SIAM J. CONTROL OPTIM.
Vol. 37, No. 4, pp. 1123–1149

CONTROL OF THE STOCHASTIC BURGERS MODEL OF TURBULENCE*

GIUSEPPE DA PRATO† AND ARNAUD DEBUSSCHE‡

Abstract. We consider a control problem for a stochastic Burgers equation. This problem is motivated by a model from the control of Turbulence (see [Choi et al., *J. Fluid Mech.*, 253 (1993), pp. 509–543]). We study a sequence of approximated Hamilton–Jacobi equations by using dynamic programming.

Key words. stochastic Burgers equations, turbulence, Hamilton–Jacobi equations, dynamic programming

AMS subject classifications. 93C20, 93C90, 93E20, 76F99

PII. S0363012996311307

1. Introduction.

It has been shown in [3] that the stochastic Burgers equation is a good and simple model with which to study turbulence phenomena. The mathematical study of this equation has been the object of several papers [2], [9], [10], [17], [21].

This model also has been used in [6] to test a numerical algorithm for reducing the cost function in the very important problem of the control of turbulence.

In this paper we consider the stochastic Burgers equation with distributed parameter controls. The cost function has the same form as in [6] and contains the analogue of the kinetic energy. The problem is as follows: minimize

$$J(z) = \mathbb{E}\left(\int_0^T \left[\left|\frac{\partial X}{\partial \xi}\right|^2_{L^2(0,1)} + \frac{1}{2}\ |z(s)|^2_{L^2(0,1)}\right] ds + \frac{1}{2}\ |X(T)|^2_{L^2(0,1)}\right),$$

where the control z is in $L^2(\Omega \times [0,T] \times [0,1])$, and $X(t,\xi),\ \xi \in [0,1],\ t \in [0,T]$, is the solution of the controlled Burgers equation

$$\text{(1.1)} \qquad \begin{cases} dX = \left(\dfrac{\partial^2 X}{\partial \xi^2} + \dfrac{\partial}{\partial \xi}(X^2)\right) dt + \sqrt{Q} z dt + \sqrt{Q} dW,\ \xi \in [0,1],\ t \geq 0, \\ X(t,0) = X(t,1) = 0,\ t \geq 0, \\ X(0,\xi) = x(\xi),\ \xi \in [0,1], \end{cases}$$

where $x \in L^2(0,1)$.

Here W is a cylindrical Wiener process on $L^2(0,1)$ (in other words $\frac{dW}{dt}$ is the "space–time white noise") and is adapted to a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ (of course the control z has to be adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$). Moreover Q is a symmetric linear operator on $L^2(0,1)$. In (1.1) the operator $\sqrt{Q}$ acts both on the noise and on the control. This is essential in our work: it enables us to use a Hopf

*Received by the editors October 28, 1996; accepted for publication (in revised form) June 10, 1998; published electronically May 13, 1999.
http://www.siam.org/journals/sicon/37-4/31130.html

†Scuola Normale Superiore di Pisa, 56126 Pisa, Italy. The research of this author was partially supported by the Italian National Project MURST "Equazioni di Evoluzione e Applicazioni Fisico-Matematiche."

‡CNRS et Université de Paris Sud, 91405 Orsay, France. This research was done while this author was visiting the Scuola Normale Superiore di Pisa.

transform on the Hamilton–Jacobi equation (see below). This might be a restriction in the applications. However, this assumption is not artificial. It can be interpreted as a control acting on the solution in the same way as the noise or as a noise acting on the control.

It is easy to see that the cost functional J cannot have finite values unless Q is a nuclear operator. This is a simple consequence of the Ito formula.

In this paper we study this control problem following the dynamic programming approach. We solve the associated Hamilton–Jacobi equation and prove that it has a solution that coincides with the value function. More precisely, let A be the unbounded operator on $L^2(0,1)$ defined by

$$Ax = \frac{\partial^2 x}{\partial \xi^2},\ D(A) = H^2(0,1) \cap H_0^1(0,1),$$

and F, g are the nonlinear functions:

$$F(x) = \frac{\partial (x^2)}{\partial \xi},\quad g(x) = \left|\frac{\partial x}{\partial \xi}\right|^2_{L^2(0,1)}.$$

Then we can associate our control problem with the Hamilton–Jacobi equation

$$\begin{cases} u_t(t,x) = \dfrac{1}{2}\ \mathrm{Tr}\ [Qu_{xx}(t,x)] + (Ax + F(x), u_x(t,x)) \\ \qquad\qquad\quad - \dfrac{1}{2}\ |\sqrt{Q}\ u_x(t,x)|^2 + g(x), \\ u(0,x) = \dfrac{1}{2}\ |x|^2_{L^2(0,1)} \end{cases} \tag{1.2}$$

for $x \in L^2(0,1)$, $t \in [0,T]$.

We prove below that there exists a solution u and that

$$u(T,x) = \inf_z J(z). \tag{1.3}$$

Moreover, for each control z and its associated solution X of (1.1), the fundamental identity holds:

$$u(T,x) + \frac{1}{2}\mathbb{E}\int_0^T |\sqrt{Q}\ u_x(T-s, X(s,x)) + z(s)|^2_{L^2(0,1)} ds = J(z). \tag{1.4}$$

We prove that the *closed loop equation*

$$\begin{cases} dX^* = \left(\dfrac{\partial^2 X^*}{\partial \xi^2} + \dfrac{\partial}{\partial \xi}(X^{*2}) - Qu_x(T-t, X^*(t))\right) dt + \sqrt{Q}\ dW, \\ X^*(t,0) = X^*(t,1) = 0,\ t \geq 0, \\ X^*(0,\xi) = x(\xi),\ \xi \in [0,1] \end{cases} \tag{1.5}$$

has a unique solution. It follows that there exists a unique optimal control given by

$$z^*(t) = -\sqrt{Q}\ u_x(T-t, X^*(t)). \tag{1.6}$$

To prove the existence of a solution to the Hamilton–Jacobi equation (1.2) we use a Hopf transform

$$u = -\ln v.$$

The function v satisfies

$$v_t(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[Qv_{xx}(t,x)] + (Ax + F(x), v_x(t,x)) - g(x)v \tag{1.7}$$

so that using the Feynmann–Kac formula we have an explicit representation for u,

$$u(t,x) = -\ln \mathbb{E}\left(\exp\left[-\frac{1}{2}\,|Y(t)|^2_{L^2(0,1)} - \int_0^t g(Y(s))ds\right]\right), \tag{1.8}$$

where Y is the solution to the uncontrolled equation

$$\begin{cases} dY = \left(\dfrac{\partial^2 Y}{\partial \xi^2} + \dfrac{\partial}{\partial \xi}(Y^2)\right)dt + \sqrt{Q}dW,\ \xi \in [0,1],\ t \geq 0, \\ Y(t,0) = Y(t,1) = 0,\ t \geq 0, \\ Y(0,\xi) = x(\xi),\ \xi \in [0,1]. \end{cases} \tag{1.9}$$

The study of second-order Hamilton–Jacobi equations has been the object of several articles. Existence and uniqueness in finite and infinite dimensions have been obtained using semigroups methods (see [1], [8], [4], [5], [13], [14], [15]) and also using the concept of viscosity solution (see [7], [12], [19], [20], [16]). However, these results do not cover our case. Indeed here we simultaneousy have a non-Lipschitz Hamiltonian $H(u) = \frac{1}{2}\,|\sqrt{Q}\,u_x|^2$, a singular term in the cost functional $g(x) = \frac{1}{2}\,|\frac{\partial}{\partial \xi}X|^2_{L^2(0,1)}$, and the nonlinear term $f(x) = \frac{\partial}{\partial \xi}(X^2)$ coming from the Burgers equation.

All the formula described above can be derived formally; we use an approximation technique to justify them. We consider an approximate problem which is finite dimensional by using a Galerkin approximation and in which g and f are replaced by bounded and Lipschitz functions. We obtain a control problem which we can solve easily and a sequence $\{u^m\}$ of approximations of the solution to (1.2). We derive several a priori estimates and prove convergence of the approximation. The main difficulty is that we are not able to obtain an a priori estimate in the space of C^1 bounded functions on u^m. We have only the estimates

$$|u^m(t,x)| \leq \frac{1}{2}\,\left(|x|^2_{L^2(0,1)} + \mathrm{Tr}\,tQ\right),$$
$$|u^m_x(t,x)| \leq Ce^{\frac{1}{2}\,(|x|^2_{L^2(0,1)} + \mathrm{Tr}\,tQ)}$$

and a similar estimate on $u^m_{xx}(t)$.

However, we are able to prove that u^m converges to a C^2 function u which is a solution of (1.2), that the formulas (1.1) and (1.3) hold, and that the closed loop equation (1.5) has a unique solution. Thus the original control problem is completely solved.

2. Preliminaries and main results. Let $H = L^2(0,1)$ endowed with the usual norm and inner product denoted by $|\cdot|$ and $(\cdot,\cdot)$. We define a linear operator A in H by setting

$$Ax = \frac{\partial^2 x}{\partial \xi^2},\ x \in D(A) = H^2(0,1) \cap H^1_0(0,1).$$

As usual, $H^k(0,1),\ k \in \mathbb{N}$, is the Sobolev space of all functions in H whose derivatives up to the order k belong to H, and $H_0^1(0,1)$ is the subspace of $H^1(0,1)$ of all functions whose traces at 0 and 1 vanish.

The operator A is self-adjoint and strictly negative and has a compact inverse. We can define $(-A)^s$ and $D((-A)^s)$ for any $s \in \mathbb{R}$. For $s = \frac{1}{2}$, we have $D((-A)^{1/2}) = H_0^1(0,1)$ and its norm and inner product are denoted by

$$\|x\| = |(-A)^{1/2}x|,\ ((x,y)) = \left((-A)^{1/2}x, (-A)^{1/2}y\right),\ x, y \in H_0^1(0,1).$$

The sequence of eigenvalues of A is

$$\lambda_k = -k^2\pi^2,\ k \in \mathbb{N},$$

and it is associated with the orthonormal basis of eigenvectors $\{e_k\}_{k\in\mathbb{N}}$,

$$e_k = \sqrt{2/\pi}\ \sin k\xi,\ k \in \mathbb{N},\ \xi \in [0,1].$$

For any positive integer m we denote by P_m the orthogonal projector on the space spanned by $e_1, \dots, e_m$.

We also consider a linear operator Q which is assumed to be symmetric, nonnegative, and of trace class; and a cylindrical Wiener process W on H associated with a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. (The reader is referred to [11] for precise definitions.) Let $L_W^2(\Omega \times [0,T]; H)$ be the space of all square integrable and adapted processes with values in $L^2(0,T;H)$. For $x \in H_0^1(0,1)$ we set

$$F(x) = \frac{\partial}{\partial\xi}(x^2).$$

The control problem we want to study is

$$\text{(2.1)} \qquad \begin{cases} \text{Minimize} \\ J(z) = \mathbb{E}\left(\int_0^T \left(\|X(t)\|^2 + \frac{1}{2}\,|z(t)|^2\right)dt + \frac{1}{2}\,|X(T)|^2\right) \\ \text{over all } \ z \in L_W^2(\Omega \times [0,T]; H), \end{cases}$$

where X is the solution of the controlled Burgers equation

$$\text{(2.2)} \qquad \begin{cases} dX = (AX + F(X) + \sqrt{Q}\ z)dt + \sqrt{Q}\ dW, \\ X(0) = x \end{cases}$$

and the initial datum x is in H.

For any $z \in L_W^2(\Omega \times [0,T]; H)$, (2.2) has a unique solution. More precisely, its solution can be constructed as the limit of Galerkin approximations. For $m \in \mathbb{N}$, we define F_m by

$$F_m(x) = \frac{\partial}{\partial\xi}\left(f_m(x)\right),\ x \in H_0^1(0,1),$$

where

$$f_m(\alpha) = \frac{m\alpha^2}{m + \alpha^2},\ \alpha \in \mathbb{R},$$

and we consider the following Galerkin approximation of (2.2):

$$\begin{cases} dX_m = (AX_m + P_mF_m(X_m) + P_m\sqrt{Q}\, z_m)dt + P_m\sqrt{Q}\, dW, \\ X_m(0) = x_m, \end{cases} \tag{2.3}$$

where $z_m \in L^2_W(\Omega \times [0,T]; P_mH)$ and $x_m \in P_mH$. The existence and uniqueness of X_m follow from the classical theory of finite dimensional stochastic differential equations.

We will need a lemma, whose proof is given in the appendix.

LEMMA 2.1. *Let* $\{x_m\}_{m\in\mathbb{N}}, \{z_m\}_{m\in\mathbb{N}}$ *be such that* $x_m \to x$ *in* H *and* $z_m \to z$ *in* $L^2_W(\Omega \times [0,T]; H)$ *and almost surely in* $L^2([0,T] \times H)$. *Let* X_m *be the solution of* (2.3). *Then* $\{X_m\}_{m\in\mathbb{N}}$ *is convergent to the unique solution* X *of* (2.2) *in*

$$L^2(\Omega \times [0,T]; H^1_0(0,1)) \cap L^2(\Omega; C([0,T]; H))$$

and almost surely in $L^2([0,T] \times H)$.

As mentioned in the introduction, we can formally associate the following Hamilton–Jacobi equation with the control problem (2.1)–(2.2):

$$\begin{cases} u_t(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[Qu_{xx}(t,x)] + (Ax + F(x), u_x(t,x)) \\ \qquad\qquad - \frac{1}{2}\,|\sqrt{Q}\,u_x(t,x)|^2 + \|x\|^2, \\ u(0,x) = \frac{1}{2}\,|x|^2, \end{cases} \tag{2.4}$$

for $t \in [0,T], x \in H$. Using the Hopf transformation

$$u = -\ln v,$$

v formally satisfies

$$v_t(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[Qv_{xx}(t,x)] + (Ax + F(x), v_x(t,x)) - \|x\|^2 v, \tag{2.5}$$

and so, by the Feynmann–Kac formula,

$$v(t,x) = \mathbb{E}\left(\exp\left[-\frac{1}{2}\,|Y(t)|^2 - \int_0^t \|Y(s)\|^2 ds\right]\right), \tag{2.6}$$

where Y is the solution to the uncontrolled Burgers equation

$$\begin{cases} dY = (AY + F(Y))dt + \sqrt{Q}\,dW, \\ Y(0) = x. \end{cases} \tag{2.7}$$

It is classical that Y is two times differentiable with respect to x. More precisely, we have the following lemma, whose proof is given in the appendix.

LEMMA 2.2. *The function* v *defined by* (2.6)–(2.7) *is two times differentiable with respect to* $x \in H$. *For any* $(x,h) \in H \times H$, *its derivative at* x *in the direction of* h *is given by*

$$v_x(t,x)h = -\mathbb{E}\left[\left((Y(t), \eta^h(t)) + 2\int_0^t ((Y(s), \eta^h(s)))ds\right) e^{-\frac{1}{2}|Y(t)|^2 - \int_0^t \|Y(s)\|^2 ds}\right], \tag{2.8}$$

where η^h *is the solution of*

$$\left\{\begin{array}{l}\dfrac{d\eta^h}{dt} = A\eta^h + 2\dfrac{\partial}{\partial \xi}\left(Y\eta^h\right),\\[2mm] \eta^h(0) = h.\end{array}\right. \tag{2.9}$$

Moreover, its second derivative is given by

$$\begin{aligned} v_{xx}(t,x)(h,h) = -\mathbb{E}\Bigg[\Bigg(&|\eta^h(t)|^2 + 2\int_0^t \|\eta^h(s)\|^2 ds + (Y(t),\zeta^h(t)) \\ &+ 2\int_0^t ((Y(s),\zeta^h(s)))ds\Bigg) e^{-\frac{1}{2}|Y(t)|^2 - \int_0^t \|Y(s)\|^2 ds}\Bigg] \\ +\mathbb{E}\Bigg[\Bigg((Y(t),\eta^h(t)) + 2&\int_0^t ((Y(s),\eta^h(s)))ds\Bigg)^2 e^{-\frac{1}{2}|Y(t)|^2 - \int_0^t \|Y(s)\|^2 ds}\Bigg], \end{aligned} \tag{2.10}$$

where ζ^h *is the solution of*

$$\left\{\begin{array}{l}\dfrac{d\zeta^h}{dt} = A\zeta^h + 2\dfrac{\partial}{\partial \xi}\left(Y\zeta^h + (\eta_h)^2\right),\\[2mm] \zeta^h(0) = 0.\end{array}\right. \tag{2.11}$$

We will also consider the Galerkin approximation of (2.9),

$$\left\{\begin{array}{l}\dfrac{d\eta_m^h}{dt} = A\eta_m^h + P_m\dfrac{\partial}{\partial \xi}\left(f_m'(Y_m)\eta_m^h\right),\\[2mm] \eta_m^h(0) = P_m h,\end{array}\right. \tag{2.12}$$

and of (2.11),

$$\left\{\begin{array}{l}\dfrac{d\zeta_m^h}{dt} = A\zeta_m^h + P_m\dfrac{\partial}{\partial \xi}\left(f_m'(Y_m)\zeta_m^h + f_m''(Y_m)(\eta_m^h)^2\right),\\[2mm] \zeta_m^h(0) = 0,\end{array}\right. \tag{2.13}$$

where Y_m is the solution to

$$\left\{\begin{array}{l}dY_m = (AY_m + F_m(Y_m))dt + P_m\sqrt{Q}\,dW,\\[2mm] Y_m(0) = x_m\end{array}\right. \tag{2.14}$$

and $x_m \in P_m H,\ h \in P_m h$.

LEMMA 2.3. *Let* $\{x_m\}_{m\in\mathbb{N}}$ *be such that* $x_m \to x$ *in H; then*

$$\sup_{h\in H,|h|=1} \left|\eta_m^{P_m h} - \eta^h\right|^2_{L^2(0,T;H_0^1(0,1))} \to 0,$$

$$\sup_{h\in H,|h|=1} \left|\eta_m^{P_m h} - \eta^h\right|_{C([0,T];H)} \to 0,$$

$$\sup_{h\in H,|h|=1} \left|\zeta_m^{P_m h} - \zeta^h\right|^2_{L^2(0,T;H_0^1(0,1))} \to 0,$$

$$\sup_{h\in H,|h|=1} \left|\zeta_m^{P_m h} - \zeta^h\right|_{C([0,T];H)} \to 0$$

almost surely when $m \to \infty$. The proof of this lemma is given in the appendix.

In section 4 we will prove, by an approximation technique, that v given by (2.6) is a strict solution of (2.5). By strict solution we mean that v is a C^2 function with respect to x; that for any $x \in D(A)$, $t \to v(t,x)$ is a C^1 function; and that (2.5) holds for any $(t,x) \in D(A) \times [0,T]$. We will also obtain that $u = \ln v$ is a strict solution of (2.4).

Then, again by approximation, we show that the fundamental identity (1.4) holds.

It remains to be proved that the closed loop equation (1.5) has a unique solution X^*. The difficulty here is that we have only a rather bad estimate on u_x. We will consider this problem in section 5.

The main result of this paper, whose proof is presented in sections 4 and 5, is the following.

THEOREM 2.4. *Let v be defined by* (2.6)–(2.7) *and $u = -\ln v$; then u is a strict solution to the Hamilton–Jacobi equation* (2.4). *Moreover for any $z \in L^2_W(\Omega \times [0,T]; H)$, we have*

$$u(T,x) + \frac{1}{2}\mathbb{E}\int_0^T |\sqrt{Q}\, u_x(T-s, X(s,x)) + z(s)|^2 ds = J(z),$$

where X is the solution of (2.2) *and J is defined by* (2.1).

The control problem (2.1) *has a unique solution given by*

$$z^*(t) = -\sqrt{Q}\, u_x(T-t, X^*(t)),$$

where X^* is the unique solution to the closed loop equation

$$\begin{cases} dX^* = (AX^* + F(X^*)dt - Qu_x(T-t, X^*(t)))\, dt + \sqrt{Q}\, dW, \\ X^*(0) = x. \end{cases}$$

Remark 2.5. In fact we prove a little bit more. Indeed, we show that the optimal control z^* and the optimal state X^* are the limits of an approximated finite dimensional problem.

3. Approximations.

We already have introduced the Galerkin approximation (2.3) of (2.2). We also need to approximate the terms $\|\cdot\|^2$ and $\frac{1}{2}|\cdot|^2$ in the functional J. If $l \in \mathbb{N}$, we set

$$\varphi_l(x) = \frac{1}{2}\,\frac{l|x|^2}{l+|x|^2},\ x \in H, \quad g_l(x) = \frac{l\|x\|^2}{l+\|x\|^2},\ x \in H_0^1(0,1).$$

The approximated control problem is

$$\text{(3.1)} \qquad \begin{cases} \text{Minimize} \\ J_{l,m}(z_m) = \mathbb{E}\left(\displaystyle\int_0^T (g_l(X_m(t)) + \frac{1}{2}\,|z_m(t)|^2)dt + \varphi_l(x_m(T))\right) \\ \text{over all } \ z_m \in L^2_W(\Omega \times [0,T]; P_m H), \end{cases}$$

where X_m is the solution of (2.3).

We define for $l,m \in \mathbb{N}$, $x_m \in P_m H$, $t \in [0,T]$

$$v^{l,m}(t,x_m) = \mathbb{E}\left(e^{-\varphi_l(Y_m(t)) - \int_0^t g_l(Y_m(s))ds}\right),$$

where Y_m is the solution of (2.14). It defines a two times continuously differentiable function with respect to $x_m \in P_m H$, and for $h \in P_m H$ we have

$$v^{l,m}_{x_m}(t,x)h = -\mathbb{E}\Bigg[\bigg(\big(D_x\varphi_l(Y_m(t)),\eta^h_m(t)\big) + \int_0^t \big(D_x g_l(Y_m(s)),\eta^h_m(s)\big)\,ds\bigg) e^{-\varphi_l(Y_m(t))-\int_0^t g_l(Y_m(s))ds}\Bigg], \tag{3.2}$$

where η^h_m is the solution of (2.12) and

$$\begin{aligned} v^{l,m}_{x_m x_m}(t,x)(h,h) = &-\mathbb{E}\Bigg[\bigg(\big(D_x\varphi_l(Y_m(t)),\zeta^h_m(t)\big) + \int_0^t \big(D_x g_l(Y_m(s)),\zeta^h_m(s)\big)\,ds \\ &+D^2_x\varphi_l(Y_m(t))(\eta^h_m(t),\eta^h_m(t)) + \int_0^t D^2_x g_l(Y_m(s))(\eta^h_m(s),\eta^h_m(s))ds\bigg) \\ &\times e^{-\varphi_l(Y_m(t))-\int_0^t g_l(Y_m(s))ds}\Bigg] \\ &+\mathbb{E}\Bigg[\bigg(\big(D_x\varphi_l(Y_m(t)),\eta^h_m(t)\big) + \int_0^t \big(D_x g_l(Y_m(s)),\eta^h_m(s)\big)\,ds\bigg)^2 \\ &\times e^{-\varphi_l(Y_m(t))-\int_0^t g_l(Y_m(s))ds}\Bigg], \end{aligned} \tag{3.3}$$

where ζ^h_m is the solution of (2.13). By the Feynman–Kac formula we know that $v^{l,m}$ satisfies the equation

$$\begin{cases} v^{l,m}_t = \dfrac{1}{2}\,\mathrm{Tr}\,[P_m Q v^{l,m}_{x_m,x_m}] + \big(Ax_m + P_m F_m(x_m), v^{l,m}_{x_m}\big) - g_l(x_m)v^{l,m}, \\ v^{l,m}(0,x_m) = e^{-\varphi_l(x_m)} \end{cases} \tag{3.4}$$

on $P_m H \times [0,T]$. Also, clearly

$$v^{l,m}(t,x_m) \ge e^{-l(\frac{1}{2}+T)}. \tag{3.5}$$

Therefore the function

$$u^{l,m} = -\ln v^{l,m}$$

is two times continuously differentiable and it can be checked that it is a solution of the Hamilton–Jacobi equation associated with (3.1):

$$\begin{cases} u^{l,m}_t(t,x) &= \dfrac{1}{2}\,\mathrm{Tr}\,[P_m Q u^{l,m}_{x_m,x_m}] + \big(Ax_m + P_m F_m(x_m), u^{l,m}_{x_m}\big) \\ &- \dfrac{1}{2}\,|P_m\sqrt{Q}\,u^{l,m}_{x_m}|^2 + g_l(x_m), \\ u^{l,m}(0,x_m) &= \varphi_l(x_m). \end{cases} \tag{3.6}$$

A standard computation using Ito's formula shows that

$$\begin{aligned} u^{l,m}(T,x_m) &+ \frac{1}{2}\int_0^T \left|\sqrt{Q}\, u^{l,m}_{x_m}(T-t,X_m(t))+z_m(t)\right|^2 dt \\ &= \varphi_l(X_m(T))+\int_0^T \left(g_l(X_m(t))+\frac{1}{2}\,|z_m(t)|^2\right)dt \\ &+ \int_0^T \left(u^{l,m}_{x_m}(T-t,X_m(t)),P_m\sqrt{Q}\,dW(t)\right). \end{aligned} \tag{3.7}$$

Taking the expectation, we obtain the fundamental identity

$$u^{l,m}(T,x_m)+\frac{1}{2}\mathbb{E}\int_0^T \left|\sqrt{Q}\, u^{l,m}_{x_m}(T-t,X_m(t))+z_m(t)\right|^2 dt = J_{l,m}(z_m). \tag{3.8}$$

We deduce that if $X^*_{l,m}$ is the solution to the closed loop equation

$$\begin{cases} dX^*_{l,m}=\left(AX^*_{l,m}+P_mF_m(X^*_{l,m})-P_mQu^{l,m}_{x_m}(T-t,X^*_{l,m})\right)dt+P_m\sqrt{Q}\,dW, \\ X^*_{l,m}(0)=x_m, \end{cases} \tag{3.9}$$

then there exists a unique optimal control $z^*_{l,m}$ for (3.1) which is given by the feedback formula

$$z^*_{l,m}(t)=-\sqrt{Q}\,u^{l,m}_{x_m}(T-t,X^*_{l,m}). \tag{3.10}$$

We will see below (see Lemma 4.1) that $v^{l,m}_{x_m}(T-t,X^*_{l,m})$ is a globally Lipschitz and bounded function so that by (3.5) the same holds for $u^{l,m}_{x_m}(T-t,X^*_{l,m})$ and we know that $X^*_{l,m}$ exists and is unique. We also have

$$J_{l,m}(z^*_{l,m})=u^{l,m}(T,x_m)=\inf_{z_m} J_{l,m}(z_m)=-\ln\mathbb{E}\left(e^{-\varphi_l(y_m(T))-\int_0^T g_l(Y_m(s))ds}\right), \tag{3.11}$$

where Y_m satisfies (2.14).

We will also use the function

$$v^m(t,x_m)=\mathbb{E}\left(e^{-\frac{1}{2}|Y_m(t)|^2-\int_0^t \|Y_m(s)\|^2ds}\right), \tag{3.12}$$

where Y_m is the solution of (2.14), with first and second derivatives given by

$$\begin{aligned} &v^m_{x_m}(t,x_m)h \\ &= -\mathbb{E}\left[\left((Y_m(t),\eta^h_m(t))+2\int_0^t((Y_m(s),\eta^h_m(s)))ds\right)e^{-\frac{1}{2}|Y_m(t)|^2-\int_0^t \|Y_m(s)\|^2ds}\right], \end{aligned} \tag{3.13}$$

$$\begin{aligned} &v^m_{x_mx_m}(t,x)(h,h)=-\mathbb{E}\Bigg[\Bigg((Y_m(t),\zeta^h_m(t))+2\int_0^t((Y_m(s),\zeta^h_m(s)))ds \\ &+|\eta^h_m(t)|^2+2\int_0^t\|\eta^h_m(s)\|^2ds\Bigg)\,e^{-\frac{1}{2}|Y_m(t)|^2-\int_0^t \|Y_m(s)\|^2ds}\Bigg] \\ &+\mathbb{E}\left[\left((Y_m(t),\eta^h_m(t))+2\int_0^t((Y_m(s),\eta^h_m(s)))ds\right)^2 e^{-\frac{1}{2}|Y_m(t)|^2-\int_0^t \|Y_m(s)\|^2ds}\right], \end{aligned} \tag{3.14}$$

where η_m^h and ζ_m^h are defined by (2.12), (2.13).

In the next two sections, c denotes any constant depending only on the data A, Q, T. We always use the same symbol c although the constants have different values. Sometimes, we use a constant depending on $\omega \in \Omega$, or $m \in \mathbb{N}, \dots$, in which case we will write $C(\omega)$ or $k_m, \dots$.

Also, when f is a C^1 (resp., C^2) function from H or $P_m H$ to $\mathbb{R}$, we will identify its first (resp., second) differential f_x (resp., f_{xx}) with the gradient (resp., the Hessian) of f; i.e., we use the two notations

$$f_x(x)h = (f_x(x), h),\ x, h \in H$$

and

$$f_{xx}(x)(h,h) = (f_{xx}(x)h, h),\ x, h \in H,$$

respectively.

4. Passing to the limit. We take the limit in our approximation in two steps. We first proceed to the limit $l \to \infty$, then, using a priori estimates on the Galerkin approximation, we take the limit $m \to \infty$. We first bound $v^{l,m}$ uniformly in l.

LEMMA 4.1. *For any $m \in \mathbb{N}$ there exists a constant k_m depending on m and on A, Q, T such that for any $x_m \in P_m H, t \in [0,T]$*

$$\text{(i)}\quad |v^{l,m}_{x_m}(t,x_m)| \le k_m,$$

$$\text{(ii)}\quad |v^{l,m}_{x_m x_m}(t,x_m)|_{\mathcal{L}(P_m H)} \le k_m.$$

Proof. We have the following inequalities:

$$|(-A)^{1/2} D_x g_l(y)|^2 \le 4 g_l(y),\quad y \in H_0^1(0,1), \tag{4.1}$$

$$|D_x \varphi_l(y)|^2 \le 4\varphi_l(y),\quad y \in H, \tag{4.2}$$

$$|D_x^2 g_l(y)(\eta,\eta)| \le 6\|\eta\|^2,\quad y, \eta \in H_0^1(0,1), \tag{4.3}$$

$$|D_x^2 \varphi_l(y)(\eta,\eta)| \le 6|\eta|^2,\quad y, \eta \in H. \tag{4.4}$$

Since f_m' is bounded by $\sqrt{m}$ and (2.12) is a linear system of ordinary differential equations, there exists a constant $c(m,T)$ such that

$$|\eta_m^h(t)| \le c(m,T)|h|,\ h \in H. \tag{4.5}$$

Similarly we have for the solution of (2.13)

$$|\zeta_m^h(t)| \le c(m,T)|h|^2,\ h \in H. \tag{4.6}$$

By (3.2), (4.1), (4.2), and the Cauchy–Schwarz inequality, for $x_m, h \in P_m H$, $t \in [0,T]$,

$$\begin{aligned}
&|v^m_{x_m}(t,x_m)h| \\
&\le c(m,T)\mathbb{E}\left[\left(\varphi_l(Y_m(t)) + \int_0^t g_l(Y_m(s))ds\right)^{1/2} e^{-\varphi_l(Y_m(t)) - \int_0^t g_l(Y_m(s))ds}\right]|h| \\
&\le C(m,T)|h|,
\end{aligned}$$

since $\sqrt{x}e^{-x}$ is bounded. This proves (i). Similarly (ii) follows from (3.3), (4.1)–(4.4), and elementary inequalities. □

Using (4.1)–(4.4) and the dominated convergence theorem it can be seen that for any $x_m \in P_m H,\ t \in [0,T]$,

$$\begin{aligned} &v^{l,m}(t,x_m) \to v^m(t,x_m),\\ &v^{l,m}_{x_m}(t,x_m) \to v^m_{x_m}(t,x_m) \text{ in } P_m H,\\ &v^{l,m}_{x_m x_m}(t,x_m) \to v^m_{x_m x_m}(t,x_m) \text{ in } \mathcal{L}(P_m H)\end{aligned} \tag{4.7}$$

when $l \to \infty$. Also, using Lemma 4.1 and with another application of the dominated convergence theorem, it follows that v_m is a solution of

$$\begin{cases} v^m_t = \dfrac{1}{2}\operatorname{Tr}\left[P_m Q v^m_{x_m,x_m}\right] + \left(Ax_m + P_m F_m(x_m), v^m_{x_m}\right) - \|x_m\|^2 v^m,\\ v^m(0,x_m) = e^{-\frac{1}{2}|x_m|^2}. \end{cases} \tag{4.8}$$

From Lemma 4.1 we deduce the following estimates on

$$u^{l,m} = -\ln v^{l,m}.$$

LEMMA 4.2. *For any $m \in \mathbb{N}$, there exists a constant k_m depending on m and on A, Q, T such that for any $x_m \in P_m H, t \in [0,T]$*

(i) $|u^{l,m}(t,x_m)| \le \frac{1}{2}\ \left(|x_m|^2 + T\ Tr\, Q\right)$,

(ii) $|u^{l,m}_{x_m}(t,x_m)| \le k_m\ e^{\frac{1}{2}\ \left(|x_m|^2 + T\ Tr\ Q\right)}$,

(iii) $|u^{l,m}_{x_m x_m}(t,x_m)|_{\mathcal{L}(P_m H)} \le k_m e^{\frac{1}{2}\ \left(|x_m|^2 + T\ Tr\ Q\right)} + k_m^2 e^{|x_m|^2 + T\ Tr\ Q}$.

Proof. By Jensen's inequality we have

$$v^{l,m}(t,x_m) \ge e^{-\mathbb{E}(\varphi_l(Y_m(t)) + \int_0^t g_l(Y_m(s))ds)} \ge e^{-\mathbb{E}(\frac{1}{2}|Y_m(t)|^2 + \int_0^t \|Y_m(s)\|^2 ds)}.$$

By Ito's formula we have

$$\begin{aligned}\frac{1}{2}|Y_m(t)|^2 + \int_0^t \|Y_m(s)\|^2 ds = &\int_0^t \left(Y_m(s), \sqrt{Q}\, dW(s)\right)\\ &+ \frac{1}{2}\ \left(|x_m|^2 +\ t \operatorname{Tr}(P_m Q)\right),\end{aligned}$$

since $(F_m(Y_m), Y_m) = 0$. Thus

$$v^{l,m}(t,x_m) \ge e^{-\frac{1}{2}\left(|x_m|^2 + t \operatorname{Tr}(P_m Q)\right)} \ge e^{-\frac{1}{2}\left(|x_m|^2 + T \operatorname{Tr}(P_m Q)\right)}.$$

Now (i) follows from the definition of $u^{l,m}$, and (ii), (iii) from the chain rule. □

Let us define

$$u^m = -\ln v^m.$$

Then by (4.7) for any $x_m \in P_m H,\ t \in [0,T]$,

$$\begin{aligned} &u^{l,m}(t,x_m) \to u^m(t,x_m),\\ &u^{l,m}_{x_m}(t,x_m) \to u^m_{x_m}(t,x_m) \text{ in } P_m H,\\ &u^{l,m}_{x_m x_m}(t,x_m) \to u^m_{x_m x_m}(t,x_m) \text{ in } \mathcal{L}(P_m H),\end{aligned} \tag{4.9}$$

and by (4.8) u_m is a solution of

$$\left\{\begin{aligned} u_t^m(t,x) &= \frac{1}{2}\,\mathrm{Tr}\,[P_m Q u^m_{x_m,x_m}] + \big(Ax_m + P_m F_m(x_m), u^m_{x_m}\big)\\ &\qquad -\frac{1}{2}\,|P_m\sqrt{Q}u^m_{x_m}|^2 + \|x_m\|^2,\\ u^m(0,x_m) &= \frac{1}{2}|x_m|^2. \end{aligned}\right. \tag{4.10}$$

Using Ito's formula we have for any $z_m \in L^2(\Omega\times[0,T];P_mH)$, $x_m \in P_mH$,

$$\begin{aligned} u^m(T,x_m) &+ \frac{1}{2}\int_0^T \left|\sqrt{Q}\,u^m_{x_m}(T-t,X_m(t)) + z_m(t)\right|^2 dt\\ &= \frac{1}{2}\,|X_m(T)|^2 + \int_0^T \left(\|X_m(t)\|^2 + \frac{1}{2}\,|z_m(t)|^2\right)dt\\ &+ \int_0^T \left(u^m_{x_m}(T-t,X_m(t)), P_m\sqrt{Q}\,dW(t)\right). \end{aligned} \tag{4.11}$$

We now derive some a priori estimates uniform in m in order to take the limit $m\to\infty$.

LEMMA 4.3. *There exists a constant k_1 depending on A,Q,T such that for any $x_m \in P_mH, t\in[0,T]$*

$$\text{(i)}\quad |v^m_{x_m}(t,x_m)| \le k_1,$$
$$\text{(ii)}\quad |v^m_{x_mx_m}(t,x_m)|_{\mathcal{L}(P_mH)} \le k_1.$$

Remark 4.4.

- We are not able to give an a priori estimate on $v^{l,m}$ independently of m. This explains why we take the limit in two steps.
- We do not have a lower bound on v^m such as in (3.5) for $v^{l,m}$. Thus we do not know whether u^m has a bounded derivative. Formally u^m and v^m are associated to the control problem in which the cost functional $J_{l,m}$ is replaced by

$$J_m(z_m) = \mathbb{E}\left(\int_0^T \left(\|X_m(t)\|^2 + \frac{1}{2}\,|z_m(t)|^2\right)dt + \frac{1}{2}\,|X_m(t)|^2\right). \tag{4.12}$$

  We shall prove in section 5 that the corresponding closed loop equation has a unique solution.

Proof of Lemma 4.3. Let us first note that

$$f''_m(\alpha) \le 2,\ \alpha\in\mathbb{R}. \tag{4.13}$$

Let $h\in P_mH$. We take the scalar product of (2.12) with η^h_m and obtain

$$\begin{aligned} \frac{1}{2}\,\frac{d}{dt}|\eta^h_m|^2 + \|\eta^h_m\|^2 &= \left(P_m\frac{\partial}{\partial\xi}(f'_m(Y_m)\eta^h_m), \eta^h_m\right)\\ &= \frac{1}{2}\int_0^1 f''_m(Y_m)\left(\frac{\partial}{\partial\xi}Y_m\right)(\eta^h_m)^2 d\xi\\ &\le \|Y_m\|\,|\eta^h_m|^2_{L^4(0,1)} \end{aligned} \tag{4.14}$$

by integration by parts and Hölder's inequality. Using interpolation and the Sobolev embedding theorem, we have

$$|\eta_m^h|^2_{L^4(0,1)} \le c|\eta_m^h|^{3/2}\ \|\eta_m^h\|^{1/2}. \tag{4.15}$$

Hence, using Young's inequality,

$$\frac{1}{2}\ \frac{d}{dt}\ |\eta_m^h|^2 + \|\eta_m^h\|^2 \le c\|Y_m\|^{4/3}|\eta_m^h|^2 + \frac{1}{2}\ \|\eta_m^h\|^2,$$

and, by Gronwall's lemma,

$$\begin{cases} |\eta_m^h(t)|^2 \le e^{c\int_0^t \|Y_m(s)\|^{4/3}ds}|h|^2, \\ \displaystyle\int_0^t \|\eta_m^h(s)\|^2 ds \le e^{c\int_0^t \|Y_m(s)\|^{4/3}ds}|h|^2. \end{cases} \tag{4.16}$$

We infer from (3.13) and the Cauchy–Schwarz inequality that

$$|v_{x_m}^m(t,x_m)h| \le \mathbb{E}\left[\left(|Y_m(t)|^2 + 2\int_0^t \|Y_m(s)\|^2 ds\right)^{1/2} \right.$$
$$\left. e^{-\frac{1}{2}|Y_m(t)|^2 - \int_0^t \|Y_m(s)\|^2 ds + c\int_0^t \|Y_m(s)\|^{4/3}ds}\right]|h|$$

and (i) follows from elementary inequalities.

For the second estimate we take the scalar product of (2.13) with ζ_m^h and obtain

$$\frac{1}{2}\ \frac{d}{dt}|\zeta_m^h|^2 + \|\zeta_m^h\|^2 = \left(P_m\frac{\partial}{\partial\xi}(f_m'(Y_m)\zeta_m^h), \zeta_m^h\right) + \left(P_m\frac{\partial}{\partial\xi}(f_m''(Y_m)(\eta_m^h)^2), \zeta_m^h\right)$$

and use

$$\left|\left(P_m\frac{\partial}{\partial\xi}(f_m'(Y_m)\zeta_m^h), \zeta_m^h\right)\right| \le C\|Y_m\|^{4/3}|\zeta_m^h|^2 + \frac{1}{4}\ \|\zeta_m^h\|^2$$

and

$$\left|\left(P_m\frac{\partial}{\partial\xi}(f_m''(Y_m)(\eta_m^h)^2), \zeta_m^h\right)\right| = \left|\int_0^1 f_m''(Y_m)(\eta_m^h)^2\frac{\partial}{\partial\xi}\zeta_m^h d\xi\right|$$
$$\le 2|\eta_m^h|^2_{L^4(0,1)}\|\zeta_m^h\| \le C|\eta_m^h|^4_{L^4(0,1)} + \frac{1}{4}\ \|\zeta_m^h\|^2.$$

We deduce

$$\frac{d}{dt}\ |\zeta_m^h|^2 + \|\zeta_m^h\|^2 \le c\|Y_m\|^{4/3}|\zeta_m^h|^2 + c\|\eta_m^h\|^4_{L^4(0,1)}$$

and by (4.15), (4.16), and Gronwall's lemma

$$\begin{cases} |\zeta_m^h(t)|^2 \le e^{c\int_0^t \|Y_m(s)\|^{4/3}ds}|h|^4, \\ \displaystyle\int_0^t \|\zeta_m^h(s)\|^2 ds \le e^{c\int_0^t \|Y_m(s)\|^{4/3}ds}\ |h|^4. \end{cases} \tag{4.17}$$

By the Cauchy–Schwarz inequality, (3.14), (4.16), (4.17), we obtain

$$|v^m_{x_m x_m}(t,x_m)(h,h)| \le c\mathbb{E}\left[\left(1+|Y_m(t)|^2+\int_0^t \|Y_m(s)\|^2 ds\right)\right.$$
$$\left. e^{-\frac{1}{2}|Y_m(t)|^2-\int_0^t\|Y_m(s)\|^2ds+c\int_0^t\|Y_m(s)\|^{4/3}ds}\right]|h|^2,$$

and (ii) follows. □

Applying Lemma 2.1 with $z_m=0$ and $x_m=P_m x$, we easily prove that for each $x\in H, t\in[0,T]$

$$v^m(t,P_m x)\to v(t,x) \tag{4.18}$$

when $m\to\infty$. Also, we have for any $x\in H,\ t\in[0,T]$,

$$\begin{aligned}
&|v_{x_m}(t,P_m x)-v_x(t,x)|\\
&=\sup_{|h|=1}\,(v_{x_m}(t,P_m x),P_m h)-(v_x(t,x),h)\\
&=\sup_{|h|=1}\mathbb{E}\Bigg[\left((Y_m(t),\eta_m^{P_m h}(t))+2\int_0^t((Y_m(s),\eta_m^{P_m h}(s)))ds\right)e^{-\frac12|Y_m(t)|^2-\int_0^t||Y_m(s)||^2ds}\\
&\qquad-\left((Y(t),\eta(t))+2\int_0^t((Y(s),\eta(s)))ds\right)e^{-\frac12|Y(t)|^2-\int_0^t||Y(s)||^2ds}\Bigg]\\
&\le\mathbb{E}\Bigg[\sup_{|h|=1}\Bigg|\left((Y_m(t),\eta_m^{P_m h}(t))+2\int_0^t((Y_m(s),\eta_m^{P_m h}(s)))ds\right)e^{-\frac12|Y_m(t)|^2-\int_0^t||Y_m(s)||^2ds}\\
&\qquad-\left((Y(t),\eta(t))+2\int_0^t((Y(s),\eta(s)))ds\right)e^{-\frac12|Y(t)|^2-\int_0^t||Y(s)||^2ds}\Bigg|\Bigg].
\end{aligned}$$

It follows from Lemma 2.2 and Lemma 2.3 that the quantity inside of the expectation of the right-hand side above almost surely goes to zero. We infer from the dominated convergence theorem and estimate (4.16) that

$$v^m_{x_m}(t,P_m x)\to v_x(t,x)\text{ in }H. \tag{4.19}$$

By a similar argument, we prove that for any $x\in H,\ t\in[0,T]$,

$$v^m_{x_m x_m}(t,P_m x)\to v_{xx}(t,x)\text{ in }\mathcal{L}(H) \tag{4.20}$$

when $m\to\infty$. (The expressions of v,v_x,v_{xx} are given in (2.6), (2.8), (2.10).) Integrating (4.8), we have for $x\in H,t\in[0,T]$,

$$\begin{aligned}
&v^m(t,P_m x)=e^{-\frac12|P_m x|^2}\\
&+\int_0^t\Big[\frac12\operatorname{Tr}\,\big(P_m Q v^m_{x_m x_m}(s,P_m x)\big)+\big(P_m Ax+P_m F_m(P_m x),v^m_{x_m}(s,P_m x)\big)\\
&\qquad-\|P_m x\|^2 v^m(s,P_m x)\Big]ds.
\end{aligned}$$

We choose $x \in D(A)$. Using Lemma 4.3, we have for any $s \in [0,T]$

$$\left| \frac{1}{2} \text{ Tr } \left(P_m Q v^m_{x_m x_m}(s, P_m x)\right) + \left(P_m A x + P_m F_m(P_m x), v^m_{x_m}(s, P_m x)\right) \right.$$
$$\left. - \|P_m x\|^2 v^m(s, P_m x) \right|$$
$$\leq \frac{1}{2}\, k_1 \text{ Tr } Q + k_1 \left(|Ax|^2 + c|x|^{1/2}\|x\|^{3/2}\right) + \|x\|^2.$$

We have used inequalities

$$0 \leq v^m(s, P_m x) \leq 1$$

and the following consequence of Agmon's inequality:

$$|P_m F_m(P_m x)| \leq |F_m(P_m x)| \leq |f_m'(P_m x) \frac{\partial}{\partial \xi} P_m x|$$
$$\leq 2|P_m x|_{L^\infty(0,1)} \|P_m x\| \leq c|P_m x|^{1/2} \|P_m x\|^{3/2} \leq c|x|^{1/2}\|x\|^{3/2}.$$

We deduce from (4.18), (4.19), (4.20), and the dominated convergence theorem that for $x \in D(A), t \in [0,T]$,

$$v(t,x) = e^{-\frac{1}{2}|x|^2}$$
$$+ \int_0^t \left[\frac{1}{2} \text{ Tr } \left(Q v^m_{xx}(s,x)\right) + (Ax + F(x), v_x(s,x)) - \|x\|^2 v(s,x)\right] ds.$$

Since v, v_x, v_{xx} are continuous with respect to t, it follows that $t \mapsto v(t,x)$ is a C^1 function for $x \in D(A)$, and for $x \in D(A), t \in [0,T]$,

$$v_t(t,x) = \frac{1}{2} \text{ Tr } [Q v_{xx}(t,x)] + (Ax + F(x), v_x(t,x)) - \|x\|^2 v(t,x),$$

so that v is a strict solution of (2.5). The following lemma is an easy consequence of Lemma 4.2, Lemma 4.3, and (4.9).

LEMMA 4.5. *For any $x_m \in P_m H, t \in [0,T]$ we have*

(i) $0 \leq |u^m(t, x_m)| \leq \frac{1}{2} \left(|x_m|^2 + T \text{ Tr } Q\right)$,

(ii) $|u^m_{x_m}(t, x_m)| \leq k_1\, e^{\frac{1}{2}\left(|x_m|^2 + T \text{ Tr } Q\right)}$,

(iii) $|u^m_{x_m x_m}(t, x_m)|_{\mathcal{L}(P_m H)} \leq k_1 e^{\frac{1}{2}\left(|x_m|^2 + T \text{ Tr } Q\right)} + k_1^2 e^{|x_m|^2 + T \text{ Tr } Q}$.

From (4.18), (4.19), and (4.20) we have for $x \in H, t \in [0,T]$,

$$
\begin{aligned}
&u^m(t, P_m x) \to u(t,x),\\
&u^m_{x_m}(t, P_m x) \to u_x(t,x) \text{ in } H, \qquad (4.21)\\
&u^m_{x_m x_m}(t, P_m x) P_m \to u_{xx}(t,x) \text{ in } \mathcal{L}(H).
\end{aligned}
$$

Arguing as above we see that $t \mapsto u(t,x)$ is a C^1 function for $x \in D(A)$, and for $x \in D(A), t \in [0,T]$,

$$u_t(t,x) = \frac{1}{2} \text{ Tr } [Q u_{xx}(t,x)] + (Ax + F(x), u_x(t,x)) - \frac{1}{2}\, |\sqrt{Q} u_x(t,x)|^2 + \|x\|^2,$$

and u is a strict solution of the Hamilton–Jacobi equation (2.4).

We now want to take the limit $m \to \infty$ in (4.11).

LEMMA 4.6. *Let $x \in H$ and $z \in L^2_W(\Omega \times [0,T]; H)$, let X be the solution of* (2.2) *and X_m the solution of* (2.3), *with $x_m = P_m x, z_m = P_m z$. Then*

$$u^m_{x_m}(T-t, X_m(t)) \to u_x(T-t, X(t))$$

in $L^2(0,T;H)$ $\mathbb{P}$–almost surely.

Proof. It is shown in the proof of Lemma 2.1 that X_m is almost surely bounded in $L^\infty(0,T;H)$ and it converges almost surely in $L^2(0,T;H)$ to X. By the mean value theorem and Lemma 4.5(iii) it follows

$$u^m_{x_m}(T-t, X_m(t)) - u^m_{x_m}(T-t, P_m X(t)) \to 0$$

in $L^2(0,T;H)$ almost surely. Also by (4.21)

$$u^m_{x_m}(T-t, P_m X(t)) - u_x(T-t, X(t)) \to 0$$

in H for any $t \in [0,T]$, and by Lemma 4.5(ii) and the dominated convergence theorem we deduce that this convergence holds in $L^2(0,T;H)$. ◻

Let $x \in H$ and $z \in L^2_W(\Omega \times [0,T]; H)$. We take $x_m = P_m x, z_m = P_m z$ in (4.11). Then thanks to Lemma 2.1, (4.21), and Lemma 4.6 we can take the limit and deduce that

$$\begin{aligned} u(T,x) &+ \frac{1}{2}\int_0^T \left|\sqrt{Q}\, u_x(T-t,X(t)) + z(t)\right|^2 dt \\ &= \frac{1}{2}\,|X(T)|^2 + \int_0^T \left(\|X(t)\|^2 + \frac{1}{2}\,|z(t)|^2\right) dt \\ &+ \int_0^T \left(u_x(T-t,X(t)), \sqrt{Q}\, dW(t)\right). \end{aligned} \tag{4.22}$$

From (3.8) we have

$$\begin{aligned} &\mathbb{E}\left(\int_0^T |\sqrt{Q}\, u^{l,m}_{x_m}(T-t, X_m(t))|^2 dt\right) \\ &\le 4J_{l,m}(z_m) + 2\mathbb{E}\int_0^T |z_m|^2 dt. \end{aligned}$$

By Ito's formula, for $t \in [0,T]$

$$\begin{aligned} &\frac{1}{2}\mathbb{E}|X_m(t)|^2 + \int_0^t ||X_m(s)||^2 ds \\ &= \mathbb{E}\int_0^t (X_m(s), z_m(s))ds + \frac{1}{2}|x_m|^2 + \frac{1}{2}t \text{ Tr } P_m Q \\ &\le \frac{1}{2}\int_0^t |X_m(s)|^2 ds + \frac{1}{2}s\int_0^t |z(s)|^2 ds + \frac{1}{2}|x|^2 + \frac{1}{2}T \text{ Tr } Q. \end{aligned}$$

By the Gronwall lemma it follows easily that

$$J_{l,m}(z_m) \le c\left(|x|^2 + T \text{ Tr } (Q) + \mathbb{E}\int_0^T |z|^2 dt\right)$$

and

$$\mathbb{E}\left(\int_0^T |\sqrt{Q}\; u^{l,m}_{x_m}(T-t,X_m(t))|^2dt\right) \le c\left(|x|^2+T\;\mathrm{Tr}\;(Q)+\mathbb{E}\int_0^T|z|^2dt\right).$$

It follows that $\sqrt{Q}\; u^{l,m}_{x_m}(T-t,X_m(t))$ is bounded in $L^2(\Omega\times[0,T])$. Since it converges pointwise to $\sqrt{Q}\; u_x(T-t,X(t))$ we have by Fatou's lemma

$$\mathbb{E}\left(\int_0^T |\sqrt{Q}\; u_x(T-t,X(t))|^2dt\right) \le c\left(|x|^2+T\;\mathrm{Tr}\;(Q)+2\mathbb{E}\int_0^T|z|^2dt,\right)$$

and $\sqrt{Q}\; u_x(T-t,X(t))$ belongs to $L^2(\Omega\times[0,T])$. Therefore we can take the expectation in (4.22) and obtain the fundamental identity

$$u(T,x)+\frac{1}{2}\;\mathbb{E}\int_0^T|\sqrt{Q}\; u_x(T-t,X(t))+z(t)|^2dt = J(z). \tag{4.23}$$

5. Existence of a solution to the closed loop equation. We now consider the closed loop equation

$$\begin{cases} dX^* = (AX^*+F(X^*)dt - Qu_x(T-t,X^*(t)))\,dt+\sqrt{Q}\;dW,\\ X^*(0)=x.\end{cases} \tag{5.1}$$

We first note that thanks to Lemma 4.5 and (4.21)

$$|u_x(t,x)| \le k_1\; e^{\frac{1}{2}\;\left(|x|^2+T\;\mathrm{Tr}\;Q\right)},\; x\in H, \tag{5.2}$$

$$|u_{xx}(t,x)|_{\mathcal{L}(H)} \le 2k_1^2\; e^{\left(|x|^2+T\;\mathrm{Tr}\;Q\right)},\; x\in H. \tag{5.3}$$

Hence u_x is locally Lipschitz in x. This is the main ingredient in the proof of the following result.

LEMMA 5.1. *There exists at most one solution of* (5.1) *with trajectories in*

$$L^\infty(0,T;H)\cap L^2(0,T;H^1_0(0,1)).$$

Proof. Let X_1, X_2 be two solutions of (5.1) and $X = X_1 - X_2$. We have

$$\frac{dX}{dt} = AX + F(X_1) - F(X_2) - Q\left(u_x(T-t,X_1(t)) - u_x(T-t,X_2(t))\right).$$

It follows that $\frac{dX}{dt}\in L^2(0,T;H^{-1}(0,1))$ and

$$\frac{1}{2}\;\frac{d}{dt}|X|^2+\|X\|^2 = (F(X_1)-F(X_2),X) - (Qu_x(T-t,X_1)-Qu_x(T-t,X_2),X)$$
$$\le \frac{1}{2}\;\|X\|^2 + c\left(|X_1|^2_{L^\infty(0,1)}+|X_2|^2_{L^\infty(0,1)}\right)|X|^2 + 4k_1^4e^{2\left(M_1^2+T\;\mathrm{Tr}\;Q\right)}|X|^2,$$

where

$$M_1 = \max\{|X_1|_{L^\infty(0,T;H)}, |X_2|_{L^\infty(0,T;H)}\}.$$

By the Sobolev embedding theorem

$$|X_i|_{L^\infty(0,1)} \le c\|X_i\|, \quad i=1,2.$$

Thus by Gronwall's lemma

$$|X(t)|^2 \le e^{c\int_0^T (\|X_1\|^2+\|X_2\|^2)ds + M_2 T}|X(0)|^2,$$

with

$$M_2 = 4k_1^4 e^{2\left(M_1^2 + T \text{ Tr } Q\right)}.$$

The result follows since $X(0)=0$. ∎

We prove the existence of X^* by approximation. Let $X^*_{l,m}$ be the solution of (3.9) with $x_m = P_m x$.

Lemma 5.2. *There exists a constant k_2 depending only on A, Q, T such that for any $l, m \in \mathbb{N}$*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X^*_{l,m}(t)|^2\right) \le k_2\left(|x|^2 + \text{ Tr } Q\right).$$

Proof. First we have

$$J_{l,m}(z^*_{l,m}) \le J_{l,m}(0) \le \frac{1}{2}\left(|x_m|^2 + t \text{ Tr } Q\right)$$

by Ito's formula, where $z^*_{l,m}$ is defined in (3.10). It follows that

$$\mathbb{E}\left(\int_0^T |z^*_{l,m}(t)|^2 dt\right) \le |x_m|^2 + t \text{ Tr } Q \le |x|^2 + t \text{ Tr } Q. \tag{5.4}$$

By Ito's formula

$$\begin{aligned}
&\frac{1}{2}|X^*_{l,m}(t)|^2 + \int_0^t \|X^*_{l,m}(s)\|^2 ds \\
&= \int_0^t \left(\sqrt{Q}\, z^*_{l,m}(s), X^*_{l,m}(s)\right) ds + \int_0^t \left(X^*_{l,m}(s), \sqrt{Q}\, dW(s)\right) \\
&+\frac{1}{2}\left(|x_m|^2 + t \text{ Tr } (P_m Q)\right) \\
&\le c\int_0^T |z^*_{l,m}(s)|^2 ds + \frac{1}{2}\int_0^T \|X^*_{l,m}(s)\|^2 ds \\
&+ \sup_{t\in[0,T]} \int_0^t \left(X^*_{l,m}(s), \sqrt{Q}\, dW(s)\right) + \frac{1}{2}\left(|x|^2 + t \text{ Tr } Q\right).
\end{aligned}$$

The result follows by the application of a Martingale inequality. ∎

We deduce that there exists $\overline{X}_m$ in $L^2(\Omega; L^\infty(0,T;H))$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |\overline{X}_m(t)|\right) \le k_2\left(|x|^2 + \text{ Tr } Q\right)$$

and

$$X^*_{l,m} \rightharpoonup \overline{X}_m \text{in } L^2(\Omega; L^\infty(0,T;H)), \text{ weak star.}$$

We now derive a pathwise estimate for solutions of (3.9).

LEMMA 5.3. *Let $k(\omega)$ be a random variable. For any $m \in \mathbb{N}$, there exist random times t_k^m and constants c_k^m such that if $\widetilde{X}_{l,m}$ is a solution of* (3.9) *satisfying*

$$|\widetilde{X}_{l,m}(0)| \le k, \ \mathbb{P}\text{–almost surely},$$

then

$$\sup_{t\in[0,t_k^m]} |\widetilde{X}_{l,m}(t)|^2 + \int_0^{t_k} \|\widetilde{X}_{l,m}(s)\|^2 ds \le c_k^m, \ \mathbb{P}\text{–almost surely}$$

Proof. Let

$$W_A^m(t) = \int_0^t e^{(t-s)A} P_m \sqrt{Q}\, dW(s)$$

and

$$\overline{X}_{l,m} = \widetilde{X}_{l,m} - W_A^m.$$

Then

$$\frac{d}{dt}\overline{X}_{l,m} = A\overline{X}_{l,m} + P_m F_m(\overline{X}_{l,m} + W_A^m) - Q u_{x_m}^{l,m}(T-t, \overline{X}_{l,m} + W_A^m).$$

Using similar arguments as in the proofs of Lemmas 2.1 and 4.2 we can prove

$$\begin{aligned}\frac{d}{dt}|\overline{X}_{l,m}|^2 \quad &+ \quad \|\overline{X}_{l,m}\|^2 \le c|W_A^m|^{4/3}_{L^4(0,1)} |\overline{X}_{l,m}|^2 + ck_m e^{2|\overline{X}_{l,m}|^2 + 2|W_A^m|^2 + T \text{ Tr } Q} \\ &+ \quad c|W_A^m|^4_{L^4(0,1)}.\end{aligned}$$

We set

$$\begin{aligned} F_m(t) &= e^{-c\int_0^t |W_A^m(s)|^{4/3}_{L^4(0,1)} ds} |\overline{X}_{l,m}|^2, \\ g_m &= 2e^{c\int_0^t |W_A^m(s)|^{4/3}_{L^4(0,1)} ds}, \\ h_m &= ck_m e^{\sup_{t\in[0,T]} |W_A^m(t)|^2 + T \text{ Tr } Q}, \\ k_m &= c \sup_{t\in[0,T]} |W_A^m(t)|^4. \end{aligned}$$

It is easy to obtain

$$\frac{d}{dt}\, F_m \le h_m e^{g_m F_m} + k_m$$

so that

$$e^{-g_m F_m(t)} \ge -\left(h_m + k_m\right) t + e^{-g_m k^2},$$

and if we take

$$t_k^m = \frac{1}{2(h_m + k_m)}\ e^{-g_m k^2}$$

we have

$$F_m(t) \le \frac{1}{g_m}\ \ln 2 + k^2$$

for $t \in [0, t_k^m]$. Now the proof can be completed easily. □

It is not difficult to use the estimate in Lemma 5.3 and to prove that, for almost all $\omega \in \Omega$, a subsequence $\{X_{l,m}^*\}$ converges to X_m^* a solution of

$$(5.5)\quad \begin{cases} dX_m^* = (AX_m^* + P_m F_m(X_m^*) - P_m Q u_{x_m}^m(T-t, X_m^*))dt + P_m\sqrt{Q}\, dW, \\ X_m^*(0) = x_m \end{cases}$$

on the interval $[0, t_m^k]$ whenever $|x_m| \le k$. Arguing as in Lemma 5.1, (5.5) has at most one solution so that the whole sequence converges.

We take

$$k = |\overline{X}_m|_{L^\infty(0,T;H)}.$$

Since $\{X_{l,m}^*\}$ converges pointwise to X_m^* and in $L^2(\Omega; L^\infty(0,T;H))$ weak star to $\overline{X}_m$, we have $X_m^* = \overline{X}_m$ $\mathbb{P}$–almost surely on $[0, t_k^m]$. It follows $|X_m^*(t_k^m)| \le k$, so that our construction can be reiterated and X_m^* can be prolonged to a solution of (5.5) on the interval $[0,T]$. Moreover, by Lemma 5.2 if $x_m = P_m x$

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_m^*|^2\right) \le k_2\left(|x|^2 + T\ \mathrm{Tr}\ Q\right).$$

Arguing as in the proof of Lemma 5.3 and using Lemma 4.5 and the uniform boundedness of W_A^m in $L^\infty(0,T;L^4(0,1))$, we prove the following pathwise estimate on X_m^*.

LEMMA 5.4. *Let $k(\omega)$ be a random variable; there exists a random time t_k and a constant c_k such that if $\widetilde{X}_m$ is a solution of* (5.5) *satisfying*

$$|\widetilde{X}_m(0)| \le k, \text{ almost surely,}$$

then

$$\sup_{t\in[0,t_k]} |\widetilde{X}_m(t)|^2 + \int_0^{t_k} \|\widetilde{X}_m(s)\|^2 ds \le c_k, \text{ almost surely.}$$

Now we can repeat the argument that we have used to construct X_m^* and obtain X^*, a solution of (5.1) on $[0,T]$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X^*(t)|^2\right) \le k_2\left(|x|^2 + T\ \mathrm{Tr}\ Q\right).$$

It remains to prove that

$$z^*(t) = -\sqrt{Q}\ u_x(T-t, X^*(t))$$

is an admissible control, i.e., that $z^* \in L^2_W(\Omega \times [0,T]; H)$.

Arguing as in Lemma 4.6 we have

$$u^{l,m}_{x_m}(T-t, X^*_{l,m}(t)) \overset{l\to\infty}{\to} u^m_{x_m}(T-t, X^*_m(t)) \overset{m\to\infty}{\to} u_x(T-t, X^*(t))$$

in $L^2(0,T;H)$ $\mathbb{P}$–almost surely. Thus by (5.4) we have $u_x(T-t, X^*(t)) \in L^2_W(\Omega \times [0,T]; H)$ and

$$\mathbb{E}\left(\int_0^T |z^*(t)|^2 dt\right) \le |x|^2 + T \operatorname{Tr} Q.$$

This ends the proof of Theorem 2.4. □

Appendix A.

A.1. Proof of Lemma 2.1. For any $m \in N$ we set

$$W^m_A(t) = \int_0^t e^{(t-s)A} P_m \sqrt{Q}\, dW(s);$$

it is the unique solution of

$$\begin{cases} dW^m_A = AW^m_A dt + P_m\sqrt{Q}\, dW, \\ W^m_A(0) = 0. \end{cases}$$

Also

$$W_A(t) = \int_0^t e^{(t-s)A} \sqrt{Q}\, dW(s)$$

is the unique solution (see [11]) of

$$\begin{cases} dW_A = AW_A dt + \sqrt{Q}\, dW, \\ W_A(0) = 0. \end{cases}$$

It is not difficult to see that W^m_A converges to W_A in $L^4([0,T]\times[0,1])$ almost surely.

Let X_m be the solution to (2.3). We set

$$\overline{X}_m = X_m - W^m_A;$$

thus

$$\text{(A.1)} \qquad \begin{cases} \dfrac{d\overline{X}_m}{dt} = A\overline{X}_m + P_m F_m(\overline{X}_m + W^m_A) + P_m\sqrt{Q}\, z_m, \\ \overline{X}_m(0) = x_m. \end{cases}$$

To derive an a priori estimate, we take the scalar product of (A.1) by $\overline{X}_m$. Using integration by parts, interpolation inequality, Sobolev embedding theorem, and Young's inequality, we have

$$
\begin{aligned}
&\left(P_m F_m(\overline{X}_m + W_A^m), \overline{X}_m\right) = -\int_0^1 \left(f_m(\overline{X}_m + W_A^m) - f_m(\overline{X}_m)\right) \frac{\partial}{\partial \xi}\overline{X}_m d\xi \\
&\le 2\int_0^1 |2\overline{X}_m + W_A^m|\, |W_A^m| \left|\frac{\partial}{\partial\xi}\overline{X}_m\right| d\xi \\
&\le 2\left(2|\overline{X}_m|_{L^4(0,1)} + |W_A^m|_{L^4(0,1)}\right)|W_A^m|_{L^4(0,1)}\|\overline{X}_m\| \\
&\le c|W_A^m|_{L^4(0,1)}|\overline{X}_m|^{3/4}\|\overline{X}_m\|^{5/4} + \frac{1}{8}\,\|\overline{X}_m\|^2 + 8|W_A^m|^4_{L^4(0,1)} \\
&\le c|W_A^m|^{8/3}_{L^4(0,1)}|\overline{X}_m|^2 + \frac{1}{4}\,\|\overline{X}_m\|^2 + 8|W_A^m|^4_{L^4(0,1)}.
\end{aligned}
$$

We deduce

$$
\frac{d}{dt}\,|\overline{X}_m|^2 + \|\overline{X}_m\|^2 \le c|W_A^m|^{8/3}_{L^4(0,1)}|\overline{X}_m|^2 + 16|W_A^m|^4_{L^4(0,1)} + c|\sqrt{Q}\,z_m|^2
$$

and

$$
\begin{aligned}
&|\overline{X}_m(t)|^2 + \int_0^t \|\overline{X}_m(s)\|^2 ds \\
&\le e^{c\int_0^t |W_A^m(s)|^{8/3}_{L^4(0,1)}ds}|x_m|^2 \\
&\quad + \int_0^t e^{c\int_s^t |W_A^m(r)|^{8/3}_{L^4(0,1)}dr}\left(16|W_A^m(s)|^4_{L^4(0,1)} + c + |\sqrt{Q}\,z_m|^2\right)\,ds.
\end{aligned}
$$

This proves that for fixed $\omega \in \Omega$, $\{\overline{X_m}\}$ is a bounded sequence in $L^\infty(0,T;L^2(0,1))$ and $L^2(0,T;H_0^1(0,1))$. By standard arguments based on compactness and the uniqueness of the limit (see [18]), we deduce that $\{\overline{X}_m\}$ converges almost surely to $\overline{X}$ in $L^2([0,T]\times[0,1])$, the unique solution of

$$
\begin{cases}
\dfrac{d\overline{X}}{dt} = A\overline{X} + F(\overline{X} + W_A) + \sqrt{Q}\,z, \\
\overline{X}(0) = x.
\end{cases}
$$

We set $X = \overline{X} + W_A$ and have

$$
\begin{cases}
dX = (AX + F(X) + \sqrt{Q}\,z)dt + \sqrt{Q}\,dW, \\
X(0) = x.
\end{cases}
$$

We apply Ito's formula to $|X^m|^2$ and take the expectation

$$
\begin{aligned}
\text{(A.2)}\qquad &\frac{1}{2}\,\mathbb{E}|X^m(t)|^2 + \mathbb{E}\int_0^t \|X^m(s)\|^2 ds = \frac{1}{2}\,|x_m|^2 \\
&\quad + \mathbb{E}\left(\int_0^t \left(\sqrt{Q}\,z_m, X^m\right)ds + \frac{1}{2}\,t\mathrm{Tr}\,[P_m Q]\right).
\end{aligned}
$$

Hence

$$
\mathbb{E}|X^m(t)|^2 + \mathbb{E}\int_0^t \|X^m\|^2 ds \le\; |x_m|^2 + c\mathbb{E}\left(\int_0^t |\sqrt{Q}\,z_m|^2 ds + t\mathrm{Tr}\,Q\right),
$$

which proves that X^m is bounded in $L^2(\Omega, L^2(0,T; H_0^1(0,1)))$ and $X^m(t)$ in $L^2(\Omega, L^2(0,1))$. It is classical that this implies

$$X^m \rightharpoonup X \text{ in } L^2(\Omega, L^2(0,T; H_0^1(0,1)) \text{ weak}, \qquad X^m(t) \rightharpoonup X(t) \text{ in } L^2(\Omega, L^2(0,1)) \text{ weak}. \tag{A.3}$$

Since z_m converges to z in $L^2(\Omega, L^2(0,T; L^2(0,1)))$ strongly, we also have

$$\mathbb{E}\int_0^t \left(\sqrt{Q}\, z_m, X^m\right) ds \to \mathbb{E}\int_0^t \left(\sqrt{Q}\, z, X\right) ds. \tag{A.4}$$

By Ito's formula for $|X|^2$, we also have

$$\frac{1}{2}\,\mathbb{E}|X(t)|^2 + \mathbb{E}\int_0^t \|X(s)\|^2 ds = \frac{1}{2}\,|x|^2 + \mathbb{E}\int_0^t \left(\sqrt{Q}\, z, X\right) ds + \frac{1}{2}\, t\, \mathrm{Tr}\, Q,$$

and by $(A.2)$, $(A.4)$

$$\frac{1}{2}\,\mathbb{E}|X^m(t)|^2 + \mathbb{E}\int_0^t \|X^m(s)\|^2 ds \to \frac{1}{2}\,\mathbb{E}|X(t)|^2 + \mathbb{E}\int_0^t \|X(s)\|^2 ds$$

so that convergences in $(A.3)$ hold in the strong topology.

Let us write Ito's formula for $\frac{1}{2}\,|X^m - P_m X|^2$:

$$\begin{aligned}
&\frac{1}{2}\,|X^m - P_m X|^2 + \int_0^t \|X^m - P_m X\|^2 ds \\
&= \int_0^t \left(\sqrt{Q}\,(z^m - P_m z), X^m - P_m X\right) ds \\
&\quad + \int_0^t (P_m F_m(X^m) - P_m F(X), X^m - P_m X)\, ds \\
&\le c\int_0^t |\sqrt{Q}\,(z^m - P_m z)|^2 ds + c\int_0^t |f_m(X^m) - X^2|^2 ds + \frac{1}{2}\int_0^t \|X^m - P_m X\|^2 ds.
\end{aligned}$$

We deduce

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]} |X^m - P_m X|\right) \le c\mathbb{E}\left(\int_0^T |\sqrt{Q}\,(z_m - P_m z)|^2 ds\right)^{1/2} \\
&+ c\mathbb{E}\left(\int_0^T |f_m(X^m) - X^2|^2 ds\right)^{1/2}.
\end{aligned}$$

By standard estimates based on Ito's formula it can be seen that $\{X^m\}$ is bounded in $L^p(\Omega, C([0,T]; L^2(0,1)))$ for any $p \ge 1$. By Sobolev's embedding theorem and the strong convergence of X^m to X in $L^2(\Omega, L^2(0,T; H_0^1(0,1)))$, we can prove

$$\mathbb{E}\left[\left(\int_0^T |f_m(X^m) - X^2|^2 ds\right)^{1/2}\right] \to 0,$$

implying

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X^m - P_m X|\right) \to 0.$$

Since X^m is bounded in any $L^p(\Omega, C([0,T]; L^2(0,1)))$, the conclusion follows. ∎

A.2. Proof of Lemma 2.2. The existence of η^h and ζ^h solutions of (2.9) and (2.11) is classical. Let Y^x (resp., Y^{x+h}) be the solution of (2.7) with initial datum $x \in H$ (resp., $x+h \in H$). We set

$$r = Y^{x+h} - Y^x - \eta^h.$$

r satisfies the equation

$$\begin{aligned}\frac{dr}{dt} &= Ar + \frac{\partial}{\partial \xi}\left((Y^{x+h})^2 - (Y^x)^2 - 2Y^x\eta^h\right)\\ &= Ar + \frac{\partial}{\partial \xi}\left((Y^{x+h} - Y^x)^2 + 2Y^x r\right).\end{aligned}$$

By similar arguments as in the proof of Lemma 4.3, we have

$$\text{(A.5)}\quad |Y^{x+h}(t) - Y^x(t)|^2 + \int_0^t \|Y^{x+h}(s) - Y^x(s)\|^2 ds \le ce^{c\int_0^t \|Y^x(s)\|^{4/3}ds}\, |h|^2$$

and

$$\begin{aligned}&|r(t)|^2 + \int_0^t \|r(s)\|^2 ds\\ &\le ce^{c\int_0^t \|Y^x(s)\|^{4/3}ds} \int_0^t |Y^{x+h}(s) - Y^x(s)|^3 \|Y^{x+h}(s) - Y^x(s)\| ds.\end{aligned}$$

It follows that

$$\text{(A.6)}\qquad |r(t)|^2 + \int_0^t \|r(s)\|^2 ds \le ce^{c\int_0^t \|Y^x(s)\|^{4/3}ds}|h|^4.$$

We have

$$\begin{aligned}&|v(t,x+h) - v(t,x) - v_x(t,x)h|\\ &= \mathbb{E}\Bigg(e^{-\frac12|Y^{x+h}(t)|^2 - \int_0^t \|Y^{x+h}(s)\|^2 ds} - e^{-\frac12|Y^x(t)|^2 - \int_0^t \|Y^x(s)\|^2 ds}\\ &\qquad + \left((Y^x(t),\eta^h(t)) + 2\int_0^t (Y^x(s),\eta^h(s))ds\right) e^{-\frac12|Y^x(t)|^2 - \int_0^t \|Y^x(s)\|^2 ds}\Bigg)\\ &= \mathbb{E}\Bigg(\Big(e^{-\frac12(|Y^{x+h}(t)|^2 - |Y^x(t)|^2) - \int_0^t (\|Y^{x+h}(s)\|^2 - \|Y^x(s)\|^2)ds}\\ &\qquad -1 + \left(Y^x(t), Y^{x+h}(t) - Y^x(t)\right) + 2\int_0^t \left(Y^x(s), Y^{x+h}(s) - Y^x(s)\right) ds\\ &\qquad -(Y^x(t), r(t)) - 2\int_0^t (Y^x(s), r(s))\, ds\Big) e^{-\frac12|Y^x(t)|^2 - \int_0^t \|Y^x(s)\|^2 ds}\Bigg).\end{aligned}$$

By (A.5), (A.6), and elementary inequalities we obtain

$$\begin{aligned}&|v(t,x+h) - v(t,x) - v_x(t,x)h|\\ &\le c\mathbb{E}\Bigg[\left(1 + |Y^x(t)|^2 + \int_0^t \|Y^x(s)\|^2 ds\right)\\ &\qquad e^{-\frac12|Y^x(t)|^2 - \int_0^t \|Y^x(s)\|^2 ds + c\int_0^t \|Y^x(s)\|^{4/3}ds}\Bigg]\, |h|^2.\end{aligned}$$

This proves the differentiability of v. The proof that v is twice differentiable is similar. ∎

A.3. Proof of Lemma 2.3. We define

$$e_m^h = \eta_m^{P_m h} - P_m \eta^h.$$

By integration by parts, Hölder's inequality, and Agmon's inequality

$$|x|_{L^\infty(0,1)} \leq c|x|^{1/2}\|x\|^{1/2},\ x \in H_0^1(0,1),$$

we have

$$\frac{1}{2}\ \frac{d}{dt}\ |e_m^h|^2 + \|e_m^h\|^2 = -\int_0^1 \Bigg[\left((f_m'(Y_m) - f_m'(Y)\right)\eta_m^h$$
$$+f_m'(Y)e_m^h + f_m'(Y)\left((P_m - I)\right)\eta^h + \left(f_m'(Y) - 2Y)\eta^h\right) \frac{\partial}{\partial \xi} e_m^h \Bigg] d\xi$$
$$\leq \frac{1}{2}\ \|e_m^h\|^2 + c|\eta_m^{P_m h}|\ \|\eta_m^{P_m h}\|\ |Y_m - Y|^2 + c|Y|\|Y\|(|e_m^h|^2 + |(I - P_m)\eta^h|^2)$$
$$+c|\eta^h|\ \|\eta^h\|\ |f_m'(Y) - 2Y|^2.$$

Thus by Gronwall's lemma

$$|e_m^h|^2 + \int_0^t \|e_m^h(s)\|^2 ds \leq c e^{c\int_0^t |Y(s)|\ \|Y(s)\| ds}$$
$$\left(\int_0^t |\eta_m^{P_m h}|\ \|\eta_m^{P_m h}\|\ |Y_m - Y|^2 ds + \int_0^t |Y(s)|\ \|Y(s)\|\ |(I - P_m)\eta^h|^2 ds \right.$$
$$\left. + \int_0^t |\eta^h|\ \|\eta^h\|\ |f_m'(Y) - 2Y|^2 ds \right).$$

We write

$$|(I - P_m)\eta^h|^2 \leq c\|(I - P_m)\|^2_{\mathcal{L}(D((-A)^{1/4}),H)} |\eta^h|\ \|\eta^h\|,$$

and since Y is almost surely in

$$L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1)),$$

by (4.16) and a similar estimate on η we have

$$|e_m^h(t)|^2 + \int_0^t \|e_m^h(s)\|^2 ds \leq c(\omega) \Bigg(\sup_{t\in[0,T]} |Y_m - Y|^2$$
$$+|I - P_m|_{\mathcal{L}(D((-A)^{1/4}),H)} + c\left(\int_0^T |f_m'(Y) - 2Y|^4 ds \right)^{1/2} \Bigg) |h|^2.$$

We have

$$|f_m'(Y) - 2Y|^4 = \left| \frac{2}{m}\ \frac{Y^3}{(1 + \frac{1}{m}Y^2)^2} - \frac{2}{m^2}\ \frac{Y^4}{(1 + \frac{1}{m}Y^2)^2} \right|^4$$
$$\leq \frac{c}{m^2}\ |Y|^8_{L^4(0,1)} \leq \frac{c}{m^2}\ |Y|^6 \|Y\|^2,$$

so that

$$|e_m^h|^2 + \int_0^t \|e_m^h(s)\|^2 ds \le c(\omega)\left(\sup_{t\in[0,T]} |Y_m - Y|^2 \right.$$
$$\left. + |I - P_m|_{\mathcal{L}(H)} + \frac{c}{m} \right) |h|^2.$$

The first part of the lemma follows by Lemma 2.1. The proof of the second part goes along the same line. ∎

REFERENCES

[1] V. Barbu and G. Da Prato, *Hamilton-Jacobi Equations in Hilbert Spaces*, Pitman Res. Notes Math. Ser. 86, Longman, Harlow, UK, 1982.

[2] L. Bertini, N. Cancrini, and G. Jona-Lasinio, *The stochastic Burgers equation*, Comm. Math. Phys., 165 (1994), pp. 211–232.

[3] D.H. Chambers, R.J. Adrian, P. Moin, D.S. Stewart, and H.J. Sung, *Karhunuen-Loeve expansion of Burgers model of turbulence,* Phys. Fluids, 31 (1988), p. 2573.

[4] P. Cannarsa and G. Da Prato, *Some results on nonlinear optimal control problems and Hamilton-Jacobi equations in infinite dimensions*, J. Funct. Anal., 90 (1990), pp. 27–47.

[5] P. Cannarsa and G. Da Prato, *Direct solution of a second order Hamilton-Jacobi equation in Hilbert spaces*, in Stochastic Partial Differential Equations and Applications, Pitman Res. Notes Math. Ser. 268, G. Da Prato and L. Tubaro, eds., Longman, Harlow, UK, 1992, pp. 72–85.

[6] H. Choi, R. Temam, P. Moin, and J. Kim, *Feedback control for unsteady flow and its application to the stochastic Burgers equation*, J. Fluid Mech., 253 (1993), pp. 509–543.

[7] M.G. Crandall, H. Ishi, and P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67.

[8] G. Da Prato, *Some results on Bellman equation in Hilbert spaces*, SIAM J. Control Optim., 23 (1985), pp. 61–71.

[9] G. Da Prato, A. Debussche, and R. Temam, *Stochastic Burgers equation*, NoDEA Nonlinear Differential Equations Appl., 1 (1994), pp. 389–402.

[10] G. Da Prato and D. Gątarek, *Stochastic Burgers equation with correlated noise*, Stochastics Stochastics Rep., 52 (1995), pp. 29–41.

[11] G. Da Prato and J. Zabczyk, *Stochastic Evolution Equations in Infinite Dimensions*, Cambridge University Press, London, UK, 1992.

[12] W.H. Fleming and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, Berlin, New York, 1993.

[13] F. Gozzi, *Regularity of solutions of a second order Hamilton–Jacobi equation and application to a control problem*, Comm. Partial Differential Equations, 20 (1995), pp. 775–826.

[14] F. Gozzi, *Global regular solutions of second order Hamilton-Jacobi equations in Hilbert spaces with locally Lipschitz nonlinearities,* J. Math. Anal. Appl., 198 (1996), pp. 399–443.

[15] F. Gozzi and E. Rouy, *Regular solutions of second order stationary Hamilton–Jacobi equations*, J. Differential Equations, 130 (1996), pp. 201–234.

[16] F. Gozzi, E. Rouy, and A. Swiech, *Second Order Hamilton–Jacobi Equations in Hilbert Spaces and Stochastic Boundary Control*, preprint 46, Scuola Normale Superiore di Pisa, Pisa, Italy, 1996.

[17] Y. Kifer, *The Burgers Equation with a Random Force and a General Model for Directed Polymers in Random Environments*, preprint, 1995.

[18] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier–Villars, Paris, 1969.

[19] P.L. Lions, *Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions, Part* I: *The case of bounded stochastic evolution*, Acta Math., 161 (1988), pp. 243–278, *Part* II: *Optimal control of Zakai's equation*, in Stochastic Partial Differential Equations and Applications, Lecture Notes in Math. 1390, G. Da Prato

and L. Tubaro, eds., Springer-Verlag, New York, 1989, *Part* III: *Uniqueness of viscosity solutions for general second order equations*, J. Funct. Anal., 86 (1989), pp. 1–18.

[20] A. Swiech, *Viscosity Solutions of Fully Nonlinear Partial Differential Equations with "Unbounded" Terms in Infinite Dimensions*, Ph.D. thesis, University of California, Santa Barbara, 1993 .

[21] M. Tessitore and J. Zabczyk, *Invariant Measures for Stochastic Heat Equation*, preprint 16, Scuola Normale Superiore di Pisa, Pisa, Italy.