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Compactness of solutions to some geometric fourth-order equations

By *Andrea Malchiodi* at Trieste

Abstract. We prove compactness of solutions to some fourth order equations with exponential nonlinearities on four manifolds. The proof is based on a refined bubbling analysis, for which the main estimates are given in integral form. Our result is used in a subsequent paper to find critical points (via minimax arguments) of some geometric functional, which give rise to conformal metrics of constant Q-curvature. As a byproduct of our method, we also obtain compactness of such metrics.

1. Introduction

Consider a compact four-dimensional manifold (M,g) with Ricci tensor Ric_g and scalar curvature R_g. The *Q-curvature* and the *Paneitz operator*, introduced in [7], [41] and [42], are defined respectively by

$$Q_g = -\frac{1}{12}(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2), \tag{1}$$

$$P_g(\varphi) = \Delta_g^2\varphi + \mathrm{div}\left(\frac{2}{3}R_g g - 2\,\mathrm{Ric}_g\right)d\varphi, \tag{2}$$

where φ is any smooth function on M, see also the survey [19].

The Q-curvature and the Paneitz operator arise in several contexts in the study of four-manifolds and of particular interest is their role, and their mutual relation, in conformal geometry. In fact, given a metric $\tilde{g} = e^{2w}g$, the following equations hold:

$$P_{\tilde{g}} = e^{-4w}P_g, \quad P_g w + 2Q_g = 2Q_{\tilde{g}}e^{4w}. \tag{3}$$

A first connection to the topology of a manifold is a Gauss-Bonnet type formula. If W_g denotes the Weyl's tensor of M, then one has

$$\int_M \left(Q_g + \frac{|W_g|^2}{8}\right) dV_g = 4\pi^2\chi(M),$$

where dV_g stands for the volume element in (M,g) and $\chi(M)$ is the Euler characteristic of M. In particular, since $|W_g|^2$ is a pointwise conformal invariant, it follows that $\int_M Q_g\,dV_g$ is a *global* conformal invariant.

To mention some geometric applications we recall three results proven by Gursky, [31], and by Chang, Gursky and Yang, [13], [14] (see also [30]). If a manifold of positive Yamabe class satisfies $\int_M Q_g\,dV_g > 0$, then its first Betti number vanishes. Moreover there exists a conformal metric with positive Ricci tensor, and hence M has finite fundamental group. Furthermore, under the additional quantitative assumption $\int_M Q_g\,dV_g > \frac{1}{8}\int_M |W_g|^2\,dV_g$, M must be diffeomorphic to the four-sphere or to the projective space. In particular the last result is a conformally invariant improvement of a theorem by Margerin, [39], which assumed pointwise pinching conditions on the Ricci tensor in terms of W_g.

Finally, we also point out that the Paneitz operator and the Q-curvature (together with their higher-dimensional analogues, see [5], [6], [27], [29]) appear in the study of Moser-Trudinger type inequalities, log-determinant formulas and the compactification of locally conformally flat manifolds, see [4], [7], [8], [15], [16], [17].

As for the uniformization theorem, one can ask whether every four-manifold (M,g) carries a conformal metric $\tilde{g}$ for which the corresponding Q-curvature $Q_{\tilde{g}}$ is a constant. Writing $\tilde{g} = e^{2w}g$, by (3) the problem is equivalent to finding a solution of the equation

$$P_g w + 2Q_g = 2\bar{Q}e^{4w}, \tag{4}$$

where $\bar{Q}$ is a real constant. In view of the regularity results in [47], solutions of (4) can be found as critical points of the following functional:

$$\mathrm{II}(u) = \langle P_g u, u\rangle + 4\int_M Q_g u\,dV_g - k_P \log\int_M e^{4u}\,dV_g, \quad u \in H^2(M), \tag{5}$$

where we are using the notation

$$\langle P_g u, v\rangle = \int_M \left(\Delta_g u \Delta_g v + \frac{2}{3}R_g \nabla_g u \cdot \nabla_g v - 2(\mathrm{Ric}_g\,\nabla_g u, \nabla_g v)\right) dV_g, \quad u, v \in H^2(M),$$

and where

$$k_P = \int_M Q_g\,dV_g. \tag{6}$$

Problem (4) has been solved in [17] for the case in which P_g is a positive operator and $k_P < 8\pi^2$ ($8\pi^2$ is the value of k_P on the standard sphere). Under these assumptions by the Adams inequality, see (16), the functional II is bounded from below and coercive, hence solutions can be found as global minima. The result has also been extended in [9] to higher-dimensional manifolds (regarding higher-order operators and curvatures) using a geometric flow. A first sufficient condition to ensure these hypotheses was given by Gursky in [31]. He proved that if the Yamabe invariant is positive and if $k_P > 0$, then P_g is positive definite and moreover $k_P \leqq 8\pi^2$, with the equality holding if and only if M is conformally equivalent to S^4. Other more general sufficient conditions are given in [32]. The solvability of (4) also turns out to be useful in the study of some interesting class of fully non-linear equations, as it has been shown in [14], with the remarkable geometric consequences mentioned above.

We are interested here in the more general case when P_g has no kernel and $k_P \neq 8k\pi^2$ for $k = 1, 2, \dots$. These conditions are generic, and in particular include manifolds with negative curvature or negative Yamabe class, for which k_P can be bigger than $8\pi^2$.

In the case under investigation the functional II can be unbounded from below, and hence it is necessary to find extrema which are possibly saddle points. As we shall explain later, in order to find these critical points it is useful to study compactness of solutions to perturbations of (4).

Therefore we consider the following sequence of problems:

$$P_g u_l + 2Q_l = 2k_l e^{4u_l} \quad \text{in } M, \tag{7}$$

where $(k_l)_l$ are constants and where

$$Q_l \to Q_0 \quad \text{in } C^0(M). \tag{8}$$

Without loss of generality, we can assume that the sequence $(u_l)_l$ satisfies the volume normalization

$$\int_M e^{4u_l}\, dV_g = 1, \quad \text{for all } l, \tag{9}$$

which implies that we must choose $k_l = \int_M Q_l\, dV_g$.

Our main result is the following.

Theorem 1.1. *Suppose* $\ker P_g = \{\text{constants}\}$ *and that* $(u_l)_l$ *is a sequence of solutions of* (7), (9), *with* $(Q_l)_l$ *satisfying* (8). *Assume also that*

$$k_0 := \int_M Q_0\, dV_g \neq 8k\pi^2, \quad \textit{for } k = 1, 2, \dots . \tag{10}$$

Then $(u_l)_l$ *is bounded in* $C^\alpha(M)$ *for any* $\alpha \in (0, 1)$.

The main application of Theorem 1.1 concerns the case $Q_0 = Q_g$. Indeed, if a sequence of solutions to (7)–(9) can be produced, its weak limit will be a critical point of the functional II and a solution of (4). This is indeed verified in [26] under the assumptions of Theorem 1.1 (with $Q_0 = Q_g$). As a consequence one finds conformal metrics with constant Q-curvature for a large class of four manifolds. We have indeed the following result, announced in the preliminary note [25] with some sketch of the ideas of the proof.

Theorem 1.2 ([26]). *Suppose* $\ker P_g = \{\text{constants}\}$, *and assume that* $k_P \neq 8k\pi^2$ *for* $k = 1, 2, \dots$. *Then equation* (4) *has a solution.*

The proof requires a minimax scheme which becomes more and more involved as k increases and when P_g possesses negative eigenvalues. This scheme extends the one in [24], which in our case would correspond to $P_g \geqq 0$ and $k_0 \in (8\pi^2, 16\pi^2)$.

The way we use Theorem 1.1 in [26] is the following. First, for ρ in a neighborhood of 1, we introduce the modified functional

$$\mathrm{II}_\rho(u) = \langle P_g u, u \rangle + 4\rho \int_M Q_g u \, dV_g - k_P \rho \log \int_M e^{4u} \, dV_g, \quad u \in H^2(M),$$

and, using the minimax scheme, we prove existence of Palais-Smale sequences at some level c_ρ. It turns out that the function $\rho \mapsto c_\rho$ is a.e. differentiable and, following an idea in [45] (used also in [24], [33], [46]), we prove existence of critical points of II_ρ for those values of ρ at which c_ρ is differentiable. Then we are led to consider (7) for $Q_l = \rho_l Q_g$, where $(\rho_l)_l$ is a suitable sequence tending to 1.

Theorem 1.1 applies also to any sequence of smooth solutions of (4). Therefore, as another application, we have the following result, which extends a compactness theorem in [17].

Corollary 1.3. *Suppose* $\ker P_g = \{\text{constants}\}$ *and that* $k_p \neq 8k\pi^2$ *for* $k = 1, 2, \ldots$. *Suppose* $(u_l)_l$ *is a sequence of solutions of* (4) *satisfying* (9). *Then, for any* $m \in \mathbb{N}$, $(u_l)_l$ *is bounded in* $C^m(M)$.

Corollary 1.3 has a counterpart in [35] (see also [21]), where compactness of solutions is proved for a mean field equation on compact surfaces.

The case when k_P is an integer multiple of $8\pi^2$ is more delicate, and should require an asymptotic analysis as in [3], [20], [21], [35] (see also the references therein). An interesting particular case of this situation is the standard sphere. Being an homogeneous space, the Q-curvature is already constant and indeed all the solutions of (4) on S^4, which have been classified in [18], arise from conformal factors of Möbius transformations. Henceforth, a natural problem to consider is to prescribe the Q-curvature as a given function f on S^4. Some results in this direction are given in [10], [38] and [48]. Typically, the methods are based on blow-up or asymptotic analysis combined with Morse theory, in order to deal with a possible loss of compactness.

The Paneitz operator and the Q-curvature can be considered as natural extensions to four-manifolds of, respectively, the Laplace-Beltrami operator Δ_g and the Gauss curvature K_g on two-dimensional surfaces. In fact, similarly to P_g and Q_g, these transform according to the equations

$$(11) \qquad \Delta_{\tilde{g}} = e^{-2w} \Delta_g, \quad -\Delta_g w + K_g = K_{\tilde{g}} e^{2w},$$

where, again, $\tilde{g} = e^{2w} g$. Hence, in the case of a flat domain $\Omega \subseteqq \mathbb{R}^2$, one is led to study equations of the form

$$(12) \qquad -\Delta v_l = K_l(x) e^{2v_l} \quad \text{in } \Omega.$$

In [12] the authors proved, among other things, that if $(K_l)_l$ are non-negative, uniformly bounded in $L^\infty(\Omega)$ and if $\int_\Omega e^{2u_l} \leqq C$, then either $(v_l)_l$ stays bounded in $L^\infty_{\mathrm{loc}}(\Omega)$, or

$v_l \to -\infty$ on the compact subsets of Ω, or $K_l e^{2v_l}$ concentrates at a finite number of points in Ω, namely $K_l e^{2v_l} \rightharpoonup \sum_{i=1}^{j} \alpha_i \delta_{x_i}$ (δ_{x_i} stands for the Dirac mass at x_i). In the latter case, they also proved that each α_i is greater or equal than 4π. This result was specialized in [36] where, assuming that $K_l \to K_0$ in $C^0(\bar{\Omega})$ and using the sup+inf inequalities in [11], [44], the authors proved that each α_i is indeed an integer multiple of 4π. Chen showed then in [23] that the case of a multiple bigger than 1 may indeed occur. On the other hand, if Ω is replaced by a compact surface (subtracting a constant term to the right-hand side, to get solvability of the equation), then each α_i is precisely 4π, see [35]. The same result is obtained in [40] for approximate solutions in domains, but with an extra assumption on the L^∞ norm of the error terms.

Our argument for the proof of Theorem 1.1, which we outline below, relies on proving a quantization result for the volume of blowing-up solutions as in [36]. The main idea is to show that at every blow-up point the volume is a multiple of $8\pi^2/k_0$. Then, proving also that there is no residual volume amount, we reach a contradiction with (9) since we are assuming that k_0 is not an integer multiple of $8\pi^2$. However, instead of using pointwise estimates on the solutions, as in [12] or [36], our results are mainly given in integral form, see Remark 1.4.

Except for the last subsection, we work under the assumption

$$k_0 \in \left(8k\pi^2, 8(k+1)\pi^2\right), \quad k \in \mathbb{N}, \tag{13}$$

since this case contains most of the difficulties.

The plan of the paper (and the strategy of the proof) is the following. In Section 2 we collect some preliminary facts including a modified version of the Adams inequality, to deal with the presence of negative eigenvalues, and some L^p estimates on the first, second and third derivatives of the solutions.

In Section 3 we derive a compactness criterion based on the amount of concentration of the nonlinear term, see Proposition 3.1, and then we study the asymptotic profile of u_l near the concentration points. In particular we prove that the minimal volume accumulation is $8\pi^2/k_0$, see (39).

In Section 4, which is the core of our analysis, we introduce the notion of *simple blow-up* (adopting the terminology used by R. Schoen) and we show in Proposition 4.2 that at such blow-ups the accumulation is exactly $8\pi^2/k_0$. In order to prove this we use some integral form of the Harnack inequality, see in particular Subsection 4.1, combined with a careful ODE analysis for the function $r \mapsto \bar{u}_{r,l}$. Here $\bar{u}_{r,l}$ denotes, naively, the average of u_l on an annulus A_r of radii r and $2r$ centered near a concentration point.

Finally, in Section 5 we show how a general blow-up situation can be essentially reduced to the case of finitely-many simple blow-ups. In particular, we prove that at any general blow-up point the amount of concentration is an integer multiple of $8\pi^2/k_0$. Recalling the normalization (9) and that $k_0 \neq 8k\pi^2$ for any integer k, we reach then a contradiction to the fact that $(u_l)_l$ is unbounded in some C^α norm. In Subsection 5.2 we consider the case $k_0 < 8\pi^2$, which is easier and requires only the analysis of Section 3.

In our proof we exploit crucially the fact that we are working on a compact manifold, since we often make use of the Green's representation formula. We also point out that our assumptions on M are generic and do not require the metric to be locally conformally flat or Einstein.

Remark 1.4. It is an open problem to understand whether the functional II itself (see (5)) possesses bounded Palais-Smale sequences, or equivalently if it is possible to find solutions of (4) without introducing the perturbed functional II_ρ.

The reason why we kept most of our estimates in integral form is that many of them could be applied to functions of class H^2 only (not necessarily smooth or bounded) and we hope that some could be useful to understand the question. At the moment, in particular, the counterpart of Proposition 4.2 is missing for Palais-Smale sequences and we need the full rigidity of equation (7). For related topics see [40].

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2. Notation and preliminaries

In this brief section we collect some useful preliminary facts, and in particular we state a version of the Moser-Trudinger inequality involving the Paneitz operator. In the following $B_r(p)$ stands for the metric ball of radius r and center p. We also denote by $|x-y|$ the distance of two points $x, y \in M$. $H^2(M)$ is the Sobolev space of functions on M which are in $L^2(M)$ together with their first and second derivatives. Large positive constants are always denoted by C, and the value of C is allowed to vary from formula to formula and also within the same line.

As already mentioned, throughout most of the paper we will work under the assumption (13). When the operator P_g is positive definite, by the Poincaré inequality the H^2 norm is equivalent to the following one:

$$\|u\|^2 = \langle P_g u, u\rangle + \int_M u^2\, dV_g, \quad u \in H^2(M). \tag{14}$$

Being M four-dimensional, $H^2(M) \hookrightarrow L^p(M)$ for all $p > 1$. We have indeed the following limit-case embedding, proved in [1] and [8] for the operator Δ^2 and extended in [17] for the Paneitz operator.

Proposition 2.1. *If $P_g \geqq 0$, there exists a positive constant C depending on M such that*

$$\int_M e^{\frac{32\pi^2(u-\bar{u})^2}{\langle P_g u, u\rangle}}\, dV_g \leqq C, \quad \textit{for every } u \in H^2(M), \tag{15}$$

where $\bar{u} = \frac{1}{\mathrm{Vol}(M)} \int_M u \, dV_g$ *denotes the average of u on M. The last formula implies*

$$(16) \qquad \log \int_M e^{4(u-\bar{u})} \, dV_g \leqq C + \frac{1}{8\pi^2} \langle P_g u, u \rangle, \quad \textit{for every } u \in H^2(M).$$

Here we are interested in the case in which P_g might possess some negative eigenvalues. We denote by $V \subseteqq H^2(M)$ the direct sum of the eigenspaces corresponding to negative eigenvalues of P_g. Of course the dimension of V is finite, say $\bar{k}$, and since P_g has no kernel and is self-adjoint we can find an orthonormal basis of eigenfunctions $\hat{v}_1, \dots, \hat{v}_{\bar{k}}$ of V with the properties

$$(17) \qquad P_g \hat{v}_i = \lambda_i \hat{v}_i, \quad i = 1, \dots, \bar{k}, \quad \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{\bar{k}} < 0 < \lambda_{\bar{k}+1} \leqq \cdots,$$

where the λ_i's are the eigenvalues of P_g. Having introduced the subspace V, we need a modified version of the Adams inequality.

Lemma 2.2. *Suppose P_g possesses some negative eigenvalues, that*

$$\ker P_g = \{\text{constants}\},$$

and let V denote the direct sum of the negative eigenspaces of P_g. Then there exists a constant C such that

$$(18) \qquad \int_M e^{\frac{32\pi^2 (u-\bar{u})^2}{\langle P_g u, u \rangle}} \, dV_g \leqq C, \quad \textit{for every function } u \in H^2(M) \textit{ with } \hat{u} = 0.$$

Here $\hat{u}$ denotes the component of u in V. As a consequence one has

$$(19) \qquad \log \int_M e^{4(u-\bar{u})} \, dV_g \leqq C + \frac{1}{8\pi^2} \langle P_g u, u \rangle,$$

$$\textit{for every function } u \in H^2(M) \textit{ with } \hat{u} = 0.$$

Proof. The proof is a variant of the arguments of [8] and [17]. If $\hat{v}_1, \dots, \hat{v}_{\bar{k}}$ and $\lambda_1, \dots, \lambda_{\bar{k}}$ are as in (17), we introduce the following positive-definite pseudo-differential operator P_g^+:

$$P_g^+ u = P_g u - 2 \sum_{i=1}^{\bar{k}} \lambda_i \left(\int_M u \hat{v}_i \, dV_g \right) \hat{v}_i.$$

Basically, we are reversing the sign of the negative eigenvalues of P_g. The operator P_g^+ admits the following Green's function:

$$G^+(x, y) = G(x, y) - 2 \sum_{i=1}^{\bar{k}} \lambda_i \hat{v}_i(x) \hat{v}_i(y),$$

where $G(x, y)$ corresponds to P_g. Then the arguments of [17] (see also [1], [8]), which are based on representations for pseudo-differential operators, can be adapted to the case of P_g^+, yielding

$$\int_M e^{\frac{32\pi^2(u-\bar{u})^2}{\langle P_g^+ u, u\rangle}} \, dV_g \leqq C, \quad \text{for every } u \in H^2(M).$$

Applying the last formula to functions for which $\hat{u} = 0$, we obtain (18). Finally, from the elementary inequality $4ab \leqq 32\pi^2 a^2 + \frac{1}{8\pi^2} b^2$, applied with $a = (u - \bar{u})$ and $b = \langle P_g u, u\rangle$, we also deduce (19). $\square$

Theorem 1.1 is proved by contradiction. We claim that unboundedness in some C^α norm is equivalent (under the assumption (13), which implies $k_l > 0$ for l large) to the following condition:

$$(20) \qquad \|u_l - \bar{u}_l\| \to +\infty \quad \text{as } l \to +\infty.$$

In order to prove this we first notice that, by (9) and the Jensen inequality, $\bar{u}_l$ is uniformly bounded from above. Assuming that $\|u_l - \bar{u}_l\|$ is uniformly bounded (which implies, in the above notation, that also $\|u_l - \bar{u}_l - \hat{u}_l\|$ is uniformly bounded), then by (19) the right-hand side of (7) is also uniformly bounded in $L^p(M)$ for every $p > 1$. By elliptic regularity, then $(u_l)_l$ would be uniformly bounded in $W^{4,p}(M)$, and hence in $C^\alpha(M)$ for any $\alpha \in (0,1)$ by the Sobolev embeddings.

Hence from now on we assume that there exists a sequence $(u_l)_l$ satisfying (7)–(9) and (20).

We prove now a preliminary integrability result on the first, second and third derivatives of u_l.

Lemma 2.3. *Let $(u_l)_l$ be a sequence of solutions of (7)–(9), with $(Q_l)_l$ satisfying (8), and let $p \geqq 1$. Then there is a constant C depending only on p, M and k_0 such that, for r sufficiently small and for any $x \in M$ there holds*

$$\int_{B_r(x)} |\nabla^3 u_l|^p \, dV_g \leqq C r^{4-3p}, \qquad \int_{B_r(x)} |\nabla^2 u_l|^p \, dV_g \leqq C r^{4-2p}, \qquad \int_{B_r(x)} |\nabla u_l|^p \, dV_g \leqq C r^{4-p},$$

where, respectively, $p < 4/3$, $p < 2$ and $p < 4$.

Proof. We write

$$P_g u_l = f_l,$$

where

$$(21) \qquad f_l = 2k_l e^{4u_l} - 2Q_l.$$

We have the following representation formula:

$$(22) \qquad u_l(x) = \bar{u}_l + \int_M G(x,y) f_l(y) \, dV_g(y), \quad \text{for a.e. } x \in M,$$

where, by the results in [17], $G : M \times M \backslash \mathrm{diag}$ is symmetric and satisfies

(23) $$\left|G(x,y) - \frac{1}{8\pi^2}\log\frac{1}{|x-y|}\right| \leqq C, \quad x, y \in M,\ x \neq y,$$

while for its derivatives there holds

(24) $$|\nabla G(x,y)| \leqq C\frac{1}{|x-y|}, \quad |\nabla^2 G(x,y)| \leqq C\frac{1}{|x-y|^2},$$
$$|\nabla^3 G(x,y)| \leqq C\tfrac{1}{|x-y|^3}.$$

The last two estimates in (24) are not shown in [17] but they can be derived with the same approach, by an expansion of G at higher order using the parametrix, see also [2]. Similarly (this formula will be used later in the paper), one also finds that

(25) $$\nabla_x G(x,y) = \frac{1}{8\pi^2}\nabla_x \log\frac{1}{|x-y|} + O(1).$$

Recalling the definition of f_l in (21), we obtain

$$|\nabla^3 u_l|(x) \leqq C \int_M \frac{1}{|x-y|^3}|f_l(y)|\,dV_g(y), \quad \text{for a.e. } x \in M.$$

Then, from the Jensen's inequality it follows that

$$|\nabla^3 u_l|^p(x) \leqq C \int_M \left(\frac{\|f_l\|_{L^1(M)}}{|x-y|^3}\right)^p \frac{|f_l(y)|}{\|f_l\|_{L^1(M)}}\,dV_g(y), \quad \text{for a.e. } x \in M.$$

The Fubini's Theorem implies

$$\int_{B_r(\bar{x})} |\nabla^3 u_l|^p(x)\,dV_g(x) \leqq C \sup_{y\in M} \int_{B_r(\bar{x})} \frac{1}{|x-y|^{3p}}\,dV_g(x) \leqq C \int_{B_x(\bar{x})} \frac{1}{|x-\bar{x}|^{3p}}\,dV_g(x).$$

The last integral is finite provided $3p < 4$, as in our assumptions, and can be estimated using polar coordinates, giving

$$\int_{B_r(\bar{x})} |\nabla^3 u_l|^p(x)\,dV_g(x) \leqq C(p,M)r^{4-3p}.$$

This concludes the proof of the first inequality in the statement of the lemma. The remaining two follow similarly. □

3. The bubbling phenomenon

In this section we study the local behavior of unbounded sequences of solutions at a concentration point. In Subsection 3.1 we give compactness criteria when the amount of concentration is below a certain threshold. Then, in Subsection 3.2, we reduce ourselves to the preceding situation using a scaling argument. As a byproduct we describe the asymptotic profile of u_l, proving that it has the form of a *standard bubble*, and we show that the amount of volume concentration at any blow-up point is greater or equal than $8\pi^2/k_0$.

3.1. Concentration-compactness. In this subsection we give a concentration-compactness criterion for solutions of the equation $P_g v = h$ on M. In the case of the sphere a similar result has been shown in [9], and our proof basically goes along the same line. However we prefer to write the details, since some of them will be needed in the following.

Proposition 3.1. *Let $(h_l)_l \subseteq L^1(M)$ be a sequence of functions satisfying $\int_M |h_l|\, dV_g \leqq C$ for every l. Let v_l be solutions of $P_g v_l = h_l$ on M. Then, up to a subsequence, either for every l*

$$\int_M e^{\alpha(v_l - \bar{v}_l)}\, dV_g \leqq C, \quad \textit{for some } C > 0 \textit{ and some } \alpha > 4,$$

or there exist points $x_1, \dots, x_L \in M$ such that, for any $r > 0$ and any $i \in \{1, \dots, L\}$ there holds

$$\liminf_{l \to +\infty} \int_{B_r(x_i)} |h_l|\, dV_g \geqq 8\pi^2. \tag{26}$$

Proof. Assume the second alternative does not occur, namely

$$\text{for every } x \in M \text{ there exists } r_x > 0 \text{ such that } \int_{B_{r_x}(x)} |h_l|\, dV_g \leqq 8\pi^2 - \delta_x, \tag{27}$$

for some $\delta_x > 0$ and for l sufficiently large. We cover M with j balls $B_i := B_{\frac{r_{x_i}}{2}}(x_i)$, $i = 1, \dots, j$. Using (22) and setting $B_{r_{x_i}}(x_i) = \tilde{B}_i$, for a.e. $x \in B_i$ we can write

$$v_l(x) - \bar{v}_l = \int_{\tilde{B}_i} h_l(y) G(x, y)\, dV_g(y) + \int_{M \setminus \tilde{B}_i} h_l(y) G(x, y)\, dV_g(y). \tag{28}$$

Hence if $\alpha > 0$, for a.e. $x \in B_i$ we have

$$\exp\big[\alpha\big(v_l(x) - \bar{v}_l\big)\big] = \exp\left[\int_{\tilde{B}_i} \alpha G(x, y) h_l(y)\, dV_g(y)\right] \times \exp\left[\int_{M \setminus \tilde{B}_i} \alpha G(x, y) h_l(y)\, dV_g(y)\right] \tag{29}$$

Since G is smooth outside the diagonal, and since $\int_M |h_l|\, dV_g$ is uniformly bounded, there exists a positive constant C (independent of l) such that

$$\exp\left[\int_{M \setminus \tilde{B}_i} \alpha G(x, y) h_l(y)\, dV_g(y)\right] \leqq C, \quad \text{for any } x \in B_i.$$

Then by (29) we have

$$\int_{B_i} \exp\big[\alpha\big(v_l(x) - \bar{v}_l\big)\big]\, dV_g(x) \leqq C \int_{B_i} \exp\left[\int_M \alpha |G(x, y)|\, |h_l(y)| \chi_{\tilde{B}_i}\, dV_g(y)\right] dV_g(x). \tag{30}$$

Now, as in [12], we can use the Jensen's inequality to get

$$\exp\left[\int_M \alpha |G(x, y)|\, |h_l(y)| \chi_{\tilde{B}_i}\, dV_g(y)\right] \leqq \int_M \exp[\alpha \|h_l \chi_{\tilde{B}_i}\|_{L^1(M)} |G(x, y)|] \frac{|h_l \chi_{\tilde{B}_i}|(y)}{\|h_l \chi_{\tilde{B}_i}\|_{L^1(M)}}\, dV_g(y),$$

and hence, by the Fubini Theorem and (30)

$$\int_{B_i} \exp\left[\alpha\left(v_l(x) - \bar{v}_l\right)\right] dV_g(x) \leqq C \sup_{y \in M} \int_M \exp[\alpha \|h_l \chi_{\tilde{B}_i}\|_{L^1(M)} |G(x, y)|] \, dV_g(x).$$

By (23), there holds

$$\int_M \exp[\alpha \|h_l \chi_{\tilde{B}_i}\|_{L^1(M)} |G(x, y)|] \, dV_g(x) \leqq C \int_M \left(\frac{1}{|x-y|}\right)^{\frac{\alpha \|h_l \chi_{\tilde{B}_i}\|_{L^1(M)}}{8\pi^2}} dV_g(x).$$

The last integral is finite if

$$\text{(31)} \qquad \frac{\alpha \|h_l \chi_{\tilde{B}_i}\|_{L^1(M)}}{8\pi^2} < 4 \quad \Leftrightarrow \quad \alpha \int_{\tilde{B}_i} |h_l| \, dV_g < 32\pi^2.$$

By (27), this is satisfied for some $\alpha > 4$ provided we take l sufficiently large. We have shown that $\int_{B_i} e^{\alpha(v_l - \bar{v}_l)} \, dV_g < +\infty$ for every $i = 1, \dots, L$. Since M is covered by finitely many B_i's, the conclusion follows. □

Remark 3.2. Using the same proof, it is possible to extend Proposition 3.1 to the case in which also the metric on M depends on l, and converges to some smooth g in $C^m(M)$ for any integer m. We have to use this variant in the next subsection.

3.2. Asymptotic profile. We consider now the alternative in Proposition 3.1 for which compactness does not hold, applied to the case $h_l = 2k_l e^{4u_l} - Q_l$. We assume that there exist $\rho \in (0, \pi^2/k_0)$, radii $(r_l)_l$, $(\hat{r}_l)_l$ and points $(x_l)_l \subseteqq M$ with the following properties:

$$\text{(32)} \qquad \hat{r}_l \to 0, \quad \frac{r_l}{\hat{r}_l} \to 0, \quad \int_{B_{r_l}(x_l)} e^{4u_l} \, dV_g = \rho, \quad \int_{B_{r_l}(y)} e^{4u_l} \, dV_g < \frac{\pi^2}{k_0},$$
$$\text{for every } y \in B_{\hat{r}_l}(x_l).$$

Remark 3.3. If the second alternative in Proposition 3.1 holds, an example of the situation described in (32) is the following. Choose r_l, x_l satisfying

$$\text{(33)} \qquad \int_{B_{r_l}(x_l)} e^{4u_l} \, dV_g = \sup_{x \in M} \int_{B_{r_l}(x)} e^{4u_l} \, dV_g = \rho.$$

Then $r_l \to 0$ as $l \to +\infty$, and we can take $\hat{r}_l = r_l^{\frac{1}{2}}$.

Given a small $\delta > 0$, we consider the exponential maps

$$\exp_l : B_\delta^{\mathbb{R}^4} \to M, \quad \exp_l(0) = x_l,$$

where $B_\delta^{\mathbb{R}^4} = \{x \in \mathbb{R}^4 : |x| < \delta\}$. We also define the metric $\tilde{g}_l$ on $B_\delta^{\mathbb{R}^4}$ by $\tilde{g}_l := (\exp_l)^* g$, and the functions $\tilde{u}_l : B_\delta^{\mathbb{R}^4} \to \mathbb{R}$ by

$$\tilde{u}_l = u_l \circ \exp_l.$$

Now in $\mathbb{R}^4$ we consider the dilation $T_l : x \mapsto r_l x$, and we define another sequence

$$(34) \qquad \hat{u}_l(x) = \tilde{u}_l(T_l x) + \log r_l, \quad x \in B^{\mathbb{R}^4}_{\frac{\delta}{r_l}}.$$

Using a change of variables, one easily verifies that the function $\tilde{u}_l$ solves the equation

$$P_{\tilde{g}_l}\tilde{u}_l(x) + 2Q_l(x) = 2k_l e^{4\tilde{u}_l(x)}, \quad x \in B^{\mathbb{R}^4}_{\delta}.$$

Hence, setting $\hat{g}_l = r_l^{-2} T_l^* \tilde{g}_l$ and using the conformal properties of the Paneitz operator we obtain that $\hat{u}_l$ satisfies

$$(35) \qquad P_{\hat{g}_l}\hat{u}_l(x) + 2r_l^4 Q_l(T_l x) = 2k_l e^{4\hat{u}_l(x)}, \quad x \in B^{\mathbb{R}^4}_{\frac{\delta}{r_l}}.$$

Note that the metrics $\hat{g}_l$ converge in $C^m_{\mathrm{loc}}(\mathbb{R}^4)$ to the flat metric $(dx)^2$ for any integer m. Also, since $(Q_l)_l$ are uniformly bounded functions on M, one also finds

$$r_l^4 Q_{\tilde{g}_l}(T_l \cdot) \to 0 \quad \text{in } C^0_{\mathrm{loc}}(\mathbb{R}^4).$$

By (32), using a change of variables we obtain

$$(36) \qquad \rho = \int_{B_{r_l}(x_l)} e^{4u_l}\, dV_g = \int_{\frac{1}{r_l}(\exp_l)^{-1} B_{r_l}(x_l)} e^{4\hat{u}_l}\, dV_{\hat{g}_l},$$

where $o_l(1) \to 0$ as $l \to +\infty$. Note also that the sets $\dfrac{1}{r_l}(\exp_l)^{-1} B_{r_l}(x_l) \subseteqq \mathbb{R}^4$ approach the unit ball $B_1^{\mathbb{R}^4}$ as $l \to +\infty$. Moreover, by the last inequality in (32) and by our choice of ρ, it is easy to derive that

$$(37) \qquad \int_{B^{\mathbb{R}^4}_{\frac{1}{2}}(y)} e^{4\hat{u}_l}\, dV_{\hat{g}_l} < \frac{\pi^2}{k_0}, \quad \text{for every } y \in B^{\mathbb{R}^4}_{\frac{\tilde{r}_l}{2r_l}}.$$

Regarding the functions $\hat{u}_l$, we have the following convergence result.

Proposition 3.4. *Suppose $\rho \in (0, \pi^2/k_0)$, $(r_l)_l$, $(\tilde{r}_l)_l$, $(x_l)_l$ and $(u_l)_l$ satisfy* (32), *and let $(\hat{u}_l)_l$ be defined by* (34). *Then there exists $\lambda > 0$, $x_0 \in \mathbb{R}^4$ and $\alpha \in (0,1)$ such that*

$$\hat{u}_l \to \hat{u}_\infty \quad \text{in } C^\alpha_{\mathrm{loc}}(\mathbb{R}^4) \text{ and in } H^2_{\mathrm{loc}}(\mathbb{R}^4)$$

for some $\alpha \in (0,1)$, where the function $\hat{u}_\infty$ is given by

$$(38) \qquad \hat{u}_\infty(x) = \log \frac{2\lambda}{1 + \lambda^2|x - x_0|^2} - \frac{1}{4}\log\left(\frac{1}{3}k_0\right), \quad x \in \mathbb{R}^4.$$

Moreover, if $b_l \to +\infty$ sufficiently slowly, one has

$$(39) \qquad \int_{B_{b_l r_l}(x_l)} e^{4u_l}\, dV_g \to \frac{8\pi^2}{k_0} \quad \text{as } l \to +\infty.$$

Proof. Given $R > 0$, we define a smooth cut-off function Ψ_R satisfying

$$\begin{cases} \Psi_R(x) = 1, & \text{for } |x| \leqq R/2, \\ \Psi_R(x) = 0, & \text{for } |x| \geqq R. \end{cases}$$

We also set

$$a_l = \frac{1}{|B_R^{\mathbb{R}^4}|} \int_{B_R^{\mathbb{R}^4}} \hat{u}_l \, dV_{\hat{g}_l}, \quad v_l = \Psi_R \hat{u}_l + (1 - \Psi_R) a_l = a_l + \Psi_R(\hat{u}_l - a_l),$$

$$\hat{v}_l = v_l - a_l.$$

We notice that the functions v_l coincide with a_l outside $B_R^{\mathbb{R}^4}$ and that $\hat{v}_l$ is identically zero outside $B_R^{\mathbb{R}^4}$. By Lemma 2.3 and some scaling argument one finds

$$(40) \qquad \int_{B_{2R}^{\mathbb{R}^4}} (|\nabla^3 \hat{u}_l|^p + |\nabla^2 \hat{u}_l|^p + |\nabla \hat{u}_l|^p) \, dV_{\hat{g}_l} \leqq C_R, \quad l \in \mathbb{N}, \ p \in \left(1, \frac{4}{3}\right),$$

and hence by the Poincaré inequality (recall that the $\hat{v}_l$'s have a uniform compact support) it follows that

$$(41) \qquad \int_{B_R^{\mathbb{R}^4}} |\hat{v}_l|^p \, dV_{\hat{g}_l} \leqq C_R, \quad l \in \mathbb{N}, \ p \in \left(1, \frac{4}{3}\right).$$

By (35) there holds

$$(42) \qquad \begin{aligned} P_{\hat{g}_l} \hat{v}_l &= (\Delta_{\hat{g}_l})^2 [\Psi_R(\hat{u}_l - a_l)] + L_l[\Psi_R(\hat{u}_l - a_l)] \\ &= \Psi_R P_{\hat{g}_l} \hat{u}_l + \tilde{L}_l(\hat{u}_l - a_l) = 2k_l \Psi_R e^{4\hat{u}_l} + \hat{f}_l, \end{aligned}$$

where

$$\hat{f}_l = \tilde{L}_l(\hat{u}_l - a_l) - 2r_l^4 Q_l(T_l \cdot).$$

Here $(L_l)_l$ are linear operators which contain derivatives of order 1 and 2 with uniformly bounded and smooth coefficients. Also, $(\tilde{L}_l)_l$ are linear operators which contain derivatives of order 0, 1, 2 and 3 with uniformly bounded and smooth coefficients. As a consequence, by (40) and (41) one has

$$(43) \qquad \int_{B_{2R}^{\mathbb{R}^4}} |\hat{f}_l|^p \, dV_{\hat{g}_l} \leqq C_R, \quad l \in \mathbb{N}, \ p \in \left(1, \frac{4}{3}\right).$$

Hence using (37) and Remark 3.2 one finds

$$(44) \qquad \int_{B_R^{\mathbb{R}^4}} e^{4q\hat{v}_l} \, dV_{\hat{g}_l} \leqq C, \quad \text{for some } q > 1$$

and for some fixed constant C. Remark 3.2 applies indeed to the case of a compact manifold while in the present situation we are working in $\mathbb{R}^4$ (endowed with the metric $\hat{g}_l$). But since all the functions $\hat{v}_l$ vanish identically outside $B_R^{\mathbb{R}^4}$, we can embed a fixed neighborhood of $(B_{2R}^{\mathbb{R}^4}, \hat{g}_l)$ into a compact manifold, a torus for example, such that its metric (coinciding with $\hat{g}_l$ on $B_{2R}^{\mathbb{R}^4}$) converges to the flat one.

On the other hand, from (37) we deduce

$$a_l = \frac{1}{|B_R^{\mathbb{R}^4}|} \int_{B_R^{\mathbb{R}^4}} \hat{u}_l \, dV_{\hat{g}_l} \leqq \frac{1}{4|B_R^{\mathbb{R}^4}|} \int_{B_R^{\mathbb{R}^4}} e^{4\hat{u}_l} \, dV_{\hat{g}_l} \leqq C,$$

and from (36), since $v_l = \hat{u}_l$ in $B_R^{\mathbb{R}^4}$

$$C^{-1} \leqq \int_{B_R^{\mathbb{R}^4}} e^{4v_l} \, dV_{\hat{g}_l} \leqq e^{4a_l} \int_{B_R^{\mathbb{R}^4}} e^{4\hat{v}_l} \, dV_{\hat{g}_l} \leqq Ce^{4a_l}.$$

This implies $a_l \geqq -C$, and hence we find

$$|a_l| \leqq C.$$

As a consequence of this estimate and (44) we get the following uniform improved integrability for $\hat{u}_l$ (recall the definition of v_l and $\hat{v}_l$)

$$\int_{B_R^{\mathbb{R}^4}} e^{4q\hat{u}_l} \, dV_{\hat{g}_l} \leqq C, \quad \text{for some } q > 1.$$

This estimate, joint with (40), (42), (43) and standard elliptic regularity results, yields that $\hat{u}_l$ is bounded in $W^{4,q}(B_{\frac{R}{2}}^{\mathbb{R}^4})$. Hence, by the arbitrarity of R, $(\hat{u}_l)_l$ converge strongly in $C^\alpha_{\text{loc}}(\mathbb{R}^4)$ for some $\alpha \in (0,1)$ and strongly in $H^2_{\text{loc}}(\mathbb{R}^4)$ to a function $\hat{u}_\infty \in C^\alpha_{\text{loc}}(\mathbb{R}^4) \cup H^2_{\text{loc}}(\mathbb{R}^4)$.

Now we prove that $\hat{u}_\infty$ has the form in (38). First of all, we test equation (35) on a smooth function φ with compact support. Integrating by parts we obtain

$$\langle P_{\hat{g}_l} \hat{u}_l, \varphi \rangle + 2r_l^4 \int_{\mathbb{R}^4} Q_l(T_l \cdot) \varphi \, dV_{\hat{g}_l} = 2k_l \int_{\mathbb{R}^4} e^{4\hat{u}_l} \varphi \, dV_{\hat{g}_l}.$$

As l tends to infinity we get

$$\langle P_{\mathbb{R}^4} \hat{u}_\infty, \varphi \rangle = 2k_0 \int_{\mathbb{R}^4} e^{4\hat{u}_\infty} \varphi \, dV_{\mathbb{R}^4} + o_l(1).$$

Hence the limit function $\hat{u}_\infty$ satisfies

$$\Delta^2_{\mathbb{R}^4} \hat{u}_\infty = 2k_0 e^{4\hat{u}_\infty} \quad \text{in } \mathbb{R}^4, \tag{45}$$

and, by semicontinuity

$$\int_{\mathbb{R}^4} e^{4\hat{u}_\infty} \, dV_{\mathbb{R}^4} \leqq 1, \tag{46}$$

since by (9) and some scaling there holds $\int_{B_{\frac{\delta}{r_l}}^{\mathbb{R}^4}} e^{4\hat{u}_l} \, dV_{\hat{g}_l} \leqq 1$.

The solutions of (45)–(46), with $k_0 > 0$, have been classified in [37], and one of the following two possibilities occur:

(a) $\hat{u}_\infty$ is of the form (38), or

(b) $\Delta_{\mathbb{R}^4} \hat{u}_\infty$ has the following asymptotic behavior:

$$-\Delta_{\mathbb{R}^4} \hat{u}_\infty(x) \to a > 0, \quad \text{for } |x| \to +\infty. \tag{47}$$

Following [43], we show that the second alternative does not happen. In fact, assuming (b), for R large we have

$$(48)\qquad \lim_{l\to+\infty} \int_{B_R^{\mathbb{R}^4}} (-\Delta_{\hat{g}_l}\hat{u}_l)\, dV_{\hat{g}_l} = \int_{B_R^{\mathbb{R}^4}} (-\Delta_{\mathbb{R}^4}\hat{u}_\infty)\, dV_{\mathbb{R}^4} \sim \frac{\omega_3}{4} aR^4,$$

where $\omega_3 = |S^3| = 2\pi^2$. Scaling back to M (recall that the dilation factor is r_l), we obtain

$$(49)\qquad \lim_{l\to+\infty} \int_{B_{Rr_l}(x_l)} (-\Delta u_l)\, dV_g \sim \bar{C}aR^4 r_l^2,$$

for some $\bar{C} > 0$. On the other hand, by Lemma 2.3 we get

$$(50)\qquad \int_{B_{Rr_l}(x_l)} (-\Delta u_l)\, dV_g \leqq \hat{C}_0 r_l^2 R^2.$$

Taking R sufficiently large, from (49) and (50) we reach a contradiction.

Hence the alternative (a) holds and $\hat{u}_\infty$ arises as a conformal factor of a stereographic projection of S^4 onto $\mathbb{R}^4$, which must satisfy

$$(51)\qquad \int_{\mathbb{R}^4} e^{4\hat{u}_\infty}\, dV_{\mathbb{R}^4} = \frac{8\pi^2}{k_0}.$$

This concludes the proof. □

4. Simple blow-ups

In this section we consider an unbounded sequence of solutions $(u_l)_l$ and we examine a particular class of blow-up points, which we call *simple*, in analogy with a definition introduced by R. Schoen. In Proposition 4.2 below we give some quantitative estimate on the concentration at simple blow-up points. Then in the next section we show that every general blow-up phenomenon can be essentially reduced to the study of finitely many simple blow-ups. In the following $i(M)$ denotes the injectivity radius of M.

Definition 4.1. If $(u_l)_l$ satisfies (7) and (9), we say that the three sequences $(x_l)_l \subseteqq M$, $r_l \to 0$, $(s_l)_l \subseteqq \mathbb{R}_+$, $|s_l| \leqq i(M)$ are a *simple blow-up* for $(u_l)_l$ if the following properties hold:

$$(52)\quad \frac{s_l}{r_l} \to +\infty, \exists R_l \to +\infty \text{ s.t. } \left\| \hat{u}_l - \log\frac{2}{1+|\cdot|^2} - \frac{1}{4}\log\left(\frac{1}{3}k_0\right)\right\|_{H^4(B_{R_l}^{\mathbb{R}^4})\cap C^\alpha(B_{R_l}^{\mathbb{R}^4})} \to 0,$$

$$(53)\quad \forall \rho > 0\ \exists C_\rho > 0 \text{ s.t. if } \int_{B_s(y)} e^{4u_l}\, dV_g \geqq \rho \text{ with } B_s(y) \subseteqq B_{s_l}(x_l)\backslash B_{R_l r_l}(x),$$

$$\text{then } s \geqq C_\rho^{-1}|y - x_l|,$$

where $\hat{u}_l$ is defined in (34).

The main result of this section is the following proposition.

Proposition 4.2. *Suppose $(x_l)_l$, $(r_l)_l$, $(s_l)_l$ are a simple blow-up for $(u_l)_l$. Then there exists a fixed $C > 0$ such that*

$$(54) \qquad \int_{B_{C^{-1}s_l}(x_l)} e^{4u_l}\, dV_g = \frac{8\pi^2}{k_0} + o_l(1),$$

where $o_l(1) \to 0$ as $l \to +\infty$.

Remark 4.3. (a) We notice that, if $\hat{u}_l$ satisfies the assertion in Proposition 3.4, it is always possible to modify $(x_l)_l$ and $(r_l)_l$ in order to get $x_0 = 0$ and $\lambda = 1$.

(b) Proposition 4.2 is basically an improvement of formula (53) to a sequence of sets with larger size.

The proof of Proposition 4.2 is based on the analysis of the next two subsections. In the first we prove some Harnack inequality in integral form while in the second, defining

$$(55) \qquad A_{r,l} = \{x \in M : r < |x - x_l| < 2r\},$$

we study the average of u_l on $A_{r,l}$ as a function of r.

4.1. Integral Harnack-type inequalities. In this subsection we prove some integral Harnack-type inequalities for the functions $(u_l)_l$ near simple blow-ups. Although it is maybe possible to get pointwise estimates on the solutions, for our purposes it is sufficient to obtain integral volume estimates. We need first a preliminary result involving the average of the Green's function G on annuli. Given $\rho \in (0, \pi^2/k_0)$, let C_ρ be the corresponding constant in (53) (which we can suppose bigger than 1), and we define the following sets:

$$(56) \qquad A'_{r,l} = \left\{x \in M : \frac{5}{4}r < |x - x_l| < \frac{7}{4}r\right\} \subseteqq A_{r,l}, \quad r \in (R_l r_l, s_l),$$

$$(57) \qquad \mathscr{B}_r(x) = B_{\frac{r}{16C_\rho}}(x) \subseteqq A'_{r,l}, \quad \tilde{\mathscr{B}}_r(x) = B_{\frac{r}{8C_\rho}}(x) \subseteqq A'_{r,l}, \quad x \in A'_{r,l}.$$

Lemma 4.4. *Suppose $(x_l)_l \subseteqq M$, $(s_l)_l \subseteqq \mathbb{R}_+$, $|s_l| \leqq i(M)$, and let $A_{r,l}$, $A'_{r,l}$, $\tilde{\mathscr{B}}_r(x)$ be defined respectively in* (55), (56) *and* (57). *Then there exists a positive constant $C = C(C_\rho)$, independent of r and l such that, setting*

$$f_{r,l}(y) = \frac{1}{|A_{r,l}|} \int_{A_{r,l}} G(z, y)\, dV_g(z),$$

there holds

$$(58) \qquad \begin{cases} \left|f_{r,l}(y) - \dfrac{1}{8\pi^2}\log\dfrac{1}{r}\right| \leqq C, & \text{for every } x \in A'_{r,l},\ y \in \tilde{\mathscr{B}}_r(x), \\ |f_{r,l}(y) - G(x,y)| \leqq C, & \text{for every } x \in A'_{r,l},\ y \in M \setminus \tilde{\mathscr{B}}_r(x), \end{cases} \quad r \leqq i(M).$$

Proof. We first notice that the following inequality holds:

$$\left|\bar{f}_r(y) - \log\frac{1}{r}\right| \leqq \bar{C}, \quad |y| \leqq 4r, \tag{59}$$

where

$$A_r = \{x \in \mathbb{R}^4 : r < |x| < 2r\}, \quad \bar{f}_r(y) = \frac{1}{|A_r|_{\mathbb{R}^4}} \int_{A_r} \log\frac{1}{|z-y|_{\mathbb{R}^4}} dV_{\mathbb{R}^4}.$$

Here $|A_r|_{\mathbb{R}^4}$ stands for the Lebesgue measure of A_r and $|z-y|_{\mathbb{R}^4}$ denotes the Euclidean distance.

The inequality is indeed trivial for $r = 1$ since $\bar{f}_1(y)$ is bounded on $B_4^{\mathbb{R}^4}$, while for a general r it is sufficient to use a scaling argument. We use (23), the exponential map and standard geometric estimates on M (see (69) below for the volume element) to write

$$\begin{aligned} 8\pi^2 f_{r,l}(y) &= \frac{1}{|A_{r,l}|} \int_{A_{r,l}} \log\frac{1}{|y-z|} dV_g(z) + O(1) \\ &= \left(1 + O(r^2)\right) \frac{1}{|A_r|_{\mathbb{R}^4}} \int_{A_r} \log\frac{1}{|y-z|_{\mathbb{R}^4}} \left(1 + O(r^2)\right) dV_{\mathbb{R}^4} + O(1) \\ &= \left(1 + O(r^2)\right) \bar{f}_r(y) + O(1), \quad y \in B_{4r}(x_l). \end{aligned}$$

Jointly with (59), this proves the first estimate in (58).

The second one is trivial for $y \in B_{4r}(x_l) \backslash \tilde{\mathscr{B}}_r(x)$, by the preceding argument. For $y \in M \backslash B_{4r}(x_l)$, we notice that

$$C^{-1} \leqq \frac{|z-y|}{|x-y|} \leqq C, \quad \text{for } z \in A_{r,l},\ x \in A'_{r,l},$$

and we use again (23). This concludes the proof. □

Next, we prove some inequality involving the integral of the function e^{4u_l} and the average of u_l on small annuli. We recall the definitions of $A_{r,l}$ and $A'_{r,l}$ in (55) and (56), and those of $\mathscr{B}_r(x)$, $\tilde{\mathscr{B}}_r(x)$ in (57).

Lemma 4.5. *Suppose that $(x_l)_l \subseteq M$, $r_l \to 0$, $(s_l)_l \subseteq \mathbb{R}_+$, $|s_l| \leqq i(M)$ are a simple blow-up for $(u_l)_l$. Suppose $R_l \to +\infty$, and define*

$$\bar{u}_{r,l} = \frac{1}{|A_{r,l}|} \int_{A_{r,l}} u_l \, dV_g, \quad R_l r_l < r < s_l.$$

Then, if l is sufficiently large, there exists a positive constant C (independent of l and r) such that

$$\int_{A'_{r,l}} e^{4u_l} \, dV_g \leqq C |A_{r,l}| e^{4\bar{u}_{l,r}}, \quad R_l r_l < r < s_l.$$

Proof. Using (22) and recalling the definition of f_l (see (21)) and that of $f_{r,l}$ (see Lemma 4.4), we have

$$\bar{u}_{r,l} = \bar{u}_l + \int_M f_{r,l}(y) f_l(y)\, dV_g(y).$$

For $x \in A'_{r,l}$, we divide the last integral into $\tilde{\mathscr{B}}_r(x)$ and its complement, to obtain

$$\exp\bigl(4(\bar{u}_{r,l} - \bar{u}_l)\bigr) = \exp\left(4 \int_{\tilde{\mathscr{B}}_r(x)} f_{r,l}(y) f_l(y)\, dV_g(y)\right) \exp\left(4 \int_{M \setminus \tilde{\mathscr{B}}_r(x)} f_{r,l}(y) f_l(y)\, dV_g(y)\right).$$

Using Lemma 4.4 and the fact that $(f_l)_l$ is bounded in $L^1(M)$, we then find

$$\exp\bigl(4(\bar{u}_{r,l} - \bar{u}_l)\bigr) \geqq C^{-1} \exp\left(\frac{1}{2\pi^2} \log \frac{1}{r} \int_{\tilde{\mathscr{B}}_r(x)} f_l(y)\, dV_g(y)\right) \exp\left(4 \int_{M \setminus \tilde{\mathscr{B}}_r(x)} G(x, y) f_l(y)\, dV_g(y)\right).$$

Hence, integrating on $A_{r,l}$ we obtain

$$\int_{A_{r,l}} e^{4(\bar{u}_{r,l} - \bar{u}_l)}\, dV_g \geqq C^{-1} |A_{r,l}| \left(\frac{1}{r}\right)^{\frac{\int_{\tilde{\mathscr{B}}_r(x)} f_l\, dV_g}{2\pi^2}} \exp\left(4 \int_{M \setminus \tilde{\mathscr{B}}_r(x)} G(x, y) f_l(y)\, dV_g(y)\right). \tag{60}$$

On the other hand, again by (22), for $x \in A'_{r,l}$ and a.e. $z \in \mathscr{B}_r(x)$ we have also

$$u_l(z) - \bar{u}_l = \int_{M \setminus \tilde{\mathscr{B}}_r(x)} G(z, y) f_l(y)\, dV_g(y) + \int_{\tilde{\mathscr{B}}_r(x)} G(z, y) f_l(y)\, dV_g(y).$$

Then, exponentiating and integrating on $\mathscr{B}_r(x)$ we get

$$\begin{aligned} &\int_{\mathscr{B}_r(x)} e^{4(u_l(z) - \bar{u}_l)}\, dV_g(z) \qquad (61)\\ &= \int_{\mathscr{B}_r(x)} \exp\left(4 \int_{M \setminus \tilde{\mathscr{B}}_r(x)} G(z, y) f_l(y)\, dV_g(y)\right) \exp\left(4 \int_{\tilde{\mathscr{B}}_r(x)} G(z, y) f_l(y)\, dV_g(y)\right) dV_g(z) \\ &\leqq \underbrace{\sup_{z \in \mathscr{B}_r(x)} \exp\left(4 \int_{M \setminus \tilde{\mathscr{B}}_r(x)} G(z, y) f_l(y)\, dV_g(y)\right)}_{\mathrm{J}} \underbrace{\int_{\mathscr{B}_r(x)} \exp\left(4 \int_{\tilde{\mathscr{B}}_r(x)} G(z, y) f_l(y)\, dV_g(y)\right) dV_g(z)}_{\mathrm{JJ}}. \end{aligned}$$

Now we write

$$\begin{aligned} \int_{M \setminus \tilde{\mathscr{B}}_r(x)} G(z, y) f_l(y)\, dV_g(y) = &\int_{M \setminus \tilde{\mathscr{B}}_r(x)} G(x, y) f_l(y)\, dV_g(y) \\ &+ \int_{M \setminus \tilde{\mathscr{B}}_r(x)} \bigl(G(z, y) - G(x, y)\bigr) f_l(y)\, dV_g(y). \end{aligned}$$

Using (23), for $z \in \mathscr{B}_r(x)$ and $y \in M \setminus \tilde{\mathscr{B}}_r(x)$, we have

$$G(z, y) - G(x, y) = O(1) + \frac{1}{8\pi^2} \log \frac{|z - y|}{|x - y|} = O(1).$$

As a consequence we deduce

$$\mathrm{J} \leqq C \exp\left(4 \int_{M \setminus \tilde{\mathscr{B}}_r(x)} G(x,y) f_l(y)\, dV_g(y)\right). \tag{62}$$

We now turn to JJ. Since $z \in \mathscr{B}_r(x)$ and $y \in \tilde{\mathscr{B}}_r(x)$, $G(z,y)$ is positive (for r sufficiently small), and hence

$$\int_{\tilde{\mathscr{B}}_r(x)} G(z,y) f_l(y)\, dV_g(y) \leqq \int_{\tilde{\mathscr{B}}_r(x)} G(z,y) |f_l|(y)\, dV_g(y).$$

Using the Jensen inequality, as in the proof of Proposition 3.1, we obtain

$$\exp\left(4 \int_{\tilde{\mathscr{B}}_r(x)} G(z,y) f_l(y)\, dV_g(y)\right) \leqq \int_{\tilde{\mathscr{B}}_r(x)} \exp\big(4G(z,y) \|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}\big) \frac{|f_l(y)|}{\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}}\, dV_g(y).$$

Again (23) implies

$$\begin{aligned} \mathrm{JJ} &\leqq \int_{\mathscr{B}_r(x)} dV_g(z) \int_{\tilde{\mathscr{B}}_r(x)} \exp\big(4G(z,y)\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}\big) \frac{|f_l(y)|}{\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}}\, dV_g(y) \\ &\leqq C \int_{\mathscr{B}_r(x)} dV_g(z) \int_{\tilde{\mathscr{B}}_r(x)} \left(\frac{1}{|z-y|}\right)^{\frac{\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}}{2\pi^2}} \frac{|f_l(y)|}{\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}}\, dV_g(y). \end{aligned}$$

Now, the Fubini theorem and some elementary computations yield

$$\mathrm{JJ} \leqq C \sup_{y \in M} \int_{\mathscr{B}_r(x)} dV_g(z) \left(\frac{1}{|z-y|}\right)^{\frac{\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}}{2\pi^2}} \leqq C r^{4 - \frac{\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}}{2\pi^2}}. \tag{63}$$

In the last inequality we have used the fact that $\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}$ is uniformly small since we are dealing with a simple blow-up, see (53), and since we have chosen $\tilde{\mathscr{B}}_r(x)$ suitably. This implies that the last constant C is independent of r and l. From (61), (62) and (63) it follows that

$$\int_{\mathscr{B}_r(x)} e^{4(u_l(z) - \bar{u}_l)}\, dV_g(z) \leqq C r^{4 - \frac{\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}}{2\pi^2}} \exp\left(4 \int_{M \setminus \tilde{\mathscr{B}}_r(x)} G(x,y) f_l(y)\, dV_g(y)\right).$$

Now the assertion of the lemma follows from the last formula, (60) and the observation that, since $f_l = 2k_l e^{4u_l} - 2Q_l$, it is $\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))} = \int_{\tilde{\mathscr{B}}_r(x)} f_l\, dV_g + O(r^4)$, and hence

$$r^{4 - \frac{\|f_l\|_{L^1(\tilde{\mathscr{B}}_r(x))}}{2\pi^2}} \leqq C|A_{r,l}| \left(\frac{1}{r}\right)^{\frac{\int_{\tilde{\mathscr{B}}_r(x)} f_l\, dV_g}{2\pi^2}} \quad \text{independently of } r \text{ and } l.$$

This concludes the proof. □

Next we show some further estimates involving the Laplacian of u_l. Recall that we have set $f_l = 2k_l e^{4u_l} - Q_l$, see (21).

Lemma 4.6. *Suppose that $(x_l)_l \subseteqq M$, $(\Sigma_l)_l, (S_l)_l \subseteqq \mathbb{R}_+$, $i(M) \geqq S_l > \Sigma_l > 0$, and that $(u_l)_l$ satisfies* (7) *and* (9). *Suppose also that*

$$\int_{B_{S_l}(x_l)\setminus B_{\Sigma_l}(x_l)} e^{4u_l}\,dV_g \leqq \varepsilon.$$

Then, for any $R > 0$ sufficiently large and any $r \in [\Sigma_l + R, S_l - R]$, one has

$$\int_{A_{r,l}} |x - x_l|^2\big(-\Delta u_l(x)\big)\,dV_g(x) = \left(\frac{15}{8}\int_{B_{\frac{r}{R}}(x_l)} f_l\,dV_g + o_R(1) + O(\varepsilon R^2) + o_r(1)\right)r^4,$$

where $o_R(1) \to 0$ as $R \to +\infty$ and $o_r(1) \to 0$ as $r \to 0$.

Proof. We can write (7) in the following form:

$$-\Delta(-\Delta u_l) = f_l + F_l(u_l),$$

where F_l is a linear expression in ∇u_l and $\nabla^2 u_l$ with uniformly bounded coefficients. If $\hat{G}$ is the Green's function for the (negative) Laplacian on M, then it is a standard fact that

$$\hat{G}(x,y) = \big(1 + o(1)\big)\frac{1}{4\pi^2|x-y|^2}, \quad (x,y) \in M \times M\setminus\mathrm{diag}, \tag{64}$$

where $o(1) \to 0$ as $|x - y| \to 0$, see for example [2]. Hence, using the representation formula, for a.e. $x \in A_{r,l}$ we obtain

$$\begin{aligned} -\Delta u_l(x) &= \int_M \hat{G}(x,y) f_l(y)\,dV_g(y) + \int_M \hat{G}(x,y)F_l(u_l)(y)\,dV_g(y) \\ &:= v_{1,l}(x) + v_{2,l}(x). \end{aligned} \tag{65}$$

Given $R > 0$ large but fixed and for $|x - x_l| = r \in [\Sigma_l + R, S_l - R]$, we write

$$\begin{aligned} v_{1,l}(x) = &\int_{B_{\frac{r}{R}}(x_l)} \hat{G}(x,y) f_l(y)\,dV_g(y) + \int_{B_{Rr}(x_l)\setminus B_{\frac{r}{R}}(x_l)} \hat{G}(x,y) f_l(y)\,dV_g(y) \\ &+ \int_{M\setminus B_{Rr}(x_l)} \hat{G}(x,y) f_l(y)\,dV_g(y). \end{aligned}$$

From the asymptotics in (64) and some scaling argument we obtain (for $x \in A_{r,l}$)

$$\int_{B_{\frac{r}{R}}(x_l)} \hat{G}(x,y) f_l(y)\,dV_g(y) = \big(1 + o_r(1) + o_R(1)\big)\frac{1}{4\pi^2 r^2}\int_{B_{\frac{r}{R}}} f_l\,dV_g,$$

$$\left|\int_{M\setminus B_{Rr}(x_l)} \hat{G}(x,y) f_l(y)\,dV_g(y)\right| \leqq \frac{C}{(Rr)^2},$$

where $o_r(1) \to 0$ as $r \to 0$ and $o_R(1) \to 0$ as $R \to +\infty$. Moreover, by our assumptions and (21), we have

$$\int_{B_{Rr}(x_l)\setminus B_{\frac{r}{R}}(x_l)} f_l(y)\,dV_g(y) \leqq C\varepsilon, \quad f_l(x) \geqq -C,$$

where C is independent of r, and l. Using the Fubini theorem and reasoning as in the proof of Lemma 2.3 it follows that

$$\left| \int_{A_{r,l}} dV_g(x) \int_{B_{Rr}(x_l)\setminus B_{\frac{r}{R}}(x_l)} \hat{G}(x,y) f_l(y)\, dV_g(y) \right| \leqq C\varepsilon R^2 r^2.$$

The last formulas imply

$$\begin{aligned} (66) \qquad \int_{A_{r,l}} |x-x_l|^2 v_{1,l}(x)\, dV_g(x) &= \left(\frac{1+o_r(1)+o_R(1)}{4\pi^2} \int_{B_{\frac{r}{R}}} f_l\, dV_g + O(\varepsilon R^2) + O\left(\frac{1}{R^2}\right) \right) |A_{r,l}| \\ &= \left(\frac{15}{8} \int_{B_{\frac{r}{R}}(x_l)} f_l\, dV_g + o_R(1) + O(\varepsilon R^2) + o_r(1) \right) r^4. \end{aligned}$$

To study the integral of $v_{2,l}$, we use again the representation formula and we write

$$\begin{aligned} |v_{2,l}(x)| \leqq{}& C \int_{B_{r^2}(x_l)} \frac{1}{|x-y|^2} \left(|\nabla^2 u_l|(y) + |\nabla u_l|(y) \right) dV_g(y) \\ &+ C \int_{M\setminus B_{\sqrt{r}}(x_l)} \frac{1}{|x-y|^2} \left(|\nabla^2 u_l|(y) + |\nabla u_l|(y) \right) dV_g(y) \\ &+ \underbrace{C \int_{B_{\sqrt{r}}(x_l)\setminus B_{r^2}(x_l)} \frac{1}{|x-y|^2} \left(|\nabla^2 u_l|(y) + |\nabla u_l|(y) \right) dV_g(y)}_{\mathrm{JJJ}}. \end{aligned}$$

To estimate the first and the second integral, we notice that $|x-y| \geqq C^{-1} r$ and $|x-y| \geqq C^{-1}\sqrt{r}$ for respectively $y \in B_{r^2}(x_l)$ and $y \in B_{\sqrt{r}}(x_l)$ (recall that $x \in A_{r,l}$). Hence using Lemma 2.3 it follows that

$$\int_{B_{r^2}(x_l)} \frac{1}{|x-y|^2} \left(|\nabla^2 u_l|(y) + |\nabla u_l|(y) \right) dV_g(y) \leqq Cr^2,$$

$$\int_{M\setminus B_{\sqrt{r}}(x_l)} \frac{1}{|x-y|^2} \left(|\nabla^2 u_l|(y) + |\nabla u_l|(y) \right) dV_g(y) \leqq \frac{C}{r}.$$

To estimate the third integral we use the Hölder's inequality to find, for $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\mathrm{JJJ} \leqq C\left(\int_{B_{\sqrt{r}}(x_l)\setminus B_{r^2}(x_l)} \frac{1}{|x-y|^{2p}} dV_g(y) \right)^{\frac{1}{p}} \left(\int_{B_{\sqrt{r}}(x_l)\setminus B_{r^2}(x_l)} \left(|\nabla^2 u_l|(y) + |\nabla u_l|(y) \right)^{p'} dV_g(y) \right)^{\frac{1}{p'}}.$$

Again by and Lemma 2.3 it follows that for $p > 2$ (and hence for $p' < 2$) it is $\mathrm{JJJ} \leqq Cr^{\frac{6}{p}-4}$. If we choose $p \in (2,3)$, then $\frac{6}{p} - 4 > -2$, which implies $\mathrm{JJJ} < o_r(1) r^2$, and hence also

$$(67) \qquad \int_{A_{r,l}} v_{2,l}\, dV_g = o_r(1) r^2.$$

Then, choosing first R sufficiently large and then l sufficiently large, (65), (66) and (67) conclude the proof. $\square$

4.2. Radial behavior. The next step consists in studying the dependence on r of the function $\bar{u}_{r,l}$ defined in Lemma 4.5. It is well known that in geodesic coordinates the metric coefficients g_{ij} have the expression

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l + O(|x|^3), \tag{68}$$

where R_{ikjl} are the components of the curvature tensor, see for example [34], and the volume element satisfies

$$dV_g = \sqrt{\det g}\, dV_{\mathbb{R}^4} = \left(1 + O(|x|^2)\right) dV_{\mathbb{R}^4} \tag{69}$$

$$\text{with } \nabla\sqrt{\det g} = O(|x|) \text{ and } \nabla^2\sqrt{\det g} = O(1).$$

Using the exponential map at x_l, we can use coordinates r, θ in a neighborhood of x_l, where $r = |x| > 0$ and $\theta \in S^3$. In these coordinates the volume element dV_g and the surface element $d\sigma_g$ take the form

$$dV_g = r^3 \tilde{f}(r,\theta)\, dr\, d\theta, \quad d\sigma_g = \tilde{f}(r,\theta)\, d\theta,$$

where $\tilde{f}$ is a smooth bounded function on $\{r > 0\}$. Using these coordinates, considering a regular function h, and letting $A_{\tilde{r}} = B_{2\tilde{r}}(x_l) \backslash B_{\tilde{r}}(x_l)$, one has

$$\int_{A_{\tilde{r}}} h\, dV_g = \int_{\tilde{r}}^{2\tilde{r}} r^3\, dr \int_{S^3} h(r,\theta) f(r,\theta)\, d\theta, \quad \frac{\partial h}{\partial v}(r,\theta) = \frac{\partial h}{\partial r}(r,\theta),$$

where v denotes the exterior unit normal to $\partial B_{\tilde{r}}(x_l)$.

We also use the coordinates z, θ, where $z = \log r$. In these new coordinates we obtain

$$dV_g = e^{4z} f(z,\theta)\, dz\, d\theta, \quad d\sigma_g = e^{3z} f(z,\theta)\, d\theta,$$

where $f(z,\theta) = \tilde{f}(e^z,\theta)$, and

$$\int_{A_{\tilde{r}}} h\, dV_g = \int_s^{s+\beta} dz \int_{S^3} h(z,\theta) f(z,\theta) e^{4z}\, d\theta, \quad \frac{\partial h}{\partial v}(z,\theta) = e^{-z} \frac{\partial h}{\partial z}(z,\theta).$$

Here we have set $\beta = \log 2$ and $s = \log \tilde{r}$. From (69) we also find

$$f(z,\theta) = 1 + O(e^{2z}), \quad \frac{\partial f}{\partial z}(z,\theta) = O(e^{2z}), \quad \frac{\partial^2 f}{\partial z^2}(z,\theta) = O(e^{2z}). \tag{70}$$

Now we can write

$$(71)\qquad \frac{\partial}{\partial s}\int_{A_{\tilde{r}}} h\,dV_g = \int_{S^3} h(z,\theta)e^{4z}f(z,\theta)\,d\theta|_{z=s}^{s+\beta}$$

$$= \int_s^{s+\beta}\int_{S^3} \frac{\partial}{\partial z}\big(h(z,\theta)e^{4z}f(z,\theta)\big)\,d\theta\,dz$$

$$= \int_s^{s+\beta}\int_{S^3} \frac{\partial h}{\partial z} e^{4z}f(z,\theta)\,d\theta\,dz$$

$$+ \int_s^{s+\beta}\int_{S^3} h(z,\theta)\left(4f(z,\theta)e^{4z} + e^{4z}\frac{\partial f}{\partial z}(z,\theta)\right)d\theta\,dz.$$

Taking a second derivative with respect to s, from the above formulas we obtain

$$\frac{\partial^2}{\partial s^2}\int_{A_{\tilde{r}}} h\,dV_g = \int_{S^3}\frac{\partial h}{\partial z}(z,\theta)e^{4z}h(z,\theta)\,d\theta|_{z=s}^{s+\beta} + 4\frac{\partial}{\partial s}\int_{A_{\tilde{r}}} h\,dV_g$$

$$+\frac{\partial}{\partial s}\left(\int_s^{s+\beta}\int_{S^3} h(z,\theta)e^{4z}\frac{\partial f}{\partial z}(z,\theta)\,d\theta\,dz\right)$$

$$= \int_{\partial A_{\tilde{r}}} e^{2z}\frac{\partial h}{\partial v}d\sigma_g + 4\frac{\partial}{\partial s}\int_{A_{\tilde{r}}} h\,dV_g + \frac{\partial}{\partial s}\left(\int_s^{s+\beta}\int_{S^3} h(z,\theta)e^{4z}\frac{\partial f}{\partial z}(z,\theta)\,d\theta\,dz\right).$$

Using the coordinates (r,θ) and integrating by parts we derive

$$\int_{\partial A_{\tilde{r}}} e^{2z}\frac{\partial h}{\partial v}d\sigma_g = \int_{\partial A_{\tilde{r}}} r^2\frac{\partial h}{\partial v}d\sigma = \int_{A_{\tilde{r}}} r^2\Delta h\,dV_g - \int_{A_{\tilde{r}}} h\Delta r^2\,dV_g + \int_{\partial A_{\tilde{r}}} h\frac{\partial r^2}{\partial v}d\sigma_g$$

$$= \int_{A_{\tilde{r}}} r^2\Delta h\,dV_g - 8\int_{A_{\tilde{r}}} h\,dV_g + 2\int_{\partial A_{\tilde{r}}} he^{4z}\,d\sigma_g + \int_{A_{\tilde{r}}}(\Delta r^2 - 8)h\,dV_g.$$

By the last two formulas we finally get the following equation:

$$(72)\qquad \frac{\partial^2}{\partial s^2}\int_{A_{\tilde{r}}} h\,dV_g = 6\frac{\partial}{\partial s}\int_{A_{\tilde{r}}} h\,dV_g - 8\int_{A_{\tilde{r}}} h\,dV_g + \int_{A_{\tilde{r}}} r^2\Delta h\,dV_g$$

$$+ \int_{A_{\tilde{r}}}(\Delta r^2 - 8)h\,dV_g + \frac{\partial}{\partial s}\left(\int_s^{s+\beta}\int_{S^3} h(z,\theta)e^{4z}\frac{\partial f}{\partial z}(z,\theta)\,d\theta\,dz\right).$$

Next we want to apply (72) to the case of $h = u_l$, and derive a differential equation involving the average $\bar{u}_{r,l}$ of u_l on the annuli $A_{r,l}$.

Lemma 4.7. *Suppose that* $(x_l)_l \subseteq M$, $(s_l)_l \subseteq \mathbb{R}_+$, $i(M) \geqq s_l > 0$, *and that* $(u_l)_l$ *satisfies* (7) *and* (9). *Then, for every* l *and every* $r < s_l$ *we let*

$$W_l(z) = \frac{1}{\mathrm{Vol}(A_{r,l})}\int_{A_{r,l}} u_l\,dV_g, \qquad z = \log r,$$

where $A_{r,l}$ is defined in (55). *Then the functions $W_l(z)$ solve the following equation*:

$$W_l''(z) + 2\big(1 + O(e^{2z})\big) W_l'(z) = \frac{\int_{A_{r,l}} r^2 \Delta_g u_l \, dV_g}{\mathrm{Vol}(A_{r,l})} + O(e^{2z}), \tag{73}$$

$$\textit{for } z \in \big(\log(a_l r_l), \log s_l\big).$$

We first notice that $W_l(z)$ coincides with $\bar{u}_{r,l}$ up to the change of variables $r \mapsto z = \log r$.

Proof. We first let

$$\tilde{W}_l(z) = \int_{A_{r,l}} u_l \, dV_g, \quad Y_l(z) = \int_{A_{r,l}} dV_g, \quad z = \log r.$$

We have clearly

$$W_l'(z) = \left(\frac{\tilde{W}_l(z)}{Y_l(z)}\right)' = \frac{\tilde{W}_l'(z) Y_l(z) - Y_l'(z) \tilde{W}_l(z)}{Y_l^2(z)},$$

and

$$W_l''(z) = \frac{Y_l^2(z)[\tilde{W}_l''(z) Y_l(z) - Y_l''(z) \tilde{W}_l(z)] - 2 Y_l(z) Y_l'(z)[\tilde{W}_l'(z) Y_l(z) - Y_l'(z) \tilde{W}_l(z)]}{Y_l^4(z)}.$$

Using the last two formulas and (72) with $A_{\tilde{r}} = A_{r,l}$ and $h = u_l$, after some calculation (which also uses (71) with h replaced by $\frac{h}{f} \frac{\partial f}{\partial z}$) we obtain

$$\begin{aligned} W_l''(z) &= 6 W_l'(z) - 2 \frac{Y_l'(z)}{Y_l(z)} W_l'(z) + \frac{\int_{A_{r,l}} r^2 \Delta_g u_l \, dV_g}{Y_l(z)} \\ &\quad + \left[\int (\Delta_g r^2 - 8) u_l + \int \frac{\partial}{\partial z}\left(\frac{u_l \frac{\partial f}{\partial z}}{f}\right) e^{4z} f + \int u_l \frac{\frac{\partial f}{\partial z}}{f}\left(4 f e^{4z} + e^{4z} \frac{\partial f}{\partial z}\right)\right] \frac{\int e^{4z} f}{Y_l(z)^2} \\ &\quad - \left[\int (\Delta_g r^2 - 8) + \int \frac{\partial}{\partial z}\left(\frac{\frac{\partial f}{\partial z}}{f}\right) e^{4z} f + \int \frac{\frac{\partial f}{\partial z}}{f}\left(4 f e^{4z} + e^{4z} \frac{\partial f}{\partial z}\right)\right] \frac{\int u_l e^{4z} f}{Y_l(z)^2}. \end{aligned}$$

We notice that, adding and subtracting the average of $\bar{u}_{r,l}$ to u_l, some cancellation occurs. Moreover, from (70) and (71) we get

$$\frac{Y_l'(z)}{Y_l(z)} = \frac{\int \left(4 e^{4z} f + e^{4z} \frac{\partial f}{\partial z}\right)}{Y_l(z)} = 4 + O(e^{2z}).$$

Therefore, using these remarks we obtain

$$
\begin{aligned}
W_l''(z) = {} & -2\big(1+O(e^{2z})\big)W_l'(z) + \frac{\int_{A_{r,l}} r^2 \Delta_g u_l \, dV_g}{Y_l(z)} \\
& + \left[\int (\Delta_g r^2 - 8)(u_l - \bar{u}_{r,l}) + \int \frac{\partial}{\partial z}\left(\frac{(u_l - \bar{u}_{r,l})\frac{\partial f}{\partial z}}{f} \right) e^{4z} f \right. \\
& \left. + \int (u_l - \bar{u}_{r,l}) \frac{\frac{\partial f}{\partial z}}{f}\left(4fe^{4z} + e^{4z}\frac{\partial f}{\partial z}\right) \right] \frac{\int e^{4z} f}{Y_l(z)^2} \\
& - \left[\int (\Delta_g r^2 - 8) + \int \frac{\partial}{\partial z}\left(\frac{\frac{\partial f}{\partial z}}{f} \right) e^{4z} f + \int \frac{\frac{\partial f}{\partial z}}{f}\left(4fe^{4z} + e^{4z}\frac{\partial f}{\partial z}\right) \right] \frac{\int (u_l - \bar{u}_{r,l}) e^{4z} f}{Y_l(z)^2}.
\end{aligned}
$$

We next estimate the terms in the last three lines of this expression. We begin by noticing that $(\Delta r^2 - 8) = O(r^2)$, which can be deduced from elementary computations in local coordinates. This and the Poincaré inequality imply

$$
\left| \int (\Delta_g r^2 - 8)(u_l - \bar{u}_{r,l}) \, dV_g \right| \leqq Ce^{3z} \int_{A_{r,l}} |\nabla u_l| \, dV_g, \quad z = \log r.
$$

From Lemma 2.3 then one finds

$$
\left| \int (\Delta_g r^2 - 8)(u_l - \bar{u}_{r,l}) \, dV_g \right| \leqq Ce^{6z}.
$$

Similarly, using (70) and also the fact that $\frac{\partial u_l}{\partial z} = \frac{\partial u_l}{\partial r} \frac{\partial r}{\partial z} = O(e^z |\nabla u_l|)$, we obtain

$$
\begin{aligned}
\left| \int \frac{\partial}{\partial z}\left(\frac{(u_l - \bar{u}_{r,l})\frac{\partial f}{\partial z}}{f} \right) e^{4z} f \right| & \leqq \int_{A_{r,l}} O(e^{2z}) |u_l - \bar{u}_{r,l}| \, dV_g + \int_{A_{r,l}} O(e^{3z}) |\nabla u_l| \, dV_g \\
& \leqq Ce^{6z}.
\end{aligned}
$$

Reasoning in the same way for the remaining terms we finally deduce

$$
W_l''(z) + 2\big(1+O(e^{2z})\big)W_l'(z) = \frac{\int_{A_{r,l}} r^2 \Delta_g h \, dV_g}{Y_l(z)} + O(e^{2z}).
$$

Then the last four estimates imply the first equation in (73). □

Remark 4.8. Using (71) with $A_{\tilde{r}} = A_{r,l}$, and with $h = u_l$ (or with $h = 1$ to compute Y_l'), we obtain

$$W_l'(z) = \frac{\left[\int_{A_{r,l}} u_l(4fe^{4z} + e^{4z}\frac{\partial f}{\partial z}e^{4z}f) + \int_{A_{r,l}} \frac{\partial u_l}{\partial z}\right] \int_{A_{r,l}} fe^{4z} - \left[\int_{A_{r,l}} 4e^{4z}f + e^{4z}\frac{\partial f}{\partial z}\right] \int_{A_{r,l}} u_l fe^{4z}}{\left(\int_{A_{r,l}} fe^{4z}\right)^2}.$$

If we denote again by $\bar{u}_{r,l}$ the average of u_l in the annulus $A_{r,l}$, adding and subtracting $\bar{u}_{r,l}$ from u_l in the last formula we get some cancellations and we are left with

$$W_l'(z) = \frac{\left[\int_{A_{r,l}} (u_l - \bar{u}_{r,l})\left(e^{4z}\frac{\partial f}{\partial z}\right) + \int_{A_{r,l}} \frac{\partial u_l}{\partial z}e^{4z}f\right] \int_{A_{r,l}} fe^{4z}}{\left(\int_{A_{r,l}} fe^{4z}\right)^2} - \frac{\left[\int_{A_{r,l}} e^{4z}\frac{\partial f}{\partial z}\right] \int_{A_{r,l}} (u_l - \bar{u}_{r,l})fe^{4z}}{\left(\int_{A_{r,l}} fe^{4z}\right)^2}.$$

As a byproduct of this formula and the Poincaré inequality we deduce

$$|W_l'(z)| \leqq C\frac{\int_{A_{r,l}} |u_l - \bar{u}_{r,l}|\, dV_g}{Y_l(z)} + Cr\frac{\int_{A_{r,l}} |\nabla u_l|\, dV_g}{Y_l(z)} \leqq Cr\frac{\int_{A_{r,l}} |\nabla u_l|\, dV_g}{Y_l(z)}.$$

Then, applying Lemma 2.3, we find

$$|W_l'(z)| \leqq C. \tag{74}$$

In the next lemma we study the solutions of (73) in the case of a simple blow-up. When $x_0 = 0$ and $\lambda = 1$, the function $\hat{u}_\infty$, see (38), is of the form

$$\hat{u}_\infty(x) = \log\left(\frac{2}{1+|x|^2}\right) + \frac{1}{4}\log\frac{3}{k_0}.$$

From straightforward computations one finds

$$\int_{A_r} \hat{u}_\infty\, dV_{\mathbb{R}^4} = 2\pi^2\left[\frac{15}{4}r^4\log 2 + 4r^4\log\left(\frac{1}{1+4r^2}\right) + \frac{15}{8}r^4 - \frac{3}{4}r^2 + \frac{1}{4}\log(1+4r^2)\right.$$
$$\left. - \frac{1}{4}r^4\log\left(\frac{1}{1+r^2}\right) - \frac{1}{4}\log(1+r^2)\right].$$

Scaling back to u_l, using (52) and some elementary estimates one deduces (for $t > 0$ large and fixed)

$$\begin{aligned} W_l(\log r_l + t) &= -2t + \bar{C} - \log r_l + O(e^{-2t}) + o_l(1), \\ W_l'(\log r_l + t) &= -2 + O(e^{-2t}) + o_l(1), \end{aligned} \tag{75}$$

where $\bar{C}$ is some explicit positive constant.

Now we prove some upper bounds for the function W_l. Notice from (75) that W_l at $z = \log r_l + t$ (t large and fixed) has slope close to -2. Given $\gamma \in (1, 2)$, we consider an affine function $h^\gamma_{t,l}$ which coincides with W_l for $z \sim \log r_l$ and which has slope $-\gamma > -2$. The next lemma asserts that indeed $W_l(z) < h^\gamma_{t,l}(z)$ until z gets close to $\log s_l$. This is helpful to get integral estimates on e^{4u_l}, which is done at the end of the section.

Lemma 4.9. *Suppose $(x_l)_l$, $(r_l)_l$, $(s_l)_l$ are a simple blow-up for $(u_l)_l$, and let $(W_l)_l$ be given by Lemma 4.7. Given $\gamma \in (1,2)$ and $t > 0$, consider the following functions:*

$$h^\gamma_{t,l}(z) = -\gamma(z - \log r_l - t) + W_l(\log r_l + t).$$

Then there exist $t_l \to +\infty$ arbitrarily slowly and $C_\gamma > 0$ such that for l large

$$W_l(z) \leqq h^\gamma_{t_l,l}(z), \quad z \in [\log r_l + t_l, \log s_l - C_\gamma].$$

Proof. Recall that $(W_l)_l$ are solutions of (73) satisfying the *initial* conditions (75) for any large and fixed t. If $t_l \to +\infty$ sufficiently slowly, we can also replace t by t_l in (75), namely we can also assume that

$$(76) \qquad W_l(\log r_l + t_l) = -2t_l + \bar{C} - \log r_l + o_l(1), \quad W_l'(\log r_l + t_l) = -2 + o_l(1).$$

Suppose by contradiction that there exist $\bar{s}_l \in [\log r_l, \log s_l]$, with $\log s_l - \bar{s}_l \to +\infty$ such that W_l intersects $h^\gamma_{t_l,l}$ for the first time. We notice that, by the asymptotics in (75), it must also be $\bar{s}_l - \log r_l - t_l \to +\infty$ if $t_l \to +\infty$ sufficiently slowly. Then we have

$$W_l(\bar{s}_l) = h^\gamma_{t_l,l}(\bar{s}_l), \quad W_l'(\bar{s}_l) \geqq -\gamma.$$

We now choose a sequence of real numbers $(H_l)_l$ by means of the following condition:

$$H_l = \sup\{H \in \mathbb{R} : h^{\frac{\gamma+2}{2}}_{t_l,l} + H < W_l \text{ in } [\log r_l + t_l, \bar{s}_l]\}.$$

By (75) it must be $H_l \to -\infty$ as $l \to +\infty$ (provided $t_l \to +\infty$ sufficiently slowly), and there exist $\tilde{s}_l$ such that

$$(77) \qquad W_l(\tilde{s}_l) = h^{\frac{\gamma+2}{2}}_{t_l,l}(\tilde{s}_l) + H_l, \quad W_l'(\tilde{s}_l) = -\frac{\gamma+2}{2}, \quad W_l''(\tilde{s}_l) \geqq 0.$$

Moreover, by (74) and (75), $\tilde{s}_l$ satisfies

$$(78) \qquad |\bar{s}_l - \tilde{s}_l| \to +\infty \quad \text{as } l \to +\infty, \quad |\tilde{s}_l - \log r_l - t_l| \to +\infty \quad \text{as } l \to +\infty.$$

Next we claim that, for $C > 0$ sufficiently large, the following property holds:

$$(79) \qquad \int_{B_{\frac{e^{\tilde{s}_l}}{C}}(x_l) \setminus B_{e^{t_l} r_l}(x_l)} e^{4u_l}\, dV_g \to 0 \quad \text{as } l \to +\infty.$$

In order to prove this claim, let us recall that by our choice of $\bar{s}_l$, it is $W_l(z) \leqq h^\gamma_{t_l,l}(z)$ for every $z \in [\log r_l + t_l, \bar{s}_l]$. Dividing the region $B_{\frac{e^{\tilde{s}_l}}{C}}(x_l) \setminus B_{e^{t_l} r_l}(x_l)$ into concentric annuli $A'_{r,l}$ (see (56)) of suitable radii, we get

$$\int_{B_{\frac{e^{\bar{s}_l}}{C}}(x_l)\setminus B_{e^{t_l}r_l}(x_l)} e^{4u_l}\,dV_g \leqq \sum_{j=0}^{j_l}\int_{A'_{\hat{r}_{l,j},l}} e^{4u_l}\,dV_g,$$

where

$$\hat{r}_{l,j} = \frac{4}{5}e^{t_l}r_l\left(\frac{7}{5}\right)^j, \quad \left(\frac{7}{5}\right)^{j_l} \in \left(\frac{5}{4}\frac{e^{\bar{s}_l}}{Ce^{t_l}r_l}, \frac{5}{2}\frac{e^{\bar{s}_l}}{Ce^{t_l}r_l}\right).$$

Given $\gamma \in (1,2)$, from Lemma 4.5 it follows that

$$\int_{A'_{\hat{r}_{l,j},l}} e^{4u_l}\,dV_g \leqq C|A_{\hat{r}_{l,j},l}|e^{4\bar{u}_{l,\hat{r}_{l,j}}} \leqq C\hat{r}_{l,j}^4 e^{4W_l(\log\hat{r}_{l,j})} \leqq C\hat{r}_{l,j}^4 e^{4h^{\gamma}_{t_l,l}(\log\hat{r}_{l,j})}, \quad j = 1,\dots,j_l.$$

From the expression of $h^{\gamma}_{t_l,l}$ and (76) we deduce

$$\begin{aligned}
\hat{r}_{l,j}^4 e^{4h^{\gamma}_{t_l,l}(\log\hat{r}_{l,j})} &\leqq C\hat{r}_{l,j}^4 \exp\bigl[4\bigl(-\gamma(\log\hat{r}_{l,j} - \log r_l - t_l) - 2t_l + \bar{C} - \log r_l + o_l(1)\bigr)\bigr] \\
&= C\hat{r}_{l,j}^4 \exp[-4\gamma\log\hat{r}_{l,j} + 4(\gamma-1)\log r_l + 4(\gamma-2)t_l + \bar{C} + o_l(1)] \\
&\leqq C\left(\frac{r_l}{\hat{r}_{l,j}}\right)^{4(\gamma-1)} e^{4(\gamma-2)t_l} = C\left(\frac{5}{4e^{t_l}}\right)^{4(\gamma-1)} e^{4(\gamma-2)t_l}\left(\frac{5}{7}\right)^{4(\gamma-1)j}.
\end{aligned}$$

Hence it follows that

$$\int_{B_{\frac{e^{\bar{s}_l}}{C}}(x_l)\setminus B_{e^{t_l}r_l}(x_l)} e^{4u_l}\,dV_g \leqq C\left(\frac{5}{4e^{t_l}}\right)^{4(\gamma-1)} e^{4(\gamma-2)t_l}\sum_{j=0}^{\infty}\left(\frac{5}{7}\right)^{4(\gamma-1)j} \to 0,$$

since $\gamma \in (1,2)$ and since $t_l \to +\infty$. This proves (79).

We can now apply Lemma 4.6 with $\Sigma_l = e^{t_l}r_l$, $S_l = \dfrac{e^{\bar{s}_l}}{C}$, and $\log r = \tilde{s}_l$. Also, by (78) and (79), we can choose $\varepsilon = \varepsilon_l \to 0$ and $R = R_l \to +\infty$ sufficiently slowly. Therefore, from Lemma 4.6 and Proposition 3.4 (see in particular (39)) we deduce that

$$\int_{A_{e^{\tilde{s}_l},l}} |x - x_l|^2\bigl(-\Delta u_l(x)\bigr)\,dV_g(x) = \left(\frac{15}{8}\int_{B_{\frac{e^{\tilde{s}_l}}{R_l}}(x_l)} f_l\,dV_g + o_l(1)\right)e^{4\tilde{s}_l} \geqq \bigl(30\pi^2 + o_l(1)\bigr)e^{4\tilde{s}_l}.$$

On the other hand, from (73) and the last two conditions in (77) we find

$$\begin{aligned}
\int_{A_{e^{\tilde{s}_l},l}} |x - x_l|^2\bigl(-\Delta u_l(x)\bigr)\,dV_g(x) &= \bigl[-W_l''(\tilde{s}_l) - 2\bigl(1 + O(e^{2\tilde{s}_l})\bigr)W_l'(\tilde{s}_l) + O(e^{2\tilde{s}_l})\bigr]\,Y_l(\tilde{s}_l) \\
&\leqq [\gamma + 2 + o_l(1)]\left(\frac{15\pi^2}{2} + o_l(1)\right)e^{4\tilde{s}_l}.
\end{aligned}$$

Since $\gamma < 2$, from the last two inequalities we get a contradiction. This concludes the proof of the lemma. □

We are finally in position to prove Proposition 4.2.

Proof of Proposition 4.2. It is sufficient to apply Lemma 4.9 and to reason as for the proof of (79). In fact, in this way we get

$$\int_{B_{\frac{e^{s_l}}{C}}(x_l)\setminus B_{e^{t_l}r_l}(x_l)} e^{4u_l}\,dV_g \to 0 \quad \text{as } l \to +\infty.$$

Moreover, choosing $b_l = e^{t_l}$ in (39) and $t_l \to +\infty$ sufficiently slowly, we also have

$$\int_{B_{b_l r_l}(x_l)} e^{4u_l}\,dV_g \to \frac{8\pi^2}{k_0} \quad \text{as } l \to +\infty.$$

The last two formulas yield the conclusion. □

5. Proof of Theorem 1.1

We prove first the theorem under the assumption (13), and we postpone the remaining cases to a second subsection.

5.1. Proof under the assumption (13). In this subsection we show how a general blow-up phenomenon can be essentially reduced to the case of finitely-many simple blow-ups. We divide the proof into three steps, and we always assume that $(u_l)_l$ is a sequence satisfying (9) and (20). We recall that the integer k is defined by the condition $k_0 \in \left(8k\pi^2, 8(k+1)\pi^2\right)$.

Step 1. There exist an integer $j \leqq k$, sequences $(x_{1,l})_l, \ldots, (x_{j,l})_l \subseteqq M$ and radii $(r_{1,l})_l, \ldots, (r_{j,l})_l, (\tilde{r}_{1,l})_l, \ldots, (\tilde{r}_{j,l})_l \to 0$ satisfying the properties (for some $\alpha \in (0,1)$)

$$(80) \qquad \frac{\tilde{r}_{i,l}}{r_{i,l}} \to +\infty \text{ (slowly) as } l \to +\infty, \quad B_{\tilde{r}_{i,l}} \cap B_{\tilde{r}_{h,l}} = \emptyset \text{ for } i \neq h,$$

$$(81) \quad \forall R > 0 \quad \hat{u}_{l,i} \to \log\frac{2}{1+|x|^2} - \frac{1}{4}\log\left(\frac{1}{3}k_0\right) \text{ in } H^4(B_R^{\mathbb{R}^4}) \cap C^\alpha(B_R^{\mathbb{R}^4}) \text{ as } l \to +\infty,$$

$$(82) \quad \forall \rho > 0\ \exists C_\rho > 0 \text{ s.t. if } \int_{B_s(y)} e^{4u_l}\,dV_g \geqq \rho \text{ with } B_s(y) \subseteqq M \setminus \bigcup_{i=1}^{j} B_{\tilde{r}_{i,l}}(x_{i,l}),$$

$$\text{then } s \geqq C_\rho^{-1} d_l(y),$$

where $d_l(y) = \min_{i=1,\ldots,j} |y - x_{i,l}|$. Here $\hat{u}_{l,j}$ denotes the function obtained using the procedure in Section 3, but scaling around the point $x_{i,l}$ with dilation factor $r_{i,l}$.

In order to prove Step 1, we consider a small number $\rho > 0$, say $\rho \in (0, \pi^2/k_0)$, and we define sequences $(x_{1,l})_l \subseteqq M$, $(r_{1,l})_l \subseteqq \mathbb{R}_+$ satisfying

$$\int_{B_{r_{1,l}}(x_{1,l})} e^{4u_l}\,dV_g = \max_{x \in M} \int_{B_{r_{1,l}}(x)} e^{4u_l}\,dV_g = \rho.$$

If (20) holds, it must be $r_{1,l} \to 0$ as $l \to +\infty$. In fact, if it were $r_{1,l} \geqq C^{-1}$, we could apply Proposition 3.1 to get uniform L^p bounds on $e^{4(u_l - \bar{u}_l)}$ for some $p > 1$. This fact and the Jensen inequality would yield

$$1 = e^{4\bar{u}_l} \int_M e^{4(u_l - \bar{u}_l)} \, dV_g \leqq C e^{4\bar{u}_l}, \quad \bar{u}_l \leqq C,$$

and hence uniform bounds on e^{4u_l} in $L^p(M)$. This would imply, by elliptic regularity results, uniform bounds in $H^2(M)$ on $(u_l)_l$, which is a contradiction to our assumptions.

Then, if $\tilde{r}_{1,l}/r_{1,l}$ tends to infinity sufficiently slowly, $(r_{1,l})_l$ and $(\tilde{r}_{1,l})_l$ satisfy (32), so Proposition 3.4 applies yielding the existence of a bubble, giving (81) for $i = 1$ and

$$\int_{B_{\tilde{r}_{1,l}}(x_{1,l})} e^{4u_l} \, dV_g = \frac{8\pi^2}{k_0} + o_l(1).$$

If (82) holds for $j = 1$, Step 1 is proved.

If (82) does not hold, there exists $\rho_1 > 0$, which can be assumed belonging to $(0, \pi^2/k_0)$, and there exist sequences $(y_l)_l \subseteqq M$, $\bar{r}_l \subseteqq \mathbb{R}_+$ such that

$$\int_{B_{\bar{r}_l}(y_l)} e^{4u_l} \, dV_g \geqq \rho_1, \quad B_{\bar{r}_l}(y_l) \subseteqq M \backslash B_{\tilde{r}_{1,l}}(x_{1,l}), \quad \frac{\bar{r}_l}{|y_l - x_{1,l}|} \to 0 \quad \text{as } l \to +\infty. \tag{83}$$

Now we define $r_{2,l}$ and $x_{2,l}$ such that

$$\int_{B_{r_{2,l}}(x_{2,l})} e^{4u_l} \, dV_g = \max_{B_{r_{2,l}}(y) \subseteqq M \backslash B_{\tilde{r}_{1,l}}(x_{1,l})} \int_{B_{r_{2,l}}(y)} e^{4u_l} \, dV_g = \rho_1.$$

By Proposition 3.4 it is easy to see that if $\tilde{r}_{1,l}/r_{1,l} \to +\infty$ sufficiently slowly, then we have

$$\frac{\tilde{r}_{1,l}}{|x_{1,l} - x_{2,l}|} \to 0, \quad \frac{r_{2,l}}{|x_{1,l} - x_{2,l}|} \to 0 \quad \text{as } l \to +\infty, \tag{84}$$

which in particular implies $r_{2,l} \to 0$ as $l \to +\infty$. Therefore, by the last formula we can find $\hat{r}_{2,l} \subseteqq \mathbb{R}_+$ such that

$$\int_{B_{r_{2,l}}(y)} e^{4u_l} \, dV_g \leqq \rho_1 \quad \text{for every } y \in B_{\hat{r}_{2,l}}(x_{2,l}), \quad \frac{\hat{r}_{2,l}}{|x_{1,l} - x_{2,l}|} \to 0 \quad \text{as } l \to +\infty.$$

Then Proposition 3.4 applies yielding the existence of a second bubble.

Continuing in this way, we see immediately that j cannot exceed k, since every bubble contributes an amount of $8\pi^2/k_0$ to the volume and since we are assuming (9). This concludes the proof of Step 1.

Step 2. If in Step 1 it is $j = 1$, then there holds

$$\int_M e^{4u_l}\,dV_g = \frac{8\pi^2}{k_0} + o_l(1). \tag{85}$$

In this case, if we choose $s_l = \frac{1}{2} i(M)$ for every l, where $i(M)$ is the injectivity radius of M, then by (82), $(x_{1,l})_l$, $(r_{1,l})_l$, $(s_l)_l$ are a simple blow-up for u_l. Therefore Proposition 4.2 applies and, since $(s_l)_l$ is uniformly bounded from below, there exists $C > 0$ such that for l large

$$\int_{B_{C^{-1}}(x_{1,l})} e^{4u_l} = \frac{8\pi^2}{k_0} + o_l(1). \tag{86}$$

We prove first the following property:

$$\bar{u}_l \to -\infty \quad \text{as } l \to +\infty. \tag{87}$$

In fact, using the Green's representation formula, for a.e. $x \in M$ we obtain

$$u_l(x) = \bar{u}_l + \int_M G(x,y)\big(2k_l e^{4u_l}(y) - 2Q_l\big)\,dV_g(y) \geqq \bar{u}_l - C + \int_M G(x,y) 2k_l e^{4u_l}(y)\,dV_g(y).$$

By (81) and (51), given any small $\tilde{\varepsilon} > 0$, there exists $R_{\tilde{\varepsilon}}$ such that, for l sufficiently large

$$\int_{B_{R_{\tilde{\varepsilon}} r_{1,l}}(x_{1,l})} 2k_l e^{2u_l} \geqq 16\pi^2 - 2\pi^2\tilde{\varepsilon}.$$

Hence the last two formulas and (23) imply

$$e^{4u_l(x)} \geqq C^{-1} e^{4\bar{u}_l} \frac{1}{|x - x_{1,l}|^{8-\tilde{\varepsilon}}}, \quad \text{for } |x - x_{1,l}| \geqq 2R_{\tilde{\varepsilon}} r_{1,l},$$

from which it follows that

$$\begin{aligned} \int_M e^{4u_l}\,dV_g &\geqq \int_{B_{i(M)}(x_{1,l}) \setminus B_{2R_{\tilde{\varepsilon}} r_{1,l}}(x_{1,l})} e^{4u_l}\,dV_g \\ &\geqq C^{-1} e^{4\bar{u}_l} \int_{2R_{\tilde{\varepsilon}} r_{1,l}}^{i(M)} s^{\tilde{\varepsilon}-5}\,ds \geqq C^{-1} e^{4\bar{u}_l} (R_{\tilde{\varepsilon}} r_{1,l})^{\tilde{\varepsilon}-4}. \end{aligned} \tag{88}$$

If $\tilde{\varepsilon}$ is sufficiently small, the last factor tends to $+\infty$ as $l \to +\infty$. Therefore (87) follows from (9).

Now, by (82), we can cover $M \setminus B_{C^{-1}}(x_{1,l})$ with a finite number of balls $B_{r_i}(y_i)$, $i = 1, \dots, \ell$ such that for every i there holds $\int_{B_{2r_i}(y_i)} e^{4u_l}\,dV_g \leqq \pi^2/k_0$. Reasoning as in the proof of Proposition 3.1 one then finds

$$\int_{M \setminus B_{C^{-1}}(x_{1,l})} e^{4u_l} \leqq C e^{4\bar{u}_l} \sup_{y \in M, i=1,\dots,\ell} \int_M \left(\frac{1}{|x-y|}\right)^{\frac{4\|e^{4u_l}\|_{L^1(B_{2r_i}(y_i))}}{8\pi^2}} \leqq C e^{4\bar{u}_l} \to 0.$$

Then (86) and the last formula conclude the proof of Step 2.

Step 3. If j in Step 1 is arbitrary, there holds

(89)
$$\int_M e^{4u_l}\,dV_g = \frac{8\pi^2}{k_0} j + o_l(1).$$

If $j > 1$ we reason as in [36], and we analyze the clustering of accumulation points. By relabelling the indices, we can assume that

(90)
$$|x_{1,l} - x_{2,l}| = \inf_{i \neq h} |x_{i,l} - x_{h,l}| \to 0 \quad \text{as } l \to +\infty.$$

Of course, if $\inf_{i \neq h} |x_{i,l} - x_{h,l}| \not\to 0$, then we could reason as in Step 2 a finite number of times. Assuming (90), we consider the set $X_{1,l} \subseteqq \{x_{1,l}, \dots, x_{h,l}\}$ of accumulation points for which the distance from $x_{1,l}$ is comparable to $|x_{1,l} - x_{2,l}|$, namely for which there exists $C > 0$ (independent of l) such that

$$|x_{i,l} - x_{1,l}| \leqq C|x_{1,l} - x_{2,l}|, \quad i = 2, \dots, h = \operatorname{card}(X_{1,l}).$$

By our choices of the points $x_{1,l}, \dots, x_{h,l}$ and by (90), one easily checks that the three sequences $(x_{i,l})_l$, $(r_{i,l})_l$ and $C^{-1}|x_{1,l} - x_{2,l}|$, $i = 1, \dots, h$, are a simple blow-up if C is sufficiently large, and Proposition 4.2 applies yielding

(91)
$$\int_{B_{C^{-1}|x_{1,l}-x_{2,l}|}(x_{i,l})} e^{4u_l}\,dV_g = \frac{8\pi^2}{k_0} + o_l(1), \quad i = 1, \dots, h.$$

Our next claim is that there is no further concentration in a neighborhood of $X_{1,l}$ of size comparable to $|x_{1,l} - x_{2,l}|$. More precisely we have the following result.

Lemma 5.1. *In the above notation, for any large and fixed C there holds*

(92)
$$\int_{B_{C|x_{1,l}-x_{2,l}|}(x_{1,l})} e^{4u_l}\,dV_g = \frac{8\pi^2}{k_0} \operatorname{card}(X_{1,l}) + o_l(1).$$

Proof. In order to prove this claim we use a variant of the argument in Step 2. First of all, for ρ small and fixed, we can cover the set $B_{C|x_{1,l}-x_{2,l}|}(x_{1,l}) \setminus \bigcup_{i=1,\dots,h} B_{C^{-1}|x_{1,l}-x_{2,l}|}(x_{i,l})$ with ℓ_l balls $B_{\rho_{n,l}}(y_{n,l})$, $n = 1, \dots, \ell_l$, with the following properties:

(93)
$$\ell_l \leqq C, \quad C^{-1}|x_{1,l} - x_{2,l}| \leqq \rho_{n,l} \leqq C|x_{1,l} - x_{2,l}|,$$
$$\int_{B_{2\rho_{n,l}}(y_{n,l})} e^{4u_l}\,dV_g \leqq \rho, \quad n = 1, \dots, \ell_l.$$

Reasoning as in the proof of Proposition 3.1 one finds

$$\int_{B_{\rho_{n,l}}(y_{n,l})} e^{4u_l}\,dV_g \leqq C \int_{B_{\rho_{n,l}}(y_{n,l})} dV_g(x) \exp\left[4 \int_{M \setminus B_{2\rho_{n,l}}(y_{n,l})} G(x,y) 2k_l e^{4u_l(y)}\,dV_g(y)\right]$$
$$\times \int_{B_{2\rho_{n,l}}(y_{n,l})} \left(\frac{1}{|x-y|}\right)^{\frac{k_l \rho}{\pi^2}} e^{4\bar{u}_l}\,dV_g(y).$$

From (23) and (91), after some computation we get

$$(94)\qquad \int_{B_{\rho_{n,l}}(y_{n,l})} e^{4u_l}\,dV_g \leqq C \int_{B_{\rho_{n,l}}(y_{n,l})} \left[4 \int_{M\setminus(B_{2\rho_{n,l}}(y_{n,l})\cup B_{\frac{C^{-1}|x_{1,l}-x_{2,l}|}{2}}(x_{1,l}))} G(x,y)2k_l e^{4u_l(y)}\,dV_g(y)\right]$$

$$\times |x_{1,l}-x_{2,l}|^{-8+o_l(1)}|x_{1,l}-x_{2,l}|^{4-\frac{k_l\rho}{\pi^2}} e^{4\bar{u}_l}\,dV_g(x)$$

$$\leqq C \sup_{x\in B_{\rho_{n,l}}(y_{n,l})} \left[8 \int_{M\setminus(B_{2\rho_{n,l}}(y_{n,l})\cup B_{\frac{C^{-1}|x_{1,l}-x_{2,l}|}{2}}(x_{1,l}))} G(x,y)k_l e^{4u_l(y)}\,dV_g(y)\right]$$

$$\times |x_{1,l}-x_{2,l}|^{-\frac{k_l\rho}{\pi^2}+o_l(1)} e^{4\bar{u}_l},$$

since $\rho_{n,l}$ is bounded from above by $C|x_{1,l}-x_{2,l}|$.

On the other hand, if $\tilde{\varepsilon}$ and $R_{\tilde{\varepsilon}}$ are as in Step 2, we also have

$$u_l(x) \geqq -C + \bar{u}_l + \int_{M\setminus B_{\frac{C^{-1}|x_{1,l}-x_{2,l}|}{2}}(x_{1,l})} G(x,y)2k_l e^{4u_l(y)}\,dV_g(y)$$

$$+ \int_{B_{R_{\tilde{\varepsilon}}r_{1,l}}(x_{1,l})} G(x,y)2k_l e^{4u_l(y)}\,dV_g(y), \quad \text{a.e. } x\in B_{\frac{C^{-1}|x_{1,l}-x_{2,l}|}{4}}(x_{1,l})\setminus B_{2R_{\tilde{\varepsilon}}r_{1,l}}(x_{1,l}).$$

Reasoning as for (88), we then deduce that

$$1 \geqq \int_{B_{\frac{C^{-1}|x_{1,l}-x_{2,l}|}{4}}(x_{1,l})\setminus B_{2R_{\tilde{\varepsilon}}r_{1,l}}(x_{1,l})} e^{4u_l}\,dV_g \geqq C^{-1}e^{4\bar{u}_l}(R_{\tilde{\varepsilon}}r_{1,l})^{\tilde{\varepsilon}-4}$$

$$\times \inf_{z\in B_{\frac{C^{-1}|x_{1,l}-x_{2,l}|}{4}}(x_{1,l})} \left[8 \int_{M\setminus(B_{2\rho_{n,l}}(y_{n,l})\cup B_{\frac{C^{-1}|x_{1,l}-x_{2,l}|}{2}}(x_{1,l}))} G(z,y)k_l e^{4u_l(y)}\,dV_g(y)\right].$$

Now we notice that by (93) and (23) one has

$$|G(z,y)-G(x,y)| \leqq C,$$

$$x\in B_{\rho_{n,l}}(y_{n,l}),\ y\in M\setminus\left(B_{2\rho_{n,l}}(y_{n,l})\cup B_{\frac{C^{-1}|x_{1,l}-x_{2,l}|}{2}}(x_{1,l})\right), \text{ and for } z\in B_{\frac{C^{-1}|x_{1,l}-x_{2,l}|}{4}}(x_{1,l}).$$

From (94) and the last two formulas it follows that

$$\int_{B_{\rho_{n,l}}(y_{n,l})} e^{4u_l}\,dV_g \leqq C|x_{1,l}-x_{2,l}|^{-\frac{k_l\rho}{\pi^2}+o_l(1)}(R_{\tilde{\varepsilon}}r_{1,l})^{\tilde{\varepsilon}-4} \to 0 \quad \text{as } l\to+\infty,$$

since $\dfrac{r_{1,l}}{|x_{1,l}-x_{2,l}|}\to 0$ by (84). Then the conclusion follows from (91) and the fact that $B_{C|x_{1,l}-x_{2,l}|}(x_{1,l})\setminus \bigcup_{i=1,\dots,h} B_{C^{-1}|x_{1,l}-x_{2,l}|}(x_{i,l})$ is covered by a finite (and uniformly bounded) number of balls $B_{\rho_{n,l}}(y_{n,l})$. $\square$

Now we let

$$d_{1,l} = \inf\{|x_{1,l} - x_{i,l}| : x_{i,l} \notin X_{1,l}\}.$$

Note that, by our definition of $X_{1,l}$, we have $\dfrac{d_{1,l}}{|x_{1,l} - x_{2,l}|} \to +\infty$ as $l \to +\infty$. We prove next the following result, which improves the estimate in formula (92) to a larger set.

Lemma 5.2. *There exists $C > 0$ such that for l large*

$$\int_{B_{C^{-1}d_{1,l}}(x_{1,l})} e^{4u_l}\, dV_g = \frac{8\pi^2}{k_0} \operatorname{card}(X_{1,l}) + o_l(1). \tag{95}$$

Proof. The proof follows closely the arguments of Proposition 4.2, hence we will be sketchy. We use the same notation as in Section 4 for the functions $(W_l)_l$ and the annuli $A_{r,l}$, except for the fact that now we take $x_{1,l}$ as centers, hence replacing the points x_l.

First of all we notice that, by the arbitrarity of C in Lemma 5.1, there exists $Z_l \to +\infty$ such that

$$\int_{B_{e^{4Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l}) \setminus B_{C|x_{1,l}-x_{2,l}|}(x_{1,l})} e^{4u_l}\, dV_g \to 0 \quad \text{as } l \to +\infty. \tag{96}$$

Using the Jensen inequality in the annulus $B_{e^{4Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l}) \setminus B_{e^{Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l})$, it follows that

$$\sup_{z \in [Z_l + \log|x_{1,l}-x_{2,l}|,\, 4Z_l + \log|x_{1,l}-x_{2,l}|]} \big(z + W_l(z)\big) \to -\infty \quad \text{as } l \to +\infty. \tag{97}$$

Our next goal is to prove that also

$$W_l'(z) = -2 \operatorname{card}(X_{1,l}) + o_l(1), \tag{98}$$

$$\text{for } z \in [2Z_l + \log|x_{1,l} - x_{2,l}|, 3Z_l + \log|x_{1,l} - x_{2,l}|].$$

In order to show this, we notice that by the second formula in Remark 4.8 and by some manipulation (reasoning as in the proof of Lemma 4.7), there holds

$$W_l'(z) = \frac{\int_{A_{r,l}} \frac{\partial u_l}{\partial z} f e^{4z}}{\int_{A_{r,l}} f e^{4z}} + O(e^{2z}), \quad \text{for } z \in [Z_l + \log|x_{1,l} - x_{2,l}|, 4Z_l + \log|x_{1,l} - x_{2,l}|],\ r = e^z.$$

Using the Green's representation formula we obtain

$$\begin{aligned} \frac{\partial u}{\partial r}(x) = & \int_{B_{e^{Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l})} \frac{\partial_x G(x,y)}{\partial r} f_l(y)\, dV_g(y) + \int_{M \setminus B_{e^{4Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l})} \frac{\partial_x G(x,y)}{\partial r} f_l(y)\, dV_g(y) \\ & + \int_{B_{e^{4Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l}) \setminus B_{e^{Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l})} \frac{\partial_x G(x,y)}{\partial r} f_l(y)\, dV_g(y). \end{aligned}$$

From (25), Lemma 5.1 and (96) it follows that, for $Z_l \to +\infty$ sufficiently slowly

$$\int_{B_{e^{Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l})} \frac{\partial_x G(x,y)}{\partial r} f_l(y)\, dV_g(y) = -\frac{2\operatorname{card}(X_{1,l})}{|x-x_{1,l}|} + o_l(1).$$

Also, reasoning as in the proof of Lemma 2.3 and using (96) one finds that

$$\left| \int_{A_{r,l}} dx \int_{B_{e^{4Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l}) \setminus B_{e^{Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l})} \frac{\partial_x G(x,y)}{\partial r} f_l(y)\, dV_g(y) \right| = o(1)|x-x_{1,l}|^3.$$

Finally, since $Z_l \to +\infty$ one also derives

$$\int_{M \setminus B_{e^{4Z_l}|x_{1,l}-x_{2,l}|}(x_{1,l})} \frac{\partial_x G(x,y)}{\partial r} f_l(y)\, dV_g(y) = o_l(1)\frac{1}{|x-x_{1,l}|}.$$

Recalling that $\frac{\partial u_l}{\partial z} = r\frac{\partial u_l}{\partial r}$, with $r = \operatorname{dist}(x, x_{1,l})$, the last three formulas yield (98).

Now, for $\gamma \in (1,2)$ we consider the following sequence of functions

$$h_l^{\gamma}(z) = -\gamma(z - \log|x_{1,l}-x_{2,l}| - 2Z_l) + W_l(\log|x_{1,l}-x_{2,l}| + 2Z_l).$$

Exactly as in the proof of Proposition 4.2 one can show that

$$W_l(z) \leqq h_l^{\gamma}(z), \quad z \in [\log|x_{1,l}-x_{2,l}| + 2Z_l, \log d_{1,l} - C_\gamma].$$

As above, we define

$$\hat{r}_{l,j} = \frac{4}{5} e^{2Z_l}|x_{1,l}-x_{2,l}|\left(\frac{7}{5}\right)^j, \quad \left(\frac{7}{5}\right)^{j_l} \in \left(\frac{5}{4}\frac{d_{1,l}}{Ce^{2Z_l}|x_{1,l}-x_{2,l}|}, \frac{5}{2}\frac{d_{1,l}}{Ce^{2Z_l}|x_{1,l}-x_{2,l}|}\right),$$

and we obtain

$$\int_{A'_{\hat{r}_{l,j},l}} e^{4u_l}\, dV_g \leqq C|A_{\hat{r}_{l,j},l}| e^{4\bar{u}_{l,\hat{r}_{l,j}}} \leqq C\hat{r}_{l,j}^4 e^{4W_l(\log \hat{r}_{l,j})} \leqq C\hat{r}_{l,j}^4 e^{4h_l^{\gamma}(\log \hat{r}_{l,j})}, \quad j = 1, \dots, j_l.$$

From the expression of h_l^{γ} and (97) we deduce

$$\begin{aligned}
\hat{r}_{l,j}^4 e^{4h_l^{\gamma}(\log \hat{r}_{l,j})} &\leqq C\hat{r}_{l,j}^4 \exp\big[4\big(-\gamma(\log \hat{r}_{l,j} - \log|x_{1,l}-x_{2,l}| - 2Z_l) + W_l(\log|x_{1,l}-x_{2,l}| + 2Z_l)\big)\big] \\
&\leqq o_l(1)\hat{r}_{l,j}^4 \exp[-4\gamma \log \hat{r}_{l,j} + 4(\gamma-1)\log|x_{1,l}-x_{2,l}| + 8(\gamma-2)Z_l] \\
&\leqq o_l(1)\left(\frac{|x_{1,l}-x_{2,l}|}{\hat{r}_{l,j}}\right)^{4(\gamma-1)} = o_l(1)\left(\frac{5}{7}\right)^{4(\gamma-1)j}.
\end{aligned}$$

As before we then find

$$\int_{B_{\frac{d_{1,l}}{C}}(x_l) \setminus B_{C|x_{1,l}-x_{2,l}|}(x_l)} e^{4u_l}\, dV_g \leqq o_l(1)\sum_{j=0}^{\infty}\left(\frac{5}{7}\right)^{4(\gamma-1)j} \to 0.$$

This formula, joint with (92), yields the conclusion of the lemma. □

The proof of Step 3 follows from the arguments of Lemmas 5.1, 5.2, repeating the procedure for all the clusters of the points of $\{x_{1,l}, \dots, x_{j,l}\} \setminus X_{1,l}$.

The proof of the theorem is now an easy consequence of (9) and (85), since k_0 is not an integer multiple of $8\pi^2$.

5.2. The case $k_0 < 8\pi^2$. In this final subsection we consider the cases in which P_g possesses some negative eigenvalues and $k_0 < 8\pi^2$. We prove first the following result, which regards boundedness of the V-component of sequences of solutions.

Lemma 5.3. *Suppose P_g possesses some negative eigenvalues, and suppose that* $\ker P_g = \{\text{constants}\}$. *Let $(u_l)_l \subseteq H^2(M)$ be a sequence satisfying (7)–(9). Let us write $u_l = \hat{u}_l + \tilde{u}_l$ with $\hat{u}_l \in V$ and $\tilde{u}_l \perp V$, where V denotes the direct sum of the negative eigenspaces of P_g. Then there holds*

$$\|\hat{u}_l\|_{H^2(M)} \leqq C,$$

for some positive constant C independent of l.

Proof. Let $\hat{v}_1, \dots, \hat{v}_{\bar{k}}$ be as in (17). Then, by standard elliptic regularity theory, each $\hat{v}_i$ is smooth on M. Testing (7) on $\hat{u}_l$ we obtain

$$\langle P_g \hat{u}_l, \hat{u}_l \rangle + 4 \int_M Q_l \hat{u}_l \, dV_g + 4k_l \int_M e^{4u_l} \hat{u}_l \, dV_g = 0.$$

Using (9), the fact that on V the L^∞-norm is equivalent to the H^2-norm, and the Poincaré inequality, from the last formula we deduce that

$$-\langle P_g \hat{u}_l, \hat{u}_l \rangle = O(1) \|\hat{u}_l\|_{H^2(M)}.$$

Since P_g is negative-definite on V, the conclusion follows. □

Next, we consider separately the following three possibilities, one of which will always occur for $k_0 < 8\pi^2$ and for l sufficiently large.

Case 1: $k_l < 0$. First of all, using the Jensen inequality we find immediately that $\bar{u}_l \leqq C$, for some constant C independent of l. Then, multiplying (7) by u_l and integrating on M, using the Poincaré inequality and Lemma 5.3, we find

$$\begin{aligned} \langle P_g u_l, u_l \rangle &= 2k_l \int_M e^{4u_l} u_l \, dV_g - 2k_l \bar{u}_l + O(\langle P_g u_l, u_l \rangle^{\frac{1}{2}}) + C \\ &\leqq C + (-2k_l)\bar{u}_l + O(\langle P_g u_l, u_l \rangle^{\frac{1}{2}}) \leqq C + O(\langle P_g u_l, u_l \rangle^{\frac{1}{2}}). \end{aligned}$$

Again by Lemma 5.3, this implies uniform bounds on $\|u_l - \bar{u}_l\|$ and hence, by (19), uniform L^p bounds on e^{4u_l} for any $p > 1$. Then the conclusion follows from standard elliptic regularity results.

Case 2: $0 \leqq k_l \leqq 2\pi^2$. Since we are assuming (9), we easily see that the alternative (26) in Proposition 3.1 cannot occur. Therefore, reasoning as in the previous case, we obtain uniform L^p bounds on e^{4u_l} for some $p > 1$.

Case 3: $2\pi^2 \leqq k_l < \frac{1}{2}(k_0 + 8\pi^2) < 8\pi^2$. In this case it is $k_0 > 0$. Assuming $(u_l)_l$ unbounded, Proposition 3.4 applies, and (39) gives a contradiction to (9), since $k_0 < 8\pi^2$.

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Sissa, Via Beirut 2-4, 34014 Trieste, Italy
e-mail: malchiod@sissa.it

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