SIAM J. CONTROL AND OPTIMIZATION
Vol. 21, No. 4, July 1983

0363-0129/83/2104-0002 $01.25/0

SOLUTION OF THE BELLMAN EQUATION ASSOCIATED WITH AN INFINITE DIMENSIONAL STOCHASTIC CONTROL PROBLEM AND SYNTHESIS OF OPTIMAL CONTROL*

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Abstract. We prove the existence and uniqueness of the dynamic programming equation for control diffusion processes in Hilbert spaces.

Introduction. Consider the optimal control problem:

Minimize

$$\text{(P)}\qquad E\left(\int_0^T (g(t,x(t)))+\frac{1}{2}|u(t)|^2\right)dt+\phi_0(x(T))$$

over all u in M_w^2 $(0,T;H)$ subject to

$$\text{(0.1)}\qquad \begin{aligned} &dx=(Ax+u)\,dt+\sqrt{\varepsilon}\,dW_t, \qquad \varepsilon>0,\\ &x(0)=x_0.\end{aligned}$$

Here A is the infinitesimal generator of a contraction C_0-semigroup in a real separable Hilbert space H with the norm $|\cdot|$ $(\Omega,\mathcal{F},P)$ is a probability space, W_t is a H-valued Brownian motion on $(\Omega,\mathcal{F},P)$ and E is the expectation.

The function $g:[0,T]\times H\to\mathbb{R}$ is continuous and convex as a function of x for every $t\in[0,T]$.

This paper is concerned with a direct approach to the dynamic programming equation associated with problem (P), namely (see for instance [4], [6]):

$$\text{(0.2)}\qquad \begin{aligned} &\phi_t(t,x)+\frac{1}{2}|\phi_x(t,x)|^2-\langle Ax,\phi_x(t,x)\rangle-\frac{\varepsilon}{2}\operatorname{Tr}(S\phi_{xx}(t,x))=g(t,x),\\ &\phi(0,x)=\phi_0(x)\end{aligned}$$

where S is the covariance of W_1. In few words the idea (already used in [2]) consists in approximating the term $\frac{1}{2}|\phi_x|^2$ by $\alpha^{-1}(\phi-\phi_\alpha)$, where ϕ_α is the convex regularization of ϕ and after to let α tend to zero. We have previously studied in [3] this problem in the particular case where g_0 and $g(\cdot,x)$ have a sublinear growth. We remark that when g is quadratic (0.2) reduces to a Riccati equation and the corresponding control problem has been studied by several authors (see for instance [6]).

The contents of the paper are outlined below. Sections 1 and 2 are concerned with notation and preliminary results for spaces of differential functions and convex functions frequently used in the text. Section 3 studies a linearized version of problem (0.2). Section 4 gives the main result on existence and uniqueness for problem (0.2). Furthermore it is shown that for $\varepsilon\to 0$ the solution to (0.2) converges to the solution of the Hamilton–Jacobi equation

$$\text{(0.3)}\qquad \begin{aligned} &\phi_t(t,x)+\tfrac{1}{2}|\phi_x(t,x)|^2-\langle Ax,\phi_x(t,x)\rangle=g(t,x),\\ &\phi(0,x)=\phi_0(x)\end{aligned}$$

* Received by the editors January 25, 1982, and in revised form June 30, 1982.

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which has been studied in [2] by a different method. This result resembles the classical approach of Hamilton–Jacobi equations in finite dimensional spaces [9]. Finally in § 5 we study the synthesis for problem (P) proving the existence and uniqueness of a smooth feedback control.

1. Notation and preliminary results. Throughout in the sequel H is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. For $k = 0, 1, \cdots$, denote by $C^k(H)$ the space of all k times continuously differentiable (Fréchet) functions $\phi : H \to \mathbb{R}$ which are bounded on bounded subset on H along with their derivatives up to order k. For $k = 0$ we shall simply write $C(H)$. Other notation such as $C^{(k)}([0, T] \times H)$, $k = 0, 1$, and $\mathscr{L}(H)$ is obvious. We set

$$|\phi|_{h,n} = \operatorname{Sup} \{|\phi^{(h)}(x)(1+|x|^{2n})^{-1}|; x \in H\}, \tag{1.1}$$

$$\|\phi\|_{h,n} = \operatorname{Sup} \{|\phi^{(h)}(x) - \phi^{(h)}(y)| |x-y|^{-1} \cdot (1 + (|x| \vee |y|)^{2n})^{-1}; x \neq y \in H\} \tag{1.2}$$

where $\phi^{(h)}$ stands for the derivative of order h and $|x| \vee |y| = \max(|x|, |y|)$.

LEMMA 1. *For any* $\phi \in C^{k+1}(H)$ *one has*

$$\|\phi\|_{k,n} = |\phi|_{k+1,n}. \tag{1.3}$$

Proof. For $|y| \leqq 1$ we have

$$|\phi^{(k+1)}(x) \cdot y|(1+|x|^{2n})^{-1}$$
$$= \lim_{t \to 0} |\phi^{(k)}(x+ty) - \phi^{(k)}(x)| \cdot |t|^{-1}[1 + (|x| \vee |x+ty|)^{2n}]^{-1} \leqq \|\phi\|_{k,n}|y|,$$

which implies that $|\phi|_{k+1,n} \leqq \|\phi\|_{k,n}$. Conversely we have

$$|\phi^{(k)}(x) - \phi^{(k)}(y)| |x-y|^{-1}[1 + (|x| \vee |y|)^{2n}]^{-1}$$
$$\leqq \int_0^1 |\phi^{(k+1)}((1-\lambda)x + \lambda y)\, d\lambda [1 + (|x| \vee |y|)^{2n}]^{-1}.$$

Since $|(1-\lambda)x + \lambda y| \leqq |x| \vee |y|$, the latter implies $\|\phi\|_{k,n} \leqq |\phi|_{k+1,n}$ as claimed.

We shall also use the following notation:

$$X = \{\phi \in C(H);\ |\phi|_{0,n_0} < +\infty\},$$
$$Y = \{\phi \in C^1(H); |\phi|_{0,n_0} + |\phi|_{1,n_1} < +\infty\},$$
$$Z = \{\phi \in C^2(H); |\phi|_{0,n_0} < +\infty, |\phi|_{1,n_1} < +\infty, |\phi|_{2,n_2} < +\infty, \|\phi\|_{2,n_3} < +\infty\}$$

where $n_0 \geqq n_1 \geqq n_2 \geqq n_3 \geqq 0$ are fixed integers. The spaces X, Y and Z are endowed with the norms

$$|\phi|_X = |\phi|_{0,n_0}, \qquad |\phi|_Y = |\phi|_{0,n_0} + |\phi|_{1,n_1}, \tag{1.4}$$

$$|\phi|_Z = |\phi|_{0,n_0} + |\phi|_{1,n_1} + |\phi|_{2,n_2} + \|\phi\|_{2,n_3}. \tag{1.5}$$

We note for the purposes of § 4 the following lemma.

LEMMA 2. *For each* $M > 0$ *the set*

$$\Lambda = \{\phi \in Z; |\phi|_Z \leqq M\} \tag{1.6}$$

is closed in X. *Furthermore if* $\{\phi_n\} \subset \Lambda$ *is convergent in* X *to* ϕ *then*

$$\langle y, \phi_{n,x}(x)\rangle \to \langle y, \phi_x(x)\rangle \tag{1.7}$$

uniformly on every bounded subset of H. Finally, if $S \in \mathscr{L}(H)$ is a nuclear symmetric operator then

$$\operatorname{Tr}(S\phi_{n,xx}(x)) \to \operatorname{Tr}(S\phi_{xx}(x)) \tag{1.8}$$

uniformly on bounded subsets of H.

Proof. Let $\{\phi_n\} \subset Z$, $|\phi_n|_Z \leqq M$ and $\phi \in X$ be such that $\phi_n \to \phi$ in X. We have to show that $\phi \in Z$ and $|\phi|_Z \leqq M$. For any $R > 0$ there exist $M_{i,R}$, $i = 0, 1, 2, 3$, such that

$$\begin{aligned} &\sup_{x \in B_R} |\phi^{(i)}(x)| \leqq M_{i,R}, \qquad i = 0, 1, 2, \\ &\sup_{\substack{x,y \in B_R \\ x \neq y}} \frac{|\phi^{(2)}(x) - \phi^{(2)}(y)|}{|x-y|} \leqq M_{3,R}, \end{aligned} \tag{1.9}$$

where $B_R = \{x \in H;\, |x| \leqq R\}$.

Let now $R > 0$ be fixed and $x, y \in B_R$, $h \in [-1, 1]$. We set

$$\psi_{n,x,y}(h) = \psi_n(h) = \phi_n(x + hy) \tag{1.10}$$

and notice that

$$\psi_n'(h) = \langle y, \phi_{nx}(x + hy)\rangle. \tag{1.11}$$

By (1.9) we have

$$\begin{aligned} &|\psi_n(h)| \leqq M_{0,2R}, \ |\psi_n'(h)| \leqq M_{1,2R}|y|, \\ &|\psi_n'(h) - \psi_n'(k)| \leqq M_{2,2R}|h-k|\,|y|^2, \qquad h, k \in [-1, 1]. \end{aligned} \tag{1.12}$$

By the Ascoli–Arzelà theorem there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{\psi'_{n_k}\}$ is uniformly convergent as $k \to \infty$. It follows that

$$\left.\begin{aligned} &\psi_{n_k}(h) \to \phi(x + hy), \\ &\psi'_{n_k}(h) \to \frac{d}{dh}(\phi(x + hy)), \end{aligned}\right\} \quad \text{uniformly in } [-1, 1] \tag{1.13}$$

and consequently ϕ is Gateaux differentiable (we shall denote by $D\phi(x)$ the Gateaux derivative of ϕ at x). We have

$$\psi'_{n_k}(0) = \langle y, \phi_{n_k x}(x)\rangle \to \langle y, D\phi(x)\rangle. \tag{1.14}$$

We can show now that $D\phi(x)$ is continuous in x which will imply $D\phi = \phi_x$. We have indeed

$$|\psi'_{n_k,x,y}(0) - \psi'_{n_k,z,y}(0)| = |\langle y, \phi_{n_k x}(x) - \phi_{n_k}(z)\rangle| \leqq M_{2,R}|x - z|\,|y|, \tag{1.15}$$

from which, recalling (1.14) and letting x tend to $+\infty$ we get

$$|\langle y, D\phi(x) - D\phi(z)\rangle| \leqq M_{2,R}|x - z|\,|y| \quad \forall y \in H, \tag{1.16}$$

$$|D\phi(x) - D\phi(z)| \leqq M_{2,R}|x - z|. \tag{1.17}$$

Consequently $\phi \in C^1(H)$, $D\phi = \phi_x$ and

$$\langle y, \phi_{n_k x}(x)\rangle \xrightarrow{k \to \infty} \langle y, \phi_x(x)\rangle \quad \forall y \in H. \tag{1.18}$$

The latter implies (1.7) by a standard argument. Now we set

$$\zeta_{n,x,y,u}(h) = \zeta_n(h) = \langle u, \phi_{nx}(x + hy)\rangle. \tag{1.19}$$

It follows that

$$\zeta_n'(h) = \phi_{nxx}(x+hy)(u, y), \tag{1.20}$$

and by (1.9)

$$|\zeta_n'(h) - \zeta_n'(k) \leqq M_{3R}|h-k||u||y|^2. \tag{1.21}$$

Using once again the Ascoli–Arzelà theorem we may conclude that there exists a subsequence $\{n_k'\}$ of $\{n_k\}$ such that

$$\begin{aligned} \zeta_{n_k'}(h) &= \langle u, \phi_{n_kx}(x+hy)\rangle \to \langle u, \phi_x(x+hy)\rangle, \\ \zeta_{n_k'}'(h) &\to \frac{d}{dh}\langle u, \phi_x(x+hy)\rangle. \end{aligned} \tag{1.22}$$

We set

$$\left[\frac{d}{dh}\langle u, \phi_x(x+hy)\rangle\right]_{h=0} = E\phi(x)(u, y). \tag{1.23}$$

From (1.22) it follows that

$$\phi_{nxx}(x)(u, y) \to E\phi(x)(u, y). \tag{1.24}$$

To prove that $E\phi = \phi_{xx}$ it suffices to show that E is continuous in x. We have

$$\begin{aligned} |\zeta_{n,x,y,u}'(0) - \zeta_{n,z,y,u}'(0)| &= |\phi_{nxx}(x)(u, y) - \phi_{nxx}(z)(u, y)| \\ &\leqq M_{3,R}|x-z||u||y| \end{aligned} \tag{1.25}$$

and, recalling (1.24), we get for $k \to \infty$

$$|(E\phi(x) - E\phi(z))(u, y)| \leqq M_{3,R}|x-z||u||y|. \tag{1.26}$$

It follows that $E\phi(x) = \phi_{xx}(x)$ and

$$\phi_{n_k'xx}(x)(u, y) \to \phi_{xx}(x)(u, y). \tag{1.27}$$

It is also clear that

$$\phi_{nxx}(x)(u, y) \to \phi_{xx}(x)(u, y) \tag{1.28}$$

uniformly on bounded sets on H.

To prove that $|\phi|_Z \leqq M$ we proceed as follows. Let $\{e_i\}$ be an orthonormal basis in H such that

$$Se_i = \lambda_i e_i, \qquad \sum_{i=0}^{\infty} |\lambda_i| < \infty. \tag{1.29}$$

We have

$$\operatorname{Tr}(S\phi_{n,xx}(x)) = \sum_{i=0}^{\infty} \lambda_i \phi_{n,xx}(e_i, e_i) \tag{1.30}$$

which along with (1.28) and some simple calculations implies the claimed conclusion.

In the sequel we shall denote by $B([0, T]; C^h(H))$ the space of all $\phi: [0, T] \times H \to \mathbb{R}$ such that $(\partial^i\phi/\partial x^i)(z_1, z_2, \cdots, z_i)$ is continuous in $[0, T] \times H$, for $z_1, z_2, \cdots, z_i \in H$;

besides for any $R>0$ we have

$$\operatorname*{Sup}_{\substack{t\in[0,T]\\|x|\leqq R}}\left|\frac{\partial^i\phi(t,x)}{\partial x^i}\right|<+\infty,\qquad i=0,1,\cdots,h,\quad t\in[0,T].$$

We set moreover

$$B([0,T];X)=\{\phi\in B([0,T];C(H));\ \operatorname*{Sup}_{t\in[0,T]}|\phi(t,\cdot)|_X<+\infty\},$$

$$B([0,T];Y)=\{\phi\in B([0,T];C^1(H));\ \operatorname*{Sup}_{t\in[0,T]}|\phi(t,\cdot)|_Y<+\infty\},$$

$$B([0,T];Z)=\{\phi\in B([0,T];C^2(H));\ \operatorname*{Sup}_{t\in[0,T]}|\phi(t,\cdot)|_Z<+\infty\}.$$

Let $\{\Omega,\mathcal{F},P\}$ be a complete probability space and let W be a H-valued Brownian motion. Let $\{e_k\}$ be an orthonormal basis in H and assume that W is given by

$$W(t)=\sum_{i=0}^{\infty}\sqrt{\lambda_i}\,\beta_i(t)e_i, \tag{1.31}$$

where $\lambda_i\geqq 0, i=1,2,\cdots,\sum_{i=0}^{\infty}\lambda_i<\infty$ and $\{\beta_i(t)\}$ are scalar Brownian motions mutually independent. Let S be the nuclear positive operator defined by

$$Se_i=\lambda_i e_i,\qquad i=1,\cdots. \tag{1.32}$$

We note (see [6]) that

$$\operatorname{Cov}(W_t)=tS. \tag{1.33}$$

In the sequel we shall denote by $L^2_w(0,T;H)$ (resp. $M^2_w(0,T;H)$) the space of all nonanticipative mappings $x:[0,T]\times\Omega\to H$ with respect to W, such that

$$P\left(\int_0^T|x(t)|^2\,dt<\infty\right)=1 \tag{1.34}$$

resp.

$$E\left(\int_0^T|x(s)|^2\,ds<+\infty\right). \tag{1.35}$$

For other concepts and fundamental results on Brownian motion we refer the reader to [6], [10], [11], [12].

2. Preliminaries on convex functions. In this section we recall for later use some definitions and elementary properties of some spaces of convex functions. For general concepts and results on convex analysis we refer to [1] and [5].

We shall denote by K the set of all convex functions $\phi\in C(H)$. For any $\phi\in K$ denote by ϕ_α the function

$$\phi_\alpha(x)=\inf\{(2\alpha)^{-1}|x-y|^2+\phi(y);\ y\in H\},\qquad \alpha>0 \tag{2.1}$$

and recall that $\phi_\alpha\in K\cap C^1(H)$.

For any $\phi\in K$ denote by $\partial\phi:H\to H$ the subdifferential of ϕ, i.e.,

$$\partial\phi(x)=\{x^*\in H;\ \langle x^*,x-y\rangle\geqq\phi(x)-\phi(y),\ \forall y\in H\}.$$

If $\phi \in C^1(H)$ then $\partial\phi$ is single valued and $\partial\phi = \phi'$ (ϕ' is the derivative of ϕ). The map $\partial\phi : H \to H$ is maximal monotone, i.e.,

$$\langle x^* - y^*, x - y\rangle \geqq 0 \quad \text{for all } x^* \in \partial\phi(x),\ y^* \in \partial\phi(y)$$

and the range $R(1+\alpha\,\partial\phi)$ is all of H (1 is the identity operator).

In particular this implies that

$$x_\alpha = (1+\alpha\ \partial\phi)^{-1}\cdot x$$

exists for all $\alpha > 0$ and moreover $|x_\alpha - \bar{x}_\alpha| \leqq |x - \bar{x}|$ for any $x, \bar{x} \in H$. Also we have

$$\phi_\alpha(x) = \phi(x_\alpha) + (2\alpha)^{-1}|x - x_\alpha|^2 \quad \forall \alpha > 0, x \in H \tag{2.2}$$

and

$$F_\alpha(x) = F(x_\alpha) = \alpha^{-1}(x - x_\alpha), \tag{2.3}$$

where $F = \partial\phi$ and $F_\alpha = \phi'_\alpha$.

Assume now that $\phi \in K \cap C^2(H)$. Since $x = x_\alpha + \alpha F(x_\alpha)$ we have

$$1 = x'_\alpha + \alpha F'(x_\alpha)\cdot x'_\alpha,$$

where x'_α is the derivative of the operator $x \to x_\alpha$. Hence

$$x'_\alpha = (1+\alpha F'(x_\alpha))^{-1}. \tag{2.4}$$

In the next lemma we gather for later use some immediate properties of x_α and F_α.

LEMMA 3. *For any $\phi \in K \cap C^2(H)$ and $x, y \in H$ we have*

$$|x_\alpha| \leqq |x| + \alpha|F(0)|, \tag{2.5}$$

$$|F_\alpha(x)| \leqq |F(x)|, \tag{2.6}$$

$$|F'_\alpha(x)| \leqq |F'(x)|, \tag{2.7}$$

$$|F'_\alpha(x) - F'_\alpha(y)| \leqq |F'(x_\alpha) - F'(y_\alpha)|, \tag{2.8}$$

Proof. The proof is well known but we sketch it for the reader's convenience.

Since F is monotone, we have the inequality

$$0 \leqq \langle F(x_\alpha) - F(0), x_\alpha\rangle = \alpha^{-1}\langle x - x_\alpha, x_\alpha\rangle - \langle F(0), x_\alpha\rangle$$

which implies

$$|x_\alpha|^2 \leqq |x_\alpha|(|x| + \alpha|F(0)|)$$

and (2.5) follows. To prove (2.6) we notice that

$$\begin{aligned}|F_\alpha(x)| &\leqq |\alpha^{-1}(x - (1+\alpha F)^{-1}x)| \\ &= \alpha^{-1}|(1+\alpha F)^{-1}[(1+\alpha F)x - x]| \leqq |F(x)|.\end{aligned}$$

As regards (2.7) it follows by (2.4) because F' is a positive operator. Finally, again by (2.4), we have

$$|x'_\alpha - y'_\alpha| \leqq \alpha|F'(x_\alpha) - F'(y_\alpha)|$$

while by (2.3)

$$|F'_\alpha(x) - F'_\alpha(y)| = \alpha^{-1}|x'_\alpha - y'_\alpha| \leqq |F'(x_\alpha) - F'(y_\alpha)|$$

which yields (2.8) as claimed.

For any $\varepsilon > 0$ and $\phi \in C^1(H) \cap K$ we set

$$R_{\phi,\alpha}(x) = \alpha^{-1}(\phi(x) - \phi_\alpha(x)) - \tfrac{1}{2}|\phi'(x)|^2. \tag{2.9}$$

LEMMA 4. *We have*

$$|R_{\phi,\alpha}(x)| \leqq |F(x)| \int_0^1 |F(x - \alpha t F(x_\alpha)) - F(x)|\, dt + \frac{1}{2}|F(x) - F(x_\alpha)|^2 \tag{2.10}$$

and

$$|R'_{\phi,\alpha}(x)| \leqq |F(x)| \int_0^1 |F'(x - \alpha t F(x_\alpha)) - F'(x)|\, dt + |F'(x)||F(x_\alpha) - F(x)| \tag{2.11}$$

where $F = \phi'$.

Proof. To prove (2.10) it suffices to notice the equality

$$\begin{aligned} R_{\phi,\alpha}(x) &= \frac{1}{\alpha}(\phi(x) - \phi(x_\alpha)) - \frac{1}{2}(|F(x_\alpha)|^2 + |F(x)|^2) \\ &= \int_0^1 \langle F(x - \alpha t F(x_\alpha)), F(x_\alpha)\rangle\, dt - \frac{1}{2}(|F(x_\alpha)|^2 + |F(x)|^2) \\ &= \int_0^1 \langle F(x - \alpha t F(x_\alpha)) - F(x), F(x_\alpha)\rangle\, dt - \frac{1}{2}|F(x) - F(x_\alpha)|^2. \end{aligned}$$

Finally (2.11) follows from the identity

$$\begin{aligned} R'_{\phi,\alpha}(x) &= \frac{1}{\alpha}(F(x) - F_\alpha(x)) - F'(x) \cdot F(x) \\ &= \int_0^1 F'(x - \alpha t F(x_\alpha)) \cdot F(x_\alpha)\, dt - F'(x) \cdot F(x) \\ &= \int_0^1 [F'(x - \alpha t F(x_\alpha)) - F'(x)] \cdot F(x_\alpha)\, dt + F'(x)(F(x_\alpha) - F(x)). \end{aligned}$$

LEMMA 5. *Assume that* $\phi, \bar\phi \in K \cap C^2(H)$. *Then for all* $x \in H$

$$\phi_\alpha(x) - \bar\phi_\alpha(x) \leqq \phi(\bar x_\alpha) - \bar\phi(\bar x_\alpha), \tag{2.12}$$

$$|x_\alpha - \bar x_\alpha| \leqq \alpha |F(x_\alpha) - \bar F(x_\alpha)|, \tag{2.13}$$

$$|F_\alpha(x) - \bar F_\alpha(x)| \leqq |F(x_\alpha) - \bar F(x_\alpha)|. \tag{2.14}$$

Proof. By (2.1) we have

$$\phi_\alpha(x) - \bar\phi_\alpha(x) = \inf\left\{\phi(y)\frac{1}{2\alpha}|x-y|^2;\, y \in H\right\} - \bar\phi(\bar x_\alpha) - \frac{\alpha}{2}|\bar F(\bar x_\alpha)|^2,$$

which clearly implies (2.12). Next we have

$$x_\alpha - \bar x_\alpha = (1 + \alpha\bar F)^{-1}(x + \alpha(\bar F(x_\alpha) - F(x_\alpha))) - (1 + \alpha \bar F)^{-1} x$$

and by (2.2)

$$|x_\alpha - \bar x_\alpha| \leqq \alpha |\bar F(x_\alpha) - F(x_\alpha)|.$$

Finally by the identity

$$F_\alpha(x)-\bar{F}_\alpha(x)=\alpha^{-1}(x_\alpha-\bar{x}_\alpha)$$

we find (2.14) as claimed.

LEMMA 6. *Assume that* $\phi\in C^1(H)\cap K$. *Then for every* $k=0,1,2,\cdots$ *there exists a continuous positive function* C_k *such that*

$$1+|x_\alpha|^k\leqq(1+|x|^{2k})(1+\alpha C_k(|F(0)|)) \tag{2.15}$$

for all $x\in H$ *and* $\alpha>0$.

Proof. It is a simple consequence of (2.5).

PROPOSITION 1. *Assume that* $\phi,\bar{\phi}\in C^2(H)\cap K$. *Then for all* $n=0,1,\cdots$ *we have*

$$|\phi_\alpha|_{0,n}\leqq(1+\alpha C_n(|F(0)|))|\phi|_{0,n}, \tag{2.16}$$

$$|\phi_\alpha|_{1,n}\leqq|\phi|_{1,n},\qquad |\phi_\alpha|_{2,n}\leqq|\phi|_{2,n}, \tag{2.17}$$

$$\|\phi_\alpha\|_{2,n}\leqq(1+\alpha C_n(|F(0)|))\|\phi\|_{2,n}, \tag{2.18}$$

$$|\phi_\alpha-\bar{\phi}_\alpha|_{0,n}\leqq(1+\alpha C_n(|F(0)|))|\phi-\bar{\phi}|_{0,n}, \tag{2.19}$$

$$|\phi_\alpha-\bar{\phi}_\alpha|_{1,n}\leqq(1+\alpha C_n(|\bar{F}(0)|))|\phi-\bar{\phi}|_{1,n}. \tag{2.20}$$

Moreover, if $|\phi|_z, |\bar{\phi}|_z\leqq\lambda$ *then there exists* $C(\lambda)>0$ *such that*

$$\begin{aligned}&|\phi_\alpha-\bar{\phi}_\alpha|_{0,n_0}+|\phi_\alpha-\bar{\phi}_\alpha|_{1,n_1}+|\phi_\alpha-\bar{\phi}_\alpha|_{2,n_1+n_3}\\&\quad\leqq(1+\alpha C(\lambda))\{|\phi-\bar{\phi}|_{0,n_0}+|\phi-\bar{\phi}|_{1,n_1}+|\phi-\bar{\phi}|_{2,n_1+n_3}\}.\end{aligned} \tag{2.21}$$

Proof. From (2.12) and (2.15) the below inequalities follow

$$\begin{aligned}\frac{\phi_\alpha(x)-\bar{\phi}_\alpha(x)}{1+|x|^{2n}}&\leqq\frac{\phi(x_\alpha)-\bar{\phi}(\bar{x}_\alpha)}{1+|\bar{x}_\alpha|^{2n}}\cdot\frac{1+|\bar{x}_\alpha|^{2n}}{1+|x|^{2n}}\\&\leqq|\phi-\bar{\phi}|_{0,n}(1+\alpha C_n(|\bar{F}(0)|)),\end{aligned}$$

which imply (2.20) and (2.16). Estimates (2.17) are immediate consequences of (2.6) and (2.7); the other inequalities are simple (although tedious) consequences of properties of ϕ_α.

PROPOSITION 2. *Assume that* $\phi\in C^2(H)\cap K$ *and that* n_0, n_1, n_2, *are nonnegative integers such that*

$$n_0\geqq 2n_1(1+n_2),\qquad n_0\geqq n_1\geqq n_2. \tag{2.22}$$

Then there exists a continuous increasing mapping $\gamma:\mathbb{R}^2\to\mathbb{R}_+$ *such that*

$$|R_{\phi,\alpha}|_{0,n_0}\leqq\alpha\gamma(|\phi|_{1,n_1},|\phi|_{2,n_2}). \tag{2.23}$$

Proof. We have

$$\begin{aligned}|F(x-\alpha tF(x_\alpha))-F(x)|&\leqq|\phi|_{2,n_2}\alpha|F(x)|\{1+[|x|\vee|x-\alpha tF(x_\alpha)|]^{2n_2}\}\\&\leqq\alpha|\phi|_{2,n_2}|\phi|_{1,n_1}(1+|x|^{2n_1})\{1+[|x|+|F(x)|]^{2n_2}\}\\&\leqq\alpha|\phi|_{2,n_2}|\phi|_{1,n_1}(1+|x|^{2n_1})\{1+[|x|+|\phi|_{1,n_1}(1+|x|^{2n_1})]\}^{2n_2}\\&\leqq\alpha\gamma_1(|\phi|_{1,n_1},|\phi|_{2,n_2})(1+|x|^{2n_1+4n_1n_2}).\end{aligned} \tag{2.24}$$

Moreover, one has

$$
\begin{aligned}
|F(x)-F(x_\alpha)| &\leqq |\phi|_{2,n_2}|x-x_\alpha|\{1+[|x|\vee|x_\alpha|]^{2n_2}\} \\
&\leqq \alpha|\phi|_{2,n_2}|\phi|_{1,n_1}(1+|x|^{2n_1})\{1+[|x|+|F(0)|]^{2n_2}\} \\
&\leqq \alpha\gamma_2(|\phi|_{1,n_1},|\phi|_{2,n_2})(1+|x|^{2n_1+2n_2}).
\end{aligned} \tag{2.25}
$$

From (2.24) and (2.25) the conclusion follows, by virtue of (2.10).

PROPOSITION 3. *Assume that* $\phi \in C^2(H)\cap K$ *and* $n_0 \geqq n_1 \geqq n_2 \geqq n_3 \geqq 0$. *Let* m_1 *be a positive integer such that*

$$m_1 \geqq (2n_1+2n_1n_3)\vee(n_1+2n_2). \tag{2.26}$$

Then there exists a continuous increasing mapping $\eta:\mathbb{R}^3\to\mathbb{R}_+$ *such that*

$$|R_{\phi,\alpha}|_{1,m_1} \leqq \alpha\eta(|\phi|_{1,n_1},|\phi|_{2,n_2},\|\phi\|_{2,n_3}). \tag{2.27}$$

Proof. We have

$$
\begin{aligned}
&|F'(x-\alpha tF(x_\alpha))-F'(x)| \\
&\quad\leqq \alpha\|\phi\|_{2,n_3}\cdot|\phi|_{1,n_1}(1+|x|^{2n_1})\{1+[|x|+|F(x)|]^{2n_3}\} \\
&\quad\leqq \alpha\|\phi\|_{2,n_3}|\phi|_{1,n_1}(1+|x|^{2n_1})\{1+[|x|+|\phi|_{1,n_1}(1+|x|^{2n_1})]^{2n_3}\} \\
&\quad\leqq \alpha\eta_1(|\phi|_{1,n_1},|\phi|_{2,n_2},\|\phi\|_{2,n_3})(1+|x|^{2n_1+4n_1n_3}).
\end{aligned} \tag{2.28}
$$

Recalling (2.11), (2.25) we get

$$
\begin{aligned}
|R'_{\phi,\alpha}(x)| \leqq \alpha\,\eta_3(&|\phi|_{1,n_1},|\phi|_{2,n_2},\|\phi\|_{2,n_3}) \\
&\cdot\{1+|x|^{4n_1+4n_1n_3}+|x|^{2n_1+4n_2}\},
\end{aligned}
$$

as claimed.

3. The linearized problem. We shall study here the linear Cauchy problem:

$$
\begin{aligned}
&\phi_t(t,x)-\langle Ax,\phi_x(t,x)\rangle-\frac{\varepsilon}{2}\operatorname{Tr}(S\phi_{xx}(t,x))=0, \\
&\phi(0,x)=\phi_0(x),
\end{aligned} \tag{3.1}
$$

where $\phi_0\in Z$, $\varepsilon>0$ and $A: D(A)\subset H\to H$ is the infinitesimal generator of a C_0-semigroup e^{At} of contractions on H, i.e.,

$$|e^{tA}|\leqq 1 \quad \text{for all } t\geqq 0. \tag{3.2}$$

By a solution to problem (3.1) we mean a function $\phi\in B([0,T];Z)$ which belongs to $C^1[0,T]$ for each $x\in D(A)$ and satisfies (3.1) for all $x\in D(A)$ and all $t\in[0,T]$. Consider the approximating problem

$$
\begin{aligned}
&\phi^n_t(t,x)-\langle A_nx,\phi^n_x(t,x)\rangle-\frac{\varepsilon}{2}\operatorname{Tr}(S\phi^n_{xx}(t,x))=0, \\
&\phi^n(0,x)=\phi_0(x),
\end{aligned} \tag{3.3}
$$

where $A_n=n^2(n-A)^{-1}=nA(n-A)^{-1}$ (the Yosida approximation of A).

It is well known that $\exp(tA_n)x\to\exp(tA)x$ for every x in H and $A_nx\to Ax$ for every x in $D(A)$ (see for instance [6]). Moreover, since A_n is bounded, it is easy to prove many properties in equations involving A_n (for example Itô's formula for $\phi^n(t,u)$) and afterward to pass to limit as n goes to infinity.

LEMMA 7. *For every $\phi_0 \in Z$, problem* (3.3) *has a unique solution $\phi^n \in C^1([0,T]\times H)\cap B([0,T];Z)$ given by the formula*

$$\phi^n(t,x)=E\phi_0\left(e^{tA_n}x+\sqrt{\varepsilon}\int_0^t e^{sA_n}\,dW_{T-s}\right). \tag{3.4}$$

Proof. For existence we first remark that $\zeta_t = W_T - W_{T-t}$ is a Brownian motion. We have:

$$\phi_x^n(t,x)=e^{tA_n^*}E\phi_{0,x}\left(e^{tA_n}x-\sqrt{\varepsilon}\int_0^t e^{sA_n}\,d\zeta(s)\right), \tag{3.5}$$

$$\phi_{xx}^n(t,x)=e^{tA_n^*}E\phi_{0,xx}\left(e^{tA_n}x-\sqrt{\varepsilon}\int_0^t e^{sA_n}\,d\zeta(s)\right)e^{tA_n}. \tag{3.6}$$

We notice that if $\phi\in C^2(H)$ then $\phi''(x)\in\mathscr{L}(H,\mathscr{L}(H,\mathbb{R}))$ and so we may write $(\phi''(x)\cdot y)\cdot z=\phi''(x)\cdot(y,z)=\langle\phi''(x)\cdot y,z\rangle$. In this sense we have $(\phi P)''(x)=P^*\phi''(x)P$ for all $P\in\mathscr{L}(H)$. To prove that Φ is differentiable with respect to t we notice that for each $h>0$ one has

$$\begin{aligned}
&\phi_0\left(e^{(t+h)A_n}x-\sqrt{\varepsilon}\int_0^{t+h}e^{sA_n}\,d\zeta_s\right)-\phi_0\left(e^{tA_n}x-\sqrt{\varepsilon}\int_0^t e^{sA_n}\,d\zeta_s\right)\\
&\quad=\left\langle e^{(t+h)A_n}x-e^{tA_n}x-\sqrt{\varepsilon}\int_t^{t+h}e^{sA_n}\,d\zeta_s,\ \phi_{0x}\left(e^{tA_n}x-\sqrt{\varepsilon}\int_0^t e^{sA_n}\,d\zeta_s\right)\right\rangle\\
&\quad\quad+\frac{1}{2}\phi_{0xx}\left(e^{tA_n}x-\sqrt{\varepsilon}\int_0^t e^{sA_n}\,d\zeta_s\right)\\
&\quad\quad\quad\cdot\left[e^{(t+h)A_n}x-e^{tA_n}x-\sqrt{\varepsilon}\int_t^{t+h}e^{sA_n}\,d\zeta_s\right]^2\\
&\quad\quad+\int_0^1\left\{(1-a)\phi_{0xx}\left((1-a)\left[e^{tA_n}x-\sqrt{\varepsilon}\int_0^t e^{sA_n}\,d\zeta_s\right]\right.\right.\\
&\quad\quad\quad\left.+a\left[e^{(t+h)A_n}x-\sqrt{\varepsilon}\int_0^{t+h}e^{sA_n}\,d\zeta_s\right]\right)\\
&\quad\quad\quad\left.\cdot\left[e^{(t+h)A_n}x-e^{tA_n}x-\sqrt{\varepsilon}\int_t^{t+h}e^{sA_n}\,d\zeta_s\right]^2\right\}da.
\end{aligned} \tag{3.7}$$

To calculate $\phi_t(t,x)$ we remark that

$$\lim_{h\to0}\frac{1}{h}\left(e^{(t+h)A_n}x-e^{tA_n}x\right)=A_n\,e^{tA_n}x \tag{3.8}$$

and

$$E\left\langle\int_t^{t+h}e^{sA_n}\,d\zeta_s,\ \phi_{0x}\left(e^{tA_n}x-\sqrt{\varepsilon}\int_0^t e^{sA_n}\,d\zeta_s\right)\right\rangle=0, \tag{3.9}$$

since ζ_s have independent increments. Moreover, one has

$$\lim_{h\to0}\frac{1}{h}\left|e^{(t+h)A_n}x-e^{tA_n}x\right|^2=0, \tag{3.10}$$

$$
\begin{aligned}
&\lim_{h\to 0} E\left\{\frac{1}{2h}\phi_{0xx}\left(e^{tA_n}x-\sqrt{\varepsilon}\int_0^t e^{sA_n}\,d\zeta_s\right)\left[\int_t^{t+h} e^{sA_n}\,d\zeta_s\right]^2\right\} \\
(3.11)\qquad &=\lim_{h\to 0}\left\{\frac{\varepsilon}{2}\int_t^{t+h}\operatorname{Tr}\left(S\,e^{sA_n^*}\phi_{0xx}\left(e^{-tA_n}x-\sqrt{\varepsilon}\int_0^t e^{sA_n}\,d\zeta_s\right)e^{sA_n}\,ds\right)\right\} \\
&=\frac{\varepsilon}{2}\operatorname{Tr}\left[S\,e^{tA_n}\phi_{0xx}\left(e^{tA_n}x-\sqrt{\varepsilon}\int_0^t e^{sA_n}\,d\zeta_s\right)e^{tA_n}\right].
\end{aligned}
$$

Observe also that the last integral in (3.7) goes to 0 for $h\to 0$ by the Lebesgue dominated convergence theorem.

It follows that

$$
(3.12)\qquad D_t^+\phi^n(t,x)=\langle A_nx,\phi_x^n(t,x)\rangle+\frac{\varepsilon}{2}\operatorname{Tr}\left(S\phi_{xx}^n(t,x)\right),
$$

where D_t^+ means the right derivative. Since the right-hand side of (3.12) is continuous we conclude by a standard result, that (3.3) holds.

Uniqueness. Let ϕ be a solution to problem (3.3). We set $\psi(t,x)=\phi(T-t,x)$. Then ψ is a solution to the backward problem

$$
(3.13)\qquad
\begin{aligned}
&\psi_t(t,x)+\langle A_nx,\psi_x(t,x)\rangle+\frac{\varepsilon}{2}\operatorname{Tr}\left(S\psi_{xx}(t,x)\right)=0,\\
&\psi(T,x)=\phi_0(x).
\end{aligned}
$$

Let $u=u(s,t,x)$ be the solution to the stochastic differential equation

$$
(3.14)\qquad du=A_nu\,ds+\sqrt{\varepsilon}\,dW_s,\qquad u(t)=x,
$$

i.e.,

$$
(3.15)\qquad u(s,t,x)=e^{(s-t)A_n}x+\sqrt{\varepsilon}\int_t^s e^{(s-\sigma)A_n}\,dW_\sigma=u(s).
$$

By the Itô formula

$$
d\psi(s,u)=\left[\psi_s(s,u)+\frac{\varepsilon}{2}\operatorname{Tr}\left(S\psi_{xx}(s,u)\right)\right]ds+\psi_x(s,u)\,du,
$$

from which, by integrating in $[t,T]$ and taking the expectation we obtain

$$
\psi(t,x)=E\psi(t,u(t,t,x))=E\psi(T,u(T,t,x))=E\phi_0(u(T,t,x)),
$$

and therefore $\psi=\phi^n$ as claimed. The following corollary follows via a standard variation of constants formula.

COROLLARY 1. *Under the assumptions of Lemma 7, the problem*

$$
(3.16)\qquad
\begin{aligned}
&\phi_t^n(t,x)-\langle A_nx,\phi_x^n(t,x)\rangle-\frac{\varepsilon}{2}\operatorname{Tr}\left(S\phi_{xx}^n(t,x)\right)=\zeta(t,x),\\
&\phi^n(0,x)=\phi_0(x)
\end{aligned}
$$

has for every $\zeta\in B([0,T];Z)$ a unique solution ϕ^n given by

$$
(3.17)\qquad
\begin{aligned}
\phi^n(t,x)=&\,E\phi_0\left(e^{tA_n}x+\sqrt{\varepsilon}\int_0^t e^{sA_n}\,dW_{T-s}\right)\\
&+E\int_0^t\zeta\left(s,e^{(t-s)A_n}x+\sqrt{\varepsilon}\int_0^{t-s}e^{\sigma A_n}\,dW_{T-\sigma}\right)ds.
\end{aligned}
$$

PROPOSITION 4. *For every* $\phi_0 \in Z$ *problem* (3.1) *has a unique solution* $\phi \in B([0, T]; Z)$ *given by*

$$\phi(t, x) = E\phi_0\left(e^{tA}x + \sqrt{\varepsilon}\int_0^t e^{sA}\, dW_{T-s}\right). \tag{3.18}$$

Moreover, for each nonnegative integer m there exists $\omega_m > 0$ *such that*

$$|\phi(t, \cdot)|_{i,m} \leqq e^{\varepsilon\omega_m t}|\phi_0|_{i,m}, \qquad i = 0, 1, \cdots, \tag{3.19}$$

$$\|\phi(t, \cdot)\|_{i,m} \leqq e^{\varepsilon\omega_{m+1} t}\|\phi_0\|_{i,m}, \qquad i = 0, 1, \cdots. \tag{3.20}$$

Proof. Let ϕ^n given by (3.4). Inasmuch as for each $x \in H$, $e^{tA_n}x \to e^{tA}x$ uniformly on compacts we see that

$$\begin{aligned}&\phi^n(t, x) \to \phi(t, x),\\ &\mathrm{Tr}\,(S\phi^n_{xx}(t, x) \to \mathrm{Tr}\,(S\phi_{xx}(t, x),\end{aligned} \qquad \text{uniformly in } [0, T],\ T > 0.$$

Moreover, since for all $x \in D(A)\langle A_n x, \phi^n_x(t, x)\rangle \to \langle Ax, \phi_x(t, x)\rangle$ uniformly in $[0, T]$ we infer that $\phi^n_t(t, x) \to \phi_t(t, x)$ uniformly on $[0, T]$ and therefore ϕ satisfies (3.1) for all $t \in [0, T]$ and $x \in D(A)$ (we note that in this case ϕ is differentiable as a function of t for each $x \in D(A)$).

For uniqueness let $\eta \in B$ $([0, T]; Z)$ be another solution to problem (3.1). For $x \in H$ we set $x_n = (n - A)^{-1}nx$ and notice the equation

$$\begin{aligned}\eta_t(t, x_n) - \langle A_n x_n, \eta_x(t, x_n)\rangle - \frac{\varepsilon}{2}\mathrm{Tr}\,(S\eta_{xx}(t, x_n))\\ = \langle (A - A_n)x_n, \eta_x(t, x_n)\rangle.\end{aligned}$$

Then by Corollary 1,

$$\begin{aligned}\eta(t, x_n) &= E\phi_0\left(e^{tA_n}x_n + \sqrt{\varepsilon}\int_0^t e^{sA_n}\, dW_{T-s}\right)\\ &\quad + E\int_0^t \left\langle (A - A_n)x_n, \eta_x\left(e^{(t-s)A_n}x_n + \sqrt{\varepsilon}\int_0^{t-s} e^{sA_n}\, dW_{T-\sigma}\right) ds\right\rangle.\end{aligned}$$

Letting n tend to $+\infty$ we get $\eta = \phi$ as claimed. To prove (3.19) we shall restrict ourselves to the case $i = 0$ the other cases being similar. Since $M_t = \int_0^t e^{sA}\, dW_{T-s}$ is a martingale it follows (see for instance [12]) that for each $m \in N$ there exists $\gamma_m > 0$ such that

$$E\left|\int_0^t e^{sA}\, dW_{T-s}\right|^{2m} \leqq \gamma_m t^m. \tag{3.21}$$

We have

$$|\phi(t, x)| \leqq |\phi_0|_{0,m} E\left(1 + \left|e^{tA}x + \sqrt{\varepsilon}\int_0^t e^{sA}\, dW_{T-s}\right|^{2m}\right).$$

It follows that there exist real bounded functions a_e, $e = 2, 3, \cdots, 2m$ such that

$$|\phi(t, x)| \leqq |\phi_0|_{0,m} E\left\{1 + \sum_{e=2}^{2m} a_e(|x|)\sqrt{\varepsilon}\left|\int_0^t e^{-sA}\, dW_{t-s}\right|^e\right\}(1 + |x|^{2m}).$$

By virtue of (3.21) there exists real bounded functions b_e such that

$$|\phi(t,x)| \leqq |\phi_0|_{0,m}(1+|x|^{2m})\left(1+\sum_{e=2}^{2m} b_e(\varepsilon t)^{e/2}\right),$$

so (3.19) is proved. The proof of (3.20) is completely similar, so it will be omitted. We shall consider now the nonhomogeneous Cauchy problem,

$$(3.22)\qquad \begin{aligned} &\phi_t(t,x) - \langle Ax, \phi_x(t,x)\rangle - \frac{\varepsilon}{2}\operatorname{Tr}(S\phi_{xx}(t,x)) = g(t,x),\\ &\phi(0,x) = \phi_0(x), \end{aligned}$$

where $\phi_0 \in Z$ and $g \in B([0,T];Z)$.

By a solution to (3.22) we mean a function $\phi \in B([0,T];Z)$ which belongs to $W^{1,\infty}(0,T)$ as a function cf t (for each $x \in D(A)$) and satisfies (3.22) for all $x \in D(A)$ and a.e. $t \in]0,T[$.

For later use, we notice the following existence result.

PROPOSITION 5. *For every $\phi_0 \in Z$ and $g \in B([0,T];Z)$ problem (3.22) has a unique solution $\phi \in B([0,T];Z)$ given by the formula*

$$(3.23)\qquad \begin{aligned} \phi(t,x) = {}& E\phi_0\left(e^{tA}x + \sqrt{\varepsilon}\int_0^t e^{sA}\,dW_{T-s}\right)\\ &+E\int_0^t g\left(s, e^{(t-s)A}x + \sqrt{\varepsilon}\int_0^{t-s} e^{\sigma A}\,dW_{T-\sigma}\right)ds. \end{aligned}$$

Proof. Existence follows from Proposition 4. To prove uniqueness, arguing as in the proof of Lemma 7 it suffices to assume that A is bounded.

Let $\phi \in B([0,T];Z)$ be a solution to (3.22) where $g=0$ and $\phi_0=0$ and let $\psi(t,x) = \phi(T-t,x)$. Finally set $\psi^n(t,x) = \psi(t,P^n x)$, where

$$(3.24)\qquad P^n x = \sum_{i=1}^n \langle x, e_i\rangle e_i.$$

Clearly $\psi^n \in C^1([0,T]\times H;\mathbb{R}) \cap C([0,T];Z)$, so we may apply the Itô formula to $\psi^n(s,u)$ (see for instance [11]) and get

$$(3.25)\qquad d\psi^n(s,u) = \left(\psi^n_s(s,u) + \frac{\varepsilon}{2}\operatorname{Tr}(S\psi^n_{xx}(s,u))\right)ds + \psi^n_u(s,u)\,dW_s.$$

Integrating and taking the expectation we obtain

$$(3.26)\quad 0 = E\psi^n(T,u(T)) = \psi^n(t,x) + E\int_t^T \left[\psi^n_s(s,u(s)) + \frac{\varepsilon}{2}\operatorname{Tr}(S\psi^n_{xx}(s,u(s)))\right]ds,$$

where u is the solution to (3.1) (with $A_n = A$).

As n goes to infinity we get

$$\psi(t,x) = -E\int_t^T \left(\psi_s(s,u(s)) + \frac{\varepsilon}{2}\operatorname{Tr}(S\psi_{xx}(s,u(s)))\right)ds = 0,$$

as claimed.

4. The main results. Consider the Cauchy problem

$$\phi_t(t,x)+\frac{1}{2}|\phi_x(t,x)|^2-\langle Ax,\phi_x(t,x)\rangle-\frac{\varepsilon}{2}\operatorname{Tr}(S\phi_{xx}(t,x))=g(t,x),$$
(4.1)
$$\phi(0,x)=\phi_0(x),\qquad \varepsilon>0,$$

under the following assumptions:

(4.2)
a) A is the infinitesimal generator of a C_0 semigroup of contractions;
b) $\phi_0\in Z\cap K$; $g\in B([0,T];Z)\cap\mathcal{K}$;
c) $n_0\geqq 2n_1(1+n_2)$;

where $\mathcal{K}=\{\phi\in B([0,T];C(H));\ \phi(t)\in K,\ \forall t\in[0,T]\}$.

We shall consider the approximating problem

$$\phi_t^\alpha(t,x)+\frac{1}{\alpha}(\phi^\alpha(t,x)-\phi_\alpha^\alpha(t,x))-\langle Ax,\phi_x^\alpha(t,x)\rangle$$
(4.3)
$$-\frac{1}{2}\varepsilon\operatorname{Tr}(S\phi_{xx}^\alpha(t,x))=g(t,x),$$
$$\phi^\alpha(0,x)=\phi_0(x),$$

where $\alpha\in]0,1]$ and ϕ_α^α is defined by (2.1), i.e.,

$$\phi_\alpha^\alpha(t,x)=\inf\left\{\phi^\alpha(t,y)+\frac{1}{2\alpha}|x-y|^2;\ y\in H\right\}.$$

A weak form of problem (4.3) is given by the following integral equation:

$$\phi^\alpha(t,x)=\exp(-\alpha^{-1}t)E\phi_0\left(e^{tA}x+\sqrt{\varepsilon}\int_0^t e^{\sigma A}\,dW_{T-s}\right)$$
(4.4)
$$+E\int_0^t\exp(-\alpha^{-1}(t-s))(\alpha^{-1}\phi_\alpha^\alpha+g)$$
$$\cdot\left(s,e^{(t-s)A}+\sqrt{\varepsilon}\int_0^{t-s}e^{\sigma A}\,dW_{T-s}\right)ds.$$

PROPOSITION 6. *Under assumption* (4.2) *for every* $\alpha>0$, (4.4) *has a unique solution* $\phi^\alpha\in B([0,T];Z)$. *Moreover* ϕ^α *satisfies* (4.3) *for all* $(t,x)\in[0,T]\times D(A)$.

Finally there exist $\tilde\omega_i\geqq 0$, $i=0,1,2,3$ *such that*

(4.5)
$$|\phi^\alpha(t,\cdot)|_{i,n_i}\leqq\exp(\varepsilon\tilde\omega_i t)|\phi_0|_{i,n_i}+\int_0^t\exp(\varepsilon\tilde\omega_i(t-s))|g(s,\cdot)|_{i,n_i}\,ds,\qquad i=0,1,2,$$

(4.6)
$$\|\phi^\alpha(t,\cdot)\|_{2,n_3}\leqq\exp(\varepsilon\tilde\omega_3 t)\|\phi_0\|_{2,n_3}+\int_0^t\exp(\varepsilon\tilde\omega_3(t-s))\|g(s,\cdot)\|_{2,n_3}\,ds.$$

Proof. Set

(4.7)
$$\phi^0(t,x)=e^{-t/\alpha}E\phi_0\left(e^{tA}x+\sqrt{\varepsilon}\int_0^t e^{sA}\,dW_{T-s}\right)+E\int_0^t e^{-(t-s)/\alpha}g\left(s,e^{(t-s)A}x+\sqrt{\varepsilon}\int_0^{t-s}e^{\sigma A}\,dW_{T-\sigma}\right)ds,$$

$$\phi^{n+1}(t,x)=\phi^0(t,x) \tag{4.8}$$
$$+\frac{1}{\alpha}E\int_0^t e^{-(t-s)/\alpha}\phi_\alpha^n\left(s,e^{(t-s)A}x+\int_0^{t-s}e^{\sigma A}\,dW_{T-\sigma}\right)ds.$$

Using inequalities (2.16)–(2.20) it is not difficult to find $i=0,1,2,3$, such that

$$|\phi^n(t,\cdot)|_{i,n_i}\leqq \exp(\varepsilon\tilde{\omega}_i t)|\phi_0|_{i,n_i} \tag{4.9}$$
$$+\int_0^t \exp(\varepsilon\tilde{\omega}_i(t-s))|g(s,\cdot)|_{i,n_i}\,ds,\qquad i=0,1,2,$$

$$\|\phi^n(t,\cdot)\|_{2,n_3}\leqq \exp(\varepsilon\tilde{\omega}_3 t)\|\phi_0\|_{2,n_3} \tag{4.10}$$
$$+\int_0^t \exp(\varepsilon\tilde{\omega}_3(t-s))\|g(s,\cdot)\|_{2,n_3}\,ds.$$

By (4.9) and (4.10) we see that the set

$$\Gamma=\{\phi^n(t,\cdot);\, n\in N,\, t\in[0,T]\}$$

is bounded in Z.

Set now

$$|\phi|_\Sigma=|\phi|_{0,n_0}+|\phi|_{1,n_1}+|\phi|_{2,n_1+n_3}. \tag{4.11}$$

By (2.21) it follows (Γ being bounded in Z) that $\{\phi^n(t,\cdot)\}$ is a Cauchy sequence, with respect to the norm $|\cdot|_\Sigma$, uniformly in t. This implies that $\{\phi^n\}$ converges in $B([0,T];C^2(H))$ to a function ϕ^α. By (4.9) and (4.10) it follows that $\phi^\alpha\in B([0,T];Z)$ and also that (4.5) and (4.6) hold.

Finally, using Proposition 5 it is easy to check that ϕ^α is the unique solution to (4.3) for all $(t,x)\in[0,T]\times D(A)$.

THEOREM 1. *Under assumptions* (4.2) *the Cauchy problem* (4.1) *has a unique solution* $\phi\in B([0,T];X)\cap B([0,T];C^1(H))$ *such that* $\phi(\cdot,x)\in W^{1,\infty}(0,T)$ *for all* $x\in D(A)$. *Moreover, the map* $(\phi_0,g)\to\phi$ *is Lipschitz from* $XXB([0,T];Z)$ *to* $B([0,T];X)$. *Finally the set* $\{\phi(t,\cdot),t\in[0,T]\}$ *is bounded in* Z.

Proof. We shall obtain the solution ϕ to (4.1) as the limit for $\alpha\to0$ of ϕ^α. The first step is the proof of the convergence of ϕ^α. Let $\alpha,\beta>0$. By (2.10) it follows that

$$\phi_t^\beta+\frac{1}{\alpha}(\phi^\beta-\phi_\alpha^\beta)-\langle Ax,\phi_x^\beta\rangle-\frac{\varepsilon}{2}\operatorname{Tr}(S\phi_{xx}^\beta)=g+R_{\phi_\beta,\alpha}-R_{\phi_\beta,\beta},$$
$$\phi^\beta(0,x)=\phi_0(x),$$

and by Proposition 5, we get

$$\phi^\beta(t,x)=e^{-t/\alpha}E\phi_0\left(e^{tA}x+\sqrt{\varepsilon}\int_0^t e^{sA}\,dW_{T-s}\right)$$
$$+E\int_0^t e^{-(t-s)/\alpha}\left(\frac{1}{\alpha}\phi_\alpha^\beta+R_{\phi_\beta,\alpha}-R_{\phi_\beta,\beta}\right) \tag{4.12}$$
$$\cdot\left(s,e^{(t-s)A}x+\sqrt{\varepsilon}\int_0^{t-s}e^{\sigma A}\,dW_{T-\sigma}\right)ds.$$

By (2.20), (2.23) and (3.19) we see that

$$
\begin{aligned}
&|\phi^{\alpha}(t,\cdot)-\phi^{\beta}(t,\cdot)|_{0,n_0} \\
&\leqq \int_0^t \exp(-(t-s)(\alpha^{-1}+\omega_0)) \\
(4.13)\qquad &\cdot [\alpha^{-1}(1+\alpha C_{n_0}(\phi^{\beta}(s,\cdot)))|\phi^{\alpha}(s,\cdot)-\phi^{\beta}(s,\cdot)|_{0,n_0}, \\
&\qquad + (\alpha+\beta)\gamma(|\phi^{\beta}|_{1,n_1}, |\phi^{\beta}|_{2,n_2})]\, ds.
\end{aligned}
$$

By (4.5), (4.6) and the Gronwall lemma it follows that there exists $C>0$ such that

$$
(4.14)\qquad |\phi^{\alpha}(t,\cdot)-\phi^{\beta}(t,\cdot)|_{0,n_0} \leqq C(\alpha+\beta),
$$

and all $\{\phi^{\alpha}(t,\cdot)\}$ belong to a closed bounded subset of Z. Hence there exists $\phi \in B([0,T];X)$ such that $\phi^{\alpha}\to\phi$ in $B([0,T];X)$.

We shall study the convergence of ϕ_x^{α}. Choose $m_1 \geqq (2n_1+2n_1n_3)\vee(n_1+2n_2)$. Recalling (4.12) and (3.19) it follows by Proposition 3 that

$$
\begin{aligned}
|\phi^{\alpha}(t,\cdot)-\phi^{\beta}(t,\cdot)|_{1,m_1} &\leqq \int_0^t \exp(-(t-s)(\alpha^{-1}+\omega_0)) \\
(4.15)\qquad &\cdot [\alpha^{-1}(1+\alpha C(\phi^{\beta}(s,\cdot)))|\phi^{\alpha}(s,\cdot)-\phi^{\beta}(s,\cdot)|_{1,m_1} \\
&+ (\alpha+\beta)\eta(|\phi^{\beta}(s,\cdot)|_{1,n_1}, |\phi(s,\cdot)|_{2,n_2}, |\phi^{\beta}(s,\cdot)|_{2,n_3})]\, ds.
\end{aligned}
$$

Using once again the Gronwall lemma we get

$$
(4.16)\qquad |\phi^{\alpha}(t,\cdot)-\phi^{\beta}(t,\cdot)|_{1,m_1} \leqq C_1(\alpha+\beta),
$$

and therefore $\phi \in B([0,T]; C^1(H))$. By Lemma 2 it follows that for every $t\in[0,T]$

$$
(4.17)\qquad \mathrm{Tr}\,(S\phi_{xx}^{\alpha}(t,x)) \to \mathrm{Tr}\,(S\phi_{xx}^{\alpha}(t,x))
$$

uniformly on every B_R. Recalling now that

$$
\phi_t^{\alpha} + \frac{1}{2}|\phi_x^{\alpha}|^2 - \langle Ax, \phi_x^{\alpha}\rangle - \frac{\varepsilon}{2}\mathrm{Tr}\,(S\phi_{xx}^{\alpha}) = g - R_{\phi^{\alpha},\alpha},
$$

and keeping in mind estimate (2.21) we may infer that for every $x\in D(A)\phi_t^{\alpha}(t,x)$ is bounded in $L^{\infty}(0,T)$ and as $\alpha\to 0$

$$
\phi_t^{\alpha}(t,x) \to -\frac{1}{2}|\phi_x(t,x)|^2 + \langle Ax, \phi_x(t,x)\rangle + \frac{\varepsilon}{2}\mathrm{Tr}\,(S\phi_{xx}(t,x)) + g(t,x)
$$

uniformly on B_R, for any $t\in[0,T]$, where R is arbitrary. We have therefore proved that ϕ satisfies the conditions of Theorem 1. Let ϕ^i, $i=1,2$ be two solutions to problem (4.1) corresponding to (ϕ_0^i, g^i). We have

$$
\begin{aligned}
&\phi_t^i + \alpha^{-1}(\phi^i - \phi_{\alpha}^i) - \langle Ax, \phi_x^i\rangle - \frac{\varepsilon}{2}\mathrm{Tr}\,(S\phi_{xx}^i) = g^i + R_{\phi^i,\alpha}, \\
&\phi^i(t,x) = \phi_0^i.
\end{aligned}
$$

This yields, recalling Proposition 5,

$$
\begin{aligned}
\phi^i(t,x) &= \exp(-\alpha^{-1}t)E\phi_0^i\left(e^{tA}x + \sqrt{\varepsilon}\int_0^t e^{sA}\, dW_{T-s}\right) \\
&+ E\int_0^t \exp(-\alpha^{-1}(t-s))(\alpha^{-1}\phi_{\alpha}^i + g^i)\left(s, e^{(t-s)A} + \sqrt{\varepsilon}\int_0^{t-s} e^{\sigma A}\, dW_{T-\sigma}\right) ds
\end{aligned}
$$

and by (2.20), (2.23), it follows via Gronwall's lemma that

(4.18)
$$|\phi^1(t,\cdot)-\phi^2(t,\cdot)|_{0,n_0}\leqq C\left(|\phi_0^1-\phi^2|_{0,n_0}+\int_0^t |g^1(s,\cdot)-g^2(s,\cdot)|_{0,n_0}\,ds\right),\qquad 0\leqq t\in T$$

where C is independent of ε. In particular, we may conclude that the solution ϕ to (4.1) is unique and the proof of Theorem 1 is complete.

Remark. If Z happens to be a dense subset of X then for any $\phi_0\in X$ and $g\in B([0,T];X)$ problem (4.1) has a unique weak solution $\phi\in B([0,T];X)$.

Now we shall study the convergence of $\{\phi^\varepsilon\}$ for $\varepsilon\to 0$.

THEOREM 2. *Under the assumptions of Theorem* 1, $\phi^\varepsilon\to\phi$ *in* $B([0,T];X)\cap B([0,T];C^1(H))$, *where* $\phi(\cdot,x)\in W^{1,\infty}(0,T)$ *for all* $x\in D(A)$ *is the solution to the Hamilton–Jacobi equation*:

(4.19)
$$\phi_t(t,x)+\tfrac{1}{2}|\phi_x(t,x)|^2-\langle Ax,\phi_x(t,x)\rangle=g(t,x)\quad \text{a.e. } t\in[0,T],\quad x\in D(A)$$
$$\phi(0,x)=\phi_0(x).$$

Proof. First we observe that in estimates (4.5), (4.6) the constants can be taken independent of ε. For $\varepsilon,\lambda>0$ we have

$$\phi_t^\lambda+\frac{1}{2}|\phi_x^\lambda|^2=\langle Ax,\phi_x^\lambda\rangle-\frac{\varepsilon}{2}\operatorname{Tr}(S\phi_{xx}^\lambda)=g+\frac{\lambda-\varepsilon}{2}\operatorname{Tr}(S\phi_{xx}^\lambda).$$

Then by (4.5) and (4.18) we see that $\{\phi^\varepsilon\}$ is a Cauchy sequence in $B([0,T];X)$ and therefore for $\varepsilon\to0$, $\phi^\varepsilon\to\phi$ in $B([0,T];X)$. Moreover, arguing as in the proof of inequality (4.16) we show that $\phi^\varepsilon\to\phi$ in $B([0,T];C^1(H))$. Again by estimates (4.5) and (4.6) it follows that $\{\phi^\varepsilon(t,\cdot)\}$ remain in a bounded subset of Z and for $\varepsilon\to0$, $t\in[0,T]$ and $x\in D(A)$.

$$\phi_t^\varepsilon(t,x)\to-\tfrac{1}{2}|\phi_x(t,x)|^2+\langle Ax,\phi_x^\varepsilon(t,x)\rangle+g(t,x),$$

thereby completing the proof of Theorem 2.

We shall give now another approximation result which will be useful in the next section. Again we denote by $A_n=n^2(n-A)^{-1}-n$ the Yosida approximation of A.

PROPOSITION 7. *Assume that hypotheses of Theorem* 1 *hold for any* $n\in\mathbb{N}$ *let* ϕ^n *be the solution to the problem*

(4.20)
$$\phi_t^n+\frac{1}{2}|\phi_x^n|^2+\langle Ax,\phi_x^n\rangle-\frac{\varepsilon}{2}\operatorname{Tr}(S\phi_{xx}^n)=g,$$
$$\phi^n(0,x)=\phi_0(x),$$

and let ϕ *be the solution of problem* (4.1). *Then* $\phi^n\to\phi$ *in* $B([0,T];X)\cap B([0,T];C^1(H))$.

Proof. Let $\phi^{n,\alpha}$ be the solution to

(4.21)
$$\phi_t^{n,\alpha}+\frac{1}{\alpha}(\phi^{n,\alpha}-\phi_\alpha^{n,\alpha})+\langle A_nx,\phi_x^{n,\alpha}\rangle-\frac{\varepsilon}{2}\operatorname{Tr}(S\phi_{xx}^{n,\alpha})=g,$$
$$\phi^{n,\alpha}(0,x)=\phi_0(x).$$

Proceeding as in the proof of Theorem 1 we see that

(4.22)
$$\lim_{n\to\infty}\phi^{n,\alpha}=\phi^\alpha\quad \text{in } B([0,T];X)\cap B([0,T];C^1(H))\quad \text{for all } \alpha>0.$$

Hence for $n \to \infty$

$$|\phi-\phi^n|_X \leqq |\phi-\phi^\alpha|_X+|\phi^\alpha-\phi^{n,\alpha}|_X+|\phi^{n,\alpha}-\phi^n|_X \to 0.$$

5. Synthesis of optimal control. Consider the Cauchy problem:

$$\begin{gathered}\psi_t(t,x)-\tfrac{1}{2}|\psi_x(t,x)|^2+\langle Ax,\psi_x(t,x)\rangle \\ +\frac{\varepsilon}{2}\operatorname{Tr}(S\psi_{xx}(t,x))+V(t,x)=0, \\ \psi(T,x)=\phi_0(x),\end{gathered} \tag{5.1}$$

where $V(t,x)=g(T-t,x)$. Remark that (5.1) is obtained from (4.1) by setting $\psi(t,x)=\phi(T-t,x)$. Throughout this section we shall assume that assumptions (4.2) are satisfied. Then by Theorem 1, problem (5.1) has a unique solution $\psi\in B([0,T];C^1(H))$ such that $\psi(\cdot,x)\in W^{1,\infty}(0,T)$ for every $x\in D(A)$. Notice also that by virtue of Proposition 7 the solution to the problem

$$\begin{gathered}\psi_t^n(t,x)-\tfrac{1}{2}|\psi_x^n(t,x)|^2+\langle A_nx,\psi_x^n(t,x)\rangle \\ +\frac{\varepsilon}{2}\operatorname{Tr}(S\psi_{xx}^n(t,x))+V(t,x)=0, \\ \psi^n(T,x)=\phi_0(x)\end{gathered} \tag{5.2}$$

is convergent to ψ in the following sense:

$$\psi^n\to\psi \quad \text{in } B([0,T];X), \tag{5.3}$$

$$\psi_x^n\to\psi_x \quad \text{uniformly in } [0,T]\times B_R. \tag{5.4}$$

We shall use these facts to prove the following lemma.

LEMMA 8. *Let $u\in M_w^2(0,T;H)$ and let ζ be the mild solution to the stochastic equation*

$$d\zeta=(A\zeta+u)\,ds+\sqrt{\varepsilon}\,dW_s,\qquad \zeta(t)=x,\quad t\leqq s\leqq T. \tag{5.5}$$

If ψ is the solution to (5.1) *then the equality*

$$\begin{aligned}\psi(t,x)&+\frac{1}{2}E\int_t^T|\psi_x(s,\zeta(s))+u(s)|^2\,ds \\ &=E\int_t^T\left(V(s,\zeta(s))+\frac{1}{2}|u(s)|^2\right)ds+\phi_0(\zeta(T))\end{aligned} \tag{5.6}$$

holds for all $(t,x)\in[0,T]\times H$.

Proof. Let ζ_n be the solution to

$$d\zeta_n=(A_n\zeta_n+u)\,ds+\sqrt{\varepsilon}\,dW_s,\qquad \zeta_n(t)=x, \tag{5.7}$$

and let $\psi_n:[0,T]\times H\to\mathbb{R}$ be the function defined by

$$\psi_n(t,x)=\int_0^T\psi^n(s,x)\rho_n(t-s)\,ds,$$

where $\{\rho_n\}$ is a family of C^∞-real valued functions such that

$$\operatorname{supp}(\rho_n)\subset\left]-\frac{1}{n},\frac{1}{n}\right[,\qquad \rho_n(t)=\rho_n(-t),\quad \rho_n\geqq 0$$

and $\int_{-\infty}^{+\infty} \rho_n(t)\,dt = 1$. Clearly $\psi_n \in B([0, T]; Z)$ and $(\psi_n)_t \in B([0, T]; X)$. By (5.3) and (5.4) it follows that

$$\psi_n \to \psi \quad \text{in } B([0, T]; X). \tag{5.8}$$

By standard results on infinite dimensional stochastic equations (see for instance [7]) we know that $\zeta_n(t) \to \zeta(t)$ uniformly on $[0, T]$ with probability 1 and therefore

$$E((\psi_n)_x(t, y_n(t))) \to E\psi_x(t, y(t)) \quad \text{uniformly on } [0, T]. \tag{5.9}$$

Next by the Itô formula,

$$d\psi_n(s, \zeta_n) = (\psi_n)_s(s, \zeta_n)\,ds + \langle A_n\zeta_n\,ds + u\,ds + \sqrt{\varepsilon}\,dW_s, (\psi_n)_x(s, \zeta_n)\rangle$$
$$+\frac{\varepsilon}{2}\operatorname{Tr}(S\psi_{n,xx}(s, \zeta_n))\,ds.$$

Then, integrating on $[t, T]$ and taking the expectation gives

$$\psi_n(t, x) + \frac{1}{2}E\int_t^T |\psi_{n,x}(s, \zeta_n) + u(s)|^2\,ds$$
$$= E\int_t^T \left(V(s, \zeta_n) + \frac{1}{2}|u(s)|^2\right) ds + \phi_0(\zeta_n(T)).$$

Then, if we let n tend to $+\infty$, by (5.8) and (5.9), (5.6) follows as claimed.

The relevance of the solution ψ to (5.1) for the optimal control problem (P), is explained in Theorem 3 below.

THEOREM 3. *Assume that conditions* (4.2) *are satisfied. Then the solution* ψ *to* (5.1) *is the optimal value function of problem* (P), *i.e., for every* $t \in [0, T]$ *one has*

$$\psi(t, x) = \inf\left\{E\int_t^T \left(V(s, \zeta(s)) + \frac{1}{2}|u(s)|^2\right) ds + \phi_0(\zeta(T));\right.$$
$$\left. d\zeta = (A\zeta + u)\,ds + \sqrt{\varepsilon}\,dW_s,\ \zeta(t) = x,\ u \in M_w^2(0, T; H)\right\}. \tag{5.10}$$

Moreover, the solution ζ^+ *to the problem*

$$d\zeta = (A\zeta - \psi_x(t, \zeta))\,dt + \sqrt{\varepsilon}\,dW_t, \qquad t \in [0, T],$$
$$\zeta(0) = x \tag{5.11}$$

is an optimal trajectory to problem (P) *corresponding to the optimal control* u^+ *given by*

$$u^+(t) = -\psi_x(t, \zeta^+(t)) \quad \text{a.e. } t \in]0, T[. \tag{5.12}$$

The optimal control u^+ *is unique.*

In few words, Theorem 3 says that under assumption (4.2) $u(t) = -\psi_x(t, \zeta(t))$ is an *optimal feedback control* for the stochastic control problem (P) (see [8] for definitions and classical results on these topics).

Proof of Theorem 3. By Lemma 8 (formula (5.6)) we see that for each $(t, x) \in [0, T] \times H$, $\psi(t, x) = \Phi(t, x)$ where Φ is the optimal value functions of problem (P).

Now let (ζ^+, u^+) be a pair given by (5.10). Since $\psi_x \in B([0, T]; C(H))$ and it is monotone in x (as the derivative of a convex function) (5.9) has a unique solution ζ^+ (see [7, Thms. 4 and 7]; remark that hypothesis (24) in Theorem 7 is satisfied in our situation because ψ_x is monotone). By (5.6) we see that for every $t \in [0, T]$

$$\psi(t, x) = E\left\{\int_t^T \left(g(s, \zeta^+(s)) + \frac{1}{2}|u^+(s)|^2\right) ds\right\} + \phi_0(\zeta(T)),$$

and therefore u^+ is optimal in problem (P).

Assume now that $(\tilde{u}, \tilde{y})$ is another optimal pair. Again by formula (5.6) it follows that

$$E \int_t^T |\psi_x(s, \tilde{y}(s)) + \tfrac{1}{2}\tilde{u}(s)|^2 \, ds = 0$$

which implies $\tilde{u} = -\psi_x(s, \tilde{\zeta}(s))$. since the solution to (5.9) is unique we infer that $\tilde{\zeta} = \zeta^+$ and $\tilde{u} = u^+$ as claimed.

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