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OPTIMAL CONTROL OF LINEAR SYSTEMS WITH ALMOST PERIODIC INPUTS*

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Abstract. In this paper we consider linear time-invariant and periodic systems with periodic forcing terms. We propose new quadratic control problems, both deterministic and stochastic. We also consider stochastic control with partial observation and show that the separation principle holds. Our mathematical models cover both finite and infinite dimensional systems.

Key words. optimal control, filtering, linear periodic systems

AMS(MOS) subject classifications. 93C, 49B

1. Introduction. Recently much effort has been devoted to the study of periodic systems and periodic optimization problems [6], [10], [11], [16], [22], [25], [27]. An obvious reason for this is that there are many periodic systems in nature [16], [25], [27]. But another important aspect is that periodic controls are easy to implement compared with general time-varying controls and that they sometimes even produce better performances. In [10] one of the authors has considered the quadratic control problem for a periodic system in infinite dimension and shows that under some stabilizability condition the optimal control is given by a feedback control which involves the periodic solution of a Riccati equation. We have then considered similar problems for stochastic differential equations [12]. We have shown that under partial observation the separation principle holds.

Periodic functions are easy to handle, but they lack some important properties. As we can see from simple examples [15], [16], sums of periodic functions are not periodic in general but almost periodic. Almost periodic functions are generalizations of periodic functions in some sense and were introduced by H. Bohr in 1920s. Since then almost periodic functions and differential equations related to them have been extensively studied [1], [15], [16]. It is known that almost periodic functions naturally appear in many physical systems for example in celestial mechanics or in stable electronic circuits [15], [16], [27]. So it is important to study systems with almost periodic functions.

In this paper we consider linear infinite dimensional time-invariant and periodic systems with almost periodic forcing terms. We consider both deterministic and stochastic cases and propose new quadratic control problems which are natural for almost periodic functions. With slightly different formulation, we also consider stochastic control with partial observation. We shall show that the separation principle holds.

2. The semigroup model. In this section we consider linear systems described by a strongly continuous semigroup and solve quadratic control problems.

2.1. Almost periodic solutions of a differential equation. Let Y be a real separable Hilbert space with inner product $\langle\,,\rangle$ and norm $|\,|$. Let $f(t)$, $t\in\mathbb{R}$ be a continuous function in Y. It is said to be almost periodic if from every sequence a_n we can extract a subsequence a_n' such that $\lim_{n\to\infty} f(t+a_n')$ exists uniformly on the real line $\mathbb{R}$. We

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denote by $AP(Y)$ the Banach space of all continuous almost periodic functions in Y with sup norm. Periodic functions are almost periodic but the converse is not true; see for example $\cos t+\cos\sqrt{2}t$. Note that almost periodic functions are bounded. It is also easy to see that $AP(\mathbb{R})$ (scalar functions) forms an algebra. Let $f, g \in AP(Y)$. Then the mean value $\lim_{T\to\infty} 1/T\int_0^T |f(t)|^2\,dt$ exists. Thus $\lim_{T\to\infty} 1/T\int_0^T \langle f(t), g(t)\rangle\,dt$ defines an inner product on $AP(Y)$ which we denote by $\langle f, g\rangle_{ap}$. The corresponding norm is denoted by $|\,|_{ap}$. Let $L^2_{ap}(Y)$ be the completion of $AP(Y)$ with respect to this inner product. See for details of almost periodic functions [1], [15], [16].

Now we consider the differential equation

$$y' = Ay + f, \tag{2.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on Y [8], [23], [26] and $f \in AP(Y)$. If $Y=\mathbb{R}^n$ and $A\in\mathbb{R}^{n\times n}$, then e^{tA} is the usual matrix exponential function. If A is stable i.e., e^{tA} is exponentially stable, then

$$y(t) = \int_{-\infty}^{t} e^{(t-s)A} f(s)\,ds \tag{2.2}$$

is well defined and is almost periodic. In fact, let a_n be an arbitrary sequence. Then

$$y(t+a_n) = \int_{-\infty}^{t+a_n} e^{(t+a_n-s)A} f(s)\,ds = \int_{-\infty}^{t} e^{(t-\tau)A} f(\tau+a_n)\,d\tau.$$

Now we can easily obtain the uniform convergence of a subsequence of $y(t+a_n)$ since f is almost periodic. In general it does not satisfy (2.1) but we can find a sequence $f_n \in AP(Y)$ such that $f_n \to f$ in $AP(Y)$ and

$$y_n(t) = \int_{-\infty}^{t} e^{(t-s)A} f_n(s)\,ds$$

is the solution of (3.1) with $f=f_n$ converging to y in $AP(Y)$.

A continuous function y on $\mathbb{R}$ is called a mild solution of (2.1) if

$$y(t) = e^{(t-s)A} y(s) + \int_s^t e^{(t-r)A} f(r)\,dr$$

for any $t \geqq s$. Then $y(t)$ given by (2.2) is a unique mild solution of (2.1) in $AP(Y)$. In fact if z is another solution, then $z(t)-y(t)=e^{(t-s)A}[z(s)-y(s)]$. Letting $s\to-\infty$ and noting that z, y are bounded, we obtain $z(t)-y(t)=0$ for any t. A more general condition for the existence of an almost periodic mild solution to (2.1) is that e^{tA} satisfies an exponential dichotomy [15], [16] i.e., there exists a projection operator Π such that

(i) $Y_1 \triangleq \Pi Y \subset D(A)$ and $A_1 \triangleq A\Pi$ is a bounded operator on Y_1 with

$$|e^{-tA_1}| \leqq M_1 e^{-a_1 t}, \quad t\geqq 0 \quad \text{for some } M_1>0, \quad a_1>0.$$

(ii) $A: D(A)\cap Y_2 \to Y_2$, where $Y_2=(I-\Pi)Y$ and $A_2 \triangleq A(I-\Pi)$ generates an exponentially stable semigroup on Y_2. Then

$$y(t) = \int_{-\infty}^{t} e^{(t-s)A_2}(I-\Pi) f(s)\,ds - \int_t^{\infty} e^{(t-s)A_1}\Pi f(s)\,ds$$

is a unique mild solution of (2.1) in $AP(Y)$.

The conditions (i), (ii) are fulfilled if A satisfies (a), (b) below.

(a) The spectrum decomposition assumption of Kato [8] of the following type: the spectrum $\sigma(A)$ of A has a decomposition

$$\sigma(A) = \sigma_1(A) \cup \sigma_2(A)$$

such that $\sigma_1(A) \subset \{\lambda \in C: \text{Re } \lambda \geqq \delta\}$, $\sigma_2(A) \subset \{\lambda \in C: \text{Re } \lambda < -\delta\}$ for some $\delta > 0$ and there is a rectifiable simple closed curve Γ that encloses an open set containing $\sigma_1(A)$ in its interior and $\sigma_2(A)$ in its exterior.

Now define

$$\Pi = \frac{1}{2\pi i} \int_\Gamma R(\lambda, A)\, d\lambda,$$

where $R(\lambda, A)$ is the resolvent of A. Then A satisfies the properties (i), (ii) except the exponential stability of e^{tA_2}. To assure this, it is sufficient to assume;

(b) e^{tA_2} satisfies the spectrum determined growth assumption [8] i.e.,

$$\sup \text{Re } \sigma(A_2) = \lim_{t\to\infty} \frac{\log |e^{tA_2}|}{t}.$$

Then we have

$$|e^{tA_2}| \leqq M_2\, e^{-\delta t}, \quad t \geqq 0 \quad \text{for some } M_2 > 0.$$

Note that (b) is satisfied for analytic semigroups or compact semigroups. Note also that if $Y = \mathbb{R}^n$, then (i), (ii) holds if A has no pure imaginary eigenvalue.

2.2. Quadratic control: the deterministic case. Now we consider the system

$$y' = Ay + Bu + f, \tag{2.3}$$

where $u \in L^2_{\text{ap}}(U)$, U is a real separable Hilbert space, $B \in L(U, Y)$ and $f \in L^2_{\text{ap}}(Y)$. Let U_{ad} be the class of all controls $u \in L^2_{\text{ap}}(U)$ such that (2.3) has an almost periodic mild solution. We wish to minimize over U_{ad} the cost functional

$$J(u) = \lim_{T\to\infty} \frac{1}{T} \int_0^T [|My|^2 + \langle Nu, u\rangle]\, dt, \tag{2.4}$$

where $M \in L(Y)$ and $0 < N \in L(U)$ has bounded inverse N^{-1}. We may write (2.4) as

$$J(u) = |My|^2_{\text{ap}} + |N^{1/2}u|^2_{\text{ap}}. \tag{2.5}$$

This setup guarantees that admissible controls and their responses are necessarily bounded, which is often a requirement in practice. We may relax this as in Remark 2.1 but then we need a formulation as in § 2.6.

It is not clear if there exist any admissible controls. However, if $A - BF$ is stable for some $F \in L(Y, U)$, then the feedback law

$$u = -Fy + v, \qquad v \in L^2_{\text{ap}}(U)$$

is admissible and in fact

$$y(t) = \int_{-\infty}^t e^{(t-s)(A-BF)}[Bv(s) + f(s)]\, ds$$

is the unique mild solution of (2.3) in $AP(Y)$. Because of the presence of f it is reasonable to assume the existence of such F. Here we shall assume slightly more:

$$\begin{array}{ll} \text{(i)} & (A, B) \text{ is stabilizable,} \\ \text{(ii)} & (M, A) \text{ is detectable.} \end{array} \tag{2.6}$$

See [21], [30], for these definitions in finite dimension and [24], [31], for the infinite dimensional case.

PROPOSITION 2.1 ([31], [24]). *Suppose* (2.6) *holds. Then there exists a unique* $0 \leqq Q \in L(Y)$ *satisfying the algebraic Riccati equation*

$$QA + A^*Q + M^*M - QBN^{-1}B^*Q = 0 \tag{2.7}$$

in the inner product sense [24]. *Moreover,* $A - BN^{-1}B^*Q$ *is stable.*

The immediate consequence of this proposition is that $r(t)$ given by

$$r(t) = \int_t^\infty e^{(s-t)(A^* - QBN^{-1}B^*)} Qf(s)\, ds \tag{2.8}$$

is in $AP(Y)$ and is the unique mild solution of

$$r' + (A^* - QBN^{-1}B^*)r + Qf = 0. \tag{2.9}$$

The solution to our control problem is given by the following theorem.

THEOREM 2.1. *Assume* (2.6). *Then there is a unique optimal control for* (2.3), (2.4), *and it is given by the feedback law*

$$\bar{u} = -N^{-1}B^*(Qy + r) \tag{2.10}$$

and

$$J(\bar{u}) = 2\langle r, f\rangle_{\mathrm{ap}} - |N^{-(1/2)}B^*r|^2_{\mathrm{ap}}, \tag{2.11}$$

where $0 \leqq Q$ *and* r *are the unique solutions of* (2.7) *and* (2.8) *respectively.*

Proof. Note that $\bar{u}$ is admissible since $A - BN^{-1}B^*Q$ is stable. Let u be an arbitrary admissible control and y its response. We differentiate $\langle Qy(t), y(t)\rangle + 2\langle r(t), y(t)\rangle$ formally and remove the terms involving A using (2.7). Then, integrating from 0 to T, we obtain

$$\begin{aligned}\langle Qy(T), y(T)\rangle &+ 2\langle r(T), y(T)\rangle - \langle Qy(0), y(0)\rangle - 2\langle r(0), y(0)\rangle \\ &= -\int_0^T [|My|^2 + \langle Nu, u\rangle]\, dt + \int_0^T |N^{1/2}[u + N^{-1}B^*(Qy + r)]|^2\, dt \\ &\quad + \int_0^T [2\langle r, f\rangle - \langle N^{-1}B^*r, B^*r\rangle]\, dt.\end{aligned}$$

Dividing by T and letting $T \to \infty$, we obtain

$$J(u) = |N^{1/2}[u + N^{-1}B^*(Qy + r)]|^2_{\mathrm{ap}} + 2\langle r, f\rangle_{\mathrm{ap}} - |N^{-1/2}B^*r|^2_{\mathrm{ap}} \tag{2.12}$$

where we have used the boundedness of $y(T)$ and $r(T)$. As is well known, this formal procedure can be justified by introducing approximating systems of (2.1), (2.9) with strict solutions [3] and then by passing to the limit. See [3], [9] for arguments based on Yosida approximations of A and [18], [19] for arguments using resolvent operator of A. Now the optimality of $\bar{u}$ and (2.11) follows easily from (2.12). Since $A - BN^{-1}B^*Q$ is stable, the uniqueness of $\bar{u}$ also follows.

2.3. Almost periodic processes. Let (Ω, F, P) be a probability space. Let $y(t)$, $t \in \mathbb{R}$ be a measurable stochastic process in Y. We say that $y(t)$ is weakly almost periodic if $y(t) \in L^2(\Omega, F, P; Y)$ (square integrable) and $Ey(t)$ and $\mathrm{cov}\,[y(t)]h$, for any $h \in Y$ (mean and covariance) are almost periodic.

Now we consider

$$dy = (Ay + f)\, dt + G\, dw, \tag{2.13}$$

where A and f are taken as in § 2.1, H is a real separable Hilbert space, $w(t)$, $t \geqq 0$ is an H-valued Wiener process with $\operatorname{cov}[w(t)] = tW$, $W \geqq 0$, a nuclear operator on H, and $G \in L(H, Y)$ is strongly continuous and $G(t)h$ for any $h \in H$ is almost periodic. We extend $w(t)$ on $\mathbb{R}$ by setting

$$w(t) = w_1(-t), \qquad t < 0$$

where $w_1(t)$, $t \geqq 0$ is a Wiener process in H with $\operatorname{cov}[w_1(t)] = tW$ but is independent of $w(t)$, $t \geqq 0$. With this convention we are able to consider almost periodic solutions of (2.13). Let $F_t = \sigma\{w(s), s \leqq t\}$, $t \in \mathbb{R}$. If A is stable then

$$y(t) = \int_{-\infty}^{t} e^{(t-s)A} f(s)\, ds + \int_{-\infty}^{t} e^{(t-s)A} G(s)\, dw(s) \tag{2.14}$$

is quadratic mean continuous, F_t-adapted and almost periodic. We define a mild solution of (2.13) as in (2.1). Then (2.14) is the unique mild solution of (2.13). It has a property similar to that of (2.7). We denote by $M^2_{\text{ap}}(Y)$ the space of F_t-adapted almost periodic processes in Y. We define $M^2_{\text{ap}}(U)$ in a similar manner.

2.4. Quadratic control under complete observation. We consider a stochastic version of the control system (2.3):

$$dy = (Ay + Bu + f)\, dt + G(t)\, dw. \tag{2.15}$$

Let U_{ad} be the set of all controls $u \in M^2_{\text{ap}}(U)$ for which (2.15) has mild solutions in $M^2_{\text{ap}}(Y)$. We wish to minimize over U_{ad} the cost functional

$$J(u) = \lim_{T \to \infty} \frac{1}{T} E \int_0^T [|My|^2 + \langle Nu, u \rangle]\, dt. \tag{2.16}$$

If $A - BK$ is stable for some $K \in L(Y, U)$, then the feedback law

$$u = -Ky + v, \qquad v \in M^2_{\text{ap}}(U)$$

is admissible and in fact

$$y(t) = \int_{-\infty}^{t} e^{(t-s)(A-BK)} [Bv(s) + f(s)]\, ds + \int_{-\infty}^{t} e^{(t-s)(A-BK)} G(s)\, dw(s)$$

is the unique mild solution in $M^2_{\text{ap}}(Y)$. We have a result analogous to Theorem 2.1.

THEOREM 2.2. *Assume* (2.6). *Then there is a unique optimal control for* (2.15), (2.16), *and it is given by the feedback law*

$$\bar{u} = -N^{-1} B^*(Qy + r) \tag{2.17}$$

and

$$J(\bar{u}) = 2\langle r, f \rangle_{\text{ap}} - |N^{-(1/2)} B^* r|^2_{\text{ap}} + \langle \operatorname{tr} GWG^*Q, 1 \rangle_{\text{ap}} \tag{2.18}$$

where $0 \leqq Q$ *and* r *are unique solutions of* (2.7), (2.8) *respectively,* tr *denotes the trace of nuclear operators and the last term in* (2.18) *is the scalar product of real valued almost periodic functions.*

Proof. Let y be the mild solution corresponding to an admissible control u. As in the proof of Theorem 2.1 it is possible to justify the formal application of Ito's formula to $\langle Qy(t), y(t) \rangle + 2\langle r(t), y(t) \rangle$ [3], [18], [19]. Then we obtain

$$\begin{aligned}
&\langle Qy(T), y(T) \rangle + 2\langle r(T), y(T) \rangle - \langle Qy(0), y(0) \rangle - 2\langle r(0), y(0) \rangle \\
&\quad = \int_0^T \{|N^{1/2}[u + N^{-1}B^*(Qy + r)]|^2 - |My|^2 - \langle Nu, u \rangle \\
&\qquad + 2\langle r, f \rangle - \langle N^{-1}B^* r, r \rangle + \operatorname{tr} GWG^*Q\}\, dt + 2 \int_0^T \langle Qy + r, G\,dw \rangle.
\end{aligned}$$

Now, taking expectations, dividing by T and letting $T\to\infty$, we obtain

$$J(u)=\|N^{1/2}[u+N^{-1}B^*(Qy+r)]\|_{\mathrm{ap}}^2+2\langle r,f\rangle_{\mathrm{ap}}-|N^{-(1/2)}B^*r|_{\mathrm{ap}}^2+\langle \mathrm{tr}\, GWG^*Q, 1\rangle_{\mathrm{ap}},$$

where $\|\cdot\|_{\mathrm{ap}}$ denotes the norm in $M^2_{\mathrm{ap}}(U)$. The optimality of $\bar{u}$ and (2.18) follow immediately.

2.5. Quadratic control under partial observation. This subsection is devoted to a more general situation where the system is nondirectly observable and hence feedback controls are not feasible. We first recall usual quadratic problems under partial observations. Given signal and observation processes

$$dy=(Ay+Bu+f)\,dt+G(t)\,dw,\qquad y(0)=y_0, \tag{2.19}$$

$$dz=Cy\,dt+V\,dv,\qquad z(0)=0, \tag{2.20}$$

one wishes to minimize

$$J_0(u)=E\int_0^T[|My|^2+\langle Nu,u\rangle]\,dt \tag{2.21}$$

over all controls $u\in L^2((0,T)\times\Omega;U)$ such that $u(t)$ is adapted to $\sigma\{z(s),0\leqq s\leqq t\}$ and (2.19), (2.20) have solutions where $C\in L(Y,R^m)$, $V\in\mathbb{R}^{m\times m}$ nonsingular, v is an m-dimensional Wiener process, $y_0\in L^2(\Omega,F,P;Y)$ is Gaussian with mean $\bar{y}_s$ and covariance P_0 and y_0, $w(t)$, $v(t)$ are independent. To solve this problem, one needs to consider the filtering problem

$$dy=Ay\,dt+G(t)\,dw,\qquad y(0)=y_0, \tag{2.22}$$

$$dz=Cy\,dt+V\,dv,\qquad z(0)=0. \tag{2.23}$$

The optimal filter $\hat{y}(t)$ of $y(t)$ given $\{z(s),0\leqq s\leqq t\}$ is defined in terms of projections [11], [20] and is given by [4], [8], [11], [20]

$$d\hat{y}=A\hat{y}\,dt+P(t)C^*(VV^*)^{-1}\,d\eta,\qquad \hat{y}(0)=\bar{y}_0, \tag{2.24}$$

where η is the innovation process [11] given by

$$d\eta=dz-C\hat{y}\,dt \tag{2.25}$$

and P is the solution of the Riccati equation

$$P'-AP-PA^*-GWG^*+PC^*(VV^*)^{-1}CP=0,\qquad P(0)=P_0. \tag{2.26}$$

Admissible controls for (2.19)-(2.21) are all controls $u\in L^2((0,T)\times\Omega;U)$ such that $u(t)\in L^2(\Omega,H_t,P;U)\cap L^2(\Omega,Z_t,P;U)$ a.e. t, where $H_t=\sigma\{\eta(s),0\leqq s\leqq t\}$ and $Z_t=\sigma\{z(s),0\leqq s\leqq t\}$ see [5], [7], [20]. Define $\hat{y}$ by

$$d\hat{y}=(A\hat{y}+Bu+f)\,dt+P(t)C^*(VV^*)^{-1}\,d\eta,\qquad \hat{y}(0)=\bar{y}_0; \tag{2.27}$$

then for each admissible control u we have

$$J_0(u)=\int_0^T \mathrm{tr}\, MP(t)M^*\,dt+E\int_0^T[|M\hat{y}|^2+\langle Nu,u\rangle]\,dt. \tag{2.28}$$

If we denote by $\hat{J}_0(u)$ the second term on the right-hand side of (2.28), then the original problem (2.19)-(2.21) is essentially reduced to the problem of complete observation, (2.27) and $\hat{J}(u)$.

Unfortunately following these steps it is not clear how we can formulate a quadratic control problem for (2.19) and (2.20) in terms of almost periodic functions. So we shall consider a quadratic problem which is slightly different from the previous ones but is nevertheless a natural modification of them.

We take the set of admissible controls

$$U_{ad}=\{u\in L^\infty(0,T;L^2(\Omega,F,P;U)): u(t)\in L^2(\Omega,H_t,P;U)\cap L^2(\Omega,Z_t,P;U) \text{ a.e. } t \text{ such that its response } y\in C_B(0,\infty;L^2(\Omega,F,P;Y))\}, \tag{2.29}$$

where C_B denotes the space of bounded continuous functions. We then wish to minimize

$$J(u)=\overline{\lim_{T\to\infty}}\frac{1}{T}E\int_0^T[|My|^2+\langle Nu,u\rangle]\,dt. \tag{2.30}$$

The requirement in (2.29) is in the spirit of almost periodic functions. In (2.30) we take $\overline{\lim}$ since the limit no longer exists in general.

Remark 2.1. We may replace quadratic problems in §§ 2.2 and 2.4 by problems of the type above. We may also drop boundedness conditions for u and y. Such a problem was considered in [17] for the complete observation case (see also Wonham [28]).

In the sequel we take C, V and G constant and assume the following:

$$\text{(i)}\quad (A^*,C) \text{ is stabilizable;} \qquad \text{(ii)}\quad (W^{1/2}G^*,A^*) \text{ is detectable.} \tag{2.31}$$

Then by [8], [21] the solution of the Riccati equation (2.26) converges strongly to $0\leqq P_\infty\in L(Y)$ (see Corollary 3.1). Thus, in this case we have

$$J(u)=\operatorname{tr} MP_\infty M^*+\overline{\lim_{T\to\infty}}\frac{1}{T}\hat{J}_0(u).$$

We denote by $\hat{J}(u)$ the second term above. Now consider an auxiliary control problem of minimizing $\hat{J}(u)$ over all $\hat{U}_{\mathrm{ad}}$ subject to (2.27), where

$$\hat{U}_{\mathrm{ad}}=\{u\in L^\infty(0,T;L^2(\Omega,F,P;U)): u(t)\in L^2(\Omega,H_t,P;U) \text{ a.e. } t \text{ such that } \hat{y}\in C_B(0,\infty;L^2(\Omega,F,P;Y))\}. \tag{2.32}$$

If (2.6) is satisfied, then as Theorem 2.2 holds we can show that the optimal control is given by

$$\bar{u}=-N^{-1}B^*(Q\hat{y}+r) \tag{2.33}$$

and

$$\hat{J}(\bar{u})=2\langle r,f\rangle_{\mathrm{ap}}-|N^{-1/2}B^*r|^2_{\mathrm{ap}}+\operatorname{tr} P_\infty C^*(VV^*)^{-1}CP_\infty Q^*, \tag{2.34}$$

where Q and r are given as in Theorem 2.1. It is well known [2], [5], [7], [14] that the control $\bar{u}$ is also in U_{ad} i.e., admissible for the control problem defined by (2.19), (2.20) and (2.30). Summing up, we have the separation principles as follows.

THEOREM 2.3. *Assume* (2.6) *and* (2.31). *Then there is a unique optimal control for the problem defined by* (2.19), (2.20), (2.29) *and* (2.30) *and it is given by* (2.33). *Moreover*

$$J(\bar{u})=2\langle r,f\rangle_{\mathrm{ap}}-|N^{-1/2}B^*r|^2_{\mathrm{ap}}+\operatorname{tr} MP_\infty M^*+\operatorname{tr} P_\infty C^*(VV^*)^{-1}CP_\infty Q^*. \tag{2.35}$$

If we set $f=0$, then the control problem (2.19), (2.20), (2.30) is the infinite horizon problem with average cost. In this case we have the following corollary.

COROLLARY 2.1. *The unique optimal control is given by the feedback law on the filter*

$$\bar{u}=-N^{-1}B^*Q\hat{y}$$

and

$$J(\bar{u}) = \text{tr } MP_\infty M^* + \text{tr } PC^*(VV^*)^{-1}CPQ.$$

This is the separation principle on infinite horizon. See [2], [5], [7], [14], [20], [30] for the usual separation principle.

2.6. Examples. We give two simple examples.

Example 2.1. *A deterministic problem.* In § 2.2 we take $Y = U = \mathbb{R}$ and set $A = 3$, $B = 4$, $M = N = 1$ and $f(t) = \sin t$. Then (2.3) is

$$y' = 3y + 4u + \sin t.$$

Then the solution of (2.7) which is nonnegative is $Q = \frac{1}{2}$. Then

$$r(t) = \frac{1}{52}[\cos t + 5 \sin t].$$

It is easy to obtain

$$2\langle r, f\rangle_{\text{ap}} = \frac{5}{52}, \qquad |B^* r|^2_{\text{ap}} = \frac{4}{52}.$$

Thus the optimal control is given by

$$\bar{u} = -2y - \frac{1}{13}(\cos t + 5 \sin t)$$

and

$$J(\bar{u}) = \frac{1}{52}.$$

Here f is periodic, but we may add, for example, $\sin\sqrt{2}t$. Then it becomes almost periodic and we can compute $\bar{u}$ and $J(\bar{u})$ in a similar manner.

Example 2.2. Consider the stochastic parabolic equation

$$dy = \left(\frac{\partial^2}{\partial x^2} y + u\right) dt + \sin x \sin at\, dw, \qquad a \in \mathbb{R}, \quad \text{constant},$$

$$y(t, 0) = y(t, \pi) = 0.$$

In (2.15) we take

$$Y = U = L^2(0, \pi), \qquad Ay = \frac{d^2}{dx^2} y, \qquad D(A) = H^2(0, \pi) \cap H^1_0(0, \pi), \qquad H = \mathbb{R}$$

and $W(t)$ a real standard Wiener process. We take

$$J(u) = \lim_{T \to \infty} \frac{1}{T} E \int_0^T [|y|^2 + |u|^2]\, dt.$$

Then the algebraic Riccati equation (2.7) becomes

$$QA + AQ - Q^2 + I = 0,$$

whose solution is $Q = \sqrt{A^2 + I} + A$ i.e.,

$$Qy = \sum_{n=1}^{\infty} (\sqrt{n^4 + 1} - n^2)\langle y, e_n\rangle e_n, \qquad e_n = \sqrt{\frac{2}{\pi}} \sin nx.$$

The optimal control is

$$\bar{u} = -(\sqrt{A^2+I}+A)y$$

and its response $\bar{y}$ is given by

$$\bar{y}(t) = \int_{-\infty}^{t} e^{-\sqrt{A^2+I}(t-s)} \sin x \sin as\, dw(s)$$

$$= \int_{-\infty}^{t} e^{-\sqrt{2}(t-s)} \sin as\, dw(s) \sin x.$$

Thus

$$\bar{u}(t) = -(\sqrt{2}-1) \int_{-\infty}^{t} e^{-\sqrt{2}(t-s)} \sin as\, dw(s) \sin x$$

and

$$J(\bar{u}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sin^2 at \langle Q \sin x, \sin x\rangle\, dt = \frac{\pi}{2}(\sqrt{2}-1).$$

3. The evolution operator model. In § 2 we have taken A time-invariant, but here we replace it by $A(t)$, $t \in \mathbb{R}$ with the following properties:

(i) $A(t) = A(t+\theta)$, $t \in \mathbb{R}$ for some $\theta > 0$.

(ii) There exists an evolution operator $U(t, s)$, $t \geqq s \geqq 0$ such that for the initial value problem

$$y' = A(t)y + g(t), \qquad y(0) = y_0, \quad g \in L^2(0, \theta; Y)$$

has a unique mild solution

$$y(t) = U(t, s)y_0 + \int_0^t Y(t, s)g(s)\, ds. \tag{3.1}$$

(iii) If n is large, then $n \in \rho(A(t))$ and $A_n(t) = n^2[n - A(t)]^{-1} - nI$ is well defined. Moreover, $y_n(t) \to y(t)$ in $C([0, \theta], Y)$ where y_n is the strict solution of the approximating systems

$$y_n' = A_n(t)y_n + g, \qquad y_n(0) = y_0.$$

(iv) $A^*(t)$ has properties similar to (i)–(iii).

The conditions (3.1) (i)–(iii) are fulfilled if the usual hypotheses of Tanabe and Kato–Tanabe [23], [26] are satisfied. Note that

$$U(t+\theta, s+\theta) = U(t, s) \quad \text{for any } t > s. \tag{3.2}$$

We replace (2.1) by

$$y' = A(t)y + f. \tag{3.3}$$

If $U(t, s)$ is exponentially stable i.e., $|U(t, s)| \leqq C_1\, e^{-a(t-s)}$, $t \geqq s$ for some $C_1 > 0$, $a > 0$, then using (3.2) we can show that

$$y(t) = \int_{-\infty}^{t} U(t, s)f(s)\, ds$$

is almost periodic and it is the unique mild solution of (3.3). We can construct strict solutions of approximating systems of (3.3) which converge to $y(t)$.

Below we shall consider quadratic problems as in § 2. Since most of the arguments are similar, we omit details of proofs.

3.1. Quadratic control: the deterministic case. We follow § 2.2 and consider

$$y'=A(t)y+B(t)u+f(t), \tag{3.4}$$

$$J(u)=\lim_{T\to\infty}\frac{1}{T}\int_0^T [|My|^2+\langle Nu, u\rangle]\,dt, \tag{3.5}$$

where B, M and N are strongly continuous θ-periodic operators and we assume that $0<N$ has bounded inverse N^{-1}. We take admissible controls as in § 2.2. If $A(t)-B(t)K(t)$, with $K\in L(Y,U)$ θ-periodic strongly continuous, generates an exponentially stable evolution operator (we shall write this as $U_{A-BK}(t-s)$), then the feedback law

$$u=-K(t)y+v(t), \qquad v\in L^2_{ap}(U)$$

is admissible. As with (2.6) we assume

(3.6)
(i) (A,B) is stabilizable, i.e., there exists a θ-periodic strongly continuous operator $K\in L(Y,U)$ such that $U_{A-BK}(t,s)$ is exponentially stable;
(ii) (M,A) is detectable, i.e., there exists a θ-periodic strongly continuous operator $L\in L(Y)$ such that $U_{A^*-M^*L^*}(t,s)$ is exponentially stable.

PROPOSITION 3.1 [10]. *Assume* (3.6). *Then there exists a unique θ-periodic solution to the Riccati equation*

$$Q'+QA+A^*Q+M^*M-QBN^{-1}B^*Q=0. \tag{3.7}$$

Moreover, $A-BN^{-1}B^*Q$ *generates an exponentially stable evolution operator* $U_{A^*-BN^{-1}B^*Q}(t,s)$.

If, further, $U_{A^*-QBN^{-1}B^*}(t,s)$ *is exponentially stable, then*

$$r(t)=\int_t^\infty U_{A^*-QBN^{-1}B^*}(s,t)Q(s)f(s)\,ds \tag{3.8}$$

is a unique almost periodic solution of

$$r'+[A^*(t)-Q(t)B(t)N^{-1}(t)B^*(t)]r+Q(t)f(t)=0. \tag{3.9}$$

THEOREM 3.1. *Assume* (3.6) *and that* $U_{A^*-QBN^{-1}B^*}(t-s)$ *is exponentially stable. Then the unique optimal control for* (3.4), (3.5) *is given by the feedback law*

$$\bar{u}=N^{-1}B^*(Qy+r) \tag{3.10}$$

and

$$J(\bar{u})=2\langle r,f\rangle_{\mathrm{ap}}-|N^{-1/2}B^*r|^2_{\mathrm{ap}} \tag{3.11}$$

where Q and r are given in Proposition 3.1 *and in* (3.8) *respectively.*

Proof. Similar to the proof of Theorem 2.1. But we approximate (3.4), (3.9) by systems involving $A_n(t)$ and then employ limit arguments [3], [10].

3.2. Quadratic control with complete observation. We replace (2.15), (2.16) by the following:

$$dy=[A(t)y+B(t)u+f(t)]\,dt+G(t)\,dw, \tag{3.12}$$

$$J(u)=\lim_{T\to\infty}\frac{1}{T}E\int_0^T [|My|^2+\langle Nu, u\rangle]\,dt, \tag{3.13}$$

where A, B, M and N are given as in § 3.1 and G as in § 2.4. We define admissible controls as in § 2.4.

THEOREM 3.2. *Assume* (3.6) *and that* $U_{A^*-QBN^{-1}B^*}(t,s)$ *is exponentially stable. Then the feedback control*

$$\bar{u} = -N^{-1}B^*(Qy+r) \tag{3.14}$$

is the unique optimal control and

$$J(\bar{u}) = 2\langle r, f\rangle_{\mathrm{ap}} - |N^{-1/2}B^*r|^2_{\mathrm{ap}} + \langle \operatorname{tr} GWG^*Q, 1\rangle_{\mathrm{ap}}, \tag{3.15}$$

where Q and r are unique solutions of (3.7) *and* (3.8).

Proof. Similar to the proof of Theorem 2.2.

3.3. Quadratic control with partial observation. The signal and observation processes are respectively

$$dy = [A(t)y + B(t)u + f(t)]\,dt + G(t)\,dw, \qquad y(0) = y_0, \tag{3.16}$$

$$dz = C(t)y\,dt + V(t)\,dv, \qquad z(0) = 0, \tag{3.17}$$

and the cost functional is

$$J(u) = \overline{\lim_{T\to\infty}} \frac{1}{T} E \int_0^T [|My|^2 + \langle Nu, u\rangle]\,dt, \tag{3.18}$$

where A, B, M and N are given as in § 3.1 and G, C and V are θ-periodic strongly continuous operators.

We consider the filtering problem as in § 2.5 and define the innovations process similarly. Then, taking admissible controls as in (2.29), we parallel the developments to 2.5. We assume (2.31) in the periodic sense, i.e., in the sense of (3.6). Then from [10] there exists a unique θ-periodic solution to the Riccati equation

$$P' - AP - PA^* - GWG^* + PC^*(VV^*)^{-1}CP = 0. \tag{3.19}$$

Moreover, by Lemma 3.1 below the solution of (3.19) with $P(0) = P_0$ converges orbitally to the periodic solution as $t\to\infty$. Thus we have

$$J(u) = \frac{1}{\theta}\int_0^\theta \operatorname{tr} M(t)\bar{P}(t)M^*(t)\,dt + \overline{\lim_{T\to\infty}} \frac{1}{T}\hat{J}_0(u),$$

where $\hat{J}_0(u)$ is defined as (2.28) and $\hat{y}$ is now given by

$$d\hat{y} = [A(t)\hat{y} + B(t)u + f(t)]\,dt + P(t)C^*(VV^*)^{-1}\,d\eta, \qquad \hat{y}(0) = \bar{y}_0.$$

Thus we have the following.

THEOREM 3.3. *Assume* (3.6) *and that* $U_{A^*-QBN^{-1}B^*}(t,s)$ *is exponentially stable. Assume further that* (2.31), *in the periodic sense, holds. Then*

$$\bar{u} = -N^{-1}B^*(Q\hat{y} + r)$$

is the unique optimal control for (3.16)–(3.18) *and*

$$J(\bar{u}) = 2\langle r, f\rangle_{\mathrm{ap}} - |N^{-1/2}B^*r|^2_{\mathrm{ap}}$$
$$+\frac{1}{\theta}\int_0^\theta \operatorname{tr}[M(t)\bar{P}(t)M^*(t) + \bar{P}(t)C^*(t)(V(t)V^*(t))^{-1}C(t)\bar{P}(t)Q(t)]\,dt,$$

where $\bar{P}$, Q are the unique θ-periodic solutions of (3.7) *and* (3.19) *respectively and r is given by* (3.8).

Now we shall prove the following lemma.

LEMMA 3.1. *Let P be the solution of* (3.19) *with $P(0)=P_0$ and let $\bar{P}$ be the unique θ-periodic solution of* (3.19) *with $\bar{P}(0)=\bar{P}_0$. Then*

$$P(t+n\theta)\to\bar{P}(t) \text{ strongly for any } t\geqq 0 \text{ as } n\to\infty. \tag{3.20}$$

Proof. We shall show this in three steps. We denote by $\tilde{P}(t)$ the solution of (3.19) with $\tilde{P}(0)=0$.

(i) $P_0\leqq\bar{P}_0$. By a comparison theorem, see for instance [10], we know that

$$\tilde{P}(t)\leqq P(t)\leqq\bar{P}(t).$$

By [10] $\tilde{P}(t+n\theta)\to\bar{P}(t)$ for any $t\geqq 0$ as $n\to\infty$. Thus $P(t+n\theta)\to\bar{P}(t)$ as $n\to\infty$.

(ii) $P_0\geqq\bar{P}_0$. Note that $P(t)\geqq\bar{P}(t)$ for any $t\geqq 0$. Set $Q(t)=P(t)-\bar{P}(t)$, then

$$Q'-(A-BN^{-1}B^*\bar{P})^*Q-Q(A-BN^{-1}B^*\bar{P})+QBN^{-1}B^*Q=0.$$

Let $\bar{U}(t,s)$ be the evolution operator generated by $A-BN^{-1}B^*\bar{P}$. By differentiating we obtain

$$\frac{d}{ds}\langle Q(t-s)\bar{U}(s,0)y,\ \bar{U}(s,0)y\rangle=\langle Q(t-s)BN^{-1}B^*Q(t-s)\bar{U}(s,0)y,\ \bar{U}(s,0)y\rangle$$

$$=|N^{-1/2}B^*Q(t-s)\bar{U}(s,0)y|^2,\qquad y\in Y.$$

Integrating this form 0 to t, we obtain

$$\langle Q(0)\bar{U}(t,0),\ \bar{U}(t,0)y\rangle-\langle Q(t)y,y\rangle=\int_0^t|N^{-1/2}BQ(t-s)\bar{U}(s,0)y|^2\,ds,$$

which implies

$$0\leqq\langle Q(t)y,y\rangle\leqq\langle Q(0)\bar{U}(t,0)y,\ \bar{U}(t,0)y\rangle\to 0\quad\text{as } t\to\infty\ [10].$$

Hence $P(t)-\bar{P}(t)\to 0$ strongly as $t\to\infty$.

(iii) General case. Choose n large enough so that $P_0\leqq nI$ and $\bar{P}_0\leqq nI$.

Let $R(t)$ be the solution of (3.19) with $R(0)=nI$. Then by (ii) $R(t)-\bar{P}(t)\to 0$ as $t\to\infty$. By the comparison theorem we have

$$\tilde{P}(t)\leqq P(t)\leqq R(t).$$

Now

$$\tilde{P}(t)-\bar{P}(t)\leqq P(t)-\bar{P}(t)\leqq R(t)-\bar{P}(t).$$

Hence

$$P(t+n\theta)-\bar{P}(t+n\theta)\to 0\quad\text{strongly as } n\to\infty.$$

Remark 3.1. Note that (3.20) is the global asymptotic orbital stability of $\bar{P}(t)$, i.e., the trajectory of $P(t)$ approaches to that of $\bar{P}(t)$ asymptotically.

COROLLARY 3.1. *Let A, G, C and V be constant in* (2.26). *Then under condition* (2.31), *$P(t)\to P_\infty$, where $0\leqq P_\infty$ is the unique solution of*

$$AP+PA^*+GWG^*-PC^*(VV^*)^{-1}CP=0$$

and $A^*-P_\infty C^*(VV^*)^{-1}C$ is stable.

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