



Stretching of polymers and turbulence: Fokker Planck equation, special stochastic scaling limit and stationary law

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Abstract

The aim of this work is understanding the stretching mechanism of stochastic models of turbulence acting on a simple model of dilute polymers. We consider a turbulent model that is white noise in time and activates frequencies in a shell $N \leq |k| \leq 2N$ and investigate the scaling limit as $N \rightarrow \infty$, under suitable intensity assumption, such that the stretching term has a finite limit covariance. The polymer density equation, initially an SPDE, converges weakly to a limit deterministic equation with a new term. Stationary solutions can be computed and show power law decay in the polymer length.

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1. Introduction

Polymers are complex molecular systems, that can be vaguely thought as a chain of springs. In a turbulent fluid they are usually found in two states called *coil* and *stretched*. The coil state is like a spherical or ellipsoidal rolled chain, which may be more or less elongated, but still in roll position. The stretched state is when the chain is elongated, more similar to a straight line than a sphere. Turbulence with small stretching intensity produces only a small perturbation of the coil state, while strongly stretching turbulence may lead to the stretched state; when the polymer pass from one state to the other we speak of coil-stretch transition.

In this work, we consider in dimension 2 or 3 the following *Hookean* model:

$$\begin{cases} dR_t &= \nabla u(X_t, t)R_t dt - \frac{1}{\beta}R_t dt + \sqrt{2}\sigma d\mathcal{W}_t, \\ dX_t &= u(X_t, t)dt, \end{cases} \tag{1}$$

where X_t is the polymer position (the center of mass) and R_t is the end-to-end vector, representing the orientation and elongation of the chain, see e.g. [1, Section 4.2]. The polymer is embedded into a fluid having velocity $u(t, x)$, which stretches R_t by $\nabla u(x, t)$. The equation for R_t contains also a damping (restoring) term with relaxation time β and Brownian fluctuations $\sqrt{2\sigma}dW_t$ where, to simplify the notations we have denoted by σ^2 the product $\frac{kT}{\beta}$, k being Boltzmann constant and T being the temperature.

The statistics of the polymer length r , say the diameter, have been investigated by several authors in the physical literature, see for instance [2–5] and other references mentioned below. In the coil state, the distribution of r is found to be power law

$$f(r) \sim r^{-1-\alpha} \quad \text{for relatively large } r \tag{2}$$

with the exponent α positive (so that f is normalizable). The exponent α depends on the stretching properties of the turbulent flow: the highest is the stretching intensity, the lowest is α . At $\alpha = 0$ one has the coil-stretch transition. In the stretched state, the precise mathematics depends on the idealizations of the model. If we had introduced a superlinear damping instead of the linear damping $-\frac{1}{\beta}R_t$, this would produce a sort of cut-off at very high lengths (e.g. FENE model, see Remark 12), so that the behavior (2) would be true only in a range

$$r_0 \ll r \ll r_1$$

and globally the function f would still be a pdf. In our idealization of linear damping, the stretching may overcome the damping and lead to infinite length in the asymptotic regime, which is the idealized signature of stretch state, see subsection 4.1.

Clearly, one would like to predict the exponent α based on turbulence features. The theory developed on physical grounds by [2] and [6] tells us that α is related to the Lyapunov exponents of the turbulent flow. Precisely, if $\phi_t(x)$ is the Lagrangian flow associated to the turbulent fluid, and we define

$$\mathcal{L}(q) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} [|D\phi_t(x)|^q]$$

then α satisfies $\frac{\alpha}{2\beta} = \mathcal{L}(\alpha)$. However, the computation of $\mathcal{L}(\alpha)$ is not trivial in general.

The theory just mentioned does not make use of scale separation (and not even delta correlation in time of the turbulent fluid), hence it is quite general (much more than ours). In our paper we consider a special class of turbulent flows, delta-correlated in time, having a precise dyadic scale ℓ , namely based on Fourier components of modes $|k| \in [\ell^{-1}, 2\ell^{-1}]$. When ℓ is very small, precisely in the scaling limit as $\ell \rightarrow 0$, we discover a power law of the form (2). This provides another proof (or a more rigorous proof, although in a particular regime) of the emergence of power law in polymer length r . In addition, we find a very direct relation between α and the turbulent flow characteristics, thanks essentially to the special structure of the flow and the separation of scales intrinsic in the scaling limit; the non-trivial informations on the Lyapunov exponents are not needed anymore.

For the convenience of the reader, let us explain heuristically the main result of this work then precise results (see section 4) and rigorous proofs are provided in the main body of the paper.

1.1. Sketch of the main results

We consider a dilute (non-interacting) family of polymers subject to equations (1), thus described by the kinetic equation for the density $f^N(x, r, t)$ of polymers with position x and length r at time t

$$\begin{cases} \partial_t f^N(x, r, t) + \operatorname{div}_x(u^N(x, t)f^N(x, r, t)) + \operatorname{div}_r((\nabla u^N(x, t)r - \frac{1}{\beta}r)f^N(x, r, t)) \\ = \sigma^2 \Delta_r f^N(x, r, t) \\ f^N|_{t=0} = f_0. \end{cases} \tag{3}$$

Here $u^N(x, t)$ is the fluid velocity. We assume that $u^N(x, t)$ is made of two components, a deterministic large-scale one $u_L(x, t)$ and a stochastic one, modeling small-scale turbulence, of the form $\sum_{k \in K} \sigma_k^N(x) \partial_t W_t^k$, acting in Stratonovich form, that will be described below in detail; hence

$$u^N(x, t) = u_L(x, t) + \circ \sum_{k \in K} \sigma_k^N(x) \partial_t W_t^k.$$

Here we just stress the fact that the coefficients $\sigma_k^N(x)$ of the turbulent part depend on a parameter N so that, when N increases, they represent smaller and smaller space scales, precisely Fourier frequencies $N \leq |k| \leq 2N$, providing a separation of scale regime essential for our analysis. The noise acts on $f^N(x, r, t)$ in transport form, but incorporating also the stretching action by the term $\nabla u^N(x, t)r$.

Our aim in this work is twofold. Firstly, present a rigorous mathematical framework and proofs to study a stochastic Fokker-Planck equations with transport noise and general class of initial data. These equations has an hyperbolic nature with respect to the space variable $x \in \mathbb{T}^2$ and polymer length vector variable $r \in \mathbb{R}^2$. Thus, we should combine techniques for weighted spaces to handle the r -variable, spatial homogeneity and mirror symmetry of the covariance operator to handle the stochastic part and prove existence of “quasi-regular weak solution”, see Definition 5. Then, combine commutators techniques to handle the space variable, using a density result based on Wiener chaos decomposition to prove uniqueness in this class of solution, see Theorem 8. We refer to section 5 for discussion about the mathematical challenges related to these equations.

Secondly, understand the stretching power on the polymer, in the limit as $N \rightarrow \infty$ when the noise activates frequencies in a shell $N \leq |k| \leq 2N$. The final result is a limit model, of Fokker-Planck type, with a new diffusion term in the radial variable of the polymer, with non-homogeneous and degenerate coefficients. Its radially symmetric stationary solutions are explicit and have power-law tails. Let us describe quickly the scaling limit result, see Theorem 9 for the precise result. Under the assumptions described in section 2, we prove that $f^N(x, r, t)$ weakly converges to the solution of a deterministic equation of the form (the results below contain also a deterministic term in the velocity, which is omitted here in the Introduction for notational simplicity)

$$\begin{cases} \partial_t \bar{f}(x, r, t) - \operatorname{div}_r(\frac{1}{\beta}r \bar{f}(x, r, t)) = \sigma^2 \Delta_r \bar{f}(x, r, t) + \frac{1}{2} \operatorname{div}_r(A(r)\nabla \bar{f}(x, r, t)) \\ \bar{f}|_{t=0} = f_0 \end{cases} \tag{4}$$

where $A(r) = k_T(3|r|^2 I - 2r \otimes r)$ and $k_T = \frac{\pi \log(2)}{8} a^2$ where a is an intensity parameter of the noise. The new diffusion term in the r -variable is one of the most important novelty of this work. It captures the statistical properties of the stretching mechanism. We compute the explicit rotation-invariant solution of the associated stationary equation and find it has a power law decay for large $|r|$:

$$\bar{f}(|r|) \sim |r|^{-\frac{2}{k_T \beta}}$$

indicating large values with high probability. The constants in the power are directly associated to those of the stochastic model of u^N . This fact was predicted in the physical literature based on other models and assumptions, see e.g. [2]. In our model we identify a simple link between the power of the tail and parameters of the turbulence model, see subsection 4.1.

We wish to draw the reader’s attention to the following: introduce the mean of the structure tensor $\mathbf{T}(t, x) := \int_{\mathbb{R}^2} r \otimes r f^N(t, x, r) dr$ then \mathbf{T}^N satisfies the following closed system of PDEs

$$\begin{cases} \partial_t \mathbf{T}^N + u^N \cdot \nabla \mathbf{T}^N = (\nabla u^N) \mathbf{T}^N + \mathbf{T}^N (\nabla u^N)^t - \frac{2}{\beta} (\mathbf{T}^N - k_T \mathbf{I}) \\ \mathbf{T}^N|_{t=0} = \mathbf{T}_0. \end{cases} \tag{5}$$

The last equation (5) is a macroscopic Oldroyd-B model (see e.g. [7, Section 2.8]) and the tensor \mathbf{T} characterize the viscoelastic (non-Newtonian part) of the flow (1). Many rheological behavior can be detected such as shear viscosity, normal stress difference and overshoot phenomenon in contrast with Newtonian flows, see e.g. [1] and we refer e.g. to [7–9] for other types of non-Newtonian flows. As we discuss in subsection 4.1, the explicit rotation-invariant solution of the stationary equation associated with the limit equation has a power-law density and therefore it is not sufficient to study the system (5) only, this is another reason we base our analysis on FP equation (3). We will comment about the limit equation associated with (5) in Remark 11. Finally, although the results and proofs are presented in 2D, similar results hold in 3D after some cosmetic changes and mostly the form of the stochastic turbulent velocity, see subsection 7.4.

This work is a key step in a research project on the effects of small-scale turbulence on large-scale motion, started a few years ago after the development of a new technique. In order to understand the novelties it may be useful to review these recent developments and stress similarities and differences.

1.2. Literature review

We divide the presentation in two subsections, the scalar and the vector case. The second one is characterized by a peculiar issue, the presence of stretching, which is by no means an incremental detail over the scalar case; and the present work aims to give relevant information on the stretching case.

¹ The subscript N to stress the dependence on N because of the presence of u^N .

1.2.1. The scalar case

The foundational work [10] investigated a problem that, to some extent, could be considered as a variant of diffusion approximation results in stochastic homogenization theory: a stochastic first order differential operator in Stratonovich form gives rise, in a suitable scaling limit, to a deterministic diffusion term. The stochastic equation had the form (we do not describe in detail the spatial domain, the finite or countable set where the index k varies and other details which have no relevance for the purpose of this section, see [10])

$$\begin{cases} d\rho_t^N &= \sum_k \theta_k^N e_k \cdot \nabla \rho_t^N \circ dW_t^k \\ \rho^N|_{t=0} &= \rho_0 \in L^2 \end{cases}$$

with suitable vector fields e_k (a suitable subset of a complete orthonormal system in L^2) and the family of real valued coefficients $\theta^N := (\theta_k^N)$. Under the assumptions

$$\lim_{N \rightarrow \infty} \|\theta^N\|_{\ell^2}^2 = \kappa, \quad \lim_{N \rightarrow \infty} \|\theta^N\|_{\ell^\infty} = 0,$$

it has been proved that, in a weak sense (namely against test functions or in suitable negative order Sobolev spaces) the solution ρ^N converges in mean square to the unique deterministic solution ρ_t of the heat equation

$$\begin{cases} \partial_t \rho_t &= \kappa \Delta \rho_t \\ \rho|_{t=0} &= \rho_0 \end{cases}$$

(possibly with the constant κ modified by a constant related to space dimension and the choice of the orthonormal system). The technical assumption mentioned above on θ^N can be reformulated in terms of the noise covariance function:

$$Q_N(x, y) = \sum_k \left(\theta_k^N\right)^2 e_k(x) \otimes e_k(y) = \mathbb{E}[W(x, 1) \otimes W(y, 1)]$$

where $W(x, t) = \sum_k \theta_k^N e_k(x) W_t^k$ and the associated linear operator on L^2 vector fields $v(x)$

$$(\mathbb{Q}_N v)(x) = \int Q_N(x, y) v(y) dy.$$

The assumption is that the trace of \mathbb{Q}_N converges to κ and the operator norm to zero. Heuristically, it means that the function $Q_N(x, y)$ tends to a non zero value along the diagonal and to zero outside, in other words

$$\begin{cases} \lim_{N \rightarrow \infty} Q_N(x, x) &= \kappa \\ \lim_{N \rightarrow \infty} Q_N(x, y) &= 0 \text{ for } x \neq y \end{cases} \tag{6}$$

namely the variance of the noise remains constant at the limit but the space correlation goes to zero.

We have said that this is related to results in homogenization theory [11], but it is important to notice that the model and strategy are different from classical homogenization and especially they are very efficient for generalizations to nonlinear problems. The result of [10] has been extended, indeed, to 2D Euler equations, 2D Navier-Stokes equations, and other nonlinear models like Keller-Siegel and nonlinear heat equations, see [12] and [13] for the basic results in this direction, obtained both with compactness methods and more quantitatively with estimates on stochastic convolutions. The results have been supplemented by the analysis of Gaussian fluctuations and Large Deviations [14]. Mixing and dissipation enhancement results can also be obtained by this approach, see [15–18], [13, section 9] and [19,20]. The fact that the limit equation contains an additional strongly elliptic term sometimes has a feedback effect of regularization on the approximating stochastic problems (the closeness in some norm provides additional a priori bounds), leading to delay of blow-up results [21]. See also [22–31] for various other applications, examples and numerical approximations. The problem is of interest also for Large Eddy Simulations and Boussinesq assumption, see [32] for a Smagorinsky type result. The scaling limit has been also transposed to particle systems, see for instance [33,34]. This list of works is certainly non complete but it gives the feeling of the fertility of the scaling limit result proved in [10].

Let us also mention that transport and transport-stretching noise (the second being the topic of next subsection) is currently used in many other researches and for different purposes; above we have restricted the attention to works dealing with the special scaling limit to an additional diffusion. The list of references on other research directions on transport-stretching noise is too long, let us only mention as examples [35–40].

1.2.2. The vector fields case

The works discussed above dealt with scalar problems, where transport noise induces an additional elliptic operator. When the turbulent fluid acts on vector fields, the action has two aspects: not only transport but also *stretching*: namely the differential $D\phi_t$ of the Lagrangian flow ϕ_t of a fluid produces a modification of the length of vectors v , $v \mapsto D\phi_t v$. Since we consider divergence free fluids, the deformation tensor Du of the velocity field u has zero trace, the symmetrization typically has positive eigenvalues, indicating that an increase of length should be expected for many directions v .

Understanding deterministic and random stretching is much more difficult and open, with respect to transport. Two indications of this difficulty are that stretching is considered the most important problem in view of the open problem of blow-up for the 3D Navier-Stokes equations [41]; and that in contrast to a wide range of results on mixing and dissipation enhancement produced by chaotic transport of scalars, there is very few on the effect of complex stretching [42]. The problem of random stretching has been approached also by the scaling limit described above, and the research of the present paper is along this line. Let us see what previous papers did on this topic.

The results are more fragmentary than the scalar case. One of the first works on the action of vector fields has been done in [43] on the 3D Navier-Stokes equations in vorticity form, for the vorticity $\omega = \nabla u$, u being the velocity of the fluid. However, precisely because the effect of the random stretching term $(\omega \cdot \nabla e_k)$, more precisely

$$\sum_k \theta_k^N (\omega \cdot \nabla e_k) \circ dW_t^k$$

was (and it stills until now) not well understood, this term was neglected and only the transport

$$\sum_k \theta_k^N (e_k \cdot \nabla \omega) \circ dW_t^k$$

was considered (with the need of a projection operator, see [43] for motivation and details). The result is similar to the scalar case [21], namely a delay of blow-up, relevant because of the relevance of the equation, but still incomplete because it does not explain what would happen in the presence of random stretching. See also a generalization to rough path noise [44] and to magnetic hydrodynamics [19].

Concerning the 3D Navier-Stokes equations, a very important progress appeared recently [45]. It is concerned with the Navier-Stokes equations in velocity form and the noise, of transport type, is at that level (energy-preserving instead of circulation-preserving as it should be the transport-stretching noise at the level of vorticity equation, see the foundational work [38]). The stretching action of such noise is a bit hidden but the mathematical progress is very important since it shows that a reasonable noise (with some form of stretching) may still regularize the 3D Navier-Stokes type equations (the result needs an hyperviscosity but of lower degree than the one of the deterministic theory).

Then three works appeared with explicit inclusion of a random stretching term, all of them basically assuming that the noise satisfies the classical scaling (6) of the scalar case. The first one, [46] deals with a Navier-Stokes type system called 2D-3C, having 3D features but also 2D simplifications, which allows one to control the strength of the stretching term; the physics is motivated by a limit of fast rotating systems. The second one is [47] on a passive magnetic field in 3D thin domains, where the smallness of one of the three dimensions is a key ingredient for the control of stretching; the mathematics is quite intricate and thus the result is developed for the simplified case of a passive vector field. The third one is [48], again on the passive magnetic field, but in a full three dimensional domain. Stretching is full, in a sense, here but the key ingredient is controlling the solution in a negative Sobolev space, physically speaking at the level of the magnetic potential. As already said, but crucial for the comparison with the present paper, all these works try to assume that the noise scales, in N , as in (6), which in particular it means that we expect a Laplacian coming for the transport, in the limit equation. Heuristically, when the Laplacian-due-to-transport persists, stretching could be infinite (at least a priori). The ways to overcome this blow-up of stretching, in the above papers, has been either to constrain it by a 2D-3C structure, or by a smallness parameter related to thin nomains, or controlling the solution in a negative Sobolev space.

Let us come to the different scaling in N used in the present paper, which also appears in [49] for very different purposes (a precise comparison is made below). Here we have stretching of a passive vector quantity R_t , that in Lagrangian form is written as

$$\sum_k \theta_k^N \nabla e_k (X_t^N) R_t^N \circ dW_t^k$$

(X_t being the position of the polymer), and in Eulerian form as

$$\sum_k \theta_k^N (\nabla e_k(x) r) \cdot \nabla f^N(x, t) \circ dW_t^k$$

(∇e_k is a matrix acting on the vectors R_t^N and r respectively). Then we are faced also with the covariance matrix

$$C_{N,r}(x, y) = \sum_k \left(\theta_k^N\right)^2 (\nabla e_k(x) r) \otimes (\nabla e_k(y) r).$$

Clearly, since (roughly speaking) $|\nabla e_k(x)| \sim |k|$, we cannot hope that the classical scaling (6) of the scalar case holds and simultaneously also the covariance matrix $C_{N,r}(x, y)$ has a finite limit in N . For this reason we introduce a different scaling w.r.t. (6), where $Q_N(x, y)$ goes to zero outside and also on the diagonal with the right speed but $C_{N,r}(x, y)$ satisfies, very heuristically,

$$\begin{aligned} \lim_{N \rightarrow 0} C_{N,r}(x, x) &= A(r) \\ \lim_{N \rightarrow 0} C_{N,r}(x, y) &= 0 \text{ for } x \neq y \end{aligned}$$

where $A(r)$ is the matrix function described in this paper (equal to a multiple of $3|r|^2 I - 2r \otimes r$). The consequence of this scaling is that no diffusion in the x -variable appears, opposite to the scalar case; but a diffusion in r appears, with diffusion matrix $A(r)$. In other words, the effect of transport becomes irrelevant in the scaling limit, while the effect of stretching has a limit which is a diffusion in the length variable r , with a special diffusion matrix.

Finally, let us compare the present work with [49]. The work is devoted to a stochastic PDE for a passive magnetic field, subject to a model of turbulence fluid. In [49] it is shown that it is possible to investigate a Vlasov-type equation, suitably associated to the equation of the magnetic field, prove a diffusion limit result for it and deduce informations on the increase of magnetic field, in the scaling regime that controls the derivatives of the stretching used also here. The two works stem from a similar physical intuition, but the two problems are very different, the present one starting from a Lagrangian description of a polymer, while [49] from the Eulerian fluid dynamic description of a magnetic field. Moreover, the mathematical techniques to deal with them are completely different: in [49] the Vlasov-type equation is investigated in the space of measures, in the spirit of Young measure solutions, and the technique to prove well posedness and scaling limit is ultimately based on the properties of the solutions of the original Eulerian SPDE. On the contrary, in the present paper we analyze the Fokker-Planck equation directly in a space of functions, with non trivial technical details related to weighted spaces (not used before in the above mentioned literature on these scaling limit problems).

Structure of the paper

The manuscript is organized as follows: in section 2, we present the functional and stochastic settings. Then we introduce in section 3 the stochastic FP in Itô form after presenting some properties of the covariance operator. Section 4 collects the main results of our work, the computation of the explicit rotation-invariant solution of the associated stationary equation of (13). In section 5, we highlight the ideas of the proof and some formal calculations as well as the technical challenges. Section 6 is devoted to the proof of the existence and uniqueness of *quasi-regular* solution to the stochastic FP (12). In section 7, we prove the convergence of the stochastic FP (12) to the limit PDE (13).

2. Notations, definitions and problem formulation

2.0.1. Notations and functional setting

We will consider the periodic boundary conditions with respect to the spacial variable x , namely x belongs to the 2-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. On the other hand, the end-to-end vector variable r belongs to \mathbb{R}^2 . Let $m \in \mathbb{N}^*$ and introduce the following Lebesgue and Sobolev spaces with polynomial weight, namely

$$L^2_{r,m}(\mathbb{T}^2 \times \mathbb{R}^2) := \{f : \mathbb{T}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} : \int_{\mathbb{T}^2 \times \mathbb{R}^2} |f(x, r)|^2 (1 + |r|^2)^{m/2} dx dr := \|f\|^2_{L^2_{r,m}} < \infty\},$$

$$H^l_{r,m}(\mathbb{T}^2 \times \mathbb{R}^2) := \{f : \mathbb{T}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} : \sum_{|\gamma|+|\beta| \leq l} \int_{\mathbb{T}^2 \times \mathbb{R}^2} |\partial_x^\gamma \partial_r^\beta f(x, r)|^2 (1 + |r|^2)^{m/2} dx dr = \|f\|^2_{H^l_{r,m}} < \infty\},$$

where $l \in \mathbb{N}^*$. Now, let us precise the functional setting to study (12). We will use the following notations

$$H^2_{r,4}(\mathbb{T}^2 \times \mathbb{R}^2) := V, \quad L^2_{r,2}(\mathbb{T}^2 \times \mathbb{R}^2) := H.$$

We recall the definition of inner products defined on the spaces V and H .

$$(h, g)_V := \sum_{|\gamma|+|\beta| \leq 2} \int_{\mathbb{T}^2 \times \mathbb{R}^2} \partial_x^\gamma \partial_r^\beta h(x, r) \partial_x^\gamma \partial_r^\beta g(x, r) (1 + |r|^2)^2 dx dr, \quad \forall g, h \in V;$$

$$(h, g)_H := \int_{\mathbb{T}^2 \times \mathbb{R}^2} h(x, r) g(x, r) (1 + |r|^2) dx dr, \quad \forall g, h \in H;$$

$$(h, g) := \int_{\mathbb{T}^2 \times \mathbb{R}^2} h(x, r) g(x, r) dx dr, \quad \forall g, h \in L^2(\mathbb{T}^2 \times \mathbb{R}^2).$$

Since $L^\infty(0, T; H)$ is not separable, it is convenient to introduce the following space:

$$L^2_{w-*}(\Omega; L^\infty(0, T; H)) = \{u : \Omega \rightarrow L^\infty(0, T; H) \text{ is weakly-* measurable and } \mathbb{E} \|u\|^2_{L^\infty(0, T; H)} < \infty\},$$

where weakly-* measurable stands for the measurability when $L^\infty(0, T; H)$ is endowed with the σ -algebra generated by the Borel sets of weak-* topology, we recall that (see [50, Thm. 8.20.3])

$$L^2_{w-*}(\Omega; L^\infty([0, T]; H)) \simeq \left(L^2(\Omega; L^1([0, T]; H')) \right)'$$

For $y = (y_1, y_2) \in \mathbb{R}^2$, y^\perp stands for $(-y_2, y_1)$. We recall the following notations:

- $(\nabla_x g)_{i=1,2} = (\frac{\partial g}{\partial x_i})_{i=1,2}$; $(\nabla_r g)_{i=1,2} = (\frac{\partial g}{\partial r_i})_{i=1,2}$ for scalar function g .
- $(\nabla_x g)_{i,j=1,2} = (\frac{\partial g^i}{\partial x_j})_{i,j=1,2}$; $(\nabla_r g)_{i,j=1,2} = (\frac{\partial g^i}{\partial r_j})_{i,j=1,2}$ for vector valued function g .
- $\Delta_x g = \sum_{i=1}^2 \frac{\partial^2 g}{\partial^2 x_i}$; $\Delta_r g = \sum_{i=1}^2 \frac{\partial^2 g}{\partial^2 r_i}$ for scalar function g .

and $\text{div}_{x/r} g = \nabla_{x/r} \cdot g$ for vector valued function g . We don't stress the subscript in ∇_x when it is clear from the context.

In order to prove a uniqueness results, we will need some regularization kernel. More precisely, let $\delta > 0$ and ρ be a smooth density of a probability measure on \mathbb{R}^2 , compactly supported in $B(0, 1)$ and define the approximation of identity for the convolution on \mathbb{R}^2 as $\rho_\delta(y) = \frac{1}{\delta^2} \rho(\frac{y}{\delta})$ (We also assume that ρ is radially symmetric). Since we are working on $\mathbb{T}^2 \times \mathbb{R}^2$, we recall that for any integrable function g on \mathbb{T}^2 , g can be extended periodically to a locally integrable function on the whole \mathbb{R}^2 and convolution $\rho_\delta * g$ is meaningful and $\rho_\delta * g$ is still a C^∞ -periodic function.

Finally, throughout the article, we denote by $C, C_i, i \in \mathbb{N}$, generic constants, which may vary from line to line.

2.0.2. Assumptions on the noise

Consider $\mathbb{Z}_0^2 := \mathbb{Z}^2 - \{(0, 0)\}$ divided into its four quadrants (write $k = (k_1, k_2)$)

$$K_{++} = \{k \in \mathbb{Z}_0^2 : k_1 \geq 0, k_2 > 0\}; \quad K_{-+} = \{k \in \mathbb{Z}_0^2 : k_1 < 0, k_2 \geq 0\}$$

$$K_{--} = \{k \in \mathbb{Z}_0^2 : k_1 \leq 0, k_2 < 0\}; \quad K_{+-} = \{k \in \mathbb{Z}_0^2 : k_1 > 0, k_2 \leq 0\}$$

and set

$$K_+ = K_{++} \cup K_{+-}; \quad K_- = K_{-+} \cup K_{--} \text{ and } K = K_+ \cup K_-.$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ be a complete filtered probability space. Define

$$\sigma_k^N(x) = \theta_k^N \frac{k^\perp}{|k|} \cos k \cdot x, \quad k \in K_+, \quad \sigma_k^N(x) = \theta_k^N \frac{k^\perp}{|k|} \sin k \cdot x, \quad k \in K_-$$

where

$$\theta_k^N = \frac{a}{|k|^2}, \quad N \leq |k| \leq 2N, \quad N \in \mathbb{N}^*; \quad \theta_k^N = 0 \quad \text{elsewhere.}$$

This is the main assumption about the shell structure of the noise; and a is a positive constant measuring the intensity. Since θ_k^N depends only on $|k|$, sometimes we write $\theta_{|k|}$. Notice that

$$\begin{aligned} \partial_i \sigma_k^\alpha(x) &= -\theta_k \frac{k_i (k^\perp)^\alpha}{|k|} \sin k \cdot x, & k \in K_+, \quad \alpha, i = 1, 2, \\ \partial_i \sigma_k^\alpha(x) &= \theta_k \frac{k_i (k^\perp)^\alpha}{|k|} \cos k \cdot x, & k \in K_-. \end{aligned}$$

Let us also consider a family $(W_t^k)_t^{k \in \mathbb{Z}_0^2}$ of independent Brownian motions on the probability space (Ω, \mathcal{F}, P) . On the same probability space we shall soon assume that there exists another independent 2-dimensional Brownian motion $(W_t)_t$.

2.0.3. Dense subsets in the space $L^2(\Omega)$

The purpose of this part is to recall a density result, which will play a crucial role in subsection 6.3 and subsection 6.3.1. Namely, we will prove that uniqueness holds a particular class of solution, see Definition 5. It is sometimes called Wiener uniqueness and was used for instance by [51–55]. For that, let \mathcal{G}_t be the filtration associated with $(W_t^k)_t^{k \in \mathbb{Z}_0^2}$ namely $\mathcal{G}_t = \sigma\{W_s^k; s \in [0, t], k \in \mathbb{Z}_0^2\}$, and denote by $\overline{\mathcal{G}}_t$ its completed filtration.² For $T > 0$, let us introduce

$$\begin{aligned} \mathcal{H} &= L^2(\Omega, \overline{\mathcal{G}}_T, P), \quad M_n = \{k \in \mathbb{Z}_0^2; |k| \leq n\} \\ G &= \bigcup_{n \in \mathbb{N}} G_n; \quad G_n = \{g = (g_k)_{k \in M_n}; g_k \in L^2(0, T); \quad \forall k \in M_n\}. \end{aligned}$$

For $n \in \mathbb{N}$, $g \in G_n$, we set

$$\begin{aligned} e_g(t) &= \exp\left(\sum_{k \in M_n} \int_0^t g_k(s) dW^k(s) - \frac{1}{2} \sum_{k \in M_n} \int_0^t |g_k(s)|^2 ds\right), \text{ for } t \in [0, T]; \\ \mathcal{D} &= \{e_g(T); \quad g \in G\}. \end{aligned}$$

From Itô formula, we get $de_g(t) = \sum_{k \in M_n} g_k(t) e_g(t) dW^k(t)$. Based on the Wiener chaos decomposition, we recall the following result, see [56, Ch. 1].

Lemma 1. \mathcal{D} is dense in \mathcal{H} .

2.0.4. Lagrangian description and stochastic Fokker-Planck equation

Let $X_t \in \mathbb{R}^2$ and $R_t \in \mathbb{R}^2$ be the position and end-to-end vector of the polymer. In the Introduction we have generically stated that they satisfy (1). Here we shall be more specific. Let (X_t, R_t) satisfying

$$\begin{cases} dR_t &= \nabla u(X_t, t) R_t dt - \frac{1}{\beta} R_t dt + \sqrt{2} \sigma dW_t, \\ dX_t &= u(X_t, t) dt, \end{cases} \tag{7}$$

² We assume that \mathcal{F}_0 contains all the P -null subset of Ω .

where ∇ is the gradient with respect to x -variable. We assume that the velocity field u is the sum of a large scale divergence-free component $u_L(x, t)$ (deterministic, with a reasonable smoothness specified below) plus a stochastic small-scale component, precisely given by the noise coefficients introduced above, in other words

$$u(x, t) := u^N(x, t) = u_L(x, t) + \circ \sum_{k \in K} \sigma_k^N(x) \partial_t W^k, \tag{8}$$

where we choose the Stratonovich multiplication both in virtue of Wong-Zakai principle (a white noise is the idealization of smooth noise) and because of conservation laws. Since the Brownian motions $(W_t)_t$, due to thermal force (microscopic level), in (7) acts on a very short time scale and also that the polymer length is much smaller than the viscous length of the turbulent flow, we write firstly the FP equations with respect to it, with quenched velocity. Then we use the random velocity given by (8) in the resulting FP equation, with Brownian motions $(W_t^k)_t^{k \in K}$ having a different time scale (mesoscopic/macrosopic level). Formally speaking, we may associate a stochastic Fokker-Planck equation in Stratonovich form:

$$\left\{ \begin{aligned} \partial_t f^N(x, r, t) &+ \operatorname{div}_x(u_L(x, t) f^N(x, r, t)) + \operatorname{div}_r((\nabla u_L(t, x)r - \frac{1}{\beta} r) f^N(x, r, t)) \\ &= \sigma^2 \Delta_r f^N(x, r, t) \\ &\quad - \sum_{k \in K} \sigma_k^N \cdot \nabla_x f^N(x, r, t) \circ \partial_t W^k - \sum_{k \in K} (\nabla \sigma_k^N r) \cdot \nabla_r f^N(x, r, t) \circ \partial_t W^k \\ f^N|_{t=0} &= f_0, \end{aligned} \right. \tag{9}$$

where we have used the properties $\operatorname{div}_x(\sigma_k^N) = 0, k \in K, \operatorname{div}_x(u_L) = 0$ and $\operatorname{div}_r(\nabla \sigma_k^N r) = 0$.³ The derivation (rigorously in weak form) of equation (9) from (7) and (8) is done by Itô formula applied to the composition of a test function with the process (X_t, R_t) . Then taking conditional expectation instead of the classical expectation, so that the martingale term corresponding to the noise (8) remains. We refer to [57,58] for similar computations about environmental noise in the case of interacting particles and mean field theory and [59,60] for general mathematical results about Fokker-Planck equation. Therefore, (9) will be used as the main equation we study in this work.

Before we formulate rigorously the meaning of the equation (9), let us rewrite the previous equation (formally) from the Stratonovich to the Itô form. The question is the form of the Itô-Stratonovich corrector. This computation requires some additional care with respect to previously known cases developed in the literature, hence we devote to it a separate section.

3. Itô-Stratonovich correctors

Let ψ be a given smooth function and set $Q(x, y) := \sum_{k \in K} \sigma_k^N(x) \otimes \sigma_k^N(y)$. In this part, we will write the Itô form associated with (9) when Q is space-homogeneous and has a mirror symmetry property.

³ $\operatorname{div}_r(\nabla \sigma_k^N r) = 0$ is consequence of $\operatorname{div}_x(\sigma_k^N) = 0$.

Denote by $L_k \psi = -(\sigma_k^N \cdot \nabla_x \psi + (\nabla \sigma_k^N r) \cdot \nabla_r \psi)$ then the corrector term is given by $\frac{1}{2} \sum_{k \in K} L_k L_k \psi$. Recall that $\text{div}_x(\sigma_k^N) = 0$ and let us compute $L_k L_k \psi$.

$$\begin{aligned} L_k L_k \psi &= -(\sigma_k^N \cdot \nabla_x L_k \psi + (\nabla \sigma_k^N r) \cdot \nabla_r L_k \psi) \\ &= (\sigma_k^N \cdot \nabla_x (\sigma_k^N \cdot \nabla_x \psi + (\nabla \sigma_k^N r) \cdot \nabla_r \psi) + (\nabla \sigma_k^N r) \cdot \nabla_r (\sigma_k^N \cdot \nabla_x \psi + (\nabla \sigma_k^N r) \cdot \nabla_r \psi)) \\ &= \text{div}_x((\sigma_k^N \otimes \sigma_k^N) \nabla_x \psi) + \text{div}_r(((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r \psi) \\ &\quad + \sigma_k^N \cdot \nabla_x ((\nabla \sigma_k^N r) \cdot \nabla_r \psi) + (\nabla \sigma_k^N r) \cdot \nabla_r (\sigma_k^N \cdot \nabla_x \psi). \end{aligned}$$

Lemma 2. Assume the noise is space-homogeneous i.e. $Q(x, y) = Q(x - y)$ and $Q(x, x) = Q(0)$, be a constant matrix, then

$$\begin{aligned} I(\psi) = I^1(\psi) + I^2(\psi) &= \sum_{k \in K} \sigma_k^N \cdot \nabla_x ((\nabla \sigma_k^N r) \cdot \nabla_r \psi) + (\nabla \sigma_k^N r) \cdot \nabla_r (\sigma_k^N \cdot \nabla_x \psi) \\ &= 2 \sum_{l, \gamma, i=1}^2 \partial_{x_\gamma} Q_{i,l}(0) \partial_{x_l} (r_\gamma \partial_{r_i} \psi). \end{aligned}$$

If moreover Q has the mirror symmetry property i.e. $Q(x) = Q(-x)$, then $I(\psi) = 0$.

Proof. Let ψ be a smooth function (we drop the dependence of σ_k on N here for the simplicity of notation). We have

$$\begin{aligned} I^1(\psi) &= \sum_{k \in K} \sum_{l, \gamma, i=1}^2 \sigma_k^l \partial_{x_l} (\partial_{x_\gamma} \sigma_k^i r_\gamma \partial_{r_i} \psi) \\ &= \sum_{k \in K} \sum_{l, \gamma, i=1}^2 \sigma_k^l (\partial_{x_l} \partial_{x_\gamma} \sigma_k^i) r_\gamma \partial_{r_i} \psi + \sum_{k \in K} \sum_{l, \gamma, i=1}^2 \sigma_k^l \partial_{x_\gamma} \sigma_k^i \partial_{x_l} (r_\gamma \partial_{r_i} \psi). \end{aligned}$$

First, let us compute the second term in the last equation. We have

$$\sum_{k \in K} \sigma_k^l(y) \partial_{x_\gamma} \sigma_k^i(x) = \partial_{x_\gamma} \sum_{k \in K} \sigma_k^l(y) \sigma_k^i(x) = \partial_{x_\gamma} Q_{i,l}(x - y),$$

which gives (we recall that Q is space-homogeneous)

$$\sum_{k \in K} \sigma_k^l(x) \partial_{x_\gamma} \sigma_k^i(x) = \partial_{x_\gamma} Q_{i,l}(0). \tag{10}$$

Thus $\sum_{k \in K} \sum_{l, \gamma, i=1}^2 \sigma_k^l \partial_{x_\gamma} \sigma_k^i \partial_{x_l} (r_\gamma \partial_{r_i} \psi) = \sum_{l, \gamma, i=1}^2 \partial_{x_\gamma} Q_{i,l}(0) \partial_{x_l} (r_\gamma \partial_{r_i} \psi)$. Next, let us compute I^2

$$\begin{aligned}
 I^2(\psi) &= \sum_{k \in K} (\nabla \sigma_k^N r) \cdot \nabla_r (\sigma_k^N \cdot \nabla_x \psi) = \sum_{k \in K} \sum_{l, \gamma, i=1}^2 \partial_{x_\gamma} \sigma_k^i r_\gamma \partial_{r_i} (\sigma_k^l \partial_{x_l} \psi) \\
 &= \sum_{k \in K} \sum_{l, \gamma, i=1}^2 \partial_{x_\gamma} \sigma_k^i \sigma_k^l \partial_{x_l} (r_\gamma \partial_{r_i} \psi) = \sum_{l, \gamma, i=1}^2 \partial_{x_\gamma} Q_{i,l}(0) \partial_{x_l} (r_\gamma \partial_{r_i} \psi).
 \end{aligned}$$

Now, let us prove that the first part of I^1 vanishes. Namely

$$\sum_{k \in K} \sum_{l, \gamma, i=1}^2 \sigma_k^l (\partial_{x_l} \partial_{x_\gamma} \sigma_k^i) r_\gamma \partial_{r_i} \psi = \sum_{\gamma, i=1}^2 \left(\sum_l \sum_{k \in K} \sigma_k^l \partial_{x_l} \partial_{x_\gamma} \sigma_k^i \right) r_\gamma \partial_{r_i} \psi = 0.$$

It is sufficient to show that $\sum_l \sum_{k \in K} \sigma_k^l \partial_{x_l} \partial_{x_\gamma} \sigma_k^i = 0$. Indeed, notice that

$$\sum_l \sum_{k \in K} \sigma_k^l \partial_{x_l} \partial_{x_\gamma} \sigma_k^i = \sum_l \partial_{x_\gamma} \sum_{k \in K} \sigma_k^l \partial_{x_l} \sigma_k^i - \sum_l \sum_{k \in K} \partial_{x_\gamma} \sigma_k^l \partial_{x_l} \sigma_k^i = - \sum_l \sum_{k \in K} \partial_{x_\gamma} \sigma_k^l \partial_{x_l} \sigma_k^i,$$

where we used similar arguments to the one used to obtain (10) to get

$$\sum_{k \in K} \sigma_k^l(x) \partial_{x_l} \sigma_k^i(x) = \partial_{x_l} Q_{i,l}(0) \text{ and } \partial_{x_\gamma} \sum_{k \in K} \sigma_k^l \partial_{x_l} \sigma_k^i = 0.$$

On the other hand, note that

$$\sum_l \sum_{k \in K} \partial_{x_\gamma} \sigma_k^l \partial_{x_l} \sigma_k^i = \sum_l \partial_{x_l} \sum_{k \in K} \partial_{x_\gamma} \sigma_k^l \sigma_k^i - \sum_{k \in K} \partial_{x_\gamma} \left(\sum_l \partial_{x_l} \sigma_k^l \right) \sigma_k^i = \sum_l \partial_{x_l} \sum_{k \in K} \partial_{x_\gamma} \sigma_k^l \sigma_k^i,$$

since $\text{div}_x(\sigma_k) = 0$. Again, note that $\sum_{k \in K} \partial_{x_\gamma} \sigma_k^l \sigma_k^i = \partial_{x_\gamma} Q_{l,i}(0)$. Therefore

$$\partial_{x_l} \sum_{k \in K} \partial_{x_\gamma} \sigma_k^l \sigma_k^i = \partial_{x_l} (\partial_{x_\gamma} Q_{l,i}(0)) = 0.$$

Summing up, we get $I(\psi) = 2 \sum_{l, \gamma, i=1}^2 \partial_{x_\gamma} Q_{i,l}(0) \partial_{x_l} (r_\gamma \partial_{r_i} \psi)$. If $Q(x) = Q(-x)$ and Q is smooth function, we see that $\partial_{x_\gamma} Q_{i,l}(0) = 0$ and the second part of Lemma 2 follows.

Moreover, the correctors have a special form.

Lemma 3. *The following equalities hold*

$$\begin{aligned}
 \frac{1}{2} \text{div}_x \left(\sum_{k \in K} (\sigma_k^N \otimes \sigma_k^N) \nabla_x f \right) &= \frac{1}{2} \sum_{k \in K_{++}} (\theta_{|k|}^N)^2 \Delta_x f := \alpha_N \Delta_x f, \\
 \sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) &= A(r) + O\left(\frac{1}{N}\right) P(r),
 \end{aligned}$$

where $A(r) = k_T(3|r|^2 I - 2r \otimes r) = k_T(|r|^2 I + 2r^\perp \otimes r^\perp)$, $k_T = \frac{\pi \log(2)}{8} a^2$, $r = (r_1, r_2)$ and P is a polynomial of second degree.

Proof. Let us simplify the expressions of

$$\mathcal{S}(f) := \frac{1}{2} \operatorname{div}_r \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r f \right) \text{ and } \mathcal{B}(f) := \frac{1}{2} \operatorname{div}_x \left(\sum_{k \in K} (\sigma_k^N \otimes \sigma_k^N) \nabla_x f \right).$$

First, we consider the term \mathcal{B} . Recall that

$$\sigma_k^N(x) = \theta_{|k|}^N \frac{k^\perp}{|k|} \cos k \cdot x, \quad k \in K_+, \quad \sigma_k^N(x) = \theta_{|k|}^N \frac{k^\perp}{|k|} \sin k \cdot x, \quad k \in K_-.$$

We have

$$\begin{aligned} \sum_{k \in K} \sigma_k^N \otimes \sigma_k^N &= \sum_{k \in K_+} (\theta_{|k|}^N)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} \cos^2 k \cdot x + \sum_{k \in K_-} (\theta_{|k|}^N)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} \sin^2 k \cdot x \\ (k \in K_- \rightarrow -k \in K_+) &= \sum_{k \in K_+} (\theta_{|k|}^N)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} \\ (k \in K_{+-} \rightarrow k^\perp \in K_{++}) &= \sum_{k \in K_{++}} (\theta_{|k|}^N)^2 \left(\frac{k^\perp \otimes k^\perp}{|k|^2} + \frac{k \otimes k}{|k|^2} \right) \quad (K_+ = K_{++} \cup K_{+-}). \end{aligned}$$

Thus $\sum_{k \in K} \sigma_k^N \otimes \sigma_k^N = \sum_{k \in K_{++}} (\theta_{|k|}^N)^2 I$; $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and therefore

$$\frac{1}{2} \operatorname{div}_x \left(\sum_{k \in K} (\sigma_k^N \otimes \sigma_k^N) \nabla_x f \right) = \frac{1}{2} \sum_{k \in K_{++}} (\theta_{|k|}^N)^2 \Delta_x f := \alpha_N \Delta_x f.$$

On the other hand, let us present some properties of \mathcal{S} . We have (with slight abuse of notation we use σ_k instead of σ_k^N)

$$\begin{aligned} \left(\sum_{k \in K} (\nabla \sigma_k r) \otimes (\nabla \sigma_k r) \right)_{i,l} &= \sum_{j,\alpha=1}^2 \sum_{k \in K} (\partial_{x_j} \sigma_k^i r_j \partial_{x_\alpha} \sigma_k^l r_\alpha) \\ &= \sum_{k \in K_+} (\theta_{|k|}^N)^2 (k \cdot r)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} \sin^2 k \cdot x + \sum_{k \in K_-} (\theta_{|k|}^N)^2 (k \cdot r)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} \cos^2 k \cdot x \\ (k \in K_- \rightarrow -k \in K_+) &= \sum_{k \in K_+} (\theta_{|k|}^N)^2 (k \cdot r)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} = a^2 \sum_{\substack{k \in K_+ \\ N \leq |k| \leq 2N}} \frac{1}{|k|^6} (k \cdot r)^2 k^\perp \otimes k^\perp. \end{aligned}$$

Now, let us compute $\sum_{\substack{k \in K_+ \\ N \leq |k| \leq 2N}} \frac{1}{|k|^6} (k \cdot r)^2 k^\perp \otimes k^\perp$. Note that

$$\sum_{\substack{k \in K_+ \\ N \leq |k| \leq 2N}} \frac{1}{|k|^6} (k \cdot r)^2 k^\perp \otimes k^\perp = \frac{1}{N^2} \sum_{\substack{k \in K_+ \\ 1 \leq \frac{|k|}{N} \leq 2}} \frac{N^6}{|k|^6} \frac{1}{N^2} (k \cdot r)^2 \frac{1}{N^2} (k^\perp \otimes k^\perp).$$

Note that $h_r(x) = \frac{1}{x^6} (x \cdot r)^2 (x^\perp \otimes x^\perp)$ is smooth function for $1 \leq |x| \leq 2$. By using Riemann sum, we get

$$\frac{1}{N^2} \sum_{\substack{k \in K_+ \\ 1 \leq \frac{|k|}{N} \leq 2}} \frac{N^6}{|k|^6} \frac{1}{N^2} (k \cdot r)^2 \frac{1}{N^2} (k^\perp \otimes k^\perp) = \int_D h_r(x) dx + O\left(\frac{1}{N}\right) P(r),$$

where $D = \{x = (|x| \cos(\varphi), |x| \sin(\varphi)) : 1 \leq |x| \leq 2 \text{ and } \varphi \in D_\pi :=]0, \frac{\pi}{2} [\cup] \frac{3\pi}{2}, 2\pi [\}$ and P is a polynomial of second degree. On the other hand, we have

$$\int_D h_r(x) dx = \int_1^2 \frac{1}{z} \int_{D_\pi} (r_1 \cos(\varphi) + r_2 \sin(\varphi))^2 \begin{pmatrix} \sin^2(\varphi) & -\sin(\varphi) \cos(\varphi) \\ -\sin(\varphi) \cos(\varphi) & \cos^2(\varphi) \end{pmatrix} d\varphi dz.$$

Let us compute the following integral

$$I(r) := \int_{D_\pi} (r_1 \cos(\varphi) + r_2 \sin(\varphi))^2 \begin{pmatrix} \sin^2(\varphi) & -\sin(\varphi) \cos(\varphi) \\ -\sin(\varphi) \cos(\varphi) & \cos^2(\varphi) \end{pmatrix} d\varphi.$$

A standard integration with respect to φ gives

$$I(r) = \frac{\pi}{8} \begin{pmatrix} 3|r|^2 - 2r_1^2 & -2r_1 r_2 \\ -2r_1 r_2 & 3|r|^2 - 2r_2^2 \end{pmatrix}; r = (r_1, r_2).$$

Since $\int_1^2 \frac{1}{z} dz = \log(2)$, we get the final expression of $A(r)$.

Remark 4. When $\theta_k^N = \frac{a}{|k|^2}$ if $N \leq |k| \leq 2N$, $N \in \mathbb{N}^*$ and $\theta_k^N = 0$ elsewhere, we get

$$0 < \frac{\pi}{64} \frac{a^2}{N^3} \leq \alpha_N \leq \frac{\pi}{4} \frac{a^2}{N^3} < a^2 \text{ and } \alpha_N \rightarrow 0. \tag{11}$$

Based on these two lemmata, the Itô form (still formulated only formally) of the stochastic Fokker Planck equation (9), by assuming that the noise is space-homogeneous and satisfies mirror symmetry property, is given by⁴

$$\left\{ \begin{aligned} &df^N + \operatorname{div}_x(u_L(x, t) f^N)dt + \operatorname{div}_r((\nabla u_L(t, x)r - \frac{1}{\beta}r) f^N)dt \\ &= \sigma^2 \Delta_r f^N dt - \sum_{k \in K} \sigma_k^N \cdot \nabla_x f^N dW^k - \sum_{k \in K} (\nabla \sigma_k^N r) \cdot \nabla_r f^N dW^k \\ &\quad + \alpha_N \Delta_x f^N dt + \frac{1}{2} \operatorname{div}_r(\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r f^N)dt \\ &f^N|_{t=0} = f_0, \end{aligned} \right. \tag{12}$$

where

$$\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) = A(r) + O(\frac{1}{N})P(r).$$

4. Main results

Following [53–55]. We introduce the concept of “quasi-regular weak solution” to (12), where we prove the well-posedness. Notice that uniqueness in this class (sometimes called Wiener uniqueness) is weaker than pathwise uniqueness.

Assumptions on large scale component

Let $T > 0$. In the following, we assume that $u_L \in C([0, T], C^2(\mathbb{T}^2; \mathbb{R}^2))$ such that $\operatorname{div}_x(u_L) = 0$.

Definition 5. (Quasi-regular weak solution) Let $f_0 \in H$ and $N \in \mathbb{N}$. We say that f^N is quasi-regular weak solution to (12) if f^N is $(\mathcal{F}_t)_t$ -adapted and

1. $f^N \in L^2_{w-*}(\Omega; L^\infty([0, T]; H)), \nabla_r f^N \in L^2(\Omega \times [0, T]; H),$
2. P a.s. in $\Omega: f^N \in C_w([0, T]; H),$ ⁵
3. P-a.s: for any $t \in]0, T]$ the following equation holds:

$$\begin{aligned} &\int_{\mathbb{T}^2 \mathbb{R}^2} f^N(t) \phi dr dx - \int_{\mathbb{T}^2 \mathbb{R}^2} f_0 \phi dr dx \\ &- \int_0^t \int_{\mathbb{T}^2 \mathbb{R}^2} f^N(s) \left(u_L(s) \cdot \nabla_x \phi + (\nabla u_L(s)r - \frac{1}{\beta}r) \cdot \nabla_r \phi \right) dr dx ds \\ &= -\sigma^2 \int_0^t \int_{\mathbb{T}^2 \mathbb{R}^2} \nabla_r f^N(s) \cdot \nabla_r \phi dr dx ds + \alpha_N \int_0^t \int_{\mathbb{T}^2 \mathbb{R}^2} f^N(s) \cdot \Delta_x \phi dr dx ds \end{aligned}$$

⁴ Recall that f depends on t, x, r, ω and N but we don't stress the dependence on the above variables for the simplicity of notation, that is, with slight abuse of notation $f^N := f^N(t, x, r, \omega)$.

⁵ $C_w([0, T]; H)$ denotes the Bochner space of weakly continuous functions with values in H .

$$\begin{aligned}
 & + \sum_{k \in K} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (f^N(s) \sigma_k^N \cdot \nabla_x \phi + f^N(s) (\nabla \sigma_k^N r) \cdot \nabla_r \phi) dr dx dW^k(s) \\
 & - \frac{1}{2} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r f^N(s) \cdot \nabla_r \phi \right) dr dx ds, \quad \text{for any } \phi \in U.
 \end{aligned}$$

4. (Regularity in Mean) For all $n \in \mathbb{N}^*$ and each function $g \in G_n$, the deterministic function $V^N(t, x, r) = \mathbb{E}[f^N(t, x, r) e_g(t)]$ is a measurable function, which belongs to $L^\infty([0, T]; H) \cap C_w([0, T]; H)$ and $\nabla_r V^N \in L^2([0, T]; H)$ and satisfies the following equation

$$\begin{aligned}
 \frac{dV^N}{dt} + \operatorname{div}_x([u_L - h_n]V^N) + \operatorname{div}_r([\nabla u_L r] - \frac{1}{\beta}r)V^N \\
 = \sigma^2 \Delta_r V^N + \alpha_N \Delta_x V^N + \frac{1}{2} \operatorname{div}_r \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r V^N \right),
 \end{aligned}$$

in a weak sense (see Proposition 17), where

$$\sum_{k \in K_n} g_k \sigma_k^N = h_n \text{ and } \sum_{k \in K_n} g_k (\nabla \sigma_k^N r) = y_n,$$

where $K_n = \{k \in K : \min(n, N) \leq |k| \leq \max(2N, n)\}$.

Remark 6. The point (3) in Definition 5 is satisfied for larger class of test functions, namely $\phi \in V$ thanks to the regularity properties of f^N .

The main results are given by the following theorems.

Theorem 7. *There exists at least one solution f^N to (12) in the sense of Definition 5. Moreover, $(f^N)_N$ and $(\nabla_r f^N)_N$ are bounded in $L^2_{w-*}(\Omega; L^\infty([0, T]; H))$ and $L^2(\Omega \times [0, T]; H)$ respectively.*

Proof. See subsection 6.2.

Theorem 8. *Under the assumptions of Theorem 7, let $f_i^N, i = 1, 2$, be two quasi-regular weak solutions of (12) with the same initial data f_0 . Assume that $(f_i^N(t), \varphi)$ is $\overline{\mathcal{G}}_t$ -adapted, for both $i = 1, 2$, for any $\varphi \in V$. Then $f_1^N = f_2^N$.*

Proof. See subsection 6.3.1.

Concerning the scaling limit as $N \rightarrow +\infty$, we have

Theorem 9. *There exists a new probability space, denoted by the same way (for simplicity) (Ω, \mathcal{F}, P) , $\overline{f} \in L^2_{w-*}(\Omega; L^\infty([0, T]; H))$, $\nabla_r \overline{f} \in L^2(\Omega; L^2([0, T]; H))$ such that the following convergence holds (up to a sub-sequence): $f^N \rightarrow \overline{f}$ in $C([0, T]; U') \quad P - a.s.$ and*

$$f^N \rightharpoonup \bar{f} \text{ in } L^2_{w-*}(\Omega; L^\infty([0, T]; H)), \quad \nabla_r f^N \rightharpoonup \nabla_r \bar{f} \text{ in } L^2(\Omega; L^2([0, T]; H)).$$

Moreover \bar{f} is the unique solution of the following problem: *P*-a.s.

$$\begin{aligned} & \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \bar{f}(x, r, t) \phi(x) \psi(r) dr dx - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_0(x, r) \phi(x) \psi(r) dr dx \tag{13} \\ &= \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \bar{f}(x, r, s) \left(u_L(x, s) \cdot \nabla_x \phi(x) \psi(r) + (\nabla u_L(s, x)r - \frac{1}{\beta}r) \cdot \nabla_r \psi(r) \phi(x) \right) dr dx ds \\ & - \sigma^2 \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r \bar{f}(x, r, s) \cdot \nabla_r \psi(r) \phi(x) dr dx ds \\ & - \frac{1}{2} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} A(r) \nabla_r \bar{f}(x, r, s) \cdot \nabla_r \psi(r) \phi(x) dr dx ds, \end{aligned}$$

for any $\phi \in C^\infty(\mathbb{T}^2)$ and $\psi \in C_c^\infty(\mathbb{R}^2)$ and $A(r)$ is given in Lemma 3.

Proof. See subsection 7.2.1 for the existence proof and Lemma 31 for the uniqueness.

Remark 10. By taking into account the regularity of \bar{f} , (13) can be written as

$$\begin{cases} \partial_t \bar{f}(x, r, t) & + \operatorname{div}_x(u_L(x, t) \bar{f}(x, r, t)) + \operatorname{div}_r((\nabla u_L(t, x)r - \frac{1}{\beta}r) \bar{f}(x, r, t)) \\ & = \sigma^2 \Delta_r \bar{f}(x, r, t) + \frac{1}{2} \operatorname{div}_r(A(r) \nabla_r \bar{f}(x, r, t)) \\ f|_{t=0} & = f_0, \end{cases} \tag{14}$$

in \mathcal{Y}' -sense, where $\mathcal{Y} = \{\varphi \in H : \nabla_r \varphi \in H, \nabla_x \varphi \in L^2(\mathbb{T}^2 \times \mathbb{R}^2)\}$. Moreover, since $\bar{f} \in L^\infty(0, T; H)$ and $\bar{f} \in C([0, T]; U')$, we get $\bar{f} \in C_w([0, T]; H)$ *P*-a.s.(see [61, Lem. 1.4 p. 263]).

- Since (14) is Fokker-Planck equation of (15) then $\bar{f} \geq 0$ and $\frac{d}{dt} \int_{\mathbb{T}^2 \times \mathbb{R}^2} \bar{f}(t, x, r) dx dr = 0$ if $f_0 \geq 0$ and $f_0 \in L^1(\mathbb{T}^2 \times \mathbb{R}^2)$.

Remark 11. (The limit FP, Lagrangian description and macroscopic equation)

- From Lemma 3, we get for each $r \in \mathbb{R}^2$

$$A(r) = \frac{a^2 \pi \log 2}{8} |r|^2 \frac{r \otimes r}{|r|^2} + \frac{2a^2 \pi \log 2}{8} |r|^2 \left(\frac{r^\perp \otimes r^\perp}{|r|^2} \right).$$

Therefore $A(r) = Q(r)Q(r)$, where $Q(r) = \frac{a\sqrt{\pi \log 2}}{2\sqrt{2}} \left(\frac{r \otimes r}{|r|} + \sqrt{2} \frac{r^\perp \otimes r^\perp}{|r|} \right)$. Notice that $Q(r)$ is a symmetric, non negative matrix for each $r \in \mathbb{R}^2$ and the function $r \rightarrow Q(r)$ is Lipschitz with

linear growth.

- Since for each $i \in \{1, 2\}$, $\sum_{j=1}^2 \partial_j A_{i,j}(r) = 0$, if \tilde{W} and $\tilde{\tilde{W}}$ are 2D independent Brownian motions, then the stochastic differential equation (SDE)

$$\begin{cases} d\bar{X}_t = u_L(t, \bar{X}_t)dt; & \bar{X}_0 = x \\ d\bar{R}_t = (\nabla u_L(t, \bar{X}_t)\bar{R}_t - \frac{1}{\beta}\bar{R}_t)dt + \mathcal{Q}(\bar{R}_t)d\tilde{\tilde{W}}_t + \sqrt{2}\sigma d\tilde{W}_t; & \bar{R}_0 = r, \end{cases} \tag{15}$$

has (14) as Fokker-Planck equation.

- Define $\bar{\mathbf{T}} = \mathbb{E}_r(\bar{R}_t \otimes \bar{R}_t)$, a standard computation gives that $\bar{\mathbf{T}}$ satisfies

$$\begin{cases} \partial_t \bar{\mathbf{T}} + u_L \cdot \nabla \bar{\mathbf{T}} = (\nabla u_L)\bar{\mathbf{T}} + \bar{\mathbf{T}}(\nabla u_L)^t - \frac{2}{\beta}(\bar{\mathbf{T}} - kT\mathbf{I}) + \mathbb{E}A(\bar{R}_t) \\ \mathbf{T}|_{t=0} = \mathbf{T}_0, \end{cases} \tag{16}$$

which means that \mathbf{T}^N solution to (5) converges weakly to $\bar{\mathbf{T}}$ in appropriate (weak) sense and the turbulent velocity generates the term $\mathbb{E}A(\bar{R}_t)$, as $N \rightarrow +\infty$, at the macroscopic level.

Let us briefly interpret the previous results. The additional term $\mathcal{Q}(\bar{R}_t)d\tilde{\tilde{W}}_t$ in the stochastic equation contributes to a higher dispersion of the values of \bar{R}_t , increasing the variance; and not in a ‘‘Gaussian’’ way, since the diffusion matrix depends on \bar{R}_t and its quadratic form evaluated in the \bar{R}_t direction increases with \bar{R}_t . This is coherent with the power law result for the stationary solution illustrated in the next subsection. Similarly, the additional term $\mathbb{E}_r A(\bar{R}_t)$ in the equation for the tensor $\mathbb{E}_r(\bar{R}_t \otimes \bar{R}_t)$ is non-negative definite, mostly positive definite, hence it contributes to a higher value of $\partial_t \bar{\mathbf{T}}$ hence to an increase of \bar{R}_t , coherent with the picture above: turbulent stretching statistically increases polymer length.

Remark 12. It is worth mentioning that we consider the Hookean model. If we consider the ‘‘Finitely Extensible Nonlinear Elastic (FENE)’’ model, namely replace $\frac{1}{\beta}R_t$ in (1) by $\frac{1}{\beta} \frac{R_t}{1-|R_t|^2/b}$, where b is a parameter denote the maximal length of the polymer. Then, the analysis must be considered in a bounded domain with respect to r -variable with the appropriate boundary conditions, see e.g. [62]. It is interesting to extend our result to the FENE case. It seems that the scaling limit equation should have similar features as the current one. However, since it requires significant changes, this will be considered in a future work.

4.1. Stationary solutions and power-law tail

The aim of this section is to discuss ‘‘formally’’ that the limit PDE (13) exhibits interesting features of the power-law tail predicted in the physical literature, see for example [2]. More rigorous results and discussions will be the subject of a future work. We consider the limit equation (13) in the simplest case $u_L = 0$ and look for stationary solutions $\bar{f}(r)$, independent of x . We look for rotation-invariant and non-negative solutions $\bar{f}(r) = g(|r|)$, so that \bar{f} should satisfy

$$k_T \operatorname{div}_r \left(\frac{1}{2} \left(3|r|^2 Id - 2r \otimes r \right) \nabla \bar{f}(r) + \frac{\sigma^2}{k_T} \nabla \bar{f}(r) + \frac{r}{\beta k_T} \bar{f}(r) \right) = 0.$$

Then, since $\nabla \bar{f}(r) = \frac{r}{|r|} g'(|r|)$ and $k_T > 0$ we get

$$\operatorname{div}_r \left(\left(\frac{|r|^2}{2} + \frac{\sigma^2}{k_T} \right) \nabla \bar{f}(r) + \frac{r}{\beta k_T} \bar{f}(r) \right) = 0. \tag{17}$$

Set $\frac{1}{\beta k_T} = \alpha$ and $p(r) = \frac{|r|^2}{2} + \frac{\sigma^2}{k_T}$. Then we can write (17) as follows

$$\operatorname{div}_r \left(p^{1-\alpha}(r) \nabla [p^\alpha(r) \bar{f}(r)] \right) = 0. \tag{18}$$

If \bar{f} is smooth function, we can multiply (18) by $p^\alpha(r) \bar{f}$ and integrate over \mathbb{R}^2 to get

$$\int_{\mathbb{R}^2} p^{1-\alpha}(r) |\nabla (p^\alpha(r) \bar{f}(r))|^2 dr = 0.$$

Since $\frac{\sigma^2}{k_T} > 0$, the last equality implies the existence of $C \in \mathbb{R}$ such that $p^\alpha(r) \bar{f}(r) = C$. Thus, since we are looking for positive nontrivial solution, we obtain

$$\bar{f}(|r|) = C p^{-\alpha}(r) = C \left(\frac{|r|^2}{2} + \frac{\sigma^2}{k_T} \right)^{-\frac{1}{\beta k_T}}, C > 0.$$

In relation to physical literature, e.g. [2, eqn. (11)]. Let us just write the last formula only in $|r|$, namely

$$\bar{f}(|r|) \sim |r|^{-\frac{2}{\beta k_T}} = |r|^{-\left(\frac{2}{\beta k_T} - 1\right) - 1} = |r|^{-\gamma - 1}, \quad \gamma = \frac{2}{\beta k_T} - 1. \tag{19}$$

Notice that the power γ depends on the interpretable constants, the relaxation time of the polymer β and the number k_T proportional to the square-intensity of the turbulent eddies. Let us make some comments about the formula (19).

- The case $\gamma = \frac{2}{\beta k_T} - 1 < 0 \Leftrightarrow 2 < \beta k_T$: \bar{f} is not integrable and larger values of $|r|$ are more probable, which means that the most of polymers have large size and the polymers are in *stretched state*. This state holds either when the relaxation time β is large or the intensity k_T of the turbulent fluid is large.
- The case $\gamma = \frac{2}{\beta k_T} - 1 > 0 \Leftrightarrow \beta k_T < 2$: \bar{f} is integrable and thus the normalized \bar{f} is PDF, in other words, most polymers have an equilibrium size and the Polymers are in *coil state*. The last expression of \bar{f} confirms that when either the relaxation time is small (namely the polymer is fast in recovering its equilibrium position) or the intensity k_T of the turbulent fluid is small then the polymers tend to be in *coil state*.
- The case $\gamma = \frac{2}{\beta k_T} - 1 = 0 \Leftrightarrow \beta k_T = 2$: The power γ changes the sign and as discussed in [2], this can be interpreted as the criterion for the coil-stretch transition.

- As mentioned in Introduction, we recall that the power tail is related to Lyapunov exponent λ of the turbulent flow, which is not trivial to compute in general but one would like also to predict it. From [2], one extracts the following link between λ and γ :
 - i. If λ increases then γ decreases.
 - ii. γ changes its sign at $\lambda = \frac{1}{\beta}$.
 - iii. As λ tends to zero, γ tends to infinity.

From (ii), we get that γ changes the sign at $\frac{k_T}{2} = \frac{1}{\beta} = \lambda$. Moreover, as λ is related to the turbulent flow and by combining (i) and (iii), we can say that the Lyapunov exponent λ is our model of turbulent flow is $\lambda = \frac{k_T}{2}$. However, we do not prove such equality “just heuristic!”. Finally, in the case of the point (iii), we get strong suppression of the tail, which is quite natural since in a weak flow the molecules are only weakly stretched.

5. About proofs and mathematical challenges

For the convenience of the reader, let us summarize the outline of the proof and the main technical difficulties we face in proving the main results. The details will be provided in section 6 and section 7. Note that (12) is stochastic Fokker Planck, has a hyperbolic nature with respect to the space variable x , it cannot regularize the initial condition. Moreover, its well-posedness is not a standard result and we need to construct a solution in an appropriate sense, see Definition 5.

The first step concerns the construction of weak solutions (see Definition 5, point (3)) by using Galerkin approximation scheme. We proceed as follows: we construct an orthonormal basis of H (the weight used to give sense to the terms including the coefficient with r -variable, $r \in \mathbb{R}^2$). It’s worth to mention we consider a general setting to prove the existence of solution and up to cosmetic modifications, the same result follows by using other weights e.g. $(1 + |r|^2)^m$, $m > 1$ which corresponds to $L^2_{r,m}(\mathbb{T}^2 \times \mathbb{R}^2) \hookrightarrow L^1(\mathbb{T}^2 \times \mathbb{R}^2)$ if $m > 2$. The first problem concerns the proof of some a priori estimates, for that we add the extra term \mathcal{Y}^m given by (29) to control the term coming from the interaction between the two part of the stochastic integral, in order to get the desired estimates. Then, we may pass to the limit as $m \rightarrow +\infty$ and construct solution to our problem, after showing that \mathcal{Y}^m vanishes as $m \rightarrow +\infty$ (we exploit that Q is space-homogeneous and has mirror symmetry property). Consequently, we construct weak solution, see Theorem 7.

After that, we face the problem of uniqueness due to lack of regularity with respect to the space variable x and the presence of noise, which motivates the introduction of the notion “quasi-regular weak solution”, which is characterized by the point (4) of Definition 5 to show uniqueness in this subclass of solution, see Theorem 8 (we take the advantage of the linearity of (12)). On the other hand, we should use commutators estimates and techniques associated with hyperbolic equations to prove in the beginning the uniqueness of solution to the equation satisfied by some appropriate mean (Definition 5, point (4)) and we combine that with some density arguments in the L^2 -space of random variables to deduce the well-posedness in the sense of Definition 5. We recall that this notion of uniqueness is weaker than the classical notion of pathwise uniqueness, see [55, Rmk. 7]. Another key point concerns the uniform estimate with respect to N , which follows directly from the construction of the solution since Itô formula is classical at the level of Galerkin approximation in contrast with infinite dimension where the solution f^N is not smooth and one needs to be careful in the application of Itô formula.

Finally, after obtaining some estimates regarding to N , we can pass to the limit as $N \rightarrow +\infty$ (in weak sense) by considering the new regime when the noise covariance goes to zero but a

suitable covariance built on derivatives of the noise converges to a non zero limit and we obtain a degenerate PDE with respect to the variable r , see Theorem 9 where the stationary solution of the limit equation (13) has a power-law, with an explicit power depending on the friction parameter β and the number k_T (related to the turbulent kinetic energy), see subsection 4.1.

5.0.1. Formal computations and estimates

Let us present some formal computations related to (12). Namely, we compute $\|f^N\|^2$, by applying Itô formula (formally), we get

$$\begin{aligned} \|f^N(t)\|^2 - \|f_0\|^2 &= 2 \int_0^t (f^N(s), u_L \cdot \nabla_x f^N(s) + (\nabla_x u_L r - \frac{1}{\beta} r) \cdot \nabla_r f^N(s)) ds \\ &- 2\sigma^2 \int_0^t \|\nabla_r f^N(s)\|^2 ds - 2\alpha_N \int_0^t \|\nabla_x f^N(s)\|^2 ds - \int_0^t (A_k^N(x, r) \nabla_r f^N(s), \nabla_r f^N(s)) ds \\ &+ 2 \sum_{k \in K} \int_0^t [(f^N(s), \sigma_k^N \cdot \nabla_x f^N(s)) + (f^N(s), (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s))] dW^k(s) \\ &+ \sum_{k \in K} \int_0^t \|\sigma_k^N \cdot \nabla_x f^N(s)\|^2 + \|(\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)\|^2 ds \\ &+ 2 \sum_{k \in K} \int_0^t (\sigma_k^N \cdot \nabla_x f^N(s), (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)) ds. \end{aligned}$$

Since $\text{div}_x(u_L) = \text{div}_r(\nabla_x u_L r) = \text{div}_x(\sigma_k^N) = \text{div}_r(\nabla \sigma_k^N r) = 0$, one has

$$\begin{aligned} (f^N, u_L \cdot \nabla_x f^N) &= (f^N, \sigma_k^N \cdot \nabla_x f^N) = 0, \\ (f^N, (\nabla_x u_L r) \cdot \nabla_r f^N) &= (f^N, (\nabla \sigma_k^N r) \cdot \nabla_r f^N) = 0. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \sum_{k \in K} \int_0^t \|\sigma_k^N \cdot \nabla_x f^N(s)\|^2 ds &= \sum_{k \in K} \int_0^t (\sigma_k^N \cdot \nabla_x f^N(s), \sigma_k^N \cdot \nabla_x f^N(s)) ds \\ &= \int_0^t (\sum_{k \in K} \sigma_k^N(x) \otimes \sigma_k^N(x) \nabla_x f^N(s), \nabla_x f^N(s)) ds \\ &= \sum_{k \in K_{++}} (\theta_{|k|}^N)^2 \int_0^t (\nabla_x f^N(s), \nabla_x f^N(s)) ds = 2\alpha_N \int_0^t \|\nabla_x f^N(s)\|^2 ds. \end{aligned}$$

Hence $-2\alpha_N \int_0^t \|\nabla_x f^N(s)\|^2 ds + \sum_{k \in K} \int_0^t \|\sigma_k^N \cdot \nabla_x f^N(s)\|^2 ds = 0$. Similarly, we have

$$\sum_{k \in K} \int_0^t \|(\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)\|^2 ds = \int_0^t (A_k^N \nabla_r f^N(s), \nabla_r f^N(s)) ds,$$

where A_k^N is given by

$$A_k^N := \sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right); \tag{20}$$

thus $-\int_0^t (A_k^N \nabla_r f^N(s), \nabla_r f^N(s)) ds + \sum_{k \in K} \int_0^t \|(\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)\|^2 ds = 0$.

Summing up, we get

$$\begin{aligned} & \|f^N(t)\|^2 + 2\sigma^2 \int_0^t \|\nabla_r f^N(s)\|^2 ds - \|f_0\|^2 \\ & \leq -\frac{2}{\beta} \int_0^t (f^N(s), r \cdot \nabla_r f^N(s)) ds + 2 \sum_{k \in K} \int_0^t (\sigma_k^N \cdot \nabla_x f^N(s), (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)) ds. \end{aligned}$$

On the other hand, notice that

$$\sum_{k \in K} (\sigma_k^N \cdot \nabla_x f^N(s), (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)) = -\left(\sum_{k \in K} (\nabla \sigma_k^N r) \cdot \nabla_r (\sigma_k^N \cdot \nabla_x f^N(s)), f^N(s) \right) = 0, \tag{21}$$

since the covariance matrix Q is space-homogeneous and has mirror symmetry property, see Lemma 2. By using (21), we obtain

$$\|f^N(t)\|^2 + 2\sigma^2 \int_0^t \|\nabla_r f^N(s)\|^2 ds \leq \|f_0\|^2 - \frac{2}{\beta} \int_0^t (f^N(s), r \cdot \nabla_r f^N(s)) ds.$$

Let us compute the last term of the RHS. We have

$$\begin{aligned} (f^N, r \cdot \nabla_r f^N) &= \sum_{i=1}^2 \int_{\mathbb{T}^2 \times \mathbb{R}^2} f^N(x, r) r_i \cdot \partial_{r_i} f^N(x, r) dr dx \\ &= \frac{1}{2} \sum_{i=1}^2 \int_{\mathbb{T}^2 \times \mathbb{R}^2} r_i \partial_{r_i} (f^N)^2(x, r) dr dx = -\|f^N\|^2. \end{aligned}$$

Consequently, $\|f^N(t)\|^2 + 2\sigma^2 \int_0^t \|\nabla_r f^N(s)\|^2 ds \leq \|f_0\|^2 + \frac{2}{\beta} \int_0^t \|f^N(s)\|^2 ds$. Grönwall lemma ensures the following: P-a.s.

$$\forall t \geq 0: \quad \|f^N(t)\|^2 + 2\sigma^2 \int_0^t \|\nabla_r f^N(s)\|^2 ds \leq \|f_0\|^2 e^{\frac{2}{\beta}t}. \tag{22}$$

Note that the last inequality (22) is sufficient, *a priori*, to construct a weak solution and also ensure the uniqueness, since (12) is a linear equation. Unfortunately, (22) is not rigorous. Indeed, if f^N satisfies (22) only then we are not able to apply directly Itô formula to (12) and we need either to approximate or regularize (12) in appropriate way and prove the existence of f^N and (22), after passing to the limit with respect to the approximation or regularization parameters. It turns out that it is not sufficient to consider $f_0 \in L^2(\mathbb{T} \times \mathbb{R}^2)$ and we need more regular initial data, namely $f_0 \in H$ to construct a weak solution, see Definition 5. Uniqueness is more delicate, since (12) has a hyperbolic character with respect to x -variable and we will prove uniqueness in particular class of solution, see Theorem 8. For the convenience of the reader, let us explain the arguments that failed when we tried to obtain (22) in rigorous way.

5.0.2. *A priori estimates via Galerkin approximation*

The first step concerns the projection of (12) onto a finite dimensional space, see (28). Then, we apply finite dimensional Itô formula which leads to the presence of the term

$$\sum_{k \in K} (\sigma_k^N \cdot \nabla_x P_m[(\nabla \sigma_k^N r) \cdot \nabla_r f_m], w_j)_H \quad (f_m = P_m f^N) \tag{23}$$

instead of $\sum_{k \in K} (\sigma_k^N \cdot \nabla_x f^N(s), (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s))$ and (21) is not valid anymore. We remedy this issue by subtraction this term (23) at the level of Galerkin approximation to get an appropriate estimate, see Lemma 15 and we need to show that $\lim_m (\sigma_k^N \cdot \nabla_x P_m[(\nabla \sigma_k^N r) \cdot \nabla_r f_m], w_j)_H = 0$ to recover the original problem (12). It's worth to mention that since $r \in \mathbb{R}^2$ and it acts as coefficient then we need to use initial data $f_0 \in H$ to prove that the extra term (23) vanishes as $m \rightarrow +\infty$, see (46) and one obtains the desired estimates of Lemma 15. We refer to section 6 for more details.

5.0.3. *Uniqueness via commutators and quasi-regular weak solution*

Once the existence of weak solution satisfying the points (1), (2) and (3) of Definition 5 is established, one seeks to prove the uniqueness of this class of solutions. Since (12) has an hyperbolic nature, one uses the commutators estimates in order to prove pathwise uniqueness. Let us explain why using this technique does not give uniqueness of this class of solution, for that we mention only the terms that cause problems. Let $\delta > 0$ and $\varphi \in C_c^\infty(\mathbb{T}^2 \times \mathbb{R}^2)$, if we denote $X = (x, r) \in \mathbb{T}^2 \times \mathbb{R}^2$, $\rho_\delta(X) = \rho_\delta(x)$ and $\varphi_\delta := \rho_\delta * \varphi$ then we need to pass to the limit as $\delta \rightarrow 0$ in the equality

$$\begin{aligned}
 & \frac{1}{2} \mathbb{E} \|[f^N(t)]_\delta\|^2 + \mathbb{E} \int_0^t \langle [u_L(s) \cdot \nabla_x f^N(s)]_\delta + [\operatorname{div}_r(\nabla u_L(s)r - \frac{1}{\beta}r)f^N(s)]_\delta, [f^N(s)]_\delta \rangle ds \\
 &= \mathbb{E} \int_0^t \langle \sigma^2[\Delta_r f^N(s)]_\delta + \alpha_N[\Delta_x f^N(s)]_\delta, [f^N(s)]_\delta \rangle ds \\
 &+ \frac{1}{2} \mathbb{E} \int_0^t \langle [\operatorname{div}_r \sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r f^N(s)]_\delta, [f^N(s)]_\delta \rangle ds \\
 &+ \frac{1}{2} \sum_{k \in K} \mathbb{E} \int_0^t \|[\sigma_k^N \cdot \nabla_x f^N(s)]_\delta + [(\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)]_\delta\|^2 ds.
 \end{aligned}$$

We can estimate all the terms in convenient way except the terms

$$\begin{aligned}
 & \mathbb{E} \int_0^t \langle \alpha_N[\Delta_x f^N(s)]_\delta, [f^N(s)]_\delta \rangle ds + \frac{1}{2} \sum_{k \in K} \mathbb{E} \int_0^t \|[\sigma_k^N \cdot \nabla_x f^N(s)]_\delta\|^2 ds \tag{24} \\
 &+ \sum_{k \in K} \mathbb{E} \int_0^t \|[\sigma_k^N \cdot \nabla_x f^N(s)]_\delta, [(\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)]_\delta \rangle ds.
 \end{aligned}$$

The last two terms come from the term $\frac{1}{2} \sum_{k \in K} \mathbb{E} \int_0^t \|[\sigma_k^N \cdot \nabla_x f^N(s)]_\delta + [(\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)]_\delta\|^2 ds$.

A priori, we expect that the sum of the first two terms in (24) vanishes as $\delta \rightarrow 0$ (balance of energy of Itô Stratonovich corrector) but in order to get that, we need more regularity of f^N , namely $\nabla_x f^N \in L^2(\mathbb{T}^2 \times \mathbb{R}^2)$, which is not our case. Concerning the last term in (24), it is expected to disappear due to the space homogeneity and the mirror symmetry of the covariance Q but also it requires $\nabla_x f^N \in L^2(\mathbb{T}^2 \times \mathbb{R}^2)$ as well. Consequently, we take the advantage of the linearity of the equation (12) and we prove uniqueness in weaker sense, see Theorem 8.

6. Existence and uniqueness of solutions to (12)

Our aim in this section is to prove the existence and uniqueness of solution to (12) in the sense of Definition 5, namely Theorem 7 and Theorem 8. We divide the proof into several steps. First, we introduce the Galerkin approximation. Then, we prove some estimates in the appropriate spaces. Next, we show the existence of analytically weak solution to (12). Finally, we prove the uniqueness of the class of quasi-regular weak solutions after showing the existence and uniqueness of the solution to an appropriate mean equation associated with (12), see Proposition 17 and Lemma 21.

6.0.1. Galerkin basis and approximation

We need to construct an orthonormal basis of H such that all the terms in our approximation scheme are meaningful. For that, note that $V \xrightarrow{\text{cont.}} H$. On the other hand, $(V, (\cdot, \cdot)_V)$ is a separable

Hilbert space. By using [63, Lem. C.1.], there exists a Hilbert space $(U, (\cdot, \cdot)_U)$ such that $U \hookrightarrow V$, U is dense in V and the embedding $U \hookrightarrow V$ is compact. Thus the embedding $U \hookrightarrow H$ is also compact and we can construct an orthonormal basis for H by using the eigenvectors of the compact embedding operator. More precisely, there exists an orthonormal basis $\{w_i\}_{i \in \mathbb{N}}$ of H such that $w_i \in U$ and satisfies

$$(v, w_i)_U = \lambda_i (v, w_i)_H, \quad \forall v \in U, \quad i \in \mathbb{N}, \tag{25}$$

where the sequence $\{\lambda_i\}_{i \in \mathbb{N}}$ of the corresponding eigenvalues fulfils the properties: $\lambda_i > 0, \forall i \in \mathbb{N}$. Note that $\{\tilde{w}_i = \frac{1}{\sqrt{\lambda_i}} w_i\}$ is an orthonormal basis for U , see [63, Subject. 2.3] for similar argument of construction. Now, let $m \in \mathbb{N}^*$ and denote by $H_m = \text{span}\{w_1, \dots, w_m\}$ and the operator P_m defined from U' to H_m defined by $P_m : U' \rightarrow H_m; \quad u \mapsto P_m u = \sum_{i=1}^m \langle u, w_i \rangle_{U', U} w_i$. In particular, the restriction of P_m to H , denoted by the same way, is the (\cdot, \cdot) -orthogonal projection from H to H_m and given by

$$P_m : H \rightarrow H_m; \quad u \mapsto P_m u = \sum_{i=1}^m (u, w_i)_H w_i. \tag{26}$$

We notice that $\|P_m u\|_H \leq \|u\|_H, \forall u \in H$, then $\|P_m\|_{L(H, H)} \leq 1$.

Remark 13. It is worth to mention that the restriction of P_m to U is also an orthogonal projection, thanks to (25) and thus $\|P_m\|_{L(U, U)} \leq 1$.

We have the following continuous embedding $U \hookrightarrow V \hookrightarrow H \hookrightarrow L^2(\mathbb{T}^2 \times \mathbb{R}^2)$. Since U is dense subset of $L^2(\mathbb{T}^2 \times \mathbb{R}^2)$, we can consider the following compact Lions-Gelfand triple, namely

$$U \underset{\text{dense}}{\hookrightarrow} L^2(\mathbb{T}^2 \times \mathbb{R}^2) \equiv L^2(\mathbb{T}^2 \times \mathbb{R}^2) \hookrightarrow U'$$

To simplify the notation, the duality between U and U' will be denoted $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{U', U}$. Thus, we have the following equality

$$\langle f, u \rangle = (f, u), \quad \forall f \in L^2(\mathbb{T}^2 \times \mathbb{R}^2), \forall u \in U. \tag{27}$$

We use Faedo-Galerkin method, we introduce the approximation $f_m(t) = \sum_{j=1}^m g_{mj}(t) w_j$ and set $f_m(0) = P_m f_0 \in H_m$. We consider the following finite dimensional SDE

$$(f_m(t), w_j)_H - (f_0, w_j)_H - \int_0^t (f_m(s), u_L \cdot \nabla_x w_j)_H ds - \int_0^t (f_m(s), \mathcal{Y}^m w_j)_H ds; \quad 1 \leq j \leq m \tag{28}$$

$$\begin{aligned}
 &= - \int_0^t ((\nabla_x u_L r) \cdot \nabla_r f_m(s), w_j)_H + \frac{2}{\beta} (f_m(s), w_j)_H + \frac{1}{\beta} (r \cdot \nabla_r f_m(s), w_j)_H ds \\
 &\quad - \int_0^t [\sigma^2 (\nabla_r f_m(s), \nabla_r w_j)_H + 2\sigma^2 (\nabla_r f_m(s), r w_j) + \alpha_N (\nabla_x f_m(s), \nabla_x w_j)_H] ds \\
 &\quad - \frac{1}{2} \int_0^t (A_k^N \nabla_r f_m(s), \nabla_r w_j)_H ds - \int_0^t (A_k^N \nabla_r f_m(s), r w_j) ds, \\
 &\quad + \sum_{k \in K} \int_0^t ((f_m(s), \sigma_k^N \cdot \nabla_x w_j)_H - ((\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), w_j)_H) dW^k(s),
 \end{aligned}$$

where

$$\mathcal{Y}^m \phi := \sum_{k \in K} P_m \left[(\nabla \sigma_k^N r) \cdot \nabla_r (P_m(\sigma_k^N \cdot \nabla_x \phi)) + 2 \frac{(\nabla \sigma_k^N r) \cdot r}{1 + |r|^2} P_m(\sigma_k^N \cdot \nabla_x \phi) \right], \forall \phi \in U. \quad (29)$$

Note that (28) is a linear system of SDE, by classical result (see e.g. [64, Chapter V]), we get the existence and uniqueness of \mathcal{F}_t -adapted solution $f_m \in C([0, T], L^2(\Omega; H_m))$.

Remark 14. Note that

$$\begin{aligned}
 (f_m, \mathcal{Y}^m w_j)_H &= \sum_{k \in K} (f_m, P_m \left[(\nabla \sigma_k^N r) \cdot \nabla_r (P_m(\sigma_k^N \cdot \nabla_x w_j)) + 2 \frac{(\nabla \sigma_k^N r) \cdot r}{1 + |r|^2} P_m(\sigma_k^N \cdot \nabla_x w_j) \right])_H \\
 &= \sum_{k \in K} (f_m, (\nabla \sigma_k^N r) \cdot \nabla_r (P_m(\sigma_k^N \cdot \nabla_x w_j)))_H + 2 (f_m, \frac{(\nabla \sigma_k^N r) \cdot r}{1 + |r|^2} P_m(\sigma_k^N \cdot \nabla_x w_j))_H \\
 &= \sum_{k \in K} (\sigma_k^N \cdot \nabla_x P_m [(\nabla \sigma_k^N r) \cdot \nabla_r f_m], w_j)_H.
 \end{aligned}$$

Indeed, using that $\text{div}(\nabla \sigma_k^N r) = 0$ and integrating by parts with respect to the r variable, we get

$$\begin{aligned}
 \sum_{k \in K} (f_m, (\nabla \sigma_k^N r) \cdot \nabla_r (P_m(\sigma_k^N \cdot \nabla_x w_j)))_H &= - \sum_{k \in K} ((\nabla \sigma_k^N r) \cdot \nabla_r f_m, (P_m(\sigma_k^N \cdot \nabla_x w_j)))_H \\
 &\quad - 2 \sum_{k \in K} ((\nabla \sigma_k^N r) \cdot r f_m, (P_m(\sigma_k^N \cdot \nabla_x w_j))).
 \end{aligned}$$

Since $((\nabla \sigma_k^N r) \cdot r f_m, (P_m(\sigma_k^N \cdot \nabla_x w_j))) = (f_m, \frac{(\nabla \sigma_k^N r) \cdot r}{1 + |r|^2} P_m(\sigma_k^N \cdot \nabla_x w_j))_H$, we get

$$\begin{aligned}
 (f_m, \mathcal{Y}^m w_j)_H &= - \sum_{k \in K} ((\nabla \sigma_k^N r) \cdot \nabla_r f_m, (P_m(\sigma_k^N \cdot \nabla_x w_j)))_H \\
 &= \sum_{k \in K} (\sigma_k^N \cdot \nabla_x P_m[(\nabla \sigma_k^N r) \cdot \nabla_r f_m], w_j)_H,
 \end{aligned}$$

where we used first that P_m is self-adjoint with respect to $(\cdot, \cdot)_H$ scalar product and then integrate by parts with respect to the x variable.

6.1. A priori estimates

We apply the finite dimensional Itô formula to the process f_m to get

$$\begin{aligned}
 \|f_m(t)\|_H^2 - \|P_m f_0\|_H^2 &= 2 \int_0^t (f_m(s), u_L \cdot \nabla_x f_m(s))_H ds \tag{30} \\
 &- 2 \int_0^t ((\nabla_x u_L r) \cdot \nabla_r f_m(s), f_m(s))_H + \frac{2}{\beta} \|f_m(s)\|_H^2 + \frac{1}{\beta} (r \cdot \nabla_r f_m(s), f_m(s))_H ds \\
 &+ 2 \int_0^t (f_m(s), \mathcal{Y}^m f_m)_H ds - 2 \int_0^t [\sigma^2 \|\nabla_r f_m(s)\|_H^2 + 2\sigma^2 (\nabla_r f_m(s), r f_m(s)) + \alpha_N \|\nabla_x f_m(s)\|_H^2] ds \\
 &- \int_0^t (A_k^N \nabla_r f_m(s), \nabla_r f_m(s))_H ds - 2 \int_0^t (A_k^N \nabla_r f_m(s), r f_m(s)) ds \\
 &+ 2 \sum_{k \in K} \int_0^t ((f_m(s), \sigma_k^N \cdot \nabla_x f_m(s))_H - ((\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), f_m(s))_H) dW^k(s) \\
 &+ \sum_{k \in K} \int_0^t \|P_m \sigma_k^N \cdot \nabla_x f_m(s)\|_H^2 + \|P_m (\nabla \sigma_k^N r) \cdot \nabla_r f_m(s)\|_H^2 ds \\
 &+ 2 \sum_{j=1}^m \sum_{k \in K} \int_0^t (\sigma_k^N \cdot \nabla_x f_m(s), w_j)_H ((\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), w_j)_H ds.
 \end{aligned}$$

Since $\text{div}_x(u_L) = \text{div}_x(\sigma_k^N) = 0$, one has $(f_m, u_L \cdot \nabla_x f_m)_H = (f_m, \sigma_k^N \cdot \nabla_x f_m)_H = 0$. On the other hand, note that

$$\sum_{k \in K} \int_0^t \|\sigma_k^N \cdot \nabla_x f_m(s)\|_H^2 ds = \sum_{k \in K_{++}} (\theta_{|k|}^N)^2 \int_0^t (\nabla_x f_m(s), \nabla_x f_m(s))_H ds$$

$$= 2\alpha_N \int_0^t \|\nabla_x f_m(s)\|_H^2 ds,$$

where we used that $(\sigma_k^N \cdot \nabla_x f_m, \sigma_k^N \cdot \nabla_x f_m)_H = (\sigma_k^N \otimes \sigma_k^N \nabla_x f_m, \nabla_x f_m)_H$. Therefore

$$-2\alpha_N \int_0^t \|\nabla_x f_m(s)\|_H^2 ds + \sum_{k \in K} \int_0^t \|P_m \sigma_k^N \cdot \nabla_x f_m(s)\|_H^2 ds \leq 0,$$

since $\|P_m \sigma_k^N \cdot \nabla_x f_m(s)\|_H^2 \leq \|\sigma_k^N \cdot \nabla_x f_m(s)\|_H^2$ due to that P_m is the $(\cdot, \cdot)_H$ -orthogonal projection from H to H_m . Similarly, we have

$$\sum_{k \in K} \int_0^t \|(\nabla \sigma_k^N r) \cdot \nabla_r f_m(s)\|_H^2 ds = \int_0^t (A_k^N \nabla_r f_m(s), \nabla_r f_m(s))_H ds,$$

where A_k^N is given by (20), thus

$$- \int_0^t (A_k^N \nabla_r f_m(s), \nabla_r f_m(s))_H ds + \sum_{k \in K} \int_0^t \|P_m (\nabla \sigma_k^N r) \cdot \nabla_r f_m(s)\|_H^2 ds \leq 0.$$

On the other hand, note that

$$\begin{aligned} & \sum_{j=1}^m \sum_{k \in K} \int_0^t (\sigma_k^N \cdot \nabla_x f_m(s), w_j)_H ((\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), w_j)_H ds \\ &= \sum_{k \in K} \int_0^t \left(\sum_{j=1}^m (\sigma_k^N \cdot \nabla_x f_m(s), w_j)_H w_j, (\nabla \sigma_k^N r) \cdot \nabla_r f_m(s) \right)_H ds \\ &= - \sum_{k \in K} \int_0^t (f_m(s), P_m \left[(\nabla \sigma_k^N r) \cdot \nabla_r (P_m (\sigma_k^N \cdot \nabla_x f_m(s))) + 2 \frac{(\nabla \sigma_k^N r) \cdot r}{1 + |r|^2} P_m (\sigma_k^N \cdot \nabla_x f_m(s)) \right])_H ds \\ &= - \int_0^t (f_m(s), \mathcal{Y}^m f_m(s)) ds. \end{aligned}$$

Summing up, we get

$$\|f_m(t)\|_H^2 - \|P_m f_0\|_H^2 + 2\sigma^2 \int_0^t \|\nabla_r f_m(s)\|_H^2 ds = -2 \int_0^t ((\nabla_x u_L r) \cdot \nabla_r f_m(s), f_m(s))_H \quad (31)$$

$$\begin{aligned}
 &+ \frac{2}{\beta} \|f_m(s)\|_H^2 + \frac{1}{\beta} (r \cdot \nabla_r f_m(s), f_m(s))_H ds - 4\sigma^2 \int_0^t (\nabla_r f_m(s), r f_m(s)) ds \\
 &- 2 \int_0^t (A_k^N \nabla_r f_m(s), r f_m(s)) ds - 2 \sum_{k \in K} \int_0^t ((\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), f_m(s))_H dW^k(s).
 \end{aligned}$$

Let us estimate the terms of the right hand side in (31). First, note that

$$\begin{aligned}
 \int_0^t ((\nabla_x u_L r) \cdot \nabla_r f_m(s), f_m(s))_H ds &= - \int_0^t (f_m(s), f_m(s) (\nabla_x u_L r) \cdot r) ds \\
 &\leq \| \nabla_x u_L \|_\infty \int_0^t \| f_m(s) \|_H^2 ds.
 \end{aligned} \tag{32}$$

A standard integration by parts gives

$$\int_0^t \frac{1}{\beta} (r \cdot \nabla_r f_m(s), f_m(s))_H ds = - \frac{1}{\beta} \int_0^t (\|f_m(s)\|_H^2 + (|r|^2 f_m(s), f_m(s)) ds,$$

and $|\int_0^t \frac{1}{\beta} (r \cdot \nabla_r f_m(s), f_m(s))_H ds| \leq \frac{2}{\beta} \int_0^t \|f_m(s)\|_H^2 ds$. Let us estimate $\int_0^t (\nabla_r f_m(s), r f_m(s)) ds$.

Note that $\int_0^t (\nabla_r f_m(s), r f_m(s)) ds = - \int_0^t (f_m(s), f_m(s)) ds$, thus

$$-4\sigma^2 \int_0^t (\nabla_r f_m(s), r f_m(s)) ds = 4\sigma^2 \int_0^t \|f_m(s)\|_H^2 ds. \tag{33}$$

Next, the term $-2 \int_0^t (A_k^N \nabla_r f_m(s), r f_m(s)) ds$. Since $\text{div}_r [\nabla \sigma_k^N r] = 0$, we get

$$\begin{aligned}
 \int_0^t (A_k^N \nabla_r f_m(s), r f_m(s)) ds &= \sum_{k \in K} \int_0^t ((\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), (\nabla \sigma_k^N r) \cdot r f_m(s)) ds \\
 &= - \sum_{k \in K} \int_0^t \left(((\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), (\nabla \sigma_k^N r) \cdot r f_m(s)) + ((\nabla \sigma_k^N r) f_m(s), f_m(s) \nabla_r [(\nabla \sigma_k^N r) \cdot r]) \right) ds,
 \end{aligned}$$

and $2 \int_0^t (A_k^N \nabla_r f_m(s), r f_m(s)) ds = - \sum_{k \in K} \int_0^t ((\nabla \sigma_k^N r) f_m(s), f_m(s) \nabla_r [(\nabla \sigma_k^N r) \cdot r]) ds$. On the other hand, note that

$$\begin{aligned} & \sum_{k \in K} \int_0^t ((\nabla \sigma_k^N r) f_m(s), f_m(s) \nabla_r [(\nabla \sigma_k^N r) \cdot r]) ds \\ &= \sum_{k \in K_+} \int_0^t (\theta_{|k|}^N)^2 |k|^2 \left(\frac{k}{|k|} \cdot r \frac{k^\perp}{|k|} \sin k \cdot x f_m(s), f_m(s) \sin k \cdot x \nabla_r \left[\frac{k}{|k|} \cdot r \frac{k^\perp}{|k|} \cdot r \right] \right) ds \\ & \quad + \sum_{k \in K_-} \int_0^t (\theta_{|k|}^N)^2 |k|^2 \left(\frac{k}{|k|} \cdot r \frac{k^\perp}{|k|} \cos k \cdot x f_m(s), f_m(s) \cos k \cdot x \nabla_r \left[\frac{k}{|k|} \cdot r \frac{k^\perp}{|k|} \cdot r \right] \right) ds \\ &= \sum_{k \in K_+} \int_0^t (\theta_{|k|}^N)^2 |k|^2 \left(\frac{k}{|k|} \cdot r \frac{k^\perp}{|k|} f_m(s), f_m(s) \nabla_r \left[\frac{k}{|k|} \cdot r \frac{k^\perp}{|k|} \cdot r \right] \right) ds \\ &= \sum_{k \in K_+} \int_0^t (\theta_{|k|}^N)^2 |k|^2 \left(\left(\frac{k}{|k|} \cdot r \frac{k^\perp \cdot k}{|k|^2} f_m(s), f_m(s) \frac{k^\perp}{|k|} \cdot r \right) \right. \\ & \quad \left. + \left(\frac{k}{|k|} \cdot r \frac{k^\perp \cdot k^\perp}{|k|^2} f_m(s), f_m(s) \frac{k}{|k|} \cdot r \right) \right) ds \\ &\leq 2 \sum_{k \in K_+} \int_0^t (\theta_{|k|}^N)^2 |k|^2 \|r\| f_m(s)^2 ds \leq \sum_{\substack{k \in K_+ \\ N \leq |k| \leq 2N}} \frac{2}{|k|^2} \int_0^t \|f_m(s)\|_H^2 ds \leq C \int_0^t \|f_m(s)\|_H^2 ds, \end{aligned}$$

thus

$$\left| \sum_{k \in K} \int_0^t ((\nabla \sigma_k^N r) f_m(s), f_m(s) \nabla_r [(\nabla \sigma_k^N r) \cdot r]) ds \right| \leq C \int_0^t \|f_m(s)\|_H^2 ds, \tag{34}$$

where $C > 0$ (independent of N) satisfies $\sum_{\substack{k \in K_+ \\ N \leq |k| \leq 2N}} \frac{2}{|k|^2} \leq C$. Concerning the stochastic integral, note that $((\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), f_m(s))_H = -(f_m(s), f_m(s) (\nabla \sigma_k^N r) \cdot r)$. Recall that

$$(\nabla \sigma_k^N r)(x) = -k \cdot r \theta_{|k|}^N \frac{k^\perp}{|k|} \sin k \cdot x, \quad k \in K_+; \quad (\nabla \sigma_k^N r)(x) = k \cdot r \theta_{|k|}^N \frac{k^\perp}{|k|} \cos k \cdot x, \quad k \in K_- \tag{35}$$

and

$$\sum_{k \in K} \int_0^t (-((\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), f_m(s))_H) dW^k(s) = 2 \sum_{k \in K} \int_0^t ((f_m(s), f_m(s)(\nabla \sigma_k^N r) \cdot r) dW^k(s).$$

By using the Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned} & 2\mathbb{E} \sup_{q \in [0, t]} \left| \int_0^q \sum_{k \in K} (f_m(s), f_m(s)(\nabla \sigma_k^N r) \cdot r) dW^k(s) \right| \\ & \leq 2\mathbb{E} \left[\int_0^t \sum_{k \in K} |(f_m(s), f_m(s)(\nabla \sigma_k^N r) \cdot r)|^2 ds \right]^{1/2}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \int_0^t \sum_{k \in K} |(f_m(s), f_m(s)(\nabla \sigma_k^N r) \cdot r)|^2 ds \\ & = \sum_{k \in K_+} (\theta_{|k|}^N)^2 |k|^2 \int_0^t \left(-\frac{k}{|k|} \cdot \frac{r}{(1 + |r|^2)} (1 + |r|^2)^{1/2} f_m(s), r \cdot \frac{k^\perp}{|k|} \sin k \cdot x (1 + |r|^2)^{1/2} f_m(s) \right)^2 ds \\ & + \sum_{k \in K_-} (\theta_{|k|}^N)^2 |k|^2 \int_0^t \left(\frac{k}{|k|} \cdot \frac{r}{(1 + |r|^2)} (1 + |r|^2)^{1/2} f_m(s), r \cdot \frac{k^\perp}{|k|} \cos k \cdot x (1 + |r|^2)^{1/2} f_m(s) \right)^2 ds \\ & \leq \sum_{k \in K} (\theta_{|k|}^N)^2 |k|^2 \int_0^t \|(1 + |r|^2)^{1/2} f_m(s)\|^2 \|(1 + |r|^2)^{1/2} f_m(s)\|^2 ds \\ & \leq \sum_{k \in K} (\theta_{|k|}^N)^2 |k|^2 \int_0^t \|(1 + |r|^2)^{1/2} f_m(s)\|^4 ds = \sum_{\substack{k \in K \\ N \leq |k| \leq 2N}} (\theta_{|k|}^N)^2 |k|^2 \int_0^t \|f_m(s)\|_H^2 \|f_m(s)\|_H^2 ds \\ & \leq \sum_{\substack{k \in K \\ N \leq |k| \leq 2N}} \frac{1}{|k|^2} \int_0^t \|f_m(s)\|_H^2 \|f_m(s)\|_H^2 ds \leq C \int_0^t \|f_m(s)\|_H^2 \|f_m(s)\|_H^2 ds \\ & \leq C \sup_{q \in [0, t]} \|f_m(q)\|_H^2 \int_0^t \|f_m(s)\|_H^2 ds, \end{aligned}$$

which gives

$$\int_0^t \sum_{k \in K} |(f_m(s), f_m(s)(\nabla \sigma_k^N r)) \cdot r|^2 ds \leq C \sup_{q \in [0,t]} \|f_m(q)\|_H^2 \int_0^t \|f_m(s)\|_H^2 ds, \tag{36}$$

where we used $\sum_{\substack{k \in K \\ N \leq |k| \leq 2N}} \frac{1}{|k|^2} \leq C$, with $C > 0$ independent of N . Therefore

$$\begin{aligned} & 2\mathbb{E} \sup_{q \in [0,t]} \left| \int_0^q \sum_{k \in K} (f_m(s), f_m(s)(\nabla \sigma_k^N r)) \cdot r dW^k(s) \right| \\ & \leq 2\mathbb{E} \left[\int_0^t \sum_{k \in K} |(f_m(s), f_m(s)(\nabla \sigma_k^N r)) \cdot r|^2 ds \right]^{1/2} \\ & \leq 2\mathbb{E} \left[C \sup_{q \in [0,t]} \|f_m(q)\|_H^2 \int_0^t \|f_m(s)\|_H^2 ds \right]^{1/2} \\ & \leq \epsilon \mathbb{E} \sup_{q \in [0,t]} \|f_m(q)\|_H^2 + \frac{C}{\epsilon} \mathbb{E} \int_0^t \|f_m(s)\|_H^2 ds, \end{aligned}$$

for any $\epsilon > 0$ (to be chosen later). By using (31) and gathering the previous estimates, we get

$$\begin{aligned} & (1 - \epsilon) \mathbb{E} \sup_{q \in [0,t]} \|f_m(q)\|_H^2 + 2\sigma^2 \mathbb{E} \int_0^t \|\nabla_r f_m(s)\|_H^2 \\ & \leq \|P_m f_0\|_H^2 + [2\|\nabla_x u_L\|_\infty + \frac{8}{\beta} + 4\sigma^2 + 2C + \frac{C}{\epsilon}] \mathbb{E} \int_0^t \|f_m(s)\|_H^2 ds. \end{aligned}$$

By choosing $\epsilon = \frac{1}{2}$ and setting

$$\lambda = 2[2\|\nabla_x u_L\|_\infty + \frac{8}{\beta} + 4\sigma^2 + 4C], \tag{37}$$

we obtain

$$\mathbb{E} \sup_{q \in [0,t]} \|f_m(q)\|_H^2 + 4\sigma^2 \mathbb{E} \int_0^t \|\nabla_r f_m(s)\|_H^2 \leq 2\|f_0\|_H^2 + \lambda \mathbb{E} \int_0^t \|f_m(s)\|_H^2 ds.$$

Finally, Grönwall lemma ensures

$$\forall t \geq 0 : \mathbb{E} \sup_{q \in [0, t]} \|f_m(q)\|_H^2 + 4\sigma^2 \mathbb{E} \int_0^t \|\nabla_r f_m(s)\|_H^2 ds \leq 2\|f_0\|_H^2 e^{\lambda t}.$$

As a conclusion, we get

Lemma 15. *For every $m \in \mathbb{N}^*$, there exists a unique solution $f_m \in C([0, T], L^2(\Omega; H_m))$ to (28), which is adapted to the filtration and satisfy*

$$\forall t \geq 0 : \mathbb{E} \sup_{q \in [0, t]} \|f_m(q)\|_H^2 + 4\sigma^2 \mathbb{E} \int_0^t \|\nabla_r f_m(s)\|_H^2 ds \leq 2\|f_0\|_H^2 e^{\lambda t}, \tag{38}$$

where λ is given by (37).

In order to handle the stochastic integral as $N \rightarrow +\infty$, it is convenient to show the following.

Lemma 16. *The unique solution $f_m \in C([0, T], L^2(\Omega; H_m))$ to (28) satisfies*

$$\forall t \in [0, T] : \mathbb{E} \sup_{q \in [0, t]} \|f_m(q)\|_H^{2p} + (2\sigma^2)^p \mathbb{E} \left(\int_0^t \|\nabla_r f_m(s)\|_H^2 ds \right)^p \leq C \|f_0\|_H^{2p} e^{Ct}, \forall 1 < p < +\infty \tag{39}$$

where $C > 0$ is independent of N and m .

Proof. From (31), by using (32), (33) and (34) we get

$$\begin{aligned} \|f_m(t)\|_H^2 + 2\sigma^2 \int_0^t \|\nabla_r f_m(s)\|_H^2 ds &\leq \|f_0\|_H^2 + \frac{\lambda}{2} \int_0^t \|f_m(s)\|_H^2 ds \\ &\quad + 2 \left| \sum_{k \in K} \int_0^t (-(\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), f_m(s))_H dW^k(s) \right|. \end{aligned}$$

Let $p > 1$, we have

$$\begin{aligned} \|f_m(t)\|_H^{2p} + (2\sigma^2)^p \left(\int_0^t \|\nabla_r f_m(s)\|_H^2 ds \right)^p &\leq C_p [\|f_0\|_H^{2p} + \left(\frac{\lambda}{2}\right)^p t^{p-1} \int_0^t \|f_m(s)\|_H^{2p} ds \\ &\quad + 2^p C_p \left| \sum_{k \in K} \int_0^t (-(\nabla \sigma_k^N r) \cdot \nabla_r f_m(s), f_m(s))_H dW^k(s) \right|^p. \end{aligned}$$

Now, by using Burkholder-Davis-Gundy inequality and (36), we get

$$\begin{aligned}
 & 2^p C_p \mathbb{E} \sup_{q \in [0, t]} \left| \int_0^q \sum_{k \in K} (f_m(s), f_m(s) (\nabla \sigma_k^N r) \cdot r) dW^k(s) \right|^p \\
 & \leq 2^p C_p \mathbb{E} \left[\int_0^t \sum_{k \in K} |(f_m(s), f_m(s) (\nabla \sigma_k^N r) \cdot r)|^2 ds \right]^{p/2} \\
 & \leq 2^p C_p C^{p/2} t^{(p-1)/2} \mathbb{E} \left[\sup_{q \in [0, t]} \|f_m(q)\|_H^{2p} \int_0^t \|f_m(s)\|_H^{2p} ds \right]^{1/2} \\
 & \leq \epsilon \mathbb{E} \sup_{q \in [0, t]} \|f_m(q)\|_H^{2p} + \frac{(2^p C_p)^2 C^p t^{p-1}}{\epsilon} \mathbb{E} \int_0^t \|f_m(s)\|_H^{2p} ds,
 \end{aligned}$$

for any $\epsilon > 0$ (to be chosen later). Consequently, there exists $C > 0$ independent of m and N such that

$$\begin{aligned}
 & (1 - \epsilon) \mathbb{E} \sup_{q \in [0, t]} \|f_m(q)\|_H^{2p} + (2\sigma^2)^p \mathbb{E} \left(\int_0^t \|\nabla_r f_m(s)\|_H^2 ds \right)^p \\
 & \leq C [\|f_0\|_H^{2p} + (C + \frac{C}{\epsilon}) \mathbb{E} \int_0^t \|f_m(s)\|_H^{2p} ds].
 \end{aligned}$$

Hence, (39) follows by choosing $\epsilon = \frac{1}{2}$ and applying Grönwall lemma.

6.2. Existence of solution to (12)

We have the following equation of $f_m(t) = f_m(x, r, t)$:

$$\left\{ \begin{aligned}
 & df_m(t) + P_m \operatorname{div}_x (u_L f_m(t)) dt + P_m \operatorname{div}_r ((\nabla u_L r - \frac{1}{\beta} r) f_m(t)) dt \\
 & = \sum_{k \in K} P_m (\sigma_k^N \cdot \nabla_x P_m [(\nabla \sigma_k^N r) \cdot \nabla_r f_m]) dt + \sigma^2 P_m \Delta_r f_m(t) dt \\
 & - \sum_{k \in K} P_m \sigma_k^N \cdot \nabla_x f^N(t) dW^k - \sum_{k \in K} P_m (\nabla \sigma_k^N r) \cdot \nabla_r f_m(t) dW^k \\
 & + \alpha_N P_m \Delta_x f_m(t) dt + \frac{1}{2} P_m \operatorname{div}_r (\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r f_m(t)) dt, \\
 & f_m|_{t=0} = P_m f_0.
 \end{aligned} \right. \tag{40}$$

From (40) we get for $1 \leq j \leq m$:

$$\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m(t) w_j dr dx - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} P_m f_0 w_j dr dx \tag{41}$$

$$\begin{aligned}
 & - \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m(s) \left(u_L(s) \cdot \nabla_x w_j + (\nabla u_L(s)r - \frac{1}{\beta}r) \cdot \nabla_r w_j \right) dr dx ds \\
 & = - \sum_{k \in K} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r f_m(s), (\nabla \sigma_k^N r) P_m[\sigma_k^N \cdot \nabla_x w_i] dr dx ds \\
 & - \sigma^2 \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r f_m(s) \cdot \nabla_r w_j dr dx ds + \alpha_N \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m(s) \cdot \Delta_x w_j dr dx ds \\
 & + \sum_{k \in K} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} [f_m(s) \sigma_k^N \cdot \nabla_x w_j + f_m(s) (\nabla \sigma_k^N r) \cdot \nabla_r w_j] dr dx dW^k(s) \\
 & - \frac{1}{2} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r f_m(s) \right) \cdot \nabla_r w_j dr dx ds.
 \end{aligned}$$

Let us pass to the limit in the last equation.

1st step

By using Lemma 15, we are able to get the following convergences (up to a subsequence)

$$f_m \rightharpoonup \tilde{f} \text{ in } L^2(\Omega; L^2([0, T]; H)), \tag{42}$$

$$f_m \rightharpoonup^* \tilde{f} \text{ in } L^2_{w-*}(\Omega; L^\infty([0, T]; H)), \tag{43}$$

$$\nabla_r f_m \rightharpoonup \nabla_r \tilde{f} \text{ in } L^2(\Omega; L^2([0, T]; H)). \tag{44}$$

On the other hand, from Lemma 16, we obtain also

$$\forall t \in [0, T]: \mathbb{E} \sup_{q \in [0, t]} \|\tilde{f}(q)\|_H^{2p} + (2\sigma^2)^p \mathbb{E} \left(\int_0^t \|\nabla_r \tilde{f}(s)\|_H^2 \right)^p \leq \mathbf{C} \|f_0\|_H^{2p} e^{\mathbf{C}t}, \forall 1 < p < +\infty \tag{45}$$

where $\mathbf{C} > 0$ is independent of N and m .

2nd step

For $\phi \in U$, note that $\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m(x, r, t) \phi(x, r) dx dr$ is adapted with respect to $(\mathcal{F}_t)_t$ and recall

that the space of adapted processes is a closed convex subspace of $L^2(\Omega \times [0, T])$, hence weakly closed. Therefore $\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}(x, r, t) \phi(x, r) dx dr$ is also adapted and its Itô integral is well defined and bounded. Now, let us consider the following mapping

$$\mathcal{L} : L^2(\Omega \times [0, T]; H) \rightarrow L^2(\Omega \times [0, T]; \mathbb{R})$$

$$f_m \mapsto \sum_{k \in K} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m(s) (\sigma_k^N \cdot \nabla_x \phi + (\nabla \sigma_k^N r) \cdot \nabla_r \phi) dr dx dW^k(s),$$

which is linear and bounded. Therefore, by using (42), we infer that

$$\begin{aligned} & \sum_{k \in K} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m(s) (\sigma_k^N \cdot \nabla_x \phi + (\nabla \sigma_k^N r) \cdot \nabla_r \phi) dr dx dW^k(s) \\ & \rightarrow \sum_{k \in K} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}(s) (\sigma_k^N \cdot \nabla_x \phi + (\nabla \sigma_k^N r) \cdot \nabla_r \phi) dr dx dW^k(s) \text{ in } L^2(\Omega \times [0, T]). \end{aligned}$$

3rd step

For $\phi \in H_m$ and $t \in [0, T]$, let us set

$$B_m(t) := \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m(t) \phi dr dx - \sum_{k \in K} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m(s) (\sigma_k^N \cdot \nabla_x \phi + (\nabla \sigma_k^N r) \cdot \nabla_r \phi) dr dx dW^k(s).$$

From (41), we write (in distributional sense with respect to t)

$$\begin{aligned} \frac{d}{dt} B_m &= \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m \left(u_L \cdot \nabla_x \phi + (\nabla u_L r - \frac{1}{\beta} r) \cdot \nabla_r \phi \right) dr dx \\ &- \sum_{k \in K} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r f_m (\nabla \sigma_k^N r) P_m [\sigma_k^N \cdot \nabla_x \phi] dr dx - \sigma^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r f_m \cdot \nabla_r \phi dr dx ds \\ &+ \alpha_N \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m \Delta_x \phi dr dx - \frac{1}{2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r f_m \right) \cdot \nabla_r \phi dr dx. \end{aligned}$$

Let $A \in \mathcal{F}$ and $\xi \in \mathcal{D}(0, T)$,⁶ by multiplying the last equation by $\mathbb{I}_A \xi$ and integrating over $\Omega \times [0, T]$ we derive

$$\begin{aligned} & - \int_A \int_0^T [B_m \frac{d\xi}{ds}] ds dP \\ &= \int_A \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m(s) \left(u_L(s) \cdot \nabla_x \phi + (\nabla u_L(s) r - \frac{1}{\beta} r) \cdot \nabla_r \phi \right) \xi(s) dr dx ds dP \end{aligned}$$

⁶ $\mathcal{D}(0, T)$ denotes the space of C^∞ -functions with compact support in $]0, T[$.

$$\begin{aligned}
 & - \int_A \int_0^T \sum_{k \in K} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r f_m(s) (\nabla \sigma_k^N r) P_m [\sigma_k^N \cdot \nabla_x \phi] \xi(s) dr dx ds dP \\
 & - \int_A \int_0^T \sigma^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r f_m(s) \cdot \nabla_r \phi \xi(s) dr dx ds dP + \int_A \int_0^T \alpha_N \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_m(s) \cdot \Delta_x \phi \xi(s) dr dx ds dP \\
 & - \int_A \int_0^T \frac{1}{2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r f_m(s) \right) \cdot \nabla_r \phi \xi(s) dr dx ds dP, \quad \forall \phi \in H_m.
 \end{aligned}$$

Now, let us prove the following.

$$\int_A \int_0^T \sum_{k \in K} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r f_m(s) (\nabla \sigma_k^N r) P_m [\sigma_k^N \cdot \nabla_x \phi] \xi(s) dr dx ds dP \rightarrow 0 \text{ as } m \rightarrow +\infty. \tag{46}$$

Indeed, it's sufficient to pass to the limit with $(w_i)_{i \in \mathbb{N}}$ as test functions. Thus, for $1 \leq i \leq m$, we recall

$$\begin{aligned}
 & - \int_A \int_0^T \sum_{k \in K} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r f_m(s) (\nabla \sigma_k^N r) P_m [\sigma_k^N \cdot \nabla_x \phi] \xi(s) dr dx ds dP \\
 & = - \sum_{k \in K} \mathbb{E} \int_0^T (\nabla_r f_m(s), (\nabla \sigma_k^N r) P_m [\sigma_k^N \cdot \nabla_x w_i]) 1_A \xi(s) ds \\
 & = \mathbb{E} \int_0^T \left\langle \sum_{k \in K} P_m (\sigma_k^N \cdot \nabla_x P_m [(\nabla \sigma_k^N r) \cdot \nabla_r f_m(s)]), w_i \right\rangle 1_A \xi(s) ds
 \end{aligned}$$

where \mathbb{E} denotes the expectation. On the other hand, the following convergence holds

$$\sum_{k \in K} (\nabla \sigma_k^N r) P_m [\sigma_k^N \cdot \nabla_x w_i] \rightarrow \sum_{k \in K} (\nabla \sigma_k^N r) \sigma_k^N \cdot \nabla_x w_i \text{ in } L^2(\mathbb{T}^2 \times \mathbb{R}^2).$$

Indeed, denote by $\| \cdot \|_2$ the norm in $L^2(\mathbb{T}^2 \times \mathbb{R}^2)$, we get

$$\begin{aligned}
 \left\| \sum_{k \in K} (\nabla \sigma_k^N r) (P_m - I) [\sigma_k^N \cdot \nabla_x w_i] \right\|_2^2 & \leq \sum_{k \in K} (\theta_{|k|}^N)^2 |k|^2 \|r\| (P_m - I) [\sigma_k^N \cdot \nabla_x w_i] \Big\|_2^2 \\
 & \leq \sum_{k \in K} (\theta_{|k|}^N)^2 |k|^2 \| (P_m - I) [\sigma_k^N \cdot \nabla_x w_i] \|_H^2 \\
 & \leq \sum_{k \in K} (\theta_{|k|}^N)^4 |k|^2 \| P_m - I \|_{L(H,H)}^2 \| \nabla_x w_i \|_H^2
 \end{aligned}$$

$$\begin{aligned} &\leq \sum_{k \in K} (\theta_{|k|}^N)^4 |k|^2 \|P_m - I\|_{L(H,H)}^2 \|w_i\|_U^2 \\ &\leq C \|P_m - I\|_{L(H,H)}^2 \|w_i\|_U^2 \rightarrow 0, \end{aligned}$$

where $C > 0^7$ and since P_m is an orthogonal projection on H . On the other hand, $\nabla_r f_m$ converges weakly to $\nabla_r \tilde{f}$ in $L^2(\Omega \times [0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)$, thanks to (44). Therefore

$$\begin{aligned} &\lim_m - \int_A \int_0^T \sum_{k \in K} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r f_m(s) (\nabla \sigma_k^N r) P_m [\sigma_k^N \cdot \nabla_x \phi] \xi(s) dr dx ds dP \\ &= - \sum_{k \in K} \int_A \int_0^T (\nabla_r \tilde{f}(s), (\nabla \sigma_k^N r) \sigma_k^N \cdot \nabla_x w_i) \xi(s) ds dP \\ &= \int_A \int_0^T (\tilde{f}(s), \sum_{k \in K} (\nabla \sigma_k^N r) \cdot \nabla_r \sigma_k^N \cdot \nabla_x w_i) \xi(s) ds dP = 0 \quad \forall i \in \mathbb{N}. \end{aligned}$$

Indeed, for given function ψ we have $\sum_{k \in K} (\nabla \sigma_k^N r) \cdot \nabla_r (\sigma_k^N \cdot \nabla_x \psi) = \sum_{k \in K} \sum_{l, \gamma, i=1}^2 \partial_{x_\gamma} \sigma_k^i r_\gamma \partial_{r_i} (\sigma_k^l \partial_{x_l} \psi)$

and

$$\sum_{k \in K} \sum_{l, \gamma, i=1}^2 \partial_{x_\gamma} \sigma_k^i r_\gamma \partial_{r_i} (\sigma_k^l \partial_{x_l} \psi) = \sum_{k \in K} \sum_{l, \gamma, i=1}^2 \partial_{x_\gamma} \sigma_k^i \sigma_k^l \partial_{x_l} (r_\gamma \partial_{r_i} \psi) = \sum_{l, \gamma, i=1}^2 \partial_{x_\gamma} Q_{i,l}(0) \partial_{x_l} (r_\gamma \partial_{r_i} \psi).$$

Since the covariance matrix Q satisfies $Q(x) = Q(-x)$ then $\partial_{x_\gamma} Q_{i,l}(0) = 0$. As a result we get

$$\begin{aligned} &- \int_A \int_0^T [B \frac{d\xi}{ds}] ds dP \\ &= \int_A \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}(s) \left(u_L(s) \cdot \nabla_x \phi + (\nabla u_L(s) r - \frac{1}{\beta} r) \cdot \nabla_r \phi \right) \xi(s) dr dx ds dP \\ &- \int_A \int_0^T \sigma^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r \tilde{f}(s) \cdot \nabla_r \phi \xi(s) dr dx ds dP + \int_A \int_0^T \alpha_N \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}(s) \cdot \Delta_x \phi \xi(s) dr dx ds dP \end{aligned}$$

⁷ Recall that $\sum_{k \in K} (\theta_{|k|}^N)^4 |k|^2 = \sum_{k \in K} \frac{1}{|k|^6} \leq C$.

$$- \int_A \int_0^T \frac{1}{2} \int_{\mathbb{T}^2 \mathbb{R}^2} \int \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r \tilde{f}(s) \right) \cdot \nabla_r \phi \xi(s) dr dx ds dP, \quad \forall \phi \in U,$$

where

$$B(t) := \int_{\mathbb{T}^2 \mathbb{R}^2} \tilde{f}(t) \phi dr dx - \sum_{k \in K} \int_0^t \int_{\mathbb{T}^2 \mathbb{R}^2} \tilde{f}(s) (\sigma_k^N \cdot \nabla_x \phi + (\nabla \sigma_k^N r) \cdot \nabla_r \phi) dr dx dW^k(s).$$

Then, taking into account the regularity of \tilde{f} , we infer that the distributional derivative $\frac{dB}{dt}$ belongs to the space $L^2(\Omega \times [0, T])$. Recalling that $B \in L^2(\Omega \times [0, T])$, we conclude that

$$B(\cdot) \in L^2(\Omega; C([0, T])).$$

Considering the properties of Itô’s integral, we deduce

$$\int_{\mathbb{T}^2 \mathbb{R}^2} \tilde{f}(t) \phi dr dx \in L^2(\Omega; C([0, T])),$$

which means that $\tilde{f} \in L^2(\Omega; C([0, T]; U'))$ and therefore $\tilde{f} \in L^2(\Omega; C_w([0, T]; H))$, thanks to (43) and [61, Lemma. 1.4 p. 263]. We finish the proof by showing some continuous convergence in time. Indeed, let $\xi \in C^\infty([0, t])$ for $t \in]0, T]$ and note that the following integration by parts formula holds

$$\int_0^t \frac{dB}{ds}(s) \xi(s) ds = - \int_0^t B(s) \frac{d\xi}{ds} ds + B(t) \xi(t) - \int_{\mathbb{T}^2 \mathbb{R}^2} f_0 \phi dr dx \xi(0). \tag{47}$$

Now, by standard arguments (see e.g. [65, proof of Prop. 3.]) we get for any $t \in]0, T]$,

$$\int_{\mathbb{T}^2 \mathbb{R}^2} \tilde{f}_m(t) \phi dr dx \rightarrow \int_{\mathbb{T}^2 \mathbb{R}^2} \tilde{f}(t) \phi dr dx \text{ in } L^2(\Omega, H), \text{ as } m \rightarrow \infty,$$

and $\tilde{f}(0) = f_0$ in H -sense. In conclusion, there exists a solution in the sense of Definition 5 ($\tilde{f} = f^N$ to stress the dependence N , since we will pass to the limit as $N \rightarrow +\infty$ in section 7.)

$$\begin{aligned} & \int_{\mathbb{T}^2 \mathbb{R}^2} f^N(t) \phi dr dx - \int_{\mathbb{T}^2 \mathbb{R}^2} f_0 \phi dr dx \\ & - \int_0^t \int_{\mathbb{T}^2 \mathbb{R}^2} f^N(s) \left(u_L(s) \cdot \nabla_x \phi + (\nabla u_L(s) r - \frac{1}{\beta} r) \cdot \nabla_r \phi \right) dr dx ds \end{aligned}$$

$$\begin{aligned}
 &= -\sigma^2 \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r f^N(s) \cdot \nabla_r \phi \, dr dx ds + \alpha_N \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f^N(s) \cdot \Delta_x \phi \, dr dx ds \\
 &+ \sum_{k \in K} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} [f^N(s) \sigma_k^N \cdot \nabla_x \phi + f^N(s) (\nabla \sigma_k^N r) \cdot \nabla_r \phi] \, dr dx dW^k(s) \\
 &- \frac{1}{2} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r f^N(s) \right) \cdot \nabla_r \phi \, dr dx ds,
 \end{aligned}$$

for any $\phi \in U$. In particular, f^N is adapted with respect to the given filtration.

6.3. Uniqueness of quasi-regular weak solution

In order to prove (4) in Definition 5 and the uniqueness of “quasi-regular weak solution” to (12), we need first to prove the existence and uniqueness of solution V^N to an appropriate mean equation. Namely, we prove the following.

Proposition 17. *For any $t \in [0, T]$, there exists $V^N(t) = \mathbb{E}[f^N(t)e_g(t)]$ such that*

1. $V^N \in L^\infty([0, T]; H)$, $\nabla_r V^N \in L^2([0, T]; H)$ and $V^N \in C_w([0, T]; H)$.
2. For any $t \in [0, T]$, it holds

$$\begin{aligned}
 &\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} V^N(t) \phi \, dx dr - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_0 \phi \, dx dr \\
 &= \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} V^N(s) \left([u_L(s) - h_n] \cdot \nabla_x \phi + ([\nabla u_L(s)r - y_n] - \frac{1}{\beta} r) \cdot \nabla_r \phi \right) \, dr dx ds \\
 &- \int_0^t \sigma^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r V^N(s) \cdot \nabla_r \phi \, dr dx ds + \int_0^t \alpha_N \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} V^N(s) \cdot \Delta_x \phi \, dr dx ds \\
 &- \int_0^t \frac{1}{2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r V^N(s) \right) \cdot \nabla_r \phi \, dr dx ds, \quad \forall \phi \in V,
 \end{aligned}$$

Remark 18. A priori, the last point holds for any $\phi \in U$ and by taking into account the regularity of the solution V^N , it holds as well for all $\phi \in V$.

Proof of Proposition 17.

Let $g \in G_n$, by using Itô formula to the product, we get

$$\left\{ \begin{aligned} d[e_g(t) f_m(t)] &+ e_g(t) P_m \operatorname{div}_x(u_L f_m(t)) dt + e_g(t) P_m \operatorname{div}_r((\nabla u_L r - \frac{1}{\beta} r) f_m(t)) dt \\ &= \sum_{k \in K} e_g(t) P_m (\sigma_k^N \cdot \nabla_x P_m [(\nabla \sigma_k^N r) \cdot \nabla_r f_m]) dt + \sigma^2 e_g(t) P_m \Delta_r f_m(t) dt \\ &- \sum_{k \in K} e_g(t) P_m \sigma_k^N \cdot \nabla_x f^N(t) dW^k - \sum_{k \in K} e_g(t) P_m (\nabla \sigma_k^N r) \cdot \nabla_r f_m(t) dW^k \\ &+ \alpha_N e_g(t) P_m \Delta_x f_m(t) dt \\ &+ \frac{1}{2} e_g(t) P_m \operatorname{div}_r(\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r f_m(t)) dt, \\ &+ \sum_{k \in M_n} f_m(t) g_k(t) e_g(t) dW^k(t) \\ &- [\sum_{k \in M_n} g_k(t) e_g(t) dW^k(t), \sum_{k \in K} e_g(t) P_m \sigma_k^N \cdot \nabla_x f^N(t) \\ &+ e_g(t) P_m (\nabla \sigma_k^N r) \cdot \nabla_r f_m(t) dW^k] \\ e_g(t) f_m|_{t=0} &= P_m f_0. \end{aligned} \right.$$

Denote $K_n = \{k \in K : \min(n, N) \leq |k| \leq \max(2N, n)\}$ and set $V_m(t) = \mathbb{E}(f_m(t) e_g(t))$ then

$$\begin{aligned} d[V_m(t)] &+ P_m \operatorname{div}_x(u_L V_m(t)) dt + P_m \operatorname{div}_r((\nabla u_L r - \frac{1}{\beta} r) V_m(t)) dt \\ &= \sum_{k \in K} P_m (\sigma_k^N \cdot \nabla_x P_m [(\nabla \sigma_k^N r) \cdot \nabla_r V_m(t)]) dt + \sigma^2 P_m \Delta_r V_m(t) dt \\ &+ \alpha_N P_m \Delta_x V_m(t) dt + \frac{1}{2} P_m \operatorname{div}_r(\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r V_m(t)) dt, \\ &- \sum_{k \in K_n} P_m g_k \sigma_k^N \cdot \nabla_x V_m(t) + P_m g_k (\nabla \sigma_k^N r) \cdot \nabla_r V_m(t) \\ V_m|_{t=0} &= P_m f_0. \end{aligned}$$

Denote $\sum_{k \in K_n} g_k \sigma_k^N = h_n$ and $\sum_{k \in K_n} g_k (\nabla \sigma_k^N r) = y_n$. Then V_m satisfies

$$\left\{ \begin{aligned} \frac{dV_m}{dt} &+ P_m \operatorname{div}_x([u_L - h_n] V_m(t)) + P_m \operatorname{div}_r([(\nabla u_L r) - y_n] - \frac{1}{\beta} r) V_m(t) \\ &= \sum_{k \in K} P_m (\sigma_k^N \cdot \nabla_x P_m [(\nabla \sigma_k^N r) \cdot \nabla_r V_m]) + \sigma^2 P_m \Delta_r V_m(t) \\ &+ \alpha_N P_m \Delta_x V_m(t) + \frac{1}{2} P_m \operatorname{div}_r(\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r V_m(t)), \\ V_m|_{t=0} &= P_m f_0. \end{aligned} \right. \tag{48}$$

(48) is linear system of ODE, by using a classical result (see e.g. [64, Chapter V]) we get

Lemma 19. *There exists a unique $V_m \in C([0, T]; H_m)$ to (48).*

Lemma 20. *For every $m \in \mathbb{N}^*$, there exists a unique solution $V_m = \mathbb{E}[e_g f_m] \in C([0, T]; H_m)$ to (48), which satisfy*

$$\forall t \geq 0: \sup_{q \in [0,t]} \|V_m(q)\|_H^2 + 4\sigma^2 \int_0^t \|\nabla_r V_m(s)\|_H^2 ds \leq 2\|f_0\|_H^2 e^{\bar{\lambda}(t)}, \tag{49}$$

where

$$\bar{\lambda}(t) = [2\|\nabla_x u_L\|_\infty + \frac{8}{\beta} + 4\sigma^2 + 2C]t + \int_0^t (\|g(s)\|^2 + 1)ds < +\infty.$$

Proof. The proof consists of arguments analogous to the proof of Lemma 15 but for the reader's convenience, let us sketch it. By applying Itô formula with $\|\cdot\|_H^2$ to the process V_m satisfying (48), we obtain

$$\begin{aligned} & \|V_m(t)\|_H^2 - \|P_m f_0\|_H^2 \\ &= 2 \int_0^t (V_m(s), u_L \cdot \nabla_x V_m(s))_H ds - 2 \int_0^t (V_m(s), h_n \cdot \nabla_x V_m(s))_H ds \\ &\quad - 2 \int_0^t ((\nabla_x u_L r) \cdot \nabla_r V_m(s), V_m(s))_H ds \\ &\quad + 2 \int_0^t (V_m(s), y_n \cdot \nabla_r V_m(s))_H ds + \int_0^t \frac{2}{\beta} \|V_m(s)\|_H^2 ds + \frac{1}{\beta} \int_0^t (r \cdot \nabla_r V_m(s), V_m(s))_H ds \\ &\quad + 2 \sum_{k \in K} \int_0^t (V_m(s), \sigma_k^N \cdot \nabla_x P_m[(\nabla \sigma_k^N r)] \nabla_r V_m(s))_H ds \\ &\quad - 2 \int_0^t [\sigma^2 \|\nabla_r V_m(s)\|_H^2 + 2\sigma^2 (\nabla_r V_m(s), r V_m(s)) + \alpha_N \|\nabla_x V_m(s)\|_H^2] ds \\ &\quad - \int_0^t (A_k^N(x, r) \nabla_r V_m(s), \nabla_r V_m(s))_H ds - 2 \int_0^t (A_k^N(x, r) \nabla_r V_m(s), r V_m(s)) ds \end{aligned}$$

The last equation has similar terms as (30) (without the stochastic integrals and Itô correctors) but with the two new terms

$$2 \int_0^t (V_m(s), h_n \cdot \nabla_x V_m(s))_H ds + 2 \int_0^t (V_m(s), y_n \cdot \nabla_r V_m(s))_H ds.$$

By noticing that $\text{div}_x(h_n) = 0$, the first term vanishes. Concerning the second one, note that $|y_n| \leq |r| \sum_{|k| \leq n} |g_k| = |r| \|g\|$ and

$$\int_0^t (V_m(s), y_n \cdot \nabla_r V_m(s))_H ds = - \sum_{k \in K_n} (V_m(s), g_k(\nabla \sigma_k^N r) \cdot r V_m(s)) ds,$$

which ensures that $\int_0^t |(V_m(s), y_n \cdot \nabla_r V_m(s))_H| ds \leq \int_0^t (\|g(s)\|^2 + 1) \|V_m(s)\|_H^2 ds$. Thus, the other terms can be estimated by similar arguments as in subsection 6.1 and we obtain Lemma 20.

By using Lemma 20, we are able to get the following convergences (up to a subsequence)

$$V_m \rightharpoonup \tilde{V} \text{ in } L^2([0, T]; H), \tag{50}$$

$$V_m \rightharpoonup^* \tilde{V} \text{ in } L^\infty([0, T]; H), \tag{51}$$

$$\nabla_r V_m \rightharpoonup \nabla_r \tilde{V} \text{ in } L^2([0, T]; H), \tag{52}$$

moreover, by using the linearity of the expectation we get $\tilde{V} = \mathbb{E}[e_g \tilde{f}]$ (recall that $V_m = \mathbb{E}(e_g f_m)$ and $e_g \in L^2(\Omega)$). Now, we have all the ingredients in hand to argue as in subsection 6.2 and obtain Proposition 17.

Lemma 21. *Let $N \in \mathbb{N}^*$. Then, the solution V^N given by Proposition 17 is unique.*

Proof of Lemma 21.

Let V_1^N and V_2^N be two solutions, with the same initial data, given by Proposition 17 and denote by V^N their difference. Then for any $t \in]0, T]$ and $\phi \in V$, we have

$$\begin{aligned} & \int_{\mathbb{T}^2 \mathbb{R}^2} \int V^N(t) \phi dx dr \tag{53} \\ &= \int_0^t \int_{\mathbb{T}^2 \mathbb{R}^2} \int V^N(x, r, s) \left([u_L(x, s) - h_n] \cdot \nabla_x \phi + ([\nabla u_L(s, x)r - y_n] - \frac{1}{\beta} r) \cdot \nabla_r \phi \right) dr dx ds \\ & - \int_0^t \sigma^2 \int_{\mathbb{T}^2 \mathbb{R}^2} \int \nabla_r V^N(x, r, s) \cdot \nabla_r \phi dr dx ds + \int_0^t \alpha_N \int_{\mathbb{T}^2 \mathbb{R}^2} \int V^N(x, r, s) \cdot \Delta_x \phi dr dx ds \\ & - \int_0^t \frac{1}{2} \int_{\mathbb{T}^2 \mathbb{R}^2} \int \left(\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r V^N(x, r, s) \right) \cdot \nabla_r \phi dr dx ds. \end{aligned}$$

Since the above equation is understood in weak form, we need first to consider an appropriate regularization of V^N , denoted by $[V^N]_\delta$. Then, take the L^2 -inner product of the above equation with $[V^N]_\delta$ and finally pass to the limit with respect to the regularization parameters δ .

Step 1: Regularization

Let $\varphi \in C_c^\infty(\mathbb{T}^2 \times \mathbb{R}^2)$, if we denote $X = (x, r) \in \mathbb{T}^2 \times \mathbb{R}^2$ and $\rho_\delta(X) = \rho_\delta(x)$, then $\varphi_\delta := \rho_\delta * \varphi$ is an appropriate test function in (53), namely we get

$$\begin{aligned} & (V^N(t), \varphi_\delta) \\ &= \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} V^N(s) \left([u_L(s) - h_n] \cdot \nabla_x \varphi_\delta + ([\nabla u_L(s)r - y_n] - \frac{1}{\beta}r) \cdot \nabla_r \varphi_\delta \right) dr dx ds \\ & - \int_0^t \sigma^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r V^N(s) \cdot \nabla_r \varphi_\delta dr dx ds + \int_0^t \alpha_N \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} V^N(s) \cdot \Delta_x \varphi_\delta dr dx ds \\ & - \int_0^t \frac{1}{2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r V^N(s) \right) \cdot \nabla_r \varphi_\delta dr dx ds. \end{aligned}$$

Since ρ is radially symmetric then the operator of convolution with ρ_δ is self-adjoint on $L^2(\mathbb{T}^2 \times \mathbb{R}^2)$. Thus

$$\begin{aligned} & ([V^N(t)]_\delta, \varphi) + \int_0^t \left([(u_L(s) - h_n) \cdot \nabla_x V^N(s)]_\delta + [\operatorname{div}_r((\nabla u_L(s)r - y_n - \frac{1}{\beta}r)V^N(s))]_\delta, \varphi \right) ds \\ &= \int_0^t \sigma^2 ([\Delta_r V^N(s)]_\delta, \varphi) ds + \int_0^t \alpha_N ([\Delta_x V^N(s)]_\delta, \varphi) ds \\ &+ \int_0^t \frac{1}{2} [\operatorname{div}_r \left(\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r V^N(s) \right)]_\delta, \varphi ds = \int_0^t ([d(s)]_\delta, \varphi) ds. \end{aligned}$$

Consider the following space $X := \{\varphi \in H; \nabla_r \varphi \in H\}$ and note that $X \hookrightarrow L^2(\mathbb{T}^2 \times \mathbb{R}^2) \hookrightarrow X'$ is Gelfand triple. Since

$$V^N \in L^\infty([0, T]; H), \nabla_r V^N \in L^2([0, T]; H),$$

and by using the regularization properties of ρ , one gets that $[d(\cdot)]_\delta \in L^2([0, T]; X')$. Therefore, we can set $\varphi = [V^N(\cdot)]_\delta$ to get

$$\begin{aligned} & \frac{1}{2} \|[V^N(t)]_\delta\|^2 + \int_0^t \left([(u_L(s) - h_n) \cdot \nabla_x V^N(s)]_\delta \right. \\ & \left. + [\operatorname{div}_r((\nabla u_L(s)r - y_n - \frac{1}{\beta}r)V^N(s))]_\delta, [V^N(s)]_\delta \right) ds \end{aligned} \tag{54}$$

$$\begin{aligned}
 &= \int_0^t \sigma^2([\Delta_r V^N(s)]_\delta, [V^N(s)]_\delta) ds + \int_0^T \alpha_N([\Delta_x V^N(s)]_\delta, [V^N(s)]_\delta) ds \\
 &+ \int_0^t \frac{1}{2} [\operatorname{div}_r (\sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r V^N(s))]_\delta, [V^N(s)]_\delta ds.
 \end{aligned}$$

Now, let us pass to the limit in the last equality (54). The proof is based on commutator estimates and using the properties of $(\sigma_k)_{k \in K}$.

Step 2: Passage to the limit as $\delta \rightarrow 0$

First, recall that $V^N(\cdot) \in L^2(\mathbb{T}^2 \times \mathbb{R}^2)$, by properties of convolution product, we get $\lim_{\delta \rightarrow 0} [V^N(r, \cdot)]_\delta = V^N(r, \cdot)$ in $L^2(\mathbb{T}^2)$ uniformly for a.e. $r \in \mathbb{R}^2$ and we get $\lim_{\delta \rightarrow 0} \|[V^N(t)]_\delta\|^2 = \|V^N(t)\|^2$. Next, we will prove the following

$$\lim_{\delta} \int_0^t \langle [u_L(s) \cdot \nabla_x V^N(s)]_\delta, [V^N(s)]_\delta \rangle ds = 0. \tag{55}$$

Since $\operatorname{div}_x u_L = 0$, we get

$$\begin{aligned}
 &\int_0^t \langle [u_L(s) \cdot \nabla_x V^N(s)]_\delta, [V^N(s)]_\delta \rangle ds \\
 &= \int_0^t \langle \rho_\delta * [u_L(s) \cdot \nabla_x V^N(s)] - u_L(s) \cdot \nabla_x \rho_\delta * [V^N(s)], \rho_\delta * [V^N(s)] \rangle ds.
 \end{aligned}$$

Let us introduce the commutator $r_\delta(s) = \rho_\delta * [u_L(s) \cdot \nabla_x V^N(s)] - u_L(s) \cdot \nabla_x \rho_\delta * [V^N(s)]$. Thus, (55) is a consequence of Friedrichs' type lemma (see e.g. [66, Vol. 3, p. 9]), namely the following holds: a.e. $s \in [0, T]$

$$\|r_\delta(s)\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)} \leq C \|\nabla u_L(s)\|_\infty \|V^N(s)\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)}, \tag{56}$$

$$\lim_{\delta} r_\delta(s) = 0 \text{ in } L^2(\mathbb{T}^2 \times \mathbb{R}^2), \tag{57}$$

where $C > 0$ independent of δ . Indeed, let us show (56), note that

$$r_\delta(s, x, r) = - \int_{\mathbb{R}^2} (u_L(x, s) - u_L(y, s)) \cdot \nabla_x \rho_\delta(x - y) V^N(s, y, r) dy.$$

Consider the following change of variables $z = \frac{x - y}{\delta}$ to get

$$\begin{aligned} r_\delta(s, x, r) &= - \int_{\mathbb{R}^2} \frac{u_L(y + \delta z, s) - u_L(y, s)}{\delta} \cdot \nabla_x \rho(z) V^N(s, y, r) dz \\ &= - \int_{\mathbb{R}^2} \int_0^1 \nabla u_L(y + \alpha \delta z, s) z d\alpha \cdot \nabla_x \rho(z) V^N(s, y, r) dz. \end{aligned}$$

Thus $|r_\delta(s, x, r)| \leq \|\nabla u_L(s)\|_\infty \int_{\mathbb{R}^2} |z| |\nabla_x \rho(z)| |V^N(s, y, r)| dz$. Since $x = y + \delta z$, we get

$$\begin{aligned} \|r_\delta(s)\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)}^2 &\leq \|\nabla u_L(s)\|_\infty^2 \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |z| |\nabla_x \rho(z)| |V^N(s, y, r)| dz)^2 dy dr \\ &\leq \|\nabla u_L(s)\|_\infty^2 \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} z \cdot \nabla_x \rho(z) V^N(s, y, r) dz)^2 dy dr \\ &\leq \|\nabla u_L(s)\|_\infty^2 \int_{\mathbb{R}^2} |z|^2 |\nabla_x \rho(z)|^2 dz \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} |V^N(s, y, r)|^2 dy dr, \\ &\leq C^2 \|\nabla u_L(s)\|_\infty^2 \|V^N(s)\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)}^2, \end{aligned}$$

since $\text{supp}[\rho] \subset B(0, 1)$ and denoted by $C^2 = \int_{\mathbb{R}^2} |z|^2 |\nabla_x \rho(z)|^2 dz < +\infty$. Concerning (57), we have

$$L^2(\mathbb{T}^2 \times \mathbb{R}^2)\text{-} \lim_{\delta \rightarrow 0} r_\delta(s) = -V^N(s) \left(\int_{\mathbb{R}^2} \nabla u_L(s) z \cdot \nabla_x \rho(z) dz \right).$$

Indeed, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \left| \int_{\mathbb{R}^2} \int_0^1 \nabla u_L(y + \alpha \delta z, s) z d\alpha \cdot \nabla_x \rho(z) V^N(s, y, r) dz \right. \\ &\quad \left. - V^N(s, x, r) \left(\int_{\mathbb{R}^2} \nabla u_L(s, x) z \cdot \nabla_x \rho(z) dz \right) \right|^2 dx dr \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \left| \int_{\mathbb{R}^2} \int_0^1 \nabla u_L(y + \alpha \delta z, s) z d\alpha \cdot \nabla_x \rho(z) V^N(s, y, r) dz \right. \\ &\quad \left. - \int_{\mathbb{R}^2} V^N(s, x, r) \nabla u_L(s, x) z \cdot \nabla_x \rho(z) dz \right|^2 dx dr \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{\mathbb{R}^2} \int_{T^2} \int_{\mathbb{R}^2} \int_0^1 |\nabla u_L(y + \alpha \delta z, s) z \cdot \nabla_x \rho(z) V^N(s, y, r) dz \\
 &\quad - V^N(s, x, r) \nabla u_L(s, x) z \cdot \nabla_x \rho(z)|^2 d\alpha dz dx dr \\
 &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{T^2} \int_0^1 |\nabla u_L(y + \alpha \delta z, s) V^N(s, y, r) \\
 &\quad - V^N(s, y + \delta z, r) \nabla u_L(s, y + \delta z)|^2 d\alpha dy dr |z|^2 |\nabla_x \rho(z)|^2 dz \\
 &\leq 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{T^2} \int_0^1 |\nabla u_L(y + \alpha \delta z, s) V^N(s, y, r) - V^N(s, y, r) \nabla u_L(s, y)|^2 d\alpha dy dr |z|^2 |\nabla_x \rho(z)|^2 dz \\
 &+ 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{T^2} \int_0^1 |V^N(s, y, r) \nabla u_L(s, y) \\
 &\quad - V^N(s, y + \delta z, r) \nabla u_L(s, y + \delta z)|^2 d\alpha dy dr |z|^2 |\nabla_x \rho(z)|^2 dz \\
 &= I_\delta^1 + I_\delta^2.
 \end{aligned}$$

Using the continuity of translations in $L^2(\mathbb{T}^2)$ for the function $V^N \nabla u_L$, we get $\limsup_{\delta \rightarrow 0} I_\delta^2 = 0$.

Concerning I_δ^1 , note that

$$I_\delta^1 \leq 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{T^2} \int_0^1 |\nabla u_L(y + \alpha \delta z, s) - \nabla u_L(s, y)|^2 |V^N(s, y, r)|^2 d\alpha dy dr |z|^2 |\nabla_x \rho(z)|^2 dz.$$

On the other hand, by mean-value theorem we get

$$|\nabla u_L(y + \alpha \delta z, s) - \nabla u_L(s, y)| \leq \alpha \delta |z| \|u_L\|_{C^2}$$

Thus $I_\delta^1 \leq \delta^2 \|u_L\|_{C^2}^2 \int_{\mathbb{R}^2} \int_{T^2} |V^N(s, y, r)|^2 dy dr \int_{\mathbb{R}^2} |z|^4 |\nabla_x \rho(z)|^2 dz \rightarrow 0$. Finally, since ρ is smooth density of a probability measure, we get $\int_{\mathbb{R}^2} z_i \partial_j \rho(z) dz = -\delta_{ij}$ and so

$$\int_{\mathbb{R}^2} \nabla u_L(s) z \cdot \nabla_x \rho(z) dz = -\operatorname{div} u_L = 0,$$

which gives (57). The next step is proving the following

$$\lim_{\delta} \int_0^t \langle [h_n(s) \cdot \nabla_x V^N(s)]_\delta, [V^N(s)]_\delta \rangle ds = 0. \tag{58}$$

We recall that $\sum_{k \in K_n} g_k \sigma_k^N = h_n$ and $\operatorname{div}_x h_n = 0$, hence

$$\begin{aligned} & \int_0^t \langle [h_n(s) \cdot \nabla_x V^N(s)]_\delta, [V^N(s)]_\delta \rangle ds \\ &= \int_0^t \langle \rho_\delta * [h_n(s) \cdot \nabla_x V^N(s)] - h_n(s) \cdot \nabla_x \rho_\delta * [V^N(s)], \rho_\delta * [V^N(s)] \rangle ds. \end{aligned}$$

Let us introduce the commutator

$$r_\delta^h(s) = \rho_\delta * [h_n(s) \cdot \nabla_x V^N(s)] - u_L(s) \cdot \nabla_x \rho_\delta * [V^N(s)].$$

(58) is a consequence of the following: a.e. $s \in [0, T]$

$$\|r_\delta^h(s)\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)} \leq C \|g\| \|V^N(s)\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)}, \tag{59}$$

$$\lim_{\delta} r_\delta^h(s) = 0 \text{ in } L^2(\mathbb{T}^2 \times \mathbb{R}^2), \tag{60}$$

where $C > 0$ independent of δ . Indeed, similarly to (56), we get

$$r_\delta^h(s, x, r) = - \int_{\mathbb{R}^2} (h_n(x, s) - h_n(y, s)) \cdot \nabla_x \rho_\delta(x - y) V^N(s, y, r) dy$$

thus, we obtain ($\|\cdot\|_\infty$ denotes the L^∞ -norm with respect to the x -variable)

$$\|r_\delta^h(s)\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)}^2 \leq C^2 \|\nabla h_n(s)\|_\infty^2 \|V^N(s)\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)}^2,$$

since $\operatorname{supp}[\rho] \subset B(0, 1)$ and denoted by $C^2 = \int_{\mathbb{R}^2} |z|^2 |\nabla_x \rho(z)|^2 dz < +\infty$. On the other hand, note that

$$\|\nabla_x h_n\|_\infty \leq \sum_{k \in K_n} |g_k| \|\nabla \sigma_k^N\|_\infty \leq \sum_{k \in K_n} \frac{1}{|k|} |g_k| \leq \sum_{|k| \leq n} |g_k| := \|g\| \in L^2(0, T).$$

The proof (60) is analogous to the proof (57) and we omit this detail.

Let us prove that $\lim_{\delta} \int_0^t \langle [\operatorname{div}_r((\nabla u_L(s)r) V^N(s))]_\delta, [V^N(s)]_\delta \rangle ds = 0$. Since $\operatorname{div}_r(\nabla u_L(s)r) = 0$, we get

$$\int_0^t \langle [\operatorname{div}_r((\nabla u_L(s)r) V^N(s))]_\delta, [V^N(s)]_\delta \rangle ds$$

$$\begin{aligned}
 &= \int_0^t \langle \operatorname{div}_r(\rho_\delta * ((\nabla u_L(s)r)V^N(s))) - \operatorname{div}_r((\nabla u_L(s)r)\rho_\delta * V^N(s)), \rho_\delta * V^N(s) \rangle ds \\
 &= - \int_0^t \langle \rho_\delta * ((\nabla u_L(s)r)V^N(s)) - (\nabla u_L(s)r)\rho_\delta * V^N(s), \rho_\delta * \nabla_r V^N(s) \rangle ds.
 \end{aligned}$$

On the other hand, note that

$$\begin{aligned}
 &(\rho_\delta * ((\nabla u_L(\cdot)r)V^N)) (s, x, r) - (\nabla u_L(s, x)r)(\rho_\delta * V^N)(s, x, r) \\
 &= \int_{\mathbb{R}^2} [\nabla u_L(x - y, s) - \nabla u_L(x, s)]r\rho_\delta(y)V^N(s, x - y, r)dy.
 \end{aligned}$$

By mean-value theorem we get $|\nabla u_L(x - y, s) - \nabla u_L(x, s)| \leq |y| \|u_L\|_{C^2}$ and

$$\begin{aligned}
 &|(\rho_\delta * ((\nabla u_L(\cdot)r)V^N)) (s, x, r) - (\nabla u_L(s, x)r)(\rho_\delta * V^N)(s, x, r)| \\
 &\leq \|u_L\|_{C^2} \int_{\mathbb{R}^2} |y||r|\rho_\delta(y)|V^N(s, x - y, r)|dy \leq \delta \|u_L\|_{C^2} \int_{\mathbb{R}^2} \rho_\delta(y)|r||V^N(s, x - y, r)|dy,
 \end{aligned}$$

since $\operatorname{supp}[\rho] \subset B(0, 1)$. Therefore, we get

$$\begin{aligned}
 &\int_0^t |\langle \rho_\delta * ((\nabla u_L(s)r)V^N(s)) - (\nabla u_L(s)r)\rho_\delta * V^N(s), \rho_\delta * \nabla_r V^N(s) \rangle| ds \\
 &\leq \delta \|u_L\|_{C^2} \int_0^t \left\| \int_{\mathbb{R}^2} \rho_\delta(y)|r||V^N(s, x - y, r)|dy \right\| \|\rho_\delta * \nabla_r V^N(s)\| ds \\
 &\leq \delta \|u_L\|_{C^2} \int_0^t \|\rho_\delta * |r||V^N(s)\| \|\rho_\delta * \nabla_r V^N(s)\| ds \\
 &\leq \delta \|u_L\|_{C^2} \int_0^t \|V^N(s)\|_H \|\nabla_r V^N(s)\| ds \rightarrow 0 \text{ as } \delta \rightarrow 0.
 \end{aligned}$$

Concerning the term $\int_0^t \langle [\operatorname{div}_r(y_n V^N(s))]_\delta, [V^N(s)]_\delta \rangle ds$. We recall that $\sum_{k \in K_n} g_k(\nabla \sigma_k^N r) = y_n$.

Since $\operatorname{div}_r(y_n) = 0$, we have

$$\int_0^t \langle [\operatorname{div}_r(y_n V^N(s))]_\delta, [V^N(s)]_\delta \rangle ds = \int_0^t \langle [\operatorname{div}_r(y_n V^N(s))]_\delta - \operatorname{div}_r(y_n[V^N(s)]_\delta), [V^N(s)]_\delta \rangle ds$$

$$\begin{aligned}
 &= \int_0^t \langle \operatorname{div}_r \rho_\delta * (y_n V^N(s)) - \operatorname{div}_r (y_n \rho_\delta * V^N(s)), \rho_\delta * V^N(s) \rangle ds \\
 &= - \int_0^t \langle \rho_\delta * (y_n V^N(s)) - y_n \rho_\delta * V^N(s), \rho_\delta * \nabla_r V^N(s) \rangle ds.
 \end{aligned}$$

On the other hand, note that

$$\begin{aligned}
 &(\rho_\delta * (y_n V^N))(s, x, r) - y_n(\rho_\delta * V^N)(s, x, r) \\
 &= \int_{\mathbb{R}^2} [y_n(x - y, s) - y_n(x, s)] \rho_\delta(y) V^N(s, x - y, r) dy.
 \end{aligned}$$

By mean-value theorem we get

$$\begin{aligned}
 |y_n(x - y, s) - y_n(x, s)| &= \left| \sum_{k \in K_n} (g_k(\nabla \sigma_k^N r)(x - y, s) - g_k(\nabla \sigma_k^N r)(x, s)) \right| \\
 &\leq \sum_{k \in K_n} |g_k| \|D^2 \sigma_k^N\|_\infty |y| |r| \leq |y| |r| \|g\| \text{ since } \|D^2 \sigma_k^N\|_\infty \leq 1,
 \end{aligned}$$

and

$$\begin{aligned}
 &|(\rho_\delta * ((\nabla u_L(\cdot) r) V^N))(s, x, r) - (\nabla u_L(s, x) r)(\rho_\delta * V^N)(s, x, r)| \\
 &\leq \|g\| \int_{\mathbb{R}^2} |y| |r| \rho_\delta(y) |V^N(s, x - y, r)| dy \leq \delta \|g\| \int_{\mathbb{R}^2} \rho_\delta(y) |r| |V^N(s, x - y, r)| dy,
 \end{aligned}$$

since $\operatorname{supp}[\rho] \subset B(0, 1)$. Therefore, we get

$$\begin{aligned}
 &\int_0^t |\langle \rho_\delta * ((\nabla u_L(s) r) V^N(s)) - (\nabla u_L(s) r) \rho_\delta * V^N(s), \rho_\delta * \nabla_r V^N(s) \rangle| ds \\
 &\leq \delta \int_0^t \|g\| \|\rho_\delta * |r| |V^N(s)|\| \|\rho_\delta * \nabla_r V^N(s)\| ds \\
 &\leq \delta \int_0^t \|g\| \|V^N(s)\|_H \|\nabla_r V^N(s)\| ds \rightarrow 0 \text{ as } \delta \rightarrow 0.
 \end{aligned}$$

Now, notice that $[\operatorname{div}_r(r V^N(s))]_\delta = \operatorname{div}_r(r[V^N(s)]_\delta)$. Hence

$$\frac{1}{\beta} \int_0^t \langle [\operatorname{div}_r(r V^N(s))]_\delta, [V^N(s)]_\delta \rangle ds = \frac{1}{\beta} \int_0^t \| [V^N(s)]_\delta \|^2 ds \leq \frac{1}{\beta} \int_0^t \| V^N(s) \|^2 ds.$$

Next, we have $\langle \sigma^2 [\Delta_r V^N(s)]_\delta, [V^N(s)]_\delta \rangle = \langle \sigma^2 \Delta_r [V^N(s)]_\delta, [V^N(s)]_\delta \rangle = -\sigma^2 \|\nabla_r [V^N(s)]_\delta\|^2$, thus

$$\begin{aligned} & \int_0^t \langle \sigma^2 [\Delta_r V^N(s)]_\delta, [V^N(s)]_\delta \rangle ds \\ &= -\sigma^2 \int_0^t \|\nabla_r V^N(s)\|_\delta^2 ds \rightarrow -\sigma^2 \int_0^t \|\nabla_r V^N(s)\|^2 ds \text{ as } \delta \rightarrow 0, \end{aligned}$$

since $\nabla_r V^N \in L^2(0, T; H)$. We have $\langle [\Delta_x V^N(s)]_\delta, [V^N(s)]_\delta \rangle = \langle \Delta_x [V^N(s)]_\delta, [V^N(s)]_\delta \rangle$ which gives

$$\int_0^t \alpha_N \langle [\Delta_x V^N(s)]_\delta, [V^N(s)]_\delta \rangle ds = - \int_0^t \alpha_N \|\nabla_x [V^N(s)]_\delta\|^2 ds \leq 0.$$

Concerning the last term, first we prove the following

$$\begin{aligned} & \int_0^t \langle [\operatorname{div}_r \sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r V^N(s)]_\delta, [V^N(s)]_\delta \rangle ds \\ &+ \sum_{k \in K} \int_0^t \|\nabla_r [(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta\|^2 ds \xrightarrow{\delta \rightarrow 0} 0. \end{aligned}$$

Indeed, we have

$$\begin{aligned} & \int_0^t \langle [\operatorname{div}_r \sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r V^N(s)]_\delta, [V^N(s)]_\delta \rangle ds \\ &+ \sum_{k \in K} \int_0^t \|\nabla_r [(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta\|^2 ds \\ &= - \sum_{k \in K} \int_0^t \langle [(\nabla \sigma_k^N r) (\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta, \nabla_r [V^N(s)]_\delta \rangle ds \\ &+ \sum_{k \in K} \int_0^t \|\nabla_r [(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta\|^2 ds \\ &= - \sum_{k \in K} \int_0^t \langle [(\nabla \sigma_k^N r) (\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta - (\nabla \sigma_k^N r) [(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta, \nabla_r [V^N(s)]_\delta \rangle ds \end{aligned}$$

$$\begin{aligned}
 & - \sum_{k \in K} \int_0^t \langle (\nabla \sigma_k^N r) [(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta, \nabla_r [V^N(s)]_\delta \rangle ds \\
 & + \sum_{k \in K} \int_0^t \langle [(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta, [(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta \rangle ds \\
 = & - \sum_{k \in K} \int_0^t \langle [(\nabla \sigma_k^N r)(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta - (\nabla \sigma_k^N r)[(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta, \nabla_r [V^N(s)]_\delta \rangle ds \\
 & - \sum_{k \in K} \int_0^t \langle [(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta, (\nabla \sigma_k^N r) \cdot \nabla_r [V^N(s)]_\delta - [(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta \rangle ds \\
 = & J_\delta^1 + J_\delta^2.
 \end{aligned}$$

Let us prove that $\limsup_{\delta \rightarrow 0} |J_\delta^1| = 0$. Recall that $(\nabla \sigma_k^N r)$ has the form $\theta_{|k|}^N |k| \frac{k}{|k|} \cdot r \frac{k^\perp}{|k|} h(k \cdot x)$ ⁸ for any $k \in \mathbb{Z}_0^2$. Thus, $|\nabla \sigma_k^N r| \leq \theta_{|k|}^N |k| |r|$. On the other hand, we have

$$\begin{aligned}
 |J_\delta^1| \leq & \sum_{k \in K} \int_0^t | \langle [(\nabla \sigma_k^N r)(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta \\
 & - (\nabla \sigma_k^N r)[(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta, \nabla_r [V^N(s)]_\delta \rangle | ds
 \end{aligned}$$

and

$$\begin{aligned}
 & \langle [(\nabla \sigma_k^N r)(\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)]_\delta - (\nabla \sigma_k^N r)[(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta \rangle(x, r) \\
 = & - \int_{\mathbb{R}^2} [(\nabla \sigma_k^N(x)r) - (\nabla \sigma_k^N(x-y)r)] (\nabla \sigma_k^N(x-y)r) \cdot \nabla_r V^N(s, x-y, r) \rho_\delta(y) dy.
 \end{aligned}$$

By mean-value theorem, we get

$$|(\nabla \sigma_k^N(x)r) - (\nabla \sigma_k^N(x-y)r)| \leq \theta_{|k|}^N |k|^2 \cdot |r| |y| \leq 2N |k| \theta_{|k|}^N \cdot |r| |y|.$$

Therefore

$$\begin{aligned}
 & | \langle [(\nabla \sigma_k^N r)(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta - (\nabla \sigma_k^N r)[(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta \rangle(x, r) | \\
 & \leq \int_{\mathbb{R}^2} 2N |k| \theta_{|k|}^N \cdot |r| |y| |(\nabla \sigma_k^N(x-y)r) \cdot \nabla_r V^N(s, x-y, r)| \rho_\delta(y) dy
 \end{aligned}$$

⁸ $h(\cdot) = \cos(\cdot)$ or $h(\cdot) = -\sin(\cdot)$, see (35).

$$\leq 2\delta N(\theta_{|k|}^N)^2 |k|^2 |r| \int_{\mathbb{R}^2} |r| |\nabla_r V^N(s, x - y, r)| \rho_\delta(y) dy.$$

Thus, we get

$$\begin{aligned} & | \langle [(\nabla\sigma_k^N r)(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta - (\nabla\sigma_k^N r)[(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta, \nabla_r[V^N(s)]_\delta \rangle | \\ &= | \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} [(\nabla\sigma_k^N r)(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta - (\nabla\sigma_k^N r)[(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta \nabla_r[V^N(s)]_\delta dx dr | \\ &\leq \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} |(\nabla\sigma_k^N r)(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta - (\nabla\sigma_k^N r)[(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta| |\nabla_r[V^N(s)]_\delta| dx dr \\ &\leq 2\delta N(\theta_{|k|}^N)^2 |k|^2 \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} |r| \int_{\mathbb{R}^2} |r| |\nabla_r V^N(s, x - y, r)| \rho_\delta(y) dy |\nabla_r[V^N(s)]_\delta| dx dr \\ &\leq 2\delta N(\theta_{|k|}^N)^2 |k|^2 \|\rho_\delta * |r| |\nabla_r V^N(s)|\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)}^2 \\ &\leq 2\delta N(\theta_{|k|}^N)^2 |k|^2 \|r| |\nabla_r V^N(s)|\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)}^2 \leq 2\delta N(\theta_{|k|}^N)^2 |k|^2 \|\nabla_r V^N(s)\|_H^2. \end{aligned}$$

Hence, we deduce $|J_\delta^1| \leq 2\delta N \sum_{k \in K} (\theta_{|k|}^N)^2 |k|^2 \int_0^t \|\nabla_r V^N(s)\|_H^2 ds \rightarrow 0$ as $\delta \rightarrow 0$. Concerning J_δ^2 ,

we recall that

$$|J_\delta^2| \leq \sum_{k \in K} \int_0^t | \langle [(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta, (\nabla\sigma_k^N r) \cdot \nabla_r[V^N(s)]_\delta - [(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta \rangle | ds.$$

On the other hand, we have

$$\begin{aligned} & |(\nabla\sigma_k^N r) \cdot \nabla_r[V^N(s)]_\delta - [(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta| \\ &= | \int_{\mathbb{R}^2} [(\nabla\sigma_k^N(x)r) - (\nabla\sigma_k^N(x-y)r)] \cdot \nabla_r V^N(s, x - y, r) \rho_\delta(y) dy | \\ &\leq \delta |k|^2 \theta_{|k|}^N \int_{\mathbb{R}^2} |r| |\nabla_r V^N(s, x - y, r)| \rho_\delta(y) dy, \end{aligned}$$

consequently, we get

$$\begin{aligned} & | \langle [(\nabla\sigma_k^N r) \cdot \nabla_r f^N(s)]_\delta, (\nabla\sigma_k^N r) \cdot \nabla_r[V^N(s)]_\delta - [(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta \rangle | \\ &\leq \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} |(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta| |(\nabla\sigma_k^N r) \cdot \nabla_r[V^N(s)]_\delta - [(\nabla\sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta| dx dr \end{aligned}$$

$$\begin{aligned}
 &\leq \delta |k|^2 \theta_{|k|}^N \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} |[(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta| \int_{\mathbb{R}^2} |r| |\nabla_r V^N(s, x - y, r)| \rho_\delta(y) dy dx dr \\
 &\leq \delta |k|^3 (\theta_{|k|}^N)^2 \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |r| |\nabla_r V^N(s, x - y, r)| \rho_\delta(y) dy dx dr \\
 &\leq \delta |k|^3 (\theta_{|k|}^N)^2 \int_{\mathbb{R}^2} |r| |\nabla_r V^N(s, x - y, r)| \rho_\delta(y) dy \Big\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)}^2 \\
 &\leq \delta |k|^3 (\theta_{|k|}^N)^2 \|\nabla_r V^N(s)\|_H^2 \leq 2\delta N |k|^2 (\theta_{|k|}^N)^2 \|\nabla_r V^N(s)\|_H^2
 \end{aligned}$$

and

$$|J_\delta^2| \leq 2\delta N \sum_{k \in K} (\theta_{|k|}^N)^2 |k|^2 \mathbb{E} \int_0^t \|\nabla_r V^N(s)\|_H^2 ds \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Finally, we get

$$\begin{aligned}
 &\lim_{\delta \rightarrow 0} \int_0^t \langle [\operatorname{div}_r \sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) \nabla_r V^N(s)]_\delta, [V^N(s)]_\delta \rangle ds \\
 &= - \sum_{k \in K} \int_0^t \|(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)\|^2 ds \leq 0,
 \end{aligned}$$

where we used $\nabla_r V^N \in L^2(0, T; H)$ and

$$\sum_{k \in K} \int_0^t \|[(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)]_\delta\|^2 ds \xrightarrow{\delta \rightarrow 0} \sum_{k \in K} \int_0^t \|(\nabla \sigma_k^N r) \cdot \nabla_r V^N(s)\|^2 ds.$$

In conclusion, by passing to the limit as $\delta \rightarrow 0$ in (54) and using the above estimates, we get

$$\frac{1}{2} \|V^N(t)\|^2 + \sigma^2 \int_0^t \|\nabla_r V^N(s)\|^2 ds \leq \frac{1}{\beta} \int_0^t \|V^N(s)\|^2 ds. \tag{61}$$

The last inequality (61) and Grönwall lemma ensure that $V^N \equiv 0$ in $L^\infty(0, T; H)$ -sense and $\nabla V^N \equiv 0$ in $L^2(0, T; H)$ -sense as well, which ends the proof of uniqueness.

6.3.1. Uniqueness of quasi-regular weak solutions of (12)

We will use Lemma 21 to prove that the solution to (12) in the sense of Definition 5 is unique (in particular we use the point (4)). First, note that the set of *quasi-regular weak* solutions forms a linear subspace of $L^2(0, T; H)$, since (12) is a linear equation, and the regularity conditions is

a linear constraint. Therefore, it is enough to show that a quasi-regular weak solution $f^N \equiv 0$ if the initial data $f_0 \equiv 0$.

Let $g \in G_n$, by using Lemma 21 we proved that $\mathbb{E}[f^N(t)e_g(t)] = 0$ in $L^\infty(0, T; H)$ -sense. Our aim is to prove that $f^N \equiv 0$. We recall that from Lemma 21: for any $t \in [0, T]$, we have

$$(\mathbb{E}[f^N(t)e_g(t)], \varphi) = 0, \forall \varphi \in V \text{ and for any } g \in \mathcal{D}.$$

Now, let G be a random variable, which can be written as a linear combination of finite number of $e_g(t)$, it follows (by linearity)

$$(\mathbb{E}[f^N(t)G], \varphi) = 0, \forall \varphi \in V.$$

Next, by density of \mathcal{D} in $L^2(\Omega, \overline{\mathcal{G}}_t)$, the last equality holds for any $G \in L^2(\Omega, \overline{\mathcal{G}}_t)$, namely

$$\mathbb{E}[(f^N(t), \varphi)G] = 0, \forall \varphi \in V, \quad \forall G \in L^2(\Omega, \overline{\mathcal{G}}_t).$$

Since $(f^N(t), \varphi)$ is $\overline{\mathcal{G}}_t$ -adapted, we get $(f^N(t), \varphi) = 0$, for any $\varphi \in V$. Recall that V is dense subspace in H , thus, we deduce that $f^N \equiv 0$ and the uniqueness holds.

7. Diffusion scaling limit as $N \rightarrow +\infty$

Our aim in this section is to show that the unique solution of stochastic FP equation (12), in the sense of Definition 5, converges weakly to the unique solution of (13), under the following scaling of the noise coefficients: $\theta_{|k|}^N = \frac{a}{|k|^2}$ if $N \leq |k| \leq 2N$ and $\theta_{|k|}^N = 0$ else. First, note that

$$\lim_{N \rightarrow +\infty} \sum_{k \in K_{++}} \left(\theta_{|k|}^N\right)^2 = 0; \quad \lim_{N \rightarrow +\infty} \sup_{k \in K} \left(\theta_{|k|}^N\right)^2 = 0 \text{ and } \lim_{N \rightarrow +\infty} \sup_{k \in K} |k|^2 \left(\theta_{|k|}^N\right)^2 = 0. \quad (62)$$

7.1. Time regularity and tightness of laws

We begin by showing the following result about regularity in time of $(f^N)_N$.

Lemma 22. *Let $2 < p < +\infty$, there exists $C > 0$, independent of N such that*

$$\mathbb{E} \|f^N\|_{\mathcal{C}^\eta([0, T], U')}^p \leq C \text{ for any } 0 < \eta < \min\left(\frac{p-2}{p}, \frac{1}{2}\right).$$

Proof. Let $0 < h < 1$ and $t \in [0, T - h]$. From Definition 5, point (3), the following equality holds in U' -sense:

$$f^N(t+h) - f^N(t) = - \int_t^{t+h} ((u_L \cdot \nabla_x f^N(s)) - \operatorname{div}_r[(\nabla_x u_L r - \frac{1}{\beta} r) f^N(s)]) ds$$

$$\begin{aligned}
 &+ \int_t^{t+h} [\sigma^2 \Delta_r f^N(s) + \alpha_N (\Delta_x f^N(s))] ds + \frac{1}{2} \int_t^{t+h} \operatorname{div}_r (A_k^N \nabla_r f^N(s)) ds \\
 &- \sum_{k \in K} \int_t^{t+h} (\sigma_k^N \cdot \nabla_x f^N(s) + (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)) dW^k(s) := I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

On the one hand, we have P-a.s.

$$\begin{aligned}
 \|I_1\|_{U'} &\leq \int_t^{t+h} \|u_L \cdot \nabla_x f^N(s) + \operatorname{div}_r[(\nabla_x u_L r - \frac{1}{\beta} r) f^N(s)]\|_{U'} ds \\
 &\leq [\|u_L\|_\infty + \|\nabla_x u_L\|_\infty + \frac{1}{\beta}] \int_t^{t+h} \|f^N(s)\|_H ds \\
 &\leq h[\|u_L\|_\infty + \|\nabla_x u_L\|_\infty + \frac{1}{\beta}] \sup_{t \in [0, T]} \|f^N(t)\|_H,
 \end{aligned}$$

since u_L is a smooth function. Concerning I_2 , note that

$$\begin{aligned}
 \|I_2\|_{U'} &\leq \int_t^{t+h} \|\sigma^2 \Delta_r f^N(s) + \alpha_N \Delta_x f^N(s)\|_{U'} ds \leq (\sigma^2 + \alpha_N) \int_t^{t+h} \|f^N(s)\|_H ds \\
 &\leq (\sigma^2 + 1) \int_t^{t+h} \|f^N(s)\|_H ds \leq h(\sigma^2 + 1) \sup_{t \in [0, T]} \|f^N(t)\|_H.
 \end{aligned}$$

Moreover $\|I_3\|_{U'} \leq \int_t^{t+h} \|\operatorname{div}_r (A_k^N \nabla_r f^N(s))\|_{U'} ds$, we recall that

$$\|\operatorname{div}_r (A_k^N \nabla_r f^N(s))\|_{U'} = \sup_{\|\phi\|_{U'} \leq 1} \langle \operatorname{div}_r (A_k^N \nabla_r f^N(s)), \phi \rangle$$

but $\langle \operatorname{div}_r (A_k^N \nabla_r f^N), \phi \rangle = (\operatorname{div}_r (A_k^N \nabla_r f^N), \phi) = (f^N, \operatorname{div}_r (A_k^N \nabla_r \phi))$, therefore

$$\begin{aligned}
 \int_t^{t+h} \|\operatorname{div}_r (A_k^N \nabla_r f^N(s))\|_{U'} ds &\leq \sum_{k \in K} |k|^2 (\theta_{|k|}^N)^2 \int_t^{t+h} \|f^N(s)\|_H ds \\
 &\leq h \sup_{t \in [0, T]} \|f^N(t)\|_H \sum_{k \in K} |k|^2 (\theta_{|k|}^N)^2.
 \end{aligned}$$

Concerning the stochastic integral I_4 , let $\phi \in U$ and note that

$$\begin{aligned} & \sum_{k \in K} \int_t^{t+h} \langle (\sigma_k^N \cdot \nabla_x f^N(s) + (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s), \phi) dW^k(s) \\ &= \sum_{k \in K} \int_t^{t+h} (\sigma_k^N \cdot \nabla_x f^N(s) + (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s), \phi) dW^k(s) \\ &= - \sum_{k \in K} \int_t^{t+h} (f^N(s), \sigma_k^N \cdot \nabla_x \phi + (\nabla \sigma_k^N r) \cdot \nabla_r \phi) dW^k(s). \end{aligned}$$

We recall

$$\begin{aligned} & \left\| \sum_{k \in K} \int_t^{t+h} (\sigma_k^N \cdot \nabla_x f^N(s) + (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)) dW^k(s) \right\|_U \\ &= \sup_{\|\phi\|_U \leq 1} \left| \sum_{k \in K} \int_t^{t+h} (f^N(s), \sigma_k^N \cdot \nabla_x \phi + (\nabla \sigma_k^N r) \cdot \nabla_r \phi) dW^k(s) \right|. \end{aligned}$$

Let $\phi \in U$ such that $\|\phi\|_U \leq 1$ and $1 < p < +\infty$. Burkholder-Davis-Gundy inequality ensures

$$\begin{aligned} & \mathbb{E} \left| \sum_{k \in K} \int_t^{t+h} (f^N(s), \sigma_k^N \cdot \nabla_x \phi + (\nabla \sigma_k^N r) \cdot \nabla_r \phi) dW^k(s) \right|^p \\ & \leq \mathbb{E} \left[\int_t^{t+h} \sum_{k \in K} (f^N(s), \sigma_k^N \cdot \nabla_x \phi + (\nabla \sigma_k^N r) \cdot \nabla_r \phi)^2 ds \right]^{p/2} \\ & \leq \left[\sum_{k \in K} (|k|^2 + 1) \left(\theta_{|k|}^N \right)^2 \right]^{p/2} \mathbb{E} \left[\int_t^{t+h} (f^N(s), \nabla_x \phi + |r| |\nabla_r \phi|)^2 ds \right]^{p/2} \\ & \leq C_1 \mathbb{E} \left[\int_t^{t+h} \|f^N(s)\|^2 \|\phi\|_U^2 ds \right]^{p/2} \leq C_1 \mathbb{E} \left[\int_t^{t+h} \|f^N(s)\|^2 \|\phi\|_U^2 ds \right]^{p/2} \\ & \leq C_1 \mathbb{E} \left[\int_t^{t+h} \|f^N(s)\|^2 ds \right]^{p/2}, \end{aligned}$$

where $C_1 := \left[\sum_{k \in K} (|k|^2 + 1) \left(\theta_{|k|}^N \right)^2 \right]^{p/2}$. Thus, we obtain

$$\begin{aligned} \mathbb{E} \left\| \sum_{k \in K} \int_t^{t+h} (\sigma_k^N \cdot \nabla_x f^N(s) + (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)) dW^k(s) \right\|_{U'}^p &\leq C_1 \mathbb{E} \left[\int_t^{t+h} \|f^N(s)\|^2 ds \right]^{p/2} \\ &\leq C_1 h^{p/2} \mathbb{E} \sup_{q \in [0, T]} \|f^N(q)\|^p. \end{aligned} \tag{63}$$

Recall the definition of $W^{s,p}(0, T; U')$, the Sobolev space of all $u \in L^p(0, T; U')$ such that

$$\int_0^T \int_0^T \frac{\|u(t) - u(r)\|_{U'}^p}{|t - r|^{1+sp}} dt dr < +\infty,$$

endowed with the norm

$$\|u\|_{W^{s,p}(0,T;U')}^p = \|u\|_{L^p(0,T;U')}^p + \int_0^T \int_0^T \frac{\|u(t) - u(r)\|_{U'}^p}{|t - r|^{1+sp}} dt dr.$$

Now, denote by $I(f^N)(\cdot) = \sum_{k \in K} \int_0^\cdot (\sigma_k^N \cdot \nabla_x f^N(s) + (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)) dW^k(s)$.

Thanks to (45) and Burkholder-Davis-Gundy inequality, one has

$$\mathbb{E} \|I(f^N)\|_{L^p(0,T;U')}^p \leq \mathbf{M}, \quad \mathbf{M} > 0 \text{ independent of } m \text{ and } N.$$

Concerning the second part, note that

$$\begin{aligned} \mathbb{E} \int_0^T \int_0^T \frac{\|I(f^N)(t) - I(f^N)(r)\|_{U'}^p}{|t - r|^{1+sp}} dt dr &= \int_0^T \int_0^T \frac{\mathbb{E} \|I(f^N)(t) - I(f^N)(r)\|_{U'}^p}{|t - r|^{1+sp}} dt dr \\ &\leq C_1 \mathbb{E} \sup_{q \in [0, T]} \|f^N(q)\|^p \int_0^T \int_0^T \frac{|t - r|^{p/2}}{|t - r|^{1+sp}} dt dr \\ &\leq C_1 \mathbb{E} \sup_{q \in [0, T]} \|f^N(q)\|^p \int_0^T \int_0^T |t - r|^{p(\frac{1}{2}-s)-1} dt dr \\ &\leq C, \end{aligned}$$

if $p(\frac{1}{2} - s) > 0$, which holds for any $s \in]0, \frac{1}{2}[$.

Let $p > 2$, then $0 < sp - 1 < \frac{p-2}{2}$. Denote by $\mathcal{C}^\eta([0, T], U')$ the space of η -Hölder continuous functions with values in U' and we recall that (see e.g. [67])

$$W^{s,p}(0, T; U') \hookrightarrow C^\eta([0, T], U') \quad \text{if } 0 < \eta < sp - 1.$$

Let us take $s \in [0, \frac{1}{2}[$ such that $sp > 1$. For $\eta \in]0, sp - 1[$, it follows from the previous estimates

$$\mathbb{E} \left\| \sum_{k \in K} \int_0^\cdot (\sigma_k^N \cdot \nabla_x f^N(s) + (\nabla \sigma_k^N r) \cdot \nabla_r f^N(s)) dW^k(s) \right\|_{W^{s,p}(0,T;U')}^p \leq C + \mathbf{M}.$$

Thus $\mathbb{E} \|I(f^N)\|_{C^\eta([0,T],U')}^p \leq C + \mathbf{M}$. Consequently, we obtain

$$\mathbb{E} \|f^N\|_{C^\eta([0,T],U')}^p \leq C + \mathbf{M} \text{ for any } 0 < \eta < \min\left(\frac{p-2}{p}, \frac{1}{2}\right),$$

which gives that $(f^N)_N$ is bounded in $L^p(\Omega, C^\eta([0, T], U'))$, $\eta \in]0, \min(\frac{p-2}{p}, \frac{1}{2})[$ with $2 < p < +\infty$.

As a consequence of Lemma 15, Lemma 16, Lemma 22 and the lower semi-continuity of the weak convergence, we obtain

Proposition 23. *Let $p \geq 2$, there exists $\mathbf{K} > 0$ such that the unique solution $(f^N)_N$ to (12) is bounded by \mathbf{K} in*

$$L^p(\Omega, L^\infty(0, T; H)) \cap L^p(\Omega, C^\eta([0, T], U')), \quad 0 < \eta < \min\left(\frac{p-2}{p}, \frac{1}{2}\right).$$

Moreover, $(\nabla_r f^N)_N$ is bounded in $L^2(\Omega, L^2(0, T; H))$.

7.1.1. Tightness of laws

Denote by μ_{f^N} the law of f^N , μ_W the law of $W := (W^k)_{k \in \mathbb{Z}_0^2}$ and their joint law μ_N defined on $C([0, T], U') \times C([0, T]; H_0)$.

Lemma 24. *The set $\{\mu_{f^N}; N \in \mathbb{N}\}$ is tight on $C([0, T], U')$.*

Proof. First, we have $H \xhookrightarrow{\text{compact}} U'$, since $U \xhookrightarrow{\text{compact}} H$. Let \mathbf{A} be a subset $C([0, T]; U')$. Following [68, Thm. 3] (the case $p = \infty$), \mathbf{A} is relatively compact in $C([0, T]; U')$ if the following conditions hold.

1. \mathbf{A} is bounded in $L^\infty(0, T; H)$.
2. Let $h > 0$, $\|f(\cdot + h) - f(\cdot)\|_{L^\infty(0, T-h; U')} \rightarrow 0$ as $h \rightarrow 0$ uniformly for $f \in \mathbf{A}$.

The following embedding is compact

$$\mathbf{Z} := L^\infty(0, T; H) \cap C^\eta([0, T], U') \hookrightarrow C([0, T], U'), \quad 0 < \eta < \min\left(\frac{p-2}{p}, \frac{1}{2}\right).$$

Indeed, let \mathbf{A} be a bounded set of \mathbf{Z} . First, note that (1) is satisfied by assumptions. Concerning the second condition, let $h > 0$ and $f \in \mathbf{A}$, by using that $f \in C^\eta([0, T], U')$ we infer

$$\|f(\cdot + h) - f(\cdot)\|_{L^\infty(0, T-h; U')} = \sup_{r \in [0, T-h]} \|f(r+h) - f(r)\|_{U'} \leq Ch^\eta \rightarrow 0, \text{ as } h \rightarrow 0,$$

where $C > 0$ is independent of f . Let $R > 0$ and set $B_{\mathbf{Z}}(0, R) := \{v \in \mathbf{Z} \mid \|v\|_{\mathbf{Z}} \leq R\}$. Then $B_{\mathbf{Z}}(0, R)$ is a compact subset of $C([0, T], U')$, by using Proposition 23 the following holds

$$\begin{aligned} \mu_{f^N}(B_{\mathbf{Z}}(0, R)) &= 1 - \mu_{f^N}(B_{\mathbf{Z}}(0, R)^c) = 1 - \int_{\{\omega \in \Omega, \|f^N\|_{\mathbf{Z}} > R\}} 1 dP \\ &\geq 1 - \frac{1}{R^p} \int_{\{\omega \in \Omega, \|f^N\|_{\mathbf{Z}} > R\}} \|f^N\|_{\mathbf{Z}}^p dP \\ &\geq 1 - \frac{1}{R^p} \mathbb{E} \|f^N\|_{\mathbf{Z}}^p = 1 - \frac{\mathbf{K}^p}{R^p}, \quad \text{for any } R > 0, \quad \text{and any } N \in \mathbb{N}. \end{aligned}$$

Therefore, for any $\delta > 0$ we can find $R_\delta > 0$ such that

$$\mu_{f^N}(B_{\mathbf{Z}}(0, R_\delta)) \geq 1 - \delta, \text{ for all } N \in \mathbb{N}.$$

Thus the family of laws $\{\mu_{f^N}; N \in \mathbb{N}\}$ is tight on $C([0, T], U')$.

Notice that the family of Brownian motions $W := (W^k)_{k \in \mathbb{Z}_0^2}$ can be seen as cylindrical Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_t)$ with values in appropriate separable Hilbert space H_0 , more precisely $W^k = We_k, k \in \mathbb{Z}_0^2$, where $(e_k)_{k \in \mathbb{Z}_0^2}$ is complete orthonormal system in a separable Hilbert space \mathbb{H} and recall that the sample paths of W take values in a larger Hilbert space H_0 such that $\mathbb{H} \hookrightarrow H_0$ defines a Hilbert–Schmidt embedding. Hence, P -a.s. the trajectories of W belong to the space $C([0, T], H_0)$, see e.g. [69, Chapter 4]. By taking into account that the law μ_W is a Radon measure on $C([0, T]; H_0)$, we obtain

Lemma 25. *The set $\{\mu_W\}$ is tight on $C([0, T]; H_0)$.*

7.1.2. *Prokhorov and Skorokhod’s representation’s theorem*

Thanks to Lemma 24 and Lemma 25, by Skorokhod’s representation’s theorem (see e.g. [70, Thm. 1.10.4, p. 59]), by passing to the limit up to subsequences (denoted by the same way), we can find a new probability space, denoted by the same way “for simplicity” (Ω, \mathcal{F}, P) and processes

$$\left(\tilde{f}^N, W^N := \{W^{N,k}\}_{k \in \mathbb{Z}_0^2}\right), \quad \left(\bar{f}, \bar{W} := \{\bar{W}^k\}_{k \in \mathbb{Z}_0^2}\right),$$

such that:

(a) $\mathcal{L}(\tilde{f}^N, W^N := \{W^{N,k}\}_{k \in \mathbb{Z}_0^2}) = \mathcal{L}(f^N, W := \{W^k\}_{k \in \mathbb{Z}_0^2})^9$ on $C([0, T], U') \times C([0, T], H_0)$.

(b) the following convergences hold

$$\tilde{f}^N \rightarrow \bar{f} \text{ in } C([0, T]; U') \quad P - a.s. \tag{64}$$

$$W^N \rightarrow \bar{W} \text{ in } C([0, T]; H_0) \quad P - a.s. \tag{65}$$

On the other hand, thanks to Proposition 23, $\mathcal{L}(f^N)(\mathcal{X}) = 1$ where $\mathcal{X} \xrightarrow[cont.]{\hookrightarrow} C([0, T], U')$, with

$$\mathcal{X} = \{g : g \in L^\infty(0, T; H) \cap C^\eta([0, T], U'); \nabla_r g \in L^2(0, T; H)\}.$$

By using the point (a) above, one gets $\mathcal{L}(\tilde{f}^N)(\mathcal{X}) = 1$ and (\tilde{f}^N, W^N) satisfies the point (3) of Definition 5. Moreover, $(\tilde{f}^N)_N$ satisfies the estimates of Proposition 23 in the new probability space. Thus, we have the following result.

Lemma 26. *There exists $\bar{f} \in L^2(\Omega; L^2([0, T]; H))$, $\nabla_r \bar{f} \in L^2(\Omega; L^2([0, T]; H))$ such that*

$$\begin{aligned} \tilde{f}^N &\rightharpoonup \bar{f} \text{ in } L^2(\Omega; L^2([0, T]; H)), \\ \nabla_r \tilde{f}^N &\rightharpoonup \nabla_r \bar{f} \text{ in } L^2(\Omega; L^2([0, T]; H)). \end{aligned}$$

Moreover, $\bar{f} \in L^2_{w-*}(\Omega; L^\infty([0, T]; H))$ and $\tilde{f}^N \rightharpoonup^* \bar{f}$ in $L^2_{w-*}(\Omega; L^\infty([0, T]; H))$.

Proof. By using Proposition 23, diagonal extraction argument and Banach–Alaoglu theorem in the spaces $L^2(\Omega; L^2([0, T]; H))$ and $L^2_{w-*}(\Omega; L^\infty([0, T]; H))$, there exist $\bar{f}, \nabla_r \bar{f} \in L^2(\Omega; L^2([0, T]; H))$ such that

$$\tilde{f}^N \rightharpoonup \bar{f} \text{ in } L^2(\Omega; L^2([0, T]; H)), \tag{66}$$

$$\nabla_r \tilde{f}^N \rightharpoonup \nabla_r \bar{f} \text{ in } L^2(\Omega; L^2([0, T]; H)). \tag{67}$$

Moreover, $\bar{f} \in L^2_{w-*}(\Omega; L^\infty([0, T]; H))$ and the following convergence holds

$$\tilde{f}^N \rightharpoonup^* \bar{f} \text{ in } L^2_{w-*}(\Omega; L^\infty([0, T]; H)), \tag{68}$$

since $(\tilde{f}^N)_N$ is bounded in $L^2(\Omega; L^\infty([0, T]; H))$ and $L^2_{w-*}(\Omega; L^\infty([0, T]; H)) \simeq (L^2(\Omega; L^1([0, T]; H')))'$.

7.2. Passage to the limit as $N \rightarrow +\infty$

In the following, we will establish some lemmas to pass to the limit as $N \rightarrow +\infty$. For that, consider $\phi \in C^\infty(\mathbb{T}^2)$, $\psi \in C^\infty_c(\mathbb{R}^2)$,¹⁰ and $A \in \mathcal{F}$.

⁹ Given a random variable ξ with values in space E , its law $\mathcal{L}(\xi)(\Gamma) = P(\xi \in \Gamma)$ for any Borel subset Γ of E .

¹⁰ Note that $\phi \otimes \psi \in V$, thus it is an appropriate test function in Definition 5, point (3).

Lemma 27. Let $t \in]0, T[$, the following convergence holds:

$$\lim_N \alpha_N \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_x \tilde{f}^N(x, r, s) \cdot \nabla_x \phi(x) \psi(r) dr dx ds dP = 0.$$

Proof. Thanks to Fubini’s theorem and by using integration by part with respect to x , we get

$$\begin{aligned} & \left| \alpha_N \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_x \tilde{f}^N(x, r, s) \cdot \nabla_x \phi(x) \psi(r) dr dx ds dP \right| \\ &= \left| \alpha_N \int_A \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \tilde{f}^N(x, r, s) \Delta_x \phi(x) \psi(r) dx dr ds dP \right| \\ &\leq C \alpha_N \|\tilde{f}^N\|_{L^2(\Omega \times]0, T[; L^2(\mathbb{T}^2 \times \mathbb{R}^2))} \|\Delta_x \phi\|_\infty \|\psi\|_\infty, \end{aligned}$$

where C is a constant depends only on the measure of $supp(\psi)$, T and the volume of \mathbb{T}^2 . We recall from (11) that $\lim_N \alpha_N = 0$ and the result follows.

Lemma 28. Let $t \in]0, T[$, we have

$$\begin{aligned} & \lim_N -\frac{1}{2} \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r \tilde{f}^N(x, r, s) \right) \cdot \nabla_r \psi(r) \phi(x) dr dx ds dP \\ &= -\frac{1}{2} \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} A(r) \nabla_r \bar{f}(x, r, s) \cdot \nabla_r \psi(r) \phi(x) dr dx ds dP. \end{aligned}$$

Proof. By using Lemma 3, we get

$$\begin{aligned} & -\frac{1}{2} \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r \tilde{f}^N(x, r, s) \right) \cdot \nabla_r \psi(r) \phi(x) dr dx ds dP \\ &= -\frac{1}{2} \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} A(r) \nabla_r \tilde{f}^N(x, r, s) \cdot \nabla_r \psi(r) \phi(x) dr dx ds dP + R_N, \end{aligned} \tag{69}$$

where R_N satisfies

$$|R_N| \leq \frac{C}{N} \int_A \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |P(r)| |\tilde{f}^N(x, r, s)| \cdot |D_r^2 \psi(r) \phi(x)| |dr dx ds dP$$

$$\leq \frac{C}{N} \|\tilde{f}^N\|_{L^2(\Omega \times [0, T]; H)} \|D_r^2 \psi\|_\infty \|\phi\|_\infty \rightarrow 0 \text{ as } N \rightarrow +\infty.$$

Now, by passing to the limit in (69) and using Lemma 26, the conclusion follows.

Concerning the stochastic integral part, we have the following result.

Lemma 29. *Let $t \in]0, T]$, the following convergence holds*

$$\begin{aligned} & \lim_N \mathbb{E} \int_0^t \sum_{k \in K} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}^N(x, r, s) \left(\sigma_k^N \cdot \nabla_x \phi(x) \psi(r) + (\nabla \sigma_k^N r) \cdot \nabla_r \psi(r) \phi(x) \right) dr dx dW^{N,k}(s) 1_A \\ & = 0. \end{aligned}$$

Proof. Let $A \in \mathcal{F}$, $\phi \in C^\infty(\mathbb{T}^2)$ and $\psi \in C_c^\infty(\mathbb{R}^2)$, by using Itô isometry we get

$$\begin{aligned} & \mathbb{E} \left(\int_0^t \sum_{k \in K} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}^N(x, r, s) \left(\sigma_k^N \cdot \nabla_x \phi(x) \psi(r) + (\nabla \sigma_k^N r) \cdot \nabla_r \psi(r) \phi(x) \right) dr dx dW^{N,k}(s) 1_A \right)^2 \\ & = \mathbb{E} \int_0^t \sum_{k \in K} \left[\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} 1_A \tilde{f}^N(x, r, s) \left(\sigma_k^N \cdot \nabla_x \phi(x) \psi(r) + (\nabla \sigma_k^N r) \cdot \nabla_r \psi(r) \phi(x) \right) dx dr \right]^2 ds \\ & = \mathbb{E} \int_0^t \sum_{k \in K} \left[\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} (1_A \tilde{f}^N(x, r, s) \nabla_x \phi(x) \psi(r)) \cdot \sigma_k^N \right. \\ & \quad \left. + (1_A \tilde{f}^N(x, r, s) \nabla_r \psi(r) \phi(x)) \cdot (\nabla \sigma_k^N r) dx dr \right]^2 ds \\ & \leq 2 \mathbb{E} \int_0^t \sum_{k \in K} \left[\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} (1_A \tilde{f}^N(x, r, s) \nabla_x \phi(x) \psi(r)) \cdot \sigma_k^N dx dr \right]^2 ds \\ & \quad + 2 \mathbb{E} \int_0^t \sum_{k \in K} \left[\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} (1_A \tilde{f}^N(x, r, s) \nabla_r \psi(r) \phi(x)) \cdot (\nabla \sigma_k^N r) dx dr \right]^2 ds := 2(I_1^N + I_2^N). \end{aligned}$$

By using the definition of σ_k^N , we get

$$I_1^N = \mathbb{E} \int_0^t \sum_{k \in K} \left[\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} (1_A \tilde{f}^N(x, r, s) \nabla_x \phi(x) \psi(r)) \cdot \sigma_k^N dx dr \right]^2 ds$$

$$\begin{aligned}
 &= \mathbb{E} \int_0^t \sum_{k \in K_+} (\theta_{|k|}^N)^2 \left[\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} (1_A \tilde{f}^N(x, r, s) \nabla_x \phi(x) \psi(r)) \cdot \frac{k^\perp}{|k|} \cos k \cdot x dx dr \right]^2 ds \\
 &+ \mathbb{E} \int_0^t \sum_{k \in K_-} (\theta_{|k|}^N)^2 \left[\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} (1_A \tilde{f}^N(x, r, s) \nabla_x \phi(x) \psi(r)) \cdot \frac{k^\perp}{|k|} \sin k \cdot x dx dr \right]^2 ds.
 \end{aligned}$$

By using that $(\frac{k^\perp}{|k|} \cos k \cdot x, \frac{k^\perp}{|k|} \sin k \cdot x)_{k \in K}$ is an (incomplete) orthonormal system in $L^2(\mathbb{T}^2; \mathbb{R}^2)$, we obtain by using (62)

$$\begin{aligned}
 |I_1^N| &\leq \sup_{k \in K} (\theta_{|k|}^N)^2 \mathbb{E} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} (\tilde{f}^N(x, r, s) \nabla_x \phi(x) \psi(r))^2 dx dr ds \\
 &\leq \|\nabla_x \phi \psi\|_\infty^2 \sup_{k \in K} (\theta_{|k|}^N)^2 \mathbb{E} \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \tilde{f}^N(x, r, s)^2 dx dr ds \\
 &\leq \mathbf{K}^2 \|\nabla_x \phi \psi\|_\infty^2 \sup_{k \in K} (\theta_{|k|}^N)^2 \rightarrow 0 \text{ as } N \rightarrow +\infty.
 \end{aligned}$$

Concerning I_2^N , we have

$$\begin{aligned}
 I_2^N &= \mathbb{E} \int_0^t \sum_{k \in K} \left[\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} (1_A \tilde{f}^N(x, r, s) \nabla_r \psi(r) \phi(x)) \cdot (\nabla \sigma_k^N r) dx dr \right]^2 ds \\
 &= \mathbb{E} \int_0^t \sum_{k \in K_+} (\theta_{|k|}^N)^2 |k|^2 \left[\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \frac{k}{|k|} \cdot r (1_A \tilde{f}^N(x, r, s) \nabla_r \psi(r) \phi(x)) \cdot \frac{k^\perp}{|k|} \sin k \cdot x dx dr \right]^2 ds \\
 &+ \mathbb{E} \int_0^t \sum_{k \in K_-} (\theta_{|k|}^N)^2 |k|^2 \left[\int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \frac{k}{|k|} \cdot r (1_A \tilde{f}^N(x, r, s) \nabla_r \psi(r) \phi(x)) \cdot \frac{k^\perp}{|k|} \cos k \cdot x dx dr \right]^2 ds.
 \end{aligned}$$

Thus, by using the same argument as above we obtain

$$\begin{aligned}
 |I_1^N| &\leq \sup_{k \in K} [(\theta_{|k|}^N)^2 |k|^2] \mathbb{E} \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} |r|^2 (\tilde{f}^N(x, r, s) \nabla_r \psi(r) \phi(x))^2 dx dr ds \\
 &\leq \sup_{k \in K} [(\theta_{|k|}^N)^2 |k|^2] \|\nabla_r \psi \phi\|_\infty^2 \|\tilde{f}^N\|_{L^2(\Omega \times [0, T]; H)}^2 \rightarrow 0 \text{ as } N \rightarrow +\infty.
 \end{aligned}$$

Lemma 30. For any $t \in]0, T[$, the following convergence holds (up to a subsequence)

$$\lim_N \int_A \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}^N(x, r, t) \phi(x) \psi(r) dr dx dP = \int_A \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \bar{f}(x, r, t) \phi(x) \psi(r) dr dx dP.$$

Proof. From (64), $\tilde{f}^N \rightarrow \bar{f}$ in $C([0, T]; U')$ P -a.s. Since $(\tilde{f}^N)_N$ is bounded in $L^2_{w-*}(\Omega; L^\infty([0, T]; H))$, Vitali’s convergence theorem ensures the convergence of \tilde{f}^N to \bar{f} in $L^1(\Omega; C([0, T]; U'))$.

7.2.1. Proof of Theorem 9

Let $\phi \in C^\infty(\mathbb{T}^2)$ and $\psi \in C_c^\infty(\mathbb{R}^2)$, Let $t \in]0, T]$ and $A \in \mathcal{F}$. From Definition 5, point (3) and by multiplying by I_A and integrating over $\Omega \times [0, t]$, we derive

$$\begin{aligned} & \int_A \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}^N(t) \phi \psi dr dx dP - \int_A \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_0 \phi \psi dr dx dP \\ & - \int_A \int_0^t \sum_{k \in K} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}^N(s) \left(\sigma_k^N \cdot \nabla_x \phi \psi + (\nabla \sigma_k^N r) \cdot \nabla_r \psi \phi \right) dr dx dW^{N,k}(s) dP \\ & = \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}^N(s) \left(u_L(s) \cdot \nabla_x \phi \psi + (\nabla u_L(s) r - \frac{1}{\beta} r) \cdot \nabla_r \psi \phi \right) dr dx ds dP \\ & - \sigma^2 \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r \tilde{f}^N(s) \cdot \nabla_r \psi \phi dr dx ds dP - \alpha_N \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_x \tilde{f}^N(s) \cdot \nabla_x \phi \psi dr dx ds dP \\ & - \frac{1}{2} \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(\sum_{k \in K} \left((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r) \right) \nabla_r \tilde{f}^N(s) \right) \cdot \nabla_r \psi \phi dr dx ds dP \\ & := J_1^N + J_2^N + J_3^N + J_4^N. \end{aligned}$$

We pass to the limit as $N \rightarrow +\infty$ in the RHS of the last equation. By using Lemma 26, one has

$$\begin{aligned} & \lim_N \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{f}^N(s) \left(u_L(s) \cdot \nabla_x \phi \psi + (\nabla u_L(s) r - \frac{1}{\beta} r) \cdot \nabla_r \psi \phi \right) dr dx ds dP \\ & = \int_A \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \bar{f}(s) \left(u_L(s) \cdot \nabla_x \phi \psi + (\nabla u_L(s) r - \frac{1}{\beta} r) \cdot \nabla_r \psi \phi \right) dr dx ds dP. \end{aligned}$$

Additionally, we have

$$\lim_N \int_A \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r \tilde{f}^N(s) \cdot \nabla_r \psi \phi dr dx ds dP = \int_A \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_r \bar{f}(s) \cdot \nabla_r \psi \phi dr dx ds dP.$$

From Lemma 27 and Lemma 28, we get

$$\lim_N (J_3^N + J_4^N) = -\frac{1}{2} \int_A \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} A(r) \nabla_r \bar{f}(s) \cdot \nabla_r \psi \phi dr dx ds dP.$$

Finally, we use Lemma 29 and Lemma 30 to pass to the limit in LHS to complete the proof.

7.3. Uniqueness of the limit problem (13)

Let us conclude this section by showing that the limit equation (13) has at most one solution.

Lemma 31. *The solution \bar{f} to (13) is unique.*

Proof. Let \bar{f}_1 and \bar{f}_2 be two solutions to (13) and denote by \bar{f} be their difference. Then P -a.s for any $t \in]0, T]$ and $\phi \in Y$, we have

$$\begin{aligned} & \langle \bar{f}(t), \phi \rangle - \int_0^t \langle \bar{f}(s), u_L(s) \cdot \nabla_x \phi + (\nabla u_L(s)r - \frac{1}{\beta}r) \cdot \nabla_r \phi \rangle ds \tag{70} \\ &= - \int_0^t \sigma^2 \langle \nabla_r \bar{f}(s), \nabla_r \phi \rangle ds - \frac{1}{2} \int_0^t \langle A(r) \nabla_r \bar{f}(s), \nabla_r \phi \rangle ds. \end{aligned}$$

Since the above equation is in weak form, we need first to consider an appropriate regularization to get an equation for $\|\bar{f}(t)\|_\delta^2$ for any $t \in [0, T]$. We use analogous argument to “Step 1” in the proof of Lemma 21 to obtain

$$\begin{aligned} & \frac{1}{2} \|\bar{f}(t)\|_\delta^2 + \int_0^t \langle [u_L(s) \cdot \nabla_x \bar{f}(s)]_\delta + [\text{div}_r(\nabla u_L(s)r - \frac{1}{\beta}r) \bar{f}(s)]_\delta, [\bar{f}(s)]_\delta \rangle ds \tag{71} \\ &= \int_0^t \langle \sigma^2 [\Delta_r \bar{f}(s)]_\delta, [\bar{f}(s)]_\delta \rangle ds + \frac{1}{2} \int_0^t \langle [\text{div}_r A(r) \nabla_r \bar{f}(s)]_\delta, [\bar{f}(s)]_\delta \rangle ds. \end{aligned}$$

Notice that $[\text{div}_r A(r) \nabla_r \bar{f}(s)]_\delta, [\bar{f}(s)]_\delta = \langle \text{div}_r A(r) \nabla_r [\bar{f}(s)]_\delta, [\bar{f}(s)]_\delta \rangle$ hence

$$\begin{aligned} & \frac{1}{2} \int_0^t \langle [\operatorname{div}_r A(r) \nabla_r \bar{f}(s)]_\delta, [\bar{f}(s)]_\delta \rangle ds \\ &= -\frac{3k_T}{2} \int_0^t \| |r| \nabla_r [\bar{f}(s)]_\delta \|^2 ds + k_T \int_0^t \| r \cdot \nabla_r [\bar{f}(s)]_\delta \|^2 ds \\ &\leq -\frac{k_T}{2} \int_0^t \| [|r| \nabla_r \bar{f}(s)]_\delta \|^2 ds \rightarrow -\frac{k_T}{2} \int_0^t \| |r| \nabla_r \bar{f}(s) \|^2 ds \text{ as } \delta \rightarrow 0. \end{aligned}$$

Next, arguments similar to that used in “Step 2” of the proof of Lemma 21 allow to pass to the limit as $\delta \rightarrow 0$ in (71) and we get

$$\frac{1}{2} \|\bar{f}(t)\|^2 + \sigma^2 \int_0^t \|\nabla_r \bar{f}(s)\|^2 ds + \frac{k_T}{2} \int_0^t \| |r| \nabla_r \bar{f}(s) \|^2 ds \leq \frac{1}{\beta} \int_0^t \|\bar{f}(s)\|^2 ds. \tag{72}$$

The last inequality (72) and Grönwall lemma completes the proof of Lemma 31.

Remark 32. Another way to prove that uniqueness to (13) holds is to notice that (15) has (13) as FP equation associated. We have uniqueness in law of weak solutions of the SDE (15) due to the properties of $\mathcal{Q}(r)$. The latter ensures uniqueness of solutions of (13) due to [71, Thm. 2.5].

7.4. About the 3D case

It is worth drawing the reader’s attention to the fact that with cosmetic changes to what was made above, it is possible to prove a similar result in a three-dimensional setting. However, we preferred to present the result in 2D because we could present the stochastic turbulent velocity explicitly, see subsection 2.0.2. The main difference between 2D and 3D will be in the form of the stochastic turbulent velocity. In 3D, we can formulate the result using Fourier decomposition. Let us give an example of noise where we get a similar result in 3D.

We introduce the partition $\mathbb{Z}_0^3 = \Gamma_{3,+} \cup \Gamma_{3,-}$ ¹¹ such that $\Gamma_{3,+} = -\Gamma_{3,-}$ and we consider a family of real valued independent Brownian motions $(B_t^{k,j})_t^{k \in \mathbb{Z}_0^3, j \in \{1, 2\}}$ defined on the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$, that is $\mathbb{E}(B_t^{k,j} B_s^{l,m}) = \min(t, s) \delta_{k,l} \delta_{j,m}$. Then, we introduce a sequence of complex-valued Brownian motions adapted to $(\mathcal{F}_t)_t$ defined as follows

$$W_t^{k,j} = \begin{cases} B_t^{k,j} + i B_t^{-k,j} & \text{if } k \in \Gamma_{3,+} \\ B_t^{-k,j} - i B_t^{k,j} & \text{if } k \in \Gamma_{3,-}. \end{cases}$$

Note that $W_t^{-k,j} = \overline{W_t^{k,j}}$ ¹² and satisfies

¹¹ $\mathbb{Z}_0^3 = \mathbb{Z}^3 - \{(0, 0, 0)\}$.

¹² For a complex number z , \bar{z} denotes its complex conjugate.

$$\mathbb{E}(W_1^{k,j}, \overline{W_1^{l,m}}) = \begin{cases} 2 & \text{if } k=l, m=j \\ 0 & \text{otherwise;} \end{cases} \quad [W_1^{k,j}, W_1^{l,m}]_t = \begin{cases} 2t & \text{if } k=-l, m=j \\ 0 & \text{otherwise.} \end{cases}$$

Let $N \in \mathbb{N}^*$, define $\theta_{k,j}^N = \frac{a}{|k|^{5/2}} \mathbf{1}_{\{N \leq |k| \leq 2N\}}$, for a positive constant a . Then, for each $k \in \mathbb{Z}_0^3$, $j \in \{1, 2\}$ we denote by $\sigma_{k,j}^N(x) = \theta_{k,j}^N a_{k,j} e^{ik \cdot x}$, where $\{\frac{k}{|k|}, a_{k,1}, a_{k,2}\}$ is an orthonormal system of \mathbb{R}^3 for $k \in \Gamma_{3,+}$ and $a_{k,j} = a_{-k,j}$ if $k \in \Gamma_{3,-}$. Set

$$\mathbf{W}^N(t, x) = \sum_{k \in \mathbb{Z}_0^3, j \in \{1,2\}} \theta_{k,j}^N a_{k,j} e^{ik \cdot x} W^{k,j}(t). \tag{73}$$

Thanks to the assumptions on $\theta_{k,j}^N, a_{k,j}$ and $W^{k,j}$, we can prove that \mathbf{W}^N is a real, mean-zero and divergence free random distribution. Moreover, $\mathbf{W}^N(t, \cdot) \in L^2(\Omega, L^2(\mathbb{T}^3, \mathbb{R}^3))$, see [10, Sect. 2.2.]. On the other hand, it is not difficult to check that the covariance operator associated with \mathbf{W}^N is space-homogeneous and has a mirror symmetry property. Now, by considering $u^N = u_L + \circ \partial_t \mathbf{W}^N$ instead of (8) and then replace in (9) we obtain a stochastic FP in 3D similar to (12). Then, by cosmetic changes to what has been done above and noticing that

$$\sum_{\substack{k \in \mathbb{Z}_0^3, j \in \{1,2\} \\ N \leq |k| \leq 2N}} \left((\nabla \sigma_{k,j}^N) \otimes \overline{(\nabla \sigma_{k,j}^N)} \right) = \sum_{\substack{k \in \mathbb{Z}_0^3 \\ N \leq |k| \leq 2N}} \frac{(r \cdot k)^2}{|k|^5} \left(I - \frac{k \otimes k}{|k|^2} \right) = \mathcal{M}(r) + O\left(\frac{1}{N}\right)P(r),$$

where $\mathcal{M}(r) = \frac{8\pi \log 2}{15} (2|r|^2 I - r \otimes r)$. We can prove that the initial stochastic FP (12) in 3D converges to a limit PDE as (14) where the matrix $A(r)$ is replaced by $\mathcal{M}(r)$.

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Data availability

No data was used for the research described in the article.

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