

D-FINITENESS, RATIONALITY, AND HEIGHT

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ABSTRACT. Motivated by a result of van der Poorten and Shparlinski for univariate power series, Bell and Chen prove that if a multivariate power series over a field of characteristic 0 is D-finite and its coefficients belong to a finite set, then it is a rational function. We extend and strengthen their results to certain power series whose coefficients may form an infinite set. We also prove that if the coefficients of a univariate D-finite power series “look like” the coefficients of a rational function, then the power series is rational. Our work relies on the theory of Weil heights, the Manin–Mumford theorem for tori, an application of the Subspace Theorem, and various combinatorial arguments involving heights, power series, and linear recurrence sequences.

1. INTRODUCTION

Let \mathbb{N} denote the set of positive integers, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $m \in \mathbb{N}$, and consider the ring $K[[x_1, \dots, x_m]]$ of power series in m variables over a field K of characteristic 0. Very broadly speaking, there are several highly interesting results of the following form: if a power series f satisfies the property \mathcal{P}_1 and its coefficients satisfy the property \mathcal{P}_2 which is usually of an arithmetic nature, then property \mathcal{P}_3 holds. For example (when $m = 1$), the Pisot’s d th Root Conjecture, settled by Zannier [Zan00], states that if K is a number field, $f(x) = \sum_{i \geq 0} a_i x^i \in K[[x]]$ is a rational function, and a_i is the d th power of an element of K , then there exists a rational function $g(x) = \sum_{i \geq 0} b_i x^i$ such that $a_i = b_i^d$ for every i . There is a similar Pisot’s Hadamard Quotient Conjecture solved by Pourchet [Pou79] and van der Poorten [vdP88] (see Rumely’s note [Rum88] for more details and the paper by Corvaja–Zannier [CZ02a] for a stronger version). Note that \mathcal{P}_1 in the above results is the property that the given power series is a rational function. In this paper, we are interested in the situation when \mathcal{P}_1 is the so-called *D-finiteness property*.

Let $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}_0^m$, and let $\mathbf{x} = (x_1, \dots, x_m)$ be the vector of the indeterminates x_1, \dots, x_m . We write $\mathbf{x}^{\mathbf{n}}$ to denote the monomial $x_1^{n_1} \dots x_m^{n_m}$ having the total degree $\|\mathbf{n}\| := n_1 + \dots + n_m$. We also write $\frac{\partial^{\|\mathbf{n}\|}}{\partial \mathbf{x}^{\mathbf{n}}}$ to denote the operator

$$\left(\frac{\partial}{\partial x_1}\right)^{n_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{n_m}$$

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on $K[x_1, \dots, x_m]$. A power series $f(\mathbf{x}) \in K[[\mathbf{x}]]$ is said to be *D-finite* (over $K(\mathbf{x})$) if all the derivatives $\frac{\partial^{|\mathbf{n}|} f}{\partial \mathbf{x}^{\mathbf{n}}}$ for $\mathbf{n} \in \mathbb{N}_0^m$ span a finite-dimensional vector space over $K(\mathbf{x})$. Univariate power series satisfying linear differential equations (such as the exponential function, hypergeometric series, etc.) have played an important role in mathematics for hundreds of years. Since the 1960s certain p -adic and cohomological aspects of univariate power series solutions of algebraic differential equations have been developed by Dwork, Katz, and others (see [DGS94] and the references therein).

In 1980, Stanley wrote an expository paper [Sta80] introducing univariate D-finite power series and many of their properties from a combinatorial point of view. After that, multivariable D-finite power series were introduced by Lipshitz [Lip89] and they have become an important part in enumerative combinatorics, especially in the theory of generating functions [Sta99]. From the linear partial differential equations satisfied by a D-finite series $f(\mathbf{x})$, one can show that the coefficients of f satisfy certain linear recurrence relations with polynomial coefficients. In particular, if $f(\mathbf{x}) \in \bar{\mathbb{Q}}[[\mathbf{x}]]$ is D-finite, the coefficients of f belong to a number field.

Let h denote the absolute logarithmic Weil height on $\bar{\mathbb{Q}}$. We have the following results of van der Poorten–Shparlinski [vdPS96] and Bell–Chen [BC17].

Theorem 1.1 (van der Poorten–Shparlinski 1996). *Let $f(x) = \sum_{n \in \mathbb{N}_0} a_n x^n \in \bar{\mathbb{Q}}[[x]]$ be a univariate D-finite power series with rational coefficients. If $\lim_{n \rightarrow \infty} \frac{h(a_n)}{\log \log n} = 0$, then the sequence $(a_n)_{n \in \mathbb{N}_0}$ is periodic.*

Theorem 1.2 (Bell–Chen 2017). *Let K be a field of characteristic 0, and let $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_0^m} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]]$ be a D-finite power series in m variables. If the coefficients of f belong to a finite set, then f is rational.*

In fact, a slightly more precise version of Theorem 1.1 was proved by van der Poorten–Shparlinski [vdPS96, pp. 147–148]. Their method uses a technical construction of a certain auxiliary function. Although they stated their result for power series with rational coefficients, it seems that the proof should remain valid over an arbitrary number field.

After a specialization argument, Theorem 1.1 implies that if the coefficients of a univariate D-finite power series over a field of characteristic 0 belong to a finite set, then the series is rational. Theorem 1.2 is a very recent result of Bell–Chen [BC17] generalizing the above consequence for multivariate power series. The proof of Theorem 1.2 in [BC17] uses induction on the number of variables m and various combinatorial arguments involving the notion of syndetic subsets of \mathbb{N} .

Our first main result strengthens and generalizes both Theorems 1.2 and 1.1 in one stroke. More specifically, we treat multivariate power series, replace the function $\log \log(n)$ in Theorem 1.1 by the more dominant function $\log(n)$, and let one conclude that certain nonrational power series are not D-finite even when the coefficients do not belong to a finite set. We have the following theorem.

Theorem 1.3. *Let $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_0^m} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in \bar{\mathbb{Q}}[[\mathbf{x}]]$. Assume that f is D-finite and that*

$$(1) \quad \lim_{\|\mathbf{n}\| \rightarrow \infty} \frac{h(a_{\mathbf{n}})}{\log \|\mathbf{n}\|} = 0.$$

Then the following hold:

- (a) f is a rational function.
- (b) If f is not a polynomial, its denominator, up to scalar multiplication, has the form

$$\prod_{i=1}^{\ell} (1 - \zeta_i \mathbf{x}^{\mathbf{n}_i}),$$

where $\ell \geq 1$, ζ_i is a root of unity, $\mathbf{n}_i \in \mathbb{N}_0^m \setminus \{0\}$ for $1 \leq i \leq \ell$, and the $1 - \zeta_i \mathbf{x}^{\mathbf{n}_i}$'s are ℓ distinct irreducible polynomials.

- (c) The coefficients $(a_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}_0^m}$ belong to a finite set.

By specialization arguments, we have the following extension of the theorem by Bell–Chen.

Corollary 1.4. *Let K , m , and f be as in Theorem 1.2. If the coefficients of f belong to a finite set, then parts (a) and (b) of Theorem 1.3 hold.*

Note that condition (1) excludes rational functions such as $\frac{1}{1-2x} = \sum_{n \geq 0} 2^n x^n$. In fact, the coefficients of a rational function have the form $P_1(n)\alpha_1^n + \dots + P_k(n)\alpha_k^n$ and the logarithmic height is comparable to n (unless all the α_i 's are a root of unity). Our next result proves that if a power series is D-finite and its coefficients “look like” the coefficients of a rational function, then the series is indeed rational. In fact, we will consider the above form $P_1(n)\alpha_1^n + \dots + P_k(n)\alpha_k^n$ in which the polynomials P_i can vary according to n as long as their degrees are bounded and their coefficients belong to a fixed number field and have small heights compared to $\log(n)$.

Theorem 1.5. *Let $d \in \mathbb{N}_0$, let $k \in \mathbb{N}$, and let $\alpha_1, \dots, \alpha_k \in \bar{\mathbb{Q}}^*$. Let K be a number field. For $n \geq 0$, let a_n be of the form*

$$a_n = (c_{n,1,0} + c_{n,1,1}n + \dots + c_{n,1,d}n^d)\alpha_1^n + \dots + (c_{n,k,0} + c_{n,k,1}n + \dots + c_{n,k,d}n^d)\alpha_k^n$$

such that $c_{n,i,j} \in K$ for $1 \leq i \leq k$ and $0 \leq j \leq d$, and $\lim_{n \rightarrow \infty} \frac{\max_{i,j} h(c_{n,i,j})}{\log n} = 0$. If $f(x) = \sum_{n \geq 0} a_n x^n$ is D-finite then f is rational.

Roughly speaking, Theorem 1.3 treats D-finite power series in which the heights of the coefficients grow very slowly, while Theorem 1.5 considers those where the coefficients are similar to those of a typical rational functions (and hence $h(a_n)$ is approximately linear in n). We now consider D-finite series in which $h(a_n)$ can be large. The typical example is the exponential function $\exp(x) = \sum_{n \geq 0} a_n x^n$ with $h(a_n) = \log(n!) \sim n \log(n)$. Our next result shows that the heights of the coefficients of a univariate D-finite power series cannot go beyond the function $n \log(n)$.

Theorem 1.6. *Let $f(x) = \sum_{n \geq 0} a_n x^n \in \bar{\mathbb{Q}}[[x]]$ be D-finite. For each $n \geq 0$, we consider the affine point $h(a_0, \dots, a_n)$ and its Weil height. We have*

$$(2) \quad \limsup_{n \rightarrow \infty} \frac{h(a_n)}{n \log n} \leq \limsup_{n \rightarrow \infty} \frac{h(a_0, \dots, a_n)}{n \log n} < \infty.$$

The organization of this paper is as follows. In the next section, we give a definition of the Weil height h and various results needed for the proofs of the above theorems. Then we prove Theorem 1.3 and present specialization arguments for Corollary 1.4. After that, we prove Theorems 1.5 and 1.6.

2. HEIGHT

A large part of this section is taken from [KMN], which, in turn, follows from earlier work of Evertse [Eve84] and Corvaja–Zannier [CZ02b, CZ04]. Let $M_{\mathbb{Q}} = M_{\mathbb{Q}}^{\infty} \cup M_{\mathbb{Q}}^0$, where $M_{\mathbb{Q}}^0$ is the set of p -adic valuations and $M_{\mathbb{Q}}^{\infty}$ is the singleton consisting of the usual archimedean valuation. More generally, for every number field K , write $M_K = M_K^{\infty} \cup M_K^0$, where M_K^{∞} is the set of archimedean places and M_K^0 is the set of finite places. For every $w \in M_K$, let K_w denote the completion of K with respect to w and denote $d(w) = [K_w : \mathbb{Q}_v]$, where v is the restriction of w to \mathbb{Q} . Following [BG06, Chapter 1], for every $w \in M_K$ restricting to v on \mathbb{Q} , we normalize $|\cdot|_w$ as follows:

$$|x|_w = |\mathrm{N}_{K_w/\mathbb{Q}_v}(x)|_v^{1/[K:\mathbb{Q}]}.$$

Let $m \in \mathbb{N}$, and for every vector $\mathbf{u} = (u_0, \dots, u_m) \in K^{m+1} \setminus \{\mathbf{0}\}$ and $w \in M_K$, let $|\mathbf{u}|_w := \max_{0 \leq i \leq m} |u_i|_w$. For $P \in \mathbb{P}^m(\mathbb{Q})$, let K be a number field such that P has a representative $\mathbf{u} \in K^{m+1} \setminus \{\mathbf{0}\}$ and define

$$H(P) = \prod_{w \in M_K} |\mathbf{u}|_w.$$

Define $h(P) = \log(H(P))$. For $\alpha \in \bar{\mathbb{Q}}$, write $H(\alpha) = H([\alpha : 1])$ and $h(\alpha) = \log(H(\alpha))$. The following properties of the height function are well known [Zan18, Proposition 1.2].

Proposition 2.1.

- (a) For every $a \in \bar{\mathbb{Q}}^*$ and $m \in \mathbb{Z}$, $h(a^m) = |m|h(a)$.
- (b) For every $r \in \mathbb{N}$ and $a_1, \dots, a_r \in \bar{\mathbb{Q}}$, $h(a_1 + \dots + a_r) \leq h(a_1) + \dots + h(a_r) + \log r$.
- (c) For every $a, b \in \bar{\mathbb{Q}}$, $h(ab) \leq h(a) + h(b)$. Hence, if $b \neq 0$, then $h(ab) \geq h(a) - h(b)$.
- (d) Let $P(t) = a_d t^d + \dots + a_1 t + a_0 \in \bar{\mathbb{Q}}[t]$. There exist constants $C_0(d)$ and $C_1(d)$ depending only on d such that if $a_d \neq 0$, then

$$|h(P(\alpha)) - dh(\alpha)| \leq C_1(d) \max_{0 \leq i \leq d} h(a_i) + C_0(d)$$

for every $\alpha \in \bar{\mathbb{Q}}$.

Proof. Parts (a), (b), and (c) are in any standard introduction to Weil heights such as [HS00, Part B] or [BG06, Chapters 1–2]. For part (d), see [HS11, Proposition 6] and [HS00, Remark B.2.7]. \square

Now we present an important application of the Subspace Theorem taken from [KMN, Section 2]. The Subspace Theorem is one of the milestones of diophantine geometry in the last 50 years. The first version was obtained by Schmidt [Sch70], and further versions were obtained by Schlickewei and Evertse [Sch92, Eve96, ES02]. In the following application, a sublinear function means a function $F : \mathbb{N} \rightarrow (0, \infty)$ such that $\lim_{n \rightarrow \infty} \frac{F(n)}{n} = 0$. Let $k \in \mathbb{N}$; a tuple of nonzero algebraic numbers $(\alpha_1, \dots, \alpha_k)$ is said to be nondegenerate if α_i/α_j is not a root of unity for $i \neq j$. We have the following proposition.

Proposition 2.2. *Let $k \in \mathbb{N}$, let $(\alpha_1, \dots, \alpha_k)$ be a nondegenerate tuple of nonzero algebraic numbers, let F be a sublinear function, and let K be a number field. Then*

there are only finitely many tuples $(n, b_1, \dots, b_k) \in \mathbb{N} \times (K^*)^k$ satisfying

$$b_1 \alpha_1^n + \dots + b_k \alpha_k^n = 0 \text{ and } \max_{1 \leq i \leq k} h(b_i) < F(n).$$

Proof. This follows from a result of Evertse [Eve84, Theorem 1]. For more details, see [KMN, Section 2]. □

3. PROOFS OF THEOREM 1.3 AND COROLLARY 1.4

We will refer to the following property of power series with algebraic coefficients throughout the paper.

Definition 3.1. Let $m \in \mathbb{N}$, let $\mathbf{x} = (x_1, \dots, x_m)$, and let $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_0^m} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$. We say that f satisfies property \mathcal{P} if

$$\lim_{\|\mathbf{n}\| \rightarrow \infty} \frac{h(a_{\mathbf{n}})}{\log \|\mathbf{n}\|} = 0.$$

For the rest of this section, let $m \in \mathbb{N}$ and $\mathbf{x} = (x_1, \dots, x_m)$. The proof of Theorem 1.3 consists of three parts. The first part is to use properties of the Weil height to establish rationality of f . The key idea is that the coefficients of f satisfy certain linear recurrence relations with polynomial coefficients and property \mathcal{P} allows the polynomial coefficients to be the dominant terms in such relations. In fact, we will prove an effective version of part (a) of Theorem 1.3 which will be used in the specialization arguments for the proof of Corollary 1.4. The second part of the proof is to prove part (b) by using the substitution $(x_1, \dots, x_m) = (t^{u_1}, \dots, t^{u_m})$ for $u_1, \dots, u_m \in \mathbb{N}$ in order to apply known results about univariate rational functions; it turns out that this part has a surprising connection to the beautiful Manin–Mumford conjecture for tori in diophantine geometry. Finally, once we know that f is a rational function whose denominator has the special form given in part (b), we can use induction and certain combinatorial arguments to finish the proof. We start with the following simple lemma.

Lemma 3.2. *Let K be a field of characteristic 0, and let $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_0^m} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]]$. We have:*

- (a) *f is D-finite over $K(\mathbf{x})$ if and only if f satisfies a system of linear partial differential equations, one for each $i = 1, \dots, m$, of the form*

$$\left(P_{i,d_i}(\mathbf{x}) \left(\frac{\partial}{\partial x_i} \right)^{d_i} + \dots + P_{i,1}(\mathbf{x}) \frac{\partial}{\partial x_i} + P_{i,0}(\mathbf{x}) \right) f(\mathbf{x}) = 0,$$

where $P_{i,j}(\mathbf{x}) \in K[\mathbf{x}]$ for every $0 \leq j \leq d_i$ and $P_{i,d_i}(\mathbf{x}) \neq 0$.

- (b) *Let F be a field containing K . Then f is D-finite over $F(\mathbf{x})$ if and only if it is D-finite over $K(\mathbf{x})$.*

Proof. Part (a) is [Lip89, Proposition 2.2]; although the author stated it for \mathbb{C} , the proof works verbatim for an arbitrary field K of characteristic 0. For part (b), if f is D-finite over $F(\mathbf{x})$, then the coefficients of the $P_{i,j}$'s give a nontrivial solution over F of a homogeneous system of (infinitely many) linear equations with coefficients in K . Hence this system must have a nontrivial solution over K , and this proves D-finiteness over $K(\mathbf{x})$. □

3.1. Proof of part (a) of Theorem 1.3. We now prove the following effective version of part (a) of Theorem 1.3 (see Remark 3.4).

Theorem 3.3. *Let $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_0^m} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in \bar{\mathbb{Q}}[[\mathbf{x}]]$ be D -finite. Assume that f satisfies a system of linear partial differential equations, one for each $i = 1, \dots, m$, of the form*

$$\left(P_{i,d_i}(\mathbf{x}) \left(\frac{\partial}{\partial x_i} \right)^{d_i} + \dots + P_{i,1}(\mathbf{x}) \frac{\partial}{\partial x_i} + P_{i,0}(\mathbf{x}) \right) f(\mathbf{x}) = 0,$$

where $P_{i,j}(\mathbf{x}) \in \bar{\mathbb{Q}}[\mathbf{x}]$ for every $0 \leq j \leq d_i$ and $P_{i,d_i}(\mathbf{x}) \neq 0$. Let M be an upper bound on the heights of the coefficients, and let D be an upper bound on the total degrees of all the $P_{i,j}$. Then there exist effectively computable positive constant δ and η depending only on m, M, D , and $\max_{1 \leq i \leq m} d_i$ such that the following holds. If N satisfies $\frac{h(a_{\mathbf{n}})}{\log \|\mathbf{n}\|} < \delta$ for every $\mathbf{n} \in \mathbb{N}_0^m$ with $\|\mathbf{n}\| \geq N$, then

$$f(\mathbf{x}) = \frac{P(\mathbf{x})}{P_{1,d_1}(\mathbf{x}) \cdots P_{m,d_m}(\mathbf{x})},$$

where $P(\mathbf{x})$ is a polynomial of total degree at most $N + \eta$.

Remark 3.4. Theorem 3.3 implies that if $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_0^m} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in \bar{\mathbb{Q}}[[\mathbf{x}]]$ is D -finite and irrational, then there exists $\delta > 0$ which is effectively computable and depends only on the system of linear partial differential equations satisfied by f such that $\limsup_{\|\mathbf{n}\| \rightarrow \infty} \frac{h(a_{\mathbf{n}})}{\|\mathbf{n}\|} \geq \delta$. An important example of such an f in the univariate case is $\log(1 + x^d) = x^d - \frac{x^{2d}}{2} + \frac{x^{3d}}{3} - \dots$ in which $h(a_n) = \log n - \log d$ when $d \mid n$ and $h(a_n) = 0$ otherwise. It is interesting to know, at least in the univariate case, if $h(a_n) \geq \log n + O(1)$ for every n in a positive density subset of \mathbb{N} ; such a result is best possible in view of the example $\log(1 + x^d)$ above.

For $1 \leq i \leq m$, let $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ be the i th elementary basis vector in \mathbb{N}_0^m . For every $j \in \mathbb{N}$, let $B_j(x) = x(x - 1) \cdots (x - j + 1) \in \mathbb{Z}[x]$, and let $B_0(x) = 1$. So we have

$$\left(\frac{\partial}{\partial x_i} \right)^j f(\mathbf{x}) = \sum_{\mathbf{n}=(n_1, \dots, n_m) \in \mathbb{N}_0^m} B_j(n_i) a_{\mathbf{n}} \mathbf{x}^{\mathbf{n} - j\mathbf{e}_i}.$$

To prove Theorem 3.3, we prove the following result that handles one linear partial differential equation at a time.

Proposition 3.5. *Let $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_0^m} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in \bar{\mathbb{Q}}[[\mathbf{x}]]$, and fix $i \in \{1, \dots, m\}$. Assume that f satisfies the linear partial differential equation*

$$(3) \quad \left(P_{i,d_i}(\mathbf{x}) \left(\frac{\partial}{\partial x_i} \right)^{d_i} + \dots + P_{i,1}(\mathbf{x}) \frac{\partial}{\partial x_i} + P_{i,0}(\mathbf{x}) \right) f(\mathbf{x}) = 0,$$

where $P_{i,j}(\mathbf{x}) \in \bar{\mathbb{Q}}[\mathbf{x}]$ for $0 \leq j \leq d_i$ and $P_{i,d_i}(\mathbf{x}) \neq 0$. Let M_i be an upper bound on the height of the coefficients, and let D_i be an upper bound on the total degrees of the $P_{i,j}$'s for $0 \leq j \leq d_i$. Let $\epsilon_i > 0$; then there exist effectively computable positive constants δ_i and η_i depending only on m, M_i, D_i, d_i , and ϵ_i such that the following holds. If N satisfies $\frac{h(a_{\mathbf{n}})}{\log \|\mathbf{n}\|} < \delta_i$ for every $\mathbf{n} \in \mathbb{N}_0^m$ with $\|\mathbf{n}\| \geq N$, then for every $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{N}_0$, if $\|\mathbf{r}\| \geq N + \eta_i$ and $r_i \geq \epsilon_i \|\mathbf{r}\|$, then the coefficient of $\mathbf{x}^{\mathbf{r} + d_i \mathbf{e}_i}$ in $P_{i,d_i} f$ is 0.

Proof. If $d_i = 0$, then $P_{i,0}f = 0$ and there is nothing to prove, so we may assume $d_i > 0$. For $0 \leq j \leq d_i$, let $S_{i,j} \subset \mathbb{N}_0^m$ be the “support” of $P_{i,j}$; this means the finite set of the multidegrees of monomials having nonconstant coefficients in $P_{i,j}$. For $0 \leq j \leq d_i$, write

$$P_{i,j}(\mathbf{x}) = \sum_{\mathbf{n} \in S_{i,j}} p_{i,j,\mathbf{n}} \mathbf{x}^{\mathbf{n}}.$$

Let $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{N}_0^m$; the coefficient of $\mathbf{x}^{\mathbf{r}}$ on the left-hand side of (3) is

$$(4) \quad \sum_{j=0}^{d_i} \sum_{\mathbf{n}=(n_1, \dots, n_m) \in S_{i,j}} p_{i,j,\mathbf{n}} B_j(r_i + j - n_i) a_{\mathbf{r}+j\mathbf{e}_i-\mathbf{n}} = 0.$$

Note that our convention here is to put $a_{\mathbf{u}} = 0$ if $\mathbf{u} \in \mathbb{Z}^m \setminus \mathbb{N}_0^m$. Since $\|\mathbf{n}\| \leq D_i$ for every $\mathbf{n} \in S_{i,j}$, there exists a constant C_2 depending only on d_i and D_i such that for every $0 \leq j \leq d_i$ and every $\mathbf{n} = (n_1, \dots, n_m) \in S_{i,j}$, $B_j(r_i + j - n_i)$ is a polynomial of degree j in r_i and the heights of its coefficients are bounded above by C_2 .

Now assume that $\|\mathbf{r}\| \geq \max\{N + d_i + D_i, 2\}$ so that for $0 \leq j \leq d_i$ and every $\mathbf{n} \in S_{i,j}$, the vector $\mathbf{r} + j\mathbf{e}_i - \mathbf{n}$ is either in $\mathbb{Z}^m \setminus \mathbb{N}_0^m$ or the sum of its coordinates is at least $N + d_i$, and we have

$$h(a_{\mathbf{r}+j\mathbf{e}_i-\mathbf{n}}) \leq \delta_i \log(\|\mathbf{r}\| + j - \|\mathbf{n}\|) \leq \delta_i \log(2\|\mathbf{r}\|) \leq 2\delta_i \log \|\mathbf{r}\|.$$

Observe that the cardinality of each $S_{i,j}$ is at most $(D_i + 1)^m$. By gathering the coefficients of common powers of r_i and using Proposition 2.1, we can write the left-hand side of (4) as

$$(5) \quad \alpha_{d_i} r_i^{d_i} + \alpha_{d_i-1} r_i^{d_i-1} + \dots + \alpha_0 = 0,$$

where $\alpha_{d_i} = \sum_{\mathbf{n} \in S_{i,d_i}} p_{i,d_i,\mathbf{n}} a_{\mathbf{r}+d_i\mathbf{e}_i-\mathbf{n}}$ and the following holds. There exist constants C_3 and C_4 depending only on m , M_i , D_i , and d_i such that $h(\alpha_j) \leq C_3 \delta_i \log \|\mathbf{r}\| + C_4$ for $j = 0, \dots, d_i$. By Proposition 2.1(d), we have C_5 and C_6 such that if $\alpha_{d_i} \neq 0$, then

$$(6) \quad \begin{aligned} C_5 \delta_i \log \|\mathbf{r}\| + C_6 &\geq |h(\alpha_{d_i} r_i^{d_i} + \alpha_{d_i-1} r_i^{d_i-1} + \dots + \alpha_0) - d_i h(r_i)| \\ &= d_i \log(r_i) \\ &\geq d_i \log(\epsilon_i \|\mathbf{r}\|), \end{aligned}$$

where the last inequality is under the further assumption that $r_i \geq \epsilon_i \|\mathbf{r}\|$. However, (6) cannot hold when δ_i is sufficiently small and $\|\mathbf{r}\|$ is sufficiently large, for instance, when $C_5 \delta_i \leq d_i/2$ and $\|\mathbf{r}\| > e^{2C_6}/\epsilon_i^2$. Hence, under this further assumption, we must have $\alpha_{d_i} = 0$. Notice that $\alpha_{d_i} = \sum_{\mathbf{n} \in S_{i,d_i}} p_{i,d_i,\mathbf{n}} a_{\mathbf{r}+d_i\mathbf{e}_i-\mathbf{n}}$ is exactly the coefficient of $\mathbf{x}^{\mathbf{r}+d_i\mathbf{e}_i}$ in $P_{i,d_i}f$ and we finish the proof. \square

Proposition 3.5 is the key step in our proof of Theorem 3.3.

Proof of Theorem 3.3. We apply Proposition 3.5 for each $i = 1, \dots, m$ with $\epsilon_i = 1/2m$, let δ be the minimum of the resulting δ_i 's, and let η' be the maximum of the resulting η_i 's. We now take

$$\eta'' := \tilde{\eta} + (2m - 1)(d_1 + \dots + d_m).$$

Let $\mathbf{r} \in \mathbb{N}_0^m$ with $\|\mathbf{r}\| > N + \eta''$. There exists $i \in \{1, \dots, m\}$ such that $r_i \geq \|\mathbf{r}\|/m$. Hence the vector $\mathbf{r}' := \mathbf{r} - d_i\mathbf{e}_i$ satisfies $r'_i \geq \|\mathbf{r}'\|/2m$ and $\|\mathbf{r}'\| > N + \tilde{\eta} \geq N + \eta_i$. By Proposition 3.5, the coefficient of $\mathbf{x}^{\mathbf{r}}$ in $P_{i,d_i}f$ is zero. Therefore, if we choose

$\eta := \eta'' + (m - 1)D$, then $P_{1,d_1} \dots P_{m,d_m} f$ is a polynomial of total degree at most $N + \eta$. □

3.2. Proof of part (b) of Theorem 1.3. We will use the following simple result for univariate rational functions.

Proposition 3.6. *Let $G(t) = \sum_{n \geq 0} g_n x^n \in \bar{\mathbb{Q}}[[t]]$ be a rational function that is not a polynomial. Assume that $h(g_n) = o(n)$; then every root of the denominator of G is a root of unity. Moreover, if*

$$\limsup_{n \rightarrow \infty} \frac{h(g_n)}{\log n} \leq L < \infty,$$

then every root of the denominator of G has multiplicity at most $L + 1$.

Proof. Let $\alpha_1, \dots, \alpha_\ell$ be all the (distinct) roots of the denominator of G ; we have $\alpha_i \in \bar{\mathbb{Q}}^*$ for every i . Then there exist $P_1(X), \dots, P_\ell(X) \in \bar{\mathbb{Q}}[X] \setminus \{0\}$ such that for all sufficiently large n , we have

$$g_n = P_1(n)\alpha_1^n + \dots + P_\ell(n)\alpha_\ell^n.$$

For the first assertion, assume $h(g_n) = o(n)$, and we prove that all the α_i 's are roots of unity. This can be done easily using induction on $r := \ell + \sum_{i=1}^\ell \deg(P_i)$ and working with the sequence $g_{n+1} - \alpha_\ell g_n$ which lowers the value of r .

For the second assertion, let D denote the maximum of the degrees of the P_i 's. Then for n belonging to an appropriate arithmetic progression, $g_n = \sum_{i=1}^\ell P_i(n)\alpha_i^n$ is a polynomial in n with degree D . Hence $D \leq L$ and this finishes the proof. □

We will use the following version of the Manin–Mumford conjecture for tori. For $n \geq 1$, by a torsion coset of \mathbb{G}_m^n , we mean a torsion translate of an algebraic subgroup. For a closed subvariety V of \mathbb{G}_m^n , a torsion coset in V means a torsion coset of \mathbb{G}_m^n that is contained in V .

Theorem 3.7. *Let $n \geq 1$, and let V be a closed subvariety of \mathbb{G}_m^n defined over \mathbb{C} . Then the following hold:*

- (a) *Every torsion coset in V is contained in a maximal torsion coset in V .*
- (b) *There are only finitely many maximal torsion cosets in V , and their union is the Zariski closure of torsion points in V .*

Proof. This is given in [BG06, Chapter 3] following earlier work of Laurent [Lau84], Bombieri–Zannier [BZ95], and Schmidt [Sch96]. In fact, the number of maximal torsion cosets can be bounded by an explicit expression involving only n and the maximum of the degrees of polynomials defining V . □

Lemma 3.8. *Let $P(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n] \setminus \{0\}$. Then there exist finitely many proper vector subspaces $W_1, \dots, W_k \subsetneq \mathbb{Q}^n$ such that for every $(u_1, \dots, u_n) \in \mathbb{N}^n$ outside $\bigcup_{j=1}^k W_j$ we have that $P(t^{u_1}, \dots, t^{u_n})$ is a nonzero polynomial in t .*

Proof. If we do the substitution $x_i = t^{u_i}$ and get $P(t^{u_1}, \dots, t^{u_n}) = 0$, then two distinct monomials in $P(x_1, \dots, x_n)$ yield the same t^k and this gives rise to a non-trivial linear relation among the u_i 's. □

Proof of part (b) of Theorem 1.3. We have proved that f is a rational function. Suppose that f is not a polynomial and write $f = \frac{F}{G}$, where F and G are coprime

polynomials in $\bar{\mathbb{Q}}[x_1, \dots, x_m]$ and G is nonconstant. We first prove that every irreducible factor of G has the form $1 - \zeta \mathbf{x}^{\mathbf{n}}$, where ζ is a root of unity and $\mathbf{n} \in \mathbb{N}_0^m$.

Since the property \mathcal{P} still holds after replacing f by its product with a polynomial, we may assume that G is irreducible. Fix an embedding of $\bar{\mathbb{Q}}$ into \mathbb{C} ; the condition $h(a_{\mathbf{n}}) = o(\log \|\mathbf{n}\|)$ implies that f is convergent in the polydisc \mathcal{D} given by $|x_i| < 1$ for $1 \leq i \leq m$. For a polynomial $P \in \mathbb{C}[x_1, \dots, x_m]$, let $\mathcal{Z}(P)$ denote the zero set of P . We have that $\mathcal{Z}(G) \cap \mathcal{D}$ is contained in $\mathcal{Z}(F) \cap \mathcal{D}$ since $F = fG$ as analytic functions on \mathcal{D} . If $G(0, \dots, 0) = 0$, we get that $\mathcal{Z}(G) \cap \mathcal{Z}(F) \cap \mathcal{D} = \mathcal{Z}(G) \cap \mathcal{D}$ is a nonempty open set of $\mathcal{Z}(G)$ containing $(0, \dots, 0)$; this gives a contradiction since $\mathcal{Z}(F) \cap \mathcal{Z}(G)$ has strictly smaller dimension than $\mathcal{Z}(G)$. Hence $G(0, \dots, 0) \neq 0$.

Since G is not one of the coordinate functions x_i , the closed subvariety V of $(\mathbb{C}^*)^m$ defined by $G = 0$ has dimension $m - 1$ and our goal is to prove that V is a torsion coset. Assume that each of the finitely many maximal torsion cosets in V has codimension at least 2 and we will arrive at a contradiction. Each such maximal torsion coset satisfies at least 2 independent equations of the form

$$\begin{aligned} x_1^{\gamma_1} \dots x_m^{\gamma_m} &= 1, \\ x_1^{\delta_1} \dots x_m^{\delta_m} &= 1. \end{aligned}$$

Therefore we can eliminate x_m if necessary to conclude that the maximal torsion coset is contained in the subgroup defined by an equation of the form

$$x_1^{\kappa_1} \dots x_{m-1}^{\kappa_{m-1}} = 1.$$

Since there are only finitely many maximal torsion cosets, we obtain a finite set \mathcal{X} of nonzero vectors in \mathbb{Z}^{m-1} such that for every torsion point $(\xi_1, \dots, \xi_m) \in V$, there is a vector $(\kappa_1, \dots, \kappa_{m-1}) \in \mathcal{X}$ satisfying

$$(7) \quad \xi_1^{\kappa_1} \dots \xi_{m-1}^{\kappa_{m-1}} = 1.$$

Let K be a number field containing the coefficients of F and G . By relabelling the x_i 's when necessary, we may assume that G has the form

$$G = G_0 + G_1 x_m + \dots + G_d x_m^d,$$

where $d \geq 1$, each G_i is in $K[x_1, \dots, x_{m-1}]$, and $G_d \neq 0$. Since F and G are coprime, there exist polynomials $\tilde{F}, \tilde{G}, \tilde{H}$ in $K[x_1, \dots, x_{m-1}]$ with $\tilde{H} \neq 0$ such that

$$\tilde{F}F + \tilde{G}G = \tilde{H}.$$

By Lemma 3.8, there is a union W of finitely many proper subspaces of \mathbb{Q}^{m-1} such that for every $(u_1, \dots, u_{m-1}) \in \mathbb{N}^{m-1} \setminus W$ we have

$$(8) \quad G_d(t^{u_1}, \dots, t^{u_{m-1}}) \tilde{H}(t^{u_1}, \dots, t^{u_{m-1}}) \neq 0.$$

By adding to W the subspaces of \mathbb{Q}^{m-1} each of which is the orthogonal complement to some $(\kappa_1, \dots, \kappa_{m-1}) \in \mathcal{X}$, we may assume the additional property that

$$u_1 \kappa_1 + \dots + u_{m-1} \kappa_{m-1} \neq 0$$

for every $(\kappa_1, \dots, \kappa_{m-1}) \in \mathcal{X}$.

Fix one such (u_1, \dots, u_{m-1}) and let

$$B = \max\{|u_1 \kappa_1 + \dots + u_{m-1} \kappa_{m-1}| : (\kappa_1, \dots, \kappa_{m-1}) \in \mathcal{X}\}.$$

Let S be a finite subset of M_K containing M_K^∞ such that the ring of S -integers $\mathcal{O}_{K,S}$ is an UFD and the coefficients of F and G are in $\mathcal{O}_{K,S}$. From $F = fG$ and

the fact that $\mathcal{O}_{K,S}[x_1, \dots, x_m]$ is an UFD, we conclude that the coefficients of f are in $\mathcal{O}_{K,S}$, too. Let $u_m \in \mathbb{N}$ that will be chosen to be sufficiently large.

Consider the following rational function in t :

$$\frac{F(t^{u_1}, \dots, t^{u_m})}{G(t^{u_1}, \dots, t^{u_m})} = f(t^{u_1}, \dots, t^{u_m}) =: \sum_{n \geq 0} \tau_n t^n,$$

where $\tau_n = \sum_{\mathbf{n}} a_{\mathbf{n}}$ in which \mathbf{n} ranges over all $\mathbf{n} = (n_1, \dots, n_m)$ with $n_1 u_1 + \dots + n_m u_m = n$; there are $O(n^{m-1})$ such \mathbf{n} 's. Equation (8) implies

$$(9) \quad \gcd(F(t^{u_1}, \dots, t^{u_m}), G(t^{u_1}, \dots, t^{u_m})) \mid \tilde{H}(t^{u_1}, \dots, t^{u_{m-1}}).$$

Hence when u_m is sufficiently large so that

$$\deg(G(t^{u_1}, \dots, t^{u_m})) = du_m + \deg(G_d(t^{u_1}, \dots, t^{u_{m-1}})) > \deg(\tilde{H}(t^{u_1}, \dots, t^{u_{m-1}})),$$

$f(t^{u_1}, \dots, t^{u_m})$ is not a polynomial and its denominator is

$$(10) \quad \frac{G(t^{u_1}, \dots, t^{u_m})}{\gcd(F(t^{u_1}, \dots, t^{u_m}), G(t^{u_1}, \dots, t^{u_m}))}.$$

Since the $a_{\mathbf{n}}$'s are in $\mathcal{O}_{K,S}$ and $h(a_{\mathbf{n}}) = o(\log \|\mathbf{n}\|)$, we have:

- τ_n is in $\mathcal{O}_{K,S}$ for every n .
- $|\tau_n|_v \leq \max\{|a_{\mathbf{n}}|_v : \|\mathbf{n}\| \leq n\} = n^{o(1)}$ for every $v \in S \cap M_K^0$.
- $|\tau_n|_v \leq n^{m-1+o(1)}$ for every $v \in M_K^\infty$.

Therefore $h(\tau_n) \leq (m-1+o(1)) \log n$. Proposition 3.6 implies that the denominator of $f(t^{u_1}, \dots, t^{u_m})$ has the form

$$(11) \quad (t - \zeta_1)^{e_1} \dots (t - \zeta_\ell)^{e_\ell},$$

where $\ell \geq 1$, the ζ_i 's are ℓ distinct roots of unity, and $1 \leq e_i \leq m$.

From the expressions (10) and (11) for the denominator of $f(t^{u_1}, \dots, t^{u_m})$ and (9), we have

$$G(t^{u_1}, \dots, t^{u_m}) = \gcd(F(t^{u_1}, \dots, t^{u_m}), G(t^{u_1}, \dots, t^{u_m})) \prod_{j=1}^{\ell} (t - \zeta_j)^{e_j}$$

and

$$\begin{aligned} \ell m &\geq \sum_{j=1}^{\ell} e_j = \deg(G(t^{u_1}, \dots, t^{u_m})) - \deg(\gcd(F(t^{u_1}, \dots, t^{u_m}), G(t^{u_1}, \dots, t^{u_m}))) \\ &\geq du_m + \deg(G_d(t^{u_1}, \dots, t^{u_{m-1}})) - \deg(\tilde{H}(t^{u_1}, \dots, t^{u_{m-1}})). \end{aligned}$$

When u_m is large, we have that ℓ is large, and hence among the distinct roots of unity $\zeta_1, \dots, \zeta_\ell$ there is ζ_i with order greater than B . Since $G(\zeta_i^{u_1}, \dots, \zeta_i^{u_m}) = 0$, the point $(\zeta_i^{u_1}, \dots, \zeta_i^{u_m})$ is a torsion point of V . But we have

$$(\zeta_i^{u_1})^{\kappa_1} \dots (\zeta_i^{u_{m-1}})^{\kappa_{m-1}} = \zeta_i^{u_1 \kappa_1 + \dots + u_{m-1} \kappa_{m-1}} \neq 1$$

for every $(\kappa_1, \dots, \kappa_{m-1}) \in \mathcal{H}$ since $0 < |u_1 \kappa_1 + \dots + u_{m-1} \kappa_{m-1}| \leq B$, which is less than the order of ζ_i . This contradicts (7). Therefore, V itself is a torsion coset. Since $G(0, \dots, 0) \neq 0$, we conclude that G is a scalar multiple of $1 - \zeta \mathbf{x}^{\mathbf{n}}$.

Now we no longer assume that G is irreducible. The above arguments prove that every irreducible factor of G has the form $1 - \zeta \mathbf{x}^{\mathbf{n}}$. To finish the proof of part (b) of Theorem 1.3, it remains to show that every irreducible factor of G has multiplicity 1. As before, by considering the product of f with a polynomial, we may assume

that $G = (1 - \zeta \mathbf{x}^{\mathbf{n}})^r$ in which $r \in \mathbb{N}$ and $1 - \zeta \mathbf{x}^{\mathbf{n}}$ is irreducible. Let s denote the order of ζ . Then we can write,

$$f = \frac{F}{(1 - \zeta \mathbf{x}^{\mathbf{n}})^r} = \frac{P}{(1 - \mathbf{x}^{\mathbf{n}'})^r},$$

where $\mathbf{n}' := s\mathbf{n}$ and $P(\mathbf{x}) \in \bar{\mathbb{Q}}[\mathbf{x}]$ that is not divisible by $1 - \mathbf{x}^{\mathbf{n}'}$. Assume that $r \geq 2$ and we will arrive at a contradiction. Write

$$P(\mathbf{x}) = \sum_{\mathbf{k} \in S(P)} p_{\mathbf{k}} x^{\mathbf{k}},$$

where $S(P) := \{\mathbf{k} \in \mathbb{N}_0^m : p_{\mathbf{k}} \neq 0\}$ is the support of P . On \mathbb{N}_0^m , define the equivalence relation: $\mathbf{k} \sim \mathbf{k}'$ if and only if $\mathbf{k} - \mathbf{k}' \in \mathbb{Z}\mathbf{n}'$. The equivalence class of $\mathbf{k} \in \mathbb{N}_0^m$ is denoted $\bar{\mathbf{k}}$. Fix $\mathbf{k}^* \in S(P)$, and let $\alpha \in \mathbb{N}$ be sufficiently large. By computing the Taylor series of $\frac{P}{(1 - \mathbf{x}^{\mathbf{n}'})^r}$ directly, we have that the coefficient $a_{\mathbf{k}^* + \alpha \mathbf{n}'}$ is a polynomial of degree $r - 1$ in α whose leading coefficient has the form

$$c \sum_{\mathbf{k} \in \bar{\mathbf{k}}^* \cap S(P)} p_{\mathbf{k}},$$

where c is a nonzero constant. By the assumption on the height of the coefficients of f , we must have

$$\sum_{\mathbf{k} \in \bar{\mathbf{k}}^* \cap S(P)} p_{\mathbf{k}} = 0.$$

Since this is true for every $\mathbf{k}^* \in S(P)$, we have that P is divisible by $1 - \mathbf{x}^{\mathbf{n}'}$, a contradiction. Hence $r = 1$ and we finish the proof. \square

3.3. Proof of part (c) of Theorem 1.3. We use induction on the number of variables m . The case $m = 1$ follows from Proposition 3.6. Now consider $m \geq 2$ and assume that the conclusion holds for all power series with less than m variables.

If x_m does not appear in the denominator of f , then we can write f as a *finite* sum:

$$\sum_{n \geq 0} x_m^n f_n(x_1, \dots, x_{m-1}),$$

in which each f_n is D-finite and satisfies property \mathcal{P} for power series in $m - 1$ variables. Then we are done by the induction hypothesis. So we may assume that x_m appears in the denominator of f . By part (b), we can write

$$f = \frac{P(x_1, \dots, x_m)}{Q(x_1, \dots, x_{m-1}) \prod_{i=1}^{\ell} (1 - \zeta_i \mathbf{x}^{\mathbf{n}_i})},$$

where the $1 - \zeta_i \mathbf{x}^{\mathbf{n}_i}$'s are all the irreducible factors of the denominator of f in which x_m appears and $Q(x_1, \dots, x_{m-1}) \in \bar{\mathbb{Q}}[x_1, \dots, x_{m-1}]$ is the product of the remaining irreducible factors.

Write $\mathbf{n}_i = (n_{i,1}, \dots, n_{i,m})$ for $1 \leq i \leq \ell$; hence $n_{i,m} > 0$ for every i . Denote $r_{i,j} = n_{i,j}/n_{i,m}$ for $1 \leq i \leq \ell$ and $1 \leq j \leq m - 1$. Consider the change of variables $x_m = y_m$, $x_{m-1} = y_{m-1}$, and

$$x_j = y_j (y_{j+1} \dots y_{m-1})^{u_j}$$

for $1 \leq j \leq m - 2$, where the u_j 's will be chosen as follows. We start with a sufficiently large $u_{m-2} \in \mathbb{N}$. Then $u_{m-3} \in \mathbb{N}$ with a sufficiently large u_{m-3}/u_{m-2} , and so on until $u_1 \in \mathbb{N}$ with a sufficiently large u_1/u_2 such that the following

holds. First we consider the formal monomials $M_i = x_1^{r_{i,1}} \dots x_{m-1}^{r_{i,m-1}}$ with rational exponents for $1 \leq i \leq \ell$. Then after the change of variables into the y_j 's, each M_i becomes a formal monomial in the y_j 's denoted by $y_1^{e_{i,1}} \dots y_{m-1}^{e_{i,m-1}}$ for $1 \leq i \leq \ell$. With our choice of the u_1, \dots, u_{m-2} , we have the following: for $1 \leq i, j \leq \ell$, if

$$(r_{i,1}, \dots, r_{i,m-1}) \geq (r_{j,1}, \dots, r_{j,m-1})$$

with respect to the lexicographic ordering on \mathbb{Q}^{m-1} induced by the usual ordering \geq on \mathbb{Q} , then

$$e_{i,1} \geq e_{j,1}, \dots, e_{i,m-1} \geq e_{j,m-1}.$$

The power series obtained from f after the change of variables into the y_j 's satisfies property \mathcal{P} and its coefficients belong to a finite set if and only if the a_n 's do so.

Therefore, after a change of variables of the above form if necessary, we may assume that for $1 \leq i \neq j \leq \ell$ either $r_{i,k} \geq r_{j,k}$ for every $k \in \{1, \dots, m-1\}$ or $r_{i,k} \leq r_{j,k}$ for every $k \in \{1, \dots, m-1\}$. Moreover, by considering $f(x_1^L, \dots, x_{m-1}^L, x_m)$ where $L := \text{lcm}(n_{1,m}, \dots, n_{\ell,m})$, from now on we may assume that each $r_{i,k} \in \mathbb{N}_0$ for every $1 \leq i \leq \ell$ and $1 \leq k \leq m-1$.

Let $R(x_m) = \prod_{i=1}^{\ell} (1 - \zeta_i \mathbf{x}^{\mathbf{n}_i})$ be regarded as a polynomial in x_m with coefficients in $\mathbb{Q}[x_1, \dots, x_{m-1}]$, and we have $D := \deg(R) = n_{1,m} + \dots + n_{\ell,m}$. Then we can write

$$f(x) = \frac{P(x_1, \dots, x_m)}{Q(x_1, \dots, x_{m-1}) \prod_{i=1}^{\ell} (1 - \zeta_i \mathbf{x}^{\mathbf{n}_i})} = \sum_{N \geq 0} g_N(x_1, \dots, x_{m-1}) x_m^N.$$

Each g_N is a power series in the variables x_1, \dots, x_{m-1} and satisfies property \mathcal{P} , hence each g_N is a rational function and its denominator has the special form given in part (b). We also have that the coefficients of each g_N belong to a finite set by the induction hypothesis. Such a finite set depends a priori on N , and our goal is to prove that there is a common finite set containing the coefficients of all the g_N 's.

Observe that there is N_1 such that the sequence $(g_N)_{N \geq N_1}$ satisfies a linear recurrence relation whose characteristic polynomial is $x_m^D R(1/x_m)$. Since $1 - \zeta_i \mathbf{x}^{\mathbf{n}_i}$ is irreducible for every i , we have that \mathbf{n}_i is not a nontrivial integral multiple of a vector in \mathbb{N}_0^m . In particular, $\mathbf{n}_i = \mathbf{n}_j$ if and only if $(r_{i,1}, \dots, r_{i,m-1}) = (r_{j,1}, \dots, r_{j,m-1})$. Moreover, for $i \neq j$, since $1 - \zeta_i \mathbf{x}^{\mathbf{n}_i}$ and $1 - \zeta_j \mathbf{x}^{\mathbf{n}_j}$ are distinct, if $\mathbf{n}_i = \mathbf{n}_j$, then we obviously have that $n_{i,m} = n_{j,m}$ and $\zeta_i \neq \zeta_j$. Therefore we have exactly D distinct characteristic roots denoted $\gamma_1, \dots, \gamma_D$, each of which has the form

$$\zeta_i^{1/n_{i,m}} M_i = \zeta_i^{1/n_{i,m}} x_1^{r_{i,1}} \dots x_{m-1}^{r_{i,m-1}}$$

for $1 \leq i \leq \ell$ and where each $\zeta_i^{1/n_{i,m}}$ denotes one of the $n_{i,m}$ th roots of ζ_i . The difference of two different characteristic roots is either a constant multiple of some M_i or has the form

$$(12) \quad \zeta_i^{1/n_{i,m}} x_1^{r_{i,1}} \dots x_{m-1}^{r_{i,m-1}} - \zeta_j^{1/n_{j,m}} x_1^{r_{j,1}} \dots x_{m-1}^{r_{j,m-1}}$$

with $(r_{i,1}, \dots, r_{i,m-1}) \neq (r_{j,1}, \dots, r_{j,m-1})$. Since we have that either $r_{i,k} \geq r_{j,k}$ for every k or $r_{i,k} \leq r_{j,k}$ for every k , the form (12) has the form

$$(13) \quad \xi_1 P_1 (1 - \xi_2 P_2),$$

where ξ_1 and ξ_2 are roots of unity, and P_1 and P_2 are monomials in x_1, \dots, x_{m-1} .

By the theory of linear recurrence sequences, we have

$$g_N = \sum_{i=1}^D s_i \gamma_i^N$$

for $N \geq N_1$, where the s_i 's are the unique solution of the system of linear equations

$$s_1 \gamma_1^{N_1+j} + \dots + s_D \gamma_D^{N_1+j} = g_{N_1+j} \text{ for } j \in \{0, \dots, D-1\}.$$

The determinant of the matrix $(\gamma_i^{N_1+j})_{1 \leq i \leq D, 0 \leq j \leq D-1}$ is in $\mathbb{Q}[x_1, \dots, x_{m-1}]$ and is equal to the product of a root of unity, a monomial, and polynomials of the form $(1 - \zeta M)$, where ζ is a root of unity and M is a monomial in x_1, \dots, x_{m-1} . Hence by Cramer's rule and the properties of the g_{N_1+j} for $0 \leq j \leq D-1$, each s_i is a rational function whose denominator is the product of a monomial and polynomials of the form $(1 - \zeta M)$ as above. Replacing f by its product with an appropriate monomial in x_1, \dots, x_{m-1} to cancel out the monomials in the denominators of the s_i 's, we may assume that the denominator of each s_i is the product of polynomials of the form $1 - \zeta M$.

Let $\ell' \leq \ell$ denote the number of distinct tuples among the tuples $(r_{i,1}, \dots, r_{i,m-1})$ for $1 \leq i \leq \ell$. By relabelling those tuples, we may assume that $(r_{i,1}, \dots, r_{i,m-1})$ for $1 \leq i \leq \ell'$ are all the distinct tuples and $r_{i,k} \leq r_{j,k}$ for every $k \in \{1, \dots, m-1\}$ and $i \leq j$. Let N_2 be the lcm of the orders of the roots of unity $\zeta_i^{1/n_{i,m}}$ for $1 \leq i \leq \ell$. For $0 \leq \tau \leq N_2 - 1$, we restrict to the arithmetic progression $\{NN_2 + \tau : N \geq 0\}$ and get

$$g_{NN_2+\tau} = t_{\tau,1} M_1^{NN_2+\tau} + \dots + t_{\tau,\ell'} M_{\ell'}^{NN_2+\tau}$$

for all N such that $NN_2 + \tau \geq N_1$, where each $t_{\tau,k}$ is a rational function in x_1, \dots, x_{m-1} whose denominator is the product of polynomials of the form $1 - \zeta M$ as above. Since the denominator of each $t_{\tau,k}$ has the mentioned form, it can be expressed as a power series in x_1, \dots, x_{m-1} .

Fix a $\tau \in \{0, \dots, N_2 - 1\}$. Let $(e_1, \dots, e_{m-1}) \in \mathbb{N}_0^{m-1}$, and let c be the coefficient of $x_1^{e_1} \dots x_{m-1}^{e_{m-1}}$ in $t_{\tau,1}$. We choose N so that $NN_2 + \tau \geq N_1$ and

$$e_1 + \dots + e_{m-1} + (NN_2 + \tau) \deg(M_1) < (NN_2 + \tau) \deg(M_2).$$

More specifically, let

$$N = \max \left(\left\lceil \frac{e_1 + \dots + e_{m-1}}{N_2} \right\rceil, \left\lceil \frac{N_1}{N_2} \right\rceil \right) + 1;$$

and we have that c is the coefficient of $x_1^{e_1} \dots x_{m-1}^{e_{m-1}} (x_1^{r_{1,1}} \dots x_{m-1}^{r_{1,m-1}})^{NN_2+\tau}$ in $g_{NN_2+\tau}$ which is also the coefficient of $x_1^{e_1} \dots x_{m-1}^{e_{m-1}} (x_1^{r_{1,1}} \dots x_{m-1}^{r_{1,m-1}})^{NN_2+\tau} x_m^{NN_2+\tau}$ in f . Since f satisfies \mathcal{P} , this implies that $h(c) = o(\log(e_1 + \dots + e_{m-1}))$, and hence $t_{\tau,1}$ satisfies property \mathcal{P} as well. Having proved that $t_{\tau,1}, \dots, t_{\tau,i}$ satisfy property \mathcal{P} , we use the equation

$$g_{NN_2+\tau} - t_{\tau,1} M_1^{NN_2+\tau} - \dots - t_{\tau,i} M_i^{NN_2+\tau} = t_{\tau,i+1} M_{i+1}^{NN_2+\tau} + \dots + t_{\tau,\ell'} M_{\ell'}^{NN_2+\tau}$$

and similar arguments to conclude that $t_{\tau,i+1}$ satisfies property \mathcal{P} .

In conclusion, we have that $t_{\tau,i}$ satisfies \mathcal{P} for every $\tau \in \{0, \dots, N_2 - 1\}$ and $i \in \{1, \dots, \ell'\}$. By the induction hypothesis, the coefficients of all those $t_{\tau,i}$'s belong to a finite set. Hence there is a finite set containing the coefficients of g_N for $N \geq N_1$. Since the coefficients of each g_N for $N < N_1$ also contains in a finite set, we finish the proof.

3.4. Proof of Corollary 1.4. We prove Corollary 1.4 using standard specialization arguments. This gives another proof of Theorem 1.2 in addition to the combinatorial method of Bell–Chen.

Let $f(\mathbf{x}) \in K[[\mathbf{x}]]$ be D-finite, and assume that the coefficients of f belong to a finite set. By Lemma 3.2, we may assume $K = \bar{K}$, and for each $i = 1, \dots, m$, f satisfies a linear partial differential equation as in the statement of this lemma. Let R be the $\bar{\mathbb{Q}}$ -subalgebra of K generated by the coefficients of f and the $A_{i,j}$'s, and let V be the affine algebraic variety with coordinate ring R . For every point $\zeta \in V(\bar{\mathbb{Q}})$, let $A_{i,j,\zeta}$ and f_ζ denote the corresponding specialization in $\bar{\mathbb{Q}}[[\mathbf{x}]]$. We will consider ζ outside the proper Zariski closed subset defined by $A_{1,d_1} \dots A_{m,d_m} = 0$ so that the specializations of the given differential equations remain nontrivial.

By Noether normalization, there exist $y_1, \dots, y_s \in R$ algebraically independent over $\bar{\mathbb{Q}}$ such that R is finite over $\bar{\mathbb{Q}}[y_1, \dots, y_s]$, and this gives a finite surjective morphism $\pi : V \rightarrow \mathbb{A}^s$. The set of points $(\alpha_1, \dots, \alpha_s) \in \bar{\mathbb{Q}}^s$, where each α_i is a root of unity is Zariski dense in \mathbb{A}^s . Each of the coefficients of f and the $A_{i,j}$'s is a zero of a monic polynomial with coefficients in $\bar{\mathbb{Q}}[y_1, \dots, y_s]$. Hence there is a positive constant M depending only on R and a Zariski dense set of $\zeta \in V(\bar{\mathbb{Q}})$ such that the following holds. For each $i = 1, \dots, m$, f_ζ satisfies the equation

$$\left(A_{i,d_i,\zeta} \left(\frac{\partial}{\partial x_i} \right)^{d_i} + \dots + A_{i,1,\zeta} \frac{\partial}{\partial x_i} + A_{i,0,\zeta}(\mathbf{x}) \right) f_\zeta = 0$$

with $A_{i,d_i,\zeta} \neq 0$, and the heights of the coefficients of f_ζ and the $A_{i,j,\zeta}$ are bounded above by M . By Theorem 3.3, $A_{1,d_1,\zeta} \dots A_{m,d_m,\zeta} f_\zeta$ is a polynomial and its total degree is bounded independently of ζ . Since this holds for every ζ in a Zariski dense subset of $V(\bar{\mathbb{Q}})$, we have that $A_{1,d_1} \dots A_{m,d_m} f$ is a polynomial, and this finishes the proof that f is rational.

Similarly, by Theorem 1.3, we have that for a Zariski dense set of points $\zeta \in V(\bar{\mathbb{Q}})$, the denominator of f_ζ has the special form specified in part (b) of Theorem 1.3. Therefore the denominator of f has such a special form as well.

4. PROOF OF THEOREM 1.5

Let $d, k, \alpha_1, \dots, \alpha_k, K, (a_n)_{n \geq 0}$ be as in the statement of Theorem 1.5. Assume that $f(x) = \sum_{n \geq 0} a_n x^n$ is D-finite. By Lemma 3.2, we have that f satisfies the equation

$$(14) \quad \left(P_D(x) \left(\frac{\partial}{\partial x} \right)^D + \dots + P_1(x) \frac{\partial}{\partial x} + P_0(x) \right) f(x) = 0,$$

where $P_j(x) \in \bar{\mathbb{Q}}[x]$ for every $0 \leq j \leq D$ and $P_D(x) \neq 0$. If $D = 0$, there is nothing to prove, so we may assume $D > 0$. For $0 \leq j \leq D$, let S_j be the support of $P_j(x)$, let $B_j(x) \in \mathbb{Z}[x]$ be as in Section 3.1, and write $P_j(x) = \sum_{n \in S_j} p_{j,n} x^n$. As in the proof of Proposition 3.5, for every $r \in \mathbb{N}_0$, the coefficient of x^r in the left-hand side of (14) is

$$(15) \quad \sum_{j=0}^D \sum_{n \in S_j} p_{j,n} B_j(r + j - n) a_{r+j-n} = 0$$

with the convention that $a_n = 0$ if $n \in \mathbb{Z} \setminus \mathbb{N}$. We now assume that r is sufficiently large so that $r + j - n \geq 0$ for every $j \in \{0, \dots, d\}$ and $n \in S_j$. Then we apply the

given formula for a_{r+j-n} to (15) to obtain

$$(16) \quad \sum_{j=0}^D \sum_{n \in S_j} p_{j,n} B_j(r+j-n) \sum_{s=1}^k \sum_{t=0}^d c_{r+j-n,s,t} (r+j-n)^t \alpha_s^{r+j-n} = 0.$$

This equation can be written as $\sum_{s=1}^k \beta_{r,s} \alpha_s^r = 0$, where

$$(17) \quad \beta_{r,s} := \sum_{j=0}^D \sum_{n \in S_j} \sum_{t=0}^d p_{j,n} B_j(r+j-n) c_{r+j-n,s,t} \frac{(r+j-n)^t}{\alpha_s^{n-j}}.$$

By Proposition 2.1 and the given properties of the $c_{r+j-n,s,t}$'s, we have

$$(18) \quad \lim_{r \rightarrow \infty} \frac{h(\beta_{r,s})}{(\log r)^{D+d+1}} = 0$$

for every $s = 1, \dots, k$. Consider the equivalence relation \sim on $\{1, \dots, k\}$ defined by $i \sim j$ if and only if α_i/α_j is a root of unity. Assume there are γ equivalence classes and let s_1, \dots, s_γ be the representatives. The equation $\sum_{s=1}^k \beta_{r,s} \alpha_s^r = 0$ can be rewritten as

$$(19) \quad \sum_{\ell=1}^{\gamma} \left(\sum_{i \sim s_\ell} \beta_{r,i} \frac{\alpha_i^r}{\alpha_{s_\ell}^r} \right) \alpha_{s_\ell}^r = 0.$$

Note that the tuple $(\alpha_{s_\ell})_\ell$ is nondegenerate and the height of each coefficient $\sum_{i \sim s_\ell} \beta_{r,i} \frac{\alpha_i^r}{\alpha_{s_\ell}^r}$ is $o((\log r)^{D+d+1})$. By Proposition 2.2, we must have that

$$(20) \quad \sum_{i \sim s_\ell} \beta_{r,i} \frac{\alpha_i^r}{\alpha_{s_\ell}^r} = 0$$

for all $\ell = 1, \dots, \gamma$ for all sufficiently large r . We now apply the same trick as in the proof of Proposition 3.5, where each $\beta_{r,i}$ is a linear combination of $1, r, \dots, r^{d+D}$ in which the height of each coefficient is $o(\log r)$. Arguing as in the proof of Proposition 3.5, (20) implies

$$(21) \quad \sum_{i \sim s_\ell} \sum_{n \in S_D} p_{D,n} \frac{c_{r+D-n,i,d} \alpha_i^r}{\alpha_i^{n-D} \alpha_{s_\ell}^r} = 0$$

for all $\ell = 1, \dots, \gamma$ for all sufficiently large r . By multiplying both sides of (21) by $\alpha_{s_\ell}^r$ and summing over all $\ell = 1, \dots, \gamma$, we obtain

$$(22) \quad \sum_{s=1}^k \sum_{n \in S_D} p_{D,n} c_{r+D-n,s,d} \alpha_s^{r+D-n} = 0$$

for all sufficiently large r . Put $g(x) = \sum_{n \geq 0} (\sum_{s=1}^k c_{n,s,d} \alpha_s^n) x^n$ and observe that the left-hand side of (22) is exactly the coefficient of x^{r+D} in $P_D g$. Therefore g is a rational function. Consider the operator $\bar{Q}[[x]] \rightarrow \bar{Q}[[x]]$ given by

$$F \mapsto x \frac{\partial F}{\partial x}.$$

By applying this operator to g for d many times, we can show that

$$\tilde{g}(x) = \sum_{n \geq 0} \left(\sum_{s=1}^k c_{n,s,d} n^d \alpha_s^n \right) x^n$$

is a rational function. This yields two things. First, Theorem 1.5 holds when $d = 0$. Second, the power series

$$f(x) - \tilde{g}(x) = \sum_{n \geq 0} \left(\sum_{s=1}^k \sum_{t=0}^{d-1} c_{n,s,t} n^t \alpha_s^n \right) x^n$$

is D-finite so that we can finish the proof by using induction.

Remark 4.1. From the above proof, we have that if f is D-finite, then for each $t \in \{0, \dots, d\}$, the power series $\sum_{n \geq 0} \left(\sum_{s=1}^k c_{n,s,t} \alpha_s^n \right) x^n$ is a rational function.

5. PROOF OF THEOREM 1.6

Since $h(a_n) \leq h(a_0, \dots, a_n)$, it remains to show that

$$\limsup_{n \rightarrow \infty} \frac{h(a_0, \dots, a_n)}{n \log n} < \infty.$$

Let K be a number field containing the coefficients of f . As before, we have that the coefficients (a_n) eventually satisfy a linear recurrence relation with polynomial coefficients. In other words, there exist $M \in \mathbb{N}_0$ and polynomials $R_0(t), \dots, R_M(t) \in \mathbb{Q}[t]$ with $R_M \neq 0$ such that

$$(23) \quad R_M(n)a_{n+M} + \dots + R_0(n)a_n = 0$$

for all sufficiently large n .

Let $v \in M_K$. If v is nonarchimedean, we have

$$|a_{n+M}|_v \leq \max_{0 \leq i \leq M-1} \left| \frac{R_i(n)}{R_M(n)} a_{n+i} \right|_v,$$

which implies

$$\max_{0 \leq i \leq n+M} \log^+ |a_i|_v \leq \max_{0 \leq i \leq n+M-1} \log^+ |a_i|_v + \max_{0 \leq i \leq M-1} \log^+ \left| \frac{R_i(n)}{R_M(n)} \right|_v.$$

If v is archimedean, we have:

$$|a_{n+M}|_v \leq |M|_v \max_{0 \leq i \leq M-1} \left| \frac{R_i(n)}{R_M(n)} a_{n+i} \right|_v,$$

which implies

$$\max_{0 \leq i \leq n+M} \log^+ |a_i|_v \leq \log |M|_v + \max_{0 \leq i \leq n+M-1} \log^+ |a_i|_v + \max_{0 \leq i \leq M-1} \log^+ \left| \frac{R_i(n)}{R_M(n)} \right|_v.$$

Summing over all v , we have

$$h(a_0, \dots, a_{n+M}) - h(a_0, \dots, a_{n+M-1}) = O(\log n),$$

and this yields the desired result.

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