

Magnetic skyrmions: From lumps to supercompactons

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The magnetic skyrmion is described by one control parameter and one length scale. We study the two extreme limits of the control parameter, infinitely large and vanishing, and find that the magnetic skyrmion becomes a “restricted” magnetic skyrmion and an $O(3)$ sigma model lump, respectively. Depending on the potential under consideration, the restricted limit manifests differently. In the case of the Zeeman term, the restricted magnetic skyrmion becomes a “supercompacton” that develops a discontinuity, whereas for the Zeeman term to the power $\frac{3}{2}$ it becomes a normal compacton. In both the lump and the restricted limit the solution is given in exact explicit form. We observe that the case of the Zeeman term squared, which can also be understood as a special combination of the Zeeman term and the easy-plane potential (realizable in the laboratory), the analytically exact solution for all values of the coupling, including the Bogomol’nyi-Prasad-Sommerfield case, is also of the lump type. Finally, we notice that certain materials (e.g., $\text{Fe}_{1-x}\text{Co}_x\text{Si}$ or $\text{Mn}_{1-x}\text{Fe}_x\text{Ge}$) have a rather large control parameter ϵ of order 100, making the restricted limit a suitable rough approximation.

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I. INTRODUCTION

Topological soliton applications cover many fields of theoretical physics, from high-energy physics to condensed matter theory [1–6]. Some topological solitons that have garnered significant attention in the field of condensed matter physics in recent years are known as *magnetic skyrmions* named after the famous works of Skyrme [7,8]. These magnetic skyrmions are the main focus of this work. From a condensed matter point of view, a magnetic skyrmion is a stable two-dimensional nanoparticle describing a localized winding of the magnetization in certain magnetic materials [9–11]. These topological defects are subject of intense experimental and theoretical investigations [12–25] and have shown potential for technological spintronic applications [26,27]. They were observed for the first time in 2009 by Muhlbauer *et al.* [28] in a chiral magnet of MnSi, and subsequently in a plethora of other chiral magnets [29–34]. Magnetic skyrmions have risen to prominence as up-and-coming candidates for next-generation high-density efficient information encoding [35,36] or for quantum computing applications [37].

The spin texture of a magnetic skyrmion is a stable configuration that originates from chiral interactions, known as Dzyaloshinskii-Moriya (DM) interactions. The Hamiltonian interaction term involving two atomic spins in the chiral magnet was proposed by Dzyaloshinskii [38] as an additional contribution to the usual Heisenberg Hamiltonian. Subsequently, Moriya [39,40] demonstrated that this type of interaction originates from the relativistic spin-orbit coupling. It was suggested [41] to consider the mean field approximation (MFT) for the field theory model in the continuum limit, where both the Heisenberg exchange term (also called Dirichlet term E_2) and the DM one (E_1) are competing and the Zeeman energy (the external potential E_0) is included to control the system.

In this work we consider the theory of magnetic skyrmions in two spatial dimensions (2D) and apply certain techniques used in the past in the context of the baby Skyrme model [42]. We show that a dimensionless parameter enables us to control the relation between the Dirichlet term, the one with two derivatives, and the rest of the energy functional that contains the potential and the DM term. By changing this parameter we can flow from an almost Bogomol’nyi-Prasad-Sommerfield (BPS) theory, close to the $O(3)$ sigma model which admits lumps as solutions, to a “restricted” magnetic skyrmion model with just E_1 and E_0 . The restricted model is analytically solvable. In the case of the ordinary Zeeman potential, which is quadratic in the perturbative excitation fields near the vacuum, a feature never encountered before in the context of the baby Skyrme model is present: the solution is not only

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a compacton¹ but it is also discontinuous. We may call it a “supercompacton.” The discontinuity is resolved as soon as we turn on a small coefficient for the Dirichlet term. We also confirm the result of Ref. [20] that the magnetic skyrmion being a solution to the variational equations of motion only corresponds to a stationary point of the energy functional if a certain boundary term is subtracted off. This boundary term can be written in a form of a vorticity [19,20]. Indeed, this proves important for determining the correct lump sizes in the limit of the control parameter (ϵ) going to zero.

Finally, we observe that this lump-type solution corresponds to the solution of the critically coupled version of magnetic skyrmions with the Zeeman term squared, as described by Barton-Singer *et al.* [19]. It is not limited to this scenario: it is also the solution to the full second-order variational equation of motion for *all* values of the coupling, first observed by Döring and Melcher [50]. We find that for this potential, discarding the boundary term changes the energy functional to become dependent on ϵ , hence revealing a phase transition between the homogeneous phase and the skyrmionic phase.

On the experimental side, we also confront the dimensionless parameter with the various experimental realizations of the magnetic skyrmion and find that the (near) “restricted” magnetic skyrmion, corresponding to the case of a large control parameter, is actually realized in some materials, like $\text{Fe}_{1-x}\text{Co}_x\text{Si}$ or $\text{Mn}_{1-x}\text{Fe}_x\text{Ge}$. We also confirm that it is possible to experimentally realize the Zeeman squared term, by using magnets with a certain anisotropy term and adjusting the magnetic field of the Zeeman term, so as to complete the square. At this critically tuned value of the magnetic field, the exact analytic lump-type solution is thus realizable in the laboratory.

This paper is organized as follows. In Sec. II we review the magnetic skyrmion in the continuum formulation with various potentials in our notation. In Sec. II A, we review the hedgehog ansatz and discuss the symmetry-breaking properties leading to the fixing of the phase parameter. In Secs. II B and II C we introduce the sigma model lump and restricted limits of the magnetic skyrmion. In Sec. II D we review the case of the critical coupling and in Sec. II E we show that the lump solution is a solution to the model with the Zeeman term squared for any coupling. In Sec. II F we provide numerical solutions that interpolate between the lump and restricted limits, verifying the analytical results obtained. In Sec. II G we compare the control parameter to experimental parameters of various chiral magnets. We conclude in Sec. III with a discussion.

II. 2D MAGNETIC SKYRMIONS

A magnetic skyrmion is a stable *vortexlike* configuration of magnetic spin vectors [9–11]. The relevant physical quantity

in a two-dimensional magnet is the direction of the magnetization vector $\mathbf{n}$. The latter is the order parameter for the system of spins $\mathbf{S}_i$, localized on the sites of two-dimensional lattices. The simplest model describing such a system is given by the Heisenberg Hamiltonian

$$\mathcal{H}_H = - \sum_{i < j} \mathbb{J}_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (1)$$

where the interaction is only between nearest-neighbor spins, and $\mathbb{J}_{ij}$ is the exchange interaction constant (which is negative in ferromagnetic systems). We suppose that $\mathbb{J}_{ij} = J$ for i and j being nearest-neighbor sites in the crystal. In the mean field approximation, a proper dynamical variable is the expectation value of the spins, i.e., the unit magnetization vector $\mathbf{n} \in S^2$. Its dynamics is governed by the continuum Ginzburg-Landau (GL) effective theory (see, e.g., Ref. [6]). This is the static energy of the O(3) sigma model. Furthermore, the Hamiltonian of the Heisenberg model can be supplemented by the potential-like term $-\mathbf{B} \cdot \mathbf{n}$ that describes the Zeeman interaction with an external magnetic field, which we choose to be aligned along the third Cartesian axis $\mathbf{B} = B\mathbf{e}_3$.

In cubic crystals, without an inversion center, such as MnSi, symmetry analysis reveals an additional term that can be incorporated into the GL description, known as the Dzyaloshinskii-Moriya (DM) interaction [38–40]:

$$\mathcal{H}_{DM} = - \sum_{ij} \mathbf{D}_{ij} \cdot (\mathbf{S}_i \times \mathbf{S}_j), \quad (2)$$

where $\mathbf{D}_{ij}$ is the chiral coupling parameter, chosen here to be equal to $D\hat{\mathbf{e}}_{j-i}$ for i and j being nearest-neighbor sites in the crystal, $\hat{\mathbf{e}}_{j-i}$ is a unit vector pointing from the lattice site i to the lattice site j and D depends on the material.² The corresponding energy in the continuum limit (see the Appendix), at the leading order in the long-wavelength approximation, reduces to the functional

$$\begin{aligned} E[\mathbf{n}] &= \int d^2x J \left(\frac{1}{2} \partial_i \mathbf{n} \cdot \partial_i \mathbf{n} + \kappa \epsilon_{iab} \partial^i n^a n^b + h(1 - n_3) \right) \\ &\equiv E_2 + E_1 + E_0, \end{aligned} \quad (3)$$

where $\mathbf{n} = (n^1, n^2, n^3)$ is a unit magnetization vector ($\mathbf{n} \cdot \mathbf{n} = 1$), $i, j = 1, 2$ are planar indices, $a, b = 1, 2, 3$ are (target) spatial indices, $\kappa = \frac{2D}{J}$ is the effective DM coupling, and $h = \frac{|B|}{J}$ is the parameter playing the role of the mass or Zeeman coupling strength. The DM term E_1 breaks parity but is invariant under a simultaneous O(3) rotation of spin and O(2) rotation of spatial coordinates, a property inherited from its origin as a spin-orbit interaction.

In presence of a generic potential term E_0 , the energy functional takes the form

$$\begin{aligned} E[\mathbf{n}] &= \int d^2x \left(\frac{1}{2} \partial_i \mathbf{n} \cdot \partial_i \mathbf{n} + \kappa \epsilon_{iab} \partial^i n^a n^b + V_p(\mathbf{n}) \right. \\ &\quad \left. + \frac{\lambda}{2} (\mathbf{n} \cdot \mathbf{n} - 1) \right), \end{aligned} \quad (4)$$

¹Compactons are solitonic solutions with compact support: they have a nontrivial profile function for $r < R$ and vanishing energy outside, i.e., for $r > R$. Often the fields are continuous but their derivatives are not at $r = R$. Compactons exist in the baby Skyrme model [43–45], the BPS-Skyrme model [46], as Q-balls in the gauged (ungauged) $\mathbb{C}P^N$ model [47] ([48]) and in K -field theories [49], etc.

²The chosen DM term here is denoted the Bloch-type DM term in the literature [6,19,21].

where we have introduced the Lagrange multiplier λ to enforce the unit-length constraint ($\mathbf{n} \cdot \mathbf{n} = 1$) and we simply measure the energy in units of J . We choose to work with the class of potentials $E_0 = \int V_p(\mathbf{n}) d^2x$, that are parametrized by

$$V_p(\mathbf{n}) = h(1 - \mathbf{n} \cdot \mathbf{N})^p, \quad (5)$$

which corresponds to the usual massive Zeeman term for $p = 1$. Note that $\mathbf{N} = N^a \mathbf{e}_a$ is the vacuum field. The perturbative fields have mass $m = \sqrt{h}$ if $p = 1$ and $m = 0$ if $p > 1$. We will concentrate on the cases $p = 1, \frac{3}{2}, 2$ in this paper. The case $p = 2$ corresponds to a particular linear combination of the Zeeman term and the so-called easy-plane term

$$h(1 - \mathbf{n} \cdot \mathbf{N})^2 = 2h(1 - \mathbf{n} \cdot \mathbf{N}) + h(\mathbf{n} \cdot \mathbf{N})^2 - h. \quad (6)$$

As was shown in Ref. [20], the energy functional is only stationary with respect to the solutions to the variational equations of motion if a certain boundary term is subtracted off:

$$\tilde{E}[\mathbf{n}] = E[\mathbf{n}] - \Omega[\mathbf{n}], \quad (7)$$

$$\Omega[\mathbf{n}] = \kappa \int d^2x \epsilon_{iab} \partial^i n^a N^b, \quad (8)$$

where $\Omega[\mathbf{n}]$ is called the vorticity. The vorticity is not part of the DM term, but a similar term that is locally different, yet giving the same boundary integral for finite-energy configurations (i.e., configuration with $\lim_{|x| \rightarrow \infty} \mathbf{n} = \mathbf{N}$) is

$$\hat{\Omega}[\mathbf{n}] = \kappa \int d^2x \epsilon_{ia3} \partial^i (n^a n^3). \quad (9)$$

Subtracting either the vorticity Ω or the boundary term $\hat{\Omega}$ from the energy functional $E[\mathbf{n}]$ will give rise to a well-defined variational problem. The first option is given in Eq. (7) as $\tilde{E}[\mathbf{n}]$ and we will call the second option $\hat{E}[\mathbf{n}] = E[\mathbf{n}] - \hat{\Omega}[\mathbf{n}]$.

We can write the DM term in an invariant form. For this we need a tensor T^{ai} such that the DM term can be written as

$$\mathcal{E}_1 = \kappa \epsilon_{abc} T^{ai} \partial_i n^b n^c. \quad (10)$$

The transformations are SO(3) for the internal indices a, b, c and SO(2) for the spatial index i . Any tensor related by SO(3) and SO(2) transformations to this is essentially equivalent. Those are the tensors that leave invariant the maximal amount of symmetry SO(2)_{diag}. We can express the tensors of this form in an invariant way. The choice is essentially determined once we pair $i = 1, 2$ with two corresponding orthogonal unit vectors T^{ai} in $\mathbb{R}^3$, i.e.,

$$T^{a1} T^{a1} = T^{a2} T^{a2} = 1, \quad T^{a1} T^{a2} = 0. \quad (11)$$

So the meaning of $\mathbf{T}$ is a pair of orthogonal unit vectors in $\mathbb{R}^3$. The energy functional (3) corresponds to the choice

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (12)$$

which in indices is $T^{ai} = \delta^{ai}$ and corresponds to the Bloch-type DM term. A more general DM term can thus be written using a different tensor obeying the conditions (11). In

particular, Refs. [19,21] use

$$\mathbf{T} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \\ 0 & 0 \end{pmatrix}, \quad (13)$$

where $\alpha = 0$ corresponds to the Bloch-type DM term and $\alpha = \frac{\pi}{2}$ to the Néel-type DM term.

The equations of motion are independent of boundary terms and read as

$$\partial_i^2 n^a + hp N^a (1 - \mathbf{n} \cdot \mathbf{N})^{p-1} - 2\kappa \epsilon_{aib} \partial^i n^b - \lambda n^a = 0, \quad (14)$$

$\mathbf{n} \cdot \mathbf{n} = 1$, where the Lagrange multiplier is

$$\lambda = n^a (\partial_i^2 n^a + hp N^a (1 - \mathbf{n} \cdot \mathbf{N})^{p-1} - 2\kappa \epsilon_{aib} \partial^i n^b). \quad (15)$$

The vector $\mathbf{N} = \mathbf{e}_3 = (0, 0, 1)$ corresponds to the vacuum. Due to the following condition

$$\lim_{r \rightarrow \infty} \mathbf{n} = \mathbf{N}, \quad (16)$$

the base space of the vector field $\mathbf{n} : \mathbb{R}^2 \rightarrow S^2$ can be compactified to the sphere S^2 . Therefore, due to the nontrivial homotopy group $\pi_2(S^2) = \mathbb{Z}$, there are topologically non-trivial solutions (with finite energy) that can be classified according to the degree of the map $\mathbf{n} : S^2 \rightarrow S^2$, i.e., the winding number $Q \in \mathbb{Z}$. The topological degree or the skyrmion number Q is given by

$$Q = \int d^2x \mathcal{Q} = \frac{1}{4\pi} \int d^2x \mathbf{n} \cdot \partial_1 \mathbf{n} \times \partial_2 \mathbf{n}. \quad (17)$$

The target space S^2 can be parametrized by the angles (ϑ, φ) :

$$\mathbf{n} = \begin{pmatrix} \sin \vartheta \cos \varphi \\ \sin \vartheta \sin \varphi \\ \cos \vartheta \end{pmatrix}, \quad \vartheta \in [0, \pi], \quad \varphi \in [0, 2\pi) \quad (18)$$

and the energy functional (7) becomes

$$\begin{aligned} \tilde{E}[\vartheta, \varphi] = \int d^2x & \left(\frac{1}{2} (\partial_i \vartheta)^2 + \frac{1}{2} \sin^2 \vartheta (\partial_i \varphi)^2 + h(1 - \cos \vartheta)^p \right. \\ & + \kappa (1 - \cos \vartheta) [\sin \varphi (\partial_1 \vartheta - \sin \vartheta \partial_2 \varphi) \\ & \left. - \cos \varphi (\partial_2 \vartheta + \sin \vartheta \partial_1 \varphi)] \right), \quad (19) \end{aligned}$$

whereas for reference, the energy functional with the boundary term (9) subtracted off instead simplifies as

$$\begin{aligned} \hat{E}[\vartheta, \varphi] = \int d^2x & \left(\frac{1}{2} (\partial_i \vartheta)^2 + \frac{1}{2} \sin^2 \vartheta (\partial_i \varphi)^2 \right. \\ & + h(1 - \cos \vartheta)^p + \kappa \sin^2 \vartheta \\ & \left. \times (\sin \varphi \partial_1 \vartheta - \cos \varphi \partial_2 \vartheta) \right). \quad (20) \end{aligned}$$

The simplification occurs because the boundary term (9) is already contained in the original DM term of Eq. (4).

Derrick's argument allows for the existence of these 2D magnetic solitons. The idea is to make a rescaling of the field $\mathbf{n}(x) \mapsto \mathbf{n}(x/R)$, and see how the energy terms scale with R ,

i.e.,

$$E_n(R) \equiv E[\mathbf{n}(x/R)] = E_{2,n}(1) + E_{1,n}(1)R + E_{0,n}(1)R^2, \quad (21)$$

where the three terms correspond to the kinetic energy, the DM term, and the potential term, respectively. If the DM term is negative, $E_n(R)$ starts, for small R , decreasing linearly with R until it increases quadratically due to the potential contribution. This change allows for the existence of stationary points corresponding to

$$R_* = \frac{E_{1,n}(1)}{2E_{0,n}(1)}, \quad (22)$$

which is roughly the length scale at which a soliton can be stabilized. By using the order of magnitude estimate

$$E_{2,n}(1) \propto 1, \quad E_{1,n}(1) \propto \kappa, \quad E_{0,n}(1) \propto h, \quad (23)$$

we have

$$R_* \sim \frac{\kappa}{h}. \quad (24)$$

The length scale of the soliton thus increases with κ and decreases with h . We thus define two new parameters:

$$\ell = \frac{\kappa}{h}, \quad \epsilon = \frac{\kappa^2}{h}, \quad (25)$$

where ℓ controls the size of the skyrmion (22) while ϵ , the dimensionless parameter of the theory, controls the relation between E_2 and $E_1 + E_0$. We call the latter the control parameter of the theory (i.e., ϵ). In order to see the scaling behavior of the theory, we can write

$$E_n = \tilde{E}_{2,n} + \kappa \tilde{E}_{1,n} + h \tilde{E}_{0,n}, \quad (26)$$

and perform the scaling $\mathbf{n}(x) \mapsto \mathbf{n}(x/\ell)$, which yields

$$E_n = \tilde{E}_{2,n} + \epsilon \tilde{E}_{1,n} + \epsilon \tilde{E}_{0,n}. \quad (27)$$

Clearly, in the limit of small ϵ , there is only the kinetic term, whereas in the limit of large ϵ , there is only a small contribution from the kinetic term. In these rescaled coordinates, the soliton size is of order one. On the other hand, if we instead perform the following rescaling $\mathbf{n}(x) \mapsto \mathbf{n}(x\kappa)$, we obtain

$$E_n = \tilde{E}_{2,n} + \tilde{E}_{1,n} + \frac{1}{\epsilon} \tilde{E}_{0,n}. \quad (28)$$

In these rescaled coordinates, it is clear that the model only depends on ϵ , but now the size of the soliton is no longer of order one, but of order ϵ .

The symmetry group $\text{SO}(3)_{\text{int}}$ (internal) of the sigma model is broken to $\text{SO}(2)_{\text{int}}$ by the potential, viz., by the vector $\mathbf{N}$ in E_0 . So, by considering $E_2 + E_0$, we have a symmetry group of spatial and internal rotations $\text{SO}(2)_{\text{spt}} \times \text{SO}(2)_{\text{int}}$. The DM term locks together spatial and internal rotations, so only a diagonal combination of the two remains invariant, $\text{SO}(2)_{\text{diag}}$. Rotations of $\text{SO}(2)_{\text{diag}}$ act as

$$\mathbf{n}(x_1, x_2) \mapsto R_{12}(\zeta) \mathbf{n}(\cos \zeta x_1 - \sin \zeta x_2, \sin \zeta x_1 + \cos \zeta x_2), \quad (29)$$

$\zeta \in [0, 2\pi)$. Moreover, there are the two internal parities as a discrete group $P_{\text{spt}} \times P_{\text{int}}$. The DM interaction is responsible

for the breaking to the diagonal group P_{diag} . Up to an additional internal rotation of π , the parity transformations act as

$$\mathbf{n}(x_1, x_2) \mapsto \bar{\mathbf{n}}(x_1, -x_2), \quad \bar{\mathbf{n}} = (n^1, -n^2, n^3). \quad (30)$$

Therefore, the total invariance group of the model is given by

$$G = \text{O}(2)_{\text{diag}} \ltimes T_2, \quad (31)$$

where T_2 corresponds to the two-dimensional translations, $\text{O}(2)_{\text{diag}}$ contains the diagonal parity, and $\ltimes$ denotes a semidirect product.

A. Hedgehog solution

The two-dimensional base space can be parametrized by the polar coordinates (r, ϕ) where $r > 0$ and $\phi \in [0, 2\pi]$ as $x_1 + ix_2 = re^{i\phi}$. We seek solutions within the class of hedgehog maps and can express the hedgehog ansatz as

$$\vartheta = \vartheta(r), \quad \varphi = -Q\phi + \delta, \quad (32)$$

where $\vartheta(r)$ is the profile function of the soliton. Since the field must approach the vacuum $\mathbf{N} = (0, 0, 1)$ at spatial infinity, it satisfies the boundary conditions: $\vartheta(0) = \pi$ and $\vartheta(\infty) = 0$. The phase δ corresponds to the internal orientation of the skyrmion, i.e., a generic rotation in x_1 - x_2 plane. Substituting the ansatz (32) into the energy functional (3) and integrating over φ , we find that

$$\begin{aligned} E = 2\pi \int r dr \left\{ \frac{1}{2}(\vartheta')^2 + \frac{Q^2}{2r^2} \sin^2 \vartheta + h(1 - \cos \vartheta)^p \right. \\ \left. + \frac{\kappa}{\pi(Q+1)} \sin(Q\pi) \sin(\delta - Q\pi) \right. \\ \left. \times \left(\vartheta' - \frac{Q}{2r} \sin(2\vartheta) \right) \right\}, \end{aligned} \quad (33)$$

and the topological charge (17) reduces to

$$\begin{aligned} Q = -\frac{Q}{2} \int \sin(\vartheta) \vartheta' dr = -\frac{Q}{2} \int_{\pi}^0 \sin(\vartheta) d\vartheta \\ = -\frac{Q}{2} [\cos \vartheta]_0^{\pi} = Q. \end{aligned} \quad (34)$$

We note that the contribution of the DM interaction energy is nontrivial only for the rotationally invariant configuration with topological charge $Q = -1$. Indeed we have that

$$\frac{\kappa}{\pi(Q+1)} \sin(Q\pi) \sin(\delta - Q\pi) = \begin{cases} \kappa \sin \delta & \text{if } Q = -1, \\ 0 & \text{if } Q \neq -1. \end{cases} \quad (35)$$

Assuming that the chiral coupling constant κ is positive, the energy functional (33) has a global minimum for the phase $\delta = \pi/2$. It means that the magnetic skyrmion possesses an intrinsic internal orientation. The energy functional (33) for the profile function $\vartheta(r)$ in the topological sector $Q = -1$, with the minimal-energy orientation $\delta = \pi/2$, reduces to

$$\begin{aligned} E = 2\pi \int r dr \left[\frac{1}{2}(\vartheta')^2 + \frac{\sin^2 \vartheta}{2r^2} + \kappa \left(\vartheta' + \frac{\sin(2\vartheta)}{2r} \right) \right. \\ \left. + h(1 - \cos \vartheta)^p \right], \end{aligned} \quad (36)$$

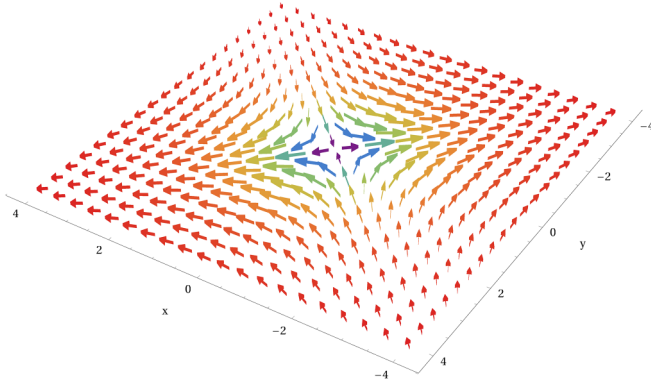


FIG. 1. The magnetization vector (18) for a typical magnetic skyrmion solution.

and the vector plot of the corresponding magnetization vector (18) is shown in Fig. 1. The energy functional with the vorticity and boundary term subtracted off reads as

$$\tilde{E} = 2\pi \int r dr \left[\frac{1}{2}(\vartheta')^2 + \frac{\sin^2 \vartheta}{2r^2} + \kappa(1 - \cos \vartheta) \right. \\ \left. \times \left(\vartheta' - \frac{\sin \vartheta}{r} \right) + h(1 - \cos \vartheta)^p \right], \quad (37)$$

$$\hat{E} = 2\pi \int r dr \left(\frac{1}{2}(\vartheta')^2 + \frac{\sin^2 \vartheta}{2r^2} + \kappa \sin^2(\vartheta) \vartheta' \right. \\ \left. + h(1 - \cos \vartheta)^p \right), \quad (38)$$

respectively. The associated variational equation is

$$\vartheta'' + \frac{1}{r} \vartheta' + \frac{2\kappa}{r} \sin^2 \vartheta - \frac{1}{2r^2} \sin(2\vartheta) \\ - hp \sin \vartheta (1 - \cos \vartheta)^{p-1} = 0. \quad (39)$$

Remarkably, for the possible existence of these solutions, a negative contribution to the energy of the DM interaction is a necessity. Many analytical works studied mathematical properties proving the theoretical existence of these stable solutions in chiral magnets [16,18–21,23,51]. If we had chosen the boundary conditions for ϑ corresponding to those of an antiskyrmion, $\vartheta(0) = -\pi$ and $\vartheta(\infty) = 0$, the kinetic energy and potential energy would remain invariant, but the DM contribution to the energy would flip sign, making the antiskyrmion unstable. We thus understand that the chirality of the DM term selects either the skyrmion or the antiskyrmion according to the sign of $\kappa \sin \delta$.

It will prove useful to consider the asymptotic behavior of the field ϑ found by linearizing (39). In particular, we find

$$\vartheta = CK_1(\sqrt{hr}), \quad p = 1, \\ \vartheta = \frac{C}{r}, \quad p > 1, \quad (40)$$

where C is an undetermined constant and K_1 is the modified Bessel function of the second kind, that tends to zero exponentially. Notice that for the cases with $p > 1$, the profile function

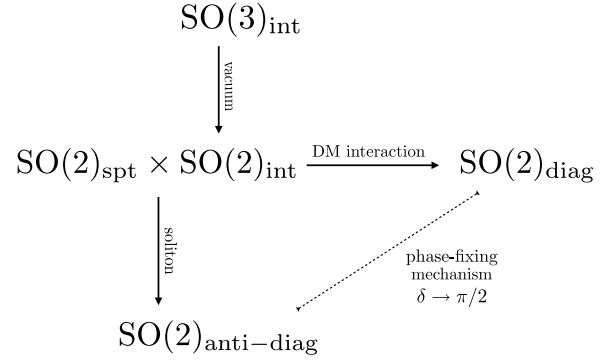


FIG. 2. Schematic of the symmetry-breaking mechanism for the compact symmetry group of the theory.

ϑ tends to its vacuum value with a power law and hence much slower than the cases with a Zeeman term.

At this point, we can analyze the symmetry group of the theory in the background of the soliton. The group of translations in the plane T_2 of Eq. (36) is broken by the soliton (32) and this symmetry breaking is responsible for two moduli defining the position of the magnetic skyrmion. On the other hand, the rotational contribution is more complicated. If we consider the Lagrangian density in the vacuum without the DM term, the solitonic background, as we can verify looking at Eq. (32), breaks the $SO(2)_{\text{spt}} \times SO(2)_{\text{int}}$ group to $SO(2)_{\text{antidiag}}$ (antidiagonal). It means that the unbroken group is $SO(2)$ whose internal angle would be responsible for another modulus, i.e., the orientation δ of the skyrmion. However, the soliton exhibits a similar symmetry-breaking effect as well in the DM interaction. Therefore, by considering both the background of the soliton and the DM interaction together, we observe that further symmetry breaking occurs for the two separate $SO(2)_{\text{diag}}$ groups, resulting in the fixing of the orientation angle δ . As we have previously discussed, the energy is minimized when $\delta = \pi/2$. Figure 2 shows a scheme of the symmetry-breaking effect responsible of fixing the δ phase. Finally, the moduli space $\mathcal{M}_1$ of a two-dimensional magnetic skyrmion has real dimension

$$\dim_{\mathbb{R}}(\mathcal{M}_1) = 2, \quad (41)$$

due only to the translational zero modes.

B. O(3) sigma-model limit

If we keep $\ell \equiv \kappa/h$ fixed and send $\epsilon \equiv \kappa^2/h \rightarrow 0$ we obtain the equation for the sigma-model lump with just the E_2 contribution, which has the equation of motion

$$\vartheta'' + \frac{1}{r} \vartheta' - \frac{1}{2r^2} \sin(2\vartheta) = 0. \quad (42)$$

The solution is

$$\vartheta(r) = 2 \arctan \frac{a}{r}, \quad (43)$$

with a an arbitrary scale of the soliton [size modulus of the O(3) sigma-model lump]. We then evaluate the energy functional for small but finite values of ϵ . Since it was shown in Ref. [20] that $\tilde{E}$ of Eq. (7) is the energy functional leading to

a well-defined variational problem, we will plug in the above lump solution to that energy functional:

$$\tilde{E}(a) = 4\pi - 8\pi\kappa a + \frac{2^p}{p-1} h\pi a^2, \quad (44)$$

and in turn the lump size

$$a = \frac{p-1}{2^{p-2}} \ell. \quad (45)$$

Inserting the lump size back into the energy (44), we find

$$\tilde{E} = 4\pi - 4\pi \frac{(p-1)}{2^{p-2}} \epsilon. \quad (46)$$

We notice that the correction to the energy is linearly proportional to ϵ . This estimate, for $p=2$, predicts that the skyrmion phase is the stable phase (compared with the homogeneous ground state) when $\epsilon > 1$.³ For the $p = \frac{3}{2}$ case, the corresponding critical coupling is $\epsilon^{\text{crit}} = \sqrt{2}$ in the lump approximation (in this case, the solution is not of lump type for ϵ large).

For $p=1$ the solution for the lump (43) evaluated on E_0 diverges [see Eq. (44)]. Introducing a cutoff ($\int_0^\infty dr \rightarrow \int_0^\Lambda dr$) gives

$$\tilde{E}(a) \simeq 4\pi - 8\pi\kappa a + 2\pi a^2 h \log\left(1 + \frac{\Lambda^2}{a^2}\right). \quad (47)$$

Assuming that $\Lambda \gg a$, we can approximate $\log(1 + \Lambda^2/a^2) \simeq 2 \log(\Lambda/a)$. Hence, extremization with respect to a yields

$$a = -\frac{\ell}{\omega(-\frac{\sqrt{e}\ell}{\Lambda})}, \quad (48)$$

where ω is the Lambert W function, i.e., the inverse of the function $f(W) = We^W$, and $\sqrt{e} \simeq 1.649$. Expanding a in small $\ell = \kappa/h$ yields

$$a \simeq \frac{\Lambda}{\sqrt{e}} - \ell + O(\ell^2 \Lambda^{-1}). \quad (49)$$

Unfortunately, it is not possible to remove the infinity leaving behind a positive result, so this estimate should be considered unphysical.

Alternatively, we can consider extracting the “effective” lump size from the $\epsilon \rightarrow 0$ limit, numerically, and regularizing the energy yielding

$$\tilde{E}_{\text{reg}}(a) \simeq 4\pi - 4\pi c\epsilon, \quad (50)$$

with $a = c\ell$. The above considerations suggest that the lump size a goes to zero in the $\epsilon \rightarrow 0$ limit, probably logarithmically or with a slow power law.

C. Restricted model limit

In the case of ℓ fixed and $\epsilon \rightarrow \infty$ we expect that the stable skyrmion is essentially given by a stabilization between E_1

and E_0 , while E_2 is negligible with respect to the other two terms. Thus, the energy functional takes the simplified form⁴

$$E[\mathbf{n}] = \int d^2x \left(\kappa \epsilon_{iab} \partial^i n^a \partial^b n^b + V_p(\mathbf{n}) + \frac{\lambda}{2} (\mathbf{n} \cdot \mathbf{n} - 1) \right). \quad (51)$$

We may call this model the “restricted” magnetic Skyrme model. The equations of motion are

$$h p N^a (1 - \mathbf{n} \cdot \mathbf{N})^{p-1} - 2\kappa \epsilon_{aib} \partial^i n^b - \lambda n^a = 0, \quad (52)$$

where the Lagrange multiplier is

$$\lambda = n^a [h p N^a (1 - \mathbf{n} \cdot \mathbf{N})^{p-1} - 2\kappa \epsilon_{aib} \partial^i n^b]. \quad (53)$$

One interesting aspect of these equations is that they are first order in the derivative. Thus, they can easily be solved analytically. Unlike the restricted baby Skyrme model [42–45], here we do not have an infinite-dimensional symmetry group. The restricted equation for the hedgehog ansatz has no derivatives and can be written as

$$\frac{\kappa}{r} \sin^2 \vartheta - \frac{h p}{2} \sin \vartheta (1 - \cos \vartheta)^{p-1} = 0. \quad (54)$$

Solving for r as a function of ϑ , we obtain

$$r = \frac{2\ell \sin \vartheta}{p(1 - \cos \vartheta)^{p-1}}, \quad (55)$$

where $\ell = \frac{\kappa}{h}$. Some examples are as follows:

$$p=1 : \vartheta(r) = \pi - \arcsin \rho(r), \quad \rho(r) = \frac{r}{2\ell} \quad (56)$$

$$p=\frac{3}{2} : \vartheta(r) = 2 \arccos \rho(r), \quad \rho(r) = \frac{3\sqrt{2}r}{8\ell} \quad (57)$$

⁴One may wonder why we restrict the potential to the form V_p of Eq. (5) and hence do not consider, e.g., $V \propto [1 - (n^3)^2]$. Although such easy-axis potential is often utilized in magnetic systems, they do not allow for the large- ϵ limit because the ground state becomes inhomogeneous [11,52] and requires a different analysis than carried out here.

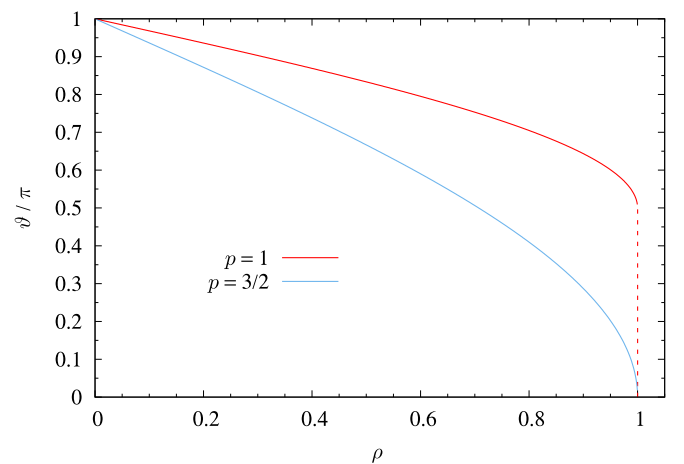


FIG. 3. Compacton (blue) for $p = \frac{3}{2}$ (57) supercompacton (red) for $p=1$ (56). Notice that $\rho(r)$ has a different numerical prefactor of r/ℓ in each case.

³We thank B. Barton-Singer for pointing this out to us.

$$p = 2 : \vartheta(r) = 2 \arctan \frac{1}{\rho(r)}, \quad \rho(r) = \frac{r}{\ell} \quad (58)$$

with $\rho = \rho(r) \in [0, 1)$. For $p = 1$, an interesting feature is present in the theory, a compact and discontinuous solution that we may call a *supercompacton*. As shown in Fig. 3, it differs from the usual compacton of the baby Skyrme model by possessing a discontinuity. For $p < \frac{3}{2}$, the solution is discontinuous and covers only half of the sphere. For $p = \frac{3}{2}$ it is a compacton. For $p > \frac{3}{2}$, $\vartheta(r)$ goes to 0 smoothly as $r \rightarrow \infty$.

For a generic potential the condition for having a supercompacton is not necessarily related to the behavior close to the minimum of the potential. Let us consider, as potential, a generic function $V(\vartheta)$ with minimum at $\vartheta = 0$ whose expansion near the vacuum is given by

$V(\vartheta) = \vartheta^{2p} + \dots$. The solution of the restricted model, in implicit form, is

$$r(\vartheta) = \frac{2\kappa \sin^2 \vartheta}{V'(\vartheta)}. \quad (59)$$

A supercompacton is defined as having a discontinuity in the profile function $\vartheta(r)$. This condition is met when the function $r(\vartheta)$ reaches a maximum at $\vartheta > 0$. Consequently, it is not necessarily related to the power law near the vacuum. In the event that the function $r(\vartheta)$ is monotonic and attains a (finite) maximum at $\vartheta = 0$, the restricted solution is not a supercompacton.

D. Critical coupling

The critical coupling for the case of $p = 2$ can readily be seen from a Bogomol'nyi completion [19]

$$\begin{aligned} E &= 2\pi \int r dr \left[\frac{1}{2}(\vartheta')^2 + \frac{\sin^2 \vartheta}{2r^2} + \kappa \left(\vartheta' + \frac{\sin(2\vartheta)}{2r} \right) + h(1 - \cos \vartheta)^2 \right] \\ &= 2\pi \int r dr \left[\frac{1}{2} \left(\vartheta' - \frac{\sin \vartheta}{r} + \kappa(1 - \cos \vartheta) \right)^2 + \frac{\sin \vartheta}{r} \vartheta' + \frac{\kappa}{r} \partial_r(r \sin \vartheta) \right] \\ &= \pi \int r dr \left(\vartheta' - \frac{\sin \vartheta}{r} + \kappa(1 - \cos \vartheta) \right)^2 + 4\pi Q + 4\pi \Omega, \end{aligned} \quad (60)$$

which holds for $h = \frac{\kappa^2}{2}$ and

$$Q = \frac{1}{2} \int dr \sin(\vartheta) \vartheta' = -1, \quad (61)$$

$$\Omega = \frac{\kappa}{2} \int dr \partial_r(r \sin \vartheta) = 2, \quad (62)$$

which are the topological charge and a boundary term, respectively, where $\vartheta(0) = \pi$ and $\vartheta(\infty) = 0$. The Bogomol'nyi completion can also be understood as the BPS limit of the gauged $\mathbb{C}P^1$ model with a particular choice of constant gauge field $A_t^a = -\kappa \delta_t^a$, that generates the DM term [19]. The energy (60) shows the Bogomol'nyi or BPS equation

$$\vartheta' - \frac{\sin \vartheta}{r} + \kappa(1 - \cos \vartheta) = 0, \quad (63)$$

as well as the topological energy bound

$$E \geq 4\pi(Q + \Omega) = 4\pi, \quad (64)$$

with equality that holds only when the BPS equation is satisfied. The BPS equation is solved by

$$\vartheta = 2 \arctan \left(\frac{2}{\kappa r} \right). \quad (65)$$

This is the $O(3)$ sigma-model lump solution with lump size $a = \frac{2}{\kappa}$. We also notice that the BPS bound on the energy is 4π , just like the lump energy. Finally, we have used the asymptotic behavior of the solution $\vartheta \sim \frac{4}{\kappa r}$ at $r \rightarrow \infty$ in evaluating the boundary term Ω in Eq. (62).

We note that the energy $\tilde{E}$ of Eq. (7), which has the vorticity subtracted off, leads to a different Bogomol'nyi bound [20]:

$$\tilde{E} \geq -4\pi. \quad (66)$$

E. Analytic solution at any coupling

We note that the Bogomol'nyi completion requires us to select the critical coupling $h = \frac{\kappa^2}{2}$ enabling us to complete the square in the energy functional. This choice of coupling corresponds to $\epsilon = 2$. However, another interesting fact about the Zeeman squared potential ($p = 2$) is that inserting the ansatz $\vartheta = 2 \arctan \frac{a}{r}$ into the full second-order Euler-Lagrange equation gives [19,21,50]

$$\frac{8a^2 r (\kappa - ah)}{(a^2 + r^2)^2} = 0, \quad (67)$$

which beautifully has the solution $a = \frac{\kappa}{h} = \ell$. This size does not contradict the BPS solution since the first-order BPS equation required $h = \frac{\kappa^2}{2}$ which in the full second-order solution corresponds to the lump size $a = \frac{\kappa}{h} = \frac{2}{\kappa}$. It is interesting, however to insert the exact solution into the energy functional (36), which gives [19,20]

$$E = 2\pi \int r dr \frac{4h^2 \kappa^2}{(h^2 r^2 + \kappa^2)^2} = 4\pi. \quad (68)$$

But this is nothing but the lump energy. We see that the $p = 2$ case is a rather special case, where the energy is always the lump energy and the solution is always the lump solution

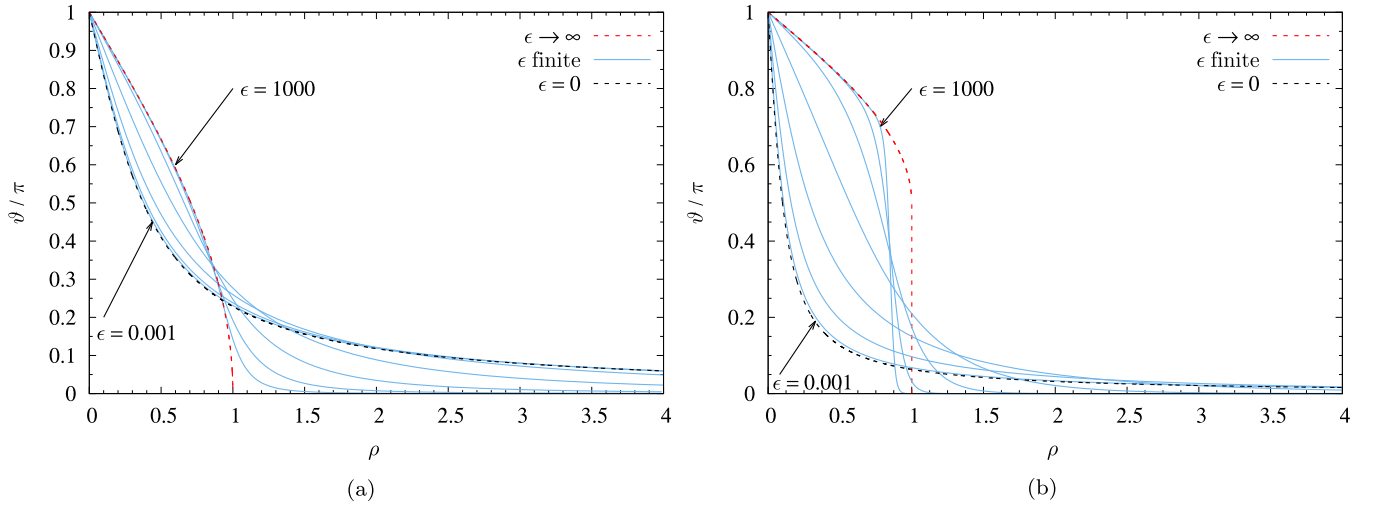


FIG. 4. Profile function $\vartheta(\rho)$ for (a) $p = \frac{3}{2}$ and (b) $p = 1$ with $\epsilon = 0.001, 0.01, 0.1, 1, 10, 100, 1000$. The red-dashed curve corresponds to the restricted limit, whereas the black-dashed curve is the sigma-model lump limit with lump sizes (a) $a = \frac{\ell}{\sqrt{2}}$ and (b) $a = \ell/5$. The latter is chosen for illustrative purposes.

with the lump size $a = \ell = \frac{\kappa}{h}$. However, only at the critical coupling, the second-order equation of motion reduces to the first-order BPS equation (63).⁵

The $p = 2$ case is thus applicable to many more physical systems than just the critically coupled one [19,21,50]. Having the analytic solution in hand may be advantageous for further studies, including interaction and scattering, etc.

Using the energy functional $\tilde{E}$ of Eq. (7), where the vorticity is subtracted off, the energy becomes instead

$$\tilde{E} = 2\pi \int r \, dr \frac{4h\kappa^2(h - \kappa^2)}{(h^2r^2 + \kappa^2)^2} = 4\pi(1 - \epsilon). \quad (69)$$

We may interpret the change of sign of the energy at $\epsilon = \epsilon^{\text{crit}} = 1$ as the skyrmionic phase being energetically preferable for $\epsilon > 1$ and the homogeneous phase on the other hand preferable for $\epsilon < 1$. Luckily, the BPS case resides in the skyrmionic phase at $\epsilon = 2$.

Since the $p = 2$ case is an exact solution for all values of ϵ , there is no need to perform numerical computations for this case.

F. Numerical solutions

For the general theory, we can solve Eq. (39) using a standard shooting method, beginning from the origin with the initial condition $\vartheta(0) = \pi$. The form of the solution close to the origin is $\vartheta(r) \simeq \pi - \gamma r$. By varying the angular coefficient γ , we find only one solution in the topological sector $Q = -1$ as already shown in the literature [16,18–21,23,51].

Let us now study how the theory behaves by varying the control parameters ℓ and ϵ . We plot the solutions ϑ for a variety of couplings ϵ for $p = \frac{3}{2}$ [Fig. 4(a)] and $p = 1$ [Fig. 4(b)]. In each case, we plot ϑ as a function of ρ which is proportional to r/ℓ , such that ℓ dependence is removed from the

solutions and ρ is defined in Eqs. (56) and (57). We see that as ϵ goes from 0 to ∞ the solution flows from the sigma-model lump solutions (black dashed lines) to the restricted (compacton) solution (red dashed lines) in both cases. In the $p = \frac{3}{2}$ case, the lump solution (43) with a lump size (45) is shown and the restricted solution is that of Eq. (57). We notice that the numerical solutions tend to the lump solution with the correct lump size (45), where a boundary term has been discarded when determining the lump size from the energy functional. In the $p = 1$ case, on the other hand, we have set the lump size to $a = \frac{\ell}{5}$ by hand for visualization purposes and the restricted supercompacton solution that the solutions flow to for large ϵ is given in Eq. (56). We see again that the lump solution matches well with the numerical solutions, but the lump size is set by hand for comparison with the solution at small $\epsilon = 10^{-3}$. We will shortly argue that the lump size should go to zero in the limit $\epsilon \rightarrow 0$. Note that the supercompacton has a discontinuity and covers only half of the 2-sphere, from the south pole to the equator. At finite ϵ , the discontinuity is resolved and topological soliton covers the entire sphere. As $\epsilon \rightarrow \infty$, a step-function-type discontinuity is created and the missing half of the sphere is mapped into a very narrow region.

Figure 5 shows the topological charge density that integrates to $Q = -1$, in the $p = \frac{3}{2}$ case [Fig. 5(a)] and in the $p = 1$ case [Fig. 5(b)].

Figure 6 shows the energy densities for the sigma-model lump limit ($\epsilon \rightarrow 0$) in Figs. 6(a) and 6(b) with and without the boundary term for $p = \frac{3}{2}$ and 1, respectively, and for the restricted limit ($\epsilon \rightarrow \infty$) in Figs. 6(c) and 6(d) for $p = \frac{3}{2}$ and 1, respectively, in Figs. 6(e) and 6(f) $p = \frac{3}{2}$ and 1, respectively, with the boundary term (9) subtracted off. In both the charge density as well as in the energy density, the discontinuity that develops in the $p = 1$ case for $\epsilon \rightarrow \infty$ is clearly seen in Figs. 5(b) and 6(d), respectively. We note that the energy density is divergent (singular) in the $p = \frac{3}{2}$ case at $\rho = 1$ when the boundary term is taken into account [see Fig. 6(c)], but is convergent (finite) when the boundary term is dropped [see

⁵The fact that this analytic solution holds for all values of the coupling is somewhat similar to the situation in the baby Skyrme model for $p = 4$ [42,53].

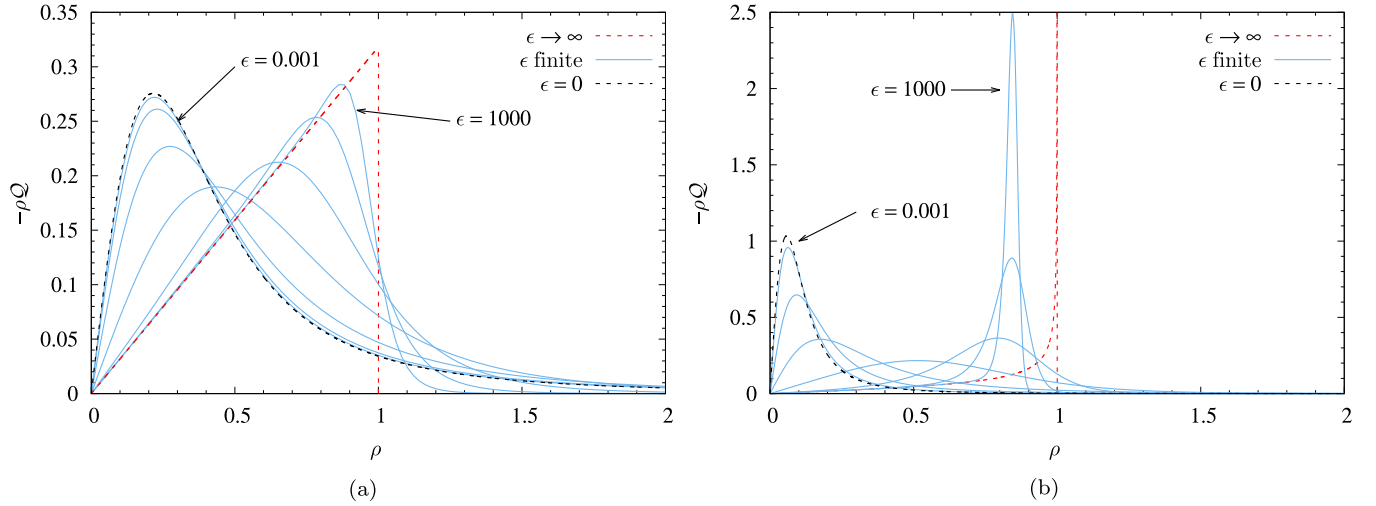


FIG. 5. Topological charge density $\mathcal{Q}$ for (a) $p = \frac{3}{2}$ and (b) $p = 1$ with $\epsilon = 0.001, 0.01, 0.1, 1, 10, 100, 1000$. The topological charge densities of the lumps and restricted skyrmions are shown with dashed black and red curves, respectively. In the $p = 1$ case, the size of the lump is selected for illustrative purpose, corresponding to the solutions in Fig. 4.

Fig. 6(e)]. We also note that the total energy (the integral) is unchanged in the $p = 1$ case [see Figs. 6(d) and 6(f)], but the energy density is slightly modified by the presence of the boundary term.

We show the total energy of the numerical solutions in the lump limit ($\epsilon \rightarrow 0$) for the $p = \frac{3}{2}, 1$ cases in Fig. 7(a). We notice that the $p = \frac{3}{2}$ case is well described by the energy functional (46) with the correct lump size (45). This confirms that discarding the boundary term (9) is appropriate for computing the energies in the lump limit, which is also the case where the variational problem is well defined [20]. In the $p = 1$ case, we show the regularized energy (47) with a dashed red curve and lump size a as chosen in Fig. 4. We show the sign change of the energy in the $p = \frac{3}{2}, 1$ case in Fig. 7(b) which appears to take place around $1 < \epsilon < 2$, which is consistent with the lump prediction (46) which for $p = \frac{3}{2}$ should take place at $\epsilon = \epsilon^{\text{crit}} = \sqrt{2}$.

Finally, we consider the difficult problem of estimating the lump size in the $\epsilon \rightarrow 0$ limit. Since we can see from Fig. 4 that the profiles match the lump solution if the lump size is fitted correctly, we can extract this lump size from the numerical solutions by fitting the size of the lump solution to the numerical ones. Figure 8 shows the numerically extracted lump sizes and a logarithmic fit finding an approximate behavior of the lump size as $c \approx 1.291(-\log \epsilon)^{-0.918}$. Clearly, the lump size goes very slowly to zero.

G. Comparison with experiments

We have confirmed that the magnetic skyrmion model, up to rescaling of energy and length, is essentially a one-parameter model (27) and (28). This parameter ϵ controls the relation between E_2 and $E_1 + E_0$. In both the large- ϵ and the small- ϵ limits, analytic approximations are available (see Secs. IIB and IIC). It may be interesting to explore various phenomenological realizations of chiral magnets and determine the order of magnitude of the parameter ϵ that can be realized.

The typical skyrmion size, determined by the DM interaction, in chiral magnets, e.g., MnSi [28,33], is about $\ell \simeq 5\text{--}100$ nm. From the phase diagram for MnSi [26], the range for the external magnetic field in the skyrmion phase is $B \sim 0.1\text{--}0.3$ T. Another important experimental parameter is λ , the period of the spiral, determined by the ratio of the dipolar and exchange interactions:

$$\lambda = \frac{2\pi J}{D}. \quad (70)$$

Such parameter is well known for different chiral magnets (see, e.g., Table 1 in Ref. [33]), in the case of MnSi: $\lambda = 18$ nm. From Eq. (25), we get

$$\begin{aligned} J &= \frac{\lambda D}{2\pi} = \frac{\lambda \ell |B|}{4\pi}, \\ \epsilon &= \frac{4D^2}{J|B|} = 4\pi \frac{\ell}{\lambda}. \end{aligned} \quad (71)$$

Using the minimal values for the previous parameters (for MnSi) defined on a range of values, i.e., $B \sim 10^{-1}$ T, $\ell \sim 10$ nm, and $\lambda \simeq 10$ nm, we find

$$\begin{aligned} J &\simeq 100 \text{ nm}^2 \times \text{T} \simeq 10^{-14} \text{ J}, \\ \epsilon &\simeq 10, \end{aligned} \quad (72)$$

where J is the unit *joule*, not to be confused with the interaction strength of the Heisenberg Hamiltonian. Note that in general ℓ can change significantly, viz., one order of magnitude. For instance, if we fix $\ell = 100$ nm and $\lambda \in O(10^1)$ nm as before, we find that ϵ changes one order of magnitude, i.e., $\epsilon \simeq 100$. For materials other than MnSi, e.g., $\text{Fe}_{1-x}\text{Co}_x\text{Si}$ or $\text{Mn}_{1-x}\text{Fe}_x\text{Ge}$, λ can also be of the order $O(10^2)$ nm. Therefore, considering the other two cases where λ is fixed as $\lambda \sim 10^2$ nm and ℓ takes its maximum and minimum values, we find the relevant ranges for ϵ :

$$\begin{aligned} 1 &\lesssim \epsilon \lesssim 10, & \ell &\simeq 10 \text{ nm}, \\ 10 &\lesssim \epsilon \lesssim 100, & \ell &\simeq 100 \text{ nm}. \end{aligned} \quad (73)$$

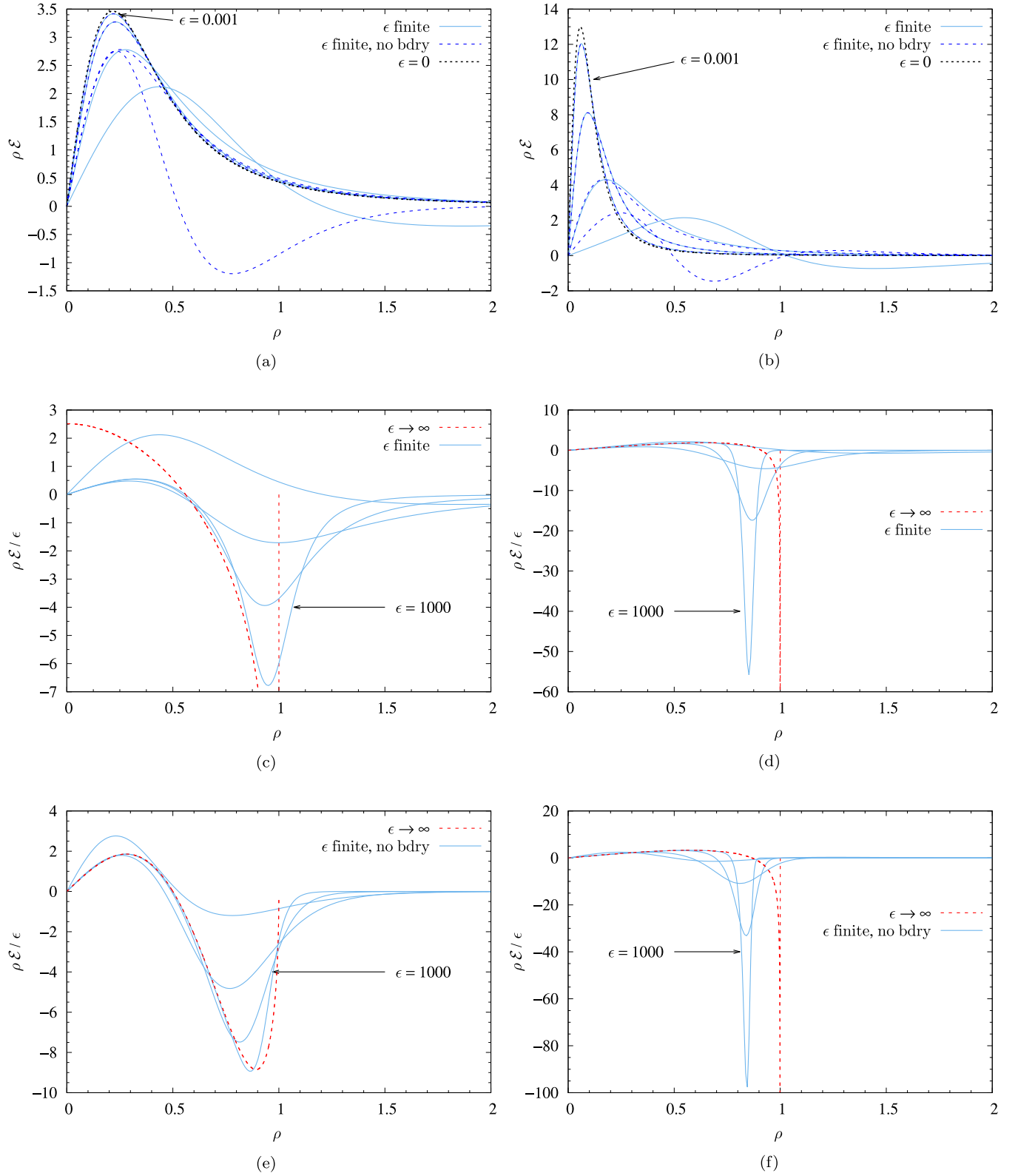


FIG. 6. Energy density: (a) $p = \frac{3}{2}$, (b) $p = 1$ for $\epsilon = 0.001, 0.01, 0.1, 1$ (solid blue lines) and with the boundary term (9) subtracted off (dashed dark-blue lines). Energy density: (c) $p = \frac{3}{2}$, (d) $p = 1$ for $\epsilon = 1, 10, 100, 1000$ and finally (e) $p = \frac{3}{2}$, (f) $p = 1$ and $\epsilon = 1, 10, 100, 1000$ with the boundary term (9) subtracted off.

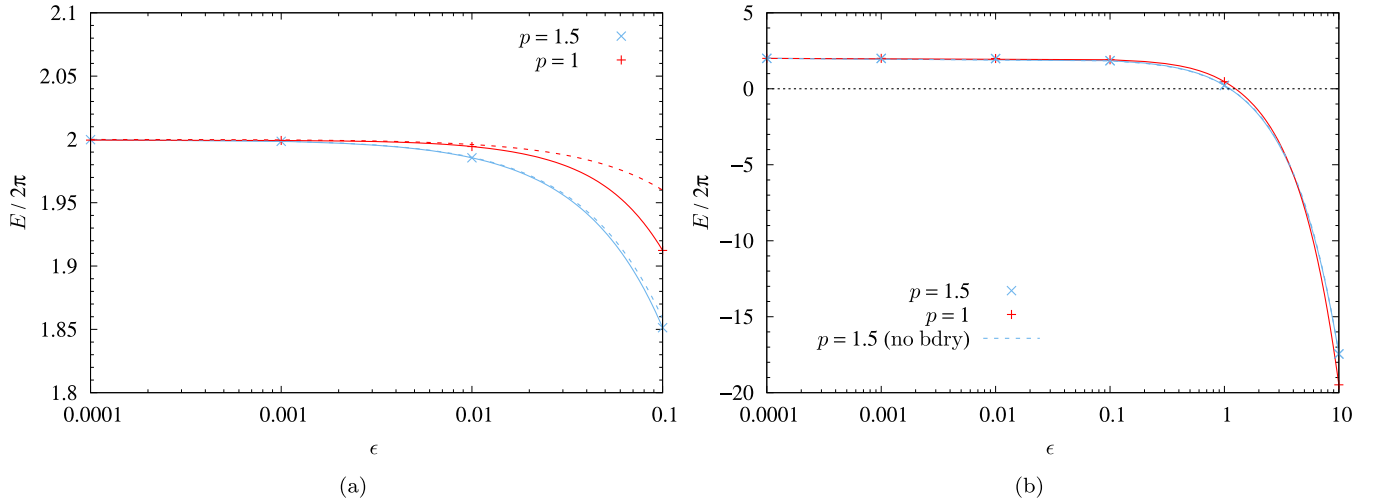


FIG. 7. Energy in the lump limit ($\epsilon \rightarrow 0$) as a function of ϵ for $p = \frac{3}{2}$ and 1. (a) The blue dashed line corresponds to the analytical energy formula (46) for the lump in the limit of small but nonvanishing ϵ in the case of $p = \frac{3}{2}$. The red dashed line instead is the energy of Eq. (50) with the lump size taken from Fig. 4. (b) The energy shown up to larger values of ϵ revealing the sign change near $1 < \epsilon < 2$. The case $p = \frac{3}{2}$ is shown for the full energy (solid blue line) and with the boundary term (9) subtracted off (dashed blue line).

Finally, the approximating limits studied in this work can be seen to be relevant for describing certain experimental 2D magnetic skyrmions within the ranges given in the above equation.

In Secs. IID and IIE, we saw that in the special physical case of $p = 2$, which describes chiral magnets with anisotropy interactions, any spherical magnetic skyrmion can be represented by the analytical lump solution, regardless of its DM coupling κ . However, the potential $V_{p=2}$ corresponds to the sum of a magnetic field and anisotropy (easy-plane) contributions. To realize this potential, the couplings of the anisotropy term must be exactly half of that of the Zeeman term. It turns out to be possible [15]. Furthermore, since the critical coupling regime corresponds to the condition $\forall \ell, \epsilon = 2$, which satisfies the inequalities (73), we suggest that, in principle, a BPS magnetic skyrmion could be realized experimentally.

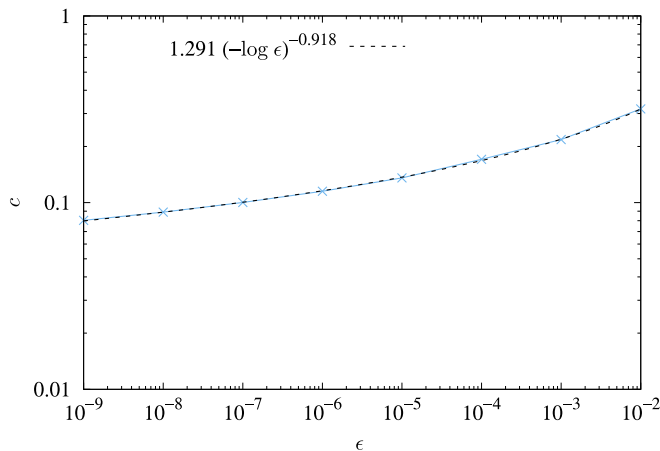


FIG. 8. The lump size $c = \frac{a}{\ell}$ estimated by fitting the lump solution to the numerical solutions. The dashed black curve is a logarithmic fit.

III. CONCLUSION

In this paper, we have studied magnetic skyrmions in chiral magnets with three different potentials, corresponding to $p = 1, \frac{3}{2}$, and 2 in the limits of the control parameter $\epsilon = \frac{\kappa^2}{h} = \frac{4D^2}{|B|J}$ becoming very large and going to zero. The $p = 1$ case is simply the Zeeman potential term and the $p = 2$ case is a particular combination of the easy-plane (anisotropy) and the Zeeman potentials. The $p = \frac{3}{2}$ case is probably somewhat more academic. The ϵ vanishing limit reduces the skyrmion to a sigma-model lump with a lump size that we accurately estimated, except for the case of the Zeeman term ($p = 1$). The large- ϵ limit instead corresponds to the “restricted” magnetic skyrmions, for which we found the exact analytic solutions as well. In the $p = \frac{3}{2}$ case, the restricted solution is a compacton, similar in nature to those found in the baby Skyrme [43–45] and 3D Skyrme models [46]. However, in the $p = 1$ case corresponding to the Zeeman term, we find a new type of restricted solution, which we dub a *supercompacton*, since it becomes discontinuous. We provide numerical solutions showing that they slowly tend to the supercompacton limit for very large values of ϵ . We have also reviewed the beautiful result of the critically coupled magnetic skyrmion of Barton-Singer-Ross-Schroers [19], where an analytic solution is available, hence providing many studies with a clear advantage. Their solution is known to be an exact solution for any value of the coupling ϵ [19,21,50], but we find here that dropping the boundary term that leads to the correct lump size for this solution also changes the energy from the lump energy for all values of ℓ to the energy functional $4\pi(1 - \epsilon)$. This energy functional changes sign at $\epsilon = 1$ suggesting $\epsilon = 1$ being the critical coupling for having the skyrmionic phase. We finally emphasize that our study, within the defined restricted limit, provides analytical solutions for magnetic skyrmions in certain magnetic material, as shown in Sec. IIG.

Future directions that would be interesting to study would be to utilize the exact analytic solutions for dynamical

problems such as acceleration, interaction, and scattering. Extending the planar materials to 3D materials may also be interesting, where numerical computations have found braiding magnetic skyrmion solutions [54]. Using various limits, perhaps analytic insight can be found also in such cases.

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APPENDIX: CONTINUUM LIMIT

The Heisenberg term of the GL energy density can be derived from the continuum limit of the lattice Hamiltonian

$$\mathcal{H}_H = -J \sum_{i,\mu} \mathbf{n}_i \cdot \mathbf{n}_{i+a\hat{\mathbf{e}}_\mu}, \quad (\text{A1})$$

where $a\hat{\mathbf{e}}_\mu$ is the vector connecting the i lattice site with its neighboring sites $i + a\hat{\mathbf{e}}_\mu$, on a square ($\mu = x, y$) lattice. Assuming a small lattice constant a , one can perform a Taylor expansion on $\mathbf{n}_{i+a\hat{\mathbf{e}}_\mu}$, i.e.,

$$\mathbf{n}_{i+a\hat{\mathbf{e}}_\mu} \simeq \mathbf{n}_i + a\partial_\mu \mathbf{n}_i + (a^2/2)\partial_\mu^2 \mathbf{n}_i + \dots \quad (\text{A2})$$

and insert it into Eq. (A1). One finds that first term $\mathbf{n}_i \cdot \mathbf{n}_i$ is equal to the unity and can be discarded as a constant term.

The second term, proportional to $\mathbf{n}_i \cdot \partial_\mu \mathbf{n}_i$, vanishes due to nonlinear constraint $\mathbf{n} \cdot \mathbf{n} = 1$. However, the third term

$$\mathcal{H}_H \approx -\frac{1}{2}Ja^2 \sum_i \mathbf{n}_i \cdot \partial_\mu^2 \mathbf{n}_i \quad (\text{A3})$$

in the continuum limit $a \rightarrow 0$, and after an integration by parts, leads to a nonvanishing contribution, i.e.,

$$E_2[\mathbf{n}] = \frac{1}{2}J \int d^2x (\partial_\mu \mathbf{n} \cdot \partial_\mu \mathbf{n}), \quad \mathbf{n} \cdot \mathbf{n} = 1. \quad (\text{A4})$$

The DM interaction at the lattice level can be written as

$$\mathcal{H}_{DM} = -D \sum_{i,\mu} \hat{\mathbf{e}}_\mu \cdot (\mathbf{n}_i \times \mathbf{n}_{i+a\hat{\mathbf{e}}_\mu}), \quad (\text{A5})$$

where we expressed the DM vector as $\mathbf{D} = D\hat{\mathbf{e}}_\mu$. Upon Taylor expansion, the first derivative of $\mathbf{n}_{i+a\hat{\mathbf{e}}_\mu}$ provides the non-null contribution

$$\mathcal{H}_{DM} \approx Da \sum_{i,\mu} \mathbf{n}_i \cdot (\hat{\mathbf{e}}_\mu \times \partial_\mu \mathbf{n}_i) = Da \sum_i \mathbf{n}_i \cdot (\nabla \times \mathbf{n}_i), \quad (\text{A6})$$

where at the second step we have applied the vector definition for the curl operator. Thus, in the continuum limit, one finds

$$E_1[\mathbf{n}] = Da^{-1} \int d^2x \mathbf{n} \cdot (\nabla \times \mathbf{n}), \quad (\text{A7})$$

which corresponds to the usual DM energy functional. Finally, let us consider the potential-like term $-\mathbf{B} \cdot \mathbf{n}$ that describes the Zeeman interaction with an external magnetic field, which we choose to be aligned along the third direction $\mathbf{B} = B\mathbf{e}_3$. Since we have arbitrarily chosen the vacuum field $\mathbf{N} = \mathbf{e}_3$, and we need the potential to vanish when evaluated on the vacuum, the energy contribution can be written as $E_0[\mathbf{n}] = B(1 - \mathbf{N} \cdot \mathbf{n}) = B(1 - n_3)$. Such potential contribution can be generalized to a class of potentials V_p of the form (5). At this point, by considering all the three contributions to the whole energy functional $E[\mathbf{n}]$ one gets Eq. (3).

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