COHOMOLOGY RINGS OF EXTENDED POWERS AND OF FREE INFINITE LOOP SPACES

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Abstract. We calculate mod-p cohomology of extended powers, and their group completions which are free infinite loop spaces. We consider the cohomology of all extended powers of a space together and identify a Hopf ring structure with divided powers within which cup product structure is more readily computable than on its own. We build on our previous calculations of cohomology of symmetric groups, which are the cohomology of extended powers of a point, the well-known calculation of homology, and new results on cohomology of symmetric groups with coefficients in the sign representation. We then use this framework to understand cohomology rings of related spaces such as infinite extended powers and free infinite loop spaces.

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1. Introduction

Homotopy orbit spaces with respect to the symmetric group action on iterated products, known as extended powers, play many key roles in algebraic topology. Steenrod introduced them as a central character in the study of cohomology operations [Ste53]. Extended powers of spectra play an essential role in Nishida's proof of his Nilpotence Theorem [Nis75]. By the Barratt-Priddy-Quillen theorem [BP72], after group completion the infinite extended powers functor provides a model for the free infinite loop space functor Q. May and his collaborators demonstrated that the stable version of extended powers not only were essential in defining structured ring spectra but also provided access to a wide range of phenomena which had previously been ad-hoc [BMMS86]. In addition to necessarily playing a role in derived algebra, modern applications of extended powers and free infinite loop spaces range

2020 *Mathematics Subject Classification.* Primary 20J06, 20B30.

The first two authors acknowledge the MUR Excellence Department Project MatMod@TOV awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C23000330006.

widely from, for example, the calculus of functors [AM99] to the stable cohomology of mapping class groups [Gal04].

We focus on the cohomology rings of these spaces, as this ring structure is the most basic unstable homotopical structure. These rings have long been implicitly understood through homology and coproduct calculations [CLM76]. But calculations with such require application of Adem relations for Kudo-Araki-Dyer-Lashof operations, so are difficult to work with. Here, as in our work on symmetric groups and alternating groups [Gue17, GSS12, GS21], we find that cohomology ring structure is best understood when coupled with a transfer (or induction) product, first defined by Strickland and Turner [ST97]. Our techniques in this setting for example have led to further development of the Curtis-Wellington spectral sequence [Hun].

Our work is thus parallel to as well as building on the well-established homology calculations. In [CLM76], the homology of extended powers all together forms an algebra over the Dyer-Lashof algebra, which behaves similarly to and in fact can be related to the cohomology rings of spaces over the Steenrod Algebra [May70]. Our presentations capture cohomology ring structure as part of a larger structure as well, ultimately presenting cohomology as "universal component super-Hopf rigs with additive divided powers" which then have small, if any, sets of relations. Prior to the elaboration of such Hopf ring structures, calculations of mod-two cohomology rings of classifying spaces of symmetric groups – the extended powers of a point – had cumbersome descriptions [AM04, Fes02], and calculations were not made at all for odd primes.

We develop these needed algebraic notions in Section 2, and we encourage the reader to take a look at the statement of first main results, namely Theorems 2.34 through 2.38 to understand the goal of the algebraic work before going through the details of that section. Such statements are remarkably simple, given the complexity for example of individual component rings. In the odd primary setting, such brevity is not possible without use of trivial and twisted coefficients together. We follow these statements with a development of additive bases in their algebraic frameworks. After extending well-known results about homology to the twisted coefficient setting, as needed for our cleanest statements, we prove the theorems about ground rings, namely Theorems 2.34 and 2.35, in Section 3. We then use those to prove our first main theorems, namely Theorems 2.37 and 2.38, in Section 4. Our strategy is to use the two products and divided powers structures to produce a set of classes in cohomology which pairs perfectly with the standard basis in homology. This perfect pairing both establishes the set of cohomology classes as a basis and validates the algebraic presentation which governs them.

After treating the cohomology of extended powers with disjoint basepoints, we calculate the cohomology of DX, the extended powers of a topological space X, $D_\infty X$, the stable extended power of X, CX, the free E_∞-algebra on X, and QX, the free infinite loop space generated by X, in Section 5. The precise definition of these objects is given below. For CX and QX these reproduce Dũng's description of the cohomology ring for QX when $p = 2$ and give new results for odd primes.

While we capitalize on finding the right algebraic structures and rely on standard techniques from algebraic topology and group cohomology as well as previous homology calculations, geometry has inspired and guided us. We focus on $C(X)$, the free E_∞-space on X. We choose ES_n to be the (ordered) configuration space of n points in $\mathbb{R}^\infty$

$$E\Sigma_n = \{(x_1, \dots, x_n) \in (\mathbb{R}^\infty)^n : x_i \neq x_j \forall i \neq j\},$$

with an action of Σ_n given by permutation of coordinates.

Definition 1.1. Let X be a pointed topological space. Let $\varphi_{i,n} : E\Sigma_n \to E\Sigma_{n-1}$ be the map that forgets the i^{th} point of the configuration. Let $s_{i,n} : X^{n-1} \to X^n$ be the map that adds the basepoint as the i^{th} coordinate. Define CX, the **space of configurations in $\mathbb{R}^\infty$ with labels in X**, as $\coprod_{n\geq 0} E\Sigma_n \times_{\Sigma_n} X^n/\sim$, where $\sim$ is the equivalence relation generated by $(\varphi_{i,n}a, b) \sim (a, s_{i,n}b)$ for all $a \in E\Sigma_n$ and $b \in X^{n-1}$.

We consider the finite-dimensional versions of this model, with a limited number of points in some $\mathbb{R}^N \subset \mathbb{R}^\infty$, in which case if X is a manifold then so is this quotient. In that setting, we can use geometric chains and cochains [FMS22]. The homology story is well-known through Kudo-Araki-Dyer-Lashof operations, which are manifestly geometric. If $f : P \to X$ represents a mod-two geometric homology class y then we can map $S^i \times_{\mathbb{Z}/2} (P \times P)$ to CX by sending $v \times p_1 \times p_2$ to the configuration of two points at $v, -v$ with labels $f(p_1)$ and $f(p_2)$. The resulting geometric homology class represents the operation $q_i(y)$. On the cohomology side, the analogous geometric cochain would be the submanifold of CX where two points in the underlying configuration share their first coordinate with labels both in some geometric cochain $W \to X$. This geometry guided us in formulating our main structures, with cup product corresponding to intersection as usual, transfer product a type of "union", and divided powers a "repeating".

We see many possible investigations building on this work. Following Remark 2.39, it would be interesting and probably reflective of deeper structure to fully understand divided powers operations on cohomology, as we only employ them on a limited set of classes. The pairing between homology and cohomology is not Kronecker, and is likely to be useful since the two settings differ in which structures are most readily expressed. We have made partial progress on computing Steenrod operations, a natural "self-serving" calculation to consider. Calculating cup product more organically as we have done here could help in extending these calculations to generalized cohomology theories, with even understanding whether divided powers operations exist for the transfer product an interesting first question. We suspect not, as the transfer product is analogous to the Pontrjagin product on the homology of Eilenberg-MacLane spaces, which has a divided powers structure [Car55, HHP+], but the homology of other infinite loop spaces instead has Dyer-Lashof operations. Related to this, there is possibility of further binding these structures through higher algebra enhancements of transfer product, in which case all of this intricate structure could likely be tied back to the homology of $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$.

2. Hopf ring with additive divided powers structure on the cohomology of DX

In this section we introduce the algebraic structure of component Hopf ring with additive divided powers. We show in Corollary 2.31 that this structure applies to the cohomology of extended powers, and then give an explicit presentation of this cohomology in Theorem 2.35.

2.1. **Preliminaries.** Extended powers are our first main objects of study, and we assume throughout that X is Hausdorff and compactly generated, and that any basepoint has a neighborhood deformation retract.

Definition 2.1. If spaces E and Y are both S_n-spaces, with action preserving the base point of Y, then $E \rtimes_{S_n} Y$ is defined as the quotient $((E)_+ \wedge Y)/S_n$, where $(E)_+$ is E with a disjoint basepoint and S_n acts diagonally.

Define the n^{th} **extended power** of X by $D_nX = ES_n \rtimes_{S_n} X^{\wedge n}$. Let $DX =$

$EG \times_G X$, so that $X^n{}_{hS_n}$

$_n(D_nX)$. Recall that if X is a G-space we let X_{hG} denote the homotopy quotient $_n = D_n(X_+)$. We will also consider this unbased version of D_n, for which we use the symbol $\tilde{D}$ to differentiate the two. Hence $\tilde{D}_n(X) = X_{hS_n}$.

For $p = 2$ let $H_{EP}{}^*(X) := {}_{n\geq 0}\ H^*(D_nX;\ F_2)$.

For odd primes let $H_{EP}{}^*(X) := F_p \oplus {}_{n\geq 1} H^*(D_nX; F_p \oplus sgn)$, where F_p denotes the trivial representation and sgn is the sign representation. These are bi- or tri-graded by component, degree, and in the odd prime setting a mod-two grading induced by an obvious grading on $F_p \oplus sgn$.

Observe that $\tilde{D}(X) \cong D(X_+) \cong C(X_+)$.

We use both algebraic and geometric variants of the extended powers topological construction, using the algebra to provide the main framework for our arguments. Almost by definition, cohomology of D_nX is the equivariant cohomology of the n-fold tensor product of the cochains of X. But the following basic computation shows one can consider only the cohomology of X (see for example [May70, Lemma 1.1]).

Proposition 2.2. *There are natural isomorphisms*

$$H^*(D_nX; F_p) \cong H^*(S_n; H^*(X; F_p)^{\otimes n}),$$

and similarly for homology, compatible with standard pairings.

Our main results present the cohomology rings, with F_p coefficients, of these three related spaces, namely DX, the extended power of a topological space X, CX, the free E_∞-space over X, and QX, the free infinite-loop space over X. Because the homology of these spaces, as co-algebras, is known and only depends on the homology of X [CLM76], our first step is to develop the algebraic functors which will take as their input the cohomology ring of X and produce the cohomology rings of these spaces.

2.2. **Component Hopf rings with additive divided powers.** We work primarily in a bigraded setting in a strong sense, namely products vanish on elements which differ in the first grading.

Definition 2.3. Let $(H, \Delta, \cdot)$ be a bialgebra over R. We say that H is a **bigraded component bialgebra** if $H = {}_{n,d\in N\times N} H_{n,d}$, where

component bialgebra if $H =$

- the product sends $H_{m,i} \otimes H_{m,j}$ to $H_{m,i+j}$
- the product of x and y is zero if $x \in H_{m,i}$, $y \in H_{n,j}$ with $n \quad m$
- the coproduct is standardly bigraded, sending $H_{p,k}$ to ${}_{n+m=p,k=i+j} H_{n,i} \otimes H_{m,j}$.
- the coproduct is standardly bigraded, sending $H_{p,k}$ to ${}_{n+m=p,k=i+j} H_{n,i} \otimes H_{m,j}$.

The first grading of an element is called its **component**, while the second is called its **dimension**. For such H, we let $\overline{H}$ denote the direct sum of $H_{n,d}$ for

$n > 0$ (and all $d \geq 0$). We require the connectedness condition $A = \text{im}(\eta) \quad A$, where $\eta : R \to A$ is the image of the unit map.

Similarly, a **super-component bialgebra** is a bialgebra graded over $\mathbb{N} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$ satisfying these same axioms.

Recall that a rig (ring without negatives) satisfies all ring axioms except the existence of additive inverses.

Definition 2.4. A **Hopf rig** H is a rig object in the category of coalgebras, so that there are two products, a "multiplication product" $\cdot$ and an "addition product" 8, both giving bialgebras with one coproduct Δ, and a distributivity axiom that $a \cdot (b \, 8 \, c) = \sum_{\Delta a = \sum a' \otimes a''} (a' \cdot b) \, 8 \, (a'' \cdot c)$.

A **bigraded component Hopf rig** (respectively **super-component Hopf rig**) is one for which $(H, , \Delta)$ is a bigraded component bialgebra (respectively super-component bialgebra), and 8 preserves both gradings. In this setting, we modify the distributivity axiom above by introducing a coefficient $(-1)^{|a''||b|}$ corresponding to the usual Koszul sign.

We require the structural morphisms to be graded (co)commutative with respect to the dimension. When this commutativity condition is not necessarily satisfied, we speak of a **non-commutative bigraded component Hopf rig**.

The Hopf rigs we study have antipodes, and are thus Hopf rings. But we will not need the antipode structures, and only mention them in passing. Hopf rings first appeared in topology in the study of homology of infinite loop spaces which represent cohomology with product structure [Mil70, RW80, Wil00]. But following Strickland and Turner [ST97], we have found them essential in describing the cohomology of symmetric groups. An interesting related case of Hopf rings arises when one introduces an induction or symmetrization product on rings of symmetric polynomials. Readers who would like to see concrete examples can see these treated in the second section of [GSS12].

We will show that the cohomology of DX not only is a Hopf rig, but is also endowed with divided powers operations.

Definition 2.5. A **divided powers algebra** is a triple (A, I, γ), where A is an algebra, I □ A is a proper ideal, and $\gamma = \{\gamma_n\}_{n \in \mathbb{N}}$ is a family of functions from I to I, also denoted $x^{[n]} := \gamma_n(x)$, which satisfies the following relations, whenever $x, y \in I$ and $\lambda \in A$:

(1) $x^{[0]} = 1$, and $x^{[1]} = x$ (0, 1 **Cases**)

(2) $(x+y)^{[n]} = \sum_{i=0}^{n} x^{[i]} y^{[n-i]}$ (**Binomial**)

(3) $(\lambda x)^{[n]} = \lambda^n x^{[n]}$ (**Distributivity over Multiplication**)

(4) $x^{[n]} x^{[m]} = \binom{n+m}{n} x^{[n+m]}$ (**Law of Exponents**)

(5) $(x^{[m]})^{[n]} = \frac{(nm)!}{(m!)^n n!} x^{[nm]}$. (**Composition**)

We also refer to such triples as **divided powers structures**, on the algebra A. Introduced in [Car55], these have been extensively studied and applied to various contexts, for example in [Rob65], from which we borrow notation, and [Ber74, Haz78]. In algebraic topology, the homology of Eilenberg-MacLane spaces is divided powers algebras [EML54, Tho57].

These relations imply that the formal series $f(t) = \sum_{i=0}^{\infty} x^{[i]} t^i \in A[[t]]$ satisfies $f(0) = 1$ and $f(t+s) = f(t)f(s)$. The set $\mathrm{Exp}(A)$ of $f \in A[[t]]$ satisfying these conditions constitutes an R-module. The left adjoint of Exp viewed as a functor from

R- algebras to R-modules is the free divided powers functor DP, defined explicitly as follows.

Definition 2.6. For an R-module M, we let $DP(M)$, the **free divided powers algebra** generated by M, be generated as an algebra by the set $\{x^{[n]}\}$, where $x \in M$, with relations (1)-(5) from Definition 2.5 imposed. The ideal I is given by the collection of $x^{[n]}$ with $n \geq 1$ and divided powers maps are defined through the Binomial, Scalar Multiplication and Composition relations of Definition 2.5.

If M is graded, then naturally so are $DP(M)$ and the universal map. Over a field, if M is finite-dimensional, there is a standard additive basis of $DP(M)$ which we can associate to a choice of basis $\{x_1, \ldots, x_r\}$ for M, namely $\left\{\prod_{i=1}^{r} x_i^{[n_i]}\right\}_{n_i \in \mathbb{N}}$.

Definition 2.7. For a divided powers algebra (A, I, γ), the **tensor product** $A \otimes A$ has a natural divided powers structure $(A \otimes A, A \otimes I + I \otimes A, \gamma_{\otimes})$ defined for $x \in I$ and $y \in A$ by $(x \otimes y)^{[n]} = x^{[n]} \otimes y^n$ and $(y \otimes x)^{[n]} = y^n \otimes x^{[n]}$.

A **divided powers bialgebra** is a divided powers structure $(H, \overline{H}, \gamma)$ on a bialgebra H, with $\overline{H}$ kernel of the counit, such that the coproduct $\Delta\colon H \to H \otimes H$ is a morphism of divided powers structures.

In the graded setting, we require divided powers operations to be compatible with degrees, in the sense that $\deg(x^{[k]}) = k \deg(x)$. In characteristic not equal to two, elements of odd degree have zero squares and thus higher powers, so in this setting $x^{[k]} = 0$ if $\deg(x)$ is odd and $k \geq 2$. This is also the case for the odd degree classes in the super-bigraded component setting. By a classical theorem of Milnor and Moore, a graded Hopf algebra H that is bicommutative in the graded sense can always be split into an even and odd part where the odd factor is an exterior algebra generated by primitive elements in odd degree. A graded divided powers structure on H is thus equivalent to a divided powers structure in the standard sense on the even part, and on the odd part the divided powers with exponent larger than one are zero.

Admitting a divided powers structure is a very restrictive condition on a Hopf algebra. Such algebras arise as dual Hopf algebras to free, primitively generated Hopf algebras or more generally as duals of enveloping algebras of Lie algebras [Sch67, And71].

Given a counital R-coalgebra C, let $\varepsilon\colon C \to R$ be the counit and $\overline{C} = \ker(\varepsilon)$. The coproduct $\Delta\colon C \to C \otimes C$ extends uniquely to $DP(\overline{C}) \to DP(\overline{C}) \otimes DP(\overline{C})$, which by abuse of notation we still denote Δ. This is a coassociative and cocommutative coproduct, and defines a bialgebra structure on $DP(\overline{C})$.

Definition 2.8. Denote by DP_{HA} the **free divided powers Hopf algebra** functor from R-coalgebras to Hopf algebras with divided powers over R, with $DP_{HA}(C) = DP(\overline{C})$ as an algebra, the coproduct induced by that of C, and extended to morphisms by the universal property.

Then $DP_{HA}(C)$ is left adjoint to the forgetful functor from divided powers bialgebras to coalgebras.

Our main structure we use comprises the two structures detailed above.

Definition 2.9. A **bigraded component Hopf ring with additive divided powers** is a septad $(A, 8, \cdot, \Delta, \{\gamma_n\})$ such that

- $(A, 8, \Delta, \{\gamma_n\})$ is a divided powers bigraded component Hopf algebra
- $(A, 8, \cdot, \Delta)$ is a bigraded component Hopf rig

We also require that each component algebra $(A_n, \cdot)$ is unital, and that the unit of A_n is the n^{th} divided power of the unit of A_1.

A **component super-Hopf ring with additive divided powers** is a septad as above where A has an additional $\mathbb{Z}/2\mathbb{Z}$-grading $A = A_{even} \oplus A_{odd}$, preserved by Δ and $\cdot$ and such that $A_{even} 8 A_{odd} = 0$, satisfying the following conditions:

- $(A, 8, \Delta, \{\gamma_n\})$ is a divided powers Hopf algebra
- $(A, \cdot, \Delta)$ is a component super-algebra
- the Hopf ring distributivity axiom holds in A

We will prove below that the Hopf rings of our interest have divided powers operations for the addition product satisfying Definition 2.9. However, they are not compatible with the multiplication product in a simple way, as we show in Remark 2.39.

2.3. **The algebraic extended powers functor.** Recall that our first main goal is to describe the algebraic functor which takes the cohomology ring of a space X and produces the cohomology of its extended powers. In light of Proposition 2.2, this can be viewed in terms of group cohomology. For optimal results, at odd primes we take cohomology with coefficients which incorporate sign representation.

Definition 2.10. Let V be a representation of S_n over a field. Define the **cohomology of extended powers of a space X with coefficients in V**, denoted $H^*(D_nX; V)$, to be the group cohomology $H^*(\mathsf{S}_n; H^*(X)^{\otimes n} \otimes V)$.

In our application V will be either trivial when $p = 2$ or for odd primes will incorporate the sign representation. Clearly the notation is consistent in the case of trivial coefficients by Proposition 2.2.

Definition 2.11. By abuse, for any odd prime p and any n let ρ be the representation of S_n given by $\rho = \mathbb{F}_p \oplus sgn$, where $\mathbb{F}_p$ denotes the trivial representation and sgn denotes the sign representation.

We grade ρ by having the trivial representation in degree zero (even) and the sign representation in degree one (odd). We use canonical isomorphisms $\mathbb{F}_p \otimes V \cong V$ and $sgn \otimes sgn \cong \mathbb{F}_p$ to define a graded algebra structure on ρ.

In our setting, the coefficients we are using have multiplicative structure which is then reflected in cohomology.

Definition 2.12. A **product series of algebras** over a field k is a collection $\{A_n\}$ with A_n $k[\mathsf{S}_n]$-algebra, with isomorphisms of $k[\mathsf{S}_i \times \mathsf{S}_j]$-algebras $\chi_{i,j} : A_{i+j} \to A_i \otimes_k A_j$ that are coherent in the sense that the following two conditions are satisfied:

(1) for all $i, j, k > 0$ the following diagram commutes in the category of $k[\mathsf{S}_i \times \mathsf{S}_j \times \mathsf{S}_k]$-modules:

$$\begin{array}{ccc} A_{i+j+k} & \xrightarrow{\chi_{i,j+k}} & A_i \otimes A_{j+k} \\ \downarrow{\scriptstyle \chi_{i+j,k}} & & \downarrow{\scriptstyle \mathrm{id} \otimes \chi_{j,k}} \\ A_{i+j} \otimes A_k & \xrightarrow{\chi_{i,j} \otimes \mathrm{id}} & A_i \otimes A_j \otimes A_k \end{array}$$

(2) for all $n, m > 0$, the following diagram commutes

$$\begin{array}{ccc} A_{n+m} & \xrightarrow{\sigma_{n,m}} & A_{n+m} \\ \downarrow{\scriptstyle x_{n,m}} & & \downarrow{\scriptstyle x_{m,n}} \\ A_n \otimes A_m & \xrightarrow{\tau} & A_m \otimes A_n \end{array},$$

where τ exchanges the two factors and $\sigma_{n,m} \in \Sigma_{n+m}$ is the permutation given by

$$\sigma_{n,m}(i) = \begin{cases} m+i & \text{if } 1 \le i \le n \\ i-n & \text{if } n+1 \le i \le n+m \end{cases}.$$

A **super-product series of algebras** is a collection $\{A_n\}$ with A_n $k[S_n]$-module with the following additional structure:

- a grading $A_n = A_{n,0} \oplus A_{n,1}$ of each A_n over $\mathbb{Z}/2\mathbb{Z}$,
- a product $A_n \otimes A_n \to A_n$ that makes each A_n a $k[S_n]$-super-algebra,
- and $k[S_i \times S_j]$-module isomorphism $\chi_{i,j,e} : A_{i+j,e} \to A_{i,e} \otimes_k A_{j,e}$ for all $i, j \in \mathbb{N}$ and $e \in \mathbb{Z}/2\mathbb{Z}$, that are coherent in the sense that the two conditions above are satisfied and such that $\chi_{i,j,0} \oplus \chi_{i,j,1} : A_{i,j} \to A_i \otimes_k A_j$ are super-algebra morphisms.

We note that a product series of algebras can be regarded as a super-product series of algebras concentrated in $\mathbb{Z}/2\mathbb{Z}$-degree 0.

The product series we use are built from a single algebra, ultimately the cohomology ring of a space.

Definition 2.13. Let A be an algebra. Define TA to be the product series of algebras with $TA_n = A^{\otimes n}$ with canonical restrictions. For odd primes define $T_\rho A$ to be the sequence of modules with $T_\rho A_n = \rho \otimes A^{\otimes n}$ and structure maps given by the identity map on ρ tensored with canonical restrictions.

Definition 2.14. Let $\{A_n\}$ be a product series of algebras. Let $f \in H^*(S_n; A_n)$ and $g \in H^*(S_m; A_m)$, so f is represented by a homomorphism from a resolution $W_*^{S_n}$ of k over $k[S_n]$ to a suspension of A_n. Define the preliminary multiplicative structures on cohomology of symmetric groups with coefficients in $\{A_n\}$ as follows.

- A coproduct $\tilde{\Delta} = \sum_{i,j} \tilde{\Delta}_{i,j}$, where $\tilde{\Delta}_{i,j}$: $H^*(S_{i+j}; A_{i+j}) \to H^*(S_i; A_i) \otimes H^*(S_j; A_j)$ is induced by the composition

$$\hom_{S_{i+j}}(W_*^{S_{i+j}}; A_{i+j}) \xrightarrow{\text{res}} \hom_{S_i \times S_j}(W_*^{S_i} \otimes W_*^{S_j}; A_{i+j}) \xrightarrow{x_{i,j}}$$

$$\hom_{S_i \times S_j}(W_*^{S_i} \otimes W_*^{S_j}; A_i \otimes A_j) \xrightarrow{\cong} \hom_{S_i}(W_*^{S_i}; A_i) \otimes \hom_{S_j}(W_*^{S_j}; A_j),$$

where res is the restriction to the resolution of a subgroup;
- A transfer product $\tilde{\odot} : H^*(S_n; A_n) \otimes H^*(S_m; A_m) \to H^*(S_{n+m}; A_{n+m})$ is induced by

$$\hom_{S_n}(W_*^{S_n}; A_n) \otimes \hom_{S_m}(W_*^{S_m}; A_m) \xrightarrow{\cong} \hom_{S_n \times S_m}(W_*^{S_n} \otimes W_*^{S_m}; A_n \otimes A_m) \xrightarrow{x^{-1}_{n,m}}$$

$$\hom_{S_n \times S_m}(W_*^{S_n} \otimes W_*^{S_m}; A_{n+m}) \xrightarrow{\text{tr}} \hom_{S_{n+m}}(W_*^{S_{n+m}}; A_{n+m}),$$

where tr denotes the usual transfer map;

- a cup product $\cdot$ is defined in the standard way on each component, using restriction along the diagonal on a resolution of k over $k[S_n]$ and the product on A_n

$$\hom_{S_n}(W_*^{S_n};A_n)\otimes\hom_{S_n}(W_*^{S_n};A_n) \xrightarrow{\cong} \hom_{S_n\times S_n}(W_*^{S_n}\otimes W_*^{S_n};A_n\otimes A_n)\longrightarrow$$

$$\hom_{S_n}(W_*^{S_n};A_n\otimes A_n) \longrightarrow \hom_{S_n}(W_*^{S_n};A_n),$$

and is zero between distinct components;

More generally, if we start with a super-product series of algebras $\{A_n\}$ with $A_n = \bigoplus_{e\in\mathbb{Z}/2\mathbb{Z}} A_{n,e}$, we similarly define:

- a coproduct $\tilde{\Delta}$ as the direct sum of the coproducts associated with the addend product series $A_{*,e}$;
- a transfer product that restricts to that defined above on each addend $A_{*,e}$, and such that $f\tilde{\odot}g$ is zero if $f\in H^*(S_n;A_{n,e})$ and $g\in H^*(S_m;A_{m,e'})$ with $e\neq e'$;
- if each A_n is an algebra, a cup product $\cdot$ as above.

These structures are preliminary because of some additional signs which we will include below.

Definition 2.15. If $\{A_n\}$ is a product series of algebras or super-product series of algebras, define $H^*(S_\cdot;A_\cdot)$ to be the direct sum $\mathbb{F}_p\oplus\bigoplus_{n\geq 1}H^*(S_n;A_n)$, obtained from $\bigoplus_{n\geq 0}H^*(S_n;A_n)$ by replacing in component 0 the representation $\bigoplus_{e\in\mathbb{Z}/2\mathbb{Z}}A_{0,e}\cong\bigoplus_e\mathbb{F}_p$ via the epimorphism $\sum_e\colon\bigoplus_e\mathbb{F}_p\to\mathbb{F}_p$, with coproduct $\tilde{\Delta}$ and product $\tilde{\odot}$

Theorem 2.16. $H^*(S_\cdot;A_\cdot)$ *forms a non-commutative Hopf algebra, with structures defined in Definition* 2.14. *Moreover,* $\left(H^*(S_\cdot;A_\cdot),\tilde{\Delta},\cdot\right)$ *is a bigraded component bialgebra.*

Proof. This theorem has been essentially proved by Strickland and Turner [ST97] for generalized cohomology theories, and in particular for cohomology with coefficients in the trivial representations. Their proof can be reinterpreted diagrammatically in a group-theoretic setting as explained by Giusti–Salvatore–Sinha [GSS12]. In our case, the modification in component zero is introduced only to ensure that the unit behaves correctly with both the coproduct and the counit. With this exception, those diagrams also prove the statement for coefficients in a general super-product series of algebras, with the exception of those yielding the commutativity of $\tilde{\odot}$ and the cocommutativity of $\tilde{\Delta}$, that involve conjugation by elements of the symmetric groups.

The associativity of $\tilde{\Delta}$ and $\tilde{\odot}$ is shown using the following commutative diagram of finite coverings:

$$\begin{array}{ccc}
B(S_n\times S_m\times S_l) & \longrightarrow & B(S_{n+m}\times S_l)\\
\downarrow & & \downarrow\\
B(S_n\times S_{m+l}) & \longrightarrow & B(S_{n+m+l})
\end{array}$$

When taking cohomology with coefficient in the product series we obtain the following commutative diagram:

$$\begin{array}{ccccc}
H^*(\mathsf{S}_{n+m+l}; A_{n+m+l}) & \longrightarrow & H^*(\mathsf{S}_{n+m} \times \mathsf{S}_l; A_{n+m+l}) & \longrightarrow & H^*(\mathsf{S}_{n+m}; A_{n+m}) \otimes H^*(\mathsf{S}_l; A_l) \\
\downarrow & & \downarrow & & \downarrow \\
H^*(\mathsf{S}_n \times \mathsf{S}_{m+l}; A_{n+m+l}) & \longrightarrow & H^*(\mathsf{S}_n \times \mathsf{S}_m \times \mathsf{S}_l; A_{n+m+l}) & \longrightarrow & H^*(\mathsf{S}_n \times \mathsf{S}_m; A_{n+m}) \otimes H^*(\mathsf{S}_l; A_l) \\
\downarrow & & \downarrow & & \downarrow \\
H^*(\mathsf{S}_n; A_n) \otimes H^*(\mathsf{S}_{m+l}; A_{m+l}) & \longrightarrow & H^*(\mathsf{S}_n; A_n) \otimes H^*(\mathsf{S}_m \times \mathsf{S}_l; A_m \otimes A_l) & \longrightarrow & H^*(\mathsf{S}_n; A_n) \otimes H^*(\mathsf{S}_m; A_m) \otimes H^*(\mathsf{S}_l; A_l).
\end{array}$$

The outer square looks as follows and encodes the co-commutativity of $\tilde{\Delta}$.

$$\begin{array}{ccc}
H^*(\mathsf{S}_n; A_n) \otimes H^*(\mathsf{S}_m; A_m) \otimes H^*(\mathsf{S}_l; A_l) & \xleftarrow[\tilde{\Delta}\otimes 1]{} & H^*(\mathsf{S}_{n+m} \times \mathsf{S}_l; A_{n+m} \otimes A_l) \cong H^*(\mathsf{S}_{n+m}; A_{n+m}) \otimes H^*(\mathsf{S}_l; A_l) \\
{\scriptstyle 1\otimes\tilde{\Delta}}\uparrow & & \uparrow{\scriptstyle \tilde{\Delta}} \\
H^*(\mathsf{S}_n; A_n) \otimes H^*(\mathsf{S}_{m+l}; A_{m+l}) & \xleftarrow[\tilde{\Delta}]{} & H^*(\mathsf{S}_{n+m+l}; A_{n+m+l}).
\end{array}$$

Moreover, the representations considered in the diagram at the bottom are all canonically identified with the restriction of the S_{n+m+l}-representation A_{n+m+l}. Therefore, there is also an induced diagram of transfer maps, obtained by replacing restrictions with transfer maps in the commutative diagrams above:

$$\begin{array}{ccc}
H^*(\mathsf{S}_n; A_n) \otimes H^*(\mathsf{S}_m; A_m) \otimes H^*(\mathsf{S}_l; A_l)) & \xrightarrow{\tilde{\ }\otimes 1} & H^*(\mathsf{S}_{n+m} \times \mathsf{S}_l; A_{n+m} \otimes A_l) \cong H^*(\mathsf{S}_{n+m}; A_{n+m}) \otimes H^*(\mathsf{S}_l; A_l) \\
\downarrow{\scriptstyle 1\otimes\tilde{\ }} & & \downarrow{\scriptstyle \tilde{\ }} \\
H^*(\mathsf{S}_n; A_n) \otimes H^*(\mathsf{S}_{m+l}; A_{m+l}) & \longrightarrow & H^*(\mathsf{S}_{n+m+l}; A_{n+m+l}).
\end{array}$$

The fact that both $\tilde{\delta}$ and $\cdot$ are morphisms of coalgebras with the preliminary coproduct is proved in a similar way using the following commutative diagrams for all $n, m \in \mathbb{N}$, where i denotes the standard inclusions, $\tau : \mathsf{S}_b \times \mathsf{S}_c \to \mathsf{S}_c \times \mathsf{S}_b$ the switching map, and d the diagonal maps:

$$\begin{array}{ccc}
\coprod B(\mathsf{S}_a \times \mathsf{S}_b \times \mathsf{S}_c \times \mathsf{S}_d) & \xrightarrow{\coprod B(i_{a,b} \times i_{c,d})} & B(\mathsf{S}_n \times \mathsf{S}_m) \\
\downarrow{\scriptstyle \coprod B(i_{a,c} \times i_{b,d}) \circ B(1 \times \tau \times 1)} & & \downarrow{\scriptstyle B(i_{n,m})} \\
\coprod B(\mathsf{S}_p \times \mathsf{S}_q) & \xrightarrow{\coprod B(i_{p,q})} & B(\mathsf{S}_{n+m})
\end{array}$$

$$\begin{array}{ccc}
B(\mathsf{S}_n \times \mathsf{S}_m) & \xrightarrow{B(1 \times \tau \times 1) \circ B(d_{\mathsf{S}_n} \times d_{\mathsf{S}_m})} & B((\mathsf{S}_n \times \mathsf{S}_m)^2) \\
\downarrow{\scriptstyle i_{n,m}} & & \downarrow{\scriptstyle i_{n,m} \times i_{n,m}} \\
B(\mathsf{S}_{n+m}) & \xrightarrow{d_{\mathsf{S}_{n+m}}} & B(\mathsf{S}_{n+m}^2)
\end{array}$$

where the first unions are over all $p + q = n + m$ and the second union is over all $a + b = n,\ c + d = m,\ a + c = p,\ b + d = q$.

It is well-known in general that is associative. Since, under our hypotheses, all the morphisms involved in the definition of the coproduct are algebra maps with respect to the cup product, so is $\tilde{\Delta}$. D

In the cases of our interest, the statement of Theorem 2.16 can be improved to show the commutativity of a twisted version of the coproduct and the products and Hopf rig distributivity.

Definition 2.17. Given a graded commutative algebra A over the field $\mathbb{F}_p$, define the **algebraic extended powers functor** by

$$EP(A) = \mathbb{F}_p \oplus \bigoplus_{d\geq 0, n>0} H^d(\mathsf{S}_n; T_\rho A_n)$$

for odd primes. Replace $T_\rho A$ by TA to obtain the definition when $p = 2$.

By Proposition 2.2, for a pointed topological space X, $H_{EP}{}^*(X) \cong EP(\tilde{H}(X))$.

We observe three different gradings on $EP(A)$. The first two are the cohomological grading d, and component n, defined before. We also use the decomposition $T_\rho A_n \cong A^{\otimes n} \oplus (sgn \otimes A^{\otimes n})$ to induce a decomposition on $EP(A)$.

Definition 2.18. Over $\mathbb{F}_p$ with p odd, consider the decomposition

$$EP(A) \cong \mathbb{F}_p \bigoplus_{n>0, d\geq 0} H^d(\mathsf{S}_n; A^{\otimes n}) \oplus \bigoplus_{n>0, d\geq 0} H^d(\mathsf{S}_n; sgn \otimes A^{\otimes n}).$$

We define **sign degree** e of an element to be zero or even if it is in the second summand of this decomposition and to be one or odd if it is in the third summand. Equivalently, we define $EP(A)^{d,n,e}$ to be the summand $H^d(\mathsf{S}_n; A^{\otimes n})$ when $e = 0$ or the summand $H^d(\mathsf{S}_n; sgn \otimes A^{\otimes n})$ when $e = 1$. We define the **total degree** to be $t = ne + d$. As the sign degree, the total degree is only defined modulo 2.

By convention, if $p = 2$, then the sign degree is always even/zero. In order to ultimately account for signs we need the following.

Definition 2.19. A bi-graded component super-Hopf ring is a super-Hopf ring endowed with the component grading n, the $\mathbb{Z}/2\mathbb{Z}$-grading e arising from the "super-" structure, and an additional grading d over $\mathbb{N}$ where Δ preserves all three gradings, 8 preserves n and d, $\cdot$ preserves d and e in each component, and the axioms of super-Hopf ring are considered in the graded sense with respect to the total degree $t = ne + d$.

Definition 2.20. Let $EP(A)$ be the extended powers of an algebra, as above. Let $\tilde{\Delta}$ and $\tilde{8}$ be the maps constructed in Definition 2.14. We define:

- For all x such that, using Sweedler's notation, $\tilde{\Delta}(x) = x_{(1)} \otimes x_{(2)}$, with $x_{(1)}, x_{(2)}$ all tri-homogeneous, the modified coproduct is given by

$$\Delta(x) = \sum (-1)^{d(x_{(2)})n(x_{(1)})e(x_{(1)})} x_{(1)} \otimes x_{(2)}.$$

- For all x, y tri-homogeneous, the modified transfer product $x \, 8 \, y = (-1)^{d(y)n(x)e(x)} x \tilde{8} y$.

We can finally state our main result of this section.

Theorem 2.21. *Let A be a graded commutative algebra of finite type over $\mathbb{F}_p$. Then $EP(A)$, with the coproduct Δ and the products 8 and $\cdot$, is a super-component Hopf rig bigraded by the component n and the total degree t. If $p = 2$, it is a component Hopf ring bigraded by the component n and the cohomological dimension d.*

Proof. Theorem 2.16 guarantees that $EP(A)$ is a non-commutative Hopf algebra with the non-modified coproduct and transfer product $\tilde{\Delta}$ and $\tilde{\circledast}$. Clearly the additional signs do not disrupt the (co)associativity of the transfer product $\circledast$ and the coproduct Δ, that still form a Hopf algebra, and Δ and $\cdot$ still form a component bialgebra. On the addend corresponding to the trivial representation, the remaining Hopf rig axioms are proved diagrammatically as in [GS21, Theorem 2.4]. On the addend corresponding to the sign representation, one can adapt their proof by additionally keeping track of coefficients. However, the cocommutativity of $\tilde{\Delta}$, the commutativity of $\tilde{\circledast}$ and the Hopf ring distributivity axiom fail because, contrary to constant coefficients, inner automorphisms do not induce the identity on cohomology with coefficients in non-trivial representations. For instance, the diagram that would encode the commutativity of $\tilde{\circledast}$ and the cocommutativity of $\tilde{\Delta}$ is the following:

$$\begin{array}{ccc} S_n \times S_m & \xrightarrow{\tau} & S_m \times S_n \\ \downarrow & & \downarrow \\ S_{n+m} & \xrightarrow{\gamma} & S_{n+m}, \end{array}$$

where τ is the obvious switching isomorphism and γ is the conjugation by a permutation $\sigma_{n,m} \in S_{n+m}$ depending on n and m. In cohomology with trivial coefficients, such conjugation maps induce the identity, but in cohomology with coefficients in sgn they induce the multiplication by $(-1)^{nm}$. Therefore $x \tilde{\circledast} y = (-1)^{n(x)n(y)e(x)e(y)+d(x)d(y)} y \tilde{\circledast} x$. The additional signs introduced with the modified transfer product guarantee that $x \circledast y = (-1)^{t(x)t(y)} y \circledast x$. The same argument shows that Δ is cocommutative. By analyzing the corresponding diagram, one proves similarly that $(EP(A), \Delta, \circledast)$ satisfy the graded version of Hopf ring distributivity, with respect to the total degree. ∎

Remark 2.22. Additively, $H^*(S_n; \mathrm{sgn} \otimes A^{\otimes n})$ is isomorphic to $H^*(S_n; \Sigma(A)^{\otimes n})$, where Σ is the suspension functor on graded vector spaces. Hence, for p odd, $DP(A) \cong \bigoplus_{n\geq 0} H^*(S_n; A^{\otimes n}) \oplus H^*(S_n; \Sigma(A)^{\otimes n})$ as vector spaces. The total degree on $EP(A)$ is defined in such a way that this isomorphism becomes a degree-preserving linear map. Seen through this lens, the correction signs appearing in the modified coproduct and transfer product become less mysterious and arise from the Koszul sign conventions on $Hom(W_*^{S_n}; \Sigma(A))$.

For $p = 2$, $H_{EP}{}^*(X)$ differs from the cohomology of DX by an innocuous completion. We see something similar at odd primes if we take only the $H^*(D_nX; \mathbb{F}_p)$ summands. Our work ahead is to understand the algebraic extended powers functor as a free functor to the category of Hopf rings with additive divided powers, which we next develop.

Remark 2.23. As we remarked in the first section, if X is a manifold then cochains of finite dimensional approximations of CX can be defined by submanifolds, which informally we consider through "conditions" on the underlying points of the configuration or on their labels in X.

Cup product as usual is given by intersecting, or in this case requiring that two sets of conditions hold. Transfer product defines a condition on $n + m$ points by asking that a condition is satisfied on some subcollection of n points and another is satisfied on m points. The transfer product of a class with itself is divisible by two because the condition is satisfied whether some n points or its complement is considered "first." The divided powers operation repeats a condition on n points k times to define a condition on nk points.

See Theorem 4.9 of [GSS12] where we discuss the representatives for Hopf ring generators as subvarieties defined by 2^n points sharing a coordinate, and [GS14] which gives a development through cellular models for one-point compactifications of configuration spaces. A chain-level model for the transfer product is given which would also work for the divided powers operations. (Note that the cup product result claimed but not proven in [GS14] is not correct.)

2.4. **Divided powers on cohomology of symmetric groups with coefficients in a product series.** We now develop a divided powers structure on $H_{EP}{}^*(X)$, as a special case of divided power structure on cohomology of symmetric groups with coefficients in a product series of algebras. In order to define it we need restriction and transfer maps for subgroups of symmetric groups which are defined by partitions.

Definition 2.24. A **labeled multipartition** π of a set S is the labeling of the leaves of a rooted tree by the elements of S, possibly with additional labeling of internal vertices. The subsets defined by considering all of the leaf labels over a fixed internal vertex are called the **blocks** of the labeled multipartition. The **depth** of a block is the number of edges between its corresponding internal vertex and the root vertex.

Given a labeled multipartition π of $\{1, \ldots, n\}$ define its **automorphism group** S_π to be the automorphisms of the tree which preserve any additional labels of internal vertices. We identify S_π canonically with a subgroup of S_n through the action on leaves.

A labeled multipartition is determined up to additional labels by its blocks. The additional labels of internal vertices are sometimes used to make such structures more rigid, ruling out automorphisms between blocks. Since we only use them in this way, we do not give explicit additional labels or name the labeling set; we only indicate when labels are shared or they differ.

There are some basic constructions which give rise to the multipartitions and thus automorphism subgroups we consider.

Definition 2.25. If $\pi_1, \ldots, \pi_k$ are labeled multipartitions of $S_1, \ldots, S_k$ define the **union** $\pi_1 \cup \cdots \cup \pi_k$ to be the multipartition of $\coprod S_i$ in which the trees defining the π_i are grafted to a (new) root with k edges, and the internal vertices corresponding to the original roots are given distinct labels.

If π is a labeled multipartition of S define the **multiple** $m \cdot \pi$ to be the labeled multipartition of $\coprod_{i=1}^{m} S$ in which m copies of the tree defining π are grafted to a multipartition of S in which m copies of the tree defining π are grafted to a single root and given the same label.

Let $\underline{n}$ denote the trivial labeled partition of $\{1, \ldots, n\}$, defined by a rooted tree with no internal vertices.

It is convenient to use the standard finite sets $\{1, \ldots, n\}$ for all labels, in which case we identify the abstract disjoint union $\{1, \ldots, n_1\} \coprod \cdots \coprod \{1, \ldots, n_k\}$ with the set $\{1, \ldots, n_i\}$ through the standard "ordering on the page." We identify $\coprod_m \{1, \ldots, n\} \cong \{1, \ldots, mn\}$ similarly.

The fact that the union of labeled multipartitions has distinct labels while those of a multiple multipartition are repeated is a key distinction, in a sense giving rise to our divided powers operations. In particular, $\mathsf{S}_{\underline{n}\cup\underline{m}}$ is isomorphic to $\mathsf{S}_n \times \mathsf{S}_m$, including when $n = m$, while $\mathsf{S}_{2\cdot\underline{n}}$ is isomorphic to $\mathsf{S}_n \wr \mathsf{S}_2$.

We consider standard set partitions as labeled multipartitions through trees whose internal vertices all have depth one, and identical labels. Recall that the set of partitions of a set is a poset under refinement and that this poset has meet and join operators, which we denote $\wedge$ and $\vee$, that provide it with a lattice structure. The following properties of S_π are straightforward.

Lemma 2.26. *Let $\pi = \{S_i\}$ be a partition of $\{1,\dots,n\}$.*

1. *$\mathsf{S}_{m\cdot\underline{k}}$ is isomorphic to the wreath product $\mathsf{S}_k \wr \mathsf{S}_m$. More generally, if $m_\pi(i)$ is the number of parts of π whose cardinality is i, S_π is conjugate to $\bigtimes_{i=0}^{n} \mathsf{S}_i \wr \mathsf{S}_{m_\pi(i)}$ in S_n.*
2. *For any $\sigma \in \mathsf{S}_n$, the conjugate $\sigma \mathsf{S}_\pi \sigma^{-1}$ coincides with $\mathsf{S}_{\sigma\pi}$.*
3. *The map $[\sigma] \quad \sigma\pi$ is a bijection between the cosets $\mathsf{S}_n/\mathsf{S}_\pi$ and the set of partitions that are permutations of π.*

Partition subgroups play key roles in defining the divided powers.

Definition 2.27. Let $\{A_n\}$ be a product series of algebras. For $x \in H^*(\mathsf{S}_n; A_n)$ we define $x^{[k]} \in H^*(\mathsf{S}_{kn}; A_{kn})$ to be

$$x^{[k]} := tr^{\mathsf{S}_{kn}}_{\mathsf{S}_n \wr \mathsf{S}_k}(x^{\otimes k}),$$

where we identify A_{kn} with $A_n^{\otimes k}$ via the structural isomorphism.

Proposition 2.28. *If $\{A_n\}$ is a product series of algebras then $_n H^*(\mathsf{S}_n; A_n)$ with cup and transfer product from Definition 2.14, coproduct from Definition 2.20 and divided powers from Definition 2.27 satisfies the axioms for divided powers structures.*

Before proving this, we set aside a basic fact from group cohomology which we repeatedly use, which follows from the standard fact that restriction followed by transfer from a subgroup is multiplication by the index of a subgroup.

Lemma 2.29. *If $K \subset H \subset G$ are finite-index inclusions of groups then*

$$tr^G_K \circ res^H_K = [K : H] tr^G_H.$$

Proof of Proposition 2.28. Let $x \in H^*(\mathsf{S}_n; A_n)$ be represented by a homomorphism f from a resolution of S_n to A_n, and similarly $y \in H^*(\mathsf{S}_m; A_m)$ be represented by g.
g.

The 0, 1-Cases axioms of Definition 2.5 are immediate. For the Binomial axiom, we calculate $(x + y)^{[r]}$ by taking the transfer (induction) of $(f + g)^{\otimes r}$, which is a sum over i of shuffles of $f^{\otimes i}$ and $g^{\otimes(r-i)}$. But the transfer on such shuffles is exactly $x^{[i]}\ 8\ y^{[r-i]}$.

Checking the Exponent Axiom, $x^{[m]}\ 8\ x^{[r]}$ and $x^{[m+r]}$ are both images under transfer of $f^{\otimes m+r}$, but the former is induced up from the subgroup $(\mathsf{S}_{m\cdot\underline{n}}) \times (\mathsf{S}_{r\cdot\underline{n}})$, while the latter is induced up from $\mathsf{S}_{(m+r)\cdot\underline{n}}$. By Lemma 2.29, the former is obtained from the latter by multiplication by the quotient of the indices of these subgroups, namely $\binom{m+r}{m}$.

We next verify the Distributivity Axiom, which in the Hopf rig setting is that $(x\ 8\ y)^{[k]} = x^{k}\ 8\ y^{[k]}$. By definition $(x\ 8\ y)^{\otimes k}$ is a composite of induction maps of $(f \otimes g)^{\otimes k}$ from $\mathsf{S}_{k\cdot(\underline{n}\cup\underline{m})}$ ultimately to $\mathsf{S}_{k(n+m)}$. Similarly, $x^{k}\ 8\ y^{[k]}$ is obtained by inducing $f^{\otimes k} \otimes g^{\otimes k}$ from $S_{\cup_k \underline{n} \cup k\cdot\underline{m}}$. After conjugating so that $(f \otimes g)^{\otimes k}$ becomes $f^{\otimes k} \otimes g^{\otimes k}$, both subgroups are then subgroups of $S_{k\cdot\underline{n}\cup k\cdot\underline{m}}$, for which $f^{\otimes k} \otimes g^{\otimes k}$ is invariant. Lemma 2.29 applies in both cases, so that each result is the induction from $S_{k\cdot\underline{n}\cup k\cdot\underline{m}}$ multiplied by the index of the corresponding subgroup. The Distributivity axiom follows as these two indices are the same.

We verify the Composition Axiom $(x^{[k]})^{[h]} = \frac{(hk)!}{(k!)^h h!} x^{[hk]}$ similarly. By definition $x^{[hk]}$ is given by inducing $f^{\otimes hk}$ from $S_{hk\cdot n}$ to S_{nhk}, while $(x^{[k]})^{[h]}$ is the induction of the same cochain from $S_{h\cdot(k\cdot\underline{n})}$. Since $S_{h\cdot(k\cdot\underline{n})} \subset S_{hk\cdot\underline{n}}$, Lemma 2.29 applies to show that they differ by a multiplicative coefficient equal to the index of this inclusion, which is exactly $\frac{(hk)!}{(k!)^h h!}$. □

We next turn to compatibility of divided powers with coproduct. As our divided powers structures are defined by transfers, and our coproduct is defined by restriction, the Cartan-Eilenberg Double Coset Formula will be of use. Recall that if H and K are subgroups of a finite group G and R is a set of representatives in G for the set of double cosets $H\backslash G/K$, then, on cohomology with coefficients in a given G- representation,

$$\rho^G_H \circ \mathrm{tr}^G_K = \sum_{r\in R} \mathrm{tr}^H_{H\cap c_r K} \circ \rho^{c_r K}_{H\cap c_r K} \circ c_r^*,$$

where c_r is the isomorphism of K with rKr^{-1} and ρ and tr denote restriction and transfer maps respectively. This is proved for example as Theorem I.6.2 of [AM04].

Proposition 2.30. *If* $\{A_n\}$ *is a product series of algebras, then on* $\bigoplus_n H^*(S_n; A_n)$ *the divided powers of Definition* 2.27 *commute with the coproduct of Definition* 2.20.

Proof. We start with a more explicit expression of divided powers of a coproduct. If $x \in H^*(S_n; A_n)$, set $\Delta x = \sum x_{(i,n-i)}$, where $x_{(i,n-i)} \in H^*(S_i; A_i) \otimes H^*(S_{n-i}; A_{n-i})$. By the Binomial Axiom, which was just established in Proposition 2.28,

$$(\Delta x)^{[k]} = \sum_{j_0+\cdots+j_n=k} \prod_{i=0}^{n} \left(x_{(i,n-i)}\right)^{[j_i]}.$$

If $a + b = n$ the (a, b) component of this consists of terms with $\sum j_i \cdot i = a$. We show that $\left(\Delta(x^{[k]})\right)_{(a,b)}$ agrees with this.

show that $\left(\Delta(x^{[k]})\right)_{(a,b)}$ agrees with this.

Let $x \in H^*(S_n; A_n)$ be represented by a homomorphism f from a resolution of S_n to A_n. By definition $(x^{[k]})_{(a,b)}$ will be represented by $\rho^{S_{nk}}_{S_{\underline{a}\cup\underline{b}}} \circ \mathrm{tr}^{S_{nk}}_{S_{k\cdot\underline{n}}}(f^{\otimes k})$. Applying the Cartan-Eilenberg Double Coset Formula, this coincides with

$$\sum_{r\in R} \mathrm{tr}^{S_{\underline{a}\cup\underline{b}}}_{S_{\underline{a}\cup\underline{b}}\cap c_r S_{k\cdot n}} \circ \rho^{c_r S_{k\cdot\underline{n}}}_{S_{\underline{a}\cup\underline{b}}\cap c_r S_{k\cdot n}} \circ c_r^{\#}(f^{\otimes k}),$$

where R is a set of representatives for the set of double cosets $(S_{\underline{a}\cup\underline{b}})\backslash S_{nk}/S_{k\cdot\underline{n}})$.

To understand this double coset space, recall that the cosets of symmetric groups modulo the automorphism groups S_π correspond to partitions with the same shape as π. Thus $S_{nk}/S_{k\cdot\underline{n}}$ corresponds to partitions of $\{1, \ldots, nk\}$ with k parts of cardinality n, while $S_{\underline{a}\cup\underline{b}}\backslash S_{nk}$ corresponds to partitions into two sets with cardinalities a and b, or equivalently bicolorings. Thus R corresponds to bicolored partitions of $\{1, \ldots, nk\}$ with k parts of cardinality n. Equivalently R is given by multipartitions governed by a tree with k edges attached to the root, and each of those with at most two edges labeled a and/or b, with a total of n leaf edges over them.

All labels in the multipartition are identical. The intersections $S_{\underline{a}\cup b} \cap c_r S_{k\cdot \underline{n}}$ are the automorphisms of such multipartitions. These are conjugate to $S_{\cup_i m_i \cdot (\underline{i} \cup \underline{n-i})}$, where m_i is the number of times there are i leaves in the group labeled by a within a k-block.

We then identify the restriction $\rho^{c_r S_{k\cdot \underline{n}}}_{S_{\underline{a}\cup \underline{b}} \cap c_r S_{k\cdot \underline{n}}} \circ c_r^{\#}(f^{\otimes k})$ with $\bigotimes_{i=0}^{n} f_{(i,n-i)}^{\otimes m_i}$, where $f_{(i,n-i)}$ represents $x_{(i,n-i)}$. To calculate $\mathrm{tr}^{S_{a\cup b}}_{S_{\underline{a}\cup \underline{b}} \cap c_r S_{k\cdot \underline{n}}}$ applied to this class, we factor the inclusion of $S_{\underline{a}\cup b} \cap c_r S_{k\cdot \underline{n}}$ in $S_{\underline{a}\cup \underline{b}}$. Since $S_{\underline{a}\cup b} \cap c_r S_{k\cdot \underline{n}}$ is conjugate to $S_{\cup_i m_i \cdot (\underline{i} \cup \underline{n-i})}$ it is contained in a conjugate of $S_{\cup_i (\underline{i m_i} \cup \underline{(n-i) m_i})}$. The transfer map up to this subgroup is a product of transfer maps defining divided powers, so $\bigotimes_{i=0}^{n} f_{(i,n-i)}^{\otimes m_i}$ is sent to $\bigotimes_{i=0}^{n} (f_{(i,n-i)})^{[m_i]}$. The transfer map from $S_{\cup_i (\underline{i m_i} \cup \underline{(n-i) m_i})}$ to $S_{\underline{a}\cup b}$ is the transfer product on the tensor product of the cohomology with itself, so $\bigotimes_{i=0}^{n} (f_{(i,n-i)})^{[m_i]}$ is sent to $\bigodot_{i=0}^{n} (f_{(i,n-i)})^{[m_i]}$, which agrees with our determination of $\left((\Delta x)_{[k]}\right)^{(a,b)}$ above. □

Corollary 2.31. *$H_{EP}{}^*(X)$ is a bigraded component Hopf ring with additive divided powers.*

2.5. **Statement of first main results.** We state here our first main structure theorem, that we will prove in the remaining sections.

We require two more definitions before stating our first main theorems.

Definition 2.32. An element of a divided powers algebra of even degree when p odd, or any degree when $p = 2$, is **standard non-nilpotent** if all $x^{[m]}$ are non-zero, with $\Delta x^{[m]} = \sum_{i+j=m} x^{[i]} \otimes x^{[j]}$ and $x^{[i]} \cdot x^{[j]} = \binom{i+j}{i} x^{[i+j]}$.

Definition 2.33. Let C be a category, S a subcategory, and $F : C \to D$ a functor. We say some object $x \in S$ is universal among S with respect to F if its image under F is initial in the full subcategory generated by $F(C)$.

We first give an immediate reformulation of the calculations of Giusti–Salvatore–Sinha utilizing divided powers structure.

Theorem 2.34 (From [GSS12])**.** *The mod-two cohomology of extended powers of S^0, namely $H^*_{EP}(S^0) \cong \bigoplus H^*(BS_n)$, is a component Hopf rig with additive divided powers which contains standard non-nilpotent classes $\gamma_i \in H^{2^i-1}(BS_{2^i})$. It is universal among such objects, with respect to the functor which forgets divided powers.*

In other words, there are no relations required to understand $H^*_{EP}(S^0)$ as a Hopf ring other than those given in the axioms of component Hopf rig with additive divided powers. This thus determines the cohomology ring structure. In particular, the subrings generated by $\gamma_i^{[2^j]}$ for $i + j = n$ are polynomial subrings of the cohomology of BS_n. But there are many relations for classes involving the transfer product. At the moment, we do not fully understand the divided powers structure—see Remark 2.39—so while we use divided powers to generate classes we forget them in order to have a unique characterization.

The following is the odd primes counterpart of Theorem 2.34, that we will prove in Section 3. For $p > 2$, let $T_\rho(H^*(X; F_p)) = \{T_{\rho,n}(H^*(X; F_p))\}_{n\in N}$ be the superproduct series of algebras defined above. We consider the sign degree e that determines the super-structure and the degree d induced by the cohomological dimension of $H^*(X; F_p)$.

Theorem 2.35. *The mod-p cohomology of extended powers of S^0, namely $H^*_{EP}(S^0) \cong H^*(B\mathfrak{S}_n;\rho)$, is a trigraded component super-Hopf rig with additive divided powers which contains*

- *standard non-nilpotent classes γ_k in grading $(2(p^k-1), p^k, 0)$,*
- *primitive classes λ_k in grading $(p^k-1, p^k, 1)$,*
- *standard non-nilpotent classes γ'_k in grading $(p^k-2, p^k, 1)$,*

with cup product relations $\lambda_k^2 = \gamma_k$. It is universal among such objects with respect to the functor which forgets divided powers.

Remark 2.36. The sub-Hopf rig of $H_{EP}{}^*(S^0)$ corresponding to the addend of even sign degree has been calculated by Guerra [Gue17]. The generating classes γ_k, $\alpha_{i,k}$ and $\beta_{i,j,k}$ appearing there can be retrieved from our presentation as γ_k, $(-1)^{\frac{p-1}{2}(k-i)}\lambda_k\gamma'^{[p^{k-i}]}_i$ and $(-1)^{\frac{p-1}{2}(j-i)}\gamma'_j\gamma'^{[p^{k-i}]}_i$, respectively.

For any nonempty X, the cohomology of $D(X_+)$ is an algebra over the cohomology of $D(S^0)$, through the projection map which sends X to the non-base point of S^0. This is injective, with any choice point in X giving a splitting. We use this algebra structure for the two following results, that will be proved in Section 4. For odd primes, we compute a bigger Hopf ring with cohomology taken with coefficients in ρ, but one can easily recover the ordinary mod p cohomology of $\tilde{D}(X)$ as a Hopf ring by extracting only the homogeneous part of even sign degree.

Theorem 2.37. *The mod-two cohomology of extended powers of X_+ is the universal component super-Hopf rig with additive divided powers over the cohomology of extended powers of S^0 which contains the cohomology classes of X as classes which are standard non-nilpotent. Relations are*

$$x^{[n]} \cdot y^{[n]} = (x \cdot y)^{[n]} \text{ for } x, y \in H^*(X) \text{ and}$$
$$x^{[nm]} \cdot y^{[n]} = (x^{[m]} \cdot y)^{[n]} \text{ for } x \in H^*(X), y \in H^*(D_m(S^0)).$$

It is universal among such objects with respect to the functor which forgets divided powers.

Theorem 2.38. *For $p > 2$, the mod-p cohomology of extended powers of X_+ is a component super-Hopf rig with additive divided powers over the cohomology of extended powers of S^0 which contains two copies of the cohomology of X, which we denote by x_e degree $(d, 1, e)$ for $e \in \{0, 1\}$. These classes are standard non-nilpotent if $d + e$ is even and primitive if $d + e$ is odd, with*

$$x_e^{[n]} \cdot y_{e'}^{[n]} = (x \cdot y)^{[n]}_{e+e'},$$
$$x_e^{[nm]} \cdot z^{[n]} = (x_e^{[m]} \cdot z)^{[n]} \text{ and}$$
$$x_e \odot y_{e'} = 0 \text{ if } e \neq e',$$

if $x, y \in H^*(X; \mathbb{F}_p)$, $z \in H^*(D_m(S^0); \mathbb{F}_p)$, and $e, e' \in \{0, 1\}$, with the sum $e + e'$ understood modulo two. It is universal among such objects with respect to the functor which forgets divided powers.

Remark 2.39. Theorems 2.37 and 2.38 explicitly embed the cohomology of the symmetric groups and the cohomology of X in $H_{EP}{}^*(X)$. The relations provide a compatibility identity between divided powers and cup product. We can interpret this structure as a "bigraded component Hopf ring with divided powers over the cohomology of symmetric groups generated by $H^*(X)$". More generally, one could define a "bigraded component Hopf ring with divided powers" as a bigraded component Hopf ring with additive divided powers A such that a similar compatibility condition between divided powers and cup product holds for all $x, y \in A$ divided powers of primitive elements. However, this simple identity does not extend to classes x not arising from the cohomology of X. For example, with $p = 2$ and $X = \{*\}$, $H_{EP}{}^*(X_+)$ becomes the ordinary mod 2 cohomology of the symmetric groups and the divided powers operations $\_^{[k]}$ agree with the cohomological transfer maps associated to $S_k \wr S_n \to S_{nk}$. A procedure to compute these transfer maps is described by Kechagias [Kec09]. Combining Kechagias's algorithm with the computations of the restriction to elementary abelian subgroups achieved by Giusti–Salvatore–Sinha [GSS12] one obtains the following equality

$$(\gamma_1)^{[2]} \cdot (\gamma_1^2)^{[2]} = \gamma_2^2 + (\gamma_1^3)^{[2]}.$$

As the right hand side is not simply $(\gamma_1^3)^{[2]}$, the relationship between the divided powers operations and the cup product in $H_{EP}{}^*(X)$ is not as expected and appears likely to be complicated in general.

Remark 2.40. Recall from Remark 2.23 that the cohomology of extended powers of manifolds all have simple geometric cochain models. The γ_k for $\mathbb{F}_2$ are represented by "2^k points which share a coordinate". Formally one considers the manifold consisting of triples (x, X, Y) where $x \in \mathbb{R}$, X is a configuration of 2^k distinct unordered points in $x \times \mathbb{R}^{N-1} \subset \mathbb{R}^N$, and Y is a configuration of unordered points in the complement of X in $\mathbb{R}^N$. This manifold maps properly to the configuration space, generically an embedding but finite-to-one if some point in Y is contained in $x \times \mathbb{R}^{N-1}$ or some 2^k points in Y contained in an $x' \times \mathbb{R}^{N-1}$. It thus defines a geometric cocycle, which represents γ_k as shown in Theorem 4.9 of [GSS12]. Similarly, the γ_k for odd primes are represented by the geometric cochain defined by "p^k points in $\mathbb{C}^N$ which share a complex coordinate." The cup products $\gamma_k \cdot x^{[k]}$ on the kth extended power are represented by labeled points which share a coordinate and are labeled by the representative of x. One must name which extended power because the cup product of manifestations of these classes on higher extended powers, given by transfer products with unit classes, can differ from this (as some points in a configuration could share their coordinate while others share their label).

We have not developed geometric representatives for the $\gamma_k^!$ and λ_k, or their even sign degree products as considered earlier in [Gue17]. And as mentioned before, one must take care with geometry and cup products since one would expect ${\gamma_1^{[2]}}^3$ to be represented by the geometric cochain defined by four points which consist of two sets of two points each of which share three coordinates. But it is represented by the union of this along with a copy of the geometric cochain defined by four points which share two coordinates.

2.6. **Explicit construction of (relatively) universal component Hopf rings with divided powers.** In this subsection we analyze better the structures elaborated in the main theorems just stated, abstracting as follows.

Definition 2.41. Let A be a component super-Hopf ring with additive divided powers containing ρ as a subalgebra of A_1, its $\cdot$-algebra in component 1, and let V be a connected graded algebra. Let $H_{alg}(A, V)$, which we generally shorten to

H_{alg}, be the super-Hopf ring with additive divided powers universal among those satisfying the following five properties.

(1) There is a map $\pi : A \to H_{alg}$ that is both a super-Hopf ring map and a morphism of divided powers structures preserving the $\cdot$-unit of each component;
(2) There is a ρ-algebra homomorphism ι from $V \otimes \rho$ to the subspace of elements of component 1 inside H_{alg} (that is, an algebra morphism $\iota : V \to H_{alg}$ in component 1 if $p = 2$);
(3) For all $x, x' \in V \otimes \rho$ both with even total degree $\iota(x)^{[n]} \cdot \iota(x')^{[n]} = \iota(xx')^{[n]}$;
(4) For all $x \in V \otimes \rho$ with even total degree and for all $y \in A$ such that $n(y) = m$,
$$\iota(x)^{[lm]} \cdot \pi(y)^{[l]} = (\iota(x)^{[m]} \cdot \pi(y))^{[l]};$$
(5) If $p > 2$, $x^{[n]} \odot x'^{[m]} = 0$ if $x \in V \otimes \mathsf{F}_p \subseteq V \otimes \rho$ and $x' \in V \otimes \mathrm{sgn} \subseteq V \otimes \rho$.

We wish to describe, in terms of additive bases of A and V, a basis for H_{alg}. In Section 4.1 we will specialize it to our case of interest, where

$$A = \mathsf{F}_p \oplus \bigoplus_{n \geq 1} H^*(\mathsf{S}_n; \rho) \quad \text{and} \quad V = H^*(X; \mathsf{F}_p).$$

In the remainder of this section, we will assume that $p > 2$, because the treatment for $p = 2$ is similar and much simpler. Let V_{even} (respectively V_{odd}) be the subspaces of $V \otimes \rho$ of even (respectively odd) total degree.

As A, with the product $\odot$ and the coproduct Δ alone is a bicommutative divided powers Hopf algebra, by a classical result [And71] it must be the free divided powers Hopf algebra on its subspace of primitive elements. That is, $A \cong DP(P(A))$. Since A is a component super-Hopf ring, $P(A)$ is a non-unital component super-algebra with $\cdot$, that inherits a triple grading from A.

We consider the tensor product of algebras $V_{even} \otimes P(A)$, that we tri-grade with the following rule: if $v \in V_{even}$ is homogeneous of degree $d(v)$ and $y \in P(A)$ is tri-homogeneous of tri-degree $(d(y), n(y), e(y))$, then $v \otimes y \in V \otimes P(A)$ is tri-homogeneous of tri-degree $(d(v)n(y) + d(y), n(y), e(y) + d(v))$. Intuitively, we interpret a pure tensor $v \otimes y \in V_{even} \otimes P(A)$ as the element $\iota(v)^{[n(y)]} \cdot \pi(y)$. This justifies our choice of degrees. Recall that the sign degree e is defined only modulo 2.

From a graded module M we can construct an augmented non-unital coalgebra $M^{pr} = \mathsf{F}_p \oplus M$ with the F_p addend in tri-degree $(0, 0, 0)$ and a coproduct $\Delta(x) = 1 \otimes x + x \otimes 1$ for all $x \in M$. This is the "primitive coalgebra extension" of the module M.

We then decompose $V_{even} \otimes P(A)$ as the direct sum $(V_{even} \otimes P(A))_+ \oplus (V_{even} \otimes P(A))_-$, where $(V_{even} \otimes P(A))_+$ is the subspace generated by elements $v \otimes x \in V_{even} \otimes P(A)$ in which $e(v) = e(x) \bmod 2$ and $(V_{even} \otimes P(A))_-$ consists of such with $e(v) \neq e(x) \bmod 2$). Then we let

$$(H, \odot, \Delta) \cong DP_{HA}((V_{even} \otimes P(A))_+^{pr}) \oplus DP_{HA}((V_{even} \otimes P(A))_-^{pr})/\sim,$$

where $\sim$ is the equivalence relation that identifies the units of the two addends. Additive bases for V and $P(A)$ induce a basis for $DP_{HA}((V_{even} \otimes P(A))_\pm^{pr})$, and consequently on H, consisting of elements of the form $\bigodot_i (v_i \otimes y_i)^{[n_i]}$ with v_i, y_i basis elements with $n_i = 1$ if $v_i \otimes y_i$ has odd total degree and $v_i \otimes y_i$ belong all to $(V \otimes P(A))_+$ or all to $(V \otimes P(A))_-$.

For now, H is only a component divided powers super-Hopf algebra. Under this isomorphism we next identify maps $\pi : A \to H$ and $\imath : V \otimes \rho \to H$, as well as a second product $\cdot : H \otimes H \to H$ which provide it with a super-Hopf ring structure.

First, we define $\pi : A \to H$ as the unique divided powers super-Hopf algebra morphism extending the linear map on primitives $\pi : y \in P(A) \mapsto 1_V \otimes y \in V_{even} \otimes P(A) \subseteq P(H)$. Second, let $\imath : V \otimes \rho \to H$. If $v \in V_{even}$, then we let $\imath(v) = v \otimes 1_1$, where $1_1 \in A$ is the $\cdot$-product unit of the 1-component of A. If $v \in V_{odd}$, then let $s \in \rho$ be a generator of the sgn addend. Since $vs \in V_{even}$, we can define $\imath(v)$ as $\imath(vs) \otimes s$, where $s \in \rho \subseteq A_1$ is considered as a primitive element of A.

We construct the second product on H step-by-step. As always, $\cdot$ between elements of different components will be zero. As an intermediate step, we define the product $\imath(v)^{[n]} \cdot x$ for all $v \in V_{even}$ and $x \in H$ with $n(x) = n \geq 1$. The case $n = 1$ reduces to the product on $V \otimes \rho$, so we can assume that $n \geq 2$. If $x = (w \otimes y)^{[m]}$ is a divided power of a primitive element $w \otimes y \in V_{even} \otimes P(A)$, then we let $\imath(v)^{[n]} \cdot x = (\imath(v)(w \otimes y))^{[m]}$. We can then extend it to general $x \in H$ by using Hopf ring distributivity, because H is generated under $\circledast$ by the divided powers of its primitive elements. Explicitly, every element of H is a linear combination of products $\circledast_{i=1}^{r} x_i^{[m_i]}$ for some $x_i = (w_i \otimes y_i) \in V_{even} \otimes P(A)$ and $m_i \geq 1$. We let

$$\imath(v)^{[\sum_i n(x_i) m_i]} \cdot \circledast_i x_i^{[m_i]} = \circledast_i (\imath(v)(w_i \otimes y_i))^{[m_i]}$$

and we extend to all H by linearity.

Proposition 2.42. *There is a unique component super-Hopf ring structure on H such that the multiplication by $\imath(v)^{[n]}$ for $v \in V_{even}$ has the form defined above, and satisfying the five conditions of Definition 2.41.*

Proof. Let $v, w \in V_{even}$ and $y, z \in P(A)$ be tri-homogeneous elements. Note that, by the definition of the multiplication by divided powers of elements of V_{even} above, we must have $v \otimes y = \imath(v)^{[n(y)]} \cdot \pi(y)$ and $w \otimes z = \imath(w)^{[n(z)]} \cdot \pi(z)$. Hence, if H satisfies the third and fourth conditions we must have

$$\begin{aligned}(v \otimes y)^{[a]} \cdot (w \otimes z)^{[b]} &= (\imath(v)^{[n(y)]} \cdot \pi(y))^{[a]} \cdot (\imath(w)^{[n(z)]} \cdot \pi(z))^{[b]} \\ &= (-1)^{t(y)d(w)n(z)ab}\, \imath(v)^{[n(y)a]} \cdot \imath(w)^{[n(z)b]} \pi(y)^{[a]} \pi(z)^{[b]} \\ &= (-1)^{t(y)d(w)n(z)ab}\, \imath(vw)^{[n(y)a]} \cdot \pi(y)^{[a]} \cdot \pi(z)^{[b]}\end{aligned}$$

for all a, b such that $n(y)a = n(z)b$. Moreover, if π is a morphism of Hopf rings and of divided powers structure, $\pi(y)^{[a]} \cdot \pi(z)^{[b]} = \pi(y^{[a]} \cdot z^{[b]})$. In conclusion

$$(v \otimes y)^{[a]} \cdot (w \otimes z)^{[b]} = (-1)^{t(y)d(w)n(z)ab}\, \imath(vw)^{[n(y)a]} \cdot \pi(y^{[a]} \cdot z^{[b]}).$$

Consequently, the conditions above uniquely determine the values of $\cdot$ on $\circledast$-indecomposables. Therefore, if an extension of $\cdot$ exists, it is necessarily unique.

To prove existence, we only need to check that the formula above provides a well-defined bilinear product satisfying Hopf ring distributivity. This is straightforward. For instance, one can fix bases of A and V, define the $\cdot$ product as above on the induced basis, extend it to all H by bilinearity and directly check the axioms of Hopf rings using the basis. □

We now prove that H is our desired object.

Proposition 2.43. *The super-Hopf rig with additive divided powers H explicitly constructed above is the universal object H_{alg} of Definition 2.41.*

Proof. Let B be a super-Hopf rig with additive divided powers with maps $\pi_B\colon A \to B$, $\iota_B\colon V \otimes \rho \to B$ satisfying our desired hypotheses. DP_{HA} and $\underline{\ }^{pr}$ are left adjoints of the forgetful functor from divided powers Hopf algebras to coalgebras and the primitives $P\colon R$-*coalg* $\to R$-*mod*, respectively. Therefore, a morphism of divided powers Hopf algebras $f\colon H_{alg} \to B$ such that $f\pi = \pi_B$ and $f\iota = \iota_V$ is uniquely determined by its restriction to $V_{even} \otimes P(A)$. If, in addition, f is a super-Hopf rig morphism, then

$$f(v \otimes y) = f(\iota(v)^{[n(y)]} \cdot \pi(y)) = \iota_B(v)^{[n(y)]} \cdot \pi_B(y)$$

for all $v \in V_{even}$ and $y \in P(A)$. Therefore, such a morphism is unique (if it exists).

To prove existence, we let $f\colon H_{alg} \to B$ be the divided powers super-Hopf algebra morphism adjoint of the linear map $v \otimes y \in V_{even} \otimes P(A) \mapsto \iota_B(v)^{[n(y)]} \cdot \pi_B(y) \in P(B)$. We only need to check that f preserves the $\cdot$ product, and by Hopf ring distributivity it is enough to prove this on 8-indecomposables, that is divided powers of elements of $P(H_{alg}) = V_{even} \otimes P(A)$. By our construction of the $\cdot$ product in H_{alg}, we immediately see that $f(\iota(v)^{[a]} \cdot x) = \iota_B(v)^{[a]} \cdot f(x)$ for all $v \in V_{even}$ and $x \in H_{alg}$ and $a \geq 2$. Moreover, since π_B is a super-Hopf rig morphism, we have that

$$\begin{aligned} f\left((v \otimes y)^{[a]} \cdot (w \otimes z)^{[b]}\right) &= (-1)^{t(y)d(w)n(z)ab} f\left(\iota(vw)^{[n(y)a]} \cdot \pi(y^{[a]} \cdot z^{[b]})\right) \\ &= (-1)^{t(y)d(w)n(z)ab}\, \iota_B(vw)^{[n(y)a]} \cdot \pi_B(y^{[a]} \cdot z^{[b]}) \\ &= \left(\iota_B(v)^{[n(y)]} \cdot \pi_B(y)\right)^{[a]} \cdot \left(\iota_B(w)^{[n(w)]} \cdot \pi_B(z)\right)^{[b]} \\ &= f(v \otimes y)^{[a]} \cdot f(w \otimes z)^{[b]}. \end{aligned}$$

D

Since we want to specialize to the case $A = H_{EP}{}^*(S^0)$ and $V = H^*(X; \mathbb{F}_p)$ and as shown in Remark 2.39 the relation between cup product and divided powers in $H_{EP}{}^*(S^0)$ is complicated, it is preferable to rephrase the construction above in terms of indecomposables. In a Hopf algebra with divided powers, the indecomposables are elements of the form $x^{[p^k]}$ with x primitive and $k \geq 0$. Consequently, as a graded bicommutative Hopf algebra alone, $DP_{HA}(V \otimes P(A)^{pr})$ is generated by

$$\{(v \otimes y)^{[p^k]}\}_{v \in V, y \in P(A), k \geq 0} = V \otimes Q(A).$$

We obtain an isomorphism of indecomposables $V_{even} \otimes Q(A) \cong QH_{alg}$ given by $v \otimes y \mapsto v^{[n(y)]} \cdot y$ for all $v \in V$ and $y \in A$, and this realizes H_{alg}, under the product 8 alone, as the free graded commutative algebra generated by $V_{even} \otimes Q(A)$, quotiented by the image of the Frobenius and the ideal generated by the relation (5). As a result, any pair of additive bases for V and $Q(A)$ induce an additive basis for H_{alg}.

In summary, we have the following.

Lemma 2.44. *Let e_0 (respectively e_1) in ρ be a non-zero element in the constant representation (respectively sign representation) addend of ρ. Let $Q(A)$ be the space of 8-indecomposables of A. Let B_V and B_A be totally ordered additive bases of V_{even} and $Q(A)$ respectively. Define B^{ρ}_V as the set of elements $\{v \otimes e_i\}_{v \in \mathsf{B}_V, i \in \{0,1\}}$. Order*

B_V *in any way (e.g. lexicographically). Let* B *be the set of elements of the form* $\prod_{i=1}^{r} \iota(v_i)^{[n(y_i)]} \cdot \pi(y_i) \in H_{alg}$ *with* $v_1 \leq \cdots \leq v_r \in \mathsf{B}^{\rho}$, $y_1 \leq \cdots \leq y_r \in \mathsf{B}_{A,V}$ *such that:*

- *the sign degree of all the* $\otimes$*-factors is the same*
- *the multiplicity of every* $\otimes$*-factor of even total degree is at most* $p-1$,
- *and the multiplicity of every* $\otimes$*-factor of odd total degree is at most* 1.

Then B *is a basis for* H_{alg} *as an* F_p*-vector space.*

3. The homology and cohomology of symmetric groups with twisted coefficients

Homology of extended powers and free infinite loop spaces are algebras over the Kudo-Araki-Dyer-Lashof algebra, an algebra of homology operations that constitute the basic building blocks of the homology of the extended powers of a point – that is, the homology of symmetric groups. This structure is well-understood by the work of Cohen–Lada–May [CLM76]. Our strategy in this paper is to show that our descriptions which build on cup product structure pair perfectly with their descriptions. But for odd primes the cleanest descriptions of cohomology require considering coefficients in ρ, the $\mathsf{F}_p[\mathsf{S}_n]$-representation which is the sum of the trivial and sign representations, introduced in the previous section.

Thus, this section is divided in two parts. In the first subsection we provide a description of the homology of $\tilde{D}X$ with coefficients in ρ, in terms of Kudo-Araki-Dyer-Lashof operations. We do not make any claim of originality here, as we only review classical results by Cohen–Lada–May [CLM76] for the trivial summand and their recent extension to sign-twisted coefficients by Bernard [Ber]. In the second subsection we dualize these results to obtain a description of $\bigoplus_{n\geq 0} H^*(\mathsf{S}_n;\rho)$ as a Hopf ring, and prove Theorem 2.35. While the summand of sign degree zero is known by our previous work, namely [GSS12] and [Gue17], the description of the sign degree one summand is new.

3.1. The homology of extended powers in terms of KADL operations.

Since we work with field coefficients in a setting which is finite dimensional in each grading, the homology of DX with local coefficients given by the S_n-representation A_n of Definition 2.17 is isomorphic to the bigraded linear dual of

$$\bigoplus_{n\geq 0} H^*(D_n(X); A_n) = H_{EP}{}^*(X).$$

Hence it will be a Hopf co-ring, endowed with two coproducts $\Delta_{\odot}$, $\Delta_{\cdot}$, and a product $*$ dual to $\odot$, $\cdot$, and Δ respectively, satisfying all the axioms of a Hopf ring with the directions of all morphisms reversed.

The product $*: H_i(D_nX) \otimes H_j(D_mX) \to H_{i+j}(D_{n+m}X)$. also has a group-homology interpretation. Let R_{S_n} be a resolution of F_p as an $\mathsf{F}_p[\mathsf{S}_n]$-module, so that by Proposition 2.2 the homology of $R_{\mathsf{S}_n} \otimes_{\mathsf{S}_n} \tilde{H}_*(X)^{\otimes n}$ is that of D_nX. The product $\cdot$ is induced, up to sign, by the tensor product of the map of resolutions $R_{\mathsf{S}_n} \otimes R_{\mathsf{S}_m} \to R_{\mathsf{S}_{n+m}}$ induced by the inclusion $\mathsf{S}_n \times \mathsf{S}_m \hookrightarrow \mathsf{S}_{n+m}$ with the isomorphism $\tilde{H}_*(X)^{\otimes n} \otimes \tilde{H}_*(X)^{\otimes m} \cong \tilde{H}_*(X)^{\otimes n+m}$. Geometrically, the product on $D(X)$ and $C(X)$ can be defined through an embedding $\mathbb{R}^\infty \sqcup \mathbb{R}^\infty \to \mathbb{R}^\infty$, through which one can take the image of the union of configurations with labels in X. Since it arises from a homotopy commutative multiplication on spaces we also call this product the Pontrjagin product.

Since the coproduct and the transfer product on cohomology form a bialgebra, then for example if the evaluations of cohomology classes a_i on homology x_j are Kronecker – that is, $\langle a_i, x_j\rangle = \delta_{i,j}$ – and the a_i are primitive, then the evaluation $\langle \ _i a_i, *_i x_i\rangle$ will be one. But if some classes are repeated then coefficients are introduced, as for example $\langle \ _n a, x^{*n}\rangle = n!$. $\langle a^{[n]}, x^{*n}\rangle$ is one, as it is given by the tensor product at the chain and cochain level. Therefore, the divided powers operations in cohomology "fill in" to produce duals in these cases.

Just as the inclusions $\mathsf{S}_n \times \mathsf{S}_m \hookrightarrow \mathsf{S}_{n+m}$ give rise to the Pontrjagin product, the inclusions of wreath products $\mathsf{S}_n \wr \mathsf{S}_k \hookrightarrow \mathsf{S}_{nk}$ give rise to the algebraic Kudo-Araki-Dyer-Lashof operations.

Definition 3.1 (Compare [Ber]). Use $R_{\mathsf{S}_n} \otimes (R_{\mathsf{S}_n})^{\otimes k}$ as a resolution for $\mathsf{S}_n \wr \mathsf{S}_k \hookrightarrow \mathsf{S}_{nk}$. Let W_* be the standard minimal $\mathbb{F}_p$ resolution of C_p, the cyclic group of order p and e_i a generator for W_i. To every chain $e \in W_*$ we associate an operation $q(e)^{\#} : R_{\mathsf{S}_n} \to R_{\mathsf{S}_{pn}}$ as sending a chain c to the image of $e \otimes c^p$ under the map of resolutions induced by the inclusion of $C_p \wr \mathsf{S}_n \hookrightarrow \mathsf{S}_{np}$.

We consider the linear morphisms

$$q(e)\colon H_*(\mathsf{S}_n; \bar{H}_*(X)^{\otimes n} \otimes \rho) \to H_*(\mathsf{S}_{pn}; \bar{H}_*(X)^{\otimes pn} \otimes \rho)$$

induced by the map

$$c \otimes (x_1 \otimes \cdots \otimes x_n) \mapsto q(e)^{\#}(c) \otimes (x_1 \otimes \cdots \otimes x_n)^{\otimes p}.$$

The **Kudo-Araki-Dyer-Lashof (KADL)** operations are linear morphisms

$$q_i = q(e_{i(p-1)})\colon H_m(\mathsf{S}_n; \bar{H}_*(X)^{\otimes n} \otimes \rho) \to H_{pm+i(p-1)}(\mathsf{S}_{pn}; \bar{H}_*(X)^{\otimes pn} \otimes \rho).$$

Geometrically, q_i on a homology class of X (the $n = 1$ case) when i is odd is represented in CX by a family of configurations with labels where the p points in the configuration are on the $i(p-1)$-sphere related by the action of a p-th root of unity, with labels all in the same cycle on X.

We also remark that our indexing differs from [Ber, Definition 6.3] by a factor of $(p-1)$.

For $p = 2$ we consider r-tuples $(i_1, \cdots, i_r)$. For odd primes, let β be the Bockstein homomorphism. We associate to a $2r$-tuple $I = (\varepsilon_1, i_1, \ldots, \varepsilon_r, i_r)$, with $\varepsilon_k \in \{0, 1\}$ and all $i_k \geq 0$, the homology operation $q_I = \beta^{\varepsilon_1} q_{i_1} \circ \cdots \circ \beta^{\varepsilon_r} q_{i_r}$. We say that I is **admissible** if $i_k \leq i_{k+1} - \varepsilon_{k+1}$ (respectively $i_k \leq i_{k+1}$ when $p = 2$) for all $1 \leq k < r$ and, in case $p > 2$, $i_k \equiv i_{k+1} - \varepsilon_{k+1} \bmod 2\ \forall 1 \leq k < r$. We say that I is **strongly admissible** if it is admissible and $i_1 > 0$. This is a reformulation of Bernard's admissibility conditions, in Section 7 of [Ber], with the lower indices notation.

Theorem 3.2 (Cohen–Lada–May [CLM76]). *The mod-p homology of CX is a free graded commutative algebra with respect to the Pontrjagin product, generated by $q_I(x)$ where x ranges over a graded basis for the reduced homology of X and I ranges over strongly admissible sequences, with the additional requirement that for p odd i_r and the homological degree of x have the same parity, where $I = (\varepsilon_1, i_1, \ldots, \varepsilon_r, i_r)$.*

Similarly, the homology of CX with twisted coefficients given by the mod p sign representation for p odd can be computed as an algebra using the twisted versions of Dyer-Lashof operations.

Theorem 3.3 (Bernard [Ber])**.** *For p odd, the homology of $\tilde{D}(X) = C(X_+)$ with coefficients in* sgn*, the mod p sign representation, is the free graded commutative algebra with respect to the homology product generated by $q_I(x)$, where x ranges over a graded basis for the homology of X and I ranges over strongly admissible sequences such that i_r and the homological degree of x have different parity, with $I = (\varepsilon_1, i_1, \dots, \varepsilon_r, i_r)$.*

Remark 3.4. Admissible, but not strongly admissible, KADL operations on CX can be retrieved by the identity $q_0(x) = x^{*p}$. Iterated non-admissible sequences of KADL operations can be computed by means of Adem relations. Both in [CLM76] and in [Ber], an "upper indices" notation is used, because homological degrees behave better and Adem relations have a better form. We use a "lower indices" notation, because it makes the argument for the computation of the dual module and the construction of Hopf ring generators more transparent. The two conventions differ only by coefficients and reindexing, and are essentially equivalent. We refer to the two papers cited above for the precise relations between the two notations.

Definition 3.5. Let p be an odd prime and let $k \geq 0$. Define R^t_k be the F_p-vector space spanned by the admissible KADL operations q_I where $I = (\varepsilon_1, i_1, \dots, \varepsilon_k, i_k)$ has length $2k$. Let $\mathsf{R}^t = {}_{k\geq 0} \mathsf{R}^t_k$.

For $p = 2$, let R_k be the F_2-vector space spanned by the admissible KADL operations q_I, where $I = (i_1, \dots, i_k)$ has length k.

In light of our main application, we consider an admissible operation q_I as acting on a 0-dimensional class. In this case, the subspace $\mathsf{R}_k \subseteq \mathsf{R}^t_k$ spanned by q_I such that i_k is even corresponds to the classical untwisted KADL operations, and the subspace $\mathsf{R}^{\mathsf{l}}_k \subseteq \mathsf{R}^t_k$ spanned by q_I such that i_k is odd corresponds to the twisted KADL operations. Precisely, by sending q_I to $q_I(\iota)$, for ι 0 $\in \overline{H_0(S^0)}$, we identify R_k as a subspace of $H_*(\mathsf{S}_{p^k}; \mathsf{F}_p) \subseteq H_*(D(S^0); \mathsf{F}_p)$ and $\mathsf{R}^{\mathsf{l}}_k$ as a subspace of $H_*(\mathsf{S}_{p^k}; \mathrm{sgn}) \subseteq H_*(D(S^0); \mathrm{sgn})$. The component of an operation $q_I \in \mathsf{R}^t_k$ is $n(q_I) = p^k$, and its homological degree is $d(q_I) = {}^{k}\,{}_{j=0}^{1-j} p\ (i_{j+1}(p-1) - \varepsilon_{j+1})$.

As a consequence of Theorem 3.2, $H_*(DS^0; \mathsf{F}_p) = H_*(CS^0; \mathsf{F}_p)$ has a basis given by Pontryagin monomials in strongly admissible KADL operations in $\mathsf{R} = {}_k \mathsf{R}_k$. We denote this basis with B, and we call it **Nakaoka basis**. Similarly, by Theorem 3.3, $H_*(DS^0; \mathrm{sgn})$ has a basis given by $*$-monomials in strongly admissible KADL operations in $\mathsf{R}^{\mathsf{l}} = {}_k \mathsf{R}^{\mathsf{l}}_k$, that we denote with $\mathsf{B}_{\mathrm{sgn}}$. In both B and $\mathsf{B}_{\mathrm{sgn}}$, operations with odd degree must not appear twice in the same monomial.

For $p = 2$, $\mathsf{R} = {}_k \mathsf{R}_k$ embeds as a subspace of $H_*(CS^0; \mathsf{F}_2)$ by sending q_I to $q_I(\iota)$, whose component is $n(q_I) = 2^k$ and whose homological degree is $d(q_I) = {}_{j=0}^{k-1} 2^j i_{j+1}$. Theorem 3.2 provides a Nakaoka monomial basis as above.

3.2. **Cohomology of symmetric groups with coefficients in the sign representation mod-p.** To complete the description of $H^*(DX; \mathsf{F}_p)$ as a Hopf ring in the odd primary case, we must first analyze the cohomology of the symmetric groups with coefficients in the sign representation, mod p. When we consider the cohomology of $X^n_{h\mathsf{S}_n}$, through say cellular chain models, a d-cell in X gives rise to a chain of dimension dn, and a permutation $\sigma \in \mathsf{S}_n$ acts on this chain as multiplication by $(-1)^{d\,\mathrm{sgn}(\sigma)}$. So for example $H^{*+nd}(D_n(S^d); \mathsf{F}_p)$ has a summand isomorphic to $H^*(\mathsf{S}_n; \mathrm{sgn}^d)$. While additive structures are isomorphic in the d even and odd cases, the product structures here differ significantly.

A first step in understanding a Hopf ring is through its ⊗-indecomposables, which by Hopf ring distributivity form a ring under the ⋅-product. In the bigraded component setting, this ring is a coproduct of the ⊗-indecomposables on each component. For the cohomology of symmetric groups these rings of ⊗-indecomposables form the foundation of our previous calculations [GSS12, Gue17], but the calculation of these rings goes back to the work of Cohen–Lada–May [CLM76], where they were called R_k^*.

Thus the focus of this section is Lemma 3.8, which consists of the calculation of the dual algebra of $\mathsf{R} \quad \mathsf{R}^{\mathsf{l}}$. While this computation for twisted operations is similar to the untwisted case, it has not been computed previously. Our calculation here is a straightforward generalization of arguments in Section I.3 of [CLM76], and as suggested in that text we use lower index notation.

Let R_k^* and $\mathsf{R}^{\mathsf{l}*}_k$ be the spans of linear duals $q_I^\vee$ of the operations q_I of length k respectively in $H^*(\mathsf{S}_{p^k}; \mathsf{F}_p)$ and $H^*(\mathsf{S}_{p^k}; \mathrm{sgn} \otimes \mathsf{F}_p)$ with respect to the Nakaoka bases. Because of the naturality of twisted KADL operations, the proof of Theorem 4.13 in [GSS12] also shows that the subspace of primitives with respect to the coproduct dual to the transfer product $P(\bigoplus_{n\geq 0} H_*(\mathsf{S}_n; A_n))$ is $\mathsf{R}^t = \mathsf{R} \quad \mathsf{R}^{\mathsf{l}}$. Because the transfer product forms a bialgebra with the coproduct dual to $*$, the dual classes $\{q_I^\vee\}$ for I admissible are a basis for our Hopf ring indecomposables. Moreover, each component of this module of ⊗-indecomposables is a graded commutative ring under cup product.

Recall that the coproduct $\Delta\colon H_*(C_p) \to H_*(C_p) \otimes H_*(C_p)$ dual to cup product in the homology of the cyclic group C_p induces a unique coproduct ψ on R^t such that

which encodes a Cartan formula $q_I(x \otimes y) = \sum_i \eta_i q_{J_i}(x) \otimes q_{K_i}(y)$. We refer to

$\psi(q(e)) = (q \otimes q)(\Delta(e))$. The coproduct ψ has the form $\psi(q_I) = \sum_i \eta_i q_{J_i} \otimes q_{K_i}$,
which encodes a Cartan formula $q_I(x \otimes y) = \sum_i \eta_i q_{J_i}(x) \otimes q_{K_i}(y)$. We refer to

[CLM76] for the precise construction and related calculations.

While the isomorphism $H^*(\mathsf{S}_n; \mathrm{sgn}) \cong H^*(D_n S^1; \mathsf{F}_p)$ guides us, we need to make a finer distinction for this calculation. While the transfer product and the coproduct are preserved by this isomorphism, the same does not happen for the cup product. By letting an operation q_I act on $[S^1]$ in $H_*(DS^1; \mathsf{F}_p)$ and composing with that isomorphism, we can nevertheless embed $\mathsf{R}^{\mathsf{l}}_k$ into $H_*(\mathsf{S}_{p^k}; \mathrm{sgn})$. Combining this with what we know about the homology of the symmetric group with trivial coefficients, we obtain a map $\kappa : \mathsf{R} \oplus \mathsf{R}^{\mathsf{l}} \to \bigoplus_n H_*(\mathsf{S}_n; \mathsf{F}_p \oplus \mathrm{sgn})$. By duality this provides a map of F_p-modules $\kappa^* : H^*(\mathsf{S}_{p^k}; \mathsf{F}_p \oplus \mathrm{sgn}) \to \mathsf{R}_k^* \oplus \mathsf{R}^{\mathsf{l}*}_k$. We compare the products in those rings.

Proposition 3.6. *If* $x \in H^d(\mathsf{S}_{p^k}; \mathrm{sgn}^\varepsilon)$ *and* $y \in H^{d^{\mathsf{l}}}(\mathsf{S}_{p^k}; \mathrm{sgn}^{\varepsilon^{\mathsf{l}}})$, *then*

$$\psi^*(\kappa^*(x) \otimes \kappa^*(y)) = (-1)^{\varepsilon(d^{\mathsf{l}} + \frac{p-1}{2} k\varepsilon^{\mathsf{l}})} \kappa^*(x \cdot y).$$

Proof. Let X be a space, and let $n = p^k$. Fix a graded basis B of the homology of X. We observe that the following diagram commutes, where d denotes diagonal maps, τ the obvious shuffle map, and q is the obvious quotient map.

$$\begin{array}{ccc} \tilde{D}_n(X) & \xrightarrow{\quad d_{\tilde{D}_n(X)} \quad} & \tilde{D}_n(X)^2 \\ \Big\downarrow{\scriptstyle d_{E(\mathsf{S}_n)} \times_{\mathsf{S}_n} d_{X^n}} & & \Big\downarrow{\scriptstyle \tau} \\ (E(\mathsf{S}_n) \times E(\mathsf{S}_n)) \times_{\mathsf{S}_n} (X^n \times X^n) & \xrightarrow{\quad q \quad} & (E(\mathsf{S}_n) \times E(\mathsf{S}_n)) \times_{(\mathsf{S}_n \times \mathsf{S}_n)} X^{2n} \end{array}$$

Consider $q_I(x) \in H_*(\tilde{D}_n(X))$, where $x \in H_*(X)$. Assume that $\Delta(x) = \sum_i \lambda_i x'_i \otimes x''_i$ in $H_*(X)$, for some $x'_i, x''_i \in B$. Write $\psi(q_I) = \sum_l \eta_l q_{J'_l} \otimes q_{J''_l}$. Let ϕ be the coproduct in homology dual to the cup product of $H^*(\mathsf{S}_n; \mathsf{F}_p \oplus \mathrm{sgn})$.

Taking homology and keeping track of the image of $q_I(x)$ in the upper path in the diagram above, we obtain

$$\tau_* \circ d_{\tilde{D}_n(X)_*}(q_I(x)) = \tau_* \left(\sum_{i,k} \lambda_i \eta_l q_{J'_l}(x'_i) \otimes q_{J''_l}(x''_i) \right)$$

$$= \sum_{i,l} (-1)^{|q_{J''_l}||x'_i|} \lambda_i \eta_l q_{J'_l} \otimes q_{J''_l} \otimes (x'_i)^{\otimes n} \otimes (x''_i)^{\otimes n}.$$

Similarly, for the lower path, we have

$$q_*(d_{E(\mathsf{S}_n)} \times_{\mathsf{S}_n} d_{X^n})_*(q_I(x))$$

$$= \sum_{i_1,\ldots,i_n} \prod_{j=1}^{n} \lambda_{i_j} (-1)^{\sum_{1 \le j < l \le n} |x''_{i_j}||x'_{i_l}|} d_{E(\mathsf{S}_n)*}(q_I) \otimes_{\mathsf{S}_n} \left(\bigotimes_{j=1}^{n} x'_{i_j} \otimes \bigotimes_{j=1}^{n} x''_{i_j} \right)$$

We can split this last summation as the sum over the set of n-tuples where all i_k are equal to each other (say i) and the sum over the set on n-tuples which are not all equal. The first part can be rewritten as

$$\sum_i \lambda_i (-1)^{\frac{n(n-1)}{2}|x'_i||x''_i|} d_{E(\mathsf{S}_n)*}(q_I) \otimes (x'^{\otimes n}_i \otimes x''^{\otimes n}_i),$$

which maps to $\sum_i \lambda_i (-1)^{\frac{n(n-1)}{2}|x'_i||x''_i|} \phi(q_I) \otimes x'^{\otimes n}_i \otimes x''^{\otimes n}_i$ under q. We claim that the second part is zero. Recall that there is an isomorphism $H_*((E(\mathsf{S}_n) \times E(\mathsf{S}_n)) \times_{\mathsf{S}_n} X^{2n}; \mathsf{F}_p) \cong H_*(\mathsf{S}_n; (H_*(X; \mathsf{F}_p)^{\otimes n})^{\otimes 2})$. The S_n-subrepresentation of $(H_*(X; \mathsf{F}_p)^{\otimes n})^{\otimes 2}$ generated by $(x'_{i_1} \otimes \ldots \otimes x'_{i_n}) \otimes (x''_{i_1} \otimes \cdots \otimes x''_{i_n})$ with $i_1, \ldots, i_n$ not all equal is isomorphic to the induced S_n-representation of a G-representation for a Young subgroup $G \subsetneq \mathsf{S}_n$. The corresponding terms in the sum above amount to $\mathrm{tr}^{\mathsf{S}_n}_G \rho^{\mathsf{S}_n}_G(q_I)$, which is multiplication by the index $[\mathsf{S}_n : G]$, which is zero modulo p.

In conclusion,

$$\sum_i \lambda_i (-1)^{|x'_i||q_{J''_l}|} \psi(q_I) \otimes (x'_i)^{\otimes n} \otimes (x''_i)^{\otimes n}$$

$$= \sum_i \lambda_i (-1)^{\frac{n(n-1)}{2}|x'_i||x''_i|} \phi(q_I) \otimes (x'_i)^{\otimes n} \otimes (x''_i)^{\otimes n}.$$

Since $n = p^k$, $\frac{n(n-1)}{2} \equiv \frac{p-1}{2}k \mod 2$. Hence, the equality above is true for all spaces X and for all classes $x \in H_*(X; \mathsf{F}_p)$ if and only if

$$\sum_l \eta_l \kappa(q_{J'_l}) \otimes \kappa(q_{J''_l}) = \sum_l \eta_l (-1)^{\varepsilon(|q_{J''_l}| + \frac{p-1}{2}k\varepsilon')} \phi\kappa(q_I)$$

for $\varepsilon, \varepsilon' \in \{0, 1\}$. Dualizing this we obtain the desired formula. □

To summarize, the coproduct dual to the cup product, when restricted on R^t, has the form

$l=1$ 2 J K

$$\phi(q_I) = \sum_{J+K=I} (-1)^{\sum_{l=1}^{n} \left(\frac{p-1}{2} j_l k_l + \delta_l j_l\right)} q_J \otimes q_K ,$$

where $J = (\varepsilon_1, j_1, \ldots, j_n)$ and $K = (\delta_1, k_1, \ldots, k_n)$.

The difficulty with this formula is that the sum is over all ways to decompose the sequence I, not just admissible sequences. Adem relations are needed to make calculations. The dual algebra is in the end manageable, though to this day (to our knowledge) even in the trivial coefficient setting the pairing between the Nakaoka basis on homology and the standard basis on cohomology from analysis such as we give below is not known.

To calculate the product dual to ϕ in the sign representation setting, we continue to follow the trivial coefficient treatment of [CLM76, Section I.3]. We consider sequences of operations with minimal entries.

Let $S \subseteq \{1, \dots, k\}$ and define $I_S[k]$ and $I^{!}_S[k]$ respectively as the sets of the admissible $2k$-tuples $I = (\varepsilon_1, i_1, \dots, \varepsilon_k, i_k)$ such that i_j is equal to $|S \cap \{1, \dots, k-j\}|$ (respectively $(|S \cap \{1, \dots, k-j\}| + 1)$) mod 2 and $\varepsilon_j = 1$ if and only if $k+1-j \in S$. Note that $\{q_I : I \in I_S[k]\} \subseteq \mathsf{R}_k$ and $\{q_I : I \in I^{!}_S[k]\} \subseteq \mathsf{R}^{!}_k$.

For all $k \in \mathsf{N}$ and $S \subseteq \{1, \dots, k\}$, we can define a partial order $\leq$ on $I_S[k]$ and $I^{!}_S[k]$ by comparison of all entries. This partially ordered set possesses all meets, obtained by taking the minimum on each entry. In particular, $I_S[k]$ and $I^{!}_S[k]$ have a minimal element, that we denote $L_{S,k}$ and $L^{!}_{S,k}$ respectively.

Although we will not need this, it is straightforward to write down the elements $L_{S,k}$ and $L^{!}_{S,k}$ explicitly. For instance, $L_{\{1,2\},3} = (0, 0, 1, 1, 1, 2)$. Up to switching to upper-index notation, the $(2k)$-tuples $L_{S,k}$ coincide with those defined in Section I.3 of [CLM76].

The following is straightforward combinatorics, following the same argument as Lemmas I.3.3 and I.3.4 in [CLM76].

Lemma 3.7. *For all $j, k \in \mathsf{N}$ such that $1 \leq j \leq k$, let*

$$I_{j,k} = (\underbrace{0, \dots, 0}_{2(k-j)}, \underbrace{0, 2, \dots, 0, 2}_{2j}) \in I_\varnothing[k].$$

The weighted sum $(n_1, \dots, n_k) \mapsto L^{!}_{S,k} + \sum_{j=1}^{k} n_j I_{j,k}$ is a bijection from $\mathsf{N}^k \to I^{!}_S[k]$ for all subsets S of $\{1, \dots, k\}$.

We can now compute the algebra dual to (R^t, ϕ).

Lemma 3.8. *Assume that p is odd. We consider the following dual classes in $(\mathsf{R}^t)^\vee$ with respect to the admissible sequences basis:*

- $\xi_{j,k} = q^\vee_{I_{j,k}}$ *for* $1 \leq j \leq k$
- $\xi^{!}_k = q^\vee_{L^{!}_{\varnothing,k}}$ *for* $k \geq 1$
- $\tau^{!}_{j,k} = q^\vee_{L^{!}_{\{j\},k}}$ *for* $1 \leq j \leq k$

As an $\mathsf{F}_p[\xi_{1,k}, \dots, \xi_{k,k}]$-module, $(\mathsf{R}^t)^\vee_k$ is isomorphic to $\mathsf{F}_p[\xi_{1,k}, \dots, \xi_{k,k}] \otimes (M_k \quad M^{!}_k)$, where

- *M_k is the F_p-vector space with basis $\{q^\vee_{L_{S,k}}\}_{S \subseteq \{1,\dots,k\}}$, as in Cohen–Lada–May, and*
- *$M^{!}_k$ is the F_p-vector space with basis $\{q^\vee_{L^{!}_{S,k}}\}_{S \subseteq \{1,\dots,k\}}$.*

As a graded commutative algebra, $(\mathsf{R}^t)^\vee$ is generated by $\xi_{j,k}$, $\xi^{!}_k$ and $\tau^{!}_{j,k}$ with $1 \leq j \leq k$, under the relations

$${\xi^{!}_k}^2 = \xi_{k,k}.$$

Proof. Theorem I.3.7 in [CLM76] states that $\mathsf{R}^\vee_k$ is additively isomorphic to $\mathsf{F}_p[\xi_{1,k}, \dots, \xi_{k,k}] \otimes M_k$. The exact argument used to prove that result, with Lemma

3.7 in place of Lemma I.3.4 of [CLM76], yields an additive isomorphism $\mathsf{R}^{!\vee}_k \cong \mathsf{F}_p[q^\vee_{I_{j,k}}]_{1\leq j\leq k} \otimes M^!_k$. In particular, $\mathsf{F}_p[\xi_{1,k},\ldots,\xi_{k,k}]$ is the dual of the quotient coalgebra $\mathsf{R}_\varnothing[k]$. Let $+$ denote component-wise addition of $2k$-tuples. One can check that

$$L_{S,k} = \begin{cases} \sum_{s\in S} L^!_{\{s\},k} & \text{if } |S| \text{ is even} \\ \sum_{s\in S} L^!_{\{s\},k} + L^!_{\varnothing,k} & \text{if } |S| \text{ is odd} \end{cases},$$

$$L^!_{S,k} = \begin{cases} \sum_{s\in S} L^!_{\{s\},k} + L^!_{\varnothing,k} & \text{if } |S| \text{ is even} \\ \sum_{s\in S} L^!_{\{s\},k} & \text{if } |S| \text{ is odd} \end{cases}.$$

This is true because the function $- + L^!_{\{f\},k} : I_S[k] \to I^!_{S\cup\{f\}}[k]$ (respectively $- + L^!_{\{f\},k} : I^!_S[k] \to I_{S\cup\{f\}}[k]$) is an order-preserving bijection for all $S \subseteq \{1,\ldots,f-1\}$ of even (respectively odd) cardinality, thus it must preserve minimal elements. Since application of Adem relations to a sequence q_I produces elements q_J with $J > I$ and $L_{S,k}, L^!_{S,k}$ are minimal, the argument in the proof of [CLM76, I.3.7] can be used to prove that the pairing between products of the form $\prod_{s\in S}\tau^!_{s,k}$ (if $|S|$ is even) or $\xi^!_k \prod_{s\in S}\tau^!_{s,k}$ (if $|S|$ is odd) and the $q_{L^!_{S,k}}$ is perfect. This implies by Lemma 3.7 that $\xi^!_k$ and $\tau^!_{j,k}$ with $1 \leq j \leq k$ generate $(\mathsf{R}^t_k)^\vee$ as an $\mathsf{F}_p[\xi_{1,k},\ldots,\xi_{k,k}]$-algebra. We can check the relations $\xi^{!\,2}_k = \xi_{k,k}$ between the generators also following [CLM76]. Explicitly, since $I_{k,k} = L^!_{\varnothing,k} + L^!_{\varnothing,k}$, $(\xi^{!\,2}_k, q_{I_{k,k}}) = 1$. Tracking degrees, the only possible summand of $\xi^{!\,2}_k$ is $q^\vee_{I_{k,k}}$, hence the relation holds. By comparing dimensions, we see that these provide a presentation of $(\mathsf{R}^t_k)^\vee$ as a graded commutative algebra. □

Remark 3.9. We recover the elements $\tau_{j,k}$ and $\sigma_{i,j,k}$ defined at page 28 in Cohen, Lada and May's book by the identities $\tau_{j,k} = \pm\xi^!_k\tau^!_{j,k}$ and $\sigma_{i,j,k} = \pm\tau^!_{i,k}\tau^!_{j,k}$. These are indecomposables in $\mathsf{R}^\vee_k$, but not in $(\mathsf{R}^t)^\vee_k$.

We now provide an additive basis for the twisted cohomology of the symmetric groups. In Definition 3.10, gathered blocks are elements of the universal Hopf ring of the statement of Theorem 2.35.

Definition 3.10. Given a subset $S \subset \{1,\ldots,k\}$, define $\gamma^!_{S,k} = q^\vee_{L^!_{S,k}}$, where linear duals are taken with respect to the Nakaoka monomial basis. Similarly, define $\gamma_{S,k} = q^\vee_{L_{S,k}}$. We also define:

- $\lambda_k = \gamma^!_{\varnothing,k}$
- $\gamma_{j,m} = (q^{*m}_{I_{j,j}})^\vee$
- $\gamma_k = \gamma_{k,1}$
- $\gamma^!_{i,m} = (q^{*m}_{L^!_{\{i\},i}})^\vee$
- $\gamma^!_k = \gamma^!_{k,1}$

Let $S = \{s_1 < s_2 \cdots < s_r\} \subset \{1,\ldots,k\}$ and let $D = \{n_i\}_{i=1}^k$ be a finite sequence of non-negative integers of length k. If $|S|$ is odd let $m = 1$, but if it is even let $m \geq 1$ and assume that $k \in S$ or $n_k > 0$.

Set

$$\Gamma_{S,D,m} = \prod_{i=1}^{k} \left(\gamma^{[mp^{k-i}]}_{\varnothing,i}\right)^{n_i} \prod_{j=1}^{r} (\gamma^!_{s_j})^{[mp^{k-s_j}]}\lambda_k^{\varepsilon(r)},$$

where $\varepsilon(r)$ is 0 if r is even and 1 if r is odd.

Similarly, we define

$$\Gamma'_{S,D,m} = \prod_{i=1}^{k} (\gamma^{[mp^{k-i}]}_{\varnothing,i})^{n_i} \prod_{j=1}^{r} (\gamma'_{s_j})^{[mp^{j-s_j}]} \lambda_k^{1-\varepsilon(r)},$$

for $S \subseteq \{1,\ldots,k\}$, with S and n_i as above, but now if $|S|$ is even then $m = 1$ and if $|S|$ is odd then $k \in S$ or $n_k > 0$.

We call $\Gamma_{S,D,m}$ and $\Gamma'_{S,D,m}$ **gathered blocks** with **profile** (S, D) in $_nH^*(\mathsf{S}_n; \mathsf{F}_p)$ or $_nH^*(\mathsf{S}_n; \mathrm{sgn})$ respectively.

We define our preferred **Hopf monomials** in $H_{EP}{}^*(S^0)$ as a ⊙ product of divided powers of primitive gathered blocks with pairwise different profiles, all belonging to $_nH^*(\mathsf{S}_n; \mathsf{F}_p)$ or all belonging to $_nH^*(\mathsf{S}_n; \mathrm{sgn})$.

The product of a collection of $\gamma^{[p^{n_i}]}_{S_i}$ only depends, up to sign, on the union of the S_i as well as the list, with multiplicity, of the n_i.

We observe that $\Gamma_{S,D,m}$ and $\Gamma'_{S,D,m}$ are primitive if and only if $m = 1$.

These gathered blocks will be the building blocks of Hopf monomial bases for the cohomology we compute, and can be assembled into "skyline diagrams." The reader is encouraged to compare this with the notion of gathered block in the classical mod p cohomology of the symmetric groups, as defined in Guerra's paper [Gue17] at page 964.

Lemma 3.11. *The universal Hopf ring described by the statement of Theorem 2.35 is spanned as an F_p-vector space by Hopf monomials.*

Proof. By unraveling the definition of the universal object, we directly see that a complete set of relations for the considered component super-Hopf rig between the generators $\gamma^{[n]}_k$, $\gamma'^{[n]}_i$, λ_k is the following:

- $(\lambda_k)^2 = \gamma_k$
- the ⊙ product of a class in $H^*(\mathsf{S}_n; \mathsf{F}_p)$ and a class in $H^*(\mathsf{S}_n; \mathrm{sgn})$ is 0.
- $\Delta(\lambda_k) = 1 \otimes \lambda_k + \lambda_k \otimes 1$
- $\Delta(\gamma^{[n]}_k) = \sum_i \gamma^{[i]}_k \otimes \gamma^{[n-i]}_k$
- $\Delta(\gamma'^{[n]}_k) = \sum_i \gamma'^{[i]}_k \otimes \gamma'^{[n-i]}_k$
- $\gamma^{[n]}_k \odot \gamma'^{[m]}_k = \binom{n+m}{n} \gamma'^{[m+n]}_k$
- $\gamma^{[n]}_k \odot \gamma^{[m]}_k = \binom{n+m}{n} \gamma^{[m+n]}_k$
- the ⊙ product of elements with different sign degree is 0

Since our Hopf rig is generated by the classes above, monomials in both products (· and ⊙) generate it as a vector space. By Hopf ring distributivity we can reduce to ⊙-products of ·-monomials of generators. Moreover, we can further restrict to considering ·-monomials of generators all belonging to the same component and we can discard monomial in which λ_k appears at least twice thanks to the relation $\lambda_k^2 = \gamma_k$. All the relations above preserve profile of such ·-monomials and can be used to equate any product which has the same profile as a gathered block. Hence, Hopf monomials generate our Hopf ring as a vector space. □

We can now prove the main result of this section.

Proof of Theorem 2.35. Since q_0 is the p^{th} power in the mod p homology of $D(S^0)$ with coefficients in ρ, then $\gamma_{j,p^{k-j}} = (q_{I_{j,k}})^\vee$ and $\gamma'_{k-i} = \gamma'$.

i,p $\{i\},k$

We use Proposition 3.6 and Lemma 3.8 to deduce our cup product relations. The structure of $\mathsf{R}_k^\vee$ is described by Theorem I.3.7 in [CLM76]. In this reference, the product relations involve dividing by $\gamma_{\varnothing,k}$, but they are equivalent to the identities stated in [Gue17], of whose our relations (as far as $\mathsf{R}_k^\vee$ is concerned) are straightforward reformulations that use lower indexes. For the cohomology with coefficients in the sign representation, a similar argument using Lemma 3.8 yields our description.

We first observe that $\gamma_{j,p^{k-j}}$, $\gamma^!_{i,p^{k-i}}$ and λ_k are liftings of the generators $\xi_{j,k}$, $\tau^!_{i,k}$ and $\xi^!_k$ of $(\mathsf{R}^t_k)^\vee$.

We first prove by induction on $m \geq 1$ that $\gamma^{![m]}_k = \gamma^!_{k,m}$. The base of the induction ($m = 1$) is obvious, so we assume $m > 1$. Since the product $*$ in homology is linear dual to the coproduct Δ in $H_{EP}{}^*(S^0)$, a straightforward calculation by induction on i yields the coproduct formula:

$$\Delta(\gamma^!_{k,m}) = \sum_{i+j=m} \gamma^!_{k,i} \otimes \gamma^!_{k,j}.$$

Similarly, using the compatibility between coproduct and divided powers, we deduce by induction that

$$\Delta(\gamma'^{![m]}_k) = \sum_{j=0}^{m} \gamma'^{![j]}_k \otimes \gamma'^{![m-j]}_k.$$

By combining these two coproduct identities with the induction hypothesis we deduce that the difference between $\gamma'^{![m]}_k$ and $\gamma'^!_{k,m}$ must be primitive. If m is not a power of p, then there are no 8-indecomposables in the right component. If $m = p^l$, the primitives are determined by Lemma 3.8 and there is none in the correct dimension. In all cases, this difference must be zero. This shows that $\gamma^!_k$ is standard non-nilpotent. A similar argument shows that $\gamma^{[m]}_k = \gamma_{k,m}$ and γ_k is standard non-nilpotent.

It is obvious from its definition that λ_k is primitive. The cup-product relation $(\lambda_k)^2 = \gamma_k$ holds because both sides are primitive, hence 8-indecomposables, and because this identity holds in $(\mathsf{R}^t_k)^\vee$. Then our cup-product relations are a reformulation of those in $(\mathsf{R}^t_k)^\vee$.

Let H be the universal Hopf rig described in the statement of Theorem 2.35. Since we have checked that all the required relations hold in $H_{EP}{}^*(S^0)$, there must be a unique Hopf rig map $\pi \colon H \to H_{EP}{}^*(S^0)$ compatible with our choice of generators. As the ring of 8-indecomposables in $\bigoplus_{n\geq 0} H^*(\mathsf{S}_n; \rho)$ is the linear dual of R^t, which is generated by the images of $\gamma_i^{[p^k]}$, $\gamma^{![p^k]}_i$ and λ_k by Lemma 3.8, $H_{EP}{}^*(S^0)$ is generated by the classes defined at the beginning of this proof and, consequently, π is surjective. Hence, by Lemma 3.11, the images of Hopf monomials generate $H_{EP}{}^*(S^0)$ as an F_p-vector space. To prove that π is an isomorphism, it is enough to check that they form a basis, which can be done by comparing dimension degree-wise with the Nakaoka basis in homology, as done in [Gue17, Theorem 2.7]. D

As a byproduct of this proof we directly obtain the following.

Corollary 3.12. *The set of Hopf monomials in $H_{EP}{}^*(S^0)$, defined in Definition* 3.10, *is a tri-graded additive basis for $H_{EP}{}^*(S^0)$ as an F_p-vector space.*

4. Cohomology rings of extended powers with basepoints

We are now ready to establish our main calculations. In Section 2 we developed the algebraic framework which governs the cohomology rings of extended powers, when considered together. Such a framework demands twisted coefficients, so in Section 3 we extended well-known results about homology to that setting. We now show that the cohomology generated within the framework of component Hopf rings with divided powers pairs perfectly with homology. We develop an additive basis for the former before proving the pairing result.

4.1. Additive basis for the cohomology of DX_+.

We describe explicit additive bases of the super-Hopf ring with additive divided powers given Theorems 2.37 and 2.38. We use these to pair with homology. Our basis is useful to perform concrete calculations, because we describe an algorithm to compute the two products and the coproduct in terms of it.

We primarily combine results from Section 2.6 with Lemma 3.11. We apply Lemma 2.44 when $A = H_{EP}{}^*(S^0)$ and $V = H^*(X; \mathsf{F}_p)$, so that a basis for $Q(A)$ is given by gathered blocks of component equal to a power of p by Corollary 3.12. The basis provided by Lemma 2.44 consists of such gathered blocks, with the extra datum of a class in the cohomology of X. Informally, by the Hopf ring relations in A, the transfer product of two such gathered blocks with the same profile is again another gathered block. Therefore, we can merge $\otimes$-factors with the same profile and the same extra cohomology class of X and remove the constraint on the component. We formalize this as follows.

Definition 4.1. Let X be a topological space and let B be a graded additive basis for $H^*(X; \mathsf{F}_p)$. A **decorated gathered block** on B is a pair (x, b), where b is a gathered block in $\bigoplus_{n\geq 0} H^*(\mathsf{S}_n; A_n)$ and $x \in \mathsf{B}$. The **profile** of the decorated

a gathered block in

gathered block (x, b) is the profile of b, and its **decoration** is x. Its *sign degree* is 0 if the degree of x and the sign degree of b have the same parity, is 1 if they have different parity. A **decorated Hopf monomial** on B is a formal expression of the form $b_1 \otimes \cdots \otimes b_r$, where $b_1, \ldots, b_r$ are decorated gathered blocks on B, up to permutation of the b_i, where no two b_is have the same profile and decoration, and where all the b_is have the same sign degree.

Our previous results assemble as follows.

Proposition 4.2. *Let X be a topological space and let B be a graded additive basis for $H^*(X; \mathsf{F}_p)$. Let $A = \bigoplus_{n\geq 0} H^*(\mathsf{S}_n; A_n)$ and let $V = H^*(X; \mathsf{F}_p)$. Let H_{alg} be the universal object among Hopf ring with additive divided powers satisfying the conditions of the statement of Theorem 2.38. Let M be the set of decorated Hopf monomials on B. We map M to H_{alg} by realizing a decorated gathered block (x, b) as $\iota(x)^{[n(b)]}\ \pi(b)$, and a decorated Hopf monomial as the $\otimes$product in H_{alg} of the constituent gathered blocks. Then M is an additive basis for H_{alg} as an F_p-vector space.*

We can give a graphical description of our additive basis using a mild generalization of Giusti, Salvatore, and Sinha's skyline diagrams. Moreover, all our structural morphisms can be understood graphically. We first consider undecorated skyline diagrams, which correspond to classes in $H_{EP}{}^*(S^0)$.

First, we recall some definitions from [GSS12] and [Gue17].

Definition 4.3. A **skyline diagram** is a display of columns placed one next to the other horizontally. Each column is comprised of rectangular **boxes** with the same width stacked one on top of the other. The possible dimensions of these boxes depend on a prime p:

- If $p = 2$, a column can be made of any number of boxes of width $n2^k$ and height $1 - 2^{-k}$, for some $k, n \in \mathbb{N}$, $n \geq 1$.
- If $p > 2$, a column can be made of any number of hollow rectangles of width np^k and height $2(1 - p^{-k})$ and at most one solid box of width np^k and height dependent on a subset $S \subseteq \{1, \dots, k\}$. To simplify notation, we let $|S| = 2a + \varepsilon$, with $a \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$. For any choice of k and S, we have two possible heights: $\sum_{s\in S} p^{s-k} - 2(|S| + \varepsilon)p^{-k} + \varepsilon$ and $\sum_{s\in S} p^{s-k} - 2(|S| + 1 - \varepsilon)p^{-k} + 1 - \varepsilon$.

  If a box has height given by the first formula, we say that it is of **even type**, while we say that it is of **odd type** otherwise. Boxes of even type are always defined if the subset has even cardinality $|S| = 2a$, while they are allowed for $|S| = 2a + 1$ only if $n = 1$. Boxes of odd type are always defined if the subset has odd cardinality $|S| = 2a + 1$, while they are allowed for $|S| = 2a$ only if $n = 1$.

  We say that a column is of even type if its solid box is of even type or is absent, and we say that it is of odd type if its solid box is of odd type.

We require that a skyline diagram does not contain two columns made of boxes with matching heights, and that its constituent columns are all of the even type or all of odd type. We do not keep track of the order in which columns are placed or in which boxes are stacked inside each column, so two skyline diagrams that differ by a permutation of their columns are considered the same.

Each fundamental box should be interpreted as a class in $\bigoplus_{n\geq 0} H^*(S_n; F_p)$ or $\bigoplus_{n\geq 0} H^*(S_n; \mathrm{sgn})$, in the component corresponding to its width. Precisely:

- if $p = 2$, the rectangle of dimensions $n2^k$ and $1 - 2^{-k}$ corresponds to $\gamma_k{}^{[n]}$
- if $p > 2$, we regard the hollow rectangle whose width is np^k and whose height is $2(1 - p^{-k})$ as $\gamma_k{}^{[n]}$
- if $p > 2$, the solid box of even (respectively odd) type with width np^k associated to the subset S corresponds to the gathered block $\Gamma_{S,0,n}$ (respectively $\Gamma'_{S,0,n}$), as defined in Definition 3.10.

A column is understood as the cup product of the constituent fundamental boxes, and a general skyline diagram is interpreted as the transfer product of its columns. With these correspondences, the elements of the additive bases for $\bigoplus_{n\geq 0} H^*(S_n; F_2)$ and $\bigoplus_{n\geq 0} H^*(S_n; F_p)$ ($p > 2$) obtained in the papers mentioned above are described as those skyline diagrams (with columns of even type if $p > 2$). For example, in Figure 1 we depict the skyline diagrams corresponding to $\Gamma_{\emptyset,\{3\},1} \otimes \Gamma_{\emptyset,\{0,1\},1} = \gamma_1^3 \otimes \gamma_2 \in H^*(S_6; F_2)$ and $\Gamma_{\{1\},\{1,0\},1} \otimes \Gamma_{\emptyset,\{2\},1} = \gamma_1{}^{[3]}\gamma_1'{}^{[3]}\lambda_2 \otimes \gamma_1^2 \in H^*(S_{12}; F_3)$.

It is sometimes useful to consider vertical dashed lines dividing some boxes into equal parts. Explicitly, boxes corresponding to $\gamma_k{}^{[n]}$ are divided into n parts of width p^k and, if $p > 2$, even-dimensional solid boxes of even type and odd-dimensional solid boxes of odd type corresponding to a subset S are divided into parts of width $p^{\max(S)}$.

$\gamma_1^3 \otimes \gamma_2$

$\gamma_1^{[3]}\gamma_1'^{[3]}\lambda_2 \otimes \gamma_1^2$

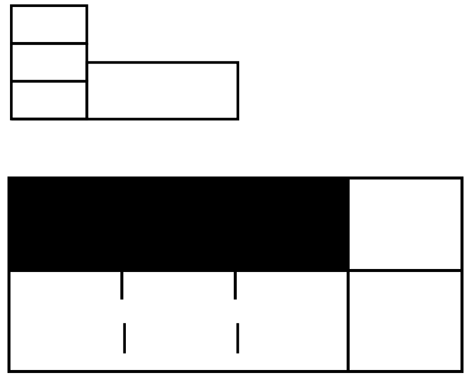

Figure 1. Examples of skyline diagrams in $H^*(\mathsf{S}_6; \mathsf{F}_2)$ (top) and $H^*(\mathsf{S}_{12}; \mathsf{F}_3)$ (bottom)

Proposition 4.4. *Hopf monomials in $H_{EP}{}^*(S^0)$ are in bijective correspondence with skyline diagrams as defined above.*

Proof. We already noted that solid blocks correspond to gathered blocks $\Gamma_{S,D,n}$ or $\Gamma'_{S,D,n}$ and hollow rectangles correspond to $\gamma_k^{[n]}$. Hence, the set of all possible columns is in bijective correspondence with the set of gathered blocks, with taking the cup product of the classes corresponding to the stacking of constituent boxes. Similarly, a skyline diagram gives rise to a Hopf monomial in $H_{EP}{}^*(S^0)$ by taking the transfer product of the gathered blocks corresponding to its constituent columns.

To define this precisely, one needs to fix an ordering of the boxes and the columns for each skyline diagram, because cup product and transfer product are only commutative up to sign. With that subtlety, it is straightforward to check directly from the definitions that this is a bijection. □

We next describe our generalization of skyline diagrams for $H_{EP}{}^*(X_+)$.

Definition 4.5. Let p be a prime number and V be a graded algebra over F_p. A **column decorated in** V is a column diagram as in Definition 4.3, together with an element $x \in V$. Graphically, we depict the decoration of the column as an additional rectangle labeled x with width equal to its width and height equal to the degree of x, placed beneath that column. When $p > 2$, we say that a decorated column is of even type if the type of underlying undecorated column and the dimension of the decoration are both even or both odd. We say that it is of odd type otherwise.

A **skyline diagram decorated in** V is a diagram constituted by columns decorated in V placed one next to each other horizontally, in which there are not two columns made of blocks of corresponding heights and having the same decoration and in which the constituent columns are all of even type or all of odd type.

If $\mathcal{B}$ is an additive basis for V as a graded F_p-vector space, then we define a **skyline diagram decorated in** $\mathcal{B}$ as a skyline diagram decorated in V whose decorations are elements of that basis.

We now assume $V = H^*(X; \mathbb{F}_p)$.

Proposition 4.6. *Let p be a prime number and let X be a topological space and $\mathcal{B}$ be a basis for $H^*(X; \mathbb{F}_p)$. There is a bijective correspondence between the set of skyline diagrams decorated in $\mathcal{B}$ and Hopf monomials with respect to the given basis in $H_{EP}{}^*(X_+)$ as in Definition 4.1. Stacking boxes on top of each other corresponds to taking the cup product of the associated cohomology classes (restricted to the relevant component) and placing columns next to each other horizontally corresponds to taking their transfer product.*

Proof. If c is a column with decoration x, we associate with it the decorated gathered block $\Gamma(c)\, x^{[w]}$, where w is the width of the column and $\Gamma(c)$ is the gathered block in $H_{EP}{}^*(S^0)$ corresponding to c. We then associate to a decorated skyline diagram the transfer product of the decorated gathered blocks corresponding to its constituent decorated columns. It is easy to deduce from Proposition 4.4 that this is a bijection. D

By taking only decorated skyline diagrams whose columns are all of even type, we obtain a basis for $\bigoplus_{n\geq 0} H^*(\tilde{D}_n(X); \mathbb{F}_p)$.

Our next aim is to describe a way to compute the structural morphisms via skyline diagrams. The rough graphical interpretation of the cup product of gathered blocks is "splitting and piling the corresponding columns one on top of the other." However, if $p > 2$, this procedure does not always yield an "allowable" column. Hence, we must use our cup product relations in $H^*(\mathfrak{S}; \mathbb{F}_p)$ or $H^*(\ _n\mathfrak{S}\mathbb{F}_p \ \mathrm{sgn})$ to write that stack of rectangles as a multiple of an allowable column as follows.

Proposition 4.7. *The cup product of classes in $H_{EP}{}^*(X_+)$ represented by single columns of width $np^k, p \nmid n$, proceeds as follows:*

- *We stack the hollow boxes of the two columns, and we substitute the two solid parts (if any) with a single solid rectangle whose height is the sum of the height of the two original solid boxes*
- *Assume that the two solid boxes correspond to two increasing subsets $S = \{s_1 < \ldots < s_k\}$ and $S' = \{s'_1, \ldots, s'_l\}$. We multiply the result by a scalar coefficient λ. If the two sequences have a common entry, $\lambda = 0$. Otherwise, it is equal to ± 1, the sign being that of the permutation that puts the sequence $(s_1, \ldots, s_k, s'_1, \ldots, s'_l)$ in increasing order.*
- *When $n = 1$ and the given columns are odd-dimensional and of even type or even-dimensional and of odd type, we add a hollow rectangle of height $2(1\ p^k)$ and lower the solid part accordingly to preserve the total height of the column.*

The cup product of columns decorated in $H^*(X)$ is obtained by cup-multiplying the underlying non-decorated columns and choosing as decoration the cup product of the original decorations.

We are now ready to describe the two products ⋅, $\otimes$, and the coproduct Δ graphically:

- We obtain the transfer product of two skyline diagrams by juxtaposing them horizontally. If there are two columns of width n and m having the same decoration and boxes with the same heights (i.e., gathered blocks with the same profile), we merge them into a single column whose width is $n+m$, with a coefficient $\binom{n+m}{n}$, or 0 if $p > 2$ and the two decorated columns are odd-dimensional.

- The coproduct of a Hopf monomial is computed via the fact that the generators are standard non-nilpotent of primitive and the bialgebra property of Δ and $\otimes$. Graphically, this corresponds to drawing vertical dashed lines inside the rectangles as explained above, separating each column into two along vertical lines (dashed or not) of full height in all the possible ways, and arranging the new columns into two diagrams. Each new column is assigned the decoration of the old column from which it arises. Equivalently, one can divide rectangles corresponding to $x^{[n]}$, $x \in H_{EP}{}^*(X_+)$ evenly in parts of width 1 using internal dashed lines.
- The cup product of two skyline diagrams is obtained by first drawing all the dashed lines internal to the fundamental boxes. Then, divide the two diagrams into columns, either using the original boundaries or along the dashed lines that run entirely through the height of an existing column. Finally, if possible match the columns arising from the first diagram bijectively with those coming from the second, respecting width, and cup-multiply the matched columns as in Proposition 4.7 and use them to create a new skyline diagram (with a suitable coefficient if $p > 2$).
- We remark that there is no clear graphical description of the divided power operations.

4.2. **Analyzing the pairing.** To prove our main theorem we need a preliminary technical lemma.

Lemma 4.8. *Let $\iota_X \colon H^*(X; \mathbb{F}_p) \otimes \rho \to H_{EP}{}^*(X_+)$ be the homomorphism that identifies the domain with the component-1 algebra of the codomain. Let $\pi : H_{EP}{}^*(S^0) \to H_{EP}{}^*(X_+)$ the morphism naturally induced by the terminal map $X \to \{*\}$. For all $m \in \mathbb{N}$, for all $x \in H^*(X; \mathbb{F}_p)$ the following statements are true.*

(1) For all $\alpha_i \in H_*(X; \mathbb{F}_p)$, the Kronecker pairing between homology and cohomology satisfies

$$\langle \iota_X(x)^{[n]}, \alpha_1 * \cdots * \alpha_n \rangle = \pm \prod_{i=1}^{n} \langle \iota_X(x), \alpha_i \rangle,$$

while

$$\langle \iota_X(x)^{[n]}, q_{I_1}(\alpha_1) * \cdots * q_{I_r}(\alpha_r) \rangle = 0$$

if $I_1, \dots, I_r$ are admissible sequences of KADL operations with at least one q_{I_i} different from $q_0 \dots q_0$.

(2) For all $x, x' \in H^*(X; \mathbb{F}_p) \otimes \rho$ (both of even total degree if $p > 2$), $\iota_X(x)^{[n]} \cdot \iota_X(x')^{[n]} = \iota_X(xx')^{[n]}$

(3) For all $y \in H^*(\mathfrak{S}_n; \rho)$, $(\iota_X(x)^{[n]} \cdot \pi(y))^{[m]} = \iota_X(x)^{[nm]} \cdot \pi(y)^{[m]}$.

Proof. We provide the proof for $p > 2$. The case $p = 2$ can be proved in the same way, without the complication of the sign representation. The statement for $n = 1$ is trivial, thus we assume that $n \geq 2$.

The first statement follows from the formula for the coproduct of divided powers and the fact that in component $n = 1$ the Kronecker pairing coincides with the Kronecker pairing between $H^*(X; \mathbb{F}_p)$ and $H_*(X; \mathbb{F}_p)$. For the last statement, exploiting again the coproduct formula of divided powers, we reduce to the case $r = 1$.

Thus we only need to prove that $(\iota_X(x)^{[p^k]}, q_I(\alpha)) = 0$ if I is an admissible sequence of KADL operations of length k different from $q_0 \dots q_0$. To prove this, we pick a suitable cochain representative of $\iota_X(x)^{[m]}$. Assume that $x \in H^d(X; \mathbb{F}_p) \otimes \rho$. Then for all $m \geq 2$, the class $\iota_X(x)^{[m]}$ is in $H_{EP}{}^*(X)^{dm,m,e}$, where $e = 0$ if d is even, $e = 1$ if d is odd. Almost by construction, $\iota_X(x)^{[m]}$ is represented by the cochain

$$W_0^{S_m} \xrightarrow{\varepsilon} \mathbb{F}_p \xrightarrow{x^{\otimes m}} H^*(X; \mathbb{F}_p)^{\otimes m} \otimes \mathrm{sgn}^e,$$

where $W_*^{S_m} \to \mathbb{F}_p \to 0$ is a free resolution of $\mathbb{F}_p$ as an $\mathbb{F}_p[S_m]$-module. This is a well-defined cocycle because $x^{\otimes m}$ is invariant under the action of S_m in $H^*(X; \mathbb{F}_p)^{\otimes m} \otimes \mathrm{sgn}^e$. If q_I is different from $q_0 \dots q_0$, then $q_I(\alpha) \in H_*(S_{p^k}; H_*(X; \mathbb{F}_p)^{\otimes p^k} \otimes_{S_{p^k}} \rho)$ is represented by a cycle of the form $y \otimes_{S_{p^k}} \alpha^{\otimes p^k}$, with $y \in W_l\, S_{p^k}$ and $l > 0$. Such a cycle must pair trivially with the cochain representative for $\iota_X(x)^{[p^k]}$ above.

Similarly, $\iota_X(x^\prime)^{[n]}$ can be represented at the cochain level by $W_0^{S_n} \xrightarrow{\varepsilon} \mathbb{F}_p \xrightarrow{x^{\prime\otimes n}} H^*(X; \mathbb{F}_p)^{\otimes n} \otimes \mathrm{sgn}^{e(x^\prime)}$. Since the diagonal of the resolution W_* must preserve the augmentation ε, composing the tensor product of the cochains representing $\iota_X(x)^{[n]}$ and $\iota_X(x^\prime)^{[n]}$ with the diagonal we obtain $W_0^{S_n} \xrightarrow{\varepsilon} \mathbb{F}_p \xrightarrow{(xx^\prime)^{\otimes n}} H^*(X; \mathbb{F}_p)^{\otimes n} \otimes \mathrm{sgn}^{e(x)+e(x^\prime)}$, which represents both $\iota_X(x)^{[n]} \cdot \iota_X(x^\prime)^{[n]}$ and $\iota_X(xx^\prime)^{[n]}$.

We can assume that y is tri-homogeneous and that $k \geq 2$. If $t(y)$ is odd, then the statement is trivial because both $(x^{[n]} \cdot y)^{[k]}$ and $x^{[nk]} \cdot y^{[k]}$ are zero, so we assume that $t(y)$ is even. Represent y by an $S_{n(y)}$-equivariant cocycle $f : W_{d(y)}^{S_{n(y)}} \to \mathrm{sgn}^{e(y)}$. Let $\mathrm{tr} : W_*^{S_{n(y)k}} \to W_*^{S_k} \otimes (W_*^{S_{n(y)}})^{\otimes k}$ be a chain-level representative of the transfer map. Using our representative of $\iota_X(x)^{[m]}$ and combining with the diagonal map, we immediately have that both $\iota_X(x)^{[n(y)k]} \cdot \pi(y)^{[k]}$ and $(\iota_X(x)^{[n(y)]} \pi(y))^{[k]}$ are represented by the same cocycle

$$W_*^{S_{nk}} \xrightarrow{\mathrm{tr}} W_*^{S_k} \otimes (W_*^{S_{n(y)}})^{\otimes k} \xrightarrow{\varepsilon \otimes f^{\otimes k}} \mathrm{sgn}^{e(y)}$$

$$\xrightarrow{x^{\otimes kn(y)}} H^*(X; \mathbb{F}_p)^{\otimes kn(y)} \otimes_{S_{kn(y)}} \mathrm{sgn}^e \otimes \mathrm{sgn}^{e(y)} \xrightarrow{\mathrm{id} \otimes \mu_{kn(y)}} H^*(X; \mathbb{F}_p)^{\otimes kn(y)} \otimes_{S_{kn(y)}} \mathrm{sgn}^{e+e(y)}.$$

D

We can now prove the main result of this paper.

Proof of Theorems 2.37 *and* 2.38. We only prove the theorem for $p > 2$, because the assertion for $p = 2$ is similar and significantly simpler.

Let H_{alg} be the component super-Hopf rig with additive divided powers having the algebraic presentation given. Since we know from Lemma 4.8 (statements (2) and (3)) that there are classes in $H_{EP}{}^*(X)$ satisfying the relations stated by the theorem, there is a morphism of component super-Hopf rigs preserving the divided powers structure $H_{alg} \to H_{EP}{}^*(X_+)$. We show that this map is an isomorphism.

First, as above we invoke well-known results about structure of Hopf algebras [MM65, And71] to see that the graded linear dual of a bicommutative divided powers super-Hopf algebra H is the free graded commutative algebra primitively generated by $P(H)^\vee$, the dual space of the subspace of primitives $P(H) \subseteq H$. To prove that the map $H_{alg} \to H_{EP}{}^*(X_+)$ is an isomorphism, it is enough to check that the induced pairing between $P(H_{alg})$ and the space Q of indecomposables of

$(H_{EP}{}^*(X_+))^\vee$ is perfect. We achieve this by performing an ad-hoc analysis, which is the most delicate part of the proof.

We need to have explicit descriptions of the space Q of indecomposables of $H_{EP}{}^*(X_+)^\vee$ and the space P of primitives of H_{alg}. The graded dual of $H_{EP}{}^*(X_+)$ is described in terms of KADL operations, extended to the sign representation case as described in Section 3. Precisely, Q has a basis B_* given by classes of the form

$$\beta^{\varepsilon_1} q_{i_1} \beta^{\varepsilon_2} q_{i_2} \ldots \beta^{\varepsilon_r} q_{i_r}(\alpha),$$

where $(\varepsilon_1, i_1, \ldots, \varepsilon_r, i_r)$ is a strongly admissible sequence of KADL operations, $i_{j+1} - \varepsilon_{j+1}$ have the same parity for all j, and where α varies on a given additive basis of $H_*(X; \mathsf{F}_p)$. Since $H_{EP}{}^*(X_+)$ is tri-graded and its structure maps preserve degrees, $(H_{EP}{}^*(X_+))^\vee$ and Q are also tri-graded. The homological dimension of a basis element $\beta^{\varepsilon_1} q_{i_1} \beta^{\varepsilon_2} q_{i_2} \ldots \beta^{\varepsilon_r} q_{i_r}(\alpha)$, when α is tri-homogeneous, is computed from $d(\alpha)$ via the usual dimension formulas for KADL operations. Its component is $p^r n(\alpha)$. Its sign degree is 0 if $d(\alpha)$ and i_r have the same parity, 1 otherwise.

$P = P(H_{alg})$ can be described directly in terms of the Hopf monomial basis of H_{alg}. Indeed, by the construction of Section 4.1, a basis B^* of P is given by the set of classes of the form $b \cdot x^{[p^n]}$, where $b \in H^*(\mathsf{S}_{p^n}; \rho)$ is a primitive gathered block in the twisted or untwisted cohomology of a symmetric group S_{p^n} and x belongs to a given additive basis of $H^*(X; \mathsf{F}_p)$. The cohomological degree, the component and the sign degree of $b \cdot x^{[p^n]}$ in H_{alg} are $d(b) + d(x)p^n$, p^n, $e(b) + d(x)$ (modulo 2) respectively.

Let $P_{\{*\}}$ and $Q_{\{*\}}$ be the module of primitives of $H_{EP}{}^*(S^0)$ and that of indecomposables for $H_{EP}{}^*(S^0)^\vee$, respectively. By the description above there are isomorphisms $P_{\{*\}} \otimes H^*(X; \mathsf{F}_p) \cong P$ and $Q_{\{*\}} \otimes H_*(X; \mathsf{F}_p) \cong Q$ given by $b \otimes x \mapsto b \cdot x^{[n]}$ and $q_I \otimes \alpha \mapsto q_I(\alpha)$, respectively. We claim that, under these isomorphisms, the pairing between P and Q is equivalent to the tensor product of the Kronecker pairings $P_{\{*\}} \otimes Q_{\{*\}} \to \mathsf{F}_p$ and $H^*(X; \mathsf{F}_p) \otimes H_*(X; \mathsf{F}_p) \to \mathsf{F}_p$. This claim reduces the calculation to the special case $X = \{*\}$ already discussed in Section 3 and would thus complete the proof.

To prove this claim, we first use May's formulas for the coproduct $\Delta_\cdot$ dual to cup product of KADL operations and we obtain that, for all $b \in P_{\{*\}}$, $x \in H^*(X; \mathsf{F}_p)$, $q_I \in Q_{\{*\}}$, $\alpha \in H_*(X; \mathsf{F}_p)$,

$$(b \cdot x^{[n]}, q_I(\alpha)) = \sum_{J+K=I} \sum_{\Delta_\cdot(\alpha)=\alpha_{(1)} \otimes \alpha_{(2)}} \pm (b \otimes x^{[n]}, q_J(\alpha_{(1)}) \otimes q_K(\alpha_{(2)})).$$

By Statement (1) of Lemma 4.8, $(x^{[n]}, q_K(\alpha_{(2)}))$ is zero unless $q_K = q_0 \ldots q_0$ and the homological degree of $\alpha_{(2)}$ is the same as the cohomological degree of x. Moreover, b being the pullback of a class from $H_{EP}{}^*(S^0)$, $(b, q_J(\alpha_{(1)}))$ is 0 unless the homological degree of $\alpha_{(1)}$ is 0. These conditions can only be satisfied for a single addend: $\alpha = \alpha_{(2)}$, $\alpha_{(1)}$ is a zero-dimensional class, $q_J = q_I$ and $q_K = q_0 \ldots q_0$, for which we obtain $\pm(b, q_I)(x, \alpha)$. D

5. Further cohomology ring calculations

We use our description of $H^*(\tilde{D}X; \mathsf{F}_p)$ to obtain a presentation of the related algebras $H^*(DX; \mathsf{F}_p)$, $H^*(CX; \mathsf{F}_p)$ and $H^*(QX; \mathsf{F}_p)$. These results are all new except for $H^*(QX; \mathsf{F}_2)$ for which an equivalent result has been obtained by Dũng

[Dun92]. The techniques used in that paper are different from ours, though, and we believe that our approach leads to a simpler proof.

5.1. **General extended powers.** First, we describe $H_{EP}{}^*(X)$ when X is not necessarily obtained from a topological space by adjoining a disjoint basepoint. The following result is a consequence of our main theorems.

Corollary 5.1. *Let X be a pointed topological space. Let B' be a graded basis for $\tilde{H}^*(X; \mathbb{F}_p)$ and let $B = B' \cup \{1_X\}$ be the basis of the unreduced cohomology of X obtained by adding the unit of $H^*(X_{[*]}; \mathbb{F}_p)$, where $X_{[*]}$ is the connected component of X containing the basepoint. $H_{EP}{}^*(X)$ is isomorphic to the sub-Hopf rig with additive divided powers of $H_{EP}{}^*(X_+)$ consisting of the unit and the linear span of decorated Hopf monomials (with respect to the basis B) with decorations different from 1_X.*

Proof. The map $\pi : X_+ \to X$ that sends $*$ to the basepoint of X induces a map $\pi^*: H_{EP}{}^*(X) \to H_{EP}{}^*(X_+)$ that is both a morphism of Hopf rings and of divided powers structures. The dual map π_* in homology is an epimorphism of algebras whose kernel is the ideal generated by $q_I(*)$, with I admissible and non-empty. Consequently, π^* must be injective. From the proof of Theorem 2.38 given in the previous section, we see that the Kronecker pairing between $H_{EP}{}^*(X_+)$ and the homology of $D(X_+)$ with coefficient in ρ restricts to a perfect pairing between the primitive gathered blocks not decorated with 1_X and indecomposables of the homology of $D(X_+)$ not in $\ker(\pi_*)$. We deduce that the image of π^* must be the sub-Hopf ring identified in the statement of the corollary. D

5.2. **Infinite extended powers.** Given a pointed topological space X with basepoint $*$, we take into consideration the space $D_\infty X$ defined as

$$\tilde{D}_\infty X = \{p \times_{S_\infty} (x_1, \dots, x_n, \dots) : |\{n : x_n \neq *\}| < \infty\} \subseteq E(S_\infty) \times_{S_\infty} X^{\mathbb{N}},$$

where $S_\infty = \varinjlim_n S_n$ is the infinite symmetric group. (In Dũng's paper, this space is denoted by $S \wr X$.) The calculation of the cohomology of $\tilde{D}_\infty X$ is a straightforward consequence of our main theorems.

Corollary 5.2. *Let X be a topological space and let B be a basis of $H^*(X; \mathbb{F}_p)$ as an $\mathbb{F}_p$-algebra that contains $1_{H^*(X)}$ and extends an additive basis of the reduced cohomology of X. For all $n \leq m$, let $\rho_{n,m}: H^*(\tilde{D}_m X; \mathbb{F}_p) \to H^*(\tilde{D}_n X; \mathbb{F}_p)$ be the vector space homomorphism that maps a Hopf monomial x into 0 if x does not have a constituent block of the form $1_{H^*(X)}^{[r]}$ with $r \geq m-n$, or into $y \odot 1_{H^*X}^{[r-m+n]}$ if $x = y \odot 1_{H^*X}^{[r]}$, where y is another Hopf monomial not containing any divided power of 1_{H^*X} as an $\odot$-factor.*

Then, all the maps $\rho_{n,m}$ are ring homomorphisms and form an inverse system of $\mathbb{F}_p$-algebras. Moreover, the mod p cohomology ring of $\tilde{D}_\infty X$ is naturally isomorphic to its inverse limit

$$\varprojlim_n H^*(\tilde{D}_n X; \mathbb{F}_p).$$

Proof. By construction $\tilde{D}_\infty X$ is the direct limit of the topological spaces $\tilde{D}_n X$. The embedding of $\tilde{D}_n X$ into $\tilde{D}_m X$ for $n \leq m$ is given by the functions

$$f_{n,m}: \tilde{D}_n X = E(S_n) \times_{S_n} X^n \to E(S_m) \times_{S_m} X^m = \tilde{D}_m X, \quad (n \leq m)$$

corresponding to the standard monomorphism $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_m$, and the inclusion $X^n \hookrightarrow X^m$ that maps X^n identically onto the first n coordinates of X^m and makes the last $m-n$ coordinates equal to the basepoint.

For all $n \leq m$, the induced morphism $f^*_{n,m}\colon H^*(\tilde{D}_m X; \mathbb{F}_p) \to H^*(\tilde{D}_n X; \mathbb{F}_p)$ is equal to $\rho_{n,m}$. This fact follows directly from Theorem 2.37 and Theorem 2.38 because we can identify $f^*_{n,m}$ with the composition of the iterated coproduct

$$\begin{aligned} H^*(\tilde{D}_m X; \mathbb{F}_p) &\to H^*(\tilde{D}_n X; \mathbb{F}_p) \otimes H^*(\tilde{D}_1 X; \mathbb{F}_p)^{\otimes m-n} \\ &\cong H^*(\tilde{D}_n X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)^{\otimes m-n} \end{aligned}$$

with $\mathrm{id}_{H^*(\tilde{D}_n X;\mathbb{F}_p)} \otimes \varepsilon^{\otimes m-n}$, where $\varepsilon\colon H^*(\tilde{D}_1 X; \mathbb{F}_p) \cong H^*(X; \mathbb{F}_p) \to \mathbb{F}_p$ is the augmentation given by evaluation against the class of the basepoint of X.

Since the maps $\rho_{n,m}$ are epimorphisms, taking the cohomology of the given limit of spaces yields the desired isomorphism between $H^*(\tilde{D}_\infty X; \mathbb{F}_p)$ and $\varprojlim_n H^*(\tilde{D}_n X; \mathbb{F}_p)$. □

Corollary 5.2 provides a practical way to calculate cup products in the cohomology of $\tilde{D}_\infty X$ by restriction to $\tilde{D}_n X$ with n finite. Suppose X is connected, and $\alpha_1, \alpha_2 \in H^*(\tilde{D}_\infty X; \mathbb{F}_p)$ are cohomology classes of dimension d_1 and d_2 respectively. The map $\rho_{n,m}$ is an isomorphism in dimension less than or equal to d_1+d_2, provided that $n, m \geq d_1 + d_2$. This fact is easily seen directly, but we can also view it as a consequence of homological stability. Therefore the natural restriction morphism $\rho_n\colon H^{d_1+d_2}(\tilde{D}_\infty X; \mathbb{F}_p) \to H^{d_1+d_2}(\tilde{D}_n X; \mathbb{F}_p)$ is an isomorphism for all $n \geq d_1 + d_2$. Hence, to calculate $\alpha_1 \cdot \alpha_2$ is sufficient to apply $\rho_{d_1+d_2}$ and perform the computation in $H^*(\tilde{D}_{d_1+d_2}; \mathbb{F}_p)$.

We use this method to extract a complete set of generators and relations for $H^*(\tilde{D}_\infty X; \mathbb{F}_p)$. In what follows, until the end of this section, we will always assume that $\mathcal{B}'$ is an additive basis of $\tilde{H}^*(X; \mathbb{F}_p)$ as an $\mathbb{F}_p$-vector space, and that $\mathcal{B} = \mathcal{B}' \cup \{1\}$ is the basis of the unreduced cohomology of X obtained by adding the unit $1 \in H^*(X_{[*]}; \mathbb{F}_p)$, where $X_{[*]}$ is the connected component of X containing the basepoint. In this section, we will often consider only connected spaces X. In those cases, 1 will be the unit of the cohomology ring of X.

Definition 5.3. Let $\mathcal{M}$ be the skyline diagrams basis of $H^*(\tilde{D}X; \mathbb{F}_p)$ constructed from $\mathcal{B}$. The **effective width** of an element $x \in \mathcal{M}$ is the width of the skyline diagram obtained from x by removing any column with decoration 1 and height 0. In terms of gathered monomials, the effective width of x is the component of the unique gathered monomial y such that $x = y \odot 1^{[r]}$ for some $r \geq 0$. We say that $x \in \mathcal{M}$ is **pure** is its width is equal to its effective width or, equivalently, if x does not have any constituent block of the form $1^{[r]}$.

Lemma 5.4. *Let $x \in H^*(\tilde{D}_n X; \mathbb{F}_p)$ be a pure Hopf monomial. There is a class $x \odot 1^{[*]}$ in $\varprojlim_n H^*(D_n X; \mathbb{F}_p) \cong H^*(D_\infty X; \mathbb{F}_p)$ which maps to $x \odot 1^{[r]}$ in $H^*(D_m X)$ when $m = n + r$.*

Proof. It is sufficient to check that the following equalities hold to see there is such a class in the inverse limit

- $\rho_{n-1,n}(x) = 0$
- $\forall m > 0 : \rho_{n+m-1,n+m}(x \otimes 1^{[m]}) = x \otimes 1^{[m-1]}$.

To prove the first identity, remember that $\rho_{n-1,n}$ is equal to the iterated coproduct composed with augmentation ε on one tensor factor. Let V be the $\mathbb{F}_p$-subspace of $H^*(\tilde{D}X; \mathbb{F}_p)$ generated by pure Hopf monomials. Since the coproduct preserves the decorations of columns, $\Delta(V) \subseteq V \otimes V$. Thus $\rho_{n-1,n}$ is equal to a sum of terms of the form $\varepsilon(y)z$, where $y \in H^*(X; \mathbb{F}_p)$ and $z \in H^*(\tilde{D}_{n-1}X; \mathbb{F}_p)$ are pure Hopf monomials. By definition, pure Hopf monomials in the cohomology of X ,.. $\tilde{D}_1 X$ are the elements of $\mathsf{B}^!$, and, as a consequence, they belong to the kernel of ε. Hence $\rho_{n-1,n}(x) = 0$.

In order to prove the latter identity, we observe that $\rho_{n+m-1,n+m}(x \otimes 1^{[m]})$ is the image of x under the morphism

$$H^*(\tilde{D}_nX; \mathbb{F}_p) \xrightarrow{\otimes 1^{[m]}} H^*(\tilde{D}_nX; \mathbb{F}_p) \otimes H^*(\tilde{D}_mX; \mathbb{F}_p) \longrightarrow H^*(\tilde{D}_{n+m}X; \mathbb{F}_p)$$

$$\xrightarrow{\Delta_{n+m-1,1}} H^*(\tilde{D}_{n+m-1}X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \xrightarrow{\mathrm{id} \otimes \varepsilon} H^*(\tilde{D}_{n+m-1}X; \mathbb{F}_p).$$

Since Δ and $\otimes$ form a bialgebra, this is equal to the sum of the following two composition of maps, where τ exchanges two factors:

$$H^*(\tilde{D}_nX; \mathbb{F}_p) \xrightarrow{-\otimes 1^{[m]}} H^*(\tilde{D}_nX; \mathbb{F}_p) \otimes H^*(\tilde{D}_mX; \mathbb{F}_p) \xrightarrow{\mathrm{id} \otimes \Delta_{m-1,1}} H^*(\tilde{D}_nX; \mathbb{F}_p) \otimes H^*(\tilde{D}_{m-1}X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$$

$$\xrightarrow{\otimes \mathrm{id}} H^*(\tilde{D}_{n+m-1}X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \xrightarrow{\mathrm{id} \otimes \varepsilon} H^*(\tilde{D}_{n+m}X; \mathbb{F}_p),$$

$$H^*(\tilde{D}_nX; \mathbb{F}_p) \xrightarrow{-\otimes 1^{[m]}} H^*(\tilde{D}_nX; \mathbb{F}_p) \otimes H^*(\tilde{D}_mX; \mathbb{F}_p) \xrightarrow{\Delta_{n-1,1} \otimes \mathrm{id}} H^*(\tilde{D}_{n-1}X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \otimes H^*(\tilde{D}_mX; \mathbb{F}_p)$$

$$\xrightarrow{(\otimes \mathrm{id}) \circ \tau} H^*(\tilde{D}_{n+m-1}X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \xrightarrow{\mathrm{id} \otimes \varepsilon} H^*(\tilde{D}_{n+m}X; \mathbb{F}_p).$$

The first composition is equal to $- \otimes 1^{[m-1]}$, because $\Delta_{m-1,1}(1^{[m]}) = 1^{[m-1]} \otimes 1$ and $\varepsilon(1) = 1$. Pure Hopf monomials belong to the kernel of the second one, because, as we have already seen, the coproduct preserves the subspace generated by them and every pure monomial in $H^*(X; \mathbb{F}_p)$ lies in the kernel of ε. Thus, $\rho_{n+m-1,n+m}(x \otimes 1^{[m]}) = x \otimes 1^{[m-1]}$. $\square$

Lemma 5.5. *The set* $\{x \otimes 1^{[*]} : x \in \mathsf{M}$ *pure*$\}$ *is linearly independent in* $H^*(\tilde{D}_\infty X; \mathbb{F}_p)$. *Moreover, if* X *is connected, it is a basis for that vector space.*

Proof. Given a finite set of elements of the form $x \otimes 1^{[*]}$, for $N \in \mathbb{N}$ large enough, their restrictions to $H^*(\tilde{D}_NX; \mathbb{F}_p)$ are linearly independent, which implies the linear independence of the original elements. If X is connected, given $k \in \mathbb{N}$, for N large enough dependent on k the map $\rho_N \colon H^k(\tilde{D}_\infty X; \mathbb{F}_p) \to H^k(\tilde{D}_NX; \mathbb{F}_p)$ is an isomorphism, thus the given set is a basis for $H^k(\tilde{D}_\infty X; \mathbb{F}_p)$, because it restricts to a basis for $H^k(\tilde{D}_NX; \mathbb{F}_p)$. $\square$

If X is not connected, we can reduce the computation of $H^*(\tilde{D}_\infty X;\mathbb{F}_p)$ to the connected case as follows. Let $\{X_a\}_{a\in\pi_0(X)}$ be the set of the connected components of X. Let $[*]$ denote the element of $\pi_0(X)$ corresponding to the component containing the basepoint. Then there is a natural homotopy equivalence

$$\tilde{D}_\infty X \simeq \prod_{a\in\pi_0(X)\setminus\{[*]\}} \tilde{D}X_a \times \tilde{D}_\infty X_{[*]},$$

which induces an isomorphism of the cohomology algebras

$$H^*(\tilde{D}_\infty X;\mathbb{F}_p) \cong \bigoplus_{\{n_a\}_a\in\mathbb{N}^{\pi_0(X)\setminus\{[*]\}}} \bigotimes_a H^*(\tilde{D}_{n_a}X_a;\mathbb{F}_p) \otimes H^*(\tilde{D}_\infty X_{[*]};\mathbb{F}_p).$$

We next analyze cup product computations for the classes $x\odot 1^{[*]}$. As a technical tool, we will exploit the existence of an increasing filtration $\{F_nH^*(\tilde{D}_\infty X;\mathbb{F}_p)\}_{n\geq 0}$ of $\mathbb{F}_p$-vector spaces, where $F_nH^*(\tilde{D}_\infty X;\mathbb{F}_p)$ is the linear span of the elements $x\odot 1^{[*]}$ arising from pure Hopf monomials x with a width lower than or equal to n. Lemma 5.5 guarantees that $\bigcup_{n\geq 0} F_nH^*(\tilde{D}_\infty X;\mathbb{F}_p) = H^*(\tilde{D}_\infty X;\mathbb{F}_p)$ when X is connected.

Lemma 5.6. *Let $b_1,\dots,b_r$ be primitive gathered block in $H^*(\tilde{D}X;\mathbb{F}_p)$ and let $n_1,\dots,n_r$ be strictly positive integers. Then*

$$\prod_{i=1}^{r} (b_i^{[n_i]}\odot 1^{[*]}) - b_1^{[n_1]}\odot\cdots\odot b_r^{[n_r]}\odot 1^{[*]} \in F_{w-1}H^*(\tilde{D}_\infty X;\mathbb{F}_p),$$

where w is the sum of the widths of $b_1^{[n_1]},\dots,b_r^{[n_r]}$.

Proof. The argument boils down to restricting to $H^*(\tilde{D}_N X;\mathbb{F}_p)$ with $N\in\mathbb{N}$ large enough and perform the calculations there, where the claim is a straightforward consequence of our Hopf ring presentation.

Explicitly, we observe that Δ preserves the effective width. Formally, if the effective width of a Hopf monomial x is w, then its coproduct can be written as $\Delta(x) = \sum_i x'_i\otimes x''_i$ where the sum of the effective widths of x'_i and x''_i is w. Moreover, our cup product formulas imply that given a non-trivial gathered block $b\in H^*(\tilde{D}_nX;\mathbb{F}_p)$ and a Hopf monomial (not necessarily pure) $x\in H^*(\tilde{D}_nX;\mathbb{F}_p)$ belonging to the same component, the cup product $x\cdot b$ is always the sum of pure Hopf monomials (when it is not zero). These two facts together with the Hopf ring distributivity formula guarantee that, given $x\in H^*(\tilde{D}_nX;\mathbb{F}_p)$ with effective width w, and a non-trivial gathered block $b\in H^*(\tilde{D}_mX;\mathbb{F}_p)$ whose width is less than or equal to $n-w$, the following equality holds:

$$x\cdot(b\odot 1^{[n-m]}) = \sum_i (x'_i b)\odot x''_i,$$

where the effective width of x''_i is less than or equal to w, and we have equality in a single case in which $x'_i = 1^{[m]}$ and $x''_i = \rho_{n-m,n}(x)$. Therefore the difference $x\cdot(b\odot 1^{[n-m]}) - b\odot\rho_{n-m,n}(x)$ is a linear combination of Hopf monomials whose effective widths are strictly less than n.

We can now prove our claim by induction on r. For $r = 1$ the statement is trivial. For $r > 1$, the previous argument with $x = b_1^{[n_1]}\odot\cdots\odot b_{r-1}^{[n_{r-1}]}\odot 1^{[N]}$ and $b = b_r^{[n_r]}\odot 1^{[M]}$, with $N, M\in\mathbb{N}$ big enough, completes the induction. □

Lemma 5.7. *Let $b \in H^*(\tilde{D}X; \mathbb{F}_p)$ be a gathered block. If $p > 2$ we also assume that b is even-dimensional. Then $(b \odot 1^{[*]})^p = b^p \odot 1^{[*]}$.*

Proof. First, we prove an auxiliary formula. For any $a \in \mathbb{N}$, define $v_j(a) \in \{0, 1\}$ as the coefficient of 2^j in the dyadic expansion of a. Given $n, k, N \in \mathbb{N}$, let $\mathcal{A}_{n,k,N}$ be the set of 2^k-tuples $(a_1, \dots, a_{2^k}) \in \mathbb{N}^{2^k}$ that satisfy the following two conditions:

- $\forall 0 \leq j < k : \sum_{i=1}^{2^k} a_i v_j(2^k - i) = n$
- $\sum_{i=1}^{2^k} a_i = n + N.$

For all $n, k \in \mathbb{N}$, for all $N \in \mathbb{N}$ large enough, and for all $b \in H^*(\tilde{D}X; \mathbb{F}_p)$ primitive gathered block of width w, the following equality holds:

$$(b^{[n]} \odot 1_X^{[Nw(b)]})^k = \sum_{a \in \mathcal{A}_{n,k,N}} \bigodot_{i=1}^{2^k} (b^{[a_i]})^{\cdot \sum_{j=0}^{k-1} v_j(2^k - i)}.$$

We prove this formula by induction on k. For $k = 1$ it is trivial, and the induction step is a straightforward application of Hopf ring distributivity.

We now construct a permutation $\sigma \in \Sigma_{2^k}$ that maps $\mathcal{A}_{n,k,N}$ into itself, when acting on $\mathbb{N}^{2^k}$ by permutation of the 2^k coordinates. Given $\tau \in \Sigma_k$, we define $\bar{\tau} \in \Sigma_{2^k}$ by the formula

$$\forall 1 \leq i \leq 2^k : \bar{\tau}(i) = 2^k - \sum_{j=0}^{k-1} 2^j v_{\tau(j+1)-1}(2^k - i).$$

We let $\sigma = \bar{\tau}$ where $\tau \in \Sigma_k$ is a k-cycle. Observe that:

- σ has order k and its fixed points in $\{1, \dots, 2^k\}$ are 1 and 2^k;
- $\sigma(\mathcal{A}_{n,k,N}) = \mathcal{A}_{n,k,N}$;
- Two 2^k-tuples in $\mathcal{A}_{n,k,N}$ that belong to the same σ-orbit give rise to addends in the formula above that differ only by a permutation of the transfer product factors.

These remarks imply that the fixed points of σ in $\mathcal{A}_{n,k,N}$ are the 2^k-tuples that are constant on the σ-orbits of $\{1, \dots, 2^k\}$.

We specialize to the case $k = p$ to complete the proof. Since, under this condition, σ has order p, the sum of all the addends in the previous equality that do not correspond to fixed points of σ in $\mathcal{A}_{n,p,N}$ is zero, because of the commutativity of $\odot$. Moreover, the function $\sum_{j=0}^{k-1} v_j(2^k - \_)$ is constant on the orbits of σ in $\{1, \dots, 2^k\}$. Therefore, in the addends corresponding to a 2^k-tuple $\underline{a} \in \mathcal{A}_{n,k,N}$ such that $\sigma(\underline{a}) = \underline{a}$, the factor $(b^{[a_i]})^{\cdot \sum_{j=0}^{k-1} v_j(2^k - i)}$ appears at least p times if $i \notin \{1, 2^k\}$. Due to the properties of divided power algebras, all these addends are zero, except when $a_i = 0$ for all $i \notin \{1, 2^k\}$. This leaves only one addend, easily seen to be equal to $(b^{[n]})^p \odot 1^{[Nw]}$. Passing to the limit completes the proof. □

We are finally ready to produce our presentation for $H^*(\tilde{D}_\infty X; \mathbb{F}_p)$. In order to do so, we need to impose some restrictions on the basis $\mathcal{B}$ for $H^*(X; \mathbb{F}_p)$. Recall that the Frobenius homomorphism makes $H^*(X; \mathbb{F}_p)$ an abelian restricted Lie algebra over $\mathbb{F}_p$. If it is of finite type, then it can be decomposed as a direct sum of monogenerated abelian restricted Lie algebras, of the form $\mathbb{F}_p\{x, x^p, \dots, x^{p^k}\}$ with $\deg(x) = d \in \mathbb{N}$, $k \in \mathbb{N}$, and $x^{p^{k+1}} = 0$. The bases $\{x, x^p, \dots, x^{p^k}\}$ on those subspaces and the existence of that decomposition determine a basis $\mathcal{B}_F$ of $H^*(X; \mathbb{F}_p)$.

Theorem 5.8 (After Dũng [Dun92] for $p = 2$). *Let X be a connected space of finite type. Let B_F be the basis constructed above. Then $H^*(\tilde{D}_\infty X; \mathbb{F}_p)$ is the graded commutative $\mathbb{F}_p$-algebra generated by the set of classes of the form $b \otimes 1^{[*]}$, where b is a gathered block (or decorated column, in the language of skyline diagrams) whose width is a power of p satisfying at least one of the following conditions:*

- *the decoration of the column is not a p^{th} power in $H^*(X; \mathbb{F}_p)$*
- *at least one of the constituent rectangles of the column appears a number of times that is not divisible by p*
- *the column has a non-trivial solid part (if $p > 2$)*

Moreover, a complete set of relations for these generators is given by equalities of the form $(b \otimes 1_{[*]})^{p^{h(b)}} = 0$, with b even-dimensional. If x denotes the decoration of b, we define $h(b)$ through the following formula:

$$h(b) = \begin{cases} \min\{n \in \mathbb{N} : x^{p^n} = 0\} & \text{if } p = 2 \text{ or } b \text{ has no solid part,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Lemma 5.7 guarantees that the given relations hold in $H^*(\tilde{D}_\infty X; \mathbb{F}_p)$. Thus we need to prove two facts, namely that the given classes generate that cohomology algebra, and that there are no other independent relations.

In order to prove the first claim, we use the filtration $\{F_n H^*(\tilde{D}_\infty X; \mathbb{F}_p)\}_{n\geq 0}$ of the cohomology of $\tilde{D}_\infty X$ by effective width defined previously. We prove by induction on n that $F_n H^*(\tilde{D}_\infty X; \mathbb{F}_p)$ is contained in the subspace V generated by our selected elements $b \otimes 1^{[*]}$. For $n = 0$, the claim is obvious. So assume that $n > 0$ and that $F_{n-1} H^*(\tilde{D}_\infty X; \mathbb{F}_p) \subseteq V$.

First, for any gathered block b whose width is a power of p, $b \otimes 1^{[*]} \in V$, even if b does not satisfy any of the three required conditions. Indeed, because of Lemma 5.7, any such element is the $(p^k)^{th}$ power, for some $k \in \mathbb{N}$, of some other class of the form $b' \otimes 1^{[*]}$ arising from a gathered block b' that meets one of those conditions.

Second, for any pure Hopf monomial x of width n, x can be written as a multiple of the transfer product of some gathered blocks $b_1, \dots, b_r$ whose widths are powers of p. By Lemma 5.6 we have that

$$x \in \bigodot_{i=1}^{r} (b_i \otimes 1^{[*]}) + F_n H^*(\tilde{D}_\infty X; \mathbb{F}_p) \subseteq V$$

and this proves our first claim.

In order to prove that the given relations suffice to describe $H^*(\tilde{D}_\infty X; \mathbb{F}_p)$ completely, it is sufficient to check that the Poincaré series of this cohomology algebra and the graded commutative $\mathbb{F}_p$-algebra $A_\infty(X)$ defined by the given presentation are equal.

On the one hand, we observe that

$$A_\infty(X) = \bigotimes_{b, \deg(b) \text{ even}} \frac{\mathbb{F}_p[b \otimes 1^{[*]}]}{(b \otimes 1^{[*]})^{h(b)}} \otimes \Lambda\{b, \deg(b) \text{ odd}\},$$

where Λ is the exterior algebra functor and the tensor product is over gathered blocks b satisfying the conditions in the statement of this theorem.

On the other hand, the subspace (isomorphic to $H^*(\tilde{D}_\infty X; \mathbb{F}_p)$ by Lemma 5.5) generated by pure Hopf monomials in $H^*(\tilde{D}X; \mathbb{F}_p)$ is a subalgebra under the transfer product, and is isomorphic to

$$\bigotimes_{b,\deg(b)\text{ even}} \frac{\mathbb{F}_p[b]}{(b^p)} \otimes \Lambda\{b, \deg(b) \text{ odd}\}.$$

Here the tensor product is performed over all the gathered blocks whose width is a power of p that are different from $1^{[p^k]}$, regardless of the stated conditions.

As we already observed, for every generator $b \otimes 1^{[*]}$ and every $1 \leq k < h(b)$, the power $(b \otimes 1^{[*]})^{p^k}$ is again a class of the form $b' \otimes 1^{[*]}$, where b' is another non-trivial gathered block whose width is a power of p. Conversely, any such element can be obtained by iteratively applying the Frobenius homomorphism to one of the generators of $A_\infty(X)$. This implies that in any degree d, $(A_\infty(X))_d$ and $H^d(\tilde{D}X; \mathbb{F}_p)$ have the same dimension. □

5.3. **Free E_∞ spaces.** We move to the cohomology of CX, the free E_∞-space generated by X, as defined in Cohen–May–Lada [CLM76].

Let X_+ be the pointed topological space obtained from X by adding a disjoint basepoint $*$. Observe that $C(X_+) \cong D(X_+) \cong \tilde{D}X$. The map $p\colon X_+ \to X$ that sends $*$ to the basepoint of X induces a surjective function $Cp\colon \tilde{D}X \cong C(X_+) \to CX$. This map factors through the projection $\tilde{D}X \to \tilde{D}_\infty X$.

The description of the functor $H_*(C-; \mathbb{F}_p)$ given in [CLM76, part I, Theorem 4.1] and the surjectivity of the map $p_*\colon H_*(X_+; \mathbb{F}_p) \to H_*(X; \mathbb{F}_p)$ imply that the induced morphism in homology $(Cp)_*\colon H_*(\tilde{D}X; \mathbb{F}_p) \to H_*(CX; \mathbb{F}_p)$ and, as a direct consequence, $H_*(\tilde{D}_\infty X; \mathbb{F}_p) \to H_*(CX; \mathbb{F}_p)$ are epimorphisms. Dually, this means that $H^*(CX; \mathbb{F}_p) \to H^*(\tilde{D}_\infty X; \mathbb{F}_p)$ is a monomorphism. Therefore, we identify $H^*(CX; \mathbb{F}_p)$ with a subring of $H^*(\tilde{D}_\infty X; \mathbb{F}_p)$.

At the level of homology, $(Cp)_*\colon H_*(\tilde{D}X; \mathbb{F}_p) \to H_*(CX; \mathbb{F}_p)$ is the unique morphism of algebras over the Dyer–Lashof algebra that extends the map $p_*\colon H_*(X_+; \mathbb{F}_p) \to H_*(X; \mathbb{F}_p)$. Let $[x_0] \in H_*(X; \mathbb{F}_p)$ being the class in $H_0(X; \mathbb{F}_p)$ corresponding to the basepoint $x_0 \in X$. Then, using homology operations, the kernel can be described as the ideal generated by $q_I([x_0])$, with I non-empty, along with $1 - [x_0]$, Dualizing yields the following.

Corollary 5.9. *Let X be a connected space of finite type. Then $H^*(CX; \mathbb{F}_p)$ is naturally isomorphic to the subring of $H^*(\tilde{D}_\infty X; \mathbb{F}_p)$ generated by classes of the form $(b \otimes 1^{[*]})$ satisfying the criteria stated in Theorem 5.8 and the additional condition that the decoration of b is different from 1.*

Moreover, the relations $(b \otimes 1^{[*]})^{p^{h(b)}} = 0$ among those generators are sufficient to give a complete description of $H^*(CX; \mathbb{F}_p)$ as an algebra.

Although Corollary 5.9 holds only for connected spaces X, we can reduce the general calculation of $H^*(CX; \mathbb{F}_p)$ to this particular case. It is enough to note that, if $\{X_a\}_{a\in\pi_0(X)}$ is the set of the connected components of X, then there is a natural homeomorphism $CX \cong \prod_{a\in\pi_0(X)\setminus\{[*]\}} \tilde{D}(X_a) \times C(X_{[*]})$, and to apply the Künneth isomorphism.

5.4. **Free infinite loop spaces.** We conclude this section with an analogous description of the cohomology of QX. When X is connected, CX and QX are also connected and the natural map $CX \to QX$ induces an isomorphism in cohomology. When X is not connected, CX is not connected, and its commutative H-space structure makes $\pi_0(CX)$ a commutative monoid. Using the description above, we immediately see that $\pi_0(CX) \cong \mathbb{N}^{\pi_0(X)\setminus\{[*]\}}$ and that the component corresponding to $\underline{k} = \{k_a\}_{a\in\pi_0(X)\setminus\{[*]\}}$ is homeomorphic to $\prod_a \tilde{D}_{k_a}(X_a) \times C(X_{[*]})$. We name this component $(CX)_{\underline{k}}$.

We recall the existence of a natural isomorphisms of Hopf algebras $H_*(QX;\mathbb{F}_p) \cong H_*(CX;\mathbb{F}_p) \otimes_{\pi_0(CX)} \mathbb{F}_p[G]$, where $G = \mathbb{Z}^{\pi_0(X)\setminus\{[*]\}}$ is the group completion of $\pi_0(CX)$, and $\mathbb{F}_p[G]$ is its group algebra over $\mathbb{F}_p$. This isomorphism is classically well-known and is discussed in [CLM76, part I, Theorem 4.2].

Let Π be the poset $\mathbb{N}^{\pi_0(X)\setminus\{[*]\}} = \pi_0(CX)$ with order

$$\{k_a\}_a \leq \{n_a\}_a \Leftrightarrow \forall a \in \pi_0(X) \setminus \{[*]\} : k_a \leq n_a.$$

For all $\underline{k}, \underline{n} \in \Pi$ with $\underline{k} \leq \underline{n}$, let $f_{\underline{k},\underline{n}} : H_*((CX)_{\underline{k}};\mathbb{F}_p) \to H_*((CX)_{\underline{n}};\mathbb{F}_p)$ be defined by multiplication with $[x_{\underline{n}-\underline{k}}]$, the class of a point $x_{\underline{n}-\underline{k}} \in (CX)_{\underline{n}-\underline{k}}$. These maps define a direct system, whose limit is isomorphic to each component of $H_*(CX;\mathbb{F}_p) \otimes_{\pi_0(CX)} \mathbb{F}_p[G]$. The topological counterpart of this is the homotopy equivalence $QX \simeq Q_0X \times G$, where Q_0X is the connected component of QX containing the basepoint. Indeed, the direct limit above is isomorphic to the homology of Q_0X.

Therefore describing the cohomology of QX is equivalent to calculating the dual of that direct limit, i.e. the inverse limit of the inverse system made by the dual spaces. Note that the morphisms $f^*_{\underline{k},\underline{n}} : H^*((CX)_{\underline{n}};\mathbb{F}_p) \to H^*((CX)_{\underline{k}};\mathbb{F}_p)$ are analogous to the maps $\rho_{n,m}$ used to compute $H^*(\tilde{D}_\infty X;\mathbb{F}_p)$ in Lemma 5.5, but with blocks of the form $1^{[r]}$ replaced with any gathered block of dimension zero. Explicitly, they are of the form $1^{[r]}_{H^*(X_a;\mathbb{F}_p)}$ for some $a \in \pi_0(X)$ if the basis $\mathcal{B}$ contains all the units $1_{H^*(X_a;\mathbb{F}_p)}$. Thus, we can replicate the argument used to compute the cohomology of $\tilde{D}_\infty X$ to determine $H^*(QX;\mathbb{F}_p)$, by replacing pure Hopf monomials with Hopf monomials that do not contain columns of height zero or with decoration 1. We give below the precise statements.

Definition 5.10. Let x be a Hopf monomial in $H^*(\tilde{D}X;\mathbb{F}_p)$. We say that x is **full-width** if it does not have any gathered block of dimension 0.

Lemma 5.11. *Assume that the chosen basis $\mathcal{B}$ of $H^*(X;\mathbb{F}_p)$ contains $1_a = 1_{H^*(X_a;\mathbb{F}_p)}$ for all $a \in \pi_0(X)$. Then, to every full-width Hopf monomial without constituent blocks with $1_{[*]}$ as decoration, we can associate an element $x_\infty \in \varprojlim_{\underline{k}\in\Pi} H^*((CX)_{\underline{k}};\mathbb{F}_p) \cong H^*(Q_0X;\mathbb{F}_p)$ defined by the formula*

$$(x_\infty)|_{H^*((CX)_{\underline{k}};\mathbb{F}_p)} = \begin{cases} x \otimes \bigotimes_{a\in\pi_0(X)\setminus\{[*]\}} 1_a^{[k_a-n_a]} \otimes 1^{[*]} & \text{if } \underline{k} \geq \underline{n}_x \\ 0 & \text{otherwise,} \end{cases}$$

where $\underline{n}_x$ is the unique element of $\Pi = \pi_0(CX)$ such that $x \otimes 1^{[*]} \in (CX)_{\underline{n}_x}$. These elements x_∞ form a basis of $H^*(Q_0X;\mathbb{F}_p)$ as an $\mathbb{F}_p$-vector space.

Theorem 5.12. *Chose the basis $\mathcal{B}$ of X as we did for Theorem 5.8. Moreover, assume that $1_{H^*(X_a;\mathbb{F}_p)} \in \mathcal{B}$ for all $a \in \pi_0(X)$. Consider the set of elements b_∞, where b is a full-width gathered block of width equal to a power of p, whose decoration is different from $1_{[*]}$, and that satisfies at least one of the following conditions:*

- *the decoration of the column is not a p^{th} power in $H^*(X;\mathbb{F}_p)$*

- *at least one of the constituent rectangles of the column appears a number of times that is not divisible by* p
- *the column has a non-trivial solid part (if* $p > 2$*)*

This set generates $H^*(Q_0X; \mathbb{F}_p)$ *as a graded commutative algebra, with relations given by* $(b_\infty)^{p^{h(b)}} = 0$*, where we define* $h(b)$ *as in Theorem* 5.8*.*

Proof. Since the obvious analogs of Lemma 5.6 and Lemma 5.7 also hold in this case, the statement can be proved with the same argument used for Theorem 5.8, by replacing CX with Q_0X and $b \otimes 1^{[*]}$ with b_∞. □

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