x_{\tau_n}(r))dr \leq \liminf_{n \rightarrow \infty} \int_t^s |Dsl_{\tau_n}(r)|^2 dr \leq 2T(E(\bar{x}) - \inf E).$$

From this follows that the L^2 norm of

$$f(t) := \liminf_{n \rightarrow \infty} |\nabla E|(x_{\tau_n}(t))$$

on $[0, \infty)$ is finite, thus $\{f < \infty\}$ has full Lebesgue measure, and for each $t \in \{f < \infty\}$ there exists a subsequence $\{t_{n_k}\}_{k \in \mathbb{N}} \rightarrow 0$ such that $\sup_{k \in \mathbb{N}} |\nabla E|(x_{\tau_{n_k}}(t)) < \infty$. Thus by Assumption 2.2.13 $E(x_{\tau_{n_k}}(t)) \rightarrow E(x(t))$ and the lower semicontinuity of E guarantees $E(x(s)) \leq \liminf_{k \rightarrow \infty} E(x_{\tau_{n_k}}(s))$ for every $s \geq t$. Thus passing to the limit $k \rightarrow \infty$ in (2.2.6) gives

$$E(x(s)) + \frac{1}{2} \int_t^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_t^s |\nabla E|^2(x(r))dr \leq E(x(t)) \quad \forall t \in \{f < \infty\}, \forall s \geq t.$$

Finally, passing to the limit $k \rightarrow \infty$ in equation (2.2.6) with $t = 0$ gives

$$E(x(s)) + \frac{1}{2} \int_0^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_0^s |\nabla E|^2(x(r))dr \leq E(\bar{x}) \forall s \geq 0,$$

and the proof is complete. $\square$

In this generality equality in EDI is false, as the main problem is that the map $\tau \mapsto E(x(\tau))$ can fail to be absolutely continuous. Consider the following counterexample:

- Choose $C \subseteq [0, 1]$ the Cantor set, endowed with Euclidean distance, and a function

$$C^0 \cap L^1 \ni f : [0, 1] \rightarrow [1, \infty], \quad f|_C = +\infty, \quad f|_{[0,1] \setminus C} \in C^\infty;$$

let $g : [0, 1] \rightarrow [0, \infty)$ be the “devil staircase” built over C (i.e. g is a continuous function with $g|_C = [0, 1]$, $g(0) = 0$, $g(1) = 1$ and constant on every connected component of $[0, 1] \setminus C$). Define $E, \tilde{E} : [0, 1] \rightarrow \mathbb{R}$ by

$$E(x) := -g(x) - \int_0^x f(y)dy, \quad \tilde{E}(x) := - \int_0^x f(y)dy$$

and Assumptions 2.2.8 and 2.2.13 are easily verified by both E and $\tilde{E}$.

Build a gradient flow starting from 0, and it is possible to check that in both cases the minimizing movement scheme converges to absolutely continuous curves $x, \tilde{x} : [0, \infty) \rightarrow [0, 1]$ satisfying

$$x'(t) = -|\nabla E|(x(t)), \quad a.e. t$$

$$\tilde{x}'(t) = -|\nabla \tilde{E}|(\tilde{x}(t)), \quad a.e. t$$

respectively.

For any $x \in [0, 1]$ equality $|\nabla E|(x(t)) = |\nabla \tilde{E}|(\tilde{x}(t)) = f(x)$ hold, and combined with the fact $f \geq 1$ gives that both equations admit a unique solution, thus $x = \tilde{x}$. The effect of g is not evident on these solutions, and it is easy to check that EDE holds for $\tilde{E}$, but not for E .

2.2.2 Geodesically convex case

Geodesically convex functions are generalization to metric spaces of convex functions on linear spaces:

Definition 2.2.15. *Given a metric space (X, d) , a functional $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$, and $\lambda > 0$, E is “ λ -geodesically convex” if for any $x, y \in X$ there exists a constant speed geodesic $\gamma : [0, 1] \rightarrow X$, $\gamma(0) = x$, $\gamma(1) = y$ such that*

$$E(\gamma(t)) \leq (1-t)E(x) + tE(y) - \frac{\lambda}{2}t(1-t)d(x, y)^2. \quad (2.2.7)$$

In this subsection we will analyze gradient flows by assuming that the function E will be geodesically convex:

Assumption 2.2.16. *Let (X, d) be a metric space, $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$, assume that E is lower semi-continuous, λ -geodesically convex for some $\lambda \in \mathbb{R}$; moreover assume that the sublevels of E are boundedly compact.*

Under this hypothesis the main goal is to prove the existence of gradient flows in EDE sense.

Lemma 2.2.17. *Let (X, d) be a metric space, $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$ verifying Assumption 2.2.16, then for any $x \in \overline{D(E)}$*

$$|\nabla E|(x) = \sup_{y \neq x} \left(\frac{E(x) - E(y)}{d(x, y)} + \frac{\lambda}{2}d(x, y) \right)^+. \quad (2.2.8)$$

Proof. It is easy to observe that

$$|\nabla E|(x) = \limsup_{y \rightarrow x} \left(\frac{E(x) - E(y)}{d(x, y)} + \frac{\lambda}{2}d(x, y) \right)^+ \leq \sup_{y \neq x} \left(\frac{E(x) - E(y)}{d(x, y)} + \frac{\lambda}{2}d(x, y) \right)^+.$$

For the converse inequality, fix $y \neq x$, and let $\gamma : [0, 1] \rightarrow X$ a constant speed geodesic with $\gamma(0) = x$, $\gamma(1) = y$, and due to Assumption 2.2.16, inequality (2.2.7) is satisfied with some λ . Then

$$\begin{aligned} |\nabla E|(x) &\leq \limsup_{t \rightarrow 0} \left(\frac{E(x) - E(\gamma(t))}{d(x, \gamma(t))} \right)^+ = \left(\limsup_{t \rightarrow 0} \frac{E(x) - E(\gamma(t))}{d(x, \gamma(t))} \right)^+ \\ &\leq \left(\limsup_{t \rightarrow 0} \left(\frac{E(x) - E(y)}{d(x, y)} + \frac{\lambda}{2}(1-t)d(x, y) \right) \right)^+ = \left(\frac{E(x) - E(y)}{d(x, y)} + \frac{\lambda}{2}d(x, y) \right)^+ \end{aligned}$$

and the proof is complete. $\square$

It is possible to prove that Assumption 2.2.16 implies Assumptions 2.2.8 and 2.2.13, thus existence of gradient flows in EDI sense is achieved. In order to get existence in EDE sense, the following result is useful:

Proposition 2.2.18. *Let (X, d) be a metric space, $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$ a λ -geodesically convex and lower semicontinuous functional. Then for any absolute continuous curve $x : [0, \infty) \rightarrow X$ such that $E(x(t)) < \infty$ for any t it holds*

$$|E(x(s)) - E(x(t))| \leq \int_t^s |\dot{x}(r)| |\nabla E(x(r))| dr, \quad \forall t < s. \quad (2.2.9)$$

Proof. Assume that the right hand side is finite, otherwise the claim is trivial. Upon reparametrization, assume $|\dot{x}(t)| = 1$ for a.e. t , thus x is 1-Lipschitz, and $t \mapsto |\nabla E|(x(t))$ is L^1 function.

It suffices to prove that $t \mapsto E(x(t))$ is absolutely continuous, and use inequality

$$\begin{aligned} \limsup_{h \rightarrow 0} \frac{E(x(t+h)) - E(x(t))}{h} &\leq \limsup_{h \rightarrow 0} \frac{E(x(t)) - E(x(t+h))^+}{|h|} \\ &\leq \limsup_{h \rightarrow 0} \frac{E(x(t)) - E(x(t+h))^+}{d(x(t), x(t+h))} \cdot \limsup_{h \rightarrow 0} \frac{d(x(t), x(t+h))}{|h|} \\ &\leq |\nabla E(x(t))| |\dot{x}(t)| \end{aligned}$$

valid for any $t \in [0, 1]$. Define functions $f, g : [0, 1] \rightarrow \mathbb{R}$ by

$$f(t) := E(x(t)), \quad g(t) := \sup_{s \neq t} \frac{(f(t) - f(s))^+}{|s - t|};$$

let D be the diameter of the compact set $\{x(t)\}_{t \in [0, 1]}$, and combining the 1-Lipschitz property with (2.2.8) yields

$$g(t) \leq \sup_{s \neq t} \frac{(E(x(t)) - E(x(s)))^+}{d(x(s), x(t))} \leq |\nabla E|(x(t)) + \frac{\lambda \wedge 0}{2} D.$$

Therefore the thesis follows if implication

$$g \in L^1 \implies |f(s) - f(t)| \leq \int_t^s g(r) dr, \quad \forall t < s$$

holds.

Fix $M > 0$ and define $f^M := f \wedge M$. Fix $\varepsilon > 0$, let $\rho_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ be a mollifier with support in $[-\varepsilon, \varepsilon]$, and define

$$\begin{aligned} f_\varepsilon^M &: [\varepsilon, 1 - \varepsilon] \rightarrow \mathbb{R}, \quad f_\varepsilon^M(t) := f^M * \rho_\varepsilon(t), \\ g_\varepsilon^M &: [\varepsilon, 1 - \varepsilon] \rightarrow \mathbb{R}, \quad g_\varepsilon^M(t) := \sup_{s \neq t} \frac{(f_\varepsilon^M(t) - f_\varepsilon^M(s))^+}{|s - t|}. \end{aligned}$$

From smoothness of f_ε^M and the fact $g_\varepsilon^M \geq (f_\varepsilon^M)'$ it holds

$$|f_\varepsilon^M(s) - f_\varepsilon^M(t)| \leq \int -t^s g_\varepsilon^M(r) dr,$$

and then

$$\begin{aligned} g_\varepsilon^M(t) &\leq \sup_s \frac{1}{|s-t|} \int_{\mathbb{R}} (f^M(t-r) - f^M(s-r))^+ \rho_\varepsilon(r) dr \leq \sup_s \frac{1}{|s-t|} \int_{\mathbb{R}} (f(t-r) - f(s-r))^+ \rho_\varepsilon(r) dr \\ &= \sup_s \int_{\mathbb{R}} \frac{(f(t-r) - f(s-r))^+}{|(s-r) - (t-r)|} \rho_\varepsilon(r) dr \\ &\leq \int_{\mathbb{R}} g(t-r) \rho_\varepsilon(r) dr = g * \rho_\varepsilon(t) \end{aligned}$$

which implies that the family $\{g_\varepsilon^M\}_\varepsilon$ is dominated in $L^1(0,1)$, and the family $\{f_\varepsilon^M\}_\varepsilon$ converges uniformly to some $\tilde{f}^M$ as $\varepsilon \rightarrow 0$. For the limit function it holds

$$|\tilde{f}^M(s) - \tilde{f}^M(t)| \leq \int_t^s g(r) dr.$$

We know that $f^M = \tilde{f}^M$ on some $\mathcal{A} \subseteq [0,1]$ with $\mathcal{L}^1([0,1] \setminus \mathcal{A}) = 0$ and the goal is to prove that equality holds on $[0,1] \setminus \mathcal{A}$ too. As f^M is lower semicontinuous, $f^M \leq \tilde{f}^M$ is guaranteed. If it holds $f^M(t_0) < c < C < \tilde{f}^M(t_0)$ for some t_0 , then there exists $\delta > 0$ such that $\tilde{f}_{[t_0-\delta, t_0+\delta] \cap \mathcal{A}}^M > C$, and

$$\int_0^1 g(t) dt \geq \int_{[t_0-\delta, t_0+\delta] \cap \mathcal{A}} g(t) dt \geq \int_{[t_0-\delta, t_0+\delta] \cap \mathcal{A}} \frac{C-c}{|t-t_0|} dt = \infty$$

which is a contradiction. Thus if $g \in L^1(0,1)$, then

$$|f^M(t) - f^M(s)| \leq \int_t^s g(r) dr, \quad \forall t < s \in [0,1], M > 0$$

and taking the limit $M \rightarrow \infty$ concludes the proof. $\square$

The next result allows to pass from existence in EDI sense to the one in EDE sense.

Theorem 2.2.19. *Let (X, d) be a metric space, $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$ verifying Assumption 2.2.16, and $\bar{x} \in \overline{D(E)}$. Then all results of Theorem 2.2.14 are valid. Moreover, any gradient flow in EDI sense is gradient flow in EDE sense too.*

Proof. All results of Theorem 2.2.14 are valid as Assumption 2.2.16 implies both Assumptions 2.2.8 and 2.2.13.

By Theorem 2.2.14 the limit curve is absolutely continuous, and satisfies

$$E(x(s)) + \frac{1}{2} \int_0^s |\dot{x}(r)| dr + \frac{1}{2} \int_0^s |\nabla E|^2(x(r)) dr \leq E(\bar{x}), \quad s \geq 0.$$

In particular $t \mapsto |\dot{x}(t)|$ and $t \mapsto |\nabla E|(x(t))$ belong to $L_{loc}^2(0, +\infty)$; using Proposition 2.2.9

$$|E(\bar{x}) - E(x(s))| \leq \int_0^s |\dot{x}(r)| |\nabla E|(x(r)) dr \leq \frac{1}{2} \int_0^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_0^s |\nabla E|^2(x(r)) dr,$$

thus $t \mapsto E(x(t))$ is locally absolutely continuous and it holds

$$E(x(s)) + \frac{1}{2} \int_0^s |\dot{x}(r)|^2 dr + \frac{1}{2} \int_0^s |\nabla E|^2(x(r)) dr = E(\bar{x}), \quad s \geq 0;$$

the same equation written with t in place of s is

$$E(x(t)) + \frac{1}{2} \int_0^t |\dot{x}(r)|^2 dr + \frac{1}{2} \int_0^t |\nabla E|^2(x(r)) dr = E(\bar{x}), \quad t \geq 0;$$

and subtracting the last two equations the thesis follows. $\square$

Geodesic convexity ensures more regularity properties, listed in the following result (see [3] for more details):

Proposition 2.2.20. *Let (X, d) be a metric space, $E : X \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying Assumption 2.2.16 for some $\lambda \in \mathbb{R}$, and $x : [0, \infty) \rightarrow X$ limit of a sequence of discrete solutions. Then*

1. *for every $t > 0$ the limit*

$$|\dot{x}^+(t)| := \lim_{h \rightarrow 0} \frac{d(x(t+h), x(t))}{h}$$

exists,

2. *for every $t > 0$ it holds*

$$\frac{d_+}{dt} E(x(t)) = -|\nabla E|^2(x(t)) = -|\dot{x}^+(t)| |\nabla E|(x(t)),$$

3. *the map $t \mapsto \exp(-2\lambda^- t) E(x(t))$ is convex; $t \mapsto e^{\lambda t} |\nabla E|(x(t))$ is non increasing, right continuous and satisfies*

$$\frac{1}{2} |\nabla E|^2(x(t)) \leq \exp(-2\lambda^- t) E(x(t)) (E(x(0)) - E_t(x(0))),$$

$$t |\nabla E|^2(x(t)) \leq (1 + 2\lambda^+ t) e^{2-\lambda t} (E(x(0)) - \inf E)$$

with $E_t : X \rightarrow \mathbb{R}$ defined as

$$E_t(x) := \inf_{y \in X} E(y) + \frac{d(x, y)^2}{2t},$$

4. *if $\lambda > 0$, then E admits a unique minimum $x_{\min}$ and it holds*

$$\frac{\lambda}{2} d(x(t), x_{\min})^2 \leq E(x(t)) - E(x_{\min}) \leq e^{-2\lambda t} (E(x(0)) - E(x_{\min})).$$

Similarly to the EDI context, uniqueness is generally not true, as shown by the following counterexample:

- Consider $\mathbb{R}^2$ endowed with the L^∞ norm, $E : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $E(x_1, x_2) := x_1$, and $\bar{x} := (0, 0)$. Then $|\nabla E| = 1$ and any Lipschitz curve $t \mapsto x(t) = (x_1(t), x_2(t))$ satisfying

$$\begin{cases} x_1(t) = -t & \forall t \\ |x_2(t)|' \leq 1 & \text{a.e. } t > 0 \end{cases}$$

satisfies also

$$E(x(t)) = -t, \quad |\dot{x}(t)| = 1$$

thus such curve satisfies EDE too.

2.2.3 Limit of discrete solutions as $n \rightarrow \infty$

In this subsection our goal is to analyze limit sets of discrete solutions, as defined in (2.2.4). Given a metric space (X, d) , a functional E , an initial datum $\bar{x} \in \overline{D(E)}$ and a time step $\tau > 0$ consider the sequence

$$\begin{cases} x(0) := \bar{x} \\ x(n+1) \in \operatorname{argmin} E(\cdot) + \frac{d(\cdot, x(n))^2}{2\tau} \end{cases}$$

and our goal is to investigate properties of $x(k)$ as $k \rightarrow \infty$. We will assume that Assumption 2.2.8 holds. However without further hypothesis on the metric space existence of such limits is generally false, and we will assume (X, d) sequentially compact.

Under these assumptions the following result holds:

Proposition 2.2.21. *Let (X, d) be a compact metric space, $E : X \rightarrow \mathbb{R}$ lower semicontinuous and bounded from below, and suppose Assumption 2.2.8 holds. Consider a the recursive sequence*

$$\begin{cases} x(0) := \bar{x} \\ x(n+1) \in \operatorname{argmin} E(\cdot) + \frac{d(\cdot, x(n))^2}{2\tau} \end{cases}.$$

Then every x^ such that there exists a subsequence $\{x(a(h))\}_{h \in \mathbb{N}} \subseteq \{x(k)\}_{k \in \mathbb{N}}$ converging to x^* is stationary, i.e. $|\nabla E|(x^*) = 0$.*

Proof. The proof is done by contradiction: suppose there exists a subsequence $\{x(a(h))\}_{h \in \mathbb{N}}$ converging to some point x^* not stationary. Thus there exists $c > 0$ and a sequence $\{y_k\}_{k \in \mathbb{N}}$ converging to x^* such that

$$E(x^*) - E(y_k) \geq cd(x^*, y_k) > 0 \quad \forall k \in \mathbb{N}. \quad (2.2.10)$$

It is easy to observe that $\{E(x(k))\}_{k \in \mathbb{N}}$ is decreasing, thus $x(k) \neq y_h$ for any $k, h \in \mathbb{N}$. Consider an index $a(h)$: from minimality properties it must hold

$$E(x(a(h) + 1)) + \frac{d(x(a(h) + 1), x(a(h)))^2}{2\tau} \leq E(y_k) + \frac{d(x(a(h)), y_k)^2}{2\tau} \quad \forall h \in \mathbb{N},$$

thus combining with (2.2.10) yields

$$\begin{aligned} E(x(a(h) + 1)) + \frac{d(x(a(h) + 1), x(a(h)))^2}{2\tau} &\leq E(y_k) + \frac{d(x(a(h)), y_k)^2}{2\tau} \\ &\leq E(x^*) - cd(x^*, y_k) + \frac{d(x(a(h)), y_k)^2}{2\tau} \end{aligned}$$

for any h . Note that the distance between $x(a(h))$ and $x(a(h) + 1)$ goes to 0, thus in the following estimates the role of such points are somewhat interchangeable. Then passing to the limit $h \rightarrow 0$ and using the lower semicontinuity of E concludes the proof. $\square$

Notice that in the proof $\tau > 0$ fixed is crucial: indeed passing to the limit as $\tau \downarrow 0$ this result can be false; however, if E is also convex, then passing to the limit $\tau \downarrow 0$ this result can be proven. Moreover using the same argument the following stronger result can be proven:

Proposition 2.2.22. *Let (X, d) be a sequentially compact metric space, $E : X \rightarrow \mathbb{R}$ lower semicontinuous and bounded from below, and suppose Assumption 2.2.8 holds. Consider a the recursive sequence*

$$\begin{cases} x(0) := \bar{x} \\ x(n+1) \in \operatorname{argmin} E(\cdot) + \frac{d(\cdot, x(n))^2}{2\tau} \end{cases}.$$

Then for every set x^ such that there exists a subsequence $\{x(a(h))\}_{h \in \mathbb{N}} \subseteq \{x(k)\}_{k \in \mathbb{N}}$ converging to x^* it holds*

$$\limsup_{y \rightarrow x^*} \frac{(E(x^*) - E(y))^+}{d(x^*, y)^\alpha} = 0$$

for any $\alpha < 2$.

Chapter 3

Maximal and average distance problems

In the previous chapter we have presented a review of gradient flow theory in a general metric setting, with weak assumptions on both functional and distance. In this chapter we will introduce the “average distance” and the “maximal distance” functionals, and discuss associated problems. The main focus will be on the average distance problem.

Section 3.1 will recall basic properties first proven by Buttazzo, Oudet and Stepanov. Section 3.2 will recall some geometric properties of solutions, and Section 3.3 will recall results concerning asymptotic behavior (for large and small length constraints) and regularity. Most results from these three sections were proven by Buttazzo, Oudet and Stepanov in several works (see for instance [14], [16], [17]), but include contributions from other authors (including Santambrogio, Tilli and Slepčev, mainly in the part concerning regularity). Section 3.4 deals with similar properties in higher dimension cases (results are mainly from [44] by Paolini and Stepanov). Section 3.5 presents some side notes by the author, about cases in which the total mass is infinite.

3.1 Maximal and average distance functional

The average distance problem was first introduced in [14] and [16], where some geometric and analytic properties were studied.

We present first the main objects analyzed in this chapter: given $\mathbb{N} \ni N \geq 2$ and a domain $\Omega \subseteq \mathbb{R}^N$, denote with $\mathcal{A}(\Omega)$ the set of compact, pathwise connected subsets $\mathcal{X} \subseteq \Omega$ with $\dim_{\mathcal{H}} \mathcal{X} = 1$ and $\mathcal{H}^1(\mathcal{X}) < \infty$. Moreover, given $l > 0$, define

$$\mathcal{A}_l(\Omega) := \{\mathcal{X} \in \mathcal{A}(\Omega) : \mathcal{H}^1(\mathcal{X}) \leq l\}.$$

Finally denote with $\text{dist}_{\Omega}(\cdot, \cdot)$ the geodesic distance in Ω , and to simplify notation, “ $\text{dist}_{\Omega}(x, K)$ ” (where x is a point and K a closed set) will be used instead of “ $\min_{y \in K} \text{dist}_{\Omega}(x, y)$ ”.

In all the chapter we will assume that the domain Ω is closure of a connected, bounded and open set.

Now we can present the “average distance problem”:

Problem 3.1.1. Given quantities $\mathbb{N} \ni N \geq 2$, $l > 0$, a measure μ on Ω and a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, solve

$$\min_{\mathcal{X} \in \mathcal{A}_l(\Omega)} F_{\Omega, \mu, A}(\mathcal{X}),$$

where

$$F_{\Omega, \mu, A} : \mathcal{A}(\Omega) \rightarrow [0, \infty), \quad F_{\Omega, \mu, A}(\mathcal{X}) := \int_{\Omega} A(\text{dist}_{\Omega}(x, \mathcal{X})) d\mu(x).$$

This formulation is often referred as “constrained problem”, and was originally introduced in [14]. An alternative formulation (see for instance [13] and [31]) is

Problem 3.1.2. Given quantities $\mathbb{N} \ni N \geq 2$, $l > 0$, a measure μ on Ω , a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, and a parameter $\lambda > 0$, solve

$$\min_{\mathcal{X} \in \mathcal{A}_l(\Omega)} F_{\Omega, \mu, A}(\mathcal{X}) + \lambda \mathcal{H}^1(\mathcal{X}).$$

This formulation is often referred as “penalized problem”, in which the length constraint $\mathcal{H}^1(\mathcal{X}) \leq l$ is replaced by the penalization term $\lambda \mathcal{H}^1(\mathcal{X})$. Both formulations exhibit little difference in most arguments, thus unless explicitly stated, or made clear in the context, the expression “average distance problem” will refer to both of them.

The average distance problem has several interpretations. An easy one arises from urban planning:

- Ω is a city, with population distribution given by μ ,
- $\mathcal{X}$ is a transport network to be built,
- A gives the relation between the distance from the transport network and the cost to reach it.

Thus $F_{\Omega, \mu, A}(\mathcal{X})$ is the total cost to reach the transport network, which coincided with the average cost if μ is a probability measure. The constraint/penalization on length accounts for the cost to build the network. Solving Problem 3.1.1 is equivalent to find the “best” network satisfying length constraints, which minimizes the average cost (or upon multiplying for a constant, the total cost) for the whole citizenship to reach it.

An alternative interpretation is found in the field of cloud data approximation:

- Ω a region of the space, with data distribution given by μ ,
- $\mathcal{X}$ is an one dimension set used to approximate the entire data cloud,

In this case, $F_{\Omega, \mu, A}(\mathcal{X})$ represents the error of such approximation, while $\lambda \mathcal{H}^1(\mathcal{X})$ represents the cost due to its complexity.

Thus solving Problem 3.1.2 is equivalent to find the “best” approximation which minimizes the sum of approximation error and complexity cost. Despite the rather simple formulation, actually solving the average distance problem (in both constrained and penalized formulation) is extremely difficult, generally not possible without strong hypothesis on the domain, and computationally not feasible.

A related problem is the “maximal distance problem”:

Problem 3.1.3. Given quantities $\mathbb{N} \ni N \geq 2$ and $l > 0$, the “maximal distance problem” is solving

$$\min_{\mathcal{X} \in \mathcal{A}_l(\Omega)} F^*(\mathcal{X}),$$

where

$$F_\Omega^*(\mathcal{X}) := \max_{\Omega} \text{dist}_\Omega(x, \mathcal{X}).$$

Similarly to the average distance problem, the maximal distance problem has an easy interpretation from urban planning to:

- Ω is a city, with population distribution given by μ ,
- $\mathcal{X}$ is a transport network to be built,
- A gives the relation between the distance from the transport network and the cost to reach it.

Solving Problem 3.1.3 is equivalent to find the “best” network satisfying length constraints which minimizes the maximal cost (considered among all citizens) to reach it.

For the maximal distance functional, the role of both μ and A is less relevant: indeed Problem 3.1.3 does not involve any measure, and it is a more geometric problem. The function A plays no role as long as it is increasing. Thus in the following, when discussing the maximal distance problem, the measure and the function will be assumed Lebesgue measure and identity function respectively. Notice that while both functionals have dependence on the domain Ω , in the following we will omit writing it explicitly when no risk of confusion arises, i.e. “ $F_{\mu,A}$ ” instead of “ $F_{\Omega,\mu,A}$ ” and “ F^* ” instead of “ F_Ω^* ”. Moreover, when Ω, μ, A are given, the expression “solution of the average/maximal distance problem” will be used to denote a set solution of $\min_{\mathcal{A}_l(\Omega)} F_{\mu,A}$ or $\min_{\mathcal{A}_l(\Omega)} F^*$ for some l (or λ for Problem 3.1.2).

In this generality little can be said about such solutions, thus some restriction on the measure μ and function A is required. The first condition is that the measures μ does not charge ridges, i.e. given an arbitrary $W \in \mathcal{A}(\Omega)$, the set

$$\mathcal{R}_W := \{x \in \Omega : \text{there exist distinct } y_1, y_2 \in W \text{ such that } \text{dist}_\Omega(x, y_1) = \text{dist}_\Omega(x, y_2) = \text{dist}(x, W)\}$$

is μ -negligible. This is a quite weak condition, as from [39] these ridges are $(\mathcal{H}^1, 1)$ -rectifiable. Thus any measure absolutely continuous with respect to the Lebesgue measure does not charge ridges.

Some restrictions on the function A must be imposed too. As done in [14] and [44] assume:

- (α_1) $A : [0, \text{diam } \Omega] \rightarrow \mathbb{R}$ is Lipschitz continuous with Lipschitz constant Λ , $A(0) = 0$, monotone increasing,
- (α_2) for any $c > 0$ there exists $\lambda = \lambda(c) > 0$ such that $|A(x) - A(y)| \geq \lambda|x - y|$ whenever $|x - y| \in [c, \text{diam } \Omega]$.

From the above conditions (satisfied by several regular functions, like $A(x) := x^p$ for any $p \geq 1$) follows A injective on $[c', \text{diam } \Omega]$ for any $c' \in (0, \text{diam } \Omega)$. In most cases, if a result is true with $A = id$, then it is true with A satisfying above conditions.

3.1.1 Link with optimal transport problem

Problem 3.1.1, albeit having a very different formulation with optimal transport problem, can be seen as a Kantorovich problem in presence of “free regions”:

Definition 3.1.4. *Given a domain Ω , a cost function $c : \Omega \times \Omega \longrightarrow [0, \infty]$, a subset $\Sigma \subseteq \Omega$ is a “Dirichlet region” (for the cost c) if for any points $x, y \in \Sigma$ such that there exists a path $\alpha : [0, 1] \longrightarrow \Sigma$, $x, y \in \alpha([0, 1])$, $c(x, y) = 0$ holds.*

In other words, a Dirichlet region is a subset where “transport is free”. In this context, geodesic distance $\text{dist}_\Omega(\cdot, \cdot)$ is not significant as it does not consider Dirichlet regions, and a natural modification is the semi-distance

$$\text{dist}_{\Omega, \Sigma} : \Omega \times \Omega \longrightarrow [0, \infty], \quad \text{dist}_{\Omega, \Sigma}(x, y) := \inf_{\xi_1, \xi_2 \in \Sigma} \text{dist}_\Omega(x, \xi_1) + \text{dist}_\Omega(y, \xi_2).$$

A natural generalization of optimal plans in presence of non empty Dirichlet regions $\Sigma \subseteq \Omega$ can be given:

Definition 3.1.5. *Given a domain Ω , a Dirichlet region $\Sigma \subseteq \Omega$, Borel measures μ, ν , a Borel measure γ on $\Omega \times \Omega$ is a transport plan between μ and ν if*

$$\pi_\#^+ \gamma - \pi_\#^- \gamma = \mu - \nu \quad \text{on } \Omega \setminus \Sigma,$$

where $\pi^\pm$ denotes the projection on the first and the second component respectively.

Combining this definition with the semi-distance $\text{dist}_{\Omega, \Sigma}$ yields the new Kantorovich problem

$$\min \left\{ \int_{\Omega \times \Omega} A(\text{dist}_{\Omega, \Sigma}(x, y)) d\gamma(x, y) \right\} \quad (3.1.1)$$

where $A : [0, \text{diam } \Omega] \longrightarrow [0, \infty]$ is a given function, and the minimum is taken for γ varying among transport plans (as in Definition 3.1.5) between μ and ν .

In this formulation it is not required $\mu(\Omega) = \nu(\Omega)$: indeed denoting with μ the restriction of Lebesgue measure on Ω , and $\nu = 0$ (3.1.1) becomes Problem 3.1.1. The following definition will be useful:

Definition 3.1.6. *Given a domain Ω , $\Sigma \subseteq \Omega$, let $\Omega' \subseteq \Omega$ be the set of points with unique projection on Σ . Then given $x \in \Omega'$ there exists a unique $z \in \Sigma$ such that $\text{dist}_\Omega(x, \Sigma) = \text{dist}_\Omega(x, z)$, and the “transport ray” passing through x is the set*

$$\{y \in \Omega' : \text{dist}_\Omega(y, \Sigma) = \text{dist}_\Omega(y, z)\} \ni x.$$

Moreover, z will be referred as the endpoint of such transport ray.

3.1.2 Link with q -compliance problem

In this subsection we analyse the link between Problem 3.1.1 and the q -compliance problem, in two dimension case.

Given a domain $\Omega \subseteq \mathbb{R}^2$, for any $\Sigma \in \mathcal{A}(\Omega)$ and $q > 0$ denote u_Σ the solution of problem

$$\begin{cases} -\Delta_q u = 1 & \text{in } \Omega \setminus \Sigma \\ u = 0 & \text{on } \partial\Omega \cup \Sigma \end{cases}$$

where Δ_q denotes the q -Laplacian (i.e. $\Delta_q u := \operatorname{div}(|\nabla u|^{q-2} \nabla u)$). For given Σ the solution u_Σ can be obtained by minimizing

$$E_{q,\Sigma}(u) := \frac{1}{q} \int_{\Omega \setminus \Sigma} |\nabla u|^q dx - \int_{\Omega} u dx$$

among $u \in W_0^{1,q}(\Omega \setminus \Sigma)$.

The q -compliance energy is defined as

$$\mathcal{C}_q(\Sigma) := \left(1 - \frac{1}{q}\right) \int_{\Omega} u_\Sigma(x) dx,$$

and given a parameter $l > 0$, the associated q -compliance problem is

$$\min_{\Sigma \in \mathcal{A}_l(\Omega)} \mathcal{C}_q(\Sigma). \quad (3.1.2)$$

The link with Problem 3.1.1 is stated in the following result:

Theorem 3.1.7. *In the metric space $(\mathcal{A}(\Omega), d_{\mathcal{H}})$ taking the limit $q \rightarrow \infty$, the q -compliance energy $\mathcal{C}_q$ Γ -converges to*

$$\mathcal{F} : \mathcal{A}(\Omega) \longrightarrow [0, \infty), \quad \mathcal{F}(\Sigma) := \int_{\Omega} \operatorname{dist}(x, \partial\Omega \cup \Sigma) dx.$$

For the proof we refer to [15].

3.1.3 Basic properties

Problem 3.1.1 explicitly involves finding minimum of certain functionals, and the first problem is existence of such minimum. This is rather easy, and mainly consequence of Golab theorem:

Theorem 3.1.8. *Given a domain $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$), a parameter $l \geq 0$, and non negative measures μ, ν consider the average distance problem*

$$\min_{\Sigma \in \mathcal{A}_l(\Omega)} \int_{\Omega} A(\operatorname{dist}_{\Omega}(x, \Sigma)) dx. \quad (3.1.3)$$

If A is continuous, then (3.1.3) admits solutions.

Proof. Consider a minimizing sequence $\{\Sigma_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}_l(\Omega)$ for the average distance functional: according to Blaschke theorem, upon subsequence (for simplicity we do not relabel), $\Sigma_n \rightarrow \Sigma \in \mathcal{A}(\Omega)$ in the sense of Hausdorff convergence, and Golab theorem yields $\mathcal{H}^1(\Sigma) \leq l$.

As Hausdorff convergence implies $\text{dist}_\Omega(x, \Sigma_n) \rightarrow \text{dist}_\Omega(x, \Sigma)$ for any $x \in \Omega$, we obtain

$$\text{dist}_{\Omega, \Sigma_n}(x, y) \rightarrow \text{dist}_{\Omega, \Sigma}(x, y) \quad \forall x, y \in \Omega;$$

since $\text{dist}_{\Omega, \Sigma_n}(\cdot, \cdot)$ are Lipschitz continuous with respect to the Euclidean distance, with the same Lipschitz constant, the convergence is uniform.

Denote $\{\gamma_n\}_{n \in \mathbb{N}}$ the associated transport plans, i.e. for any $n \in \mathbb{N}$ it holds

$$\int_{\Omega} A(\text{dist}(x, \Sigma_n)) dx = \int_{\Omega \times \Omega} A(\text{dist}_{\Omega, \Sigma_n}(x, y)) d\gamma_n(x, y)$$

and

$$\pi_{\#}^+ \gamma_n - \pi_{\#}^- \gamma_n = \mu - \nu \quad \text{in } \Omega \setminus \Sigma_n.$$

The sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ can be assumed bounded, and upon subsequence (again without relabeling) $\gamma_n \rightharpoonup \gamma$ *-weakly in the space of Borel measures over Ω , thus

$$\pi_{\#}^+ \gamma - \pi_{\#}^- \gamma = \mu - \nu \quad \text{in } \Omega \setminus \Sigma.$$

Indeed for every test function ψ it holds

$$\int_{\Omega} \psi d(\pi_{\#}^+ \gamma - \pi_{\#}^- \gamma) = \lim_{n \rightarrow \infty} \int_{\Omega} \psi d(\pi_{\#}^+ \gamma_n - \pi_{\#}^- \gamma_n) = \int_{\Omega} \psi d(\mu - \nu),$$

Finally from

$$\int_{\Omega} A(\text{dist}(x, \Sigma)) dx \leq \int_{\Omega \times \Omega} A(\text{dist}_{\Omega, \Sigma}(x, y)) d\gamma(x, y) = \lim_{n \rightarrow \infty} \int_{\Omega \times \Omega} A(\text{dist}_{\Omega, \Sigma_n}(x, y)) d\gamma_n(x, y)$$

follows the minimality of Σ . □

Existence of solutions for the penalized formulation is proven with similar argument.

Another basic property of solutions of the average distance functional is that they attain the maximum length allowed:

Lemma 3.1.9. *Given a domain $\Omega \subseteq \mathbb{R}^N$, a non negative measure μ , a function $A : [0, \text{diam } \Omega] \rightarrow \mathbb{R}$, for any elements $\Sigma_1, \Sigma_2 \in \mathcal{A}(\Omega)$ with $\Sigma_1 \subseteq \Sigma_2$ inequality*

$$F_{\mu, A}(\Sigma_2) \leq F_{\mu, A}(\Sigma_1)$$

holds. In other words, $F_{\mu, A}$ is not decreasing with respect to the inclusion.

Moreover, suppose $\mathcal{H}^1(\Sigma_2 \setminus \Sigma_1) > 0$. Then inequality

$$F_{\mu, A}(\Sigma_2) < F_{\mu, A}(\Sigma_1)$$

holds.

Proof. The proof is very simple: $\Sigma_1 \subseteq \Sigma_2$ gives

$$\text{dist}_\Omega(x, \Sigma_2) \leq \text{dist}_\Omega(x, \Sigma_1) \quad \forall x \in \Omega,$$

thus

$$A(\text{dist}_\Omega(x, \Sigma_2)) \leq A(\text{dist}_\Omega(x, \Sigma_1)) \quad \forall x \in \Omega,$$

and integrating on Ω

$$\int_\Omega A(\text{dist}_\Omega(x, \Sigma_2)) df(x) \leq \int_\Omega A(\text{dist}_\Omega(x, \Sigma_1)) df(x) \quad \forall x \in \Omega.$$

For the second part, $\Sigma_1 \subsetneq \Sigma_2$ implies there exists an open set B such that for any $z \in B$ the inequality $\text{dist}_\Omega(z, \Sigma_1) > \text{dist}_\Omega(z, \Sigma_2)$ holds, so using the strict monotonicity of A and integrating on Ω concludes the proof. $\square$

This result has a first consequence: under these hypothesis on Ω, μ, A , for any $l > 0$

$$\text{argmin}_{\mathcal{A}_l(\Omega)} F_{\mu, A} \subseteq \mathcal{A}_l(\Omega) \setminus \bigcup_{0 \leq j < l} \mathcal{A}_j(\Omega).$$

3.2 Geometric properties

In the previous section we have proven that existence of solutions for the average distance problem is quite simple, and requires very little assumption. In this section our goal is to analyse geometric properties of such solutions in two dimension case. Some preliminary definition is useful:

Definition 3.2.1. Let $S \subseteq \mathbb{R}^n$ be a given set, S is a “loop” if it is homeomorphic to $S^1 \subseteq \mathbb{R}^2$.

Definition 3.2.2. Let $S \subseteq \mathbb{R}^n$ be a set, $x \in S$ an arbitrary point, the “multiplicity” (or “order”) of x does not exceed the cardinal number $\mathfrak{n}$ if for $\varepsilon \rightarrow 0$ the set $(S \cap B(x, \varepsilon)) \setminus \{x\}$ has not more than $\mathfrak{n}$ connected components. Denoted with $\mathcal{N}$ the set of cardinal numbers $\mathfrak{n}$ for which the order of x does not exceed $\mathfrak{n}$, the minimum element of $\mathcal{N}$ will be referred as “multiplicity” (or “order”) of x , and denoted with $\text{ord}_x S$.

Moreover, it is convenient to distinguish the following class of points:

Definition 3.2.3. Let Ω be a given domain, $S \in \mathcal{A}(\Omega)$ a given element, the point $x \in S$ is “noncut” point if $S \setminus \{x\}$ is connected by arc.

It is easy to observe that endpoints are always noncut points; moreover, if a non endpoint point is noncut, then the set S must contain a loop.

Given a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ a function A , a measure $\mu \in L^p$ with $p \geq 1$ (in order to simplify notations in the following the expression “ $\mu \in L^p$ ” will mean that the Radon density $\frac{d\mu}{d\mathcal{L}^n}$ belongs to $L^p(\Omega, \mathcal{L}^n)$), and $l > 0$ we will prove that any solution

$$\Sigma_{\text{opt}} \in \text{argmin}_{\Sigma \in \mathcal{A}_l(\Omega)} \int_\Omega A(\text{dist}_\Omega(x, \Sigma)) d\mu(x)$$

satisfies:

1. Absence of loops: there are no subsets $E \subseteq \Sigma_{\text{opt}}$ homeomorphic to $S^1 \subseteq \mathbb{R}^2$.
2. Absence of crosses: if $p \geq 4/3$ and $n = 2$, then for every point $x \in \Sigma_{\text{opt}}$, the multiplicity of x is at most 3, and their number is finite.
3. Ahlfors regularity: if $p \geq 2$ (if $n \geq 3$), or $p \geq 4/3$ (if $n = 2$), then there exists $c_-, c_+ \in (0, \infty)$ such that

$$c_- \leq \frac{\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho))}{\rho} \leq c_+$$

for any $x \in \Sigma_{\text{opt}}$, $\rho > 0$.

Some further properties will be discussed later in this Chapter.

3.2.1 Absence of loops

The first property is the absence of loops, i.e. any solution Σ_{opt} of the average distance problem does not contain subsets homeomorphic to S^1 . The proof is done by contradiction, and consists of two parts:

- first, if there exists $E \subseteq \Sigma_{\text{opt}}$ homeomorphic to S^1 , a suitable set $I_\varepsilon \in \mathcal{A}_\varepsilon(\Omega) \subseteq E$ is removed, and the difference

$$F_{\mu,A}(\Sigma_{\text{opt}} \setminus I_\varepsilon) - F_{\mu,A}(\Sigma_{\text{opt}}) \tag{3.2.1}$$

estimated,

- then a suitable J_ε with $\mathcal{H}^1(J_\varepsilon) = \mathcal{H}^1(I_\varepsilon)$ is added at a suitable point of $\Sigma_{\text{opt}} \setminus I_\varepsilon$ and the difference

$$F_{\mu,A}(\Sigma_{\text{opt}} \setminus I_\varepsilon) - F_{\mu,A}(\Sigma_{\text{opt}} \setminus I_\varepsilon \cup J_\varepsilon) \tag{3.2.2}$$

estimated.

A preliminary result from [16] is required:

Lemma 3.2.4. *Given a domain $\Omega \subseteq \mathbb{R}^2$, $\Sigma \in \mathcal{A}(\Omega)$ consisting of more than one point, let $x \in \Sigma$ be a noncut point of Σ . Then there exists a sequence of open sets $\{D_k\}_{k \in \mathbb{N}} \subseteq \Sigma$ such that:*

- $x \in D_k$ for k sufficiently large,
- $\Sigma \setminus D_k$ is connected for any k ,
- $\text{diam } D_k \downarrow 0$ for $k \rightarrow \infty$,
- D_k is connected for any k .

Proof. Fix x , and consider $z \in \Sigma \setminus \{x\}$; two points y, y' are said to be connected through Γ (compact) in Σ if $\{y, y'\} \subseteq \Gamma$. Define the sets

$$X_k := \{y : y \text{ connected to } z \text{ through some } \Gamma \subseteq \Sigma \setminus B(x, 1/k)\},$$

$$O_k := \Sigma \setminus X_k.$$

Observe that X_k are closed by construction, as given $y_k \rightarrow y$ with $\{y_k\} \subseteq X_k$, denoting with Γ_k a set connecting y_k to z , upon subsequence $\Gamma_k \rightarrow \Gamma$, connecting y to z . Thus O_k are open. As $z \neq x$, it follows $x \in O_k$ for k sufficiently large.

It remains to prove that $\text{diam } O_k \rightarrow 0$ as $k \rightarrow \infty$. Assume the contrary holds, i.e. there exists a sequence $\{y_k\}$ for which for any set Γ_k connecting y_k to z it holds

$$\Gamma_k \cap \overline{B(x, 1/k)} \neq \emptyset.$$

Choose an arbitrary accumulation point y of $\{y_k\}$. Local connectedness implies that there exists C_k connecting y_k to y with $C_k \cap B(x, r/2) = \emptyset$, for some $r > 0$, thus for any set Γ connecting y to z it holds $x \in \Gamma$. Recall that $y \notin B(x, r)$, thus $x \in \Sigma \setminus \{x\}$, with the latter space being locally pathwise connected, as it is open and completely metrizable. But every such arc connecting y and z must pass through x thus leading to a contradiction.

Denote with D_k the connected component of O_k containing x , and simple topological considerations yield that D_k is relatively open in Σ . Thus all the points follow quite straightforward, from properties proven for O_k , except for “ $\Sigma \setminus D_k$ is connected for any k ”.

To prove the latter, assume the contrary holds, i.e. for some $y \in O_k \setminus D_k$ for any set Γ connecting y to z it holds $\Gamma \cap D_k \neq \emptyset$. Let γ an arbitrary arc connecting y to z , with $\gamma(0) = y$, $\gamma(1) = z$ and denote with

$$\bar{t} := \sup\{t \in [0, 1] : \gamma(s) \notin D_k \forall s \in [0, t)\}, \quad \bar{x} := \gamma(\bar{t}).$$

Consider an arc $[y, \bar{x}] \subseteq \gamma$ and $[y, \bar{x}) := [y, \bar{x}] \setminus \{\bar{x}\}$. It is straightforward to check $[y, \bar{x}) \subseteq O_k$. Using a similar argument, it can be checked that $\bar{x} \in O_k$, thus giving $\gamma([0, \bar{t}]) \subseteq O_k$. Since O_k open, there exists $t' > \bar{t}$ such that $\gamma([0, t']) \subseteq O_k$, thus belongs to the same connected component of O_k , and since by definition one has $\gamma([0, t']) \subseteq O_k \neq \emptyset$, this gives that this said connected component is D_k , and $y \in D_k$, which is a contradiction. $\square$

Lemma 3.2.5. *Given a domain $\Omega \subseteq \mathbb{R}^2$, a measure $\mu \in L^p(\Omega)$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, $\Sigma \in \mathcal{A}(\Omega)$ containing a subset E homeomorphic to $S^1 \subseteq \mathbb{R}^2$. Then for $\mathcal{H}^1$ -a.e. point $x \in \Sigma$ there exists $I_\varepsilon(x) \ni x$ contained in Σ such that*

$$F_{\mu, A}(\Sigma \setminus I_\varepsilon(x)) - F_{\mu, A}(\Sigma) \leq K\varepsilon^{1+\frac{1}{q}}$$

for some $K > 0$ not dependent on ε , with q denoting the conjugate exponent of p .

Proof. Define

$$E^* := \left\{ x \in E : \lim_{r \rightarrow 0^+} \frac{\mathcal{H}^1(\Sigma \cap B(x, r))}{2r} = 1 \right\},$$

and as Σ is $(\mathcal{H}^1, 1)$ -rectifiable, by Besicovitch-Marstrand-Mattila theorem (see [2] for more details) $\mathcal{H}^1(E^*) = \mathcal{H}^1(E)$ follows. For every $x \in E^*$ and $\varepsilon > 0$ denote with $T(x, \varepsilon)$ the union of transport rays of the Monge-Kantorovich problem of transporting $\mathcal{L}^2_\Omega$ on its projection over Σ which end in $\Sigma \cap B(x, \varepsilon)$. As E is homeomorphic to S^1 , clearly $\Sigma \setminus B(x, \varepsilon)$ is connected, and satisfies $\mathcal{H}^1(\Sigma \setminus B(x, \varepsilon)) \leq \mathcal{H}^1(\Sigma)$. The following estimate holds:

$$\begin{aligned} \int_{\Omega} \text{dist}(x, \Sigma \setminus B(x, \varepsilon)) d\mu(x) &= \int_{\Omega \setminus T(x, \varepsilon)} \text{dist}(x, \Sigma \setminus B(x, \varepsilon)) d\mu(x) + \int_{T(x, \varepsilon)} \text{dist}(x, \Sigma \setminus B(x, \varepsilon)) d\mu(x) \\ &\leq \int_{\Omega \setminus T(x, \varepsilon)} \text{dist}(x, \Sigma) d\mu(x) + \int_{T(x, \varepsilon)} \text{dist}(x, \Sigma) + \varepsilon d\mu(x) \\ &= \int_{\Omega} \text{dist}(x, \Sigma) d\mu(x) + \varepsilon \mu(T(x, \varepsilon)) \end{aligned} .$$

Moreover, denoting with ψ the projection of μ to Σ , one has

$$\limsup_{\varepsilon \rightarrow 0^+} \frac{\psi(B(x, \varepsilon))}{\varepsilon} < \infty$$

for $\mathcal{H}^1$ -a.e. $x \in \Sigma$, and

$$\mathcal{L}^2(T(x, \varepsilon)) = \psi(B(x, \varepsilon))$$

which implies

$$\mathcal{L}^2(T(x, \varepsilon)) \leq K\varepsilon$$

for some $K = K(x)$ not dependent on ε . Then applying Hölder inequality yields

$$\mu(T(x, \varepsilon)) \leq \|\mu\|_{L^p(T(x, \varepsilon))}^{1/p} (\mathcal{L}^2(T(x, \varepsilon)))^{1/q} \leq K\varepsilon^{1/q},$$

and combining with

$$\text{dist}_{\Omega}(x, \Sigma \setminus B(x, \varepsilon)) \leq \text{dist}_{\Omega}(x, \Sigma) + \varepsilon \quad \forall x \in \Omega$$

the proof is complete. □

Lemma 3.2.5 can be further generalized:

Lemma 3.2.6. *Using the same notations of Lemma 3.2.5, with the only modification of $\mu \in L^1(\Omega)$ (instead of $\mu \in L^p(\Omega)$ as in Lemma 3.2.5), there exists $x \in \Sigma$ and $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ inequality*

$$F_{\mu, A}(\Sigma \setminus T_{\varepsilon}(x)) - F_{\mu, A}(\Sigma) \leq C^* \varepsilon^2$$

where $T_{\varepsilon}(x) \subseteq \Sigma$ is some suitable neighborhood of x (with respect to the induced topology) and $C > 0$ a constant not dependent on ε .

Proof. As Ω is compact, it is clear that $L^{s_1}(\Omega) \subseteq L^{s_2}(\Omega)$ if $s_1 \leq s_2$, thus Lemma 3.2.5 is sufficient to yield the thesis if $\mu \in L^p(\Omega)$ for some $p \geq 2$.

Denote with Ω' the set of points of Ω with unique projection on Σ , and since Ω_k is the set of differentiable points of $x \mapsto \text{dist}_\Omega(x, \Sigma)$ then $\mathcal{L}^n(\Omega') = \mathcal{L}^n(\Omega)$, and condition $\mu \ll \mathcal{L}^n$ allows to consider Ω' instead of Ω without loss of generality.

Choose an arbitrary point $X \in E$ (not influential for the rest of the proof), and define

$$\phi : [0, 1] \longrightarrow S^1, \quad \psi : S^1 \longrightarrow E, \quad (3.2.3)$$

thus $\varphi := \phi \circ \psi : [0, 1] \longrightarrow E$ is a parameterization, with $\varphi(0) = \varphi(1) = X$; then choose an arbitrary $\mu \in L^1(\Omega)$, and for any $x \in E$ denote with $J_\varepsilon(x) := \{y \in E : d_E(y, x) < \varepsilon/2\}$, with d_E denoting the path distance on E , and

$$U_\varepsilon(x) := \{z \in \Omega' : \text{dist}(z, \Sigma) = \text{dist}(z, J_\varepsilon(x))\}.$$

Then, define $N_\varepsilon := \left\lceil \frac{\mathcal{H}^1(E)}{\varepsilon} \right\rceil + 1$, and for any ε there exists a finite set of points $\{x_{j,\varepsilon}\}_{j=1}^{N_\varepsilon} \subseteq E$ such that

$$\bigcup_{j=1}^{N_\varepsilon} J_\varepsilon(x_j) = E;$$

define $U := \bigcup_{j=1}^{N_\varepsilon} U_\varepsilon(x_j)$, and by definition U is the set of points with projection on E . Now two configurations may arise:

1. $\mu(U) = 0$, which implies that the set of points projecting on E is μ -negligible, thus $F_{\mu,A}(\Sigma) = F_{\mu,A}(\Sigma \setminus E)$. From topological considerations there exists $\bar{j} \in \{1, \dots, N_\varepsilon\}$ such that $\Sigma \setminus J_\varepsilon(x_{\bar{j}})$ is connected for ε sufficiently small, thus the competitor $\Sigma' := \Sigma \setminus J_\varepsilon(x_{\bar{j}})$ satisfies

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \varepsilon, \quad F_{\mu,A}(\Sigma) = F_{\mu,A}(\Sigma');$$

2. $\mu(U) > 0$: as the projection on Σ of any point in Ω' is unique, for $k \neq h$ sets $J_\varepsilon(x_k)$ and $J_\varepsilon(x_h)$ are disjoint, yielding

$$\mu(U) = \mu\left(\bigcup_{j=1}^{N_\varepsilon} U_\varepsilon(x_j)\right) = \sum_{j=1}^{N_\varepsilon} \mu(U_\varepsilon(x_j)) \geq N_\varepsilon \min_{1 \leq j \leq N_\varepsilon} \mu(U_\varepsilon(x_j)).$$

This implies

$$\min_{1 \leq j \leq N_\varepsilon} \mu(U_\varepsilon(x_j)) \leq \mu(U) \left(\left\lceil \frac{\mathcal{H}^1(E)}{\varepsilon} \right\rceil + 1 \right)^{-1},$$

or equivalently there exists j^* such that $\mu(U_\varepsilon(x_{j^*})) \leq C^* \varepsilon$ for some constant $C^* > 0$ not dependent on ε . Then

$$\text{dist}_\Omega(z, \Sigma) = \text{dist}_\Omega(z, \Sigma \setminus J_\varepsilon(x_{j^*})) \iff z \notin U_\varepsilon(x_{j^*})$$

and considering that the difference $\text{dist}_\Omega(z, \Sigma) = \text{dist}_\Omega(z, \Sigma \setminus J_\varepsilon(x_{j^*}))$ is at most ε , it holds

$$F_{\mu,A}(\Sigma \setminus J_\varepsilon(x_{j^*})) \leq F_{\mu,A}(\Sigma) + C^* \varepsilon^2.$$

Finally, while generally $\Sigma \setminus J_\varepsilon(x_{j^*}) \notin \mathcal{A}(\Omega)$, denote with $J_\varepsilon^0(x_{j^*})$ the interior part of $J_\varepsilon(x_{j^*})$, and $\Sigma \setminus J_\varepsilon^0(x_{j^*})$ verifies

$$\Sigma \setminus J_\varepsilon^0(x_{j^*}) \in \mathcal{A}(\Omega), \quad F_{\mu,A}(\Sigma \setminus J_\varepsilon^0(x_{j^*})) \leq F_{\mu,A}(\Sigma) + C^* \varepsilon^2,$$

thus the proof is complete. $\square$

The next result, first proven in [16], is useful in estimating (3.2.2)

Lemma 3.2.7. *Given a domain Ω , a Borel measure $\mu \ll \mathcal{L}^n$ ($n \geq 2$), a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$ and a sequence of closed sets $\{\Sigma_k\}_{k \in \mathbb{N}}$. Assume there exists a Borel set $B \subseteq \Omega$ such that $\Sigma := \Sigma_k \cap B$ is independent of k . Denote*

$$T' := \{x \in \Omega : \exists k_0 = k_0(x) \text{ such that } 0 < \text{dist}_\Omega(x, \Sigma) < \text{dist}_\Omega(x, \Sigma_k \setminus \Sigma) \text{ for any } k \geq k_0\}$$

and assume $\mu(T') > 0$. Then for any $\varepsilon > 0$ there exists a closed segment $I_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ such that $\Sigma \cap I_\varepsilon \neq \emptyset$ and there exists constants $C, \varepsilon > 0$ not dependent on ε for which inequality

$$F_{\mu,A}(\Sigma_k \cup I_\varepsilon) \leq F_{\mu,A}(\Sigma_k) - C\varepsilon^{(n+1)/2}$$

holds for any $\varepsilon < \varepsilon_0$.

Notice that this result is useful only for $n = 2$, as for $n \geq 3$ a stronger estimate holds (see Lemma 3.4.6 for more details).

Proof. Denote with Ω_k the set of points with unique projection on Σ_k , and let $k_k : \Omega_k \rightarrow \Sigma_k$ be the projection map. Again, similar to the argument found in the proof of Lemma 3.2.6, since Ω_k is the set of differentiable points of $x \mapsto \text{dist}_\Omega(x, \Sigma_k)$ then $\mathcal{L}^n(\Omega_k) = \mathcal{L}^n(\Omega)$, and condition $\mu \ll \mathcal{L}^n$ allows to consider $\Omega' := \bigcap_{k \in \mathbb{N}} \Omega_k$ instead of Ω without loss of generality.

Choose an arbitrary point $x \in T' \cap \Omega'$, and consider the transport ray $R_K(x)$ of x on Σ_k , and put $l_k := \mathcal{H}^1(R_k(x))$. By hypothesis this ray ends on Σ for any $k \geq k_0$, and $R_k(x) \subseteq R_h(x)$ for any $h \geq k \geq k_0$; denote with $R(x)$ the transport ray of x on Σ , $R_k(x) \uparrow R(x)$, and $l_k \uparrow l := \mathcal{H}^1(R(x))$. Denote with O_k the endpoint of $R_k(x)$ not belonging to Σ , then

$$\Sigma_k \cap B(O_k, l_k) \neq \emptyset.$$

Without loss of generality impose a coordinate system with origin in O_k , with x_n -axis directed along $R(x)$, and such that the endpoint P of $R(x)$ on Σ has coordinates $P := (0, \dots, 0, -l_k)$. Define $P_\varepsilon := (0, \dots, 0, -l_k + \varepsilon l_k)$, $I_\varepsilon := [P, P_\varepsilon]$, the segment between P and P_ε . For each $m \in \mathbb{N}$ and $\varepsilon < l_m$ define

$$\Lambda_{m,\varepsilon} := \{(Y, x_n) \in B(0, l_m) : x_n \leq 0, \text{dist}_\Omega((Y, x_n), P_\varepsilon) - \text{dist}_\Omega((Y, x_n), \partial B(0, l_m)) \leq -\varepsilon l_m/4\}.$$

If there exists a choice of x such that

$$\mu(\Lambda_{k,\varepsilon}) \geq C\varepsilon^{(n+1)/2} \quad (3.2.4)$$

for some constant $C > 0$ not dependent on ε and for any ε sufficiently small then the thesis follows: indeed if this is the case consider an arbitrary $j \geq j(x) \geq k_0$ such that

$$l_h \geq l/2 \quad \forall h \geq j$$

and for $z \in \Lambda_{h,\varepsilon}$ inequality

$$\text{dist}_\Omega(z, \Sigma_h \cup I_\varepsilon) \leq \text{dist}_\Omega(z, \partial B(0, l_h)) - \varepsilon l_h/4 \leq \text{dist}(z, \Sigma) - \varepsilon l/8$$

holds, which implies

$$F_{\mu,A}(\Sigma_h \cup I_\varepsilon) - F_{\mu,A}(\Sigma) \leq -\varepsilon l \mu(\Lambda_{h,\varepsilon})/8$$

for any $h \geq j$. Thus (3.2.4) need to be proven. Choose $m \in (0.1/2)$ and define

$$\Pi_{m,k,\varepsilon} := \{z = (Y, x_n) \in B(0, l_k) : |x_n + l_k/2| \leq l_k(1 - 4m^2)^{1/2}/2, |Y| \leq ml_k\varepsilon^{1/2}\}.$$

For every $z = (Y, x_n) \in \Pi_{m,k,\varepsilon}$ it holds

$$\begin{aligned} \text{dist}_\Omega(z, P_\varepsilon) - \text{dist}_\Omega(z, \partial B(0, l_k)) &\leq (|Y|^2 + (x_n + l_k - \varepsilon l_k)^{1/2}) - l_k + (x_n^2 + |Y|^2)^{1/2} \\ &\leq (m^2 l_k^2 \varepsilon + (x_n + l_k - \varepsilon l_k)^2)^{1/2} - l_k + (m^2 l_k \varepsilon + x_n^2)^{1/2} \\ &\leq -l_k + |x_n| + \frac{m^2 l_k^2 \varepsilon}{2|x_n|} + (x_n + l_k) + \frac{\varepsilon(m^2 l_k^2 - 2l(x_n + l_k))}{2(x_n + l_k)} + \alpha \varepsilon^2 \\ &= \frac{\varepsilon}{2} \left(\frac{m^2 l_k^2}{x_n + l_k} - 2l_k + \frac{m^2 l_k^2}{|x_n|} \right) + \alpha \varepsilon^2 \end{aligned}$$

where $\alpha = \alpha(m, l) > 0$ is a constant not dependent on k . It easy to verify

$$\frac{m^2 l_k^2}{x_n + l_k} - 2l_k + \frac{m^2 l_k^2}{|x_n|} \leq -l_k \quad \forall z \in \Pi_{m,k,\varepsilon}$$

and assuming $k \geq j$ it holds

$$\text{dist}_\Omega(z, P_\varepsilon) - \text{dist}_\Omega(z, \partial B(0, l_k)) \leq \varepsilon l_k/4$$

whenever $\varepsilon < l/8\alpha$, implying $\Pi_{m,k,\varepsilon} \subseteq \Lambda_{k,\varepsilon}$ for such ε .

Assume now x is chosen in such way that $\mathcal{H}^1$ -a.e. $z \in R_k(x)$ is a Lebesgue point of $\frac{d\mu}{d\mathcal{L}^n}$ (which is true for μ -a.e. $x \in T' \cap \Omega'$ in view of [17]). Fixed an arbitrary $k \geq j$,

$$\liminf_{\varepsilon \rightarrow 0^+} \frac{\mu(\Pi_{m,k,\varepsilon})}{\varepsilon^{(n-1)/2}} \geq C_m(R_k(x)) := C \int_{l_{k,m}^-}^{l_{k,m}^+} \frac{d\mu}{d\mathcal{L}^n}(Y, x_n) d\mathcal{H}^1(x_n) \quad (3.2.5)$$

holds, with $l_{k,m}^\pm := \frac{l_k}{2} \pm \frac{l_k(1-4m^2)^{1/2}}{2}$ and $C > 0$ is a constant not dependent on k . Indeed this steers from

$$\begin{aligned} \mu(\Pi_{m,k,\varepsilon}) &= \int_{l_{k,m}^-}^{l_{k,m}^+} d\mathcal{H}^1(x_n) \int_{|Y| < ml_k \varepsilon^{1/2}} \frac{d\mu}{d\mathcal{L}^n}(Y, x_n) d\mathcal{H}^{n-1}(Y) \\ &= \frac{1}{2\varepsilon^{1/2}} \int_{-\varepsilon^{1/2}}^{\varepsilon^{1/2}} dh \int_0^l d\mathcal{H}^1(x_n) \int_{|Y| < ml_k \varepsilon^{1/2}} f_k(Y, x_n + h) d\mathcal{H}^{n-1}(x_n)(Y) \\ &= \int_0^l f_{k,\varepsilon}(x_n) d\mathcal{H}^1(x_n) \end{aligned}$$

where

$$\begin{aligned} f_k(Y, x_n) &:= \frac{d\mu}{d\mathcal{L}^n}(Y, x_n) \chi_{[l_{k,m}^-, l_{k,m}^+]}(x_n), \\ f_{k,\varepsilon}(x_n) &:= \frac{1}{2\varepsilon^{1/2}} \int_{x_n - \varepsilon^{1/2}}^{x_n + \varepsilon^{1/2}} \int_{|Y| < ml_k \varepsilon^{1/2}} f_k(Y, s) d\mathcal{H}^{n-1}(Y) d\mathcal{H}^1(s), \end{aligned}$$

then in view of assumptions on $R(x)$, for $\mathcal{H}^1$ -a.e. on $R(x)$ the convergence

$$\frac{f_k(x_n)}{\varepsilon^{(n-1)/2}} \rightarrow C_n(ml_k)^{n-1} f(0, x_n)$$

holds, where $C_n > 0$ is a constant depending only on n . Using Fatou's lemma and $l_k \geq l/2$ estimate (3.2.5) follows.

Observe $C_m(R_k(x)) > 0$ for some x , then choose $\bar{k} \geq j$ and $\bar{m} = \bar{m}(\bar{k}) \in (0, 1/2)$ such that $C_m(R_k(x)) > 0$. Then (3.2.5) implies

$$\mu(\Pi_{\bar{m}, \bar{k}, \varepsilon}) \geq C_{\bar{m}}(R_{\bar{k}}(x)) \varepsilon^{(n-1)/2}$$

for any $\varepsilon < \varepsilon_0$, with $\varepsilon_0 > 0$ dependent only on $\bar{k}$.

In view of $l_k \geq l_{\bar{k}}$ inclusion $\Pi_{\bar{m}, \bar{k}, \varepsilon} \subseteq \Pi_{m, k, \varepsilon}$ holds whenever

$$l_{k,m}^+ \geq l_{\bar{k}, \bar{m}}^+, \quad l_{k,m}^- \leq l_{\bar{k}, \bar{m}}^-, \quad \bar{m} l_{\bar{k}} \leq m l_k. \quad (3.2.6)$$

Denoting

$$\delta_k := l_{\bar{k}}/l_k$$

the inequalities (3.2.6) can be written as

$$\begin{aligned} 4\bar{m}^2 \delta_k^2 &\leq \leq 1 - (\delta_k(1 - 4\bar{m}^2)^{1/2} + 1 - \delta_k) \\ &= \delta_k(1 - (1 - 4\bar{m}^2)^{1/2})(2 + \delta_k(1 - 4\bar{m}^2)^{1/2} - \delta_k). \end{aligned} \quad (3.2.7)$$

If this system has to be solvable,

$$\delta_k(1 + (1 - 4\bar{m}^2)^{1/2}) \leq 2 + \delta_k(1 - 4\bar{m}^2)^{1/2} - \delta_k \quad (3.2.8)$$

must hold, which is equivalent to $\delta_k \leq 1$, always valid since l_k is non decreasing in k . Therefore there exists $m = m(k)$ satisfying (3.2.4).

Finally combining

$$\Pi_{\bar{m}, \bar{k}, \varepsilon} \subseteq \Pi_{m, k, \varepsilon}$$

and $\mu(\Pi_{\bar{m}, \bar{k}, \varepsilon}) \geq C_{\bar{m}}(R_{\bar{k}}(x))\varepsilon^{(n-1)/2}$ yields

$$\mu(\Pi_{m, k, \varepsilon}) \geq C_{\bar{m}}(R_{\bar{k}}(x))\varepsilon^{(n-1)/2} \geq C\varepsilon^{(n-1)/2}$$

for some constant $C = C(\bar{k}) > 0$ not dependent on k and ε and valid for any $\varepsilon < \varepsilon_0$. At last

$$\mu(\Pi_{m, k, \varepsilon}) \subseteq \Lambda_{m, k, \varepsilon}$$

yields

$$\mu(\Lambda_{m, k, \varepsilon}) \geq C\varepsilon^{(n-1)/2}$$

for the same C and ε , and the proof is complete. $\square$

Now we are able to prove that solutions of the average distance problem do not contain loops:

Theorem 3.2.8. *Given a domain $\Omega \subseteq \mathbb{R}^2$, a measure $\mu \in L^1(\Omega)$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, let Σ_{opt} be an arbitrary solution of the average distance problem. Then no subsets $S \subseteq \Sigma_{\text{opt}}$ can be homeomorphic to S^1 .*

Proof. The proof is now simple: if there exists $E \subseteq \Sigma_{\text{opt}}$ homeomorphic to S^1 , then applying Lemma 3.2.6 yields the existence of a competitor $\Sigma' \in \mathcal{A}(\Omega)$ satisfying

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \varepsilon, \quad F_{\mu, A}(\Sigma') \leq F_{\mu, A}(\Sigma) + C^*\varepsilon^2$$

for any ε sufficiently small, and some constant $C^* > 0$ not dependent on ε .

Then applying Lemma 3.2.7 yields the existence of $\Sigma'' \in \mathcal{A}(\Omega)$ satisfying

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + \varepsilon, \quad F_{\mu, A}(\Sigma'') \leq F_{\mu, A}(\Sigma') - C_*\varepsilon^{3/2},$$

for some $C_* > 0$ not dependent on ε , thus for ε sufficiently small it holds

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma_{\text{opt}}), \quad F_{\mu, A}(\Sigma'') < F_{\mu, A}(\Sigma_{\text{opt}})$$

contradicting the optimality of Σ_{opt} . $\square$

For similar results in higher dimensional domains we refer to Section 3.4 later in this Chapter.

3.2.2 Triple points and endpoints

In two dimensional domains, another property satisfied by solutions of the average distance problem, under particular hypothesis on the measure, is that they have only a finite number of endpoints. Considering that a non endpoint cannot be a noncut point unless a loop is present, this is equivalent to state that the number of noncut points is finite. All domains considered in this subsection will be two dimension domains.

Proposition 3.2.9. *Given a domain Ω , a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, let Σ_{opt} be a solution of the average distance problem. Assume there exists $y \in \Sigma_{\text{opt}}$ such that $\mu(V(y)) > 0$. Then there exists $C > 0$ such that for every noncut point $x \in \Sigma_{\text{opt}}$ it holds $\mu(V(x)) \geq C$, and the number for noncut points is finite.*

Proof. The first part would follow from the following claim:

- let $x \in \Sigma_{\text{opt}}$ be a noncut point, then

$$\mu(V(x)) \geq \sup_{y \in \Sigma_{\text{opt}}} \frac{\mu(V(y))}{2\pi}.$$

Choose an arbitrary point $y \in \Sigma_{\text{opt}}$ and let $\{D_k\}_{k \in \mathbb{N}}$ a sequence satisfying conditions of Lemma 3.2.4, with $D_k \ni x$ for any k . Without loss of generality suppose $\varepsilon_k := \text{diam } D_k$ sufficiently small such that $y \notin D_k$ for any k . Obviously $\mathcal{H}^1(D_k) \geq \varepsilon_k$, and in the following the index k will be omitted, as this does not generate confusion.

Define $\Sigma'_\varepsilon := \Sigma_{\text{opt}} \setminus D_k$, $\Sigma''_\varepsilon := \Sigma'_\varepsilon \cup \partial B(y, \varepsilon/2\pi)$, clearly $\Sigma'_\varepsilon, \Sigma''_\varepsilon \in \mathcal{A}(\Omega)$ and $\mathcal{H}^1(\Sigma_{\text{opt}}) \geq \mathcal{H}^1(\Sigma''_\varepsilon)$. Then

$$F_{\mu, A}(\Sigma'_\varepsilon) \leq F_{\mu, A}(\Sigma_{\text{opt}}) + \varepsilon \mu(V(D_k)) \quad \forall k \in \mathbb{N}$$

and

$$F_{\mu, A}(\Sigma''_\varepsilon) \leq F_{\mu, A}(\Sigma'_\varepsilon) - \varepsilon(\mu(V(y)) - \mu(B(y, \varepsilon/2\pi))),$$

yielding

$$F_{\mu, A}(\Sigma''_\varepsilon) \leq F_{\mu, A}(\Sigma_{\text{opt}}) + \varepsilon \mu(V(D_k)) - \varepsilon(\mu(V(y)) - \mu(B(y, \varepsilon/2\pi))) \quad \forall k \in \mathbb{N}.$$

As $\varepsilon \rightarrow 0$, $D_k \rightarrow \{x\}$ and $\mu(B(y, \varepsilon/2\pi)) \rightarrow 0$ as $k \rightarrow \infty$, the minimality of Σ_{opt} forces $\mu(V(x)) \geq \mu(V(y))/2\pi$.

For the second part, i.e. finiteness of noncut points, consider x_1, x_2 noncut points, from the first claim it holds

$$\mu(V(x_1)) \leq 2\pi \mu(V(x_2)), \quad \mu(V(x_2)) \leq 2\pi \mu(V(x_1))$$

thus

$$\mu(V(x_1)) \leq 2\pi \mu(V(x_2)) \leq 4\pi^2 \mu(V(x_1)).$$

As by hypothesis there exists $y \in \Sigma_{\text{opt}}$ with $\mu(V(y)) > 0$, the first claim yields $\mu(V(x_1)) \geq \mu(V(y))/2\pi > 0$, and $\mu(V(x_2)) \geq \mu(V(y))$ follows. From the arbitrariness of x_1, x_2 follows that for any noncut point x it holds $\mu(V(x)) \geq \mu(V(y))/2\pi$ thus there are at most

$$\frac{2\pi\mu(\Omega)}{\mu(V(y))}$$

noncut points, and the proof is complete. $\square$

Proposition 3.2.9 states that there exists a finite number of noncut points if there exists $y \in \Sigma_{\text{opt}}$ with $\mu(V(y)) > 0$: the next result deal with this existence problem:

Proposition 3.2.10. *Given a domain Ω , a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, and $\Sigma \in \mathcal{A}(\Omega)$, there exists $y \in \Sigma$ such that $\mu(V(y)) > 0$.*

Notice that combining Proposition 3.2.9 and 3.2.10 yields:

Theorem 3.2.11. *Given a domain Ω , a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, and $\Sigma_{\text{opt}} \in \mathcal{A}(\Omega)$ solution of the average distance problem, then Σ_{opt} has finite endpoints.*

The proof of Proposition 3.2.10 requires auxiliary construction. Given a domain Ω , and $\Sigma \in \mathcal{A}(\Omega)$, for any couple of points $x, z \in \Sigma$ define

$$D(x, z) := \{y \in \Sigma : y \text{ is connected through a path } \gamma \text{ to } x \text{ and } z \notin \gamma\}.$$

If Σ does not contain loops, then every couple of points on Σ is connected by a unique arc, thus the following order can be imposed: given $x \in \Sigma$, and an arc $\gamma \subseteq \Sigma$ starting in x , $z_1, z_2 \in \gamma$

$$z_1 \leq_{\gamma, x} z_2$$

if $[x, z_1] \subseteq [x, z_2]$, where $[x, z]$ denotes the arc connecting x and z .

Similarly, $z_1 <_{\gamma, x} z_2$ if $z_1 \leq_{\gamma, x} z_2$ and $z_1 \neq z_2$. In this context a new natural distance can be introduced:

$$d_\Sigma : \Sigma \times \Sigma \rightarrow [0, \mathcal{H}^1(\Sigma)], \quad d_\Sigma(x_1, x_2) := \mathcal{H}^1([x_1, x_2]).$$

The set $D(x, z)$ satisfies several properties:

Proposition 3.2.12. *Given a domain Ω , $\Sigma \in \mathcal{A}(\Omega)$ consisting of more than one point, let $x \in \Sigma$ be arbitrary point. Then for all $z \in \Sigma \setminus \{x\}$ it holds:*

- $D(x, z)$ is connected and contains x ,
- $\Sigma \setminus D(x, z)$ is connected and closed.

Assume that Σ does not contain loops, then for any arc γ starting in x it holds:

- $D(x, z_1) \subseteq D(x, z_2)$ and $\mathcal{H}^1(D(x, z_1)) < \mathcal{H}^1(D(x, z_2))$ whenever $z_1 <_{\gamma, x} z_2$,

- $\bigcap_{k \in \mathbb{N}} D(x, z_k) = \{x\}$ whenever $z_k \rightarrow x$ as $k \rightarrow \infty$ and x is an endpoint.

The proof is from [16].

Proof. The proof will be split in several passages.

- To prove: $D(x, z)$ is connected and contains x .

This follows from the definition of $D(x, z)$.

- To prove: $\Sigma \setminus D(x, z)$ is connected and closed.

Notice that $z \notin D(x, z)$; let γ be an arc connecting $y \in \Sigma \setminus D(x, z)$ to z . It follows $\gamma \subseteq \Sigma \setminus D(x, z)$ as the contrary would give the existence of v connected to x by some arc γ' not passing through z . Denoting with $[y, v] \subseteq \gamma$ an arc connecting y to v , $[y, v] \circ \gamma'$ is an arc connecting y to z without passing through z , contradicting the choice of y .

It remains to prove that $\Sigma \setminus D(x, z)$ is closed. Consider a sequence $\{y_h\} \rightarrow y$ and suppose by contradiction that $y \notin \Sigma \setminus D(x, z)$, i.e. y is connected to x by some arc γ not containing z , and assume $y \neq z$, as otherwise there is nothing to prove. Let $\varepsilon > 0$ such that $z \notin B(y, \varepsilon)$ and for any h sufficiently large one has $y_h \in B(y, \varepsilon)$, and arcwise connectedness gives the existence of $\gamma_h \subseteq \Sigma \cap B(y, \varepsilon)$ connecting y_h to y and clearly not containing z . Thus y_h is connected to x through an arc not containing z , which contradicts $y_h \notin D(x, z)$.

- To prove: $D(x, z_1) \subseteq D(x, z_2)$ and $\mathcal{H}^1(D(x, z_1)) < \mathcal{H}^1(D(x, z_2))$ whenever $z_1 <_{\gamma, x} z_2$,

Let $\gamma \subseteq \Sigma$ be an arc through x and choose $z_1 \leq_{\gamma, x} z_2$; consider an arbitrary $y \in D(x, z_1)$ and consider an arc γ' connecting x to y . The set $\gamma \cap \gamma'$ is such that if $v \in \gamma \cap \gamma'$ then for any u , if it holds $u <_{\gamma, x} v$ or $u <_{\gamma', x} v$, then $u \in \gamma \cap \gamma'$. Thus either $\gamma \cap \gamma' = \{x\}$ or it is an arc $[x, y'] \subseteq \gamma$. Clearly $y' <_{\gamma, x} z_1$ for some $z_1 \in \gamma$, hence $y' <_{\gamma, x} z_2$ and $z_2 \notin \gamma'$. The arbitrariness of y gives $D(x, z_1) \subseteq D(x, z_2)$.

Denote with $\Delta := [z_1, z_2]$, and by construction it is a piece of γ between z_1 and z_2 . Consider an arbitrary $z \in \Delta$, and $z \in D(x, z_2)$; suppose $z \in D(x, z_1)$, i.e. there exists an arc $\gamma_1 \subseteq \Sigma$ connecting z and x and not containing z_1 , which is a contradiction, as there would exist two arcs between z and x . Thus $z \notin D(x, z_1)$, $\mathcal{H}^1(\Delta) > 0$ and the thesis is proven.

- To prove: $\bigcap_{k \in \mathbb{N}} D(x, z_k) = \{x\}$ whenever $z_k \rightarrow x$ as $k \rightarrow \infty$ and x is an endpoint.

Assume x is an endpoint, and consider a sequence $z_k \rightarrow x$, and assume by contradiction that there exists $y \in \bigcap_{k \in \mathbb{N}} D(x, z_k) \setminus \{x\}$. Let $\gamma' := [x, y]$, and $\gamma \cap \gamma' \neq \{x\}$ since this would give $\text{ord}_x \Sigma \geq 2$, thus $\gamma \cap \gamma'$ is an arc, and this implies $y \notin D(x, z_k)$ for any k sufficiently large, which is a contradiction. $\square$

Some auxiliary results concerning solutions of average distance problem are required. The next two results are from [16].

Lemma 3.2.13. *Let Σ_{opt} be a solution of the average distance problem, and $\{x_k\}_{k \in \mathbb{N}} \subseteq \Sigma_{\text{opt}}$ a sequence of noncut points and $\{z_k\}_{k \in \mathbb{N}} \subseteq \Sigma_{\text{opt}}$ such that $\varepsilon_k := \text{diam } D(x_k, z_k) \rightarrow 0$ as $k \rightarrow \infty$. Then for any $\sigma > (n-1)/2$ it holds*

$$\lim_{k \rightarrow \infty} \frac{\mu(V(D(x_k, z_k)))}{\varepsilon_k^\sigma} = \infty.$$

Proof. Denote with $\psi(\cdot) := \mu(V(\cdot))$. Upon subsequencne (which will not be relabeled) assume $x_k \rightarrow x \in \Sigma_{\text{opt}}$ for the sake of brevity denote with $D_k := D(x_k, z_k)$ and the index in $\varepsilon_k := \text{diam } D_k$ will be omitted. Denote with $\Sigma_\varepsilon := \Sigma \setminus D_k$, and recall that Σ_ε is compact in view of Proposition 3.2.12. Moreover it holds

$$\int_{\bar{\Omega}} \text{dist}(x, \Sigma_\varepsilon) d\varphi_s \leq \int_{\bar{\Omega}} \text{dist}(x, \Sigma_{\text{opt}}) d\varphi_s + \varepsilon \psi(D_k).$$

On the other hand one has $\psi(\bar{\Omega}) > \psi(B(x, r))$ for some r , as otherwise it would mean that a single point is optimal, which is not the case. Thus there exists Σ'_ε such that

$$\mathcal{H}^1(\Sigma'_\varepsilon) = \mathcal{H}^1(\Sigma_{\text{opt}}),$$

and

$$\int_{\bar{\Omega}} \text{dist}(x, \Sigma'_\varepsilon) d\varphi_s \leq \int_{\bar{\Omega}} \text{dist}(x, \Sigma_\varepsilon) d\varphi_s - C\varepsilon^{(n+1)/2}$$

for some C independent of k for any ε sufficiently small (independently of k). Arguing by contradiction yields $\varepsilon \psi(D_k) = o(\varepsilon^{(n+1)/2})$, which contradicts the optimality of Σ_{opt} . $\square$

Lemma 3.2.14. *Given a domain $\Omega \subseteq \mathbb{R}^n$, measures $\mu, \nu \in L^p(\Omega)$ with $p > 2n/(n+1)$, a solution Σ_{opt} of the average distance functional, and an endpoint $x \in \Sigma_{\text{opt}}$, let γ be an injective arc in Σ starting at x . Then there exists $\sigma > 1$ such that $(n-1)q\sigma/2n < 1$ (q is the conjugate exponent of p) and for all $z \in \gamma$ the set H' of points $y \in D(x, z)$ for which there exists $z' \in \gamma$ such that $y \in D(x, z')$ and*

$$l_y \geq (\mathcal{H}^1(D(x, z')))^{(n-1)q\sigma/2n}$$

where l_y denotes the maximum length of transport ray ending in y , satisfies $\mu(V(H)) > 0$.

Proof. Due to the assumptions on p one has $q < 2n/(n-1)$ and there exists $\sigma > 1$ such that $(n-1)q\sigma/2n < 1$. Suppose the statement does not hold with this σ , i.e. there exists $z \in \gamma$ such that $\psi(B_z) = \psi(D(x, z))$ where

$$B_z := \{y \in D(x, z) : l_y < (\mathcal{H}^1(D(x, z')))^{(n-1)q\sigma/2n} \forall z' \in \gamma, y \in D(x, z')\}.$$

In this case there exist $z' \in \gamma \cap D(x, z)$ and $y \in D(x, z')$. Denote with $\varepsilon := \mathcal{H}^1(D(x, z'))$ and the arbitrariness of z' allows to make ε as small as required. Consider an arbitrary v with $y = k(v)$ (this notation means that v belongs to the closure of some transport ray endpoint in y), and it holds

$$|v - z'| \leq |v - y| + |y - z'| \leq l_y + \varepsilon < \varepsilon^{(n-1)q\sigma/2n} + \varepsilon.$$

Condition $(n-1)q\sigma/2n < 1$ gives that in the above inequality the addend ε is negligible for $\varepsilon \ll 1$. Therefore

$$\begin{aligned}\psi(D(x, z')) &= \psi(D(x, z') \cap B_z) \\ &= \varphi_s(\{v : k(v) \in D(x, z') \cap B_z\}) \\ &\leq \varphi_s(B(z', (n-1)q\sigma/2n))\end{aligned}$$

and using Hölder inequality gives

$$\psi(D(x, z')) \leq C \|\varphi_s\|_{L^p} \varepsilon^{(n-1)\sigma/2}$$

contradicting Lemma 3.2.13, thus concluding the proof. $\square$

Another construction, valid only in the two dimension case, is required: endow $\mathbb{R}^2$ with a coordinate system, let be given Ω , the domain and a set $\Sigma \in \mathcal{A}(\Omega)$. Consider the Monge-Kantorovich problem of transporting $\mathcal{L}^2_\Omega$ on $\mathcal{H}^1_\Sigma$. Let T be the transport set (union of transport rays without endpoints) and define

$$\begin{aligned}T^+ &:= \{x \in T : \pi_1(x) > \pi_1(k(x)) \text{ or } \pi_1(x) = \pi_1(k(x)), \pi_2(x) > \pi_2(k(x))\} \\ T^- &:= T \setminus T^+\end{aligned}$$

where $\pi_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the projection on the i -th coordinate, and $k(x)$ is the projection of x on Σ , uniquely defined if $x \in T$. Let $y \in \Sigma_{\text{opt}}$ be an endpoint of some transport ray, such that any point projecting on y belongs to the same line l , and denote with $\theta(y) \in [0, \pi/2]$ the angle between l and e_2 , the second unit coordinate vector.

Moreover, the following notations will be used:

$$\psi^\pm(\cdot) := \mu(k^{-1}(\cdot) \cap T^\pm), \quad \psi(\cdot) := \mu(V(\cdot)).$$

Lemma 3.2.15. *Given a domain $\Omega \subseteq \mathbb{R}^2$, $\Sigma_{\text{opt}} \in \mathcal{A}(\Omega)$ solution of the average distance problem, let $x \in \Sigma_{\text{opt}}$ be a noncut point, and let $D \subseteq \Omega$ be an open set such that for ψ -a.e. point $y \in D \cap \Sigma_{\text{opt}}$, $\theta(y)$ is well defined, while $x \notin D$ and $\psi^+(D) \geq \psi^-(D)$. Then for every $f \in C^1_0(D)$, 1-Lipschitz and vanishing on Σ_{opt} , it holds*

$$\int_D f \sin \theta d(\psi^+ - \psi^-) \leq \psi(\{x\}).$$

The proof is from [16].

Proof. Denote with D_k the sets from Lemma 3.2.4, and suppose $\varepsilon_k := \text{diam } D_k$ (in the following the index will be omitted for the sake of brevity) to be sufficiently small such that $D \cap D_k = \emptyset$. Let $\Sigma_\varepsilon := \Sigma_{\text{opt}} \setminus D_k$, then

$$\int_\Omega \text{dist}(z, \Sigma_\varepsilon) d\varphi_s \leq \int_\Omega \text{dist}(z, \Sigma) d\varphi_s + \varepsilon \psi(D_k).$$

Assume without loss of generality that

$$\bar{co}(\text{supp}(\varphi^+) \cup \text{supp}(\varphi^-)) \subseteq \subseteq \Omega.$$

Consider a smooth function $f \in C_0^1(D)$ as in the hypothesis and define the diffeomorphism

$$\Phi_\delta : \bar{\Omega} \longrightarrow \bar{\Omega}, \quad \Phi_\delta(x_1, x_2) := (x_1 - \delta f(x_1, x_2), x_2),$$

for δ sufficiently small. Then

$$\mathcal{H}^1(\Phi_\delta(\Sigma_\varepsilon)) \leq \mathcal{H}^1(\Sigma_\varepsilon) + C_f \delta$$

for some constant $C_f > 0$, and choose $\delta := \varepsilon/C_f$. Define $\Sigma_\varepsilon := \Phi_\delta(\Sigma_\varepsilon)$ and it follows

$$\mathcal{H}^1(\Sigma'_\varepsilon) \leq \mathcal{H}^1(\Sigma_\varepsilon) + C_f \delta \leq \mathcal{H}^1(\Sigma_{\text{opt}}) - \varepsilon + C_f \delta = \mathcal{H}^1(\Sigma_{\text{opt}}),$$

as $\mathcal{H}^1(D_k) \geq \varepsilon$. For the average distance term it holds

$$\begin{aligned} \int_{\bar{\Omega}} \text{dist}(z, \Sigma'_\varepsilon) d\varphi_s &\leq \int_{\bar{\Omega}} \text{dist}(z, \Sigma'_\varepsilon) d\varphi_s - C\varepsilon + o(\varepsilon) \\ &\leq \int_{\bar{\Omega}} \text{dist}(z, \Sigma_{\text{opt}}) d\varphi_s - C\varepsilon + o(\varepsilon) + \varepsilon\psi(D_k) \end{aligned}$$

with

$$C := \int_D f \sin \theta d(\psi^+ - \psi^-).$$

The optimality of Σ_{opt} forces $C \leq \psi(D_k)$, and passing to the limit $k \rightarrow \infty$ the proof is complete. $\square$

Now it is possible to prove Proposition 3.2.10:

Proof. (of Proposition 3.2.10).

The proof is done by contradiction, and is split in several steps. Define $\psi(\cdot) := \mu(V(\cdot))$, and suppose the opposite, i.e. $\psi(x) = 0$ for any $x \in \Sigma_{\text{opt}}$.

Step 1:

From assumptions on the measure, one has that θ is ψ -a.e. well defined: indeed denote with E the set on which θ is not defined, and it is possible to decompose E into

$$E = E_0 \cup E_1$$

where E_0 is the set of points Σ_{opt} which are not endpoints of any transport ray (i.e. the set of points for which no point of $\Omega \setminus \Sigma_{\text{opt}}$ projects onto), and E_1 is the set of points which are endpoints of multiple non collinear transport rays. Thus $V(E_0) \subseteq \Sigma_{\text{opt}}$, forcing $\psi(E_0) = 0$; E_1 is at most countable and assumption $\psi(y) = 0$ for any $y \in \Sigma_{\text{opt}}$ implies $\psi(E_1) = 0$.

In view of Lemma 3.2.15 it holds $\sin \theta \psi^+(e) = \sin \theta \psi^-(e)$ for every Borel set $e \subseteq \Sigma_{\text{opt}}$, which implies

$$\psi^+(e \cap \{\theta \neq 0\}) = \psi^-(e \cap \{\theta \neq 0\})$$

and exchanging the coordinates yields

$$\psi^+ = \psi^-.$$

Step 2:

Fix an arbitrary endpoint $x \in \Sigma_{\text{opt}}$ and an arbitrary point $z \in \Sigma_{\text{opt}}$; since θ is well defined on $D(x, z)$, and combining Lemma 3.2.14 and $\psi^+ = \psi^-$ yields that for a non ψ negligible set of $y \in D(x, z)$ the angle $\theta(y)$ is defined and y is endpoint of exactly two transport rays $R_y^\pm \subseteq T^\pm$ belonging to the same line l , which satisfies

$$l_y^\pm \geq (\mathcal{H}^1(D(x, z)))^{q\sigma/4}$$

for some $\sigma > 1$, where $l_y^\pm := \mathcal{H}^1(R_y^\pm)$, and $q\sigma/4 < 1$. Denote with C_z the set of such $y \in D(x, z)$. It holds:

- for ψ -a.e. $y \in C_z$ are not endpoints of Σ_{opt} once $\mathcal{H}^1(D(x, z))$ is sufficiently small.

To prove the claim above, define

$$C(y') := \{y \in D(x, z) \text{ endpoint} : [x, y] \cap [x, z] = [x, y']\}$$

for every $y' \in [x, z]$ with order at least 3. Clearly every endpoint of $D(x, z)$ belongs to a $C(y')$ for some $y' \in [x, z]$; moreover $C(y'_1) \neq C(y'_2)$ whenever $y'_1 \neq y'_2$, as Σ_{opt} does not contains loops.

If it holds

$$|\{y \in C(y') : l_y^\pm \geq (\mathcal{H}^1(D(x, z')))^{q\sigma/4} \text{ for some } z' \in [x, z], y \in D(x, z')\}| \leq 1,$$

for z' sufficiently close to x , then the claim is proven. Indeed the set of points with order at least 3 is at most countable, since Σ_{opt} does not contain loops. This would imply that the set C_z^{end} of endpoints of C_z is at most countable, and due to assumptions on ψ , $\psi(C_z^{\text{end}}) = 0$.

Denote

$$\delta(y) := \inf\{\mathcal{H}^1(D(x, z)) : z' \in [x, z], y \in D(x, z')\}$$

and it is clear that if $y \in C(y')$ then

$$\delta(y) := \inf\{\mathcal{H}^1(D(x, z')) : y' <_{\gamma, x} z' \in [x, z]\}$$

and thus for any $y \in C(y')$ $\delta(y)$ is equal to a constant δ . Let $y_1 \in C(y') \cap C_z$, without loss of generality assume that the origin of the coordinate system is at y_1 the x axis coincides with the line of transport rays $R_{y_1}^\pm$. Then

$$\Sigma_{\text{opt}} \cap (B((0, \delta^{q\sigma/4}), \delta^{q\sigma/4}) \cup B((0, -\delta^{q\sigma/4}), \delta^{q\sigma/4})) = \emptyset.$$

Let $z' \geq_{\gamma, x} y'$ such that $\mathcal{H}^1(D(x, z')) \leq 2\delta$, then for any $\xi \in D(x, z')$ it holds

$$|\xi - y_1| \leq \text{diam } D(x, z') \leq \mathcal{H}^1(D(x, z')) \leq 2\delta$$

and hence $D(x, z') \subseteq B(y_1, 2\delta)$. Assume z is close to y so that $\delta \leq \mathcal{H}^1(D(x, z'))$ can be as small as required and $2\delta \ll \delta^{q\sigma/4}$. Moreover, without loss of generality suppose $x \in \{x_2 < 0\}$ which yields $D(x, z') \subseteq x_2 < 0$.

Suppose there exists another $y_2 \in C(y') \cap C_z$ distinct from y_1 . Then Σ_{opt} must be outside the union of balls centred in $R_{y_2}^\pm$ with radii $\delta^{q\sigma/4}$ and touching y_2 . Since $y \in B(y_1, 2\delta)$ then x must be

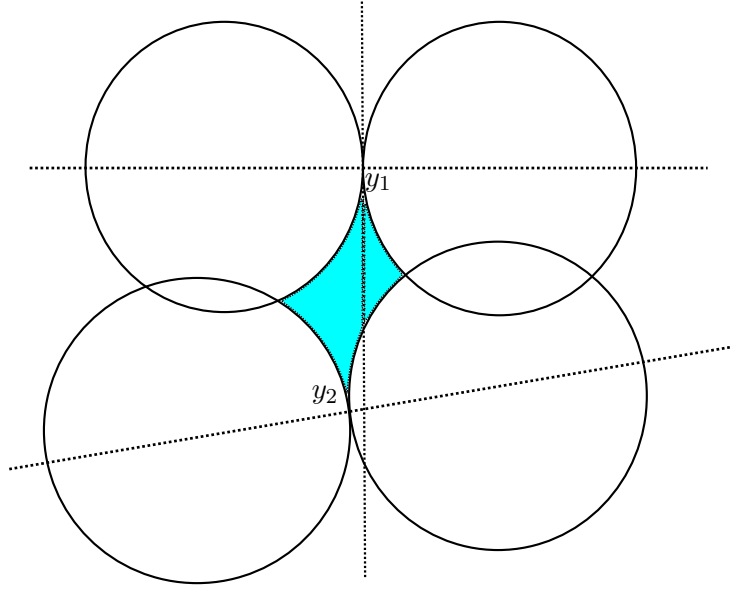


Figure 3.2.1: This is a schematic representation of the configuration.

inside the shaded region in Figure 3.2.1, as otherwise the arc $[x, y_1]$ must pass through y_2 , which is assumed an endpoint.

Then Σ_{opt} must belong to this region as otherwise there exists $x' \in \Sigma_{\text{opt}}$ outside, and arc $[x, x']$ must pass through y_1 or y_2 which are assumed be endpoints. The diameter of this region cannot exceed 2δ , thus $\text{diam } \Sigma_{\text{opt}} \leq \delta$. Letting $z \rightarrow_\gamma x$ $\delta \rightarrow 0$ therefore $\text{diam } \Sigma_{\text{opt}} = 0$, which would mean Σ_{opt} consists of one point, contradiction.

Step 3:

Let $x \in \Sigma_{\text{opt}}$ be an endpoint, $\gamma \subseteq \Sigma_{\text{opt}}$ be an arc starting at x , using results from Step 2 it follows that for any sequence $\{z'_h\}_{h \in \mathbb{N}} \subseteq \gamma$ with $x_h \rightarrow_\gamma x$ there exists a sequence $\{y_h\}_{h \in \mathbb{N}}$ satisfying

- $y_h \in D(x, z'_h)$ and are not endpoints,
- $\theta(y_h)$ is defined and y_h is endpoint of exactly two transport rays belonging to the same line l starting from outside $B(y_h, r_h)$ where

$$r_h := (\mathcal{H}^1(D(x, h'_h)))^{q\sigma/4}.$$

Then for any h there exists an endpoint x_h of Σ_{opt} such that $D(x_h, y_h) \subseteq D(x, z'_h)$: indeed either $y_h \in \gamma$, in which case $x_h := z$ is acceptable choice, or $y_h \notin \gamma$, in which case $\gamma \cap [0, y'_h] = [x, y'_h]$ for some $y'_h <_{\gamma, x} z'_h$ and x_h can be taken any of the endpoints of Σ_{opt} belonging to $D(y_h, y'_h)$.

Denote $D_h := D(x_h, y_h)$, and $\varepsilon_h := \text{diam } D_h$, omitting the index h when no risk of confusion arises. Note that

$$\varepsilon_h = \text{diam } D_h \leq \mathcal{H}^1(D_h) \leq \mathcal{H}^1(D(x, z'_h)) \rightarrow 0$$

as $h \rightarrow \infty$. Without loss of generality assume the coordinate system is placed with axis x_1 coinciding with l_h and the origin at y_h . Denote $r_\varepsilon := \varepsilon_h^{q\sigma/4} \leq r_h$ and consider points $P_\varepsilon^\pm := (\pm r_\varepsilon, 0)$. Clearly

$$\Sigma_{\text{opt}} \cap (B(P_\varepsilon^+, r_\varepsilon) \cup B(P_\varepsilon^-, r_\varepsilon)) = \emptyset$$

while $D_h \subseteq B(y_h, \varepsilon)$, thus

$$D_h \subseteq B(y_h, \varepsilon) \setminus (B(P_\varepsilon^+, r_\varepsilon) \cup B(P_\varepsilon^-, r_\varepsilon)).$$

Without loss of generality assume $D_h \subseteq \{x_2 < 0\}$. Consider points

$$A_\varepsilon^\pm := \partial B(y_h, \varepsilon) \cap \partial B(P_\varepsilon^\pm, r_\varepsilon) \cap Q^\pm$$

where $Q^\pm$ are respectively the fourth and the third quadrant in the coordinate system. Then D_h belongs to the curvilinear triangle with vertexes y_h and $A_\varepsilon^\pm$. By direct computation

$$A_\varepsilon^\pm = (\pm \varepsilon^2/2r, \varepsilon(1 - \varepsilon^2/4r_\varepsilon^2)^{1/2}) = (o(\varepsilon), \varepsilon + o(\varepsilon))$$

due to the choice of r_ε and assumptions on p .

Denote with $R^\pm$ the transport rays starting at $P_\varepsilon^\pm$ and passing through $A_\varepsilon^\pm$, and with $C_\varepsilon^\pm$ the cones with vertexes $P_\varepsilon^\pm$ and formed by axis x_1 and rays $R^\pm$; define also

$$D_\varepsilon^\pm := C_\varepsilon^\pm \setminus B(y_h, r_\varepsilon)$$

$$D_\varepsilon^0 := x_2 < 0 \setminus (D_\varepsilon^+ \cup D_\varepsilon^- \cup B(y_h, r_\varepsilon)).$$

Define the measures

$$\mu_\varepsilon^\pm := \mu|_{k^{-1}(D_h) \cap T^\pm}, \quad \mu_\varepsilon := \mu_\varepsilon^+ + \mu_\varepsilon^- = \mu|_{k^{-1}(D_h)}.$$

Disintegrating the measures yields

$$\mu_\varepsilon^\pm(e) = \int_{D_h} (\mu_\varepsilon^\pm)'(t, e) d\psi^\pm(t),$$

where $D_h \ni t \mapsto (\mu_\varepsilon^\pm)'(t, \cdot)$ is measurable and probability measures $(\mu_\varepsilon^\pm)'(t, \cdot)$ are concentrated on $k^{-1}(t) \cap T^\pm$ for $\psi^\pm$ -a.e. and in view of results from Step 1, for ψ -a.e. $t \in D_h$.

Note that

$$\mu_\varepsilon(D_\varepsilon^0) \leq \mu_\varepsilon(B(y_h, r_\varepsilon)).$$

Indeed for ψ -a.e. $e \in D_h$ the set $k^{-1}(t)$ is contained in a line l_t . It is clear that if l_t passes through D_ε^0 then both $l_t^\pm := l_t \cap T^\pm$ intersect the horizontal segment $(P_\varepsilon^-, P_\varepsilon^+)$. Suppose without loss of generality $l_t^+ \cap (P_\varepsilon^-, y_h] \neq \emptyset$. Then $k^{-1}(t) \cap T^\pm \subseteq (P_t, t)$, where $\{P_t\} := l_t^- \cap (P_\varepsilon^-, y_h]$. Since $\mu_\varepsilon^\pm(t, \cdot)$ are concentrated on $l_t^\pm$, then

$$(\mu_\varepsilon^+)'(t, l_t^+) = (\mu_\varepsilon^+)'(t, (T_t, t)) = (\mu_\varepsilon^+)'(t, B(y_h, r_\varepsilon)) = 1$$

while

$$(\mu_\varepsilon^-)'(t, l_t^-) = (\mu_\varepsilon^-)'(t, D_\varepsilon^0) = (\mu_\varepsilon^-)'(t, B(y_h, r_\varepsilon)) = 1,$$

therefore $(\mu_\varepsilon^-)'(t, D_\varepsilon^0) \leq (\mu_\varepsilon^+)'(t, B(y_h, r_\varepsilon))$ and symmetrically $(\mu_\varepsilon^+)'(t, D_\varepsilon^0) \leq (\mu_\varepsilon^-)'(t, B(y_h, r_\varepsilon))$.

Now construct the set Σ_ε in the following way: remove D_h from Σ_{opt} , then add a segment I_ε centred at y_h along x_1 axis and having length ε . $\Sigma_\varepsilon \in \mathcal{A}(\Omega)$ and satisfies $\mathcal{H}^1(\Sigma_\varepsilon) \leq \mathcal{H}^1(\Sigma_{\text{opt}})$, and observe that

$$\text{dist}_\Omega(\cdot, \Sigma_\varepsilon) \leq \text{dist}_\Omega(\cdot, \Sigma_{\text{opt}}) + \varepsilon.$$

On the other hand by direct computation

$$\text{dist}_\Omega(z, \Sigma_\varepsilon) = \text{dist}_\Omega(z, z_\varepsilon^\pm) \leq \text{dist}_\Omega(z, D_h) - \varepsilon/4$$

for any $z \in D_\varepsilon^+ \cup D_\varepsilon^-$, where $z_\varepsilon^\pm$ are the endpoints of I_ε in $Q^\pm$ respectively. Therefore

$$F_{\mu,A}(\Sigma_\varepsilon) \leq F_{\mu,A}(z, \Sigma_{\text{opt}}) + \varepsilon \mu_\varepsilon(D_\varepsilon^0 \cup B(y_h, r_\varepsilon)) - \varepsilon \mu_\varepsilon(D_\varepsilon^+ \cup D - \varepsilon^-)/4.$$

Considering that

$$\mu_\varepsilon(D_\varepsilon^+ \cup D - \varepsilon^-) + \mu_\varepsilon(D_\varepsilon^0 \cup B(y_h, r_\varepsilon)) = \psi(D_h),$$

it holds

$$F_{\mu,A}(\Sigma_\varepsilon) < F_{\mu,A}(\Sigma_{\text{opt}}),$$

contradicting the optimality of Σ_{opt} . □

In two dimensional domain, under suitable assumptions on measure and function, another property satisfied by solutions of the average distance problem is that any point has multiplicity at most 3, and only a finite number of points can have multiplicity 3. The proof is similar to that done for the absence of loops, but more stringent conditions on the measure are required.

Lemma 3.2.16. *Given a domain Ω , a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, let Σ_{opt} be a solution of the average distance problem. Then it holds:*

1. $|\{x \in \Sigma_{\text{opt}} : \text{ord}_x \Sigma_{\text{opt}} \geq 3\}| < \infty$,
2. for any $z \in \Sigma_{\text{opt}}$, $\text{ord}_z \Sigma_{\text{opt}} \leq 3$.

Before the proof, the notion of “depth” of a point is required:

Definition 3.2.17. *Given a domain Ω and a set $\mathcal{X} \in \mathcal{A}(\Omega)$, the “depth” of a point $z \in \mathcal{X}$ is not exceeding the cardinal number $\mathfrak{n}$ if there exists an arc connecting z to an endpoint l containing not more than $\mathfrak{n}$ points with order at least 3; the “depth” of z is $\mathfrak{n}$ if $\mathfrak{n}$ is the minimal cardinal for which the depth of z does not exceed $\mathfrak{n}$.*

Another result is useful:

Theorem 3.2.18. *Given a domain Ω , an element $X \in \mathcal{A}(\Omega)$ a constant $L > 0$, and a point $P \in X$ with order $\text{ord}_P X \geq L$, then there exists arcs $\{\gamma_i\}_{i=1}^L$ with positive length and an endpoint in P and $\gamma_i \cap \gamma_j = \{P\}$ whenever $i \neq j$.*

This result is known as “Menger n -Beinsatz”. The proof can be found in [30].

Proof. (of Lemma 3.2.16)

As Σ_{opt} is connected, no point has order 0. Let k be the number of endpoints of Σ_{opt} , and the first claim is:

- for any $x \in \Sigma_{\text{opt}}$, the depth of x is at most $k - 1$.

Consider an arbitrary point $x_0 \in \Sigma_{\text{opt}}$, and an arc γ_0 connecting it to an arbitrary endpoint, and let $\{x_i\}_{i \geq 1}$ be the set of points with order at least 3. From Menger n -Beinsatz for any x_i there exists an arc γ_i starting in x_1 and intersecting γ_0 only in x_i . Pick an arbitrary internal point x'_i on this arc, and consider the connected component of $\Sigma_{\text{opt}} \setminus \{x_i\}$ containing x'_i . Choose an arbitrary endpoint l_i on $\Sigma_{\text{opt}} \setminus \{x_i\}$, and from Theorem 3.2.8 $l_i \neq l_0$. Using the same argument $l_j \neq l_i$ whenever $j \neq i$, thus the depth of x_0 does not exceed $k - 1$.

Denote with $\mathcal{B}_j$ the set of points of Σ_{opt} with depth j , and with $\mathcal{B}$ the number of points with order at least 3,

$$\mathcal{B} = \bigcup_{j=0}^{k-1} \mathcal{B}_j$$

and considering that $\mathcal{B}_0 \leq k$, $\mathcal{B}_{i+1} \leq \mathcal{B}_i$ for any i , $|\mathcal{B}| < \infty$.

The second claim is:

- every point has finite order, not exceeding k .

Consider an arbitrary point $x \in \Sigma_{\text{opt}}$, and suppose the contrary i.e. the order of x is at least $k + 1$. From Menger n -Beinsatz there exist $k + 1$ distinct arcs $\{\gamma_i\}_{i=1}^{k+1}$ starting at x and pairwise disjoint outside of x . Taking an arbitrary internal point $x_i \in \gamma_i$, the connected component of $\Sigma_{\text{opt}} \setminus \{x\}$ containing $\{x_i\}$ must contain another endpoint l_i , and using the same argument found before, $l_i \neq l_j$ whenever $i \neq j$. This implies there exist at least $k + 1$ endpoints, contradiction.

Now the second point of the thesis can be proven. Suppose the contrary i.e. there exists $x \in \Sigma_{\text{opt}}$ with $\text{ord}_x \Sigma_{\text{opt}} \geq 4$. From Menger n -Beinsatz there exist arcs $\{\gamma_i\}_{i=1}^4$ starting at x and disjoint outside x . Choose $\varepsilon > 0$ sufficiently small such that $\gamma_i \cap \partial B(x, \varepsilon) \neq \emptyset$ for $i = 1, \dots, 4$, and choose points $a_1, \dots, a_4$ such that $a_i \in \gamma_i \cap \partial B(x, \varepsilon)$.

Without loss of generality there exist two points $a_{i'}, a_{j'}$ such that the minimum arc of $\partial B(x, \varepsilon)$ between them has length at most $\varepsilon\pi/2$. Then define $St(a_{i'}, a_{j'}, x, \varepsilon)$ the Steiner graph connecting $a_{i'}$, $a_{j'}$ and x in $B(x, \varepsilon)$, and by direct computation $\mathcal{H}^1(St(a_{i'}, a_{j'}, x, \varepsilon)) = \varepsilon(2 - \beta)$, where $\beta > 0$ does not depend on any other parameter.

Then the competitor

$$\Sigma' := \Sigma_{\text{opt}} \setminus (\gamma_{i'} \cup \gamma_{j'}(\partial B(x, \varepsilon))) \cup St(a_{i'}, a_{j'}, x, \varepsilon)$$

clearly satisfies $\Sigma' \in \mathcal{A}_{\mathcal{H}^1(\Sigma_{\text{opt}}) - \varepsilon\beta}(\Omega)$, and the points z for which $\text{dist}_\Omega(z, \Sigma_{\text{opt}}) \neq \text{dist}_\Omega(z, \Sigma')$ are those projecting on $\Sigma_{\text{opt}} \cap \partial B(x, \varepsilon)$. Denoting with $\Gamma_\varepsilon(x)$ this set, for $\varepsilon \rightarrow 0$, $\mathcal{L}^n(\Gamma_\varepsilon) \rightarrow 0$, thus $\mu(\Gamma_\varepsilon) \rightarrow 0$.

Combining with the fact that the difference $|\text{dist}_\Omega(z, \Sigma_{\text{opt}}) - \text{dist}_\Omega(z, \Sigma')|$ is at most ε , the thesis follows. $\square$

Notice that Proposition 3.2.9 is crucial: indeed its proof relies on geometric properties specific to $\mathbb{R}^2$, which cannot be extended to higher dimensions without significantly changing the proof. Such possible extension (for the constrained problem) is listed as one of the most interesting open questions concerning the average distance problem in the review paper [31]. A partial answer has been given in a work in progress of the author in collaboration with Slepčev:

Theorem 3.2.19. *Given a domain $\Omega \subseteq \mathbb{R}^N$ with $N \geq 2$, a probability measure $\mu \ll \mathcal{L}^N$, there exists a closed set $A \subseteq [0, \infty)$ with $\min A = 0$, $\sup A = \infty$ such that for any $L \in A$, for any solution Σ_{opt} of*

$$\min_{\mathcal{A}_L(\Omega)} F_\mu(\cdot),$$

the number of endpoints of Σ_{opt} is at most $1/\lambda$ for some $\lambda = \lambda(L) > 0$. As consequence the number of triple points is at most $1/\lambda$.

However very little is known about such set A .

3.3 Regularity and asymptotic behavior

In the previous Section we have shown that solutions of the average distance problem, under suitable assumptions on the measure (and for some of them, on the dimension), must satisfy certain geometric properties. In this Section we will investigate regularity of such solutions, and their distance from the border. Only a weak regularity property is proven (see for instance [14]) in this case.

3.3.1 Ahlfors regularity

The main analytic property satisfied by solutions of the average distance problem, is the “Ahlfors regularity”:

Definition 3.3.1. *A set $S \subseteq \mathbb{R}^n$ with $\dim_{\mathcal{H}} S = 1$ is Ahlfors regular if there exists $c, C, \rho_0 > 0$ such that*

$$c \leq \frac{\mathcal{H}^1(S \cap B(x, \rho))}{\rho} \leq C$$

for any $x \in S$, $\rho \in (0, \rho_0)$.

This is a weak regularity property, but an interesting property is that Ahlfors regular sets are uniform rectifiable.

The proof of Ahlfors regularity is split in two parts: lower bound estimates and upper bound estimates. Notice immediately that the former is almost trivial: indeed given a domain $\Omega \subseteq \mathbb{R}^n$ with $n \geq 2$, a solution Σ_{opt} of the average distance problem belongs to $\mathcal{A}(\Omega)$ by definition, and for any $x \in \Sigma_{\text{opt}}$, $\rho < \text{diam } \Sigma_{\text{opt}}/2$ there exists $y \in \Sigma_{\text{opt}} \cap (\Omega \setminus B(x, \rho))$, and since Σ_{opt} is connected, $\Sigma_{\text{opt}} \cap \partial B(x, \rho) \neq \emptyset$. Choose a point $z \in \Sigma_{\text{opt}} \cap \partial B(x, \rho)$ such that z is connected to x by an arc $\gamma \subseteq \Sigma_{\text{opt}} \cap \overline{B(x, \rho)}$, and clearly $\mathcal{H}^1(\Sigma_{\text{opt}} \cap \partial B(x, \rho)) \geq \mathcal{H}^1(\gamma) \geq \text{dist}_\Omega(x, z) \geq \rho$, thus

$$1 \leq \frac{\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho))}{\rho} \quad \forall x \in \Sigma_{\text{opt}}, \rho \in (0, \text{diam } \Sigma_{\text{opt}}/2). \quad (3.3.1)$$

Notice that lower bound estimate relies only on $\Sigma_{\text{opt}} \in \mathcal{A}(\Omega)$. The upper bound estimate will require more stringent conditions:

Proposition 3.3.2. *Given a domain $\Omega \subseteq \mathbb{R}^2$, a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, let Σ_{opt} be a solution of the average distance problem. Then there exists $C, \rho_0 > 0$ such that*

$$\frac{\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho))}{\rho} \leq C \quad \forall x \in \Sigma_{\text{opt}}, \rho \in (0, \rho_0).$$

Proof. The proof is achieved by contradiction: suppose there exists $x \in \Sigma_{\text{opt}}$ and $\{\rho_j\}_{j \in \mathbb{N}} \downarrow 0$ such that

$$\lim_{j \rightarrow \infty} \frac{\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho_j))}{\rho_j} \geq 2\pi + 2.$$

Without loss of generality suppose that for any j it holds

$$\frac{\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho_j))}{\rho_j} > 2\pi + 1. \quad (3.3.2)$$

The goal is to find a competitor $\Sigma' \in \mathcal{A}_{\mathcal{H}^1(\Sigma_{\text{opt}})}(\Omega)$ satisfying $F_{\mu, A}(\Sigma') < F_{\mu, A}(\Sigma_{\text{opt}})$. Denote $\Sigma'_j := \Sigma_{\text{opt}} \setminus (\Sigma_{\text{opt}} \cap B(x, \rho_j)) \cup \partial B(x, \rho_j)$, and from (3.3.2) it follows

$$\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho_j)) > (2\pi + 1)\rho_j.$$

As clearly $\mathcal{H}^1(\partial B(x, \rho_j)) = 2\pi\rho_j$, this implies $\mathcal{H}^1(\Sigma'_j) \leq \mathcal{H}^1(\Sigma_{\text{opt}}) - \rho'_j$ for any j .

Choose an arbitrary point $y \in \Omega$, without loss of generality suppose y belongs to the set Ω' of points having unique projection on Σ_{opt} and Σ'_j for any $j \in \mathbb{N}$, as such set has full measure. Denote with $k : \Omega' \rightarrow \Sigma_{\text{opt}}$ the projection on Σ_{opt} , and the following cases are possible:

- if $k(y) \notin \Sigma_{\text{opt}} \cap B(x, \rho_j)$ then $k(y) \in \Sigma'_j$, thus $\text{dist}_\Omega(y, \Sigma_{\text{opt}}) \geq \text{dist}_\Omega(y, \Sigma'_j)$,
- if $k(y) \in \Sigma_{\text{opt}} \cap B(x, \rho_j)$ and $y \notin \overline{B(x, \rho'_j)}$ then the transport ray passing through y must intersect $\partial B(x, \rho'_j)$, thus $\text{dist}_\Omega(y, \Sigma_{\text{opt}}) \geq \text{dist}_\Omega(y, \Sigma'_j)$,

- if $k(y) \in \Sigma_{\text{opt}} \cap B(x, \rho_j)$ and $y \in \overline{B(x, \rho_j)}$ then $\text{dist}_\Omega(y, \Sigma_{\text{opt}}) + \rho'_j \geq \text{dist}_\Omega(y, \Sigma'_j)$.

Thus using Hölder inequality we have

$$\begin{aligned} F_{\mu,A}(\Sigma'_j) &\leq F_{\mu,A}(\Sigma_{\text{opt}}) + \rho'_j \mu(B(x, \rho'_j)) \\ &= F_{\mu,A}(\Sigma_{\text{opt}}) + \rho'_j \|\mu\|_{L^p(B(x, \rho'_j))}^{1/p} (\pi(\rho'_j)^2)^{1/q} \\ &= F_{\mu,A}(\Sigma_{\text{opt}}) + (\|\mu\|_{L^p(B(x, \rho'_j))}^{1/p} \pi^{1/q}) (\rho'_j)^{1+2/q} \end{aligned}$$

where q is the conjugate exponent of p . From Lemma 3.2.7 there exists $\Sigma''_j \in \mathcal{A}_{\mathcal{H}^1(\Sigma'_j) + \rho'_j}(\Omega)$ such that

$$F_{\mu,A}(\Sigma''_j) \leq F_{\mu,A}(\Sigma'_j) - K(\rho'_j)^{3/2}$$

once ρ'_j is sufficiently small, where K is a constant not dependent on ρ'_j . As by hypothesis $p > 4/3$, i.e. $q < 4$, thus

$$\begin{aligned} F_{\mu,A}(\Sigma''_j) &\leq F_{\mu,A}(\Sigma'_j) - K(\rho'_j)^{3/2} \\ &\leq F_{\mu,A}(\Sigma_{\text{opt}}) + (\|\mu\|_{L^p(B(x, \rho'_j))}^{1/p} \pi^{1/q}) (\rho'_j)^{1+2/q} - K(\rho'_j)^{3/2} \end{aligned}$$

and for all j sufficiently large it holds $F_{\mu,A}(\Sigma''_j) < F(\Sigma_{\text{opt}})$. Thus Ahlfors regularity is proven, being $2\pi + 2$ an admissible upper bound. $\square$

Combining (3.3.1) and Proposition 3.3.2 it follows:

Theorem 3.3.3. *Given a domain $\Omega \subseteq \mathbb{R}^2$, a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, let Σ_{opt} be solution of the average distance problem. Then Σ_{opt} is Ahlfors regular.*

3.3.2 Asymptotic behavior for $\mathcal{H}^1(\Sigma_{\text{opt}}) \rightarrow \infty$

In this subsection our goal is to analyze some asymptotic behavior of solutions of the average distance problem when the allowed length goes to infinity.

The first result concerns the asymptotic behavior of the average distance functional:

Proposition 3.3.4. *Given a domain $\Sigma \subseteq \mathbb{R}^n$ with $n \geq 2$, define*

$$V(l) := \min_{\mathcal{A}_l(\Omega)} F_{\mathcal{L}^n, id}.$$

Then there exist constants $c, C > 0$ such that

$$c \leq V(l) l^{\frac{1}{n-1}} \leq C$$

for all l sufficiently large.

A preliminary lemma is required:

Lemma 3.3.5. Let $Q \subseteq \mathbb{R}^n$ ($n \geq 2$) be a cube, divided by a uniform grid parallel to its edges into small cubes with side ε . Let β be a Lipschitz curve of length l intersecting exactly k such cubes. Then there exist $c_1, c_2 > 0$ not dependent on ε and l for which

$$k \leq c_1 l / \varepsilon + c_2.$$

Proof. Notice that in a union of $2^n + 1$ cubes with side ε there exist two cubes for which the distance between them is at least ε , thus if β intersects k cubes, then

$$l \geq [k / (2^n + 1)] \varepsilon$$

where $[\cdot]$ denotes the integer part mapping. □

Proof. (of Proposition 3.3.4). The proof is split into two steps.

Step 1:

Let $Q \subseteq \Omega$ be a cube, and divide Q with an uniform grid parallel to its sides, such that each small cube of the grid has side ε . Clearly for any $l > 0$

$$\int_{\Omega} \text{dist}_{\Omega}(x, \Sigma_{\text{opt}}^l) dx \geq \int_Q \text{dist}_{\Omega}(x, \Sigma_{\text{opt}}^l) dx$$

where Σ_{opt}^l is an arbitrary element of $\text{argmin}_{\mathcal{A}_l(\Omega)} F_{\mathcal{L}^n, id}$.

Fix an arbitrary $l > 0$, if $\Sigma_{\text{opt}} \in \text{argmin}_{\mathcal{A}_l(\Omega)} F_{\mathcal{L}^n, id}$, then $\mathcal{H}^1(\Sigma_{\text{opt}}) = l$. For each such cube Q_{ε} with side ε not intersecting Σ_{opt} the following estimate holds:

$$\int_{Q_{\varepsilon}} \text{dist}_{\Omega}(x, \Sigma_{\text{opt}}) dx \geq \alpha^n (1 - \alpha) \varepsilon \mathcal{L}^n(Q_{\varepsilon})$$

for any $\alpha \in [0, 1]$. Maximizing in α yields

$$\int_{Q_{\varepsilon}} \text{dist}_{\Omega}(x, \Sigma_{\text{opt}}) dx \geq C \varepsilon^{n+1}$$

where $C > 0$ is a constant not depending on ε .

Denote with k the number of cubes with side ε intersecting Σ_{opt} , since clearly Q contains $\mathcal{L}^n(Q) \varepsilon^{-n}$ cubes with side ε , there exists $\mathcal{L}^n(Q) \varepsilon^{-n} - k$ such cubes not intersecting Σ_{opt} , thus

$$\int_{\Omega} \text{dist}_{\Omega}(x, \Sigma_{\text{opt}}) dx \geq C (\mathcal{L}^n(Q) \varepsilon^{-n} - k) \varepsilon^{n+1}.$$

From Lemma 3.3.5 we have

$$l \geq [k / (2^n + 1)] \varepsilon,$$

and it holds

$$\int_{\Omega} \text{dist}_{\Omega}(x, \Sigma_{\text{opt}}) dx \geq c_1 \varepsilon - c_2 \varepsilon^n l - c_3 \varepsilon^{n+1}$$

where c_i , $i = 1, 2, 3$ are constants not dependent on ε and l . Choosing $\varepsilon := kl^{1/(1-n)}$ with $C \in (0, (c_1/c_2)^{1/(n-1)})$ yields

$$\int_{\Omega} \text{dist}_{\Omega}(x, \Sigma_{\text{opt}}) dx \geq cl^{1/(1-n)}$$

for some constant $c > 0$ not dependent on ε and l .

Step 2:

Consider an $(n-1)$ -hyperplane π intersecting Ω by an open set T . Impose an uniform grid parallel to coordinate axis directions on T , with each “cell” ($n-1$ dimension cubes) having edge length ε . Thus the total length of the grid is at most C/ε , with some $C > 0$ not dependent on ε . Let Σ be the union of this grid with line segments perpendicular to π passing through nodes of T and staying in Ω . The total length of all such segments is at most K/ε^{n-1} , thus for small ε it holds $\mathcal{H}^1(\Sigma) \leq K_1/\varepsilon^{n-1}$. In this construction $\text{dist}_{\Omega}(x, \Sigma) \leq K_2\varepsilon$ for any $x \in \Omega$, where $K_2 > 0$ is independent of ε , thus

$$\int_{\Omega} \text{dist}_{\Omega}(x, \Sigma) dx \leq K_2\varepsilon.$$

Finally putting $l := 1/\varepsilon^{n-1}$ concludes the proof. $\square$

Another interesting property of solutions of the average distance problem is that under regularity conditions on the domain's border, the distance between Σ_{opt} and $\partial\Omega$ can be bounded from below, when the length constraint on Σ_{opt} goes to 0.

Proposition 3.3.6. *Given a domain $\Omega \subseteq \mathbb{R}^n$ with $\partial\Omega$ C^2 regular, there exist $l, d_0 > 0$ depending only on Ω and n such that for any $l < l_0$ any element $\Sigma_{\text{opt}}^l \in \text{argmin}_{\mathcal{A}_l(\Omega)} F_{\mathcal{L}^n, id}^l$ satisfies $d_{\mathcal{H}}(\Sigma_{\text{opt}}^l, \partial\Omega) > d_0$.*

Proof. The functionals

$$F_{\mathcal{L}^n, id}^l(\mathcal{X}) := \begin{cases} \int_{\Omega} \text{dist}_{\Omega}(x, \mathcal{X}) dx & \text{if } \mathcal{H}^1(\mathcal{X}) \leq l \\ \infty & \text{otherwise} \end{cases}$$

Γ -converges to

$$F_{\mathcal{L}^n, id}^0(\mathcal{X}) := \begin{cases} \int_{\Omega} \text{dist}_{\Omega}(x, P) dx & \text{if } \mathcal{X} = \{P\} \text{ consists of one point} \\ \infty & \text{otherwise} \end{cases}$$

for $l \downarrow 0$. Indeed for any sequence $\{\Sigma_k\}_{k \in \mathbb{N}} \rightarrow \Sigma$ in the Hausdorff sense, and $l_k := \mathcal{H}^1(\Sigma_k) \downarrow 0$, then Σ consists of one point, and

$$F_{\mathcal{L}^n, id}^0(\Sigma) = \lim_{k \rightarrow \infty} F_{\mathcal{L}^n, id}^{l_k}(\Sigma_k).$$

Suppose the thesis is false, i.e. there exists $\{\Sigma_k\}_{k \in \mathbb{N}}$ with $\mathcal{H}^1(\Sigma_k) \rightarrow 0$ and $d_{\mathcal{H}}(\Sigma_k, \partial\Omega) \rightarrow 0$. Upon subsequence, assume $\{\Sigma_k\}_{k \in \mathbb{N}} \rightarrow \{P\} \in \partial\Omega$, thus P is optimal for the functional

$$F(Q) := \int_{\Omega} |x - Q| dx.$$

Endow Ω with a coordinate system with origin at P , x_n axis directed in such way that all points of Ω have positive x_n coordinate, and $(x_1, \dots, x_{n-1})$ are in the supporting hyperplane of Ω at P . Then for each $i = 1, \dots, n$ it holds

$$\frac{\partial}{\partial x_i} F(0, \dots, 0) = - \int_{\Omega} \frac{x_i}{|x - P|} dx < 0,$$

contradiction. □

3.3.3 Maximal regularity

A rather difficult problem for solutions of the average distance problem is the regularity: indeed very little is known, apart from being countable union of Lipschitz curves (finite union of Lipschitz curves in two dimension case, or when considering the penalized problem). Thus it would be interesting to determine the “maximal regularity”, i.e. the most stringent regularity property satisfied by any solution. Two results impose an upper bound on this: first in [55] it has been proven that for some $C^{1,1}$ regular curve γ there exists a domain Ω_γ such that γ is a minimizer of the average distance problem, with $\mu := \mathcal{L}^2$. This leaves the question if such minimizers must be C^1 regular. In [52] an example of minimizer which is not C^1 regular has been constructed.

1. the “maximal regularity” cannot exceed $C^{1,1}$ (from [55]).

Proposition 3.3.7. *Given a $C^{1,1}$ regular curve $\gamma : [0, l] \rightarrow \mathbb{R}^2$ (endowed with the measure $\mathcal{L}^2$), parameterized w.r.t. arclength, for any $l \in (0, R)$ with $R > 0$ satisfying*

- $|\gamma''(t)| \leq 1/R$ for $\mathcal{L}^1$ -a.e. $t \in [0, l]$.
- $l \leq \pi R$, which implies γ injective,

there exists a domain Ω for which

$$\gamma([0, l]) \in \operatorname{argmin}_{\mathcal{A}_l(\Omega)} F_{\mathcal{L}^2}.$$

Proof. Define $\Sigma_\gamma := \gamma([0, l])$ and

$$\Omega := \{x \in \mathbb{R}^2 : \operatorname{dist}_\Omega(x, \Sigma_\gamma) \leq \lambda\}.$$

The proof relies on two important facts:

$$\mathcal{L}^2(\{x \in \mathbb{R}^2 : 0 < \operatorname{dist}_\Omega(x, \Sigma_\gamma) < s\}) = 2ls + \pi s^2 \quad \forall s$$

and

$$\mathcal{L}^2(\{x \in \mathbb{R}^2 : 0 < \operatorname{dist}_\Omega(x, \Sigma) < s\}) \leq 2ls + \pi s^2 \quad \forall s$$

for any $\Sigma \in \mathcal{A}_l(\Omega)$ (see [55], and [51] for a proof of the last two estimates). Define $D := \text{diam } \Omega$, and for any competitor $\Sigma \in \mathcal{A}_l(\Omega)$ it holds

$$\begin{aligned} \int_{\Omega} \text{dist}_{\Omega}(x, \Sigma) &= \int_0^D \mathcal{L}^2(\{x \in \mathbb{R}^2 : \text{dist}_{\Omega}(x, \Sigma) > s\}) ds \\ &= D\mathcal{L}^1(\Omega) - \int_0^D \mathcal{L}^2(\{x \in \mathbb{R}^2 : 0 < \text{dist}_{\Omega}(x, \Sigma_{\gamma}) < s\}) ds \\ &\geq D\mathcal{L}^1(\Omega) - \int_0^D \min\{\mathcal{L}^1(\Omega), 2ls + \pi s^2\} ds \end{aligned}$$

and equality holds if $\Sigma = \Sigma_{\gamma}$, i.e. Σ_{γ} is a minimizer. $\square$

2. The “maximal regularity” is weaker than C^1 (from [52]).

The construction starts from a discrete configuration.

1. Basic configuration: in $\mathbb{R}^2$, define parameters $m_1 = m_3 := 0.38$, $m_2 := 0.24$, $\lambda := 0.36$ (a posteriori we can replace this 0.36 with any value in $(0.24, 0.38)$, to guarantee that the two “heavier” masses still attract the minimizer, while the “lighter” mass will not generate another branch), points $x_1 := (-1, 0)$, $x_2 := (0, 1)$, $x_3 := (1, 0)$, and the probability measure

$$\bar{\mu} := \sum_{i=1}^3 m_i \delta_{x_i}.$$

The first step proves that the minimizer of

$$E_{\mu}(\cdot) := \int_{\mathbb{R}^2} \text{dist}(x, \cdot) d\bar{\mu}(x) + \lambda \mathcal{H}^1(\cdot)$$

is the set

$$\bar{\Sigma}_{\text{opt}} := \{t \in [0, 1] : (1-t)x_1 + tv_2\} \cup \{t \in [0, 1] : (1-t)x_3 + tv_2\},$$

where $v_2 := \left(0, \frac{1}{2\sqrt{2}}\right)$.

2. Counterexample: let η be a mollifier, i.e. smooth, radially symmetric, positive on $B(0, 1)$, null outside, $\eta(0, 0) = 1$ and $\int_{\mathbb{R}^2} \eta dx = 1$. For $\delta > 0$ define $\eta_{\delta}(x) := \delta^{-2} \eta(x/\delta)$, $\rho_{i,\delta}(\cdot) := m_i \eta_{\delta}(\cdot - x_i)$, and the measure $\mu_{\delta} := (\rho_{1,\delta} + \rho_{2,\delta} + \rho_{3,\delta}) \cdot \mathcal{L}^2$.

A background measure $\tilde{\mu} := \eta_{3/2} \cdot \mathcal{L}^2$ is required, and consider the smooth measure

$$\mu_{q,\delta} := (1-q)\mu_{\delta} + q\tilde{\mu}.$$

The following result holds:

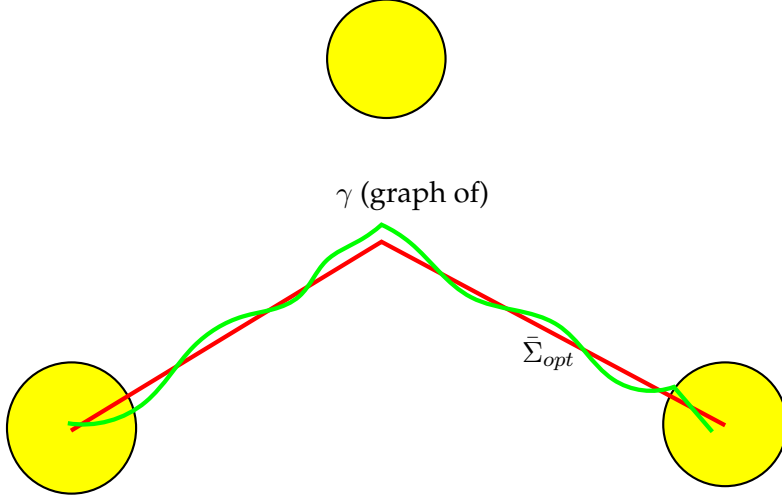


Figure 3.3.1: The colored balls denote the area on which μ_δ is supported. The red set $\bar{\Sigma}_{opt}$ is the minimizer when the considered measure is $\bar{\mu}$, while the green set is a minimizer when the considered measure is $\mu_{q,\delta}$, for q, δ small.

Theorem 3.3.8. *There exist $q, \delta > 0$ for which one of the minimizers of*

$$E(\cdot) := \int_{\mathbb{R}^2} d_{\mathcal{H}}(x, \cdot) d\mu_{q,\delta} + 0.36\mathcal{H}^1(\cdot)$$

is a simple curve. Denoting with $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ a constant speed parameterization, $\gamma' : [0, 1] \rightarrow \mathbb{R}^2$ is BV, and γ'' is a measure with an atom of size at least 1 at some point $s \in (0, 1)$.

The value 0.36 as constant multiplying $\mathcal{H}^1(\cdot)$ can be replaced by any value in $(1/3, m_1 \wedge m_3)$, as this guarantees that the minimizer is a simple curve. For the proof we refer to [52].

Notice that these two results can be considered in view of the following, proven in [50] for the two dimensional case (as it involves using Proposition 3.2.9, not yet proven for higher dimensional domains), stating that given a minimizer Σ , $C^{1,1}$ holds near triple points, while corners may arise only near atoms, i.e.:

- if $x \in \Sigma$ is a triple point, then using Menger n -Beinsatz there exist (locally) three curves $\gamma_1, \gamma_2, \gamma_3$ intersecting only in x , which are $C^{1,1}$ regular,
- if $x \in \Sigma$ is such that the mass projecting on it is zero, then it cannot be a corner.

In ([38]) it has been proven a stronger regularity result:

Proposition 3.3.9. *Given a domain $\Omega \subseteq \mathbb{R}^N$ with $N \geq 2$, a probability measure $\mu \ll \mathcal{L}^N$, a parameter $\lambda > 0$, any solution Σ_{opt} of the penalized problem*

$$\min_{\mathcal{A}(\Omega)} F_\mu(\cdot) + \lambda\mathcal{H}^1(\cdot)$$

is a finite union of curves $\{\gamma_k\}_{k=1}^j$ (without loss of generality assume γ_k parameterized by arclength), with $j = j(\mu, \lambda, \Omega)$, such that for any k the BV norm $\|\gamma'_k\|_{BV} \leq C = C(\mu, \lambda, \Omega)$.

For the proof we refer to Theorem 6.4.1, in Chapter 6.

3.4 Higher dimension case

In the previous Section we have discussed geometric properties of solutions of the average distance functional, with some results proven only in the two dimension case. In particular, the proof of the absence of loops, absence of crosses and Ahlfors regularity relied on construction specific to two dimension case.

In this Section our goal is to generalize those results to higher dimension domains, imposing more stringent conditions if necessary. Moreover, we will prove that (similarly to the two dimension case) solutions of the maximal distance problem do not contain loops, and satisfy Ahlfors regularity.

3.4.1 Average distance functional solutions

The absence of loops can be generalized to higher dimension cases with minimal modifications. The idea of the proof is the same as in two dimension case, but estimates differ. The main results were developed in [44].

The next five results are from [44].

Lemma 3.4.1. *Given $\rho > 0$, define the set*

$$K_\rho := \bigcup_{k=1}^n \{te_k : t \in [-\rho, \rho]\},$$

where $\{e_k\}_{k=1}^n$ is a standard orthonormal base of $\mathbb{R}^n$. Then it hold:

- K_ρ is connected and contains $0 \in \mathbb{R}^n$,
- $\mathcal{H}^1(K_\rho) = 2n\rho$,
- given $y \in \mathbb{R}^n$ with $|y| \geq n^{1/2}\rho$ then it holds

$$d_{\mathcal{H}}(\{y\}, K_\rho) \leq |y| - \rho/(2n^{1/2}).$$

Proof. Claims “ K_ρ is connected and contains $0 \in \mathbb{R}^n$ ” and “ $\mathcal{H}^1(K_\rho) = 2n\rho$ ” are very easy to check.

- To prove: given $y \in \mathbb{R}^n$ with $|y| \geq n^{1/2}\rho$ then it holds

$$d_{\mathcal{H}}(\{y\}, K_\rho) \leq |y| - \rho/(2n^{1/2}).$$

Upon scaling in ρ , assume $\rho = 1$, $y = (y_1, \dots, y_n)$ with all coordinates nonnegative and $y_1 = \max_i y_i$. Then

$$\begin{aligned} d_{\mathcal{H}}(\{y\}, K_{\rho}) &\leq ((y_1 - 1)^2 + y_2^2 + \dots + y_n^2)^{1/2} \\ &= (|y|^2 + 1 - 2y_1)^{1/2} \leq (|y|^2 + 1 - 2|y|/n^{1/2})^{1/2} \end{aligned}$$

Using Lemma 3.4.2 concludes the proof. $\square$

Lemma 3.4.2. *Given $\alpha, \beta > 0$ with $\alpha^2 \leq 4\beta$, and $x \geq 2\beta/\alpha$, then it holds*

$$(x^2 - \alpha x + \beta)^{1/2} \leq x - \alpha/4.$$

Proof. It suffices to notice that under such hypothesis, $f(x) := x - (x^2 - \alpha x + \beta)^{1/2}$ is non decreasing, and it holds (by direct computation) $f(2\beta/\alpha) \geq 2\beta/\alpha - (2\beta/\alpha - \alpha/4) = \alpha/4$. $\square$

Lemma 3.4.3. *Let $\rho > 0$, $\beta \in [0, 1]$, $r \geq 9n\rho$, and $a, b \in [0, \rho]$ be given. Let $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ and consider $R := [a, b] \times \overline{B}((0, 0), \beta\rho)$. Then there exists a set $X = X^+ \cup X^-$ such that*

1. $(-a, 0) \in X^-, (b, 0) \in X^+$,
2. $X^{\pm}$ are compact and connected,
3. if $y \in \mathbb{R}^n$ is such that $|y| \geq 2r$, then it holds

$$d_{\mathcal{H}}(\{y\}, X) \leq d_{\mathcal{H}}(\{y\}, R) - \beta\rho/2 - 3\rho^2/(2r),$$

$$4. \mathcal{H}^1(X^{\pm}) \leq 8n^{3/2}(\beta + \rho/r)\rho,$$

$$5. X \subseteq B((0, 0), r/n^{1/2}).$$

Proof. Statements 1 and 2 are clearly true, while 4 and 5 are straightforward to check, being 4 a direct consequence of Lemma 3.4.1, and 5 a consequence of the estimate

$$\rho + \lambda < \frac{r}{9n} + 8\sqrt{n}\rho \leq \frac{r}{9n} + \frac{8n}{9\sqrt{n}} \leq \frac{r}{\sqrt{n}}.$$

It remains to prove

- if $y \in \mathbb{R}^n$ is such that $|y| \geq 2r$, then it holds

$$d_{\mathcal{H}}(\{y\}, X) \leq d_{\mathcal{H}}(\{y\}, R) - \beta\rho/2 - 3\rho^2/(2r).$$

Let $y := (y_1, y') \in \mathbb{R} \times \mathbb{R}^{n-1}$ satisfying $|y| > 2r$. There are two cases to consider.

CASE 1. Assume $y_1 \in [0, \rho]$ (the case $y_1 \in [-\rho, 0]$ is symmetric). Then it holds

$$\begin{aligned} \text{dist}(y, K_\lambda(b, 0)) &\leq (|y_1 - b|^2 + \text{dist}^2(y', K_\lambda))^{1/2} \\ &\leq (\rho^2 + \text{dist}^2(y', K_\lambda))^{1/2}. \end{aligned}$$

Since $|y| \geq 2r \geq 18n\rho$, while $|y_1| \leq \rho$, it follows $|y'| \geq 17n\rho \geq \sqrt{n}\lambda$, and using Lemma 3.4.1 it holds

$$\begin{aligned} \text{dist}(y, K_\lambda(b, 0)) &\leq (\rho^2 + (|y'| - \frac{\lambda}{2\sqrt{n}})^2)^{1/2} \\ &= (\rho^2 + (|y'| - 2\beta\rho - 2\rho^2/r)^2)^{1/2}. \end{aligned}$$

Let $t := |y'| - \beta\rho - \rho^2/r$ so that

$$\begin{aligned} \text{dist}(y, K_\lambda(b, 0)) &\leq (\rho^2 + (t - \beta\rho - \rho^2/r)^2)^{1/2} \\ &= (t - 2(\beta\rho + \rho^2/r)t + \rho^2 + (\beta\rho + \rho^2/r)^2)^{1/2}. \end{aligned}$$

Applying Lemma 3.4.2 gives

$$\text{dist}(y, K_\lambda(b, 0)) \leq t - \frac{\beta\rho + \rho^2/r}{2} = |y'| - \frac{3}{2}(\beta\rho + \rho^2/r)$$

whenever

$$t \geq \frac{\rho^2 + (\beta\rho + \rho^2/r)^2}{\beta\rho + \rho^2/r}.$$

This is true, since $|y| \geq 2r$ it follows (by direct computation)

$$t \geq 6n\rho + r \geq 2\rho + r,$$

and

$$\frac{\rho^2 + (\beta\rho + \rho^2/r)^2}{\beta\rho + \rho^2/r} \leq 2\rho + r.$$

Since $|y'| = \text{dist}(y, R) + \beta\rho$, then we obtain

$$\text{dist}(y, K_\lambda(0, b)) \leq \text{dist}(y, R) - \frac{1}{2}\beta\rho - \frac{3}{2}\rho^2/r.$$

CASE 2. Consider the case $y_1 \geq \rho$ ($y_1 \leq -\rho$ is symmetric). Since $|y| \geq \rho$ we have

$$\begin{aligned} |y - (b, 0)| &\geq |y| - \rho \geq r - \rho \geq 8n\rho \\ &\geq 4n(\beta\rho + \rho^2/r) = \sqrt{n}\lambda, \end{aligned}$$

so using Lemma 3.4.1 gives

$$\text{dist}(y, K_\lambda(b, 0)) \leq \text{dist}(y, R) - \frac{1}{2}\beta\rho - \frac{3}{2}\rho^2/r,$$

concluding the proof. $\square$

Lemma 3.4.4. *Given $\Sigma \in \mathcal{A}(\Omega)$ containing a loop E , then $\mathcal{H}^1$ -a.e. $x \in E$ is a noncut point (i.e. $E \setminus \{x\}$ is connected).*

Proof. If x is not a noncut point, then there exists $L_x \ni x$ with $L_x \cap E = \{x\}$, $L_x \neq \{x\}$ and $\mathcal{H}^1(L_x) > 0$. Moreover $L_x \cap L_y = \emptyset$ whenever $x \neq y$. Using $\mathcal{H}^1(\Sigma) < \infty$, it follows that for at most countably many x such L_x can exist. $\square$

Lemma 3.4.5. *Let $\Omega \subseteq \mathbb{R}^n$ be the domain, μ a given measure, $\Sigma \in \mathcal{A}(\Omega)$ containing a loop $E \subseteq \Sigma$. Then given $\beta \in (0, 1]$, for $\mathcal{H}^1$ -almost any point $x \in E$, for any $r > 0$ there exists $\rho \in (0, r)$ and $\Sigma' \in \mathcal{A}$ such that:*

- $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \rho/2 + (16n^{3/2} + 2)\beta\rho$,
- $\Sigma \setminus \Sigma' \subseteq B(x, \rho)$, $\Sigma' \setminus \Sigma \subseteq B(x, 32n\rho)$,
- $\text{dist}_\Omega(y, \Sigma') \leq \text{dist}_\Omega(y, \Sigma)$ for any $y \notin B(x, 64n^{3/2}\rho)$,
- $\text{dist}_\Omega(y, \Sigma') \leq \text{dist}_\Omega(y, \Sigma) + \rho$ for any $y \in B(x, 64n^{3/2}\rho)$.

Proof. Let $\gamma : [0, 1] \rightarrow \Sigma$ be a Lipschitz parameterization of E , and $\bar{x} := \gamma(\bar{t})$ with $\bar{t} \in (0, t)$ and γ differentiable in $\bar{t}$, $\bar{x}$ a noncut point (i.e. $\Sigma \setminus \{x\}$ is connected), and $\lim_{\rho \rightarrow 0} \beta_\Sigma(\bar{x}, \rho) = 0$, where $\beta_\Sigma(\bar{x}, \rho) := \inf_\Pi \beta_{\Sigma, \Pi}(\bar{x}, \rho)$, with the infimum taken among straight lines Π passing through $\bar{x}$, and $\beta_{\Sigma, \Pi}(\bar{x}, \rho) := \sup_{y \in \Sigma \cap B(\bar{x}, \rho)} \text{dist}(y, \Pi)/\rho$. Existence of such β_Σ and $\beta_{\Sigma, \Pi}$ has been proven in [40].

In view of Lemma 3.4.4, it follows that $\mathcal{H}^1$ -a.e. $x \in E$ has such property. Endow the configuration with an orthogonal coordinate system (on $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$) with $\bar{x} = (0, 0)$ and $\gamma'(\bar{t}) = (|\gamma'(\bar{t})|, 0)$.

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, choose $\rho_0 \in (0, r)$ such that

$$\beta(\bar{x}, \rho) \leq \beta \quad \forall \rho \leq \rho_0.$$

Denote with D_k a neighborhood of x in Σ such that $\text{diam } D_k \leq \rho_0$, and let $\rho > 0$ be such that $\overline{B(\bar{x}, \rho)}$ is the smallest ball containing D_k . Hence $D_k \subseteq \overline{B(\bar{x}, \rho)} \cap \Sigma \subseteq [-\rho, \rho] \times \overline{B(0, \beta\rho)}$.

Denote with a, b the smallest number such that $D_k \subseteq [-a, b] \times \overline{B(0, \beta\rho)}$ and choose $x_1 \in \Sigma \cap \overline{B((-a, 0), \beta\rho)}$, $x_2 \in \Sigma \cap \overline{B((b, 0), \beta\rho)}$; choose $r' := 32n^{3/2}\rho$ and in view of Lemma 3.4.3 there exist such $X = X^+ \cup X^-$ (given by Lemma 3.4.3). Let S^- be the segment connecting x_1 to $(-a, 0)$, and S^+ the segment connecting x_2 to $(b, 0)$ respectively. Define

$$\Sigma' := \Sigma \setminus D_k \cup X \cup S^- \cup S^+.$$

By construction it follows Σ' connected, $\Sigma \setminus \Sigma' \subseteq D_k \subseteq \overline{B(\bar{x}, \rho)}$. On the other hand $\Sigma \setminus \Sigma' \subseteq \overline{B(\bar{x}, 32n\rho)}$, by Lemma 3.4.3, while $S^\pm \subseteq \overline{B(\bar{x}, (1 + \beta)\rho)}$.

Observing that $\mathcal{H}^1(S^\pm) \leq \beta\rho$, $\mathcal{H}^1(D_k) \geq \rho$, it follows

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \rho/2 + C_2\beta\rho$$

where $C_2 := 16n^{3/2} + 2$.

Finally the statements on $\text{dist}_\Omega(y, \Sigma')$ follow from Lemma 3.4.3 and $\Sigma \setminus \Sigma' \subseteq \overline{B(\bar{x}, \rho)}$ respectively. $\square$

Another result estimating the “gain” for the average distance functional is required. While Lemma 3.2.7 is valid for the higher dimension cases, a sharper estimate holds:

Lemma 3.4.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, $l > 0$ a given value, μ a given measure, Borel sets $H, K \subseteq \Omega$ such that $\mu(K) > 0$ and*

$$r := \inf\{\text{dist}_\Omega(x, H) : x \in K\} > 0.$$

Then for any compact set $\Sigma \subseteq H$ with $\mathcal{H}^1(\Sigma) \leq l$ there exists for any ε sufficiently small a set $\Sigma' \supseteq \Sigma$ such that

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) + 2n\varepsilon, \quad F_{\mu, A}(\Sigma') \leq F_{\mu, A}(\Sigma) - \frac{\lambda(r)\mu(K)}{32nl}\varepsilon^2,$$

where $\lambda(\cdot)$ is the constant for which

$$|A(x) - A(y)| \geq \lambda(c)|x - y|$$

for any x, y with $|x - y| \in [c, \text{diam } \Omega]$.

For the proof we refer to [44].

These preliminary results are sufficient to prove the absence of loops:

Theorem 3.4.7. *Given a domain $\Omega \in \mathbb{R}^n$ with $n \geq 3$, a measure $\mu \in L^1(\Omega)$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, any solution Σ_{opt} of the average distance problem does not contain loops.*

Proof. Choose an arbitrary Σ_{opt} solution of the average distance problem, and define $l := \mathcal{H}^1(\Sigma_{\text{opt}})$. Suppose $l > 0$, otherwise Σ_{opt} would consist of only one point. Since $\mu(\Sigma_{\text{opt}}) = 0$, there exists a compact set K not intersecting Σ_{opt} such that $\mu(K) > 0$. Define

$$R := \frac{1}{2} \min\{\text{dist}_\Omega(y, \Sigma_{\text{opt}}) : y \in K\} > 0$$

and

$$H := \{z \in \Omega : \text{dist}_\Omega(z, \Sigma_{\text{opt}}) < R\}.$$

Suppose there exists a subset $E \subseteq \Sigma_{\text{opt}}$ homeomorphic to $S^1 \subseteq \mathbb{R}^2$. Put $\beta := 4(16n^{3/2} + 2)^{-1}$; it holds (see [2] for further details)

$$\lim_{\rho \rightarrow 0^+} \frac{\mu(B(x, \rho))}{\rho} = 0 \quad \mathcal{H}^1 - a.e. \ x \in E.$$

Choose $r > 0$ (the exact value will be determined later): from Lemma 3.4.5 there exists $x' \in E$ with $\frac{\mu(B(x', t))}{t} \rightarrow 0$ as $t \rightarrow 0^+$, and $\rho \in (0, r)$ such that there exists $\Sigma' \in \mathcal{A}(\Omega)$ satisfying

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_{\text{opt}}) - \rho/2 + 4(16n^{3/2} + 2)\beta\rho = \mathcal{H}^1(\Sigma_{\text{opt}}) - \rho/4 \quad (3.4.1)$$

and

$$\begin{aligned} F_{\mu,A}(\Sigma') &\leq F_{\mu,A}(\Sigma_{\text{opt}}) + \int_{B(x', 64n^{3/2}\rho)} A(\text{dist}_{\Omega}(y, \Sigma_{\text{opt}}) + \rho) - A(\text{dist}_{\Omega}(y, \Sigma_{\text{opt}})) d\mu(y) \\ &\leq F_{\mu,A}(\Sigma_{\text{opt}}) + 64n^{3/2}\Lambda\rho^2 \frac{\mu(B(x', 64n^{3/2}\rho))}{64n^{3/2}\rho}, \end{aligned}$$

where Λ denotes the Lipschitz constant of A .

Applying Lemma 3.4.6 yields the existence of $\Sigma'' \in \mathcal{A}(\Omega)$ satisfying

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + 2n\varepsilon \leq \mathcal{H}^1(\Sigma_{\text{opt}})$$

and

$$\begin{aligned} F_{\mu,A}(\Sigma'') &\leq F_{\mu,A}(\Sigma') - C_1\varepsilon^2 \\ &\leq F_{\mu,A}(\Sigma_{\text{opt}}) + 64n^{3/2}\Lambda\rho^2 \frac{\mu(B(x', 64n^{3/2}\rho))}{64n^{3/2}\rho} - \frac{C_1}{16n^2}\rho^2 \end{aligned}$$

where $C_1 := \frac{\lambda(r)\mu(K)}{32nl}$, and $\lambda(\cdot)$ denoted the function for which

$$|A(x) - A(y)| \geq \lambda(c)|x - y| \quad \forall c > 0, \forall x, y \in \Omega : |x - y| \in [c, \text{diam } \Omega].$$

Then choosing $r > 0$ satisfying

$$64n^{3/2}\Lambda \frac{\mu(B(x', 64n^{3/2}\rho))}{64n^{3/2}\rho} < \frac{C_1}{16n^2} \quad \forall \rho \in (0, r)$$

and passing to the limit $\rho \rightarrow 0$ yields

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma_{\text{opt}}), \quad F_{\mu,A}(\Sigma'') < F_{\mu,A}(\Sigma_{\text{opt}}),$$

which is a contradiction. □

Theorem 3.3.3 proves that in the two dimension case, under summability conditions on the measure, solutions of the average distance problem are Ahlfors regular. This can be generalized to higher dimensions, under slightly different conditions on the measure. Some preliminary results are required. All the proofs can be found in [44].

Lemma 3.4.8. *Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 3$) be a given domain, $\Sigma \in \mathcal{A}(\Omega)$, then for any $x \in \Sigma$ there exists $\Sigma' \in \mathcal{A}(\Omega)$ such that for any $\rho > 0$*

- $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x, \rho)) + C\left(\left(\frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 1\right)\rho,$
- $\Sigma \setminus \Sigma' \subseteq B(x, 2\rho), \Sigma' \setminus \Sigma \subseteq B(x, 8\sqrt{n}\rho),$
- $\text{dist}_{\Omega}(z, \Sigma') < \text{dist}_{\Omega}(z, \Sigma)$ for any $z \notin B(x, 4n\rho),$

- $\text{dist}_\Omega(z, \Sigma') \leq \text{dist}_\Omega(z, \Sigma) + \rho$ for any $z \in B(x, 4n\rho)$.

where C is a positive constant depending only on n .

Lemma 3.4.9. *Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 3$) be a given domain, $\Sigma \in \mathcal{A}(\Omega)$ and suppose there exists $r > 0$ such that for any $x \in \Sigma$, $0 < \rho < r$ the inequality*

$$\frac{\mathcal{H}^1(\Sigma \cap B(x, \rho))}{\rho} \leq a \frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))^\alpha}{2\rho} + b$$

holds for some fixed $a > 0, b \geq 0, \alpha \in (0, 1)$. Then there exists a constant $K = K(a, b, \alpha, r, \mathcal{H}^1(\Sigma))$ such that

$$\frac{\mathcal{H}^1(\Sigma \cap B(x, \rho))}{\rho} \leq K.$$

Now it is possible to prove that solutions of the average distance problem are Ahlfors regular, under suitable conditions on the measure.

Theorem 3.4.10. *Let be $\Omega \subseteq \mathbb{R}^n$ ($n \geq 3$) a given domain, $\mu \in L^p$, $p \geq \frac{n}{n-1}$ a given measure, $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$ a given function, and $\Sigma_{\text{opt}} \in \text{argmin}_{\mathcal{A}_L(\Omega)} F_{\mu, A}$ for some $L \geq 0$. Then Σ_{opt} is Ahlfors regular.*

Proof. First suppose $L > 0$, otherwise Σ_{opt} is a single point.

1. $\mu(\Sigma_{\text{opt}}) = 0$, thus there exists a compact set K with $\mu(K) > 0$ and $K \cap \Sigma_{\text{opt}} = \emptyset$. This can be chosen as $K := \Omega \setminus (\Sigma_{\text{opt}})_{2c}$, with $c \in (0, \text{diam } \Sigma_{\text{opt}})$ and $(\Sigma_{\text{opt}})_{2c} := \{y \in \Omega : \text{dist}_\Omega(y, \Sigma_{\text{opt}}) < 2c\}$; choose a small $\rho > 0$;
2. let be Σ' the competitor given in Lemma 3.4.8, and using Hölder inequality yields

$$F_{\mu, A}(\Sigma') \leq F_{\mu, A}(\Sigma_{\text{opt}}) + 2\Lambda\rho\mu(B(x, 4n\rho)) \leq F_{\mu, A}(\Sigma_{\text{opt}}) + 2\Lambda\rho\|\mu\|_{L^p}^{1/p} \mathcal{L}^n(B(x, 4n\rho))^{1/q}$$

where $\mathcal{L}^n(B(x, 4n\rho))$ clearly has order $O(\rho^n)$,

3. inequality

$$\mathcal{H}^1(\Sigma') \geq \mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho)) - \rho H\left(\frac{\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, 2\rho))}{2\rho} + 1\right) \quad (3.4.2)$$

holds, and two cases arise:

- (a) if

$$\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho)) - \rho H\left(\frac{\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, 2\rho))}{2\rho} + 1\right) \leq 0$$

Lemma 3.4.9 concludes $\frac{\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho))}{\rho} \leq K'$ for some $K' > 0$.

- (b) if $\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho)) - \rho H\left(\frac{\Sigma_{\text{opt}} \cap B(x, 2\rho)}{2\rho} + 1\right) > 0$ then for ρ sufficiently small inclusion $\Sigma' \subseteq \{z \in \Omega : \text{dist}_\Omega(x, \Sigma_{\text{opt}}) < c\}$ holds. Applying Lemma 3.4.6 yields to the existence of a set $\Sigma'' \in \mathcal{A}(\Omega)$ such that

$$F_{\mu,A}(\Sigma'') \leq F_{\mu,A}(\Sigma') - H'(\mathcal{H}^1(\Sigma' \cap B(x, \rho)) - \rho H''\left(\left(\frac{\mathcal{H}^1(\Sigma' \cap B(x, 2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 1\right))^2 \quad (3.4.3)$$

where H', H'' are positive constants not dependent on ρ and x . Combining

$$F_{\mu,A}(\Sigma') \leq F_{\mu,A}(\Sigma_{\text{opt}}) + O(\rho^{\frac{n}{q}+1}) \quad (3.4.4)$$

with (3.4.3) and the optimality of Σ_{opt} (i.e. $F_{\mu,A}(\Sigma_{\text{opt}}) \leq F_{\mu,A}(\Sigma'')$) yields

$$\mathcal{H}^1(\Sigma' \cap B(x, \rho)) - \rho H''\left(\left(\frac{\mathcal{H}^1(\Sigma' \cap B(x, 2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 1\right) \leq H^* \rho^{\frac{n}{2q}-\frac{1}{2}} \leq H^*(\text{diam } \Sigma_{\text{opt}})^{\frac{n}{2q}-\frac{1}{2}}$$

with H^* independent of x and ρ , and applying Lemma 3.4.9 yields the thesis.

Thus the proof is complete. $\square$

3.4.2 Maximal distance functional solutions

As presented at the beginning of this Chapter, a problem related to the average distance problem is the “maximal distance problem”, in which the maximum displacement from a set with prescribed maximum length is to be minimized. Solutions of this problem exhibit some similar properties. The maximal distance problem will be discussed only marginally. The absence of loops, under suitable hypothesis, holds too, and has simpler proof:

Theorem 3.4.11. *Given a domain $\Omega \in \mathbb{R}^n$ with $n \geq 3$, any solution Σ_{opt}^* of the maximal distance problem does not contain loops.*

Proof. The proof is done by contradiction: suppose there exists a solution Σ_{opt}^* containing a loop E ; define $r := F^*(\Sigma_{\text{opt}}^*)/(65n^{3/2})$, $\beta := 1/(4(16n^{3/2} + 2))$, and consider a point $x^* \in \Sigma_{\text{opt}}^*$ and $\rho < r$ for which Lemma 3.4.5 yields the existence of Σ' satisfying:

- $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_{\text{opt}}^*) - \rho/2 + (16n^{3/2} + 2)\beta\rho$,
- $\text{dist}_\Omega(y, \Sigma') \leq \text{dist}_\Omega(y, \Sigma_{\text{opt}}^*)$ for any $y \notin B(x^*, 64n^{3/2}\rho)$,
- $\text{dist}_\Omega(y, \Sigma') \leq \text{dist}_\Omega(y, \Sigma_{\text{opt}}^*) + \rho$ for any $y \in B(x^*, 64n^{3/2}\rho)$.

Then from choices for r, β , $\mathcal{H}^1(\Sigma') < \mathcal{H}^1(\Sigma_{\text{opt}}^*)$ and

- if $\text{dist}_\Omega(y, x^*) \geq 64n^{3/2}\rho$ then $\text{dist}_\Omega(y, \Sigma') \leq \text{dist}_\Omega(y, \Sigma_{\text{opt}}^*)$,

- if $\text{dist}_\Omega(y, x^*) < 64n^{3/2}\rho$ then

$$\text{dist}_\Omega(y, \Sigma') \leq \text{dist}_\Omega(y, \Sigma_{\text{opt}}^*) + \rho \leq 64n^{3/2}\rho + \rho \leq 65n^{3/2}r \leq F^*(\Sigma_{\text{opt}}^*),$$

and the proof is complete. $\square$

Solutions of the maximal distance problem exhibit Ahlfors regularity too:

Theorem 3.4.12. *Let be $\Omega \subseteq \mathbb{R}^n$ ($n \geq 3$) a given domain, and $\Sigma_{\text{opt}}^* \in \text{argmin}_{\mathcal{A}_L(\Omega)} F^*$ for some $L \geq 0$. Then Σ_{opt}^* is Ahlfors regular.*

Proof. Define $r := F^*(\Sigma_{\text{opt}}^*)/6n$, and consider a point $x^* \in \Sigma_{\text{opt}}^*$ and $\rho < r$ for which Lemma 3.4.8 gives the existence of Σ' satisfying those conditions:

- if $y \notin B(x^*, 4n\rho)$ then it holds

$$\text{dist}_\Omega(y, \Sigma') \leq \text{dist}_\Omega(y, \Sigma_{\text{opt}}^*) \leq F^*(\Sigma_{\text{opt}}^*),$$

- if $y \in B(x^*, 4n\rho)$ then it holds

$$\text{dist}_\Omega(y, \Sigma') \leq \text{dist}_\Omega(y, \Sigma_{\text{opt}}^*) + 2\rho \leq 6n\rho \leq 6nr = F^*(\Sigma_{\text{opt}}^*),$$

thus

$$F^*(\Sigma') \leq F^*(\Sigma_{\text{opt}}^*)$$

and the optimality of Σ_{opt}^* yields

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_{\text{opt}}^*).$$

Using Lemma 3.4.8 gives

$$\frac{\mathcal{H}^1(\Sigma_{\text{opt}}^* \cap B(x^*, \rho))}{\rho} - C\left(\frac{\mathcal{H}^1(\Sigma_{\text{opt}}^* \cap B(x^*, 2\rho))}{2\rho}\right)^\alpha + 1 \leq 0$$

for some constant $C > 0$ not depending on x^*, ρ , and Lemma 3.4.9 gives the existence of K such that

$$\frac{\mathcal{H}^1(\Sigma_{\text{opt}}^* \cap B(x^*, \rho))}{\rho} \leq K,$$

concluding the proof. $\square$

3.5 Counterexamples

In this Section we will present counterexamples to results about geometric properties of solutions of the average distance problem, when key summability properties on the measure are not assumed. In particular, Theorem 3.2.8 and Lemma 3.2.16 will be shown to be false if more general measures are considered.

In Theorem 3.4.7 we have proven that given a domain Ω and a measure $\mu \in L^1(\Omega)$, no optimal set can contain a loop. In this subsection we will prove that without such assumption on the measure, this result does not hold. A counterexample will be constructed by exploiting this lack of summability.

Let $\Omega := \bar{B}((0, 0), 2) \subseteq \mathbb{R}^2$ be the domain, $\mu := f \cdot \mathcal{L}^2$ the measure, where (in polar coordinates)

$$f(r, \theta) := \frac{1}{|r - 1|},$$

and $A := id$ the identity function.

First, clearly $f \notin L^1(\bar{B}((0, 0), 2))$:

$$\int_{\bar{B}((0, 0), 2)} f(x) dx \geq 2\pi \int_{1/2}^1 \frac{r}{|r - 1|} dr \geq \pi \int_{1/2}^1 \frac{1}{|r - 1|} dr = \infty.$$

Then consider $\Sigma_{\text{opt}} := \{(r, \theta) \in \bar{B}((0, 0), 2) : r = 1\}$ the unit circle, and it holds

$$\int_{\bar{B}((0, 0), 2)} \text{dist}_{\Omega}(x, \Sigma_{\text{opt}}) f(x) dx = 2\pi \int_0^2 |r - 1| \frac{r}{|r - 1|} dr < \infty$$

Consider the family

$$G_{\delta, \eta}(\beta) := \{(r, \theta) \in \bar{B}((0, 0), 2) : r \in [1 - \delta, 1 + \delta], \theta \in [\beta - \eta, \beta + \eta]\},$$

where $\delta \in [0, 1]$, $\eta \in [0, \pi]$, $\beta \in [0, 2\pi]$.

Clearly for any δ, η, β ,

$$\int_{G_{\delta, \eta}(\beta)} f(x) dx \geq \eta \int_{1-\delta}^{1+\delta} \frac{\rho}{|\rho - 1|} d\rho = \infty$$

Thus given any $\mathcal{X} \in \mathcal{A}(\Omega)$, if there exists $\varepsilon > 0$, δ, η, β such that $\text{dist}_{\Omega}(G_{\delta, \eta}(\beta), \mathcal{X}) \geq \varepsilon$, then inevitably $F_{f, \mathcal{L}^2, id}(\mathcal{X}) = \infty$.

Thus in order to achieve $F_{f, \mathcal{L}^2, id}(\mathcal{X}) < \infty$, for any $\varepsilon > 0$, δ, η, β , $\mathcal{X}$ should have distance from $G_{\delta, \eta}(\beta)$ not more than ε . As $\mathcal{X}$ is connected and compact, it should intersect $G_{\delta, \eta}(\beta)$ for any δ, η, β , thus it must intersect

$$\bigcup_{\beta \in [0, 2\pi]} \bigcap_{\delta \in [0, 1]} \bigcap_{\eta \in [0, \pi]} G_{\delta, \eta}(\beta),$$

which ultimately leads to $\Sigma_{\text{opt}} \subseteq \mathcal{X}$.

From this we can see that $\mathcal{A}_l(\Omega)$ does not contain any set S with finite energy for any $l < 2\pi$, while for $l \geq 2\pi$ any set with finite energy should contain Σ_{opt} .

It has been proven that in the two dimension case a cross cannot be present in optimal sets if the measure $\mu \in L^p(\Omega)$, $p > 4/3$. In this subsection we will construct a counterexample showing that the same result does not hold when $\mu \notin L^1(\Omega)$.

The case $f \in L^p$ with $p \in [1, 4/3]$ remains not clear.

Let $\Omega := [-1, 1] \times [-1, 1]$ be the domain, and define $X := (\{0\} \times [-1/2, 1/2]) \cup ([-1/2, 1/2] \times \{0\})$. Then consider the measure $\mu := f \cdot \mathcal{L}^2$, where

$$f : [-1, 1] \times [-1, 1] \longrightarrow [0, \infty), \quad f(x, y) := \frac{1}{\text{dist}((x, y), X)},$$

and $A := id$ the identity function.

Again, it holds

$$\int_{-1}^1 \int_{-1}^1 \frac{1}{\text{dist}_\Omega((x, y), X)} dx dy = \infty,$$

while

$$\int_{-1}^1 \int_{-1}^1 \text{dist}_\Omega((x, y), X) \frac{1}{\text{dist}_\Omega((x, y), X)} dx dy < \infty.$$

Given $a \in [-1/2, 1/2]$, $\varepsilon \in [0, 1/4]$ consider the family

$$L_\varepsilon(a) := \{(x, y) \in [-1, 1] \times [-1, 1] : x \in [(a - \varepsilon) \vee -1/2, (a + \varepsilon) \wedge 1/2], y \in [-\varepsilon, \varepsilon]\},$$

$$H_\varepsilon(a) := \{(x, y) \in [-1, 1] \times [-1, 1] : y \in [(a - \varepsilon) \vee -1/2, (a + \varepsilon) \wedge 1/2], x \in [-\varepsilon, \varepsilon]\},$$

and clearly for any a, ε we have $F_{f, \mathcal{L}^2, id}(L_\varepsilon(a)) = F_{f, \mathcal{L}^2, id}(H_\varepsilon(a)) = \infty$.

So, similarly to the previous subsection, if some set $\mathcal{X} \in \mathcal{A}$ satisfies $F_{f, \mathcal{L}^2, id}(\mathcal{X}) < \infty$, then for any $\eta > 0$, a, ε , $\mathcal{X}$ must have distance from $L_\varepsilon(a)$ not more than η , which translates to

$$\bigcup_{a \in [-1/2, 1/2]} \bigcap_{\varepsilon \in [0, 1/4]} L_\varepsilon(a) = [-1/2, 1/2] \times \{0\} \subseteq \mathcal{X};$$

with the same argument applied to $H_\varepsilon(a)$ leads to $\{0\} \times [-1/2, 1/2] \subseteq \mathcal{X}$, and thus $X \subseteq \mathcal{X}$.

Then any set with finite energy should contain X , which is homeomorphic to a cross.

Chapter 4

Quasi static evolutions

In this chapter we study the evolution for a class of problems, the minimizing movement problems associated to the average distance functional. Similarly to the average distance problem, they arise from urban planning/network optimization problems, when some additional variables (frequently a time variable is present) and constraints are considered. Given $\Omega \subseteq \mathbb{R}^N, \mu, A$ as in the average distance problem, and an initial datum $S_0 \in \mathcal{A}(\Omega)$, consider the recursive sequence

$$\begin{cases} w(0) := S_0 \\ w(n) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu, A}(\cdot) + \lambda \mathcal{H}^1(\cdot \Delta w(n-1)) \end{cases}, \quad (4.0.1)$$

where $\lambda > 0$ is a given constant. Here Δ denotes the symmetric difference. The choice of using the penalization $\lambda \mathcal{H}^1(\cdot \Delta w(n-1))$ may seem arbitrary, as a priori one can use more general penalization terms like $\eta(\mathcal{H}^1(\cdot \Delta w(n-1)))$, where η is a given function satisfying natural conditions like:

- $\eta(0) = 0$,
- η non decreasing.

However, as will emerge from the arguments used in the proofs, the arguments used for functions η of the form $\eta(t) = \lambda t$ can be easily generalized to functions of the form $\tilde{\eta}(t) = kt^h$ ($k > 0, h \geq 1$).

An important variant is (given a time step $\varepsilon > 0$)

$$\begin{cases} w(0) := S_0 \\ w(n) \in \operatorname{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0) + n\varepsilon}(\Omega)} F_{\mu, A} \end{cases}. \quad (4.0.2)$$

The main difference between formulations (4.0.2) and (4.0.1) is in the constraints: in the former the length is prescribed at any step, while in the latter no constraints on the length are imposed, but the “penalization term” $\lambda \mathcal{H}^1(\cdot \Delta w(n-1))$ interdicts optimality for sets with large length. Most results proven for one case hold for the other too, and proofs for both cases often use the same argument. Thus unless otherwise specified, results proven for one case will be valid for the other formulation too.

In the following, when we will write “ $\mathcal{X} \in \operatorname{argmin} \mathcal{G}$ ”, where $\mathcal{X}$ is an element and $\mathcal{G}$ a functional, we will mean that $\mathcal{X}$ is an arbitrary element of $\operatorname{argmin} \mathcal{G}$ ($\# \operatorname{argmin} \mathcal{G} > 1$ is possible in general).

An important sub-class, are the “irreversible” evolutions:

$$\begin{cases} w(0) := S_0 \\ w(n) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \lambda \mathcal{H}^1(\cdot \Delta w(n-1)) \\ w(n) \supseteq w(n-1), \end{cases} \quad (4.0.3)$$

and

$$\begin{cases} w(0) := S_0 \\ w(n) \in \operatorname{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+n\varepsilon}(\Omega)} F_{\mu,A} \\ w(n) \supseteq w(n-1) \end{cases} \quad (4.0.4)$$

In this case it is explicitly stated that every set in the evolution must contain all previous sets: this property will be called “irreversibility” in the following. This property can be used to model irreversible physical phenomena, like fracture propagation or membrane debonding (see for instance [11] and [10]), or transportation network expansion where removing existing network is highly uneconomical. As seen in the following this property, can significantly alter qualitative properties of solutions.

For both problems (4.0.2) and (4.0.1) a solution will be a sequence $\{w(k)\}_{k=0}^{\infty}$ of elements of $\mathcal{A}(\Omega)$, verifying the constraints and minimality properties imposed. In this chapter our goal is to extend geometric/regularity properties satisfied by solutions of the average distance functional. In particular we will prove that the absence of loops is valid even in higher dimensions, along with some weak analytic regularity (Ahlfors regularity).

The main results are in Section 4.1, in which results from the static case are adapted to the evolutionary case (most results concerning evolutions, in particular those stating some geometric properties, are from works by the author). Moreover it presents a counterexample (from a paper by the author), when key properties are not assumed. Section 4.2 contains some side observations.

4.1 Evolution of solutions of the average distance problem

In this Section our goal is to study solutions of the quasi static evolution related to the average distance problem. We will prove that many properties satisfied by solutions of the average distance problem can be retrieved.

4.1.1 Geometric and analytic properties

Theorems 3.2.8 and 3.4.7 proved the absence of loops for solutions of the average distance problem. In this subsection the proof will be adapted to prove absence of loops for solutions of (4.0.2) and (4.0.1). These results have been discussed in [33] for the two dimension case, and [35] for higher dimension case.

As we are considering evolutions like (4.0.2) or (4.0.1), it may be possible that at some step k the difference $w(k) \setminus w(k-1)$ is not connected.

If this is the case, it holds

$$w(k) \setminus w(k-1) = \bigcup_{i \in \mathcal{I}} \mathcal{C}_i$$

where $\mathcal{C}_i$ are its connected components and $\mathcal{J}$ is a suitable set of indices. As $\mathcal{H}^1(w(k) \setminus w(k-1)) < \infty$, for at most countable indexes $h \in \mathcal{J}$ the component $\mathcal{C}_h$ verifies $\mathcal{H}^1(\mathcal{C}_h) > 0$, thus we can split the passage (here the arrow does not indicate any sort of convergence, the expression $w(k-1) \rightarrow w(k)$ just says “passing from configuration $w(k-1)$ to configuration $w(k)$ ”)

$$w(k-1) \rightarrow w(k)$$

in

$$w(k-1) \rightarrow w(k-1) \cup \mathcal{C}_{i_1} \rightarrow w(k-1) \cup \mathcal{C}_{i_1} \cup \mathcal{C}_{i_2} \rightarrow w(k-1) \cup \mathcal{C}_{i_1} \cup \mathcal{C}_{i_2} \cup \mathcal{C}_{i_3} \rightarrow \dots$$

where $\{i_s\}_{s=1}^\infty$ are indexes for which $\mathcal{H}^1(\mathcal{C}_{i_s}) > 0$, and analyze each single passage separately, i.e.

$$\begin{aligned} F_{\mu,A}(w(k-1)) - F_{\mu,A}(w(k)) &= F_{\mu,A}(w(k-1)) - F_{\mu,A}(w(k-1) \cup \mathcal{C}_{i_1}) \\ &\quad + \sum_{j=1}^\infty \left(F_{\mu,A}(w(k-1) \cup \bigcup_{r=1}^j \mathcal{C}_{i_r}) - F_{\mu,A}(w(k-1) \cup \bigcup_{r=1}^{j+1} \mathcal{C}_{i_r}) \right). \end{aligned}$$

Notice that by construction $w(k-1) \cup \bigcup_{r=1}^j \mathcal{C}_{i_r} \rightarrow w(k)$ as sets, thus in this way it is possible to analyze each passage

$$F_{\mu,A}(w(k-1) \cup \bigcup_{r=1}^j \mathcal{C}_{i_r}) - F_{\mu,A}(w(k-1) \cup \bigcup_{r=1}^{j+1} \mathcal{C}_{i_r}),$$

and sum all such terms in order to compute

$$F_{\mu,A}(w(k-1)) - F_{\mu,A}(w(k)).$$

Thus if $w(k) \setminus w(k-1)$ is not connected, i.e. the passage from $w(k-1)$ to $w(k)$ is obtained by adding a non connected set, then it is possible to split it into at most countably many steps, in which a connected set is added at each step. Notice that unlike $w(k)$, which by definition is chosen among minimizers of some energy (this will be explained later), “intermediate” steps are not required to satisfy any minimality property.

The absence of loops for solutions of (4.0.2) was first proven in two dimension case (see [33] for more details), then generalized to higher dimension cases (see [35] for more details). The proof reported here deals with the general case.

Lemma 4.1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, μ a given measure, A a given function, $S_0 \in \mathcal{A}(\Omega)$ with $F_{\mu,A}(S_0) < \infty$ and not containing loops, and $h > 0$ a given positive value. Then any element*

$$\Sigma_{\text{opt}} \in \{S \in \text{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+h}(\Omega)} F_{\mu,A} : S \supseteq S_0\}$$

is such that $\Sigma_{\text{opt}} \setminus S_0$ does not contain loops.

Proof. Suppose there exists an element

$$\Sigma_{\text{opt}} \in \{S \in \text{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+h}(\Omega)} F_{\mu,A} : S \supseteq S_0\}$$

such that the difference $I := \Sigma_{\text{opt}} \setminus S_0$ contains a loop $E \subseteq I$. From Lemma 3.1.9 follows that such Σ_{opt} must verify $\mathcal{H}^1(\Sigma_{\text{opt}}) = \mathcal{H}^1(S_0) + h$. The goal will be finding a competitor $\Sigma' \in \mathcal{A}_{\mathcal{H}^1(S_0)+h}(\Omega)$ satisfying $F_{\mu,A}(\Sigma') < F_{\mu,A}(\Sigma_{\text{opt}})$.

The idea used here is similar to that used in [44] to prove the absence of loops in minimizers of the average distance problem (and in [33] for the two dimension case).

As $\mu(\Sigma_{\text{opt}}) = 0$ by hypothesis, there exists a not μ -negligible compact set K such that $\Sigma_{\text{opt}} \cap K = \emptyset$, and put

$$R := \frac{1}{2} \min\{\text{dist}(y, \Sigma_{\text{opt}}) : y \in K\} > 0.$$

We have supposed the existence of loop $E \subseteq \Sigma_{\text{opt}}$, thus $\mu(E) = 0$, and

$$\lim_{r \rightarrow 0^+} \frac{\mu(B(x, r))}{r} = 0$$

for $\mathcal{H}^1$ -almost every $x \in E$ (see [2] for further details).

Let be $\beta := \frac{1}{64n^{3/2} + 8}$, and t a free parameter for now. Applying Lemma 3.4.5 yields the existence of:

- $\rho \in (0, t)$ and $\Sigma' \in \mathcal{A}(\Omega)$ such that

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_{\text{opt}}) - \rho/4.$$

Choose $x^* \in E$ such that $\lim_{r \rightarrow 0^+} \frac{\mu(B(x^*, r))}{r} = 0$, this leads to

$$\begin{aligned} F_{\mu,A}(\Sigma') &\leq F_{\mu,A}(\Sigma_{\text{opt}}) + \int_{B(x^*, 64n^{3/2}\rho)} (A(\text{dist}_{\Omega}(w, \Sigma_{\text{opt}}) + \rho) - A(\text{dist}_{\Omega}(w, \Sigma_{\text{opt}}))) d\mu(w) \\ &\leq F_{\mu,A}(\Sigma_{\text{opt}}) + \rho\mu(B(x^*, 64n^{3/2}\rho))\Lambda \\ &= F_{\mu,A}(\Sigma_{\text{opt}}) + 64n^{3/2}\rho^2 \frac{\mu(B(x^*, 64n^{3/2}\rho))}{64n^{3/2}\rho} \Lambda. \end{aligned} \tag{4.1.1}$$

Lemma 3.4.6 applied to Σ' gives the existence of a competitor Σ'' verifying

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + 2n\varepsilon \leq \mathcal{H}^1(\Sigma_{\text{opt}}) + 2n\varepsilon - \rho/4$$

and choosing $\varepsilon := \rho/8n$ this yields

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma_{\text{opt}}).$$

For the average distance functional

$$F_{\mu,A}(\Sigma'') \leq F_{\mu,A}(\Sigma') - \frac{\lambda(R)\mu(K)}{32n\mathcal{H}^1(\Sigma')} \frac{\rho^2}{64n^2} \tag{4.1.2}$$

holds. Combining (4.1.1) and (4.1.2), for ρ sufficiently small, Σ'' satisfies $\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma_{\text{opt}})$ and $F_{\mu,A}(\Sigma'') < F_{\mu,A}(\Sigma_{\text{opt}})$. Finally, the competitor Σ'' contains S_0 , thus is admissible. $\square$

Lemma 4.1.2. *Let Ω be a given domain, μ a given measure, A a given function, $S_0 \in \mathcal{A}(\Omega)$ with $F_{\mu,A}(S_0) < \infty$ and not containing loops, and $h > 0$ a given positive value. Consider an arbitrary element*

$$\Sigma_{\text{opt}} \in \{S \in \operatorname{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+h}(\Omega)} F_{\mu,A} : S \supseteq S_0\}.$$

Suppose there exists a loop $E \in \Sigma_{\text{opt}}$, and let $\varphi : \mathbb{R}^2 \supset S^1 \rightarrow E$ be an arbitrary homeomorphism. Then the set $V := \varphi^{-1}(E \cap (\Sigma_{\text{opt}} \setminus S_0))$ has non empty interior.

Proof. From Lemma 4.1.1 it follows that $E \not\subseteq \Sigma_{\text{opt}} \setminus S_0$. As by hypothesis $E \not\subseteq S_0$, then both $E \cap S_0$ and $E \cap \Sigma_{\text{opt}} \setminus S_0$ are non empty. So $V := \varphi^{-1}(E \cap (\Sigma_{\text{opt}} \setminus S_0)) \neq \emptyset$. Without loss of generality we can work with another homeomorphism ϕ satisfying:

1. $\phi : [0, 1] \rightarrow E$, $\phi(0) = \phi(1) = P \in E \cap S_0$,
2. $\phi|_{(0,1)} : (0, 1) \rightarrow E \setminus \{P\}$ is an homeomorphism.

This choice is due to technical reasons only, as it is easier to work with ϕ . Proving that V has non empty interior is equivalent to prove $W := \phi^{-1}(E \cap (\Sigma_{\text{opt}} \setminus S_0))$ has non empty interior. Suppose the opposite, i.e. W has empty interior (that is, as both ϕ and ϕ^{-1} are homeomorphism, $E \cap (\Sigma_{\text{opt}} \setminus S_0)$ has empty interior). From assumption (2) on ϕ this means $(E \cap (\Sigma_{\text{opt}} \setminus S_0)) \setminus \{P\}$ has empty interior in $E \setminus \{P\}$, or equivalently $(E \cap S_0) \setminus \{P\}$ dense in $E \setminus \{P\}$.

Since $E \setminus \{P\}$ dense in E , this leads to

$$\overline{(E \cap S_0) \setminus \{P\}} = \overline{E \setminus \{P\}} = E$$

which ultimately yields

$$\overline{E \cap S_0} = E$$

and considering that E, S_0 are closed sets, $E \cap S_0 = E$ follows, contradicting the hypothesis. $\square$

Now we can prove the absence of loops for solutions of (4.0.2) and (4.0.1):

Theorem 4.1.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a given domain, μ a given measure, A a given function, $\varepsilon > 0$ a given time step $S_0 \in \mathcal{A}(\Omega)$ an initial datum with $F_{\mu,A}(S_0) < \infty$ and not containing loops, and consider*

$$\begin{cases} w(0) := S_0 \\ w(n+1) \in \operatorname{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+(n+1)\varepsilon}(\Omega)} F_{\mu,A} \\ w(n+1) \supseteq w(n) \end{cases} \quad (4.1.3)$$

Then for any $n \geq 0$ the set $w(n)$ does not contain loops. Similarly solutions of (4.0.2) with the same initial datum S_0 do not contain loops.

Proof. The proof is done by induction on n :

- by hypothesis $w(0) := S_0$ does not contain loops,
- suppose $w(n)$ does not contain loops.

The goal is to prove that $w(n+1)$ does not contain loops. Suppose the contrary, i.e. there exists a loop $S \subseteq w(n+1)$: this may lead to two possibilities:

1. $S \subseteq w(n+1) \setminus w(n)$,

or

2. $S \cap w(n+1) \setminus w(n)$ and $S \cap w(n)$ are non empty,

with the third possibility $S \subseteq w(n)$ excluded by inductive hypothesis.

Notice that by construction $w(n+1) \supseteq w(n)$, and

$$w(n+1) \in \{\mathcal{X} \in \operatorname{argmin}_{\mathcal{A}_{\mathcal{H}^1(w(n))+\varepsilon}} F_f : \mathcal{X} \supseteq w(n)\},$$

so hypothesis of Lemma 4.1.1 and 4.1.2 are applicable to both possibility (1) and (2). Applying Lemma 4.1.1 would lead immediately $S \not\subseteq w(n+1) \setminus w(n)$, thus possibility (1) is excluded.

Let be $\phi : [0, 1] \rightarrow S$ an homeomorphism like that chosen in the proof of Lemma 4.1.2; applying the latter, $\phi^{-1}(E \cap (w(n+1) \setminus w(n)))$ is not empty, thus contains an open ball $(t^* - \rho, t^* + \rho) \subseteq \phi^{-1}(E \cap (w(n+1) \setminus w(n)))$, with $\rho > 0$. The image $\phi((t^* - \rho, t^* + \rho))$ is an open connected arc in $E \cap (w(n+1) \setminus w(n))$. Then it is possible to apply Lemmas 3.4.5 and 3.4.6, similarly to what done in [44], and construct a competitor $\Sigma' \in \{\mathcal{X} \in \mathcal{A}_{\mathcal{H}^1(w(n))+\varepsilon}(\Omega) : \mathcal{X} \supseteq w(n)\}$ with $F_{\mu,A}(\Sigma') < F_{\mu,A}(w(n+1))$, contradicting the optimality of $w(n+1)$.

The proof for solutions of (4.0.2), (4.0.1) and (4.0.3) use the same argument, with very small modifications. $\square$

In [44] it has been proven that minimizers of the average distance functional exhibit Ahlfors regularity, when the considered measure verifies some summability properties. In this section we aim to extend these results to solutions of (4.0.2) and (4.0.4), by adapting the proof. We present now some preliminary results about Ahlfors regularity.

The following condition regarding the domain will be assumed through the Chapter:

Assumption 4.1.4. *The domain Ω satisfies: for any point $x \in \Omega$ (where Ω is the domain being considered) there exists $\delta > 0$ such that:*

- $B(x, \delta) \subseteq \Omega$ if x is in the interior,
- $\overline{B(x, \delta) \cap \Omega}$ is convex, and there exists a constant $\eta > 0$ such that $\mathcal{L}^2(\overline{B(x, \delta) \cap \Omega}) \geq \eta \delta^2$ for any $x \in \partial\Omega$.

Lemma 4.1.5. *Given natural numbers n and k , $x \in \mathbb{R}^n$, $\rho > 0$, points $\{z_i\}_{i=1}^k \subseteq \overline{B(x, \rho)}$, there exists $\Sigma \in \mathcal{A}(\overline{B(x, \rho)})$ such that*

- $z_i \in \Sigma$ for $i = 1, \dots, k$,
- $\mathcal{H}^1(\Sigma) \leq C^* k^{\frac{n-1}{n}} \rho$, where C^* depends only on n .

The proof can be found in [44].

Proof. Upon translation and rescaling suppose $x = (0, \dots, 0)$, $\rho = 1/2$ and $\{z_i\}_{i=1}^k \subseteq [0, 1]^n$. Let Γ_j be a uniform one dimensional grid with step j ($\{(x_1, \dots, x_n) : jx_i \in \mathbb{N} \text{ for at least } n-1 \text{ indexes}\}$): it holds

$$H^1(\Gamma_j) \leq n(j+1)^{n-1}, \quad \max_{y \in [0,1]^n} \text{dist}(y, \Gamma_j) \leq \frac{\sqrt{n}}{2j}. \quad (4.1.4)$$

Let $z_{i,j}$ be an arbitrary projection of z_i on Γ_j for $j = 1, \dots, k$, and put

$$\Gamma_j^* := \Gamma_j \cup \bigcup_{i=1}^k \{sz_i + (1-s)z_{i,j} : s \in [0, 1]\}. \quad (4.1.5)$$

It is obvious that $z_i \in \Gamma_j^*$ for any i, j ; from (4.1.4) inequality

$$\mathcal{H}^1(\Gamma_j^*) \leq n(j+1)^{n-1} + \frac{k\sqrt{n}}{2j}$$

follows, and the choice $j := \lfloor k^{1/n} \rfloor$ gives

$$\mathcal{H}^1(\Gamma_{\lfloor k^{1/n} \rfloor}^*) \leq n(\lfloor k^{1/n} \rfloor + 1)^{n-1} + \frac{k\sqrt{n}}{2\lfloor k^{1/n} \rfloor},$$

which concludes the proof. $\square$

Remark 4.1.6. Let $M \subseteq \mathbb{R}^n$ be a convex set, and assume there exists a homeomorphism $\varphi : M \rightarrow \overline{B(x, \rho)}$ verifying:

- there exists $m_1, m_2 > 0$ such that

$$m_1 \text{dist}_{\mathbb{R}^n}(\varphi(z_1), \varphi(z_2)) \leq \text{dist}_M(z_1, z_2) \leq m_2 \text{dist}_{\mathbb{R}^n}(\varphi(z_1), \varphi(z_2)) \quad (4.1.6)$$

for any $z_1, z_2 \in M$.

Notice that (4.1.6) implies a bi-Lipschitz behavior.

Then the conclusion of Lemma 4.1.5 can be applied for points $\{y_i\}_{i=1}^K \subseteq M$: indeed given k points $\{z_i\}_{i=1}^k$ of M , upon translation and rescaling, we can apply Lemma 4.1.5 to points $\{\varphi(z_i)\}_{i=1}^k$ in the domain $[0, 1]^n$. Using the same construction, let Γ_j be the same set defined in the proof of Lemma 4.1.5, and define

$$\Gamma'_j := \Gamma_j \cup \bigcup_{i=1}^k \{s\varphi(z_i) + (1-s)z_{\varphi,i,j} : s \in [0, 1]\}$$

where $z_{\varphi,i,j}$ denote an arbitrary projection of $\varphi(z_i)$ on Γ_j .

Now it is clear that $\varphi^{-1}(\Gamma_j) \subseteq M$, as well $\varphi^{-1}(\{s\varphi(z_i) + (1-s)z_{\varphi,i,j} : s \in [0, 1]\}) \subseteq M$ for any $i = 1, \dots, k$. From (4.1.6) there exists m'_1, m'_2 such that

$$m'_1 \mathcal{H}^1(\Gamma_j) \leq \mathcal{H}^1(\varphi^{-1}(\Gamma_j)) \leq m'_2 \mathcal{H}^1(\Gamma_j)$$

and

$$m'_1(\text{dist}_{\mathbb{R}^n}(\varphi(z_i), z_{\varphi,i,j})) \leq \text{dist}_M(z_i, \varphi^{-1}(z_{\varphi,i,j})) \leq m'_2(\text{dist}_{\mathbb{R}^n}(\varphi(z_i), z_{\varphi,i,j}))$$

thus the same conclusion of Lemma 4.1.5 holds for points $\{z_i\}_{i=1}^k \subseteq M$.

The next two (technical) results are from [44].

Lemma 4.1.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, $\Sigma \in \mathcal{A}(\Omega)$, then for any $x \in \Sigma$ there exists $\Sigma' \in \mathcal{A}(\Omega)$ such that for any $\rho > 0$*

- $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x, \rho)) + C\left(\left(\frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 1\right)\rho,$
- $\Sigma \setminus \Sigma' \subseteq B(x, 2\rho), \Sigma' \setminus \Sigma \subseteq B(x, 8\sqrt{n}\rho),$
- $\text{dist}_\Omega(z, \Sigma') < \text{dist}(z, \Sigma)$ for any $z \notin B(x, 4n\rho),$
- $\text{dist}_\Omega(z, \Sigma') \leq \text{dist}(z, \Sigma) + \rho$ for any $z \in B(x, 4n\rho).$

where C is a positive constant depending only on n .

Lemma 4.1.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, $\Sigma \in \mathcal{A}(\Omega)$ and suppose there exists $r > 0$ such that for any $x \in \Sigma, 0 < \rho < r$ the inequality*

$$\frac{\mathcal{H}^1(\Sigma \cap B(x, \rho))}{\rho} \leq a \frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))^\alpha}{2\rho} + b$$

holds for some fixed $a > 0, b \geq 0, \alpha \in (0, 1)$. Then there exists a constant $K = K(a, b, \alpha, r, \mathcal{H}^1(\Sigma))$ such that

$$\frac{\mathcal{H}^1(\Sigma \cap B(x, \rho))}{\rho} \leq K.$$

Lemma 4.1.7 cannot be used when irreversibility condition is added. A slightly different result is required.

Lemma 4.1.9. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, $\Sigma_* \in \mathcal{A}(\Omega)$ Ahlfors regular, $\Sigma \supseteq \Sigma_*$, then for any $x \in \Sigma$ there exists $\Sigma' \in \mathcal{A}(\Omega), \Sigma' \supseteq \Sigma_*$, such that for any $\rho > 0$*

- $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x, \rho)) + C\left(\left(\frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 1\right)\rho,$
- $\Sigma \setminus \Sigma' \subseteq B(x, 2\rho), \Sigma' \setminus \Sigma \subseteq B(x, 8\sqrt{n}\rho),$
- $\text{dist}_\Omega(z, \Sigma') < \text{dist}(z, \Sigma)$ for any $z \notin B(x, 4n\rho),$
- $\text{dist}_\Omega(z, \Sigma') \leq \text{dist}(z, \Sigma) + \rho$ for any $z \in B(x, 4n\rho).$

where C is a positive constant depending only on n and Σ_* .

Proof. The proof uses an idea similar to that found for Lemma 4.1.7 (see [44] for instance), with corrections due to irreversibility condition.

Given a point $x \in \Sigma$, $\rho \in (0, \delta)$ (δ given by Assumption 4.1.4), put $k(x, \rho) := \sharp\{\Sigma \cap \partial B(x, \rho)\}$; from coarea formula

$$\mathcal{H}^1(\Sigma \cap B(x, 2\rho)) \geq \int_0^{2\rho} k(x, t) dx \geq \int_\rho^{2\rho} k(x, t) dt$$

which implies there exists $t \in [\rho, 2\rho]$ such that

$$k(x, t) \leq 2 \frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{\rho}.$$

Lemma 4.1.5, with Assumption 4.1.4 and Remark 4.1.6 guarantees the existence of $\Sigma_0(t) \in \mathcal{A}(\Omega)$ such that $\{\Sigma \cap B(x, t)\} \subseteq \Sigma_0(t)$, and $\mathcal{H}^1(\Sigma_0(t)) \leq C^*(n)k(x, t) \frac{n-1}{n} t$.

Let be $\Sigma_1(t) := x + \cup_{j=1}^n \{se_j : s \in [-t, t]\}$, where e_j denotes the j -th unit vector ($e_j = (0, \dots, 0, 1, 0, \dots, 0)$, with the only “1” occupying the j -th place). This set mainly serves to preserve connectedness for Σ' , which will be constructed in the following.

Some discussion about $\Sigma_1(t)$ is required, as we have only $\Sigma_1(t) \subseteq \overline{B(x, t)}$ but not $\Sigma_1(t) \subseteq \Omega$, thus we should prove $\Sigma_1(t) \cap \Omega$ is connected first. Thus given an arbitrary point $z_0 \in (\Sigma_1(t) \setminus \{x\}) \cap \Omega$, there exists $t(z_0) \in [-t, t]$ and $j(z_0) \in \{1, \dots, n\}$ such that $z_0 = x + t(z_0)e_{j(z_0)}$, and since $\overline{B(x, t)} \cap \Omega$ is convex by Assumption 4.1.4, $\{x + ue_{j(z_0)} : u \in [0, t(z_0)]\} \subseteq \Omega$ follows. This guarantees that every point $z \in \Sigma_1(t) \cap \Omega$ is connected by a path (as $\{x + ue_{j(z_0)} : u \in [0, t(z_0)]\} \subseteq \Omega$) to $x \in \Omega$, thus $\Sigma_1(t) \cap \Omega$ is connected. In the following we will write $\Sigma_1(t)$ instead of $\Sigma_1(t) \cap \Omega$.

Upon a rotation $\Sigma_0(t) \cap \Sigma_1(t) \neq \emptyset$. Put

$$\Sigma' := \Sigma \setminus B(x, t) \cup (\Sigma_* \cap B(x, t)) \cup \Sigma_0(t) \cup \Sigma_1(t),$$

and inequality

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x, t)) + \mathcal{H}^1(\Sigma_* \cap B(x, t)) + \mathcal{H}^1(\Sigma_0(t)) + \mathcal{H}^1(\Sigma_1(t))$$

follows.

By construction $\mathcal{H}^1(\Sigma_1(t)) \leq 4n^{3/2}t$; combining

$$\mathcal{H}^1(\Sigma_0(t)) \leq C^*(n)k(x, t) \frac{n-1}{n} t$$

given by Lemma 4.1.5 and

$$k(x, t) \leq 2 \frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{\rho},$$

inequality

$$\mathcal{H}^1(\Sigma_0(t)) \leq 2C^*(n) \left(\frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{2\rho} \right)^{\frac{n-1}{n}} t$$

follows, yielding

$$\begin{aligned}
\mathcal{H}^1(\Sigma') &\leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x, t)) + \mathcal{H}^1(\Sigma_* \cap B(x, t)) + 2C^*(n) \left(\frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{2\rho} \right)^{\frac{n-1}{n}} t + 4n^{3/2}t \\
&\leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x, t)) + \mathcal{H}^1(\Sigma_* \cap B(x, t)) + 4C^*(n) \left(\frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{2\rho} \right)^{\frac{n-1}{n}} \rho + 8n^{3/2}\rho \\
&= \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x, \rho)) + \mathcal{H}^1(\Sigma_* \cap B(x, t)) + \left(4C^*(n) \left(\frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{2\rho} \right)^{\frac{n-1}{n}} + 8n^{3/2} \right) \rho.
\end{aligned}$$

As Σ_* is Ahlfors regular by hypothesis, there exists $K > 0$ such that

$$\frac{\mathcal{H}^1(\Sigma_* \cap B(x, t))}{t} \leq K,$$

thus

$$\begin{aligned}
\mathcal{H}^1(\Sigma') &\leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x, \rho)) + Kt \left(4C^*(n) \left(\frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{2\rho} \right)^{\frac{n-1}{n}} + 8n^{3/2} \right) \rho \\
&\leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x, \rho)) + \left(2K + 4C^*(n) \left(\frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{2\rho} \right)^{\frac{n-1}{n}} + 8n^{3/2} \right) \rho
\end{aligned}$$

and putting $C := 2K + 4C^*(n) \left(\frac{\mathcal{H}^1(\Sigma \cap B(x, 2\rho))}{2\rho} \right)^{\frac{n-1}{n}} + 8n^{3/2}$ concludes the proof. $\square$

Now we can present the result about evolution cases:

Theorem 4.1.10. *Let be $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) a given domain, $\mu \in L^p(\Omega)$ with $p > \frac{N}{N-1}$ a given measure, $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$ a given function, $S_0 \in \mathcal{A}(\Omega)$ an Ahlfors regular initial datum, $\varepsilon > 0$ a given time step, and consider*

$$\begin{cases} w(0) := S_0 \\ w(n+1) \in \operatorname{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0) + (n+1)\varepsilon}(\Omega)} F_{\mu, A} \\ w(n+1) \supseteq w(n). \end{cases} \quad (4.1.7)$$

Then for any n the set $w(n)$ is Ahlfors regular. Similarly solutions of (4.0.1) with the same initial datum S_0 are Ahlfors regular.

Proof. The proof is done by induction. By hypothesis $w(0) := S_0$ is Ahlfors regular. Suppose that $w(n)$ is Ahlfors regular, the goal is to prove $w(n+1)$ is Ahlfors regular too.

First notice that $\mu(w(n+1)) = 0$ forces the existence of a compact set $K \subseteq \Omega$ with $\mu(K) > 0$ (similarly to what done in the proof of Theorem 3.4.10, available in [44], the choice $K := \Omega \setminus \{\omega \in \Omega : \text{dist}_\Omega(\omega, w(n+1)) < 2c\}$ is acceptable for some $c \in (0, \text{diam } w(n+1))$).

Consider a point $y \in w(n+1)$. Applying Lemma 4.1.9 (with $\Sigma_* = w(n)$) yields the existence of $\Sigma' \in \mathcal{A}(\Omega)$ verifying

- $\Sigma' \supseteq w(n)$,
- inequality

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(w(n+1)) - \mathcal{H}^1(w(n+1) \cap B(y, \rho)) + C\left(\left(\frac{\mathcal{H}^1(w(n+1) \cap B(y, 2\rho))}{2\rho}\right)^{\frac{N-1}{N}} + 1\right)\rho$$

for some $C > 0$ depending on N and $w(n)$.

Moreover

$$\begin{aligned} F_{\mu,A}(\Sigma') &\leq F_{\mu,A}(w(n+1)) + 2\Lambda\rho\mu(B(y, 4N\rho)) \\ &\leq F_{\mu,A}(w(n+1)) + 2\Lambda\|\mu\|_{L^p(B(y, 4N\rho))}^{1/p} \mathcal{L}^2(B(y, 4N\rho))^{1/q} \\ &= F_{\mu,A}(w(n+1)) + C'\rho^{\frac{N}{q}+1} \end{aligned} \quad (4.1.8)$$

for ρ sufficiently small, with Λ denoting the Lipschitz constant of A , q the conjugate exponent of p and $C' > 0$ a constant not dependent on y and ρ .

Then from the argument found in Theorem 3.4.10 follows:

$$\mathcal{H}^1(w(n+1)) - \mathcal{H}^1(\Sigma') \geq \mathcal{H}^1(w(n+1) \cap B(y, \rho)) - \rho C\left(\left(\frac{\mathcal{H}^1(w(n+1) \cap B(y, 2\rho))}{2\rho}\right)^{\frac{N-1}{N}} + 1\right)$$

and if

$$\mathcal{H}^1(w(n+1) \cap B(y, \rho)) - \rho C\left(\left(\frac{\mathcal{H}^1(w(n+1) \cap B(y, 2\rho))}{2\rho}\right)^{\frac{N-1}{N}} + 1\right) \leq 0$$

Lemma 4.1.8 (applied with $a = C$, $\alpha = \frac{N-1}{N}$, $b = C$, $r = \text{diam } w(n+1)$, $\Sigma = w(n+1)$) concludes the proof. If

$$\mathcal{H}^1(w(n+1) \cap B(y, \rho)) - \rho C\left(\left(\frac{\mathcal{H}^1(w(n+1) \cap B(y, 2\rho))}{2\rho}\right)^{\frac{N-1}{N}} + 1\right) > 0$$

then put

$$\xi := (\mathcal{H}^1(w(n+1) \cap B(y, \rho)) - \rho C\left(\left(\frac{\mathcal{H}^1(w(n+1) \cap B(y, 2\rho))}{2\rho}\right)^{\frac{N-1}{N}} + 1\right))/2N;$$

using Lemma 3.4.6 there exists $\Sigma'' \in \mathcal{A}(\Omega)$, $\Sigma'' \supseteq \Sigma'$, such that

$$F_{\mu,A}(\Sigma'') \leq F_{\mu,A}(\Sigma') - C_1\xi^2, \quad \mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + 2N\xi \quad (4.1.9)$$

with $C_1 > 0$ not dependent on y and ρ , thus combined with $\mathcal{H}^1(w(n+1)) - \mathcal{H}^1(\Sigma') \geq 2N\xi$ gives

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(w(n+1)). \quad (4.1.10)$$

Combining $\Sigma'' \supseteq \Sigma' \supseteq w(n)$, (4.1.10) and minimality property of $w(n+1)$, we get $F_{\mu,A}(\Sigma'') \geq F_{\mu,A}(w(n+1))$,

Thus the competitor Σ'' verifies $F_{\mu,A}(\Sigma'') \geq F_{\mu,A}(w(n+1))$, and from now the proof returns to be valid in both cases. Combining (4.1.8) and $F_{\mu,A}(\Sigma'') \geq F_{\mu,A}(w(n+1))$ leads to

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + 2N\xi, \quad F_{\mu,A}(w(n+1)) + C'\rho^{\frac{N}{q}+1} - C_1\xi^2 \geq F_{\mu,A}(\Sigma'') \geq F_{\mu,A}(w(n+1)).$$

By direct computation we get

$$(\mathcal{H}^1(w(n+1) \cap B(y, \rho)) - \rho C((\frac{\mathcal{H}^1(w(n+1) \cap B(y, 2\rho))}{2\rho})^{\frac{N-1}{N}} + 1))^2 \leq C' \rho^{\frac{N}{q}+1},$$

thus

$$\mathcal{H}^1(w(n+1) \cap B(y, \rho)) - \rho C((\frac{\mathcal{H}^1(w(n+1) \cap B(y, 2\rho))}{2\rho})^{\frac{N-1}{N}} + 1) \leq C' \rho^{\frac{N}{2q}-\frac{1}{2}}.$$

By hypothesis $\frac{N}{2q} \geq \frac{1}{2}$, thus forcing

$$\rho^{\frac{N}{2q}-\frac{1}{2}} \leq (\text{diam } w(n+1))^{\frac{N}{2q}-\frac{1}{2}},$$

and Lemma 4.1.8 concludes the proof. $\square$

In the proof of Ahlfors regularity a crucial role is played by Assumption 4.1.4: indeed an explicit example shows that without this condition, Ahlfors regularity can be false.

4.1.2 Counterexample to Ahlfors regularity

Now we present an example of domain which is not convex, and the results concerning Ahlfors regularity do not hold. In all this subsection the notation $\text{dist}(\cdot, \cdot)$ will denote the geodesic distance on the domain (which we will construct).

Given $\alpha \in (1, 2)$ (and this α will be fixed in all the subsection), $k \in \mathbb{N}$, impose a cartesian coordinate system in $\mathbb{R}^2$ (see Figure 4.1.1), and define sets

$$C_k := \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \in \left[\frac{1}{k^\alpha}, \frac{1}{k^\alpha} + 4^{-k} \right] \right\}$$

and L_k the rectangle of $\mathbb{R}^2$ with vertices:

- $\left(\frac{1}{k^\alpha}, 0 \right), \left(\frac{1}{k^\alpha}, \frac{1}{4^{k+1}} \right), \left(\frac{1}{(k+1)^\alpha}, 0 \right), \left(\frac{1}{(k+1)^\alpha}, \frac{1}{4^{k+1}} \right)$ if k even,
- $\left(-\frac{1}{k^\alpha}, 0 \right), \left(-\frac{1}{k^\alpha}, -\frac{1}{4^{k+1}} \right), \left(-\frac{1}{(k+1)^\alpha}, 0 \right), \left(-\frac{1}{(k+1)^\alpha}, -\frac{1}{4^{k+1}} \right)$ if k odd.

Let

$$\Omega := \bigcup_{k=1}^{\infty} C_k \cup \{(0, 0)\} \cup \bigcup_{k=1}^{\infty} L_k$$

be our domain, endowed with the geodesic distance (i.e. the distance between two points $x_1, x_2 \in \Omega$ is given by the length of the shortest path $\beta : [0, 1] \rightarrow \Omega$, $\beta(0) = x_1$, $\beta(1) = x_2$).

Lemma 4.1.11. *The set Ω is sequentially compact.*

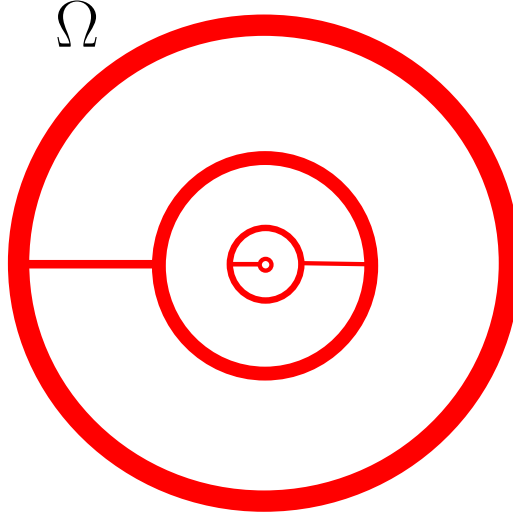


Figure 4.1.1: This domain Ω does not satisfy Assumption 4.1.4. Note that non convexity (actually this domain is not even simply connected) is strongly used in this counterexample.

Proof. The proof is straightforward, using basic topological considerations. Let $\{x_j\}_{j=0}^\infty \subseteq \Omega$ be an arbitrary sequence. If $I := \{i : x_i = (0, 0)\}$ verifies $\#I = \infty$ then $\{x_i\}_{i \in I}$ is a converging sequence.

If $\#I < \infty$, we can consider the sequence $\{x_j\}_{j=0}^\infty \setminus \{x_i\}_{i \in I}$ since removing finitely many elements from a sequence has no influence on the Cauchy condition. Thus without loss of generality we will assume $I = \emptyset$. The following dichotomy holds:

1. if there exists $M > 0$ such that $\{x_j\}_{j=0}^\infty \subseteq \bigcup_{i=1}^M (C_i \cup L_i)$, then $\{x_j\}_{j=0}^\infty$ admits a converging subsequence, since $\bigcup_{i=1}^M (C_i \cup L_i)$ is a finite union of compact sets,
2. if a similar M does not exist, then for any $K > 0$ we have

$$\{x_j\}_{j=0}^\infty \setminus \bigcup_{s=0}^K (C_s \cup L_s) \neq \emptyset,$$

or equivalently

$$\{x_j\}_{j=0}^\infty \cap \bigcup_{s=K+1}^\infty (C_s \cup L_s) \neq \emptyset,$$

and as $\bigcup_{s=K+1}^\infty (C_s \cup L_s) \subseteq B((0, 0), K^{-\alpha})$ for any $K > 0$,

$$\{x_j\}_{j=0}^\infty \cap B((0, 0), K^{-\alpha}) \neq \emptyset.$$

Thus there exists a subsequence $\{x_{j_g}\}_{g=0}^\infty \subseteq \{x_j\}_{j=0}^\infty$ converging to $(0, 0)$.

Thus Ω is sequentially compact. □

Now we provide some estimate on the distance between two points in Ω .

Lemma 4.1.12. *Given arbitrary $a, b \in \mathbb{N}$, $a < b$, for any couple of points $x_1 \in C_a$, $x_2 \in C_b$ inequality*

$$\frac{2}{3}\pi \sum_{j=a+1}^{b-2} \frac{1}{j^\alpha} \leq \text{dist}(x_1, x_2) \leq \frac{4}{3}\pi \sum_{j=a-1}^b \frac{1}{j^\alpha} \quad (4.1.11)$$

holds.

Proof. The proof is split on several passages:

- We first estimate $\text{dist}_\Omega(C_k, C_{k+1})$ for a given $k \in \mathbb{N}$.

By construction for any $k \in \mathbb{N}$ we have

$$\text{dist}_\Omega(C_k, C_{k+1}) \geq \frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha} - \frac{1}{4^{k+1}}$$

On the other hand:

- if k even, there exists $\gamma : [0, 1] \rightarrow \Omega$, $\gamma(0) = (k^{-\alpha}, 0) \in C_k \cap L_k$, $\gamma(1) = ((k+1)^{-\alpha}, 0) \in C_{k+1} \cap L_k$, as $\gamma(s) := (1-s)(k^{-\alpha}, 0) + s((k+1)^{-\alpha}, 0)$ is admissible due to convexity of L_k
- if k odd there exists $\gamma' : [0, 1] \rightarrow \Omega$, $\gamma'(0) = (-k^{-\alpha}, 0) \in C_k$, $\gamma'(1) = (-(k+1)^{-\alpha}, 0) \in C_{k+1}$, as $\gamma'(s) := (1-s)(-k^{-\alpha}, 0) + s(-(k+1)^{-\alpha}, 0)$ is admissible due to convexity of L_k .

thus in both cases $\text{dist}_\Omega(C_k, C_{k+1}) \leq k^{-\alpha} - (k+1)^{-\alpha}$, and

$$\frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha} - \frac{1}{4^{k+1}} \leq \text{dist}_\Omega(C_k, C_{k+1}) \leq \frac{1}{k^\alpha} - \frac{1}{(k+1)^{-\alpha}} \quad (4.1.12)$$

holds.

- Now we have to estimate $\text{dist}_\Omega(L_k, L_{k+1})$ for a given $k \in \mathbb{N}$.

By construction the only way to connect arbitrary points $p_0 \in L_k$ and $p_1 \in L_{k+1}$ is through a path $\beta : [0, 1] \rightarrow \Omega$ verifying $\beta([0, 1]) \cap L_k \supseteq \{p_0\}$, $\beta([0, 1]) \cap L_{k+1} \supseteq \{p_1\}$, and this path must “pass through” C_{k+1} .

As p_0 and p_1 are almost antipodal (i.e. $\text{dist}(p_0, -p_1) \leq 2 \cdot 4^{-(k+1)}$, where $-p_1$ denotes the point symmetric to p_1 with respect to $(0, 0)$), any such path β' must verify

$$\frac{2\pi}{3(k+1)^\alpha} \leq \frac{\pi}{(k+1)^{-\alpha}} - \frac{2}{4^{k+1}} \leq \mathcal{H}^1(\beta([0, 1])).$$

On the other hand, as both $p_0, p_1 \in C_{k+1}$, the path β can be chosen verifying

$$\mathcal{H}^1(\beta([0, 1])) \leq \frac{\pi}{(k+1)^\alpha} + \frac{2}{4^{k+1}} \leq \frac{4\pi}{3(k+1)^\alpha},$$

thus

$$\frac{2\pi}{3(k+1)^\alpha} \leq \text{dist}(L_k, L_{k+1}) \leq \frac{4\pi}{3(k+1)^\alpha} \quad (4.1.13)$$

Similarly, given $x_1 \in C_a$, $x_2 \in C_b$, we have $\text{dist}_\Omega(x_1, L_a) \leq \frac{4\pi}{3a^\alpha}$ and $\text{dist}(x_2, L_{b-1}) \leq \frac{4\pi}{3b^\alpha}$. Combining with (4.1.12) and (4.1.13), with simple algebraic passages, leads to

$$\frac{2}{3} \left(\frac{1}{a^\alpha} - \frac{1}{b^\alpha} + \pi \sum_{j=a+1}^{b-1} \frac{1}{j^\alpha} \right) \leq \text{dist}_\Omega(x_1, x_2) \leq \frac{4}{3} \left(\frac{1}{a^\alpha} - \frac{1}{b^\alpha} + \pi \sum_{j=a}^b \frac{1}{j^\alpha} \right)$$

and the thesis follows with simple estimates. $\square$

If we let $b \rightarrow \infty$, point x_2 converges to $(0, 0)$, and (4.1.11) reads

$$\frac{2}{3}\pi \sum_{j=a+1}^{\infty} \frac{1}{j^\alpha} \leq \text{dist}_\Omega(x_1, (0, 0)) \leq \frac{4}{3}\pi \sum_{j=a-1}^{\infty} \frac{1}{j^\alpha}. \quad (4.1.14)$$

Notice that although we had better estimates for (4.1.13), the less accurate one is sufficient for our goals.

Before proceeding with the main result, another important lemma is required.

Lemma 4.1.13. *Any element of $\mathcal{A}(\Omega) \setminus \mathcal{A}_0(\Omega)$ containing $(0, 0)$ is not Ahlfors regular.*

Proof. Let $W \in \mathcal{A}(\Omega) \setminus \mathcal{A}_0(\Omega)$ be an arbitrary element, and H the smallest index for which $W \cap C_H \neq \emptyset$ (if such H does not exist, i.e. $W \cap C_g = \emptyset$ for any $g \in \mathbb{N}$, would lead $W = \{(0, 0)\}$ contradicting $\mathcal{H}^1(W) > 0$), and choose $X \in W \cap C_H$: as $W \supseteq (0, 0)$, there exists a path $\varphi : [0, 1] \rightarrow W$ with $\varphi(0) = X$, $\varphi(1) = (0, 0)$. From (4.1.14) we have

$$\frac{2}{3}\pi \sum_{j=H+1}^{\infty} \frac{1}{j^\alpha} \leq \text{dist}(X, (0, 0)) \leq \frac{4}{3}\pi \sum_{j=H-1}^{\infty} \frac{1}{j^\alpha},$$

and using

$$\sum_{i=n}^{\infty} \frac{1}{i^\alpha} \geq \frac{1}{\alpha-1} \frac{1}{(n+1)^{\alpha-1}}, \quad (4.1.15)$$

we get

$$\frac{2}{3}\pi \frac{1}{\alpha-1} \frac{1}{(H+2)^{\alpha-1}} \leq \text{dist}(X, (0, 0)).$$

From the construction of Ω , for any $n \geq 0$ there exists $X_n \in \varphi([0, 1]) \cap C_{H+n}$. From (4.1.14) and (4.1.15) we have that

$$\frac{2}{3}\pi \frac{1}{\alpha-1} \frac{1}{(H+n+2)^{\alpha-1}} \leq \text{dist}_\Omega(X_n, (0, 0)),$$

holds for any $n \geq 1$.

Define $r_s := \frac{1}{s^\alpha}$: for any $k \geq 1$ it holds

$$\begin{aligned} \frac{\mathcal{H}^1(W \cap B((0,0), r_{H+k}))}{r_{H+k}} &\geq \frac{\mathcal{H}^1(\varphi([0,1]) \cap B((0,0), r_{H+k}))}{r_{H+k}} \\ &\geq \frac{\text{dist}_\Omega(X_k, (0,0))}{r_{H+k}} \\ &\geq \frac{1}{r_{H+k}} \frac{2}{3} \pi \frac{1}{\alpha-1} \frac{1}{(H+k+2)^{\alpha-1}} \\ &= \frac{2}{3} \pi \frac{1}{\alpha-1} \frac{(H+k)^\alpha}{(H+k+2)^{\alpha-1}}, \end{aligned}$$

leading do

$$\lim_{k \rightarrow \infty} \frac{\mathcal{H}^1(W \cap B((0,0), r_{H+k}))}{r_{H+k}} \geq \lim_{k \rightarrow \infty} \frac{2}{3} \pi \frac{1}{\alpha-1} \frac{(H+k)^\alpha}{(H+k+2)^{\alpha-1}} = \infty,$$

thus W cannot be Ahlfors regular. $\square$

Given a parameter $\varepsilon > 0$ consider the evolution

$$\begin{cases} w(0) := \{(0,0)\} \\ w(n+1) \in \operatorname{argmin}_{\mathcal{A}(\Omega)_{(n+1)\varepsilon}} F \\ w(n+1) \supseteq w(n) \end{cases} \quad (4.1.16)$$

where

$$F : \mathcal{A}(\Omega) \longrightarrow (0, \infty), \quad F(S) := \int_\Omega \text{dist}_\Omega(x, S) dx.$$

Proposition 4.1.14. *For any parameter $\varepsilon > 0$, any solution $\{w(j)\}_{j=0}^\infty$ of (4.1.16) is such that $w(1)$ is not Ahlfors regular.*

Proof. From

$$\text{dist}_\Omega(x, (0,0)) \leq \text{dist}_\Omega(x, w(1)) + \max_{y \in w(1)} \text{dist}_\Omega(y, (0,0)),$$

integrating on Ω leads to

$$\begin{aligned} F(\{(0,0)\}) - F(w(1)) &= \int_\Omega \text{dist}_\Omega(x, (0,0)) dx - \int_\Omega \text{dist}_\Omega(x, w(1)) dx \\ &\leq \int_\Omega \text{dist}_\Omega(x, w(1)) + \max_{y \in w(1)} \text{dist}_\Omega(y, (0,0)) dx - \int_\Omega \text{dist}_\Omega(x, w(1)) dx \\ &\leq \max_{y \in w(1)} \text{dist}_\Omega(y, (0,0)) \mathcal{L}^2(\Omega). \end{aligned}$$

As $\mathcal{H}^1(w(1)) = \varepsilon$ is admissible, we can choose $S \in \mathcal{A}_\varepsilon$ containing $(0,0)$ such that $\max_{y \in S} \text{dist}_\Omega(y, (0,0)) = \varepsilon$, with $\{z \in \Omega : \text{dist}_\Omega(z, (0,0)) = \varepsilon\}$ is not empty as $t \mapsto \text{dist}_\Omega(t, (0,0))$ is continuous on Ω . Let H be

the smallest index for which $S \cap C_H \neq \emptyset$: this forces $S \cap L_H \neq \emptyset$; moreover, as S intersects both C_H and C_{H+1} , $\text{diam}(S \cap L_H) \geq \frac{1}{2} \left(\frac{1}{H^\alpha} - \frac{1}{(H+1)^\alpha} \right)$, thus the set

$$\{w \in \Omega : \text{dist}_\Omega(w, (0,0)) < \text{dist}(w, S)\}$$

contains at least $\{w \in L_H : \text{dist}_\Omega(w, S) \leq \frac{1}{2} \text{dist}_\Omega(C_{H+2}, (0,0))\}$, which has positive measure. Thus $F(S) < F(\{(0,0)\})$, and $w(1) \neq \{(0,0)\}$ as by definition $w(1) \in \argmin_{S' \supseteq (0,0), \mathcal{H}^1(S') \leq \varepsilon} F(S')$. Lemma 4.1.13 concludes the proof. $\square$

4.1.3 Branching and high order points

In this subsection we will restrict the discussion to two dimensional domains, endowed with the Lebesgue measure. Moreover, irreversibility will be imposed, i.e. we will analyze only non decreasing solutions with respect to inclusion. Irreversibility will be crucial for this argument. Our main goal is to analyze topological properties of solutions of Euler schemes, in particular conditions related to branching behaviors, i.e. when a non endpoint increases its order, or an endpoint increases its order by at least 2. The measure and function A considered will be Lebesgue measure and identity function respectively. Although these restrictions actually lead to a loss of generality, the arguments presented analyze the main properties leading to such topology changing, and give a good explanation of the causes leading to such behavior.

Definition 4.1.15. *Given a domain $\Omega \subseteq \mathbb{R}^2$, an initial datum Σ_0 a time step $\varepsilon > 0$, consider a sequence $\{w(k)\}_{k=0}^\infty$ solution of (4.0.4), with time step $\varepsilon > 0$, measure $\mu := \mathcal{L}^2$ and function $A := \text{id}$. Then we say $w(k)$ exhibits a branching behavior if*

- *there exists a non endpoint $y \in w(k-1)$ such that $\text{ord}_y w(k-1) < \text{ord}_y w(k)$,*

or

- *there exists an endpoint $y \in w(k-1)$ such that $\text{ord}_y w(k-1) \leq \text{ord}_y w(k) - 2$.*

Thus an easy interpretation (valid when all points have finite order) branching behavior appears if a non endpoint increases its order, or an endpoint increases its order from 1 to at least 3.

The next result is a crucial estimate (from below) for the gain for the average distance functional in configurations containing points with non negligible Voronoi cell.

Proposition 4.1.16. *Given a domain Ω , let $\Sigma \in \mathcal{A}(\Omega)$, with a point P satisfying:*

- (α_3) *there exists $\xi > 0$ such that $\Sigma \cap B(P, \xi)$ is contained in the circular sector with center P and arc γ , with length strictly less than $\pi\xi$.*

Then it holds:

- (1) *there exist $\rho > 0$ and $\theta > 0$ and a isosceles triangle $T' \subset V(P)$ with a vertex in P , two sides with length ρ and angle $\widehat{X_1 P X_2}$ measuring θ not intersecting $\Sigma \setminus \{P\}$,*

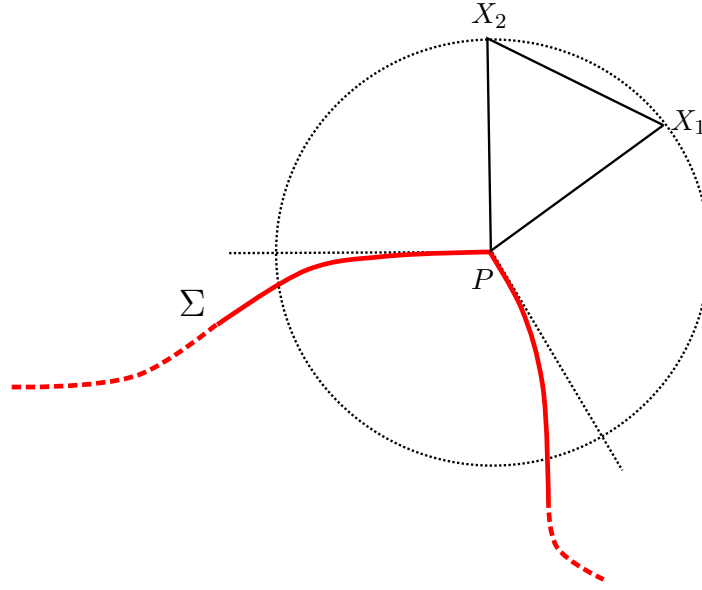


Figure 4.1.2: Condition (α_3) guarantees the existence of such triangle $X_1PX_2 \subseteq V(P)$.

(2) there exists $\varepsilon_0 > 0$ such for any $\varepsilon < \varepsilon_0$ adding a segment λ_ε in P , with $\mathcal{H}^1(\lambda_\varepsilon) = \varepsilon$ in yields

$$F_{\mathcal{L}^2, id}(\Sigma) - F_{\mathcal{L}^2, id}(\Sigma \cup \lambda_\varepsilon) \geq K\varepsilon,$$

where $K > 0$ is a constant not depending on ε .

Proof. For statement (1) (Figure 4.1.2 is a schematic representation) is a simple consequence of the fact that by construction, triangle X_1PX_2 , with $X_1P = X_2P = \rho/2$, is contained in $V(P)$.

Now pass to statement (2): choose a small $\varepsilon > 0$, adding a the segment $\lambda_\varepsilon := \{(1-t)P + tP^* : t \in [0, \varepsilon]\}$ (P^* is the projection of P on X_1X_2 , see Figure 4.1.2), by direct computation any point $z \in X_1PX_2$ with distance at least $\rho/4$ from P satisfies

$$\text{dist}_\Omega(z, \Sigma) - \text{dist}_\Omega(z, \Sigma \cup \lambda_\varepsilon) \geq \text{dist}_\Omega(z, P) - (\text{dist}_\Omega(z, P)^2 + \varepsilon^2 - 2\varepsilon \cos(\frac{\widehat{X_1PX_2}}{2})) = K(\rho, \widehat{X_1PX_2})\varepsilon \quad (4.1.17)$$

Therefore

$$F_{\mathcal{L}^2, id}(\Sigma) - F_{\mathcal{L}^2, id}(\Sigma \cup \lambda_\varepsilon) \geq K(\rho, \widehat{X_1PX_2})\varepsilon \mathcal{L}^2(X_1PX_2 \setminus B(P, \rho/4)),$$

and the proof is complete. $\square$

Theorem 4.1.17. *Given a domain Ω , let $\Sigma_0 \in \mathcal{A}(\Omega)$ be a generic element, T a positive time and $\varepsilon > 0$ a positive time step, and consider the Euler scheme*

$$\begin{cases} w(0) := \Sigma_0 \\ w(k) \in \operatorname{argmin}_{\mathcal{H}^1(\mathcal{S}') \leq \mathcal{H}^1(\Sigma_0) + k\varepsilon, w(k-1) \subseteq \mathcal{S}'} F_{\mathcal{L}^2, id}(\mathcal{S}') \end{cases}$$

in the time interval $[0, T]$. Suppose that there exist $P_0 \in \Sigma_0$ verifying condition (α_2) of Proposition 4.1.16, and suppose there exists $\eta > 0$ such that $B(P_0, \eta) \cap (w(k) \setminus w(0)) = \emptyset$ for any k . Then there is an upper bound $T_{\max}^\varepsilon$ such that for any $T > T_{\max}^\varepsilon$, a branching behavior is necessary.

Proof. Applying Proposition 4.1.16 to P_0 , there exists a constant $K(P_0) > 0$ (not depending on ε) such that for any j

$$\min_{\mathcal{H}^1(\mathcal{S}') \leq \mathcal{H}^1(w(j-1)) + \varepsilon, w(j-1) \subseteq \mathcal{S}'} F_{\mathcal{L}^2, id}(\mathcal{S}') \leq F_{\mathcal{L}^2, id}(w(j-1)) - K(P_0)\varepsilon,$$

as this gain is achieved by simply adding a segment $Seg_\varepsilon \subset T_P$ ($\mathcal{H}^1(Seg_\varepsilon) = \varepsilon$) along the bisector of $\hat{P}_0$, as in the proof of Proposition 4.1.16, which would imply a branching behavior.

If this is avoided, then for any k , $w(k)$ must be obtained from $w(d-1)$ by adding length at endpoints of $w(k-1)$, and it must hold

$$F_{\mathcal{L}^2, id}(w(k)) \leq F_{\mathcal{L}^2, id}(w(k-1)) - K(P_0)\varepsilon \quad \forall k = 1, \dots, \left\lfloor \frac{T}{\varepsilon} \right\rfloor,$$

which leads to

$$F_{\mathcal{L}^2, id}(w(k)) \leq F_{\mathcal{L}^2, id}(w(0)) - kK(P_0)\varepsilon \quad \forall k = 1, \dots, \left\lfloor \frac{T}{\varepsilon} \right\rfloor$$

and finally, for $k = \left\lfloor \frac{T}{\varepsilon} \right\rfloor$,

$$F_{\mathcal{L}^2, id}\left(\left\lfloor \frac{T}{\varepsilon} \right\rfloor\right) \leq F_{\mathcal{L}^2, id}(w(0)) - \left\lfloor \frac{T}{\varepsilon} \right\rfloor K(P_0)\varepsilon.$$

As $\frac{T}{\varepsilon} - 1 \leq \left\lfloor \frac{T}{\varepsilon} \right\rfloor \leq \frac{T}{\varepsilon}$, this leads to

$$0 \leq F_{\mathcal{L}^2, id}\left(\left\lfloor \frac{T}{\varepsilon} \right\rfloor\right) \leq F_{\mathcal{L}^2, id}(w(0)) - (T - \varepsilon)K(P_0),$$

which forces

$$T \leq \varepsilon + \frac{F_{\mathcal{L}^2, id}(\Sigma_0)}{K(P_0)}$$

and the choice $T_{\max}^\varepsilon := \varepsilon + \frac{F_{\mathcal{L}^2, id}(\Sigma_0)}{K(P_0)}$ completes the proof. $\square$

Note that the dependency of $T_{\max}^\varepsilon$ on ε is very weak, and can be easily removed by considering only ε sufficiently small, and replacing $T_{\max}^\varepsilon$ with

$$T_{\max} := 1 + \frac{F_{\mathcal{L}^2, id}(\Sigma_0)}{K(P_0)}.$$

Next we present an application of this result to determine an upper bound estimate for the branching behavior, in a discrete irreversible evolution scheme. Some preliminary results are useful to obtain sharper estimates.

Lemma 4.1.18. *Given a domain Ω , an element $\Sigma_1 \in \mathcal{A}(\Omega)$, and suppose that there exists $Q \in \Omega$ and $R > 0$ such that the ball $B(Q, R) \cap \Sigma_1 = \emptyset$. Then*

$$F_{\mathcal{L}^2, id}(\Sigma_1) \geq \frac{4\pi R^3}{27}.$$

Proof. As $B(Q, R) \cap \Sigma_1 = \emptyset$, for any $r < R$ all points $x \in B(Q, r)$ verify $\text{dist}_\Omega(x, \Sigma_1) \geq R - r$, so

$$F_{\mathcal{L}^2, id}(\Sigma_1) = \int_{\Omega} \text{dist}_\Omega(x, \Sigma_1) dx \geq \int_{B(Q, r)} \text{dist}_\Omega(x, \Sigma_1) dx \geq (R - r)\pi r^2.$$

Differentiating the expression $(R - r)\pi r^2$, its maximum value is attained by $r = \frac{2}{3}R$, which corresponds to

$$F_{\mathcal{L}^2, id}(\Sigma_1) \geq \frac{4\pi}{27}R^3$$

and the proof is complete. $\square$

Lemma 4.1.19. *Given a domain Ω , an element $\Sigma_2 \in \mathcal{A}(\Omega)$, a point $Q' \in \Sigma_2$ and suppose that its Voronoi cell $V(Q')$ has $\mathcal{L}^2(V(Q')) > 0$. Then there exists $\bar{Q} \in \Omega$ such that $B(\bar{Q}, \frac{1}{2}\text{diam}(V(Q'))) \cap \Sigma_2 = \emptyset$.*

Proof. From $V(Q') \subseteq B(Q', \text{diam}(V(Q')))$ it follows $\mathcal{L}^2(V(Q')) \leq \frac{\pi}{4}\text{diam}(V(Q'))^2$. Let be $X_1, X_2 \in V(Q')$ points such that $\text{dist}_\Omega(X_1, X_2) = \text{diam}(V(Q'))$:

$$\text{dist}_\Omega(X_1, X_2) = \text{diam}(V(Q')) \leq \text{dist}_\Omega(X_1, Q') + \text{dist}_\Omega(Q', X_2)$$

so $\min\{\text{dist}_\Omega(X_1, Q'), \text{dist}_\Omega(Q', X_2)\} \geq \frac{1}{2}\text{diam}(V(Q'))$.

Assume that $\text{dist}_\Omega(X_1, Q') \geq \frac{1}{2}\text{diam}(V(Q'))$: $X_1 \in V(Q')$ implies that for any $s < \frac{1}{2}\text{diam}(V(Q'))$, $B(X_1, s) \cap \Sigma_2 = \emptyset$ to avoid $\text{dist}_\Omega(X_1, B(X_1, s) \cap \Sigma_2) \leq s < \frac{1}{2}\text{diam}(V(Q'))$.

Choose $\bar{Q} := X_1$, and considering that $\text{diam}(V(Q')) \geq \sqrt{\frac{4}{\pi}|V(Q')|}$, the proof is complete. $\square$

Consider the following configuration: given a domain Ω , let Σ_0^{dat} be the initial datum, and suppose there exist

- a closed injective path $\gamma^* : [0, 1] \rightarrow \Omega$ such that $\gamma^*([0, 1]) \subseteq \Sigma_0^{dat}$: the domain Ω is now divided in two regions, Ω^+ and Ω^- with $\Omega = \Omega^+ \cup \Omega^-$ (they are the two connected components of $\Omega \setminus \gamma^*([0, 1])$, and correspond to the “interior” and the “exterior” part of $\gamma^*([0, 1])$ – the order is not relevant – given by the Jordan Curve Theorem);

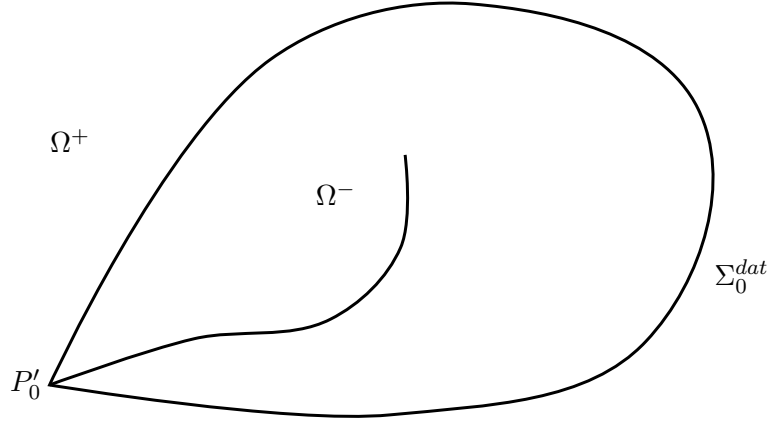


Figure 4.1.3: This is an example of possible set Σ_0^{dat} ; in this case the only point verifying the required conditions is P'_0 , and Ω^+ is the part “outside” the curve, while Ω^- is the part “inside” the curve.

- $P'_0 \in \Sigma_0^{dat}$ and a triangle $T_{P'_0} \subset V(P'_0) \cap B(P'_0, \xi')$ with $|T_{P'_0} \cap \Omega^+| > 0$, and $ext(S_0^{dat}) \subset \Omega^-$.

In the rest of this subsection suppose that Ω^- is large enough (both in diameter and in measure) so that all computations can be done without considering constraints imposed by $\text{diam}(\Omega^-)$, $\mathcal{L}^2(\Omega^-)$. This because such constraints, in the following discussion, can only diminish the mass projecting on endpoints of $w(k)$, thus can only decrease the time at which a branching behavior becomes necessary, and we are looking for an upper bound for such time.

Consider

$$\begin{cases} w(0) = w(0) := \Sigma_0^{dat} \\ w(k) \in \text{argmin}_{\mathcal{H}^1(\mathcal{X}'') \leq \mathcal{H}^1(\Sigma_0^{dat}) + k\varepsilon'} F(\mathcal{X}'') \\ w(k) \supseteq w(k-1), \end{cases}$$

where ε' is a given (small) parameter, and

$$u_{\varepsilon'} : [0, T] \longrightarrow A, u_{\varepsilon'}(t) := w\left(\left\lceil \frac{t}{\varepsilon'} \right\rceil\right).$$

The main estimate here is Theorem 4.1.20.

Notice first that the only way to exclude a branching behavior is that the difference $w(k) \setminus w(k-1)$ is always contained in Ω^- .

The notations introduced (except mute counters like k and n) will have the same meaning in the following of this subsection. There exists a positive constant $K(P'_0)$ (depending only on geometric quantities, not on ε' and estimable with the same argument found in Proposition 4.1.16) such that for any k

$$\min_{\mathcal{H}^1(\mathcal{X}'') \leq w(k-1) + k\varepsilon', w(k-1) \subset \mathcal{X}''} F_{\mathcal{L}^2, id}(\mathcal{X}'') \leq F_{\mathcal{L}^2, id}(w(k-1)) - K(P'_0)\varepsilon',$$

thus

$$F_{\mathcal{L}^2, id}(w(k)) \leq F_{\mathcal{L}^2, id}(w(0)) - kK(P'_0)\varepsilon' \quad (4.1.18)$$

i.e. $\forall t \in [0, T]$

$$F_{\mathcal{L}^2, id}(u_{\varepsilon'}(t)) := F_{\mathcal{L}^2, id}(w(\left\lfloor \frac{t}{\varepsilon'} \right\rfloor)) \leq F_{\mathcal{L}^2, id}(\Sigma_0^{dat}) - \left\lfloor \frac{t}{\varepsilon'} \right\rfloor K(P'_0)\varepsilon' \leq F_{\mathcal{L}^2, id}(\Sigma_0^{dat}) - (\frac{t}{\varepsilon'} - 1)K(P'_0)\varepsilon'.$$

But obviously

$$F_{\mathcal{L}^2, id}(u_{\varepsilon'}(t)) \geq 0$$

and combining the above inequalities gives

$$0 \leq F_{\mathcal{L}^2, id}(u_{\varepsilon'}(t)) \leq F_{\mathcal{L}^2, id}(\Sigma_0^{dat}) - (\frac{t}{\varepsilon'} - 1)K(P'_0)\varepsilon'$$

which forces

$$F_{\mathcal{L}^2, id}(\Sigma_0^{dat}) \geq (t - 1)K(P'_0).$$

Theorem 4.1.20. *For this configuration, with the above notations, an upper bound for the branching time is given by*

$$T_{\max} := 1 + \frac{F_{\mathcal{L}^2, id}(\Sigma_0^{dat})}{K(P'_0)}.$$

Notice that the partition $\Omega^+ \cup \Omega^-$ is crucial as it is not possible to “pass” from one region to another without intersecting $\gamma([0, 1])$, so it prevents $u(t)$ from visiting $T(P'_0) \cap \Omega^+$ without exhibiting branching behaviors.

A very similar result is easily obtained for the penalized case:

Theorem 4.1.21. *Under the same configuration consider*

$$\begin{cases} w(0) := \Sigma_0^{dat} \\ w(k) \in \operatorname{argmin}_{\mathcal{H}^1(\mathcal{X}'') \leq \mathcal{H}^1(\Sigma_0^{dat}) + k\varepsilon'} F_{\mathcal{L}^2, id}(\mathcal{X}'') + \lambda(\mathcal{X}'' \setminus w(k-1)) \\ w(k) \supseteq w(k-1), \end{cases}$$

and define

$$u_{\varepsilon'} : [0, T] \longrightarrow A, u_{\varepsilon'}(t) := w\left(\left\lfloor \frac{t}{\varepsilon'} \right\rfloor\right).$$

Then there exists $\lambda_0 > 0$ such that for any $\lambda < \lambda_0$ there exists an upper bound $T_{\max}$ with dependence on λ for the branching time.

For the penalized problem an upper bound on the number of endpoints can be given too. Consider the evolution

$$\begin{cases} w(0) = w(0) := \Sigma_0^{dat} \\ w(k) \in \operatorname{argmin}_{\mathcal{H}^1(\mathcal{X}'') \leq \mathcal{H}^1(\Sigma_0^{dat}) + k\varepsilon'} F_{\mathcal{L}^2, id}(\mathcal{X}'') + \lambda(\mathcal{X}'' \setminus w(k-1)) \\ w(k) \supseteq w(k-1), \end{cases}$$

where $\lambda > 0$ is a given parameter. Choose arbitrary $k \in \mathbb{N}$ and endpoint $P \in w(k) \setminus w(k-1)$. Denote with $TM(P, w(k))$ the total mass projecting on P ; it is known (see for instance [14], [16] and [17]) that there exists $r_0 > 0$ such that $w(k) \setminus B(P, r)$ is connected for any $r \in (0, r_0)$, and obviously

$$\text{dist}_\Omega(z, w(k)) \leq \text{dist}_\Omega(z, w(k) \setminus B(P, r)) + r$$

for any point z , thus using the arbitrariness of r

$$F(w(k) \setminus B(P, r)) - F(w(k)) \leq r TM(P, w(k))$$

which combined with

$$F(w(1)) + \lambda(w(k) \setminus w(k-1)) \leq F(w(k) \setminus B(P, r)) + \lambda(w(k) \setminus B(P, r) \setminus w(k-1))$$

yields $TM(P, w(k)) \geq \lambda$. Thus each “new” endpoint (i.e. endpoint of $w(k)$ not present in $w(k-1)$) has an uniform positive lower bound for the mass projecting on it, and the total number of such “new” endpoints (i.e the number of endpoints present in $w(k)$ for some k but not present in Σ_0^{dat}) cannot exceed $\mathcal{L}^2(\Omega)/\lambda$.

4.2 Limit sets

In this Section our goal is to analyze limit sets of quasi static evolutions related to the average distance functional. The first problem to deal is existence of such limit sets: indeed given a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, the space $(\mathcal{A}(\Omega), d)$ where $d(\cdot, \cdot) := \mathcal{H}^1(\cdot \Delta \cdot)$ is not sequentially compact, while $(\mathcal{A}_l(\Omega), d)$ is sequentially compact for any l . Thus a natural way to retrieve sequential compactness is to restrict the evolution to sets with limited length, but a priori such choice can cause loss of generality. Fortunately for the average distance functional this does not happen. Indeed consider arbitrary domain $\Omega \subseteq \mathbb{R}^n$, measure $\mu \in L^1$, function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$, and the evolution

$$\begin{cases} w(0) := \Sigma_0 \\ w(k+1) \in \text{argmin}_{\mathcal{A}(\Omega)} F_{\mu, A}(\cdot) + \lambda \mathcal{H}^1(\cdot \Delta w(k)) \end{cases}$$

with $\lambda > 0$ a given constant. From minimality properties of $w(j+1)$ it follows

$$F_{\mu, A}(w(j)) \geq F_{\mu, A}(w(j+1)) + \lambda \mathcal{H}^1(w(j) \Delta w(j+1)),$$

and thus

$$F_{\mu, A}(w(j_1)) - F_{\mu, A}(w(j_2)) \geq \lambda \sum_{i=j_1}^{j_2-1} \mathcal{H}^1(w(i) \Delta w(i+1)).$$

Choosing $j_1 = 0$ it follows

$$F_{\mu, A}(\Sigma_0) - F_{\mu, A}(w(j_2)) \geq \lambda \sum_{i=0}^{j_2-1} \mathcal{H}^1(w(i) \Delta w(i+1)) \geq \lambda \mathcal{H}^1(\Sigma_0 \Delta w(j_2)),$$

yielding

$$\mathcal{H}^1(\Sigma_0 \Delta w(j_2)) \leq \frac{F_{\mu,A}(\Sigma_0)}{\lambda},$$

effectively providing an upper bound for $\mathcal{H}^1(w(j_2))$ independent of j_2 (but depending on λ). Thus sequential compactness is proven true for this kind of evolution.

The following result holds:

Proposition 4.2.1. *Fix a domain $\Omega \subseteq \mathbb{R}^n$, a measure $\mu \in L^1$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$, and consider the evolution*

$$\begin{cases} w(0) := \Sigma_0 \\ w(k+1) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \lambda \mathcal{H}^1(\cdot \Delta w(k)) \end{cases}$$

with $\lambda > 0$ a given constant. Then it holds:

- if $|\nabla F_{\mu,A}|(\Sigma_0) \geq \lambda$, then for any limit set Σ^* it holds $|\nabla F_{\mu,A}|(\Sigma^*) \leq \lambda$,

Proof. Assume for the sake of contradiction that there exists a limit set Σ^* , a constant $\eta > 0$ and a sequence y_k such that

$$F_{\mu,A}(\Sigma^*) - F_{\mu,A}(y_k) \geq (\lambda + \eta) \mathcal{H}^1(\Sigma^* \Delta y_k) \quad \forall k.$$

Choose $w(h)$ with h large, then it holds

$$F_{\mu,A}(w(h+1)) + \lambda \mathcal{H}^1(w(h) \Delta w(h+1)) \leq F_{\mu,A}(y_s) + \lambda \mathcal{H}^1(w(h) \Delta y_s) \quad \forall s.$$

As

$$F_{\mu,A}(y_k) \leq F_{\mu,A}(\Sigma^*) - (\lambda + \eta) \mathcal{H}^1(y_k \Delta \Sigma^*) \quad \forall k,$$

this yields

$$\begin{aligned} F_{\mu,A}(w(h+1)) + \lambda \mathcal{H}^1(w(h) \Delta w(h+1)) &\leq F_{\mu,A}(y_s) + \lambda \mathcal{H}^1(w(h) \Delta y_s) \\ &\leq F_{\mu,A}(\Sigma^*) - (\lambda + \eta) \mathcal{H}^1(y_s \Delta \Sigma^*) + \lambda (\mathcal{H}^1(w(h) \Delta \Sigma^*) + \mathcal{H}^1(y_s \Delta \Sigma^*)) \\ &= F_{\mu,A}(\Sigma^*) - \eta \mathcal{H}^1(y_s \Delta \Sigma^*) + \lambda \mathcal{H}^1(w(h) \Delta \Sigma^*), \end{aligned}$$

or equivalently

$$\eta \mathcal{H}^1(y_s \Delta \Sigma^*) \leq |F_{\mu,A}(w(h+1)) - F_{\mu,A}(\Sigma^*)| + \lambda |\mathcal{H}^1(w(h) \Delta \Sigma^*) - \mathcal{H}^1(w(h) \Delta w(h+1))| \quad \forall s,$$

which is false once h is chosen sufficiently large as the right hand side goes to 0 as $h \rightarrow \infty$. $\square$

Limit sets of quasi static evolution inherits Ahlfors regularity, but the proof is surprisingly difficult. This is being studied in a work in progress by the author. Here is an idea of the proof.

Proposition 4.2.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a given domain, an Ahlfors regular initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$, and consider*

$$\begin{cases} w(0) := S_0 \\ w(n+1) \in \operatorname{argmin}_{\mathcal{X} \in \mathcal{A}(\Omega)} F_{\mu, A}(\mathcal{X}) + \lambda \mathcal{H}^1(\mathcal{X} \Delta w(n)) \end{cases}.$$

Then any accumulation point of $\{w(k)\}_{k \in \mathbb{N}}$ belonging to $\mathcal{A}(\Omega)$ is Ahlfors regular.

Proof. (Sketch) The lower bound estimate follow from the fact of being in $\mathcal{A}(\Omega)$. For the upper bound estimate, fix arbitrary, $k \in \mathbb{N}$ and consider a point $P \in w(k)$. Denote with $\xi(1, \rho) := ((w(k+1) \setminus w(k)) \cap B(P, \rho))$, i.e. the set added in $B(P, \rho)$ at step $k+1$, which can be empty. Denoting with $w'_1 := w(k+1) \setminus \xi(1, \rho)$. It holds

$$\mathcal{H}^1(w'_1 \Delta w(k)) = \mathcal{H}^1(w(k+1) \Delta w(k)) - \mathcal{H}^1(\xi(1, \rho))$$

as $\xi(1, \rho)$ does not intersect $w(k)$. Thus it must hold

$$F(w'_1) - F(w(k+1)) \geq \lambda \mathcal{H}^1(\xi(1, \rho)),$$

as the contrary would contradict the optimality of $w(k+1)$. This argument can be repeated for all steps: indeed denote with $\xi(j, \rho) := ((w(k+j) \setminus w(k+j-1)) \cap B(P, \rho))$ and $w'_j := w(k+j) \setminus \xi(j, \rho)$. It holds

$$\mathcal{H}^1(w'_j \Delta w(k+j-1)) = \mathcal{H}^1(w(k+j) \Delta w(k+j-1)) - \mathcal{H}^1(\xi(j, \rho))$$

and

$$F(w'_j) - F(w(k+j)) \geq \lambda \mathcal{H}^1(\xi(j, \rho)).$$

Summing over j , and considering that $\xi(j, \rho) \subseteq B(P, \rho)$ for any j , yields

$$\lambda \sum_{j=1}^{\infty} \mathcal{H}^1(\xi(j, \rho)) \leq \rho \mathcal{L}^2(\Omega)$$

which implies

$$\sum_{j=1}^{\infty} \mathcal{H}^1(\xi(j, \rho)) \leq \rho \mathcal{L}^2(\Omega) / \lambda.$$

Using the arbitrariness of k and $P \in w(k)$, this is sufficient to prove Ahlfors regularity. $\square$

Notice that this is only an idea of the proof: indeed here we have omitted several details, in particular the discussion about connectedness of the competitor w'_j , as a priori it cannot be assured that removing $\xi(j, \rho)$ will preserve connectedness.

Chapter 5

Gradient flow evolutions

In Chapter 4 we have analyzed the quasi static evolution related to the average distance functional, mainly focusing on geometric and analytic properties. Another important class is the gradient flow evolution, discussed in the abstract metric context in Chapter 2.

This chapter's structure is similar to that of Chapter 4, with main results about geometric and regularity properties presented in Section 4.1, and some side notes in Section 4.2.

The discrete form of the gradient flow will be

$$\begin{cases} w(0) := \Sigma_0 \in \mathcal{A}(\Omega) \\ w(n+1) \in \operatorname{argmin} \int_{\Omega} \operatorname{dist}(x, \cdot) d\mu + \frac{1}{2\tau} d^2(\cdot, w(n)) \end{cases}$$

where Ω, μ are respectively a given domain and measure, Σ_0 is a given initial datum, and $\tau > 0$ is a fixed (small) parameter. Note that here we have not specified which distance d is considered. This will be determined later (and will be the $\mathcal{H}^1$ -measure of the symmetric difference), and requires some care to guarantee the well-posedness of such problem.

The two sets of assumptions (Assumptions 2.2.8 and 2.2.13) must be checked.

Given a domain $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$), a measure μ and a function $A : [0, \operatorname{diam} \Omega] \rightarrow [0, \infty)$, Assumption 2.2.8 from Chapter 2 about existence of minimizers becomes:

Assumption 5.0.3. *Let $(\mathcal{A}(\Omega), d)$ be the metric space, and $F_{\mu, A}$ a nonnegative functional continuous with respect to d . Assume that there exists $\bar{\tau}$ such that for every $\tau \in [0, \bar{\tau}]$ and $\Sigma_0 \in \overline{D(F_{\mu, A})}$ the map*

$$\Sigma \mapsto F_{\mu, A}(\Sigma) + \frac{d(\Sigma, \Sigma_0)^2}{2\tau}$$

has at least a minimum.

We leave the distance d undefined for now, as the exact choice will be made later, but suppose that $(\mathcal{A}(\Omega), d)$ is sequentially compact (which will be the case for all cases considered in this Chapter). The first goal is to check if Assumption 5.0.3 holds for the average distance functional: fix the domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, the measure $\mu \ll \mathcal{L}^n$ and the function $A : [0, \operatorname{diam} \Omega] \rightarrow [0, \infty)$. Given arbitrary $\tau \in (0, \infty)$, $\Sigma^* \in \mathcal{A}(\Omega)$, consider the map

$$\Sigma \mapsto F_{\mu,A}(\Sigma) + \frac{d(\Sigma, \Sigma^*)^2}{2\tau}.$$

Consider a minimizing sequence $\{\Sigma_k\}_{k \in \mathbb{N}} \subseteq \mathcal{A}(\Omega)$ converging to $\Sigma_\infty \in \mathcal{A}(\Omega)$ (it has been assumed that $(\mathcal{A}(\Omega), d)$ is sequentially compact, and for the sake of simplicity we avoided renaming indexes) with respect to d : this implies

$$\text{dist}_\Omega(z, \Sigma_\infty) = \lim_{k \rightarrow \infty} \text{dist}_\Omega(z, \Sigma_k)$$

thus

$$F_{\mu,A}(\Sigma_\infty) = \lim_{k \rightarrow \infty} F_{\mu,A}(\Sigma_k);$$

obviously (from triangular inequality)

$$d(\Sigma_k, \Sigma^*) \leq d(\Sigma_\infty, \Sigma^*) + d(\Sigma_k, \Sigma_\infty)$$

$$d(\Sigma_\infty, \Sigma^*) \leq d(\Sigma_k, \Sigma^*) + d(\Sigma_k, \Sigma_\infty)$$

implying $d(\Sigma_k, \Sigma^*) \rightarrow d(\Sigma_\infty, \Sigma^*)$ and

$$\frac{d(\Sigma_\infty, \Sigma^*)^2}{2\tau} = \lim_{k \rightarrow \infty} \frac{d(\Sigma_k, \Sigma^*)^2}{2\tau}.$$

Thus

$$F_{\mu,A}(\Sigma_\infty) + \frac{d(\Sigma_\infty, \Sigma^*)^2}{2\tau} = \lim_{k \rightarrow \infty} F_{\mu,A}(\Sigma_k) + \frac{d(\Sigma_k, \Sigma^*)^2}{2\tau},$$

and Assumption 5.0.3 is proven valid. From arbitrariness of τ we conclude that interval $[0, \bar{\tau}]$ (using notations from Assumption 5.0.3) can be chosen $[0, \infty)$.

5.1 Piecewise constant time discretization

In this Section we consider the discrete gradient flow evolution (see Definition 2.2.4) related to the average distance functional. The distance d appearing in Assumption 5.0.3 has still to be determined yet: in our context, where the main object of analysis are Hausdorff one dimensional sets with finite length, the natural distances are $d_{\mathcal{H}}(\cdot, \cdot)$ and $\mathcal{H}^1(\cdot \Delta \cdot)$. The former will be proven unsuitable in the following, as Assumption 5.0.3 does not hold in $(\mathcal{A}(\Omega), d_{\mathcal{H}})$, where sequential compactness is not true.

Given a domain $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$), a measure $\mu \in L^1(\Omega)$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$, a parameter $\tau > 0$, consider the following discrete evolution

$$\begin{cases} w(0) := \Sigma_0 \\ w(k) \in \text{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, w(k-1))^2}{2\tau} \end{cases} \quad (5.1.1)$$

The first problem is existence of such minimum, which can be not true in the general case. Consider for instance the following example: $\Omega := \overline{B(0, 1)} \subseteq \mathbb{R}^2$, $\mu := \mathcal{L}_{|\Omega}^2$, $A := id$, $\Sigma_0 := \{(0, 0)\}$, $\tau \in (0, 1/2)$ an arbitrary value, and $d := d_{\mathcal{H}}$.

By definition a set $w(1)$ must satisfy

$$w(1) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu, A}(\cdot) + \frac{d_{\mathcal{H}}(\cdot, \Sigma_0)^2}{2\tau},$$

thus unless $w(1) = w(0)$ (which would lead $w(0) = w(1) = \dots$, i.e. there is no evolution at all), it must hold $d_{\mathcal{H}}(w(1), w(0)) > 0$.

It is clear that $w(1) \subseteq \overline{B((0, 0), d_{\mathcal{H}}(w(1), w(0)))}$, and

$$\int_{\Omega} \operatorname{dist}_{\Omega}(z, w(1)) dz > \int_{\Omega} \operatorname{dist}_{\Omega}(x, \overline{B((0, 0), d_{\mathcal{H}}(w(1), w(0)))}) dz.$$

Thus let $\{\Sigma'_k\}_{k \in \mathbb{N}} \subseteq \mathcal{A}(\Omega)$ a sequence of elements satisfying

- $\Sigma'_k \subseteq \overline{B((0, 0), d_{\mathcal{H}}(w(1), w(0)))}$ for any k ,
- $\Sigma'_k \supseteq w(1)$ for any k ,
- Σ'_k is strictly increasing (i.e. $\Sigma'_k \supseteq \Sigma'_{k-1}$ and $\mathcal{H}^1(\Sigma'_k \setminus \Sigma'_{k-1}) > 0$ for any k),
- for $k \rightarrow \infty$ the sequence converges to a Hausdorff one dimensional set Σ' dense in $\overline{B((0, 0), d_{\mathcal{H}}(w(1), w(0)))}$.

Clearly

$$\{F_{\mu, A}(\Sigma'_k)\} \downarrow \int_{\Omega} \operatorname{dist}_{\Omega}(z, \overline{B((0, 0), d_{\mathcal{H}}(w(1), w(0)))}) dz,$$

thus existence of minimizers does not hold.

Therefore the distance considered in this Chapter will be

$$d(\mathcal{X}_1, \mathcal{X}_2) := \mathcal{H}^1(\mathcal{X}_1 \Delta \mathcal{X}_2),$$

with Δ denoting the symmetric difference. From here, unless otherwise stated, the notation $d(\cdot, \cdot)$ will always refer to this specific distance.

Then the evolutions will have form

$$\begin{cases} w(0) := \Sigma_0 \\ w(k) \in \operatorname{argmin}_{\mathcal{X} \in \mathcal{A}(\Omega)} F_{\mu, A}(\mathcal{X}) + \frac{\mathcal{H}^1(\mathcal{X} \Delta w(k-1))^2}{2\tau} \end{cases}$$

Several properties from the discrete quasi static case can be retrieved:

- if Σ_0 does not contain loops, then $w(k)$ does not contain loops for any k ,
- in two dimension case, if the measure $\mu \in L^p(\Omega)$ for some $p > 4/3$ and Σ_0 contains a finite number of endpoints, then $w(k)$ has a finite number of endpoints for any k ,

- in two dimension case, if the measure $\mu \in L^p(\Omega)$ for some $p > 4/3$ and Σ_0 contains only points with order at most 3, then $w(k)$ contains points with order at most 3 for any k ,
- if the measure $\mu \in L^p(\Omega)$ for some $p > n/(n-1)$ (or $p > 4/3$ in two dimension case) and Σ_0 is Ahlfors regular, then $w(k)$ is Ahlfors regular for any k .

5.1.1 Geometric properties

In this subsection our goal is to analyze geometric and regularity properties of solutions of (5.1.1), in the discrete case (i.e. with $\tau > 0$ being given). Apart from the absence of loops, the two dimension case is significantly simpler than higher dimension cases, and will be discussed separately.

Theorem 5.1.1. *Given a domain $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$), a measure $\mu \in L^1(\Omega)$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$ not containing loops, a time step $\tau > 0$, consider the following evolution*

$$\begin{cases} w(0) := \Sigma_0 \\ w(k) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, w(k-1))^2}{2\tau} \end{cases} \quad (5.1.2)$$

Then for any k the set $w(k)$ does not contain loops.

Proof. The proof is done by induction: for $k = 0$ the set $w(0) = \Sigma_0$ does not contain loops. Suppose that for some k , $w(k-1)$ does not contain loops, but $w(k)$ contains a loop $E \subseteq w(k)$.

Choose $r > 0$ as in the proof of Theorem 3.4.7: from Lemma 3.4.5 there exists $x' \in E$ with $\frac{\mu(B(x',t))}{t} \rightarrow 0$ as $t \rightarrow 0^+$, and $\rho \in (0, r)$ such that there exists $\Sigma' \in \mathcal{A}(\Omega)$ satisfying

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(w(k)) - \rho/4, \quad F_{\mu,A}(\Sigma') \leq F_{\mu,A}(w(k)) + 64n^{3/2}\Lambda\rho^2 \frac{\mu(B(x', 64n^{3/2}\rho))}{64n^{3/2}\rho} \quad (5.1.3)$$

where Λ denotes the Lipschitz constant of A . Moreover, it holds $w(k)\Delta\Sigma' \subseteq B(x', 32n\rho)$.

Then the proof continues as in Theorem 3.4.7: for any $\varepsilon > 0$ sufficiently small, applying Lemma 3.4.6 yields the existence of a competitor $\mathcal{A}(\Omega) \ni \Sigma'' \supseteq \Sigma'$ satisfying

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + 2n\varepsilon, \quad F_{\mu,A}(\Sigma'') \leq F_{\mu,A}(\Sigma') - C\varepsilon^2,$$

with C depending only on geometric quantities. Clearly $\mathcal{H}^1(\Sigma'' \setminus \Sigma') \leq 2n\varepsilon$.

Thus Σ'' satisfies

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(w(k)), \quad F_{\mu,A}(\Sigma'') < F_{\mu,A}(w(k)).$$

Notice that Theorem 3.4.5 states that for $\mathcal{H}^1$ -a.e. $x \in E$ there exists such set Σ' , thus x' can be assumed chosen in $E \cap (w(k) \setminus w(k-1))$, which must contain an open set (in the induced topology) as both $w(k)$ and $w(k-1)$ are compact. Thus for r sufficiently small it holds $B(x', 32n\rho) \cap w(k-1) = \emptyset$ for any $\rho \in (0, r)$. Using $\Sigma' \Delta w(k) \subseteq B(x', 32n\rho)$ yields

$$\mathcal{H}^1(\Sigma' \Delta w(k-1)) \leq \mathcal{H}^1(w(k) \Delta w(k-1)) - \mathcal{H}^1(w(k) \cap B(x', 32n\rho))$$

and considering that $x' \in w(k)$, it holds $\mathcal{H}^1(w(k) \cap B(x', 32n\rho)) \geq 32n\rho$, thus

$$\mathcal{H}^1(\Sigma' \Delta w(k-1)) \leq \mathcal{H}^1(w(k) \Delta w(k-1)) - 32n\rho.$$

Using $\Sigma'' \supseteq \Sigma'$ and $\mathcal{H}^1(\Sigma'' \setminus \Sigma') \leq 2n\varepsilon$ it holds

$$d(\Sigma'', w(k-1)) \leq d(\Sigma', w(k-1)) + 2n\varepsilon.$$

Choose $\varepsilon = \rho$: this yields

$$d(\Sigma'', w(k-1)) \leq d(\Sigma', w(k-1)) + 2n\varepsilon \leq d(w(k), w(k-1)) - 32n\varepsilon + 2n\varepsilon.$$

By hypothesis it holds

$$w(k) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, w(k-1))^2}{2\tau},$$

while for Σ'' it holds

$$F_{\mu,A}(\Sigma'') \leq F_{\mu,A}(\Sigma') - C\varepsilon^2 \leq F_{\mu,A}(w(k)) + 64n^{3/2}\Lambda\varepsilon^2 \frac{\mu(B(x', 64n^{3/2}\varepsilon^2))}{64n^{3/2}\varepsilon^2} - C\varepsilon^2$$

thus $F_{\mu,A}(\Sigma'') < F_{\mu,A}(w(k))$ as $\frac{\mu(B(x', 64n^{3/2}\varepsilon^2))}{64n^{3/2}\varepsilon^2} \rightarrow 0$ for any ε sufficiently small.

Combined with

$$d(\Sigma'', w(k-1)) \leq d(w(k), w(k-1))$$

the minimality property of $w(k)$ is contradicted, concluding the proof. $\square$

Notice that in the proof inductive hypothesis on $w(k-1)$ is crucial as it serves to apply Lemma 3.4.5 and yielding the existence of a point $x' \in w(k) \setminus w(k-1)$ satisfying (5.1.3). This result has important consequences: under such hypothesis, any solution $\{w(k)\}_{k \in \mathbb{N}}$ of (5.1.1) does not contain loops. This implies that for any k the set $w(k)$ has endpoints (and potentially an infinite number of endpoints).

Ahlfors regularity can be extended too, and the proof is quite different for two dimension case and higher dimension case:

Proposition 5.1.2. *Given a domain $\Omega \subseteq \mathbb{R}^2$, a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \operatorname{diam} \Omega] \rightarrow [0, \infty)$, an arbitrary Ahlfors regular set $\Sigma_0 \in \mathcal{A}(\Omega)$, consider the following minimization problem:*

$$\min_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + d(\cdot, \Sigma_0)^2. \quad (5.1.4)$$

Then any solution is Ahlfors regular.

Proof. By hypothesis Σ_0 is Ahlfors regular, thus there exists $c_1, c_2 > 0$ such that

$$c_1 \leq \frac{\mathcal{H}^1(\Sigma_0 \cap B(x, \rho))}{\rho} \leq c_2$$

for any $x \in \Sigma_0, \rho > 0$.

Choose an arbitrary

$$\Sigma_{\text{opt}} \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + d(\cdot, \Sigma_0)^2.$$

The lower bound estimate is obvious, and follows from $\Sigma_{\text{opt}} \in \mathcal{A}(\Omega)$. For the upper bound estimate, consider an arbitrary point $z \in \Sigma_{\text{opt}}$ and suppose there exists ρ^* such that

$$\frac{\mathcal{H}^1(\Sigma_{\text{opt}} \cap B(x, \rho^*))}{\rho^*} = k > 2\pi \vee c_2.$$

For any j define

$$\Sigma' := \Sigma_{\text{opt}} \setminus (\Sigma_{\text{opt}} \cap B(z, \rho^*))$$

and

$$\Sigma'' := \Sigma' \cup \partial B(z, \rho^*).$$

Clearly $|d(\Sigma_{\text{opt}}, \Sigma_0) - d(\Sigma', \Sigma_0)| \leq k\rho^*$, and $|d(\Sigma'', \Sigma_0) - d(\Sigma', \Sigma_0)| \leq 2\pi\rho^*$ thus $|d(\Sigma_{\text{opt}}, \Sigma_0) - d(\Sigma'', \Sigma_0)| \leq (2\pi + k)\rho^*$.

Using Lemma 3.2.7 yields the existence of $\mathcal{A}(\Omega) \ni \Sigma''' \supseteq \Sigma''$ such that

$$\mathcal{H}^1(\Sigma''') \leq \mathcal{H}^1(\Sigma'') + \varepsilon, \quad F_{\mu,A}(\Sigma''') \leq F_{\mu,A}(\Sigma'') - C\varepsilon^{3/2}$$

for any ε sufficiently small, where $C > 0$ is a constant not dependent on ε . Choosing $\varepsilon := \rho^*$ and considering that $\operatorname{dist}_{\Omega}(y, \Sigma''') \leq \operatorname{dist}_{\Omega}(y, \Sigma_{\text{opt}}) + \rho^*$ for $y \in B(x, \rho^*)$, and $\operatorname{dist}_{\Omega}(y, \Sigma''') \leq \operatorname{dist}_{\Omega}(y, \Sigma_{\text{opt}})$ elsewhere, yields

$$\begin{aligned} F_{\mu,A}(\Sigma''') &\leq F_{\mu,A}(\Sigma_{\text{opt}}) + \int_{B(x, \rho^*)} A(\operatorname{dist}_{\Omega}(y, \Sigma_{\text{opt}}) + \rho^*) - A(\operatorname{dist}_{\Omega}(y, \Sigma_{\text{opt}})) d\mu(y) - C\varepsilon^{3/2} \\ &\leq F_{\mu,A}(\Sigma_{\text{opt}}) + K(\rho^*)^{\frac{2}{q}+1} - C(\rho^*)^{3/2} \end{aligned}$$

where $K > 0$ is a constant not dependent on ρ^* , and q is the conjugate exponent of p . Using hypothesis $p > 4/3$ yields $K(\rho^*)^{\frac{2}{q}+1} \ll C(\rho^*)^{3/2}$, i.e. $\lim_{\rho^* \rightarrow 0} K(\rho^*)^{\frac{2}{q}+1}/C(\rho^*)^{3/2} = 0$.

Consider the difference:

$$(F_{\mu,A}(\Sigma_{\text{opt}}) - F_{\mu,A}(\Sigma''')) + (d(\Sigma_{\text{opt}}, \Sigma_0)^2 - d(\Sigma''', \Sigma_0)^2).$$

The term $F_{\mu,A}(\Sigma_{\text{opt}}) - F_{\mu,A}(\Sigma''')$ is comparable with $(\rho^*)^{3/2}$, while $(d(\Sigma_{\text{opt}}, \Sigma_0)^2 - d(\Sigma''', \Sigma_0)^2)$ is comparable with $(\rho^*)^2$, and using arbitrariness of ρ^* concludes the proof. $\square$

Thus the following result follows:

Theorem 5.1.3. *Given a domain $\Omega \subseteq \mathbb{R}^2$, a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \operatorname{diam} \Omega] \rightarrow [0, \infty)$, an Ahlfors regular initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$, and a time step $\tau > 0$, consider the following evolution*

$$\begin{cases} w(0) := \Sigma_0 \\ w(k) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, w(k-1))^2}{2\tau} \end{cases}. \quad (5.1.5)$$

Then for any k the set $w(k)$ is Ahlfors regular.

Proof. The proof is done by induction: $w(0) = \Sigma_0$ is Ahlfors regular by hypothesis, and suppose $w(k-1)$ is Ahlfors regular, but $w(k)$ is not. Using the construction in the proof of Proposition 5.1.2 yields a competitor Σ' satisfying

$$\begin{aligned} F_{\mu,A}(\Sigma') &\leq F_{\mu,A}(w(k)) - C_1 \varepsilon^{3/2}, \\ |d(\Sigma', w(k-1))^2 - d(w(k), w(k-1))^2| &\leq C_2 \varepsilon. \end{aligned}$$

In the difference

$$(F_{\mu,A}(w(k)) - F_{\mu,A}(\Sigma')) + \frac{1}{2\tau}(d(w(k), w(k-1))^2 - d(\Sigma', w(k-1))^2)$$

and choosing ε sufficiently small yields

$$F_{\mu,A}(w(k)) - F_{\mu,A}(\Sigma') = O(\varepsilon^{3/2})$$

and

$$\frac{1}{2\tau}d(w(k), w(k-1))^2 - d(\Sigma', w(k-1))^2 = O(\varepsilon^2).$$

Thus it follows

$$F_{\mu,A}(\Sigma') + \frac{d(\Sigma', w(k-1))^2}{2\tau} < F_{\mu,A}(w(k)) + \frac{d(w(k), w(k-1))^2}{2\tau},$$

contradicting the optimality of $w(k)$. □

The absence of points having order greater than 3 in two dimension case can be extended too:

Theorem 5.1.4. *Given a domain $\Omega \subseteq \mathbb{R}^2$, a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$, and a time step $\tau > 0$, consider the following evolution*

$$\begin{cases} w(0) := \Sigma_0 \\ w(k) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, w(k-1))^2}{2\tau} \end{cases} \quad (5.1.6)$$

Then for any $k \geq 1$ the set $w(k)$ does not contain points with order greater than 3.

Similar to results about Ahlfors regularity, this stems from the more general results too:

Proposition 5.1.5. *Given a domain $\Omega \subseteq \mathbb{R}^2$, a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an arbitrary set $\Sigma_0 \in \mathcal{A}(\Omega)$ not containing points with order greater than 3, consider the following minimization problem:*

$$\min_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + d(\cdot, \Sigma_0)^2. \quad (5.1.7)$$

Then any solution does not contain points with order greater than 3.

Proof. The proof is done by contradiction: suppose the contrary, i.e. there exists

$$\Sigma_{\text{opt}} \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + d(\cdot, \Sigma_0)^2$$

containing a point $x \in \Sigma_{\text{opt}}$ with $\operatorname{ord}_x \Sigma_{\text{opt}} \geq 4$. Thus from Menger n -Beinsatz there exists arcs $\gamma_1, \dots, \gamma_4$ starting in x and mutually disjoint outside x .

Using a construction similar to that used for the proof of Lemma 3.2.16, choose $r > 0$ (the exact value is not relevant), and define $x_i := \gamma_i \cap \partial B(x, r)$ for $i = 1, \dots, 4$. There exists at least a couple x_{j_1}, x_{j_2} such that the angle $\widehat{x_{j_1} x x_{j_2}}$ has value at most $\pi/2$. Upon renaming indexes, suppose $\widehat{x_1 x x_2} \leq \pi/2$. Let $St(x_1, x, x_2)$ be a Steiner graph connecting those points, and by direct computation

$$\mathcal{H}^1(St(x_1, x, x_2)) \leq 2r \leq \mathcal{H}^1(\gamma_1 \cap B(x, r)) + \mathcal{H}^1(\gamma_2 \cap B(x, r)).$$

Then define

$$\Sigma' := \Sigma_{\text{opt}} \setminus ((\gamma_1 \cup \gamma_2) \cap \overline{B(x, r)}) \cup St(x_1, x, x_2)$$

and applying Lemma 3.2.7 yields the existence of $\mathcal{A}(\Omega) \ni \Sigma'' \supseteq \Sigma'$ such that

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + \varepsilon, \quad F_{\mu,A}(\Sigma'') \leq F_{\mu,A}(\Sigma') - C\varepsilon^{3/2}$$

for any ε sufficiently small, where $C > 0$ is a constant not dependent on ε .

For any point $z \in \Omega$ it holds $\operatorname{dist}_{\Omega}(z, \Sigma_{\text{opt}}) \leq \operatorname{dist}_{\Omega}(z, \Sigma_{\text{opt}}) + 2r$, and such points belong to a set Γ_r with $\mathcal{L}^2(\Gamma_r) \leq 2r \operatorname{diam} \Omega$, thus using hypothesis $p > 4/3$ yields

$$F_{\mu,A}(\Sigma') \leq F_{\mu,A}(\Sigma_{\text{opt}}) - Kr^{3/2}$$

for some $K > 0$ not dependent on r . Combined with

$$|d(\Sigma_{\text{opt}}, \Sigma_0) - d(\Sigma', \Sigma_0)| \leq (2 + \theta)r$$

this yields

$$F_{\mu,A}(\Sigma') + d(\Sigma', \Sigma_0)^2 < F_{\mu,A}(\Sigma_{\text{opt}}) + d(\Sigma_{\text{opt}}, \Sigma_0)^2,$$

and the proof follows from the arbitrariness of r . □

Now this result can be easily used to prove Theorem 5.1.4:

Proof. (of Theorem 5.1.4). The proof is done by induction. By hypothesis $w(0) = \Sigma_0$ does not contain points with order at least 4; suppose this is true for $w(k-1)$, but not for $w(k)$. Let $x \in w(k)$ a point with $\operatorname{ord}_x w(k) \geq 4$, and choose $r > 0$ sufficiently small. Applying Proposition 5.1.5 yields the existence of $\Sigma' \in \mathcal{A}(\Omega)$ such that

$$F_{\mu,A}(\Sigma') \leq F_{\mu,A}(w(k)) - Kr^{3/2}$$

with $K > 0$ no dependent on r , and $\Sigma' \Delta w(k) \subseteq B(x, r) \cup B(x', r)$ for some $x' \in \Omega$. Moreover $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(w(k))$, and $|\mathcal{H}^1(\Sigma') - \mathcal{H}^1(w(k))| \leq (2 + \theta)r$ where $\theta > 0$ is dependent only on geometric quantities and not on r . Thus it holds

$$|d(w(k), \Sigma_0) - d(\Sigma', \Sigma_0)| \leq (2 + \theta)r,$$

thus

$$F_{\mu,A}(\Sigma') \leq F_{\mu,A}(w(k)) - Kr^{\frac{3}{2}},$$

$$\frac{1}{2\tau}|d(w(k), w(k-1)) - d(\Sigma', w(k-1))| = O(r^2).$$

Therefore for any r sufficiently small it holds

$$F_{\mu,A}(\Sigma') + \frac{1}{2\tau}d(\Sigma', w(k-1))^2 < F_{\mu,A}(w(k)) + \frac{1}{2\tau}d(w(k), w(k-1))^2,$$

contradicting the minimality of $w(k)$. □

The proofs of Theorems 5.1.1 and 5.1.3 can be easily extended to irreversible evolutions, while the construction used in the proof of Proposition 5.1.5 strongly relies on the absence of irreversibility: indeed Theorem 5.1.4 is false if irreversibility is imposed. Moreover, it is not known how to extend this result to higher dimensional domains.

Theorem 5.1.3 can be extended to higher dimensions, but in this case the proof is significantly different: for instance a key element was Lemma 3.2.7, which is valid only in $\mathbb{R}^2$; for higher dimensions Lemma 3.4.6 holds, but provides a less sharp estimate.

An even more important problem is that the construction used in the proof for two dimensional domains is inherently non reproducible in higher dimensions: indeed the proof in the former strongly relies on the fact that a closed curve can separate the space in two connected components (Jordan curve theorem), which is not the case for higher dimensional domains.

Theorem 5.1.6. *Given a domain $\Omega \subseteq \mathbb{R}^n$ ($n \geq 3$), a measure $\mu \in L^p(\Omega)$ with $p > n/(n-1)$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an Ahlfors regular initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$, and a time step $\tau > 0$, consider the following evolution*

$$\begin{cases} w(0) := \Sigma_0 \\ w(k) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, w(k-1))^2}{2\tau} \end{cases} \quad (5.1.8)$$

Then for any k the set $w(k)$ is Ahlfors regular.

The proof can be found in [35], and it uses an argument similar to that used to prove Ahlfors regularity for the quasi static case: the main difference is the penalization term, but since all the constructions are local (i.e. $d(\cdot, w(k-1))$ is small), the term $\frac{d(\cdot, w(k-1))^2}{2\tau}$ is well approximated with a linear term in $d(\cdot, w(k-1))$. Then the estimates (and arguments) used in Theorem 4.1.10 follow (upon non influent constants).

5.1.2 Discrete variational interpolation

In the previous Section we have analyzed some geometric properties of solutions of piecewise constant discrete evolutions, first introduced in Definition 2.2.4 in a general context. As discussed in

Chapter 2, in view of results as Theorem 2.2.9 and Lemma 2.2.10 another class of discrete evolution is naturally introduced, the “variational interpolations” (see Definition 2.2.12 for instance).

Given a domain $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$), a measure μ , a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$ and a time step $\tau > 0$, a variational interpolated evolution (as defined in Definition 2.2.12) is a function

$$x_\tau : [0, \infty) \rightarrow \mathcal{A}(\Omega)$$

defined as

$$\begin{aligned} x_\tau(0) &:= \Sigma_0 \\ x_\tau((n+1)\tau) &\in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, x_\tau(n\tau))^2}{2\tau} \quad \forall n \in \mathbb{N} \\ x_\tau(t) &\in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, x_\tau(n\tau))^2}{2(t-n\tau)} \quad \forall t \in (n\tau, (n+1)\tau) \end{aligned} .$$

Results from piecewise constant time discretized solutions are all proven valid:

Theorem 5.1.7. *Given a domain $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$), a measure $\mu \in L^1(\Omega)$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$ not containing loops, consider the function $x : [0, \infty) \rightarrow \mathcal{A}(\Omega)$ defined as*

$$\begin{aligned} x_\tau(0) &:= \Sigma_0 \\ x((h+1)\tau) &\in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, x(h\tau))^2}{2\tau} \quad \forall h \in \mathbb{N} \\ x(t) &\in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, x(h\tau))^2}{2(t-h\tau)} \quad \forall t \in (h\tau, (h+1)\tau) \end{aligned} . \quad (5.1.9)$$

Then for any $t \in [0, \infty)$ the set $x(t)$ does not contain loops.

Ahlfors regularity is true too:

Theorem 5.1.8. *Given a domain $\Omega \subseteq \mathbb{R}^2$, a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an Ahlfors regular initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$, and a time step $\tau > 0$, consider the function $x : [0, \infty) \rightarrow \mathcal{A}(\Omega)$ defined by*

$$\begin{aligned} x(0) &:= \Sigma_0 \\ x(j\tau) &\in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, x((j-1)\tau))^2}{2\tau} \quad \forall j \in \mathbb{N} \\ x(t) &\in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, x(j\tau))^2}{2(t-j\tau)} \quad \forall t \in (j\tau, (j+1)\tau) \end{aligned} . \quad (5.1.10)$$

Then for any t the set $x(t)$ is Ahlfors regular.

Similarly the absence of points having order greater than 3 can be proven:

Theorem 5.1.9. *Given a domain $\Omega \subseteq \mathbb{R}^2$, a measure $\mu \in L^p(\Omega)$ with $p > 4/3$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$ not containing points with order greater than 3, and a time step $\tau > 0$, consider the function $x : [0, \infty) \rightarrow \mathcal{A}(\Omega)$ defined by*

$$\begin{aligned}
x(0) &:= \Sigma_0 \\
x(j\tau) &\in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, x((j-1)\tau))^2}{2\tau} \quad \forall j \in \mathbb{N} \\
x(t) &\in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu,A}(\cdot) + \frac{d(\cdot, x(j\tau))^2}{2(t-j\tau)} \quad \forall t \in (j\tau, (j+1)\tau)
\end{aligned} \tag{5.1.11}$$

Then for any t the set $x(t)$ does not contain points with order greater than 3.

5.1.3 Topology

For solutions of the quasi static evolution, under irreversibility condition, we have discussed branching behaviours, and presented an explicit example where branching is expected to occur after a given time. The discussion for this case is quite similar: indeed all arguments used in the quasi static case apply, with slight modification.

Consider the configuration from the quasi static case: given a domain Ω , let Σ_0^{dat} be the initial datum, and suppose there exist

- a closed injective path $\gamma^* : [0, 1] \rightarrow \Omega$ such that $\gamma^*([0, 1]) \subseteq \Sigma_0^{dat}$: the domain Ω is now divided in two regions, Ω^+ and Ω^- with $\Omega = \Omega^+ \cup \Omega^-$ (they are the two connected components of $\Omega \setminus \gamma^*([0, 1])$, and correspond to the “interior” and the “exterior” part of $\gamma^*([0, 1])$ – the order is not relevant – given by the Jordan Curve Theorem);
- $P'_0 \in \Sigma_0^{dat}$ and a triangle $T_{P'_0} \subset V(P'_0) \cap B(P'_0, \xi')$ with $|T_{P'_0} \cap \Omega^+| > 0$, and $\operatorname{ext}(S_0^{dat}) \subset \Omega^-$.

Similarly suppose that Ω^- is large enough (both in diameter and in measure) so that all computations can be done without considering constraints imposed by $\operatorname{diam}(\Omega^-), |\Omega^-|$.

Consider

$$\begin{cases} w(0) = w(0) := \Sigma_0^{dat} \\ w(k) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F(\mathcal{X}'') + \frac{1}{2\tau} d(\mathcal{X}'', w(k-1)) \\ w(k) \supseteq w(k-1) \end{cases},$$

where $\tau > 0$ is a given parameter, and define

$$u_\tau : [0, \infty) \rightarrow \mathcal{A}(\Omega), u_\tau(t) := w\left(\left\lceil \frac{t}{\tau} \right\rceil\right).$$

Using the same argument from the quasi static case, with slight modifications due to the different penalization on the distance term, it can be proven that $F(S_0^{dat})/K(P'_0) + 1$ is an upper bound for branching time, where $K(P'_0)$ can be computed using Proposition 4.1.16.

Moreover, notice that if a branching arises in the point $(0, 0)$, then this point must have order at least 4, thus contradicting the absence of crosses in two dimensional domains (Theorem 5.1.4). Notice that irreversibility is the crucial condition here, as it does not allow the argument used in the proof of Theorem 5.1.4 where the “cross” is replaced by a Steiner graph.

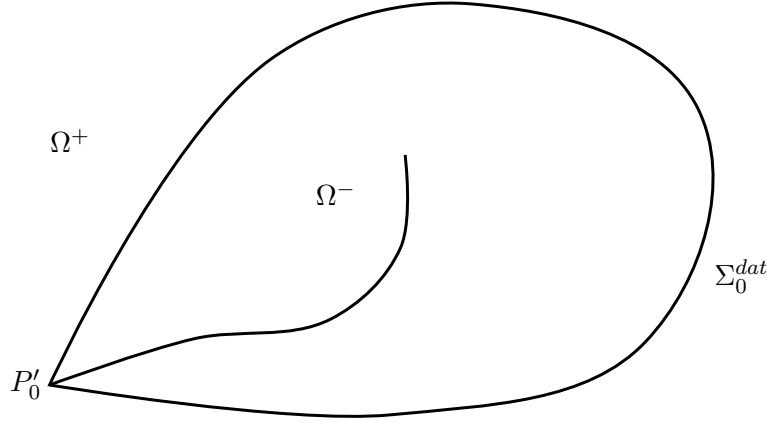


Figure 5.1.1: The same example of configurations exhibiting a branching behavior works for this kind of evolution too. The same example also provides a counterexample to the absence of crosses under irreversibility condition.

5.2 Limit $\tau \rightarrow 0^+$

Results from the previous Section are all about “discrete” evolutions, where the time discretization had step $\tau > 0$. In this Section our goal is to pass to the “continuous” case, i.e. $\tau \rightarrow 0^+$. From the discussion about gradient flow in Chapter 2, in the purely metric space (X, m) neither existence nor uniqueness is guaranteed; two sets of assumptions, one ensuring existence (already proven true in the previous Section) and the other allowing the passage to the limit $\tau \rightarrow 0$, were required.

Given a domain $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$), a measure μ and a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, Assumption 2.2.13 from Chapter 2 becomes:

Assumption 5.2.1. Let $(\mathcal{A}(\Omega), d)$ ($d(\mathcal{X}_1, \mathcal{X}_2) := \mathcal{H}^1(\mathcal{X}_1 \Delta \mathcal{X}_2)$) be the metric space, $F_{\mu, A}$ the average distance functional (with given measure μ and function A), assume the following conditions hold:

1. $F_{\mu, A}$ is bounded from below, and its sublevels are boundedly compact, i.e. $\{F_{\mu, A} \leq c\} \cap \overline{B(\mathcal{X}, r)}$ is compact for any $c \in \mathbb{R}$, $r > 0$ and $\mathcal{X} \in \mathcal{A}(\Omega)$,
2. the slope $|\nabla F_{\mu, A}| : D(F_{\mu, A}) \rightarrow [0, \infty]$ is lower semicontinuous,
3. for any sequence $\{\Sigma_k\}_{k \in \mathbb{N}}$ converging to Σ_∞ , implication

$$\sup_{k \in \mathbb{N}} \{|\nabla F_{\mu, A}|(\Sigma_k), F_{\mu, A}(\Sigma_k)\} < \infty \implies F_{\mu, A}(\Sigma_k) \rightarrow F_{\mu, A}(\Sigma_\infty)$$

is true.

The first goal is to check these assumptions.

1. $F_{\mu,A}$ is obviously bounded from below. To prove its sublevels are boundedly compact, consider arbitrary $\Sigma^* \in \mathcal{A}(\Omega)$, $r > 0$, $c \in \mathbb{R}$ and a sequence $\{\Sigma_k\}_{k \in \mathbb{N}} \subseteq \{F_{\mu,A} \leq c\} \cap \overline{B(\Sigma^*, r)}$. Condition $\{\Sigma_k\}_{k \in \mathbb{N}} \subseteq \overline{B(\Sigma^*, r)}$ implies $\mathcal{H}^1(\Sigma_k) \leq \mathcal{H}^1(\Sigma^*) + r < \infty$. Thus upon passing to subsequence we can assume $\Sigma_k \rightarrow \Sigma_\infty \in \overline{B(\Sigma^*, r)}$. This implies $F_{\mu,A}(\Sigma_k) \rightarrow F_{\mu,A}(\Sigma_\infty)$ which combined with $\{\Sigma_k\}_{k \in \mathbb{N}} \subseteq \{F_{\mu,A} \leq c\}$, yields $F_{\mu,A}(\Sigma_\infty) \leq c$. Thus $\Sigma_\infty \in \{F_{\mu,A} \leq c\} \cap \overline{B(\Sigma^*, r)}$, proving boundedly compactness,
2. the slope $|\nabla F_{\mu,A}| : D(F_{\mu,A}) \rightarrow [0, \infty]$ is defined as

$$|\nabla F_{\mu,A}(\Sigma)| := \limsup_{\mathcal{X} \rightarrow \Sigma_k} \frac{(F_{\mu,A}(\Sigma) - F_{\mu,A}(\mathcal{X}))^+}{d(\mathcal{X}, \Sigma)}.$$

However very little is known about slopes. In particular we are not able to prove lower semicontinuity.

3. for any sequence $\{\Sigma_k\}_{k \in \mathbb{N}}$ converging to Σ_∞ it always holds $F_{\mu,A}(\Sigma_k) \rightarrow F_{\mu,A}(\Sigma_\infty)$.

Now we are able to prove the following result:

Theorem 5.2.2. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ a given domain, $\mu \ll \mathcal{L}^n$ a given measure and $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$ a given function. Fix $\Sigma_0 \in \mathcal{A}(\Omega)$, $\tau > 0$, and consider a discrete solution $x : [0, \infty) \rightarrow \mathcal{A}(\Omega)$ defined via variational interpolation. Then the following results hold:*

- the set $\{x_\tau(t)\}_\tau$ is relatively compact in the set of curves in X with respect to the uniform local convergence,
- any limit curve is a gradient flow in the EDI sense, but with the slope replaced $|\nabla F_{\mu,A}|$ by the relaxed slope $|\partial^- F_{\mu,A}|$ (for the definition see [3]).

Proof. The proof is based on Theorem 2.2.14, and it is sufficient to check if its hypothesis are satisfied: we recall that for Theorem 2.2.14 it is sufficient that Assumption 2.2.13.

For the average distance functional we have checked that Assumption 5.2.1, without the lower semicontinuity part, which is Assumption 2.2.13 formulated for this specific case, holds. Thus we can apply the same argument used in the proof of Theorem 2.2.14, and the proof is complete. $\square$

Finally we discuss briefly some open problems regarding properties of solutions of evolutions schemes related to the average distance functional when the time step goes to 0.

Differently from the time-discretized version, results concerning geometric and analytic properties for minimizing movements are surprisingly hard to prove.

There are mainly two problems in passing to the limit: first, no argument used in the discrete case applies, as they rely on the fact that the set added at each step has positive length, and the estimates used for discrete solutions always exhibit a dependence on the time step, and lack uniformity when the time step goes to 0. Moreover, the argument used for Ahlfors regularity for limit sets of the quasi static case cannot be applied here.

The basic idea is quite simple, and involves constructing a competitor contradicting the minimality of some $w_j(k)$, but many surprisingly difficult problems arise when dealing with connectedness of such competitor.

Similar properties can be expected:

1. Absence of loops:

Given a domain $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$), a measure $\mu \in L^1(\Omega)$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$ not containing loops, a sequence of time steps $\{\tau_k\}_{k \in \mathbb{N}} \downarrow 0$, consider the following family of evolutions:

$$\left\{ \begin{cases} w_j(0) := \Sigma_0 \\ w_j(k) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu, A}(\cdot) + \frac{d(\cdot, w_j(k-1))^2}{2\tau_j} \end{cases} \right\}_{j \in \mathbb{N}}.$$

Given $T > 0$, define functions

$$x_j : [0, T] \rightarrow \mathcal{A}(\Omega), \quad x_j(t) := w_j([t/\tau_j])$$

and suppose there exists the limit function

$$x : [0, T] \rightarrow \mathcal{A}(\Omega), \quad x(t) := \lim_{j \rightarrow \infty} x_j(t).$$

Then there exists $T_0 > 0$ such that for any $t \in [0, T_0]$ the set $x(t)$ does not contain loops.

2. Ahlfors regularity:

Given a domain $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$), a measure $\mu \in L^1(\Omega)$, a function $A : [0, \text{diam } \Omega] \rightarrow [0, \infty)$, an Ahlfors regular initial datum $\Sigma_0 \in \mathcal{A}(\Omega)$, a sequence of time steps $\{\tau_k\}_{k \in \mathbb{N}} \downarrow 0$, consider the following family of evolutions:

$$\left\{ \begin{cases} w_j(0) := \Sigma_0 \\ w_j(k) \in \operatorname{argmin}_{\mathcal{A}(\Omega)} F_{\mu, A}(\cdot) + \frac{d(\cdot, w_j(k-1))^2}{2\tau_j} \\ w_j(k) \supseteq w_j(k-1) \end{cases} \right\}_{j \in \mathbb{N}}.$$

Given $T > 0$, define functions

$$x_j : [0, T] \rightarrow \mathcal{A}(\Omega), \quad x_j(t) := w_j([t/\tau_j])$$

and suppose there exists the limit function

$$x : [0, T] \rightarrow \mathcal{A}(\Omega), \quad x(t) := \lim_{j \rightarrow \infty} x_j(t).$$

Moreover assume there exists $\lambda > 0$ such that $|\nabla F(x_j(t))|, |\nabla F(x(t))| \geq \lambda$ for any $t \in [0, T]$ and $j \in \mathbb{N}$. Then for any $t \in [0, T]$ the set $x(t)$ is Ahlfors regular.

3. Absence of crosses:

It is expected that a similar result holds for the continuous case, without irreversibility condition, as Theorem 5.1.4 is proven false under irreversibility condition.

However, without very strong assumptions (e.g. in the irreversible evolution, assuming the existence of some sort of uniformly controllable set A_k containing $w(k)$, such that $w(j) \setminus w(k)$ never intersects A_k whenever $j > k$), no similar result can be proven actually.

Chapter 6

BV regularity of derivatives and “topological lower semicontinuity”

This chapter aims to extend regularity results, and prove that minimizers are finite unions of Lipschitz curves with BV derivatives. More precisely we show that given an arbitrary nonnegative finite measure with compact support, μ , and $\lambda > 0$, any solution $\Sigma \in \operatorname{argmin} E_\mu^\lambda$ (in all this chapter the symbol $E_\mu^\lambda(\cdot)$ will denote the sum $\int d(x, \cdot) d\mu + \lambda \mathcal{H}^1(\cdot)$) is finite union of Lipschitz curves $\{\gamma_k\}_{k=1}^N$ (without loss of generality assume that all γ_k are arc-length parameterized), such that it holds

$$\sum_k \|\gamma_k'\|_{TV} \leq \frac{1}{\lambda} |\mu(\mathbb{R}^d)|, \quad (6.0.1)$$

where $\|\cdot\|_{TV}$ denotes the total variation.

In other words we provide an estimate on the total curvature of the curves that make up Σ , where the curvature, $\kappa = \gamma_k''$ is understood as the signed measure. The fact the the total mass (times $1/\lambda$) bounds the curvature is not surprising. To motivate it let us assume that the minimizer Σ is a single smooth curve and that μ is absolutely continuous with respect to the Lebesgue measure. Let Π be the projection onto Σ (it is known from [39] that Π has unique value almost everywhere). Then the first variation for the Problem 8.1.2 gives that for any smooth vector field $v : \Sigma \rightarrow \mathbb{R}^d$ supported away from the endpoints of Σ

$$-\int_\Sigma \kappa \cdot v d\mathcal{H}^1(\Sigma) = \frac{1}{\lambda} \int_{\mathbb{R}^d} \frac{(x - \Pi(x))}{|x - \Pi(x)|} \cdot v(\Pi(x)) d\mu(x).$$

Taking supremum over all v , as above, with $|v| \leq 1$ implies (6.0.1).

The difficulties one faces in applying the reasoning above are that Σ is not regular, nor even a curve in general, and that μ is not assumed to be absolutely continuous with respect to the Lebesgue measure.

The approach we use is based on approximating a measure μ by a sequence of discrete measures $\{\mu_n\} \xrightarrow{*} \mu$, and analyzing the minimizers of Problem 8.1.2 with μ replaced by μ_n . In addition to the estimate on the BV norm, we prove a topological relation between minimizers of the approximating problem and the minimizers of the limiting problem.

This chapter is structured as follows:

- In Section 6.1 we recall the known results on the average-distance problem, introduce the discrete approximations and prove a couple of preliminary results.
- In Section 6.2 we prove an upper bound on the number endpoints of minimizers and analyze the behavior of endpoints in the approximation process. In particular we prove in Theorem 6.2.6 that if μ_n are discrete approximations of μ and Σ_n are minimizers of $E_{\mu_n}^\lambda$ which converge to Σ , a minimizer of E_μ^λ , then each endpoint of Σ is a limit of endpoints of Σ_n . This result (obtained in collaboration with Slepčev) will be crucial for the following sections.
- Section 6.3 (whose results are mainly obtained in collaboration with Slepčev) is devoted to the topological comparison between minimizers for the approximate problem and the minimizer corresponding to μ .
- In Section 6.4 (whose results are mainly obtained in collaboration with Slepčev) we prove that minimizers of the average-distance problem are finite union of Lipschitz curves with BV derivatives, and prove an a priori estimates on BV norms in Theorem 6.4.1.

6.1 Preliminary results

We restrict our attention to probability measures, μ , purely for notational simplicity. Given a compactly supported probability measure, μ , and a compact set, Σ , we need to find a "projection" of the measure μ onto Σ . The issue is that for $x \in \mathbb{R}^d$, the minimizer of $|x - y|$ over $y \in \Sigma$ is in general not unique. In fact it has been proven by the authors in [39] that the ridge of Σ (points having nonunique projection on Σ) is an $\mathcal{H}^{d-1}$ -rectifiable set. If μ is absolutely continuous with respect to the Lebesgue measure, $\mu \ll \mathcal{L}^d$, then the ridge is μ -negligible and thus the projection $\Pi : x \mapsto \operatorname{argmin}_{y \in \Sigma} |x - y|$ is unique μ -a.e. and one can define the projection of μ onto Σ by $\sigma = \Pi_\# \mu$ (the push-forward of the measure).

However if μ is not absolutely continuous with respect to Lebesgue measure then more care is needed. In particular we define the "projection" as a second marginal of a coupling (i.e. a transportation plan) rather than the push forward by a projection map.

Lemma 6.1.1. *Let μ be a probability measure and Σ a compact set. There exists a probability measure π on $\mathbb{R}^d \times \Sigma$ such that the first marginal of π is μ (that is $\pi(A \times \Sigma) = \mu(A)$ for any Borel set A) and that for π -a.e. (x, y) , $|x - y| = \min_{z \in \Sigma} |x - z|$.*

We define σ , the projection of μ onto Σ , to be the second marginal of π . Finally we note that π and σ may be nonunique.

Remark 6.1.2. *While, for given μ and Σ , the measures π and σ may be nonunique, all of the subsequent statements in this chapter hold for any π (and associated σ) chosen, unless explicitly stated otherwise.*

Proof. Let $\mathcal{P}_\Sigma$ be the set of Borel probability measures on Σ . Consider the functional $\sigma \mapsto d_W(\mu, \sigma)$ on $\mathcal{P}_\Sigma$, where d_W is the Wasserstein distance. Since Σ is compact, $\mathcal{P}_\Sigma$ is sequentially compact with respect to the weak-* convergence of measures. Given that $d_W(\mu, \cdot)$ is continuous on $\mathcal{P}_\Sigma$ with

respect to the weak-* convergence of measures we conclude that there exists $\bar{\sigma} \in \mathcal{P}_\Sigma$ minimizing the Wasserstein distance to μ . Let π be the optimal transportation plan (for the quadratic cost) between μ and $\bar{\sigma}$. We claim that π has the desired properties.

Since, by definition, the first marginal of π is μ we only need to verify that for π -a.e. (x, y) , $|x - y| = \min_{z \in \Sigma} |x - z|$. Assume that this is not the case, that is that there exists $(x, y) \in \text{supp} \pi$ and $z \in \Sigma$ such that $|x - y| > |x - z|$. Let $\delta = (|x - y| - |x - z|)/3$ and $U = B(x, \delta) \times (B(y, \delta) \cap \Sigma)$. Since $(x, y) \in \text{supp} \pi$, $\varepsilon := \pi(U) > 0$. Let $\pi_{\text{new}} = \pi - \pi|_U + \varepsilon \delta_{(x, z)}$ and let σ_{new} the the second marginal of π_{new} . Then $\sigma_{\text{new}} \in \mathcal{P}_\Sigma$ and $d_W(\mu, \sigma_{\text{new}}) < d_W(\mu, \bar{\sigma})$ which contradicts the fact that $\bar{\sigma}$ is a minimizer.

To see that π and σ may not be unique consider $\mu = \delta_0$ and $\Sigma = \partial B(0, 1)$. Then any Borel measure σ on Σ can be obtained as a "projection" by choosing π to be any coupling between μ and σ (for example the product measure $\mu \times \sigma$). $\square$

We introduce some notation and terminology:

- The measure σ defined in Lemma 6.1.1 can have atoms. For simplicity for $y \in \Sigma$ we write $\sigma(y)$ to mean $\sigma(\{y\})$.
- Sometimes it is important to emphasize which μ and Σ the measure σ corresponds to. Then we write $\sigma(\mu, \Sigma, A)$ for $\sigma(A)$, where A is a measurable subset of Σ .
- The *order* of a point $y \in \Sigma$ is defined to be the supremum of the number of connected subsets of Σ which contain y and are mutually disjoint on $\Sigma \setminus \{y\}$. If Σ is a minimizer of E_μ^λ then all points on Σ are of order 1, 2, or 3 (see Lemma 6.2.5). Also, if Σ is a minimizer then it is topologically a tree (Lemma 6.2.2) and thus the order of a point is equal to the number of connected components of $\Sigma \setminus \{y\}$.
- Point $y \in \Sigma$ of order one is called an *endpoint*. We show in Lemma 6.2.1 that for any endpoint $\sigma(\mu, \Sigma, y) \geq \lambda$. We denote the set of endpoints of Σ by $\Xi(\Sigma)$. A point of order three is called a *triple junction*. A point $y \in \Sigma$ is called a *corner* if it is of order two and $\sigma(\mu, \Sigma, y) > 0$.

We note that there is a simple bound on the length of the minimizers of E_μ^λ . Namely if Σ is a minimizer of E_μ^λ then

$$\mathcal{H}^1(\Sigma) \leq \frac{1}{\lambda} \text{diam supp}(\mu). \quad (6.1.1)$$

The reason is that for any $z \in \text{supp} \mu$ the minimality of Σ implies

$$\begin{aligned} \lambda \mathcal{H}^1(\Sigma) &\leq E_\mu^\lambda(\Sigma) \leq E_\mu^\lambda(\{z\}) \\ &= \int_{\mathbb{R}^d} d(y, \{z\}) d\mu(y) = \int_{\text{supp}(\mu)} d(y, \{z\}) d\mu(y) \leq \text{diam supp}(\mu). \end{aligned}$$

We also remark that if $\lambda > \frac{1}{2}$ then the only minimizer of E_μ^λ is a single point; see Corollary 6.2.4.

We recall the following facts on the average-distance problem. We refer to [52] for further details. Let $\mathcal{P}_r$ be the set of probability measures supported in $\bar{B}(0, r)$.

- (i) For any $\mu \in \mathcal{P}_r$ and $\lambda > 0$, the functional E_μ^λ is lower semicontinuous w.r.t. Hausdorff distance, $d_{\mathcal{H}}$.
- (ii) Given $\Sigma \in \mathcal{A}$, and $\lambda > 0$, the mapping $\mu \mapsto E_\mu^\lambda(\Sigma)$ is continuous on $\mathcal{P}_r$ w.r.t. weak-* convergence of measures.
- (iii) If $\{\mu_n\} \xrightarrow{*} \mu$ on $\mathcal{P}_r$ then for any $\lambda > 0$, $E_{\mu_n}^\lambda$ Γ -converges to E_μ^λ w.r.t. Hausdorff convergence of sets of $\mathcal{A}$.
- (iv) Consider a sequence $\{\mu_n\} \xrightarrow{*} \mu$ in $\mathcal{P}_r$. For any n choose $\Sigma_n \in \operatorname{argmin} E_{\mu_n}^\lambda$. Then along a subsequence $\Sigma_n \xrightarrow{d_{\mathcal{H}}} \Sigma$ for some $\Sigma \in \operatorname{argmin} E_\mu^\lambda$.
- (v) Given $R > 0$, consider a sequence $\{\gamma_n\} : [0, 1] \rightarrow \overline{B(0, R)}$ of Lipschitz curves with constant-speed parameterization, satisfying $\sup \mathcal{H}^1(\gamma_n) < \infty$ and $\sup \|\gamma_n'\|_{BV} < \infty$. Then upon subsequence there exists a Lipschitz curve γ such that:
 - $\{\gamma_n\} \rightarrow \gamma$ in C^α for any $\alpha \in [0, 1)$,
 - $\{\gamma_n'\} \rightarrow \gamma'$ in L^p for any $p \in [1, \infty)$,
 - $\{\gamma_n''\} \rightarrow \gamma''$ weakly in the space of signed finite Borel measures on $\mathbb{R}^d$.

We also need a basic result on the nature of path connectedness of Σ . Given points $p, q \in \Sigma$, we use the following terminology:

- a “curve between p and q ” is a continuous (not necessarily injective) mapping $\gamma : [0, 1] \rightarrow \Sigma$ with $\gamma(0) = p, \gamma(1) = q$.
- a “path between p and q ” is the image of a curve $\gamma : [0, 1] \rightarrow \Sigma$ with $\gamma(0) = p, \gamma(1) = q$.

Lemma 6.1.3. *Consider an arbitrary element $X \in \mathcal{A}$. Given distinct points $p, q \in X$ there exists a minimal (w.r.t. set inclusion) path connecting them. Moreover such path is a geodesic (in the metric sense), and as such can be parameterized by an injective curve.*

For the proof we refer to the general result about the existence of geodesics in metric spaces (Theorem 4.3.2 in [6]).

We refer to such, minimal (w.r.t. inclusion), compact, injective paths as the “minimal paths”. Notice that nothing is claimed about uniqueness of such minimal paths. The, well known, result which we state below shows that if the minimal path is not unique then the set contains a loop.

Lemma 6.1.4. *Assume $X \in \mathcal{A}$, and $p, q \in X$ are distinct points. If there exist distinct minimal paths, L_1 and L_2 , parameterized by*

$$\gamma_1, \gamma_2 : [0, 1] \rightarrow X, \quad \gamma_1(0) = \gamma_2(0) = p, \gamma_1(1) = \gamma_2(1) = q,$$

then X contains a loop.

Proof. Since the roles of L_1 and L_2 are symmetric, we can assume that there exists $t \in (0, 1)$ such that $\gamma_1(t) \in L_1 \setminus L_2$. Define

$$t_1 := \inf\{\tau \in [0, t] : \gamma_1((\tau, t)) \cap L_2 = \emptyset\},$$

$$t_2 := \sup\{\tau \in [t, 1] : \gamma_1((t, \tau)) \cap L_2 = \emptyset\}.$$

Since $L_1 \cap L_2$ is closed, $\gamma_1(t_1)$ and $\gamma_1(t_2)$ belong to L_2 . We also note that $\gamma_1(t_1)$ and $\gamma_1(t_2)$ are distinct since otherwise L_1 is not a minimal path. By Lemma 6.1.3 there exists a minimal path $\tilde{L}_2 \subseteq L_2$ connecting the points. It follows that $\gamma_1([t_1, t_2]) \cup \tilde{L}_2$ is a loop. $\square$

Since the minimizers of E_μ^λ cannot contain loops (Lemma 6.2.2), Lemmas 6.1.3 and 6.1.4 imply that for minimizers the minimal path between any two or their points is unique.

6.1.1 Discrete approximations

We first recall the setup and some results from [52].

Definition 6.1.5. *Given points $y_1, \dots, y_n \in \mathbb{R}^d$, a Steiner graph $St(y_1, \dots, y_n)$ is a set of minimal length containing $y_1, \dots, y_n$.*

We note that in general Steiner graphs for the given set of points are not unique in general. Here we list a few basic properties of Steiner graphs, their proofs and more on Steiner graphs can be found in [26] and [27].

Proposition 6.1.6. *Let G be a Steiner graph.*

- (i) *G is a tree with straight edges,*
- (ii) *The order of any point of G does not exceed 3,*
- (iii) *If $v \in G \setminus \{y_1, \dots, y_n\}$ has order 3, then the edges intersecting in v are coplanar, forming 3 angles measuring $2\pi/3$ each.*

The next definition is similar to the notion of curvature:

Definition 6.1.7. *Given a graph with straight edges Σ and a vertex $v \in \Sigma$ with degree 2, denote by w_1, w_2 its neighbors. The turning angle at v is*

$$TA(v) := \pi - \angle w_1 v w_2.$$

The turning angle for a subset $A \subseteq \Sigma$ is defined to be the sum of all turning angles at vertices of degree 2 which belong to A .

The following facts were established in [52]:

- (i) *If μ is a discrete probability measure, then for any $\lambda > 0$, any $\Sigma \in \operatorname{argmin} E_\mu^\lambda$ is a Steiner graph for a set of points (which in general are not the points in the support of μ). More precisely there exists a projection, σ , of μ onto Σ (as defined in Lemma 6.1.1) such that Σ is a Steiner graph for the support of σ .*

- (ii) Given a discrete probability measure, $\mu, \lambda > 0$, and $\Sigma \in \operatorname{argmin} E_\mu^\lambda$, for any endpoint $v \in \Sigma$ it holds that

$$\sigma(\mu, \Sigma, v) \geq \lambda. \quad (6.1.2)$$

- (iii) Given a discrete probability measure, $\mu, \lambda > 0$, $\Sigma \in \operatorname{argmin} E_\mu^\lambda$, for any $A \subseteq \Sigma$ measurable

$$TA(A) \leq \frac{\pi}{2\lambda} \sigma(\mu, \Sigma, A).$$

6.2 Endpoint estimates

We first establish an upper bound on the number of endpoints by proving a lower bound on the mass that projects on each endpoint.

Lemma 6.2.1. *Let μ be a finite, compactly supported, measure, let $\lambda > 0$, and let $\Sigma \in \operatorname{argmin} E_\mu^\lambda$. If Σ is not a single point then $\sigma(\mu, \Sigma, v) \geq \lambda$ for any endpoint $v \in \Sigma$.*

For discrete measures μ this is the statement (6.1.2) we mentioned above; here we prove it for general measures.

Proof. Choose an arbitrary endpoint $v \in \Sigma$. In [16] it has been proven that there exists $r_0 > 0$ such that $\Sigma \setminus B(v, r) \in \mathcal{A}$ for all $r \leq r_0$. Let π be a coupling between μ and σ defined in Lemma 6.1.1. Let $l(r) = \mathcal{H}^1(\Sigma \cap B(v, r))$.

Note that for π -a.e. $(x, y) \in \operatorname{supp} \pi$ and $y \in B(v, r)$

$$d(x, \Sigma \setminus B(v, r)) - d(x, y) \leq l(r).$$

Furthermore if $y \in \Sigma \setminus B(v, r)$ then

$$d(x, \Sigma \setminus B(v, r)) = d(x, y).$$

Therefore

$$\begin{aligned} F_\mu(\Sigma \setminus B(v, r)) - F_\mu(\Sigma) &= \int_{\mathbb{R}^d \times \Sigma} d(x, \Sigma \setminus B(v, r)) - d(x, y) d\pi \\ &\leq l(r) \pi(\mathbb{R}^d \times B(v, r)) = l(r) \sigma(\mu, \Sigma, \Sigma \cap B(v, r)). \end{aligned}$$

Combining this with the fact that $\mathcal{H}^1(\Sigma \setminus B(v, r)) = \mathcal{H}^1(\Sigma) - l(r)$, and using the minimality of Σ implies

$$\begin{aligned} F_\mu(\Sigma) + \lambda \mathcal{H}^1(\Sigma) &\leq F_\mu(\Sigma \setminus B(v, r)) + \lambda \mathcal{H}^1(\Sigma \setminus B(v, r)) \\ &\leq F_\mu(\Sigma) + l(r) \sigma(\mu, \Sigma, \Sigma \cap B(v, r)) + \lambda (\mathcal{H}^1(\Sigma) - l(r)). \end{aligned}$$

Passing to the limit $r \rightarrow 0^+$ gives $\sigma(\mu, \Sigma, v) \geq \lambda$. □

We show that minimizers of the average-distance problem cannot contain loops. A similar result was shown in [14] for μ absolutely continuous with respect to Lebesgue measure.

Lemma 6.2.2. *Given a finite compactly supported measure μ , a parameter $\lambda > 0$, and $\Sigma \in \operatorname{argmin} E_\mu^\lambda$, the set Σ does not contain a loop.*

Proof. Suppose that Σ contains a loop, E , and let $\varphi : [0, 1] \rightarrow E$ be a constant speed parameterization, with $\varphi(0) = \varphi(1)$ and injective in $(0, 1)$. Choose an arbitrary (large) $N \in \mathbb{N}$, and partition E into N mutually disjoint measurable sets $I_1, \dots, I_N$, with $I_j := \varphi([(j-1)/N, j/N])$. Clearly $\mathcal{H}^1(I_j) = \mathcal{H}^1(E)/N$ for any j .

Denote with $\{C_j^k\}_{k \in \mathcal{J}_j}$ the set of connected components of $\Sigma \setminus I_j$ which do not intersect E , where $\mathcal{J}_j$ is a suitable set of indexes. Choosing $N > 2/\lambda$ guarantees that there exists index j such that

$$\sigma\left(I_j \cup \bigcup_{k \in \mathcal{J}_j} C_j^k\right) < \lambda/2.$$

Consider the competitor Σ' defined in the following way:

- (i) remove I_j from Σ ,
- (ii) for all $k \in \mathcal{J}_j$, choose $p_k \in C_j^k \cap \bar{I}_j$ (such p_k exists since Σ is connected). Noticing that $\varphi(j/N) \in \Sigma \setminus I_j$, consider $T_k(x) := x + (\varphi(j/N) - p_k)$ the translation by the vector $\varphi(j/N) - p_k$, and replace C_j^k by $T_k(C_j^k)$ in $\Sigma \setminus I_j$.

By construction $\Sigma' \in \mathcal{A}$. As $\sigma(I_j) < \lambda/2$, and each T_k is a translation by a vector $\varphi(j/N) - p_k$, and $|\varphi(j/N) - p_k| \leq \mathcal{H}^1(E)/N$, it follows $F_\mu(\Sigma') \leq F_\mu(\Sigma) + \frac{\lambda}{2} \frac{\mathcal{H}^1(E)}{N}$. Since by construction $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(E)/N$, this contradicts the minimality of Σ . $\square$

Lemma 6.2.3. *Let X be a minimizer of E_μ^λ with $\mathcal{H}^1(X) > 0$. For any point $p \in X$, each connected component of $X \setminus \{p\}$ must contain an endpoint of X .*

Proof. Choose an arbitrary connected component of $X \setminus \{p\}$, and denote it by C . Let d_C be the path distance on C . Choose an arbitrary point $q \in \operatorname{argmax}_{z \in C \cup \{p\}} d_C(z, p)$, which exists by compactness of $C \cup \{p\}$. Note that since C is nonempty, $q \neq p$. We claim that q is an endpoint. For if q is not an endpoint, then choose q' belonging to the component of $C \setminus \{q\}$ which does not contain p . Due to the absence of loops, $d_C(q', p) > d_C(q, p)$, as each path connecting p and q' must contain q . This contradicts the construction of q . $\square$

Combining with Lemma 6.2.1 we establish the following corollary:

Corollary 6.2.4. *Let μ be a finitely supported probability measure. If $\lambda > \frac{1}{2}$ the only minimizer of Σ of E_μ^λ is a singleton $\Sigma = \{p\}$.*

Proof. If Σ is not a singleton then, by Lemma 6.2.3, it must have at least two endpoints. But each endpoint has at least mass λ projecting to it. Hence $\sigma(\Sigma) > 1$. Contradiction. $\square$

Another property of minimizers is that they do not contain crosses, i.e. points with order at least 4. This result was proved in two dimensions in [14]. We use a similar construction here.

Lemma 6.2.5. *Let μ be a finite compactly supported measure and let $\lambda > 0$. No minimizer, Σ , of E_μ^λ contains points of order 4 or more.*

Proof. Assume for the sake of contradiction that there exists a point z in a minimizer Σ which has order at least 4. Menger n -Beinsatz gives the existence of $\varepsilon_0 > 0$ and curves (which we assume to be parameterized by arclength) $\xi_i : [0, \varepsilon_0) \rightarrow \Sigma \cap B(z, \varepsilon_0)$, $i = 1, \dots, 4$ such that

$$(\forall i) \ \xi_i(0) = z, \quad (\forall i \neq j) \ \xi_i((0, \delta_i)) \cap \xi_j((0, \delta_j)) = \emptyset.$$

Moreover upon choosing $r_0 > 0$ sufficiently small it can be guaranteed that $\xi_i([0, \delta_i)) \cap \{w : |z - w| = r\} \neq \emptyset$ for any index i and radius $0 < r \leq r_0$.

Given $0 < r < r_0$, denote for by p_i^r , the intersections $\xi_i([0, \delta_i)) \cap B(z, r)$. Denote by $\theta_{i,j}^r$ the angle between vectors $p_i^r - z$ and $p_j^r - z$. To apply the construction from [14] we need to prove that $\liminf_{r \rightarrow 0} \min_{i \neq j} \theta_{i,j}^r < 2\pi/3$. This is a consequence of the following:

Claim: Given $k \geq 4$ unit vectors $v_1, \dots, v_k$, denote with $\beta_{i,j}$ the angle between v_i and v_j . Then $\beta := \min_{i \neq j} \beta_{i,j} < 2\pi/3$.

To prove the claim note

$$0 \leq \left(\sum_{i=1}^k v_i \right) \cdot \left(\sum_{j=1}^k v_j \right) = \sum_{i,j=1}^k v_i \cdot v_j = k + \sum_{i=1}^k \sum_{j \neq i}^k \cos \beta_{i,j} \leq k + k(k-1) \cos \beta.$$

Hence

$$-k \leq k(k-1) \cos \beta,$$

and consequently

$$\beta \leq \arccos \left(-\frac{1}{k-1} \right) \leq \arccos \left(-\frac{1}{3} \right) =: \bar{\beta} < \frac{2}{3}\pi.$$

Thus for any $r \in (0, r_0)$ there exists i, j such that angle $\theta_{i,j}^r =: \theta_r \leq \bar{\beta}$. Let us write $x_r = p_i^r$ and $y_r = p_j^r$. Now we can use the construction from the proof of absence of crosses in [14].

The competitor Σ_r is constructed by replacing the paths $[x_r, z]_\Sigma$ and $[y_r, z]_\Sigma$ (the parts of ξ_i and ξ_j between x_r and z , and y_r and z , respectively; see Figure 6.2.1) with the Steiner graph for $\{x_r, y_r, z\}$ (i.e. the union of line segments between x_r and z_r , y_r and z_r , and z and z_r , where z_r is such that the angles between the segments are 120°). More formally, the competitor Σ_r is defined as

$$\begin{aligned} \Sigma_r := \Sigma \setminus ([x_r, z]_\Sigma \cup [y_r, z]_\Sigma) &\cup \{(1-t)x_r + tz_r : t \in [0, 1]\} \\ &\cup \{(1-t)y_r + tz_r : t \in [0, 1]\} \\ &\cup \{(1-t)z_r + tz : t \in [0, 1]\}. \end{aligned}$$

Note that by construction, the only points w for which $d(w, \Sigma_r) > d(w, \Sigma)$, are those for which $\operatorname{argmin}_{y \in \Sigma} |w - y| \subseteq I_r := [x_r, z]_\Sigma \cup [y_r, z]_\Sigma \setminus \{x_r, y_r, z\}$ holds. The diameter of I_r is less than or equal to $2r$ which implies

$$F_\mu(\Sigma_r) \leq F_\mu(\Sigma) + 2r\sigma(\mu, \Sigma, I_r).$$

Since by construction $\cap_{r>0} I_r = \emptyset$, $\sigma(\mu, \Sigma, I_r) \rightarrow 0$ as $r \rightarrow 0$. Elementary geometry gives

$$\mathcal{H}^1(\Sigma_r) \leq \mathcal{H}^1(\Sigma) - \left(2 - \sqrt{3} \sin \frac{\theta_r}{2} - \cos \frac{\theta_r}{2} \right) r,$$

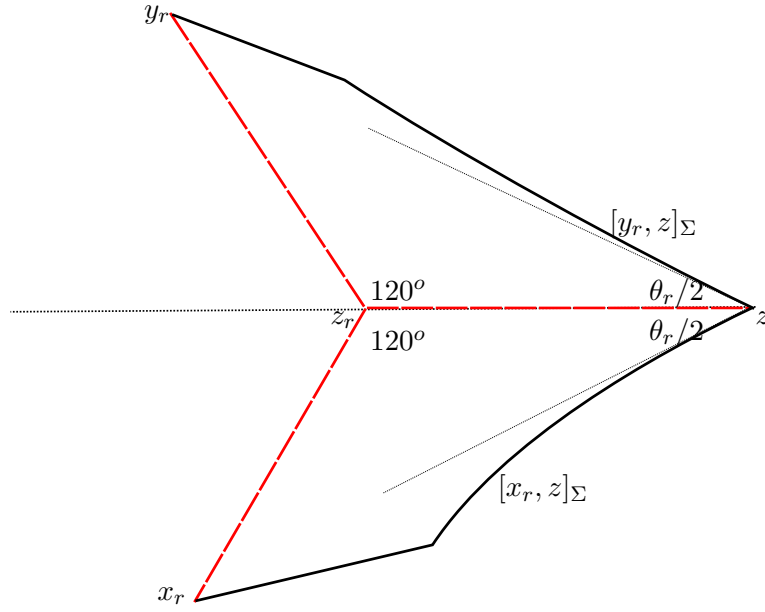


Figure 6.2.1: The modification of Σ : The paths $[x_r, z]$ and $[y_r, z]$ (continuous curves) are replaced by the union of line segments $\overline{x_r z_r}$, $\overline{y_r z_r}$, and $\overline{z_r z}$. The distances from x_r to z and from y_r to z are both equal to r .

and since $\frac{\beta}{2} < \frac{\pi}{3}$, the quantity $2 - \sqrt{3} \sin \frac{\theta_r}{2} - \cos \frac{\theta_r}{2}$ is positive. Thus

$$E_\mu^\lambda(\Sigma_r) \leq E_\mu^\lambda(\Sigma) - \lambda \left(2 - \sqrt{3} \sin \frac{\theta_r}{2} - \cos \frac{\theta_r}{2} \right) r + 2r\sigma(\mu, \Sigma, I_r),$$

which contradicts the minimality of Σ for r sufficiently small. $\square$

The main result of this section deals with the relation of endpoints of minimizers corresponding to discrete approximation of μ and the endpoints of a minimizer, Σ , corresponding to μ . Recall that by $\Xi(\Sigma)$ we denote the set of endpoints of Σ .

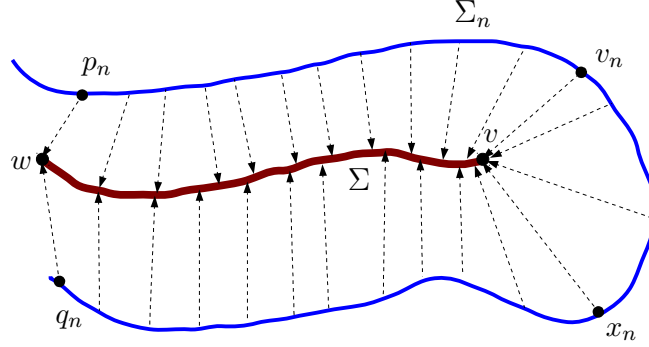
Theorem 6.2.6. *Let μ be a compactly supported probability measure. Given a sequence of probability measures $\{\mu_n\} \xrightarrow{*} \mu$, with uniformly bounded supports, for any n choose an element*

$$\Sigma_n \in \operatorname{argmin} E_{\mu_n}^\lambda.$$

Then along a subsequence $\{\Sigma_n\} \xrightarrow{d_H} \Sigma$ for some $\Sigma \in \operatorname{argmin} E_\mu^\lambda$.

By relabeling the indices we can assume that the subsequence is the whole sequence. Then for any endpoint $z \in \Xi(\Sigma)$ there exists a sequence of endpoints $z_n \in \exp(\Sigma_n)$ such that $z_n \rightarrow z$ as $n \rightarrow \infty$. This in particular implies

$$\liminf_{n \rightarrow \infty} \#\Xi(\Sigma_n) \geq \#\Xi(\Sigma).$$

Figure 6.2.2: Σ is an example of a double line.

This estimate is crucial in the next section, when we discuss the topological relation between minimizers of E_μ^λ and minimizers of $E_{\mu_n}^\lambda$, where μ_n is a discrete approximation to μ .

The proof requires us to introduce the notion of a double line and prove a preliminary technical result.

Definition 6.2.7. Consider a sequence of probability measures with uniformly bounded supports, $\{\mu_n\}$, converging to a probability measure μ w.r.t. weak-* topology, and a sequence of minimizers $\Sigma_n \in \operatorname{argmin} E_{\mu_n}^\lambda$ converging to $\Sigma \in \operatorname{argmin} E_\mu^\lambda$ w.r.t. $d_{\mathcal{H}}$. A closed subset $L \subseteq \Sigma$ is a double line if it is a minimal path connecting distinct points v and w (in Σ) and there exist points $\{v_n, x_n, p_n, q_n\} \subseteq \Sigma_n$, satisfying

- $v_n \rightarrow v, x_n \rightarrow v, p_n \rightarrow w, q_n \rightarrow w$ as $n \rightarrow \infty$,
- For any n , the minimal paths between p_n and v_n , and between x_n and q_n are disjoint.

Lemma 6.2.8. Assume the setting of the Definition 6.2.7. Σ cannot contain any double lines.

Proof. Assume there exists a double line $L \subseteq \Sigma$. The aim is to find, for some n , a competitor $\tilde{\Sigma}_n$ contradicting the optimality of Σ_n .

By considering a subsequence we can assume that the minimal paths L'_n and L''_n in Σ_n connecting p_n to v_n , and q_n to x_n respectively converge in the Hausdorff distance to $L' \subset \Sigma$ and $L'' \subset \Sigma$. Since $v, w \in L' \cap L''$ and L is the minimal path between v and w it follows that $L \subseteq L' \cap L''$. Let $L_n := L'_n \cup L''_n$. It also holds that $(\Sigma_n \setminus L_n) \cup L' \cup L'' \rightarrow \Sigma$ w.r.t. $d_{\mathcal{H}}$ as $n \rightarrow \infty$. By the lower semicontinuity of $\mathcal{H}^1$ it follows that $\liminf_{n \rightarrow \infty} \mathcal{H}^1(L'_n) \geq \mathcal{H}^1(L')$ and $\liminf_{n \rightarrow \infty} \mathcal{H}^1(L''_n) \geq \mathcal{H}^1(L'')$. We note that

$$\mathcal{H}^1(L') + \mathcal{H}^1(L'') - \mathcal{H}^1(L' \cup L'') \geq \mathcal{H}^1(L) \geq d_L(v, w) =: a > 0,$$

where d_L denotes the path distance on L . It follows that for n sufficiently large

$$\mathcal{H}^1(L_n) \geq \mathcal{H}^1(L' \cup L'') + 0.9\mathcal{H}^1(L).$$

Choose $\varepsilon > 0$, and n sufficiently large such that $\max\{|p_n - w|, |q_n - w|, |v_n - v|, |x_n - v|\} < \varepsilon$ and $\max\{d_{\mathcal{H}}(\Sigma_n, \Sigma), d_{\mathcal{H}}(L_n, L' \cup L''), d_{\mathcal{H}}(\Sigma_n \setminus L_n \cup L' \cup L'', \Sigma)\} < \varepsilon$. Denote by $\overline{xy}$ the line segment with endpoints in x and y . Define

$$A_n := (\Sigma_n \setminus L_n) \cup \overline{p_n w} \cup \overline{q_n w} \cup \overline{v_n v} \cup \overline{x_n v} \cup L' \cup L'',$$

i.e. A_n is obtained from Σ_n by first removing L_n and then replacing it with line segments $\overline{p_n w}$, $\overline{q_n w}$, $\overline{v_n v}$, $\overline{x_n v}$ and $L' \cup L''$. For any sets $A, B \subset \mathbb{R}^d$ we define

$$d_{a\mathcal{H}}(A, B) = \sup_{b \in B} \inf_{a \in A} |a - b|.$$

Note that $d_{\mathcal{H}}(A, B) = \max\{d_{a\mathcal{H}}(A, B), d_{a\mathcal{H}}(B, A)\}$. Then $d_{a\mathcal{H}}(\Sigma_n, \overline{p_n w}) \leq |p_n - w| < \varepsilon$, and similarly $d_{a\mathcal{H}}(\Sigma_n, \overline{q_n w}) < \varepsilon$, $d_{a\mathcal{H}}(\Sigma_n, \overline{v_n v}) < \varepsilon$, $d_{a\mathcal{H}}(\Sigma_n, \overline{x_n v}) < \varepsilon$. Combining with $d_{\mathcal{H}}(\Sigma_n \setminus L_n \cup L' \cup L'', \Sigma) < \varepsilon$ gives $d_{a\mathcal{H}}(A_n, \Sigma_n) < \varepsilon$. Moreover it holds

$$\begin{aligned} \mathcal{H}^1(A_n) &\leq \mathcal{H}^1(\Sigma_n) - \mathcal{H}^1(L_n) + |p_n - w| + |q_n - w| + |v_n - v| + |x_n - v| + \mathcal{H}^1(L' \cup L'') \\ &\leq \mathcal{H}^1(\Sigma_n) - 0.9\mathcal{H}^1(L) + 4\varepsilon \\ &\leq \mathcal{H}^1(\Sigma_n) - 0.9a + 4\varepsilon. \end{aligned} \tag{6.2.1}$$

The issue we still face is that A_n may not belong to $\mathcal{A}$. Namely A_n may be disconnected (for example if L_n contains triple junctions). Let C_0^n be the connected component of A_n containing $L' \cup L''$. Let $\mathcal{F}^n := \{C_j^n\}_{j \in I_n}$ be the family of connected components of A_n , other than C_0^n . Since each connected component must contain an endpoint of Σ_n , by Lemma 6.2.1, I_n must be finite. We now translate these components so that they connect to $L' \cup L''$. Consider an arbitrary $C_j^n \in \mathcal{F}^n$. As L_n is the only set present in Σ_n but not present in A_n , there exists a point $s_j^n \in \overline{L_n} \cap \overline{C_j^n}$. Define

$$T_\theta : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad T_\theta(x) := x + \theta$$

the translation by a vector θ , and let Π be the projection on $L' \cup L''$, i.e. for any x the point $\Pi(x)$ satisfies $|x - \Pi(x)| = d(x, L' \cup L'')$ (if there is more than one minimizer, then $\Pi(x)$ can be chosen arbitrarily among these). Define

$$\tilde{\Sigma}_n := \left(A_n \setminus \bigcup_{j \in I_n} C_j^n \right) \cup \bigcup_{j \in I_n} T_{\Pi(s_j^n) - s_j^n}(C_j^n).$$

It is pathwise connected and compact by construction. Notice that $d_{\mathcal{H}}(L_n, L' \cup L'') < \varepsilon$ implies $|s_j^n - \Pi(s_j^n)| < \varepsilon$. Therefore $d_{a\mathcal{H}}(\tilde{\Sigma}_n, A_n) \leq \varepsilon$, which combined with $d_{a\mathcal{H}}(A_n, \Sigma_n) < \varepsilon$ gives $d_{a\mathcal{H}}(\tilde{\Sigma}_n, \Sigma_n) \leq 2\varepsilon$. From $|d(x, \tilde{\Sigma}_n) - d(x, \Sigma_n)| \leq d_{a\mathcal{H}}(\tilde{\Sigma}_n, \Sigma_n) \leq 2\varepsilon$, integrating on $\mathbb{R}^d$ yields

$$\left| \int d(x, \tilde{\Sigma}_n) d\mu_n - \int d(x, \Sigma_n) d\mu_n \right| \leq \mu_n(\mathbb{R}^d) d_{\mathcal{H}}(\tilde{\Sigma}_n, \Sigma_n) \leq 2\varepsilon. \tag{6.2.2}$$

The last step involves estimating $\mathcal{H}^1(\tilde{\Sigma}_n)$: by construction $\tilde{\Sigma}_n$ is obtained from A_n by first removing $\bigcup_{j \in I_n} C_j^n$, then adding $\bigcup_{j \in I_n} T_{\Pi(s_j^n) - s_j^n}(C_j^n)$. Since $T_{\Pi(s_j^n) - s_j^n}(C_j^n)$ is the image of C_j^n through a

translation $\mathcal{H}^1(T_{\Pi(s_j^n)-s_j^n}(C_j^n)) = \mathcal{H}^1(C_j^n)$ for any j . Using that by definition $C_j^n \cap C_s^n = \emptyset$ if $j \neq s$ it follows that

$$\mathcal{H}^1\left(\bigcup_{j \in I_n} T_{\Pi(s_j^n)-s_j^n}(C_j^n)\right) \leq \sum_{j \in I_n} \mathcal{H}^1(C_j^n) = \mathcal{H}^1\left(\bigcup_{j \in I_n} C_j^n\right) \leq \mathcal{H}^1(\Sigma_n).$$

Using (6.2.1) this gives

$$\mathcal{H}^1(\tilde{\Sigma}_n) \leq \mathcal{H}^1(A_n) \leq \mathcal{H}^1(\Sigma_n) - 0.9a + 4\varepsilon.$$

Combining with (6.2.2) we conclude

$$E_{\mu_n}^\lambda(\tilde{\Sigma}_n) \leq E_{\mu_n}^\lambda(\Sigma_n) + 2\varepsilon - 0.9\lambda a + 4\varepsilon\lambda,$$

which for ε sufficiently small contradicts the minimality of Σ_n . $\square$

Proof. [of Theorem 6.2.6] By our assumptions there exists $R > 0$ such that for all n , $\text{supp } \mu_n \subseteq \overline{B}(0, R)$. Note that then $\Sigma_n \subseteq \overline{B}(0, R)$. Let us also note that by (6.1.1) the lengths $\mathcal{H}^1(\Sigma_n)$ are uniformly bounded. By Blaschke's theorem, along a subsequence $\Sigma_n \xrightarrow{d_{\mathcal{H}}} \Sigma$ as $n \rightarrow \infty$. By relabeling the subsequence we can assume that it is the whole sequence. The lower-semicontinuity of the $\mathcal{H}^1$ with respect to Hausdorff metric proved by Golab and the continuity of $F_\mu(\Sigma)$ in both μ (with respect to weak-* topology) and Σ (with respect to Hausdorff metric) implies that Σ is a minimizer of E_μ^λ . Furthermore for any endpoint z of Σ the convergence Σ_n to Σ in Hausdorff metric implies that there exists a sequence $z_n \in \Sigma_n$ such that $z_n \rightarrow z$ as $n \rightarrow \infty$.

If z_n are all endpoints then there is nothing to prove. We start the discussion by assuming that z_n has a subsequence of points of order 2 (triple junctions are considered later). By relabeling we can assume that all of $z_n \in \Sigma_n$ are of order 2. We denote by Σ'_n and Σ''_n the two connected components of $\Sigma_n \setminus \{z_n\}$. We also choose sequences $\{v_n\}, \{x_n\}$ both converging to z and such that $v_n \in \Sigma'_n, x_n \in \Sigma''_n$ for any n .

For any set $X \subset \mathbb{R}^d$ let $r(X) = \sup_{y \in X} |y - z|$. The following dichotomy applies:

$$(\sharp) \quad \limsup_{n \rightarrow \infty} \min\{r(\Sigma'_n), r(\Sigma''_n)\} = 0.$$

$$(*) \quad \text{There exists } \beta > 0 \text{ and a subsequence } \{n_k\}_{k=1,2,\dots} \text{ such that for all } k \text{ large enough } \min\{r(\Sigma'_{n_k}), r(\Sigma''_{n_k})\} \geq \beta.$$

If $(\sharp)$ holds then, since by Lemma 6.2.3 both Σ'_n and Σ''_n contain at least one endpoint, there exists an endpoint, $\tilde{z}_n$, at distance at most $\min\{r(\Sigma'), r(\Sigma'')\}$ to z_n . Then $\tilde{z}_n \rightarrow z$ as $n \rightarrow \infty$ and thus z is a limit of endpoints as desired.

What remains is to show that the case $(*)$ is impossible. If $(*)$ holds then, from Blaschke's compactness theorem, follows that along a further subsequence, which we relabel to be the whole sequence, both connected components converge in $d_{\mathcal{H}}$, to sets with positive length. More precisely

$$\Sigma' := \lim_{n \rightarrow \infty} \Sigma'_n \cup \{z_n\}, \quad \Sigma'' := \lim_{n \rightarrow \infty} \Sigma''_n \cup \{z_n\}$$

We observe that $z \in \Sigma' \cap \Sigma''$ and that $r(\Sigma') \geq \beta$ and $r(\Sigma'') \geq \beta$. Therefore $\mathcal{H}^1(\Sigma') \geq \beta$ and $\mathcal{H}^1(\Sigma'') \geq \beta$.

We claim that there exists a point besides z in $\Sigma' \cap \Sigma''$. The reason is that, if $\Sigma' \cap \Sigma'' = \{z\}$ then the order of z is at least two, so it cannot be an endpoint, which would contradict the assumption on z . So let $w \in \Sigma' \cap \Sigma'' \setminus \{z\}$. Denote by $\Sigma'_{z,w}, \Sigma''_{z,w}$ the minimal paths connecting z and w in Σ' and Σ'' respectively. As $\Sigma'_{z,w} \neq \Sigma''_{z,w}$ would imply the existence of a loop in Σ (in view of Lemma 6.1.4), $\Sigma'_{z,w} = \Sigma''_{z,w}$ must hold.

There exist sequences $\{p_n\}, \{q_n\}$ with $p_n \in \Sigma'_n, q_n \in \Sigma''_n$, both converging to w . Let $L = \Sigma'_{z,w}$ be the minimal path in Σ between z and w . The above shows that L is a double line. This contradicts the claim of Lemma 6.2.8.

It remains to consider the case that an endpoint $z \in \Xi(\Sigma)$ is a limit of points $z_n \in \Sigma_n$ of order 3. We note that arbitrarily close to any point of order 3 there exists a point of order 2. Thus z can be obtained as a limit of points of order 2, which is the case considered above. $\square$

6.3 Topological "lower semicontinuity"

6.3.1 Topological relation

Given $\lambda > 0$ and a compactly supported probability measure μ , consider a sequence of probability measures $\{\mu_n\} \xrightarrow{*} \mu$, and for any n choose a minimizer $\Sigma_n \in \operatorname{argmin} E_{\mu_n}^\lambda$. Then upon subsequence $\Sigma_n \xrightarrow{d_H} \Sigma \in \operatorname{argmin} E_\mu^\lambda$. The aim of this section is to analyze topological relation between Σ_n (for n sufficiently large) and Σ .

The main result is:

Theorem 6.3.1. *Given $\lambda > 0$ and a compactly supported probability measure μ , consider a sequence of probability measures $\{\mu_n\} \xrightarrow{*} \mu$, with uniformly bounded support, and for any n choose a minimizer $\Sigma_n \in \operatorname{argmin} E_{\mu_n}^\lambda$. Then, along a subsequence, $\Sigma_n \xrightarrow{d_H} \Sigma \in \operatorname{argmin} E_\mu^\lambda$. For all sufficiently large n along the subsequence, there exists a homeomorphism $\varphi_n : \Sigma \rightarrow S_n$, for some $S_n \subseteq \Sigma_n$.*

Proof. We note that the statement is trivial if Σ is a singleton. Thus we assume that Σ is not a single point. The convergence of Σ_n along a subsequence follows from Theorem 6.2.6. We again assume that the subsequence is the whole sequence. Let V be the set of all endpoints and triple junctions of Σ . By Theorem 6.2.6, Σ and Σ_n contain at most $1/\lambda$ endpoints. By induction on the number of triple junctions, it is easy to prove that in any tree the number of endpoints is greater than the number of triple junctions. Thus the number of triple junctions is also bounded by $1/\lambda$. Hence V is a finite set. Thus there exists $c > 0$ such that

$$c \leq \min_{v, \tilde{v} \in V, v \neq \tilde{v}} |v - \tilde{v}|. \quad (6.3.1)$$

Choose n sufficiently large such that $d_H(\Sigma_n, \Sigma) < c/2$. From Theorem 6.2.6 it follows that any endpoint v of Σ is limit of a sequence of endpoints $\{v_n\}$ of Σ_n . Such a sequence may be not unique; we fix one for each endpoint of Σ . By relabeling the sequence, it can be assumed that for all endpoints v of Σ and the corresponding sequence of endpoints v_n of Σ_n , it holds that $|v - v_n| < c/2$ and all n .

To continue the proof we need the following lemma:

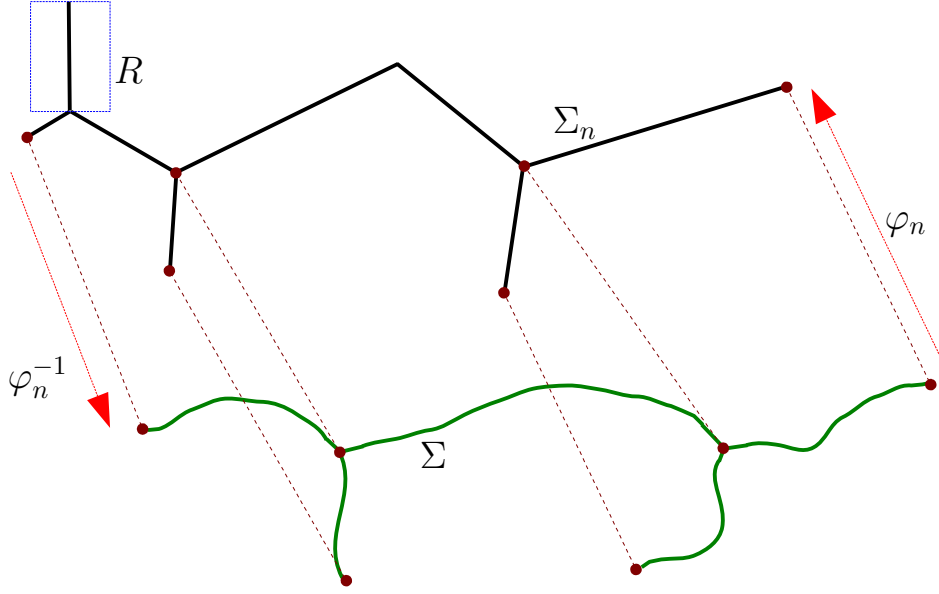


Figure 6.3.1: φ_n is an example of homeomorphism between Σ and a (proper) subset of Σ_n . The part within the dashed rectangle R is not involved in the homeomorphism.

Lemma 6.3.2. *Let Σ_n and Σ be as above. If some sequences $\{y_n\}$ and $\{w_n\}$ in Σ_n converge to distinct points $y, w \in \Sigma$ then the sequence of minimal paths $[y_n, w_n]_{\Sigma_n}$ converges in $d_{\mathcal{H}}$ to the minimal path $[y, w]_{\Sigma}$.*

Here $[y, w]_{\Sigma}$ is the minimal path in Σ containing y and w . The existence of such minimal path is ensured by Lemma 6.1.3.

Proof. Assume that this is not the case. Then there exists $\varepsilon > 0$ and a subsequence of $[y_n, w_n]_{\Sigma_n}$ such that all paths in the subsequence are at distance at least ε from $[y, w]_{\Sigma}$. By relabeling we can assume that this is the whole sequence. To obtain a contradiction it is enough to find a (further) subsequence which does converge to $[y, w]_{\Sigma}$. By compactness we know $[y_n, w_n]_{\Sigma_n}$ converges along a subsequence to some connected set $A \subset \Sigma$ which contains y and w . Since Σ is a tree, $[y, w]_{\Sigma} \subseteq A$. Let us assume that $L := A \setminus [y, w]_{\Sigma} \neq \emptyset$. Then there exists a sequence $x_n \in [y_n, w_n]_{\Sigma_n}$ such that $x_n \rightarrow x \in L$ as $n \rightarrow \infty$. Let L' be the connected component of L containing x . If $\overline{L'} \cap [y, w]_{\Sigma}$ has two or more points then Σ contains a loop, which contradicts the fact that Σ is a tree. Hence $\overline{L'} \cap [y, w]_{\Sigma}$ is a single point, denote it by p . Then $p \in [y, x]_{\Sigma}$ and $p \in [x, w]_{\Sigma}$. Thus there exist sequences $p_n \in [y_n, x_n]_{\Sigma_n}$ and $q_n \in [x_n, w_n]_{\Sigma_n}$ such that $p_n \rightarrow p$ and $q_n \rightarrow p$ as $n \rightarrow \infty$. Consequently Σ contains a double line which contradicts Lemma 6.2.8. $\square$

We return to the proof of the theorem.

Claim 1. Any triple junction $z \in \Sigma$ can be obtained as limit of a sequence of triple junctions $z_n \in \Sigma_n$. If this is not the case, there exists a triple junction $z \in \Sigma$ which cannot be obtained as limit

of triple junctions. Then there exists $\varepsilon > 0$ such that no point in $\Sigma \cap B(z, \varepsilon)$ is a limit of a sequence of triple junctions. Since z is a triple junction, there exist paths $\Gamma_i \subset B(z, \varepsilon)$, for $i = 1, 2, 3$ such that $z \in \Gamma_i$ and except for z , the paths are mutually disjoint. Choose, for $i = 1, 2, 3$, $w^i \in \Gamma_i \setminus \{z\}$. Then there exist sequences $w_n^i \in \Sigma_n$ for $i = 1, 2, 3$ such that $w_n^i \rightarrow w^i$ as $n \rightarrow \infty$. By Lemma 6.3.2, for $i, j \in \{1, 2, 3\}$ distinct, $[w_n^i, w_n^j]_{\Sigma_n} \rightarrow [w^i, w^j]_{\Sigma}$ as $n \rightarrow \infty$. Since $[w^i, w^j]_{\Sigma} \subset B(z, \varepsilon)$, none of $[w_n^i, w_n^j]_{\Sigma_n}$ contain a triple junction (for n large enough). Hence one of the points has to lie on the minimal path connecting the other two, say $w_n^2 \in [w_n^1, w_n^3]_{\Sigma_n}$ for all n large enough (along a subsequence). Thus $w^2 \in [w^1, w^3]_{\Sigma}$, which contradicts the facts that $w_2 \in \Gamma_2 \setminus \{z\}$, $[w^1, w^3] \subset \Gamma_1 \cup \Gamma_3$ and $(\Gamma_1 \cup \Gamma_3) \cap \Gamma_2 \setminus \{z\} = \emptyset$.

Similarly to the argument made for endpoints, we can assume that any triple junction $z \in \Sigma$, is a limit of a sequence of triple junctions $\{z_n\}$ such that $|z - z_n| < c/2$ for all n . This sequence may be nonunique, but we select one. From (6.3.1) follows that sequences converging to distinct endpoints/triple junctions have no overlapping elements. Let V_n be the set of endpoints and triple junctions of Σ_n which are in the sequences (selected above) converging to elements of V .

Claim 2. If $w^1, w^2 \in V$ are such that $[w^1, w^2]_{\Sigma}$ does not contain endpoints/triple junctions besides w^1 and w^2 then for all n large enough $[w_n^1, w_n^2]_{\Sigma_n}$ does not contain any elements of V_n besides w_n^1 and w_n^2 . Assume that this is not the case: that along a subsequence $V_n \cap [w_n^1, w_n^2]_{\Sigma_n}$ contains an element of V_n other than w_n^1 and w_n^2 . By considering a further subsequence we can assume that it is always from the same sequence, say $\{w_n^3\}$. That is $w_n^3 \in [w_n^1, w_n^2]_{\Sigma_n}$ for all n along a subsequence. As before we can assume that the subsequence is the whole sequence. From Lemma 6.3.2 it follows that $w^3 \in [w^1, w^2]_{\Sigma}$. Contradiction.

We are finally ready to define the desired homeomorphism. Choose a function $\varphi_n : \Sigma \rightarrow \varphi_n(\Sigma) \subseteq \Sigma_n$ such that:

- (i) if an endpoint $v \in \Sigma$ is limit of a sequence of endpoints $\{v_n\}$ with $v_n \in V_n$ then $\varphi_n(v) = v_n$,
- (ii) if a triple junction $z \in \Sigma$ is limit of a sequence of triple junctions $\{z_n\}$ with $z_n \in V_n$, then $\varphi_n(z) = z_n$,
- (iii) if $w^1, w^2 \in V$ are such that $[w^1, w^2]_{\Sigma}$ does not contain endpoints/triple junctions besides w^1, w^2 , then define $\varphi_n|_{[w^1, w^2]_{\Sigma}}$ as an arbitrary homeomorphism between $[w^1, w^2]_{\Sigma}$ and $[w_n^1, w_n^2]_{\Sigma_n}$, where $w_n^1, w_n^2 \in V_n$ and $\{w_n^1\} \rightarrow w^1, \{w_n^2\} \rightarrow w^2$, as $n \rightarrow \infty$.

The function $\varphi_n : \Sigma \rightarrow \varphi_n(\Sigma) \subseteq \Sigma_n$ is well defined, for n large enough, by Claim 2. It is injective and continuous by construction. Since Σ is compact and φ_n is a bijection, φ_n^{-1} is continuous. Thus φ_n is a homeomorphism. $\square$

6.4 BV estimates

The aim of this section is to prove a regularity result about minimizers of Problem 8.1.2. In [14], [16] and [17] it has been proven that any minimizer Σ of Problem 8.1.2 is a countable union of Lipschitz curves or a single point. Given that, by Theorem 6.2.6, the total mass arriving at each endpoint is at least λ there exists at most $1/\lambda$ endpoints. Since the number of triple junctions in a tree is less than

the number of endpoints the number of curves, forming the tree, is bounded by $1/\lambda$. Thus Σ is a finite union of Lipschitz curves or a single point. We recall that by Corollary 6.2.4 if $\lambda > 1/2$ then the only minimizer is a singleton.

The main objective of this section is to prove the following regularity result:

Theorem 6.4.1. *Given a compactly supported finite measure μ , and $\lambda > 0$, any $\Sigma \in \operatorname{argmin} E_\mu^\lambda$, which is not a single point, is finite union of Lipschitz curves $\{\gamma_k\}_{k=1}^N$ (without loss of generality assume that all γ_k are arc-length parameterized), such that*

$$\sum_k \|\gamma'_k\|_{TV} \leq \frac{1}{\lambda} |\mu(\mathbb{R}^d)|,$$

where $\|\cdot\|_{TV}$ denotes the total variation.

Note that we do not assume that μ is a probability measure.

The proof uses a discrete approximation of μ ; thus we start by proving the result for discrete measures.

Lemma 6.4.2. *Given an arbitrary positive discrete measure μ , and $\lambda > 0$, any $\Sigma \in \operatorname{argmin} E_\mu^\lambda$ is either a single point or a finite union of Lipschitz curves $\{\gamma_k\}_{k=1}^N$ (without loss of generality assume γ_k are arc-length parameterized), such that*

$$\|\gamma'_k\|_{TV} \leq \frac{1}{\lambda} \sigma(\mu, \Sigma, \gamma_k)$$

where σ is defined in Lemma 6.1.1.

Proof. Let μ be a probability measure. The result for general measures follows by scaling (See Section 2.1 in [52]). Consider an arbitrary minimizer $\Sigma \in \operatorname{argmin} E_\mu^\lambda$ which is not a single point. As we mentioned in Subsection 2.1(i) the minimizer Σ is a Steiner graph with finitely many vertices. As Steiner graphs are trees, it follows that Σ is finite union of arc-length parameterized Lipschitz curves $\{\gamma_k\}_{k=1}^N$, where each of these curves is union of line segments. Moreover, the number of curves, N , is bounded by $1/\lambda$. Choose an arbitrary $k \in \{1, \dots, N\}$. To simplify notation set $\gamma := \gamma_k$ and $L := \mathcal{H}^1(\gamma)$. Let $s_1 < s_2 < \dots < s_m$ be the values in $[0, L]$ for which $\gamma(s_i)$ is a corner.

From definition of total variation and the fact that the curve is piecewise linear

$$\|\gamma'\|_{TV([0,L])} := \sup_{0=t_0 < t_1 < \dots < t_{M-1} < t_M=L} \sum_{i=0}^{M-1} |\gamma'(t_{i+1}) - \gamma'(t_i)| = \sum_{j=1}^m |\gamma'(s_j-) - \gamma'(s_j+)|$$

where $\gamma'(\cdot-)$ and $\gamma'(\cdot+)$ denote the left and the right derivative respectively.

From the proof of Lemma 11 in [52] follows that

$$\sum_{j=1}^m |\gamma'(s_j-) - \gamma'(s_j+)| \leq \frac{1}{\lambda} \sigma(\mu, \Sigma, \gamma).$$

The inequalities above imply the desired estimate on total variation. □

We remark that combining the estimate on the total variation above and the estimate (6.1.1) on $\mathcal{H}^1(\Sigma)$ we obtain an estimate on the BV norm

$$\sum_{k=1}^N \|\gamma'_k\|_{BV} \leq \frac{1}{\lambda} (|\mu(\mathbb{R}^d)| + \text{diam supp } \mu).$$

Proof. (of Theorem 6.4.1) As in the proof of the Lemma 6.4.2, we assume that μ is a probability measure, as the general result follows by scaling. In [14], [16] and [17] (to which we refer for further details) it has been proven that minimizers of the average-distance functional are at most countable unions of Lipschitz curves, for the constrained formulation. It easy to notice that each minimizer $\Sigma \in \text{argmin } E_\mu^\lambda$ is a minimizer of $\min_{\mathcal{H}^1(\mathcal{X}) \leq \mathcal{H}^1(\Sigma)} F_\mu(\mathcal{X})$ too. In Lemma 6.2.1 it has been proven that minimizer $\Sigma \in \text{argmin } E_\mu^\lambda$ has at most $1/\lambda$ endpoints, thus Σ is finite union of Lipschitz curves.

For BV regularity, consider an arbitrary minimizer $\Sigma \in \text{argmin } E_\mu^\lambda$. If we consider a sequence of discrete approximations of μ and denote the corresponding minimizers by $\tilde{\Sigma}_n$ then $\tilde{\Sigma}_n$ converge along a subsequence to some $\tilde{\Sigma} \in \text{argmin } E_\mu^\lambda$. The problem, we need to overcome, is that $\tilde{\Sigma}$ may be different from Σ , as the minimizers are not unique in general. Note that if $\Sigma \subseteq \tilde{\Sigma}$ then there is no problem since the regularity of $\tilde{\Sigma}$ implies the regularity of Σ .

Thus we modify the measure μ in such a way that Σ is still a minimizer of the energy corresponding to the modified measure, but that ensures that any minimizer of the energy corresponding to the modified measure contains Σ . This is one of the key ideas of the chapter. Thus we introduce a perturbation μ_ε of the original measure μ : let $\varepsilon > 0$ and

$$\mu_\varepsilon := \mu + \varepsilon \frac{1}{\mathcal{H}^1(\Sigma)} \mathcal{H}^1 \llcorner \Sigma.$$

The key advantage is that, for any $\varepsilon > 0$, Σ is the minimal (w.r.t. set inclusion) minimizer of $E_{\mu_\varepsilon}^\lambda$ i.e. every minimizer $\Sigma' \in \text{argmin } E_{\mu_\varepsilon}^\lambda$ contains Σ . Indeed, as Σ is already a minimizer of E_μ^λ , given an arbitrary element $\Sigma' \in \mathcal{A}$ it holds that $E_\mu^\lambda(\Sigma) \leq E_\mu^\lambda(\Sigma')$. Hence if Σ' is a minimizer of $E_{\mu_\varepsilon}^\lambda$ then $\int_\Sigma d(x, \Sigma') d\mathcal{H}^1 \llcorner \Sigma = 0$ and thus $\Sigma \subseteq \Sigma'$.

The strategy would be the following:

- we first fix ε , and consider a discrete approximation of the perturbed measure via a sequence $\{\mu_n\} \xrightarrow{*} \mu_\varepsilon$,
- then we apply the same argument used for discrete measures, which follow without modifications,
- finally we pass to the limit $\varepsilon \rightarrow 0$, and prove that such estimates obtained at the second point are kept.

Fix $\varepsilon > 0$, and choose an approximating sequence $\{\mu_n\} \xrightarrow{*} \mu_\varepsilon$, where μ_n is a discrete measure with $\mu_n(\mathbb{R}^d) = \mu_\varepsilon(\mathbb{R}^d) = 1 + \varepsilon$. For any n choose $\Sigma_n \in \text{argmin } E_{\mu_n}^\lambda$. Along a subsequence (which we can assume to be the whole sequence), $\Sigma_n \xrightarrow{d_H} \Sigma_* \in \text{argmin } E_{\mu_\varepsilon}^\lambda$. Thus $\Sigma \subseteq \Sigma_*$.

We know that Σ_* is a union of Lipschitz curves $\{\gamma_{*,i} : i = 1, \dots, N\}$ such that endpoints of all curves are either endpoints or triple junctions of Σ_* . By Lemma 6.3.2 and the proof of Theorem 6.3.1 for each n large enough there exist curves, $\{\gamma_i^n : i = 1, \dots, N, n \geq n_0\}$, with disjoint interiors in Σ_n such that for each i , $\gamma_i^n \rightarrow \gamma_{*,i}$ in $d_{\mathcal{H}}$ as $n \rightarrow \infty$. Note that Lemma 6.4.2 gives

$$\sum_{i=1}^N \|(\gamma_i^n)'\|_{TV} \leq \frac{1+\varepsilon}{\lambda}. \quad (6.4.1)$$

Since the $\mathcal{H}^1(\Sigma_n)$ are uniformly bounded $\sum_{i=1}^N \|(\gamma_i^n)'\|_{BV}$ are also uniformly bounded. The estimates of Lemma 6.4.2 and fact (v) from the introduction (which relies on the fact that BV is compactly embedded in L^1) imply that along a subsequence $(\gamma_i^n)' \rightarrow \gamma_{*,i}'$ in L^1 as $n \rightarrow \infty$.

Since total variation is lower semicontinuous with respect to L^1 convergence we conclude that for all $i = 1, \dots, N$

$$\liminf_{n \rightarrow \infty} \|(\gamma_i^n)'\|_{TV} \geq \|\gamma_{*,i}'\|_{TV}.$$

Combining with (6.4.1), we obtain

$$\sum_{i=1}^N \|(\gamma_{*,i})'\|_{TV} \leq \frac{1+\varepsilon}{\lambda}.$$

Since the curves that make up Σ are subsets of the curves that make up Σ_* we conclude:

$$\sum_{i=1}^N \|(\gamma_i)'\|_{TV} \leq \frac{1+\varepsilon}{\lambda}.$$

Taking $\varepsilon \rightarrow 0$ yields the desired result. □

Again by combining the estimate on the total variation above and the estimate (6.1.1) on $\mathcal{H}^1(\Sigma)$ one obtains

$$\sum_{k=1}^N \|\gamma_k\|_{BV} \leq \frac{1}{\lambda} (|\mu(\mathbb{R}^d)| + \text{diam supp } \mu).$$

Chapter 7

Average distance minimization among parameterized curves

This chapter (entirely based on [37]) is mainly aimed to discuss the average distance problem (in the penalized formulation) considered among parameterized curves. This mainly arises from applied fields, e.g. data parameterization, where working with parameterized curves is computationally significantly more convenient than working with more general elements (as elements of $\mathcal{A}$, which can be very expensive computationally).

A widely used concept in some applied fields, e.g. data approximation and machine learning, is the notion of “principal curves”, and it bears strong resemblance with the average distance problem. The main difference is that for principal curves it is required for the curve to be “self-consistent”. For further reference about the principal curves we refer to [22], [23], [19], [54], [20]. However almost no connection between this and the average distance problem has been brought in literature.

The formulation is the following:

Problem 7.0.3. *Given a probability measure $\mu \geq 0$ on $\mathbb{R}^d$ with compact support, $\lambda > 0$, solve*

$$\min_{\mathcal{C}} E_{\mu}^{\lambda}$$

where

$$\mathcal{C} := \{\gamma : I \longrightarrow \mathbb{R}^d : I \subset \mathbb{R} \text{ compact interval}\}.$$

Also the symbol E_{μ}^{λ} will denote (in Chapters 7 and 8) the sum between the average distance functional and λ times the length of the parameterized curve. A precise notation of length of parameterized curve will be introduced later. Note that here we have not specified which topology is $\mathcal{C}$ endowed with. This, along with other basic issues concerning the formulation of Problem 7.0.3, will be discussed in Section 7.1. Section 7.2 will be dedicated to this issue, and contains the main results of this chapter, i.e. to prove that minimizers of Problem 7.0.3 are injective.

Note that the formulation of Problem 7.0.3 exhibits some similarity with the formulation of the problem known as “lazy traveling salesman problem” (abbreviated LTSP in [45]). However there are still significant differences, e.g. the minimizer of LTSP studied in [45] is imposed to be a closed curve, and it is easily proven to be a convex polygon, while in our case this is not required.

Moreover the techniques we use in this Chapter are quite different, and more based on tools developed to study some geometric properties of solutions of the average distance problem.

In the following, we will consider only measures of $\mathbb{R}^2$ which are nonnegative and compactly supported. The choice to work in $\mathbb{R}^2$ is dictated by the fact that Lemma 7.2.2 heavily relies on its properties, and for higher dimensions there is too little geometric rigidity to achieve similar results with our arguments. Moreover, we will often identify a curve with its parameterization function, i.e. given a curve $\gamma : I \rightarrow \mathbb{R}^2$ (with I being the domain), we will write $\mathcal{H}^1(\gamma)$, $F_\mu(\gamma)$, $E_\mu^\lambda(\gamma)$ instead of $\mathcal{H}^1(\gamma(I))$, $F_\mu(\gamma(I))$, $E_\mu^\lambda(\gamma(I))$.

Section 7.1 will present some basic facts concerning solutions, while Section 7.2 (work in collaboration with Slepčev) is mainly dedicated to prove injectivity of solutions.

7.1 Preliminaries

The main aim of this Section is to present some preliminary arguments concerning Problem 7.0.3. The first step is to endow $\mathcal{C}$ with a suitable topology.

First we recall that given a parameterized curve, there is a natural way to define its length using total variation. Note that in this chapter, the length of the curve can be different from the $\mathcal{H}^1$ measure of its graphs. We will endow $\mathcal{C}$ with the uniform convergence of parameterizations (upon time inversion), i.e. a sequence $\{\gamma_n : [0, l_n] \rightarrow \mathbb{R}^2\} \subseteq \mathcal{C}$, parameterized by arclength, converges to an element $\gamma : [0, l] \rightarrow \mathbb{R}^2$ if

- $l_n \rightarrow l$,
- upon time inversion, i.e. replacing γ_n with $\tilde{\gamma}_n$ defined as $\tilde{\gamma}_n(t) := \gamma_n(l_n - t)$, the sequence $\{\gamma_n\}$ converges to γ uniformly.

The first problem is existence of minimizers: note that if given a minimizing sequence $\{\gamma_n : [0, l_n] \rightarrow \mathbb{R}^2\}$, the union $\bigcup_{n \geq n_0} \gamma_n([0, l_n])$ are contained in a compact set (for some $n_0 \in \mathbb{N}$), then it is possible to apply Ascoli-Arzelà theorem to get existence of an accumulation point γ , which is clearly a minimizer.

Lemma 7.1.1. *Given a measure μ , a parameter $\lambda > 0$, then for any minimizing sequence $\{\gamma_n : [0, l_n] \rightarrow \mathbb{R}^2\}$ there exists a compact set $K \subset \mathbb{R}^2$ such that $\gamma_n([0, l_n]) \subset K$ for any n sufficiently large.*

Proof. It suffices to prove that given a minimizing sequence $\{\gamma_n : [0, l_n] \rightarrow \mathbb{R}^2\}$, parameterized by arclength, then $\limsup_{n \rightarrow \infty} l_n < \infty$. Note that the opposite, i.e. there exists a subsequence $\{\gamma_{n_k} : [0, l_{n_k}] \rightarrow \mathbb{R}^2\}$ with $l_{n_k} \rightarrow \infty$, would imply

$$E_\mu^\lambda(\gamma_{n_k}) := F_\mu(\gamma_{n_k}) + \lambda l_{n_k} \rightarrow \infty,$$

contradicting the fact that $\{\gamma_n : [0, l_n] \rightarrow \mathbb{R}^2\}$ is a minimizing sequence.

Then using the fact that μ is compactly supported, such curves cannot escape to infinity, due to the uniformly bounded length, as the opposite would imply $\sup_n \{F_\mu(\gamma_n)\} = \infty$ contradicting the fact that $\{\gamma_n\}$ is a minimizing sequence. $\square$

Thus we have proven:

Theorem 7.1.2. *Given a measure μ and a parameter $\lambda > 0$, then Problem 7.0.3 admits minimizers.*

7.2 Injectivity

The main aim of this section is to prove that minimizers of Problem 7.0.3 are injective curves, in two dimension case. For convenience, given a parameterized curve $\gamma : I \rightarrow \mathbb{R}^2$, the notation $N(\gamma)$ will be used to denote the set of non injectivity of γ , i.e.

$$t \in N(\gamma) \iff \exists s \neq t : \gamma(s) = \gamma(t).$$

For convenience the image (through γ) of any subset $A \subseteq N(\gamma)$ will be called “double part”. Our goal is to prove that it is empty whenever $\gamma \in \operatorname{argmin} E_\mu^\lambda$.

Lemma 7.2.1. *Given a measure μ , a parameter $\lambda > 0$ and a minimizer $\gamma \in \operatorname{argmin} E_\mu^\lambda$, assume there exists a point $p = \gamma(t) = \gamma(s)$, for some $t < s$. Then for any sequence $\{s_n^-\} \rightarrow s^-$, $\{s_n^+\} \rightarrow s^+$, the angle $\angle \gamma(s_n^-)\gamma(s)\gamma(s_n^+)$ converges to 0 as $n \rightarrow \infty$.*

Proof. Suppose by contradiction that there exist sequences $\{s_n^-\} \rightarrow s^-$, $\{s_n^+\} \rightarrow s^+$, such that the angle $\angle \gamma(s_n^-)\gamma(s)\gamma(s_n^+)$ converges to $\alpha \neq 0$ as $n \rightarrow \infty$.

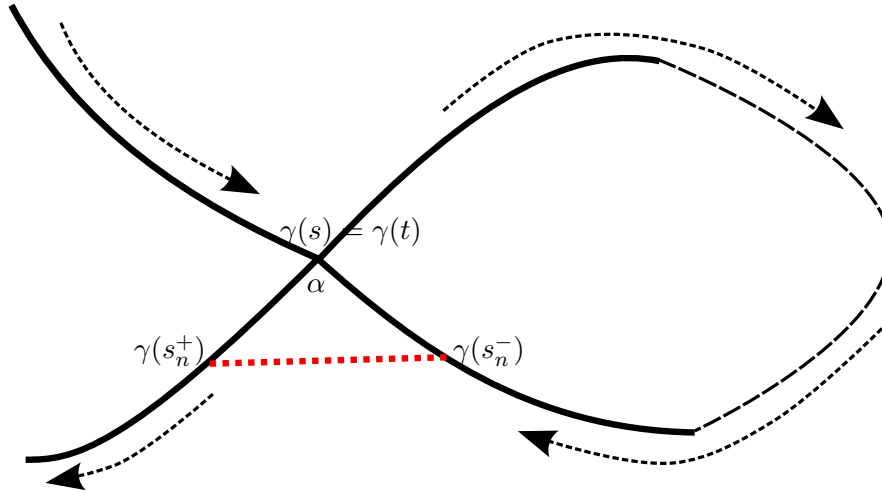


Figure 7.2.1: This is a schematic representation of the variation. The black lines belong to the (graph of) γ , while the red dotted line belong to the (graph of) competitor $\tilde{\gamma}_n$. Time increases along the direction of those arrows.

Consider the competitor $\tilde{\gamma}_n$ obtained from γ by replacing $\gamma([s_n^-, s_n^+])$ with a straight segment between $\gamma(s_n^-)$ and $\gamma(s_n^+)$. The fact that $\angle \gamma(s_n^-)\gamma(s)\gamma(s_n^+)$ converges to $\alpha \neq 0$ as $n \rightarrow \infty$ implies that denoting with $\delta_n := |\gamma(s_n^-) - \gamma(s_n^+)|$, it holds

$$|s_n^+ - s_n^-| - \delta_n \geq k|s_n^+ - s_n^-|,$$

for some $k > 0$ independent of n , or equivalently

$$\mathcal{H}^1(\gamma) \geq \mathcal{H}^1(\tilde{\gamma}_n) + k|s_n^+ - s_n^-|. \quad (7.2.1)$$

Notice that if a point z verifies

$$d(z, \gamma) < d(z, \tilde{\gamma}_n)$$

then $\operatorname{argmin}_{y \in \gamma([0, L_\gamma])} |z - y| \subseteq \gamma((s_n^-, s_n^+) \setminus \{s\})$, which yields

$$F_\mu(\tilde{\gamma}_n) - F_\mu(\gamma) \leq \mu(\{z : \operatorname{argmin}_{y \in \gamma([0, L_\gamma])} |z - y| \subseteq \gamma((s_n^-, s_n^+) \setminus \{s\})\}) |s_n^+ - s_n^-|.$$

Since $\mu(\{z : \operatorname{argmin}_{y \in \gamma([0, L_\gamma])} |z - y| \subseteq \gamma((s_n^-, s_n^+) \setminus \{s\})\}) \rightarrow 0$ as $n \rightarrow \infty$, combining with (7.2.2) gives that the minimality of γ is contradicted by $\tilde{\gamma}_n$ for n sufficiently large. $\square$

Thus the tangent line in any such point $p \in \gamma(N(\gamma))$ must be well defined. The next result deals with this case.

Lemma 7.2.2. *Given a measure μ , a parameter $\lambda > 0$ and a curve γ , assume $N(\gamma)$ contains a point $p = \gamma(t) = \gamma(s)$, for some $t < s$. Assume moreover that the tangent line is well defined in p . Then γ is not a minimizer of Problem 7.0.3.*

Before proving the Lemma 7.2.2, we recall an auxiliary result, which is a similar from [52] but applied to parameterized curves:

Lemma 7.2.3. *Given a measure μ , a parameter λ , and an arbitrary minimizer $\Sigma \in \operatorname{argmin}_A E_\mu^\lambda$, consider an arbitrary a curve $\gamma : [0, 1] \rightarrow \Sigma$ such that $\gamma([0, 1])$ does not contain points with order at least 3. Then it holds:*

1. *if for a time s the tangent line is not well defined at $\gamma(s)$, i.e. $\gamma(s)$ is a corner point, then denoting with θ the angle between the left and right tangent direction, it holds*

$$\pi - \theta \leq C\psi(\{\gamma(s)\})$$

where $\psi(A)$ denotes the mass projecting on the set A , and C is a positive constant.

2. *For any times $0 \leq s < t \leq 1$ such that the tangent lines at $\gamma(s)$ and $\gamma(t)$ are well defined, then denoting with θ the angle between these two tangent lines it holds*

$$\pi - \theta \leq C\psi(\gamma([s, t])).$$

Notice that as Σ is finite union (see for instance [38]) of curves each of which not containing points with order at least 3, the first point of Lemma 7.2.3 implies that the number of corner points is at most countable.

Proof. The proof is based on a (local) perturbation argument: given a corner $\gamma(s)$ in first order approximation the behavior of γ near this point can be approximated by its left and right tangent lines. Thus upon scaling (of an arbitrarily small neighborhood of $\gamma(s)$) it suffices to compare the following configurations:

1. $X_0 := \{(1-t)(-a, 0) + t(0, 1) : t \in [0, 1]\} \cup \{(1-t)(a, 0) + t(0, 1) : t \in [0, 1]\}$ where $a > 0$ is clearly depending on θ (this is how the configuration near $\gamma(s)$, say $\gamma([s - \varepsilon_1, s + \varepsilon_2])$ for some $\varepsilon_1, \varepsilon_2 > 0$, looks in first order approximation after scaling, with $\gamma(s)$ being mapped into $(0, 1)$),
2. $X_h := \{(1-t)(-a, 0) + t(0, 1-h) : t \in [0, 1]\} \cup \{(1-t)(a, 0) + t(0, 1-h) : t \in [0, 1]\}$, where $h > 0$ is a free (small) parameter.

If X_0 is replaced by X_h the mass projecting on $(0, 1)$, which we call M , can go on $(0, 1-h)$, thus the loss for the average distance functional is at most Mh . On the other side by directly computing (using cosine theorem) the difference between the length of the configuration X_h and the length of X_0 , using the minimality of γ , one obtains the desired lower bound on M . $\square$

Proof. (of Lemma 7.2.2) Upon scaling and translation, the configuration can be mapped into that in Figure 7.2.2.

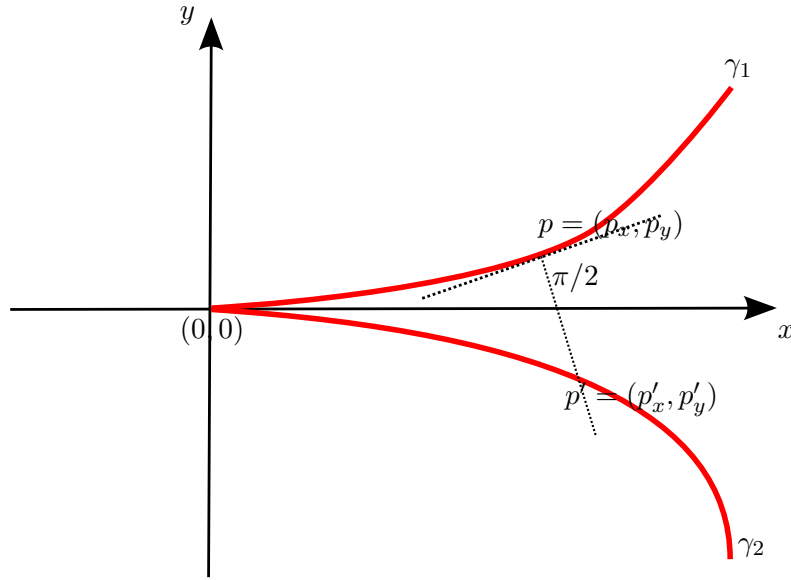


Figure 7.2.2: This is a schematic representation of the configuration.

The point p' is the intersection between the graph of the curve γ_2 the the orthogonal line to (the graph of) γ_1 at p . Using Lemma 7.2.3 it follows that a necessary condition is

$$\frac{p_y}{p_x} \leq C|p||p - p'|,$$

i.e. $|p_y| \ll |p'_y|$, where C is the (positive, but whose precise value is not influent) constant arising from the proof of Lemma 7.2.3. Using the same construction, i.e. denoting with p'' the intersection between the orthogonal line to (the graph of) γ_2 at p' and (the graph of) γ_1 , it follows $|p'_y| \ll |p''_y|$. But as the tangent line at $(0, 0)$ is well defined, and coincides with the x -axis, p_y and p''_y tend to coincide as the point p goes to $(0, 0)$, i.e.

$$p \rightarrow (0, 0) \implies |p_y|/|p''_y| \rightarrow 1.$$

Thus the proof is complete. $\square$

The key idea behind this result is quite simple: if such configuration arises, then morally “there is not enough mass to pull it to compensate for the penalization from the additional length”. Note that even if there exists a segment visited on two disjoint time intervals, on which no mass is projected (i.e. its function is merely to preserve the parameterization), we can apply the previous two lemmas on its endpoints (of the segment, not of the whole graph, if the minimizer is itself a straight segment, then it is clearly injective) and achieve a similar contradiction. Thus injectivity has been proven.

Now we investigate some more geometric properties. The next result proves some geometric properties on the graph of the curves. In particular it will prove that any minimizer γ maps endpoints of the set of times into endpoints of its graph, and the number of endpoints of its graph is finite.

Lemma 7.2.4. *Given a minimizer $\gamma \in \operatorname{argmin} E_\mu^\lambda$, assume wlog $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ parameterized by constant speed, then $\gamma(0)$ and $\gamma(1)$ are endpoints of the graph Γ_γ .*

Proof. Assume for the sake of contradiction that $\gamma(0)$ is not an endpoint. Thus there exists a sequence $\{t_n\} \subset [c, 1]$ for some $c > 0$ such that $\gamma(t_n) \rightarrow \gamma(0)$. Fix $\varepsilon > 0$ and consider the competitor defined as

$$\gamma_\varepsilon : [0, 1 - \varepsilon] \rightarrow \mathbb{R}^2, \quad \gamma_\varepsilon(t) := \gamma(t + \varepsilon).$$

By construction it holds

$$\Gamma_{\gamma_\varepsilon} \supseteq \Gamma_\gamma \setminus \gamma([0, \varepsilon)).$$

Note that the only points $z \in \mathbb{R}^2$ which can potentially satisfy

$$d(z, \Gamma_{\gamma_\varepsilon}) > d(z, \Gamma_\gamma)$$

are those satisfying

$$\operatorname{argmin}_{y \in \Gamma_\gamma} |z - y| \subseteq \gamma((0, \varepsilon)),$$

in view of the contradiction assumption. Clearly for any such z it holds

$$d(z, \Gamma_{\gamma_\varepsilon}) \leq d(z, \Gamma_\gamma) + |\dot{\gamma}| \varepsilon,$$

with $|\dot{\gamma}|$ denoting the speed of the parameterization, and

$$\mathcal{H}^1(\gamma_\varepsilon) = \mathcal{H}^1(\gamma) - |\dot{\gamma}| \varepsilon. \tag{7.2.2}$$

Note that the set $\gamma((0, \varepsilon)) \rightarrow \emptyset$ as $\varepsilon \rightarrow 0$, thus

$$\lim_{\varepsilon \rightarrow 0} \mu(\{z : \operatorname{argmin}_{y \in \Gamma_\gamma} |z - y| \subseteq \gamma((0, \varepsilon)), \}) = 0,$$

and integrating w.r.t. μ gives

$$F_\mu(\gamma_\varepsilon) - F_\mu(\gamma) \leq \varepsilon \mu(\{z : \operatorname{argmin}_{y \in \Gamma_\gamma} |z - y| \subseteq \gamma((0, \varepsilon)), \}),$$

which combined with (7.2.2) gives that γ_ε contradicts the minimality of γ for ε sufficiently small. The proof for $\gamma(1)$ is completely analogous. $\square$

The next result proves that for any minimizer, its graph may have only finitely many endpoints, and (independently of the parameterization) each of those endpoints is visited only finitely many times.

Lemma 7.2.5. *Given a minimizer $\gamma \in \operatorname{argmin} E_\mu^\lambda$, the graph Γ_γ has at most $1/\lambda$ endpoints. Moreover, for any such endpoint v the set $\gamma^{-1}(v)$ is finite.*

Proof. First we prove that for any endpoint v , the set $\gamma^{-1}(v)$ is finite. As γ is parameterized by constant speed, there exist no intervals (a, b) such that $\gamma((a, b)) = v$. Assume for the sake of contradiction that there exists a sequence $\{t_n\}$ such that $\gamma(t_n) = v$ for any n . As v is an endpoint, for each time t_n with $\gamma(t_n) = v$, it must hold

$$\lim_{s \rightarrow t_n^-} \dot{\gamma}(s) = \lim_{s' \rightarrow t_n^+} \dot{\gamma}(s'),$$

i.e. the velocity vector reverts its direction in t_n , thus increasing $|\dot{\gamma}|_{TV}$ by π . Thus having infinitely many times t_n such that $\gamma(t_n) = v$ yields

$$\|\dot{\gamma}\|_{TV} = \infty,$$

while in [38] it has been proven that for any minimizer the quantity $\|\dot{\gamma}\|_{TV}$ is bounded from above by a constant depending only on μ and λ , which is a contradiction.

To prove that the set of endpoints is finite, it suffices to find a lower bound for the mass projecting on each endpoint.

Consider an arbitrary endpoint v of Γ_γ , and choose an arbitrary time $t \in \gamma^{-1}(v)$. Fix $\varepsilon > 0$, as v is an endpoint there exist times $t_\varepsilon^- < t < t_\varepsilon^+$ such that

$$t_\varepsilon^+ - t_\varepsilon^- \leq \varepsilon, \quad \gamma(t_\varepsilon^+) = \gamma(t_\varepsilon^-).$$

Consider the competitor γ_ε defined as

$$\gamma_\varepsilon : [0, \mathcal{H}^1(\gamma) - (t_\varepsilon^+ - t_\varepsilon^-)] \longrightarrow \mathbb{R}^2, \quad \gamma_\varepsilon(t) := \begin{cases} \gamma(t) & \text{if } t \leq t_\varepsilon^- \\ \gamma(t + (t_\varepsilon^+ - t_\varepsilon^-)) & \text{if } t \geq t_\varepsilon^+ \end{cases}$$

Denote with $\psi(v)$ the mass projecting on v , it holds

$$F_\mu(\gamma_\varepsilon) \leq F_\mu(\gamma) + \varepsilon \psi(v),$$

and $E_\mu^\lambda(\gamma) \leq E_\mu^\lambda(\gamma_\varepsilon)$ yields $\psi(v) \geq \lambda$. Thus the number of endpoints is at most $1/\lambda$. $\square$

This result has a very important consequence: as the number of endpoints is finite, the number of points with order at least 3 is also finite.

Recall that we have already proven

1. Lemma 7.2.4: the graph Γ_γ contains at least 2 endpoints,
2. Lemma 7.2.5: the graph Γ_γ has a finite number of endpoints,

The last result of this chapter would be

1. there exists a neighborhood of the form $[0, \varepsilon)$ such that $[0, \varepsilon) \subsetneq N(\gamma)$.

Lemma 7.2.6. *Given a measure μ , and a parameter $\lambda > 0$, then for any minimizer $\gamma \in \operatorname{argmin}_{\mathcal{C}} E_{\mu}^{\lambda}$, the set $N(\gamma)$ does not coincide with the domain of γ .*

Proof. It suffices to prove that $N(\gamma)$ does not coincide with the whole set of times, i.e. there exists a time t such that $\gamma(t) \neq \gamma(s)$ for any $s \neq t$. Assume by contradiction that the opposite holds, i.e. $N(\gamma)$ coincides with the whole set of times, or equivalently, each point of the graph of γ is visited at least twice. Then choose an interval of the form $[0, \xi)$ with ξ arbitrary. By (contradiction) assumption $[0, \xi) \subset N(\gamma)$.

Using Lemma 7.2.5 gives that there exist only finitely many points with order at least 3, i.e. if ξ is chosen sufficiently small, the set $\gamma([0, \xi))$ is homeomorphic to $[0, \xi)$. Thus it is possible to choose a parameterization $\gamma^* : [0, \mathcal{H}^1(\gamma([0, \xi))) \rightarrow \gamma([0, \xi))$. Define the competitor γ_{ξ} as follows:

$$\gamma_{\xi}(t) := \begin{cases} \gamma^*(t) & \text{if } t < \xi \\ \gamma(t) & \text{if } t \geq \xi \end{cases}$$

As $[0, \xi) \subseteq N(\gamma)$, it follows $\mathcal{H}^1(\gamma_{\xi}) < \mathcal{H}^1(\gamma)$, and since $\Gamma_{\gamma_{\xi}} = \Gamma_{\gamma}$ by construction, the minimality of γ is contradicted. $\square$

Chapter 8

A relaxed and penalized formulation

As mentioned before, the formulation of Problem 7.0.3 could be potentially used in data parameterization, but there are several issues:

- as proven in [52], even with very regular measures μ (such that the Radon-Nykodim derivative $\frac{d\mu}{d\mathcal{L}^n}$ is C^∞), there may exist minimizers which are not C^1 regular,
- the formulation of Problem 7.0.3 imposes strong geometric rigidity on the minimizers.

More details about such issues will be described in the next section, where additional penalization terms will be introduced to take account for such issues.

Finally, even if we have proven injectivity for minimizers for Problem 7.0.3 in the previous chapter, by adding such additional terms, injectivity is not guaranteed anymore (as relaxing the problem would impose less geometric rigidity, making the arguments used in the previous chapter unusable). Nonetheless it is heavily desired, as we will discuss in the following, thus a penalization term will be added to penalize lack of injectivity.

The main aim of this chapter (based on [36]) is to introduce appropriate penalization terms, and to analyze some regularity properties of densities of mass distribution on minimizers.

Some word about notations: in this chapter we will work with parameterized injective curves. Thus given a curve $\gamma : [0, l] \rightarrow \mathbb{R}^d$, where $[0, l]$ is a suitable domain, it is natural to identify a point of the graph of γ with its counterimage through γ^{-1} . So even if ν will be a measure concentrated on the graph of γ , when no risk of confusion arises (and $\nu \ll \mathcal{H}_{|\gamma([0, l])}^1$) we can identify it with the function $\tilde{\nu} : [0, l] \rightarrow [0, \infty)$ such that the value of $\tilde{\nu}(t)$ coincides with the value of the density $d\nu/d\mathcal{H}_{|\gamma([0, l])}^1$ in $\gamma(t)$. With further abuse of notation, we will write $\nu \ll \mathcal{L}^1$ (as we will assume sufficient regularity such that $\tilde{\nu} \ll \mathcal{L}_{|[0, l]}^1$), and use the notation $d\nu/d\mathcal{L}^1$.

Section 8.1 provides a background and some motivations leading to the introduction of penalization terms, while Section 8.2 presents some regularity results. The main results are from a work by the author.

8.1 Penalization terms

As described in the Introduction, an undesirable property of solutions of Problem 7.0.3 is that its minimizers can be injective but not C^1 regular, even if μ is sufficiently regular: indeed it has been proven in [52] that there exist measures $\mu \ll \mathcal{L}^d$ (with $d\mu/d\mathcal{L}^d \in C^\infty$) for which there exists a curve $\gamma_{\text{opt}} \in \operatorname{argmin}_{\mathcal{C}_{\text{par}}} E_\mu^\lambda$ which is not C^1 -regular, and the measure γ_{opt}'' contains a Dirac mass with positive measure. In this case a set with positive μ measure is projected on a single point, which is not desirable in data approximation, as this corresponds to some loss of information.

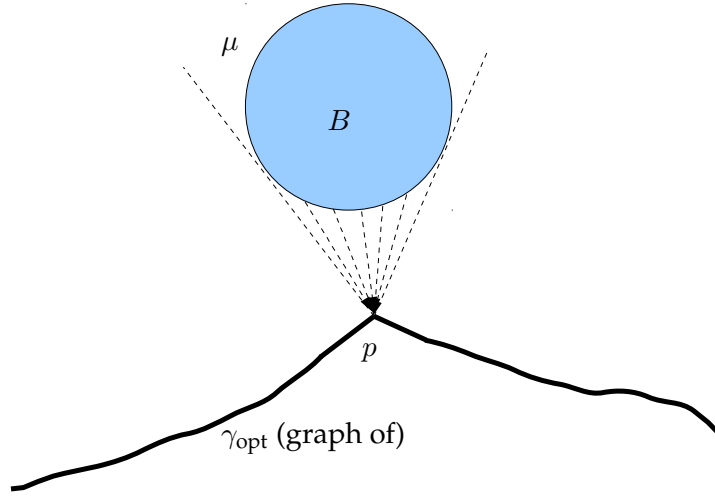


Figure 8.1.1: In this example from [52], the set B with positive μ measure is projected on the single point p . Thus $\gamma_{\text{opt}}''(\gamma_{\text{opt}}^{-1}(p))$ is an atom of positive measure. The dashed lines denote the orthogonal lines to the left and right tangent line to γ_{opt} in p . Notice that in the original example from [52] there was mass projecting on the endpoints of γ_{opt} , but we omitted representing these as not influent to our argument.

The above configuration is an extreme case of a more general issue: indeed in the formulation of Problem 7.0.3 there is no penalization for high data concentration on the graph of γ .

Denote with

$$\mathcal{C} := \{\gamma : [0, L_\gamma] \longrightarrow \mathbb{R}^d : \gamma \text{ parameterized arc-length and injective}\},$$

endowed with the convergence inherited from $\mathcal{C}_{\text{par}}$.

Notice also that in Problem 7.0.3 the very definition of F_μ forces any point to project to (one of) the closest points on the graph of γ , imposing strong geometric rigidity. A relaxed formulation will be used.

Problem 8.1.1. Given a probability measure μ on $\mathbb{R}^d$ with compact support, and parameters $\lambda, \varepsilon > 0$, $q > 1$, solve

$$\min \int_{\mathbb{R}^d \times \gamma([0, L_\gamma])} |x - y| d\Pi(x, y) + \lambda L_\gamma + \varepsilon \int_0^{L_\gamma} \nu^q ds,$$

among triples (γ, ν, Π) , where $\gamma : [0, L_\gamma] \rightarrow \mathbb{R}^d$ varies in $\mathcal{C}$, ν among probability measures on $\gamma([0, L_\gamma])$, and Π among transport plans between μ and ν .

The term $\int_0^{L_\gamma} \nu^q(s) ds$ is to be interpreted as

- $+\infty$ if $\nu \perp \mathcal{L}^1 \neq 0$,
- $\int_0^{L_\gamma} \left(\frac{d\nu}{d\mathcal{L}^1}(s) \right)^q ds$ otherwise.

This choice is justified in view of Proposition 8.1.3.

Notice that in this case, differently from Problem 7.0.3, it is not required that each x is projected to (one of) the nearest point on the graph of γ .

However there is another undesirable issue, mainly arising from lack of injectivity. Given a data cloud (represented by μ), and a triple (γ, ν, Π) solving Problem 8.1.1, there are essentially two notions of distances:

- for data points of μ , the Euclidean distance is the natural choice,
- for the projections on the parameterization γ , however the natural distance to consider is the path distance on γ , i.e. the distance between $\gamma(s)$ and $\gamma(t)$ is $|s - t|$ as γ is parameterized by arc-length.

Clearly, if γ is not injective, then there exist $s < t$ with $\gamma(s) = \gamma(t)$, and these two distances are non equivalent. This means that data points that are “close” (w.r.t. Euclidean distance) can be projected on points which are “far away” w.r.t. path distance on γ (although “close” w.r.t. Euclidean distance). Figure 8.1.2 is a possible example of this situation. This is not desirable.

Moreover, if $\gamma(s) = \gamma(t)$, then $\frac{d\nu}{d\mathcal{L}^1}(s)$ (and $\frac{d\nu}{d\mathcal{L}^1}(t)$) is not well defined.

This undesirable issue is strongly related to non injectivity. To overcome this issue, a term penalizing non injectivity will be introduced. Define

$$\eta(\gamma) := \int_0^{L_\gamma} \int_0^{L_\gamma} \left(\frac{|t - s|}{|\gamma(t) - \gamma(s)|} \right)^2 dt ds,$$

and consider the problem

Problem 8.1.2. Given probability measure μ on $\mathbb{R}^d$ with compact support, and parameters $\lambda, \varepsilon, \varepsilon' > 0$, $q > 1$, solve

$$\min \int_{\mathbb{R}^d \times \gamma([0, L])} |x - y| d\Pi(x, y) + \lambda L + \varepsilon \int_0^{L_\gamma} \nu^q ds + \varepsilon \eta(\gamma).$$

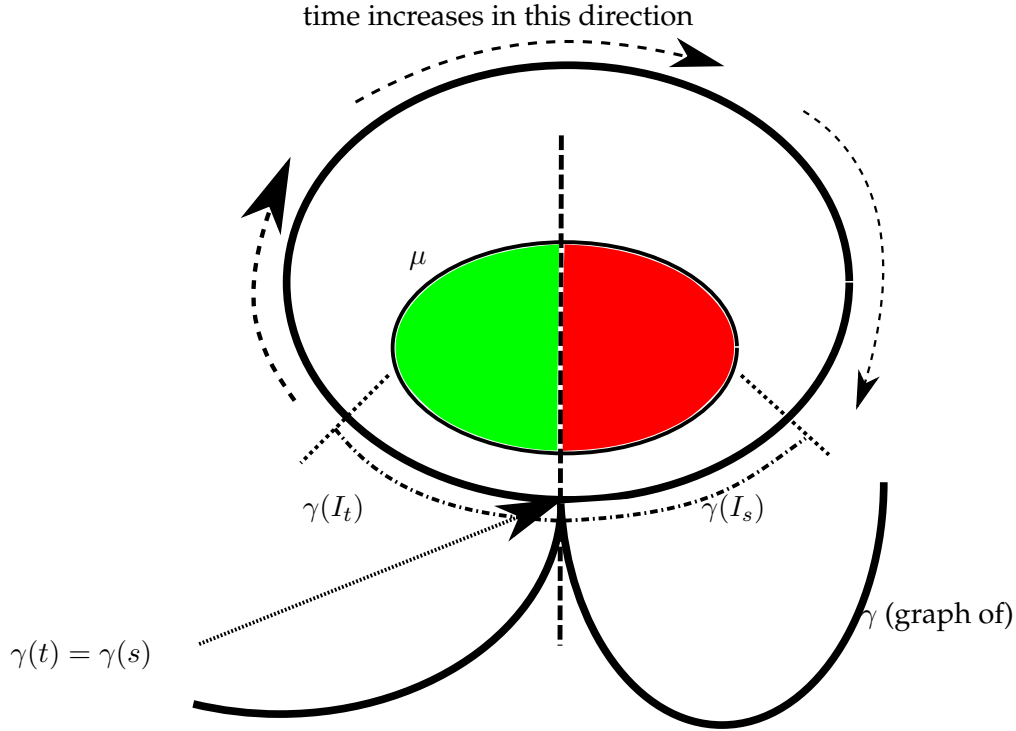


Figure 8.1.2: In this configuration, assuming $t < s$, points belonging to the red part are projected on $\gamma(I_s)$, while points belonging to the green part of the ball B are projected on $\gamma(I_t)$. The dashed line separates the two parts of B . The sets $\gamma(I_s)$ and $\gamma(I_t)$ not close in the intrinsic distance of the parameterization. The colored area is part of $\text{supp}(\mu)$.

To simplify notations, denote with

$$E_{\mu}^{\lambda, \varepsilon, \varepsilon'}(\gamma, \nu, \Pi) := \int_{\mathbb{R}^d \times \gamma([0, L_{\gamma}])} |x - y| d\Pi(x, y) + \lambda L + \varepsilon \int_0^{L_{\gamma}} \nu^q(s) ds + \varepsilon' \eta(\gamma).$$

The dependence on q will be omitted if no risk of confusion arises. The set $\mathcal{C}$ endowed with the convergence from $\mathcal{C}_{par}$ is not sequentially compact, thus the first problem is existence of minimizers for Problem 8.1.2. A preliminary result is required.

Lemma 8.1.3. *Given μ and parameters $\lambda, \varepsilon, \varepsilon' > 0, q > 1$ if a triple (γ, ν, Π) satisfies $E_{\mu}^{\lambda, \varepsilon, \varepsilon'}(\gamma, \nu, \Pi) < \infty$, then $\nu \ll \mathcal{L}^1$.*

Notice that the set $\{E_{\mu}^{\lambda, \varepsilon, \varepsilon'} < \infty\}$ is clearly non empty for any choice of $\mu, \lambda, \varepsilon, \varepsilon', q$: indeed choose $x \in \text{supp}(\mu), y$ with $|x - y| = 1$, and the element

$$\gamma : [0, 1] \longrightarrow \mathbb{R}^d, \quad \gamma(t) := (1 - t)x + ty, \quad \nu := \mathcal{L}_{[0, 1]}^1$$

and Π an optimal plan between μ and $\gamma_{\#}\nu$. The element (μ, ν, Π) belongs to $\{E_{\mu}^{\lambda, \varepsilon, \varepsilon'} < \infty\}$.

Proof. Decompose $\nu = \nu_a + \nu_s$ where $\nu_a \ll \mathcal{L}^1$, $\nu_s \perp \mathcal{L}^1$. Suppose by contradiction $\nu_s \neq 0$, i.e. there exists a $\mathcal{L}^1$ -negligible set $A \subseteq [0, L_\gamma]$ such that $\nu_s(A) = a > 0$.

Let A_n be a sequence of open sets satisfying $A_n \downarrow A$, and $\mathcal{L}^1(A_n) = 1/n$. Then it holds:

$$\int_{A_n} \nu_s^q ds \geq \int_{A_n} \left(\frac{a}{1/n} \right)^q ds = \frac{a^q}{1/n^q} 1/n = a^q n^{q-1}$$

and passing to the limit $n \rightarrow \infty$ concludes the proof. $\square$

Now it is possible to prove existence.

Proposition 8.1.4. *For any choice of $\mu, \lambda, \varepsilon, \varepsilon', q$, Problem 8.1.2 admits minimizers.*

Proof. Consider a minimizing sequence (γ_n, ν_n, Π_n) . Upon subsequence $\gamma_n \rightarrow \gamma$ (the convergence is intended in the topology of $\mathcal{C}_{par}$, and notice that a priori is not guaranteed $\gamma \in \mathcal{C}$, but only $\gamma \in \mathcal{C}_{par}$), and $\{\gamma_n^\# \nu_n\}$ is tight in view of Lemma 8.2.2. It is not restrictive to assume Π_n is an optimal plan between μ and $\gamma_n^\# \nu_n$, as otherwise replacing Π_n with Π'_n optimal plan between μ and $\gamma_n^\# \nu_n$ would yield a sequence $(\gamma_n, \nu_n, \Pi'_n)$ with $E_\mu^{\lambda, \varepsilon, \varepsilon'}(\gamma_n, \nu_n, \Pi'_n) \leq E_\mu^{\lambda, \varepsilon, \varepsilon'}(\gamma_n, \nu_n, \Pi_n)$, i.e. $(\gamma_n, \nu_n, \Pi'_n)$ is also a minimizing sequence.

Using Prokhorov theorem gives the existence of ν such that upon subsequence (which will not be relabeled) $\nu_n \rightarrow \nu$ narrowly, and endow $\gamma([0, L_\gamma])$ with probability measure $\gamma^\# \nu$. It is straightforward to check $\gamma_n^\# \nu_n \rightarrow \gamma^\# \nu$ narrowly. Choose an optimal transport plan Π_{opt} between μ and $\gamma^\# \nu$. Denoting with Π a narrow limit of $\{\Pi_n\}$ (again we do not relabel subsequences), it holds

$$\begin{aligned} \lim_n \int_{\mathbb{R}^d \times \gamma_n([0, 1])} |x - y| d\Pi_n(x, y) &= \int_{\mathbb{R}^d \times \gamma([0, 1])} |x - y| d\Pi(x, y) \\ &\geq \int_{\mathbb{R}^d \times \gamma([0, 1])} |x - y| d\Pi_{opt}(x, y). \end{aligned}$$

Being a minimizing sequence, it holds $\sup_n \int_0^{L_{\gamma_n}} \nu_n^q ds < \infty$; thus the convergence $\nu_n \xrightarrow{*} \nu$ implies $\nu \ll \mathcal{L}^1$ and

$$\int_0^{L_\gamma} \nu^q ds \leq \liminf_n \int_0^{L_{\gamma_n}} \nu^q ds,$$

and recall that

$$L_\gamma \leq \liminf_{n \rightarrow \infty} L_{\gamma_n}.$$

Thus it remains to consider the term $\varepsilon' \eta(\cdot)$. First, being a minimizing sequence it must hold

$$\sup_n \eta(\gamma_n) < \infty. \quad (8.1.1)$$

Convergence in $\mathcal{C}_{par}$ implies

$$|\gamma(t) - \gamma(s)| \leq \liminf_n |\gamma_n(t) - \gamma_n(s)|$$

for any $t, s \in [0, 1]$, thus $\eta(\gamma_n) \rightarrow \eta(\gamma)$. The final step is to prove that γ is effectively injective.

Suppose the opposite holds, i.e. there exist $t, s \in [0, 1]$ with $\gamma(t) = \gamma(s)$. Notice that in this case the integrand of $\eta(\gamma)$, $\left(\frac{|s-t|}{|\gamma(s)-\gamma(t)|}\right)^2$, would have an asymptote comparable with x^{-2} in 0, thus not integrable, and the proof is complete. $\square$

The next result imposes a very weak connection between Problem 8.1.2 and the classic average distance problem.

Lemma 8.1.5. *Given μ, λ, q as in the formulation of Problem 8.1.2, and an arbitrary sequences $\{\varepsilon_n\}, \{\varepsilon'_n\} \downarrow 0$, then it holds:*

- *for any (γ, ν, Π) , for any sequence $\{\gamma_n, \nu_n, \Pi_n\}$, with $\gamma_n \rightarrow \gamma$, $\nu_n \xrightarrow{*} \nu$ and $\Pi_n \xrightarrow{*} \Pi$, then*

$$\liminf_{n \rightarrow \infty} E_{\mu}^{\lambda, \varepsilon_n, \varepsilon'_n}(\gamma_n, \nu_n, \Pi_n) \geq E_{\mu}^{\lambda, 0, 0}(\gamma, \nu, \Pi);$$

Proof. Consider an arbitrary triple (γ, ν, Π) , and an arbitrary sequence $\{(\gamma_n, \nu_n, \Pi_n)\}$ satisfying $\{\gamma_n\} \rightarrow \gamma$, $\nu_n \xrightarrow{*} \nu$ and $\Pi_n \xrightarrow{*} \Pi$. It holds

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int |x-y| d\Pi_n(x, y) + \lambda L_{\gamma_n} + \varepsilon_n \int_0^{L_{\gamma_n}} \nu_n^q ds + \varepsilon'_n \eta(\gamma_n) \\ \geq \liminf_{n \rightarrow \infty} \int |x-y| d\Pi_n(x, y) + \lambda L_{\gamma_n} \\ \geq \int |x-y| d\Pi(x, y) + \lambda L_{\gamma}. \end{aligned}$$

$\square$

The next goal is to analyze some regularity properties of densities ν when (γ, ν, Π) is a minimizer of Problem 8.1.2. This will be the main objective of the next section.

8.2 Regularity of densities

In Proposition 8.1.3 it has been proven that whenever (γ, ν, Π) is a minimizer of Problem 8.1.2 then the measure $\nu \ll \mathcal{L}^1$. In this section some further regularity of such measure will be analyzed. Notice that as the term $\eta(\gamma)$ depends only on geometric properties of the curve, and any construction not modifying the curve γ (but alters ν and Π) does not change the value of $\eta(\gamma)$.

The main result is:

Theorem 8.2.1. *Given a measure μ , parameters $\lambda, \varepsilon, \varepsilon' > 0$, and a minimizer (γ, ν, Π) of Problem 8.1.2, then it holds:*

- $\frac{d\nu}{d\mathcal{L}^1}$ is essentially bounded (Proposition 8.2.3),
- $\frac{d\nu}{d\mathcal{L}^1}$ satisfies a very weak form of continuity (Proposition 8.2.4),

- if $\frac{d\nu}{d\mathcal{L}^1}$ is continuous, then it is Lipschitz continuous (Proposition 8.2.5).

The proof of Theorem 8.2.1 will be split into several passages, and with the arguments used in this chapter, the results must be proven in this order.

We prove first some easy (but nevertheless useful) estimates.

Lemma 8.2.2. *Given a measure μ , parameters $\lambda, \varepsilon, \varepsilon' > 0$, and a minimizer (γ, ν, Π) , then the following properties hold:*

- there exist $c_1, c_2 > 0$ such that

$$c_1 \leq L_\gamma \leq c_2$$

with c_1, c_2 independent of γ ,

- there exist $z \in \text{supp}(\mu)$ and constant Q such that $\text{supp}(\mu) \cup \gamma([0, L_\gamma])$ is contained in a ball $B(z, Q)$.

Proof. Choose an arbitrary such $z \in \text{supp}(\mu)$, and an arbitrary z' with $|z - z'| = 1$. Consider the triple (γ', ν', Π') where

$$\gamma' : [0, 1] \longrightarrow \mathbb{R}^d, \quad \gamma'(t) := (1-t)z + tz', \quad \nu' := \mathcal{L}^1_{|[0,1]}$$

and Π' is an optimal transport plan between μ and $\gamma'_\# \nu'$. Clearly it holds

$$|x - z| \leq \text{diam supp}(\mu) \quad \forall x \in \text{supp}(\mu),$$

thus

$$\int_{\mathbb{R}^d \times \gamma'([0,1])} |x - y| d\Pi'(x, y) \leq \text{diam supp}(\mu).$$

Moreover by construction $L_{\gamma'} = 1$, and $\int_0^1 \left(\frac{d\nu'}{d\mathcal{L}^1} \right) d\mathcal{H}^1 = 1$, $\eta(\gamma') = 1$. Using the minimality of (γ, ν, Π) against (γ', ν', Π') gives

$$\begin{aligned} \int_{\mathbb{R}^d \times \gamma([0, L_\gamma])} |x - y| d\Pi(x, y) + \lambda L_\gamma + \varepsilon \int_0^{L_\gamma} \left(\frac{d\nu}{d\mathcal{L}^1} \right)^q d\mathcal{L}^1 + \varepsilon' \eta(\gamma) \\ \leq \text{diam supp}(\mu) + \lambda + \varepsilon + \varepsilon', \end{aligned} \quad (8.2.1)$$

i.e.

$$\lambda L_\gamma \leq \text{diam supp}(\mu) + \lambda + \varepsilon + \varepsilon', \quad (8.2.2)$$

and

$$\varepsilon \int_0^{L_\gamma} \left(\frac{d\nu}{d\mathcal{L}^1} \right)^q d\mathcal{L}^1 \leq \text{diam supp}(\mu) + \lambda + \varepsilon + \varepsilon'. \quad (8.2.3)$$

Inequality (8.2.2) is an upper bound on the length of γ . Combining the easy fact

$$\int_0^{L_\gamma} \left(\frac{d\nu}{d\mathcal{L}^1} \right)^q d\mathcal{L}^1 \geq \int_0^{L_\gamma} \left(\frac{1}{L_\gamma} \right)^q d\mathcal{L}^1$$

with (8.2.3) gives

$$L_\gamma^{1-q} \leq \text{diam supp}(\mu) + \lambda + \varepsilon + \varepsilon',$$

which in view of hypothesis $q > 1$, represents a lower bound on the length of γ .

Combining (8.2.2) with $\text{diam supp}(\mu) < \infty$, as by hypothesis μ is compactly supported, and using the estimate

$$\inf_{x \in \text{supp}(\mu), y \in \gamma([0, L_\gamma])} |x - y| \leq \int_{\mathbb{R}^d \times \gamma([0, L_\gamma])} |x - y| d\Pi(x, y) < \infty$$

gives that $\text{supp}(\mu) \cup \gamma([0, L_\gamma])$ is contained in the ball $B(z, Q)$ for some $Q > 0$, concluding the proof. $\square$

The first result deals with essential boundedness.

Proposition 8.2.3. *Given a measure μ , parameters $\lambda, \varepsilon, \varepsilon' > 0$, and a minimizer (γ, ν, Π) , then $\frac{d\nu}{d\mathcal{L}^1} \in L^\infty$.*

Proof. Notice first that as the term $\eta(\gamma)$ depends only on the geometric properties of the curve, and not on ν or Π . As the following construction does not alter the curve, $\eta(\gamma)$ does not change.

Choose an arbitrary $M \gg 1$, and denote with $A_M := \{M \leq \frac{d\nu}{d\mathcal{L}^1} \leq M^{4/3}\}$. Clearly $\mathcal{L}^1(A_M)M \leq 1$. Assume first

- $\mathcal{L}^1(A_M) > 0$.

The goal is to find a suitable competitor (γ, ν', Π') (notice that the curve γ has not been modified), with ν' and Π' eventually depending on M , and use the minimality of (γ, ν, Π) (compared against (γ, ν', Π')) to retrieve a necessary condition.

Using Lemma 8.2.2 there exists $c, C > 0$ such that $C \geq L_\gamma \geq c$. Thus the set $B := \{\frac{d\nu}{d\mathcal{L}^1} \leq 2/c\} \subseteq [0, L_\gamma]$ has $\mathcal{L}^1$ measure at least $c/2$. Consider ν' defined as

$$\nu' := \nu - \nu|_{A_M} + \frac{\nu(A_M)}{\mathcal{L}^1(B)} \mathcal{L}^1|_B.$$

Choose an optimal plan Π' between μ and $\gamma_\# \nu'$: thus the mass transported by Π on $\gamma(A_M)$ is now transported on $\gamma(B)$ by Π' . Thus

$$W_1(\nu, \nu') \leq \mathcal{H}^1(L) \nu(\gamma(A_M))$$

with W_1 denoting the 1-Wasserstein distance and consequently

$$\begin{aligned} \left| \int_{\mathbb{R}^d \times \gamma([0, L_\gamma])} |x - y| d\Pi(x, y) - \int_{\mathbb{R}^d \times \gamma([0, L_\gamma])} |x - y| d\Pi'(x, y) \right| &\leq L_\gamma \nu(\gamma(A_M)) \\ &\leq CM^{4/3} \mathcal{L}^1(A_M). \end{aligned} \quad (8.2.4)$$

Moreover it holds

$$\begin{aligned}
\int_B \left(\frac{d\nu'}{d\mathcal{L}^1} \right)^2 d\mathcal{L}^1 - \int_B \left(\frac{d\nu}{d\mathcal{L}^1} \right)^2 d\mathcal{L}^1 &= \int_B \left(\frac{d\nu}{d\mathcal{L}^1} + \frac{\nu(A_M)}{\mathcal{L}^1(B)} \right)^2 d\mathcal{L}^1 - \int_B \left(\frac{d\nu}{d\mathcal{L}^1} \right)^2 d\mathcal{L}^1 \\
&= \frac{2\nu(A_M)}{\mathcal{L}^1(B)} \nu(B) + \left(\frac{\nu(A_M)}{\mathcal{L}^1(B)} \right)^2 \mathcal{L}^1(B) \\
&\leq \frac{2\nu(B)}{\mathcal{L}^1(B)} M^{4/3} \mathcal{L}^1(A_M) + \frac{M^{8/3} \mathcal{L}^1(A_M)^2}{\mathcal{L}^1(B)}. \tag{8.2.5}
\end{aligned}$$

Recalling that by construction $\frac{d\nu'}{d\mathcal{L}^1} = 0$ on A_M , it holds

$$\int_{A_M} \left(\frac{d\nu}{d\mathcal{L}^1} \right)^2 d\mathcal{L}^1 - \int_{A_M} \left(\frac{d\nu'}{d\mathcal{L}^1} \right)^2 d\mathcal{L}^1 \geq M^2 \mathcal{L}^1(A_M). \tag{8.2.6}$$

Combining (8.2.4), (8.2.6) and the minimality of (γ, ν, Π) (compared against (γ, ν', Π')) gives a necessary condition

$$CM^{4/3} \mathcal{L}^1(A_M) + \varepsilon \frac{2\nu(B)}{\mathcal{L}^1(B)} M^{4/3} \mathcal{L}^1(A_M) + \frac{M^{8/3} \mathcal{L}^1(A_M)^2}{\mathcal{L}^1(B)} \geq \varepsilon M^2 \mathcal{L}^1(A_M). \tag{8.2.7}$$

Notice that for M sufficiently large, $CM^{4/3} \mathcal{L}^1(A_M)$ is negligible w.r.t. $M^2 \mathcal{L}^1(A_M)$. Similarly, recall that $\mathcal{H}^1(B) \geq c/2$, thus the term $\frac{2\nu(B)}{\mathcal{L}^1(B)} M^{4/3} \mathcal{L}^1(A_M)$ has order $M^{4/3} \mathcal{H}^1(A_M)$, again negligible w.r.t. $M^2 \mathcal{H}^1(A_M)$.

The term $\frac{M^{8/3} \mathcal{L}^1(A_M)^2}{\mathcal{L}^1(B)}$ has order $M^{8/3} \mathcal{H}^1(A_M)^2$: clearly

$$\frac{M^{8/3} \mathcal{L}^1(A_M)^2}{M^2 \mathcal{L}^1(A_M)} = M^{2/3} \mathcal{L}^1(A_M) \leq M^{-1/3} \ll 1.$$

Thus for M sufficiently large, condition (8.2.7) cannot hold, and the minimality of (γ, ν, Π) is contradicted by (γ, ν', Π') .

All this argument has been done under the assumption $\mathcal{L}^1(A_M) > 0$. But if $\frac{d\nu}{d\mathcal{L}^1} \notin L^\infty$, then it is always possible to find a sequence $M_j \rightarrow +\infty$ such that $\mathcal{L}^1(A_{M_j}) > 0$ and $A_{M_j} \cap A_{M_{j'}} = \emptyset$ whenever $j \neq j'$. Thus the construction described above would give a competitor (γ, ν', Π') contradicting the minimality of (γ, ν, Π) . The only possibility is thus $\frac{d\nu}{d\mathcal{L}^1} \in L^\infty$, and the proof is complete. $\square$

The next result proves a weaker variant of continuity for such densities ν . In the following assume that the exponent q appearing in the term $\int_0^{L_\gamma} \nu^q d\mathcal{L}^1$ will be assumed $q = 2$. This mainly due to a technical fact, as noted in Remark I in the following.

Proposition 8.2.4. *Given a measure μ , parameters $\lambda, \varepsilon, \varepsilon' > 0$, and a minimizer (γ, ν, Π) , with $\gamma : [0, L_\gamma] \rightarrow \mathbb{R}^d$ parameterized by arc-length, then for any $t \in [0, L_\gamma]$ there exist no sequences of Borel sets $\{A_n\} \downarrow \{t\}$, $\{B_n\} \downarrow \{t\}$ (the convergence is intended as set convergence), and $c_1, c_2 > 0$ such that $\mathcal{L}^1(A_n) > 0$, $\mathcal{L}^1(B_n) > 0$ and $\frac{d\nu}{d\mathcal{L}^1}\Big|_{A_n} \geq c_1 > c_2 \geq \frac{d\nu}{d\mathcal{L}^1}\Big|_{B_n}$ for any n .*

Proof. Suppose by contradiction there exist such $\{A_n\} \downarrow \{t\}$, $\{B_n\} \downarrow \{t\}$ and $c_1, c_2 > 0$ such that $\frac{d\nu}{d\mathcal{L}^1}\Big|_{A_n} \geq c_1 > c_2 \geq \frac{d\nu}{d\mathcal{L}^1}\Big|_{B_n}$ for any n . Clearly such $\{A_n\}$, $\{B_n\}$ are disjoint for any n , and it can be assumed $\mathcal{L}^1(A_n) = \mathcal{L}^1(B_n)$ (simply, if $\mathcal{L}^1(A_n) > \mathcal{L}^1(B_n)$, replace A_n with $A'_n \subseteq A_n$ satisfying $\mathcal{L}^1(A'_n) = \mathcal{L}^1(B_n)$), and $t \in A_n \cap B_n$; denote with $d_n := \max\{\text{diam } A_n, \text{diam } B_n\}$, and $c := c_1 - c_2$.

Denote with $l_n := \mathcal{L}^1(A_n) = \mathcal{L}^1(B_n)$; the goal is to construct $\tilde{\nu}$ such that $(\Sigma, \tilde{\nu}, \tilde{\Pi})$ (with $\tilde{\Pi}$ an optimal plan between μ and $\gamma_\# \tilde{\nu}$) contradicts the minimality of (Σ, ν, Π) .

Denote with

$$V(A_n) := \int_{A_n} \frac{d\nu}{d\mathcal{L}^1} d\mathcal{L}^1, \quad V(B_n) := \int_{B_n} \frac{d\nu}{d\mathcal{L}^1} d\mathcal{L}^1$$

the total mass transported (by Π) on $\gamma(A_n)$ and $\gamma(B_n)$ respectively.

Consider the following modifications:

1. choose $C \subseteq A_n$ such that $\nu(C) = (V(A_n) + V(B_n))/2$, and an arbitrary $z \in B_n$. Define the measure

$$\nu' := \nu - \nu|_C + \frac{1}{2}(V(A_n) - V(B_n))\delta_z.$$

Direct computation gives $W_1(\nu, \nu') \leq d_n(V(A_n) - V(B_n))$.

2. Define the measure

$$\tilde{\nu} := \nu' - \frac{1}{2}(V(A_n) - V(B_n))\delta_z + \frac{V(A_n) - V(B_n)}{2\mathcal{L}^1(B_n)}\mathcal{L}^1_{B_n}.$$

Direct computation gives $W_1(\tilde{\nu}, \nu') \leq \frac{1}{2}(V(A_n) - V(B_n))d_n$, and $W_1(\tilde{\nu}, \nu) \leq \frac{3}{2}(V(A_n) - V(B_n))d_n$.

Choose a new optimal transport plan $\tilde{\Pi}$ between μ and $\gamma_\# \tilde{\nu}$. The dependence on n has been omitted for the sake of brevity. Using $W_1(\tilde{\nu}, \nu) \leq \frac{3}{2}(V(A_n) - V(B_n))d_n$ it holds

$$\left| \int_{\mathbb{R}^d \times \gamma([0, L_\gamma])} |x - y| d\Pi(x, y) - \int_{\mathbb{R}^d \times \gamma([0, L_\gamma])} |x - y| d\tilde{\Pi}(x, y) \right| \leq \frac{3}{2}(V(A_n) - V(B_n))d_n.$$

By direct computation it holds

$$\begin{aligned} \int_{A_n \cup B_n} \left(\frac{d\nu}{d\mathcal{L}^1} \right)^2 d\mathcal{L}^1 - \int_{A_n \cup B_n} \left(\frac{d\tilde{\nu}}{d\mathcal{L}^1} \right)^2 d\mathcal{L}^1 &\geq \frac{V(A_n)^2 + V(B_n)^2}{l_n} - \frac{(V(A_n) + V(B_n))^2}{2l_n} \\ &\geq \frac{(V(A_n) - V(B_n))^2}{2l_n}. \end{aligned} \tag{8.2.8}$$

Combining the above inequality with the minimality of (γ, ν, Π) against $(\gamma, \tilde{\nu}, \tilde{\Pi})$ gives the necessary condition

$$\varepsilon \frac{(V(A_n) - V(B_n))^2}{2l_n} \leq 2d_n V(A_n), \quad (8.2.9)$$

i.e.

$$\frac{(V(A_n) - V(B_n))^2}{l_n} \leq \frac{4d_n V(A_n)}{\varepsilon},$$

and recalling that

$$V(A_n) - V(B_n) = \int_{A_n} \frac{d\nu}{d\mathcal{L}^1} d\mathcal{L}^1 - \int_{B_n} \frac{d\nu}{d\mathcal{L}^1} d\mathcal{L}^1 \geq cl_n,$$

this gives

$$c^2 l_n \leq \frac{4d_n V(A_n)}{\varepsilon}.$$

From Proposition 8.2.3 it is known that $\frac{d\nu}{d\mathcal{L}^1} \in L^\infty$, thus $V(A_n) \leq \left\| \frac{d\nu}{d\mathcal{L}^1} \right\|_{L^\infty} l_n$, and

$$c^2 l_n \leq \frac{4d_n V(A_n)}{\varepsilon} \leq \frac{4d_n l_n}{\varepsilon} \left\| \frac{d\nu}{d\mathcal{L}^1} \right\|_{L^\infty},$$

which finally yields

$$c^2 \leq \frac{4d_n}{\varepsilon} \left\| \frac{d\nu}{d\mathcal{L}^1} \right\|_{L^\infty}$$

which is false for n sufficiently large, as by hypothesis $\{A_n\} \downarrow \{t\}$, $\{B_n\} \downarrow \{t\}$, thus $d_n \downarrow 0$. This concludes the proof. $\square$

Remark I. The choice $q = 2$ is dictated by technical reasons, as the passage (8.2.8) involves computing the difference

$$\frac{V(A_n)^q + V(B_n)^q}{2} - \left(\frac{V(A_n) + V(B_n)}{2} \right)^q. \quad (8.2.10)$$

However, we are unable to prove that for any $q > 1$ there exists a constant $M_q > 0$ depending only on q such that

$$\frac{V(A_n)^q + V(B_n)^q}{2} - \left(\frac{V(A_n) + V(B_n)}{2} \right)^q \geq M_q \left(\frac{V(A_n) - V(B_n)}{2} \right)^q.$$

This would allow to extend the result for any $q > 1$ (or for any $q > 1$ for which a similar estimate holds), by using the same argument found in the proof of Proposition 8.2.4. The next result proves that continuity of $\frac{d\nu}{d\mathcal{L}^1}$ implies Lipschitz continuity.

Proposition 8.2.5. *Given a measure μ , parameters $\lambda, \varepsilon, \varepsilon' > 0$, a minimizer (γ, ν, Π) , with $\gamma : [0, L_\gamma] \rightarrow \mathbb{R}^d$ parameterized by arc-length. Assume the function $\frac{d\nu}{d\mathcal{L}^1}$ is continuous, then it is $\frac{2}{\varepsilon}$ -Lipschitz continuous, thus $\mathcal{L}^1$ -a.e. differentiable with $\left(\frac{d\nu}{d\mathcal{L}^1} \right)' \in L^\infty$.*

Proof. Consider an arbitrary t , and two sequences $\{t_n\} \rightarrow t$, $\{s_n\} \rightarrow t$ with empty mutual intersection; from Proposition 8.2.4 it can be assumed $\frac{d\nu}{d\mathcal{L}^1}$ continuous. For any n choose a sufficiently small δ_n , and define $I_n := (t_n - \delta_n, t_n + \delta_n)$, $J_n := (s_n - \delta_n, s_n + \delta_n)$, and clearly choosing δ_n sufficiently small it can be assumed $I_n \cap J_n = \emptyset$ for any n . To simplify notations, in the following we will write δ instead of δ_n .

Denote with $V(I_n) := \int_{I_n} \frac{d\nu}{d\mathcal{L}^1} d\mathcal{L}^1$ and $V(J_n) := \int_{J_n} \frac{d\nu}{d\mathcal{L}^1} d\mathcal{L}^1$ the mass transported (by Π) on $\gamma(I_n)$ and $\gamma(J_n)$ respectively. Assume (by symmetry, and the case $\frac{d\nu}{d\mathcal{L}^1}(t) = \frac{d\nu}{d\mathcal{L}^1}(s)$ is trivial for the purposes of this proposition) $\frac{d\nu}{d\mathcal{L}^1}(t) > \frac{d\nu}{d\mathcal{L}^1}(s)$, which implies $V(I_n) > V(J_n)$ for δ sufficiently small. Consider the following modifications:

1. choose $C \subseteq I_n$ such that $\nu(C) = (V(I_n) + V(J_n))/2$, and an arbitrary $z \in J_n$. Define the measure

$$\nu' := \nu - \nu|_C + \frac{1}{2}(V(I_n) - V(J_n))\delta_z.$$

Direct computation gives $W_1(\nu, \nu') \leq \frac{1}{2}(V(I_n) - V(J_n))(|t_n - s_n| + 2\delta)$.

2. Define the measure

$$\tilde{\nu} := \nu' - \frac{1}{2}(V(I_n) - V(J_n))\delta_z + \frac{V(I_n) - V(J_n)}{2\mathcal{L}^1(J_n)}\mathcal{L}^1_{J_n}.$$

Direct computation gives $W_1(\tilde{\nu}, \nu') \leq \frac{1}{2}(V(I_n) - V(J_n))d_n$, thus $W_1(\tilde{\nu}, \nu) \leq \frac{3}{2}(V(I_n) - V(J_n))d_n$.

Choose new optimal transport plan $\tilde{\Pi}$ between μ and $\gamma_{\#}\tilde{\nu}$. Thus it holds

$$\int_{\mathbb{R}^d \times \gamma([0, L_\gamma])} |x - y| d\tilde{\Pi}(x, y) - \int_{\mathbb{R}^d \times \gamma([0, L_\gamma])} |x - y| d\Pi(x, y) \quad (8.2.11)$$

$$\leq \frac{1}{2}(V(I_n) - V(J_n))(|t_n - s_n| + 2\delta) + (V(I_n) + V(J_n))\delta. \quad (8.2.12)$$

Moreover, the term

$$\int_{I_n \cup J_n} \left(\frac{d\nu}{d\mathcal{L}^1} \right)^2 d\mathcal{L}^1 \geq \frac{V(I_n)^2 + V(J_n)^2}{\delta}$$

decreases to

$$\frac{2}{\delta} \left(\frac{V(I_n) + V(J_n)}{2} \right)^2,$$

thus the difference satisfies

$$\begin{aligned} \int_{I_n \cup J_n} \left(\frac{d\nu}{d\mathcal{L}^1} \right)^2 d\mathcal{L}^1 - \frac{2}{\delta} \left(\frac{V(I_n) + V(J_n)}{2} \right)^2 \\ \geq \frac{V(I_n)^2 + V(J_n)^2}{\delta} - \frac{2}{\delta} \left(\frac{V(I_n) + V(J_n)}{2} \right)^2 \\ \geq \frac{1}{\delta} \left(\frac{V(I_n) - V(J_n)}{2} \right)^2. \end{aligned} \quad (8.2.13)$$

Combining (8.2.11), (8.2.13) and the minimality of (Σ, ν, Π) against $(\Sigma, \tilde{\nu}, \tilde{\Pi})$ gives the necessary condition

$$\frac{1}{2}(V(I_n) - V(J_n))(|t_n - s_n| + 2\delta) + (V(I_n) + V(J_n))\delta \geq \varepsilon \frac{1}{\delta} \left(\frac{V(I_n) - V(J_n)}{2} \right)^2 \quad (8.2.14)$$

for any choice of δ sufficiently small. Using the arbitrariness of δ , and passing to the limit $\delta \downarrow 0$, (8.2.14) becomes

$$2 \left(\frac{d\nu}{d\mathcal{L}^1}(t_n) - \frac{d\nu}{d\mathcal{L}^1}(s_n) \right) |t_n - s_n| \geq \varepsilon \left(\frac{d\nu}{d\mathcal{L}^1}(t_n) - \frac{d\nu}{d\mathcal{L}^1}(s_n) \right)^2, \quad (8.2.15)$$

which implies

$$\frac{2}{\varepsilon} |t_n - s_n| \geq \frac{d\nu}{d\mathcal{L}^1}(t_n) - \frac{d\nu}{d\mathcal{L}^1}(s_n).$$

The other case $\frac{d\nu}{d\mathcal{L}^1}(t) < \frac{d\nu}{d\mathcal{L}^1}(s)$ is solved using the same argument, as the role of t and s are symmetric. This proves $\frac{2}{\varepsilon}$ -Lipschitz continuity along curves. Using Rademacher's theorem gives that $\frac{d\nu}{d\mathcal{L}^1}$ is $\mathcal{L}^1$ -a.e. differentiable, and $\left\| \left(\frac{d\nu}{d\mathcal{L}^1} \right)' \right\|_{L^\infty} \leq \frac{2}{\varepsilon}$. $\square$

Remark II. In all the discussion, including the main result in Theorem 8.2.1, the average distance term was imposed to be

$$\int |x - y| d\Pi(x, y).$$

In general one can consider a slightly more general case, in which the average distance term is replaced by

$$\int |x - y|^\alpha d\Pi(x, y), \quad \alpha \geq 1.$$

All the arguments, and the proofs, can be adapted straightforwardly: indeed in all the proofs we had to estimate the change for the average distance term when some mass is moved by some δ_n , with $\delta_n \rightarrow 0$. More precisely, for δ_n sufficiently small we have constructed a competitor $(\gamma, \tilde{\nu}, \tilde{\Pi})$ where the Wasserstein-1 distance between Π and $\tilde{\Pi}$ was bounded from above by

$$\text{mass moved} \times \text{distance}.$$

The latter was estimated to be δ_n , using triangular inequality. If the integrand becomes $|x - y|^\alpha$ (instead of $|x - y|$), then a very similar argument follows: indeed if α is integer, then for any $l, \varepsilon > 0$ it holds

$$|l + \varepsilon|^\alpha - l^\alpha = \alpha l^{\alpha-1} \varepsilon + o(\varepsilon).$$

Notice that the proof of Lemma 8.2.2 follows straightforwardly with the same arguments. For generic α it suffices to notice that the map

$$\alpha \mapsto |l + \varepsilon|^\alpha - l^\alpha$$

is nondecreasing. This shows that for any $\alpha \geq 1$, using the same constructions, the change for $\int |x - y|^\alpha d\Pi(x, y)$ has (at most) the same order as for $\int |x - y| d\Pi(x, y)$. Thus the same arguments from the proofs can be repeated straightforwardly, and the properties from Theorem 8.2.1 are proven true even if the average distance term is replaced by $\int |x - y|^\alpha d\Pi(x, y)$. Thus it holds:

Theorem 8.2.6. *Given a measure μ , parameters $\lambda, \varepsilon, \varepsilon' > 0$, $\alpha \geq 1$ and a solution (γ, ν, Π) of*

$$\min \int_{\mathbb{R}^d \times \gamma([0, L_\gamma])} |x - y|^\alpha d\Pi(x, y) + \lambda L_\gamma + \varepsilon \int_0^{L_\gamma} \left(\frac{d\nu}{d\mathcal{L}^1} \right)^2 ds + \varepsilon' \eta(\gamma),$$

with γ, ν and Π varying in the same sets as in Problem 8.1.2, then it holds:

- $\frac{d\nu}{d\mathcal{L}^1}$ is essentially bounded (Proposition 8.2.3),
- $\frac{d\nu}{d\mathcal{L}^1}$ satisfies a very weak form of continuity i.e.
 - for any $t \in [0, L_\gamma]$ there exist no sequences of Borel sets $\{A_n\} \downarrow \{t\}$, $\{B_n\} \downarrow \{t\}$ (the convergence is intended as set convergence), and $c_1, c_2 > 0$ such that $\mathcal{L}^1(A_n) > 0$, $\mathcal{L}^1(B_n) > 0$ and $\frac{d\nu}{d\mathcal{L}^1} \Big|_{A_n} \geq c_1 > c_2 \geq \frac{d\nu}{d\mathcal{L}^1} \Big|_{B_n}$ for any n ,
- if $\frac{d\nu}{d\mathcal{L}^1}$ is continuous, then it is Lipschitz continuous.

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