SOME REMARKS ON THE EQUATION $-\Delta u = \lambda(1+u)^p$ FOR VARYING λ, p AND VARYING DOMAINS *

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Abstract

We consider positive solutions of the equation $-\Delta u = \lambda(1+u)^p$ with Dirichlet boundary conditions in a smooth bounded domain Ω for $\lambda > 0$ and $p > 1$. We study the behavior of the solutions for varying λ, p and varying domains Ω in different limiting situations.

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1 Introduction

We are interested in the solutions of the two–parameter family of problems

$$(P_\lambda^p) \qquad \begin{cases} -\Delta u = \lambda(1+u)^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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where Ω is an open bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$, and $p > 1$, $\lambda > 0$. By solutions we mean here *weak* solutions in $H_0^1(\Omega)$. If $p \leq \frac{n+2}{n-2}$, by [7] it turns out that these solutions u are in $L^\infty(\Omega)$ and therefore $u \in C^{2,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$ and, up to the boundary, u is as smooth as the boundary permits.

Equation (P_λ^p) has been studied by several authors because of its wide applications to physical models. Among others, it describes problems of thermal self–ignition [14], diffusion phenomena induced by nonlinear sources [20] or a ball of isothermal gas in gravitational equilibrium as proposed by lord Kelvin [10]. We also refer to [19, 26] where different models and further references may be found. In this paper we concentrate on the problem of temperature distribution in an object heated by the application of a uniform electric current suggested in [21]. In Section 6 we discuss this model and we analyze the physical meaning of our results.

It is known [6, 8, 11, 21] that if $1 < p \leq \frac{n+2}{n-2}$, then there exists $\lambda^* = \lambda^*(\Omega, p) > 0$ such that:
- if $\lambda > \lambda^*$ there are no solutions of (P_λ^p) even in distributional sense
- if $0 \leq \lambda < \lambda^*$, problem (P_λ^p) admits at least a *minimal* solution u_λ and a *mountain–pass* solution U_λ (see next section for the definitions)
- if $\lambda = \lambda^*$ there exists a unique solution U_* of (P_λ^p), usually called the *extremal* solution [24].

The set of solutions of (P_λ^p) does not have the above stated features if $p \notin (1, \frac{n+2}{n-2}]$. We refer to Section 2 for a survey of results which highlight a strong dependence of the solutions of (P_λ^p) on λ, p and Ω. Therefore, it is an interesting problem to understand how the solutions behave when these parameters vary. This is precisely the aim of this paper.

We first restrict to subcritical and critical problems ($p \leq \frac{n+2}{n-2}$) and consider the case where $\lambda \uparrow \lambda^*$; we show that the extremal solution U_* arises from the superposition of the solutions u_λ and U_λ and therefore it is a "degenerate" solution. To see this, we use critical point theory and we give a complete description of the Nehari manifold associated to the action functional.

Next, we analyze the behavior of the solutions as $\lambda \downarrow 0$. We first give the explicit rate of uniform convergence to 0 of the minimal solution u_λ and we show that the rate of convergence is independent of p on bounded subsets of $(1, \infty)$. On the contrary, the mountain–pass solution U_λ blows up; of course, here we assume that $1 < p \leq \frac{n+2}{n-2}$. More precisely, in the critical case $p = \frac{n+2}{n-2}$ we find concentration phenomena and in the subcritical case $p < \frac{n+2}{n-2}$ we find a pointwise blow–up with rate depending on p. When dealing with the critical case we apply the technique developed by Han, Li and Schoen [17, 22, 30].

Then, we study the map $\lambda^* = \lambda^*(p, \Omega)$. We first give an alternative proof of a result of [19] which allows to determine explicitly λ^*, u_λ, U_λ and U_* when Ω is the unit ball and $p = \frac{n+2}{n-2}$. Next, we show that $\lambda^* = \lambda^*(p)$ is continuous and strictly decreasing (for a fixed domain Ω). Moreover, since (for fixed p) λ^* is minimal on balls among bounded domains having the same measure [5], we obtain uniform lower bounds for $\lambda^* = \lambda^*(\Omega, p)$.

Finally, we deal with the limiting case $p \to 1$. As the limit problem (P_λ^1) is linear, it admits at most one solution. We show that if a solution u_0 of (P_λ^1) exists, then the minimal solution u_ε and the mountain–pass solution U_ε of $(P_\lambda^{1+\varepsilon})$ also exist for $\varepsilon > 0$ small enough. Moreover, u_ε tends to u_0 while U_ε blows up exponentially as $\varepsilon \to 0$.

The outline of the paper is the following. In next section we recall some well–known results. In Section 3, by means of the Nehari manifold relative to the functional associated to (P_λ^p) $(1 < p \leq \frac{n+2}{n-2})$, we study the behavior of the solutions when $\lambda \uparrow \lambda^*$. In Section 4 we analyze the behavior of the solutions as $\lambda \downarrow 0$. In Section 5.1 we consider the critical case $p = \frac{n+2}{n-2}$ when Ω is the unit ball. In Section 5.2 we study the map $\lambda^* = \lambda^*(\Omega, p)$. In Section 5.3 we determine the behavior of both the minimal and the mountain–pass solution as $p \to 1$. In Section 6 we discuss a physical model associated to (P_λ^p) and we give a related interpretation of our results; we also state some relevant open problems. In the Appendix we recall some known results which are used in the blow–up analysis of Section 4.

2 Notations and a survey of known results

Throughout this paper we assume that $\Omega \in L$, where

$$L = \{\Omega \subset \mathbb{R}^n;\ \Omega \text{ open and bounded domain, } \partial\Omega \text{ is of class } C^{2,\alpha}\}.$$

We denote by $\lambda_1 = \lambda_1(\Omega)$ the first (positive) eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions. It is well–known that λ_1 is simple and isolated and that the corresponding eigenfunction φ_1 may be chosen positive in Ω.

We denote by $\|\cdot\|$ the Dirichlet norm in $H_0^1(\Omega)$ and by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$. The space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is the space of functions having finite Dirichlet integral over $\mathbb{R}^n$. Let $2^* = \frac{2n}{n-2}$ be the usual critical Sobolev exponent. We denote by $\mathcal{S}$ the best Sobolev constant for the embedding $\mathcal{D}^{1,2}(\mathbb{R}^n) \subset L^{2^*}(\mathbb{R}^n)$, namely

$$\mathcal{S} = \inf_{u \in \mathcal{D}^{1,2}\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}. \tag{1}$$

We assume that the minimax variational characterization of mountain–pass solutions given by Ambrosetti–Rabinowitz [1] is familiar to the reader and we recall in more precise fashion the results in [11] (when p is subcritical, i.e. $1 < p < \frac{n+2}{n-2}$) and [8, Corollary 2.5] (when p is critical, i.e. $p = \frac{n+2}{n-2}$) roughly stated in the introduction:

Theorem 1. [8, 11]
Let $\Omega \in L$, and let $1 < p \leq \frac{n+2}{n-2}$. Then, there exists $\lambda^* = \lambda^*(\Omega, p) > 0$ such that:
(i) if $\lambda > \lambda^*$ there are no solutions of (P_λ^p) even in distributional sense.
(ii) if $\lambda = \lambda^*$ there exists a unique solution U_* of (P_λ^p).
(iii) if $0 < \lambda < \lambda^*$, problem (P_λ^p) admits at least two solutions u_λ and U_λ; u_λ is minimal (in the sense that $u_\lambda(x) \leq v(x)$ for all $x \in \Omega$ and for any other solution v of (P_λ^p)) and U_λ is a mountain–pass solution.

From now on, without recalling it at each statement, we denote by u_λ, U_λ and U_* the functions defined in Theorem 1. When it is needed, we emphasize the dependence of u_λ, U_λ, U_* on p. On the other hand, we also write $\lambda^*(\Omega), \lambda^*(p)$, or simply λ^*, when there is no need to emphasize the dependence on p, Ω or both.

In general, the mountain–pass solution U_λ may not be unique, see Remark 3. In order to avoid ambiguity, we will state results concerning "the mountain–pass solution U_λ" meaning that the results hold for any solution having the same variational characterization. When Ω is a ball, the mountain–pass solution U_λ is indeed unique and (P_λ^p) admits no solutions but u_λ and U_λ

Theorem 2. [19]
Let Ω be a ball and let $1 < p \leq \frac{n+2}{n-2}$. Then, for all $\lambda < \lambda^*$ problem (P_λ^p) admits exactly two solutions.

By [26, Théorème 6], the uniqueness of U_λ is also ensured if λ belongs to a suitable left neighborhood of λ^*. It is shown there that (P_λ^p) admits exactly two solutions close to U_*, see also Corollary 1 below.

Remark 1. Let $\lambda < \lambda^*$. Then the minimal solution u_λ has also minimal $H_0^1(\Omega)$–norm. Indeed, let u be any other solution of (P_λ^p): integrating by parts to obtain

$$\int_\Omega |\nabla u_\lambda|^2 = \lambda \int_\Omega (1+u_\lambda)^p u_\lambda < \lambda \int_\Omega (1+u)^p u = \int_\Omega |\nabla u|^2$$

where the inequality follows since $u_\lambda \leq u$ and $u_\lambda \not\equiv u$. □

Theorem 1 holds in a weaker form also when $p > \frac{n+2}{n-2}$. Namely, there exists $\lambda^* > 0$ such that (P_λ^p) admits a minimal solution u_λ for all $\lambda < \lambda^*$ and no solutions if $\lambda > \lambda^*$, see [6, 11]. Moreover, the extremal solution U_* always exists in $H_0^1(\Omega)$, see [6, Lemma 5], [9, Remark 3.3]. The extremal solution U_* is unique [24] and in some cases it may not be bounded [9, 26].

We collect all these facts in the following

Theorem 3. *Let $\Omega \in L$, and let $p > 1$. Then u_λ exists for all $\lambda \in (0, \lambda^*)$, the map $\lambda \mapsto u_\lambda(x)$ is continuous in to $C^{2,\alpha}$, is strictly increasing for all $x \in \Omega$ and*

$$\lim_{\lambda \to 0} u_\lambda = 0 \qquad \text{in } C^{2,\alpha}(\overline{\Omega}).$$

Moreover, for $\lambda = \lambda^*$ there exists a unique weak solution $U_* \in H_0^1(\Omega)$ of (P_λ^p), and

$$\lim_{\lambda \to \lambda^*} u_\lambda = U_* \qquad \text{in } H_0^1(\Omega).$$

Finally, if either $n \leq 10$ or $n \geq 11$ and $p < \frac{n-2\sqrt{n-1}}{n-4-2\sqrt{n-1}}$, then $U_* \in L^\infty(\Omega)$ and $u_\lambda \to U_*$ in $C^{2,\alpha}(\overline{\Omega})$, otherwise $U_* \notin L^\infty(\Omega)$.

We now give an overview of other results concerning (P^p_λ) which explain why we will sometimes confine ourselves to the case $p \in (1, \frac{n+2}{n-2}]$. First of all, note that if $p > \frac{n+2}{n-2}$, then the Brezis–Kato result [7] no longer applies and the solutions $u \in H^1_0(\Omega)$ of (P^p_λ) may be unbounded. Indeed, in [9] is exhibited the unbounded function $U(x) = |x|^{-2/(p-1)} - 1$ which solves (P^p_λ) in the unit ball B_1 for $\lambda = \frac{2}{p-1}(n - \frac{2p}{p-1})$ but which belongs to $H^1_0(B_1)$ if $p > \frac{n+2}{n-2}$. A further analytic argument is that critical point methods fail in the supercritical case $p > \frac{n+2}{n-2}$ and, for instance, the proof of Theorem 4 would no longer be correct. This is not just a technical problem since Theorem 1 in [19] and the arguments in [9, Section 6] show that the set of solutions of (P^p_λ) *does not* obey to the statement of Theorem 1 above. We may have either uniqueness of a solution or existence of infinitely many solutions for some $\lambda < \lambda^*$. On the other hand, also in the case $0 < p \le 1$ the set of solutions of (P^p_λ) is different: for all $\lambda \in (0, \lambda^*)$ (P^p_λ) admits a unique solution [21, Corollary 4.1.3] and $(P^p_{\lambda^*})$ admits no solution [21, Corollary 4.1.2]. We also refer to Proposition 1 below for the case $p = 1$.

Remark 2. For all $\Omega \in L$, we have

$$\lambda^*(p) < \frac{(p-1)^{p-1}}{p^p}\lambda_1. \tag{2}$$

Indeed, since for all $s \ge 0$ we have $(1+s)^p \ge \frac{p^p}{(p-1)^{p-1}} s$ with the strict inequality if $s \ne \frac{1}{p-1}$, arguing as in the proof of [6, Lemma 5] (i.e. by testing (P^p_λ) with the first eigenfunction) one gets (2). We also mention that (2) may be obtained as in [20, Theorem 2]. □

Finally, we remark that the method of sub and super–solutions, see [4, Lemma 1.1], yields

$$\forall \Omega_1, \Omega_2 \in L\ , \qquad \Omega_1 \subset \Omega_2 \Longrightarrow \lambda^*(\Omega_1) \ge \lambda^*(\Omega_2),$$

and gives a positive answer (for (P^p_λ)) to a problem raised by Gelfand, see the paragraph following (15.5) p.357 in [14].

3 Behavior of the Nehari manifold for varying λ

In this section we assume $p \le \frac{n+2}{n-2}$ and we use critical point theory to describe how the solutions u_λ and U_λ of (P^p_λ) collapse to the unique extremal solution U_* as $\lambda \uparrow \lambda^*$. In order to do this, we introduce some notations. For all $\lambda \in (0, \lambda^*]$ consider the functionals defined on the space $H^1_0(\Omega)$

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{p+1}\int_\Omega |1+u|^{p+1}$$

and, with the convention that $u_{\lambda^*} = U_*$,

$$I_\lambda(u) = J_\lambda(u + u_\lambda).$$

Set also

$$Z(u) = I'_\lambda(u)[u]$$

and for all $u \in H_0^1(\Omega)$ consider the function $F_u : [0,\infty) \to \mathbb{R}$ defined by

$$F_u(t) = \frac{Z(tu)}{t} = t\int_\Omega |\nabla u|^2 + \int_\Omega \nabla u_\lambda \,\nabla u - \lambda \int_\Omega |1 + u_\lambda + tu|^{p-1}(1 + u_\lambda + tu)u.$$

In particular, since u_λ solves (P_λ^p), we have

$$F_u(0) = \int_\Omega \nabla u_\lambda \,\nabla u - \lambda \int_\Omega (1 + u_\lambda)^p \,u = 0 \qquad \forall u \in H_0^1(\Omega). \tag{3}$$

Note that for every $u \in H_0^1(\Omega)$ it is $F_u \in C^1([0,\infty))$ and

$$F'_u(t) = \int_\Omega |\nabla u|^2 - \lambda\, p \int_\Omega |1 + u_\lambda + tu|^{p-1}u^2 = \int_\Omega |\nabla u|^2 - \lambda\, p\, t^p \int_\Omega \left|\frac{1+u_\lambda}{t} + u\right|^{p-1} u^2. \tag{4}$$

Let $\Sigma = \{u \in H_0^1(\Omega);\ \|u\| = 1\}$. By [9, Lemma 2.1], u_λ is a non–degenerate minimum of J_λ, hence

$$\inf_{u\in\Sigma} F'_u(0) > 0. \tag{5}$$

Since

$$\lim_{t\to+\infty} F'_u(t) = -\infty \qquad \forall u \in \Sigma,$$

the relation (5) implies that

$$\forall u \in \Sigma \qquad \exists\, t > 0 \text{ such that } F'_u(t) = 0. \tag{6}$$

Given $u \in \Sigma$, let t_u denote the smallest positive number t for which property (6) holds true. Define

$$\mathcal{N}_\lambda = \{u_\lambda + t_u\, u \ :\ u \in \Sigma\}.$$

The set $\mathcal{N}_\lambda$ is the counterpart for problem (P_λ^p) of the *Nehari manifold*, usually introduced in the study of nonlinear homogeneous equations. Our aim in this section is to describe some qualitative properties of $\mathcal{N}_\lambda$.

Theorem 4. *Let $\Omega \in L$, let $p \in (1, \frac{n+2}{n-2}]$, and let λ^* be the extremal value for (P_λ^p). Then, as $\lambda \to \lambda^*$, $dist(u_\lambda, \mathcal{N}_\lambda) \to 0$ in $H_0^1(\Omega)$. Furthermore, as $\lambda \to 0$, $dist(u_\lambda, \mathcal{N}_\lambda) \to +\infty$ in $H_0^1(\Omega)$.*

In order to prove Theorem 4 we need the following two lemmas.

Lemma 1. *Let Ω, p and λ^* be as in Theorem 4, and let $\lambda_m \to \lambda^*$ from below. Then $\{U_{\lambda_m}\}_m$ is a bounded Palais–Smale sequence for J_{λ^*}.*

Proof. By continuity, see Theorem 3, there exists $\hat{C} > 0$ such that

$$-\hat{C} \le \sup\{J_\lambda(tu_\lambda) \,:\, t \in [0,1], \lambda \in (0,\lambda^*)\} \le \hat{C}.$$

Moreover there exists $M > 0$ sufficiently large for which $J_\lambda(M\varphi_1) < -C$ for all $\lambda \in (\lambda^*/2, \lambda^*)$, where φ_1 denotes the first eigenfunction of $-\Delta$ in Ω. As a consequence, if $\lambda \in (\lambda^*/2, \lambda^*)$, the piecewise rectilinear curve joining u_λ to 0 and then 0 to $M\varphi_1$ is a mountain pass curve for J_λ. Again by continuity we obtain

$$\sup\{J_\lambda(tu_\lambda) \,:\, t \in [0,1], \lambda \in (\lambda^*/2,\lambda^*)\} + \sup\{J_\lambda(t\varphi_1) \,:\, t \in [0,M], \lambda \in (\lambda^*/2,\lambda^*)\} \le C,$$

where C is a fixed positive constant. Thereby, using the minimax characterization of U_λ, we deduce that

$$J_\lambda(u_\lambda) < J_\lambda(U_\lambda) < C \qquad \forall \lambda \in (\lambda^*/2, \lambda^*).$$

Hence we have

$$C' < \frac{1}{2}\int_\Omega |\nabla U_\lambda|^2 - \frac{\lambda}{p+1}\int_\Omega |1+U_\lambda|^{p+1} < C, \tag{7}$$

for some other constant C'. Moreover, from the condition $J'_\lambda(U_\lambda)[U_\lambda] = 0$ we get

$$\int_\Omega |\nabla U_\lambda|^2 - \lambda\int_\Omega (1+U_\lambda)^{p+1} + \lambda\int_\Omega (1+U_\lambda)^p = 0. \tag{8}$$

We prove first the boundedness of $\|U_{\lambda_m}\|$. Suppose by contradiction that $\|U_{\lambda_m}\| \to +\infty$ as $m \to +\infty$: then from (7) we deduce $\int_\Omega (1+U_{\lambda_m})^{p+1} \to +\infty$ and

$$\|U_{\lambda_m}\|^2 = \frac{2\lambda_m}{p+1}\int_\Omega (1+U_{\lambda_m})^{p+1} + O(1). \tag{9}$$

Equation (8) and Hölder's inequality imply

$$\|U_{\lambda_m}\|^2 = \lambda_m \int_\Omega (1+U_{\lambda_m})^{p+1} + O\left[\left(\int_\Omega (1+U_{\lambda_m})^{p+1}\right)^{\frac{p}{p+1}}\right]. \tag{10}$$

From (9) and (10) we get a contradiction, since $p > 1$. This proves the boundedness of $\|U_{\lambda_m}\|$.

Since U_{λ_m} is a critical point of J_{λ_m}, we have

$$J'_{\lambda^*}(U_{\lambda_m})[v] = \int_\Omega \nabla U_{\lambda_m}\,\nabla v - \lambda^*\int_\Omega (1+U_{\lambda_m})^p v = (\lambda_m - \lambda^*)\int_\Omega (1+U_{\lambda_m})^p v \qquad \forall v \in H_0^1(\Omega).$$

Hence, from the boundedness of $\|U_{\lambda_m}\|$ and from Hölder's and Sobolev's inequalities it follows that

$$\sup_{\|v\|=1} |J'_{\lambda^*}(U_{\lambda_m})[v]| \le C_\Omega(\lambda^* - \lambda_m)\sup_{\|v\|=1}\left(\|1+U_{\lambda_m}\|_{p+1}^p \|v\|_{p+1}\right) \to 0$$

for some $C_\Omega > 0$. This, together with (7), shows that $\{U_{\lambda_m}\}_m$ is a Palais–Smale sequence for J_{λ^*} and concludes the proof of the lemma. □

Lemma 2. *Let Ω, p and λ^* be as in Theorem 4. Then, as $\lambda \uparrow \lambda^*$, $U_\lambda \to U_*$ in $H_0^1(\Omega)$.*

Proof. If $p < \frac{n+2}{n-2}$, the statement follows from Lemma 1, the fact that J_{λ^*} satisfies the Palais–Smale condition and the uniqueness of U_* (as critical point of J_{λ^*}), see [24].

Consider now the case $p = \frac{n+2}{n-2}$. Since u_λ is the minimal positive solution, using the change of variables $w = \lambda^{(n-2)/4}(u - u_\lambda)$, problem (P_λ^p) transforms into

$$\begin{cases} -\Delta w = w^{(n+2)/(n-2)} + f(x,w) & \text{in } \Omega \\ w \geq 0 \quad \text{in } \Omega \\ w = 0 \quad \text{on } \partial\Omega \end{cases} \tag{11}$$

where

$$f(x,w) =$$
$$\lambda^{\frac{n+2}{4}} \left[\left| 1 + u_\lambda(x) + \frac{w}{\lambda^{\frac{n-2}{4}}} \right|^{\frac{4}{n-2}} \left(1 + u_\lambda(x) + \frac{w}{\lambda^{\frac{n-2}{4}}} \right) - (1 + u_\lambda(x))^{\frac{n+2}{n-2}} - \lambda^{-\frac{n+2}{4}} |w|^{\frac{4}{n-2}} w \right].$$

The action functional associated to (11) is given by

$$\overline{I}_\lambda(w) = \frac{1}{2} \int_\Omega |\nabla w|^2 - \frac{1}{2^*} \int_\Omega |w|^{2^*} - \int_\Omega F(x,w) \ , \qquad w \in H_0^1(\Omega),$$

where $F(x,w) = \int_0^w f(x,s)ds$. Following the three cases in [8, p.474], the function f satisfies the hypotheses of Corollary 2.1 ($n \geq 5$), Corollary 2.2 ($n = 4$) and Corollary 2.3 ($n = 3$), for all $\lambda \in (0, \lambda^*]$. The arguments in the proofs of these corollaries imply

$$0 < \overline{I}_\lambda(w_\lambda) < \frac{\mathcal{S}^{n/2}}{n} \qquad \forall \lambda \in (0, \lambda^*),$$

where $w_\lambda = \lambda^{(n-2)/4}(U_\lambda - u_\lambda)$.

Let $\overline{I}_*$ be the extremal functional, namely the functional $\overline{I}_{\lambda^*}$ with U_* instead of u_λ. By Lemma 1, as $\lambda \to \lambda^*$, the sequence $\{w_\lambda\}$ is a bounded Palais–Smale sequence for $\overline{I}_*$. Then there exists $w_* \in H_0^1(\Omega)$ such that $w_\lambda \rightharpoonup w_*$ up to a subsequence and, by the weak continuity of $\overline{I}_*'$, w_* satisfies $\overline{I}_*'(w_*) = 0$. By uniqueness of the extremal solution [24], it follows that $w_* = 0$. We have so far obtained that

$$w_\lambda \rightharpoonup 0 \qquad \text{as } \lambda \to \lambda^*.$$

From the compactness of the functionals $w \mapsto \int_\Omega F(x,w)$ and $w \mapsto \int_\Omega f(x,w)w$, this implies

$$\int_\Omega F(x,w_\lambda) \to 0 \qquad \int_\Omega f(x,w_\lambda)w_\lambda \to 0 \qquad \text{as } \lambda \to \lambda^*. \tag{12}$$

If $\overline{I}_\lambda(w_\lambda) \to 0$, then using (12) and taking into account that $\overline{I}'_\lambda(w_\lambda)[w_\lambda] = 0$ we get

$$\frac{1}{2}\|w_\lambda\|^2 - \frac{1}{2^*}\|w_\lambda\|_{2^*}^{2^*} \to 0 \qquad \text{and} \qquad \|w_\lambda\|^2 - \|w_\lambda\|_{2^*}^{2^*} \to 0 \qquad \text{as } \lambda \to \lambda^*,$$

and hence $w_\lambda \to 0$ in $H_0^1(\Omega)$. This, together with Theorem 3 proves the statement in the case $\overline{I}_\lambda(w_\lambda) \to 0$.

It remains to consider the case where

$$\liminf_{\lambda \to \lambda^*} \overline{I}_\lambda(w_\lambda) > 0. \tag{13}$$

Using the arguments in [8, Lemma 2.1] one obtains

$$\exists \delta > 0 \text{ such that } \overline{I}_\lambda(w_\lambda) < \frac{\mathcal{S}^{n/2}}{n} - \delta \qquad \forall \lambda \in \left(\frac{\lambda^*}{2}, \lambda^*\right)$$

which, together with (13), implies that up to a subsequence

$$\overline{I}_\lambda(w_\lambda) \to c \in \left(0, \frac{\mathcal{S}^{n/2}}{n}\right) \qquad \text{as } \lambda \to \lambda^*.$$

Using again (12) and $\overline{I}'_\lambda(w_\lambda)[w_\lambda] = 0$ we get

$$\begin{cases} \frac{1}{2}\|w_\lambda\|^2 - \frac{1}{2^*}\|w_\lambda\|_{2^*}^{2^*} \to c < \frac{\mathcal{S}^{n/2}}{n} \\ \|w_\lambda\|^2 - \|w_\lambda\|_{2^*}^{2^*} \to 0. \end{cases} \tag{14}$$

The Sobolev inequality $\mathcal{S}\|w_\lambda\|_{2^*}^2 \le \|w_\lambda\|^2$ and (14) yield $\|w_\lambda\| \to 0$ as $\lambda \to \lambda^*$. This concludes the proof. □

Note that Lemma 2 and Theorem 3 entail the following result, proved in [26, Théorème 6] with an implicit function argument:

Corollary 1. *Let $\Omega \in L$, let $p \in (1, \frac{n+2}{n-2}]$, and let λ^* be the extremal value for (P_λ^p). Then, as $\lambda \uparrow \lambda^*$, we have $u_\lambda \to U_*$ and $U_\lambda \to U_*$ in $C^{2,\alpha}(\overline{\Omega})$; in particular, $\|U_\lambda - u_\lambda\|_{C^{2,\alpha}(\overline{\Omega})} \to 0$.*

Proof of Theorem 4. When $\lambda \to \lambda^*$, the statement follows from Lemma 2.

Assume now that $\lambda \to 0$. By (4), for all $u \in \Sigma$ and all $t \ge 0$ we have (here $C_i = C_i(\Omega, p)$ denote positive constants depending only on Ω and p)

$$\begin{aligned} F'_u(t) &= 1 - \lambda p \int_\Omega |1 + u_\lambda + tu|^{p-1} u^2 \\ &\ge 1 - \lambda C_1 \|1 + u_\lambda + tu\|_{2^*}^{p-1} \|u\|_{2^*}^2 && \text{(Hölder's inequality)} \\ &\ge 1 - \lambda C_2 (\|1 + u_\lambda\|_{2^*} + t\|u\|_{2^*})^{p-1} && \text{(Sobolev and Minkowski inequalities)} \\ &\ge 1 - \lambda C_3 (1 + t)^{p-1} && \text{(uniform boundedness of } u_\lambda\text{).} \end{aligned} \tag{15}$$

Let $t'_u > 0$ be the first positive value of t where $F'_u(t) = 0$; by (3) and (5) we obtain $t_u > t'_u$. Therefore from (15) we infer

$$\inf_{u\in\Sigma} t_u \geq \inf_{u\in\Sigma} t'_u \geq \frac{1}{(C_3\,\lambda)^{1/(p-1)}} - 1 \to +\infty \qquad \text{as } \lambda \to 0.$$

This proves that $dist(u_\lambda, \mathcal{N}_\lambda) \to \infty$ as $\lambda \to 0$ and the theorem follows. □

4 The limiting case $\lambda \to 0$

Throughout this section, we will denote by w_λ the unique (positive) solution of the problem

$$\begin{cases} -\Delta w_\lambda = \lambda & \text{in } \Omega \\ w_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

We first state our result about the minimal solution u_λ.

Theorem 5. *Let $\Omega \in L$, $p > 1$ and let $\lambda \in (0, \lambda^*)$. Let $u_{\lambda,p}$ be the minimal solution of (P^p_λ). Then $u_{\lambda,p}(x) > w_\lambda(x)$ for all $x \in \Omega$; moreover, for all $\overline{p} > 1$ we have*

$$\lim_{\lambda\to 0} \frac{u_{\lambda,p}(x)}{w_\lambda(x)} = 1 \qquad \textit{uniformly w.r.t. } (x,p) \in \Omega \times (1, \overline{p}]\,.$$

Proof. Fix $p > 1$ and $\lambda \in (0, \lambda^*)$. Clearly, $u \equiv 0$ is a subsolution of (16) while $u_{\lambda,p} > 0$ is a supersolution. By uniqueness of the solution of (16), this shows that $u_{\lambda,p}(x) > w_\lambda(x)$ for all $x \in \Omega$, the strict inequality being a consequence of the maximum principle.

Let $\overline{u}_\lambda$ be the minimal solution of $(P^{\overline{p}}_\lambda)$. By Theorem 3 we know that $\|\overline{u}_\lambda\|_\infty \to 0$ as $\lambda \to 0$. Therefore,

$$\forall \varepsilon > 0 \quad \exists \lambda_\varepsilon > 0 \quad \text{s.t.} \qquad \lambda < \lambda_\varepsilon \Longrightarrow \|\overline{u}_\lambda\|_\infty < \varepsilon.$$

So, fix $\varepsilon > 0$ and let $\lambda < \lambda_\varepsilon$. Then,

$$-\Delta \overline{u}_\lambda = \lambda(1 + \overline{u}_\lambda)^{\overline{p}} < \lambda(1 + \varepsilon)^{\overline{p}} = -(1 + \varepsilon)^{\overline{p}} \Delta w_\lambda.$$

This proves that $\overline{u}_\lambda(x) < (1+\varepsilon)^{\overline{p}} w_\lambda(x)$ for all $x \in \Omega$. Therefore, by Theorem 8 below (which proof is self–contained), we deduce

$$u_{\lambda,p}(x) \leq \overline{u}_\lambda(x) < (1 + \varepsilon)^{\overline{p}} w_\lambda(x) \qquad \forall (x, p) \in \Omega \times (1, \overline{p}]$$

and the result follows by arbitrariness of ε. □

Differently from the minimal solution u_λ, when $1 < p \leq \frac{n+2}{n-2}$ and $\lambda \to 0$, the behavior of the mountain–pass solutions U_λ depends strongly on the exponent p, see Theorem 6 below.

Moreover, the situation is also qualitatively different in the subcritical and critical cases. This is related to the fact that the pure power problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{17}$$

admits no mountain–pass solutions when $p = \frac{n+2}{n-2}$.

Theorem 6. *Let $\Omega \in L$, let $p \in (1, \frac{n+2}{n-2}]$ and let $\lambda \in (0, \lambda^*)$. Let U_λ be a mountain–pass solution of problem (P^p_λ).*

If $1 < p < \frac{n+2}{n-2}$ then, up to a subsequence,

$$\lim_{\lambda \to 0} \lambda^{1/(p-1)} U_\lambda \;=\; U_p \qquad \text{in } C^{2,\alpha}(\overline{\Omega}), \tag{18}$$

for some mountain–pass solution U_p of problem (17). *In particular $U_\lambda(x) \to +\infty$ for all $x \in \Omega$.*

If $p = \frac{n+2}{n-2}$ then there exists $\overline{x} \in \Omega$ such that, up to a subsequence

$$U_\lambda(x) \to \frac{1}{H(\overline{x}, \overline{x})} G(x, \overline{x}), \qquad \text{in } C^{2,\alpha}_{loc}(\overline{\Omega} \setminus \{\overline{x}\}), \tag{19}$$

where $G(\cdot,\cdot)$ denotes the Green's function in Ω and $H(\cdot,\cdot)$ its regular part. Moreover, up to a subsequence, we have

$$\lambda^{(n-2)/2} |\nabla U_\lambda|^2 \to \mathcal{S}^{n/2} \delta_{\overline{x}}; \qquad \lambda^{n/2} U_\lambda^{2^*} \to \mathcal{S}^{n/2} \delta_{\overline{x}} \tag{20}$$

in the weak sense of measures, where $\mathcal{S}$ is as in (1). *The point $\overline{x}$ is critical for the function $\varphi(x) = H(x, x)$.*

Proof. For $\lambda \in (0, \lambda^*)$, let $\varepsilon = \lambda^{1/(p-1)}$ and $V_\varepsilon = \varepsilon U_\lambda$. Then V_ε satisfies

$$\begin{cases} -\Delta V_\varepsilon = (V_\varepsilon + \varepsilon)^p & \text{in } \Omega \\ V_\varepsilon > 0 & \text{in } \Omega \\ V_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

We note that V_ε is a mountain–pass critical point of the functional $J_\varepsilon : H^1_0(\Omega) \to \mathbb{R}$

$$J_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega |u + \varepsilon|^{p+1}, \qquad u \in H^1_0(\Omega).$$

Consider first the case $1 < p < \frac{n+2}{n-2}$. Arguing as in the proof of Theorem 2 in [13], one can show that if $\varepsilon_m \to 0$ then $\{V_{\varepsilon_m}\}$ is a Palais–Smale sequence at mountain–pass level for the limit functional

$$J_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}, \qquad u \in H_0^1(\Omega).$$

Since J_0 satisfies the Palais–Smale condition (recall $p < \frac{n+2}{n-2}$), the sequence $\{V_{\varepsilon_m}\}$ converges, up to a subsequence, to a critical point of J_0, which is necessarily of mountain–pass type. This proves that $\lambda^{1/(p-1)}U_\lambda \to U_p$ in $H_0^1(\Omega)$, up to a subsequence. In view of [7], one also finds that $V_\varepsilon \to U_p$ uniformly in Ω. Then, by standard elliptic regularity we deduce that the convergence is in the $C^{2,\alpha}(\overline{\Omega})$ topology. This also proves the pointwise blow–up.

Let now $p = \frac{n+2}{n-2}$, and consider again the functional J_ε. Let M_ε (resp. M_0) be the mountain–pass level of J_ε (resp. J_0). The same arguments used in the proof of Lemma 8 in [13] show that

$$\lim_{\varepsilon\to 0} M_\varepsilon = M_0. \tag{22}$$

Let $\mathcal{S}$ be as in (1). It is well–known that $M_0 = \frac{1}{n}\mathcal{S}^{n/2}$, and that there is no solution of (17) at this level of J_0, see e.g. [31], Chapter III, Theorem 1.2. Hence the sequence $\{V_{\varepsilon_m}\}$ cannot be uniformly bounded in $L^\infty(\Omega)$. Indeed, if it were bounded, it would converge to a positive solution V_0 of (17) (with $p = \frac{n+2}{n-2}$) such that $J_0(V_0) = \frac{1}{n}\mathcal{S}^{n/2}$, which contradicts the just mentioned non–existence result for (17). Therefore the sequence $\{V_{\varepsilon_m}\}$ blows up in Ω. The convergence in (19) follows from Lemma 8 and Proposition 3 in the Appendix, while (20) follows from Proposition 2 and the subsequent discussion. The last statement of the theorem is a consequence of Proposition 4. □

In the critical case $p = \frac{n+2}{n-2}$, Theorem 6 is the counterpart of [29, Theorem 1] where the existence of solutions concentrating at *non–degenerate* critical points of φ is obtained.

Remark 3. Theorem 6 can be somehow extended to any class of non–minimal solutions, not necessarily of mountain–pass type. We quote without proof the corresponding statements.

If $p < \frac{n+2}{n-2}$, then (18) and the pointwise blow–up are still true, but U_p has to be replaced with a generic solution of (17). Note that if Ω is convex and has some symmetries and if p is sufficiently close to $\frac{n+2}{n-2}$, then (17) admits a unique solution, necessarily of mountain–pass type, see [16]. On the other hand, it is easy to construct examples of symmetric domains Ω for which problem (17) admits non–symmetric (and hence multiple) mountain pass solutions. As a consequence, by Theorem 6, for such domains also the mountain–pass solution U_λ is not unique if λ is sufficiently small.

If $p = \frac{n+2}{n-2}$, then there is convergence to a solution of (17) or concentration as in (20), but at possibly k points $\overline{x}_1, \dots, \overline{x}_k$ in Ω. The number k of blow up points cannot exceed a constant k_Ω depending on the domain Ω. In particular, if $\Omega = B_1$, $k_\Omega = 1$ and we are in the situation of Theorem 6, see also Theorem 7 below. Moreover, there exists $d_\Omega > 0$ such that $d(\overline{x}_i, \overline{x}_j) \geq d_\Omega$ for all $i \neq j$ and $d(\overline{x}_i, \partial\Omega) \geq d_\Omega$ for all i. The condition $\nabla\varphi(\overline{x}) = 0$ has to be substituted by the following. Given $(x_1, \ldots, x_k) \in \Omega^k$, with $x_j \neq x_l$ for $j \neq l$, define the symmetric matrix (M_{jl}) of order $k \times k$ by

$$M_{jl}(x_1, \ldots, x_k) = \begin{cases} H(x_j, x_j), & l = j; \\ -G(x_j, x_l) & l \neq j, \end{cases}$$

see [2]. Here $G(\cdot, \cdot)$ denotes the Green's function of Ω and $H(\cdot, \cdot)$, as before, the regular part of G. Denote by $\rho = \rho(x_1, \ldots, x_k)$ the least eigenvalue of (M_{jl}). Then the points $\overline{x}_1, \ldots, \overline{x}_k$ satisfy the properties

$$\rho(\overline{x}_1, \ldots, \overline{x}_k) \geq 0 \qquad \nabla\rho(\overline{x}_1, \ldots, \overline{x}_k) = 0.$$

Viceversa, using the arguments in [29] and [3], one could prove that if $n \geq 4$ and if $(\overline{x}_1, \ldots, \overline{x}_k)$ is a non degenerate critical point of ρ with $\rho(\overline{x}_1, \ldots, \overline{x}_k) > 0$, then for ε sufficiently small, there exists a family V_ε of solutions of (21) which blow up precisely at $(\overline{x}_1, \ldots, \overline{x}_k)$ as ε tends to zero. □

5 Results for varying p and varying domains

5.1 The case $p = \frac{n+2}{n-2}$ and $\Omega = B_1$

In this subsection we consider the particular problem

$$\begin{cases} -\Delta u = \lambda(1+u)^{(n+2)/(n-2)} & \text{in } B_1 \\ u > 0 \quad \text{in } B_1 \\ u = 0 \quad \text{on } \partial B_1. \end{cases} \tag{23}$$

By Theorem 2 we know that (23) admits exactly two solutions if $\lambda < \lambda^*$. Moreover, these solutions are radially symmetric and decreasing, see [15]. A detailed study of (23) was performed in [19, Section VI] where the extremal value λ^* was determined and the explicit solutions u_λ, U_λ and U_* were given, see (VI.3)–(VI.4) in that paper. All these results were found after several changes of variables which transformed (23) into equivalent problems. Here, we prove the same results by a more direct procedure which, in our opinion, is much simpler.

We first recall that all positive entire solutions of the equation

$$-\Delta w = w^{(n+2)/(n-2)} \qquad \text{in } \mathbb{R}^n \tag{24}$$

are radially symmetric about one point. When this point is the origin they are necessarily of the form

$$w_d(x) = \frac{[n(n-2)d]^{(n-2)/4}}{[1+d|x|^2]^{(n-2)/2}} \qquad (d > 0) \tag{25}$$

and these functions achieve the best constant $\mathcal{S}$ (defined in (1)) in Sobolev inequality, see [32].

For all $\lambda \leq n(n-2)/4$ let

$$d_\pm(\lambda) = \frac{n(n-2) - 2\lambda \pm \sqrt{n^2(n-2)^2 - 4\lambda n(n-2)}}{2\lambda}.$$

Note that $d_+(\frac{n(n-2)}{4}) = d_-(\frac{n(n-2)}{4}) = 1$. Consider also the restrictions to the unit ball B_1 of some of the functions of the family (25):

$$v_\lambda = w_{d_-(\lambda)}|_{B_1} \qquad V_\lambda = w_{d_+(\lambda)}|_{B_1} \qquad V_* = w_1|_{B_1}.$$

We may now state

Theorem 7. *There holds* $\lambda^*(B_1, \frac{n+2}{n-2}) = \frac{n(n-2)}{4}$. *Moreover, the solutions of* (23) *are*

$$U_*(x) = \left(\frac{4}{n(n-2)}\right)^{(n-2)/4} V_*(x) - 1 = \left(\frac{2}{1+|x|^2}\right)^{(n-2)/2} - 1 \qquad \text{if } \lambda = \lambda^*$$

and

$$u_\lambda = \lambda^{(2-n)/4} v_\lambda - 1 \qquad U_\lambda = \lambda^{(2-n)/4} V_\lambda - 1 \qquad \text{if } 0 < \lambda < \lambda^*.$$

Proof. By direct calculations, one can verify that u_λ and U_λ indeed solve (23) if $\lambda < \frac{n(n-2)}{4}$ and that U_* solves (23) if $\lambda = \frac{n(n-2)}{4}$. Hence, $\lambda^* \geq \frac{n(n-2)}{4}$ by Theorem 1.

Conversely, assume that (23) admits a solution u. This solution is radially symmetric in view of [15]. Then, the function $v = \lambda^{(n-2)/4}(1+u)$ is a (radial) solution of the equation

$$\begin{cases} -\Delta v = v^{(n+2)/(n-2)} & \text{in } B_1 \\ v > \lambda^{(n-2)/4} & \text{in } B_1 \\ v = \lambda^{(n-2)/4} & \text{on } \partial B_1. \end{cases} \tag{26}$$

Therefore, $v = v(r)$ solves the ordinary differential equation

$$v''(r) + \frac{n-1}{r} v'(r) + v^{(n+2)/(n-2)}(r) = 0 \tag{27}$$

with the conditions $v'(0) = 0$ and $v(1) = \lambda^{(n-2)/4}$. Moreover, $v'(1) = C < 0$ by the Hopf boundary lemma. Hence v may be extended as a smooth function to some maximal interval $[1, R)$ (with $1 < R \leq \infty$) such that $v(r) > 0$ and $v'(r) < 0$ for all $r \in [1, R)$. In fact, $R = \infty$; otherwise, we would have either $v(R) = 0$ (violating Pohozaev's non–existence result [28] for the equation (24) in the ball B_R) or $v'(R) = 0$ (contradicting the Hopf boundary lemma in the ball B_R). Therefore, the (smooth) extension $\overline{v}$ of v satisfies (27) on $[0, \infty)$ and $\overline{v}'(0) = 0$.

This shows that $\overline{v}$ (as a function of $x \in \mathbb{R}^n$) is a positive entire solution of (24) and hence, it is one of the functions of the family (25) for some $d > 0$.

We have proved that if (23) admits a solution u, then there exists $d > 0$ such that the corresponding function w_d in (25) satisfies $w_d(x) = \lambda^{(n-2)/4}$ whenever $|x| = 1$. This condition is satisfied if and only if

$$d = \frac{n(n-2) - 2\lambda \pm \sqrt{n^2(n-2)^2 - 4\lambda n(n-2)}}{2\lambda}$$

from which we infer that d exists only if $\lambda \leq \frac{n(n-2)}{4}$. This shows that $\lambda^* \leq \frac{n(n-2)}{4}$ and completes the proof. □

Remark 4. Using the explicit form of $U_\lambda(x)$, it is not difficult to verify that the map $\lambda \mapsto U_\lambda(x)$ is strictly decreasing on $(0, \frac{n(n-2)}{4})$ for all $|x| < 1$. In particular, this shows that the map $\lambda \mapsto \|U_\lambda\|_\infty = U_\lambda(0)$ is strictly decreasing.

Finally, we note that in this particular case ($p = \frac{n+2}{n-2}$ and $\Omega = B_1$), one recovers the statements of Corollary 1 and Theorems 5, 6 using explicit computations.

5.2 Some properties of the map $\lambda^* = \lambda^*(\Omega, p)$

We first show some monotonicity features of the maps $p \mapsto \lambda^*(p)$ and $p \mapsto u_{\lambda,p}$. Concerning the behavior of $\lambda^*(p)$ at infinity, we also refer to (43) below.

Theorem 8. *Let $\Omega \in L$, let $1 < p_1 < p_2$, and consider the two problems $(P_\lambda^{p_1})$ and $(P_\lambda^{p_2})$. Then*

$$\lambda^*(p_1) > \lambda^*(p_2) \qquad \text{and} \qquad u_{\lambda,p_1}(x) < u_{\lambda,p_2}(x) \quad \forall x \in \Omega \quad \forall \lambda < \lambda^*(p_2).$$

Moreover, $\lim_{p\to\infty} \lambda^*(p) = 0$.

Proof. Let $\lambda < \lambda^*(p_2)$ and let u_{λ,p_2} be the minimal solution of the problem $(P_\lambda^{p_2})$, namely

$$\begin{cases} -\Delta u_{\lambda,p_2} = \lambda(1 + u_{\lambda,p_2})^{p_2} & \text{in } \Omega \\ u_{\lambda,p_2} > 0 & \text{in } \Omega \\ u_{\lambda,p_2} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $-\Delta u_{\lambda,p_2} > \lambda(1 + u_{\lambda,p_2})^{p_1}$, i.e. u_{λ,p_2} is a supersolution of the Dirichlet problem $(P_\lambda^{p_1})$. Since $u \equiv 0$ is a subsolution, there exists a regular (minimal) solution u_{λ,p_1} of problem $(P_\lambda^{p_1})$ satisfying $u_{\lambda,p_1}(x) \leq u_{\lambda,p_2}(x)$ for all $x \in \Omega$. The strict inequality follows from the maximum principle.

The same argument applied to the extremal value $\lambda^*(p_2)$ and to the corresponding extremal solution shows that $\lambda^*(p_1) \geq \lambda^*(p_2)$, and therefore the inequality $u_{\lambda,p_1} < u_{\lambda,p_2}$ holds for all $\lambda < \lambda^*(p_2)$.

In order to prove the strict monotonicity of the map $p \mapsto \lambda^*(p)$, assume by contradiction that

$$\lambda^*(p_1) = \lambda^*(p_2) = \lambda^*. \tag{28}$$

If (28) holds we clearly have (with obvious notations)

$$-\Delta U_{*,p_2} = \lambda^*(1 + U_{*,p_2})^{p_2} > \lambda^*(1 + U_{*,p_2})^{p_1}$$

and so U_{*,p_2} is a (possibly weak) supersolution of $(P^{p_1}_{\lambda^*})$. By [6, Lemma 3] and by uniqueness of the solution of $(P^{p_1}_{\lambda^*})$, see [24], this shows that

$$U_{*,p_2}(x) \geq U_{*,p_1}(x) > 0 \qquad \text{for a.e. } x \in \Omega. \tag{29}$$

By [9, Lemma 2.3] and [9, Theorem 3.1] we have

$$\int_\Omega |\nabla u|^2 - \lambda^* p_2 \int_\Omega (1 + U_{*,p_2})^{p_2 - 1} u^2 \geq 0 \qquad \forall u \in H^1_0(\Omega). \tag{30}$$

Let $\lambda = \frac{p_1}{p_2}\lambda^* < \lambda^*$, let $\tilde{u}_\lambda$ be the minimal solution of $-\Delta \tilde{u}_\lambda = \lambda(1 + \tilde{u}_\lambda)^{p_2}$ and let $v = \frac{p_2}{p_1}\tilde{u}_\lambda$. Then v satisfies

$$-\Delta v = \frac{p_2}{p_1}(-\Delta \tilde{u}_\lambda) = \frac{p_2}{p_1}\,\lambda\, \left(1 + \frac{p_1}{p_2}\, v\right)^{p_2} = \lambda^* \left(1 + \frac{p_1}{p_2}\, v\right)^{p_2}. \tag{31}$$

We now recall the elementary inequality

$$\left(1 + \frac{p_1}{p_2}\, s\right)^{p_2} \geq (1 + s)^{p_1} \qquad \forall s \geq 0$$

which, inserted into (31), gives

$$-\Delta v \geq \lambda^*(1 + v)^{p_1}.$$

Hence v is a bounded supersolution of $(P^{p_1}_{\lambda^*})$, and U_{*,p_1} is regular since $U_{*,p_1} \leq v$. Using (29) and taking into account that $p_1 < p_2$, we have

$$p_2(1 + U_{*,p_2}(x))^{p_2 - 1} > p_1(1 + U_{*,p_1}(x))^{p_1 - 1} \qquad \forall x \in \Omega.$$

Combining the last inequality with (30) we obtain

$$\int_\Omega |\nabla u|^2 - \lambda^*\, p_1 \int_\Omega (1 + U_{*,p_1})^{p_1 - 1} u^2 > 0, \qquad \forall u \in H^1_0(\Omega) \setminus \{0\},$$

which contradicts [9, Lemma 2.3], since U_{*,p_1} is regular. The contradiction is achieved and hence (28) is false.

Finally, letting $p \to \infty$ in (2) we obtain $\lim_{p\to\infty} \lambda^*(p) = 0$. □

We now study the continuity of the map $\lambda^* = \lambda^*(\Omega, p)$. We prove

Theorem 9. *Let $\Omega \in L$; then the map $p \mapsto \lambda^*(p)$ is continuous on $(1, \infty)$.*

Proof. By Theorem 8 we know that for all $\overline{p} > 1$ the right and left limits of $\lambda^*(p)$ as $p \to \overline{p}$ exist and

$$\lim_{p \to \overline{p}^+} \lambda^*(p) \leq \lambda^*(\overline{p}) \leq \lim_{p \to \overline{p}^-} \lambda^*(p).$$

We first prove the continuity from the right. Suppose by contradiction that there exists $\overline{p} > 1$ such that

$$\lambda^*(\overline{p}) > \lim_{p \to \overline{p}^+} \lambda^*(p) =: \overline{\lambda}.$$

Let $\lambda \in (\overline{\lambda}, \lambda^*(\overline{p}))$ and let u_λ be the minimal solution of $-\Delta u_\lambda = \lambda(1 + u_\lambda)^{\overline{p}}$. Since $u_\lambda \in L^\infty(\Omega)$, we can set $M := \|u_\lambda\|_\infty$. Set also $q = \overline{p} + \frac{\log \lambda - \log \overline{\lambda}}{\log(1+M)} > \overline{p}$, then by Theorem 8

$$\lambda^*(q) < \overline{\lambda}. \tag{32}$$

From our choice of q, we infer that $\lambda(1 + M)^{\overline{p}} = \overline{\lambda}(1 + M)^q$ and with some elementary computations, one can check that

$$\lambda\,(1 + s)^{\overline{p}} \geq \overline{\lambda}\,(1 + s)^q \qquad \forall s \leq M.$$

Since $\|u_\lambda\|_\infty = M$ we obtain

$$-\Delta u_\lambda = \lambda\,(1 + u_\lambda)^{\overline{p}} \geq \overline{\lambda}\,(1 + u_\lambda)^q,$$

and hence u_λ is a supersolution of $(P^q_{\overline{\lambda}})$. Then, by [6, Lemma 3] $(P^q_{\overline{\lambda}})$ admits a solution. By Theorem 3, this implies $\lambda^*(q) \geq \overline{\lambda}$ and contradicts (32).

We now prove the continuity from the left. Assume by contradiction that there exists $\overline{p} > 1$ such that

$$\overline{\lambda} := \lim_{p \to \overline{p}^-} \lambda^*(p) > \lambda^*(\overline{p}).$$

Choose $\lambda \in (\lambda^*(\overline{p}), \overline{\lambda})$, and for $p < \overline{p}$ let u_p denote the minimal solution of $-\Delta u_p = \lambda\,(1+u_p)^p$. Testing this equation with u_p we get

$$\int_\Omega |\nabla u_p|^2 = \lambda \int_\Omega (1 + u_p)^p u_p^p. \tag{33}$$

From [9, Lemma 2.1] we also have

$$\int_\Omega |\nabla u_p|^2 - \lambda\, p \int_\Omega (1 + u_p)^{p-1} u_p^2 > 0. \tag{34}$$

From (33) and (34) we obtain

$$\int_\Omega (1 + u_p)^{p-1} u_p (1 + u_p - p\, u_p) > 0. \tag{35}$$

This implies that there exists $k > 0$ such that

$$\|u_p\|_{p+1} \leq k \qquad \forall p < \overline{p}. \tag{36}$$

Indeed, let $\Omega_p = \{x \in \Omega \, : \, u_p(x) \leq 2/(p-1)\}$ and $\Omega^p = \Omega \setminus \Omega_p$. Then, clearly,

$$\left| \int_{\Omega_p} (1+u_p)^{p-1} u_p \, (1 + u_p - p u_p) \right| \leq c_1, \tag{37}$$

for some fixed $c_1 > 0$. Furthermore, in Ω^p we have $1 + u_p - p u_p \leq -\frac{p-1}{2} u_p$, and hence

$$\int_{\Omega^p} (1+u_p)^{p-1} u_p \, (1 + u_p - p u_p) \leq -\frac{p-1}{2} \int_{\Omega^p} |u_p|^{p+1}. \tag{38}$$

If (36) were false, namely $\|u_p\|_{p+1} \to +\infty$, then (35), (37) and (38) give a contradiction by letting $p \to \overline{p}$. Hence (36) holds true and $\|u_p\|$ remains bounded by (33). From (36) it follows that also $\|u_p\|_{\overline{p}}$ remains bounded. Hence, as $p \to \overline{p}^-$, u_p converges weakly in $H^1_0(\Omega)$ and in $L^{\overline{p}}(\Omega)$ to a solution of $(P^{\overline{p}}_\lambda)$, which contradicts $\lambda > \lambda^*(\overline{p})$. □

Now let us fix $p > 1$ and let Ω vary. We recall the following continuity result

Theorem 10. [25]
The map $\lambda^* : L \mapsto (0, \infty)$ is continuous with respect to the Hausdorff distance of domains.

We wish to optimize $\lambda^* = \lambda^*(\Omega)$. By a simple rescaling, one can check that the map $\lambda^* : L \to (0, \infty)$ is homogeneous of degree -2, namely $k^2 \lambda^*(k\Omega) = \lambda^*(\Omega)$ for all $\Omega \in L$ and $k > 0$. Then, $\inf_L \lambda^* = 0$ and $\sup_L \lambda^* = +\infty$, and by Theorem 1 we know that the infimum is not attained. In order to avoid this rescaling problem, we restrict our attention to the sets Ω having the same measure ω_n as the unit ball B_1. Therefore, we introduce the family

$$\mathbb{L} = \{\Omega \in L; \, |\Omega| = \omega_n\}.$$

We first remark that for all $p > 1$ we still have

$$\sup_{\Omega \in \mathbb{L}} \lambda^*(\Omega) = +\infty. \tag{39}$$

To see this, for all $\varepsilon > 0$ consider the function

$$\phi_\varepsilon(x) = \phi_\varepsilon(x_1, ..., x_n) = x_1^2 + ... + x_{n-2}^2 + \varepsilon x_{n-1}^2 + \frac{x_n^2}{\varepsilon}.$$

Then, the ellipsoid $\Omega_\varepsilon = \{x \in \mathbb{R}^n; \, \phi_\varepsilon(x) < 1\}$ belongs to $\mathbb{L}$ and the function $v_\varepsilon(x) = 1 - \phi_\varepsilon(x)$ satisfies

$$-\Delta v_\varepsilon = 2(n + \varepsilon - 2) + \frac{2}{\varepsilon} > \frac{2}{\varepsilon} \geq \lambda 2^p \geq \lambda (1 + v_\varepsilon)^p \quad \text{in } \Omega_\varepsilon \qquad \forall \lambda \leq \frac{2^{1-p}}{\varepsilon}.$$

Hence v_ε is a supersolution of (P^p_λ) in Ω_ε for $\lambda = 2^{1-p}/\varepsilon$, and so $\lambda^*(\Omega_\varepsilon) \geq 2^{1-p}/\varepsilon$. Then (39) follows by letting $\varepsilon \to 0$.

On the contrary, $\inf_{\mathbb{L}} \lambda^*$ is attained as states the following result

Theorem 11. [5, Theorem 4.10]
Let $p > 1$. Then the functional $\lambda^* : \mathbb{L} \to (0, \infty)$ attains its minimum at B_1, $\inf_{\mathbb{L}} \lambda^* = \lambda^*(B_1)$.

Remark 5. This result of optimal design may also be stated in a different fashion. In [25] the functional λ^* is studied for a slightly different problem. It is shown there that the map $\Omega \mapsto \lambda^*(\Omega)$ is differentiable in a suitable sense. Therefore, according to Theorem 11 we can say that the derivative of $\lambda^*(\Omega)$ vanishes when $\Omega = B_1$ and whenever the variations of Ω preserve the total volume. □

Combining the previous results with an argument in [19] we obtain the following lower bounds for λ^*

Theorem 12. *For all $\Omega \in L$ and all $p > 1$ we have*

$$\lambda^*(\Omega, p) \geq 2\frac{\omega_n^2}{|\Omega|^2} \max\left\{n\frac{(p-1)^{p-1}}{p^p}, \frac{1}{p-1}\left(n - \frac{2p}{p-1}\right)\right\}. \tag{40}$$

Moreover,

$$\lambda^*(\Omega) \geq \frac{\omega_n^2}{|\Omega|^2}\frac{n(n-2)}{4} \qquad \forall p \in \left(1, \frac{n+2}{n-2}\right]. \tag{41}$$

Proof. In order to prove (40), by Theorem 11 and by rescaling it suffices to show that

$$\lambda^*(B_1, p) \geq 2 \max\left\{n\frac{(p-1)^{p-1}}{p^p}, \frac{1}{p-1}\left(n - \frac{2p}{p-1}\right)\right\} \qquad \forall p > 1. \tag{42}$$

By [19, Theorem 1] (see also [9, Section 6]), we know that for all $p > 1$ we have

$$\lambda^*(B_1, p) \geq \frac{2}{p-1}\left(n - \frac{2p}{p-1}\right).$$

On the other hand, the function $w(x) = \frac{1}{p-1}(1 - |x|^2)$ satisfies

$$-\Delta w = \frac{2n}{p-1} = 2n\frac{(p-1)^{p-1}}{p^p}\left(1 + \frac{1}{p-1}\right)^p \geq 2n\frac{(p-1)^{p-1}}{p^p}(1+w)^p,$$

so w is a supersolution for (P_λ^p) in B_1 for all $\lambda \leq 2n\frac{(p-1)^{p-1}}{p^p}$. Since $w_0 \equiv 0$ is a subsolution and $w_0 \leq w$, for any such λ there exists a solution of (P_λ^p). By Theorem 1, this shows that

$$\lambda^*(B_1, p) \geq 2n\frac{(p-1)^{p-1}}{p^p}$$

and (42) follows. For a different proof of the last inequality, see also [4, Theorem 1.1].

By Theorem 8 we have $\lambda^*(B_1, p) \geq \lambda^*(B_1, \frac{n+2}{n-2})$ for all $p \in (1, \frac{n+2}{n-2}]$. Therefore, the uniform lower bound (41) follows from Theorem 7. □

Note that the maximum in the r.h.s. of (40) coincides with its first term if p is close to 1. In particular, this happens for $p \leq \frac{n}{n-2}$ since its second term is nonpositive. Note also that by (2) and (40)

$$\forall \Omega \in L \quad \exists C_2(\Omega) > C_1(\Omega) > 0 \qquad \text{such that} \qquad C_1(\Omega) < p\lambda^*(p) < C_2(\Omega) \quad \forall p > 1. \tag{43}$$

We conclude this section with some bibliographical references on the study of the behavior of λ^* for other varying parameters. Lower bounds for λ^* for semilinear problems slightly different from (P_λ^p) were found by variational methods in [33]. The dependence of λ^* on boundary conditions was studied in [18]. Finally, further results for varying domains may be found in [4].

5.3 Behavior of solutions as $p \to 1$

In this section we study the case where $p \to 1$. Consider first the case $p = 1$. For sake of completeness we quote the proof of the following result:

Proposition 1. *Let $\Omega \in L$ and $\lambda > 0$. Then the linear equation (P_λ^1) admits a solution if and only if $\lambda < \lambda_1$. In such a case the solution is unique.*

Proof. Assume (P_λ^1) admits a solution u, multiply the equation by the first (positive) eigenfunction φ_1 and integrate by parts. We obtain

$$\lambda_1 \int_\Omega u\varphi_1 = \lambda \int_\Omega u\varphi_1 + \lambda \int_\Omega \varphi_1$$

which proves $\lambda < \lambda_1$.

Conversely, assume $\lambda < \lambda_1$. Then the functional

$$J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\lambda}{2} \int_\Omega u^2 - \lambda \int_\Omega u$$

is convex and coercive on $H_0^1(\Omega)$ (by Poincaré's inequality) and therefore it admits a minimum u which solves (P_λ^1). Since $J_\lambda(u) \geq J_\lambda(|u|)$ for all u, we may assume $u \geq 0$. Finally, the strict positivity $u > 0$ follows from the maximum principle.

In order to prove uniqueness, assume that u and v both solve (P_λ^1), for some $\lambda < \lambda_1$. Subtracting the equations we deduce that $w = u - v \in H_0^1(\Omega)$ satisfies $-\Delta w = \lambda w$. Since $\lambda < \lambda_1$, this shows that $w \equiv 0$ and completes the proof. □

Next, note that by (2) and by Theorem 8 the map $\lambda^* : p \mapsto \lambda^*(p)$ admits a limit as $p \to 1$ and

$$\lim_{p \to 1} \lambda^*(p) \leq \lambda_1. \tag{44}$$

The next result shows that in fact equality holds

Theorem 13. *Let $\Omega \in L$, then*

$$\lim_{p\to 1} \lambda^*(p) = \lambda_1.$$

Moreover, for all $0 < \lambda < \lambda_1$ there exists $\varepsilon_\lambda > 0$ such that $(P_\lambda^{1+\varepsilon})$ admits a minimal solution u_ε and a mountain–pass solution U_ε for all $\varepsilon < \varepsilon_\lambda$.

Proof. Assume that $0 < \lambda < \lambda_1$ and denote by $\overline{u} \in C_0^{2,\alpha}(\overline{\Omega})$ the unique positive solution of (P_λ^1), see Proposition 1. Consider the map

$$\begin{aligned} \Phi \,:\, C_0^{2,\alpha}(\overline{\Omega}) \times \mathbb{R} &\rightarrow C^{0,\alpha}(\overline{\Omega}) \\ (u,p) &\mapsto \Delta u + \lambda |1+u|^{p-1}(1+u). \end{aligned}$$

It is not difficult to verify that Φ is of class C^1 in a suitable neighborhood of $(u,1)$ for any *positive* $u \in C_0^{2,\alpha}(\overline{\Omega})$. In particular, $\Phi(\overline{u},1) = 0$ and there exists a neighborhood $\mathcal{U}$ of $(\overline{u},1)$ where $\Phi \in C^1(\mathcal{U})$. Moreover, the partial derivative of Φ with respect to u evaluated at $(\overline{u},1)$ is the linear operator $\ell : C_0^{2,\alpha}(\overline{\Omega}) \rightarrow C^{0,\alpha}(\overline{\Omega})$ such that $\ell(v) = \Delta v + \lambda v$. Since $\lambda < \lambda_1$, ℓ is an isomorphism. Therefore, by the implicit function Theorem, there exists a neighborhood $\mathcal{U}'$ of $p=1$ such that the equation $\Phi(u,p) = 0$ defines implicitly a family of functions $u_p = u_p(p) \in C_0^{2,\alpha}(\overline{\Omega})$ such that $\Phi(u_p,p) = \Delta u_p + \lambda|1+u_p|^{p-1}(1+u_p) = 0$ for all $p \in \mathcal{U}'$. Since $u_1 = \overline{u} > 0$ and since the map $p \mapsto u_p$ is continuous in the $C_0^{2,\alpha}(\overline{\Omega})$ topology, by restricting $\mathcal{U}'$ if necessary, we may assume that $u_p(x) \geq -1$ for all $x \in \Omega$. Therefore, u_p is a super–harmonic function and by the maximum principle $u_p(x) > 0$ for all $x \in \Omega$. We have shown that for p sufficiently close to 1 there exists a solution u_p of (P_λ^p). This proves that for such a p we have $\lambda \leq \lambda^*(p)$. By the strict monotonicity of the map $\lambda^* = \lambda^*(p)$ (see Theorem 8), and by taking a smaller p if necessary, we have $\lambda < \lambda^*(p)$. The existence of u_ε and U_ε follows from Theorem 1.

Finally, the previous argument also shows that for all $\lambda < \lambda_1$ there exists $p > 1$ such that $\lambda^*(p) \geq \lambda$. Together with Theorem 8 and (44), this proves that $\lambda^*(p) \to \lambda_1$ as $p \to 1$. □

Remark 6. Proposition 1 states that $\lambda^*(1) = \lambda_1$. Then, Theorems 9 and 13 imply that the map $\lambda^* = \lambda^*(p)$ is continuous in the *closed* interval $[1,\infty)$. □

As a by–product of the previous proof and of the maximum principle, we obtain that any solution w_λ of (P_λ^1) (for $0 < \lambda < \lambda_1$) is limit of minimal solutions of $(P_\lambda^{1+\varepsilon})$

Theorem 14. *Let $\Omega \in L$, let $0 < \lambda < \lambda_1$, let $\overline{u}$ be the unique solution of (P_λ^1) and let u_ε be the minimal solution of $(P_\lambda^{1+\varepsilon})$ when $\varepsilon < \varepsilon_\lambda$ (see Theorem 13). Then $u_\varepsilon > \overline{u}$ and, as $\varepsilon \to 0$, u_ε converges to $\overline{u}$ in $C^{2,\alpha}(\overline{\Omega})$.*

Now we study the behavior of the mountain–pass solution U_ε of $(P_\lambda^{1+\varepsilon})$ when $\varepsilon \to 0$. The following result states that

$$\|U_\varepsilon\| \approx \left(\frac{\lambda_1}{\lambda}\right)^{1/\varepsilon} \qquad \text{as } \varepsilon \to 0,$$

namely that U_ε blows up *exponentially* with respect to ε.

Theorem 15. *Let $\Omega \in L$, let $0 < \lambda < \lambda_1$, let U_ε be the mountain–pass solution of $(P_\lambda^{1+\varepsilon})$ when $\varepsilon < \varepsilon_\lambda$ (see Theorem 13). Let $A_\varepsilon = \|U_\varepsilon\|$: then $A_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and*

$$\lim_{\varepsilon\to 0} A_\varepsilon^\varepsilon = \frac{\lambda_1}{\lambda}, \qquad \lim_{\varepsilon\to 0} \; A_\varepsilon^{-1} U_\varepsilon \;=\; \varphi_1 \quad \text{in } C^{2,\alpha}(\overline{\Omega}),$$

where φ_1 is normalized so that $\|\varphi_1\| = 1$; in particular, $U_\varepsilon(x) \to +\infty$ for all $x \in \Omega$.

The proof of Theorem 15 is based on the following lemma.

Lemma 3. *Suppose the assumptions of Theorem 15 hold true. Then $A_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and $A_\varepsilon^\varepsilon$ remains bounded as $\varepsilon \to 0$.*

Proof. The function U_ε is a critical point of the functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2+\varepsilon}\int_\Omega |1+u|^{2+\varepsilon}, \qquad u \in H_0^1(\Omega).$$

Due to the fact that $\lambda < \lambda_1$, for any bounded set $B \subset H_0^1(\Omega)$ there exists $\varepsilon_B > 0$ such that the second derivative J''_ε is positive definite on B for all $\varepsilon < \varepsilon_B$. Hence, by its variational characterization, $U_\varepsilon \notin B$ if $\varepsilon < \varepsilon_B$ and this shows that $\{U_\varepsilon\}$ is not bounded in $H_0^1(\Omega)$, i.e. $A_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

In order to prove the second statement, we observe that, testing the Euler equation on the function U_ε, we get

$$A_\varepsilon^2 = \int_\Omega |\nabla U_\varepsilon|^2 = \lambda \int_\Omega |U_\varepsilon + 1|^{1+\varepsilon} U_\varepsilon. \tag{45}$$

Inserting (45) into the expression of J_ε, we deduce that

$$J_\varepsilon(U_\varepsilon) = \left(\frac{1}{2} - \frac{1}{2+\varepsilon}\right) \int_\Omega |\nabla U_\varepsilon|^2 - \frac{\lambda}{2+\varepsilon}\int_\Omega |1+U_\varepsilon|^{1+\varepsilon}.$$

Therefore, we find as $\varepsilon \to 0$

$$J_\varepsilon(U_\varepsilon) \geq \frac{\varepsilon}{2(2+\varepsilon)}\, A_\varepsilon^2 + O(A_\varepsilon^{1+\varepsilon}). \tag{46}$$

The value of $M_\varepsilon := \max_{t\geq 0} J_\varepsilon(t\,\varphi_1)$ is attained at the point $t = t_\varepsilon$ for which

$$t_\varepsilon \int_\Omega |\nabla \varphi_1|^2 = \lambda \int_\Omega |1 + t_\varepsilon\,\varphi_1|^{1+\varepsilon}\varphi_1. \tag{47}$$

By the same argument just used to show that $A_\varepsilon \to \infty$ (the positive definiteness of J''_ε), we infer that $t_\varepsilon \to \infty$. Hence (47) reads $\int |\nabla\varphi_1|^2 \approx \lambda t_\varepsilon^\varepsilon \int \varphi_1^2$, that is

$$t_\varepsilon^\varepsilon = \frac{\lambda_1}{\lambda} + o(1), \qquad \text{as } \varepsilon \to 0. \tag{48}$$

Moreover, as $\varepsilon \to 0$, by (47) we also deduce (recall $\|\varphi_1\| = 1$)

$$M_\varepsilon = J_\varepsilon(t_\varepsilon\varphi_1) \le \left(\frac{1}{2} - \frac{1}{2+\varepsilon}\right) t_\varepsilon^2 \int_\Omega |\nabla\,\varphi_1|^2 \approx \frac{\varepsilon}{4}\, t_\varepsilon^2. \tag{49}$$

Let u_ε denote the minimal solution of $(P_\lambda^{1+\varepsilon})$ and consider the path $\gamma_\varepsilon : [0,T] \to H_0^1(\Omega)$ defined by

$$\gamma_\varepsilon(t) = \begin{cases} (1-t)\, u_\varepsilon & \text{if } t \in [0,1] \\ (t-1)\,\varphi_1 & \text{if } t \in [1,T] \end{cases}$$

where $T = T(\varepsilon) > 1$ is chosen so large that $J_\varepsilon(\gamma_\varepsilon(T)) < J_\varepsilon(u_\varepsilon)$. Since γ_ε is an admissible path for the mountain–pass scheme, it must be

$$J_\varepsilon(U_\varepsilon) \le \max_{0\le t\le T} J_\varepsilon(\gamma_\varepsilon(t)) = M_\varepsilon.$$

Raising to the power $\frac{\varepsilon}{2}$, and using (46), (48) and (49), we deduce

$$\frac{\lambda_1}{\lambda} \ge \limsup_{\varepsilon\to 0} M_\varepsilon^{\varepsilon/2} \ge \limsup_{\varepsilon\to 0} [J_\varepsilon(U_\varepsilon)]^{\varepsilon/2} \ge \limsup_{\varepsilon\to 0} A_\varepsilon^\varepsilon$$

which shows that $A_\varepsilon^\varepsilon$ remains bounded and concludes the proof. □

Proof of Theorem 15. The function $v_\varepsilon = A_\varepsilon^{-1} U_\varepsilon$ satisfies the equation

$$-\Delta v_\varepsilon = \lambda\, A_\varepsilon^\varepsilon\, (A_\varepsilon^{-1} + v_\varepsilon)^{1+\varepsilon}.$$

Testing it with v_ε, we deduce in particular that

$$\int_\Omega |\nabla\, v_\varepsilon|^2 = \lambda\, A_\varepsilon^\varepsilon \int_\Omega |A_\varepsilon^{-1} + v_\varepsilon|^{1+\varepsilon} v_\varepsilon. \tag{50}$$

From (50) and from Lemma 3 we deduce that, for every sequence $\varepsilon_m \to 0$, the sequence $\{v_{\varepsilon_m}\}$ converges weakly in $H_0^1(\Omega)$ (up to a subsequence) to a nontrivial function v satisfying

$$-\Delta v = [\lambda\,(\lim_m A_{\varepsilon_m}^{\varepsilon_m})]\, v.$$

Then, since v is non–negative, it must be $\lambda(\lim_m A_{\varepsilon_m}^{\varepsilon_m}) = \lambda_1$ and v is a multiple of φ_1. From (50) one deduces that in fact v_{ε_m} converges strongly to v, and hence v coincides with φ_1. Invoking once more [7] and elliptic regularity we also get $v_{\varepsilon_m} \to \varphi_1$ in $C^{2,\alpha}(\overline{\Omega})$. Since this is true for every sequence $\varepsilon_m \to 0$, we have convergence for $\varepsilon \to 0$. Finally, from the pointwise convergence $A_\varepsilon^{-1} U_\varepsilon(x) \to \phi_1(x)$ we deduce $U_\varepsilon(x) \to +\infty$ for all $x \in \Omega$. □

6 A physical interpretation of the results and some open problems

In this section we discuss the model suggested in [21] and we give a physical interpretation of our results.

We are interested in existence and behavior of steady states u of temperature distribution in an object Ω heated by the application of a uniform electric current $I = \sqrt{\lambda} > 0$ (the gradient ∇u represents the transfer of heat). If the body Ω is homogeneous with unitary thermal conductivity, the electric resistance R is a function of the temperature u, $R = R(u)$. If the radiation is negligible, the resulting stationary equation in some dimensionless form reads

$$-\Delta u = \lambda R(u) \tag{51}$$

for which, of course, only positive solutions have to be considered. In many cases of physical interest, the resistance increases with the temperature, that is, $u \mapsto R(u)$ is monotone increasing. We assume that the temperature is kept equal to 0 on the boundary of the body so that to (51) we associate the homogeneous Dirichlet boundary condition. The resistance should be positive also at zero temperature, $R(0) > 0$. It is known that a limiting current $I^* = \sqrt{\lambda^*}$ exists beyond which positive steady states do not exist. This is precisely the content of Theorem 1. The maximal interval of values of λ for which there exists a positive temperature u solving (51) is usually improperly called the *spectrum*. Both the cases of concave and convex functions R are of some interest although they highlight very different behaviors. In the former case the spectrum is open and the stationary solution u of (51) is unique for all $\lambda \in (0, \lambda^*)$ while in the latter case the spectrum is closed and non–unique solutions of (51) exist.

In this paper we concentrate on convex resistance functions R and we deal with the particular case where $R(u) = (1 + u)^p$ which gives a unitary resistance in correspondence of zero temperature $u = 0$ and increases polynomially and superlinearly with respect to u. The parameter p characterizes the material used to fill Ω. If $p \leq \frac{n+2}{n-2}$, Theorem 1 states that for all current $I \in (0, I^*)$ there exist at least two temperatures u solving (51). Only the minimal temperature u_λ is stable (see [21, Theorem 5.1] and [9, Lemmas 2.1 and 2.4]). Theorem 3 tells us that the stable temperature increases with the current. As I tends to the extremal current I^*, Corollary 1 establishes that the stable and unstable stationary temperatures u_λ and U_λ tend to a limit value U_* and give rise to a unique solution of (51). On the contrary, these two temperatures have a very different behavior for small currents I, see Theorems 5 and 6. As the resistance R becomes more convex, Theorem 8 states that the limit current I^* becomes smaller while, for a given current I, the stable temperature u_λ becomes larger. As the resistance loses convexity, the stable and unstable temperatures behave again very differently, see Theorems 14 and 15. Finally, (39) states that for a prescribed volume ω_n of the homogeneous material considered, the limiting current I^* may be as large as desired, provided one models the body Ω in a suitable way. The current I^* is minimal when Ω has the shape of a ball, see Theorem 11.

Some open problems

• The maximal stable temperature $u_M = \|u_\lambda\|_\infty$ for (P^p_λ) is of course of great interest. A first problem is therefore to establish for which λ and which $\Omega \in \mathbb{L}$ one has $u_M \le T$ for some limiting temperature $T > 0$. Let us mention that the method we used to prove (39) shows that for all $\varepsilon > 0$ and all $\lambda \le \frac{2^{1-p}}{\varepsilon}$ we have $u_M \le 1$ when the body is the ellipsoid Ω_ε. Hence, we may have "small" stable temperatures also in correspondence of large currents I; of course, here this happens because Ω_ε is "thin", and the surface area is large with respect the volume. A good starting point to solve this problem are the upper bounds for $\lambda = \lambda(u_M)$ determined in [20, Theorem 1] where one can also find some numerical results.

• An even more interesting problem is to fix the maximal stable temperature $\|u_\lambda\|_\infty$ and its mean value $\|u_\lambda\|_1$ and to wonder about existence and uniqueness of λ and p for which these constraints are satisfied by the minimal solution u_λ of (P^p_λ) in a given domain $\Omega \in L$. This corresponds to determine the current $\sqrt{\lambda}$ and the material filling Ω since the parameter p characterizes the resistance of the material. Of course, one should assume $\|u_\lambda\|_1 \le |\Omega| \cdot \|u_\lambda\|_\infty$ by Hölder inequality and not fix $\|u_\lambda\|_\infty$ too large.

• Another natural question is the following: given a fixed amount of material (e.g. $|\Omega| = \omega_n$) for which shapes of $\Omega \in \mathbb{L}$ do we have a stationary positive temperature u in correspondence of large currents $I^* \ge \overline{I}$ (for some $\overline{I} > 0$)? In other words, for which $\Omega \in \mathbb{L}$ it is $\lambda^*(\Omega) \ge \overline{I}^2$? As we see from (39), such an Ω always exist: does it need to be "thin" in some sense (e.g. contained in a n–dimensional rectangle having very different edges)?

• Concerning the unstable (mountain–pass) stationary temperature U_λ, an interesting problem would be to compare $\|U_\lambda\|_\infty$ for different values of λ. Of course, here we assume that $p \le \frac{n+2}{n-2}$. Is it true that the map $\lambda \mapsto \|U_\lambda\|_\infty$ is decreasing? From Remark 4 we know that the answer is positive when $\Omega = B_1$ and $p = \frac{n+2}{n-2}$. Further arguments in favor of a positive answer may be found in [4, Theorem 1.2]. Indeed, the comparison between two mountain–pass solutions corresponding to different values of λ is equivalent (thanks to a rescaling) to the comparison of two mountain–pass solutions for the same value of λ but in different domains, one of them strictly containing the other. Even more interesting: do we have *pointwise monotonicity* with respect to λ of the functions U_λ? We refer again to Remark 4 for the case where $\Omega = B_1$ and $p = \frac{n+2}{n-2}$. Positive answers to these questions would bring further evidence to the "opposite" behaviors of u_λ and U_λ.

• How are the topology and the geometry of the body Ω related to the number of stationary temperatures? Are the unstable solutions of (51) all unstable in the same fashion? From a mathematical point of view, the instability may be evaluated by means of the Morse index of the (nondegenerate) critical point of the action functional associated. To this end, important contributions for slightly different problems may be found in [12, 27]. We also refer to Remark 3 for related results in the critical case $p = \frac{n+2}{n-2}$.

7 Appendix: blow–up analysis for the case $p = \frac{n+2}{n-2}$

In this section we consider in more detail problem (21) with $p = \frac{n+2}{n-2}$, namely

$$\begin{cases} -\Delta W_\varepsilon = (W_\varepsilon + \varepsilon)^{(n+2)/(n-2)} & \text{in } \Omega \\ W_\varepsilon > 0 & \text{in } \Omega \\ W_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{52}$$

Our aim is to study the behavior of the solutions when $\varepsilon \to 0$. In order to do this, one can use the blow–up analysis performed by Han [17], Schoen [30] and Li [22]. Note that, using a simple translation, equation (52) becomes $-\Delta W_\varepsilon = W_\varepsilon^{\frac{n+2}{n-2}}$ with the boundary condition $W_\varepsilon = \varepsilon$ on $\partial\Omega$. This fact will be used when we will quote some results from [22]. We recall some useful definitions.

Let $\varepsilon_i \to 0^+$, and let W_i be a sequence of solutions of (52) for $\varepsilon = \varepsilon_i$. The sequence W_i is said to *blow up* at the point $\overline{y} \in \overline{\Omega}$ if there exists a sequence of points $y_i \in \Omega$ such that $\lim_i y_i = \overline{y}$ and $\lim_i W_i(y_i) = +\infty$. The point $\overline{y} \in \Omega$ is called an *isolated* blow up point if there exists a sequence $\{y_i\}$ of local maxima of W_i tending to $\overline{y}$ with $W_i(y_i) \to +\infty$, and if there exist $\overline{r} \in (0, d(\overline{y}, \partial\,\Omega))$ and $C > 0$ such that, for i sufficiently large

$$W_i(y) \le C\,|y - y_i|^{-(n-2)/2} \qquad \forall y \in B_{\overline{r}}(y_i). \tag{53}$$

Let y_i be as above, suppose $\overline{y}$ is an isolated blow up point for $\{W_i\}$ and set

$$\overline{W}_i(r) = \frac{1}{|B_r(y_i)|} \int_{B_r(y_i)\cap\Omega} W_i\ , \qquad \overline{Z}_i(r) = r^{(n-2)/2}\, \overline{W}_i(r)\ , \qquad r \in (0, \overline{r}).$$

Suppose that for some $\varrho \in (0, \overline{r})$ independent of i, the function $\overline{Z}_i$ has precisely one critical point for large i. Then we say that $\overline{y}$ is an *isolated simple* blow up point.

The next Lemma asserts that blow up at the boundary of Ω is excluded.

Lemma 4. *Let $\{\varepsilon_i\}$ and $\{W_i\}$ be as above. Then there exists $d_\Omega > 0$ depending only on Ω with the following properties. For every i and for every solution W_i of (52) we have*

$$\nabla W_i(x) \cdot \nabla\, d(\cdot, \partial\Omega)(x) \ge 0, \qquad \textit{for all } x \in \Omega \textit{ with } d(x, \partial\Omega) < d_\Omega. \tag{54}$$

Moreover, if $\overline{y} \in \overline{\Omega}$ is a blow up point for $\{W_i\}$, then $\overline{y} \in \Omega$ and $d(\overline{y}, \partial\Omega) \ge d_\Omega$.

The proof follows from the same arguments as in [17, pp.163–164], which are based on the moving planes method in [15].

Lemma 4 allows us to consider just interior blow up. Hence, we may apply [22, Proposition 2.1] to obtain the following result

Lemma 5. *Let $\{\varepsilon_i\}$, $\{W_i\}$ and $\{y_i\}$ be as above and suppose that $\overline{y} \in \Omega$ is an isolated simple blow up point for $\{W_i\}$. Let $\{R_i\}$ and $\{\eta_i\}$ be two sequences of positive numbers such that $R_i \to +\infty$ and $\eta_i \to 0$. Then for some subsequence of $\{W_i\}$, still denoted by $\{W_i\}$, we have*

$$\left\| W_i(y_i)^{-1} W_i\left(W_i(y_i)^{-2/(n-2)} \cdot + y_i\right) - (1 + b_0 |\cdot|^2)^{(2-n)/2} \right\|_{C^2(B_{2R_i}(0))} \le \eta_i\,,$$

$$R_i\, W_i(y_i)^{-2/(n-2)} \to 0 \qquad \text{as } i \to +\infty.$$

Here $b_0 = (n(n-2))^{-1}$.

Combining [22, Proposition 3.1] and Lemma 4 we get

Lemma 6. *Let $\{\varepsilon_i\}$ and $\{W_i\}$ be as above. Then the blow up points of $\{W_i\}$ are isolated simple. If $\overline{y}_1, \ldots, \overline{y}_k$ are the blow up points of $\{W_i\}$, then there exists $d_\Omega > 0$ depending only on Ω such that $\min_{j\neq l} d(\overline{y}_j, \overline{y}_l) \ge d_\Omega$ and $\min_j d(\overline{y}_j, \partial\Omega) \ge d_\Omega$. Moreover, if $y_j^i \to \overline{y}_j$ is a sequence of points for which $W_i(y_j^i) \to +\infty$, $j = 1, \ldots, k$, then*

$$W_i(y_j^i)\, W_i(y) \to a\, |y - \overline{y}_j|^{2-n} + b_j(y) \qquad \text{in } C^{2,\alpha}_{loc}(B_{d_\Omega/2}(\overline{y}_j) \setminus \{\overline{y}_j\}),$$

where $a = (n(n-2))^{(n-2)/2}$ and $b_j(y)$ is some harmonic function in $B_{d_\Omega/2}(\overline{y}_j)$.

Lemma 5 describes the asymptotic behavior of W_i near the blow up points. Using [22, Lemma 2.4], [22, Proposition 3.1] and Lemma 6 one can prove that there is indeed no concentration of mass outside the points $\overline{y}_1, \ldots, \overline{y}_k$. More precisely, in the spirit of the concentration–compactness principle [23], the following proposition holds

Proposition 2. *Let $\varepsilon_i \to 0$. Suppose $\{W_i\}$ is a sequence of solutions of (52) with $\varepsilon = \varepsilon_i$ and such that $\max_{\overline{\Omega}} W_i \to +\infty$ as $i \to +\infty$. Then, up to a subsequence, the sequence $\{W_i\}$ concentrates at a finite number of points $\overline{y}_1, \ldots, \overline{y}_k \in \Omega$, namely*

$$|\nabla W_i|^2 \to \mathcal{S}^{n/2} \sum_{l=1}^{k} \delta_{\overline{y}_l} \qquad W_i^{2^*} \to \mathcal{S}^{n/2} \sum_{l=1}^{k} \delta_{\overline{y}_l}. \tag{55}$$

in the weak sense of measures.
Moreover, there exist $k_\Omega \in \mathbb{N}$ and $d_\Omega > 0$ (depending only on Ω) such that:
(i) the number k of concentration points cannot exceed k_Ω;
(ii) $d(\overline{y}_j, \overline{y}_l) \ge d_\Omega$ for all $j \neq l$ and $d(\overline{y}_j, \partial\Omega) \ge d_\Omega$ for all j.

In order to derive the tools needed in the proof of Theorem 6, from now on we will specialize to the case in which W_i is a mountain–pass solution of (52). From (22) and (55) it follows that $k = 1$ in this case, i.e. there is at most one blow up point $\overline{y}_1$.

Set $\tilde{W}_i(x) = W_i(y_1^i)\, W_i(x)$, where y_1^i is the sequence of local maxima converging to $\overline{y}_1$. The convergence of $\tilde{W}_i$ in Lemma 6 can be in fact extended to the whole $\overline{\Omega} \setminus \{\overline{y}_1\}$, and the limit function $b_1(x)$ is harmonic. In the compact subsets of $\Omega \setminus \{\overline{y}_1\}$, this follows from Lemma 6, a Harnack type inequality, see [22, Lemma 2.1], and standard elliptic estimates. To get convergence up to the boundary, one can use condition (54). Since $\tilde{W}_i$ vanishes on $\partial\Omega$, the limit function must be a multiple of the Green's function with pole $\overline{y}_1$. Hence we have the following result

Proposition 3. *Let $\{\varepsilon_i\}$, $\{W_i\}$ be as above and assume moreover that W_i is of mountain–pass type. Then $\{W_i\}$ has at most one blow up point $\overline{y}_1$ and*

$$W_i(y_1^i)\, W_i(y) \to a\, G(y, \overline{y}_1) \qquad \text{in } C^{2,\alpha}_{loc}(\overline{\Omega} \setminus \{\overline{y}_1\}),$$

where $a = (n(n-2))^{(n-2)/2}$.

We state now a general result, based on a Pohozaev type identity, see [22, pages 331–332]. Note that in the statement of [22, Proposition 1.1] the assumption that $A > 0$ is in fact not necessary.

Lemma 7. *Let u be a solution of problem (52), let $\overline{y}_1 \in \Omega$, let $\sigma \in (0, d(\overline{y}_1, \partial\Omega))$ and let B_σ the ball centered at $\overline{y}_1$ with radius σ. Then,*

$$\frac{n-2}{2}\,\varepsilon \int_{B_\sigma} (u+\varepsilon)^{(n+2)/(n-2)} - (n-2)\frac{\sigma}{2n}\int_{\partial B_\sigma} (u+\varepsilon)^{2^*} = \int_{\partial B_\sigma} B(\sigma, x, u, \nabla u), \qquad (56)$$

where

$$B(\sigma, x, u, \nabla u) = \frac{n-2}{2}\, u\, \frac{\partial u}{\partial \nu} - \frac{\sigma}{2}\, |\nabla u|^2 + \sigma \left| \frac{\partial u}{\partial \nu} \right|^2 .$$

Moreover, for any function $h : \Omega \to \mathbb{R}$ of the form $h(x) = a|x - \overline{y}_1|^{2-n} + A + \alpha(x - \overline{y}_1)$ (where $a > 0$, $A \in \mathbb{R}$ and α is of class C^1 with $\alpha(0) = 0$) we have

$$\lim_{\sigma \to 0} \int_{\partial B_\sigma} B(\sigma, x, h, \nabla h) = -\frac{(n-2)^2}{2}\, n\omega_n\, a\, A. \qquad (57)$$

Using Lemma 5, one can check that the asymptotic shape of the functions W_i is of the form (25) for a suitable value of d. Then, from [22, Proposition 3.1], the formula

$$\int_0^\infty \frac{r^\alpha}{(1+r^2)^\beta}\, dr \; = \; \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\beta - \frac{\alpha+1}{2}\right)}{2\Gamma\left(\beta\right)},$$

and a change of variable, one finds that

$$\lim_i W_i(y_1^i) \int_{B_\sigma} (W_i + \varepsilon_i)^{(n+2)/(n-2)} = (n(n-2))^{n/2}\, \omega_n. \qquad (58)$$

We now apply Lemma 7 to the mountain pass–solutions W_i which, by Proposition 3, has asymptotically (as $i \to \infty$) precisely the form of h with

$$a = (n(n-2))^{(n-2)/2} \qquad A = -(n(n-2))^{(n-2)/2} H(\overline{y}_1, \overline{y}_1).$$

From Proposition 3, equation (57) and the homogeneity of B it follows that

$$\begin{aligned}\lim_i W_i(y_1^i)^2 \int_{\partial B_\sigma} B(\sigma, x, W_i, \nabla W_i) &= \lim_i \int_{\partial B_\sigma} B(\sigma, x, W_i(y_1^i)\, W_i, \nabla(W_i(y_1^i)\, W_i)) \\ &= \frac{1}{2}\, n^{n-1}(n-2)^n\, \omega_n H(\overline{y}_1, \overline{y}_1) + o_\sigma(1)\end{aligned} \tag{59}$$

where $o_\sigma(1) \to 0$ as $\sigma \to 0$. Now multiply (56) by $W_i(y_1^i)^2$ and insert in (58), (59); by Proposition 3 there holds $W_i(y_1^i) W^{2^*}|_{\partial B_\sigma} \to 0$ as $i \to \infty$. Hence, letting $\sigma \to 0$ we deduce

Lemma 8. *Let $\{\varepsilon_i\}$, $\{W_i\}$, $\{y_1^i\}$ and $\overline{y}_1$ be as in Proposition 3. Then, we have*

$$\lim_i \varepsilon_i\, W_i(y_1^i) = (n(n-2))^{(n-2)/2}\, H(\overline{y}_1, \overline{y}_1).$$

Information on the location of the blow up point can be obtained following the arguments in [17, page 169]. Multiplying equation (52) by $\frac{\partial W_i}{\partial x_j}$ and integrating by parts on Ω we get

$$\frac{1}{2} \int_{\partial \Omega} |\nabla W_i|^2 \nu_j = -\frac{1}{2^*} \int_{\partial \Omega} |W_i + \varepsilon_i|^{2^*} \nu_j, \qquad j = 1, \dots, n.$$

Integrating on $\Omega \setminus B_\sigma(\overline{y_1})$, we deduce

$$\begin{aligned}\frac{1}{2} \int_{\partial \Omega} |\nabla W_i|^2 \nu_j + \frac{1}{2^*} \int_{\partial(\Omega \setminus B_\sigma(\overline{y_1}))} |W_i + \varepsilon_i|^{2^*} \nu_j + \int_{\partial B_\sigma(\overline{y_1})} \frac{\partial W_i}{\partial x_j} \frac{\partial W_i}{\partial \nu} \\ -\frac{1}{2} \int_{\partial B_\sigma(\overline{y_1})} |\nabla\, W_i|^2\, \nu_j = 0, \qquad j = 1, \dots, n.\end{aligned}$$

Here ν denotes the exterior unit normal to $\partial(\Omega \setminus B_\sigma(\overline{y}_1))$. Letting $\sigma \to 0$, using Proposition 3, the last two equations and some simple calculations, one finds

Proposition 4. *Let $\overline{y}_1$ be the concentration point given in Proposition 3. Then,*

$$\nabla \varphi(\overline{y}_1) = 0,$$

where $\varphi(\cdot) = H(\cdot, \cdot)$.

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