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2D Smagorinsky-Type Large Eddy Models as Limits of Stochastic PDEs

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Abstract

We prove that a version of Smagorinsky large eddy model for a 2D fluid in vorticity form is the scaling limit of suitable stochastic models for large scales, where the influence of small turbulent eddies is modeled by a transport-type noise.

Keywords Smagorinsky model · Eddy viscosity model · Turbulence · Transport noise · Scaling limit

Mathematics Subject Classification 60H15 · 76D05

1 Introduction

Recently, a new stochastic approach has been developed in Flandoli et al. (2021, 2022), Flandoli and Luongo (2023), Flandoli (2022), Flandoli and Luo (2023), Debussche and Pappalettera (2022), Carigi and Luongo (2023) to explain Boussinesq hypothesis that "turbulent fluctuations are dissipative on large scales" (Boussinesq 1877). The idea, better explained in Sect. 2, is that the large scales satisfy a Navier–Stokes-

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type equation with a stochastic transport term corresponding to the action of small scales. In a suitable scaling limit, we get a deterministic Navier–Stokes equation with an additional dissipative term. The turbulent viscosity is directly related to the noise (namely small-scale) covariance. All the quoted works are related to dimension 2, with the exception of Flandoli and Luo (2023) that deals with a 2D-3C model with some three dimensional feature, including a stretching term of small scales over large ones and the possibility of an AKA (anisotropic kinetic alpha) effect in the limit equation. For other approaches to justify Boussinesq hypothesis and turbulent viscosity based on Eulerian formulations of fluid dynamical systems, see for instance (Berselli 2006; Jiang et al. 2020; Wirth et al. 1995). There are also different models based on filtering the systems at the Lagrangian level rather than Eulerian one, we refer to Foias et al. (2001, 2002), Cheskidov et al. (2005), Cotter et al. (2017) for rigorous analysis and some discussions on the topic.

The previous works on the stochastic approach are, however, limited to the case of *linear* limit dissipation term, namely turbulent viscosity independent of the solution. Smagorinsky-type models are excluded from the previous analysis, and it was not clear for some time how to incorporate them into this new theory. In this paper, we solve this problem. This provides new insight into these models and their motivations.

Since our techniques are, at present, well developed for the vorticity equation, while they suffer certain difficulties for the velocity equation, we present the results for vorticity-type equations (however, as stated in (Winckelmans et al. 1996, Sect. 5), the performances of vorticity–velocity models are sometimes superior to those of velocity–pressure ones). We choose the following form, discussed for instance in Cottet et al. (2003):

$$\partial_t \omega_L + u_L \cdot \nabla \omega_L = \nu \Delta \omega_L + \operatorname{div}\left(g'\left(\omega_L\right) \nabla \omega_L\right) \tag{1}$$

(written in this way so that $\operatorname{div}\left(g'\left(\omega_L\right) \nabla \omega_L\right) = \Delta g\left(\omega_L\right)$) with the additional conditions $\omega_L = \nabla^{\perp} \cdot u_L$, $\operatorname{div} u_L = 0$ and the initial condition $\omega_L|_{t=0} = \omega_0^L$. Here, L stands for the large-scale components of fluid vorticity and velocity, see the next section for more discussions; the fields are assumed to be periodic, on a torus. The function $g(r)$ is subject to quite general assumptions which include it is non-decreasing, so that g' is not negative. The particular case treated in Cottet et al. (2003) (see also (Mansour et al. 1978; Winckelmans et al. 1996; Deng and Dong 2020)) is

$$g'(r) = (C_s \Delta)^2 |r|, \tag{2}$$

where Δ is a subgrid characteristic length scale and C_s is a non-dimensional constant which has to be calibrated, and its value may vary with the type of the flow and the Reynolds number. However, similarly to the Smagorinsky model in velocity form, it may be useful to cover more general nonlinearities, see for instance (Berselli 2006, Sect. 3.3.2). We prove that this Smagorinsky-type model is the limit of the large-scale stochastic model

$$d\omega_L + (u_L \cdot \nabla \omega_L - \nu \Delta \omega_L)\, \mathrm{d}t = -f'\left(\omega_L\right) \sum_k \sigma_k \cdot \nabla \omega_L \circ \mathrm{d}W_t^k \tag{3}$$

(where again $f'(\omega_L)\sigma_k \cdot \nabla\omega_L = \sigma_k \cdot \nabla f(\omega_L)$) with f such that $\frac{1}{4}f'(r)^2 = g'(r)$. The limit is taken along a suitable sequence of small-scale noise, namely we assume (roughly speaking) that σ_k are smaller and smaller scale (an assumption of scale separation). The notations and assumptions (like the fact that $\{W^k\}_k$ are independent Brownian motions and $\circ$ is the Stratonovich multiplication operation) will be explained in the technical sections.

The paper is organized as follows. In Sect. 2, we describe the heuristic ideas behind the stochastic model. In Sect. 3, we state our results and introduce some mathematical tools. In Sect. 4, we show the existence of martingale solutions of the problem (3). Lastly, in Sect. 5, we will show our main result about the convergence of martingale solutions of our stochastic models to a measure concentrated on the unique weak solution of the Smagorinsky model (1), see Theorem 5 for the rigorous statement.

2 The Heuristic Idea

The idea described in this section is similar to the one given in Flandoli (2022), Flandoli and Luo (2023), but we repeat it and particularize the models studied here, for completeness and to help the intuition behind the model. Consider a 2D Newtonian viscous fluid in a torus, described in vorticity form by the equations

$$\begin{aligned} &\partial_t\omega + u\cdot\nabla\omega = \nu\Delta\omega, \\ &\omega = \nabla^{\perp}\cdot u, \quad \operatorname{div} u = 0, \\ &\omega|_{t=0} = \omega_0, \end{aligned}$$

where ω is the vorticity field and u the velocity field. Assume that the initial vorticity ω_0 is the sum of a large scale component ω_0^L plus a small-scale component ω_0^S. Then, at least on a short time interval $[0,\tau]$, it is reasonable to expect that the system

$$\begin{aligned} \partial_t\omega_L + u\cdot\nabla\omega_L &= \nu\Delta\omega_L, \\ \partial_t\omega_S + u\cdot\nabla\omega_S &= \nu\Delta\omega_S, \\ \omega_L|_{t=0} = \omega_0^L, &\quad \omega_S|_{t=0} = \omega_0^S \end{aligned}$$

represents quite well the evolution of the different vortex structures, as for instance in the small vortex-blob limit to point vortices treated by Marchioro and Pulvirenti (1994). The system above is equivalent to the original one, by addition.

The next step is considering only the equation for the large scales, isolating the term which is not closed, namely depends on the small scales:

$$\partial_t\omega_L + u_L\cdot\nabla\omega_L - \nu\Delta\omega_L = -u_S\cdot\nabla\omega_L.$$

Here, u_L, with $\operatorname{div} u_L = 0$, has the property $\nabla^{\perp}\cdot u_L = \omega_L$ (namely u_L is reconstructed from ω_L by Biot–Savart law). The field u_S should correspond to ω_S by Biot–Savart law but we now introduce a stochastic closure assumption. We replace $u_S(t,x)$ by a white-in-time noise, with suitable space dependence

$$u_S(t,x) \mapsto \chi(t,x) \sum_k \sigma_k(x) \frac{\mathrm{d}W_t^k}{\mathrm{d}t},$$

where $\{\sigma_k\}_k$ are suitable divergence free vector fields, and $\chi(t,x)$ is a scalar stochastic process which will be linked to the large scales, in order to model the idea that the turbulent small scales are more active where the large scales have more intense variations (e.g., larger shear); $\{W^k\}_k$ are independent scalar Brownian motions. In the replacement, Stratonovich integrals are used, in accordance with the Wong–Zakai principle (see rigorous results in Debussche and Pappalettera (2022)). Therefore, the equation for large scales, now closed and stochastic, takes the form

$$\mathrm{d}\omega_L + (u_L \cdot \nabla\omega_L - \nu\Delta\omega_L)\,\mathrm{d}t = -\chi(t,x) \sum_k \sigma_k \cdot \nabla\omega_L \circ \mathrm{d}W_t^k.$$

Previous works developed this idea in the case when $\chi = 1$, see e.g., Flandoli and Luongo (2023); Flandoli and Pappalettera (2022); Debussche and Pappalettera (2022). Here, we assume that χ is a function of ω_L that for notational convenience will be written as

$$\chi(t,x) = f'(\omega_L(t,x))$$

for a suitable function f. As said above, the heuristic idea is that turbulence is more developed in regions of high large-scale vorticity; hence, the small-scale noise should be modulated by an increasing function f'.

This is the motivation for the stochastic model (3) presented in the Introduction. Our main purpose is showing that it leads to the Smagorinsky-type deterministic equation (1) in a suitable scaling limit of the noise.

3 Functional Setting and Main Results

Let us set some notation before stating the main contributions of this work. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus and $\mathbb{Z}^2_0 = \mathbb{Z}^2 \setminus \{0\}$ the nonzero lattice points. Let $(H^{s,p}(\mathbb{T}^2), \|\cdot\|_{H^{s,p}})$, $s \in \mathbb{R}$, $p \in (1,+\infty)$ be the Bessel spaces of zero mean periodic functions. In case of $p=2$, we simply write $H^s(\mathbb{T}^2)$ in place of $H^{s,2}(\mathbb{T}^2)$, and we denote by $\langle\cdot,\cdot\rangle_{H^s}$ the corresponding scalar products. In case also $s>0$, we denote by $\langle\cdot,\cdot\rangle_{H^{-s},H^s}$ the dual pairing between H^s and H^{-s}. Lastly, we denote by $H^{s-}(\mathbb{T}^2) = \cap_{r<s} H^r(\mathbb{T}^2)$. In case of $s=0$ we will write $H^{0-}(\mathbb{T}^2)$ as $H^-(\mathbb{T}^2)$, $L^2(\mathbb{T}^2)$ instead of $H^0(\mathbb{T}^2)$ and we will neglect the subscript in the notation for the norm and the inner product. Similarly, we introduce the Bessel spaces of zero mean vector fields

$$\mathbf{H}^{s,p} = \{(u_1,u_2)^t : u_1,u_2 \in H^{s,p}(\mathbb{T}^2)\},$$
$$\langle u,v\rangle_{\mathbf{H}^s} = \langle u_1,v_1\rangle_{H^s} + \langle u_2,v_2\rangle_{H^s}, \quad \text{for } s \in \mathbb{R}.$$

Again, in case of $s = 0$, we will write $\mathbf{L}^2$ instead of $\mathbf{H}^0$, and we will neglect the subscript in the notation for the norm and the scalar product.

Let Z be a separable Hilbert space, with associated norm $\|\cdot\|_Z$. We denote by $C^w_{\mathcal{F}}([0,T];Z)$ the space of weakly continuous adapted processes $(X_t)_{t\in[0,T]}$ with values in Z such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|X_t\|_Z^2\right] < \infty$$

and by $L^p_{\mathcal{F}}(0,T;Z)$, $p\in[1,\infty)$, the space of progressively measurable processes $(X_t)_{t\in[0,T]}$ with values in Z such that

$$\mathbb{E}\left[\int_0^T\|X_t\|_Z^p\,\mathrm{d}t\right] < \infty.$$

Following the ideas introduced in Sect. 2, we are interested in the following stochastic model with a more precise noise (cf. Galeati (2020); Flandoli et al. (2021))

$$\begin{cases} d\omega_L = (\nu\Delta\omega_L - u_L\cdot\nabla\omega_L)\,\mathrm{d}t - \sum_{k\in\mathbb{Z}_0^2}\theta_k\sigma_k\cdot\nabla f(\omega_L)\circ\mathrm{d}W^k, \\ u_L = -\nabla^\perp(-\Delta)^{-1}\omega_L, \\ \omega_L(0) = \omega_0, \end{cases} \tag{4}$$

where $f\in C^1(\mathbb{R};\mathbb{R})$, $\theta=(\theta_k)_k\in\ell^2(\mathbb{Z}_0^2)$ satisfies

$$\sum_{k\in\mathbb{Z}_0^2}\theta_k^2 = 1, \quad \theta_k=\theta_l \quad \text{if } |k|=|l|; \tag{5}$$

$\{\sigma_k\}_{k\in\mathbb{Z}_0^2}$ is the standard orthonormal basis of divergence free vector fields in $\mathbf{L}^2$ made by the eigenfunctions of the Stokes operator, i.e.,

$$\sigma_k(x) = \frac{k^\perp}{|k|}e_k(x), \quad e_k(x) = \sqrt{2}\begin{cases}\cos(2\pi k\cdot x) & \text{if } k\in\mathbb{Z}_+^2, \\ \sin(2\pi k\cdot x) & \text{if } k\in\mathbb{Z}_-^2,\end{cases}$$

where $k^\perp = (k_2,-k_1)$, $\mathbb{Z}_+^2 := \{k\in\mathbb{Z}_0^2 : (k_1>0) \text{ or } (k_1=0,\ k_2>0)\}$ and $\mathbb{Z}_-^2 := -\mathbb{Z}_+^2$; $\{W^k\}_{k\in\mathbb{Z}_0^2}$ is a family of real independent Brownian motions. Moreover, we assume that

$$|f'(x)| \lesssim 1+|x|^\alpha \tag{6}$$

for some $\alpha\in[0,1]$. This implies in particular that

$$|f(x)| \lesssim 1+|x|^{\alpha+1}. \tag{7}$$

In the sequel, we shall omit the subscript L to save notation. System (4) can be formulated easily in Itô form. Indeed, it holds

$$\sigma_k \cdot \nabla f(\omega) \circ dW^k = \sigma_k \cdot \nabla f(\omega)\, dW^k + \frac{1}{2} d\left[\sigma_k \cdot \nabla f(\omega), W^k\right],$$

where $[\cdot, \cdot]$ is the quadratic covariation of semimartingales. Since

$$\begin{aligned} d\big(\sigma_k \cdot \nabla f(\omega)\big) &= \sigma_k \cdot \nabla\big(df(\omega)\big) \\ &= \sigma_k \cdot \nabla(f'(\omega)V)\,\mathrm{d}t - \sum_l \theta_l \sigma_k \cdot \nabla\big(f'(\omega)^2 \sigma_l \cdot \nabla\omega\big) \circ \mathrm{d}W^l, \end{aligned}$$

where $V := \nu\Delta\omega - u \cdot \nabla\omega$, one has

$$\begin{aligned} d\left[\sigma_k \cdot \nabla f(\omega), W^k\right] &= -\theta_k \sigma_k \cdot \nabla\big(f'(\omega)^2 \sigma_k \cdot \nabla\omega\big)\,\mathrm{d}t \\ &= -\theta_k \operatorname{div}\big(f'(\omega)^2 (\sigma_k \otimes \sigma_k)\nabla\omega\big)\,\mathrm{d}t \end{aligned}$$

which is due to the divergence free property of σ_k; hence,

$$\begin{aligned} &-\sum_k \theta_k \sigma_k \cdot \nabla f(\omega) \circ \mathrm{d}W^k \\ &= -\sum_k \theta_k \sigma_k \cdot \nabla f(\omega)\,\mathrm{d}W^k + \frac{1}{2}\sum_k \theta_k^2 \operatorname{div}\big(f'(\omega)^2(\sigma_k \otimes \sigma_k)\nabla\omega\big)\,\mathrm{d}t \\ &= -\sum_k \theta_k \sigma_k \cdot \nabla f(\omega)\,\mathrm{d}W^k + \frac{1}{4}\operatorname{div}\big(f'(\omega)^2 \nabla\omega\big)\,\mathrm{d}t, \end{aligned}$$

where the last step is due to the fact (cf. (Flandoli and Luo 2020, Lemma 2.6) for a proof)

$$\sum_k \theta_k^2 \,(\sigma_k \otimes \sigma_k) = \frac{1}{2} I_2, \tag{8}$$

the latter being the 2×2 unit matrix. Thanks to the computations on the Itô-Stratonovich corrector above, equation (4) can be rewritten as

$$\begin{cases} d\omega = \big(\nu\Delta\omega - u \cdot \nabla\omega + \frac{1}{4}\operatorname{div}(f'(\omega)^2\nabla\omega)\big)\,\mathrm{d}t - \sum_k \theta_k \sigma_k \cdot \nabla f(\omega)\,\mathrm{d}W^k, \\ u = -\nabla^\perp(-\Delta)^{-1}\omega, \\ \omega(0) = \omega_0. \end{cases} \tag{9}$$

We introduce the real function $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \frac{1}{4}\int_0^x f'(t)^2\,\mathrm{d}t, \quad x \in \mathbb{R},$$

which satisfies $g(0) = 0$ and

$$\begin{aligned} |g(y) - g(x)| &\lesssim |y - x| + \left| y|y|^{2\alpha} - x|x|^{2\alpha} \right| \\ &\lesssim |y - x| \left(1 + |x|^{2\alpha}\right) + |y| \left| |y|^{2\alpha} - |x|^{2\alpha} \right|. \end{aligned} \tag{10}$$

From the definition of g, it follows that system (9) can be rewritten as

$$\begin{cases} d\omega = \big(\nu\Delta\omega - u \cdot \nabla\omega + \Delta g(\omega)\big)\,\mathrm{d}t - \sum_k \theta_k \sigma_k \cdot \nabla f'(\omega)\,\mathrm{d}W^k, \\ u = -\nabla^{\perp}(-\Delta)^{-1}\omega, \\ \omega(0) = \omega_0. \end{cases} \tag{11}$$

The relation between u and ω can be described in terms of the so-called Biot–Savart operator

$$K \in \mathcal{L}(H^{s,p}, \mathbf{H}^{s+1,p}) : \quad K[\omega] = -\nabla^{\perp}(-\Delta)^{-1}\omega \quad \text{for } p \in (1, +\infty),\ s \in \mathbb{R}.$$

We are now ready to define our notion of solution for system (11).

Definition 1 We say that system (11) has a weak solution if there exists a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, a sequence of independent $\mathcal{F}_t$ Brownian motions W^k and $\omega \in C^w_{\mathcal{F}}(0, T; L^2(\mathbb{T}^2)) \cap L^2_{\mathcal{F}}(0, T; H^1(\mathbb{T}^2))$ such that for any $\phi \in C^\infty(\mathbb{T}^2)$, $\mathbb{P}$-a.s. $\forall t \in [0, T]$, it holds

$$\begin{aligned} \langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \nu \int_0^t \langle \omega_s, \Delta\phi \rangle\,\mathrm{d}s + \int_0^t \langle g(\omega_s), \Delta\phi \rangle\,\mathrm{d}s \\ + \int_0^t \langle \omega_s, K[\omega_s] \cdot \nabla\phi \rangle\,\mathrm{d}s + \sum_{k \in \mathbb{Z}^2_0} \int_0^t \theta_k \langle f(\omega_s), \sigma_k \cdot \nabla\phi \rangle\,\mathrm{d}W^k_s. \end{aligned}$$

Due to the nonlinearities related to the noise in equation (11), the existence of weak solutions is a nontrivial fact which is proved in Sect. 4. Indeed, we will prove the following result.

Theorem 2 *For each $\omega_0 \in L^2(\mathbb{T}^2)$, there exists at least one weak solution of system (11) in the sense of Definition 1. Moreover,*

$$\sup_{t\in[0,T]} \|\omega_t\|^2 + 2\nu \int_0^T \|\nabla\omega_s\|^2\,\mathrm{d}s \le 2\|\omega_0\|^2 \quad \mathbb{P} - a.s.$$

Next, following the idea introduced for the first time in Galeati (2020), we consider a family $\{\theta^N\}_{N\in\mathbb{N}} \subseteq \ell^2(\mathbb{Z}^2_0)$, satisfying relation (5) such that

$$\lim_{N\to+\infty} \|\theta^N\|_{\ell^\infty} = 0, \tag{12}$$

and we call ω^N the corresponding weak solution of equation (11) with $\{\theta_k^N\}_k$ in place of $\{\theta_k\}_k$. In order to complete our plan, we want to show that the law of ω^N converges weakly to a measure supported on the unique weak solution of the Navier–Stokes equation in vorticity form with Smagorinsky correction, namely

$$\begin{cases} \partial_t \overline{\omega} = \nu \Delta \overline{\omega} + \Delta g(\overline{\omega}) - \overline{u} \cdot \nabla \overline{\omega}, \\ \overline{u} = -\nabla^\perp (-\Delta)^{-1} \overline{\omega}, \\ \overline{\omega}(0) = \omega_0. \end{cases} \tag{13}$$

Remark 3 Taking $f(r) = \frac{4}{3} C_s \Delta |r|^{1/2} r$, C_s and Δ being the same as in (2), we have $g(r) = \frac{1}{2}(C_s\Delta)^2 r^2 \mathrm{sign}(r)$, and thus $\Delta g(\overline{\omega}) = (C_s\Delta)^2 \mathrm{div}(|\overline{\omega}|\nabla\overline{\omega})$. In this way, we recover the Smagorinsky model of Cottet et al. (2003).

By a weak solution of (13), we mean the following:

Definition 4 We say that $\overline{\omega}$ is a weak solution of equation (13) if

$$\overline{\omega} \in C_w(0, T; L^2(\mathbb{T}^2)) \cap L^2(0, T; H^1(\mathbb{T}^2))$$

and for each $\phi \in C^\infty(\mathbb{T}^2)$, for all $t \in [0, T]$, one has

$$\begin{aligned} \langle \overline{\omega}_t, \phi \rangle - \langle \omega_0, \phi \rangle = \nu \int_0^t \langle \overline{\omega}_s, \Delta\phi \rangle \, \mathrm{d}s + \int_0^t \langle g(\overline{\omega}_s), \Delta\phi \rangle \, \mathrm{d}s \\ + \int_0^t \langle \overline{\omega}_s, K[\overline{\omega}_s] \cdot \nabla\phi \rangle \, \mathrm{d}s. \end{aligned}$$

In Sect. 5, indeed we will first show the uniqueness of the weak solutions of (13), then we will show our main result which reads in the following way.

Theorem 5 *Assume that $\{\theta^N\}_N \subset \ell^2$ satisfies (5) and (12). Let ω^N be a weak solution of (11) corresponding to θ^N, and Q^N its law on $C([0, T]; H^-(\mathbb{T}^2)) \cap L^2(0, T; H^{1-}(\mathbb{T}^2))$. Then, the family $\{Q^N\}_N$ is tight on $C([0, T]; H^-(\mathbb{T}^2)) \cap L^2(0, T; H^{1-}(\mathbb{T}^2))$, and it converges weakly to the Dirac measure $\delta_{\overline{\omega}}$, where $\overline{\omega}$ is the unique weak solution of equation (13).*

3.1 Preparatory Results

Before starting, we need to recall some results that we will use in Sects. 4 and 5 in order to prove Theorems 2 and 5, see Simon (1986); Bagnara et al. (2023) for more details on these results.

In the following $X,\ B,\ Y$ are separable Banach spaces such that

$$X \overset{c}{\hookrightarrow} B \hookrightarrow Y,$$

where $\overset{c}{\hookrightarrow}$ means compact embedding.

Theorem 6 *Let $p, r \in [1, +\infty]$ and $s \in \mathbb{R}$; assume that $s > 0$ if $r \geq p$ or $s > 1/r - 1/p$ if $r \leq p$. Let F be a bounded subset in $L^p(0, T; X) \cap W^{s,r}(0, T; Y)$. Then, F is relatively compact in $L^p(0, T; B)$ (in $C([0, T]; B)$ if $p = +\infty$).*

Theorem 7 *Assume that there exists $\theta \in (0, 1)$ such that*

$$\|v\|_B \leq M\|v\|_X^{1-\theta}\|v\|_Y^{\theta} \quad \forall\, v \in X.$$

Let F be bounded in $W^{s_0,r_0}(0, T; X) \cap W^{s_1,r_1}(0, T; Y)$, $r_0, \ r_1 \in [1, +\infty]$. Define

$$s_\theta = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad s_* = s_\theta - \frac{1}{r_\theta}.$$

If $s_* < 0$, then F is relatively compact in $L^p(0, T; B)$ for each $p < -1/s_*$, and if $s_* > 0$, then F is relatively compact in $C([0, T]; B)$.

Lemma 8 *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{U}$ and $\mathcal{H}$ separable Hilbert spaces. Assume $W = \sum_{k\geq 0} W_k e_k$ is an $(\mathcal{F}_t)_{t\in[0,T]}$ cylindrical Brownian motion (over $\mathcal{U}$), while $W^n = \sum_{k\geq 0} W_k^n e_k$ are $(\mathcal{F}_t^n)_{t\in[0,T]}$ cylindrical Brownian motions (over $\mathcal{U}$). Assume that G is an $(\mathcal{F}_t)_{t\in[0,T]}$ progressively measurable process which belongs to $L^2([0, T], L_2(\mathcal{U}, \mathcal{H}))$ $\mathbb{P}$-a.s., while G^n are $(\mathcal{F}_t^n)_{t\in[0,T]}$ progressively measurable processes which belong to $L^2([0, T], L_2(\mathcal{U}, \mathcal{H}))$ $\mathbb{P}$-a.s.. If*

$$W_k^n \to W_k \quad \text{in probability in } C([0, T], \mathbb{R}) \quad \forall k \geq 0, \tag{14a}$$

$$G^n \to G \quad \text{in probability in } L^2([0, T]; L_2(\mathcal{U}, \mathcal{H})), \tag{14b}$$

then

$$\sup_{t\in[0,T]} \left\| \int_0^t G^n \, dW^n - \int_0^t G \, dW \right\|_{\mathcal{H}} \to 0 \quad \text{in probability}. \tag{15}$$

In order to identify our limits, we will use the following lemma on interpolation spaces.

Lemma 9 *Let $\chi_n, \chi \in L^\infty(0, T; H^-(\mathbb{T}^2)) \cap L^2(0, T; H^{1-}(\mathbb{T}^2))$ such that*

$$\chi_n \to \chi \quad \text{in } L^\infty(0, T; H^-(\mathbb{T}^2)) \cap L^2(0, T; H^{1-}(\mathbb{T}^2)). \tag{16}$$

Then, $\forall\, \beta > 2, \ \gamma \in [0, 1)$ such that $\beta\gamma < 2$

$$\chi_n \to \chi \in L^\beta(0, T; H^\gamma(\mathbb{T}^2)).$$

Proof Let $\delta, \ \delta' > 0$ such that

$$1 - \delta > \gamma, \quad 2 - 2\delta - \beta\gamma > 0, \quad \delta' < \frac{2 - 2\delta - \beta\gamma}{\beta - 2}. \tag{17}$$

From our assumptions $\chi_n \to \chi \in L^\infty(0,T;H^{-\delta'}(\mathbb{T}^2)) \cap L^2(0,T;H^{1-\delta}(\mathbb{T}^2))$. Then, the thesis follows by interpolation inequalities and Hölder inequality. Indeed, it holds

$$\begin{aligned}\int_0^T \|\chi_n(t)-\chi(t)\|^\beta_{H^\gamma}\mathrm{d}t &\le \int_0^T \|\chi_n(t)-\chi(t)\|_{H^{1-\delta}}^{\beta\frac{\gamma+\delta'}{1-\delta+\delta'}}\|\chi_n(t)-\chi(t)\|_{H^{-\delta'}}^{\beta\frac{1-\gamma-\delta}{1-\delta+\delta'}}\mathrm{d}t\\ &\le \|\chi_n-\chi\|_{L^\infty_t H^{-\delta'}_x}^{\beta\frac{1-\gamma-\delta}{1-\delta+\delta'}}\int_0^T \|\chi_n(t)-\chi(t)\|_{H^{1-\delta}}^{\beta\frac{\gamma+\delta'}{1-\delta+\delta'}}\mathrm{d}t\\ &\lesssim \|\chi_n-\chi\|_{L^\infty_t H^{-\delta'}_x}^{\beta\frac{1-\gamma-\delta}{1-\delta+\delta'}}\int_0^T \|\chi_n(t)\|_{H^{1-}}^{\beta\frac{\gamma+\delta'}{1-\delta+\delta'}}+\|\chi(t)\|_{H^{1-}}^{\beta\frac{\gamma+\delta'}{1-\delta+\delta'}}\mathrm{d}t,\end{aligned}$$

where $\|\cdot\|_{L^\infty_t H^{-\delta'}_x}$ is the norm in $L^\infty(0,T;H^{-\delta'})$. Under our assumptions on $\delta,\ \delta'$, it follows that $\beta\frac{\gamma+\delta'}{1-\delta+\delta'}\le 2$. Therefore, we have the thesis thanks to relation (16). □

4 Existence of Solutions

Our approach for showing the existence of martingale solutions of system (11) follows by a standard compactness argument. See for example (Flandoli and Luongo 2023, Sect. 2.4) and the references therein for some discussions on this method and further examples of application.

4.1 Galerkin Approximation

We introduce a sequence of Galerkin approximations ω^n. Given the orthogonal projector $\Pi^n : L^2(\mathbb{T}^2)\to \operatorname{span}\{e_l,\ |l|\le n\}$, we look for

$$\omega^n(t)=\sum_{|l|\le n} c_l(t)\,e_l$$

such that $\forall\,\phi\in\Pi^n(L^2(\mathbb{T}^2))$, $\mathbb{P}$-a.s. $\forall\,t\in[0,T]$, it holds

$$\begin{aligned}\langle\omega^n_t,\phi\rangle &= \langle\omega^n_0,\phi\rangle+\nu\int_0^t\langle\omega^n_s,\Delta\phi\rangle\,\mathrm{d}s+\int_0^t\langle g(\omega^n_s),\Delta\phi\rangle\,\mathrm{d}s\\ &\quad+\int_0^t\langle K[\omega^n_s]\cdot\nabla\phi,\omega^n_s\rangle\,\mathrm{d}s+\sum_{k\in\mathbb{Z}^2_0}\int_0^t\theta_k\langle\sigma_k\cdot\nabla\phi,\ f(\omega^n_s)\rangle\,\mathrm{d}W^k_s,\end{aligned}$$

where $\omega^n_0=\Pi^n\omega_0$. Local existence of the solution ω^n is a classical fact due to the regularity of the coefficients appearing in the equation, see for example (Karatzas and Ioannis 1991; A Skorokhod 1982). Global existence follows from the following a priori estimates.

Lemma 10 *$\mathbb{P}$-a.s., ω^n satisfies*

$$\|\omega_t^n\|^2 + 2\nu \int_0^t \|\nabla \omega_s^n\|^2 \, \mathrm{d}s \le \|\omega_0^n\|^2 \le \|\omega_0\|^2. \tag{18}$$

Proof By Itô formula and recalling the definition of g, we have

$$\begin{aligned} d\|\omega^n\|^2 + 2\nu\|\nabla\omega^n\|^2 \, \mathrm{d}t = -2\langle K[\omega^n] \cdot \nabla\omega^n, \omega^n\rangle \, \mathrm{d}t - \frac{1}{2}\langle f'(\omega^n)^2 \nabla\omega^n, \nabla\omega^n\rangle \, \mathrm{d}t \\ - 2\sum_k \theta_k \langle \sigma_k \cdot \nabla\omega^n, f(\omega^n)\rangle \, \mathrm{d}W^k \\ + \sum_k \theta_k^2 \|\Pi^n(\sigma_k \cdot \nabla f(\omega^n))\|^2 \, \mathrm{d}t. \end{aligned}$$

The first and the third terms are identically equal to 0 due to the classical properties of the trilinear form of Navier–Stokes equations and the following relation:

$$\langle \sigma_k \cdot \nabla\omega^n, f(\omega^n)\rangle = \langle \sigma_k, \nabla F(\omega^n)\rangle = -\langle \operatorname{div} \sigma_k, F(\omega^n)\rangle = 0,$$

where the function F above is a primitive of f. Therefore, we are left to show that

$$-\frac{1}{2}\langle f'(\omega^n)^2 \nabla\omega^n, \nabla\omega^n\rangle + \sum_k \theta_k^2 \|\Pi^n(\sigma_k \cdot \nabla f(\omega^n))\|^2 \le 0.$$

The last inequality is due to

$$\begin{aligned} \sum_k \theta_k^2 \|\Pi^n(\sigma_k \cdot \nabla f(\omega^n))\|^2 &\le \sum_k \theta_k^2 \|\sigma_k \cdot \nabla f(\omega^n)\|^2 \\ &= \sum_k \theta_k^2 \int_{\mathbb{T}^2} (\nabla f(\omega^n))^* (\sigma_k \otimes \sigma_k) \nabla f(\omega^n) \, dx \\ &= \frac{1}{2}\|\nabla f(\omega^n)\|^2 = \frac{1}{2}\langle f'(\omega^n)^2 \nabla\omega^n, \nabla\omega^n\rangle, \end{aligned}$$

where in the third step we have used (8). □

Lemma 10 shows in particular that $\{\omega^n\}_{n\ge 1}$ is bounded in $L^p(\Omega; L^p(0, T; L^2)) \cap L^2(\Omega; L^2(0, T; H^1))$. In order to apply Theorem 6 and Theorem 7, we need some energy estimates in $W^{s,r}(0, T; H^{-\beta})$, $s \ge 0$, $r \ge 2$, $\beta > 0$ satisfying suitable conditions. To this end, we first prove the following Lemma.

Lemma 11 *For each $M \in \mathbb{N}$, there exists a constant C independent of n such that for all $0 \le s \le t \le T$ it holds*

$$\mathbb{E}\left[\langle \omega_t^n - \omega_s^n, e_l\rangle^M\right] \le C\left(1 + \|\omega_0\|^{M(2\vee(2\alpha+1))}\right) |l|^{2M} |t-s|^{M/2},$$

where $\alpha \in [0, 1]$ is the parameter in (6).

Proof It is enough to consider $|l| \leq n$. From the weak formulation satisfied by ω^n, it follows that

$$\begin{aligned}
\langle \omega_t^n - \omega_s^n, e_l \rangle &= \nu \int_s^t \langle \omega_r^n, \Delta e_l \rangle \, dr + \int_s^t \langle g(\omega_r^n), \Delta e_l \rangle \, dr \\
&\quad + \int_s^t \langle K[\omega_r^n] \cdot \nabla e_l, \omega_r^n \rangle \, dr + \sum_k \theta_k \langle \sigma_k \cdot \nabla e_l, f(\omega_r^n) \rangle \, \mathrm{d}W_r^k \\
&= I_{s,t}^1 + I_{s,t}^2 + I_{s,t}^3 + I_{s,t}^4.
\end{aligned}$$

The analysis of $I_{s,t}^1$ and $I_{s,t}^3$ follows arguing exactly as in (Flandoli et al. 2021, Lemma 3.4) and leads us to

$$\mathbb{E}\left[(I_{s,t}^1)^M \right] + \mathbb{E}\left[(I_{s,t}^3)^M \right] \lesssim \|\omega_0\|^M |l|^{2M} |t-s|^M + \|\omega_0\|^{2M} |l|^M |t-s|^M.$$

For what concerns $I_{s,t}^2$ with $\alpha \in [1/2, 1]$ (the case $\alpha \in [0, 1/2]$ being easier), we have by Hölder's inequality and relation (10) that

$$\begin{aligned}
\mathbb{E}\left[(I_{s,t}^2)^M \right] &\lesssim \mathbb{E}\left[\left| \int_s^t \langle g(\omega_r^n), \Delta e_l \rangle \, dr \right|^M \right] \\
&\leq \mathbb{E}\left[\left| \int_s^t \|g(\omega_r^n)\|_{L^1} \|\Delta e_l\|_{L^\infty} \, dr \right|^M \right] \\
&\lesssim |l|^{2M} \, \mathbb{E}\left[\left| \int_s^t \|1 + |\omega_r^n|^{2\alpha+1}\|_{L^1} \, dr \right|^M \right] \\
&\lesssim |l|^{2M} \left(|t-s|^M + \mathbb{E}\left[\left| \int_s^t \|\omega_r^n\|_{L^{2\alpha+1}}^{2\alpha+1} \, dr \right|^M \right] \right).
\end{aligned}$$

Next, by Sobolev embedding theorem and interpolation inequalities,

$$\begin{aligned}
\mathbb{E}\left[(I_{s,t}^2)^M \right] &\lesssim |l|^{2M} \left(|t-s|^M + \mathbb{E}\left[\left| \int_s^t \|\omega_r^n\|_{H^{\frac{2\alpha-1}{2\alpha+1}}}^{2\alpha+1} \, dr \right|^M \right] \right) \\
&\leq |l|^{2M} \left(|t-s|^M + \mathbb{E}\left[\left| \int_s^t \|\nabla \omega_r^n\|^{2\alpha-1} \|\omega_r^n\|^2 dr \right|^M \right] \right),
\end{aligned}$$

which, combined the estimates in Lemma 10, yields

$$\begin{aligned}
\mathbb{E}\left[(I_{s,t}^2)^M \right] &\leq |l|^{2M} \left(|t-s|^M + |t-s|^{M(\frac{3}{2}-\alpha)} \|\omega_0\|^{2M} \mathbb{E}\left[\left| \int_s^t \|\nabla \omega_r^n\|^2 dr \right|^{M(\alpha-\frac{1}{2})} \right] \right) \\
&\lesssim |l|^{2M} \left(|t-s|^M + |t-s|^{M(\frac{3}{2}-\alpha)} \|\omega_0\|^{M(2\alpha+1)} \right) \\
&\lesssim |l|^{2M} (1 + \|\omega_0\|^{M(2\alpha+1)}) |t-s|^{M\left(1 \wedge (\frac{3}{2}-\alpha)\right)}.
\end{aligned}$$

Lastly, we need to deal with $I^4_{s,t}$. Recall that $\theta \in \ell^2(\mathbb{Z}^2_0)$ fulfills $\|\theta\|_{\ell^2} = 1$, and $\|\sigma_k\|_{L^\infty} = \sqrt{2}$; by Burkholder–Davis–Gundy inequality and estimate (7),

$$\begin{aligned}\mathbb{E}\left[(I^4_{s,t})^M\right] &\lesssim \mathbb{E}\left[\left|\sum_k \theta_k^2 \int_s^t \langle \sigma_k \cdot \nabla e_l, f(\omega^n_r)\rangle^2 \, dr\right|^{M/2}\right] \\ &\lesssim \mathbb{E}\left[\left|\int_s^t \|f(\omega^n_r)\nabla e_l\|^2_{L^1} \, dr\right|^{M/2}\right] \\ &\lesssim |l|^M \mathbb{E}\left[\left|\int_s^t \|1 + |\omega^n_r|^{\alpha+1}\|^2_{L^1} \, dr\right|^{M/2}\right].\end{aligned}$$

Then, similarly as for the treatment of $I^2_{s,t}$, by Sobolev embedding theorem, interpolation inequalities and Lemma 10, we have

$$\begin{aligned}\mathbb{E}\left[(I^4_{s,t})^M\right] &\lesssim |l|^M \left(|t-s|^{M/2} + \mathbb{E}\left[\left|\int_s^t \|\omega^n_r\|^{2(\alpha+1)}_{L^{\alpha+1}} \, dr\right|^{M/2}\right]\right) \\ &\lesssim |l|^M \left(|t-s|^{M/2} + \mathbb{E}\left[\left|\int_s^t \|\omega^n_r\|^{2(\alpha+1)} \, dr\right|^{M/2}\right]\right) \\ &\lesssim |l|^M |t-s|^{M/2} \left(1 + \|\omega_0\|^{M(\alpha+1)}\right).\end{aligned}$$

Combining the estimates, the thesis follows. □

By Theorem 6, a set bounded in $L^2(0, T; H^1) \cap W^{s,r}(0, T; H^{-\gamma})$ is relatively compact in $L^2(0, T; H^{1-\delta})$ for each $\delta > 0$ if $s > 0, \gamma > 0, r \geq 2$. On the other side, given $\delta > 0$, if $p > \frac{r_1}{\delta(s_1 r_1 - 1)(\beta - \delta)}$, a set bounded in $L^p(0, T; L^2) \cap W^{s_1,r_1}(0, T; H^{-\beta})$ with $s_1 r_1 > 1$ is relatively compact in $C(0, T; H^{-\delta})$. Since by Lemma 10, we can take p arbitrarily large, and it is enough to show the boundedness of $\{\omega^n\}_n$ in $W^{s_1,r_1}(0, T; H^{-\beta})$ for some β. This is guaranteed by the lemma below.

Lemma 12 *If $\beta > 3 + \frac{2}{r_1}$, $s_1 < \frac{1}{2}$, $s_1 r_1 > 1$ there exists a constant C independent of n such that*

$$\mathbb{E}\left[\int_0^T \|\nabla \omega^n_s\|^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^T \|\omega^n_s\|^p \mathrm{d}s\right] + \mathbb{E}\left[\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s \frac{\|\omega^n_t - \omega^n_s\|^{r_1}_{H^{-\beta}}}{|t-s|^{1+r_1 s_1}}\right] \leq C.$$

Proof Thanks to Lemma 10, we need just to consider

$$\mathbb{E}\left[\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s \frac{\|\omega^n_t - \omega^n_s\|^{r_1}_{H^{-\beta}}}{|t-s|^{1+r_1 s_1}}\right].$$

By Fubini theorem, it follows that

$$\mathbb{E}\left[\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s \frac{\|\omega_t^n - \omega_s^n\|_{H^{-\beta}}^{r_1}}{|t-s|^{1+r_1 s_1}}\right] = \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s \frac{\mathbb{E}\left[\|\omega_t^n - \omega_s^n\|_{H^{-\beta}}^{r_1}\right]}{|t-s|^{1+r_1 s_1}}.$$

Let us understand better $\mathbb{E}\left[\|\omega_t^n - \omega_s^n\|_{H^{-\beta}}^{r_1}\right]$: by the definition of Sobolev norms and Hölder's inequality,

$$\begin{aligned}
\mathbb{E}\left[\|\omega_t^n - \omega_s^n\|_{H^{-\beta}}^{r_1}\right] &= \mathbb{E}\left[\left(\sum_{l\in\mathbb{Z}_0^2} \frac{\langle \omega_t^n - \omega_s^n, e_l\rangle^2}{|l|^{2\beta}}\right)^{r_1/2}\right] \\
&= \mathbb{E}\left[\left(\sum_{l\in\mathbb{Z}_0^2} \frac{\langle \omega_t^n - \omega_s^n, e_l\rangle^2}{|l|^{2(\beta-\frac{(1+\epsilon)(r_1-2)}{r_1})}|l|^{2(\frac{(1+\epsilon)(r_1-2)}{r_1})}}\right)^{r_1/2}\right] \\
&\le \left(\sum_{l\in\mathbb{Z}_0^2} \frac{1}{|l|^{2(1+\epsilon)}}\right)^{(r_1-2)/2} \sum_{l\in\mathbb{Z}_0^2} \mathbb{E}\left[\frac{\langle \omega_t^n - \omega_s^n, e_l\rangle^{r_1}}{|l|^{\beta r_1-(1+\epsilon)(r_1-2)}}\right].
\end{aligned}$$

Thanks to Lemma 11, we have

$$\begin{aligned}
\mathbb{E}\left[\|\omega_t^n - \omega_s^n\|_{H^{-\beta}}^{r_1}\right] &\lesssim \sum_{l\in\mathbb{Z}_0^2} \frac{\left(1+\|\omega_0\|^{r_1(2\vee(2\alpha+1))}\right)|l|^{2r_1}|t-s|^{r_1/2}}{|l|^{\beta r_1-(1+\epsilon)(r_1-2)}} \\
&\lesssim |t-s|^{r_1/2} \sum_{l\in\mathbb{Z}_0^2} \frac{1}{|l|^{r_1(\beta-3)}} \\
&\lesssim |t-s|^{r_1/2}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left[\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s \frac{\|\omega_t^n - \omega_s^n\|_{H^{-\beta}}^{r_1}}{|t-s|^{1+r_1 s_1}}\right] \lesssim \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s \frac{1}{|t-s|^{1+r_1(s_1-1/2)}} \lesssim 1.$$

The proof is complete. □

Combining Lemma 12 with Theorems 6, 7, we have the following tightness result by Markov's inequality:

Corollary 13 *The family of laws of ω^n is tight on $C([0,T]; H^-) \cap L^2(0,T; H^{1-})$.*

4.2 Passage to the Limit

Arguing as in Flandoli et al. (2021), by Skorohod's representation theorem, we can find, up to passing to subsequences, an auxiliary probability space that for simplicity we continue to call $(\Omega, \mathcal{F}, \mathbb{P})$, and processes $(\tilde{\omega}^n, W^n := \{W^{n,k}\}_{k\in\mathbb{Z}^2_0})$, $(\omega, W := \{W^k\}_{k\in\mathbb{Z}^2_0})$, such that

$$\begin{aligned}
&\tilde{\omega}^n \to \omega \quad \text{in } C([0,T]; H^-) \cap L^2(0,T; H^{1-}) \quad \mathbb{P}-a.s.\\
&W^n \to W \quad \text{in } C([0,T]; \mathbb{R}^{\mathbb{Z}^2_0}) \quad \mathbb{P}-a.s.
\end{aligned}$$

Of course the convergence above between W^n and W can be seen as the uniform convergence of cylindrical Wiener processes $W^n = \sum_{k\in\mathbb{Z}^2_0} e_k W^{n,k}$, $W = \sum_{k\in\mathbb{Z}^2_0} e_k W^k$ on a suitable Hilbert space U_0. Before going on, in order to identify ω as a weak solution of equation (11), we need further integrability properties of ω. The proof of the proposition below is analogous to Lemma 3.5 in Flandoli et al. (2021); therefore, we will omit the details in these notes.

Proposition 14 *The process ω has weakly continuous trajectories on $L^2(\mathbb{T}^2)$ and satisfies*

$$\begin{aligned}
\sup_{t\in[0,T]} \|\omega_t\|^2 &\le \|\omega_0\|^2 \quad \mathbb{P}-a.s.\\
2\nu \int_0^T \|\nabla\omega_s\|^2 \mathrm{d}s &\le \|\omega_0\|^2 \quad \mathbb{P}-a.s.
\end{aligned}$$

Now, we are ready to prove Theorem 2.

Proof of Theorem 2 Let $\phi \in \Pi^M(L^2(\mathbb{T}^2))$, by classical arguments for each $n \ge M$, $\tilde{\omega}^n$ satisfies the following weak formulation: $\mathbb{P}$-a.s. for all $t \in [0, T]$,

$$\begin{aligned}
\langle \tilde{\omega}^n_t - \omega^n_0, \phi\rangle &= \nu \int_0^t \langle \tilde{\omega}^n_s, \Delta\phi\rangle \,\mathrm{d}s + \int_0^t \langle g(\tilde{\omega}^n_s), \Delta\phi\rangle \,\mathrm{d}s\\
&\quad + \int_0^t \langle \tilde{\omega}^n_s, K[\tilde{\omega}^n_s]\cdot\nabla\phi\rangle \,\mathrm{d}s + \sum_{k\in\mathbb{Z}^2_0} \int_0^t \theta_k \langle f(\tilde{\omega}^n_s), \sigma_k\cdot\nabla\phi\rangle \,\mathrm{d}W^{n,k}_s.
\end{aligned}$$

Therefore, we will show, up to passing to a further subsequence, $\mathbb{P}$-a.s. convergence of all the terms appearing above, uniformly in time. Indeed,

$$\sup_{t\in[0,T]} |\langle \tilde{\omega}^n_t - \omega_t, \phi\rangle| \le \|\tilde{\omega}^n - \omega\|_{C([0,T];H^-)} \|\phi\|_{H^1} \to 0 \quad \mathbb{P}\text{-a.s.} \tag{19}$$

and similarly for the initial conditions. Next,

$$\sup_{t\in[0,T]} \left| \int_0^t \langle \tilde{\omega}^n_s - \omega_s, \Delta\phi\rangle \,\mathrm{d}s \right| \le \|\phi\|_{H^2} \int_0^T \|\omega_s - \tilde{\omega}^n_s\| \,\mathrm{d}s \to 0 \quad \mathbb{P}\text{-a.s.} \tag{20}$$

due to the almost surely convergence in $L^2(0,T;H^{1-})$. Moreover,

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left|\int_0^t \langle \tilde{\omega}_s^n, K[\tilde{\omega}_s^n]\cdot\nabla\phi\rangle - \langle \omega_s, K[\omega_s]\cdot\nabla\phi\rangle \,\mathrm{d}s\right| \\
&\le \int_0^T |\langle \tilde{\omega}_s^n, (K[\tilde{\omega}_s^n]-K[\omega_s])\cdot\nabla\phi\rangle|\,\mathrm{d}s + \int_0^T |\langle \tilde{\omega}_s^n - \omega_s, K[\omega_s]\cdot\nabla\phi\rangle|\,\mathrm{d}s \\
&\lesssim \|\phi\|_{W^{1,\infty}}\|\omega_0\| \int_0^T \|\tilde{\omega}_s^n - \omega_s\|\mathrm{d}s \to 0 \quad \mathbb{P}\text{-a.s.}
\end{aligned}
\tag{21}
$$

due to the almost surely convergence in $L^2(0,T;H^{1-})$. Thanks to relation (10) it follows that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left|\int_0^t \langle g(\tilde{\omega}_s^n) - g(\omega_s), \Delta\phi\rangle \,\mathrm{d}s\right| \\
&\le \|\Delta\phi\|_{L^\infty}\int_0^T \left(\left\||\tilde{\omega}_s^n - \omega_s|\left(1+|\omega_s|^{2\alpha}\right)\right\|_{L^1} + \left\||\tilde{\omega}_s^n|\,\left||\tilde{\omega}_s^n|^{2\alpha} - |\omega_s|^{2\alpha}\right|\right\|_{L^1}\right)\mathrm{d}s \\
&= \|\phi\|_{W^{2,\infty}}(I_1+I_2),
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^T \left\||\tilde{\omega}_s^n - \omega_s|\left(1+|\omega_s|^{2\alpha}\right)\right\|_{L^1}\,\mathrm{d}s, \\
I_2 &= \int_0^T \left\|\,|\tilde{\omega}_s^n|\left(|\tilde{\omega}_s^n|^\alpha - |\omega_s|^\alpha\right)\left(|\tilde{\omega}_s^n|^\alpha + |\omega_s|^\alpha\right)\right\|_{L^1}\mathrm{d}s.
\end{aligned}
$$

Let us show that $\mathbb{P}$-a.s., both I_1 and I_2 tend to 0. We can control I_1 thanks to Hölder inequality, Sobolev embedding theorem, interpolation inequalities,

$$
\begin{aligned}
I_1 &\le \int_0^T \|\tilde{\omega}_s^n - \omega_s\|\left(1+\|\omega_s\|_{L^{4\alpha}}^{2\alpha}\right)\mathrm{d}s \\
&\le \int_0^T \|\tilde{\omega}_s^n - \omega_s\|\left(1+\|\omega_s\|_{H^{\frac{2\alpha-1}{2\alpha}}}^{2\alpha}\right)\mathrm{d}s \\
&\le \int_0^T \|\tilde{\omega}_s^n - \omega_s\|\left(1+\|\omega_s\|_{H^1}^{2\alpha-1}\|\omega_s\|\right)\mathrm{d}s.
\end{aligned}
$$

By Lemma 9, we have for $\alpha \in (1/2, 1]$ (the other case being easier) that

$$
\begin{aligned}
I_1 &\lesssim \|\tilde{\omega}^n - \omega\|_{L^2(0,T;L^2)} + \|\omega_0\|\,\|\omega\|_{L^2(0,T;H^1)}^{2\alpha-1}\|\tilde{\omega}^n - \omega\|_{L^{\frac{2}{3-2\alpha}}(0,T;L^2)} \\
&\lesssim \|\tilde{\omega}^n - \omega\|_{L^2(0,T;L^2)} + \|\omega_0\|^{2\alpha}\|\tilde{\omega}^n - \omega\|_{L^{\frac{2}{3-2\alpha}}(0,T;L^2)} \xrightarrow{\mathbb{P}\text{-a.s.}} 0.
\end{aligned}
\tag{23}
$$

For what concerns I_2 similar arguments and the Hölderianity of x^α lead to

$$
\begin{aligned}
I_2 &\le \int_0^T \|\tilde{\omega}_s^n\| \left\| \left|\tilde{\omega}_s^n - \omega_s\right|^\alpha \left(\left|\tilde{\omega}_s^n\right|^\alpha + |\omega_s|^\alpha\right) \right\| \\
&\lesssim \|\omega_0\| \int_0^T \|\tilde{\omega}_s^n - \omega_s\|_{L^{4\alpha}}^\alpha \left(\|\tilde{\omega}_s^n\|_{L^{4\alpha}}^\alpha + \|\omega_s\|_{L^{4\alpha}}^\alpha\right) \mathrm{d}s \\
&\lesssim \|\omega_0\| \int_0^T \|\tilde{\omega}_s^n - \omega_s\|_{H^{\frac{2\alpha-1}{2\alpha}}}^\alpha \left(\|\tilde{\omega}_s^n\|_{H^{\frac{2\alpha-1}{2\alpha}}}^\alpha + \|\omega_s\|_{H^{\frac{2\alpha-1}{2\alpha}}}^\alpha\right) \mathrm{d}s.
\end{aligned}
$$

By interpolation and Hölder's inequality,

$$
\begin{aligned}
I_2 &\le \|\omega_0\| \int_0^T \|\tilde{\omega}_s^n - \omega_s\|_{H^{\frac{2\alpha-1}{2\alpha}}}^\alpha \left(\|\tilde{\omega}_s^n\|_{H^1}^{\frac{2\alpha-1}{2}} \|\tilde{\omega}_s^n\|^{\frac{1}{2}} + \|\omega_s\|_{H^1}^{\frac{2\alpha-1}{2}} \|\omega_s\|^{\frac{1}{2}}\right) \mathrm{d}s \\
&\le \|\omega_0\|^{3/2} \left(\|\tilde{\omega}^n\|_{L^2(0,T;H^1)}^{\frac{2\alpha-1}{2}} + \|\omega\|_{L^2(0,T;H^1)}^{\frac{2\alpha-1}{2}}\right) \|\tilde{\omega}_s^n - \omega_s\|_{L^{\frac{4\alpha}{5-2\alpha}}(0,T;H^{\frac{2\alpha-1}{2\alpha}})}^\alpha \\
&\lesssim \|\omega_0\|^{1+\alpha} \|\tilde{\omega}_s^n - \omega_s\|_{L^{\frac{4\alpha}{5-2\alpha}}(0,T;H^{\frac{2\alpha-1}{2\alpha}})}^\alpha \xrightarrow{\mathbb{P}\text{-a.s.}} 0.
\end{aligned}
\tag{24}
$$

In order to deal with the stochastic integral, we apply Lemma 8. Since we have the convergence of the Wiener processes, it is enough to show that $\mathbb{P}$-a.s., therefore in probability,

$$
\int_0^T \sum_k \theta_k^2 \langle \sigma_k \cdot \nabla\phi, f(\tilde{\omega}_s^n) - f(\omega_s)\rangle^2 \,\mathrm{d}s \to 0.
$$

The relation above is true, indeed, recall the facts that $\|\sigma_k\|_{L^\infty} = \sqrt{2}$ ($\forall\, k \in \mathbb{Z}_0^2$), $\sum_{k\in\mathbb{Z}_0^2} \theta_k^2 = 1$, and relation (6) we have

$$
\begin{aligned}
&\int_0^T \sum_k \theta_k^2 \langle \sigma_k \cdot \nabla\phi, f(\tilde{\omega}_s^n) - f(\omega_s)\rangle^2 \mathrm{d}s \\
&\le \|\phi\|_{W^{1,\infty}}^2 \int_0^T \sum_k \theta_k^2 \|\sigma_k\|_{L^\infty}^2 \|f(\tilde{\omega}_s^n) - f(\omega_s)\|_{L^1}^2 \mathrm{d}s \\
&\lesssim \|\phi\|_{W^{1,\infty}}^2 \int_0^T \left\| |\tilde{\omega}_s^n - \omega_s| + |\tilde{\omega}_s^n \left|\tilde{\omega}_s^n\right|^\alpha - \omega_s|\omega_s|^\alpha| \right\|_{L^1}^2 \mathrm{d}s \\
&\lesssim \|\phi\|_{W^{1,\infty}}^2 \left(\|\tilde{\omega}^n - \omega\|_{L_t^2 L_x^2} + \int_0^T \left\| |\tilde{\omega}_s^n - \omega_s|\, |\tilde{\omega}_s^n|^\alpha \right\|_{L^1}^2 + \left\| |\omega_s|\, |\tilde{\omega}_s^n - \omega_s|^\alpha \right\|_{L^1}^2 \mathrm{d}s \right),
\end{aligned}
$$

where $\|\cdot\|_{L^2_t L^2_x}$ is the norm in $L^2(0,T;L^2(\mathbb{T}^2))$. By Cauchy's inequality,

$$\begin{aligned}
&\int_0^T \sum_k \theta_k^2 \langle \sigma_k\cdot\nabla\phi, f(\tilde\omega^n_s)-f(\omega_s)\rangle^2 \mathrm{d}s\\
&\le \|\phi\|^2_{W^{1,\infty}}\left(\|\tilde\omega^n-\omega\|_{L^2_tL^2_x}+\int_0^T \|\tilde\omega^n_s-\omega_s\|^2\|\tilde\omega^n_s\|^{2\alpha}_{L^{2\alpha}}+\|\omega_s\|^2\|\tilde\omega^n_s-\omega_s\|^{2\alpha}_{L^{2\alpha}}\mathrm{d}s\right)\\
&\le \|\phi\|^2_{W^{1,\infty}}\left(\|\tilde\omega^n-\omega\|_{L^2_tL^2_x}+\|\omega_0\|^{2\alpha}\|\tilde\omega^n-\omega\|^2_{L^2_tL^2_x}+\|\omega_0\|^2\|\tilde\omega^n-\omega\|^{2\alpha}_{L^2_tL^2_x}\right)\\
&\to 0 \quad \mathbb{P}\text{-a.s.}
\end{aligned}\tag{25}$$

Therefore, by Lemma 8, up to passing to a subsequence, uniformly in time,

$$\sum_k\int_0^t \theta_k\langle\sigma_k\cdot\nabla\phi, f(\tilde\omega^n_s)\rangle\,\mathrm{d}W^{n,k}_s \xrightarrow{\mathbb{P}\text{-a.s.}} \sum_k\int_0^t\theta_k\langle\sigma_k\cdot\nabla\phi, f(\omega_s)\rangle\,\mathrm{d}W^k_s. \tag{26}$$

Combining relations (19), (20), (21), (22), (23), (24), (26) we have, $\mathbb{P}$-a.s. for all $t\in[0,T]$,

$$\begin{aligned}
\langle\omega_t,\phi\rangle &= \langle\omega_0,\phi\rangle+\nu\int_0^t\langle\omega_s,\Delta\phi\rangle\,\mathrm{d}s+\int_0^t\langle g(\omega_s),\Delta\phi\rangle\,\mathrm{d}s\\
&\quad+\int_0^t\langle K[\omega_s]\cdot\nabla\phi,\omega_s\rangle\,\mathrm{d}s+\int_0^t\sum_k\theta_k\langle\sigma_k\cdot\nabla\phi, f(\omega_s)\rangle\,\mathrm{d}W^k_s.
\end{aligned}\tag{27}$$

By standard density argument, we can find a zero probability set $\mathcal{N}$ such that on its complementary relation (27) holds for each $\phi\in C^\infty(\mathbb{T}^2)$. □

5 Scaling Limit

Let now $\{\theta^N\}_N$ be a sequence in $\ell^2(\mathbb{Z}^2_0)$, each satisfying the conditions (5), and moreover,

$$\lim_{N\to+\infty}\|\theta^N\|_{\ell^\infty}=0; \tag{28}$$

let ω^N be an analytically weak martingale solution in the sense of Definition 1 of

$$\begin{cases} d\omega^N=\left(\nu\Delta\omega^N-K[\omega^N]\cdot\nabla\omega^N+\Delta g(\omega^N)\right)\mathrm{d}t+\sum_k\theta^N_k\sigma_k\cdot\nabla f(\omega^N)\,\mathrm{d}W^k,\\ u^N=-\nabla^\perp(-\Delta)^{-1}\omega^N,\\ \omega^N(0)=\omega_0\end{cases}\tag{29}$$

satisfying

$$\sup_{t\in[0,T]} \|\omega_t^N\|^2 + 2\nu \int_0^T \|\nabla \omega_s^N\|^2 \, \mathrm{d}s \le 2\|\omega_0\|^2 \quad \mathbb{P}\text{-a.s.}$$

The existence of such a solution for each $N \in \mathbb{N}$ is guaranteed by Theorem 2. Of course the probability space and the Brownian motions depend on N; however, with some abuse of notation, we do not stress this dependence. Arguing as in Sect. 4, we will show the tightness of the law of ω^N in $C([0,T];H^{-}) \cap L^2(0,T;H^{1-})$. This will allow us to prove Theorem 5 following the same ideas as in Sect. 4.

5.1 Tightness

The way of showing the tightness is completely analogous to Sect. 4 thanks to Proposition 14. Therefore, we just sketch the argument. We start with the lemma below.

Lemma 15 *For each $M \in \mathbb{N}$, there exists a constant C independent of N such that for any s, t with $0 \le s \le t \le T$, it holds*

$$\mathbb{E}\left[\langle \omega_t^N - \omega_s^N, e_l\rangle^M\right] \le C\left(1 + \|\omega_0\|^{M(2\vee(2\alpha+1))}\right)|l|^{2M}|t-s|^{M/2}.$$

Proof From the weak formulation satisfied by ω^N, it follows that

$$\begin{aligned}\langle \omega_t^N - \omega_s^N, e_l\rangle &= \nu\int_s^t \langle \omega_r^N, \Delta e_l\rangle\, dr + \int_s^t \langle g(\omega_r^N), \Delta e_l\rangle\, dr \\ &\quad + \int_s^t \langle K[\omega_r^N]\cdot\nabla e_l, \omega_r^N\rangle dr + \sum_k \theta_k^N \langle \sigma_k\cdot\nabla e_l, f(\omega_r^N)\rangle\, \mathrm{d}W_r^k \\ &= I_{s,t}^1 + I_{s,t}^2 + I_{s,t}^3 + I_{s,t}^4.\end{aligned}$$

All the terms above can be treated analogously to Lemma 11, leading us to the following estimates:

$$\begin{aligned}\mathbb{E}\left[(I_{s,t}^1)^M\right] + \mathbb{E}\left[(I_{s,t}^3)^M\right] &\lesssim \|\omega_0\|^M|l|^{2M}|t-s|^M + \|\omega_0\|^{2M}|l|^M|t-s|^M, \\ \mathbb{E}\left[(I_{s,t}^2)^M\right] &\lesssim |l|^{2M}(1+\|\omega_0\|^{M(2\alpha+1)})|t-s|^{M\left(1\wedge(\frac{3}{2}-\alpha)\right)}, \\ \mathbb{E}\left[(I_{s,t}^4)^M\right] &\lesssim |l|^M|t-s|^{M/2}\big(1+\|\omega_0\|^{M(\alpha+1)}\big).\end{aligned}$$

Combining them, the thesis follows immediately. □

Thanks to the discussion before Lemma 12, in order to obtain the required tightness in $L^2([0,T];H^{1-}) \cap C([0,T];H^{-})$, we need the following result.

Lemma 16 *If $\beta > 3 + \frac{2}{r_1}$, $s_1 < \frac{1}{2}$, $s_1 r_1 > 1$ and $p > 1$, there exists a constant C independent of N such that*

$$\mathbb{E}\left[\int_0^T \|\nabla\omega_s^N\|^2 \,\mathrm{d}s + \int_0^T \|\omega_s^N\|^p \,\mathrm{d}s + \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s \frac{\|\omega_t^N - \omega_s^N\|_{H^{-\beta}}^{r_1}}{|t-s|^{1+r_1 s_1}}\right] \le C.$$

We omit its proof since it is similar to Lemma 12 based on the definition of the Sobolev norms and the estimate guaranteed by Lemma 15. Combining the lemma above with Theorems 6, 7, we have the following tightness result.

Corollary 17 *The family of laws of ω^N is tight on $C([0,T]; H^-) \cap L^2(0,T; H^{1-})$.*

5.2 Passage to the Limit

The preliminary part in order to show the convergence is analogous to Sect. 4.2. Arguing as in Flandoli et al. (2021), by Skorohod's representation theorem, we can find, up to passing to subsequences, an auxiliary probability space, that for simplicity we continue to call $(\Omega, \mathcal{F}, \mathbb{P})$, and processes $(\tilde\omega^N, \tilde W^N := \{\tilde W^{N,k}\}_{k\in\mathbb{Z}_0^2})$, $(\overline{\omega}, \tilde W := \{\tilde W^k\}_{k\in\mathbb{Z}_0^2})$, such that

$$\begin{aligned} &\tilde\omega^N \to \overline{\omega} \quad \text{in } C([0,T]; H^-) \cap L^2(0,T; H^{1-}) \quad \mathbb{P}\text{-a.s.} \\ &\tilde W^N \to \tilde W \quad \text{in } C([0,T]; \mathbb{R}^{\mathbb{Z}_0^2}) \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

The convergence above from $\tilde W^N$ to $\tilde W$ can be seen as the uniform convergence of cylindrical Wiener processes $\tilde W^N = \sum_{k\in\mathbb{Z}_0^2} e_k \tilde W^{N,k}$, $\tilde W = \sum_{k\in\mathbb{Z}_0^2} e_k \tilde W^k$ on a suitable Hilbert space U_0. Before going on, in order to identify $\overline{\omega}$ as a random variable supported on the weak solutions of equation (13), we need further integrability properties of $\overline{\omega}$. The proof of the proposition below is analogous to Proposition 14, therefore we will omit the details.

Proposition 18 *The process $\overline{\omega}$ has weakly continuous trajectories on $L^2(\mathbb{T}^2)$ and satisfies*

$$\sup_{t\in[0,T]} \|\overline{\omega}_t\|^2 \le \|\omega_0\|^2 \quad \mathbb{P}\text{-a.s.}$$

$$2\nu \int_0^T \|\nabla\overline{\omega}_s\|^2 \mathrm{d}s \le \|\omega_0\|^2 \quad \mathbb{P}\text{-a.s.}$$

Before exploiting the convergence properties of ω^N, we are interested in showing the uniqueness of weak solutions of (13). The approach we follow is the so called H^{-1}-method for active scalars, see for example Theorem 2 and Theorem 5 in Azzam and Bedrossian (2015) for other applications of this method.

Lemma 19 *There exists at most one solution of (13) in the sense of definition 4.*

Proof First note that arguing for example as in Flandoli and Luongo (2022, 2023), we can extend the weak formulation of (13) in Definition 4 to time-dependent test functions in $L^2(0,T;H^2)\cap W^{1,2}(0,T;L^2)$. Therefore, our weak formulation becomes: for any $\phi\in L^2(0,T;H^2)\cap W^{1,2}(0,T;L^2)$ and $t\in[0,T]$, it holds

$$\begin{aligned}\langle\overline{\omega}_t,\phi_t\rangle-\langle\omega_0,\phi_0\rangle&=\int_0^t\langle\overline{\omega}_s,\partial_s\phi_s\rangle\,\mathrm{d}s+\nu\int_0^t\langle\overline{\omega}_s,\Delta\phi_s\rangle\,\mathrm{d}s\\&\quad+\int_0^t\langle g(\overline{\omega}_s),\Delta\phi_s\rangle\,\mathrm{d}s+\int_0^t\langle\overline{\omega}_s,K[\overline{\omega}_s]\cdot\nabla\phi_s\rangle\,\mathrm{d}s.\end{aligned}$$

Consider now two weak solutions $\omega,\tilde{\omega}\in C_w([0,T];L^2)\cap L^2(0,T;H^1)$. Looking at the equation and exploiting the regularity of the weak solutions, it follows that actually $\omega,\tilde{\omega}\in C_w([0,T];L^2)\cap W^{1,2}(0,T;H^{-2})$. Let $w_t=\omega_t-\tilde{\omega}_t$, then

$$\psi=(-\Delta)^{-1}w\in C_w([0,T];H^2)\cap L^2(0,T;H^3)\cap W^{1,2}(0,T;L^2)$$

is a proper test function and we obtain

$$\begin{aligned}\frac{1}{2}\|w_t\|_{H^{-1}}^2=\frac{1}{2}\|\psi_t\|_{H^1}^2&\\=-\nu\int_0^t\|w_s\|^2\mathrm{d}s&-\int_0^t\int_{\mathbb{T}^2}(g(\omega_s)-g(\tilde{\omega}_s))\,(\omega_s-\tilde{\omega}_s)\,\mathrm{d}x\mathrm{d}s\\+\int_0^t\int_{\mathbb{T}^2}w_sK[\omega_s]\cdot\nabla\psi_s\,\mathrm{d}x\mathrm{d}s&+\int_0^t\int_{\mathbb{T}^2}\tilde{\omega}_sK[w_s]\cdot\nabla\psi_s\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Thanks to the fact that the function g is monotone increasing, we have

$$\int_0^t\int_{\mathbb{T}^2}(g(\omega_s)-g(\tilde{\omega}_s))\,(\omega_s-\tilde{\omega}_s)\,\mathrm{d}x\mathrm{d}s\geq 0.$$

Next, by the definition of ψ and integrating by parts,

$$\begin{aligned}\int_{\mathbb{T}^2}w_sK[\omega_s]\cdot\nabla\psi_s\,dx&=-\int_{\mathbb{T}^2}(\Delta\psi_s)\,K[\omega_s]\cdot\nabla\psi_s\,dx\\&=\int_{\mathbb{T}^2}(\nabla\psi_s)\cdot\nabla(K[\omega_s]\cdot\nabla\psi_s)\,dx\\&=\int_{\mathbb{T}^2}(\nabla\psi_s)\cdot\big((\nabla K[\omega_s])\cdot\nabla\psi_s\big)\,dx\\&\quad+\int_{\mathbb{T}^2}(\nabla\psi_s)\cdot\big(K[\omega_s]\cdot\nabla(\nabla\psi_s)\big)\,dx;\end{aligned}$$

the last integral vanishes since $K[\omega_s]$ is divergence free. Therefore, we can proceed as in the proof of (Azzam and Bedrossian 2015, Theorem 5) and obtain

$$\begin{aligned}
&\frac{1}{2}\|\psi_t\|^2_{H^1} + \nu \int_0^t \|w_s\|^2 \,\mathrm{d}s \\
&\le \int_0^t \int_{\mathbb{T}^2} |\nabla K[\omega_s]|\, |\nabla \psi_s|^2 \,\mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathbb{T}^2} |\tilde{\omega}_s|\, |K[w_s]|\, |\nabla \psi_s| \,\mathrm{d}x\mathrm{d}s \\
&\le \int_0^t \left(\|\nabla K[\omega_s]\|\, \|\nabla \psi_s\|^2_{L^4} + \|\tilde{\omega}_s\|\, \|\nabla \psi_s\|_{L^4} \|K[\mathrm{div}\nabla\psi_s]\|_{L^4} \right) \mathrm{d}s.
\end{aligned}$$

We remark that $\nabla K : L^2 \to L^2$ and $K\mathrm{div} : L^4 \to L^4$ are bounded operators, hence

$$\begin{aligned}
\frac{1}{2}\|\psi_t\|^2_{H^1} + \nu \int_0^t \|w_s\|^2 \,\mathrm{d}s &\lesssim \|\nabla K\|_{L^2\to L^2} \int_0^t \|\omega_s\|\, \|\nabla \psi_s\|^2_{L^4} \,\mathrm{d}s \\
&\quad + \|K\mathrm{div}\|_{L^4\to L^4} \int_0^t \|\tilde{\omega}_s\|\, \|\nabla \psi_s\|^2_{L^4} \,\mathrm{d}s \\
&\lesssim \int_0^t (\|\omega_s\| + \|\tilde{\omega}_s\|)\, \|\nabla\psi_s\|_{L^2} \|\nabla\psi_s\|_{H^1} \,\mathrm{d}s
\end{aligned}$$

by Sobolev embedding and interpolation. Therefore,

$$\begin{aligned}
\frac{1}{2}\|\psi_t\|^2_{H^1} + \nu \int_0^t \|w_s\|^2 \,\mathrm{d}s &\lesssim \int_0^t (\|\omega_s\| + \|\tilde{\omega}_s\|)\, \|\psi_s\|_{H^1} \|\psi_s\|_{H^2} \,\mathrm{d}s \\
&\le C_\nu \int_0^t \left(\|\omega_s\|^2 + \|\tilde{\omega}_s\|^2 \right) \|\psi_s\|^2_{H^1} \mathrm{d}s + \frac{\nu}{2} \int_0^t \|w_s\|^2 \,\mathrm{d}s.
\end{aligned}$$

Since both ω and $\tilde{\omega}$ belong to $C_w(0, T; L^2(\mathbb{T}^2))$, by Grönwall's inequality, the thesis follows. □

Now, we are ready to provide the proof of our main theorem.

Proof of Theorem 5 Let $\phi \in C^\infty(\mathbb{T}^2)$, by classical arguments for each $N \in \mathbb{N}$, $\tilde{\omega}^N$ satisfies the following weak formulation: $\mathbb{P}$-a.s. for all $t \in [0, T]$,

$$\begin{aligned}
\langle \tilde{\omega}^N_t - \omega_0, \phi \rangle =& \nu \int_0^t \langle \tilde{\omega}^N_s, \Delta\phi \rangle \,\mathrm{d}s + \int_0^t \langle g(\tilde{\omega}^N_s), \Delta\phi \rangle \,\mathrm{d}s \\
&+ \int_0^t \langle K[\tilde{\omega}^N_s] \cdot \nabla\phi, \tilde{\omega}^N_s \rangle \,\mathrm{d}s + \sum_k \int_0^t \theta^N_k \langle \sigma_k \cdot \nabla\phi, f(\tilde{\omega}^N_s) \rangle \, d\tilde{W}^{N,k}_s.
\end{aligned}$$

Up to passing to a further subsequence, we will show the $\mathbb{P}$-a.s. convergence, uniformly in time, of all the terms appearing above, except the martingale part; this is the only term that will present some differences with respect to the proof of Theorem 2. Therefore, we omit the treatments of the other terms which are similar to the proof of Theorem

2, and concentrate on the martingale part which will be shown to vanish in the limit, uniformly in time.

In order to deal with the stochastic integral, applying Burkholder–Davis–Gundy inequality and using the fact that $\{\sigma_k\}_k$ is an orthonormal family of vector fields, we obtain

$$\begin{aligned} J &:= \mathbb{E}\left[\sup_{t\in[0,T]}\left|\sum_k \int_0^t \theta_k^N \langle \sigma_k \cdot \nabla\phi, f(\tilde{\omega}_s^N)\rangle \, \mathrm{d}W_s^{N,k}\right|^2\right] \\ &\lesssim \mathbb{E}\left[\sum_k \int_0^T (\theta_k^N)^2 \langle \sigma_k \cdot \nabla\phi, f(\tilde{\omega}_s^N)\rangle^2 \, \mathrm{d}s\right] \\ &\le \|\theta^N\|_{\ell^\infty}^2 \, \mathbb{E}\bigg[\int_0^T \|f(\tilde{\omega}_s^N)\nabla\phi\|^2 \, \mathrm{d}s\bigg]. \end{aligned}$$

Then, by relation (7) and Sobolev embedding theorem,

$$\begin{aligned} J &\lesssim \|\theta^N\|_{\ell^\infty}^2 \|\phi\|_{W^{1,\infty}}^2 \mathbb{E}\bigg[\int_0^T \big\|1 + |\tilde{\omega}_s^N|^{\alpha+1}\big\|^2 \, \mathrm{d}s\bigg] \\ &\lesssim \|\theta^N\|_{\ell^\infty}^2 \|\phi\|_{W^{1,\infty}}^2 \bigg(1 + \mathbb{E}\bigg[\int_0^T \|\tilde{\omega}_s^N\|_{L^{2(\alpha+1)}}^{2(\alpha+1)} \, \mathrm{d}s\bigg]\bigg) \\ &\lesssim \|\theta^N\|_{\ell^\infty}^2 \|\phi\|_{W^{1,\infty}}^2 \bigg(1 + \mathbb{E}\bigg[\int_0^T \|\tilde{\omega}_s^N\|_{H^{\frac{\alpha}{\alpha+1}}}^{2(\alpha+1)} \, \mathrm{d}s\bigg]\bigg), \end{aligned}$$

and, using interpolation inequalities and (28) yields

$$\begin{aligned} J &\le \|\theta^N\|_{\ell^\infty}^2 \|\phi\|_{W^{1,\infty}}^2 \bigg(1 + \mathbb{E}\bigg[\int_0^T \|\tilde{\omega}_s^N\|_{H^1}^{2\alpha} \|\tilde{\omega}_s^N\|^2 \, \mathrm{d}s\bigg]\bigg) \\ &\lesssim \|\theta^N\|_{\ell^\infty}^2 \|\phi\|_{W^{1,\infty}}^2 \big(1 + \|\omega_0\|^{2(\alpha+1)}\big) \to 0. \end{aligned}$$

Summarizing the above arguments, we arrive at

$$\begin{aligned} \langle \overline{\omega}_t, \phi\rangle - \langle \overline{\omega}_0, \phi\rangle &= \nu \int_0^t \langle \overline{\omega}_s, \Delta\phi\rangle \, \mathrm{d}s + \int_0^t \langle g(\overline{\omega}_s), \Delta\phi\rangle \, \mathrm{d}s \\ &\quad + \int_0^t \langle K[\overline{\omega}_s] \cdot \nabla\phi, \overline{\omega}_s\rangle \, \mathrm{d}s \quad \mathbb{P}\text{-a.s. } \forall\, t \in [0, T]. \end{aligned} \tag{30}$$

By standard density argument, we can find a zero probability set $\mathcal{N}$ such that on its complementary equation (30) holds for each $\phi \in C^\infty$. By Corollary 17 and Lemma 19, every subsequence $\mathcal{L}(\omega^{N_k})$ admits a sub-subsequence which converges to the unique limit point $\delta_{\overline{\omega}}$, where $\overline{\omega}$ is the unique deterministic solution of (13). Then, for example by (Billingsley 2013, Theorem 2.6), the whole sequence $\mathcal{L}(\omega^N)$ converges weakly to $\delta_{\overline{\omega}}$. $\square$

As a Corollary of Lemma 19 and Theorem 5, we have the following result.

Corollary 20 *There exists a unique solution of (13) in the sense of Definition 4.*

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