

QUADRATIC FUNCTIONALS AND PARTIAL REGULARITY

BY

MARIANO GIAQUINTA

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QUADRATIC FUNCTIONALS AND PARTIAL REGULARITY^(*)

by

Mariano Giaquinta

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QUADRATIC FUNCTIONALS AND PARTIAL REGULARITY

In the following I shall describe some results concerning the partial regularity of minimizers of the simple quadratic variational integral

$$(1) \quad F[u; \Omega] = \int_{\Omega} A_{ij}^{\alpha\beta}(x, u) D_{\alpha} u^i D_{\beta} u^j \, dx$$

I shall also discuss some geometric applications, and mainly I shall outline some methods used to prove these results.

For more complete information on the regularity theory for minimizers of variational integrals and for solutions of related nonlinear elliptic systems I refer to M. Giaquinta [5].

Here I shall restrict myself to the functional (1) since it already shows some of the difficulties and permits me to present in a simple situation some of the methods used.

In (1) I used the summation convention and standard notation: Ω is a bounded open set in $\mathbb{R}^n$, u is a map from Ω into $\mathbb{R}^N$, $N \geq 1$; $Du = \{D_{\alpha} u^i\}$, $\alpha = 1, \dots, n$, $i = 1, \dots, N$ stands for the gradient matrix; the coefficients $A_{ij}^{\alpha\beta}(x, u)$ are assumed to be bounded

$$(2) \quad \left| A_{ij}^{\alpha\beta}(x, u) \right| \leq L \quad \forall x \in \Omega, \quad \forall u \in \mathbb{R}^N$$

and to satisfy the very strong ellipticity condition

$$(3) \quad A_{ij}^{\alpha\beta}(x, u) \xi_i^{\alpha} \xi_j^{\beta} \geq |\xi|^2 \quad \forall \xi \in \mathbb{R}^N.$$

A minimizer for (1) is a function $u \in W_{loc}^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$F[u; \text{supp } \phi] \leq F[u + \phi; \text{supp } \phi]$$

for all $\phi \in W^{1,2}(\Omega, \mathbb{R}^N)$ with $\text{supp } \phi \subset \subset \Omega$.

As is well known, e. g., see [5, chap. 11], in the vector valued case $N > 1$, the minimizers of (1) are in general nonregular and, in contrast to the scalar case, if $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x)$ are only measurable and bounded, the minimizers may have singularities even in a dense set. Accordingly we shall assume that the coefficients are at least continuous. Under this assumption every minimizer happens to be partial regular, which means Hölder-continuous except on a closed set of "small" Hausdorff dimension. This is the first result I want to describe.

Before doing that, let me just remark that the functional (1) is not differentiable. In fact, we are not assuming the existence of the derivatives of the coefficients. But even if coefficients $A_{ij}^{\alpha\beta}$ were of class C^1 with derivatives $A_{ij,u}^{\alpha\beta}$ uniformly bounded, the Euler-Lagrange equations

$$\int_{\Omega} \left[A_{ij}^{\alpha\beta}(x, u) D_{\alpha}^i u D_{\beta}^j \phi + A_{ij,u}^{\alpha\beta}(x, u) D_{\alpha}^i u D_{\beta}^j u^{\ell} \phi^{\ell} \right] dx = 0$$

would hold only for test functions ϕ in $W_0^1 \cap L^{\infty}(\Omega, \mathbb{R}^N)$; so it seems of no use unless we know a priori that u is bounded, which is in general not the case.

Therefore, in order to study the regularity of a minimizer u we cannot use the Euler-Lagrange equations. Actually this is not only a technical difficulty, it reflects a real difference, even in terms of regularity between minimizers and stationary points, see, e. g., [5, 6, 7].

1. THEOREM. Suppose that the coefficients $A_{ij}^{\alpha\beta}(x, u)$ are continuous (or rather, for the sake of simplicity, uniformly continuous) and satisfy conditions (2) and (3). Let $u \in W_{loc}^{1,2}(\Omega, \mathbb{R}^N)$ be a minimizer. Then there exists an open set $\Omega_0 \subset \Omega$ such that u is Hölder-continuous with any exponent $\mu \in (0, 1)$ in Ω_0 . Moreover, there exist constants $\varepsilon_0, \varepsilon_1$ depending on L , on the modulus of continuity of the A 's and on μ , such that

$$\Omega - \Omega_0 \subset \left\{ x \in \Omega : \liminf_{R \rightarrow 0^+} \frac{1}{R^{n-2}} \int_{B_R(x)} |Du|^2 dy > \varepsilon_0^2 \right\}$$

or equivalently

$$\Omega - \Omega_0 \subset \left\{ x \in \Omega : \liminf_{R \rightarrow 0^+} \int_{B_R(x)} |u - u_{x,R}|^2 dy > \epsilon_1^2 \right\}^{(*)}$$

Finally the Hausdorff dimension of the singular set $\Omega - \Omega_0$ is strictly less than $n - 2$.

I shall split the proof into four steps, see [6] .

Step 1. The following Caccioppoli's inequality, see [5, 6] holds for u : for every $B_R(x_0) \subset \subset \Omega$ we have:

$$(4) \quad \int_{B_{R/2}(x_0)} |Du|^2 dx \leq \frac{c_1}{R^2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx$$

Let $0 < t < s < R < \text{dist}(x_0, \partial\Omega)$ and let η be a function in C^∞ with $\text{supp } \eta \subset B_s(x_0)$, $\eta = 1$ on B_t , $|D\eta| \leq 2(s-t)^{-1}$. If $v = u - \eta(u - u_R)$, using (2), (3) and the minimality of u we deduce, since $\eta \equiv 1$ in B_s :

$$\begin{aligned} \int_{B_s} |Du|^2 dx &\leq \int_{B_t} |Du|^2 dx \leq \int_{B_t} A(x, u) Du Du dx \leq \int_{B_t} A(x, v) Dv Dv dx \leq \\ &\leq L \int_{B_t} |Dv|^2 dx \leq c(L) \int_{B_t \setminus B_s} |Du|^2 dx + \frac{c(L)}{(s-t)^2} \int_{B_R} |u - u_R|^2 dx . \end{aligned}$$

Adding to both sides $c(L)$ times the left-hand side we get

$$(5) \quad \int_{B_s} |Du|^2 dx \leq \frac{c(L)}{c(L)+1} \int_{B_t} |Du|^2 dx + \frac{c(L)}{(s-t)^2} \int_{B_R} |u - u_R|^2 dx$$

(*) We adopt the standard notation

$$\int_E f dx = \frac{1}{|E|} \int_E f dx , \quad u_{x,R} = \int_{B_R(x)} u(y) dy , \quad B_R(x) = \{y \in \mathbb{R}^n : |x-y| < R\}$$

for all $0 < s < t < R$. A simple argument, see lemma 1.1 of [6], now permits us to deduce from (5) (since $\frac{c(L)}{c(L)+1} < 1$) that there is a constant c depending on L such that for every $\rho < R$

$$\int_{B_\rho} |Du|^2 dx \leq \frac{c}{(R-\rho)^2} \int_{B_R} |u - u_R|^2 dx$$

which implies (4) choosing $\rho = R/2$.

Step 2. Higher integrability of the gradient. From (4) and the Sobolev-Poincaré inequality we now get

$$(6) \quad \int_{B_{R/2}} |Du|^2 dx \leq \frac{c}{R^2} \int_{B_R} |u - u_R|^2 dx \leq \frac{c(L, n)}{R^2} \left(\int_{B_R} |Du|^{2_*} dx \right)^{2/2_*}$$

where

$$2_* = \frac{2n}{n+2} < 2$$

is the exponent whose Sobolev exponent $(2_*)^*$ is equal to 2. Dividing by R^n , inequality (6) becomes

$$\left(\int_{B_{R/2}} |Du|^2 dx \right)^{1/2} \leq c(L, n) \left(\int_{B_R} |Du|^{2_*} dx \right)^{1/2_*} \quad \forall B_R \subset \subset \Omega$$

This is a reverse Hölder inequality with increasing supports.

A result of Gehring's type [4], proved in [10] (see also [5]), permits us to deduce the existence of an exponent $p > 2$ such that for any $B_R \subset \subset \Omega$ we have

$$(7) \quad \left(\int_{B_{R/2}} |Du|^p dx \right)^{1/p} \leq c(L, n, p) \left(\int_{B_R} |Du|^2 dx \right)^{1/2}$$

and even

$$\left(\int_{B_{R/2}} |Du|^p dx \right)^{1/p} \leq c(n, L, p, q) \left(\int_{B_R} |Du|^q dx \right)^{1/q}$$

for any $q > 0$.

Step 3. A comparison function. For $B_R(x_0) \subset \subset \Omega$ consider the Dirichlet problem

$$\begin{aligned} \int_{B_{R/2}} A(x_0, u_{x_0, R}) Dv Dv dx &\rightarrow \min \\ v - u &\in W_0^1(B_{R/2}, \mathbb{R}^N) \end{aligned}$$

and let v be the solution. v , as well as any derivative of v , is a solution of an elliptic system with constant coefficients, so Caccioppoli's inequality implies the standard estimate

$$\|Dv\|_{W^k(B_{R/4}, \mathbb{R}^{nN})} \leq c(L, k, R) \|Dv\|_{L^2(B_{R/2}, \mathbb{R}^{nN})}$$

Therefore choosing k sufficiently large and using also Sobolev's theorem, we have for any $\rho < R/4$

$$\begin{aligned} \int_{B_\rho} |Dv|^2 dx &\leq c(n) \rho^n \sup_{B_{R/4}} |Dv|^2 \leq c(n, k, L, R) \|Dv\|_{W^k(B_{R/4}, \mathbb{R}^{nN})}^2 \leq \\ &\leq c(n, k, L, R) \rho^n \int_{B_{R/2}} |Dv|^2 dx \end{aligned}$$

A simple rescaling argument then gives the inequality

$$(8) \quad \int_{B_\rho(x_0)} |Dv|^2 dx \leq c(n, L) (\rho/R)^n \int_{B_{R/2}(x_0)} |Dv|^2 dx$$

for any $\rho < R/4$ and, since it is trivial for $R/4 \leq \rho \leq R/2$, for any $\rho < R/2$.

Inequality (8) is a classical one due to Morrey and Campanato, see, e.g., [5].

It extends the well-known monotonicity property of subharmonic functions to solutions of elliptic systems with constant coefficients: if H is harmonic, then $|DH|^2$ is subharmonic and the average of $|DH|^2$ is nondecreasing.

Step 4. The comparison argument. The result now follows by comparing on each ball $B_{R/2}$ the minimizer u with the minimizer v defined in Step 3. From (8) we obviously have for any $\rho < R/2$

$$(9) \quad \int_{B_\rho} |Du|^2 dx \leq c(\rho/R)^n \int_{B_{R/2}} |Du|^2 dx + c \int_{B_{R/2}} |D(u-v)|^2 dx .$$

On the other hand, setting

$$F^0(p) = A(x_0, u_{x_0, R})_{pp} ,$$

we have

$$\begin{aligned} F^0(Du) - F^0(Dv) &= F^0_{p_\alpha^i}(Dv) D_\alpha(u^i - v^i) + \\ &\quad + \int_0^1 (1-t) F^0_{p_\alpha^i p_\beta^j} (tDu + (1-t)Dv) D_\alpha(u^i - v^i) D_\beta(u^j - v^j) dt \\ &\geq F^0_{p_\alpha^i}(Dv) D_\alpha(u^i - v^i) + \frac{1}{2} |D(u-v)|^2 \end{aligned}$$

and integrating on $B_{R/2}(x)$, since v satisfies the Euler - Lagrange equation

$$\int_{B_{R/2}} F^0_{p_\alpha^i}(Dv) D_\alpha \Phi^i = 0 \quad \forall \Phi \in W_0^1(B_{R/2}(x_0), \mathbb{R}^N) ,$$

we get

$$\int_{B_{R/2}} |D(u-v)|^2 dx \leq c \int_{B_{R/2}} [A(x_0, u_{x_0, R}) Du Du - A(x_0, u_{x_0, R}) Dv Dv] dx .$$

Finally, using the minimality property of u we have

$$\begin{aligned} \int_{B_{R/2}} |D(u-v)|^2 dx &\leq c \int_{B_{R/2}} [A(x_0, u_{x_0, R}) Du Du - A(x, u) Du Du] dx + \\ &\quad + c \int_{B_{R/2}} [A(x, u) Du Du - A(x, v) Dv Dv] dx + \\ (10) \quad &\quad + c \int_{B_{R/2}} [A(x, v) Dv Dv - A(x_0, u_{x_0, R}) Dv Dv] dx \leq \\ &\leq c \int_{B_{R/2}} [\omega(|x-x_0| + |u-u_{x_0, R}|) |Du|^2 + \omega(|x-x_0| + |v-u_{x_0, R}|) |Dv|^2] dx \end{aligned}$$

where ω is the modulus of continuity of the A 's, which we can assume to be concave and bounded.

Now we estimate the right-hand side of (10) using the L^p -estimate (7) and the boundedness of ω as follows.

$$\begin{aligned} \int_{B_{R/2}} \omega(|x-x_0| + |u-u_{x_0, R}|) |Du|^2 &\leq c |B_{R/2}| \left(\int_{B_{R/2}} |Du|^p dx \right)^{2/p} \left(\int_{B_{R/2}} \omega dx \right)^{1-2/p} \leq \\ &\leq c \int_{B_R} |Du|^2 dx \int_{B_R} \omega dx^{1-2/p} . \end{aligned}$$

Since ω is concave we have

$$\int_{B_R} \omega dx \leq \omega \left(R + \int_{B_R} |u - u_R| dx \right) \leq \omega \left(c \left(R^2 + R^{2-n} \int_{B_R} |Du|^2 \right)^{1/2} \right)$$

where we have used Jensen and Poincaré inequalities, so we conclude that

$$(11) \quad \int_{B_{R/2}} \omega (|x - x_0| + |u - u_{x_0, R}|) |Du|^2 dx \\ \leq c \eta \left(R^2 + R^{2-n} \int_{B_R} |Du|^2 dx \right) \int_{B_R} |Du|^2 dx$$

where η is a continuous function with $\eta(0) = 0$.

In a similar way we can estimate the second term on the right-hand side of (10), taking into account that from the L^p -theory, or as in Step 2, we can deduce that

$$\left(\int_{B_{R/2}} |Dv|^p dx \right)^{1/p} \leq c \left(\int_{B_R} |Du|^2 dx \right)^{1/2}.$$

From (9), (10) and (11) we can then conclude that for any $\rho < R/2$, and therefore for any $\rho < R$, we have

$$(12) \quad \int_{B_\rho(x_0)} |Du|^2 dx \leq c \left[(\rho/R)^n + \eta \left(R^2 + R^{2-n} \int_{B_R(x_0)} |Du|^2 \right) \right] \int_{B_R(x_0)} |Du|^2 dx.$$

Set now

$$\Phi(x_0, R) = R^{2-n} \int_{B_R(x_0)} |Du|^2 dx$$

and for $\tau \in (0, 1)$ set $\rho = \tau R$; then (12) can be rewritten as

$$\Phi(x_0, \tau R) \leq c \tau^{2\mu} \tau^{2-2\mu} \left[1 + \tau^{-n} \eta \left(R^2 + \Phi(x_0, R) \right) \right] \Phi(x_0, R)$$

with $\mu \in (0, 1)$. Choose τ in such a way that $2c\tau^{2-2\mu} = 1$; since we have

$$\eta(R^{2+\Phi(x_0, R)}) < \tau^n$$

provided R is less than some R_0 and, for some ε_0 ,

$$\Phi(x_0, R) < \varepsilon_0^2$$

we then get

$$\Phi(x_0, \tau R) \leq \tau^{2\mu} \Phi(x_0, R) \quad .$$

Therefore, by iteration we conclude that

$$\Phi(x_0, \tau^k R) \leq \tau^{2\mu k} \Phi(x_0, R) \quad ,$$

i. e. ,

$$(13) \quad \Phi(x_0, \rho) \leq c(\rho/R)^{2\mu} \Phi(x_0, R) \quad \forall \rho < R$$

provided $R < \inf(R_0, \text{dist}(x_0, \partial\Omega))$ and

$$\Phi(x_0, R) < \varepsilon_0^2$$

Since $\Phi(x_0, R)$ is a continuous function of x_0 , if $\Phi(x_0, R) < \varepsilon_0^2$ for a point x_0 , then there exists a ball $B(x_0, r)$ such that $\Phi(x, R) < \varepsilon_0^2$ for all $x \in B(x_0, r)$; hence (13) holds for every point in $B(x_0, r)$. The result then follows from a classical result by Morrey on the growth of the Dirichlet integral, see, e. g. , [18, 5].

Notice that

$$\Omega_0 = \left\{ x \in \Omega : \liminf_{R \rightarrow 0^+} R^{2-n} \int_{B_R(x)} |Du(y)|^2 dx = 0 \right\}$$

so it does not depend on μ , and, moreover, it is nonempty. Then the proof of the first part of the theorem is concluded, if we notice that, because of Poincaré and Caccioppoli inequalities

$$\liminf_{R \rightarrow 0^+} R^{2-n} \int_{B_R(x_0)} |Du|^2 dx > \varepsilon_0^2$$

if and only if

$$\liminf_{R \rightarrow 0^+} \int_{B_R} |u - u_{x_0, R}|^2 dx > \varepsilon_1^2$$

for some ε_1 depending on ε_0 .

Finally, note that

$$R^{2-n} \int_{B_R(x_0)} |Du|^2 dx \leq c \left(R^{p-n} \int_{B_R(x_0)} |Du|^p dx \right)^{2/p}$$

so the last statement of the theorem follows from the measure theoretical result, see, e. g., [5], that for any L^1_{loc} function f the α -dimensional Hausdorff measure, $\alpha < n$, of the set

$$\left\{ x \in \Omega : \limsup_{\rho \rightarrow 0^+} \rho^{-\alpha} \int_{B_\rho(x)} |f(y)| dy > 0 \right\}$$

is zero.

2. Now I want to present a different proof of the theorem in section 1 based on some ideas of E. De Giorgi and J. F. Almgren used for proving the regularity of parametric minimal surfaces. Roughly the idea is that if the quantity

$$U(x_0, R) = \int_{B_R(x_0)} |u - u_{x_0, R}|^2 dx$$

is small enough and we make a blow-up procedure, then the limit is a minimization of the quadratic approximation of (1); so it satisfies good estimates. But then the original minimizer has to satisfy similar estimates. This is expressed precisely

in the proof of the following proposition, compare [13], which, in view of the previous proof, clearly implies the theorem in section 1.

PROPOSITION. For all τ , $0 < \tau < 1$, there exist two positive constants $\varepsilon_1 = \varepsilon_1(\tau, n, N, L)$, $R_0 = R_0(\tau, n, N, L)$ such that if u is a minimizer of (1) and for some $x_0 \in \Omega$ and some $R < \inf(R_0, \text{dist}(x_0, \partial\Omega))$ we have

$$U(x_0, R) < \varepsilon_1^2$$

then

$$U(x_0, \tau R) \leq 2c\tau^2 U(x_0, R)$$

where c is the constant in (8).

Proof: Suppose the proposition were not true. Then there would exist τ , $0 < \tau < 1$, a sequence $\{x_h\} \subset \Omega$, a sequence $\varepsilon_h \rightarrow 0$, a sequence $R_h \rightarrow 0$ and a sequence of minimizers $u_{(h)}$ such that

$$U^{(h)}(x_h, R_h) = \varepsilon_h^2$$

$$U^{(h)}(x_h, \tau R_h) > 2c\tau^2 \varepsilon_h^2 .$$

Define

$$v_{(h)}(y) = \varepsilon_h^{-1} [u_{(h)}(x_h + R_h y) - u_{(h)}(x_h)] .$$

We have

(i) v_h is a minimizer of the functional

$$F^{(h)}[v; B_1] = \int_{B_1} A^{(h)}(y, v) Dv Dv \, dy$$

where

$$A^{(h)}(y, z) = A(x_h + R_h y, \varepsilon_h z + u_{(h)}(x_h))$$

$$(ii) \quad V^{(h)}(0, 1) = \int_{B_1} |V_{(h)}(y)|^2 dy = 1$$

$$(iii) \quad V^{(h)}(0, \tau) > 2c\tau^2 .$$

The meaning of the notation should be clear. Passing to a subsequence we have

$$v_{(h)} \rightharpoonup v \quad \text{weakly in } L^2(B_1(0), \mathbf{R}^N)$$

and because of Caccioppoli's inequality

$$v_{(h)} \rightarrow v \quad \text{strongly in } L^2(B_\rho(0), \mathbf{R}^N) \quad \forall \rho < 1 .$$

Moreover,

$$\varepsilon_{(h)} v_{(h)} \rightarrow 0 \quad \text{a.e. in } B_1$$

$$A_{ij}^{\alpha\beta}(x_n + R_h y, u_{(h)x_h, R_h}) \rightarrow b_{ij}^{\alpha\beta}$$

and hence, taking into account the uniform continuity of the A 's we have

$$A_{ij}^{\alpha\beta}(x_h + R_h y, \varepsilon_h v_{(h)}(y) + u_{(h)x_h, R_h}) \rightarrow b_{ij}^{\alpha\beta} \quad \text{a.e. in } B_1 .$$

We now claim that v is a minimizer for the functional

$$(14) \quad F^{(\infty)}[v; B_1] = \int_{B_1(0)} b_{ij}^{\alpha\beta} D_\alpha v^i D_\beta v^j dx$$

hence we must have (compare (8))

$$V(0, \tau) \leq c\tau^2 V(0, 1) .$$

On the other hand, from (ii) $V(0, 1) \leq 1$, and, since $v_{(h)} \rightarrow v$ strongly in $L^2(B_\tau(0), \mathbf{R}^N)$, from (iii), $V(0, \tau) \geq 2c\tau^2$. Therefore we obtain a contradiction.

It remains for us to show that v is a minimizer of the functional (14). We first show that for $R < 1$

$$(15) \quad F^{(\infty)}[v; B_R] \leq \liminf_{h \rightarrow \infty} F^{(h)}[v_{(h)}; B_R] .$$

We have

$$\begin{aligned} F^{(h)}[v_{(h)}; B_R] &= \int_{B_R} b_{ij}^{\alpha\beta} D_{\alpha} v_{(h)}^i D_{\beta} v_{(h)}^j dx + \\ &\quad + \int_{B_R} [A_{ij}^{(h)}(y, v_{(h)}(y)) - b_{ij}^{\alpha\beta}] D_{\alpha} v_{(h)}^i D_{\beta} v_{(h)}^j dy \end{aligned}$$

but from the higher integrability in Step 2 of the proof of the theorem in section 1

$$\int_{B_R} |Dv_{(h)}|^p dx \leq c(R)$$

hence

$$\begin{aligned} F^{(h)}[v_{(h)}; B_R] &\geq \int_{B_R} b_{ij}^{\alpha\beta} D_{\alpha} v_{(h)}^i D_{\beta} v_{(h)}^j dx - \\ &\quad - c(R)^{2/q} \left(\int_{B_R} |A^{(h)}(y, v_{(h)}) - b_{ij}^{\alpha\beta}|^{\frac{p}{p-2}} dy \right)^{1 - \frac{2}{p}} \end{aligned}$$

When $h \rightarrow \infty$, the last term on the right-hand side tends to zero, whereas the first term is lower semicontinuous. This proves (15).

Now let w be an arbitrary function coinciding with v outside B_R , and let $\eta \in C^1(B_1(0))$, $0 \leq \eta \leq 1$, $\eta = 0$ in B_{ρ} , $\rho < R$, and $\eta = 1$ outside B_R . The function

$$w_{(h)} = w + \eta(v_{(h)} - v)$$

coincides with $v_{(h)}$ outside B_R , and therefore

$$(16) \quad F^{(h)}[v_{(h)}; B_R] \leq F^{(h)}[w_h; B_R] \quad .$$

But

$$(17) \quad F^{(h)}[w_h; B_R] \leq \int_{B_R} A^{(h)}(x, w_{(h)}) Dw Dw + c(R) \|\eta\|_{\frac{p}{p-2}(B_R)}^p + \\ + c(R, \eta) \|v_{(h)} - v\|_{L^2(B_R, \mathbb{R}^N)}^{1 + \|v_{(h)} - v\|_{L^2(B_R, \mathbb{R}^N)}} \quad .$$

So, letting $h \rightarrow \infty$, we get from (15) - (17)

$$F^{(\infty)}[v; B_R] \leq \liminf_{h \rightarrow \infty} F^{(h)}[v_{(h)}; B_R] \leq F^{(\infty)}[w; B_R] + c(R) \|\eta\|_{L^{p/(p-2)}(B_R)}^p \quad .$$

Taking ρ close to R the last term can be made arbitrarily small, thus proving the claim.

3. THEOREM. Under the assumptions of the theorem in section 1, suppose further that the coefficients $A_{ij}^{\alpha\beta}(x, u)$ be Hölder continuous with exponent σ . Then $u \in C^{1, \sigma/(2-\sigma)}(\Omega_0, \mathbb{R}^N)$.

I refer to [5, 9] for the proof. To my knowledge it is an open question as to whether $\sigma/(2-\sigma)$ is the optimal exponent. This might be suggested by the fact that it is the optimal Hölder exponent for the derivatives of the minimizers of the simple functional

$$\int_{\Omega} (|Du|^2 + |u|^\sigma) dx$$

even in the scalar case, see [9, 19].

4. Theorems in section 1 and 3 state that in dimension $n = 2$ the singular set is empty, while in general for $n \geq 3$ it might be nonempty, see 13. So in analogy with the regularity theory of parametric minimal surfaces, it is natural to raise the question whether, in dimension $n = 3$, the singular points are at most isolated. In its generality I have no answer to this question, but we have, see [8],

THEOREM. Under the assumptions of theorem 1, suppose further that the coefficients split as

$$(18) \quad A_{ij}^{\alpha\beta}(x, u) = a^{\alpha\beta}(x) g_{ij}(u) \quad .$$

Let u be a bounded minimizer of (1). Then

- i) if $n = 3$, the singular set consists at most of isolated points;
- ii) if $n > 3$, the Hausdorff dimension of the singular set is less than or equal to $n - 3$.

I will give here the proof of i) and I refer to [8] for the proof of ii), which follows by a reduction argument similar to the one in the theory of minimal surfaces due to Federer [3]. That argument actually provides more information than ii); precisely

4.1. If in dimension $\bar{n}$ the singularities of u are at most isolated, then the Hausdorff dimension of the singular set is less than or equal to $n - \bar{n}$ for $n > \bar{n}$.

Actually, even the proof of i) uses some ideas developed in the theory of minimal surfaces, e. g., see [12].

Suppose that i) is not true. Then without loss of generality, we may suppose that u is a minimizer of (1) in $B_1(0)$ and that there is a sequence of singular points $\{x_\nu\}$ converging to the origin. Then

$$u^{(\nu)}(x) = u(R_\nu x), \quad R_\nu = 2|x_\nu| < 1$$

is a minimizer in $B_1(0)$ of

$$(19) \quad \int_{B_1(0)} A^{(\nu)}(x, u^{(\nu)}(x)) Du^{(\nu)} Du^{(\nu)} dx$$

with

$$A^{(\nu)}(x, z) = A(R_\nu x, z) \quad .$$

Since

$$|\xi|^2 \leq A^{(\nu)}(x, z) \xi \xi < L |\xi|^2$$

Caccioppoli's inequality holds for $u^{(\nu)}$: for each ball $B_r \subset B_1$

$$\int_{B_{r/2}} |Du^{(\nu)}|^2 dx \leq \frac{\gamma_1}{r^2} \int_{B_r} |u^{(\nu)} - u_r^{(\nu)}|^2 dx$$

hence, upon comparison with step 2 of the proof of the theorem in section 1, we see that there is a $p > 2$ independent of ν , such that

$$\left(\int_{B_{r/2}} |Du^{(\nu)}|^p dx \right)^{1/p} \leq \gamma_2 \left(\int_{B_r} |Du^\nu|^2 dx \right)^{1/2} \quad \forall B_r \subset \subset B_1$$

and, since u is bounded, for $R < 1$ we get

$$\int_{B_R(0)} |Du^{(\nu)}|^p dx \leq c(R),$$

where $c(R)$ is independent of ν . This implies that

$$u^{(\nu)} \longrightarrow v \quad \text{strongly in } L^2(B_R(0), \mathbb{R}^N)$$

$$\text{and} \quad Du^\nu \longrightarrow Dv \quad \text{weakly in } L^p(P_R(0), \mathbb{R}^{nN})$$

for $R < 1$.

Therefore, proceeding as in the end of the proof of the proposition in section 1, we conclude that v is a minimizer of

$$\int_{B_1} A(0, v) Dv Dv dx \quad .$$

Moreover, each $u^{(\nu)}$ is singular at 0 and at some y_ν , with $|y_\nu| = 1/2$. Passing to a subsequence, we observe that $y_\nu \rightarrow y_0$ with $|y_0| = 1/2$.

Now we claim:

4.2. The limit of singular points is a singular point,

hence v is singular at 0 and at y_0 .

4.3. v is homogeneous of degree zero,

hence v is singular at each point of the segment joining 0 with y_0 ; and this contradicts the theorem in 1 and in particular the conclusion that the singular set of v has dimension strictly less than $3 - 2 = 1$.

We now prove 4.2 and 4.3.

We notice that the $A^{(\nu)}$'s are equi-uniformly continuous with respect to ν ; hence the theorem in 1 implies that $\bar{x}$ is singular for $u^{(\nu)}$ if and only if

$$\rho^{-n} \int_{B_\rho(\bar{x})} |u^{(\nu)} - u_{\bar{x}, \rho}^{(\nu)}|^2 dx \geq \epsilon_1^2$$

ϵ_1 being independent of ν . Suppose that the limit x_0 of a sequence of singular points x_ν for $u^{(\nu)}$ is regular for v , then

$$\lim_{\nu \rightarrow \infty} \rho^{-n} \int_{B_\rho(x_\nu)} |u^{(\nu)} - u_{x_\nu, \rho}^{(\nu)}|^2 dx = \rho^{-n} \int_{B_\rho(x_0)} |v - v_{x_0, \rho}|^2 dx < \epsilon_1$$

for some ρ with $B_\rho \subset B_R \subset B_1(0)$, and therefore x_ν would be regular for $u^{(\nu)}$ provided ν is sufficiently large. This proves 4.2.

Let us prove 4.3 assuming, for the sake of simplicity, that the matrix $a^{\alpha\beta}$ is constant and therefore, by a change of independent variables, it is the identity matrix. We refer to [8] for the variable case. 4.3 is a consequence of the following monotonicity lemma whose relevance is well known in the theory of minimal surfaces where a formally similar lemma holds and which is the only place where we use the splitting assumption (18):

4.4. Let u be a minimizer of (1) in $B(0, 1)$ and let (18) hold. Then for every p and R , $0 < \rho < R < 1$ we have

$$(20) \quad \int_{\partial B_1} |u(Rx) - u(\rho x)|^2 d\mathfrak{Y}^{n-1} \leq \gamma_0 \log \frac{R}{\rho} [\Phi(R) - \Phi(\rho)]$$

where

$$\Phi(R) = R^{2-n} \int_{B_R} A(x, u) Du Du \, dx$$

First let us show how 4.3 follows from (20). Let $0 < \lambda < \mu < 1$ and apply the monotonicity lemma to $\rho = \lambda R_\nu$ and $R = \mu R_\nu$. We have

$$\int_{\partial B_1} |h^{(\nu)}(\lambda x) - h^{(\nu)}(\mu x)|^2 d\mathfrak{Y}^{n-1} \leq \gamma_0 \log \frac{\mu}{\lambda} [\Phi(\mu R_\nu) - \Phi(\lambda R_\nu)]$$

On the other hand, (20) implies also that $\Phi(t)$ is increasing and therefore tends to a finite limit when $t \rightarrow 0$. So, if we let $\nu \rightarrow \infty$, the right-hand side converges to zero and hence for almost every λ and μ we have

$$\int_{\partial B_1} |v(\lambda x) - v(\mu x)|^2 d\mathfrak{Y}^{n-1} = 0$$

which implies that v is homogeneous of degree zero.

Let us prove 4.4. For $t < 1$ let $x_t = t \frac{x}{|x|}$ and $u_t(x) = u(x_t)$. We have

$$(21) \quad F[u; B_t] \leq F[u_t; B_t]$$

and

$$F[u_t; B_t] = \int_{B_t} A_{ij}^{\alpha\beta}(u_t) \frac{t^2}{|x|^2} \left(\delta_{\alpha h} - \frac{x_\alpha x_h}{|x|^2} \right) \left(\delta_{\beta k} - \frac{x_\beta x_k}{|x|^2} \right) D_h u^i(x_t) D_k u^j(x_t) \, dx$$

Observing that for every f we have

$$\int_{B_t} |x|^{-2} f(x_t) dx = \frac{t^{-1}}{n-2} \int_{\partial B_t} f(x) d\mathfrak{F}^{n-1}$$

we get

$$F[u_t; B_t] = \frac{t}{n-2} \left\{ \int_{\partial B_t} A(u) Du Du d\mathfrak{F}^{n-1} - \int_{\partial B_t} A_{ij}^{\alpha\beta} \frac{x_\alpha x_\beta}{|x|^2} \left[2\delta_{\beta k} - \frac{x_\beta x_k}{|x|^2} \right] D_h u^i D_k u^j d\mathfrak{F}^{n-1} \right\}$$

Taking into account the special form of the coefficients and the ellipticity, we conclude that

$$(22) \quad F[u_t, B_t] \leq \frac{t}{n-2} \left\{ \int_{\partial B_t} A(u) Du Du d\mathfrak{F}^{n-1} - \int_{\partial B_t} \frac{|\langle x, Du \rangle|^2}{|x|^2} d\mathfrak{F}^{n-1} \right\}$$

$$\langle x, Du \rangle = x_\alpha D_\alpha u .$$

Now we have

$$t^{2-n} \int_{\partial B_t} A(u) Du Du = \Phi'(t) + (n-2) \frac{\Phi(t)}{t}$$

therefore from (21), (22) we get

$$\Phi'(t) \geq t^{2-n} \int_{\partial B_t} \frac{|\langle x, Du \rangle|^2}{|x|^2} d\mathfrak{F}^{n-1}$$

and integrating

$$\Phi(R) - \Phi(\rho) \geq \int_\rho^R t^{2-n} \int_{\partial B_t} \frac{|\langle x, Du \rangle|^2}{|x|^2} d\mathfrak{F}^{n-1}$$

On the other hand,

$$|u(Rx) - u(\rho x)|^2 < \left(\int_{\rho}^R \langle x, DU(tx) \rangle dt \right)^2 \leq \log \frac{R}{\rho} \int_{\rho}^R t |\langle x, Du(xt) \rangle|^2 dt$$

from which (20) follows at once after integrating on ∂B .

5. The theorem in 4 applies to harmonic mappings between manifolds $\mathcal{U}: M^n \rightarrow M^N$, provided every point of M^n has a neighborhood which is mapped by $\mathcal{U}$ into a bounded coordinate chart of M^N . In fact in this case the "energy" of $\mathcal{U}$ in local coordinates is

$$\xi[u; \Omega] = \int_{\Omega} g_{ij}(u) a^{\alpha\beta}(x) D_{\alpha} u^i D_{\beta} u^j \sqrt{a} dx$$

where g_{ij} and $(a_{\alpha\beta}) = (a^{\alpha\beta})^{-1}$ are respectively the metric tensors of M^N and M^n and $a = \det(a_{\alpha\beta})$.

6. In analogy with the well-known boundary regularity result by Hardt and Simon [14], Jost and Meier [17] have shown that, under the assumptions of the theorem in 4, if u is a minimum point among all functions in $W^{1,2}(\Omega, \mathbb{R}^N)$ with boundary values $u|_{\partial\Omega}$ and $\partial\Omega$ and $u|_{\partial\Omega}$ are regular, then u is regular in a neighborhood of $\partial\Omega$.

7. The general case of harmonic mappings between manifolds has been considered independently by Schoen and Uhlenbeck [20, 21]. Without the assumption that the image of $\mathcal{U}$ is locally contained in a chart they prove that any bounded minimizer has at most isolated singular points in dimension $n = 3$, while in general the dimension of the singular set is less than or equal to $n - 3$, and finally that no singularity can occur at the boundary, provided of course the boundary values are smooth.

8. The regularity of a harmonic map between two Riemannian manifolds M^n and M^N depends on the geometry of M^N . A classical result by Eells and Sampson [2] states that if the sectional curvature of M^N is nonpositive, then any harmonic mapping is smooth. In 1977 Hildebrandt, Kaul and Widman [15] proved existence

and regularity even in the case of positive sectional curvature but provided the image is contained in a ball $\mathcal{B}_R(g) = \{p \in M^N : \text{dist}(p, q) < R\}$ which lay within normal range of all its points and for which

$$(23) \quad R < \frac{\pi}{2\sqrt{K}}$$

where $K \geq 0$ is an upper bound for the sectional curvature of M^N . They also showed that the "equator map" $U_{(x)}^* = (x/|x|, 0)$ from the unit ball B of $\mathbb{R}^n$ into the equator of the standard sphere $S^n = \{y \in \mathbb{R}^{n+1} : |y| = 1\} \subset \mathbb{R}^{n+1}$ is a critical point for the energy. Jager and Kaul [16] then showed that U^* is an absolute minimum if $n \geq 7$, but it is even unstable if $3 \leq n \leq 6$; and Baldes [1] has considered the equator map from B into ellipsoids $\Lambda = \{(y, y_{n+1}) \in \mathbb{R}^{n+1} : |y|^2 + y_{n+1}^2/a^2 = 1\} \subset \mathbb{R}^{n+1}$, showing that U^* is strictly stable for the energy functional if $a^2 > \frac{4(n-1)}{(n-2)^2}$ and unstable if $a^2 < \frac{4(n-1)}{(n-2)^2}$.

The following question can be naturally raised: consider an energy-minimizing map from a domain Ω in some n -dimensional Riemannian manifold into the N -dimensional hemisphere $S_+^N = \{y \in \mathbb{R}^{N+1} : |y| = 1, y_N \geq 0\} \subset \mathbb{R}^{N+1}$ (note that this corresponds to having equality sign in (23)); is that map regular provided $n \leq 6$? The analogy with the Bernstein problem appears clearly.

This question has been answered positively by Schoen and Uhlenbeck [22] and independently by Giaquinta and Souček [11]. I want to give an idea of the proof, which again relies on some techniques developed in the theory of minimal surfaces, see, e.g., [12], referring to [11, 22] for more information.

Take stereographic coordinates on $S^N - \{\text{south pole}\}$. The energy functional for a map u from a domain Ω in $\mathbb{R}^n$ into S_+^N is given (apart from a factor 2) by

$$E[u; \Omega] = \int_{\Omega} \frac{|Du|^2}{(1 + |u|^2)^2} dx$$

and the map u is a minimizer with image in S_+^N if $|u| \leq 1$ and

$$E[u; \text{supp}(u - v)] \leq E[v; \text{supp}(u - v)]$$

for any v with $|v| \leq 1$ and $\text{supp}(u - v) \subset \subset \Omega$.

First of all we notice that for any bounded $v \in W^{1,2}(\Omega, \mathbb{R}^N)$ if $\tilde{v}$ is the reflected map through the equator, i.e.,

$$\tilde{v} = \begin{cases} v & \text{if } |v| \leq 1 \\ \frac{v}{|v|^2} & \text{if } |v| \geq 1 \end{cases}$$

then

$$E[v; \Omega] = E[\tilde{v}; \Omega] .$$

Therefore we have:

If $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, $|u| \leq 1$, minimizes the energy among maps v with $v = u$ on $\partial\Omega$ and image into the hemisphere, i.e., $|v| \leq 1$, then u minimizes the energy in the class $\{v \in W^{1,2}(\Omega, \mathbb{R}^N) : v \text{ bounded, } v = u \text{ on } \partial\Omega\}$. Hence we may conclude that u is a solution of the Euler - Lagrange equation for E :

$$(24) \quad \int_{\Omega} \left[\frac{Du \cdot Du}{(1 + |u|^2)^2} - 2 \frac{|Du|^2}{(1 + |u|^2)^2} \frac{u \cdot \phi}{1 + |u|^2} \right] dx = 0 \quad \forall \phi \in W_0^1 \cap L^\infty(\Omega, \mathbb{R}^N) .$$

Now assume that u has a singular point $x_0 \in \Omega$. Because of 4.1 we may suppose that x_0 is isolated. By translating x_0 to the origin and blowing up as in 4, we then produce a new energy minimizing map on the unit ball B of $\mathbb{R}^n$ (still called u) with $|u| \leq 1$, u singular at zero and homogeneous of degree zero.

Since u is 0-homogeneous, from (24), choosing $\phi = u \cdot \eta(|x|)$, we deduce that (note that u is regular in $B - \{0\}$):

$$\int_{\partial B} \frac{|Du|^2}{(1 + |u|^2)^2} \frac{1 - |u|^2}{1 + |u|^2} d\mathcal{H}^{n-1} = 0 .$$

This implies that either u is constant or $|u|$ is identically 1.

4.2 excludes the first possibility, hence we conclude that:

If our original energy minimizing map had an isolated singularity at x_0 , then we can produce a 0-homogeneous energy-minimizing map, still called u , with $|u| = 1$, singular at zero.

We calculate the first and second variation of E at u , i.e., consider

$$E(t) = E[u + t\phi; B] \quad \phi \in W_0^1 \cap L^\infty(B, \mathbb{R}^N)$$

and

$$\begin{aligned} \delta E[\phi] &:= \frac{d}{dt} E(t) \Big|_{t=0} = \frac{1}{2} \int_B [Du \cdot D\phi - u \cdot \phi |Du|^2] dx \\ \delta^2 E[\phi, \phi] &:= \frac{d^2}{dt^2} E(t) \Big|_{t=0} = \frac{1}{2} \int_B \left[|D\phi|^2 - 4u \cdot \phi Du \cdot Du - |Du|^2 |\phi|^2 \right. \\ &\quad \left. + 3(u - \phi)^2 |Du|^2 \right] dx . \end{aligned}$$

We have

$$\delta E[\phi] = 0 , \quad \delta^2 E[\phi, \phi] \geq 0 .$$

Now choose $\phi = u(x) |Du(x)| \eta(|x|)$ where $\eta(t)$ is any smooth function with compact support in $(0, 1)$. Then we get

$$(25) \quad \int_B \left\{ c^2 |D\eta|^2 - \eta^2 \left[c^4 + \frac{1}{2} \Delta c^2 - |Dc|^2 \right] \right\} dx \geq 0$$

where $c(x) = |Du(x)|$.

The result then follows from the inequality

$$(26) \quad c^4 + \frac{1}{2} \Delta c^2 - |Dc|^2 \geq \frac{c^2}{|x|^2} + \frac{c^4}{n-1} .$$

In fact (25) and (26) yield

$$\int_B \left\{ c^2 |D\eta(|x|)|^2 - \eta^2(|x|) \left(\frac{c^2}{|x|^2} - \frac{c^4}{n-1} \right) \right\} dx \geq 0 .$$

Now either $c^2 \equiv 0$ (which would imply $u = \text{const}$) or, since c^2 is homogeneous of degree -2 ,

$$\int_0^1 r^{n-3} \eta^2(r) dr - \left(1 + \frac{1}{n-1} \frac{\int_{\partial B} c^4 d\mathfrak{F}^{n-1}}{\int_{\partial B} c^2 d\mathfrak{F}^{n-1}} \right) \int_0^1 r^{n-5} \eta^2 dr \geq 0$$

for any η with compact support in $(0, 1)$. By an approximation and rescaling argument, this implies

$$\int_0^\infty r^{n-3} \eta^2(r) dr - (1+\sigma) \int_0^\infty r^{n-5} \eta^2(r) dr \geq 0$$

$$\sigma = \frac{1}{n-1} \frac{\int_{\partial B} c^4 d\mathfrak{F}^{n-1}}{\int_{\partial B} c^2 d\mathfrak{F}^{n-1}} > 0$$

for any $\eta: (0, +\infty) \rightarrow \mathbb{R}$ for which the integrals converge. Finally, choosing

$$\eta(r) = \begin{cases} r^\alpha & \text{for } r < 1 \\ r^\beta & \text{for } r \geq 1 \end{cases}$$

$$\alpha = \frac{1}{2}(4-n+\epsilon) , \quad \beta = \frac{1}{2}(4-n-\epsilon) , \quad \epsilon > 0$$

we get

$$\frac{\alpha^2}{n-4+2\alpha} - \frac{\beta^2}{n-4+2\beta} \geq (1+\sigma) \left[\frac{1}{n-4+2\alpha} - \frac{1}{n-4+2\beta} \right] \quad \forall \epsilon > 0 ,$$

i.e.,

$$n \geq 4 + 2\sqrt{1+\sigma}$$

which gives the result.

Now I still have to prove inequality (26). We have

$$\frac{1}{2} \Delta c^2 + c^4 = \sum_{i=1}^N \sum_{\alpha, \beta=1}^n (\partial_\alpha \partial_\beta u^i)^2$$

$$c^2 |Dc|^2 = \sum_{\beta=1}^n \left(\sum_{j=1}^N \sum_{\alpha=1}^n \partial_\alpha u^j \partial_\beta \partial_\alpha u^j \right)^2$$

hence

$$(27) \quad \frac{1}{2} \Delta c^2 + c^4 - |Dc|^2 = \frac{1}{2c^2} \sum_{\alpha, \beta, \gamma=1}^n \sum_{ij=1}^N \left[\partial_\alpha u^i \partial_\beta \partial_\gamma u^j - \partial_\beta u^j \partial_\alpha \partial_\gamma u^i \right]^2$$

If $x_0 \in B - \{0\}$ by orthogonal changes of coordinates we can assume that

$$x_0 = (r, 0, 0, \dots, 0) \quad r = |x_0|$$

$$u(x_0) = (1, 0, \dots, 0) .$$

Therefore, if the letters $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$, and $\bar{i}, \bar{j}$ run from 2 to n and from 2 to N respectively, we have, taking into account the 0-homogeneity of u :

$$(28) \quad \partial_1 u^i(x_0) = 0$$

$$(29) \quad \partial_1 \partial_\alpha u^i(x_0) = -\frac{1}{r} \partial_\alpha u^i(x_0)$$

Since $u^i \partial_\alpha u^i = 0$ and $\partial_\beta (u^i \partial_\alpha u^i) = 0$, we get, moreover,

$$(30) \quad \partial_\alpha u^1(x_0) = 0$$

and $\partial_\alpha u^i \partial_\beta u^i + u^i \partial_\alpha \partial_\beta u^i = 0$, i. e.,

$$(31) \quad \partial_\alpha \partial_\beta u^1(x_0) = -\frac{1}{r} \sum_{k=1}^N \partial_\alpha u^k(x_0) \partial_\beta u^k(x_0) .$$

Using (28) - (31) we see immediately that at x_0

$$\begin{aligned} \sum_{\alpha, \beta, \gamma=1}^n \sum_{ij=1}^n \left[\partial_{\alpha} u^i \partial_{\beta} \partial_{\gamma} u^j - \partial_{\beta} u^j \partial_{\alpha} \partial_{\gamma} u^i \right]^2 &= \sum_{\bar{\alpha}, \bar{\beta}, \bar{\gamma}} \sum_{\bar{i} \bar{j}} [\dots]^2 + \frac{2}{r} c^4 = \\ &= \frac{2}{n-1} c^6 + \frac{2}{r} c^4 + \sum_{\alpha, \beta, \gamma} \sum_{ij} [\dots]^2 \end{aligned}$$

and the result follows from (27).

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Mariano Giaquinta
 Istituto di Matematica Applicata "G. Sansone", Università di Firenze
 via S. Marta 3
 I. 50139 FIRENZE (Italy)

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