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The field of moduli of sets of points in $\mathbb{P}^2$

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Abstract. For every $n \geq 6$, we give an example of a finite subset of $\mathbb{P}^2$ of degree n which does not descend to any Brauer–Severi surface over the field of moduli. Conversely, for every $n \leq 5$, we prove that a finite subset of degree n always descends to a 0-cycle on $\mathbb{P}^2$ over the field of moduli.

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Let k be a field with separable closure K, and $S \subset \mathbb{P}^2(K)$ a finite subset of order n. The *field of moduli* k_S of S is the subfield of K of elements fixed by Galois automorphisms $\sigma \in \mathrm{Gal}(K/k)$ such that $\sigma(S)$ is linearly equivalent to S, i.e., such that there exist $g \in \mathrm{PGL}_3(K)$ with $g(\sigma(S)) = S$. We study the problem of whether S descends to a 0-cycle on $\mathbb{P}^2_{k(S)}$, or more generally on a Brauer–Severi surface over k_S.

A. Marinatto [18] studied the analogous problem over $\mathbb{P}^1$. He showed that, if n is odd or equal to 4, then S descends to a divisor over $\mathbb{P}^1_{k_S}$. Furthermore, he has given counterexamples where S does not descend to $\mathbb{P}^1_{k_S}$ for every $n \geq 6$ even. All of his counterexamples descend to a Brauer–Severi curve, though. In [7], we have shown that if $n = 6$, then S always descends to some Brauer–Severi curve, while there are counterexamples for every $n \geq 8$ even.

Fields of moduli of curves, possibly with marked points, received a lot of attention, see [11–13, 15–17]. Furthermore, there are results about abelian varieties, most famously Shimura's result that a generic, principally polarized, odd dimensional abelian variety is defined over the field of moduli [21], and about fields of moduli of curves in $\mathbb{P}^2$ [1–5, 20]. Here is our result.

Theorem 1. *Assume* char $k \neq 2$. *Let $S \subset \mathbb{P}^2(K)$ be a finite set of n points with field of moduli k_S. If $n \leq 5$, then S descends to a finite subscheme of $\mathbb{P}^2_{k_S}$. For every $n \geq 6$, there exists a subset $S \subset \mathbb{P}^2(\mathbb{C})$ with field of moduli equal to $\mathbb{R}$ which does not descend to $\mathbb{P}^2_{\mathbb{R}}$.*

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Notice that $\mathbb{P}^2_{\mathbb{R}}$ is the only Brauer–Severi surface over $\mathbb{R}$, hence our counterexamples do not descend to any Brauer–Severi surface over $\mathbb{R}$.

1. Notation and conventions. Given a field k, we write $\mathbb{P}^n_k$ for the projective space as a scheme over k. If k'/k is an extension, then $\mathbb{P}^n_{k'} = \mathbb{P}^n_k \times_k \operatorname{Spec} k'$ and $\mathbb{P}^n_k(k) \subset \mathbb{P}^n_k(k') = \mathbb{P}^n_{k'}(k')$. Because of this, with an abuse of notation, we sometimes drop the subscript and just write $\mathbb{P}^n(k)$ and $\mathbb{P}^n(k')$.

Let $Z \subset \mathbb{P}^2$ be a closed subscheme, and $g \in \mathrm{PGL}_3(K)$ a projective linear map. We say that g *stabilizes* Z, or that Z is g-invariant, if $g(Z) = Z$. We say that g *fixes* Z if $g(Z) = Z$ and $g|_Z : Z \to Z$ is the identity. If $G \subset \mathrm{PGL}_3(K)$ is a finite subgroup, we say that G stabilizes (resp. fixes) Z if every element $g \in G$ stabilizes (resp. fixes) Z. The *fixed locus* of g (resp. G) is the subspace of points $x \in \mathbb{P}^2$ with $gx = x$ (resp. $\forall g \in G : gx = x$).

Let $S \subset \mathbb{P}^2(K)$ be a finite subset with finite automorphism group. Up to replacing k with k_S, we may assume that k is the field of moduli. We recall some definitions from [9].

A *twisted form* of $(\mathbb{P}^2_K, S)$ over a k-scheme M is the datum of a projective bundle $P \to M$ and a closed subscheme $Z \subset P$ such that (P_K, Z_K) is étale locally isomorphic to $(\mathbb{P}^2_K, S) \times_K M_K$, i.e., there exists an étale cover $M' \to M_K$ and an isomorphism

$$(\mathbb{P}^2_K, S) \times_K M' \simeq (P, Z) \times_M M' = (P_K, Z_K) \times_{M_K} M'$$

over M'. Notice that if we do not assume that k is the field of moduli, this definition is not correct since $(\mathbb{P}^2_K, S)$ would not define in general a twisted form of $(\mathbb{P}^2_K, S)$.

The fibered category $\mathscr{G}_S$ of twisted forms of $(\mathbb{P}^2_K, S)$ is a finite gerbe over $\operatorname{Spec} k$ called the *residual gerbe* of S, see [9]. Namely, for a scheme M over k, the objects of the groupoid $\mathscr{G}_S(M)$ are twisted forms (P, Z) of $(\mathbb{P}^2_K, S)$ over M, and arrows $(P, Z) \to (P', Z')$ are given by isomorphisms $\phi : P \to P'$ over M with $\phi(Z) = Z'$. The *universal bundle* $\mathscr{P}_S \to \mathscr{G}_S$ is the fibered category defined as follows: the objects of $\mathscr{P}_S(M)$ are triples (P, Z, s) where (P, Z) is a twisted form of $(\mathbb{P}^2_K, S)$ over M, and s is a section $M \to P$ of $P \to M$, and arrows are defined analogously. The base change of $\mathscr{P}_S \to \mathscr{G}_S$ to K are the quotient stacks $[\mathbb{P}^2/\underline{\mathrm{Aut}}_K(S)] \to [\operatorname{Spec} K/\underline{\mathrm{Aut}}_K(S)]$. See [9] for more details.

Another way of constructing $\mathscr{P}_S \to \mathscr{G}_S$ is the following. Let $\mathscr{N}_S \subset \mathrm{Aut}_k(\mathbb{P}^2_K)$ be the subgroup of k-linear automorphisms τ of $\mathbb{P}^2_K$ such that $\tau(S) = S$, the fact that k is the field of moduli implies that $\mathscr{N}_S$ is an extension of $\mathrm{Gal}(K/k)$ by $\mathscr{N}_S \cap \mathrm{Aut}_K(\mathbb{P}^2_K) = \mathrm{Aut}_K(\mathbb{P}^2, S)$ (see [6, §3] for details). We have an induced action of $\mathscr{N}_S$ on $\operatorname{Spec} K$ with the natural projection $\mathscr{N}_S \subset \mathrm{Aut}_k(\mathbb{P}^2_K) \to \mathrm{Gal}(K/k)$, and the finite étale gerbe $\mathscr{G}_S$ is the quotient stack $[\operatorname{Spec} K/\mathscr{N}_S]$: the natural map $\operatorname{Spec} K \to \mathscr{G}_S$ associated with the trivial twist of S is a pro-étale, Galois covering with Galois group equal to $\mathscr{N}_S$. Similarly, we can view $\mathscr{P}_S$ as the quotient stack $[\mathbb{P}^2_K/\mathscr{N}_S]$.

Twisted forms of $(\mathbb{P}^2_K, S)$ contained in Brauer–Severi surfaces over k correspond to rational points of $\mathscr{G}_S$. If $|S|$ is prime with 3 and S descends to a 0-cycle over some Brauer–Severi surface P over k, then $P \simeq \mathbb{P}^2_k$. In fact, if D is a canonical divisor on P, then $D \cdot D$ defines a 0-cycle of degree 9 on P, hence P has index 1 (recall that the index is the greatest common divisor of the degrees of 0-cycles on P). This implies that P has a rational point and $P \simeq \mathbb{P}^2_k$, see e.g. [14, Corollary 5.3.6, Theorem 5.1.3].

Denote by $\mathbf{P}_S$ the coarse moduli space of $\mathscr{P}_S$, i.e., $\mathbb{P}^2_K/\mathscr{N}_S$, since the action of $\mathrm{Aut}_K(S)$ on $\mathbb{P}^2_K$ is faithful, the natural map $\mathscr{P}_S \to \mathbf{P}_S$ has a birational inverse $\mathbf{P}_S \dashrightarrow \mathscr{P}_S$ which, by composition, gives us a rational map $\mathbf{P}_S \dashrightarrow \mathscr{G}_S$.

2. Case $n \leq 5$. It is well known that any set of 4 points in $\mathbb{P}^2$ in general position (i.e., such that no line contains 3 of them) can be mapped by a projective linear transformation in a subset of $\{(1:0:0),(0:1:0),(0:0:1),(1:1:1)\}$. Similarly, if $n \leq 4$ and at most three points are aligned, the set can be mapped by a projective linear transformation in a subset of $\{(1:0:0),(0:1:0),(1:1:0),(0:0:1)\}$. In both these cases, we thus get that S descends to a finite subscheme of $\mathbb{P}^2_k$.

Assume that $n = 4$ and all points are contained in a line. Up to a change of coordinates, we may assume that S is contained in the line $L = \{(x : y : 0)\} \simeq \mathbb{P}^1_K$, and regard it as a divisor of degree 4 on $\mathbb{P}^1_K$. Notice that the subgroup $\mathrm{GL}_2(K) \subset \mathrm{PGL}_3(K)$ acting on the first two coordinates maps surjectively on the group $\mathrm{PGL}_2(K)$ of projective linear transformations of L, hence every linear transformation of L extends to a linear transformation of $\mathbb{P}^2_K$. By [7, Proposition 13], we may thus find $g \in \mathrm{GL}_2(K) \subset \mathrm{PGL}_3(K)$ such that $g(S) \subset L = \mathbb{P}^1_K \subset \mathbb{P}^2_K$ is Galois invariant with respect to the standard Galois action of $\mathrm{Gal}(K/k)$ on $\mathbb{P}^1_K(K) \subset \mathbb{P}^2_K(K)$; it follows that $g(S)$ descends to a finite subscheme of $\mathbb{P}^2_k$ in this case, too.

Assume $n = 5$. Let $S \subset \mathbb{P}^2(K)$ be a finite subset of degree 5 with field of moduli k, since 5 is prime with 3, it is enough to show that $\mathscr{G}_S(k) \neq \emptyset$. We split the analysis in three cases: either S contains 4 points in general position, or it is contained in the union of two lines each containing at least three points of S, or it is contained in the union of a line and a point.

2.1. S contains 4 points in general position. Since we are assuming that there are 4 points of S in general position, there are two possibilities: either all 5 points are in general position, i.e., there is no line containing 3 of them, or there is a unique line containing exactly 3 points of S. Denote by C the unique non-degenerate conic passing through all the points of S in the first case, while in the second case C is the unique line containing 3 points of S.

In any case, C is a rational curve uniquely determined by S. Because of this, $\mathscr{N}_S$ stabilizes C, consider the quotient $\mathscr{C} = [C/\mathscr{N}_S] \subset \mathscr{P}_S$ and let $\mathbf{C} = C/\mathscr{N}_S$ be the coarse moduli space of $\mathscr{C}$. Notice that, since $C \cap S \geq 3$, the subgroup of $\mathrm{Aut}_K(\mathbb{P}^2, S)$ fixing C has at most 2 elements, hence the map $\mathscr{C} \to \mathbf{C}$ is either birational or generically a gerbe of degree 2. In any case, since char $k \neq 2$ by the Lang-Nishimura theorem for tame stacks [10, Theorem 4.1] applied to a birational inverse $\mathbf{P}_S \dashrightarrow \mathscr{P}_S$ and to the generic point of $\mathbf{C} \subset \mathbf{P}_S$, we get a generic section $\mathbf{C} \dashrightarrow \mathscr{C} \subset \mathscr{P}_S$.

The curve $\mathbf{C}$ is a Brauer–Severi variety of dimension 1 over k, and any canonical divisor has degree -2, hence the index of $\mathbf{C}$ is either 1 or 2. Since $C \cap S$ has odd degree, there exists an odd d such that $C \cap S$ contains an odd number of orbits of degree d, let $O \subset C \cap S$ be their union. Clearly, O is stabilized by $\mathscr{N}_S$, hence $O/\mathrm{Aut}(\mathbb{P}^2, S) \subset C/\mathrm{Aut}(\mathbb{P}^2, S)$ descends to a divisor of odd degree of $\mathbf{C}$; this implies that $\mathbf{C}$ has index 1, which in turn implies that $\mathbf{C}$ has a rational point and $\mathbf{C} \simeq \mathbb{P}^1_k$, see e.g. [14, Corollary 5.3.6, Theorem 5.1.3]. Since we have a map $\mathbf{C} \dashrightarrow \mathscr{C} \to \mathscr{P}_S \to \mathscr{G}_S$, this implies that $\mathscr{G}_S(k) \neq \emptyset$ if k is infinite. If k is finite, the statement follows from the fact that $\mathscr{N}_S \to \mathrm{Gal}(K/k) \simeq \hat{\mathbb{Z}}$ is split and hence $\mathscr{G}_S(k) \neq \emptyset$.

2.2. S **is contained in the union of two lines.** Assume that S is contained in the union of two lines L, L' each containing at least 3 points. Up to changing coordinates, we may assume that $(0:0:1), (0:1:0), (1:0:0) \in S$. It is now clear that, up to permuting the coordinates and multiplying them by scalars, we might assume that

$$S = \{(0:0:1), (0:1:0), (1:0:0), (0:1:1), (1:0:1)\},$$

which is clearly defined over k.

2.3. S **is contained in the union of a line and a point.** Suppose that S is contained in the union of a line L and a point p, choose coordinates such that $p = (0:0:1)$ and $L = \mathbb{P}^1$ is the line $\{(s:t:0)\}$.

The field of moduli of $(\mathbb{P}^2_K, S)$ is equal to the field of moduli of $(\mathbb{P}^1_K, S \cap \mathbb{P}^1_K)$: given $\sigma \in \mathrm{Gal}(K/k)$, clearly $\sigma^*(\mathbb{P}^2_K, S) \simeq (\mathbb{P}^2_K, S)$ if and only if $\sigma^*(\mathbb{P}^1_K, S \cap \mathbb{P}^1_K) \simeq (\mathbb{P}^1_K, S \cap \mathbb{P}^1_K)$. By [7, Proposition 13], $\mathbb{P}^1_K \cap S$ descends to a closed subset of $\mathbb{P}^1_k$. It follows that S descends to a closed subset of $\mathbb{P}^2_k$.

3. Case $n \geq 6$. Let us now construct a counterexample with $k = \mathbb{R}$, $K = \mathbb{C}$ for every $n \geq 6$.

If $n \geq 6$, then either $n = 2m + 4$ or $n = 2m + 5$ for some $m \geq 1$. Given $a_1, \ldots, a_m \in \mathbb{C}^*$, $|a_i| \neq 1$, define

$$\begin{aligned} F &= \{(\pm 1 : 0 : 1), (0 : \pm 1 : 1)\}, \\ S &= \{(a_i : 1 : 0), (1 : -\bar{a}_i : 0)\}_i \cup F, \\ S' &= S \cup \{(0:0:1)\}, \end{aligned}$$

then $|S| = 2m + 4$, $|S'| = 2m + 5$. The matrix $g = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ gives a linear equivalence between S and its complex conjugate, since $g(F) = F$, $g(a_i : 1 : 0) = (1 : -a_i : 0)$ and $g(1 : -\bar{a}_i : 0) = (\bar{a}_i : 1 : 0)$. Similarly, g maps S' to its complex conjugate. It follows that both S and S' have field of moduli equal to $\mathbb{R}$. Let us show that S is not defined over $\mathbb{R}$ (the case of S' is analogous).

Let $M \in \mathrm{PGL}_3(\mathbb{C})$ be the image of $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We have that M is a non-trivial automorphism of both S and S': in fact, $M(a_i : 1 : 0) = (-a_i : -1 : 0) = (a_i : 1 : 0)$, $M(\pm 1 : 0 : 1) = (\mp 1 : 0 : 1)$, $M(0 : \pm 1 : 1) = (0 : \mp 1 : 1)$, $M(0:0:1) = (0:0:1)$.

Lemma 2. *For a generic choice of* $a_1, \ldots, a_m \in \mathbb{C}$, $|a_i| \neq 1$, *and* $m \geq 1$, $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S') = \langle M \rangle$.

Proof. For a generic choice of $a_1, \dots, a_m$, there are exactly two lines containing exactly three points of S', hence their point of intersection $(0:0:1)$ is fixed by $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S') \subset \mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S)$. Since $M \in \mathrm{Aut}(\mathbb{P}^2, S')$, it is enough to show $\mathrm{Aut}(\mathbb{P}^2, S) = \langle M \rangle$.

Let $L = \{(s:t:0)\}$ be the line at infinity. We first show that it is stabilized by $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S)$ for a generic choice of the a_i. If $m \geq 2$, this is obvious since it is the only line containing at least four points of S.

Assume $m = 1$, $a_1 = a$. Since the stabilizer of F in $\mathrm{GL}_2(\mathbb{C})$ is finite and acts $\mathbb{C}$-linearly on $\mathbb{P}^2$, for a generic a, there is no element of $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S)$ swapping $(a:1:0)$ and $(1:-\bar{a}:0)$. Assume by contradiction that L is not stabilized. We may then also assume that the orbit of $(1:-\bar{a}:0)$ intersects F (if this happens for $(a:1:0)$ but not $(1:-\bar{a}:0)$, we just change coordinates).

Since M is an element of order 2 of $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S)$ acting as a double transposition of F and no element of $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S)$ swaps $(a:1:0)$ and $(1:-\bar{a}:0)$, it follows that there exists an element $g \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S)$ swapping some $p \in F$ and $(1:-\bar{a}:0)$. In particular, g permutes the other four points $F \cup \{(a:1:0)\} \smallsetminus \{p\}$, we may thus think of g as an element of S_4. Since the four points $F \cup \{(a:1:0)\} \smallsetminus \{p\}$ are in general position, for each element $\sigma \in S_4$, there exists $\phi_\sigma \in \mathrm{PGL}_3(\mathbb{C})$ acting as σ on $F \cup \{(a:1:0)\} \smallsetminus \{p\}$, and we may write ϕ_σ as a 3×3 matrix whose entries are algebraic functions of a. Since complex conjugation is not algebraic, for a generic choice of a, we have $\phi_\sigma(p) \neq (1:-\bar{a}:0)$ for every $\sigma \in S_4$. This implies that for a generic choice of a, the automorphism g cannot exist (since $g(p) = (1:-\bar{a}:0)$), and hence L is stabilized.

If L is stabilized, then F is stabilized, too. The point $(0:0:1)$ is the only point of intersection in $\mathbb{P}^2 \smallsetminus (L \cup S)$ of two lines passing through two points of F, hence it is fixed by $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S)$. This implies that $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S) \subset \mathrm{GL}_2(\mathbb{C}) \subset \mathrm{PGL}_3(\mathbb{C})$.

The subgroup of $\mathrm{GL}_2(\mathbb{C})$ stabilizing F is $D_4 = \langle r, s \mid r^4 = s^2 = rsrs = 1 \rangle$ generated by $r = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $s = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, hence $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S) \subset D_4$. The center of D_4 is $\langle r^2 = -1 \rangle = \langle M \rangle \subset \mathrm{GL}_2(\mathbb{C})$, which is also the kernel of $D_4 \to \mathrm{PGL}_2(\mathbb{C})$. Since $D_4 / \langle M \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ is finite and acts by $\mathbb{C}$-linear automorphisms on L, for a generic choice of $a_1, \dots, a_m$, the intersection $S \cap L$ is not stabilized by any non-trivial element of $D_4 / \langle M \rangle \subset \mathrm{PGL}_2(\mathbb{C})$. It follows that $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2, S) = \langle M \rangle$. □

Write C_a for the cyclic group of degree a. Consider the natural projection $C_4 \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = C_2$ giving a non-faithful action of C_4 on $\mathbb{C}$ and the Galois equivariant action on $\mathbb{P}^2_{\mathbb{C}}$ given by $(a:b:c) \mapsto (-\bar{b}:\bar{a}:\bar{c})$. Clearly, C_4 stabilizes S, hence we get a homomorphism $C_4 \to \mathscr{N}_S$. Lemma 2 implies that $\mathscr{N}_S$ is an extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = C_2$ by C_2, hence $C_4 \to \mathscr{N}_S$ is an isomorphism and $\mathscr{G}_S \simeq [\mathrm{Spec}\ \mathbb{C}/C_4]$. To conclude, it is enough to show that $[\mathrm{Spec}\ \mathbb{C}/C_4](\mathbb{R}) = \emptyset$.

By definition, an $\mathbb{R}$-point of $[\mathrm{Spec}\ \mathbb{C}/C_4]$ corresponds to a C_4-torsor over $\mathbb{R}$ with a C_4-equivariant map to $\mathrm{Spec}\ \mathbb{C}$, see e.g. [19, Example 8.1.12]. There are two C_4-torsors over $\mathbb{R}$, the trivial one and $T = \mathrm{Spec}\ \mathbb{C} \cup \mathrm{Spec}\ \mathbb{C}$, and neither of them has C_4-equivariant morphisms to $\mathrm{Spec}\ \mathbb{C}$. This is clear for the trivial torsor, while for T, it follows from the fact that $C_2 \subset C_4$ acts non-trivially on each copy of Spec $\mathbb{C} \subset T$.

Notice that $\mathbf{P}_S(\mathbb{R}) = \mathbb{P}^2_{\mathbb{C}}/C_4(\mathbb{R})$ is non-empty: however, the only real point $(0:0:1)$ is an A_1-singularity, and hence we cannot apply the Lang-Nishimura theorem for stacks [10, Theorem 4.1] to conclude that $\mathscr{P}_S(\mathbb{R})$ is non-empty (in fact, $\mathscr{P}_S(\mathbb{R}) = \emptyset$). We mention that, for most types of 2-dimensional quotient singularities (but not A_1-singularities), the Lang-Nishimura theorem is still valid, see [9, §6] and [8] for details.

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