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# QUADRATIC CONTROL FOR LINEAR TIME-VARYING SYSTEMS*

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**Abstract.** An infinite-dimensional linear time-varying system on the interval $(-\infty, \infty)$ is considered. We introduce three quadratic problems: the infinite horizon problem, and one-sided and two-sided average cost problems. A Riccati equation on $(-\infty, \infty)$ is considered first and sufficient conditions for the existence and uniqueness of a bounded solution are given. Then by dynamic programming the quadratic problems are solved. Similar problems in the stochastic case are considered.

**Key words.** linear quadratic control, time-varying systems

**AMS(MOS) subject classifications.** 49, 49C20

**1. Introduction.** Consider the usual quadratic control problem:

$$y' = A(t)y + B(t)u, \qquad y(t_0) = y_0, \tag{1.1}$$

$$J(u) = \int_{t_0}^{T} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt \tag{1.2}$$

where $A$, $B$, $M$, and $N$, are continuous matrices on $(-\infty, \infty)$ of appropriate dimensions and where $|\ |$ and $\langle\ ,\ \rangle$ denote, respectively, the norm and the inner product of vectors. The Riccati equation associated with this problem is the following [28]:

$$Q' + A^*Q + QA + M^*M - QBN^{-1}B^*Q = 0, \tag{1.3}$$

$$Q(T) = 0. \tag{1.4}$$

There exists a unique solution to (1.3), (1.4) on $[t_0, T]$. Since $t_0$ is fixed but otherwise arbitrary, we can always find a solution on $(-\infty, T]$. Of course $Q$ may not be bounded on $(-\infty, T]$. If we wish to solve the infinite horizon problem (1.1), (1.2) with $T = +\infty$, then it turns out that we need a bounded solution of (1.3) on $[t_0, \infty)$. Since $t_0$ can vary, we require a bounded solution on $(-\infty, \infty)$. If our system is defined only on a semi-infinite interval $[T_0, \infty)$ (thus, $t_0 \geqq T_0$), then we need a bounded solution of (1.3) on $[T_0, \infty)$ (a semi-infinite interval in the *positive* direction). If all matrices are periodic with a common period $\theta$ and if $(A, B)$ is stabilizable and $(A, M)$ detectable, the existence of a $\theta$-periodic solution to (1.3) is known [31], [34]. This result remains true also in infinite dimensions [12], [14]. But the existence problem for general bounded continuous matrices seems to be new.

In this paper we consider (1.3) in infinite dimensions. We assume that $A(t)$ generates an evolution operator in a Hilbert space and that other operators are bounded and continuous. We give a necessary and sufficient condition for the existence of a bounded solution to (1.3). We have uniqueness if $(A, M)$ is detectable. If these two hypotheses are fulfilled, there exists a unique bounded solution $Q_\infty$. We show that the optimal control for (1.1), (1.2) with $T = \infty$ is given by the usual feedback control involving $Q_\infty$. We introduce two quadratic problems of different kind. If (1.1) is replaced by

$$y' = A(t)y + B(t)u + f(t), \tag{1.5}$$

$$y(t_0) = y_0, \tag{1.6}$$

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then a more natural cost functional is

$$J_1(u)=\overline{\lim_{T\to\infty}}\,\frac{1}{T}\int_{t_0}^{t_0+T}[|M(t)y|^2+\langle N(t)u,u\rangle]\,dt. \tag{1.7}$$

With (1.5) we also associate

$$J_2(u)=\overline{\lim_{T\to\infty}}\,\frac{1}{2T}\int_{-T}^{T}[|M(t)y|^2+\langle N(t)u,u\rangle]\,dt. \tag{1.8}$$

We will show that $Q_\infty$ also characterizes optimal control of these problems. This is a generalization of the average cost criterion (usually for time-invariant systems) to time-varying systems.

In § 2 we give basic assumptions on our system (1.1), (1.2).

In § 3 we establish the existence of a bounded solution to the Riccati equation (1.3). We then characterize optimal control using $Q_\infty$. We will show that the optimal closed-loop system for (1.1) is exponentially asymptotically stable. In § 4 we consider the stochastic case and obtain similar results. We also consider the partially observable case and show that the separation principle holds [20], [40].

An important special class of time-varying systems is that of periodic systems. See [7], [8], [22], [35], and [37] for various examples of periodic systems and their optimization problems. We have studied [17] the quadratic problem (1.5)-(1.7) and its stochastic version for periodic systems. We may allow for almost periodic inputs such as in [16].

This paper is an extension of the last two papers. Hence if we assume $\theta$-periodicity of our system, we recover earlier results in [16] and [17].

**2. Preliminaries.** Let $Z$ be a Hilbert space ($\langle\ ,\ \rangle$ inner product, $|\ |$ norm). We will denote by $\mathscr{L}(Z)$ the Banach space of all linear bounded operators $S:Z\to Z$ endowed with the norm $|S|=\sup\{|Sz|: z\in Z, |z|\leqq 1\}$. If $S\in\mathscr{L}(Z)$ then $S^*$ will represent its adjoint operator. $S$ is called nonnegative ($S\geqq 0$) if $S$ is self-adjoint and $\langle Sz,z\rangle\geqq 0$, for all $z\in Z$. We set $\mathscr{L}^+(Z)=\{S\in\mathscr{L}(Z): S\geqq 0\}$. If $L:D(L)\subset Z\to Z$ is a linear operator we denote by $\sigma(L)$ (respectively, $\rho(L)$) the spectrum (respectively, the resolvent set) of $L$ and by $R(\lambda, L)$, $\lambda\in\rho(L)$ the resolvent operator of $L$.

For each interval $J$ in $\mathbb{R}^1$ we denote by $C_S(J;\mathscr{L}(Z))$ the set of all mappings $S(t):J\to\mathscr{L}(Z)$ that are strongly continuous, that is, $S(t)z$ is continuous on $J$ for any $z\in Z$. If $J$ is closed and bounded, then due to the Uniform Boundedness Theorem, $C_S(J;\mathscr{L}(Z))$ is a Banach space with respect to the norm:

$$\|S\|=\sup\{|S(t); t\in J\}.$$

We set $C_S(J;\mathscr{L}^+(Z))=\{S\in C_S(J;\mathscr{L}(Z)); S(t)\geqq 0, t\in J\}$. If $X$ is another Hilbert space, we denote by $\mathscr{L}(X,Z)$ the set of all bounded linear operators from $X$ into $Z$ and by $C_S(J;\mathscr{L}(X,Z))$ the set of all strongly continuous mappings from $J$ into $\mathscr{L}(X,Z)$.

Let $Y$ be a Hilbert space. We consider the initial value problem

$$y'=A(t)y+f(t),\quad y(s)=y_0,\quad t\geqq s \tag{2.1}$$

where $y_0\in Y$ and $f\in L^2_{\text{loc}}(0,\infty;Y)$, the set of locally square integrable functions. We assume the following on $A(t)$:

(H1) (i) For any $t\in\mathbb{R}^1$, $A(t)$ is a linear operator in $Y$ with a constant domain $D$ dense in $Y$. There exist numbers $\bar{M}>0$, $\eta\in(\pi/2,\pi)$, $\delta\in\mathbb{R}^1$ such that

$$S_{\delta,\eta}=\{\lambda\in\mathbb{C}: |\arg(\lambda-\delta)|<\eta\}\subset\rho(A(t))\quad\forall t\in\mathbb{R}^1$$

and the resolvent operator satisfies

$$|R(\lambda, A(t))| \leqq \bar{M}/|\lambda - \delta| \quad \forall \lambda \in S_{\delta,\eta}.$$

(ii) There exist numbers $\alpha \in (0, 1)$ and $\bar{N}$ such that

$$|A(t)x - A(s)x| \leqq \bar{N}|t-s|^{\alpha}|A(0)x| \quad \forall x \in D.$$

*Remark* 2.1. The hypothesis (H1) has been introduced by Tanabe [36] to study the abstract equation $y' = A(t)y$ where $A(t)$'s are infinitesimal generators of analytic semigroups with constant domains (parabolic equations). In fact, in the sequel we will only need the existence of an evolution operator $U(t, s)$ relative to $A(t)$. Thus our results can be easily arranged to cover hyperbolic equations as well as parabolic equations with nonconstant domains $D(A(t))$. If $A$ is constant, (H1) is interpreted as $A$ being the infinitesimal generator of a $C_0$-semigroup. Then functional differential equations can be covered [10].

The following result is proved in [36].

PROPOSITION 2.1. *Assume* (H1). *Then there exists a family of operators* $U(t, s) \in \mathscr{L}(Y)$, $t \geqq s$ *such that*

(i) $U(s, s) = I$, $s \in \mathbb{R}^1$,

(ii) $U(\cdot, \cdot)x$ *is continuous for any* $x \in Y$,

(iii) *For* $t > s$, $U(t, s)(Y) \subset D$ *and* $U(t, s)$ *is differentiable in t with*

$$\frac{\partial U(t, s)}{\partial t} = A(t)U(t, s).$$

$U(t, s)$ is called the *evolution operator* relative to $A(t)$. It is called (*exponentially*) *stable* if there exist positive numbers $\tilde{M}$, $\omega$ such that $|U(t, s)| \leqq \tilde{M} e^{-\omega(t-s)}$ for any $t \geqq s$.

We denote by $A_n(t) = n^2R(n, A(t)) - nI$ the Yosida approximations of $A(t)$ and by $U_n(t, s)$ the evolution operator relative to $A_n(t)$ (that clearly exists since $A_n(t)$'s are bounded). By using the results in [36] it is easy to prove that

$$\lim_{n\to\infty} U_n(t, s)x = U(t, s)x \quad \forall t \geqq s, \quad \forall x \in Y \tag{2.2}$$

uniformly on the bounded sets of $\mathbb{R}^2$.

We define the *mild solution* of (2.1) by

$$y(t) = U(t, s)y_0 + \int_s^t U(t, r)f(r)\, dr. \tag{2.3}$$

It is continuous on $[s, \infty)$. Let $y_n$ be the classical solution to the problem:

$$y_n' = A_n(t)y_n + f(t), \qquad y_n(s) = y_0. \tag{2.4}$$

Then, by (2.2) $y_n(t) \to y(t)$ uniformly on any bounded subset of $[s, \infty)$.

Assume that $A(t)$ is stable. Then for each $f \in L^\infty([s, \infty); Y)$ (the set of bounded measurable functions in $Y$) $y(t)$ defined by (2.3) is bounded. We now consider for each $f \in L^\infty(\mathbb{R}^1; Y)$

$$y' = A(t)y + f(t) \tag{2.4'}$$

on $[-\infty, \infty)$. We say that $y(t)$ is a mild solution on $(-\infty, \infty)$ if it satisfies the integral equation

$$y(t) = U(t, s)y(s) + \int_s^t U(t, r)f(r)\, dr \tag{2.5}$$

for any $t \geqq s$. A mild solution $y(t)$ is called *bounded* if it is bounded on $(-\infty, \infty)$. If $A(t)$ is stable, then there exists a unique bounded mild solution of (2.4'). In fact it is given by

$$y(t) = \int_{-\infty}^{t} U(t, r) f(r)\, dr. \tag{2.6}$$

If $A$, $f$ are $\theta$-periodic, then $y$ is also $\theta$-periodic. If $A$ is $\theta$-periodic and $f$ is almost periodic, then $y$ is almost periodic [16].

## 3. Optimal quadratic control in the deterministic case.

**3.1. Bounded solutions of a Riccati equation.** We consider the usual quadratic control problem.

$$y' = A(t)y + B(t)u, \qquad y(t_0) = y_0, \tag{3.1}$$

$$J_0(u) = \int_{t_0}^{\infty} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt \tag{3.2}$$

where $A(t)$ satisfies the condition (H1) and

(H2) (i) $B \in C_S(\mathbb{R}^1, \mathscr{L}(U, Y)) \cap L^\infty(\mathbb{R}^1, \mathscr{L}(U, Y))$, $M \in C_S(\mathbb{R}^1; \mathscr{L}(Y))$, $N \in C_S(\mathbb{R}^1; \mathscr{L}^+(U))$ and there exists an $\varepsilon > 0$ such that $N(t) \geqq \varepsilon$ for any $t \in \mathbb{R}^1$.

(ii) $\sup_{t\in\mathbb{R}} [|M(t)| + |N(t)|] < \infty$.

We wish to minimize $J(u)$ over the set of admissible controls

$$\mathscr{U}_{\text{ad}}^0 = \{u \in L^2([t_0, \infty); U)\text{: the corresponding mild solution } y(t) \to 0 \text{ as } t \to \infty\}. \tag{3.3}$$

To solve this problem the following Riccati equation is useful:

$$Q'(t) + A^*(t)Q(t) + Q(t)A(t) + M^*(t)M(t) - Q(t)B(t)N^{-1}(t)B^*(t)Q(t) = 0. \tag{3.4}$$

We say that $Q$ is a mild solution of (3.4) on the interval $J \subset \mathbb{R}^1$ if $Q \in C_S(J, \mathscr{L}^+(Y))$ and if it satisfies the integral equation

$$\begin{aligned} Q(t)x &= U^*(s, t)Q(s)U(s, t)x \\ &\quad + \int_t^s U^*(r, t)[M^*(r)M(r) - Q(r)B(r)N^{-1}(r)B^*(r)Q(r)]U(r, t)x\, dr \end{aligned} \tag{3.5}$$

for any $x \in Y$ and $t \leqq s$, $t, s \in J$. If $\sup_{t\in\mathbb{R}^1} |Q(t)| < \infty$, we say that $Q$ is a *bounded solution* of (3.4). Even if $Q$ is a solution of the integral equation (3.5) we cannot in general prove that $Q(t)x$ is differentiable for $x \in Y$. Thus $Q$ is not a classical solution to (3.4). Therefore it is useful to introduce approximating systems

$$Q_n' + A_n^* Q_n + Q_n A_n + M^*M - Q_n B N^{-1} B^* Q_n = 0, \tag{3.6}$$

which have classical solutions. The following result is proved in [4].

PROPOSITION 3.1. *Assume* (H1) *and* (H2). *Let* $T \in \mathbb{R}^1$ *and* $Q_0 \in \mathscr{L}^+(Y)$. *Then there exists a unique mild solution* $Q$ *of* (3.4) *on* $(-\infty, T]$ *such that* $Q(T) = Q_0$. *Moreover, there exists a unique classical solution* $Q_n \in C_S((-\infty, T]; \mathscr{L}^+(Y))$ *to* (3.6) *with* $Q_n(T) = Q_0$ *and* $Q_n(t)x \to Q(t)x$ *as* $n \to \infty$ *for any* $x \in Y$ *uniformly on any bounded subset of* $(-\infty, T]$.

In the sequel we set $Q(t) = \Lambda(t; T, Q_0)$, $Q_n(t) = \Lambda_n(t; T, Q_0)$. The following monotonicity property of $\Lambda(\Lambda_n)$ is well known:

$$\begin{aligned} &\Lambda(t; T, Q_0) \leqq \Lambda(t; T, Q_1), \\ &\Lambda_n(t, T, Q_0) \leqq \Lambda_n(t, T, Q_1) \quad \text{if } Q_0 \leqq Q_1. \end{aligned} \tag{3.7}$$

Now we will establish a bounded solution to (3.4). Let $C_b(\mathbb{R}^1; Z)$ be the space of all bounded continuous functions from $\mathbb{R}^1$ to $Z$. The fundamental hypothesis for the existence of a bounded solution is the following:

(H3) For any $t_0 \in \mathbb{R}$ and $y_0 \in Y$ there exist $u \in C_b(\mathbb{R}^1; U)$ and $C_0 > 0$ such that

$$\int_{t_0}^{\infty} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt \leqq C_0 |y_0|^2$$

where $y$ is the mild solution to (3.1).

Hypothesis (H3) is slightly weaker than the existence of an admissible control. It is satisfied, as we will see below, if $(A, B)$ is *stabilizable*, that is, if there exists $K \in C_S(\mathbb{R}^1; \mathscr{L}(Y, U))$ bounded such that the evolution operator relative to $A - BK$ is stable (such an evolution operator does exist since $BK$ is bounded [10]).

Assume that $(A, B)$ is stabilizable so that for some $K$

$$|U_K(t, s)| \leqq M_0\, e^{-\omega(t-s)}, \qquad t \geqq s \quad \text{for some } M_0 \geqq 1 \text{ and } \omega > 0 \tag{3.8}$$

where $U_K$ is the evolution operator relative to $A - BK$. Now we will show that the hypothesis (H3) is fulfilled. Set

$$y(t) = U_K(t, t_0)y_0, \qquad u(t) = -K(t)U_K(t, t_0)y_0.$$

Then

$$\int_{t_0}^{\infty} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt \leqq \frac{M_0^2}{2\omega}(\|M\|^2 + \|N\|\,\|K\|^2)$$

where $\|\cdot\| = \sup_{t \in \mathbb{R}^1} |\cdot|$.

The main result for our Riccati equation is the following theorem.

THEOREM 3.1. *Assume* (H1) *and* (H2). *Then a nonnegative bounded solution to* (3.4) *exists if and only if* (H3) *holds.*

*Proof. If.* Assume (H3). For any $\alpha \in \mathbb{R}^1$ set $Q_\alpha = \Lambda(\cdot\,; \alpha, 0)$. By (3.7) we have

$$Q_\alpha(t) \leqq Q_\beta(t) \quad \text{if } t \in (-\infty, \alpha] \text{ and } \alpha \leqq \beta. \tag{3.9}$$

Thus $\{Q_\alpha\}$ is increasing in $\alpha$. We will now show that $\|Q_\alpha(\cdot)\|$ is bounded. Let $Q_{\alpha,n} = \Lambda_n(\cdot\,; \alpha, 0)$ and let $y_n$ be the classical solution to the initial value problem:

$$y_n' = A_n(t)y_n + B(t)u, \qquad y_n(t_0) = y_0 \tag{3.10}$$

where $u$ is the control function given in (H3). We then have

$$\begin{aligned}\frac{d}{dt}\langle Q_{\alpha,n}(t)y_n(t), y_n(t)\rangle = \|N^{1/2}(u + N^{-1}B^*Q_{\alpha,n}y_n)|^2 - |M(t)y_n(t)|^2 \\ -\langle N(t)u(t), u(t)\rangle.\end{aligned} \tag{3.11}$$

Integrating this from $t_0$ to $\alpha$ and letting $n \to \infty$, we arrive at

$$\begin{aligned}&\int_{t_0}^{\alpha} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt \\ &\quad = \langle Q_{\alpha,n}(t_0)y_0, y_0\rangle + \int_{t_0}^{\alpha} |N^{1/2}(u + N^{-1}B^*Q_{\alpha,n}y_n)|^2\, dt,\end{aligned} \tag{3.12}$$

which yields

$$\langle Q_\alpha(t_0)y_0, y_0\rangle \leqq C_0|y_0|^2 \quad \text{for any } y_0 \in Y. \tag{3.13}$$

By a classical argument we can show that there exists $Q_\infty(t)$ such that

$$\lim_{\alpha\to\infty} Q_\alpha(t_0)y_0 = Q_\infty(t_0)y_0 \quad \text{for any } t_0 \in \mathbb{R}^1 \text{ and } y_0 \in Y. \tag{3.13'}$$

To prove that $Q_\infty$ is a mild solution of (3.4) on $(-\infty, \infty)$ it suffices to let $\alpha \to \infty$ in the equality:

$$\begin{aligned} Q_\alpha(t)x = U^*(s,t)Q_\alpha(s)U(s,t) \\ + \int_t^s U^*(r,t)[M^*(r)M(r) \\ - Q_\alpha(r)BN^{-1}BQ_\alpha(r)]U(r,t)\,dr \end{aligned} \tag{3.14}$$

for $t \leqq s < \alpha$.

*Only if.* Let $Q$ be a bounded solution to (3.4). For a fixed but otherwise arbitrary $T \in \mathbb{R}^1$, let $Q_n = \Lambda_n(\cdot, T; Q(T))$. Set $K = BN^{-1}B^*$, $L = A - KQ$, $L_n = A_n - KQ_n$ and let $U_L$ and $U_{L_n}$ be evolution operators relative to $L$ and $L_n$, respectively. Then

$$Q_n' + L_n^*Q_n + Q_nL_n + M^*M + Q_nKQ_n = 0, \qquad Q_n(T) = Q(T). \tag{3.15}$$

Hence, for any $t_0 \leqq t \leqq T$, we have

$$\begin{aligned} \frac{d}{dt}\langle Q_n(t)U_{L_n}(t,t_0)y_0, U_{L_n}(t,t_0)y_0\rangle \\ = -|M(t)U_{L_n}(t,t_0)y_0|^2 - |\sqrt{K(t)}Q_n(t)U_{L_n}(t,t_0)y_0|^2. \end{aligned} \tag{3.16}$$

Integrating this from $t_0$ to $t_1$ and letting $n \to \infty$, we obtain

$$\begin{aligned} \int_{t_2}^{t_1} [|M(t)U_L(t,t_0)y_0|^2 + |\sqrt{K(t)}Q(t)U_L(t,t_0)y_0|^2]\,dt \\ + \langle Q(t_1)U_L(t_1,t_0)y_0, U_L(t_1,t_0)y_0\rangle = \langle Q(t_0)y_0, y_0\rangle. \end{aligned} \tag{3.17}$$

Now set

$$y(t) = U_L(t,t_0)y_0, \qquad u(t) = -N^{-1}(t)B^*(t)Q(t)U_L(t,t_0)y_0.$$

Then $u \in C_b(\mathbb{R}^1; U)$ and $y$ is a mild solution of (3.1). Moreover,

$$\int_{t_0}^{t_1} [|M(t)y|^2 + \langle N(t)u, u\rangle]\,dt \leqq \sup_{t_0\in\mathbb{R}} |Q(t_0)|\,|y_0|^2, \qquad t_0 \leqq t_1 \leqq T.$$

Since $t_1$ is arbitrary, we have shown (H3). □

If (H1)-(H3) hold, we will denote by $Q_\infty$ the bounded solution of (3.4) defined by (3.13). We remark that $Q_\infty$ is minimal among all solutions $Q \geqq 0$ of (3.4) on $\mathbb{R}^1$, that is,

$$Q(t) \geqq Q_\infty(t), \qquad t \in \mathbb{R}^1. \tag{3.18}$$

In fact if $Q$ is a solution of (3.4) on $\mathbb{R}^1$, we have

$$Q(\alpha) \geqq Q_\alpha(\alpha) = 0$$

so that

$$Q(t) \geqq Q_\alpha(t) \quad \text{for any } t \in (-\infty, \alpha].$$

Letting $\alpha \to \infty$ we obtain (3.18). We will call $Q_\infty$ the *minimal solution* of (3.4) on $\mathbb{R}^1$.

Next we will examine the stability property of a bounded solution $Q$ of (3.4). Set $L=A-KQ$, $K=BN^{-1}B^*$ and for a fixed $T\in\mathbb{R}^1$ let $Q_1(\cdot)=\Lambda(\cdot;T,S)$, where $S\in\mathscr{L}^+(Y)$. Then we can easily check that $Z=Q_1-Q$ is a mild solution of the equation

$$Z'+L^*Z+ZL-ZKZ=0, \qquad Z(T)=S-Q(T). \tag{3.19}$$

If $U_L$ is stable, the usual linearization arguments show that $Q$ is uniformly asymptotically stable as $t\to-\infty$ [23]. But as we will see below we can show that $Q$ is attractive from above, and hence it is maximal among all bounded nonnegative solutions of (3.4). This will imply, in particular, the uniqueness of a nonnegative bounded solution for which $L$ is stable. We say that a bounded solution $Q$ of (3.4) is *stable* if $A-KQ$ is stable.

PROPOSITION 3.2. *Assume* (H1) *and* (H2) *and let Q be a stable bounded nonnegative solution to* (3.4). *Let* $T\in\mathbb{R}^1$, $S=\mathscr{L}^+(Y)$ *be arbitrary and set* $Q_1(\cdot)=\Lambda(\cdot;T,S)$. *If* $S\geqq Q(T)$, *then*

$$\lim_{t\to-\infty}(Q_1(t)x-Q(t)x)=0 \quad \textit{for any } x\in Y.$$

$Q_1(\cdot)$ *with arbitrary* $S\geqq 0$ *is bounded on* $(-\infty,T]$.

*Moreover, if* $Q_2$ *is a bounded solution, then* $Q_2(t)\leqq Q(t)$, $t\in\mathbb{R}$.

*Proof.* Let $Z=Q_1-Q$, $Q_n=\Lambda_n(\cdot;T,Q(T))$, $Q_{1n}=\Lambda_n(\cdot;T,S)$, and define $Z_n=Q_{1n}-Q_n$, $L_n=A_n-KQ_n$. Then

$$Z_n'+L_n^*Z_n+Z_nL_n-Z_nKZ_n=0, \tag{3.20}$$

from which follows

$$\frac{d}{dt}\langle Z_n(t)U_{L_n}(t,t_0)y_0,\,U_{L_n}(t,t_0)y_0\rangle=|\sqrt{K(t)}\,Z_n(t)U_{L_n}(t,t_0)y_0|^2. \tag{3.21}$$

Integrating from $t_0$ to $t$ and letting $n\to\infty$, we obtain

$$\langle Z(t)U_L(t,t_0)y_0,\,U_L(t,t_0)y_0\rangle\geqq\langle Z(t_0)y_0,y_0\rangle, \qquad t_0\leqq t. \tag{3.22}$$

Now, if $S\geqq Q(T)$ then $Z(t_0)\geqq 0$ for any $t_0\leqq T$. Letting $t_0\to-\infty$ in (3.22), we obtain $\langle Z(t_0)y_0,y_0\rangle\to 0$ as $t_0\to-\infty$. Hence $\lim_{t\to-\infty}Z(t)x=0$ for any $x\in Y$. Assume now that $Q_2$ is another nonnegative bounded solution of (3.4). Then replacing $Q_1$ by $Q_2$ in (3.22) and letting $t\to\infty$ we find $\langle Z(t_0)y_0,y_0\rangle\leqq 0$ so that $Q_2(t)\leqq Q(t)$. □

Now we give a sufficient condition for a bounded nonnegative solution of (3.4) being stable:

(H4) There exists a $K_1\in C_S(\mathbb{R}^1;\mathscr{L}(Y))$ bounded such that $A-K_1M$ is stable.

If (H4) holds, we say that $(A,M)$ is *detectable*.

PROPOSITION 3.3. *Assume* (H1), (H2), *and* (H4). *Then any bounded nonnegative solution of* (3.4) *is stable. Thus the Riccati equation* (3.4) *has at most one bounded nonnegative solution.*

*Proof.* Let $t_0\in\mathbb{R}$ be fixed and let $y_0\in Y$. Then by (3.17) we have

$$M(t)U_L(t,t_0)y_0, \qquad \sqrt{K(t)}Q(t)U(t,t_0)y_0\in L^2(t_0,\infty;Y). \tag{3.23}$$

Let $S=A-K_1M$; then $L=S+K_1M-KQ$ so that

$$U_L(t,t_0)y_0=U_S(t,t_0)y_0-\int_{t_0}^{t}U_S(t,r)(K_1M-KQ)U_L(r,t_0)y_0\,dr. \tag{3.24}$$

Since $U_S$ is stable, it follows from (3.23), (3.24) that $U_L(t, t_0)y_0 \in L^2(t_0, \infty; Y)$. By Datko [19] $U_L$ is stable.

The uniqueness follows from the fact that a stable solution is maximal. □

In practice it may be more natural to assume that the system (3.1) is defined only on $[T_0, \infty)$ so that $T_0 \leqq T_0 < \infty$. In this case we restrict the hypothesis (H1)-(H4) on $[T_0, \infty)$. We need to modify the definitions.

For example, we say that $(A, B)$ is stabilizable if there exists $K \in C_S([T_0, \infty); L(Y, U))$ bounded such that $|U_{A-BK}(t, s)| \leqq M_0 e^{-\omega(t-s)}$, $t \geqq s \geqq T_0$ for some $M_0 \geqq 1$ and $\omega > 0$. Now all the results restricted on $[T_0, \infty)$ are true. In fact, we have the following corollaries.

COROLLARY 3.1. *Assume* (H1) *and* (H2) *on* $[T_0, \infty)$. *Then a nonnegative bounded solution of* (3.4) *on* $[T_0, \infty)$ *exists if and only if* (H3) *holds on* $[T_0, \infty)$.

COROLLARY 3.2. *Assume* (H1) *and* (H2) *on* $[T_0, \infty)$. *Let $Q$ be a stable nonnegative bounded solution of* (3.4) *on* $[T_0, \infty)$. *Then for any bounded solution* $Q_2 \geqq 0$ *of* (3.4), $Q_2(t) \leqq Q(t)$, $t \in [T_0, \infty)$, *that is $Q$ is maximal.*

COROLLARY 3.3. *Assume* (H1), (H2), *and* (H4) *on* $[T_0, \infty)$. *Then any nonnegative bounded solution of* (3.4) *is stable. Thus the Riccati equation* (3.4) *has at most one nonnegative bounded solution on* $[T_0, \infty)$.

Finally, we consider two special cases of (3.4): the periodic case and the time invariant case.

In the former we assume (H5).

(H5) There exists a number $\theta > 0$ such that $A(t+\theta) = A(t)$, $B(t+\theta) = B(t)$, $M(t+\theta) = M(t)$, and $N(t+\theta) = N(t)$ for all $t \in \mathbb{R}^1$.

In this case we say that these operators are $\theta$-periodic. We say also that the system (3.1) is $\theta$-periodic. If $Q$ is a bounded solution to (3.4), then $Q_\theta(t)$ defined by

$$Q_\theta(t) = Q(t-\theta), \qquad t \in \mathbb{R}^1$$

is also a bounded solution. Thus if (3.4) has a unique nonnegative bounded solution $Q$ (for example, if $(A, M)$ is detectable) we have $Q(t) = Q(t-\theta)$. In fact, we have Proposition 3.4.

PROPOSITION 3.4. *Assume* (H1), (H3), *and* (H5). *Then the minimal solution* $Q_\infty$ *of* (3.4) *is $\theta$-periodic. If, further,* (H4) *holds, then* $Q_\infty$ *is the unique nonnegative $\theta$-periodic solution to* (3.4) *and it is uniformly asymptotically stable.*

*Proof.* Let $n$ be an integer and set

$$V(t) = Q_{n\theta}(t-\theta), \qquad t \in (-\infty, (n+1)\theta] \tag{3.25}$$

where $Q_{n\theta} = \Lambda(\cdot\,; n\theta, 0)$. Since the coefficients of (3.4) are $\theta$-periodic, $V$ is also a solution of (3.4) on $(-\infty, (n+1)\theta]$. Moreover, $V((n+1)\theta) = Q_{n\theta}(n\theta) = 0$ so that

$$V(t) = Q_{(n+1)\theta}(t) = Q_{n\theta}(t-\theta). \tag{3.26}$$

Now, letting $n \to \infty$ we obtain $Q_\infty(t) = Q_\infty(t-\theta)$. Thus $Q_\infty$ is $\theta$-periodic. Other assertions follow from Proposition 3.2. □

Next we show the global orbital attractiveness of $Q_\infty$.

PROPOSITION 3.5. *Assume* (H1)-(H5). *Let* $S_0 \in \mathscr{L}^+(Y)$ *and set* $Q = \Lambda(\cdot\,; 0, S_0)$. *Then*

$$\lim_{n\to\infty} Q(t-n\theta)x = Q_\infty(t)x \quad \forall t \in (-\infty, 0]. \tag{3.27}$$

*Proof.* Let $m$ be an integer such that

$$S_0 \leqq mI \quad \text{and} \quad Q_\infty(0) \leqq mI.$$

Let $V(\cdot)=\Lambda(\cdot\,; 0, mI)$. Then

$$Q(t) \geqq Q_0(t) \quad \forall t \leqq 0,$$

$$Q(t-n\theta) \geqq Q_0(t-n\theta) = Q_{n\theta}(t) \quad \forall t \leqq n\theta.$$

This implies

$$Q_{n\theta}(t) \leqq Q(t-n\theta) \leqq V(t-n\theta).$$

But $Q_{n\theta}(t)x \to Q_\infty(t)x$ as $n\to\infty$ and $V(t-n\theta)x \to Q_\infty(t)x$ by Proposition 3.2. Thus (3.27) follows. □

*Remark* 3.1. In Da Prato [12] the existence of a periodic solution to (3.4) is shown under (H1), (H2), (H5), and stabilizability of $(A, B)$. Hence, Proposition 3.4 gives a weaker condition. Proposition 3.5 is also an improvement of Lemma 3.1 [17]. See [31] and [34] for finite-dimensional results.

Now consider the time-invariant case: $A$, $B$, $M$, and $N$ are independent of $t$. Then (H1) can be replaced by the hypothesis that $A$ is the infinitesimal generator of a $C_0$-semigroup $e^{tA}$. Hypothesis (H2) simply implies $B \in \mathcal{L}(U, Y)$, $M \in \mathcal{L}(Y)$, and $N, N^{-1} \in \mathcal{L}^+(U)$. Hypothesis (H4) is the usual detectability condition [39], [41]. Hence we recover the results of Zabczyk [41].

PROPOSITION 3.6. *Suppose that $A$ is the infinitesimal generator of a $C_0$-semigroup $e^{tA}$ and that $B \in \mathcal{L}(U, Y)$, $M \in \mathcal{L}(Y)$, and $N, N^{-1} \in \mathcal{L}^+(U)$. Suppose (H3) holds. Then $Q_\infty(t) = Q_\infty$ is independent of $t$ and is the minimal solution of the algebraic Riccati equation*

$$A^*Q + QA + M^*M - QBN^{-1}B^*Q = 0. \tag{3.28}$$

*If, further, $(A, M)$ is detectable, then $Q_\infty$ is the unique nonnegative solution to* (3.20) *in $\mathcal{L}^+(Y)$. Moreover, for each $Q = \Lambda(\cdot, \cdot, S_0)$ with $S_0 \in \mathcal{L}^+(Y)$*

$$\overline{\lim_{t\to-\infty}}\, Q(t)x = Q_\infty x \quad \forall x \in Y.$$

**3.2. Quadratic control on the infinite horizon.** Let $-\infty \leqq T_0 < \infty$ and let $t_0 \in [T_0, \infty)$ be arbitrary. If $T_0 = -\infty$, we mean by $[T_0, \infty)$, the whole real line $(-\infty, \infty)$. Now consider our control problem

$$y' = A(t)y + B(t)u, \qquad y(t_0) = y_0, \tag{3.1}$$

$$J_0(u) = \int_{t_0}^{\infty} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt. \tag{3.2}$$

We assume (H1)-(H4) on $[T_0, \infty)$ and wish to minimize $J_0(u)$ over

$$\mathcal{U}^0_{\mathrm{ad}} = \{u \in L^2(t_0, \infty; U)\text{: the corresponding mild solution } y(t) \to 0 \text{ as } t \to \infty\}. \tag{3.3}$$

In view of (H3), (H4), this problem is nontrivial. Let $Q_\infty$ be the unique stable nonnegative bounded solution of (3.4) on $[T_0, \infty)$. The main tool we will use is the following identity.

LEMMA 3.1. *Assume* (H1)-(H3) *on* $[T_0, \infty)$. *Let $u \in L^2([t_1, t_2]; U)$, $t_0 \leqq t_1 < t_2$ and let $y$ be the mild solution of* (3.1) *on* $[t_1, t_2]$. *Then*

$$\begin{aligned} &\int_{t_1}^{t_2} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt + \langle Q_\infty(t_2)y(t_2), y(t_2)\rangle \\ &\quad = \int_{t_1}^{t_2} |N^{1/2}(u + N^{-1}B^*Q_\infty y)|^2\, dt + \langle Q_\infty(t_1)y(t_1), y(t_1)\rangle. \end{aligned} \tag{3.29}$$

*Proof.* Let $Q_n=\Lambda_n(\cdot\,; t_2, Q_\infty(t_2))$ and let $y_n$ be the solution of the initial value problem

$$y_n'=A_n(t)y_n+B(t)u, \qquad y_n(t_1)=y(t_1).$$

Then we have

$$\frac{d}{dt}\langle Q_n(t)y_n(t), y_n(t)\rangle=|N^{1/2}(u+N^{-1}B^*Q_ny_n)|^2-|M(t)y_n|^2-\langle N(t)u, u\rangle.$$

Integrating this from $t_1$ and $t_2$ and passing to the limit $n\to\infty$, we obtain (3.29).

Now it is easy to solve our control problem.

THEOREM 3.2. *Assume* (H1)-(H4) *on* $[T_0,\infty)$. *Then the optimal control is given by the feedback law*

$$\bar u=-N^{-1}B^*Q_\infty\bar y \tag{3.30}$$

*and the optimal cost by*

$$J_0(\bar u)=\langle Q_\infty(t_0)y_0, y_0\rangle. \tag{3.31}$$

*The optimal closed loop system is stable. If, further,* (H5) *holds, then* $Q_\infty$ *is* $\theta$*-periodic. If all operators in* (3.1), (3.2) *are time invariant, then* $Q_\infty$ *is constant.*

*Proof.* We set $t_1=t_0$ and pass to the limit $t_2\to\infty$. Since $y(t_2)\to 0$ as $t_2\to\infty$ we obtain

$$J_0(u)=\int_{t_0}^{\infty}|N^{1/2}(u+N^{-1}B^*Q_\infty y)|^2\,dt+\langle Q_\infty(t_0)y_0, y_0\rangle.$$

Thus the conclusion follows immediately. □

**3.3. The optimal control problem with average cost.** Assume (H1), (H2) on $[T_0,\infty)$. Here we are concerned with a more general system

$$y'=A(t)y+B(t)u+f(t), \qquad y(t_0)=y_0 \tag{3.32}$$

where $t_0\in[T_0,\infty)$ is fixed but otherwise arbitrary and $f\in C_b([T_0,\infty), Y)$. In this case we cannot expect that the cost $J_0(u)$ is finite. Instead we take a more reasonable cost

$$J_1(u)=\overline{\lim_{T\to\infty}}\frac{1}{T}\int_{t_0}^{t_0+T}[|M(t)y|^2+\langle N(t)u, u\rangle]\,dt \tag{3.33}$$

and we wish to minimize it over

$$\mathcal{U}'_{\text{ad}}=\left\{u\colon \overline{\lim_{T\to\infty}}\frac{1}{T}\int_{t_0}^{t_0+T}|u(t)|^2\,dt<\infty \text{ and the mild solution } y(t) \text{ of (3.32) is bounded on } [t_0,\infty)\right\}. \tag{3.34}$$

We further assume (H3) and let $Q_\infty$ be the minimal bounded solution of (3.4) on $[T_0,\infty)$.

Let $L=A-KQ_\infty$, $K=BN^{-1}B^*$ and consider the following equation:

$$r'+L^*r+Q_\infty f=0. \tag{3.35}$$

Then we have a similar result to Lemma 3.1.

LEMMA 3.2. *Assume* (H1)-(H3) *on* $[T_0, \infty)$. *Let* $u \in L^2([t_1, t_2]; U)$, $t_0 \leqq t_1 < t_2$, *and let* $y$ *and* $r$ *be mild solutions of* (3.32), (3.35) *on* $|t_1, t_2|$, *respectively. Then*

$$(3.36)\quad \begin{aligned} &\int_{t_1}^{t_2} [|M(t)y|^2 + \langle N(t)u, u\rangle]\,dt + \langle Q_\infty(t_2)y(t_2), y(t_2)\rangle + 2\langle r(t_2), y(t_2)\rangle \\ &\quad = \int_{t_1}^{t_2} |N^{1/2}[u + N^{-1}B^*(Q_\infty y + r)]|^2\,dt \\ &\quad\quad + \int_{t_1}^{t_2} [2\langle r, f\rangle - |N^{-1/2}B^* r|^2]\,dt \\ &\quad\quad + \langle Q_\infty(t_1)y(t_1), y(t_1)\rangle + 2\langle r(t_1), y(t_1)\rangle. \end{aligned}$$

*Proof.* We take

$$\begin{aligned} &Q_n(\cdot) = \Lambda_n(\cdot\,; t_2, Q_\infty(t_2)), \\ &y_n' = A_n y_n + Bu + f, \qquad y_n(t_1) = y(t_1), \\ &r_n' + (A_n - KQ_n)^* r_n + Q_n f = 0, \qquad r_n(t_2) = r(t_2) \end{aligned}$$

and show

$$\begin{aligned} \frac{d}{dt}[\langle Q_n y_n, y_n\rangle + 2\langle r_n, y_n\rangle] &= |N^{1/2}[u + N^{-1}B^*(q_n y_n + r_n)]|^2 \\ &\quad - |My_n|^2 - \langle Nu, u\rangle + 2\langle r_n, f_n\rangle - |N^{-1/2}B^* r_n|^2. \end{aligned}$$

Integrating this from $t_1$ to $t_2$ and passing to the limit $n \to \infty$, we obtain (3.36).

It we further assume (H4) on $[T_0, \infty)$, then $\mathcal{U}'_{\rm ad}$ is not empty. To see this, note that there exists a unique bounded solution to (3.35) given by

$$(3.37)\qquad r(t) = \int_t^\infty U_L^*(s, t)Q_\infty(s)f(s)\,ds,$$

since $L$ is stable. Now consider the feedback control

$$(3.38)\qquad \bar{u} = -N^{-1}B^*(Q_\infty \bar{y} + r).$$

Then the closed-loop system is

$$(3.39)\qquad \bar{y}' = L\bar{y} + f - Kr, \qquad \bar{y}(t_0) = y_0.$$

Since $L$ is stable, the mild solution

$$(3.40)\qquad \bar{y}(t) = U_L(t, t_0)y_0 + \int_{t_0}^t U_L(t, s)[f(s) - K(s)r(s)]\,ds$$

is bounded on $[t_0, \infty)$. Hence $u$ is admissible.

THEOREM 3.3. *Assume* (H1)-(H4) *on* $[T_0, \infty)$. *Then the optimal control is given by the feedback law* (3.38) *and the optimal cost by*

$$(3.41)\qquad J_1(\bar{u}) = \overline{\lim_{T\to\infty}} \frac{1}{T}\int_{t_0}^{t_0+T} [2\langle r, f\rangle - |N^{-1/2}B^* N|^2]\,dt.$$

*Proof.* We take any $u \in \mathcal{U}'_{\rm ad}$ and its response $y$ in (3.36). Then, setting $t_1 = t_0$, $t_2 = t_0 + T$ and taking limit supremum as $T \to \infty$, we obtain

$$(3.42)\qquad J_1(\bar{u}) = \overline{\lim_{T\to\infty}} \frac{1}{T}\int_{t_0}^{t_0+T} \{|N^{1/2}[u + N^{-1}B^*(Qy + r)]|^2 + 2\langle r, f\rangle - |N^{-1/2}B^* r|^2\}\,dt.$$

Now the assertion follows easily. □

If we assume (H5), then we recover the periodic result in [16].

COROLLARY 3.4. *Assume* (H1)-(H4) *on* $[T_0, \infty)$.

(i) *If* (H5) *holds on* $[T_0, \infty)$ *and if f is θ-periodic, then* $Q_\infty$ *and r are θ-periodic and*

$$J_1(\bar{u}) = \frac{1}{\theta} \int_{t_0}^{t_0+\theta} [2\langle r, f\rangle - |N^{-1/2}B^* r|^2]\, dt \quad \text{for any } t_0 \leqq T_0. \tag{3.43}$$

(ii) *If all operators in* (3.32), (3.33), *and f are constant, then* $Q_\infty$ *is the unique solution of the algebraic Riccati equation* (3.28) *and*

$$J_1(\bar{u}) = 2\langle r, f\rangle - |N^{-1/2}B^* r|^2 \tag{3.44}$$

*where* $r = -(L^*)^{-1} Q_\infty f$.

**3.4. The optimal control problem with average cost II.** Here we assume (H1) and (H2). Our system is

$$y' = A(t)y + B(t)u + f(t) \tag{3.45}$$

where $f \in C_b(\mathbb{R}^1; Y)$. We wish to minimize the average cost

$$J_2(u) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt, \tag{3.46}$$

over

$$\mathcal{U}^2_{\text{ad}} = \left\{ u: \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |u(t)|^2\, dt < \infty \right. \\ \left. \text{such that there exists a bounded solution to (3.45)} \right\}. \tag{3.47}$$

If (H3) holds, then Lemma 3.2 is valid. If, further, (H4) holds, then $L$ is stable and there exists a unique bounded solution to (3.35). Thus as in Theorem 3.3 we have Theorem 3.4.

THEOREM 3.4. *Assume* (H1)-(H4). *Then optimal control is given by the feedback law*

$$\bar{u} = -N^{-1}B^*(Q_\infty \bar{y} + r) \tag{3.48}$$

*where* $Q_\infty$ *is the unique bounded nonnegative stable solution to* (3.4) *and r is the unique bounded solution on* $\mathbb{R}^1$ *of the equation*

$$r' + L^* r + Q_\infty f = 0 \tag{3.49}$$

*given by*

$$r(t) = \int_t^\infty U_L^*(s, t) Q_\infty(s) f(s)\, ds, \qquad t \in \mathbb{R}^1. \tag{3.50}$$

*The optimal cost is given by*

$$J_2(\bar{u}) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} [2\langle r, f\rangle - |N^{-1/2}B^* r|^2]\, dt. \tag{3.51}$$

*The optimal response* $\bar{y}$ *is*

$$\bar{y}(t) = \int_{-\infty}^{t} U_L(t, s)[f(s) - K(s) r(s)]\, ds \tag{3.52}$$

*and $y$ is exponentially asymptotically stable (i.e., $y(t; t_0, y_0)-y(t)\to 0$, as $t\to\infty$ where $y(t; t_0, y_0)$ is the solution of* (3.39)).

COROLLARY 3.5. *Assume* (H1)-(H4).

(i) *If* (H5) *also holds and if $f$ is $\theta$-periodic, then $Q_\infty$ and $r$ are $\theta$-periodic and*

$$J_2(\bar{u})=\frac{1}{\theta}\int_0^\theta [2\langle r,f\rangle-|N^{-1/2}B^*r|^2]\,dt=J_1(\bar{u}). \tag{3.53}$$

(ii) *If all operators and $f$ are constant, then $Q_\infty$, $r$ are constant and given as in Corollary* 3.4. *Moreover,*

$$J_2(\bar{u})=2\langle r,f\rangle-|N^{-1/2}B^*r|^2=J_1(\bar{u}). \tag{3.54}$$

Finally, we will consider another special case of Theorem 3.4. Let $AP(\mathbb{R}^1; Z)$ be the Banach space of almost periodic functions in $Z$ [1], [16], [21]. We assume $f\in AP(\mathbb{R}^1; Y)$. We assume (H1)-(H5) so that $Q_\infty$ is the unique nonnegative $\theta$-periodic solution of (3.4). Then $L$ is stable and $r(t)$, given by (3.50), is the unique almost periodic solution of (3.45). Moreover $\bar{y}$, given in (3.52), is also the unique almost periodic solution of the closed system

$$\bar{y}'=L\bar{y}+f-Kr. \tag{3.55}$$

Hence the following problem is meaningful [16]. Minimize

$$J_{\text{ap}}(u)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[|M(t)y|^2+\langle N(t)u, u\rangle]\,dt \tag{3.56}$$

over the set of admissible controls

$$\begin{aligned}\mathcal{U}_{\text{ap}}=\{u\in AP(\mathbb{R}^1, U)&: \text{there exists } y\in AP(\mathcal{R}^1; Y)\\ &\text{which is a mild solution of (3.45)}\}.\end{aligned} \tag{3.57}$$

Now we find the optimal almost periodic control given in [16].

COROLLARY 3.6. *Assume* (H1)-(H5) *and let $f\in AP(\mathbb{R}^1; Y)$. Then the optimal control is given by the feedback law*

$$\bar{u}=-N^{-1}B^*(Q_\infty\bar{y}+r) \tag{3.58}$$

*where $Q_\infty$ is the nonnegative $\theta$-periodic solution to* (3.4) *and $r$ is the unique almost periodic solution to*

$$r'+L^*r+Q_\infty f=0 \tag{3.59}$$

*given by*

$$r(t)=\int_t^\infty U_L^*(s,t)Q_\infty(s)f(s)\,ds. \tag{3.60}$$

*The optimal cost is given by*

$$J_{\text{ap}}(\bar{u})=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[2\langle r,f\rangle-|N^{-1/2}B^*r|^2]\,dt \tag{3.61}$$

*and the optimal response by*

$$\bar{y}(t)=\int_{-\infty}^{t}U_L(t,s)[f(s)-K(s)r(s)]\,ds. \tag{3.62}$$

*Remark* 3.2. The conclusion of Corollary 3.6 is still valid even if we replace (H4) by a weaker condition:

(H4′) There exist numbers $\varepsilon_1$, $\varepsilon_2 \in (0,1)$ such that $\sigma(U_L(t+\theta,t)) \subset \{\lambda \in C: |\lambda| \leqq 1-\varepsilon_1\} \cup \{\lambda \in C: |\lambda| \geqq 1+\varepsilon_2\}$, for all $t \in \mathbb{R}^1$.

Note that the $\theta$-periodic solution $Q_\infty$ still exists and $U_L$ is well defined. If (H4′) holds, we set

$$\Pi_-(t) = \frac{1}{2\pi i}\int_{C_1} R(\lambda, U_L(t+\theta, t))\, d\lambda, \qquad t \in \mathbb{R}^1, \tag{3.63}$$

$$\Pi_+(t) = I - \Pi_-(t), \qquad t \in \mathbb{R}^1 \tag{3.64}$$

where $C_1$ is the unit circle in the complex plane. Then (3.55) and (3.59) have unique almost periodic solutions given by

$$\begin{aligned}\bar{y}(t) = &\int_{-\infty}^{t} U_L(t,s)\Pi_-(s)[f(s) - K(s)r(s)]\, ds\\ &-\int_{t}^{\infty} U_L(t,s)\Pi_+(s)[f(s)-K(s)r(s)]\, ds,\end{aligned} \tag{3.65}$$

$$\begin{aligned}r(t) = &\int_{t}^{\infty} U_L^*(t,s)\Pi_+^*(s)Q_\infty(s)f(s)\, ds\\ &-\int_{-\infty}^{t} U_L^*(t,s)\Pi_-^*(s)Q_\infty(s)f(s)\, ds.\end{aligned} \tag{3.66}$$

This can be proved by using estimates

$$|U_L(t,s)\Pi_-(s)| \leqq M_-\, e^{-\omega_-(t-s)}, \quad t \geqq s \quad \text{for some } M_- > 0 \text{ and } \omega_- > 0, \tag{3.67}$$

$$|U_L(t,s)\Pi_+(s)| \leqq M_+\, e^{\omega_+(t-s)}, \quad t \leqq s \quad \text{for some } M_+ > 0 \text{ and } \omega_+ > 0. \tag{3.68}$$

For a proof of (3.67) and (3.68) see [23] when $A(t)$ has a special form $A(t) = A + \bar{L}(t)$ with $\bar{L}(t)$ dominated by $A$, and see [29] in the general case.

## 4. Optimal quadratic control in the stochastic case.

**4.1. Quadratic control under complete observation.** We can "stochasticize" all results in § 3. Let $(\Omega, F, F_t, -\infty < t < \infty, P)$ be a stochastic basis and let $(W_i)$, $i = 1, 2, \cdots, N_0$ and $W$ be independent Wiener processes in $\mathbb{R}^1$ and $H$ (Hilbert), respectively, with $\text{Cov}[W(t)] = tW$, $W \in \mathcal{L}^+(H)$ nuclear. We replace (3.1), (3.2), (3.32), (3.33), (3.45), and (3.46), respectively, by

$$dy = [A(t)y + B(t)u]\, dt + G_i(t)y\, dW_i, \qquad y(t_0) = y_0, \tag{4.1}$$

$$J_0(u) = E\int_{t_0}^{\infty} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt, \tag{4.2}$$

$$dy = [A(t)y + B(t)u + f(t)]\, dt + G_i(t)y\, dW_i, \qquad y(t_0) = y_0, \tag{4.3}$$

$$J_1(u) = \overline{\lim_{T\to\infty}}\, \frac{1}{T} E\int_{t_0}^{t_0+T} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt, \tag{4.4}$$

$$dy = [A(t)y + B(t)u + f(t)]\, dt + G_i(t)y\, dW_i + G(t)\, dW, \tag{4.5}$$

$$J_2(u) = \overline{\lim_{T\to\infty}}\, \frac{1}{2T} E\int_{-T}^{T} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt \tag{4.6}$$

where in (4.1), (4.3), and (4.5) $G_i(t)y\,dW_i$ means the sum over $i=1$ to $N_0$. The sets of admissible controls are given by

$$\mathcal{U}^0_{ad}=\{u\in M^2([t_0,\infty)\times\Omega;\ U)\text{: its response has the property } E|y(t)|^2\to 0 \text{ as } t\to\infty\}, \tag{4.7}$$

$$\mathcal{U}^1_{ad}=\left\{u\in M^2_{loc}([t_0,\infty)\times\Omega;\ U)\text{: } \overline{\lim_{T\to\infty}}\frac{1}{T}\int_{t_0}^{t_0+T}|u(t)|^2\,dt<\infty,\ E|y(t)|^2 \text{ bounded}\right\}, \tag{4.8}$$

$$\mathcal{U}^2_{ad}=\left\{u\in M^2_{loc}((-\infty,\infty)\times\Omega;\ U)\text{: } \overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}|u(t)|^2\,dt<\infty,\ \text{there exists } y(t) \text{ with } E|y(t)|^2 \text{ bounded}\right\} \tag{4.9}$$

where $M^2([t_1,t_2]\times\Omega;\ U)$ is the subspace of $L^2_W([t_1,t_2]\times\Omega;\ U)$, which consists of $F_t$-adapted processes and $M^2_{loc}$ means $M^2$ for any finite intervals. We remark that it is possible to take an infinite-dimensional $\tilde{W}$ in place of $(W_i)$ as in [25].

The Riccati equations (3.4) and (3.28) are replaced by

$$Q'(t)+A^*(t)Q(t)+Q(t)A(t)+G_i(t)Q(t)G_i^*(t) + M^*(t)M(t)-Q(t)B(t)N^{-1}(t)B^*(t)Q(t)=0, \tag{4.10}$$

$$A^*Q+QA+G_iQG_i^*+M^*M-QBN^{-1}B^*Q=0. \tag{4.11}$$

We keep the hypothesis (H1) as it is and replace (H2), (H3), respectively, by

(S2) (H2) together with
(i) $G_i\in C_S(\mathbb{R}^1;\ \mathcal{L}(Y))$, $G\in C_S(\mathbb{R}^1;\ \mathcal{L}(H,Y))$; and
(ii) $\sup_{t\in\mathbb{R}^1}[|G_i(t)|+|G(t)|]<\infty$.

(S3) For each $t_0\in\mathbb{R}^1$ and $y_0\in L^2(\Omega, F_{t_0}, P)$, there exist $u\in M^2([t_0,\infty)\times\Omega;\ U)$ and $C_0$ such that

$$E\int_{t_0}^{\infty}[|M(t)y|^2+\langle N(t)u,u\rangle\,dt\leqq C_0E|y_0|^2$$

where $y$ is the mild solution of (4.1).

To replace (H4) by a new one we need to recall some definitions. Consider the homogeneous system

$$dy=A(t)y\,dt+G_i(t)y\,dW_i,\qquad y(t_0)=y_0. \tag{4.12}$$

Since (4.12) is linear, we can easily establish a unique mild solution in $C([t_0,T];\ L_2(\Omega;\ Y))$ that is adapted to $F_t$: namely, the solution of

$$y(t)=U(t,t_0)y_0+\int_{t_0}^{t}U(t,s)G_i(s)y(s)\,dW_i(s). \tag{4.13}$$

We say that $(A;\ G_i)$ is (*exponentially*) *stable* if the mild solution of (4.12) satisfies

$$E|y(t)|^2\leqq M_1\,e^{-\omega(t-t_0)}E|y_0|^2\quad \forall y_0\in L^2(\Omega,F_{t_0},P),\quad t\geqq t_0$$

for some $M_1\geqq 1$ and $\omega>0$. Let $V(t,s)\colon L^2(\Omega,F_s,P)\to L^2(\Omega,F_t,P)$ be the stochastic fundamental solution [2], [14], [27] so that $y(t)=V(t,t_0)y_0$. Then $(A,G_i)$ is stable if and only if

$$E|V(t,t_0)y_0|^2\leqq M_1\,e^{-\omega(t-t_0)}E|y_0|^2,\qquad y_0\in L^2(\Omega,F_{t_0},P).$$

We say that $(A, B; G_i)$ is *stabilizable* if there exists a $K \in C_s(\mathbb{R}^1; \mathscr{L}(Y, U))$ bounded such that $(A-BK; G_i)$ is stable. Hypothesis (S3) is fulfilled if $(A, B; G_i)$ is stabilizable. Let $D \in C_s(\mathbb{R}^1; \mathscr{L}(Y))$ be bounded. We say that $(A, D; G_i)$ is *detectable* if there exists a $K_1 \in C_S(\mathbb{R}^1; \mathscr{L}(Y))$ bounded such that $(A-K_1D; G_i)$ is stable.

Now we replace (H4) and (H5), respectively, by

(S4) $(A, M; G_i)$ is detectable,

and

(S5) (H5) and $G_i(t+\theta)=G_i(t), \quad G(t+\theta)=G(t), \quad t\in\mathbb{R}^1.$

Now we can stochasticize almost all results in § 3 under our new hypotheses, but below we will give only main results.

THEOREM 4.1. (i) *Assume* (H1) *and* (S2). *Then a nonnegative bounded solution to* (4.10) *exists if and only if* (S3) *holds.*

(ii) *Assume* (H1), (S2), *and* (S4). *Then any bounded nonnegative solution of* (4.10) *is stable, i.e.,* $(A-BN^{-1}B^*Q; G_i)$ *is stable. Hence the Riccati equation* (4.10) *has at most one bounded nonnegative solution in* $\mathscr{L}^+(Y)$.

PROPOSITION 4.1. (i) *Assume* (H1), (S2), (S3), *and* (S5). *Then the minimal solution* $Q_\infty$ *of* (4.10) *is* $\theta$*-periodic. If, further,* (S4) *holds, then* $Q_\infty$ *is the unique* $\theta$*-periodic solution to* (4.10).

(ii) *Suppose all operators are constant. Then* $Q_\infty$ *is constant and is the minimal solution of the Riccati equation* (4.11). *If* $(A, M; G_i)$ *is detectable* [15], *then* $Q_\infty$ *is the unique solution of* (4.11) *in* $\mathscr{L}^+(Y)$ *and* $(A-BN^{-1}BQ_\infty; G_i)$ *is stable.*

*Remark* 4.1. Results similar to Propositions 3.1, 3.2, 3.5 and Corollaries 3.1–3.3 are also valid.

Now we need a result similar to Lemmas 3.1 and 3.2.

LEMMA 4.1. *Assume* (H1), (S2), *and* (S3). *Let* $Q_\infty$ *be the minimal nonnegative solution of* (4.10). *Let* $u\in M^2([t_1, t_2]\times\Omega; U)$ *and let* $y$, $r$ *be any mild solutions of* (4.5) *and*

$$r'+L^*r+Q_\infty f=0, \qquad L^*=A-BN^{-1}B^*Q_\infty, \tag{4.14}$$

*respectively, on* $[t_1, t_2]$. *Then*

$$\begin{aligned} E\int_{t_1}^{t_2} [|M(t)y|^2+\langle N(t)u, u\rangle]\,dt+E\langle Q_\infty(t_2)y(t_2), y(t_2)\rangle+2E\langle r(t_2), y(t_2)\rangle \\ = E\int_{t_1}^{t_2} |N^{1/2}[u+N^{-1}B^*(Q_\infty y+r)]|^2\,dt \\ +\int_{t_1}^{t_2} [2\langle r, f\rangle-|N^{-1/2}B^*r|^2+\operatorname{tr} GWG^*Q_\infty]\,dt \\ +E\langle Q_\infty(t_1)y(t_1), y(t_1)\rangle+2E\langle r(t_1), y(t_1)\rangle. \end{aligned} \tag{4.15}$$

*Proof.* We apply Itô's formula to $\langle Q_n(t)y_n(t), y_n(t)\rangle+2\langle r_n(t), y_n(t)\rangle$, where $Q_n$, $y_n$, $r_n$ are approximations to $Q_\infty$, $y$, $r$ given by (4.10), (4.5), and (4.14). Then we rearrange terms, take expectations, and finally pass to the limit $n\to\infty$. □

Now we can solve our three problems immediately.

THEOREM 4.2. *Assume* (H1), (S2)–(S4) *and consider the control problems* (4.1), (4.2), (4.7). *The optimal control is given by the feedback law*

$$\bar{u}=-N^{-1}B^*Q_\infty y \tag{4.16}$$

*where* $Q_\infty$ *is the unique bounded nonnegative solution of* (4.10) *and the optimal cost is*

$$J_0(\bar u) = E\langle Q_\infty(t_0)y_0, y_0\rangle. \tag{4.17}$$

*The optimal closed-loop system is stable.*

*If* (S5) *holds, then* $Q_\infty$ *is* $\theta$*-periodic. If all operators in* (4.1), (4.2) *are constant, then* $Q_\infty$ *is constant.*

*Proof.* We set $f = r = 0$, $t_1 = t_0$, $t_2 = \infty$, and $G = 0$ in (4.15). □

THEOREM 4.3. *Assume* (H1), (S2)-(S4) *and consider the control problems* (4.3), (4.4), (4.8). *The optimal control is given by the feedback law*

$$\bar u = -N^{-1}B^*(Q_\infty \bar y + r) \tag{4.18}$$

*where* $Q_\infty$ *is the unique bounded nonnegative solution of* (4.10) *and* $r$ *is the unique bounded solution of*

$$r' + L^* r + Q_\infty f = 0, \qquad L = A - BN^{-1}BQ_\infty, \tag{4.19}$$

*given by*

$$r(t) = \int_t^\infty U_L^*(s, t)Q_\infty(s)f(s)\, ds, \qquad t \geqq t_0. \tag{4.20}$$

*The optimal cost is*

$$J_1(u) = \overline{\lim_{T\to\infty}} \frac{1}{T}\int_{t_0}^{t_0+T} [2\langle r, f\rangle - |N^{-1/2}B^* r|^2 + \operatorname{tr} GWG^* Q_\infty]\, dt. \tag{4.21}$$

*If, further,* (S5) *holds, and* $f$ *is* $\theta$*-periodic, then* $Q_\infty$, $r$ *are* $\theta$*-periodic and*

$$J_1(\bar u) = \frac{1}{\theta}\int_{t_0}^{t_0+\theta} [2\langle r, f\rangle - |N^{1/2}B^* r|^2 + \operatorname{tr} GWG^* Q]\, dt. \tag{4.22}$$

THEOREM 4.4. *Assume* (H1), (S2)-(S4) *and consider the control problem* (4.5), (4.6), (4.9). *Then the optimal control is given by*

$$\bar u = -N^{-1}B^*(Q_\infty \bar y + r) \tag{4.23}$$

*where* $Q_\infty$ *is the unique bounded nonnegative solution of* (4.10) *on* $\mathbb{R}^1$ *and* $r$ *is the unique bounded solution on* $\mathbb{R}^1$ *of*

$$r' + L^* r + Q_\infty f = 0, \qquad L = A - BN^{-1}B^* Q \tag{4.24}$$

*given by*

$$r(t) = \int_t^\infty U_L^*(s, t)Q_\infty(s)f(s)\, ds, \qquad t \in \mathbb{R}^1. \tag{4.25}$$

*The optimal cost is given by*

$$J_2(\bar u) = \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} [2\langle r, f\rangle - |N^{-1/2}B^* r|^2 + \operatorname{tr} GWG^* Q_\infty]\, dt \tag{4.26}$$

*and the optimal closed-loop system by*

$$d\bar y = [(A - BN^{-1}B^* Q_\infty)\bar y + f - BN^{-1}B^* r]\, dt + G_i \bar y\, dw_i + G\, dw. \tag{4.27}$$

*It has a unique bounded solution*

$$\begin{aligned} \bar y(t) = &\int_{-\infty}^t V_Q(t, s)[f(s) - B(s)N^{-1}(s)B^*(s)r(s)]\, ds \\ &+ \int_{-\infty}^t V_Q(t, s)G(s)\, dw(s) \end{aligned} \tag{4.28}$$

*where* $V_Q(t, s)$ *is the stochastic fundamental solution associated with the homogeneous part of* (4.27). $\bar{y}(t)$ *is exponentially asymptotically stable, i.e., any solution* $y(t)$ *of* (4.27) *with* $y(0) = y_0$ *satisfies*: $y(t) - \bar{y}(t) \to 0$ *exponentially in mean square as* $t \to \infty$.

*If, further,* (S5) *holds, and* $f$ *is* $\theta$*-periodic, then* $Q_\infty$, $r$ *are* $\theta$*-periodic and*

$$J_2(\bar{u}) = \frac{1}{\theta} \int_0^\theta [2\langle r, f\rangle - |N^{-1/2}B^* r|^2 + \operatorname{tr} GWG^* Q_\infty] \, dt \tag{4.29}$$

*and* $\bar{y}(t)$, *given by* (4.28), *is the unique* $\theta$*-periodic solution of* (4.27).

Finally, we consider almost periodic controls. We say that a stochastic process $z(t)$ is (*weakly*) *almost periodic* in $Z$ if $Ez(t)$ and $\operatorname{cov}[z(t)]k$, $k \in Z$ are almost periodic. We assume that all operators except $G$ are $\theta$-periodic and that $G(t)h$, $h \in H$, $f$ are almost periodic. We wish to minimize

$$J_{\text{ap}}(u) = \lim_{T\to\infty} \frac{1}{2T} E \int_{-T}^{T} [|M(t)y|^2 + \langle N(t)u, u\rangle] \, dt \tag{4.30}$$

subject to (4.5) over

$$\mathcal{U}_{\text{ap}} = \{u\colon \text{adapted to } F_t, \text{ almost periodic such that there exists a mild solution } y \text{ of (4.5) almost periodic}\}. \tag{4.31}$$

THEOREM 4.5. *Assume* (H1), (S2)-(S4), *and* (S5) *except* $G$. *Assume that* $G(t)h$, *for all* $h \in H$ *and* $f$ *are almost periodic. Consider the control problem* (4.5), (4.30), (4.31). *Then the optimal control is given by the feedback law*

$$\bar{u} = -N^{-1}B^*(Q_\infty \bar{y} + r) \tag{4.32}$$

*where* $Q_\infty$ *is the unique stable* $\theta$*-periodic solution of* (4.10) *and* $r$ *is the unique almost periodic solution of*

$$r' + L^* r + Q_\infty f = 0 \tag{4.33}$$

*given by*

$$r(t) = \int_t^\infty U_L^*(s, t) Q_\infty(s) f(s) \, ds. \tag{4.34}$$

*The optimal cost is*

$$J_{\text{ap}}(\bar{u}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [2\langle r, f\rangle - |N^{-1/2}B^* r|^2 + \operatorname{tr} GWG^* Q_\infty] \, dt. \tag{4.35}$$

*The optimal closed-loop system is given by* (4.37) *and its unique solution is given by* (4.28).

**4.2. Quadratic control under partial observation.** Consider a special case of (4.3) and its observation

$$dy = [A(t)y + B(t)u + f(t)] \, dt + G(t) \, dw, \qquad y(t_0) = y_0, \tag{4.36}$$

$$dz = C(t)y \, dt + V(t) \, dv, \qquad z(t_0) = 0 \tag{4.37}$$

where $C(t) \in C_b(\mathbb{R}^1; L(Y, \mathbb{R}^m))$, $V(t) \in C_b(\mathbb{R}^1; \mathbb{R}^{m\times n})$, nonsingular, $v$ is an $m$-dimensional Wiener process, $y_0 \in L^2(\Omega, F_{t_0}, P)$ is Gaussian with mean $\bar{y}_0$ and covariance $P_0$, and $y_0$, $w$, $v$ are independent. We assume (H1), (S2)-(S4) and wish to minimize

$$J_1(u) = \overline{\lim_{T\to\infty}} \frac{1}{T} E \int_{t_0}^{t_0+T} [|M(t)y|^2 + \langle N(t)u, u\rangle] \, dt \tag{4.38}$$

over all controls $u$ that are adapted to $\sigma\{z(s), t_0 \leqq s \leqq t\}$. We will define the set of admissible controls later. We now recall filtering results of the system:

$$dy = A(t)y\,dt + G(t)\,dw, \qquad y(t_0) = y_0, \tag{4.39}$$

$$dz = C(t)y\,dt + V(t)\,dw, \qquad z(t_0) = 0. \tag{4.40}$$

The optimal filter $\hat{y}(t)$ of $y(t)$ given $Z_t = \sigma\{z(s), t_0 \leqq s \leqq t\}$ is the projection of $y(t)$ onto $L^2(\Omega, Z_t, P)$ [20], [26] and is given by the mild solution of

$$d\hat{y} = A(t)\hat{y}\,dt + P(t)C^*(t)[VV^*(t)]^{-1}\,d\eta, \qquad \hat{y}(t_0) = \bar{y}_0 \tag{4.41}$$

where $\eta$ is the innovation process given by

$$d\eta = dz - C(t)\hat{y}\,dt \tag{4.42}$$

and $P(t)$, the covariance of the error process $e = y - \hat{y}$, is the mild solution of

$$\begin{aligned} &\text{(a)} \quad P'(t) - A(t)P(t) - P(t)A^*(t) - G(t)WG^*(t) \\ &\qquad + P(t)C^*(t)[VV^*(t)]^{-1}C(t)P(t) = 0, \\ &\text{(b)} \quad P(t_0) = P_0. \end{aligned} \tag{4.43}$$

Following [3], [6], [9], and [17] we define the set of admissible controls

$$\begin{aligned} \mathcal{U}_{\text{pad}} = \Big\{ u \in M^2_{\text{loc}}([t_0, \infty) \times \Omega; U)\colon \lim_{T\to\infty} \frac{1}{T} E \int_{t_0}^{t+T_0} |u(t)|^2\,dt \\ < \infty,\ u(t) \in L^2(\Omega, H_t, P; U) \cap L^2(\Omega, Z_t, P; U) \\ \text{a.e. } t \text{ and } E|y(t)|^2 \text{ is bounded} \Big\}, \end{aligned} \tag{4.44}$$

where $H_t = \sigma\{\eta(s), t_0 \leqq s \leqq t\}$.

Now let $u$ be an admissible control and define $\hat{y}$ by

$$d\hat{y} = [A(t)\hat{y} + B(t)u + f(t)]\,dt + P(t)C^*(t)[VV^*(t)]^{-1}\,d\eta, \qquad \hat{y}(0) = \bar{y}_0. \tag{4.45}$$

Then it is well known [3], [6], [9], [17] that

$$\begin{aligned} &E \int_{t_0}^{t_0+T} [|M(t)y|^2 + \langle N(t)u, u\rangle]\,dt \\ &\quad = E \int_{t_0}^{t_0+T} [|M(t)y|^2 + \langle N(t)u, u\rangle]\,dt + \int_{t_0}^{t_0+T} \operatorname{tr} M(t)P(t)M^*(t)\,dt \end{aligned} \tag{4.46}$$

where $y$ is the response of (4.36). To make our problem nontrivial we assume:

(S6) (a) $(A^*, C)$ is stabilizable,
(b) $(A^*, W^{1/2}G^*)$ is detectable.

PROPOSITION 4.2. *Assume* (H1) *and* (S6). *Then there exists a unique bounded stable solution* $P_\infty$ *to* (4.43a). *The solution* $P(t)$ *of* (4.43) *is bounded on* $[t_0, \infty)$ *for any* $P_0 \geqq 0$. *If, further,* $A(t)$, $C(t)$, *and* $V(t)$ *are* $\theta$*-periodic, then* $P_\infty(t)$ *is* $\theta$*-periodic and* $P(t+n\theta) \to P_\infty(t)$ *strongly for any* $t \geqq T_0$ *as* $n \to \infty$.

*Proof.* The Riccati equation (4.4) is dual to (3.4). Hence the assertions follow from Propositions 3.2–3.5. □

*Remark* 4.2. We may replace (S6)(a) by a condition similar to (H3) for the dual control problem. Note that under (S6), $P$ is bounded and

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \int_{t_0}^{t_0+T} \operatorname{tr} M(t)P(t)M^*(t)\, dt < \infty.$$

Now consider an auxiliary problem of minimizing

$$\hat{J}_1(u) = \overline{\lim_{T\to\infty}} \frac{1}{T} \int_{t_0}^{t_0+T} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt$$

subject to (4.41) over

$$\hat{\mathscr{U}}_{\text{ad}} = \Big\{ u: \overline{\lim_{T\to\infty}} \frac{1}{T} E \int_{t_0}^{t_0+T} |u(t)|^2\, dt < \infty,\ u \text{ adapted to } H_t \text{ such that } E|\hat{y}(t)|^2 \text{ is bounded} \Big\}.$$

Assume (H1), (H3), (H4), (S2), and (S6). Then in view of Theorem 4.1 the optimal control is given by

$$\bar{u} = -N^{-1}B^*(Q_\infty \hat{y} + r)$$

where $Q_\infty$, $r$ are given as in Theorem 3.4 and

$$J_1(\bar{u}) = \overline{\lim_{T\to\infty}} \frac{1}{T} \int_{t_0}^{t_0+T} [2\langle r, f\rangle - |N^{-1/2}B^*r|^2 + \operatorname{tr} PC^*[VV^*]^{-1}CPQ_\infty]\, dt.$$

It is well known that this $\bar{u}$ lies in $\mathscr{U}_{\text{pad}}$ [6], [9]. Then we have Theorem 4.6.

THEOREM 4.6. *Assume* (H1), (H3), (H4), (S2), *and* (S6) *and consider the control problem* (4.36)-(4.38), (4.44). *Then the optimal control is given by*

$$\bar{u} = -N^{-1}B^*(Q_\infty \hat{y} + r)$$

*where $Q_\infty$ is the unique bounded stable solution of* (3.4) *and $r$ is the unique bounded solution of* (3.49) *given by* (3.50). *The optimal cost is given by*

$$J_1(\bar{u}) = \overline{\lim_{T\to\infty}} \frac{1}{T} \int_{t_0}^{t_0+T} [2\langle r, f\rangle - |N^{-1/2}B^*r|^2 + \operatorname{tr} MPM^* + \operatorname{tr} PC^*[VV^*]^{-1}CPQ_\infty]\, dt.$$

*If, further, $f(t)$, $C(t)$, and $V(t)$ are $\theta$-periodic and* (S5) *holds, then $Q_\infty$, $r$ are $\theta$-periodic and*

$$J_1(\bar{u}) = \frac{1}{\theta} \int_0^\theta [2\langle r, f\rangle - |N^{-1/2}B^*r|^2 + \operatorname{tr} MP_\infty M^* + \operatorname{tr} P_\infty C^*[VV^*]^{-1}CP_\infty Q_\infty]\, dt$$

*where $P_\infty$ is the unique $\theta$-periodic solution of* (4.43a).

We may also consider two-sided average cost as $J_2$ in § 4.1 although the problem becomes a little artificial. We replace the initial conditions of (4.36), (4.37) by $y(-T) = y_0$, $z(-T) = 0$ and minimize

$$J_2(u) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} [|M(t)y|^2 + \langle N(t)u, u\rangle]\, dt.$$

We allow only for feedback controls on the filtered process $\hat{y}$ of the form

$$u = -K(t)\hat{y} + h(t)$$

where $K(t) \in C_S(\mathbb{R}^1; \mathscr{L}(Y, U))$ is bounded and $h \in C_b(\mathbb{R}^1; U)$.

THEOREM 4.7. *Assume* (H1), (H3), (H4), (S2), *and* (S6). *The optimal control is given by*

$$\bar{u} = -N^{-1}B^*(Q_\infty \bar{y} + r)$$

*and*

$$J_2(\bar{u}) = \overline{\lim_{T\to\infty}} \frac{1}{T}\int_{-T} [2\langle r, f\rangle - |N^{-1/2}B^*r|^2 + \operatorname{tr} MP_\infty M^* + \operatorname{tr} P_\infty C^*[VV^*]^{-1}CP_\infty Q_\infty]\, dt$$

*where* $Q_\infty$, $P_\infty$, *and* $r$ *are unique bounded solutions of* (3.4), (4.43a), *and* (3.49), *respectively. If* $f$, $C$, $V$ *are* $\theta$*-periodic and* (S5) *holds, then* $Q_\infty$, $P_\infty$, $r$ *are* $\theta$*-periodic and*

$$J_2(\bar{u}) = \frac{1}{\theta}\int_0^\theta [2\langle r, f\rangle - |N^{-1/2}B^*r|^2 + \operatorname{tr} MP_\infty M^* + \operatorname{tr} P_\infty C^*[VV^*]^{-1}CP_\infty Q_\infty]\, dt.$$

*Remark* 4.1. We are not able to prove the existence of almost periodic $P$ if the coefficients of (4.43) are almost periodic.

**5. An example.** Consider the system:

$$(5.1)\quad \begin{aligned} \frac{\partial y}{\partial t} &= \sum_{i,j=1}^n \frac{\partial}{\partial x_j} a_{ij}(t,x)\frac{\partial y}{\partial x_i} + \sum_{i=1}^n b_i(t,x)\frac{\partial y}{\partial x_i} \\ &\quad + c(t,x)y + f(t,x) + u(t,x), \qquad (t,x)\in\mathbb{R}\times\Omega, \quad a_{ij} = a_{ji}, \\ y(t,x) &= 0, \qquad (t,x)\in\mathbb{R}\times\partial\Omega, \\ y(0,x) &= y_0(x), \qquad x\in\Omega \end{aligned}$$

where $a_{ij}$, $b_i$, $c$, $f$, $n$ are real functions from $\mathbb{R}\times\bar{\Omega}$ to $\mathbb{R}$ and $\Omega$ is a bounded set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

We assume the following:

(5.2)
(i) $a_{ij}$, $b$, $c$, and $f$ are continuous and bounded with their first derivatives with respect to $x\in\bar{\Omega}$ and $t\in\mathbb{R}$.
(ii) There exists $\nu > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\zeta_j \geqq \nu|\zeta|^2, \qquad \xi = (\zeta_1, \cdots, \zeta_n)\in\mathbb{R}^n, \quad x\in\bar{\Omega}, \quad t\in\mathbb{R}.$$

We set $Y = L^2(\Omega)$ and denote by $A(t)$ the linear operator in $Y$

$$(5.3)\quad \begin{aligned} A(t)y &= \sum_{i,j=1}^n \frac{\partial}{\partial x_j} a_{ij}(t,x)\frac{\partial y}{\partial x_i} + \sum_{i=1}^n b_i(t,x)\frac{\partial y}{\partial x_i} + c(t,x)y, \\ D(A(t)) &= H^2(\Omega)\cap H_0^1(\Omega). \end{aligned}$$

Then hypothesis (H1)(i) is fulfilled (see [36]) and (H1)(ii) is easily checked (see, for instance, Tanabe [36]). Moreover, the adjoint operator $A^*(t)$ is given by

$$\begin{aligned} A^*(t)y &= \sum_{i,j=1}^n \frac{\partial}{\partial x_j} a_{ij}(t,x)\frac{\partial y}{\partial x_i} - \sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i(t,x)y) + c(t,x)y, \\ D(A^*(t)) &= H^2(\Omega)\cap H_0^1(\Omega) \end{aligned}$$

so that (H1)(ii) holds.

Consider the quadratic control problem. Minimize

$$(5.4)\quad J_0(u) = \int_0^\infty dt \int_\Omega (|y(t,x)|^2 + |u(t,x)|^2)\, dx$$

subject to (5.1) over the set of admissible controls

$$U^0_{\rm ad} = \{u \in L^2(0,\infty; Y); \text{ the corresponding mild solution to (5.1)}, y(t) \to 0 \text{ as } t \to \infty\}. \tag{5.5}$$

We take $U = Y = L^2(\Omega)$, $B = N = M = I$. Then (H2)-(H4) are fulfilled. Thus the Riccati equation (3.4) has a unique bounded solution and the hypotheses of Theorem 3.2 are fulfilled. Then there exists an optimal feedback control for the infinite horizon problem (5.4), (5.5).

Consider now the stochastic system:

$$\begin{aligned} dy = \Bigg( \sum_{i,j=1}^n \frac{\partial}{\partial x_j} a_{ij}(t,x) \frac{\partial y}{\partial x_i} &+ \sum_{i=1}^n b_i(t,x) \frac{\partial y}{\partial x_i} + c(t,x) y + u(t,x) + f(t,x) \Bigg) dt \\ &+ \sum_{i=1}^n g_i(t,x) y \, dw_i + g(t,x)\, dw, \qquad (t,x) \in [0,+\infty) \times \bar\Omega, \\ y(t,x) &= 0, \qquad (t,x) \in [0,+\infty) \times \partial\Omega, \\ y(0,x) &= y_0(x), \qquad x \in \bar\Omega \end{aligned} \tag{5.6}$$

where $g$, $g_i$ are also continuous and bounded with their first derivatives.

Consider the problem. Minimize

$$J_0(u) = E \int_0^\infty dt \int_\Omega (|y(t,x)|^2 + |u(t,x)|^2)\, dt \tag{5.7}$$

over all $u \in U^0_{\rm ad}$ defined by (4.7) where $y$ is the solution of (5.6). Now we can apply Theorem 4.2 and so there exists a feedback for problem (3.9).

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