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Informal Proofs and Computability

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Contents

Introduction

A recent review of “Computability: Turing, Gödel, Church, and Beyond” [Copeland et al., 2013], a collection of eleven essays dealing with old and new philosophical directions building on the concept of computability, contains the following passage:

> While the material may not be extremely relevant to the stereotypical reader of this journal (*qua* philosopher of mathematics, at least — there is of course quite a bit of overlap between those of us that self-identify as philosophers of mathematics and those that self-identify as logicians), readers will find much that is interesting and important contained in the eleven essays that compose this volume. [Cook, 2014]

The journal in which this review has appeared is *Philosophia Mathematica*, a journal which is – as the title announces – fully devoted to contemporary philosophy of mathematics. Then, the fact that a stereotypical philosopher of mathematics might find not relevant a volume such as [Copeland et al., 2013] (that the reviewer considers “uniformly excellent”) is surprising. After all, Computability, in Sieg’s words [Sieg, 2008] “is perhaps the most significant and distinctive notion modern logic has introduced”. Why should philosophers of mathematics not care? Yet, the reviewer’s impression, although slightly disappointing, is nevertheless quite accurate. Nowadays, the notion of computability is somehow far from the centre of the philosophical debate, and of course this is much more true if one considers Computability Theory that in most cases is just off the philosophers’ radar (contrast this with the now well-established philosophy of set theory, or with the spread of model-theoretic arguments in philosophy as surveyed in [Button and Walsh, 2015]).

Thus, with the present work we aim at challenging such lack of interest. In particular, our main goal is two-fold. We aim to show: (1) that Computability Theory, once purified from a sort of theoretical misconception concerning its practice, can be regarded as a privileged case-study for investigating largely debated philosophical problems, such as that of relating *proofs* and *derivations* (see Chapter 1); and (2) that computable-theoretic tools can be fruitfully adopted for defining logical models that are more sensible to certain features of real mathematical reasoning than formal systems are (see Chapter 2). In addition, we prove new results about a notion that has been studied for decades, that of *computable reducibility*, by extending it from the context of equivalence relations to more general relations (see Chapter 3).

More in detail, this work consists of three chapters.

Chapter 1 aims at providing a philosophical analysis of the notion of "proof by Church's Thesis", which is – in a nutshell – the conceptual device that permits to rely on informal methods when working in Computability Theory. This notion allows, in most cases, to not specify the background model of computation in which a given algorithm – or a construction – is framed. In pursuing such analysis, we carefully reconstruct the development of this notion (from Post to Rogers, to the present days), and we focus on some classical constructions of the field, such as the construction of a simple set. Then, we make use of this focus in order to support the following encompassing claim (which opposes to a somewhat commonly received view): the informal side of Computability, consisting of the large class of methods typically employed in the proofs of the field, is not fully reducible to its formal counterpart.

The material of Chapter 2 corresponds to that of [Amidei et al., a] and [Amidei et al., b] (written in collaboration with Jacopo Amidei, Duccio Pianigiani, Giulia Simi, and Andrea Sorbi). In this chapter, we study *dialectical systems* (firstly introduced in [Magari, 1974]) and *quasi-dialectical systems*, two generalizations of formal systems in which axioms of the represented theory are chosen through some trial-and-error process. After having discussed the significance of these systems, we prove several mathematical results concerning their expressivity. We show that they display the same computational power, in the sense that dialectical sets and quasi-dialectical sets (appropriately defined) lie in the same Turing-degrees. Nevertheless, we conclude by proving that quasi-dialectical sets and dialectical sets are different, by showing their respective place in the Ershov hierarchy.

Chapter 3 concerns *computable reducibility*, that is classically regarded as a natural way to classify equivalence relations on ω according to their complexity. Computable reducibility is defined as follows. Let R and S be two equivalence relations. We say that R is *computably reducible* to S iff there is a computable function f s.t., for all $x, y \in \omega$, the following holds:

$$xRy \Leftrightarrow f(x)Sf(y).$$

In literature, the degree structure generated by computable reducibility has been largely investigated. In particular, one the most prominent problem in the area has been that of characterizing universal equivalence relations, i.e. relations to which all others relations, of a given complexity, can be reduced. For instance, a rich theory for universal computably enumerable equivalence relations has been formulated. Nonetheless, most results do not extend to the whole arithmetical hierarchy. In fact, while, for each n, it is easy to build a Σ^0_n equivalence relation which is universal, on the other hand, in [Ianovski et al., 2014] authors prove that there is no universal Π^0_n equivalence relation for $n \geq 2$. In this chapter we consider the problem of universality in a more general context than that of equivalence relations. First, we prove that, contrary to the case of equivalence relations, for each level of the arithmetical hierarchy there is a universal binary relation. In particular, for all $n > 2$, we prove that $\{\langle i,j\rangle \mid W_i \subseteq W_j^{(n-2)}\}$ is a universal Π^0_n binary relation. Then, we show how to make use of these latter constructions in order to obtain a similar result also for several intermediate cases between general binary relations and equivalence relations (most notably, we prove the existence of natural universal graphs, i.e. symmetric binary relations, for each level of the arithmetical hierarchy).

Remark. We adopt the common convention of calling Computability Theory what was previously known as Recursion Theory (or, Theory of Recursive Functions). See [Soare, 1996] for an extensive defence of this adoption. Furthermore, when the context is clear enough, we refer to Computability Theory as just Computability.

Chapter 1

Church-Turing thesis, in practice

1.1 Proof vs derivations

A key problem of contemporary philosophy of mathematics arises from an immediate observation: real mathematical proofs (as they appear, for instance, in ordinary mathematical journals) are somehow different from the abstract models of proofs studied in formal logic. It is customary to use the word *proofs* for the former, while referring to the latter as *derivations* (see, for instance, [Rav, 1999]). So, here is the problem: what is the nature of this gap between proofs and derivations? Is it a philosophically significant one?

Notice that the problem is made more difficult by a fundamental asymmetry of the two notions involved. On the one hand, derivations admit a rigorous definition. As is known, given a formal language $\mathcal{L}$ equipped with a proper syntax, a derivation is typically a finite sequence of well-founded formulas such that each formula of the sequence is either an axiom or it follows from the preceding ones by applying some rule of inference. On the other hand, proofs seem to be best suited for some kind of ostensive description, rather than being captured by a static finite package of conditions to be satisfied. Indeed, in learning how to acknowledge the correctness of proofs, and then of course being able to produce their own, students are typically exposed to several different accepted proofs, with some possible focus on paradigmatic cases, instead of receiving any conclusive guidelines concerning what a proof looks like.

Nonetheless, this does not necessarily mean that the concept of proof is a vague one. Quite to the contrary. With the exception of relatively few border cases, that are philosophically debated precisely in virtue of their exceptionality, in most cases mathematicians do perfectly agree on what counts as a proof of something – and agreement is even more solid on what does *not* count as a proof. Indeed, one can of course fail to notice a flaw in a proof, but when such flaw is highlighted (e.g., by a referee), then he typically accepts that the "proof" as such is not valid, instead of being unable or unwilling to do that[1]. Thus, although the task of finding one encompassing and explicit definition for the notion of mathematical proof seems to be desperate, nonetheless we shall agree with Gödel's view that a proof is, at least, "a sequence of thoughts convincing a sound mind" [Gödel, 1995].

In the light of the asymmetry sketched above, let us turn back to the problem of comparing the two notions. First, one might hastily dismiss our fundamental problem by claiming that any purported gap between proofs and derivations is either trivial or irrelevant. After all, derivations are nothing but *models* of real proofs. Therefore, it is just an immediate consequence of the very idea of modelling that many details are abstracted away, while preserving any whatsoever essential core of the modelled notion. So a gap does certainly exist but, rather than being problematic, the existence of such a gap is basically what modelling is for.

Foundational programs, or rather the kind of philosophical perspectives that stem from them, usually push this latter reasoning forward. Indeed, derivations are not only convenient models of real mathematical proofs, but it is actually fruitful to *identify* the two notions. The main benefit of this identification is of course provided by metamathematics, i.e., to our present concern, the possibility of making use of mathematical tools for answering questions that, to some extent, conceptually belong to the philosophy of mathematics[2] For instance, if we subscribe the identification between formal systems and mathematical theories (which clearly echoes the one between proofs and derivations), then we obtain that limitative results for formal systems, such as Gödel's ones, do apply also to real mathematical theories, as far as we choose to understand them in a rather informal way.

The seminal work by Lakatos has famously pioneered an opposing tradition to these latter identifications. In Lakatos' words [Lakatos, 1976a]:

> The subject matter of metamathematics is an abstraction of mathemat-

[1] Azzouni interestingly argues that one of the features that makes mathematics a peculiar social practice is the fact that this solid agreement is achieved without coercion: "Unlike politics, for example, or any of the other numerous group activities we might consider, mathematical agreement isn't coerced. Individuals can see who's wrong; at least, if someone is stubborn, others (pretty much all the competent others) see it (...). By contrast, Protestantism, with all its numerous sects – in the United States, especially – is what results when coercion isn't possible (because deviants can, say, move to Rhode Island) (...) It's sociologically very surprising that conformity in mathematics isn't achieved as in these group practices. Imagine – here's a dark Wittgensteinian fable – we tortured numerical deviants to force them to add as we do" [Azzouni, 2005].

[2] Again, this possibility of importing formal tools to the meta-level is a distinctive feature of mathematics. In this respect, Hofweber writes: "Much work done under the heading of "philosophy of mathematics" consists of proofs, precise mathematical proofs. (...) This should be a bit puzzling. Proof is the method to establish results in mathematics. But it is rather unusual that the method to achieve results in the philosophy of X, some discipline, is the same as the method for achieving results in X. For example, physics achieves results via experimentation, amongst other methods. But the philosophy of physics does not. (...) In general the method of the philosophy of X is distinct from the method of X, and this might be particularly compelling when X has a very distinct method, as does mathematics with that of precise proof" [Hofweber, 2009].

> ics in which mathematical theories are replaced by formal systems, proofs by certain sequences of well-formed formulae, definitions by 'abbreviatory devices' which are 'theoretically dispensable' but 'typographically convenient'. (...) At the same time there are problems which fall outside the range of metamathematical abstractions. Among these are all problems relating to informal mathematics and to its growth, and all problems relating to the situational logic of mathematical problem-solving. I shall refer to the school of mathematical philosophy which tends to identify mathematics with its formal axiomatic abstraction (and the philosophy of mathematics with metamathematics) as the 'formalist' school.

After Lakatos, and in particular in the two last decades, there has been a growing line of research which aims to take account of a large class of problems, emerging from mathematical practice, that are classically neglected by philosophers of mathematics, either because of the influence of formalist positions or for the tendency, within the analytic tradition, of taking very small fragment of elementary mathematics as nothing but case studies for philosophical problems arising elsewhere. In general, this line of research is often referred as Philosophy of Mathematical Practice[3].

Concerning the gap between proofs and derivations, philosophers of mathematical practice obviously endorse the idea that such gap is worth studying. To this end, several examples have been provided of some aspects of proofs, occurring in concrete context, that arguably seem to lie well outside the scope of the formalistic characterization[4]. In any case, let us stress that what is crucial, in this kind of examples, is to justify why this gap corresponds to some significant loss of information. As Larvor puts it [Larvor, 2012]:

> Philosophers of mathematical practice need to show that mathematical arguments suffer some philosophically important loss or distortion in the abstraction from 'real' mathematical proof to formal derivation. For the loss or distortion to be philosophically interesting, it must have some logical significance. Whatever gets lost or distorted must play a role in the account of how informal proofs work *as proofs*. Otherwise, opponents of practice-based philosophy can safely park the results of psychological, sociological and historical studies on the 'discovery' side of the discovery/justification distinction.

Call this position *Larvor's challenge*.

Let us take stock and announce the goal of this chapter. We aim to tackle Larvor's challenge in the context of Computability Theory, by considering the gap between

[3] The interested reader can consult [Mancosu, 2008] for an anthology of papers in Philosophy of Mathematical Practice.

[4] For one thing, scholars have largely investigated the role of diagrammatic reasoning in proof-theoretic contexts, which (if not avoidable) seems to resist to many formalist identifications (see, for instance, [Giaquinto, 2007]).

the formal presentation of an algorithm within a certain model of computation, and the ordinary language in which it is commonly formulated. In particular, we aim to propose a philosophical analysis of the notion of 'proof by Church's Thesis', i.e. the conceptual device that permits to rely on informal methods when working in Computability.

1.2 The Church-Turing thesis

We begin by recalling few aspects of the Church-Turing thesis (henceforth: CTT). Being CTT the conceptual cornerstone of Computability (and, in fact, of the whole theory of computation), it has been unsurprisingly the central topic of an incredibly vast philosophical literature (see, for instance, [Olszewski et al., 2006]). It is not our intention to resume this literature or even sketch it. Rather, we simply aim to fix, quite schematically, few ideas of which we make use afterwards. This choice is made possible by the fact that our main focus is on a somewhat neglected topic concerning CTT, a topic that remains almost philosophically untouched (although being well-established among recursion-theorists), namely its *practical* use. As a result, while we do not consider the historical roots of CTT, we spend some time in reconstructing the less known history of such practical side, about which we eventually formulate a Standard View.

1.2.1 CTT: a bit of theory

CTT concerns the 'effective calculability' of the functions on positive integers . In its most general form, it expresses the fact that such notion of calculability – i.e. a pre-theoretic, informal notion – is fully captured by any of the (extensionally equivalent) classical models of computation. Thus, it can be regarded as the amalgamation of several different theses, each one corresponding to a specific model of computation. To our purpose, it is sufficient to state two versions, not casually the ones that do conflate in the label CTT:

> **Church's thesis (1936)**
> *A function is effectively calculable if and only if it is* λ*-definable.*
>
> **Turing's thesis (1936)**
> *A function is effectively calculable if and only if it is computable by a Turing machine.*

Since these two theses are extensionally the same, it is possible, and indeed frequent, to unify them into a single statement (and make this latter general enough to include all others models of computation):

> **CTT**
> *A function is effectively calculable if and only if it is computable in one of the classical models of computation.*

As is known, two aspects of the thesis are philosophically hard to reconcile. On the one hand, there is the evidence that CTT is almost universally accepted as true (and, moreover, it appears to be compelling itself for the reasons we give below). On the other hand, it is difficult to give a complete account of what makes CTT true. Or, more precisely, it is difficult to say how we *know* that the thesis is true (if so). The problem, as one can immediately see, is epistemological. In any of its versions, CTT links two notions which display different status: a formal one, expressing the computability according to some model of computation, and that certainly describes a specific subclass of the functions on positive integers; and an informal one, corresponding to the concept of 'being calculable by some effective procedure', which – at least at first sight – appears to be "a somewhat vague intuitive one" (see [Kleene, 1952]). So, how can these two notions be matched up? This question clearly echoes the problem of comparing proofs and derivations, with the twisting difference that, here, CTT precisely states that there is no gap at all, bridging the two notions involved. But then, what is the nature of such bridge? Three main answers have been formulated.

First, in a vein similar to the formalist solution of identifying proofs and derivations, one can regard at CTT as a precise (i.e., formal) *definition* of the vague notion of 'effective calculability'. This was Church's perspective when introducing the thesis in [Church, 1936]:

> The purpose of the present paper is to propose a definition of effective calculability which is thought to correspond satisfactorily to the somewhat vague intuitive notion in terms of which problems [of the form "it is required to find an effectively calculable function"] are often stated (...). This definition is thought to be justified by the considerations which follow, so far as positive justification can ever be obtained for the selection of a formal definition to correspond to an intuitive notion.

An obvious consequence of this approach is that the thesis, so understood, would become empty. Indeed, if the notion of effective calculability receives precise meaning only when it is identified with that of 'being λ-definable', then the whole question concerning the truth of CTT vanishes. Of course this is because, in general, it is inaccurate to ask whether a definition is true (while, for instance, a more centred question would be if that definition is, say, fruitful). Nonetheless, the significance of CTT involves precisely the relation between the intuitive meaning of calculability and its formal rendering. In dismissing the former concept, by collapsing it onto the latter, too much seems to get lost.

We shall then move to a second reading of CTT, which is also the most popular among the three: CTT is, indeed, a *thesis*, or, in Post's words, "a working hyphotesis" [Post, 1936]. This idea is quite clear. Referring to the intuitive meaning of a pre-theoretic concept, CTT cannot be a mathematical statement. It is neither a definition nor a theorem. Rather, it has to be taken as "a hypothesis about the intuitive notion of effective calculability" [Kleene, 1952]. Much evidence has been offered in favour of this hypothesis. Most notably, the very fact that, starting from highly different intensional approaches, one eventually describes the same class of functions seems to suggest that our notion of computability is a very *stable* one. Furthermore, such stability is to some extent confirmed – although indirectly – by the circumstance that no attempt to disprove the thesis has ever succeeded. But there is much more. In his epoch-making work [Turing, 1936], Turing famously proposes a *conceptual analysis* for the notion of computability, by modelling how an ideal "computor" would behave when computing, thus adding fuel to the claim that any reasonable sharpening of our intuitive concept of calculability would give rise to the same class of functions. So, this latter reasoning adds a kind of a priori support for the thesis, making evidence in favour of CTT so strong that most scholars share Gödel's belief that "With this concept [computability] one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion" [Gödel, 1946]. Nonetheless, taking CTT as a hypothesis is of course compatible with the (arguably remote) chance that the thesis would be eventually disproved, e.g. by designing a function which is Turing noncomputable while being 'calculable' in some intuitive, shared sense.

A last reading of CTT aims to block this latter possibility. Indeed, it has been argued that CTT has not necessarily an hypothetical status, but rather that it can be susceptible of a rigorous mathematical proof (see [Mendelson, 1990]), or even that such a proof is already contained in [Turing, 1936] (see [Gandy, 1988]), from which one can extract a "Turing's theorem", representing an axiomatically given version of the thesis ([Sieg, 1994])[5].

As we will see, many of these epistemological concerns scarcely overlap the kind of problems we aim to unveil. In fact, we will argue that too much focusing on the epistemology of CTT has corresponded to a certain lack of interest in its practical side. Yet, from all the aforementioned debate, we shall at least note down the following four basic *assumptions* concerning the thesis, on which we will partially rely:

1. In what follows, we do not cast doubts on the validity of CTT. We assume rather that thesis is true, or at least *correct* in some profound sense, being a successful bridge from the informal to the formal side of Computability.

2. We mantain that such correctness is unusual. That is, although the problem of mirroring some pre-formal intuition is arguably at the hearth of most mathematical theories, the case of Computability stands as exceptional. To quote [Gödel, 1946] again:

[5] See [Shapiro, 2006] for a clear exposition of this third line of argument.

> In all others cases treated previously, such as demonstrability or definability, one has been able to define them only relative to a given language, and for each language it is clear that the one thus obtained is not the one looked for.

3. We hold that CTT refers to *human* calculability. This prevents a somewhat debated way of attacking the thesis, which consists in designing very unconventional constructions relying on stuff that is way out of human reach, such as computations on the edge of a Black Hole[6]. More generally, this assumption expresses the fact that CTT is meaningful inasmuch it refers to a given human practice, that of making computations constrained by human capabilities.

4. Stricly related to this latter assumption is our last one: CTT itself does not embed a problem of reference. Indeed, although the problem of singling out a specific computation by making use of some informal description is a central one (and, in fact, it is deeply intertwined with our main goal of understanding what we call 'the practical side' of CTT), it is important to notice that the standard formulation of the thesis, under any of its possible readings, lies one step further of these problems. That is, whenever we say that any function which is intuitively 'calculable' can be formally implemented in a classical model of computation, we always assume that a certain amount of work has been made in order to provide a description of that function which is enough clear to avoid misunderstanding or incorrect reference. We return to this topic after having introduced the announced practical side of CTT.

1.3 The practical side of CTT

As already mentioned, most of the philosophical attention revolving around CTT has consisted in clarifying what the thesis is about and how can we possibly ascertain it. Yet, a fundamental topic remains to be considered, namely how the thesis is *used* within the mathematical discourse. In particular, we are interested in its proof-theoretic role.

Definition 1.3.1. Call *practical side of CTT* the collection of all the appeals to CTT that are steps of some mathematical proof.

By 'mathematical proofs' we mean exactly what is labelled as 'proofs' in the proofs/derivations opposition, i.e. concrete examples of mathematical proofs, written in ordinary mathematical language (i.e., a given natural language – in most cases, english – extended with some finite collection of symbols). Therefore, we are referring neither to formal derivations nor to some abstract objects.

[6] See [Welch, 2007] for a rich survey on models of transfinite computation. On the other hand, [Davis, 2006] denies any theoretical significance to "hypercomputationalism" as such.

It is important to notice that, quite independently from our notion of practice (that we aim to keep as much intuitive as possible), Definition 1.3.1 is a rather demanding one. This is because asking for CTT to occur within real proofs is basically the strictest requirement that one can formulate for something to be, in practice, part of the mathematical discourse. So, at least theoretically, it would be perfectly reasonable to expect the practical side, defined as such, to be empty. For instance, CTT could be fundamental in grounding the significance of our theory, or even in setting the agenda of its most important problems, and yet not having any specific proof-theoretic role. To this end, imagine a scenario in which, although acknowledging the validity of CTT, recursion-theorists would consider the computability (resp. relative computability) of a given function acceptably proved only if the full description of a Turing-machine (oracle-machine) is alleged. What would get lost in a similar scenario? Keep this question in mind.

For the moment, let us just say that the practical side of CTT is far from being empty. In fact, any standard textbook in Computability contains – typically in a very initial segment – a certain amount of occurrences of the following expression: "proof by Church's Thesis"[7] So, the notion is sufficiently widespread to avoid being sensible to possibly idiosyncratic or deviant uses. In fact, the idea of proving something by Church's thesis is a familiar one among practitioners; then, can it be philosophically grounded? We argue so. Moreover, we aim to show that such practical side of CTT is, in a sense, independent from CTT itself.

In doing so, we proceed with some history of this practical side, focusing on the most relevant landmarks. Actually, we begin with what shall be regarded as the prehistory of the notion of 'proof by Church's thesis'.

1.3.1 Post: "Stripped of its formalism"

It is fairly acknowledged that the basic conceptual machinery of Computatibility (in terms of concepts, definitions, and techniques) derives from [Post, 1944], which contains, in its opening paragraph, a somewhat curious remark:

> That mathematicians generally are oblivious to the importance of this work of Gödel, Church, Turing, Kleene, Rosser and others (...) is in part due to the forbidding, diverse and alien formalisms in which this work is embodied. (...)

[7] For instance, the following is the first proof-theoretic reference to CTT in [Rogers, 1967]:

Theorem (I). *There are exactly $\aleph_0$ partial recursive functions, and there are exactly $\aleph_0$ recursive functions.*

Proof. All constants functions are recursive, by Church's Thesis. Hence there are at least $\aleph_0$ recursive functions. The Gödel numbering shows that there are at most $\aleph_0$ partial recursive functions. □

> Yet, without such formalism, this pioneering work would lose most of its cogency. But apart from the question of importance, these formalisms bring to mathematics a new and precise mathematical concept, that of the general recursive function of Hërbrand-Gödel-Kleene, or its proved equivalents in the developments of Church and Turing.
>
> It is the purpose of this lecture to demonstrate by example that this concept [that of computable function] admits of development into a mathematical theory much as the group concept has been developed into a theory of groups. Moreover, that stripped of its formalism, such a theory admits of an intuitive development which can be followed, if not indeed pursued, by a mathematician, layman though he be in this formal field. (...)
>
> We must emphasize that (...) we have obtained formal proofs of all the consequently mathematical theorems here developed informally. Yet the real mathematics involved must lie in the informal development. For in every instance the informal "proof" was first obtained; and once gotten, transforming it into the formal proof turned out to be a routine chore.

In a footnote, Post adds:

> Our present formal proofs, while complete, will require drastic systematization and condensation prior to publication.

The whole passage is philosophically striking. What Post is literally saying is that:

1. most of the proofs contained in his paper do not meet the standard of formalization fixed by their proper formal field;
2. these proofs, being developed only informally, are – in a very immediate sense – *incomplete*.

If taken seriously, these two statements offer an account of proofs which is hardly sound with most ideas concerning the special reliability of mathematical facts. To put it crudely, if Post's proofs are really both incomplete and not formal enough, then why do we trust them? As is clear, to fully answer this latter question, one has to develop some kind of convincing explanation of what makes, in general, mathematical proofs so reliable – a tremendous task. For our part, let us just borrow from [Hacking, 2014] a convenient distinction between two ideal conceptions of proofs:

> There are proofs that, after some reflection and study, one totally understands, and can get in one's mind 'all at once'. That's Descartes.
>
> There are proofs in which every step is meticulously laid out, and can be checked, line by line, in a mechanical way. That's Leibniz.

Of course these are just idealizations. Especially in more complex cases, these two conceptions blend together. That is, a global understanding of a complex proof would reasonably encompass both some kind of bird-eye grasping of the proof structure, 'all at once', and an accurate mechanical verification of all delicate details. Nonetheless, they do reflect quite common sensations of which every mathematician has experience when facing many different proofs. Thus, if not completely, such idealizations bring back part of that feeling of "inexorability" that, according to Wittgenstein, characterizes real understanding of mathematical proofs. Then, how well they match with Post's remark? Apparently quite badly.

Most problems of [Post, 1944] have the following prototypical form: find a computable function f so-and-so[8]. In tackling a similar problem, one has roughly two available approaches. On the one hand, a solution would consists in obtaining a formal implementation of f within some preferred model of computation. On the other hand, it can be regarded as sufficient to provide the description of some procedure that intuitively computes f. Post chooses to adopt the second alternative, and the remark above stands as a sort of methodological disclaimer for that choice.

Now, keep CTT aside for a moment. Where do the proofs of [Post, 1944] lie in the leibnizian/cartesian spectrum? A leibnizian conception seems to fit way better with the "forbidding, diverse and alien formalisms" of papers such as [Church, 1936], that precisely represents the standard from which Post is departing. More generally, if the reliability of proofs is due to the possibility of having them presented in the most meticulous fashion – as a Leibnizian view would require – then, of course, to work in the rigid context of a model of computation seems to be the best option[9].

Post's approach might be closer with the Cartesian conception. After all, his announced goal is to show that some fragment of Computability (or, a posteriori, all of the theory) is best suited for an *intuitive* development, instead of being carried out in a mechanical Leibnizian-like way. Nonetheless, the Cartesian idealization asks for completeness, i.e. it requires to see a given proof as a whole – to get it 'all at once' – in order to grasp its truth. In Descartes' words, mathematical proposition are "to be deduced from true and known principles by the continuous and uninterrupted action of a mind that has a clear vision of each step in the process" [Descartes, 1927]. Thus, no step can be omitted. Is Post's choice of skipping all formal programs, while proving the computability of the objects that he considers, one of such omissions?

[Fallis, 2003] argues so. In a nutshell, the goal of this latter paper is that of highlighting the presence of "intensional gaps" that mathematicians leave, at times,

[8]To be fair, Post mainly speaks of computably enumerable sets, there introduced for the first time. But since, by definition, a set is computably enumerable if it is the range of a computable function, then one can trivially translate Post's formulations in instances of our prototype.

[9]It is worth noticing that the Leibnizian ideal is by no means archeological. Quite to the contrary. Hacking reports Voevodsky's opinion that "in a few years, journal will accept only articles accompanied by their machine-verifiable equivalents". More generally – and less radically – research on proof-assistants can be (partially) motivated as a way of improving automatic verication of proofs.

in their proofs. After having classified several different types of these gaps, Fallis claims that the so-called "universally untraversed gaps" are "examples of justificatory practices that are not captured by the Cartesian story, but that are nevertheless accepted by the mathematical community". As a last example of such type of gaps, Fallis hastily hints to the case we are focusing on, that of describing computable procedures just informally. It is interesting to notice that, while other cases of theoretically significant gaps are sporadic and, by author's admission, somewhat disputable, the case of Computability shows a systematic tendency of leaving intensional gaps. We will extend this consideration in the next sections.

Here, let us stress that Post's opening paragraph mismatches with both ideal conceptions of proofs we have considered. Nevertheless, we already know a possible way out to these problems: CTT. Indeed, the thesis equates the two alternative solutions to our prototypical problem through the following argument:

> **Church-Turing Bridge (CTB)**
> *If any informal description of an algorithm can be formally implemented in each model of computation (as CTT states) then, in order to prove that something is computable, it is sufficient to describe an informal way to compute it – and then make reference to CTT.*

CTB does not appear in this form in [Post, 1944], and presumably Post's preference for a somewhat informal style in mathematical writing is not mainly motivated by such application of the thesis[10].

Still, Post was certainly aware that some version of this latter argument stands as a necessary theoretical bulwark against the difficulties we have shown above. Indeed, the central notion of [Post, 1944] is that of "generated set", that in Post's words corresponds "to say that each element of the set is at some time written down, and earmarked as belonging to the set, as a result of predetermined effective processes", hence being an informal concept. The fact that propositions technically proven for generated sets hold also for recursively enumerable sets does certainly require some sort of CTB.

Let us conclude with an important remark. Post underlines that formal proofs, although being omitted in his presentation, has nonetheless been obtained: "we have obtained formal proofs of all the consequently mathematical theorems here developed informally". We assist here to a sort of conceptual twist. On the one hand, the focus is on the informal development in which "the real mathematics (...) must lie". Moreover, from CTB we obtain that informal proofs are acceptable and, in a sense, self-sufficient. But on the other hand, Post warns that for each informal proof presented in the paper a formal counterpart has been derived. The following question naturally arises: how far can one consider the knowledge of an informal proof *independent* from the knowledge of its formal renderings?

[10]For an accurate reconstruction of Post's thought see [De Mol, 2006]

As for many other aspects of the foregoing analysis, this latter one will be expanded in Rogers' presentation of the practical side of CTT, to which is devoted the following section.

1.3.2 Rogers: "Proofs which rely on informal methods"

In the two decades spanning from 1944 to 1967, Computability Theory (called 'Theory of Recursive Functions', at the time) has seen an impressive growth, becoming one of the most active and fruitful area of mathematical logic. Post's informal approach was widely adopted, proving, in Rogers' words, that "the intuitive simplicity and naturalness of the concept of *general recursive function* permitted discourse and proof at a level of informality comparable to that occurring in more traditional mathematics". Such preference towards informality contributed to a significant shift in the mainstream objects of research: from functions computable within some specific model of computation, to sets calculable by some informal procedure, and then to 'degrees', i.e. equivalence classes of sets, intuitively encoding the same complexity (according to a given notion of reducibility). Again, [Post, 1944] has opened this latter line of research, formulating a famous long-standing problem – solved only in 1957 – of whether there are 'c.e. intermediate degrees' for Turing-reducibility, that is, c.e. degrees lying strictly between the complexity of computable sets and that of the Halting set.

In this context, [Rogers, 1967][11], whose first edition appeared in 1967, accomplishes both a descriptive and a normative goal. On the descriptive side, it offers a quite comprehensive survey of the main results of the field, and its outstanding clarity is among the reasons for which the book became a classic. On the normative side, Rogers enriches the exposition of mathematical results with several profound philosophical insights, aiming to hook the mathematical content with some sort of methodological and pre-formal justification. We are interested in the passage depicting the role of informality in Computability, that is also the one in which the expression "proof by Church's thesis" is firstly introduced:

> A number of powerful techniques have been developed for showing that partial functions with informal algorithms are in fact partial recursive and for going from an informal set of instructions to a formal set of instructions. These techniques have been developed to a point where (*a*) a mathematician can recognize whether or not an alleged informal algorithm provides a partial recursive function, much as, in other part of mathematics, he can recognize whether or not an alleged informal proof is valid, and where (*b*) a logician can go from an informal definition for an

[11] From now on, in describing the practical side of CTT, we will mostly refer to textbooks. This is a natural choice. Since, as already said, there are no philosophical studies concerning the practice of Computability, the most immediate source of observations regarding how such practice has to be intended comes from the kind of expository remarks that abound in books such as Rogers'.

> algorithm to a formal definition, much as, in other parts of mathematics, he can go from an informal to a formal proof. (...)
>
> Researchers in the area have been using informal methods with increasing confidence. (...) They permit us to avoid cumbersome detail and to isolate crucial mathematical ideas from a background of routine manipulation. (...) *We continue to claim, however, that our results have concrete mathematical status* (...). *Of course any investigator who uses informal methods and makes such a claim must be prepared to supply formal details if challenged.* Proofs which rely on informal methods have, in their favor, all the evidence accumulated in favor of Church's Thesis. Such proofs will be called *proofs by Church's Thesis.*

So, the state of art is clear: one is not committed in supplying formal algorithms, since, for any informal definition, there is a corresponding formal implementation whose existence is guaranteed by CTT (of course this is just a reformulation of CTB). Let us, then, isolate Rogers' definition:

Definition 1.3.2. In Computability, a proof is called *proof by Church's thesis* if it relies on informal methods.

Two aspects shall be noticed. First, this definition is so inclusive that it encompasses *almost all* the proofs of Computability. For example, Rogers' proofs are typically formulated without referring to any background model of computation – and so is the case of all the main results of the field. Hence, the practical side of CTT, as defined in Definition 1.3.1, is basically as large as possible! Moreover, it is unsurprising that, according to this definition, not only the specification of a given model of computation is omitted, but in most cases the very reference to CTT is left implicit. Indeed, Roger refers to the thesis only while proving quite elementary facts, i.e. those facts for which going from an informal algorithm to a formal one would be really just a matter of taking care of "cumbersome detail". Vice versa, when complex constructions are considered, and the reader is acquainted to this kind of reasoning, CTT does not occur explicitly and it is not entirely clear that informal methods – such as Rogers' version of Friedberg-Muchnik solution to Post's problem – can be *trivially* translated into, say, running Turing machines. In fact, Rogers sketches a sort of 'division of labour', in which going from an informal definition to a formal one is, typically, more a task for a logician than for a mathematician.

This latter observation leads to the second aspect. Recall that [Post, 1944] states that all the formal proofs, omitted in his presentation, have been nonetheless obtained. Rogers portrays a more delicate scenario. Anyone who employs informal methods "must be prepared to supply formal details if challenged." But, in practice, such a challenge never arises. Suppose one has designed a nontrivial construction for showing that a given object is (relatively) computable. Then, it would be very unconventional – and somehow unacceptable – to write down the construction in the form of a formal program. Rather, with the goal of isolating "crucial mathematical ideas from a background of routine manipulation", what is needed is to make a certain number of sensible choices about which parts of the construction have to be formalized, and to what extent. In other words, one has to set the borders between 'important ideas' and 'negligible details'. Nor the formal side is summoned with respect to the validity of the construction.

Thus, Rogers addresses the following familiar problem. Although we would have rigorous formal languages (embodied by models of computation) at our disposal, the development of Computability is carried out in some informal frame. Gaps, in the sense of Fallis, are everywhere. By introducing the notion of 'proof by CTT', Rogers attaches to the thesis a fundamental theoretical value, that of systematically filling these latter gaps and, in doing so, justifying the adopted convention of working only on the informal side of Computability.

Nonetheless, it is remarkable that this extensive use of CTT, although theoretically significant, does not make any real difference. Computability is, of course, as reliable as any other distinct branch of mathematics, while the Post-Rogers leaning towards informality is after all analogue to that of "more traditional mathematics". We have, here, two partially conflicting views. On the one hand, both Post and Rogers believe that departing from the formal definitions of algorithms is something that needs a solid justification (a justification that, in Rogers' perspective, is fully provided by CTT). But, on the other hand, they both insist that, rather than being a distinctive feature of Computability, informality is the norm of many mathematical fields with no principle like CTT on which one can rely. So, if employing informal tools is generally permitted in mathematics, then we might argue that making use of CTT, in order to make such tools available in Computability, is at best redundant, and at worst conceptually wrong. This is the line of thinking we consider in the next section.

1.3.3 Others: "A fancy name to a routine piece of mathematics"

Consider the following passage from [Odifreddi, 1989]:

> There is another avoidable use of the Thesis, in Recursion Theory. Giving an algorithm for a function amounts, by the Thesis, to showing that this function is recursive. Although theoretically not important, and in principle always avoidable (if the Thesis is true), this use is often quite convenient, since it avoids the need for producing a precise recursive definition of a function (which might be cumbersome in details). Strictly speaking, however, this use does not even require a Thesis: it is just an expression of a general preference, widespread in mathematics, for informal (more intelligible) arguments, whenever their formalization appears to be straightforward, and not particularly informative.

Thus, Odifreddi makes use of an observation that is already present in Post and Rogers, that of the fundamentally straightforward nature of any translation from an informal definition of an algorithm to a formal one. In particular, the real cost of providing formal details for all algorithms we are working with would consist in having much less intelligible arguments. The underlying idea is clearly referring to human cognitive limitations. Since we can handle only a fairly limited amount of information at the time, it is convenient to make a distinction between relevant and irrelevant part of a given algorithm – and of course transmit only the former. Furthermore, Odifreddi argues there are no other reasons that motivate – or even, justify – working on the informal side of a given theory. This is the case of most mathematical theories, and Computability is no exception. Hence, according to this perspective, the expression "proof by Church's thesis" would be essentially unnecessary. Preferring informal arguments is a feature common to most part of mathematical endeavour (as we currently know it), and it depends only on some very basic human constraints. If we theoretically put such constraints aside, then no significant difference between formal and informal definition can be revealed.

[Epstein and Carnielli, 2008] reinforce this latter view, by denying any relevance of the practical side of CTT:

> To invoke Church's thesis when "the proof is left to the reader" is meant amounts to giving a fancy name to a routine piece of mathematics while at the same time denigrating the actual mathematics.

What does it mean to let the notion of "proving by CTT" to be equivalent to saying that "the proof is left to the reader"? In general, it can be regarded as a radical weakening of the notion. First, it expresses the fact that each informal definition of an algorithm has a proper formal referent that can be easily identified by a sufficiently painstaking reader. So, there is no real gain in giving missing formal details. But, on the other hand, such equivalence says also something about the converse relation. That is, there is no additional benefit in working with informal definitions, apart from a better understanding of a proof.

Therefore, to sum up, against Rogers (and partially Post), [Odifreddi, 1989] and [Epstein and Carnielli, 2008] argue that the preference towards informality in Computability does not require any autonomous justification, being only an instance of the general preference for more intelligible arguments in mathematics.

1.4 The Standard View

In the preceding sections, we have traced some history of the practical side of CTT. We have shown that the idea that mathematical results in Computability can be presented without referring to any background model of computation is rooted in [Post, 1944]. Then, we have highlighted the definition of "proof by Church's thesis" from

[Rogers, 1967], there presented as a necessary justification for the bold role that informality plays in Computability. Finally, we have offered two examples representing a certain tendency of dismissing the importance of such uses of thesis, by claiming instead that the case of Computability is similar to that of more mundane areas of mathematics, and that departing from formal definitions does not really affect the development of the theory. Let us then resume the outcome of this discussion by calling 'standard view (about the practical side of CTT)' the following position:

Standard View (SV)

a) *CTT allows us to rely on informal methods (by CTB)*;

b) *Yet, these methods are in the end just a matter of convenience: informal definitions point towards formal ones, and we could theoretically substitute the former with the latter without any significant loss or gain of information*;

c) *This operation is analogous to what happens in most parts of mathematics.*

SV has arguably a very large consensus among practitioners. It fits, of course, with [Odifreddi, 1989] and [Epstein and Carnielli, 2008]. More importantly, it can also explain why the practical side of CTT is so philosophically neglected. Indeed, if supporting SV, one can easily claim that the informal aspects of Computability do collapse onto their formal counterpart. Thus, once justified CTT, there is – philosophically speaking – *nothing more to do.* Let us briefly expand this point. First, recall that CTT states that any function that is calculable by some informal procedure will turn out to be computable in any of our (extensionally equivalent) formal frameworks. Then, SV claims, this is all we need in order to bridge the gap between the informal and formal side of Computability. In particular, according to SV, the widespread convention of omitting formal definitions – in any case recoverable, if needed – would be just a matter of convenience, with no particular theoretical significance. Thus, for a philosopher, the main task would be that of justifying CTT (a challenge that many philosophers have indeed accepted), while, on the other hand, considering the way in which functions are informally described, in customary presentations, would be essentially uninteresting.

A noteworthy corollary of this latter perspective is that it makes the problem of relating proofs and derivations, otherwise especially difficult, to some extent readily solved for the case of Computability. Even philosophers that strongly adverse any identification – or, hasty correspondance – between proofs and derivations have conceded, by supporting some version of SV, that CTT permits to match informal definitions with formal ones with almost no distortion. For instace, one of the main thesis of [Rav, 1999] is that proofs display a certain semantic content that is utterly destroyed when these are translated into derivations. Yet, Rav writes what follows:

> It is has been suggested to name *Hilbert's Thesis* the hypothesis that every conceptual proof can be converted into a formal derivation in a suitable formal system: proofs on one side, derivations on the other, with Hilbert's thesis as a *bridge* between the two. One immediately observes, however, that while Church's Thesis is a two-way bridge, Hilbert's Thesis is just a one-way bridge: from a formalised version of a given proof, there is no way to restore the original proof with all its semantic elements, contextual relations and technical meanings.

Rav does not subscribe to Hilbert's Thesis. Nonetheless, he seems to argue that, when considering Computability, the situation appears to be profoundly different. Being CTT a two-way bridge, it is reasonable to expect that our work is not to be conditioned by the side on which is practically carried out. Notice that, in Rav's view, what is not feasible is to going back from derivations to proofs. On the contrary, CTT would permit "to restore (...) all [the] semantic elements, contextual relations and technical meanings" of a given informal algorithm from any of its formal characterizations. Thus, saying that CTT is a two-way bridge, in the sense of Rav, is quite the same of rephrasing point *b*) of SV: we can go from informal definitions to formal ones and back with no important distortion.

Therefore, SV embeds a certain conceptual opposition concerning the relation between Computability and other mathematical theories. On the one hand, the original detachment from Roger's definition of the practical use of CTT was partially motivated by the evidence that informal tools are commonly employed in most mathematical theories – and point *c*) of SV expresses precisely this fact. On the other hand, we have just shown that point *b*) describes precisely the kind of clear correspondence between informal and formal components that many philosophers argue does not belong to mathematical practice. In order to better grasping this tension – that we eventually aim to solve by claiming that SV is untenable – we have to focus on point *b*) of SV.

1.4.1 Clarifying point *b*) of SV

Needless to say, of the three points of which SV consists, *b*) is the most delicate. Two questions are to be answered:

1. What does it mean that informal definitions 'point towards' formal ones?
2. When does a loss (or, a gain) of information count as 'significant'?

As we will see, these two questions are linked. Let us begin with the first one, with the help of some notation. Call $\mathcal{I}$ the class of all the informal definitions for algorithms[12]. Then, let C be a model of computation, and denote by $\mathcal{F}_C$ the set of

[12] $\mathcal{I}$ does clearly correspond to a pre-theoretic object whose formalization would be far from trivial. For instance, there could be a worry concerning a sort of Berry-like paradox, inasmuch we admit a

all the programs of C. We think of SV as guaranteeing the existence of a specific map μ from $\mathcal{I}$ to $\mathcal{F}_C$ such that $\mu(x)$ can be intuitively understood as a 'reasonable formalization' of the informal definition x. What can be said about this map? First, it is natural to argue that μ cannot be injective. This is because the grammar of natural languages is much less rigid than that of formal ones. Then, it is easy to imagine that we could make very minor modifications to almost any informal definition x, thus obtaining some x', in such a way that μ would be not sufficiently fine-grained to distinguish $\mu(x)$ and $\mu(x')$. More generally, the injectivity of μ is blocked by the fact that informal algorithms are typically described at a much higher level of abstraction than that of their formal counterparts. For instance, in [Blass et al., 2009] are considered two versions of the Euclidean algorithm for finding the greatest common divisor of two positive integers. The basic operation of the first version is division, while the second one is based on repeated subtraction. Then, when considering possible formalizations of these two algorithms, authors point out the following:

> The point we wish to make here (...) is that the use of low-level programs like Turing machines might cut off the debate prematurely by making the two versions of Euclid's algorithm identical. Consider programming these versions of Euclid's algorithm to run on a Turing machine, the input and output being in unary notation. The most natural way to implement division in this context would be repeated subtraction. With this implementation of division, any difference between the two versions disappears.

So, this example shows that the simplicity of the language of Turing machines in many cases can force two different algorithms to be naturally implemented with the same program. In this regard, there is quite a large literature that addresses the problem of finding a formal model better representing the kind of abstraction that is embedded in our informal algorithms. See, for instance, the Abstract State Machines of [Gurevich, 2000][13].

Nonetheless, the very fact that there is a gap between the way in which algorithms are informally presented and most of their implementations does not contrast with

too relaxed notion on what counts as an informal description for an algorithm. Nonetheless, we can suppose to deal with sufficiently clear descriptions. This is because, although border-cases cannot arguably be expunged, we are more interested, as we will see, in a somewhat global tendency.

[13] All of this is of course related to the philosophical problem of determining if one can possibly formulate a definition for algorithms that would be *correct* in the sense of Shore: "Find, and argue conclusively for, a formal definition of algorithm and the appropriate analog of the Church- Turing thesis. Here we want to capture the intuitive notion that, for example, two particular programs in perhaps different languages express the same algorithm, while other ones that compute the same function represent different algorithms for the function. Thus we want a definition that will up to some precise equivalence relation capture the notion that two algorithms are the same as opposed to just computing the same function"[Buss et al., 2001]. See also [Dean, 2007] for a rich discussion on whether algorithms can be fairly regarded as abstract mathematical objects.

SV. Arguably, whenever the language of (a fragment of) $\mathcal{I}$ and that of some $\mathcal{F}_C$ do correspond to different level of abstractions – and, again, this is the norm – μ must embeds some sort of distortion. SV does not deny such distortions. Rather, it expresses the thesis that, for most x, some gap between x and $\mu(x)$ does certainly exist, but it turns out to be practically not significant. This clearly leads to our second question: what does it mean to be 'not significant' in the present context?

The guiding intuition is that, in accordance with SV, we want CTT to be a two-way bridge. Hence, the idea is that one should be able to restore all the information distorted by μ. We might represent this latter situation as follows. SV stases that, in addition to μ, there is some sort of inverse map μ^{-1} such that the following holds:

$$\mu^{-1}(\mu(x))Ex, \tag{1.1}$$

where E is a binary predicate, defined on $\mathcal{I}$, which corresponds to 'being essentially the same algorithm'. Let us stress that these symbols are for illustrative purposes only. E is a very pre-theoretic notion, and arguably E contains several contextual and normative aspects – in several cases, two algorithms are the same if we *choose* to consider them the same – that would make the goal of developing a definitive formalization for such predicate very problematic, or even not really feasible[14]. Yet, fortunately, our notions can be left to some extent vague. Indeed, SV concerns more the global picture of Computability than particular instances of elements of $\mathcal{I}$. In particular, SV says that a scenario as the one expressed by (1.1) is *usually* correct. That is, according to SV, Computability does not extensively rely on some kind of informal devices that, for some reason, cannot be formally translated.

To shed light on this latter point, let us consider again a question we have asked on page 14. Consider the following hypothetical scenario. Suppose recursion-theorists did not adopt Post's proposal and Computability had developed with the need of providing formal implementations for each algorithm considered. Now, there are no doubts that a similar theory would appear to be different – or rather, distorted – with respect to the standard Computability we currently deal with. For one thing, proofs would be far less readable. Yet, the main question is the following: such alternative Computability would differ from the usual one in some profound sense, shifting the general meaning of the theory? A coherent supporter of SV, we argue, has to answer: no (otherwise, the possibility of regarding CTT as a two-way bridge would be severely challenged, and informality would have some independent, theoretical value). For our part, in the rest of this chapter, we will defend a positive answer.

1.5 Against SV

In this section, we argue against SV, by claiming that it is not adherent to the real practice of Computability. In particular, although informal constructions can

[14]Indeed, [Blass et al., 2009] argue that "one cannot give a precise equivalence relation capturing the intuitive notion of 'the same algorithm.' "

be always formally implemented (since CTT does hold), such constructions are, in practice, not thought as referring to their formal implementations, but they are rather *structurally* conceived. Let us stress from the beginning that we will not consider any exotic topic. In fact, our goal is to show that SV is in conflict with very basic concepts, informing how the whole theory has to be intended.

1.5.1 A case-study: the existence of a simple set

So, let us focus on a rather easy case in which some informal procedure is actually involved. First, we need to recall what follows.

There are several ways to define a *standard numbering* of partial computable functions, all essentially equivalent. Fixed one of such numberings, it is customary to denote by φ_e its e^{th} element. A numbering ψ is said to be *acceptable* if there are two computable functions f, g such that: (i) $\varphi_{f(x)} = \psi_x$; (ii) $\psi_{g(x)} = \varphi_x$. It is immediate to see that, for any acceptable numbering ψ, there is a computable permutation of ω, π, such that, for all x, $\varphi_x = \psi_{\pi(x)}$. Acceptable numberings are important since they define the scope of most results of Computability: indeed, a theorem due to Rogers [Rogers, 1967] shows that a numbering is acceptable iff it satisfies both the Enumeration and *s-m-n* Theorems (hence preserving all the results based on these latter)[15]. To our purpose, it is interesting to notice that numberings can be regarded as very abstracts ways to speak about models of computation, leaving aside all intensional aspects of such models. In particular, any natural coding of a given classical model of computation leads to an acceptable numbering, since one can always effectively translate this latter model into that of Turing machines[16].

These properties of acceptable numberings have a practical consequence that largely overlaps our main topic. In Soare's words [Soare, 1987]:

> Since most natural numberings are acceptable and two acceptable numberings differ merely by a recursive permutation, it will not matter exactly *which* acceptable numbering we chose originally.

As is clear, not specifying the background numbering on which are work is based echoes – and, in fact, subsumes – the kind of practical use of CTT that we have extensively discussed. To better illustrate such analogy, and see why SV fails, consider the following classical notion introduced in [Post, 1944]:

Definition 1.5.1. We say that a c.e. set S is *simple* if is it co-infinite and, $A \cap W_e \neq \emptyset$, for each infinite W_e.

Thus, a set of positive integers is simple if it meets all infinite c.e. sets while remaining co-infinite. Simple sets do exist:

[15] The reader is referred to [Rogers, 1967] for the proof of this fact, and to [Odifreddi, 1989] for additional results concerning numberings.

[16] Some equivalence between classical models of computation can be found in [Odifreddi, 1989].

Theorem 1.5.2 (Post)**.** *There is a simple set.*

Proof. Let $S = \text{range}(f)$ where:

$$f(i) \text{ is the first element } \geq 2i \text{ enumerated in } W_i.$$

Since f is partial computable (by CTT), we have that S is c.e. Let W_i be an infinite c.e. set. By construction, it is immediate to see that the first element $\geq 2i$ enumerated in W_i – which does certainly exist being W_i infinite – belongs to S. Thus, S intersects any infinite c.e. set. Then, notice that, if $x < 2i$ and $x \in S$, then there must be k such that $x = f(k)$. Hence, $|S \cap \{y \mid 0 \leq y < 2i\}| \leq i$. So S is co-infinite, and therefore simple. □

We have built a simple set S. That is, our proof informally describes a way to computably list all elements of S. So far, so good. But what do we *know* about S (and thus of the algorithm by which S is listed)?

For instance, consider the following two questions:

1. Does 7 belong to S?

2. Or, is S infinite?

It is worth noticing that these two questions, in Computability, do not share the same epistemological status. On the one hand, one can immediately check that S has to be infinite, since there are infinitely many infinite c.e. sets. But, on the other hand, whether 7 (or any other positive integer) would belong to S is not determined by the construction, for it essentially depends on *how* we list c.e. sets, and thus it rests on the choice of the numbering ψ. Therefore, our construction does not extensionally fix a single set, but only up to a given (acceptable) numbering.

Now, suppose that, in accordance to SV, we want to translate our informal definition into a formal one, by writing instructions for a Turing machine that lists S. Of course, this is feasible (and CTT, if true, precisely guarantees that we can do so), nevertheless in order to complete our implementation we have to specify some numbering. Otherwise, without knowing in which order the c.e. sets are enumerated, the computation will be blocked and no S would be generated. The problem is that, under such a specification, we would consider a much less general version of S, one limited by the choice of a formalism. Thus, a significant distortion between the informal construction of S and any of its formal renderings, of the kind not permitted by SV, lies in wait.

A possible way-out for a supporter of SV would consist in claiming that S, as defined above, is essentially incomplete. In this perspective, omitting to specify a background numbering, while constructing S, would be just another gap that we leave behind for reasons of convenience. In other words, the way in which the proof is presented would be another expression of the widespread preference for clarity and generality in mathematics. Yet, from a theoretical point of view, the proof has to be intended as a sort of a prototype, to be completed by specifying a certain numbering ψ. That is, our informal proof would correspond to a general method for describing, for each acceptable numbering, a corresponding simple set. To some extent a similar interpretation is certainly correct, but it is also too limited. Indeed, the following fact trivially holds:

Fact 1.5.3. *Let ψ be an acceptable numbering and let S be the simple set constructed as above with respect to ψ. Then, S is a simple set also with respect to any other acceptable numbering.*

So, we have the two following facts hold:

1. The proof of Theorem 1.5.2 provides a method for building, for each acceptable numbering, a corresponding simple set;

2. Yet, any simple S built by specifying an acceptable numbering ψ in Post's construction is also simple with respect to any other acceptable numbering.

Thus, in a sense, the theory of simple sets is *invariant* with respect to the acceptable numbering we choose to work with, making this very choice superfluous. So, against SV, it seems, first, that S does not refer to any of its formal definitions. But furthermore, S has not to be regarded as incomplete: although our informal proof does provide a method for producing, for all numberings, a given simple set, as already said, nevertheless to collapse the meaning of such proof to this method would correspond with claiming that an implicit reference to numberings is somehow needed to make complete sense of our construction of S – and this is precisely what is denied by Fact 1.5.3. Rather, the notion of simplicity is better understood as an *absolute* one, i.e. independent from the chosen formalism.

Most properties studied in Computability do share this character of absoluteness. For one thing, the notion of Turing-degree of a set (probably the main notion of the classical theory) is of course independent of the way in which we enumerate partial computable functions. Here, let us notice that, instead of being limited to properties of sets, this idea of working up to an acceptable numbering – and so without referring to any specific formal background – is embedded in most of the methods by which sets are typically constructed. In general, the success of the Post-Rogers paradigm could have led – at least theoretically – to a messy class of informal descriptions in the definitions of algorithms. In a sense, this circumstance would have been sound with SV: if informal algorithms are only conceptual shortcuts pointing towards formal ones, then there would be no reasons for expecting any independent grammar framing them. But, historically, that was not the case. In fact, these informal constructions rapidly converged towards an acknowledged standard in the form and the logic of their exposition, and their generalizations gave rise to what are called "methods". We do not wish to enter in a discussion concerning the philosophical significance of such methods (although our current analysis might be regarded as preliminary work for this latter investigation). For the moment, let us just make one more example related to SV.

Many constructions in Computability embed what shall be regarded as an effective version of Cantor's diagonalization. Generally, the goal of these constructions is to build one or more object (maybe by accessing to some oracle) in such a way that an effective list of *requirements* is eventually satisfied. Most notably, the Friedberg-Muchnik solution to Post's problem consists in constructing, by steps, two c.e. sets A and B, ensuring that, for any computable functional Φ_e, the two following requirements would eventually hold:

$$\chi_A \neq \Phi_e^B; \quad \chi_B \neq \Phi_e^A,$$

where χ_Y of course denotes the *characteristic function* of Y.

It is important to notice that A and B so defined do not refer to any specific enumeration of the functionals Φ_e. But again, this does not mean that the construction is to be considered as incomplete – as a certain reading of SV would require – but rather that we do refer to a kind of absolute version of A and B, i.e. independent from the choice of a formalism. In the next section, we aim to provide a better characterization of these *absolute objects*.

1.5.2 Indifference choices

In discarding SV, one has to clarify how objects such as the simple set S, or the Turing-independent c.e. sets A and B, are to be thought if not as objects to be completed with missing formal details. In doing so, we borrow the notion of *indifference* from [Burgess, 2015]. Indifference, in Burgess' view, is best understood in contrast with *structuralism* in philosophy of mathematics. Under this heading, there is a variety of positions – and a large body of work[17] – corresponding to different philosophical characterizations of, roughly, the same basic idea: mathematicians are not concerned with the nature of the objects they deal with, but rather with *structures* involving them. So that, if two classes of objects display the same structure, then it does not really matter which one we choose to work with. Structuralism, at least in this naive formulation, does certainly reflect a common tendency of mathematical practice. For instance, Euclid's definitions – such as "A *point* is that which has no part" – are irrelevant to his proofs, and what really counts as the mathematical meaning of the basic objects of Euclidean geometry (points, lines, angles, etc.) are the relations they entertain with each other, as expressed by the axioms. Of course, this interpretation has been strengthened in [Hilbert, 1899], that represents both "the culmination of a trend toward structuralism within mathematics" [Shapiro, 2010] and one of the most influential starting point for the distinctive emphasis on

[17]For a classical defense of structuralism in philosophy of mathematics, the reader is referred to [Resnik, 1997].

structures that characterizes contemporary mathematics obviously much more than ancient one.

Philosophers have put considerable effort in trying to propose a coherent account embracing the role that structures play in mathematics. However, most proposals develop a kind of ontological framework (motivating the existence or, alternatively, the nonexistence of structure themselves) that is quite far from our present perspective[18]. A more practice-oriented approach builds on the tempting inclination of considering the structural resemblance – up to which class of objects shall be regarded as the same – as being formally captured by isomorphism relation. As Awodey puts it [Awodey, 2014]:

> The following statement may be called the Principle of Structuralism:
>
> (PS) Isomorphic objects are identical.
>
> From one perspective, this captures a principle of reasoning embodied in everyday mathematical practice: (...)
>
> - The Cauchy reals are isomorphic to the Dedekind reals, so as far as analysis is concerned, these are the same number field, $\mathbb{R}$.
>
> (...) Within a mathematical theory, theorem, or proof, it makes no practical difference which of two "isomorphic copies" are used, and so they can be treated as the same mathematical object for all practical purposes. This common practice is even sometimes referred to light-heartedly as "abuse of notation," and mathematicians have developed a sort of systematic sloppiness to help them implement this principle, which is quite useful in practice.

A principle such as PS encounters unsurprisingly many difficulties, because part of the specific value of considering two isomorphic objects as the same comes from the possibility of distinguishing them when needed (and, indeed, the goal of Awodey's paper is to show that PS is incompatible with the standard set-theoretic foundation, promoting rather the so-called Univalent Foundations). To our interests, it is important to notice that PS does not represent an available option in the context of Computability. After all, one of the key feature of this latter theory is that of enriching the study of mathematical structures by means of considering some (possibly invariant) computational aspects, to which classical mathematics is insensitive. Consider, for instance, the case of two noncomputably isomorphic presentations of the same structure, that immediately would reject PS.

One might obtain a better candidate for Computability by replacing isomorphism, in the PS principle, with computable isomorphism. Nonetheless, it is just false that, in the theory, any two computably isomorphic object are identical – or,

[18] Two noteworthy exception being [Carter, 2008] and [McLarty, 2008].

even, that are regarded as identical for all practical purposes. For instance, *index sets*, e.g. sets containing all indices of a given computable function, do clearly depend on some background numbering. Thus, it is trival to define a computable permutation of ω, π, such that the set $\{\pi(i) \mid i \in I\}$ is not an index set. More generally, the problem here is that the distinctive focus on absolute notions in Computability seems to be a *choice*, and hence it would be very hard to explain it if referring only to the formal side of the theory.

Burgess' notion of indifference is precisely an attempt of making sense of the phenomenon highlighted by structuralist philosophers, without being committed to heavy-duty principles such as PS:

> It should be emphasized that (...) various structuralist philosophers of mathematics have performed a real service by pointing out to philosophers a real phenomenon, a kind of *indifference* on the part of working mathematicians, namely, an indifference to exactly how one got to the point from which their own investigations begin.

According to Burgess, there is a process – that he considers a process of "rigorization" – by which a certain amount of the work that has been made in order to develop a given formal notion can thereafter be dropped by remaining indifferent with respect to *how* this very notion has been introduced. So, we can define real numbers as Cauchy sequences or as Dedekind cuts, and then, while doing analysis, being perfectly indifferent with respect to which is the underlying formalization of reals. It is not our purpose to review Burgess' long defense of this notion, that, he argues, incorporates several different phenomena emerging in mathematical practice.

Let us just notice two aspects that we find appealing of Burgess' proposal. First, contrary to most structuralist philosophers, he does not refer to some peculiar nature of mathematical objects while explaining the fact that different classes of objects are regarded as the same. Thus, he claims that the correct focus is not on the ontology of mathematics, but rather on the reasons for which mathematicians do something (i.e., identifying such classes). Secondly, Burgess acknowledges that this operation of remaining indifferent towards parts of the mathematical discourse "tells us that any aspect of old work not needed or useful for new work can be disregarded, but it does not tell us *which aspects of old work these are likely to be.*" Therefore, to some extent indifference shall be regarded as a *choice*, to be accepted or rejected as part of a given mathematical theory. This reading, as we will see, has noteworthy consequences for our context.

As is now clear, our thesis is that the Post-Rogers detachment from the need of defining formal algorithms is well-represented as an instance of such indifference. The main benefit of this interpretation is two-fold. First, it takes account of the emphasis on absolute notions in Computability – an emphasis completely disregarded by SV – without referring to some peculiar ontological feature that objects of the theory would display. That is, it is not our simple set S that is absolute (in some ontological sense to be clarified), but rather it is our way to refer to a simple set that remain indifferent with respect to which acceptable numbering we make use.

Secondly, indifference is perfectly compatible with the evidence that absolute notions, although characterizing most notions of the theory, do not represent them all (recall the case of index sets). This is because indifference, in Computability, corresponds to a choice, that of having particular focus on such absolute notions – nevertheless, this focus can be suspended or disregarded, if it is fruitful doing so.

1.6 Final remarks: Back to CTT

We have shown that SV fails to represent a central phenomenon of Computability, that of conceiving most constructions as absolute, i.e. independent from the background formalism and yet not to be regarded as incomplete. Furthermore, we have argued that this feature of absoluteness can be better grasped by appealing to a certain notion of indifference. More generally, the idea embedded in SV that working on the informal side would not shift the general meaning of the theory appears to be untenable. Let us then conclude by spending a very few words on how this latter perspective can shed some light on the aforementioned problem of relating Computability with other mathematical theories. In doing so, we have to turn back to CTT.

As already said, SV trivializes the 'proofs vs derivations problem', in the context of Computability, by interpreting CTT as a two-way bridge. That is, according to SV, Computability – contrary to almost any other mathematical theory – would permit, via CTT, to deny any significance to our practice, and with almost no philosophical cost. For our part, one of the main goal of this chapter has been precisely that of separating CTT from its practical side, i.e. we have defended the following claim: although CTT, being valid, does certainly fix a unique extensional notion of calculability, on the other hand this does mean that CTT fixes also a unique practice. Rather, the practical side of Computability (as always in mathematics) relies on a collection of choices and omissions, altogether amounting to the kind of indifference that we have sketched above. What is peculiar of Computability is that this process of highlighting and neglecting parts of formal discourse – that defines how a given theory has to be intended – is much more explicit than in other mathematical contexts. Indeed, first Post and then Rogers had to somehow justify an overall conventional level of informality in their expositions because dealing with the very unconventional fact that a fully formal approach to their theory was, at least in principle, available.

Thus, once abandoning the perspective, embedded in SV, that *after CTT* there would be philosophically nothing to say about Computability, we have arrived to the hypothesis that Computability might be even taken as an ideal context in which studying practice-oriented questions, for the relation between theory and practice (and the corresponding one between formal and informal) does appear at a very high level of clarity. Of course a lot of further work has to be done in order to confirm such hypothesis.

Chapter 2

Computable theoretic models for informal provability

2.1 Introduction

The first chapter of this work has been devoted to a philosophical analysis of the role of informal proofs in Computability. We now consider a sort of converse relation. That is to say, we aim to make use of computable-theoretic tools in order to capture a dynamic feature of real mathematical theories that formal systems fail to represent. To be more precise, formal systems represent mathematical theories in a somehow static way, in which axioms of the represented theory have to be defined from the beginning, and no further modification is permitted. As is clear, this representation is not comprehensive of all aspects of real mathematical theories. In fact, these latter, as often argued, are frequently the outcome of a much more dynamic process than the one captured by formal systems. For instance, in defining a new theory, axioms can be chosen through a trial and error process, instead of being initially selected. Dialectical and quasi-dialectical systems are two logical models – best understood as generalizations of formal systems – that are apt to characterize this dynamic feature of mathematical theories.

The material of this chapter corresponds to that of [Amidei et al., a] (written in collaboration with Jacopo Amidei, Duccio Pianigiani, Giulia Simi, and Andrea Sorbi) and [Amidei et al., b] (written in collaboration with Jacopo Amidei, Duccio Pianigiani, and Andrea Sorbi). In particular, the notion of a quasi-dialectical system, with its related results, has been developed by the author of this thesis and Jacopo Amidei.

2.1.1 Trial and error computation

The seminal works by E.M. Gold [Gold, 1965] and H. Putnam [Putnam, 1965] represent the earlier basis for many subsequent attempts at formalizing a procedure of computation allowed to "change its mind", by retracting any finite number of times its conclusion (the last "yes" or "no" is always to be the correct answer). In the mid 1970s, R. Magari and R. Jeroslow proposed two further different formal counterparts of the concept of "trial and error theory"; the set of theorems of Jeroslow's "convergent experimental logics" turns out to coincide precisely with the collection of Δ_2^0 sets, as in the case of Putnam's trial and error predicates, whereas Magari's dialectical sets (where a "dialectical set" is the set of the so-called "final theses" of a dialectical system) form a strict subclass of the Δ_2^0 sets.

Thinking of Gödel's limitative results, Magari's purpose was in particular that of introducing a kind of formal systems, endowed with a certain degree of effectivity, and basically consisting of two actions: on the one hand removing contradictions when they arise, by removing some axioms, and on the other hand adding axioms until they do not give rise to contradictions. Between June and October 1973 the issue of Magari's "dialectical systems" and of Jeroslow's "experimental logics", together with the more general question of how we acquire mathematical knowledge by the underlying method of trial and error, was taken up in a brief exchange of (unpublished) letters between Magari, G. Kreisel and R.G. Jeroslow[1]. Kreisel reiterated his position according to which the trial and error approach neglects the way we actually find and explain axioms, the process of adding new axioms being rather connected with our wish of describing some mathematical object we have in mind: axioms, in Kreisel's view, emerge from a rigorous examination of informal notions. Kreisel's criticism echoed his well-known criticisms to so-called "pragmatist" and "positivist" mathematics, in line with his concept of *informal rigor* (see [Kreisel, 1967]), according to which mathematics begins with the analysis of intuitive notions, and in laying down axioms we first proceed by identifying properties of the involved entities, and eliminating doubtful properties from the intuitive notions. In his turn, Magari, in one of his letters to Kreisel, proposed to consider *logic* a *natural science*, thus supporting the idea that "genuine trials" do exist in mathematics[2]. We believe that nowadays the best way of understanding Magari's anti-systematic positions is therefore in the light of the philosophically more aware Lakatos' "dialectical" reconstruction of history of mathematics, influenced by Popper's fallibilism: mathematics is quasi-empirical and conjectural, and not growing by accumulation of eternal truths. Bridging the gap between mathematics and natural science was indeed the goal of Lakatos' search of a common fallibilist epistemological basis for them.

As is also underlined in [Mancosu, 2008, p. 5] in the wake of Lakatos' dialectical philosophy of mathematics, several logicians called for an historically more grounded analysis of the development of mathematics (see for instance [Kitcher, 1983], and [Cellucci, 2000]). Trial and error machines have drawn attention in for-

[1]We wish to thank Paolo Pagli for having made available this epistolary to them.

[2]In [Magari, 1980] Magari assimilated what he called "metamathematics" and "applied mathematics" (the latter in a sense that perhaps does not coincide with the usual one) to the "empirical sciences".

mal learning theory, and have been considered by [Angluin and Smith, 1983] and [Osherson et al., 1991] (see also [Kelly, 1996]). [Kugel, 1986] takes into serious consideration the hypothesis that the human brain is a trial and error machine. The idea of a "multimachine theory of mind" (see [Copeland and Shagrir, 2013]) on the other hand, arose in Turing's later work, [Turing, 1947, Turing, 1996], where he outlines "the possibility of letting the machine alter its own instructions, mutating in fact from one machine into another, on the basis of some process of learning from experience, and nevertheless be such that one would have to admit that the machine was still doing very worthwhile calculations" ([Turing, 1947]). Making and correcting mistakes is intended as a manifestation of intelligence. Indeed: "If a machine is expected to be infallible, it cannot also be intelligent" ([Turing, 1947]). Making mistakes is furthermore a consequence of going through new methods: "This danger of the mathematician making mistakes is an unavoidable corollary of his power of sometimes hitting upon an entirely new method. This seems to be confirmed by the well known fact that the most reliable people will not usually hit upon really new methods." ([Turing, 1996]).

Several models of trial and error computation have been introduced more recently, for instance by [Hintikka and Mutanen, 1988]; see also S. Shapiro and T. McCarthy's concept of "projectability" ([Shapiro and Mc Carthy, 1987]), where a projection is a tentative value at a certain stage of the computation, subject to revision at a further stage. As often remarked, some complexity classes of the arithmetical hierarchy have emerged in relation with the power of machine models considered in these studies, namely in most cases the levels Σ^0_2 and Δ^0_2 of the arithmetical hierarchy; for instance, [van Leeuwen and Wiedermann, 2012] have introduced another model, the so-called "red-green machines", a modern version of Turing's "non circular machines": these machines, whose set of states is partitioned in two class (*red* and *green*), recognize the sets in the class Σ^0_2 , and accept the sets in Δ^0_2.

Up to now Magari's "dialectical systems" have been somehow neglected. With the exception of [Bernardi, 1974] [Gnani, 1974], and more recently [Montagna et al., 1996], where the concept of "dialectical system" is developed by assigning probabilities to provisional theses, Magari's work is mentioned only in [Jeroslow, 1975] and in P. Hàjek's paper on Jeroslow's experimental logics [Hájek, 1977].

In Section 2.2, we give our definition of a "dialectical system" in terms of enumeration operators. This more convenient than Magari's original approach but equivalent to it (as shown in [Amidei et al., a]). In Section 2.3 we introduce a more general notion of "dialectical system", namely that of a "quasi-dialectical system", led to this notion by a careful investigation of whether or not Magari's systems can be regarded as a satisfactory formalization of some of the most distinguishing features of empiricism in mathematics. In Section 2.4, we prove that the Turing degrees of the dialectical sets and of the quasi-dialectical sets coincide with the computably enumerable Turing degrees, and we prove that the enumeration degrees of the dialectical sets and of the quasi-dialectical sets coincide with the Π^0_1 enumeration degrees. In Section 2.5, we observe that all dialectical sets are ω-computably enumerable in the Ershov hierarchy (Theorem 2.5.5); also, for every $n \geq 2$ there exist dialectical sets that are n-c.e., but not $(n-1)$-c.e.; and there is an ω-c.e. dialectical system, which is not n-c.e., for any finite n. Finally, we show that for every ordinal notation $a \in O$ of a nonzero ordinal, there is a quasi-dialectical set which lies in the level Σ^{-1}_a of the Ershov hierarchy, but not in $\bigcup_{b<_O a} \Sigma^{-1}_b$. From this, it will follow that there are quasi-dialectical sets that are not dialectical, thus concluding that the quasi-dialectical sets do not coincide with the dialectical sets.

2.1.2 Background

On the technical side, no specific prerequisites are needed to read this chapter, except perhaps for some introductory computability theory, for which our basic references are [Cooper, 2003, Rogers, 1967, Soare, 1987]. In particular, the reader is referred to [Rogers, 1967] for Kleene's system O of ordinal notations; to [Soare, 1987] for a clear introduction to Δ^0_2 sets, the least modulus function, and the computably enumerable Turing degrees; finally, [Cooper, 2003] contains a clear and succinct account of enumeration reducibility and enumeration degrees; the Ershov hierarchy is excellently treated in a few pages in [Ash and Knight, 2000].

Due to their importance throughout this chapter, we only recall here some basic facts about enumeration operators, and Δ^0_2 sets. Any computably enumerable (c.e.) set Φ defines a mapping Φ taking sets of numbers to sets of numbers, namely,

$$\Phi(A) = \{x : (\exists \text{ finite } D)[\langle x, D\rangle \in \Phi \text{ and } D \subseteq A]\},$$

for every $A \subseteq \omega$, where ω denotes the set of natural numbers, and we identify finite sets with numbers, through their canonical indices. Such a mapping Φ is called an *enumeration operator*. A *computable approximation to* an enumeration operator Φ is a sequences $\{\Phi_s\}_{s\in\omega}$ of sets such that $\Phi_s \subseteq \Phi_{s+1}$, $\Phi = \bigcup_s \Phi_s$, and the relation $x \in \Phi_s$ is decidable in x, s. Since enumeration operators are nothing but c.e. sets, one can refer to a *uniformly* computable approximation $\{W_{e,s}\}_{e,s\in\omega}$ to the c.e. sets, where $\{W_e\}_{e\in\omega}$ is the standard indexing of the c.e. sets, the relation $x \in W_{e,s}$ is decidable in x, e, s, and for every e, $\{W_{e,s}\}_{s\in\omega}$ is a computable approximation to W_e (meaning that $W_{e,s} \subseteq W_{e,s+1}$, and $W_e = \bigcup_s W_{e,s}$) such that $W_{e,0} = \emptyset$ and every $W_{e,s}$ is finite. For more information on enumeration operators, see in particular [Cooper, 2003].

For any given set B, $B(x)$ denotes the value of the characteristic function of B on x. If $\{B_s\}_{s\in\omega}$ is a sequence of sets, and $x \in \omega$, we say that $\lim_s B_s(x)$ exists if there is t such that for every $s \geq t$, $B_s(x) = B_t(x)$. A set B lies in the class Δ^0_2 of the arithmetical hierarchy if and only if there is a computable sequence of sets $\{B_s\}$ (meaning that the relation $x \in B_s$ is decidable in x, s) such that, for every x, $\lim_s B_s(x)$ exists and $B(x) = \lim_s B_s(x)$, if and only if there is a 0-1 valued computable function $g(x, s)$ such that, for every x, $B(x) = \lim_s g(x, s)$.

2.2 Dialectical systems

In this section we give the definition of a dialectical system. Our definition is different from Magari's original one (being more intuitive, and mathematically easier to work with) but equivalent to it, as shown in [Amidei et al., a].

The basic ingredients of a *dialectical system* are a number c, called a *contradiction*; a deduction operator H that tells us how to derive consequences from a finite set A of assumptions; a *proposing function*, i.e. a computable permutation f that proposes axioms, to be accepted or rejected as provisional *theses* of the system. If up to a given stage we have accepted the axioms $f(i_1), \dots f(i_m), \dots, f(i_n)$, with $i_1 < \dots < i_n$, and at this stage we see that we can derive c from $f(i_1), \dots f(i_m)$, for a least $m \leq n$, then we temporarily reject $f(i_m)$ and what follows, still accept $f(i_1), \dots, f(i_{m-1})$, and we are willing to add (perhaps again, since it might have been already proposed and discarded earlier) $f(i_{m+1})$ to our working assumptions; on the other hand, if we see that c does not arise, then we are willing to add $f(i_{n+1})$ to our working assumptions.

Notation. In what follows, if f is the so-called proposing function, we will denote $f(i)$ with f_i.

Definition 2.2.1. A *dialectical system* is a triple $d = \langle H, f, c \rangle$, where f is a computable permutation of ω (called *the proposing function*), $c \in \omega$, and H is an enumeration operator, such that $H(\emptyset) \neq \emptyset$, $H(\{c\}) = \omega$, and H is an algebraic closure operator, i.e., H satisfies, for every $X \subseteq \omega$,

- $X \subseteq H(X)$;
- $H(X) \supseteq H(H(X))$.

Given such a d, and starting from a fixed computable approximation $\{H_s\}_{s \in \omega}$, define by induction values for several computable parameters: A_s (a finite set), r_s (a function such that for every x, $r_s(x) = \emptyset$ or $r_s(x) = \{f_x\}$), $m(s)$ (the greatest number m such that $r_s(m) \neq \emptyset$), $h(s)$ (a number). In addition, there are the derived parameters: $L_s(x) = \bigcup_{y<x} r_s(y)$; and, for every i, $\chi_s(i) = \bigcup_{j \leq i} H_s(L_s(j))$.

Stage 0

Define $m(0) = 0$, $h(0) = 0$,

$$r_0(x) = \begin{cases} \{f_0\} & \text{if } x = 0 \\ \emptyset & \text{if } x > 0, \end{cases}$$

and let $A_0 = \emptyset$.

Stage $s+1$

Assume $m(s) = m$. We distinguish the following cases:

1. there exists no $k \leq m$ such that $c \in \chi_s(k)$: in this case, let $m(s+1) = m+1$, and define
$$r_{s+1}(x) = \begin{cases} r_s(x) & x \leq m \\ \{f_{m+1}\} & \text{if } x = m+1 \\ \emptyset & \text{if } x > m+1; \end{cases}$$

2. there exists $k \leq m$ such that $c \in \chi_s(k)$: in this case, let z be the least such k, let $m(s+1) = z+1$, and define
$$r_{s+1}(x) = \begin{cases} r_s(x) & \text{if } x < z \\ \{f_{z+1}\} & \text{if } x = z+1 \\ \emptyset & \text{if } x = z \text{ or } x > z+1. \end{cases}$$

Finally define $h(s+1) = m(s+1)$ if Clause (1) applies, otherwise, $h(s+1) = m(s+1) - 1$, and let
$$A_{s+1} = \bigcup_{i<h(s+1)} \chi_{s+1}(i)(= H_{s+1}(L_{s+1}(h(s+1)))).$$

The latter equality is justified by monotonicity with respect to inclusion of H_{s+1}.

Definition 2.2.2. We call A_s the set of *provisional theses* of d at stage s. The set A_d defined as
$$A_d = \{f_x : (\exists t)(\forall s \geq t)[f_x \in A_s]\}$$
is called the set of *final theses* of d. We often write $A_s = A_{d,s}$ when we want to specify the dialectical system d. A set $A \subseteq \omega$ is called *dialectical* if $A = A_d$ for some dialectical system d: in this case, we also say that A is *represented by (or, associated to) d*.

Figure 2.1 and Figure 2.2 illustrate how we go from stage s to stage $s+1$, according to Clause (1) and Clause (2), respectively, of the definition. In the pictures, it is understood that $r(v) = \{f_v\}$, if f_v is positioned above v, otherwise $r(v) = \emptyset$.

Although the sequence $\{A_{d,s}\}_{s\in\omega}$ depends on the chosen approximation to H, the set A_d does not, as follows from Lemma 2.2.3: the proofs of (1) and (2) of this lemma are postponed until Section 2.3, where we prove the same claims for quasi-dialectical systems, of which dialectical systems are particular cases.

Lemma 2.2.3. *Let $d = \langle H, f, c\rangle$ be a revised dialectical system. For every x,*

1. *$A_d(x) = \lim_s A_s(x)$ exists;*

$$\begin{array}{cccccccc} f_0 & & & f_3 & & f_5 & & f_m \\ 0 & 1 & 2 & 3 & 4 & 5 & \cdots\cdots\cdots & m \end{array}$$

Configuration at s

$$\begin{array}{ccccccccc} f_0 & & & f_3 & & f_5 & & f_m & f_{m+1} \\ 0 & 1 & 2 & 3 & 4 & 5 & \cdots\cdots\cdots & m & m+1 \end{array}$$

Configuration at $s+1$, via clause (1)

Figure 2.1: From stage s to $s+1$ using Clause (1).

$$\begin{array}{ccccccccccc} f_0 & & & f_3 & & f_5 & & f_z & & & f_m \\ 0 & 1 & 2 & 3 & 4 & 5 & \cdots\cdots\cdots & z & z+1 & \cdots\cdots\cdots & m \end{array}$$

Configuration at s

$$\begin{array}{ccccccccc} f_0 & & & f_3 & & f_5 & & & f_{z+1} \\ 0 & 1 & 2 & 3 & 4 & 5 & \cdots\cdots\cdots & z & z+1 \end{array}$$

Configuration at $s+1$, via clause (2)

Figure 2.2: From stage s to $s+1$ using Clause (2).

2. *$r(x) = \lim_s r_s(x)$ exists; $f_x \in A_d$ if and only if $r(x) = \{f_x\}$; and, letting $L(x) = \bigcup_{y<x} r(y)$, we have*

$$f_x \in A_d \Leftrightarrow c \notin H(L(x) \cup \{f_x\});$$

3. *$r(x)$, $L(x)$, and the set A_d do not depend on the chosen computable approximation to H.*

Proof. The proofs of items (1) and (2) are postponed until Section 2.3: (1) will be a consequence of Corollary 2.3.20; (2) will be a consequence of Lemma 2.3.8, Lemma 2.3.14, and Lemma 2.3.18.

Finally (3) easily comes from (2) by induction on x. Indeed, $L(0) = \emptyset$ and $f_0 \in A_d$ if and only if $r(0) = \{f_0\}$, if and only if $c \notin H(\{f_0\})$; assuming the claim for x, then $L(x+1) = L(x) \cup r(x)$, and we have $f_{x+1} \in A_d$ if and only if $r(x+1) = \{f_{x+1}\}$, if and only if $c \notin H(L(x+1) \cup \{f_{x+1}\})$. □

The following is a useful characterization of A_d:

Lemma 2.2.4 ([Magari, 1974])**.** *For every x,*

$$f(x) \in A_d \Leftrightarrow c \notin H(A_d \cap \{f(y) : y < x\} \cup \{f(x)\}).$$

Proof. See [Magari, 1974]. □

In the rest of the chapter, *dialectical systems* will be always presented as triples $\langle H, f, c\rangle$, where H is an algebraic closure enumeration operator; or even as triples $\langle H, f, c\rangle$ where H is just an enumeration operator such that $X \subseteq H(X)$, for every set X. Notice that the request that $X \subseteq H(X)$ for all X does not in fact introduce any non-effectivity, as one can effectively go from H to the enumeration operator $H \cup \{\langle x, \{x\}\rangle : x \in \omega\}$ which satisfies the request.

As an exemplification of our working definition, notice:

Lemma 2.2.5 ([Magari, 1974])**.** *If $A \in \Pi^0_1$, and $A \neq \omega$, then there is a dialectical system d such that $A = A_d$.*

Proof. If $A = \emptyset$, then $A = A_d$ for every dialectical system $d = \langle H, f, c\rangle$ such that $c \in H(\emptyset)$. Otherwise, assume that $A \in \Pi^0_1$, $A \neq \emptyset$ and $A^c \neq \emptyset$, where A^c denotes the complement of A. Choose $a \in A$ and $c \in A^c$. Define

$$H = \{\langle y, \{x\}\rangle : x \notin A, y \in \omega\} \cup \{\langle a, \emptyset\rangle\} \cup \{\langle x, \{x\}\rangle : x \in \omega\}.$$

(We let $\langle a, \emptyset\rangle \in H$, to satisfy $H(\emptyset) \neq \emptyset$.) Notice that H is an algebraic closure operator. Indeed, let $X \subseteq \omega$: clearly $X \subseteq H(X)$; on the other hand, if $X \cap A^c \neq \emptyset$ then $H(X) = \omega$, thus $H(X) = H(H(X))$; if $X \subseteq A$, then $H(X) = X \cup \{a\}$, giving again that $H(X) = H(H(X))$. Take $d = \langle H, f, c\rangle$, where f is the identity. It is easy to see that d is our desired dialectical system. □

Whereas every co-c.e. set, different from ω, is dialectical, it is interesting to observe (see [Magari, 1974]) that no c.e. set is dialectical, unless it is decidable. In view of the fact that dialectical sets represented by dialectical systems for a formal theory (see [Bernardi, 1974] and [Magari, 1974]) correspond to completions of the theory (see [Magari, 1974]), this is not surprising, as it corresponds to the well known result of logic, that every c.e. complete theory is decidable. Notice also that ω is not a dialectical set, as we can never have $c \in A_d$, for any dialectical system d.

2.3 Quasi-dialectical systems

The goal of this section is twofold. Firstly, on a more philosophical side, we aim at investigating the relation between dialectical systems and a form of empiricism, concerning mathematical theories, that naturally embeds a certain notion of "revision". Secondly, our main strategy for such an investigation results in contrasting Magari's original systems with some modified systems (whose formulation is new) that are apt to formalize this informal idea of revision.

But the whole section is not just a philosophical detour. The main point here is to analyze to which extent dialectical systems can capture that kind of trial and error view of mathematics that is supported by Magari. The belief that such a point is a central one is shared by Magari himself. In his words[3], see [Magari, 1974]:

[3] Our translation.

> "Their [of dialectical systems] possible interest lies upon certain assumptions, which, as questionable as they might be, are nonetheless tenable, in the author's opinion, regarding the systems under consideration, for instance the assumption that the systems under consideration well schematize a way of proceeding, essentially admitted by the mathematical community."

Thus, Magari claims that the general significance of his proposal, as well as its possible conceptual fruitfulness, can be fully understood only when considering the philosophical frame in which dialectical systems are incorporated. It is fair to say that the philosophical frame that Magari has in mind consists in a (rather liberal) form of empiricism. That is, mathematical theories are by no means static compounds of eternal truth, but rather they do evolve in time, through a process in which axioms are (also) chosen by trial and error. It is clear that, at this level of vagueness, such a position is compatible with several different stances in the philosophy of mathematics (for instance, Lakatos famously opens his [Lakatos, 1976b] with reviewing many different points of view that, in his opinion, are somewhat sounded with such a view). Nonetheless, when introducing his dialectical systems, Magari did not aim at subscribing to any precise form of empiricism, but he rather intended to offer a formal notion by which a wide class of these forms can be characterized. To clarify this latter point, it is useful to spell out two methodological features that, although implicitly, emerge in Magari's proposal:

1. Dialectical systems are *extremely general*, in the sense that very few requirements are imposed to their basic components, i.e. the deduction operator H, the proposing function f, and the symbol of contradiction c. For instance, in general H does not have to encode any well-known logical rule.[4]

2. Dialectical systems are *purely syntactical objects*. That is, there is no formal semantics for them. However, as we just said, there is an intended (informal) semantics, that takes them as a convenient representation of "a way of proceeding, essentially admitted by the mathematical community". The main contribution of this section is exactly that of pushing forward the boundaries of this intended semantics.

Thus one could formulate our main question, here, as follows: do dialectical systems really match Magari's informal intuition of a mathematical theory that, in choosing its axioms, proceeds by trial and error?

At first sight, a fully positive answer appears to be constrained by the lack, within the formalization of dialectical systems, of one of the key features of trial and error processes, namely some notion of revision by which our statements, in presence of a

[4] Dialectical systems that do obey specific constraints on H, reflecting the behavior of logical connectives, are studied in [Bernardi, 1974].

possible problem, are not discarded but rather substituted. For instance, Lakatos' "monster barring" (see [Lakatos, 1976a]) provides an historical example of a way in which a mathematical hypothesis, when encountering a counterexample, might be refined, instead of being just removed. Dialectical systems seem to be unfit for such cases, since each contradiction imposes to discard the axiom, and no substitution, or refinement, is considered.

Thus, we propose to modify Magari's original definition by introducing some new systems (that we call *quasi-dialectical systems*) apt to accommodate this idea of revision. Then, we will compare them to dialectical systems in terms of their expressiveness and information content, thus verifying whether such a notion of revision can be already embedded in Magari's systems. Furthermore, as we will see, the study of quasi-dialectical systems will be mathematically interesting *per se*.

Later on, we will of course provide a formal definition for these new systems, including some informal comments that might help in understanding their behavior. For the moment, let us just focus on two very general, and somewhat preliminary, aspects of our proposed formalization.

Quasi-dialectical systems extend standard dialectical systems with two additional symbols: c^- and f^-. Roughly, the role of f^- is that of replacing a certain axiom, that has produced some kind of problem, formally encoded by c^-, with another axiom. Thus, while c represents the mathematical contradiction (in both Magari's definition and ours), c^- corresponds to a large variety of possible problems that might lead a mathematician to replace an axiom. At the very high level of generality in which our presentation is pursued (such as Magari's one), the specific nature of these kind of problems is disregarded. That is, we do not want to commit ourselves to the specific semantic status of c^-. On the contrary, our aim is to keep the intended meaning of c^- vague enough to incorporate a wide class of problems. These problems do not necessarily pertain to the formal side of the mathematical practice. Indeed, due to the generality of our proposal, they might include problems related to that kind of informal *desiderata* one can expect from an axiom, such as fruitfulness, or simplicity – or even psychological and aesthetic features, these latter being fully admissible as long as they can represent some reason to replace a given axiom.

This inclination towards generality also affects f^-. In particular, we do not ask for a unique notion of relevance between axioms and their possible substitutions via f^- that might hold for all systems. Roughly, the only condition in this respect is that f^- has to be computable. Being so liberal with respect to our definition of f^- might seem to contrast with our attempt of capturing an informal idea of revision. Indeed, once dismissed any notion of relevance from our formalization, how can we claim that our revision can be regarded as a whatsoever refinement? Again, the answer appeals to generality. Since the problems encoded by c^- are taken to be as general as possible, we do not want to restrict their possible solutions (i.e., the outputs of f^-) under any *a priori* rule of relevance.

This latter point suggests the following remark on f^-. For dialectical systems the proposing function f is given independently from the computation, i.e. the operator H: consistently with the fact that there is no restriction on the generality of f, there is no reason either to take its proposals as being somewhat linked to the behavior of the computation. On the other hand, f^- can be better understood as imposing a certain number of corrections to f: i.e., if some axiom raises a problem, we substitute this very axiom with a new one, and we force the system to go on from this new axiom.

Let us conclude by citing an additional phenomenon that represents a major difference between dialectical and quasi-dialectical systems. In general, these latter ones depend on how the operator H is approximated, in the sense that different approximations of the same system might yield to different sets of final thesis. This is due to the fact that, within a quasi-dialectical system, a set of axioms X might derive both c and c^- (and in fact this is the case for basically every non-trivial system). Therefore, in such a case, there would be some approximations in which X derives c before c^-, whereas in other ones X would derive c^- before c. As it will be clear hereinafter, this difference might also affect the corresponding sets of final thesis. In particular, we will offer below an example of a quasi-dialectical system whose set of final thesis depend on the choice of approximation to the operator H. Nonetheless, we would be able to prove that, although failing in general, invariance of the set of final thesis is preserved by an important subclass of approximations, namely those in which all the axioms are eventually proposed.

Enough for motivations, to business now!

2.3.1 Quasi-dialectical systems: the definition

The definition of a quasi-dialectical system is modelled on the definition of a dialectical system, as given in Section 2.2.

Definition 2.3.1. A *quasi-dialectical system* q is a quintuple $q = \langle H, f, f^-, c, c^- \rangle$, such that $\langle H, f, c \rangle$ is a dialectical system and q satisfies the following conditions:

1. $c^- \in \omega$;
2. f^- is a total computable function and $c^- \notin \text{range}(f^-)$;
3. f^- is *acyclic*, i.e., for every x, the f^--orbit of x is infinite, where, for any function g and number x, we define the *g-orbit of* x the set

$$\text{orb}_g(x) = \{x, g(x), g(g(x)), \dots, g^n(x), \dots\}.$$

If $c \neq c^-$ then q is called a *proper* quasi-dialectical system.

We call f^- the *revising function*, and c^- the *counterexample*[5].

Remark 2.3.2. A few words concerning Condition (3) above are in order. It seems reasonable to restrict ourselves to systems in which the operation of replacement is somewhat always enriching, in the following sense. Suppose we find some axiom unsatisfactory (again, this could be for a plenty of different reasons). Then we replace it. Later on, some problem occurs with this latter axiom, and thus we replace it too, with a third one. Now, if one aims at harmonizing the definition of f^- with some informal idea of "trial and error", in which knowledge is obtained through a process of refining subsequent proposals, then it is natural to ask that this third axiom is different from the first one we already replaced. Being acyclic just generalizes this intuition.

It is important to notice that Condition (3) is non-effective. This may seems as strongly conflicting with Magari's dialectical systems in which all the basic components are computable. Still, instead of considering this as a downside of our formalization, we argue that this only expresses one of the well-known limitations, in terms of effectivity, that arise while working with formal systems. After all, mathematicians would also enjoy to know a priori if their systems are consistent or not – but, alas, because of Gödel, they just cannot.

Our definitions of *provisional theses* and *final theses* follow the ones given by Magari for dialectical systems. The difference consists, of course, in the role played by the additional symbols f^- and c^-. Informally, if at a given stage s we can derive c^- from a set of axioms $f_0, \ldots, f_n$, then we change f_n with $f^-(f_n)$. Thus c^- can be intended as imposing a revision (a weakening, perhaps) to some axiom, instead of simply dismissing it.

This only difference has (at least at first sight) significant consequences also with respect to aspects already considered in Magari's definition. Firstly, it is clear that a quasi-dialectical system, while computing, might be ready to admit repetitions in its set of axioms. In other words, the same axiom can enter the system many times. Moreover, while in a dialectical system the "new" axiom proposed after a contradiction is always the successor (under the proposing function f) of the eliminated axiom, in our context the scenario is different. Indeed, the eliminated axiom, e.g. f_i, might have been object of some replacement of a previous axiom, f_j, i.e. $f_i = f^-(f_j)$. In this case, we would like to continue our computation from the successor of this f_j, i.e. f_{j+1}. This is because we might have discarded f_{j+1} due to some conflict with f_i. Thus, once f_i is put aside, f_{j+1} has to be tested again. Of course, the same holds if $f_j = f^-(f_k)$, for some f_k (in this case, we would take f_{k+1} as the continuation of our computation).

[5]Our choice to use the word "counterexample" to refer to c^- is largely motivated by a matter of convenience. Indeed, due to the wide variety of possible meanings of c^-, there is no single word that might represent them all.

The last ingredient of the formalization of a quasi-dialectical system is given by the function $r_s(x)$. This function parallels the analogous function $r_s(x)$ which was used in the definition of dialectical systems, where, for every s, x we had either $r_s(x) = \emptyset$, or $r_s(x) = \{f_x\}$. For quasi-dialectical systems, $r_s(x)$ will always be a (possibly empty) string of axioms: its role (much more important here than for dialectical systems) is to take note, for the axioms considered at stage s, of all their history, that is to record all possible actions of f^-, at previous stages, that eventually have led to them.

Let us make, in the next section, these latter observations more formal.

2.3.2 Provisional and final theses for a quasi-dialectical system

Let $q = \langle H, f, f^-, c, c^- \rangle$ be a quasi-dialectical system, and let us fix a computable approximation $\alpha = \{H_s\}_{s\in\omega}$ to H. (As we will see, a major difference with respect to dialectical systems is that the set of final theses depends now on which computable approximation to the enumeration operator one chooses.)

Definition 2.3.3. Define by induction values for several computable parameters (which all depend on our choice of α): A_s (a finite set), r_s (a function such that for every x, $r_s(x)$ is a finite string of numbers, which is viewed as a "vertical" string, or *stack*), $m(s)$ (a number, the greatest number such that $r_s(m(s)) \neq \langle\ \rangle$, where the symbol $\langle\ \rangle$ denotes the empty string), $h(s)$ (a number). In addition, there are the derived parameters: $\rho_s(x)$ is the top of the stack $r_s(x)$, $L_s(x) = \{\rho_s(y) : y < x \text{ and } r_s(y) \neq \langle\ \rangle\}$, and, for every $i \leq m(s)$, $\chi_s(i) = \bigcup_{j\leq i} H_s(L_s(j))$.

Stage 0

Define $m(0) = 0$, $h(0) = 0$,

$$r_0(x) = \begin{cases} \langle f_0 \rangle & x = 0 \\ \langle\ \rangle & x > 0, \end{cases}$$

and let $A_0 = \emptyset$

Stage $s + 1$

Assume $m(s) = m$. We distinguish the following cases:

1. there exists no $k \leq m$ such that $\{c, c^-\} \cap \chi_s(k) \neq \emptyset$: in this case, let $m(s+1) = m + 1$, and define

$$r_{s+1}(x) = \begin{cases} r_s(x) & \text{if } x \leq m \\ \langle f_{m+1} \rangle & \text{if } x = m+1 \\ \langle\ \rangle & \text{if } x > m+1; \end{cases}$$

2. there exists $k \leq m$ such that $c \in \chi_s(k)$, and for all $k' < k$, $c^- \notin \chi_s(k')$: in this case, let z be the least such k, let $m(s+1) = z+1$, and define

$$r_{s+1}(x) = \begin{cases} r_s(x) & x < z \\ \langle f_{z+1} \rangle & x = z+1 \\ \langle\ \rangle & x = z \text{ or } x > z+1; \end{cases}$$

3. there exists $k \leq m$ such that $c^- \in \chi_s(k)$, and for all $k' \leq k$, $c \notin \chi_s(k')$: in this case, let z be the least such k, let $m(s+1) = z+1$, and define, where $\rho_s(z) = f_y$,

$$r_{s+1}(x) = \begin{cases} r_s(x) & x < z \\ r_s(x)^\frown\langle f^-(f_y) \rangle & x = z \\ \langle f_{z+1} \rangle & x = z+1 \\ \langle\ \rangle & x > z+1. \end{cases}$$

Finally define $h(s+1) = m(s+1)$, if Clause (1) applies, otherwise $h(s+1) = m(s+1) - 1$, and let

$$A_{s+1} = \bigcup_{i < h(s+1)} \chi_{s+1}(i)(= H_{s+1}(L_{s+1}(h(s+1)-1))).$$

We call A_s the set of *provisional theses* of q at stage s. The set A^α_q defined as

$$A^\alpha_q = \{f_x : (\exists t)(\forall s \geq t)[f_x \in A_s]\}$$

is called the set of *final theses* of q with respect to α. We often write $A_s = A^\alpha_{q,s}$ when we want to specify the quasi-dialectical system q and the chosen approximation to the enumeration operator. A set $A \subseteq \omega$ is called *quasi-dialectical* if $A = A^\alpha_q$ for some quasi-dialectical system q, and approximation α to the enumeration operator of q; and sometimes, we will say in this case that A is *represented* by the pair (q, α).

Stacks, tops of stacks, and other stuff

A few remarks are needed after such a long definition.

Firstly, notice that in case a certain set of axioms derive, at a given stage, both c and c^-, then the system consider only c.

Secondly, in what follows it would be convenient to regard elements of the domain of r_s as *slots*. In view of their wide use in what follows, let us spend few words on how such slots have to be intended. At each stage s, whenever we say that x is the *slot of an axiom* f_y, we mean that $r_s(x)$ is a string having f_y as its rightmost element. The idea is that

$$r_s(x) = \langle f_x, f^-(f_x), \ldots, (f^-)^{(n)}(f_x) \rangle$$

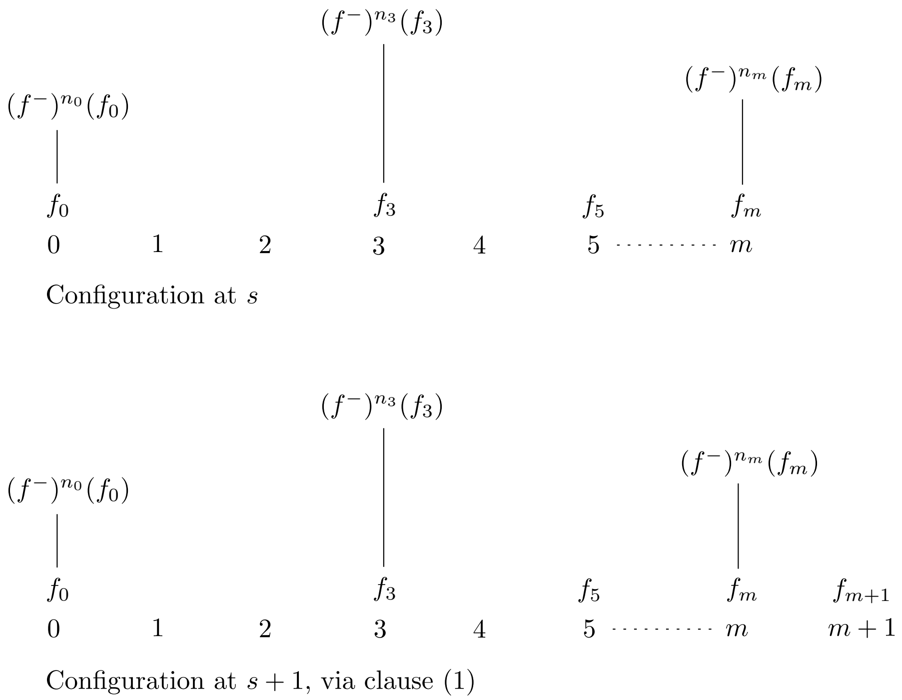

Figure 2.3: From stage s to $s+1$ using clause (1).

where the last element of the string, $f_y = (f^-)^{(n)}(f_x)$, has been obtained by stage s, through a sequence of replacements, starting from f_x, and dictated by f^-, due to case (3) of the quasi-dialectical procedure.

Figures 2.3–2.5 illustrate the various cases of Definition 2.3.3. The vertical strings above the various slots represent the various stacks $r(x)$ at the given stage. In each figure, only the relevant slots are depicted: it is of course understood that for each slot v to the right of the last one which is depicted, we have $r(v) = \langle\ \rangle$ (where for simplicity we omit to specify at which stage s the parameter $r_s(v)$ is evaluated). For every slot x, the set $L(x)$ consists of the strings which are at the top of the nonempty stacks $r(y)$, with $y < x$.

2.3.3 The dependence of the final theses from the approximations

We give an example to show how the definition of a quasi-dialectical set depends on how the relevant enumeration operator is approximated.

Example 2.3.4. Consider the quasi-dialectical system $q = \langle H, f, f^-, c, c^-\rangle$, where $f_x = x$, $f^-(x) = x + 2$, $c = 1$, $c^- = 2$, and

$$H = \{\langle y, \{2x+1\}\rangle : x, y \in \omega\} \cup \{\langle 0, \emptyset\rangle\} \cup \{\langle y, \{y\}\rangle : y \in \omega\}$$

(the axiom $\langle 0, \emptyset\rangle \in H$ is to comply with the request that in a dialectical system $H(\emptyset) \neq \emptyset$). It is easy to see that H is a closure operator: if $y \notin H(X)$, then $y \neq 0$

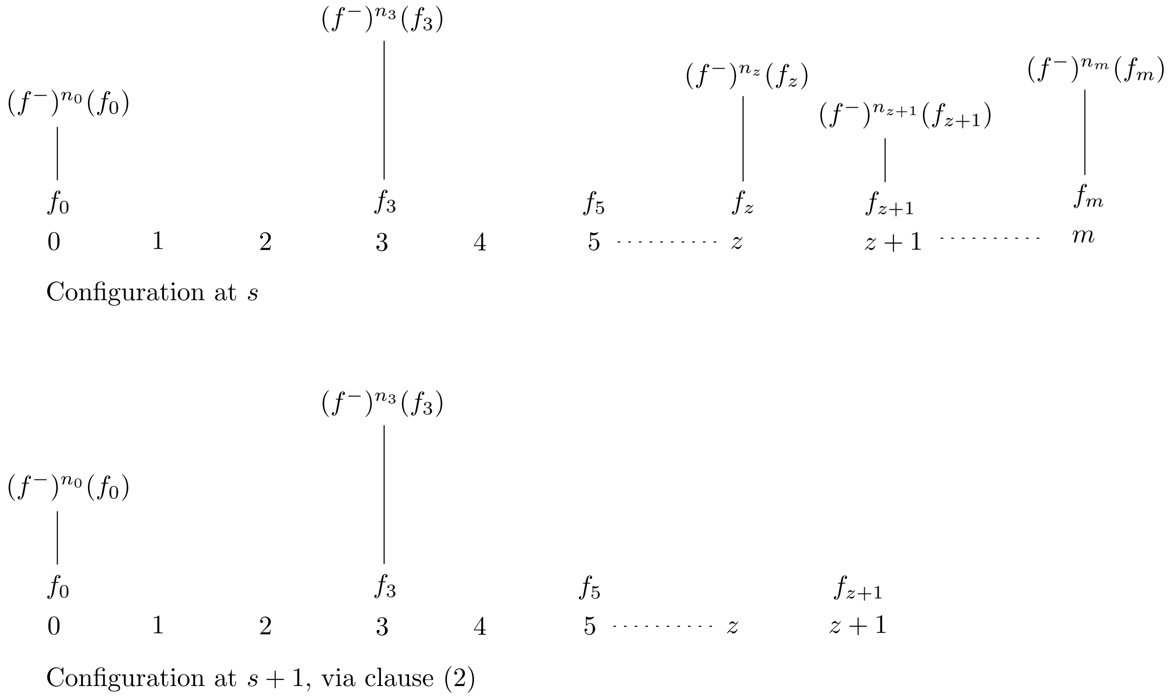

Figure 2.4: From stage s to $s+1$ using Clause (2).

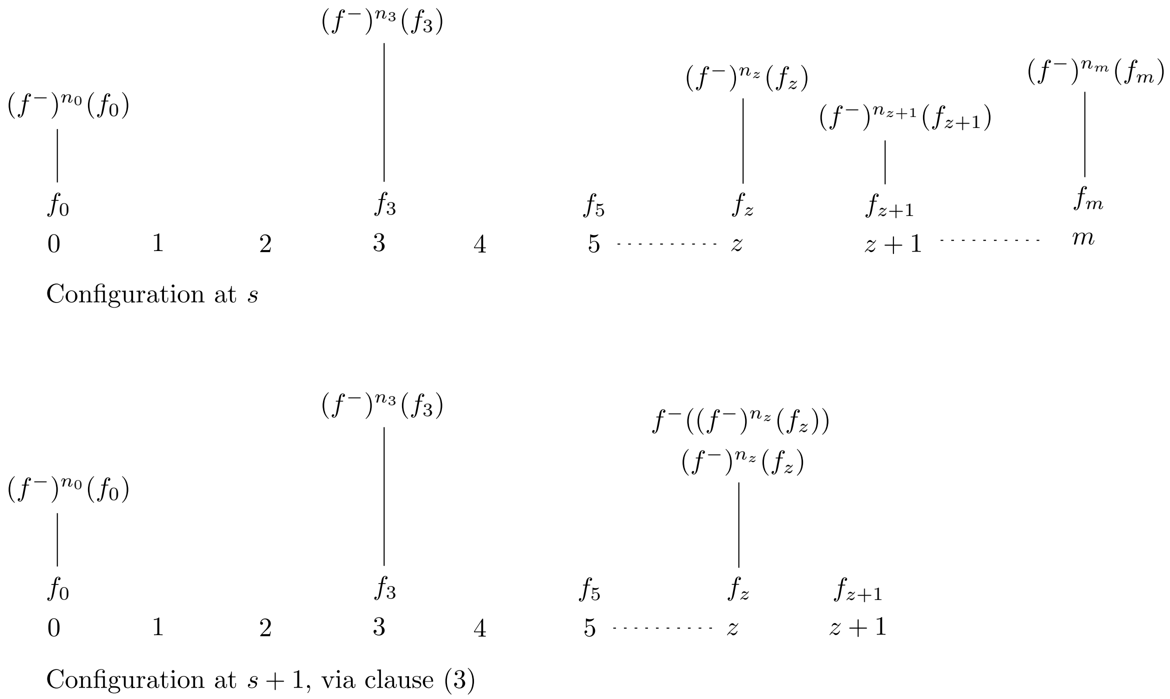

Figure 2.5: From stage s to $s+1$ using Clause (3).

and X does not contain any odd number, but then $H(X)$ does not contain any odd number either, thus $y \notin H(H(X))$. It is straightforward to see that there exist computable approximations α and β to H, such that in α for every x, the axiom $\langle c^-, \{2x+1\}\rangle$ comes before $\langle c, \{2x+1\}\rangle$, so that when processing $2x+1$, the pair (q, α) would use Clause (3) of the definition of provisional theses; on the contrary, in β for every x, the axiom $\langle c, \{2x+1\}\rangle$ comes before $\langle c^-, \{2x+1\}\rangle$, so that Clause (2) would be used. It is easy to see that these two approximations give rise to different quasi-dialectical sets, since $A_q^\alpha = \{0\}$, whereas, for instance $4 \in A_q^\beta$. Moreover, α gives rise to functions $r_s^\alpha(x)$, $\rho_s^\alpha(x)$, which have different "asymptotic" behavior from the functions $r_s^\beta(x)$, $\rho_s^\beta(x)$ yielded by β. For instance, we have that $\{\rho_s^\alpha(1) : s \in \omega\}$ is infinite. (If in addition $\alpha = \{H_s\}_{s\in\omega}$ satisfies that $\langle c^-, \{2x+1\}\rangle \in H_{x+2}$, then for every s, $r_s^\alpha(x) = \langle\ \rangle$, for all $x > 2$, i.e., for every $x > 2$, the axiom x is never "proposed".) On the contrary, $\{\rho_s^\beta(x) : s \in \omega\}$ is finite, for every x.

In view of this example, it could be objected that a quasi-dialectical system, rather than a quintuple $\langle H, f, f^-, c, c^-\rangle$ should be perhaps a quintuple $\langle \alpha, f, f^-, c, c^-\rangle$, where α is a computable approximation to an enumeration operator H. This is after all reasonable: we need first of all to be able to approximate H, in order to deduce with it.

We shall agree on the following definition:

Definition 2.3.5. An *approximated quasi-dialectical system* is a pair (q, α) where q is a quasi-dialectical system $q = \langle H, f, f^-, c, c^-\rangle$, and α is a computable approximation to H.

Hence a set A is quasi-dialectical if there is an approximated quasi-dialectical system (q, α), such that $A = A_q^\alpha$.

Lemma 2.3.6. *Every dialectical set A is quasi-dialectical, in fact there is a quasi-dialectical system q, such that for every computable approximation α to the enumeration operator of H, we have that $A = A_q^\alpha$.*

Proof. If $d = \langle H, f, c\rangle$ is a dialectical system, then let f^- be any acyclic computable function such that $c^- \notin \text{range}(f^-)$. It is clear that, letting $q = \langle H, f, f^-, c, c\rangle$, one has that $A_d = A_q$: this follows from the fact that one never uses Clause (3) of Definition 2.3.3, as in this case $c^- = c$, and Clause (2) has right to way. $\square$

2.3.4 General properties of quasi-dialectical systems

The fact that approximated quasi-dialectical systems (q, α), as the one in Example 2.3.4, do exist is not just a matter of curiosity. In fact, as we will see, pairs (q, α) that eventually propose all the axioms and pairs that – on the contrary – might fail to propose some of them do not even share the same class of representable sets. In order to distinguish these two classes of pairs, let us give a preliminary definition:

Definition 2.3.7. Let (q, α) be an approximated quasi-dialectical system, and y a slot. We say that (q, α) has a *loop over* y if $\{\rho_s(y) : s \in \omega\}$ is infinite. If (q, α) has no loops, we call it *loopless*.

Therefore, a loop can be visualized as expressing an infinite ascending stack of substitutions over some slot[6].

Example 2.3.4 shows that, for a quasi-dialectical system, having loops depends on how we approximate the enumeration operator of H.

Let us now show a result that applies to both quasi-dialectical systems with loops and loopless ones. It tells us when to expect stability for a given set of axioms.

Lemma 2.3.8. *Let (q, α) be an approximated quasi-dialectical system, and y a slot. If for each $x \leq y$, the pair (q, α) has no loop over x, then $\lim_s r_s(y)$ exists, i.e. there is a stage t such that, for every $s \geq t$, $r_s(y) = r_t(y)$.*

Proof. The proof is by induction on y. For $y = 0$ the claim is obvious. Recall that $r_0(0) = \langle f_0 \rangle$. If there is no stage t such that $\{c, c^-\} \cap H_t(\{r_t(0)\}) \neq \emptyset$, then for every s, $r_s(0) = \langle f_0 \rangle$. If there is a stage t such that $c \in H_t(\{r_t(0)\})$, then for every $s \geq t$, $r_s(0) = \langle\ \rangle$. If there is no t such that $c \in H_t(\{r_t(0)\})$, but (as q has no loop above 0) there is a last t at which $c^- \in H_t(\{r_t(0)\})$, then for every $s \geq t$, we have that $r_s(0) = r_t(0)$.

Assume that the claim is true of y, and that the pair (q, α) has no loop over any $x \leq y + 1$. Then by inductive hypothesis, there is a least stage t (necessarily, $t > 0$) such that for every $s > t$, and $x \leq y$, $r_s(x) = r_t(x)$, and let $L(y + 1) = L_t(y + 1)$. This implies also that $\{c, c^-\} \cap H(L(y + 1)) = \emptyset$. By minimality of t, we have that $L_t(y+1) \neq L_{t-1}(y+1)$. We examine all possibilities that may have led to a change at stage t. First of all, it is not possible that $h(t) = k < y-1$, as otherwise at stage $t+1$, through (1) we would change $r_{t+1}(k+2)$, contrary to the choice of t, as $k+2 \leq y$. If $h(t) = y-1$, then at t we set $r_t(y) = \langle f_y \rangle$, $r_t(y+1) = \langle\ \rangle$ but at $t+1$ we change this to $r_{t+1}(y + 1) = \langle f_{y+1} \rangle$. If $h(t) = y$, then we set $r_t(y + 1) = \langle f_{y+1} \rangle$. In all cases, we see that there is a least stage $t_0 \in \{t, t+1\}$ at which $r_{t_0}(y+1) = \langle f_{y+1} \rangle$. If there is a stage $s_0 \geq t_0$ such that $c \in H(L(y+1) \cup \{\rho_{s_0}(y+1)\})$ then for every $s \geq s_0$, we have that $r_s(y+1) = \langle\ \rangle$; otherwise, since we assume that there in no loop over $y+1$, we have that either there is no stage $s \geq s_0$ such that $c^- \in H(L(y + 1) \cup \{\rho_s(y + 1)\})$, in which case, for every $s \geq s_0$, we have $r_s(y + 1) = \langle f_{y+1} \rangle$, or there is a last stage $s_1 \geq s_0$, such that $c^- \in H(L(y+1) \cup \{\rho_{s_1}(y+1)\})$, in which case $r_s(y+1) = r_{s_1}(y+1)$, for every $s \geq s_1$. □

The proof of the previous lemma shows also:

Corollary 2.3.9. *Let (q, α) be an approximated quasi-dialectical system and y be a slot. If t is the least stage such that, for every $s \geq t$, and $x \leq y$, $r_s(x) = r_t(x)$, then at either $s_0 = t$ or $s_0 = t + 1$, we have that $r_{s_0}(y + 1) = \langle f_{y+1} \rangle$.*

[6]Following this intuition, one can even quest the choice of calling such infinite stacks "loops". In doing so, we simply want to remark the familiar intuition of a system "going into loop".

Proof. Immediate. □

Intuitively, the last result might be understood as stating that there is no loss of information – in terms of the axioms proposed – in working after the stabilization of a given $L(x)$. Indeed, the result shows that any axiom f_x is proposed after stabilization of $L(x)$.

The following section provides a full characterization of quasi-dialectical systems with loops.

2.3.5 Characterizing quasi-dialectical systems with loops

To fit loops in our intuitive interpretation is not completely straightforward. Recall Magari's idea of dialectical systems as representing the behavior of a mathematician – or even of a mathematical community – while facing possible contradictions. According to this scenario, quasi-dialectical systems with loops would describe a mathematical community in which the overall progression of the theory is indeterminately interrupted by a never-ending refinement of a single axiom – a kind of behavior that might be jokingly compared with Kafkian bureaucracy.

Nonetheless, loops are not so pathological within the theory of quasi-dialectical systems.

On one hand, due to condition (3) in Definition 2.3.1, every f^--orbit of a given axiom is infinite. Thus, in principles one can not rule out the possibility of building an infinite ascending stack over some axiom. Of course, whether or not this happens, it depends on the operator H, and how we approximate it.

On the other hand, it is worth considering quasi-dialectical systems with loops for at least two reasons. Firstly, it is not difficult to provide a full characterization of them: we give it in the current section. More importantly, even if at first sight quasi-dialectical systems with loops may appear, to some extent, stupid, it can be shown that they can represent sets (namely c.e. non-computable sets) that are not representable by standard dialectical systems. As we will see, this is permitted by the fact that the information one can encode in a loop, is not necessarily trivial. Moral of the story: not all bureaucracy is pointless.

Lemma 2.3.10. *Let (q, α) be an approximated quasi-dialectical system with loops. Then A_q^α is a c.e. set.*

Proof. Call b the least slot over which the pair (q, α) has a loop. By Lemma 2.3.8, there must be a stage t such that, for all $s \geq t$, $L_s(b) = L_t(b)$: call $X = L_t(b)$. Clearly $A_q^\alpha = H(X)$ is a c.e. set, since X is finite. The inclusion $\supseteq$ is obvious, since for every $s \geq t$, $X \subseteq L_s(h(s))$.

To show the converse, just notice that at every stage $s \geq t$ at which we add an axiom over b, we define the set of provisional theses to be $H_s(X)$. Thus no element not in $H(X)$ can be a final thesis. □

Recall that a c.e. set is said to be *simple*, if its complement is infinite, and does not contain any infinite c.e. set. As we can see through the next lemma, simplicity gives us a restraint on the kind of information that can be encoded within a loop.

Lemma 2.3.11. *Let A be a c.e. set. Then there exists an approximated quasi-dialectical system (q, α) with loops such that $A_q^\alpha = A$ if and only A is coinfinite and not simple.*

Proof. If A is coinfinite and not simple, then there exists an infinite c.e. subset $B \subseteq A^c$, such that $A^c \smallsetminus B$ contains at least two elements. Let $b = \min B$: we may assume without loss of generality that also $b = \min A^c$. Then consider a quasi-dialectical system, $q = \langle H, f, f^-, c, c^- \rangle$, where f is the identity, f^- is any 1-1 computable function such that range$(f^-) \subseteq B$, $c \neq c^-$ and $c, c^- \in A^c \smallsetminus B$, and H satisfies $H(\emptyset) = A$, $c^- \in H(\{x\})$ if and only if $x \in B$. To take an appropriate H which is also an algebraic closure operator, take

$$H = \{\langle y, \{x\}\rangle : x \in B, y \in \omega, y \neq c\} \cup \{\langle a, \emptyset\rangle : a \in A\} \cup \{\langle x, \{x\}\rangle : x \in \omega\} :$$

to show that $H(X) = H(H(X))$, notice that if $X \cap B \neq \emptyset$ then $H(X) = \omega \smallsetminus \{c\} = H(\omega \smallsetminus \{c\})$, otherwise $H(X) = A = H(A)$. It is clear that whatever approximation α we work with, we have a loop over b, and clearly for every such α, $A_q = A_q^\alpha = A$.

($\Rightarrow$): Suppose that A is c.e. and there is an approximated quasi-dialectical system (q, α) with loops, and $A_q^\alpha = A$. Let b the least slot such that there is a loop over b. It is immediate to see that $\mathrm{orb}_{f^-}(b)$ is an infinite c.e. set. We claim that $\mathrm{orb}_{f^-}(b) \subseteq A^c$. So, suppose that some $f_y \in \mathrm{orb}_{f^-}(b)$ belongs to A. As $A = A_q^\alpha$ this means that $f_y \in A_q^\alpha$. By Lemma 2.3.8, there must be a stage t such that, for all $s \geq t$, $L_s(b) = L_t(b)$: call $X = L_t(b)$. So, as in the proof of the previous lemma, we would have that $f_y \in H(X)$. But since f_y belongs to the loop over b, we must have $c^- \in H(X \cup \{f_y\})$. On the other hand, as H is a closure operator, we have $X \subseteq H(X)$, so by $\{f_y\} \subseteq H(X)$, we get

$$H(X \cup \{f_y\}) \subseteq H(H(X)) = H(X).$$

Thus, at some stage $s > t$, we would see $c^- \in H_s(X)$, contrary to the fact that $L(b)$ does not change after t. $\square$

Remark 2.3.12. Notice that the above proof shows in fact that if A is a coinfinite and not simple set, then there is a proper quasi-dialectical system q, such that for every computable approximation to the enumeration operator of H, one has that $A = A_q^\alpha$.

The conjunction of the last two lemmas give us the following characterization theorem for quasi-dialectical systems with loops:

Theorem 2.3.13. *The sets that are representable by approximated quasi-dialectical systems (q, α) with loops are exactly the c.e. sets that are coinfinite and not simple.*

Proof. Immediate. $\square$

2.3.6 A locality result for loopless quasi-dialectical systems

In this section we point out some useful properties of loopless approximated quasi-dialectical systems, which show unexpected similarities of these systems with the dialectical systems. Lemma 2.3.14 below states a sort of locality result that informally expresses the following fact: even if a quasi-dialectical system, by means of the revising function f^-, might heavily modify the order in which axioms are tested, what really counts for an axiom f_x to be a final thesis is whether or not f_x has eventually x among its slots. Thus, the expressiveness of a quasi-dialectical system without loops, by which it might proposes an axiom several times, ends up with a sort of redundancy: among all possible occurrences of f_x in the list of proposed axioms, what really counts is the one that has been proposed at slot x.

First of all, in view of Lemma 2.3.8, if the pair (q, α) is a loopless approximated quasi-dialectical system, then the corresponding parameters $r_s(x), \rho_s(x), L_s(x)$ reach a limit with respect to s, so we are justified in defining, for every x,

$$r(x) = \lim_s r_s(x) \qquad \rho(x) = \lim_s \rho_s(x) \qquad L(x) = \lim_s L_s(x).$$

Lemma 2.3.14. *Let (q, α) be a loopless approximated quasi-dialectical system. Then $f_y \in A_q^\alpha$ if and only if*

$$(\exists t)(\forall s \geq t)[r_s(y) = \langle f_y \rangle].$$

Notice that this is equivalent to saying that there exists a t such that $\rho_s(y) = f_y$ for all $s \geq t$.

Proof. ($\Leftarrow$): Let f_y be given. Under the assumption, and by Lemma 2.3.8, let t_0 be a stage such that for every $s \geq t_0$, $L_s(y+1) = L(y+1)$. Then for all $s \geq t_0$, we have that

$$f_y \in L(y+1) \subseteq L(h(s));$$

let $t_1 \geq t_0$ be such that for all $s \geq t_1$, $L(y+1) \subseteq H_s(L(y+1))$ (we use here that $X \subseteq H(X)$ for every X): then for all $s \geq t_1$ we have

$$L(y+1) \subseteq H_s(L(y+1)) \subseteq H_s(L(h(s))) = A_{q,s}^\alpha,$$

thus $f_y \in A_q^\alpha$.

($\Rightarrow$): Assume $f_y \in A_q^\alpha$, i.e. $f_y \in A_{q,s}^\alpha$, for all $s \geq t_0$, for some t_0. We first claim that there is a number i, such that, for every $s \geq t_0$, $i < h(s)$ and $f_y \in H_s(L_s(i))$. Since $f_y \in A_{q,t_0}^\alpha$, there is a least $i < h(t_0)$ such that $f_y \in H_{t_0}(L_{t_0}(i))$: we claim that this is the desired i. In order to prove the claim, assume that there is a least $s \geq t_0$ such that $f_y \in H_s(L_s(i))$, but $f_y \notin H_{s+1}(L_{s+1}(i))$: then $h(s+1) \leq i$, and thus $f_y \notin H_{s+1}(L_{s+1}(h(s+1)))$, i.e. $f_y \notin A_{q,s+1}^\alpha$, contrary to the fact that $s+1 \geq t_0$.

Therefore there is a least $x \leq i$ such that $f_y \in H(L(x))$. Now, let s_0 be the stage at which we propose f_y at slot y, as in Corollary 2.3.9. We would then remove f_y from $r(y) = \langle f_y \rangle$, only if at some later stage s we see $c \in H_s(L(y) \cup \{f_y\})$, or

$c^- \in H_s(L(y) \cup \{f_y\})$. It follows that $f_y \notin H(L(y))$: otherwise, as in the proof of the left-to-right implication of Lemma 2.3.11, we would have $\{c, c^-\} \cap H(L(y)) = H(L(y) \cup \{f_y\}) \neq \emptyset$, contrary to the fact that $L(y)$ is the limit set. Therefore $L(y) \subset L(x)$, hence $\{c, c^-\} \cap H(L(x) \cup \{f_y\}) \neq \emptyset$: this implies $f_y \notin H(L(x))$, otherwise as in the proof of Lemma 2.3.11 we would conclude that $\{c, c^-\} \cap H(L(x)) \neq \emptyset$, contrary to the fact that $L(x)$ is the limit set. We have reached a contradiction, since $f_y \in H(L(x))$.

The claim about $\rho_s(y)$ is obvious, as by f^- being acyclic, we have that $\rho_s(y) = f_y$ if and only if the length of $r_s(y)$ is 1. □

Corollary 2.3.15. *If (q, α) and (q, β) are loopless approximated quasi-dialectical systems, then $A_q^\alpha = A_q^\beta$.*

Proof. Immediate. □

Remark 2.3.16. This last corollary fixes the limits within which quasi-dialectical systems can be regarded as invariant with respect to the way in which we approximate their enumeration operators H, i.e. the set of final theses of a quasi-dialectical system remains the same as long as we consider approximations that avoid loops (clearly this fact does not hold for other parameters of the systems, such as provisional thesis, stacks, etc). Thus for any loopless approximated quasi-dialectical system, it does not count how the system is approximated.

In what follows, we will take benefit of this fact. Let us say that a quasi-dialectical system q is *loopless* if there is an approximation α such that (q, α) is a loopless approximated quasi-dialectical system. Then, we will just speak of loopless quasi-dialectical systems (and similarly of loopless quasi-dialectical sets), hence dismissing any reference to a specific approximation α, and maintaining that any approximation that avoids loops would be appropriate. In this case we write A_q to denote A_q^α, where α is *any* loopless approximation to the enumeration operator of q.

Given an approximated quasi-dialectical system (q, α), such that for every x, $\rho(x) = \lim \rho_s(x)$ and $r(x) = \lim r_s(x)$ exist, let $L = \{\rho(x) : x \in \omega \text{ and } r(x) \neq \langle\,\rangle\}$ (where of course, $\rho_s(x)$, $r_s(x)$, and consequently $L(x)$, are taken with respect to α).

Theorem 2.3.17. *If q is a loopless approximated quasi-dialectical system (in the sense of Remark 2.3.16), then $A_q = L$. Moreover, for every y,*

$$r(y) \neq \langle\,\rangle \Rightarrow \mathrm{range}(r(y)) \cap A_q = \{\rho(y)\}.$$

Proof. Lemma 2.3.14 shows that $A_q \subseteq L$. The converse is trivial, as by the very definitions, $\rho(y) \in A_{q,s}$, for every big enough s.

Suppose now that $r(y) \neq \langle\,\rangle$, and by Lemma 2.3.8, let t_0 be the least stage after which $L_s(y)$ does not change any more. Assume $t_1 > t_0$, $\rho_{t_1}(y) \neq \rho(y)$, and $\rho_{t_1}(y) = f_z \in A_q$. Now, $r(z) = \langle f_z \rangle$ by Lemma 2.2.4. Let $v = \max\{z, y\} + 1$, and let $t_2 \geq t_1$ be a stage after which for no $u \leq v$ does $r_s(u)$ change; but $f_z \in L(v)$, thus at some stage $s \geq t_2$ we would see $\{c, c^-\} \cap H(L(v)) \neq \emptyset$, contradiction, as $L(v)$ has already reached its limit. □

The following lemma can be presented as a natural companion of Lemma 2.2.4.

Lemma 2.3.18. *Let q be a loopless approximated quasi-dialectical system. Then, $f_x \in A_q$ if and only if neither $c \in H(L(x) \cup \{f_x\})$, nor $c^- \in H(L(x) \cup \{f_x\})$.*

Proof. ($\Rightarrow$): This follows from Lemma 2.3.14 which implies that $L(x+1) = L(x) \cup \{f_x\}$, so we can not have $\{c, c^-\} \cap H(L(x+1))$, because $L(x+1)$ is the limit set.

($\Leftarrow$): This follows from the fact that after we propose, at stage s_0, f_x at slot x (see Corollary 2.3.9), and under the assumption that neither $c \in H(L(x) \cup \{f_x\})$, nor $c^- \in H(L(x) \cup \{f_x\})$, we never change $r(x)$, and thus $\rho(x) = f_x$, implying $f_x \in A_q$ by Lemma 2.3.14 . □

Remark 2.3.19. The proof of the previous theorem shows in fact that if (q, α) is an approximated quasi-dialectical system, such that there are no loops over any $y < x$, then $f_x \in A_q$ if and only if neither $c \in H(L(x) \cup \{f_x\})$, nor $c^- \in H(L(x) \cup \{f_x\})$.

Corollary 2.3.20. *For every approximated quasi-dialectical system (q, α), the quasi-dialectical set A_q^α is Δ^0_2.*

Proof. If (q, α) is an approximated quasi-dialectical system with loops, then A_q^α is c.e., thus Δ^0_2. For loopless approximated quasi-dialectical sets, we give two proofs. The first proof will be used in Lemma 2.4.4 to show that the Turing degree of a quasi-dialectical set is c.e.; the second proof shows that the quasi-dialectical approximation, i.e. the computable approximation provided by the sets of provisional theses, is Δ^0_2. Thus, let q be a loopless approximated quasi-dialectical system: define the following computable sequence $\{A_s\}$ of sets:

$$A_s = \{f_y : \rho_s(y) = f_y\}.$$

It is clear from Lemma 2.3.14 that

$$f_y \in A_q \Leftrightarrow (\exists t)(\forall s \geq t)[f_y \in A_s].$$

Moreover $\lim_s A_s(f_y)$ exists for every y, as after the stage s_0 (taken as in Corollary 2.3.9) at which we propose $r_{s_0}(y) = \langle f_y \rangle$, and each $r(x)$, with $x < y$, has reached limit, once we change $\rho(y)$ we can never go back at any future stage s to $\rho_s(y) = f_y$, by f^- being acyclic. Thus $\{A_s\}_{s \in \omega}$ is a Δ^0_2 approximation to A_q.

For a different proof, we show that the quasi-dialectical approximation $\{A_{q,s}\}_{s\in\omega}$ provides a Δ^0_2 approximation to A_q. Recall that $A_{q,s} = H_s(L_s(h(s) - 1))$. On the other hand, by Lemma 2.3.8, it is easy to see that $\lim_s m(s) = \infty$, and thus $\lim_s h(s) = \infty$. Now, let f_y be given, and let t_0 be a stage such that for all $s \geq t_0$, $h(s) \geq y + 1$, $L(y) = L_s(y) = L_{t_0}(y)$. If $f_y \in H_s(L_s(h(s) - 1))$ then, by an already familiar argument, $\{c, c^-\} \cap H(L(y) \cup \{f_y\}) = \emptyset$, hence by Lemma 2, $f_y \in A_q$. □

The following easy construction (accompanying Lemma 2.3.6) proves that dialectical systems can be viewed as proper loopless quasi-dialectical systems, in the strong sense that this identity is independent of how one approximates the enumeration operator of the quasi-dialectical system.

Lemma 2.3.21. *From any dialectical system $d = \langle H, f, c\rangle$, and any number $c^- \neq c$, one can effectively build a quasi-dialectical system q, such that, if $c^- \notin A_d$, then for every approximation α to the enumeration operator of q, one has that (q, α) is loopless and $A^\alpha_q = A_d$.*

Proof. Let $d = \langle H, f, c\rangle$ be a dialectical system, and let $c^- \neq c$. Consider the quasi-dialectical system $q = \langle H^*, f, f^-, c, c^-\rangle$, where f^- is any acyclic computable function not having c^- in its range, and such that $f^-(c^-) = f^-(c) = a \in H(\emptyset)$, and let

$$H^* = \{\langle x, D\rangle : x \neq c^- \ \&\ \langle x, D\rangle \in H\} \cup \{\langle c^-, \{c^-\}\rangle, \langle c^-, \{c\}\rangle\}.$$

Let $X \subseteq \omega$. It is easy to see that $H^*(X) \subseteq H(X)$: indeed if $x \in H^*(X)$, and $x \neq c^-$ then, by definition of H^*, $x \in H(X)$; if $x = c^-$ then either $c^- \in X$, and thus $x \in H(X)$; or $c \in X$, and thus $x \in H(X)$. Also, it immediately follows from the definitions that, for $x \neq c^-$, if $x \in H(X)$ then $x \in H^*(X)$. Let us now show that H^* is a closure operator, if H is. Let $X \subseteq \omega$ be given: clearly, $X \subseteq H^*(X)$. Next, we want to show that $H^*(H^*(X)) \subseteq H^*(X)$. Assume $x \in H^*(H^*(X))$. If $x = c^-$, i.e. $c^- \in H^*(H^*(X))$ then there are two possibilities: either $c^- \in H^*(X)$ or $c \in H^*(X)$: in the former case, the claim is true; in the latter case, we have $c \in H(X)$, but then $c^- \in H(\{c\}) \subseteq H(H(X)) = H(X) \subseteq H^*(X)$. If $x \neq c^-$, then from $x \in H^*(H^*(X))$ we get $x \in H(H^*(X)) \subseteq H(H(X)) = H(X)$, giving that $x \in H^*(X)$, by definition of H^*.

Let us now consider any computable approximation to H^*, and any computable approximation to H: relatively to these approximations, we will distinguish r, L, as r^d, L^d, or r^q, L^q according to whether we deal with d or q. We now aim at showing that, if $c^- \notin A_d$, then $A_q = A_d$: by induction on x, we in fact show that

$$f_x \in A_d \Leftrightarrow f_x \in A_q,$$

and for each $y \leq x$, $r^q_s(y)$ reaches limit $r^q(y)$, and if $y \in \{c, c^-\}$, then $r^q(y) \in \{\langle\ \rangle, \langle y, a\rangle\}$.

Suppose that the claim is true of all $y < x$. Assume now that $f_x \notin A_d$. Then $c \in H(L^d(x) \cup \{f_x\})$, from which it follows that $c \in H^*(L^d(x) \cup \{f_x\})$, and hence $c \in H^*(L^q(x) \cup \{f_x\})$, as by induction $L^d(x) \subseteq L^q(x)$. Now, it might be the case that also $c^- \in H^*(L^q(x) \cup \{f_x\})$. If, after the least stage at which $L^q(x)$ stabilizes, the event $c \in H^*(L^q(x) \cup \{f_x\})$ appears earlier than the event $c^- \in H^*(L^q(x) \cup \{f_x\})$ (i.e., depending on the approximation to H^*, c is enumerated into $H^*(L^q(x) \cup \{f_x\})$ before c^- is enumerated into $H^*(L^q(x) \cup \{f_x\})$) then $r^q(x) = \langle\ \rangle$, and $f_x \notin A_q$. If $c^- \in H^*(L^q(x) \cup \{f_x\})$ appears first, then we must examine the two possibilities:

the first possibility is $c^- \in L^q(x) \cup \{f_x\}$, hence $c^- = f_x$ (as $L^q(x) \subseteq A_d$, by inductive assumption, since $a \in A_d$) and in this case it is easy to see that $r^q(x) = \langle c^-, a\rangle$; the second possibility is $c \in L^q(x) \cup \{f_x\}$, which similarly implies that $c = f_x$: thus in this case it is easy to see that $r^q(x) = \langle c, a\rangle$.

Viceversa, suppose that $f_x \notin A_q$: then (by Lemma 2 and Remark 2.3.19) $c^- \in H^*(L^q(x) \cup \{f_x\})$ or $c \in H^*(L^q(x) \cup \{f_x\})$. If $c^- \in H^*(L^q(x) \cup \{f_x\})$ appears first then, as above, $c^- \in L^q(x) \cup \{f_x\}$ or $c \in L^q(x) \cup \{f_x\}$: if $c^- = f_x$, then $r^q(x) = \langle c^-, a\rangle$ and $f_x \notin A_d$ by choice of c^-; notice that $c^- \in L^q(x)$ can not occur since by the inductive hypothesis $L^q(x) \subseteq L^d(x) \cup \{a\}$, and thus if $c^- \in L^q(x)$ then we would have that $c^- \in A_d$; on the other hand, if $c = f_x$ then $r^q(x) = \langle c, a\rangle$ and $f_x \notin A_d$. It remains to consider the case in which $c \in H^*(L^q(x) \cup \{f_x\})$ appears first: but then $r^q(x) = \langle\ \rangle$; we also have $c \in H(L^q(x) \cup \{f_x\})$, thus by the inductive assumption $L^q(x) \subseteq A_d$, we have that $f_x \notin A_d$.

This shows the inductive step. □

Corollary 2.3.22. *Every dialectical set A, such that A^c has at least two elements, is represented by a loopless proper quasi-dialectical system (and the representation is independent of any computable approximation to the enumeration operator of the quasi-dialectical system).*

Proof. Let $d = \langle H, f, c\rangle$ be a dialectical system. If $\omega \smallsetminus \{c\} \nsubseteq A_d$, then apply the previous lemma. The claim regarding $A = \omega$ is obvious. □

For further reference, let us unify some of the foregoing characterizations into a single theorem:

Theorem 2.3.23 ([Magari, 1974, Amidei et al., a])**.** *If d and (q, α) are respectively a dialectical system and a loopless approximated quasi-dialectical system, then the following hold:*

1. *A_d and A_q^α are Δ_2 sets;*
2. *for every x, $\lim_s r_s(x) = r(x)$ and $\lim_s L_s(x) = L(x)$ exist (whether the functions $r_s(x)$, $L_s(x)$ refer to d, or (q, α)) and*

$$A_d = \{f_x : r(x) = \{f_x\}\}$$
$$A_q^\alpha = \{f_x : r(x) = \langle f_x\rangle\},$$

and

$$f_x \in A_d \Leftrightarrow c \notin H(L_x \cup \{f_x\})$$
$$f_x \in A_q \Leftrightarrow \{c, c^-\} \cap H(L_x \cup \{f_x\}) = \emptyset.$$

(In fact, the assumption that (q, α) be loopless is reductant: it is just enough to assume (q, α) has no loop over any $y < x$.)

Proof. The claim that A_d is a Δ^0_2 set comes from [Magari, 1974], where it is proved that $A_d(x) = \lim_s g(x,s)$, with

$$g(x,s) = \begin{cases} 1, & \text{if } x \in A_{d,s} \\ 0, & \text{if } x \notin A_{d,s}. \end{cases}$$

The other claims come from Lemma 2.3.8, Lemma 2.3.18. □

2.4 Dialectical degrees, quasi-dialectical degrees, Turing degrees, and enumeration degrees

In this section we show that the information content of the dialectical sets coincides with that of the quasi-dialectical sets, by showing that the two classes of sets have the same Turing degrees, and the same enumeration degrees.

In the rest of this chapter, we will make use of a convention introduced with Remark 2.3.16, i.e. when dealing with a loopless approximated quasi-dialectical system we will avoid to specify which approximation we are considering. This way of doing is permitted by the fact that the set of final theses of a loopless approximated quasi-dialectical system is invariant with respect to all the loopless approximations (see Corollary 2.3.15). In this light, we say that a *loopless quasi-dialectical system* is a quasi-dialectical system for which there is a loopless computable approximation, i.e. an approximation α such that the pair (q, α) is a loopless approximated quasi-dialectical system; a *loopless quasi-dialectical* set is a set represented by a loopless approximated quasi-dialectical system. In these cases, we simply write A_q to mean A^α_q, where α is any loopless computable approximation to the enumeration operator of q. We talk about a *proper loopless quasi-dialectical system*, or a *proper loopless quasi-dialectical set*, when the relevant quasi-dialectical system is proper, i.e. $c \neq c^-$.

Definition 2.4.1. A Turing degree (enumeration degree, respectively) is called *dialectical* if it contains a dialectical set; and it is called *quasi-dialectical* if it contains a quasi-dialectical set.

2.4.1 Dialectical sets, quasi-dialectical sets, and Turing degrees

The following theorem characterizes the dialectical Turing degrees, and the quasi-dialectical Turing-degrees.

Theorem 2.4.2. *The dialectical degrees and the quasi-dialectical degrees coincide: namely, they coincide with the c.e. Turing degrees.*

Proof. The proof consists of two steps. We show (Lemma 2.4.3) that every c.e. Turing degree is a dialectical degree; and we show (Lemma 2.4.4) that every quasi-dialectical degree is a c.e. Turing degree. Since every dialectical set is quasi-dialectical (by Lemma 2.3.6; see also Lemma 2.3.22), the claim follows immediately. □

Lemma 2.4.3. *For every c.e. set A there exists a dialectical system $d = \langle H, f, c\rangle$ such that $A_d \equiv_{tt} A$.*

Proof. This is an immediate consequence of the fact that every Π^0_1 set $A \neq \omega$ is dialectical (see 2.4.8). Thus, if A is c.e. then $A \equiv_{tt} A^c$, and A^c is dialectical, where for any given set $X \subseteq \omega$, the symbol X^c denotes the complement of X. □

Lemma 2.4.4. *If (q, α) is an approximated quasi-dialectical system, then A^α_q has c.e. Turing degree.*

Proof. If (q, α) is an approximated quasi-dialectical system with loops, then A^α_q is c.e., see Lemma 2.3.10. Thus, in this case, the claim is trivial.

Let us consider the case when q is loopless. Let us recall the following facts about Δ^0_2 sets. Given a computable function $g(x, s)$ such that, for every x, $g(x, 0) = 0$, and $\lim_s g(x, s)$ exists, recall that the *least modulus function* m for g, is the function

$$m(x) = \mu s. (\forall t \geq s)[g(x, t) = g(x, s)].$$

Notice that if A is a Δ^0_2 set, such that $A(x) = \lim_s g(x, s)$ (where g is a 0-1 valued computable function; here, and in the following, given a set X of numbers, we denote by $X(x)$ the value of the characteristic function of X on x) and m is the least modulus function for g, then $A \leq_T m$. On the other hand, if B is the c.e. set

$$B = \{\langle x, s\rangle : (\exists t > s)[g(x, t) \neq g(x, s)]\}$$

then $B \equiv_T m$. So a least modulus function has always c.e. Turing degree (see e.g. [Soare, 1987]). Therefore, if A is a Δ^0_2 set, $g(x, s)$ is a 0-1 valued computable function such that $A(x) = \lim_s g(x, s)$, for all x, m is the least modulus function for g, and $m \leq_T A$, it follows that A has c.e. Turing degree.

If (q, α) is loopless, then by Corollary 2.3.20, we have that the computable sequence of sets $\{A_s\}$,

$$f_y \in A_s \Leftrightarrow \rho_s(y) = f_y,$$

is a Δ^0_2 approximation to A_q.

By Lemma 2.3.8, Lemma 2.3.14, Theorem 2.3.17, for every y, the following hold: there is a least stage t_y such that for all $s \geq t_y$, and $x \leq y$, we have that $\rho_s(x) = \rho_{t_y}(x) = \rho(x)$, and consequently $r_s(x) = r_{t_y}(x) = r(x)$; if $r(x) \neq \langle\ \rangle$ then $r(x) \cap A_q = \{\rho(x)\}$; $f_x \in A_q$ if and only if $r(x) = \{f_x\}$.

Therefore an easy induction shows that, to find such a t_y, given y, it is enough to pick the least s such that for all $x \leq y$ if $\rho_s(x) \neq \langle\ \rangle$ then $\rho_s(x) \in A_q$. In other words,

$$t_y = \begin{cases} \mu s. (\forall x < y)([\rho_s(x) \neq \langle\ \rangle \Rightarrow \rho_s(x) \in A_q \,\&\, \rho_s(y) = f_y], & \text{if } f_y \in A_q, \\ \mu s. (\forall x < y)([\rho_s(x) \neq \langle\ \rangle \Rightarrow \rho_s(x) \in A_q \,\&\, \rho_s(y) \neq f_y], & \text{if } f_y \notin A_q. \end{cases}$$

Let now m be the least modulus function for

$$g(x,s)=\begin{cases}1, & \text{if } x\in A_s,\\ 0, & \text{if } x\notin A_s.\end{cases}$$

By induction on y it is easy to see that $m(f_y)\le t_y$. (Notice that, for $y>0$, it might be $m(f_y)<t_y$ since at some stage t we could redefine $r_t(y-1)$ through Clause (3) of Stage $s+1$ in the definition of a quasi-dialectical system, and thus $r_t(y)=\langle f_y\rangle$; and at subsequent consecutive stages, we still redefine $r(y-1)$, without touching $r(y)$.) On the other hand, the mapping $y\mapsto t_y$ is $\le_T A_q$. Therefore, $m\le_T A_q$.

□

We conclude this section with the following easy consequence of Lemma 2.4.3.

Corollary 2.4.5. *Every nonzero dialectical Turing degree contains some immune dialectical set.*

Proof. Let A be a non-decidable dialectical set. By Lemma 2.4.3 there is a non-decidable c.e. set B such that $A\equiv_T B$. Let c_B be the characteristic function of B, and let

$$S=\{\sigma\in 2^{<\omega}:\sigma<c_B\}$$

where $<$ is the lexicographical order on strings, hence $\sigma<c_B$ means that there is some $i\in\text{domain}(\sigma)$ such that $\sigma(i)<c_B(i)$. Clearly, S is c.e.: to see this, let $\{b_s\}_{s\in\omega}$ be a 1-1 computable enumeration of B; let $B^s=\{b_0,\dots,b_s\}$, and let σ_s to be the longest finite initial segment of the characteristic function of B^s which ends with 1; then it is easy to see that

$$S=\{\sigma\in 2^{<\omega}:(\exists s)[\sigma<\sigma_s]\},$$

where, again, $<$ denotes lexicographical order. At this point (by suitably identifying ω with $2^{<\omega}$), take the dialectical system $d=\langle H,f,c\rangle$, where f enumerates $2^{<\omega}$ in the length-lexicographical order (in which, a string σ precedes a string τ if the length of σ is smaller than the length of τ, or the two strings have the same lengths but $\sigma<\tau$), c is any string, and H is the enumeration operator

$$H=\{\langle x,\{\sigma\}\rangle : x\in\omega\,\&\,\sigma\in S\}\cup\{\langle x,\{\sigma,\tau\}\rangle : x\in\omega\,\&\,|\sigma|=|\tau|\,\&\,\sigma<\tau\}\cup\{\langle\lambda,\emptyset\rangle\}$$

(where $|\ |$ denotes length of strings; notice that the last clause in the definition of H is to comply with the request, in the definition of dialectical systems, that $H(\emptyset)\neq\emptyset$): notice that the enumeration operator H is a closure operator. We can now see that

$$A_d=\{\sigma:\sigma\subset c_B\}:$$

this can easily be proved by induction on x, using (see Theorem 2.3.23)

$$f_x\in A_d\Leftrightarrow c\notin H(L_x\cup\{f_x\}).$$

Hence $A_d\equiv_T A$, and A_d is immune. □

2.4.2 Dialectical sets, quasi-dialectical sets, and enumeration degrees

To characterize the enumeration degrees of the dialectical sets, and of the quasi-dialectical sets, we first prove the following lemma.

Lemma 2.4.6. *If A is a loopless quasi-dialectical set then $A^c \leq_e A$.*

Proof. Let $q = \langle H, f, f^-, c, c^- \rangle$ be a loopless quasi-dialectical system, let $\{H_s\}_{s\in\omega}$ be a loopless computable approximation to H, and let $r_s(x), \rho_s(x), L_s(x)$, have the same meaning as in the definition of a quasi-dialectical set, with respect to this approximation. A closer inspection of the proof the second item of Theorem 2.3.23 easily shows that

$$f_x \in A^c \Leftrightarrow (\exists s)[\{c, c^-\} \cap H_s(L_s(x) \cup \{f_x\}) \neq \emptyset \,\&\, L_s(x) \subseteq A],$$

which provides an algorithm transforming any given enumeration of A into an enumeration of A^c, thus showing that $A^c \leq_e A$. $\square$

Corollary 2.4.7. *If A is a loopless quasi-dialectical set, then $A \equiv_e A^c \oplus A$, hence the enumeration degree of A is total (i.e. it contains the graph of some total function).*

Proof. The proof is obvious as, for every set X, $X^c \oplus X \equiv_e c_X$, where c_X is (the graph of) the characteristic function of X. $\square$

Lemma 2.4.8. *If A is a loopless quasi-dialectical set, then there is a c.e. set B such that $A \equiv_e B^c$, hence the enumeration degree of A is Π^0_1.*

Proof. We know that $A \equiv_T m$, where m is the least modulus function for the Δ^0_2 approximation to A, referred to in the proof of Lemma 2.4.4; on the other hand $m \equiv_T B$, for some c.e. set B, thus

$$A^c \oplus A \equiv_T B^c \oplus B,$$

from which, by totality of the enumeration degrees of $A^c \oplus A$ and $B^c \oplus B$, see for instance [Cooper, 2003],

$$A^c \oplus A \equiv_e B^c \oplus B;$$

finally $B^c \equiv_e B^c \oplus B$, since B is c.e., and thus $A \equiv_e B^c$, by the previous corollary. $\square$

We are now ready to characterize the enumeration degrees of the dialectical sets and of the quasi-dialectical sets.

Theorem 2.4.9. *The enumeration degrees of the dialectical sets and of the quasi-dialectical sets coincide with the Π^0_1 enumeration degrees.*

Proof. If A is a loopless quasi-dialectical set (and this includes also the case when A is dialectical), then its enumeration degree is Π^0_1 by Lemma 2.4.8. If A is represented by an approximated quasi-dialectical system with loops, then A is c.e., and thus $A \equiv_e B$, for every decidable set B: but every decidable set is Π^0_1.

On the other hand, if B is c.e., then by Lemma 2.4.4 there is a dialectical set A such that $A \equiv_T B$, hence, as in the proof of Lemma 2.4.8, $A^c \oplus A \equiv_e B^c \oplus B$. But as B is c.e., we have $B^c \oplus B \equiv_e B^c$, and by Corollary 2.4.7 we have that $A \equiv_e A^c \oplus A$, thus $A \equiv_e B^c$. □

The following corollary parallels Magari's observation in [Magari, 1974] that every c.e. dialectical set is decidable:

Corollary 2.4.10. *If A is a loopless quasi-dialectical c.e. set then A is decidable.*

Proof. If A is represented by a loopless quasi-dialectical system, then $A^c \leq_e A$ by Lemma 2.4.6: thus, if A is c.e., so is A^c. □

2.5 The distribution of dialectical sets, and of quasi-dialectical sets, within the class of limit sets

A result due to [Jockusch, 1974], states that there is no completion of Peano Arithmetic PA that is a Boolean combination of c.e. sets, i.e. there is no completion of PA in any finite level of the Ershov hierarchy. The result has been more recently generalized in [Schmerl, 2005], to any essentially undecidable theory. Since, given a formal theory T, and any pair f, c where f is a computable permutation of ω, and c is a number, it is possible to associate to T a dialectical system $d = \langle H, f, c\rangle$ such that A_d is, by coding, a completion of T (see [Magari, 1974]), a natural question is then to characterize the levels of the Ershov hierarchy that contain dialectical, or quasi dialectical sets. We show in this section that in every finite level $n \geq 2$ of the Ershov hierarchy lies a dialectical set that does not lie in any smaller level of the hierarchy; there exist dialectical sets that do not lie in any finite level; however, no dialectical set can lie outside of the class of the so-called ω-c.e. sets. As regards quasi-dialectical sets, we show that in every level of the Ershov hierarchy lies a proper quasi-dialectical set, that does not lie in any smaller level. We use these results to conclude that there are proper loopless quasi-dialectical sets that are not dialectical. This section is organized as follows: in Subsection 2.5.1 we recall the basic definitions and results concerning the Ershov hierarchy of Δ^0_2 sets. Subsection 2.5.2 shows that the dialectical sets are ω-c.e., and presents a priority-free proof of the fact that for every $n \geq 2$ there is a dialectical set which is properly Σ^{-1}_n. Subsection 2.5.3 contains a priority-free proof of the fact that for every notation a of an infinite ordinal there is a proper loopless quasi-dialectical set which is properly Σ^{-1}_a. Both the proofs in Subsections 2.5.2 and 2.5.3 build sets, which although lying in the appropriate levels of the Ershov hierarchy, are nonetheless introduced through dialectical or quasi-dialectical approximations (i.e., the approximations given by the sets of provisional theses) which in general make "too many" changes and do not directly witness memberships of these sets in the desired levels of the Ershov hierarchy. Finally, in Subsection 2.5.4, straightforward priority arguments are introduced in these proofs, to show that one can also build sets which are witnessed to lie in the appropriate levels of the Ershov hierarchy by their dialectical approximations (however, if n is odd, the dialectical approximation makes in general one more change than desired), or their quasi-dialectical approximations.

2.5.1 The Ershov hierarchy

We now give precise definitions, and a few basic facts, about the Ershov hierarchy. As is known, the Ershov hierarchy classifies the Δ^0_2 sets, through the classes Σ^{-1}_a, where a is the Kleene ordinal notation of a computable ordinal. We use standard notations and terminology for Kleene's system O of ordinal notations: in particular, for $a \in O$, the symbol $|a|_O$ represents the ordinal of which a is a notation; the symbol $<_O$ denotes the Kleene partial ordering relation on O. The Ershov hierarchy of sets was originally introduced in [Ershov, 1968a, Ershov, 1968b, Ershov, 1970]; our presentation is based on [Ash and Knight, 2000].

Definition 2.5.1. *If $a \in O$ is a notation for a nonzero computable ordinal, then a set of numbers A is said to be Σ^{-1}_a if there are computable functions $g(x,s)$ and $h(x,s)$ such that, for all x, s,*

1. *$A(x) = \lim_s g(x,s)$, with $g(x,0) = 0$;*
2. *(a) $h(x,0) = a$ and $h(x,s+1) \leq_O h(x,s)$;*
   (b) $g(x,s+1) \neq g(x,s) \Rightarrow h(x,s+1) \neq h(x,s)$.

Without loss of generality, we may assume that at each stage s, $\{x : g(x,s) = 1\}$ is finite.

We recall ([Ershov, 1968b]) that if $a <_O b$ then Σ^{-1}_a is properly contained in Σ^{-1}_b.

Definition 2.5.2. *If $a \in O$, a set A is said to be* properly Σ^{-1}_a *if*

$$A \in \Sigma^{-1}_a \smallsetminus \bigcup_{b <_O a} \Sigma^{-1}_b.$$

In order to build a set A which is properly Σ^{-1}_a, one could distinguish the two cases whether $|a|_O$ is a successor ordinal, or a limit ordinal:

1. if $|a|_O$ is a successor, say $a = 2^b$, with $|a|_O = |b|_O + 1$, then it is enough to build $A \in \Sigma^{-1}_a \smallsetminus \Sigma^{-1}_b$;
2. if $|a|_O$ is a limit, say $a = 3 \cdot 5^e$, then it is enough to build $A \in \Sigma^{-1}_a$ such that, for every n, $A \notin \Sigma^{-1}_{\phi_e(n)}$.

However, in the proof of Theorem 2.5.14 for simplicity the construction of such an A is kept uniform, relying on the following lemma. Recall that if $a \in O$ is a given notation of a non-zero ordinal, then the set $P_a = \{b \in O : b <_O a\}$ is c.e. (see for instance [Ash and Knight, 2000]), and thus there exists a computable bijection $p : \omega \times P_a \to \omega$.

Lemma 2.5.3. *The following hold:*

1. *For every $a \in O$, there is an indexing $\{V_e\}_{e\in\omega}$ of the family of all Σ_a^{-1}-sets, such that $\{\langle e, x\rangle : x \in V_e\} \in \Sigma_a^{-1}$. Moreover, from e one can effectively find a pair $\langle g_e, h_e\rangle$ of computable functions, witnessing that V_e is in Σ_a^{-1}, as in Definition 2.5.1.*

2. *Given $a \in O$, let $p : \omega\times P_a \to \omega$: be a computable bijection: there is an indexing $\{Z_{p(e,b)} : e \in \omega, b \in P_a\}$, of all sets in $\bigcup_{b<_O a} \Sigma_b^{-1}$. Moreover, from e, b one can effectively find a pair $\langle g_{p(e,b)}, h_{p(e,b)}\rangle$ of computable functions, witnessing that $Z_{p(e,b)}$ is in Σ_b^{-1}, as in Definition 2.5.1.*

Proof. Item (1) can be worked out from [Ash and Knight, 2000]. For item (2), see [Ospichev, 2014]. □

The finite levels of the Ershov hierarchy, and the ω-c.e. sets.

Since finite ordinals have only one notation, one usually writes Σ_n^{-1} instead of Σ_a^{-1}, if a is the notation of $n \in \omega$, and we say that a set A is *n-c.e.* if $A \in \Sigma_n^{-1}$, or equivalently, there is a computable function $g(x, s)$ such that

1. $A(x) = \lim_s g(x, s)$, and $g(x, 0) = 0$;

2. $|\{s : g(x, s + 1) \neq g(x, s)\}| \leq n$.

We may assume that at each stage s, $\{x : g(x, s) = 1\}$ is finite. Moreover,

Definition 2.5.4. A set A is *ω-c.e.* if there are computable functions $g(x, s)$ and $h(x)$ such that, for every x,

1. $A(x) = \lim_s g(x, s)$ and $g(x, 0) = 0$;

2. $|\{s : g(s + 1) \neq g(s)\}| \leq h(x)$, where the symbol $|X|$ denotes the cardinality of a given set X.

As in Definition 2.5.1, we may assume that at each stage s, $\{x : g(x, s) = 1\}$ is finite.

2.5.2 Dialectical sets and the Ershov hierarchy

We are now ready to characterize the levels $a \in O$ of the Ershov hierarchy containing properly Σ_a^{-1} dialectical sets. The first claim of Theorem 2.5.5 is essentially due to [Bernardi, 1974].

Theorem 2.5.5. *The following hold:*

1. *if A_d is a dialectical set, then A_d is ω-c.e.;*

2. *for every n with $2 \leq n \leq \omega$, there exists a properly n-c.e. dialectical set.*

Proof. Let us show item (1). The claim follows from the fact that if A_d is dialectic then $A_d \leq_{tt} \emptyset'$ ([Bernardi, 1974]), and on the other hand, every set $B \leq_{tt} \emptyset'$ is ω-c.e. (see [Nies, 2009]). A direct proof that A_d is ω-c.e. is as follows, where we refer to the approximation $\{A_{d,s}\}_{s\in\omega}$ to A_d, given by the sets of provisional theses. Let $\sigma(y, s)$ be the string of length $y+1$,

$$\sigma(y,s)(x) = \begin{cases} 1, & \text{if } f_x \in L_s(y+1) \\ 0 & \text{if } f_x \notin L_s(y+1). \end{cases}$$

We claim that for every y, $\sigma(y, s)$ can change at most 2^y times. The claim is true of $y = 0$. If t_0 is that least stage at which $\sigma(y, s)$ stops changing, then after t_0, $\sigma(y+1, s)$ may additionally change because of additional changes of $A_s(f_{y+1})$. But this can occur at most two more times, yielding that $\sigma(y+1, s)$ may change at most 2^{y+1} times. From this, it trivially follows that $A_s(f_y)$, which is the y-th bit of $\sigma(y, s)$, may change at most 2^y times. This ends the proof of item (1).

Let as now show (2). Let $2 \leq n < \omega$, and let $\{V_e : e \in \omega\}$ be a computable listing of the $(n-1)$-c.e. sets in the sense of Lemma 2.5.3(1), and correspondingly let $\{V_{e,s} : e, s \in \omega\}$ be a computable sequence of finite sets such that, for every e, $\{V_{e,s} : s \in \omega\}$ is an $(n-1)$-approximation to V_e: for this, take

$$V_{e,s} = \{x : g_e(x, s) = 1\},$$

where we refer to a pair $\langle g_e, h_e\rangle$ of computable functions witnessing that V_e is in Σ^{-1}_{n-1}, as in Lemma 2.5.3(1); notice that, for every x,

$$|\{s : V_{e,s}(x) \neq V_{e,s+1}(x)\}| \leq n-1.$$

We build a dialectical system d such that $A_d \neq V_e$, for all e, and $A_d \in \Sigma^{-1}_n$. Our dialectical system will be of the form $d = \langle H, f, c\rangle$, where we build H, whereas f is the identity function, i.e. $f_x = x$, and $c = 1$. To make the construction simpler to describe, the enumeration operator H that we are going to build will not be a closure operator: we will however argue in Lemma 2.5.11 that $A_d = A_{d'}$ where $d' = \langle H^\omega, f, c\rangle$, and H^ω is the enumeration operator such that, for every X, $H^\omega(X)$ is the smallest fixed point Y of H, such that $Y \supseteq X$: it is known, see e.g. [Amidei et al., a], that H^ω is a closure operator.

Informal description of the construction

The construction is by stages. At stage s we define

1. an approximation H_s to the enumeration operator H; (H_0 is a decidable set, $H_s \subseteq H_{s+1}$, $H_{s+1} \smallsetminus H_s$ is finite, and the relation $x \in H_s$ is decidable;)
2. values $g(x, s)$ of a computable function; the construction will guarantee that for every x, $\lim_s g(x, s)$ exists, and in fact $|\{s : g(x, s) \neq g(x, s+1)\}| \leq n$ (thus $A = \{x : \lim_s g(x, s) = 1\}$ is in Σ_n^{-1}), and $A \neq V_e$, for every e.

In other words, we build a set A with the desired property that A be n-c.e., but not $(n-1)$-c.e.; simultaneously, we build H, by defining stage by stage a computable approximation to H; eventually we observe that $A = A_d$, where $d = \langle H, f, c\rangle$.

Remark 2.5.6. The reader who likes to consider only computable approximations to enumeration operators, consisting of finite sets, could object that H_0, as defined below, is infinite. (This does cause any problem, since, for every decidable X, one easily sees that H_0 satisfies that $H_0(X)$ is decidable, so the construction is computable.) However, one could easily remedy to this, by putting $H_0 = \emptyset$, and delay the enumeration of our infinite H_0 (as given below), by adding step by step a suitable finite portion of it: for instance, by adding $\{\langle 0, \emptyset\rangle, \langle c, \{c\}\rangle\} \in H_1$, and by adding to our H_{s+1} below, the finite set

$$\{\langle x, \{c\}\rangle : x \leq s\} \cup \{\langle x, \{x\}\rangle : x \leq s\}.$$

This remark applies to similar cases in the proofs of Theorems 2.5.14,2.5.25,2.5.29.

Requirements

In addition to the overall requirements that $A = A_d$, and A be n-c.e., the requirements to meet are, for every $e \in \omega$:

$$P_e : A \neq V_e.$$

Strategy to meet P_e

If we were not concerned with eventually getting $A = A_d$, the strategy would be the usual strategy to build an n-c.e. set which is not $(n-1)$-c.e.: we appoint a witness b_e, with initially $b_e \in A$ (so initially, we change $A(b_e)$ (or, rather, the current value $A_s(b_e)$ of $A(b_e)$) from the value 0 to the value 1); then, every time we see that $A(b_e) = V_e(b_e)$, we respond with changing $A(b_e)$, so as to have $A(b_e) \neq V_e(b_e)$. Since $V_e(b_e)$ can change at most $n-1$ times, we have that $A(b_e)$ can change at most n times, both sets A and V_e ending up with final values $A(b_e) \neq V_e(b_e)$, as desired.

Towards getting $A = A_d$**.** So, what we really need to explain is how to simultaneously construct H, so that eventually we get $A = A_d$. To this end, a *witness* for P_e is in fact a closed interval $I(e) = [a_e, a_e + n - 1]$, where we put $b_e = a_e + n - 1$. We suppose that for every e, $a_{e+1} = a_e + n$, so that the sets $I(e)$ are pairwise disjoint. We suppose also $a_0 = 2 = c + 1$.

When we appoint $I(e)$, we momentarily put $I(e) \subseteq A$, and we go through the following module, where we count the number of cycles by the counter i_e:

1. set $i_e := n - 1$;
2. if $b_e \in V_e$, then extract b_e from A and add the axiom $\langle c, \{a_e + j, b_e : j < i_e\}\rangle \in H$; let $i_e := i_e - 1$; go to (2);
3. if $b_e \notin V_e$, then put back b_e into A; extract $a + i_e$ from A and add the axiom $\langle c, \{a_e + i_e\}\rangle \in H$ (by which $a_e + i(e)$ ends up to be out of A_d); let $i_e := i_e - 1$; go to (2).

Analysis of outcomes of the strategy for P_e

We analyze in more detail the outcomes of the strategy for P_e, with reference to how we get $A = A_d$, where $d = \langle H, f, c\rangle$.

If $i_e = n - 1$ is the final value of i_e, then we do not add any axiom in H which involves elements of $I(e)$: then clearly $b_e \in A_d$, and $a_e + j \in A_d$, for all $j < n - 1$; these values of A_d on the elements of $I(e)$ coincide with those of A;

Suppose that the value of i_e decreases to $i_e = i$ from $i_e = i + 1$. We use Theorem 2.3.23(2), an easy inductive argument on i, and the definition of H: assume by induction that up to now there is no axiom $\langle c, \{a_e + j\}\rangle \in H$, for any $j < i$; no axiom $\langle c, \{a_e + j, b_e : j < i\}\rangle \in H$; and there are already axioms $\langle c, \{a_e + j\}\rangle \in H$, for all $i < j < n - 1$.

1. if b_e is extracted from A, then we add the axiom $\langle c, \{a_e + j, b_e : j < i\}\rangle \in H$; we conclude that if this is the final value of i_e, then $b_e \notin A_d$, since $\{a_e + j : j < i\} \subseteq A_d$, and thus $c \in H(L(b_e) \cup \{b_e\})$; moreover $a_e + j \notin A_d$, for all $i \leq j < n - 1$; these values of A_d on the elements of $I(e)$ coincide with those of A;
2. if b_e is put back into A, then we add the axiom $\langle c, \{a_e + i\}\rangle \in H$, by which $a_e + i$ will be out of A_d; hence the axiom $\langle c, \{a_e + j, b_e : j < i + 1\}\rangle \in H$ does not apply, and if i is the final value of i_e, then $b_e \in A_d$, since $c \notin H(L(b_e) \cup \{b_e\})$; moreover we also have $\{a_e + j : j < i\} \subseteq A_d$, and $a_e + j \notin A_d$, for all $i < j < n - 1$; these values of A_d on the elements of $I(e)$ coincide with those of A.

The construction

The construction is by stages. We make use of the parameter $i_{e,s}$, approximating at stage s the number i_e as in the section "Strategy to meet P_e".

Definition 2.5.7. A requirement P_e *requires attention* at s, if $s > 0$, and (in the order) either $i_{e,s} =\uparrow$, or $b_e \in V_{e,s}$ if and only if $b_e \in A_{s-1}$.

Stage 0**.** Let

$$H_0 = \{\langle x, \{c\}\rangle : x \in \omega\} \cup \{\langle 0, \emptyset\rangle\} \cup \{\langle x, \{x\}\rangle : x \in \omega\}.$$

(The reason for having $0 \in H(\emptyset)$ is to comply with the definition of a dialectical system, which requires $H(\emptyset) \neq \emptyset$.) Let also $g(x, 0) = 0$, for all x. Define $i_{e,0} =\uparrow$, for every e.

Stage $s + 1$**.** Consider all $e \leq s$ such that P_e requires attention at $s + 1$.

1. If $i_{e,s} =\uparrow$, then set $i_{e,s+1} = n-1$. We put $I(e) \subseteq A_{s+1}$, by defining $g(x, s+1) = 1$, for all numbers $x \in I(e)$.

2. Otherwise:

   (a) if $b_e \in V_{e,s+1}$ (necessarily, $i_{e,s} > 0$), then add the axiom $\langle c, \{a_e + j, b_e : j \leq i_{e,s}\}\rangle \in H$, define $g(b_e, s + 1) = 0$, and define $i_{e,s+1} = i_{e,s} - 1$;

   (b) if $b_e \notin V_{e,s+1}$ (necessarily, $i_{e,s} > 0$), then add the axiom $\langle c, \{a_e + i_{e,s}\}\rangle \in H$, define $g(a_e+i_{e,s}, s+1) = 0$, $g(b_e, s+1) = 1$, and define $i_{e,s+1} = i_{e,s}-1$.

Let H_{s+1} be H_s plus the axioms for H added at stage $s+1$. Let also $g(0, s+1) = 1$. Unless explicitly redefined during stage $s + 1$, all remaining parameters and values maintain the same value as at stage s. In particular $g(c, s + 1) = 0$. Go to Stage $s + 2$.

Verification

The verification relies on the following lemmata.

Lemma 2.5.8. *A is n-c.e.*

Proof. If a number x lies in some $I(e)$, then it is clear that $A_s(x)$ can change at most n times, as has been already discussed in the section "Strategy to meet P_e". Otherwise, $x \in \{0, 1\}$: then $A_s(x)$ changes from 0 to 1 exactly once, if $x = 0$, and $A_s(x)$ never changes, if $x = 1 = c$. □

Lemma 2.5.9. *For every e, A satisfies P_e.*

Proof. We change the value $A_s(b_e)$ as many times as are necessary to diagonalize against the final value $V_e(b_e)$. □

Lemma 2.5.10. *$A = A_d$.*

Proof. Let us consider any x. If $x \in I(e)$ for some e, then it is clear by the way we update H, and the discussion in the section with title "Analysis of the outcomes of the strategy for P_e", that $A(x) = A_d(x)$. If x does not lie in any such $I(e)$, then $x \in \{0, 1\}$, and the claim is trivial. □

Lemma 2.5.11. *$A_d = A_{d'}$, where $d' = \langle H^\omega, f, c\rangle$.*

Proof. The claim follows from the following easy observation: $H^\omega = H^2$, and obviously $c \in H(H(X))$ if and only if $c \in H(X)$, by the way we have defined the axioms of H involving c. □

Finally we sketch how to prove claim (2) of the statement of the theorem, when $n = \omega$.

We start with an effective listing of all n-c.e. sets, for the various $n \geq 1$: for instance, take $Z_{\langle e,n\rangle} = V_e^n$, where $\{V_e^n\}_{e\in\omega,n\geq 1}$ is an effective listing of all n-c.e. sets.

A witness for the requirement $P_{\langle e,n\rangle}$ (with $e \geq 0$ and $n \geq 1$) is now a closed interval $I(\langle e,n\rangle) = [a_{\langle e,n\rangle}, a_{\langle e,n\rangle} + n]$. The rest of the proof is exactly as before, with the only difference that witnesses are now closed intervals of variable length. □

Remark 2.5.12. It should be noted that the proof of item (2) of the previous theorem makes use of no priority feature. Each requirement keeps its own witness forever, and there is no interference between the different strategies for the various requirements.

Remark 2.5.13. Item (2) of Theorem 2.5.5 can not be extended to include the case $n = 1$, because every c.e. dialectical set is decidable ([Magari, 1974]), and thus, every 1-c.e. dialectical set is also 0-c.e.

2.5.3 Quasi-dialectical sets and the Ershov hierarchy

The goal of this section is to prove that for every notation $a \in O$ of a nonzero computable ordinal there is a proper quasi-dialectical set, which is properly Σ_a^{-1}. The claim should be more precisely stated according to the following distinction: if $|a|_O = 1$ then there is a quasi-dialectical set A, represented by an approximated quasi-dialectical system with loops, such that A is properly Σ_a^{-1}, hence A is c.e. but not decidable; if $|a|_O \geq 2$ then there is a proper loopless quasi-dialectical set which is properly Σ_a^{-1}. It will follow from this, that there are proper loopless quasi-dialectical sets that are not dialectical.

Theorem 2.5.14. *For every notation $a \in O$, with $|a| \geq 2$, there is a proper loopless quasi-dialectical set which is properly Σ_a^{-1}.*

Proof. We rely on the possibility of building, for any given a as in the statement of the theorem, a proper quasi-dialectical system $q = \langle H, f, f^-, c, c^-\rangle$, together with a suitable loopless computable approximation α to H, which enables us to pick, when needed, pairs of numbers $y < x$ (with $f_x \neq c, c^-$), so as to satisfy the following two desiderata:

(i) no occurrences of f_x is ever permitted to the left of x, i.e., for all $z < x$, at every stage s we have that $\rho_s(z) \neq f_x$;

(ii) at no stage s do we have $c \in H_s(L_s(y + 1))$.

The elimination/recovery mechanism

If so, suppose that at some stage $s+1$, we have $f_x \in A_{q,s}$ (set of provisional theses at stage s) but we want to remove f_x from the provisional theses: we can do so, by defining at $s+1$ the axiom $\langle c, \{\rho_s(y), f_x\}\rangle \in H$. If at some bigger stage $t+1 > s+1$, we want to restore f_x in the provisional theses, it will be enough to define at $t+1$ the axiom $\langle c^-, \{\rho_s(y)\}\rangle \in H$: this has the effect of immediately getting $\rho_s(y)$ out of $A_{q,t+1}$, so that the axiom $\langle c, \{\rho_s(y), f_x\}\rangle \in H$ does not apply any more; thus, the quasi-dialectical procedure (i.e., the procedure through which the sets of provisional theses are constructed) will propose f_x again, and put it back into the set of provisional theses.

It is then clear that, by this mechanism (called the *elimination/recovery mechanism*), using the quasi-dialectical procedure, we can move f_x in and out of A_q as many times as we want.

With reference to the elimination/recovery mechanism, we fix the following terminology:

1. we call the number y the *fellow* of f_x;
2. we say that y *eliminates* f_x at stage s if $c \in H_s(\{\rho_s(y), f_x\})$,
3. we say that y *recovers* f_x at stage s, if $c^- \in H_s(\{\rho_s(y)\})$.

If $a \in O$ is a given notation, with $|a| \geq 2$, then fix a computable bijection $p : \omega \times P_a \to \omega$. Thus, by Lemma 2.5.3(2), we may refer to an indexing $\{Z_{p(e,b)} : e \in \omega, b \in P_a\}$ of all sets in $\bigcup_{b <_O a} \Sigma^{-1}_b$, such that from e, b one can effectively find a pair $\langle g_{p(e,b)}, h_{p(e,b)}\rangle$ of computable functions, witnessing that $Z_{p(e,b)}$ is in Σ^{-1}_b, as in Definition 2.5.1.

Informal description of the construction

We build a proper quasi-dialectical system $q = \langle H, f, f^-, c, c^-\rangle$, together with a suitable loopless computable approximation $\alpha = \{H_s\}_{s\in\omega}$ to H, such that $A_q \neq Z_n$, for all $n = p(e, b)$, $e \in \omega$ and $b <_O a$. Our quasi-dialectical system will be of the form $q = \langle H, f, f^-, c, c^-\rangle$, where we build H through α, whereas f is the identity function, $f^-(x) = 3x$, $c = 1$, and $c^- = 2$. To make the construction simpler to describe, the enumeration operator H that we are going to build will not be a closure operator. We will however argue in Lemma 2.5.19 that $A_q = A_{q'}$ where $q' = \langle H^\omega, f, f^-, c, c^-\rangle$: this is similar to what we have done in the proof of Theorem 2.5.5. Hopefully, q and α will allow us to pick, as needed, pairs y, x, where y is a fellow of f_x, so that we can play the above described elimination/recovery game. The construction is by stages. At stage s we define

1. an approximation H_s to the enumeration operator H;

2. values $g(x,s)$, and $h(x,s)$ of computable functions, guaranteeing that for every x, $\lim_s g(x,s)$ exists, and in fact the pair $\langle g,h\rangle$ witnesses that $A = \{x : \lim_s g(x,s) = 1\}$ is in Σ_a^{-1}, and $A \neq Z_n$, for every n. Throughout the construction, we define
$$A_s = \{x : g(x,s) = 1\}.$$

We build a set A with the desired property that $A \in \Sigma_a^{-1} \smallsetminus \bigcup_{b<_O a} \Sigma_b^{-1}$; simultaneously, we define H through a loopless $\alpha = \{H_s\}_{s\in\omega}$; eventually we observe that $A = A_q^\alpha$. Although there is no reason to conclude that H is a closure operator, nonetheless we can still construct the sets $A_{q,s}^\alpha$ of provisional theses, and thus the set A_q^α, using the approximation α to H built in the construction. For simplicity we will write $A_{q,s} = A_{q,s}^\alpha$, and $A_q = A_q^\alpha$ (also justified by the fact that α will turn out to be loopless, and easily yields a loopless approximation to the closure operator H^ω of Lemma 2.5.19).

Requirements

The requirements to meet are, for all $n = p(e,b)$, with $e \in \omega$ and $b <_O a$:
$$S : A \in \Sigma_a^{-1}$$
$$P_n : A \neq Z_n.$$

Strategy to meet P_n

As for the case of dialectical systems, the strategy to achieve $A \neq Z_n$ is obvious: we pick a witness x_n; initially we put $x_n \in A$ (notice that $f_{x_n} = x_n$); then we keep extracting and putting back x_n, responding to the movements of x_n in and out of Z_n, so that each time we diagonalize $A(x_n)$ against $Z_n(x_n)$. We keep track of changes of $A(x_n)$ by updating g and h: initially we set $g(x_n,0) = 0$ and $h(x_n,0) = a$; if at stage $s+1$ we change $A(x_n)$, we correspondingly change $g(x_n,s+1)$, and we decrease $h(x_n,s+1) <_O h(x_n,s)$, so that we do not end up at $h(x_n,t) = 1$ (recall that $|1|_O = 0$) before $h_n(x_n,t)$ does.

Towards getting $A = A_q$**.** So, what we really need to explain is again how to simultaneously construct H and $\alpha = \{H_s\}_{s\in\omega}$, so that eventually we get $A = A_q$. A *witness* for P_n, with $n = p(e,b)$, is now the two-element interval $I(n) = [y_n, x_n]$ where $y_n = 3(n+1)+1$, $x_n = 3(n+1)+2$, thus $x_n = y_n+1$, and $y_n, x_n \notin \mathrm{range}(f^-)$. We must ensure that in the limit, the values $A(x_n)$ and $A_q(x_n)$ are equal.

We go through the following module, where we use a counter i_n to count the number of cycles; for simplicity, we use the notation $z^i = f^{-(i)}(z)$:

1. set $i_n := 0$; put y_n and x_n into A;

2. if $x_n \in Z_n$, then extract x_n from A, and add the axiom $\langle c, \{y_n^i, x_n\}\rangle \in H$; define $i_n := i_n + 1$;

3. if $x_n \notin Z_n$, then we put back x_n in A, extract y_n^i from A, put y_n^{i+1} into A, and add the axiom $\langle c^-, \{y_n^i\}\rangle \in H$; define $i_n := i_n + 1$.

For A_q to catch up with A, the idea here is to have q and α play the elimination/recovery mechanism with y_n as a fellow of x_n, so that there is a sequence of stages $s_0 < s_1 < \cdots < s_{i_n}$ (where i_n is the final value of the counter), and a sequence $0 = j_0 \leq j_1 \leq \cdots \leq j_n$ (where j_n is the greatest i such that $i = 0$ or at some stage the construction has passed from y_n^{i-1} to y_n^i) such that, for every $i \leq n$, $y_n^{j_i} = \rho_{s_i}(y_n)$, and

(a) if we need to extract x_n from A at s_i, then y_n eliminates x_n at s_i;

(b) if we need to put back x_n in A at s_i, then y_n recovers x_n at s_i.

If we succeed in relating in this way the basic strategy for P_n, with the elimination/recovery mechanism, then by the discussion of this mechanism in the section dealing with this topic at the beginning of the proof, it is clear that for all $z \in \{y_n^i : i \leq j_n\} \cup \{x_n\}$ involved in the strategy for P_n, we get the same limit value $A(z) = A_q(z)$.

Analysis of outcomes of the strategy for $P_{p(e,b)}$

As in the analogous case of a P-requirement in the proof of Theorem 2.5.5, the above informal discussion regarding the movements of y_n^i and x_n, shows that we are eventually able to diagonalize $A(x_n)$ against $Z_n(x_n)$, as long as we do not exhaust the quota of allowable changes compatible with having $A \in \Sigma_a^{-1}$, i.e. as long as $h(x_n, t)$ does not reach, as a notation, the ordinal 0, before $h_n(x_n, t)$ does. Here is where we need to combine the strategy for P_n, with a suitable strategy for S, as we describe in the next paragraph.

Strategy to meet S

As promised, we define by stages two computable functions $g(x, s), h(x, s)$, witnessing that $A \in \Sigma_a^{-1}$. When, working to satisfy P_n, with $n = p(e, b)$, we first put x_n into A at a stage, say, s_0, and we define $h(x_n, s_0) = b$: up to this stage, we had $h(x_n, s) = a$. Following this stage, whenever we move x_n as above at, say, stage $s + 1$, we change the value of $g(x_n, s + 1)$, and decrease $h(x_n, s + 1)$, by defining

$$h(x_n, s + 1) = h_n(x_n, s + 1) :$$

since the action is taken because there has been a change in $g_n(x_n, s)$ which has occurred between the last stage t, for which we have $h(x_n, s) = h_n(x_n, t)$, and $s + 1$, then $h(x_n, s + 1)$ does decrease with respect to $<_O$, following the decrease of $h_n(x_n, s + 1)$. Therefore, a simple inductive argument shows that, for all s,

$$h(x_n, s) \geq h_n(x_n, s).$$

This shows that, compared with Z_n, the approximation $\{A_s\}_{s\in\omega}$ to the defined set A allows on x_n for one more change than Z_n does, so that we can get to the desired diagonalization. As regards y_n, and the other potential numbers y_n^i, which enter the strategy for P_n, we have no problem here to meet S, since we will see that each number y_n^i moves at most twice, namely it is enumerated into A, and then it may be extracted again: therefore, when y_n^i is enumerated into A, at say stage s, it will be enough to set $h(y_n^i, s) = 2$, ordinal notation of 1. (This is where the assumption that $|a|_O \geq 2$ is being used, as $h(y_n^i, s) = 2$ has to drop to 2 from a bigger notation.)

Construction

The construction is by stages. For every n, s, let

$$Z_{n,s} = \{z : g_n(z, s) = 1\}.$$

For every n, we approximate the counter i_n, with $i_{n,s}$.

Definition 2.5.15. We say that P_n *requires attention* at s, if $s > 0$, and (in the order) either $i_{n,s} =\uparrow$, or $x_{n,s} \in Z_{n,s+1}$ if and only if $x_{n,s} \in A_{s-1}$.

It will be understood that, at the end of stage $s + 1$, parameters and values (including values for $g(x, s{+}1)$ and $h(x, s{+}1)$) that have not been explicitly redefined, retain the same value as at the end of stage s.

Stage 0**.** Let

$$H_0 = \{\langle x, \{c\}\rangle : x \in \omega\} \cup \{\langle 0, \emptyset\rangle\} \cup \{\langle x, \{x\}\rangle : x \in \omega\}.$$

Let $g(x, 0) = 0$, and $h(x, 0) = a$, for all x. For every n, let $i_{n,0} =\uparrow$.

Stage $s + 1$**.** Consider all $n \leq s$ such that P_n requires attention. Then consider two cases (where $n = \langle e, b\rangle$):

1. if $i_{n,s} =\uparrow$ then set $g(y_n, s + 1) = 1$, $h(y_n, s + 1) = 2$, $g(x_n, s + 1) = 1$, $h(x_n, s + 1) = b$;

2. otherwise:

   (a) If $x_n \in Z_{n,s+1}$ then add the axiom $\langle c, \{y_n^{i_{n,s}}, x_n\}\rangle \in H$. Define $g(x_n, s + 1) = 0$, and $h(x_n, s + 1) = h_n(x_n, s + 1)$; set $i_{n,s+1} = i_{n,s} + 1$;

   (b) If $x_n \notin Z_{n,s+1}$ then add the axiom $\langle c^-, \{y_n^{i_{n,s}}\}\rangle \in H$. Define $g(x_n, s{+}1) = 1$, and $h(x_n, s + 1) = h_n(x_n, s + 1)$; define also $g(y_n^{i_{n,s}}, s + 1) = 0$, and $h(y_n^{i_{n,s}}, s + 1) = 1$; set $i_{n,s+1} = i_{n,s} + 1$.

Let H_{s+1} be H_s plus the axioms for H added at stage $s{+}1$. Finally, define $g(0, s{+}1) = 1$, $h(0, s + 1) = 1$, $g(c, s + 1) = g(c^-, s + 1) = 0$, $h(c, s + 1) = h(c^-, s + 1) = 1$. For all other $z \leq s$ such that z is in the range of $f^-(x) = 3x$, and $h(z, s) = a$, set $g(z, s + 1) = 1$ and $h(z, s + 1) = 2$.

Verification

The verification relies on the following lemmata.

Lemma 2.5.16. $A \in \Sigma_a^{-1}$.

Proof. We have defined by stages a pair $\langle g, h\rangle$ of computable functions that witness that $A \in \Sigma_a^{-1}$, as is argued in the section with the title "Strategy to meet S". □

Lemma 2.5.17. *For every n, P_n is satisfied, i.e. $A \neq Z_n$; $i_n = \lim_s i_{n,s}$ exists.*

Proof. Let n be given. It is clear that actions relative to different requirements do not interfere with each other, and thus we are able to keep changing the value of $g(x_n, s)$ (i.e., of $A_s(x_n)$) as (finitely) many times as we need in order eventually to diagonalize $A(x_n)$ against $Z_n(x_n)$, thus getting $A \neq Z_n$. It is also clear from this, that there is a stage at which we stop to change $i_{n,s}$. □

Lemma 2.5.18. $A = A_q$.

Proof. We claim that the limit value $\lim_s g(x, s)$ that the construction demands for each x, is also achieved by the sequence $\{A_{q,s}\}_{s\in\omega}$, i.e., $\lim_s g(x, s) = \lim_s A_{q,s}(x)$.

On $0, c, c^-$, the sets A and A_q clearly agree in the limit.

Let us recall that j_n is the greatest i such that $i = 0$ or at some stage the construction has passed from y_n^{i-1} to y_n^i. We now show by induction that for every n, $r(y_n) = \lim_s r_s(y_n) = \langle y_n, y_n^{j_1}, \dots, y_n^{j_n}\rangle$; and for all $u \in \text{range}(r(y_n)) \cup \{x_n\}$, $\lim_s g(u, s) = \lim_s A_{q,s}(u)$. Suppose that the claim is true of every $i < n$. Clearly, not only for $z \in I(i)$, $i < n$, can we assume that $r(z) = \lim_s r_s(z)$ exists: indeed, if z does not lie in any such $I(i)$, then $z \in \{0, 1, 2\}$, but then the claim is trivially true, or $z = 3u$, for some u: in this latter case, by definition of H, $r(z) = \langle z\rangle$, or $\rho(z) = \rho(y_i)$, for some $i < n$.

First of all, notice that neither fellows y_j, nor elements of the forms x_j chosen in witnesses $I(j)$, belong to the range of the function $f^-(x) = 3x$: therefore sets of the form $\{y_j^i : i \in \omega\}$ and $\{x_j\}$, for different j's, do not overlap, and we never define axioms for the enumeration operator H, which involve elements belonging to such sets relative to different j's. In the rest of the proof we repeatedly apply Theorem 2.3.23(2), easy inductive arguments, and the definition of H. Let t_n be the least stage at which all $r_s(z)$ for $z < y_n$ have reached limit. Starting from now on, q and α start to build the final stack on y_n, which never becomes $\langle\ \rangle$ by definition of H (no axiom of the form $\langle c, \{y_n^i\}\rangle \in H$ is ever added). By [Amidei et al., a, Corollary 3.9], there is a least stage s_0 after t_n at which $r_{s_0}(y_n) = \langle y_n\rangle$, and $r_{s_0}(x_n) = \langle x_n\rangle$: and if $i_n = 0$, then due to the absence of axioms in H involving y_n and x_n, this value $r_{s_0}(y_n)$ is clearly the last value of $r(y_n)$; moreover $y_n, x_n \in A_q$; these values of A_q on the elements of $I(n)$ coincide with those of A.

Suppose that at a stage $s_u + 1$, we have that $i_{n,s_u+1} = i_{n,s_u} + 1$, and let $i_{n,s_u} = i$; let also $r_{s_u}(y_n) = \langle y_n, y_n^{j_1}, \dots, y_n^{j_i}\rangle$. Assume by induction that up to s_u there are no axioms $\langle c^-, \{y_n^{j_i}\}\rangle \in H$, $\langle c, \{y_n^{j_i+1}, x_n\}\rangle \in H$, but there are already axioms $\langle c^-, \{y_n^j\} \in H$, for all $j < j_i$. There are two possibilities:

1. at $s_u + 1$ we extract x_n from A : in this case our action introduces the axiom $\langle c, \{y_n^{j_i}, x_n\}\rangle \in H$. The stack does not change, with value $r_{s_u+1}(y_n) = \langle y_n, y_n^{j_1}, \ldots, y_n^{j_i}\rangle$. If $i_{n,s_u+1} = i_n$ (thus $j_i = j_n$) then we would permanently get $y_n^{j_i} \in A_q$ and $x_n \notin A_q$, as $\{c, c^-\} \cap H(L(y_n) \cup \{y_n^{j_i}\}) = \emptyset$ and $c \in H(L(x_n) \cup \{x_n\})$; moreover $y_n^j \notin A_q$, for all $j < j_i$; these values of A_q on the elements used by P_n coincide with those of A; the final value of the stack would be $r(y_n) = \langle y_n, y_n^{j_1}, \ldots, y_n^{j_n}\rangle$.
2. at s_u+1 we put x_n back into A: in this case we introduce the axiom $\langle c^-, \{y_n^{j_i}\}\rangle \in H$. The new stack is $r_{s_u+1}(y_n) = \langle y_n, y_n^{j_1}, \ldots, y_n^{j_i}, y_n^{j_i+1}\rangle$. If $i_{n,s_u+1} = i_n$, then $x_n \in A_q$, as $\{c, c^-\} \cap H(L(x_n) \cup \{x_n\}) = \emptyset$, $y_n^{j_i+1} \in A_q$, and $y_n^j \notin A_q$, for all $j \leq j_i$; these values of A_q on the elements used by P_n coincide with those of A; $j_n = j_i + 1$, and the final stack would be $r(y_n) = \langle y_n, y_n^1, \ldots, y_n^{j_i}, y_n^{j_n}\rangle$.

On the other numbers, i.e. those z in the range of $f^-(x) = 3x$ which have not participated in the actions taken by any strategy, we have $A(z) = A_q(z) = 1$, thanks to the last clause at each stage $s + 1$, demanding to put into A, all such $z \leq s$ such that $h(z, s) = a$: in absence of any axiom in H involving these numbers, they will be proposed and put in A_q by the quasi-dialectical procedure. □

Lemma 2.5.19. *There is a loopless quasi-dialectical system $q' = \langle H', f, f^-, c, c^-\rangle$, where H' is a closure operator, such that $A_q = A_{q'}$.*

Proof. We have to be more careful here than in the proof of Lemma 2.5.11, since quasi-dialectical sets may depend on the chosen computable approximation to the enumeration operator. So take again $H' = H^\omega$, and take the approximation $\{H_s^\omega\}_{s\in\omega}$ obtained in the following way: we enumerate in H_s^ω all axioms enumerated into H_s; moreover, whenever at stage s we add an axiom $\langle c, \{x\}\rangle \in H$, then we add also the decidable set of axioms $\langle y, \{x\}\rangle \in H^\omega$: the important thing is that we do not enumerate axioms of the form $\langle c^-, \{\rho_s(y_n), x_n\}\rangle \in H$ *strictly before* enumerating $\langle c, \{\rho_s(y_n), x_n\}\rangle \in H$, so that there is no danger of building a stack on some x_n which is different from $\langle\ \rangle$ or $\langle x_n\rangle$. It is easy to see that we do get a loopless computable approximation α' to H^ω, such that $A_{q'}^{\alpha'} = A_q$. (The reader sensible to the problem raised in Remark 2.5.6 should easily find a way to approximate H^ω through finite sets: instead of enumerating at once an infinite set of axioms like the previous one, one can just enumerate, stage by stage, finite pieces of it at future stages.) By the proof of the previous lemma, it follows that $A_{q'}$ is loopless. □

This concludes the proof of the theorem. □

Remark 2.5.20. As for the proof of Theorem 2.5.5 (see Remark 2.5.12), it should be noted that the proof of the previous theorem is priority-free.

Remark 2.5.21. By Corollary 2.4.10, we can not include the case $|a|_O = 1$ in the statement of Theorem 2.5.14, since every c.e. set A represented by a loopless quasi-dialectical system is decidable.

Corollary 2.5.22. *For every $a \in O$ such that $|a|_O \geq 1$, there is a quasi-dialectical set*

$$A \in \Sigma_a^{-1} \smallsetminus \bigcup_{b <_O a} \Sigma_b^{-1}.$$

Proof. If $|a|_O > 1$ this follows from Theorem 2.5.14. Assume $|a|_O = 1$: we know from [Amidei et al., a, Theorem 3.12] that every coinfinite and not simple c.e. set can be represented by a quasi-dialectical system with loops: therefore there are c.e. quasi-dialectical sets which are not decidable. □

A consequence of Theorem 2.5.14 is:

Theorem 2.5.23. *There are proper loopless quasi-dialectical sets that are not dialectical.*

Proof. It is well known, and in any case easy to see, that if $a, b \in O$, and $|a|_O = |b|_O = \omega$, then $\Sigma_a^{-1} = \Sigma_b^{-1}$: for this reason, if $|a|_O = \omega$, we usually write $\Sigma_a^{-1} = \Sigma_\omega^{-1}$. On the other hand, the ω-c.e. sets are included in the Σ_ω^{-1} sets, see e.g. [Nies, 2009]. The claim is then immediate by Theorem 2.5.5 and Theorem 2.5.14: for instance, it is enough to take a proper loopless quasi-dialectical set $A \in \Sigma_a^{-1} \smallsetminus \Sigma_\omega^{-1}$, where $|a|_O = \omega + 1$. □

Theorem 2.5.23 can be obtained also as a consequence of the following:

Corollary 2.5.24. *If $\mathcal{X} = \{V_e : e \in \omega\}$ is an indexing of some class of Δ_2^0 sets, i.e. the predicate $x \in V_e$ is Δ_2^0, then there is a proper loopless quasi-dialectical set A such that $A \notin \mathcal{X}$.*

Proof. Similar to the proof of Theorem 2.5.14: in fact the proof is much easier, in that we do not have to keep track of the number of changes in the function g, giving A as a limit, since we do not have to worry about making A a Σ_a^{-1} set, for some $a \in O$. □

Theorem 2.5.23 follows from the previous corollary, by the fact that the ω-c.e. sets can be indexed as a Δ_2^0 class as in the statement of the corollary, see [Nies, 2009].

2.5.4 Stretching the proofs of Theorem 2.5.5 and Theorem 2.5.14

A legitimate curiosity is to know whether one can stretch the proofs of Theorem 2.5.5 and Theorem 2.5.14, to obtain dialectical sets A_d or quasi-dialectical sets A_q, for which the Δ_2^0 approximations $\{A_{d,s}\}_{s\in\omega}$ or $\{A_{q,s}\}_{s\in\omega}$ yielded by the sets of provisional theses (taken with respect to the computable approximation $\alpha = \{H_s\}_{s\in\omega}$ to

H, defined during the construction), already witness that the sets lie in the appropriate level of the Ershov hierarchy.

Recall that by Theorem 2.5.5(1), for every dialectical system d the Δ^0_2 approximation $\{A_{d,s}\}_{s\in\omega}$ (taken with respect to any computable approximation to the enumeration operator of d) already witnesses that A_d is ω-c.e.

Dialectical approximations. We start up with dialectical sets, and we briefly discuss the difficulties inherent in building a suitable dialectical system $d = \langle H, f, c\rangle$, together with a suitable computable approximation to H, such that for every x, the value $A_{d,s}(x)$ does not make too many changes.

With reference to the construction described in the proof of Theorem 2.5.5 (claim (2), case of n finite), there is an evident conflict arising by interactions between different strategies. Consider P_e, P_i with $e < i$. We limit our analysis to the components b_i and b_e of the respective witnesses $I(i)$ and $I(e)$, but similar considerations hold for the other components $a_i + j$ and $a_e + k$, with $j, k \leq n-1$. It could happen that we act first to satisfy P_i, so the dialectical procedure (following our definition of H and its approximations) moves b_i in and out of A_d a certain number n' of times. Then we must act for P_e. Now, following the dialectical procedure, when at a stage s we move an element b out of $A_{d,s}$, it happens that we have to keep out of $A_{d,s}$ also the elements $b' > b$: so when the dialectical procedure follows up our action for P_e it may happen that it moves again b_i. Suppose that this happens n'' times: so altogether we would have to move b_i, $n' + n''$ times, with possibly $n' + n'' > n$: too many changes!

The solution consists of course in introducing some priority within the construction, so that when we act for P_e we discard the current witness for P_i which can start afresh, and thus having the possibility of moving n times the components of the new witness, if necessary.

In this new setting, we need to approximate not only $i_{e,s}$, but also $a_{e,s}, b_{e,s}$, and therefore $I(e, s) = [a_{e,s}, a_{e,s} + n - 1]$.

When we choose $I(e, s)$, we choose it *new*, i.e., its members are bigger than all numbers so far mentioned in the construction. In particular, $b_{e,s}$ has never been a provisional thesis, and it may take a while for it to become a provisional thesis, since the dialectical procedure has to propose first a bunch of numbers and to decide on them, before proposing and momentarily accepting $b_{e,s}$; the same may happen when the dialectical procedure has momentarily discarded $b_{e,s}$, but then wants it back. (Notice that on the contrary, when we want out an element a, which is currently in the provisional theses, then we add to H a suitable axiom involving c and a, and this action takes effect immediately: for instance, we add $\langle c, \{a\}\rangle \in H$, and at this stage a is out of the provisional theses.) When in the construction below, we act to put $b_{e,s}$ back and we just need that the dialectical procedure makes it a provisional thesis, the we say that P_e is in "standby": the rigorous definition is given in the construction.

A requirement P_e is *initialized* if we set all of its parameters to be undefined. We say that P_e *requires attention* at s, if $s > 0$, and (in the order) either P_e is initialized, or P_e is in standby, or $b_{e,s} \in V_{e,s}$ if and only if $b_{e,s} \in A_{d,s-1}$.

At stage $s+1$ we act on behalf of the least e, such that P_e requires attention, and we initialize all P_i with $i > e$, by discarding their witnesses and forcing each such P_i to use a new witness when its turn to act comes again. In order to avoid that the components of the discarded witness of some P_i with $i > e$ make more moves than it is allowed, in and out of the sets of provisional theses, we *freeze* them out of the future sets $A_{d,s}$ of provisional theses, by adding the axiom $\langle c, \{a\}\rangle \in H$, for each member a of the discarded witness. Now notice that this may add an additional change for the value $A_d(a)$ with respect to the approximation $\{A_{d,s}\}_{a\in\omega}$, and, if we want this approximation to witness that $A \in \Sigma^{-1}_n$, this may not be allowed if we have already made all available n changes, and we have ended up with $A_d(a) = 1$ (necessarily, in this case, $a = b_i$). Notice however that this can not happen if n is even: in this case, if we have exhausted all allowed changes, then we have acted n times to satisfy P_i, hence $V_i(b_i)$ has changed $n-1$ times, and its final value is 1, so the final value for A_d is $A_d(b_i) = 0$, and thus freezing does not introduce any new change for $A_d(b_i)$.

So, we can state the following:

Theorem 2.5.25. *For every $n \geq 2$ we can build a dialectical system $d = \langle H, f, c\rangle$, and a computable approximation $\alpha = \{H_s\}_{s\in\omega}$ to H, such that A_d is not $(n-1)$-c.e., and if $\{A_{d,s} : s \in \omega\}$ is the approximation to A_d given by the sets of provisional theses (corresponding to α), then*

1. *if n is even then for every y,*

$$|\{s : A_{d,s}(f_y) \neq A_{d,s+1}(f_y)\}| \leq n;$$

2. *if n is odd then for every y,*

$$|\{s : A_{d,s}(f_y) \neq A_{d,s+1}(f_y)\}| \leq n + 1.$$

Proof. We build $d = \langle H, f, c\rangle$ by building H, whereas f is the identity function and $c = 1$. Given any even $n > 0$, construct H by stages as follows:

Stage 0. Initialize all P_e. Let

$$H_0 = \{\langle x, \{c\}\rangle : x \in \omega\} \cup \{\langle 0, \emptyset\rangle\} \cup \{\langle x, \{x\}\rangle : x \in \omega\}.$$

Stage $s+1$. Let e be the least number such that P_e requires attention: notice that there always is such an e, since at every stage almost all requirements are initialized.

1. If P_e is initialized at the beginning of stage $s+1$, then let $a_{e,s+1} > 1$ be the least unused number, let $I(e, s+1) = [a_{e,s+1}, a_{e,s+1} + n - 1]$, $b_{e,s+1} = a_{e,s+1} + n - 1$; declare $i_{e,s+1} = n - 1$; *put P_e in standby*;
2. if P_e is in standby, and $b_{e,s} \notin A_{d,s}$, then keep P_e in standby; if $b_{e,s} \in A_{d,s}$ then *P_e ceases to be in standby*;
3. otherwise:
   (a) if $b_{e,s} \in V_{e,s}$ (necessarily, $i > 0$), then add $\langle c, \{a_{e,s} + j, b_{e,s} : j \leq i\}\rangle \in H$; declare $i_{e,s+1} = i_{e,s} - 1$;
   (b) if $b_{e,s} \notin V_{e,s}$ (necessarily, $i > 0$), then add $\langle c, \{a_{e,s} + i\}\rangle \in H$; declare $i_{e,s+1} = i_{e,s} - 1$; *put P_e in standby.*

(Notice that thanks to the standby procedure, there is now a perfect synchronism between the action of P_e and the way the dialectical procedure moves the elements of $I(e, s)$, if P_e is no longer initialized.) After acting for P_e, initialize all P_i with $i > e$; for every $a > b_e$ such that a has been used in the construction (for instance $a \in I(i, s)$ with $i > e$) then add the axiom $\langle c, \{a\}\rangle \in H$: we call the addition of these axioms the *freezing procedure*. Let H_{s+1} be H_s plus the axioms added for H at stage $s + 1$. Go to stage $s + 2$.

The verification easily follows from:

Lemma 2.5.26. *For every e, there is a least stage s_e such that, for every $s \geq s_e$, $a_{e,s} = a_{e,s_e}$ (consequently, $I(e, s) = I(e, s_e)$ and $b_{e,s} = b_{e,s_e}$), P_e does not receive attention at stage s, and P_e is satisfied.*

Proof. By induction on e. Let t_e be the least stage after which all parameters relative to any P_i, with $i < e$, have settled down, and P_i does not require attention after t_e. So at stage $t_e + 1$, P_e requires attention, we choose the final value $[a_e, a_e + n - 1]$ of its witness. After this stage, P_e may require attention at most finitely many times. Therefore, the existence of s_e has been demonstrated. Let us call I_e, a_e, and b_e the limit values of the parameters $I(e, s), a_{e,s}, b_{e,s}$. We can repeat for the final values $I(e), a_e$ and b_e the same argument as for the witnesses for P_e in the proof of Theorem 2.5.5: in particular, as explained in the section on analysis of outcomes for the strategy for P_e in the proof of Theorem 2.5.5, the axioms which we have placed in H enable us to move $b(e)$ in and out of $A_{d,s}$, as many times we need to get eventually diagonalization of $A_d(b_e)$ against $V_e(b_e)$. □

Lemma 2.5.27. *If n is even then for every x, $A_{d,s}(x)$ can change at most n-times; if n is odd then for every x, $A_{d,s}(x)$ can change at most $n + 1$-times.*

Proof. This is clear by the discussion on interactions between strategies, which precedes the theorem. Notice that if x lies in some final value $I(e)$, then $A_{d,s}(x)$ can change at most n-times, as the components of I_e make at most the same number of moves as the components of the corresponding set $I(e)$ in the proof of Theorem 2.5.5. If $x \in I(e, s_0)$, for some e, s_0 such that $I(e, s_0)$ is later discarded, then x can move at most n times before $I(e, s_0)$ is discarded, and then x is frozen, which may bring to $n+1$ the final number of changes, if n is odd. Otherwise $A_{d,s}(x)$ can change from 0 to 1 if x is not frozen, or from 0 to 1 and back to 0 if x is frozen. $A_{d,s}(c)$ never changes. □

Lemma 2.5.28. *$A_d = A_{d'}$, where $d' = \langle H^\omega, f, c \rangle$.*

Proof. As in Lemma 2.5.11. □

This concludes the proof of Theorem 2.5.25. □

Quasi-dialectical approximations. Let us now tackle the case of quasi-dialectical sets. Since every dialectical set is a quasi-dialectical set ([Amidei et al., a, Lemma 3.6]), Theorem 2.5.25 ipso facto extends to quasi-dialectical sets. We now consider the issue of whether we can stretch the proof of Theorem 2.5.14 to get proper loopless quasi-dialectical sets whose membership in the appropriate level of the Ershov hierarchy is witnessed by a quasi-dialectical approximation.

We start with the case of the infinite levels of the Ershov hierarchy.

Theorem 2.5.29. *For every notation $a \in O$, with $|a|_O \geq \omega$, there is a proper loopless quasi-dialectical system $q = \langle H, f, f^-, c, c^- \rangle$ such that A_q is properly Σ^{-1}_a, and if $g(x, s)$ is the approximation to A_q given by the sets of provisional theses, then there is a computable $h(x, s)$ such that the pair $\langle g, h \rangle$ witnesses the fact that $A_q \in \Sigma^{-1}_a \smallsetminus \bigcup_{b <_O a} \Sigma^{-1}_b$.*

Proof. As in the case of Theorem 2.5.25 we basically insert priority in the proof of Theorem 2.5.14, with the addition of the "freezing procedure" at the end of each stage, for all discarded witnesses. Throughout the rest of the proof, we refer to notations and terminology as in Theorem 2.5.14: in particular $n = p(e, b)$, and in order to satisfy P_n, we must diagonalize A_q against $Z_{p(e,b)} \in \Sigma^{-1}_b$.

We build $q = \langle H, f, f^-, c, c^- \rangle$ by building H by stages, whereas f is the identity function, $f^-(x) = 3x$, $c = 1$, $c^- = 2$. We construct H by stages, and the quasi-dialectical procedure that we have in mind for the system $q = \langle H, f, f^-, c, c^- \rangle$ refers to the computable approximations to H, defined during the construction.

We say that a requirement is *initialized* if all parameters relative to P_n are undefined. Similarly to the proof of Theorem 2.5.25, a requirement P_n may be in *standby* if it has acted to put the component $x_{n,s}$ of its witness in the set of provisional theses and it is just waiting for the quasi-dialectical procedure to comply with this action: the main difference, compared to the proof of Theorem 2.5.25, (assuming that we work at stages after which P_n will no longer be initialized) is that when now P_n is put in standby for the first time then (as in Theorem 2.5.25) we may have to wait several stages to see x_n proposed and put into the set of provisional theses; on the other hand for future cycles of the standby procedure we have to wait only one stage for the quasi-dialectical procedure to propose a previously extracted x_n and put it back in the provisional theses.

We say that P_n *requires attention* at s, if $s > 0$, and (in the order) either P_n is initialized, or P_n is in standby, or $x_{n,s} \in A_{q,s-1}$ if and only if $x_{n,s} \in Z_{n,s}$.

Compared to the proof of Theorem 2.5.14, there is an additional parameter to consider: for every n, s, with $n = p(e, b)$, let

$$k_n(x,s) = \begin{cases} 2, & \text{if } |b|_O \text{ is finite or } (\exists u <_O b)[|u|_O \text{ limit } \& \, h_n(x,s) <_O u] \\ 1, & \text{otherwise.} \end{cases}$$

(Recall that 2 is the notation of the ordinal 1, and 1 is the notation of the ordinal 0.) It is not difficult to see that the function $k_n(x,s)$ is computable. Indeed, to compute $k_n(x,s)$, if $|b|_O$ is not finite, one checks the values $h_n(x,t)$, for $t \leq s$: if one finds the least $t < s$ such that $h_n(x,t) \geq_O u$, for some $u \in O$ with $|u|_O$ limit, and $h(x,t) <_O u$, then $k(x,s) = 2$; otherwise $k(x,s) = 1$.

Stage 0**.** Initialize all P_n. Let

$$H_0 = \{\langle x, \{c\}\rangle : x \in \omega\} \cup \{\langle 0, \emptyset\rangle\} \cup \{\langle x, \{x\}\rangle : x \in \omega\}.$$

For every x, n let $h(x, 0) = a$, $k_n(x, 0) = 1$, $i_{n,0} =\uparrow$.

Stage $s+1$**.** Let $n = p(e, b)$ be the least number such that P_n requires attention: notice that there always is such an n.

1. If P_n is initialized at the beginning of stage $s+1$, then let $y_{n,s+1}, x_{n,s+1} > 0$ be the least unused pair of numbers, such that $x_{n,s+1} = y_{n,s+1} + 1$ and $\{y_{n,s+1}, x_{n,s+1}\} \cap \text{range}(f^-) = \emptyset$; *put P_n in standby*; set $i_{n,s+1} = 0$;

2. if P_n is standby, and $x_{n,s} \notin A_{q,s+1}$ then keep P_n in standby; if $x_{n,s} \in A_{q,s+1}$ then P_n *ceases to be in standby*; we have in this case $x_{n,s} \in A_{q,s+1} \smallsetminus A_{q,s}$: if $h(x_n, s) = a$ then define $h(x_{n,s}, s+1) = b$; otherwise define $h(x_{n,s}, s+1) = h_n(x_{n,s}, s+1) +_O k_n(x_{n,s}, s)$; set $i_{n,s+1} = i_{n,s} + 1$;

3. if $x_{n,s} \in Z_{n,s+1}$, then eliminate $x_{n,s}$ by $y_{n,s}$, i.e. add the axiom $\langle c, \{\rho_s(y_{n,s}), x_{n,s}\}\rangle \in H$. This has the effect of immediately having $x_{n,s} \in A_{q,s} \smallsetminus A_{q,s+1}$. Define $h(x_{n,s}, s+1) = h_n(x_{n,s}, s+1) +_O k_n(x_{n,s}, s+1)$;

4. if $x_{n,s} \notin Z_{n,s+1}$ then recover $x_{n,s}$ by $y_{n,s}$, i.e. add $\langle c^-, \{\rho_s(y_{n,s})\}\rangle \in H$; *put P_n in standby*; set $i_{n,s+1} = i_{n,s} + 1$.

(Notice that thanks to the standby procedure, there is now a perfect synchronism between the action of P_n and the elimination/recovery mechanism for $y_{n,s}, x_{n,s}$, if P_n is no longer initialized.) After acting for P_n, initialize all P_i with $i > n$: for each $a > x_{n,s}$ such that a has been used in the construction (so that $h(a,s) \neq a$) we *freeze* a out of A_q, by adding the axiom $\langle c, \{a\}\rangle \in H$, and defining $h(a, s+1) = 1$.

If a is any number such that $a \in A_{q,s+1}$ and $h(a,s) = a$ then define $h(a,s+1) = 2$. Define also $h(0,s+1) = 1$, and $h(c,s+1) = h(c^-,s+1) = 1$. Let H_{s+1} be H_s plus the axioms added to H at stage $s+1$. All parameters that have not been explicitly redefined maintain the same values as at the previous stage. Go to stage $s+2$.

The verification easily follows from the following lemmata:

Lemma 2.5.30. *For every n, there is a least stage s_n such that, for every $s \geq s_n$, $x_{n,s} = x_{n,s_n}$ (consequently, $I(n,s) = I(n,s_n)$ and $y_{n,s} = y_{n,s_n}$), P_n does not receive attention at stage s, and P_n is satisfied.*

Proof. By induction on n. Let t_n be the least stage after which all parameters relative to any P_i, with $i < n$, have settled down, and P_i does not require attention anymore. So at stage $t_n + 1$, P_n requires attention, we choose the final value $I(n) = [y_n, x_n]$ of its witness.

After this stage, P_n may require attention only finitely many times. Therefore, the existence of s_n has been demonstrated. After the least stage at which $I(n)$ has reached its limit, the witness $I(n)$ behaves exactly as the witness $I(n)$ in the proof of Theorem 2.5.14, except for the delaying effect of the "standby" feature. Thus P_n is eventually satisfied. □

Lemma 2.5.31. *Let g be the approximation to A_q given by the sets of provisional theses, i.e.,*

$$g(x,s) = \begin{cases} 1 & \text{if } x \in A_{q,s} \\ 0 & \text{if } x \notin A_{q,s}. \end{cases}$$

Then, the pair $\langle g, h\rangle$ witnesses the fact that A_q is properly in Σ^{-1}_a.

Proof. The claim has been achieved by synchronizing the changes of g with corresponding decreases of h. Indeed, consider first the case of $h(x,s)$ where $x = x_{i,s_0} \in I(i,s_0)$, for some i, s_0, and s_0 is the least stage at with $I(i,s_0)$ is appointed as witness. We claim that whenever $g(x,s+1) \neq g(x,s)$ then $h(x,s+1) <_O h(x,s)$, and, until $I(i,s_0)$ is discarded, for all s, $h(x,s) \geq_O h_i(x,s)$, and if $h(i,s)$ is $<_O$ a notation $u <_O b$, such that $|u|_O$ is limit, then $h(x,s) >_O h_i(x,s)$. To see this, first of all notice that $h(x,0) = a >_O h_i(x,0) = b$. Next change of $g(x,s)$ is at, say, s_0, when we put $h(x,s_0) = b \geq_O h_i(x,s_0)$. Suppose now by induction that the claim is true up to stage s_1, and suppose that $g(x,s_1+1) \neq g(x,s_1)$: this is due to the fact that the strategy has responded to a change of $g_i(x_i,s)$ which has taken place between the last stage t, for which we have $h(x_i,s_1) = h_i(x_n,t)$, and s_1+1, and thus we redefine $h(x,s_1+1) = h_i(x,s_1+1) +_O k_i(x,s_1+1)$. If $h_i(x,s_1+1)$ has not dropped below a notation of a limit ordinal, then trivially $h(x,s_1+1) \geq_O h_i(x,s_1+1)$; otherwise $k_i(x,s_1+1) = 2$, and thus $h(x,s_1+1) >_O h_i(x,s_1+1)$. From now on, until $I(i,s_0)$ is discarded, it is easy to see that $h(x,s+1) >_O h_i(x,s+1)$. Moreover in the case

$k_i(x, s+1) = 2$, whether or not $k_i(x, s) = 1$ or $k_i(x, s) = 2$, it is easy to see that $h(x, s) >_O h(x, s+1)$.

If and when $I(i, s_0)$ is discarded at, say $s_2 + 1$, then we have room for freezing x, with an extra change of $h(x, s_2 + 1)$. Indeed, up to that moment either $h(x, s+1) \in \{a, b\}$, or $k_i(x, s_2) = 1$, and thus $h(x, s_2) >_O 1$ (in fact $h(x, s_2) \geq_O g_i(x, s_2) \geq_O u$, where $u <_O b$ is the notation of the greatest limit ordinal below $|b|_O$); or, $h(x, s_2) = g_i(x, s_2) +_O k_i(x, s_2)$, with $k_i(x, s_2) = 2$, and thus $h(x, s_2) >_O 1$.

As to numbers x which are never appointed as $x = x_{i,s_0}$, for any i, s_0, the claim is easy to show. Indeed, for any such x, one of the following holds: either x never enters a set of provisional theses, and thus there is no problem for a possible freezing action; or (and this is the case for instance, for numbers of the form $\rho_s(y_n)$ that enter elimination/recovery activities) x enters some set of provisional theses, at say $t_0 + 1$, at which point we set $h(x, t_0 + 1) = 2$, and thus there is room for a possible future freezing action. □

Lemma 2.5.32. *There are a proper loopless quasi-dialectical system $q' = \langle H', f, f^-, c, c^- \rangle$, where H' is a closure operator, such that $A_q = A_{q'}$.*

Proof. As in Lemma 2.5.19. □

This concludes the proof of the theorem. □

Finally, we prove:

Corollary 2.5.33. *For every finite $n \geq 2$ we can build a proper loopless quasi-dialectical system $q = \langle H, f, f^-, c, c^- \rangle$, and a computable approximation $\{H_s\}_{s \in \omega}$ to H, such that A_q is not $(n-1)$-c.e., and if $\{A_{q,s} : s \in \omega\}$ is the approximation to A_q given by the sets of provisional theses (corresponding to the built approximation to H), then*

1. *if n is even then for every y,*

$$|\{s : A_{q,s}(f_y) \neq A_{q,s+1}(f_y)\}| \leq n;$$

2. *if n is odd then for every y,*

$$|\{s : A_{q,s}(f_y) \neq A_{q,s+1}(f_y)\}| \leq n + 1.$$

Proof. The proof goes as the proof of the previous theorem, by taking $b = n - 1$, $\{Z_e\}_{e \in \omega}$ an effective listing of the $(n-1)$-c.e. sets, and

$$a = \begin{cases} n, & \text{if } n \text{ is even,} \\ n+1, & \text{if } n \text{ is odd.} \end{cases}$$

Of course for all e, x, s, we have in this case $k_e(x, s) = 1$. □

2.6 Conclusions

This chapter has been mainly concerned with comparing dialectical and quasi-dialectical systems with respect to both their information content and their deductive power. We have shown that dialectical sets and quasi-dialectical sets have the same Turing-degrees, and the same enumeration degrees. Nonetheless, the class of dialectical sets is properly contained in the class of quasi-dialectical sets, and in fact the latter is much larger than the former.

Of course many interesting problems remain untouched. In particular, recall that Magari introduced dialectical systems in order to provide a simple - yet expressive - logical model for representing the (dynamic) behavior of mathematical theories. Hence, it comes naturally to ask if such a relationship between (quasi-)dialectical systems and formal theories can be better clarified. In this regard, let us conclude by hinting at two possible directions of research – first introduced in [Magari, 1974] and [Bernardi, 1974] – one can take to investigate this problem. Firstly, given a system S (that could be either dialectical or quasi-dialectical) it is possible to dismiss some pieces of the generality of its deduction operator H, by adding particular constraints that aim at mimicking logical connectives, thus making the behavior of S somewhat closer to the one expressed by classical deduction rules. Secondly, we have already mentioned that it is possible to associate to each formal theory T, dialectical systems $d = \langle H, f, c\rangle$ such that A_d is a completion of T. Thus, one could try to study completions of (essentially undecidable) theories in terms of dialectical and quasi-dialectical sets. These lines of research will be pursued in a forthcoming work.

Chapter 3

Universal arithmetical binary relations and graphs

3.1 Introduction

One of the most fundamental tasks of contemporary mathematics is that of classifying structures in terms of their complexity. For instance, a main problem in graph theory, the so-called "subgraph isomorphism problem", consists in determining, given two graphs G and H, whether G contains a subgraph isomorphic to H. If so, one might informally say that G is structurally at least as complex as H. As is clear, similar problems arise everywhere in mathematics. To a first approximation, it can be said that most problems concerning the embedding of a structure into another fall in the same category.

Therefore, the general goal of developing a convenient formal frame, in which this kind of problems can be fruitfully encoded, comes naturally. Notice that such a goal is particularly sound with a recursion-theoretic perspective, being reducibilities the core of Classical Computability Theory, and also because in this context we can make sense, in a very precise way, of the idea of ordering objects according to their relative complexity. Thus, having this goal in mind, consider the following reducibility (with corresponding degrees introduced as usual):

Definition 3.1.1. Let R and S be two binary relations on ω. We say that R is *computably reducible* to S iff there is a computable function f such that, for all $x, y \in \omega$, the following holds:

$$xRy \Leftrightarrow f(x)Sf(y).$$

Computable reducibility has been object of study for decades, being mostly applied to the case of equivalence relations. Yet, its history is somewhat intricate: proof of this is the fact that it goes under several different names in literature, none being fixed once for all (the present one is borrowed from [Coskey et al., 2012]).

Its first definition appeared in [Ershov, 1977], where computable reducibility is formulated within the context of the theory of numberings, with the aim of studying some recursion-theoretic concepts "from a global point of view". In the Eighties, scholars continued Ershov's work by pursuing different goals, such as studying provable equivalence among formal systems (see, for instance, [Visser, 1980], [Montagna, 1982], [Bernardi and Sorbi, 1983]). This latter goal explains why their work has been mostly devoted to the case of computable enumerable equivalence relations (later called "ceers" in [Gao and Gerdes, 2001]). In particular, a problem emerged as prominent, that of characterizing universal ceers, i.e. those ceers to which all others can be computably reduced; a problem whose significance and complexity are both increased by the fact that the class of universal ceers contains distinct computable isomorphism types see [Lachlan, 1987]. More recently, the theory of universal ceers has been considerably expanded in [Andrews et al., 2014], where authors provide an almost full characterization of the universal degree for ceers.

Meanwhile, computable reducibility has been also approached with respect to two other lines of research: as representing a computable analogue of the so-called Borel reducibility, a widely studied notion in Descriptive Set Theory (see, for instance, [Gao and Gerdes, 2001] and [Coskey et al., 2012]; and in comparing those equivalence relations – such as the isomorphism relation – that can exist between computable structures (see, for instance, [Fokina et al., 2012] and [Fokina and Friedman, 2012], where our notion is referred to as "FF-reducibility"). Notice that, in these latter contexts, the equivalence relations considered are not limited to the Σ^0_1 case, but they do rather lie in higher levels of the arithmetical hierarchy, or even outside of it, as the Σ^1_1 ones considered in [Fokina and Friedman, 2012].

Thus, while authors reached the same notion from different directions, nonetheless computable reducibility has been employed mostly in the case of equivalence relations. To our knowledge, the only exception has been that of generalizing from equivalence relations to preorders on ω, as made by [Montagna and Sorbi, 1985] and [Ianovski et al., 2014]. In the present work, we aim to enlarge this perspective by determining whether universal relations exist in the context of general binary relations, and then by considering intermediate cases between this latter context and that of equivalence relations and preorders. In doing so, we have two main motivations. First, in [Ianovski et al., 2014] authors unveil a severe limitation to the existence of universal arithmetical equivalence relations: for $n \geq 2$, there is no universal Π^0_n equivalence relation. Moreover, they show that the same holds also for preorders. So it is immediate to ask whether this limitation is preserved if one weakens the kind of relation considered. This is the main question of our work. In addition, by approaching this latter question in the most general case, that of general binary relations, we show how the universal binary relations we build are – rather than artificial – very natural ones, and that their investigation is interesting for its own sake.

3.1.1 Preliminaries

We assume familiarity with the basic notions of Computability. If needed, the reader is referred to [Soare, 1987]. In particular, by A' we denote *the jump of* A, i.e. the set $\{x \mid \Phi_x^A(x)\downarrow\}$; the n-th jump of A is the result of iterating the jump n times, i.e. $A^0 = A$ and $A^{(n)} = (A^{(n-1)})'$.

All relations we consider in this work are binary relations on ω. Equivalence relations are of course reflexive symmetric transitive binary relations. In addition to the general case of binary relations, we consider two natural generalizations of equivalence relations: graphs, i.e. symmetric binary relations; and preorders, i.e. reflexive transitive binary relations. Finally, $\forall^\infty(x)$ means "for all but finitely many x".

3.1.2 Initials remarks on universality

Let us provide the definition of universality we are interested in:

Definition 3.1.2. Let $\mathcal{R}$ be a family of binary relations of given arithmetical complexity. We say that S is *universal* with respect to $\mathcal{R}$, if the two following hold:

1. $S \in \mathcal{R}$;
2. for all $R \in \mathcal{R}$, $R \leq S$.

If we ask $\mathcal{R}$ to be the class of equivalence relations lying at some level of the arithmetical hierarchy (for instance, $\mathcal{R}$ might be the class of ceers), then we do obtain the notion of universality that typically appears in the literature cited in the introduction. For equivalence relations, the problem of finding a universal one has different solutions between the Σ^0_n levels of arithmetical hierarchy and the Π^0_n ones. Indeed, it easy to see that for each n there is a universal Σ^0_n equivalence relation, while in [Ianovski et al., 2014], authors prove that there is no universal Π^0_n equivalence relation for $n \geq 2$. Let us show how the easy argument for the Σ^0_n case immediately generalizes to the case of all binary relations.

First, it can be easily shown that, for each n, there is an effective enumeration of all the Σ^0_n binary relations. This is because these latter are just the binary relations that can be computed by a Turing machine with oracle $\emptyset^{(n)}$. So let $(V_e)_{e\in\omega}$ be an acceptable list of all Σ^0_n binary relations, and let $E = \bigoplus_e V_e$: it is immediate to see that E is universal with respect to Σ^0_n equivalence relations. At times, E is called the *cylinder* of the V_e's.

While this argument is effortlessly transferred from equivalence relations to binary relations (and, in fact, it applies to any case in which $\mathcal{R}$ can be effectively enumerated), nonetheless the analogy breaks down when considering the Π^0_n case. Indeed, the following hold:

Fact 3.1.3. *Let U be a universal Σ^0_n (resp. Π^0_n) binary relation. Then the complement of U, U^c, is a universal Π^0_n (resp. Σ^0_n) binary relation.*

Proof. The proof is immediate. We limit ourselves to the case in which U is Σ_n^0: the other case is basically the same. Let R be a Π_n^0 binary relation. We aim to prove that $R \leq U^c$. There are two cases to be considered:

1. If xRy, then $x \not R^c y$. Since U is universal on Σ_n^0 relations, this means that there is a computable function g such that $g(x) \not U g(y)$. Therefore, it must be $g(x)U^c g(y)$. Hence, xRy implies $g(x)U^c g(y)$.

2. If $x \not R y$, the situation is symmetric to the latter case.

So, we have that $R \leq U^c$ via g (which is the same function that reduces R^c to U). $\square$

Notice that one cannot adapt this kind of reasoning to the case of equivalence relations because the complement of an equivalence relation R is not an equivalence relation itself.

Now, by applying Fact 3.1.3 to the existence of cylinders of binary relations for each Σ_n^0 level of the arithmetical hierarchy, we can immediately obtain the following result:

Fact 3.1.4. *There is a universal binary relation at each level of the arithmetical hierarchy.*

Is this latter result enough to put aside the topic of universality for arithmetical binary relations? Of course not. The goal is rather that of refining this kind of result with the best possible characterization of universal relations. In the context of equivalence relations, such a characterization typically leads to two complementary threads of research:

1. Providing natural examples of universal equivalence relations (i.e. much more informative than the cylinders): for instance, in [Ianovski et al., 2014] it is proven that $\{\langle i, j\rangle \mid W_i \equiv_T W_j\}$ is a universal Σ_4^0 equivalence relation;

2. Describing non-trivial properties that imply universality: for instance, in [Andrews et al., 2014] are considered some different properties a ceer may or may not satisfy, all corresponding to different isomorphism types within the universal degree for ceers.

In the present work, we put ourselves in line with 1. That is to say, we provide – for each level of the arithmetical hierarchy – a universal binary relation whose definition is, in a certain sense, natural, i.e. independent from the construction by which we prove its universality. In particular, for all $n > 2$, we prove that $\{\langle i, j\rangle \mid W_i \subseteq W_j^{(n-2)}\}$ is a universal Π_n^0 binary relation (while, by Fact 3.1.3, its complement is a universal Σ_n^0 binary relation).

3.2 Universal arithmetical binary relations

We begin by providing an example of a natural universal Σ^0_1 binary relation, whose definition shall be regarded as a low-level analogue for the kind of relations we introduce afterwards.

3.2.1 A universal computably enumerable binary relation

Consider the following c.e. binary relation on ω:

Definition 3.2.1. Let $U^{\exists}_1$ be defined as follows:

$$xU^{\exists}_1 y \Leftrightarrow x \in W_y.$$

It is clear that $U^{\exists}_1$ is c.e. Furthermore, we can prove the following theorem:

Theorem 3.2.2. *$U^{\exists}_1$ is universal with respect to c.e. binary relations.*

Proof. Let R be a c.e. binary relation. We aim to build, by steps, a computable function g such that $R \leq U^{\exists}_1$ via g. Let g be a computable function such that for all y:

$$W_{g(y)} = \{g(x) \mid xRy\}.$$

To prove that such a function exists, consider the following:

$$W_{\varphi_{f(z)}(y)} = \{\varphi_z(x) \mid xRy\}.$$

By the Fixed-Point-Theorem, we have that there is n, such that for every y

$$W_{\varphi_{f(n)}(y)} = W_{\varphi_n(y)}.$$

To conclude, just denote φ_n by g.

Now suppose xRy. Therefore, by the definition of $W_{g(y)}$ above, $g(x) \in W_{g(y)}$, and so $g(x)U^{\exists}_1 g(y)$. Conversely, if $x\not{R}y$, then, by definition of $W_{g(y)}$ again, $W_{g(y)}$ does not contain $g(x)$, and so $g(x)\not{U}^{\exists}_1 g(y)$. $\square$

Clearly, by applying Fact 3.1.3, one can obtain a universal Π^0_1 binary relation by simply negating $U^{\exists}_1$, i.e. let $U^{\forall}_1$ be so that

$$xU^{\forall}_1 y \Leftrightarrow x \notin W_y,$$

then we have the following corollary:

Corollary 3.2.3. *$U^{\forall}_1$ is a co-c.e. universal binary relation.*

Let us then move to Π^0_2 relations.

3.2.2 A universal Π^0_2 binary relation

In this section, we construct a universal Π^0_2 binary relation, thus exhibiting a first (non-trivial) difference between the general case of binary relations and that of equivalence relations.

Theorem 3.2.4. *There is a computable function f such that the binary relation so defined*

$$xU^{\forall}_2 y \Leftrightarrow W_x \subseteq W_{f(y)}$$

is universal with respect to Π^0_2 relations.

Proof. It is immediate to verify that $U^{\forall}_2$ is Π^0_2. Then, let us recall that, for each Π^0_2 binary relation R, there is a computable function h such that the following holds:

$$xRy \Leftrightarrow W_{h(x,y)} \text{ is infinite.}$$

Call such a function a Π^0_2-*approximation* to R. Furthermore, suppose at stage s a new element is listed in $W_{h(x,y)}$. If so, we say that s is an *expansionary stage* for the pair (x, y).

Construction

Let R be a Π^0_2 binary relation and let h be a Π^0_2-approximation to it. We have to show that $R \leq U^{\forall}_2$. In doing so, let $p : \omega^3 \to \omega$ be an injective computable function such that range(p) is computable (for instance, take as p the Cantor pairing function from ω^3 to ω). Then, we fix the following notation and terminology. First, denote $p(a, b, n)$ by $\langle a, b\rangle_n$, and call this latter a *witness* of (a, b). For each $\langle a, b\rangle_n$, we call a its *left-side* and b its *right-side*. Finally, we call the set $\{\langle a, b\rangle_i \mid i \in \omega\}$ *the column of* (a, b).

Now we have to define two functions, g and f, witnessing the reduction of R into $U^{\forall}_2$. For every x, let $g(x)$ be so that $W_{g(x)}$ is the following computable set:

$$W_{g(x)} = \{\langle x, b\rangle_n \mid b, n \in \omega\}.$$

So $W_{g(x)}$ contains all the witnesses that have x as their left-sides. Then, let us define f: f is a computable function from (indices of) c.e. sets to (indices of) c.e. sets. In particular, given a c.e. set W_x, $W_{f(x)}$ is enumerated as follows:

1. Find the first t such that $t \in W_x \cap \text{range}(p)$, and call y the left-side of t. If there is no such t, let $W_{f(x)}$ be $\emptyset$.

2. Then, for each pair (c, d),

   (a) If $d \neq y$, enumerate $\{\langle c, d\rangle_n \mid n \in \omega\}$ in $W_{f(x)}$;

   (b) If $d = y$, then enumerate the column of (c, d) into $W_{f(x)}$ as long as new elements are added in $W_{h(c,d)}$. That is to say, for all stages s, if s is expansionary for (c, d), then put the set $\{\langle c, d\rangle_m \mid m \leq s\}$ in $W_{f(x)}$.

Verification

We have to show that xRy iff $W_{g(x)} \subseteq W_{f(g(y))}$. First, suppose xRy and let $z \in W_{g(x)}$. From the definition of $W_{g(x)}$, it follows that there must be positive integers a, m, such that $z = \langle x, a \rangle_m$. Now, consider two cases. If $a \neq y$ then, by the construction (action 2.a), $W_{f(g(y))}$ does contain the whole column of (x, a), and therefore $\langle x, a \rangle_m$ has to be in $W_{f(g(y))}$. Otherwise, let $a = y$. In this case, since xRy, we have that $W_{h(x,y)}$ is an infinite set. So, the pair (x, y) has infinitely many expansionary stages. In particular, there must be an expansionary stage s for (x, y), with $s \geq m$. Thus, at stage s, we enter in action 2.b of the construction, and we put $\{\langle x, y \rangle_n \mid n \leq s\}$ in $W_{f(g(y))}$, hence witnessing that $z = \langle x, y \rangle_m \in W_{f(g(y))}$.

Conversely, suppose $x \not{R} y$. If so, consider the column of (x, y). By construction, we have that this whole column is contained in $W_{g(x)}$. Nonetheless, since $x \not{R} y$, we have that $W_{h(x,y)}$ is finite. This means that (x, y) has finitely many expansionary stages. Therefore, there must be n such that, for all $m > n$, we do not put any $\langle x, y \rangle_m$ into $W_{f(g(y))}$. Thus, $W_{g(x)} \not\subseteq W_{f(g(y))}$. $\square$

So, although there is no universal Π^0_2 equivalence relation (as shown in [Nies et al., 2014]), we have that $U^{\forall}_2$ is a universal Π^0_2 binary relation (to which, of course, all Π^0_2 equivalence relations are reducible). Then, one shall ask if such a disarray between these two contexts is preserved in the higher levels of the arithmetical hierarchy. In what follows, we answer positively to this latter question.

3.2.3 Approximation functions for Π^0_n relations, with $n > 2$

Our main goal is to prove that, for each $n > 2$, $U^{\forall}_n = \{\langle i, j \rangle \mid W_i \subseteq W^{(n-2)}_j\}$ is a universal Π^0_n binary relation. In doing so, we consider three cases separately: that of Π^0_3 relations, the one corresponding to Π^0_4 relations, and a last case subsuming all others Π^0_n binary relations with $n > 4$. In all these contexts, while proving that a Π^0_n relation R is reducible to $U^{\forall}_n$, we rely on a specific "approximation function" (which generalizes the notion already employed in the Π^0_2 case), i.e. a computable function that approximates our knowledge – for all pairs x, y – of whether xRy.

Thus, let us briefly introduce such approximation functions for further reference. First, recall the following characterization of Π^0_3 binary relations.

Fact 3.2.5. *R is Π^0_3 binary relation iff there is a computable function h such that*

$$xRy \Leftrightarrow \forall z(W_{h(x,y,z)} \text{ is finite})$$

and

$$x \not{R} y \Leftrightarrow \forall^{\infty} z(W_{h(x,y,z)} \text{ is infinite}).$$

Call h a Π^0_3-approximation *to R.*

Proof. See, with minor modifications, [Soare, 1987, p. 68]. □

As an easy consequence of this latter fact, the following characterization for Π^0_4 binary relations is immediately obtained:

Corollary 3.2.6. *R is a Π^0_4 binary relation iff there is a computable function h such that*

$$xRy \Leftrightarrow \forall z \forall^\infty t (W_{h(x,y,z,t)} \text{ is infinite})$$

and

$$x \not R y \Leftrightarrow \exists z \forall t (W_{h(x,y,z,t)} \text{ is finite}).$$

Call h a Π^0_4-approximation *to R.*

Proof. ($\Rightarrow$) Let R^* be the following set

$$\langle x, y \rangle \in R^* \Leftrightarrow xRy.$$

Suppose xRy. Thus, we have $\langle x, y\rangle \in R^*$. Now, since R^* is a Π^0_4 set, then there must be a Σ^0_3 binary relation S such that $\forall z(S(\langle x, y\rangle, z))$. But then, by Fact 3.2.5, we have that there is a computable function g such that $S(\langle x, y\rangle, z)$ holds iff $\forall^\infty t(W_{g(\langle x,y\rangle,z,t)}$ is infinite). Furthermore, from g, one can clearly define a computable function h such that $S(\langle x, y\rangle, z)$ holds iff $\forall^\infty t(W_{h(x,y,z,t)}$ is infinite). Therefore, if xRy, then $\forall z \forall^\infty z(W_{h(x,y,z,t)}$ is infinite).

The case in which $x \not R y$ is symmetric.

($\Leftarrow$) Obvious. □

Clearly it is possible to generalize Fact 3.2.5 much further and obtain an analogue characterization for each Π^0_n level of the arithmetical hierarchy. Indeed:

Corollary 3.2.7. *The two following facts hold:*

1. *R is a Π^0_{2k} binary relation iff there is a computable function h, such that*

$$xRy \Leftrightarrow \forall z_1 \exists z_2 \forall z_3 \ldots \forall^\infty z_{2k-2} (W_{h(x,y,z_1,z_2,z_3 \ldots z_{2k-2})} \text{ is infinite})$$

and

$$x \not R y \Leftrightarrow \exists z_1 \forall z_2 \exists z_3 \ldots \forall z_{2k-2} (W_{h(x,y,z_1,z_2,z_3 \ldots z_{2k-2})} \text{ is finite}).$$

2. R is a Π^0_{2k+1} binary relation iff there is a computable function h, such that

$$xRy \Leftrightarrow \forall z_1 \exists z_2, \ldots, \forall z_{2k-1}(W_{h(x,y,z_1,z_2 \ldots z_{2k-1})} \text{ is finite})$$

and

$$x \not R y \Leftrightarrow \exists z_1 \forall z_2, \ldots, \forall^\infty z_{2k-1}(W_{h(x,y,z_1,z_2 \ldots z_{2k-1})} \text{ is infinite}).$$

*In any case, call h a Π^0_n-*approximation *to R.*

Proof. The proof is a straightforward generalization of that of Corollary 3.2.6. □

Remark 3.2.8. For the sake of readability, in the exposition of our proofs we will refer to the information provided by h by means of using a somewhat geometric language, which is handier than its purely logical counterpart.

3.2.4 A universal Π^0_3 binary relation

It is time to define $U^\forall_3$.

Definition 3.2.9. Let $U^\forall_3$ be the following binary relation

$$xU^\forall_3 y \Leftrightarrow W_x \subseteq W'_y.$$

$U^\forall_3$ is a rather natural binary relation, representing the inclusion of c.e. sets within jumps of c.e. sets. Thus, to some extent, it might be considered as providing a sort of an analogue – structurally much weaker – of the well-known lattice of c.e. sets ordered by the inclusion $\subseteq$.

Theorem 3.2.10. *$U^\forall_3$ is a universal Π^0_3 binary relation.*

Proof. An easy calculation shows that $U^\forall_3$ is Π^0_3. Moreover, from Fact 3.2.5, we know that, if R is a Π^0_3 relation, then there is a computable function h such that the following holds:

$$xRy \Leftrightarrow \forall z(W_{h(x,y,z)} \text{ is finite}).$$

Recall that we say that h is a Π^0_3-approximation to R.

It is convenient to introduce the following terminology (which also stands as a basic case of the kind of language we shall adopt for Π^0_n relations with $n > 3$). First, we call $W_{h(x,y,m)}$ the m-th *column* of (x, y). Then, suppose that at some stage s, a new element z is listed in $W_{h(x,y,m)}$. If so, we say that s is an *expansionary stage* for the m-th column of (x, y). Thus, we can reformulate the above characterization by saying: xRy iff each column of (x, y) has finitely many expansionary stages.

Idea of the proof

We have to prove that, for any Π^0_3 binary relation R, there is a computable function g such that $R \leq U^{\forall}_3$ via g. In doing so, we build by steps, for every i, a set $W_{g(i)}$ witnessing such reduction.

The idea of the construction is to make use of the following device:

- Suppose that a stage s is expansionary for the m-th column of (i,j). If so, we aim to keep track of this information in $W_{g(j)}$ in such a way that if we act infinitely many times on the same column then a "column-witness" (denoted by $\langle i,j\rangle_m$) is eventually removed from $W'_{g(j)}$, and by this action we aim to guarantee $W_{g(i)} \nsubseteq W'_{g(j)}$.

As is clear, the main difficulty for implementing this device is the fact that we cannot act directly on $W'_{g(j)}$, but in fact we can only proceed by adding elements to $W_{g(j)}$. To overcome this problem, we are going to put in all the c.e. sets we are building (that is, in $\bigcup_{j\in\omega} W_{g(j)}$) only indices of the functional Φ_e defined below.

Definition of Φ_e

Let $p_e(x,y,z)$ be an injective padding function for the Turing functional Φ_e, i.e. for all $w \in \text{range}(p_e)$ we have that, for every A, $\Phi^A_e = \Phi^A_w$.
By the Fixed-Point-Theorem, we have that there is e such that the following computable functional Φ_e with oracle A does exist:

$$\Phi^A_e(x) = \begin{cases} 1 & \exists n(p_e(1,x,n) \notin A) \\ \uparrow & \text{otherwise.} \end{cases}$$

We aim to design our construction in such a way that, for every i, it would be $W_{g(i)} \subseteq \text{range}(p_e)$.

A terminology for Φ_e

Let us now set an helpful terminology. To be honest, this terminology is perhaps avoidable in the present context. However, when considering most complex cases (i.e., higher levels of the arithmetical hierarchy), it will provide us the basis for a notational shorthand.

- We denote $p_e(2,\langle i,j\rangle,m)$ by $\langle i,j\rangle_m$, and we call $\langle i,j\rangle_m$ a *column-witnesses* of the pair (i,j).

- We call any element of the form $p_e(1,x,m)$ an *extension* of x.

Construction

For simplicity, we choose an approximation to h such that any given stage s can be expansionary for at most one column.

We first define, by steps, a family of sets X_i.

Stage $\langle 0, 0\rangle$

For all i, let $X_i = \emptyset$.

Stage $s + 1 = \langle\langle a, i\rangle, t\rangle$

Check if there is $m < s+1$ such that the stage t is expansionary for the m-th column of (a, i). If so, put the set $\{p_e(1, \langle a, i\rangle_m, k) \mid k \leq t\}$ in X_i.

Then, for all i, let

$$W_{g(i)} := X_i \cup \{\langle i, n\rangle_m \mid n, m \in \omega\}.$$

Verification

We have to prove that $iRj \Leftrightarrow g(i)U_3^{\forall}g(j)$. This verification relies on the following lemma:

Lemma 3.2.11. *For each x and i, these two facts hold:*

a) If x is a column-witness in $W_{g(i)}$, then x has left-side i.

b) The set $\{n \mid p_e(1, x, n) \in W_{g(i)}\}$ is not empty iff x is a column-witness with right-side i.

Proof. Both facts follow immediately from the construction. On the one hand, it is immediate to notice that, for all i, a column-witness x is in $W_{g(i)}$ iff $x \in W_{g(i)} \smallsetminus X_i$ (since, by construction, X_i contains only extensions), thus being of the form $\langle i, \_\rangle_{\_}$. This proves a). On the other hand, it is enough to notice that, for all n, $p_e(1, x, n)$ belongs to $W_{g(i)}$ iff $p_e(1, x, n) \in X_i$, and – by definition of X_i – this requires x to be equal to $\langle a, i\rangle_m$ (for some a, m). This proves b). □

Suppose iRj. If so, all the columns of (i, j) have finitely many expansionary stages. Now, suppose x enters in $W_{g(i)}$ at some stage s. We aim to show that $x \in W'_{g(j)}$. Consider two cases.

1. If x is a column-witness, then by a) in Lemma 3.2.11 there must be two positive integers p, m such that $x = \langle i, p\rangle_m$. First, suppose $p \neq j$, and consider $\Phi_x^{W_{g(j)}}(x)$. Since $x \in \text{range}(p_e)$, it must be, for all A, $\Phi_x^A = \Phi_e^A$. Thus, by definition of Φ_e, we have that $\Phi_x^{W_{g(j)}}(x)$ converges iff there exists n such that $p_e(1, x, n) \notin W_{g(j)}$. But, from b) in Lemma 3.2.11 we know that the

set $\{n \mid p_e(1, x, n) \in W_{g(j)}\}$ is empty, since by hypothesis the right-side of x is $p \neq j$. Thus, for all n, $p_e(1, x, n) \notin W_{g(j)}$, and therefore it must be that $\Phi_x^{W_{g(j)}}(x)$ converges, and so $x \in W'_{g(j)}$.

On the other hand, let $x = \langle i, j\rangle_m$ for some m. If so, consider the m-th column of (i, j). From the construction, we see that any $p_e(1, \langle i, j\rangle_m, n)$ enters in $W_{g(j)}$ only if the m-th column of (i, j) has an expansionary stage t with $t \geq n$. But recall that, since iRj, all the columns of (i, j) have only finitely many expansionary stages. Thus, there must be n such that, for all $n' \geq n$, $p_e(1, x, n') \notin W_{g(j)}$. So, by definition of Φ_e (of which x is an index), this means that $\Phi_x^{W_{g(j)}}(x)$ converges, and therefore, $x \in W'_{g(j)}$.

2. If x is not a column-witness, then by b) in Lemma 3.2.11 the set $\{n \mid p_e(1, x, n) \in W_{g(i)}\}$ is empty. Thus, by definition of Φ_e, this fact is enough to guarantee that $\Phi_x^{W_{g(j)}}(x)$ converges, and so $x \in W'_{g(j)}$.

Conversely, if $i \not R j$, then there must be a column m of (i, j) with infinitely many expansionary stages. Then, let $x = \langle i, j\rangle_m$. One can immediately check that x is in $W_{g(i)} \smallsetminus X_i$. Therefore, we have to see whether x belongs also to $W'_{g(j)}$, i.e. whether $\Phi_x^{W_{g(j)}}(x)$ converges. By definition of Φ_e, this can happen only if there exists n such that $p_e(1, \langle i, j\rangle_m, n)$ does not belong to $W_{g(j)}$. But notice from the construction that, for each expansionary stage s of the m-th column of (i, j) we put in $W_{g(j)}$ all the extensions of x up to s. But, since by hypothesis the m-th column of (i, j) has infinitely many expansionary stages, it must be that, eventually, for all n, $p_e(1, \langle i, j\rangle_m, n) \in W_{g(j)}$. Thus, there is no n such that $p_e(\langle 1, x, n\rangle) \notin W_{g(j)}$. So, $x \in W_{g(i)} \smallsetminus W'_{g(j)}$, witnessing that $W_{g(i)} \not\subseteq W'_{g(j)}$. □

3.2.5 A universal Π^0_4 binary relation

Let us then move to the case of Π^0_4 binary relations. As in the previous case, we first need to make appeal to the approximation functions. In doing so, we recall Corollary 3.2.6, which provides the characterization of Π^0_4 relations that we aim to use shortly thereafter:

$R \in \Pi^0_4$ iff there is a computable function h such that

$$xRy \Leftrightarrow \forall z \forall^\infty t(W_{h(x,y,z,t)} \text{ is infinite})$$

and

$$x \not R y \Leftrightarrow \exists z \forall t(W_{h(x,y,z,t)} \text{ is finite}).$$

We call such an h a Π^0_4-*approximation* to R.

Our proof below is based on this latter characterization. In particular, we aim to adopt a three-dimensional analogue of the kind of language we already employed in the case of Π^0_3 binary relation. That is to say, in what follows we think of h as assigning to each pair (x, y) an infinite set of "floors" (that one might think as forming a cylinder), with each floor consisting of infinitely many "columns" that can be either finite or infinite. Thus, according to this language, the characterization above informally states the following: If xRy, then each floor of the pair (x, y) contains almost all infinite columns; otherwise, there is at least one floor containing only finite columns.

Let us make this latter terminology more precise. Let R be a Π^0_4 binary relation. For each pair (x, y), we call the following c.e. sets:

$$W_{h(x,y,n,0)}, W_{h(x,y,n,1)}, \dots, W_{h(x,y,n,m)}, \dots$$

the *columns* of the n-th *floor* of the pair (x, y). Furthermore – as in the case of Π^0_3 relations – suppose at some stage s a new element z is listed in $W_{h(x,y,n,m)}$. In such a case, we say that s is an expansionary stage for the m-th column of n-th floor of (x, y). Clearly, we say that a column is finite (resp. infinite) if it has finitely (infinitely) many expansionary stages.

We can now define our universal Π^0_4 relation.

Definition 3.2.12. Let $U^{\forall}_4$ be the following binary relation (that is of course an immediate generalization of $U^{\forall}_3$):

$$xU^{\forall}_4 y \Leftrightarrow W_x \subseteq W''_y.$$

Theorem 3.2.13. *$U^{\forall}_4$ is a universal Π^0_4 binary relation.*

Proof. One can easily verify that $U^{\forall}_4$ is Π^0_4. Then, let R be a Π^0_4 binary relation, and call h a Π^0_4 approximation to it. We aim to build, by steps, a computable function g witnessing the fact that $R \leq U^{\forall}_4$ via g.

Idea of the proof

The general idea is similar to that formulated for the case of Π^0_3 binary relations. We aim to make use of the information provided by the approximation function h in the following way: for each pair (i, j), we put in $W_{g(i)}$ a set of witnesses corresponding to the floors of (i, j). Then, every time a column of these floors enters an expansionary stage, we want to keep track of it in such a way that if, eventually, one of the floor of (i, j) would consist only of finite columns, then the corresponding witness would be removed from $W''_{g(j)}$, hence proving $W_{g(i)} \nsubseteq W''_{g(j)}$.

As in the case of Π^0_3 relations, our general strategy is that of defining a specific functional Φ_e, and then to design our construction so that all the elements we put in all $W_{g(i)}$ sets are just different indices of this very functional. Nonetheless, in the present context, we have an additional layer of information to take care of (namely, one has to control two kind of witnesses: floor-witnesses and column-witnesses).

Thus, in generalizing the strategy we already employed, we introduce some new elements – that we call "markers" – whose role is that of labelling all $W'_{g(i)}$ sets in a way that allows to distinguish the information that properly pertains to a given set from the one that it does not. Finally, we make use of an element μ whose role is, for all i, that of separating the case of $W_{g(i)}$ from the case of $W'_{g(i)}$. Markers are defined as follows.

Definition 3.2.14. Let $A \subseteq \omega$, and let p_e be a padding function for the functional Φ_e. We say that some z is a *marker* of A if $p_e(0, z, 0) \in A$.

Φ_e: preliminary terminology

It is useful to maintain and extend the notation and the terminology, already introduced in the case of Π^0_3 relations. Technically, since Φ_e is yet undefined, for the moment these are just notational conventions.
Let $p_e(x, y, z)$ be an injective padding function for the Turing functional Φ_e.

- We denote $p_e(3, \langle i, j\rangle, n)$ by $\langle i, j\rangle_n$. We call all such elements *floor-witnesses.*
- We denote $p_e(2, \langle i, j\rangle_n, m)$ by $\langle i, j\rangle_{n,m}$. We call all such elements *column-witnesses.*
- We denote $p_e(1, \langle i, j\rangle_{n,m}, p)$ by $\langle i, j\rangle_{n,m,p}$. We call all such elements *extensions.*
- We denote by μ an element of range(p_e) which is different from all the floor-witnesses, the column-witnesses, and the extensions. For instance, denote by μ, $p_e(4, 0, 0)$.
- We denote by F_z the set of all floor-witnesses with right-side z, and by C_z the set of all column-witnesses with right-side z, i.e.

$$F_z = \{\langle i, z\rangle_n \mid i, n \in \omega\};$$

$$C_z = \{\langle i, z\rangle_{n,m} \mid i, n, m \in \omega\}.$$

- If x is a floor-witness, i.e. $x = \langle i, j\rangle_n$, then, for each m, we say that a column-witnesses $\langle i, j\rangle_{n,m}$ is a *subwitnesses* of x. Similarly, if x is a column-witness, i.e. $x = \langle i, j\rangle_{n,m}$, then, for each p, we say that an extension $\langle i, j\rangle_{n,m,p}$ is a *subwitnesses* of x.

Φ_e: the definition

We can now provide the definition of Φ_e. By the Fixed-Point-Theorem, we have that there is e such that the following computable functional Φ_e with oracle A exists:

On input x, Φ_e^A executes the following program:

1. If $\mu \notin A$, then

   1.1 Find the first $\langle a, b\rangle_n$ in A, then

   1.1.1 If $x = \mu$, then converge to 0.

   1.1.2 If $x = p_e(0, a, 0)$ then converge to 0.

   1.1.3 If $x \in C_a \cup F_a$, then

   (a) If there is a subwitness y of x such that $y \notin A$, then converge to 0;

   (b) Otherwise, diverge.

   1.1.4 Otherwise

   (a) If, for all y, $x \neq p_e(0, y, 0)$, then converge to 0;

   (b) Otherwise, diverge.

   1.2 If A has no floor-witnesses, diverge.

2. If $\mu \in A$, then

   2.1 Find the first a such that $p_e(0, a, 0) \in A$, then

   2.1.1 If $x \in C_a \cup F_a$, then

   (a) If there is a subwitness y of x such that $y \notin A$, then converge to 0;

   (b) Otherwise, diverge.

   2.1.2 If $x \notin C_a \cup F_a$, then converge to 0.

   2.2 If A has no markers, diverge.

We aim to design our construction in such a way that all the elements of $\bigcup_{i\in\omega} W_{g(i)}$ will be indices of functional Φ_e.

Construction

Again, we choose an approximation to h such that any given stage s can be expansionary for at most one column. The construction is an immediate generalization of the one provided for the Π^0_3 case.

Stage $\langle 0, 0\rangle$

For all i, let $X_i = \emptyset$.

Stage $s+1 = \langle\langle a, i\rangle, t\rangle$

Check if there are $m, n < s+1$ such that the stage t is expansionary for the m-th column of the n-th floor of (a, i). If so, put $\{\langle a, i\rangle_{m,n,r} \mid r \leq s\}$ in X_i.

Then, for all i, let

$$W_{g(i)} := X_i \cup \{\langle i, n\rangle_m \mid n, m \in \omega\}.$$

Verification

Let us first consider a lemma concerning general properties of μ and the markers.

Lemma 3.2.15. *For all i, the two following hold:*

a) $\mu \notin W_{g(i)}$ and $\mu \in W'_{g(i)}$;

b) $W_{g(i)}$ has no markers, while $W'_{g(i)}$ has a unique marker i.

Proof. a) comes trivially from the construction and the definition of Φ_e.

With regards to b), first notice that no elements of the form $p_e(0, \_, \_)$ enter in $W_{g(i)}$, thus $W_{g(i)}$ has no markers. Then, recall that, for every z, $p_e(0, z, 0)$ is an index of the functional Φ_e, and consider $\Phi_e^{W_{g(i)}}(p_e(0, z, 0))$. We have just said that $\mu \notin W_{g(i)}$. Thus, when executing $\Phi_e^{W_{g(i)}}$ on input $p_e(0, z, 0)$ the computation enters in action 1.1, finding $\langle i, 0\rangle_0$ (which, from the construction, belongs to $W_{g(i)} \smallsetminus X_i$). Therefore, we clearly obtain that, if $z = i$, the computation converges (see action 1.1.2); otherwise, the computation diverges (see action 1.1.4.b). Therefore, $p_e(0, z, 0) \in W'_{g(i)}$ iff $z = i$. □

The verification relies on the following two lemmas:

Lemma 3.2.16. *For all x and i, if $x \in W_{g(i)}$, then*

a) either x is an extension with right-side i, i.e. there are four positive integers t, n, m, p such that $x = \langle t, i\rangle_{n,m,p}$;

b) or x is a floor-witness with left-side i, i.e. there are two positive integers t, n such that $x = \langle i, t\rangle_n$.

Proof. Let $x \in W_{g(i)}$. If x belongs to X_i, then x must be an extension with right-side i, entering in X_i at some stage $s+1$. Otherwise, if $x \in W_{g(i)} \smallsetminus X_i$, then one can immediately see that x is a floor-witness with left-side i. There are no others cases. □

Lemma 3.2.17. *The following holds for all i:*

a) $W''_{g(i)}$ contains all the extensions;

b) $W''_{g(i)}$ contains all the floor-witnesses having right-side $\neq i$.

Proof. The two facts are proved in similar ways.

a) Let x be a extension, i.e. let $x = \langle a, b\rangle_{n,m,p}$ for some a, b, n, m, p. We have to show that $\Phi_e^{W'_{g(i)}}(x)$ converges (this suffices because x is always an index of Φ_e). From Lemma 3.2.15, we know that $\mu \in W'_{g(i)}$ and i is the only marker of $W'_{g(i)}$. Furthermore, we know that x, being an extension, does not belong to $C_i \cup F_i$. Thus, when executing $\Phi_e^{W'_{g(i)}}$ on input x, we enter in action 2.1.2 (see the definition of Φ_e), and therefore the computation converges. Hence, $x \in W''_{g(i)}$.

b) Let x be a floor-witness with right-side $\neq i$. Clearly, this means that $x \notin C_i \cup F_i$. Moreover, Lemma 3.2.15 guarantees that $\mu \in W'_{g(i)}$ and i is the unique marker of $W'_{g(i)}$. Thus, by definition Φ_e, these facts imply that $\Phi_e^{W'_{g(j)}}$, on input x, enters in action 2.1.2. Therefore $\Phi_e^{W'_{g(j)}}(x)\downarrow$, and so $x \in W''_{g(j)}$.

Thus, for all i, $W''_{g(i)}$ does contain all the extensions and all the floor-witnesses with right-side $\neq i$. $\square$

Then, we have to show that R reduces to $U_4^{\forall}$ via g.

First suppose iRj, and let $x \in W_{g(i)}$. We have to see whether x is also in $W''_{g(j)}$. By Lemma 3.2.16, we know that x is either an extension with right-side i or a floor-witness with left-side i. Nonetheless, if the first case holds, then we have already proved that x belongs to $W''_{g(j)}$, since $W''_{g(j)}$ contains all the extensions by item a) of Lemma 3.2.17. So, let x be a floor-witness with left-side i, i.e. $x = \langle i, t\rangle_n$ for some t and n. Now, suppose $t \neq j$. If so, by item b) of Lemma 3.2.17, we can already conclude that x is in $W''_{g(j)}$, since $W''_{g(j)}$ contains all the floor-witnesses with right-side $\neq j$. Therefore, the only case that remains to be considered is that of x being a floor-witness of (i, j). That is to say, there is n such that $x = \langle i, j\rangle_n$. In considering this case, we make use of the following lemma.

Lemma 3.2.18. *For all m, we have*

$$\langle i, j\rangle_{n,m} \in W'_{g(j)} \text{ iff the } m\text{-th column of the } n\text{-th floor of } (i, j) \text{ is finite.}$$

Proof. ($\Rightarrow$) Suppose $\langle i, j\rangle_{n,m} \in W'_{g(j)}$. Since $\langle i, j\rangle_{n,m}$ is an index of Φ_e, this means that $\Phi_e^{W_{g(j)}}(\langle i, j\rangle_{n,m})\downarrow$. But this can occur only if there is a subwitness of $\langle i, j\rangle_{n,m}$ that is not in $W_{g(j)}$, i.e. if there is p such that $\langle i, j\rangle_{n,m,p} \notin W_{g(j)}$. This is because $\mu \notin W_{g(j)}$ (by Lemma 3.2.15), $\langle j, 0\rangle_0 \in W_{g(j)}$ (from the construction), and $\langle i, j\rangle_{n,m}$ is in $C_j \cup F_j$ (by definition of F_j). So, when running $\Phi_e^{W_{g(j)}}$ on input $\langle i, j\rangle_{n,m}$, the computation enters in action 1.1.3. Now, suppose the m-th column of the n-th floor of (i, j) is infinite, i.e. it has infinitely many expansionary stages. If so, it is immediate to see that, in our construction, we would have put eventually any $\langle i, j\rangle_{n,m,p}$ into $W_{g(j)}$. Hence, it would be $\Phi_e^{W_{g(j)}}(\langle i, j\rangle_{n,m})\uparrow$, contradicting our hypothesis.

($\Leftarrow$) Suppose that the m-th column of the n-th floor of (i,j) is finite, and consider $\Phi_e^{W_{g(j)}}(\langle i,j\rangle_{n,m})$. Three familiar facts hold: *i*) by Lemma 3.2.15, we have that $\mu \notin W_{g(j)}$ and j is the unique marker of $W_{g(j)}$; *ii*) $\langle j,0\rangle_0 \in W_{g(j)}$; *iii*) $\langle i,j\rangle_{n,m}$ is a column-witness with right-side j, i.e. $\langle i,j\rangle_{n,m} \in C_j$. Thus, as expressed by action 1.1.3 in the definition of Φ_e, we have that $\Phi_e^{W_{g(j)}}(\langle i,j\rangle_{n,m})$ converges iff there is p such that $\langle i,j\rangle_{n,m,p} \notin W_{g(j)}$. But recall that, in our construction, elements of the form $\langle i,j\rangle_{n,m,p}$ are in $W_{g(j)}$ just in case that the m-th column of n-th floor of (i,j) enters some expansionary stage t, with $t \geq p$, and by hypothesis this can happen only finitely many times. Therefore, there must be a minimum p such that $\langle i,j\rangle_{n,m,p} \notin W_{g(j)}$. Thus $\Phi_e^{W_{g(j)}}(\langle i,j\rangle_{n,m})\downarrow$, and so $\langle i,j\rangle_{n,m} \in W'_{g(j)}$. $\square$

Now, call C the set of all subwitnesses of $\langle i,j\rangle_n$ that are in $W'_{g(j)}$. The latter lemma says that C contains all and only those subwitnesses of $\langle i,j\rangle_n$ that correspond to finite columns. But since iRj holds, we know that any floor must contain just finitely many finite columns. Thus, in particular, $|C|$ has to be finite. Then, consider $\Phi_e^{W'_{g(j)}}(\langle i,j\rangle_n)$. We have already proved that j is the unique marker of $W'_{g(j)}$, and $\langle i,j\rangle_n$ clearly belongs to F_j. Thus, by action 2.1.1 in the definition of Φ_e, we have that $\Phi_e^{W'_{g(j)}}(\langle i,j\rangle_n)$ converges if and only if there is a subwitness of $\langle i,j\rangle_n$ which is not in $W'_{g(j)}$ – and therefore which is not in C, since C is defined to be set of all subwitnesses of $\langle i,j\rangle_n$ that are in $W'_{g(j)}$. But we have just proved that C is finite, and so it cannot contain all the infinite subwitnesses of $\langle i,j\rangle_n$. Thus, we can conclude that $\Phi_e^{W'_{g(j)}}(\langle i,j\rangle_n)$ does converge, and therefore $\langle i,j\rangle_n = x \in W''_{g(j)}$.

Thus, in any case, if iRj then $W_{g(i)} \subseteq W''_{g(j)}$.

Conversely, suppose $i\not R j$. So, by the characterization provided in Corollary 3.2.6, there must be a floor n of (i,j) such that all its columns are finite. Then, consider $\langle i,j\rangle_n$. By our construction, we have that $\langle i,j\rangle_n$ is in $W_{g(i)}$. We aim to prove that $\langle i,j\rangle_n \notin W''_{g(j)}$. Since all the columns of the n-th floor of (i,j) are finite, then, by Lemma 3.2.18, we have that $\langle i,j\rangle_{n,m} \in W'_{g(j)}$, for all m. Now, consider $\Phi_e^{W'_{g(j)}}(\langle i,j\rangle_n)$. As usual, Lemma 3.2.15 guarantees that j is the unique marker of $W'_{g(j)}$, and $\langle i,j\rangle_n \in F_j$. This means (by action 2.1.1) that the computation converges just in case we can find a subwitness of $\langle i,j\rangle_n$ which is not in $W'_{g(j)}$, i.e. if there is m such that $\langle i,j\rangle_{n,m} \notin W'_{g(j)}$. But we have just showed that no such m can be found. Thus $\langle i,j\rangle_n \notin W''_{g(j)}$, and so $W_{g(j)} \nsubseteq W''_{g(i)}$.

$\square$

3.2.6 Π_n^0 universality, with $n > 4$

In this section, we finally define, for all n, $U_n^{\forall}$ and we prove the corresponding universality results, hence showing a generalization of the results presented in the last two sections. To begin with, let us recall Corollary 3.2.7:

1. If R is a Π^0_{2k} binary relation, then there is a computable function h, such that

$$xRy \Leftrightarrow \forall z_1 \exists z_2 \forall z_3 \ldots \forall^\infty z_{2k-2}(W_{h(x,y,z_1,z_2,z_3 \ldots z_{2k-2})} \text{ is infinite})$$

   and

$$x \not R y \Leftrightarrow \exists z_1 \forall z_2 \exists z_3 \ldots \forall z_{2k-2}(W_{h(x,y,z_1,z_2,z_3 \ldots z_{2k-2})} \text{ is finite}).$$

2. If R is a Π^0_{2k+1} binary relation, then there is a computable function h, such that

$$xRy \Leftrightarrow \forall z_1 \exists z_2, \ldots, \forall z_{2k-1}(W_{h(x,y,z_1,z_2 \ldots z_{2k-1})} \text{ is finite})$$

   and

$$x \not R y \Leftrightarrow \exists z_1 \forall z_2, \ldots, \forall^\infty z_{2k-1}(W_{h(x,y,z_1,z_2 \ldots z_{2k-1})} \text{ is infinite}).$$

Again, we call h a Π^0_n-approximation to R.

Since the construction below is based on this latter characterization, we shall now extend the terminology already adopted for Π^0_3 and Π^0_4 relations to the n-dimensional case.

Let R be a Π^0_n relation. We think of h as assigning, to each pair (x, y), $n-1$ different layers of information organized as follows. At the bottom, we have positive integers that belong to what we have called "columns", that are c.e. sets that can be either finite or infinite. In the present context, we call such columns 2-*levels* of the pair (x, y), while their elements are called 1-*levels*. These 2-levels are then grouped together in forming classes of c.e. sets, that we call 3-*levels* (or, by maintaining the above terminology, "floors"). Similarly, partitions of 3-levels are called 4-*levels* – and so on, up to $(n-1)$-*levels*, that we also call *top-levels* of (x, y).

We can make this intuition more precise. First, given any pair (x, y), 2-levels of (x, y) (i.e., columns) are defined to be the c.e. sets of the following kind:

$$W_{h(x,y,z_1,\ldots z_{n-3},z_{n-2})},$$

which, according the above terminology, correspond to the the z_{n-2}-th column of the z_{n-3}-th floor of the . . . of the z_1-th top-level of the pair (x, y).

Then, $t+1$-levels of (x, y) are so defined

$$W_{h(x,y,z_1,\ldots,z_{n-(t+1)})} := \{W_{h(x,y,z_1,\ldots,z_{n-(t+1)},m)} \mid m \in \omega\}.$$

Also, for each t-level, we say that the $(t-1)$-levels of which it is constituted are its *sublevels*. Finally, expasionary stages are defined as above. We say that some stage s is *expansionary* for the column $W_{h(x,y,z_1,\ldots z_{n-3},z_{n-2})}$ if a new element z enters in the column at stage s.

We are now ready to give the definition of $U^\forall_n$.

Definition 3.2.19. Let $U_n^{\forall}$ be the following binary relation:

$$xU_n^{\forall}y \Leftrightarrow W_x \subseteq W_y^{(n-2)}.$$

Theorem 3.2.20. *$U_n^{\forall}$ is universal with respect to Π_n^0 binary relations.*

Proof. The structure of the proof is similar to that provided for Π_3^0 and Π_4^0 relations. In fact, since the present case clearly subsumes these latter, one might have provided only the proof for Π_n^0 relations. However, having introduced our basic machinery in a simpler context, we can now mostly focus on those aspects in which the general case differ from the previous cases.

Φ_e: preliminary terminology

We aim to keep the notation and the terminology already introduced. Let p_e be, as always, a ternary padding function for the Turing functional Φ_e. First, recall the following definition:

- Let $A \subseteq \omega$. We say that some z is a *marker* of A if $p_e(0, z, 0) \in A$.

Furthermore, we aim to extend our notation concerning witnesses by means of including all the k-levels, as defined above, of a Π_n^0 relation.

- We denote $p_e(n-1, \langle i, j\rangle, z)$ by $\langle i, j\rangle_z$. We call all such elements $(n-1)$-*witnesses* (or, *top-witnesses*) of (i, j).
- We denote $p_e(n-t, \langle i, j\rangle_{m_1,\dots,m_{t-1}}, z)$ by $\langle i, j\rangle_{m_1,\dots,m_{t-1},z}$. We call such elements t-*witnesses* of (i, j).
- We denote by μ the element $p_e(n, 0, 0)$ (thus, ensuring that μ is different from all t-witnesses).

At times, it would be convenient to maintain the language introduced in the case of Π_3^0 and Π_4^0 relations by calling *extensions* the 1-witnesses, *columns-witnesses* the 2-witnesses of (i, j), and *floors-witnesses* the 3-witnesses. Furthermore, given any t-witness $x = \langle i, j\rangle_{m_1,\dots,m_t}$, we say that, for all z, $\langle i, j\rangle_{m_1,\dots,m_t,z}$ is a subwitness of x.

- Finally, we denote by F_z^t the set of all t-witnesses having right-side z:

$$F_z^t = \{\langle i, z\rangle_{m_1,\dots,m_t} \mid i, m_1, \dots, m_t \in \omega\},$$

  and by F_{z_0,z_1}^t the set of all t-witnesses of the pair (z_0, z_1):

$$F_{z_0,z_1}^t = \{\langle z_0, z_1\rangle_{m_1,\dots,m_t} \mid m_1, \dots, m_t \in \omega\}.$$

Definition of Φ_e

We can now provide the definition of Φ_e. By the Fixed-Point-Theorem, we have that there is e such that the following computable functional Φ_e with oracle A exists:

On input x, Φ_e^A executes the following program:

1. If $\mu \notin A$, then

   1.1 Find the first $\langle a, b\rangle_n$ in A, then

   1.1.1 If $x = \mu$, then converge to 0.

   1.1.2 If $x = p_e(0, a, 0)$ then converge to 0.

   1.1.3 If $x \in \bigcup F_a^t$, with $1 < t < n$ then

   (a) If there is a subwitness y of x such that $y \notin A$, then converge to 0;

   (b) Otherwise, diverge.

   1.1.4 Otherwise

   (a) If, for all y, $x \neq p_e(0, y, 0)$, then converge to 0;

   (b) Otherwise, diverge.

   1.2 If A has no floor-witnesses, diverge.

2. If $\mu \in A$, then

   2.1 Find the first a such that $p_e(0, a, 0) \in A$, then

   2.1.1 If $x = \mu$, then converge to 0.

   2.1.2 If $x = p_e(0, a, 0)$ then converge to 0.

   2.1.3 If $x \in \bigcup F_z^t$, with $1 \leq t \leq n$, then

   (a) If there is a subwitness y of x such that $y \notin A$, then converge to 0;

   (b) Otherwise, diverge.

   2.1.4 Otherwise

   (a) If, for all y, $x \neq p_e(0, y, 0)$, then converge to 0;

   (b) Otherwise, diverge.

   2.2 If A has no marker, it diverges.

Remark 3.2.21. Notice that the set of instructions from 1.1.1 to 1.1.4 and the set of instructions from 2.1.1 to 2.1.4 correspond to the same subroutine.

Construction

The construction mimics the ones provided in the former cases. As usual, we make use of an approximation to h such that any given stage s can be expansionary for at most one column.

Stage $\langle 0, 0\rangle$

For all i, let $X_i = \emptyset$.

Stage $s + 1 = \langle\langle a, i\rangle, t\rangle$

Check if there is a $n-2$-tuple $(z_1, \ldots, z_{n-2})$, with $z_1, \ldots, z_{n-2} < s+1$, such that the stage t is expansionary for the column $W_{h(a,i,z_1,\ldots,z_{n-2})}$.
If so, put the set $\{\langle a, i\rangle_{z_1,z_2,\ldots,z_{n-2},r} \mid r \leq s\}$ in X_i.

Then, for all i, let

$$W_{g(i)} := X_i \cup \{\langle i, n\rangle_m \mid n, m \in \omega\}.$$

Verification

Let R be a Π^0_n binary relation. We aim to prove that $R \leq U$ via g. First, let us prove an analogue of Lemma 3.2.15 for the present context:

Lemma 3.2.22. *For every i, the two following hold:*

a) $\mu \notin W_{g(i)}$ and, for all $m \geq 1$, $\mu \in W^m_{g(i)}$;

b) $W_{g(i)}$ has no markers, while, for all $m \geq 1$, $W^m_{g(i)}$ has a unique marker i.

Proof. The proof is just an immediate generalization of that of Lemma 3.2.15. As an example, let us prove the inductive step of b). Suppose, we have showed that i is the unique marker of $W^{(m)}_{g(i)}$, and let us prove that i is also the unique marker of $W^{(m+1)}_{g(i)}$. In doing so, consider $\Phi_e^{W^{(m)}_{g(i)}}$ on input $p_e(0, z, 0)$ (this is enough because $p_e(0, z, 0)$ is, by definition, an index of Φ_e). We know (from item a) of the present lemma) that $\mu \in W^m_{g(i)}$, and by hypothesis we have that i is the unique marker of $W^{(m)}_{g(i)}$. This means that, on input $p_e(0, z, 0)$, $\Phi_e^{W^{(m)}_{g(i)}}$ converges if $z = i$ (by action 2.1.2); otherwise it diverges (by action 2.1.4.b). Therefore, $p_e(0, z, 0) \in W^{m+1}_{g(i)}$ iff $z = i$. $\square$

Next, we shall provide a generalization of Lemma 3.2.16 and Lemma 3.2.17.

Lemma 3.2.23. *For all x and i, the two following hold:*

a) If $x \in W_{g(i)}$, then x is either a top-witness with left-side i, or an extension with right-side i;

b) $W^{(n-2)}_{g(i)}$ contains all the extensions and all the top-witnesses having right-side $\neq i$.

Proof. The proof is basically the same of that of Lemma 3.2.16 and Lemma 3.2.17. $\square$

For what follows, we distinguish two cases.

The even case

First, suppose that $n = 2k$, i.e. we consider binary relations that lie at the even levels of the arithmetical hierarchy.

The rest of the proof is organized as follows. First, we introduce a notion of "being damaged", that says whether a given level of R satisfies the condition expressed by Corollary 3.2.7. Next, we provide two lemmas. With the first one, we show that, for each pair (i, j), iRj holds iff all the top-levels of (i, j) are undamaged. The second lemma states that, for all j, $W^{(2k-2)}_{g(j)}$ contains all and only the top-witnesses corresponding to the undamaged top-levels of R. By combining these two lemmas, we finally obtain that

$$iRj \Leftrightarrow W_{g(i)} \subseteq W^{(2k-2)}_{g(j)} = g(i)U^{\forall}_{2k}g(j).$$

So, let (i, j) be a pair. The notion of "being damaged" for a t-level of (i, j) is so defined:

Definition 3.2.24. First, let us call the columns of (i, j) *damaged*, if they are finite; otherwise, we call them *undamaged*. Then, for all t, with $2 \leq t < 2k$:

1. If a is a $2t+1$-level of (i, j), we say that a is *damaged* if all the sublevels of a are *damaged*; otherwise we say that a is *undamaged*.

2. If a is a $2t$-level of (i, j), we say that a is *damaged* if at least one of its sublevels is *damaged*; otherwise, we say that a is *undamaged*.

For instance, according to this latter definition, we have that a floor (i.e., a 3-level) of (i, j) is damaged iff all its column are damaged, i.e. finite. So, if one applies this terminology to the case of Π^0_4, one can immediately check that, if iRj, and R is Π^0_4, then all the floors of (i, j) are undamaged. By the following lemma, we aim to generalize such a fact.

Lemma 3.2.25. *Let R be a Π^0_{2k} relation. Then, iRj iff all the top-levels of (i, j) are undamaged.*

Proof. First, for Π^0_4 relations, the lemma easily follows from the characterization provided in Corollary 3.2.6. Indeed, such a corollary guarantees that, if iRj, then each floor of (i, j) does contain infinitely many columns that are infinite (i.e. infinitely many undamaged columns). Then, being a floor a 3-level of (i, j), by Definition 3.2.24, it easily follows that all these top-levels of (i, j) are undamaged.

We show that the lemma holds also for the Π^0_6 case. The remaining cases, with $2k > 6$, differ only in details.

Let R be a Π^0_6 binary relation, and let (i, j) be a pair. Suppose iRj, and suppose – by contradiction – that there is a top-level (i.e. a 5-level) of (i, j) which is damaged. Call it z_1. So, by adopting our notation, we have that the following hold: $\exists z_1(W_{h(x,y,z_1)}$ is damaged). But if so, by item 1. in Definition 3.2.24, this means that all the sublevels of z_1 (which are, by definition, all 4-levels) are damaged themselves. So, we can write $\exists z_1 \forall z_2 (W_{h(x,y,z_1,z_2)}$ is damaged). By item 2. in Definition 3.2.24, this implies that each z_2 contain a sublevel (i.e., a floor) which is damaged. Thus, we have that $\exists z_1 \forall z_2 \exists z_3 (W_{h(x,y,z_1,z_2,z_3)}$ is damaged). But, as we have just showed, a floor is damaged iff all its columns are finite. Hence, it would follow that

$$\exists z_1 \forall z_2 \exists z_3 \forall z_4 (W_{h(x,y,z_1,z_2,z_3,z_4} \text{ is finite}),$$

which, by Corollary 3.2.7 (limited to the case of a Π^0_6 relation) holds iff $x \not R y$, against our hypothesis.

The other case, in which we suppose $i \not R j$, is symmetric. □

Lemma 3.2.26. *Let R be a Π^0_{2k} relation and let (i,j) be a pair. Then, for all r, $\langle i,j \rangle_r \in W^{(2k-2)}_{g(j)}$ iff the r-th top-level of (i,j) is undamaged.*

Proof. Let us first notice that one can mimic the proof given for Lemma 3.2.18 and show that a column-witness of (i,j) is in $W'_{g(j)}$ iff the corresponding column is finite. Thus, we have that

$$x \in F^2_{i,j} \cap W'_{g(j)} \text{ iff } x \text{ is the column-witness of a damaged column.} \tag{3.1}$$

Next, consider $W''_{g(j)}$. Let x be a floor-witness of (i,j). As is clear, x is an index of Φ_e. So, in order to see whether $x \in W''_{g(j)}$, it is enough to see if $\Phi_e^{W'_{g(j)}}(x)$ converges. First, notice that, by Lemma 3.2.22, $\mu \in W'_{g(j)}$ and j is the unique marker of $W'_{g(j)}$. Furthermore, x clearly belongs to F^3_j. Therefore, when executing $\Phi_e^{W'_{g(j)}}$ on input x, the computation enters in action 2.1.3. Hence, we have that $\Phi_e^{W'_{g(j)}}(x)$ converges iff there is a subwitness of x that is not in $W'_{g(j)}$. But recall that, being x a floor-witness, all the subwitnesses of x are column-witnesses. Moreover, we have just proved that all the column-witnesses of (i,j) that correspond to damaged columns are in $W'_{g(j)}$ (see 3.1 above). Thus, the two following are equivalent:

1. $x \in F^3_{i,j} \cap W''_{g(j)}$ iff there is a subwitness of x that is not in $W'_{g(j)}$;
2. $x \in F^3_{i,j} \cap W''_{g(j)}$ iff at least one of the subwitnesses of x is a column-witness of an undamaged column.

By 1. in Definition 3.2.24, item 2. holds iff x is a floor-witness of an undamaged floor. Therefore, we have obtained:

$$x \in F^3_{i,j} \cap W''_{g(j)} \text{ iff } x \text{ is the floor-witness of an undamaged floor.} \tag{3.2}$$

A similar – yet symmetric – line of reasoning applies to the case of 4-levels. Indeed, if x is a 3-witness of (i,j), then $\Phi_e^{W''_{g(j)}}(x)$ halts iff there is a subwitness of x that is not in $W''_{g(j)}$. But then again, since all the 3-witnesses that correspond to undamaged floors are in $W''_{g(j)}$ (by (3.2)), this means that $s \in W'''_{g(j)}$ iff there is a subwitness of s that is a 3-witness of a damaged floor. That is to say, by 2 in Definition 3.2.24, that

$$x \in F^4_{i,j} \cap W'''_{g(j)} \text{ iff } x \text{ is the 4-witness of a damaged 4-level.} \tag{3.3}$$

As is clear, by iterating this kind of reasoning, one can easily obtain that the following fact holds for $p < 2k - 1$:

1. if m is odd $\Rightarrow x \in F^m_{i,j} \cap W^{(m)}_{g(j)}$ iff x is the $(m+1)$-witness of an undamaged $(m+1)$-level;
2. if m is even $\Rightarrow x \in F^m_{i,j} \cap W^{(m)}_{g(j)}$ iff x is the $(m+1)$-witness of a damaged $(m+1)$-level.

Now consider $m = 2k-1$. Since R is Π_{2k}, we have that $F^{2k-1}_{i,j}$ contains the top-witnesses of (i, j). Therefore by item 1. in the fact above, we have that a top-witness x of (i, j) belongs to $W^{(2k-2)}_{g(j)}$ iff x corresponds to an undamaged top-level of (i, j). So, the thesis is proved. □

The rest of the verification relies on the two last lemmas.

First, suppose iRj. We have to show that $W_{g(i)} \subseteq W^{(2k-2)}_{g(j)}$. Let $x \in W_{g(i)}$. We aim to prove that x is also in $W^{(2k-2)}_{g(j)}$. By 1. in Lemma 3.2.23, we know that x is either an extension or a top-witness having left-side i. Yet, if x is an extension, then, by 2. in Lemma 3.2.23, we already know that x belongs to $W^{(2k-2)}_{g(j)}$. Similarly, if x is a top-witness with right-side $\neq j$, then, again by 2. in Lemma 3.2.23, we have that $x \in W^{(2k-2)}_{g(j)}$. Thus, it remains to be considered the case in which x is a top-witness with left-side i and right-side j, i.e. there is m such that $x = \langle i, j\rangle_m$. If so, since iRj holds, by Lemma 3.2.28, we obtain that the m-th top-level of (i, j) is undamaged (as all others top-levels are). But then, Lemma 3.2.29 ensures that $\langle i, j\rangle_m$, being the top-witness of an undamaged top-level, belongs to $W^{(2k-2)}_{g(j)}$.

Therefore, in any case, if iRj, then $W_{g(j)} \subseteq W^{(2k-2)}_{g(j)}$.

Conversely, suppose $i\not R j$. If so, by Lemma 3.2.28 there must be a top-level of (i, j) that is damaged. Call it $\langle i, j\rangle_m$. On the one hand, it is immediate to see, from the construction, that $\langle i, j\rangle_m$ belongs to $W_{g(i)} \setminus X_i$. On the other hand, by Lemma 3.2.29 we have that $\langle i, j\rangle_m \notin W^{(2k-2)}_{g(j)}$, because the m-th top-level of (i, j) is damaged. Thus, $W_{g(j)} \nsubseteq W^{(2k-2)}_{g(j)}$.

The odd case

It remains to be considered the case in which $n = 2k+1$. That is to say, we have to prove that, for all k, U_{2k+1} is universal on Π^0_{2k+1} relations. Fortunately, such case is completely symmetric to the even one we have just considered. Indeed, let be the following a minimal reformulation of Definition 3.2.24:

Definition 3.2.27. Let us call the columns of (i,j) *damaged*, if they are infinite; otherwise, we call them *undamaged*. Then for all t, $t \leq 2k-1$:

1. If a is a $2t+1$-level of (i,j), we say that a is *damaged* if all the sublevels of a are *damaged*; otherwise we say that a is *undamaged*.

2. If a is a $2t$-level of (i,j), we say that a is *damaged* if at least one of its sublevels is *damaged*; otherwise, we say that a is *undamaged*.

As is clear, with respect to the even case, we have just switched the meaning of "damaged columns" and "undamaged columns" – by linking, in the current case, the former to the finite columns and the latter to the infinite ones – and we have preserved how such notions are inherited through the higher levels. It easy to see that this slight reformulation well fits with the Π^0_3 case. Indeed, recall that, if $R \in \Pi^0_3$, then iRj iff all the columns of (i,j) are finite – i.e., according to our last definition, iff all the columns of (i,j) are undamaged.

More generally, under such reformulation, one can easily show – by just copying our proofs above – that Lemma 3.2.28 and Lemma 3.2.29 are maintained, i.e. we have that

Lemma 3.2.28. *Let R be a Π^0_{2k+1} relation. Then, all the top-levels of (i,j) are undamaged iff iRj,*

and

Lemma 3.2.29. *Let R be a Π_{2k+1} relation. Then, for all r, $\langle i,j\rangle_r \in W^{(2k-2)}_{g(j)}$ iff the r-th top-level of (i,j) is undamaged,*

with the meaning of "undamaged", in these latter lemmas, setted by Definition 3.2.27.

Furthermore, having obtained these two lemmas, one can just mimic the rest of the proof described above and eventually obtain that the result also holds for the odd case, i.e., for all k, $U^{\forall}_{2k+1}$ is universal w.r.t to Π^0_{2k+1} binary relations. □

3.2.7 Σ^0_n universality

All the universal relations we have introduced so far lie, with the exception of $U^{\exists}_1$, in the Π^0_n levels of the arithmetical hierarchy. Yet, having defined these latter relations, one can trivially obtain, for all n, a universal Σ^0_n binary relation – less artificial than the corresponding cylinder – by just making use of Fact 3.1.3.

Indeed, consider first the $U^{\forall}_2$ (which appears to be the less natural case, since its definition relies on the construction of a specific computable function f). Call $U^{\exists}_2$ its complement, i.e. $U^{\exists}_2 = (U^{\forall}_2)^c$. Fact 3.1.3 immediately shows that $U^{\exists}_2$ is a universal Σ^0_2 binary relation.

In general, for each $n > 2$, let $U^{\exists}_n$ be the following binary relation:

$$xU^{\exists}_n y \Leftrightarrow W_x \not\subseteq W^{(n-2)}_y.$$

It is immediate to prove that, if R is a Σ^0_n binary relation, then R reduces to $U^{\exists}_n$.

Fact 3.2.30. *$U^{\exists}_n$ is a universal Σ^0_n binary relation.*

Proof. It is trivial to see that, for all n, $U^{\exists}_n$ is the complement of $U^{\forall}_n$. Hence, $U^{\exists}_n$ is Σ^0_n. Then, it is enough to apply Fact 3.1.3 in order to prove that all $U^{\exists}_n$ are universal with respect to to Σ^0_n binary relation. □

Finally, it might be convenient to recollect all the universality results presented so far in the following theorem:

Theorem 3.2.31. *At each level of the arithmetical hierarchy there is a natural universal binary relation. In particular, the four following facts hold:*

1. *Let $U^{\forall}_1$ be the following binary relation $\{\langle i, j\rangle \mid i \notin W_j\}$. Then, $U^{\forall}_1$ is a universal Π^0_1 binary relation;*
2. *Let $U^{\forall}_2$ be the following binary relation $\{\langle i, j\rangle \mid W_i \subseteq W_{f(j)}\}$ (with f being the computable function defined in section 3.2.2). Then, $U^{\forall}_2$ is a universal Π^0_2 binary relation;*
3. *For each $n > 2$, let $U^{\forall}_n$ be the following binary relation $\{\langle i, j\rangle \mid W_i \subseteq W^{(n-2)}_j\}$. Then, $U^{\forall}_n$ is a universal Π^0_n binary relation;*
4. *For each n, let $U^{\exists}_n$ be the complement of $U^{\forall}_n$. Then, $U^{\exists}_n$ is a universal Σ^0_n binary relation.*

3.3 Stretching the main result

Let us step back for a moment. The general problem we are considering is that of finding, within a family of relations of given arithmetical complexity, a universal one, i.e. one relation to which all the others are computably reducible. First, we have recalled a recent result from [Ianovski et al., 2014] in which the authors prove that, for $n \geq 2$, there is neither a universal Π^0_n equivalence relation nor a universal Π^0_n preorder. Nonetheless, in the present work we have shown that, for the case of general binary relations, such universal relations do exist at each level of the arithmetical hierarchy (Theorem 3.2.31).

Thus, it comes natural the idea of looking at intermediate cases, that lie in between general binary relations and equivalence relations, and see whether universality is preserved or not. Of course in building such intermediate relations, a straightforward strategy consists in dropping one or two requirements between reflexivity, symmetry and transitivity and then study the resulting relation.

In doing so, let us fix some notation.

Notation. Let S be a binary relation.

1. The *reflexive closure* of S, S^r, is the minimum reflexive relation containing S, i.e.:
$$xS^ry \Leftrightarrow x = y \vee xSy.$$

2. The *symmetric closure* of S, S^s, is the minimum symmetric relation containing S, i.e.:
$$xS^sy \Leftrightarrow xSy \vee ySx.$$

3. The *transitive closure* of S, S^*, is the minimum transitive relation containing S, i.e.:
$$xS^*y \Leftrightarrow \exists(t_0, \dots, t_n) \text{ such that } x = t_0 \;\&\; y = t_n, \text{ and, for all } 0 \leq i < n,\ t_iSt_{i+1}.$$

4. Finally, we denote by $\mathbf{R}, \mathbf{S}, \mathbf{T}$ respectively the sets of all reflexive, symmetric, and transitive binary relations on ω.

The following fact shows that is easy to make use of the universal binary relations defined above in order to solve the problem of universality for some intermediate cases.

Fact 3.3.1. *Let U be a universal binary relation with respect to a family of binary relations $\mathcal{R}$. Then,*

1. *U^r is universal with respect to $\mathcal{R} \cap \mathbf{R}$;*

2. *U^s is universal with respect to $\mathcal{R} \cap \mathbf{S}$.*

Proof. The proofs for 1. and 2. are immediate and similar. Let us prove 2.
We have to show that all the symmetric relations that are in $\mathcal{R}$ are reducible to U^s. So, let $S \in \mathcal{R}$ be a symmetric relation. Due to the universality of U, there must be a computable function g such that $S \leq U$ via g. We aim to show that g also reduces S to U^s. First, suppose xSy. If so, we have that $g(x)Ug(y)$. But U^s clearly extends U (i.e., $U \subseteq U^s$), and therefore $g(x)U^sg(y)$ also holds. So, $xSy \Rightarrow g(x)U^sg(y)$.

Conversely, suppose $x\not{S}y$ and suppose – by contradiction – that $g(x)U^sg(y)$. Since S is symmetric, from $x\not{S}y$ it immediately follows that also $y\not{S}x$. Thus, for the universality of U, it must be $g(x)\not{U}g(y)$ and $g(y)\not{U}g(x)$. But, by definition of U^s, then this means that $g(x)\not{U^s}g(y)$ holds – which gives us a contradiction with our hypothesis. $\square$

This latter fact leads immediately to a comprehensive theorem.

Theorem 3.3.2. *At each level of the arithmetical hierarchy there exist a universal reflexive binary relation, a universal symmetric binary relation (i.e. a universal graph), and a universal reflexive symmetric binary relation. In particular, the results expressed in the following table hold:*

Universal relations	Σ_n	Π_n
general binary relations	$U_n^{\exists}$	$U_n^{\forall}$
reflexive relations	$(U_n^{\exists})^r$	$(U_n^{\forall})^r$
symmetric relations	$(U_n^{\exists})^s$	$(U_n^{\forall})^s$
reflexive and symmetric relations	$(U_n^{\exists})^{r,s}$	$(U_n^{\forall})^{r,s}$

Proof. All cases shown in the table are obvious. For instance, suppose we aim to prove that, for all n, there is a universal Π_n^0 reflexive symmetric binary relation. First, by Theorem 3.2.20, we know that $U_n^{\forall}$ is a universal Π_n^0 binary relation. Then, by applying 1. in Fact 3.3.1, we obtain that $(U_n^{\forall})^r$ is universal with respect to Π_n^0 reflexive binary relation. Finally, it is enough to apply 2. in Fact 3.3.1, to show that $(U_n^{\forall})^{r,s}$ is a universal reflexive symmetric binary relation. $\square$

We are left with the case of $\mathbf{T}$. Yet, this last is again immediately solved by referring to Fact 3.3.1. Indeed, we have

Fact 3.3.3. *For each n, there is no universal Π_n transitive relation.*

Proof. Suppose that, for a given n, we would have a universal Π_n transitive relation. Call it U. By Fact 3.3.1, we would have that U^r would be a universal Π_n preorder. But, as we have just said, in [Nies et al., 2014] it is proven the nonexistence of such preorders. $\square$

Thus, it is exactly transitivity that traces the dividing line between the existence and the nonexistence of universal relations at Π_n^0 levels of the arithmetical hierarchy.

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