Research Article

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Separation functions and mild topologies

https://doi.org/10.1515/agms-2022-0149
received November 9, 2021; accepted February 2, 2023

Abstract: Given M and N Hausdorff topological spaces, we study topologies on the space $C^0(M; N)$ of continuous maps $f : M \to N$. We review two classical topologies, the "*strong*" and the "*weak*" topology. We propose a definition of "*mild topology*" that is coarser than the "*strong*" and finer than the "*weak*" topology. We compare properties of these three topologies, in particular with respect to proper continuous maps $f : M \to N$, and affine actions when $N = \mathbb{R}^n$. To define the "*mild topology*" we use "*separation functions*;" these "separation functions" are somewhat similar to the usual "*distance function* $d(x, y)$" in metric spaces (M, d), but have weaker requirements. *Separation functions* are used to define *pseudo balls* that are a global base for a T2 topology. Under some additional hypotheses, we can define "*set separation functions*" that prove that the topology is T6. Moreover, under further hypotheses, we will prove that the topology is metrizable. We provide some examples of uses of *separation functions*: one is the aforementioned case of the *mild topology* on $C^0(M; N)$. Other examples are the *Sorgenfrey line* and the topology of topological manifolds.

Keywords: continuous functions, proper maps, strong topology, weak topology, metrization, quasi metrics, asymmetric metrics, topological manifolds

MSC 2020: 54C10, 54C35, 54E35, 58D15, 58D19, 46T05, 46T10

1 Introduction

Let M and N be Hausdorff topological spaces, with topologies τ_M and τ_N. (Sometimes we may also assume that M or N be metric spaces with distances d_M, or respectively d_N; then τ_M, or respectively τ_N, will be the associated topology.)

In Section 3, we discuss a topology for the space $C^0(M; N)$ of continuous maps $f : M \to N$, which we will call "*mild topology*."

To define and discuss the properties of the mild topology, we have developed a novel method whereby we define a topology τ on a generic set X by using a family of "*separation functions*;" this method is presented in Section 2. *Separation functions* are somewhat similar to the usual *distance function* in metric spaces (M, d), but they have weaker hypotheses (so they can be more manageable in some contexts). *Separation functions* are used to define *pseudo balls* that are a global base for a T2 topology τ (this is proven in Theorem 2.3). Under some additional hypotheses, we will define "*set separation functions*" (similar to *set distance functions*) in Section 2.5 to prove that the topology is T6. Moreover, under further hypotheses, the topology τ is, in fact, metrizable (Theorems 2.21 and 2.30) – such will be the case for the *mild topology*. We will also discuss other applications of this *separation function* method. In Section 2.3, we show that the topology on the *Sorgenfrey line* (that is not metrizable) can be defined by using a suitable family of *separation functions*. In Section 2.12, we show how *separation functions* can be easily defined for topological

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manifolds starting from the atlas of the manifold. These examples show that the method of *separation functions* is a promising tool that may have further uses beyond the definition of *mild topology*.

After developing this method of *separation functions* in Section 2, we then proceed, in Section 3, to apply it to the case $X = C^0(M; N)$ to define the *mild topology*.

But first and foremost, we would like to explain why we may find useful a new topology on $C^0(M; N)$. For this purpose, in Section 1.1 we recapitulate the definitions of "*strong*" and "*weak*" topologies; in Section 1.2, assuming $N = \mathbb{R}^n$, we relate those to the usual definition of Frechét spaces $C^0(M; \mathbb{R}^n)$; then, in Sections 1.3, 1.4 and 1.5, we express some properties of these topologies.

1.1 Topologies for continuous maps

We define the "graph" of the function $f : M \to N$ as

$$\operatorname{graph}(f) \stackrel{\text{def}}{=} \{(x, y) \in M \times N : y = f(x)\}.$$

We distinguish two fundamental examples of topologies for $C^0 = C^0(M; N)$.

Definition 1.1. The *compact-open* topology is generated by sets of the form

$$\{f \in C^0 : f(K) \subseteq U\},$$

where $K \subseteq M$ is compact and $U \subseteq N$ is open. (This collection of sets is a *subbase* for the topology, but it does not always form a *base* for a topology.) It is also called the "*topology of uniform convergence on compact sets*" or the "*weak topology*" in [2]. We will write $C^0_W(M; N)$ to denote this topological space.

Definition 1.2. The *graph topology* is generated by sets of the form

$$\{f \in C^0(M; N) : \operatorname{graph}(f) \subseteq U\},$$

where U runs through all open sets in $M \times N$. It is also called "*wholly open topology*" in [4], "*fine*" or "*Whitney*" or "*strong topology*" in [2]. We will write $C^0_S(M; N)$ to denote this topological space.

An equivalent definition of the strong topology can be formulated under additional hypotheses.

Proposition 1.3. (41.6 in [4], or Chapter 2 Section 4 in [2]) *If M is paracompact and (N, d_N) is a metric space, then for $f \in C^0(M, N)$ the sets*

$$\{g \in C^0(M, N) : d_N(g(x), f(x)) < \varepsilon(x) \forall x \in M\}$$

form a base of neighborhoods for the graph topology, where ε runs through all positive continuous functions on M.

Another way to state this result[1] *is to define the distances*

$$d_\varepsilon(f, g) \stackrel{\text{def}}{=} \sup_{x \in M} \varepsilon(x) d_N(g(x), f(x));$$

then the topology generated by all these distances is the graph topology.

It is possible to define similar concepts for $C^r(M; N)$, the space of r times differentiable maps between two differentiable manifolds M, N. In this case, we do not detail the discussion.

The aforementioned topologies are invariant in the sense that the next proposition shows.

1 Exercise 3 in Chapter 2, Section 4 in [2], to be compared with the "counterexample 1.1.8" in [1].

Proposition 1.4. *If $\Phi_M : \tilde{M} \to M$ and $\Phi_N : N \to \tilde{N}$ are homeomorphisms, then the map*

$$f \mapsto \Phi_N \circ f \circ \Phi_M$$

is a homeomorphism between $C^0(M; N)$ and $C^0(\tilde{M}; \tilde{N})$, where the spaces are both endowed either with the "weak" or the "strong" topology.

1.2 When $N = \mathbb{R}^n$

Let us suppose in this section that N is the standard Euclidean space $\mathbb{R}^n$ and that M is a Hausdorff locally compact and second countable[2] topological space.

We recall that any second countable Hausdorff space that is locally compact is paracompact, so Proposition 1.3 applies in the current context; moreover, there exists a countable locally finite covering of open sets, each with compact closure.

Definition 1.5. $C^0_{\text{loc}}(M, \mathbb{R}^n)$ is the Frechét space whose topology is generated by the seminorms

$$[f]_K = \sup_{x \in K} |f(x)|,$$

for $K \subseteq M$ compact. If M is compact, then it coincides with the usual Banach space $C^0(M, \mathbb{R}^n)$ associated with the norm

$$\|f\| = \sup_{x \in M} |f(x)|.$$

Proposition 1.6. *Under the above hypotheses, $C^0_W(M; \mathbb{R}^n)$ coincides with $C^0_{\text{loc}}(M, \mathbb{R}^n)$.*

We can define also another topology.

For $K \subseteq M$, compact we define the subset

$$V_{0,K} = \{f \in C^0(M; \mathbb{R}^n) : \operatorname{supp}(f) \subseteq K\}$$

of $C^0(M; \mathbb{R}^n)$ functions with support in K. Each such $V_{0,K}$ is a closed subspace of $C^0_{\text{loc}}(M; \mathbb{R}^n)$. Then $C^0_{\text{loc}}(M; \mathbb{R}^n)$ is a Frechét space with the induced topology.

We can then define this topology.

Definition 1.7. ($C^0_c(M; \mathbb{R}^n)$ topology) The $C^0_c(M; \mathbb{R}^n)$ topology is the strict inductive limit[3] with respect to the inclusions

$$V_{0,K} \to C^0(M; \mathbb{R}^n)$$

for $K \subseteq M$ compact. A set W is open in the $C^0_c(M; \mathbb{R}^n)$ topology if, for all $K \subseteq M$ compact, $W \cap V_{0,K}$ is open in $V_{0,K}$.

1.3 Properties of weak and strong topologies

The weak topology enjoys nice metrization properties.

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2 *That is*, it admits a countable base of open sets.

3 For the definition of *strict inductive limit* and its properties, we refer to 17G at page 148 in [3].

Proposition 1.8. *Let N be metrizable with a complete metric, and let M be locally compact and second countable. Then $C^0_W(M, N)$ has a complete metric.*

This is proven in Theorem 4.1 in Chapter 2, Section 4 in [2].

We ponder on these remarks, taken from [2].

Remark 1.9. The topological space $C^0_S(M; N)$ resulting from the strong topology is the same as $C^0_W(M; N)$ if M is compact. If M is not compact, however, $C^0_S(M; N)$ can be an extremely large topology; for example, when M, N are differentiable finite dimensional manifolds (of positive dimension), then it is not metrizable and in fact it does not have a countable local base at any point, and it has uncountably many connected components.

In particular, if M is not compact, $C^0_S(M; \mathbb{R}^n)$ cannot be a topological vector space, but the connected component containing $f \equiv 0$ coincides with $C^0_c(M; \mathbb{R}^n)$.

1.4 Proper maps

Definition 1.10. A proper map $f : M \to N$ is a continuous map such that we have that $f^{-1}(K)$ is compact in M, for any $K \subseteq N$ compact.

Obviously, if M is compact then any continuous function is proper. More in general:

Lemma 1.11. *The set of proper maps $f : M \to N$ is both open and closed in the strong $C^0_S(M; N)$ topology.*

For the proof see, *e.g.*, Section 5.1 in [7] or Theorem 1.5 in Chapter 2, Section 1 in [2].

It is easily seen that, in general, the set of proper functions is neither closed nor open in any C^r_{loc} topology.

Example 1.12. Consider functions $f : \mathbb{R} \to \mathbb{R}$, and the weak topology; it is easy to show examples of sequences $f_n \to f$ such that

- none of the f_n are proper, but f is, *e.g.*,
  $$f_n(x) = n \arctan(x/n)$$
  and $f(x) = x$;
- f_n are all proper but f is not, *e.g.*,
  $$f_n(x) = \frac{1}{n} x^2$$
  and $f(x) = 0$.

The aforementioned examples hold also in C^∞_{loc}, where convergence is defined as “local uniform convergence of all derivatives.”

1.5 Drawback of strong topologies

Given the above discussion, it would seem that, when dealing with proper maps, it would be better to use a strong topology. Strong topologies have drawbacks as well.

In particular, consider the case of maps $f : M \to \mathbb{R}^n$, let $C^0 = C^0(M; \mathbb{R}^n)$; we have some natural actions:

- translations, given by the action
$$(v, f) \in \mathbb{R}^n \times C^0 \mapsto f + v \in C^0;$$
- rotations
$$(A, f) \in \mathrm{SO}(n) \times C^0 \mapsto Af \in C^0;$$
- rescalings
$$(\lambda, f) \in I \times C^0 \mapsto \lambda f \in C^0,$$
for $I = (0, \infty) \subset \mathbb{R}$
- and, in general, affine transformations
$$F, f \mapsto Ff,$$
where $Fy = Ay + v$ is given by $A \in \mathrm{GL}(\mathbb{R}^n)$, $v \in \mathbb{R}^n$.

These actions are continuous if C^0 is endowed with the weak topology, but it may fail to be continuous if C^0 is endowed with the strong topology.

There is another drawback (that we already remarked): when $N = \mathbb{R}^n$, then C^0 with the strong topology may fail to be a topological vector space.

We will show in Section 3 that the *mild topology* shares some good properties valid for the weak and the strong topology.

2 Topology by separation functions

In this section, given a generic set X, we will use "*separation functions*" to define a topology τ on X; we will then study the properties of this topology.

Definition 2.1. A family $d = (d_x)_{x\in X}$ of real positive functions
$$d_x(y) : X \to [0, \infty],$$
one for each $x \in X$, is a *family of separation functions* when
- $d_x(y) = 0$ iff $x = y$;
- given $y \in X$ and $\alpha, \beta \in \mathbb{R}$ with $0 < \beta < \alpha$ there exists a function
$$\rho_d = \rho_d(y, \alpha, \beta) > 0 \tag{2.1}$$
(called "*modulus*") such that, for all $x, z \in X$,
$$d_x(y) \le \beta \wedge d_y(z) < \rho_d \Rightarrow d_x(z) < \alpha, \tag{2.2}$$
$$d_x(y) \ge \alpha \wedge d_y(z) < \rho_d \Rightarrow d_x(z) > \beta. \tag{2.3}$$

This condition (2.2) will be called "*pseudo triangle inequality*," while condition (2.3) will be called "*pseudo reverse triangle inequality*" (Figure 1).

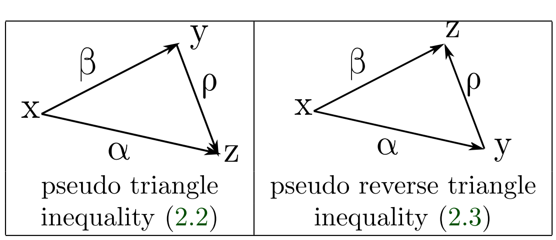


Figure 1: Representation of triangle inequalities.

These are written as $d_x(y)$ and not $d(x, y)$ to remark that they do not satisfy the axioms of "distances": they are not required to be symmetric and do not satisfy the standard triangle inequality.

Remark 2.2. An "*asymmetric distance*" (*a.k.a.* "*quasi metric*") is a function $b(x, y) : X^2 \to [0, +\infty]$ that satisfies the separation property "$b(x, y) = 0 \iff x = y$" and the standard triangle inequality, but it may fail to be symmetric. See [5,6] and references therein. An "*asymmetric distance*" immediately provides a *family of separation functions* $d_x(y) = b(x, y)$ with $\rho_d(y, \alpha, \beta) = \alpha - \beta$.

Theorem 2.3. *Given a set X with a family of separation functions, then we can define "pseudo balls"*

$$B(x, \varepsilon) = \{y \in X : d_x(y) < \varepsilon\};$$

these are then a global base for a T2 topology τ*, and each* $B(x, \varepsilon)$ *is an open neighborhood of* x *in* (X, τ).

Proof. Indeed the *pseudo triangle inequality* (2.2) shows that if $y \in B(x, \alpha)$ and $\beta = d_x(y)$ and $\rho_d = \rho_d(y, \alpha, \beta)$, then

$$B(y, \rho_d) \subseteq B(x, \alpha).$$

The *pseudo reverse triangle inequality* (2.3) shows that if $d_x(y) = \alpha > 0$ then

$$B(x, \beta) \cap B(y, \rho_d) = \emptyset.$$

□

Proposition 2.4. *Each* $d_x(y)$*, for fixed x, is continuous on* (X, τ).

Proof. Theorem 2.3 readily implies that $d_x(y)$ is upper semicontinuous: indeed we already know that

$$B(x, \varepsilon) = \{y \in X : d_x(y) < \varepsilon\}$$

is open. Let

$$V = \{z \in X : d_x(z) > \varepsilon\};$$

we want to prove that it is open. Let $y \in V$, let $\alpha = d_x(y) > \varepsilon$, and let $\varepsilon < \beta < \alpha$: the *pseudo reverse triangle inequality* (2.3) shows that

$$B(y, \rho_d) \subseteq V.$$

□

It is interesting to note that separation functions have a form of stability that makes them more manageable than distance functions.

Proposition 2.5. *Suppose that* $d_x(y)$ *is a separation function and* $\varphi : [0, \infty] \to [0, \infty]$ *is in the* Θ *class (defined in Definition* 2.11)*; let*

$$b_x(y) = \varphi(d_x(y)),$$

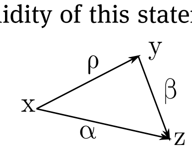
then $b_x(y)$ *is a separation function.*

Remark 2.6. A similar proposition holds for distances when φ is also subadditive. But

$$d_x(y) = |x - y|^2$$

is a *separation function* on $\mathbb{R}$, and it is not a distance.

Remark 2.7. Note that we did not assume validity of this statement.



$$\text{“Given } x \in X \text{ and } \alpha > 0 \text{ for any } 0 < \beta < \alpha \text{ there exists } \rho > 0 \text{ such that } d_x(y) < \rho \wedge d_y(z) \le \beta \Rightarrow d_x(z) < \alpha\text{.”} \tag{2.4}$$

(and similarly for a “reverse” version).

This raises an (yet) unanswered question. Let us define

$$\tilde{d}_y(x) = d_x(y):$$

under which conditions $\tilde{d}_y(x)$ is a separation function?

2.1 On the modulus

We recall that the function ρ_d defined in equation (2.1) is called “*modulus*.” We now prove that, given a family of separation functions, there exists a *maximum modulus*.

Proposition 2.8. *Having fixed a family of separation functions, there exists a maximum modulus $\hat{\rho}_d$, which can be explicitly defined as follows (for $y \in X$ and $0 < \beta < \alpha$):*

$$\hat{\rho}_d(y, \alpha, \beta) \stackrel{\text{def}}{=} \max\{r \in [0, \infty] : \forall x, z \in X, (d_x(y) \le \beta \wedge d_y(z) < r \Rightarrow d_x(z) < \alpha) \wedge (d_x(y) \ge \alpha \wedge d_y(z) < r \Rightarrow d_x(z) > \beta)\}.$$

Proof. The fact that $\hat{\rho}_d$ is a “modulus” is obvious from the definition, since the set on right-hand side (RHS) encodes the “*pseudo triangle inequality*” (2.2) and the “*pseudo reverse triangle inequality*” (2.3).

We prove that the formula defining $\hat{\rho}_d$ is correct, *i.e.*, that the set on RHS has positive maximum; to this end, we rewrite it in this form

$$\begin{aligned}
\hat{\rho}_d(y, \alpha, \beta) &\stackrel{\text{def}}{=} \max A_{(y,\alpha,\beta)}, \\
A_{(y,\alpha,\beta)} &= \bigcap_{x,z\in X} P_{(y,\alpha,\beta,x,z)} \cap R_{(y,\alpha,\beta,x,z)}, \\
P_{(y,\alpha,\beta,x,z)} &\stackrel{\text{def}}{=} \{r \in [0, \infty] : (d_x(y) > \beta \vee d_y(z) \ge r \vee d_x(z) < \alpha)\}, \\
R_{(y,\alpha,\beta,x,z)} &\stackrel{\text{def}}{=} \{r \in [0, \infty] : (d_x(y) < \alpha \vee d_y(z) \ge r \vee d_x(z) > \beta)\};
\end{aligned}$$

then we express the last two terms as

$$\begin{aligned}
P_{(y,\alpha,\beta,x,z)} &= \bigcap_{d_x(y)\le\beta\wedge d_x(z)\ge\alpha} [0, d_y(z)], \\
R_{(y,\alpha,\beta,x,z)} &= \bigcap_{d_x(y)\ge\alpha\wedge d_x(z)\le\beta} [0, d_y(z)];
\end{aligned}$$

eventually we write

$$A_{(y,\alpha,\beta)} = [0, \infty] \cap \left(\bigcap_{x,z\in X, d_x(y)\le\beta\wedge d_x(z)\ge\alpha} [0, d_y(z)] \right) \cap \left(\bigcap_{x,z\in X, d_x(y)\ge\alpha\wedge d_x(z)\le\beta} [0, d_y(z)] \right).$$

This latter is an intersection of closed intervals starting from zero (included), hence $A_{(y,\alpha,\beta)}$ is a closed interval of the form $[0, \rho]$; moreover, each interval in the RHS contains $\rho_d(y, \alpha, \beta)$, so the maximum is positive. □

Remark 2.9. It is clear by the aforementioned formulas that $\hat{\rho}_d$ is weakly increasing in α and weakly decreasing in β; so we may assume this in Definition 2.1, with no loss of generality.

If we add strict monotonicity and continuity, we obtain an interesting proposition.

Proposition 2.10. *Consider the “pseudo triangle inequality” equation* (2.2) *and these three additional conditions*: $\forall x, y, z \in X$, $\forall \alpha, \beta$ *with* $0 < \beta < \alpha$,

$$d_x(y) \le \beta \wedge d_y(z) \le \rho_d \Rightarrow d_x(z) \le \alpha, \tag{2.5}$$

$$d_x(y) < \beta \wedge d_y(z) \le \rho_d \Rightarrow d_x(z) < \alpha, \tag{2.6}$$

$$d_x(y) < \beta \wedge d_y(z) < \rho_d \Rightarrow d_x(z) < \alpha, \tag{2.7}$$

where again $\rho_d = \rho_d(y, \alpha, \beta)$ *is the same function* (2.1) *as used in equation* (2.2).

In the above four formulas (2.2), (2.5), (2.6), *and* (2.7), *we have alternated strict and loose inequalities.*

Suppose that the function $\rho_d(y, \alpha, \beta)$, *for fixed* y, *is continuous in* α, β, *strictly increasing in* α *and strictly decreasing in* β: *then the four conditions* (2.2), (2.5), (2.6), *and* (2.7) *are equivalent.*

(The proof is in Section A.1 in page 19)

This happens for distances, where $\rho_d = \alpha - \beta$; and it happens in Section 3.

A similar statement holds for the "*pseudo reverse triangle inequality*" (2.3): we skip it for brevity.

2.2 Fundamental family

In the aforementioned sections, we can assume that there are many *families of separation functions* $d_i = (d_{x,i})_{x\in X}$, for $i \in I$ a family of indexes (not depending on x); this is analogous to the framework in locally convex topological vector spaces, where we have multiple seminorms that are used to define multiple balls centered at zero (and then translated to all other points).

But, in the following considerations, we will assume for simplicity that, for each $x \in X$, there is only one *separation function*. So we have a **fundamental family of separation functions** $d = (d_x)_{x\in X}$. Then the topology satisfies the first countability axiom.

2.3 The Sorgenfrey line

The *Sorgenfrey line* is the set $X = \mathbb{R}$ with the topology generated by all the half-open intervals $[a, b)$, where $a, b \in \mathbb{R}$, $a < b$; this family is a global base for the topology. See example 51 in [10].

We can define

$$b(x, y) = \begin{cases} y - x & y \ge x \\ +\infty & y < x, \end{cases} \tag{2.8}$$

this is an "*asymmetric distance*" that generates the above topology on $\mathbb{R}$; by Remark 2.2 $d_x(y) = b(x, y)$ is a fundamental family of separation functions, with $\rho_d(y, \alpha, \beta) = \alpha - \beta$.

The *Sorgenfrey line* is a T6 space (a perfectly normal Hausdorff space); it is first countable and separable, but not second countable, so it is not metrizable.

This suggests that we need some extra hypotheses to obtain better properties.

2.4 Pseudo symmetry

We define a convenient class.

Definition 2.11. (Class Θ) We define the class Θ of functions

$$\theta : [0, \infty] \to [0, \infty]$$

that are continuous, have $\theta(0) = 0$, and are strictly increasing where they are finite. For such a θ we agree that $\theta^{-1} \in \Theta$ is so defined (with a slight abuse of notation)

$$\theta^{-1}(s) \stackrel{\text{def}}{=} \sup\{t \in [0,\infty) : \theta(t) < s\}.$$

Equivalently, if

$$D = \sup_{0\le s<\infty} \theta(s),$$

then $\theta^{-1}(s)$ is the usual inverse for $s < D$, otherwise $\theta^{-1}(s) = +\infty$[4]. This implies that

$$t = \theta^{-1}(s) \iff s = \theta(t),$$

whenever $s, t < \infty$.

Remark 2.12. This is just a convenient choice, which allows us to simplify notation and analysis. For example, the "tangent" function can be represented in Θ by defining it as

$$\theta(t) = \begin{cases} \tan(t) & s < \pi/2 \\ +\infty & s \ge \pi/2, \end{cases}$$

then its "inverse" is just $\theta^{-1}(s) = \arctan(s)$, with $\theta^{-1}(+\infty) = \pi/2$.

Definition 2.13. Let us assume that associated with X we have a fundamental family $(d_x)_{x\in X}$ of *separation functions*. We will say that this family is *pseudo symmetric* if there exists a function $\theta_d \in \Theta$ such that

$$d_y(x) \le \theta_d(d_x(y)). \tag{2.9}$$

Remark 2.14. "*Pseudo symmetry*" is useful because it tells us that the topology is equivalently generated by the inverse balls

$$\tilde{B}(x,\varepsilon) = \{y \in X : d_y(x) < \varepsilon\}.$$

This tells us that

$$\lim_{n\to\infty} x_n = x \iff d_x(x_n) \to_n 0 \iff d_{x_n}(x) \to_n 0.$$

Compare Definition 3.4 in [5].

2.5 Set separation function

Definition 2.15. Mimicking the definition in metric space, we define, for $A \subseteq X$, the *set separation function*

$$d_A(y) \stackrel{\text{def}}{=} \inf_{x\in A} d_x(y). \tag{2.10}$$

Lemma 2.16. *$d_A(y)$ is continuous.*

Proof. We know that $d_A(y)$ is upper semicontinuous, so we need to prove that it is lower semicontinuous, that is, that

$$V = \{y \in X : d_A(y) > \varepsilon\}$$

is open, for $\varepsilon \ge 0$. Let $y \in V$ and let $\alpha = d_A(y) > \varepsilon$; let then $\varepsilon < \beta < \alpha$ and let $r = \rho_d(y,\alpha,\beta)$, where ρ_d was defined in the *pseudo reverse triangle inequality* equation (2.3). We prove that

$$B(y,r) \subseteq V;$$

indeed for any $x \in A$ we have $d_x(y) \ge \alpha$: so, if $d_y(z) < r$, then $d_x(z) > \beta$ and hence $d_A(z) \ge \beta$. ☐

4 Indeed any $s \ge D$ cannot be equal to $\theta(t)$ for $t < \infty$, since θ is strictly increasing.

Remark 2.17. Since

$$\{y \in X : d_A(y) = 0\} \supseteq A$$

and the left-hand side (LHS) is closed, in general we have

$$\{y \in X : d_A(y) = 0\} \supseteq \overline{A}\,.$$

To obtain equality we add pseudo symmetry.

Hypothesis 2.18. From here on we assume that we have a fundamental family $(d_x)_{x\in X}$ of *separation functions* that is *pseudo symmetric*.

Lemma 2.19. *Assuming that Hypothesis* 2.18 *holds then*

$$\{y \in X : d_A(y) = 0\} = \overline{A}\,.$$

Proof. We need to prove that

$$\{y \in X : d_A(y) = 0\} \subseteq \overline{A}\,.$$

Let then $d_A(y) = 0$, which means that there is a sequence $(x_n)_n \subseteq A$ such that $d_{x_n}(y) \to_n 0$. Then by pseudo symmetry we obtain $y \in A$. □

Corollary 2.20. *In particular, when Hypothesis* 2.18 *holds the topological space is T6, a.k.a. a perfectly normal Hausdorff space.*

2.6 Metrization

We have then a first metrization theorem, following the Urysohn metrization theorem.

Theorem 2.21. *Assuming that Hypothesis* 2.18 *holds and the topological space* (X, τ) *is second countable, then it is metrizable.*

Proof. Indeed by Corollary 2.20 the space is T6. □

2.7 Diameter of pseudo balls

Definition 2.22. Let

$$\operatorname{diam}(A) \stackrel{\text{def}}{=} \sup_{x,y\in A} d_x(y)$$

be the *diameter* of a set $A \subseteq X$.

Lemma 2.23. *If Hypothesis* 2.18 *holds, then*

$$\lim_{\varepsilon\to 0} \operatorname{diam}(B(y,\varepsilon)) = 0.$$

If ρ_d does not depend on y, then the limit above is uniform in y.

Proof. The map $\varepsilon \mapsto \operatorname{diam}(B(y,\varepsilon))$ is monotonic. Let us fix $\alpha > 0$ and $\beta = \alpha/2$; we will use the *pseudo triangle inequality* that is equation (2.2). There is $\varepsilon > 0$ small enough such that $\theta_d(\varepsilon) < \beta$ and $\varepsilon < \rho_d(y,\alpha,\beta)$. Choose any $x, z \in B(y,\varepsilon)$. By pseudo symmetry

$$x \in B(y, \varepsilon) \iff d_y(x) < \varepsilon \Rightarrow d_x(y) \le \theta_d(\varepsilon) < \beta,$$

so by (2.2) $d_x(z) < \alpha$. ☐

2.8 Uniform modulus

"Uniform modulus" is the case when the modulus ρ_d does not depend on y. We saw in Lemma 2.23 that in this case we obtain stronger results. In the following, we discuss further results that use uniformity and pseudo symmetry.

Hypothesis 2.24. From here on we assume that we have a fundamental family $d = (d_x)_{x\in X}$ of *separation functions* that is *pseudo symmetric*, and where the modulus ρ_d does not depend on y.

We recall that the topology τ was defined in Theorem 2.3.

Lemma 2.25. *Assuming that Hypothesis* 2.24 *holds, then the topological space* (X, τ) *is separable if and only if it is second countable.*

Proof. One implication is well known.

We recall that the family of $B(x, \alpha)$, for $x \in X$ and $\alpha > 0$, is a global base for the topology τ, see Theorem 2.3. Let $(x_n)_{n\in\mathbb{N}}$ be a dense subset of X, we prove that

$$B(x_n, 1/m)$$

is a global base, for $n, m \ge 1$ integers; to this end for any $B(x, \alpha)$ we will prove that there are n, m such that

$$x \in B(x_n, 1/m) \subseteq B(x, \alpha).$$

Indeed, we choose m such that

$$\operatorname{diam} B(y, 1/m) < \alpha \tag{2.11}$$

for any y. Then, by pseudo symmetry, there is an n such that $d_{x_n}(x) < 1/m$: this implies that $x \in B(x_n, 1/m)$. Then, by (2.11), for any $z \in B(x_n, 1/m)$, we have $d_x(z) < \alpha$, so we conclude that $z \in B(x, \alpha)$. ☐

2.9 Forward and reverse

Proposition 2.26. *If Hypothesis* 2.24 *holds, then the "pseudo triangle inequality"* (2.2) *and the "pseudo reverse triangle inequality"* (2.3) *are equivalent.*

Proof. We prove that if (2.2) holds then (2.3) holds. (The other implication is similar.) Consider the formula

$$d_x(z) \ge \alpha \wedge d_z(y) < \tilde{\rho}_d \Rightarrow d_x(y) > \beta \tag{2.12}$$

that is equivalent to (2.3), since it was obtained by switching y, z and replacing the modulus. We prove that for $0 < \beta < \alpha$ there exists $\tilde{\rho}_d = \tilde{\rho}_d(\alpha, \beta) > 0$ such that $\forall\, x, y, z \in X$, (2.12) holds.

The condition (2.2) equivalently tells us that

$$d_x(y) > \beta \vee d_y(z) \ge \rho_d \vee d_x(z) < \alpha.$$

If

$$d_x(z) \ge \alpha \wedge d_z(y) < \theta^{-1}(\rho_d),$$

then (by pseudo symmetry (2.9))

$$d_y(z) \le \theta_d(d_z(y)) < \rho_d;$$

hence $d_x(y) > \beta$. We then define $\tilde{\rho}_d = \theta_d^{-1} \circ \rho_d$ with the convention in Definition 2.11. ◻

2.10 Delta complement

We extract a lemma from the proof of the following Theorem 2.29.

Definition 2.27. We assume Hypothesis 2.18. Given $C \subset X$ closed and $\delta > 0$ we define

$$F_\delta(C) = \{x \in X : d_C(x) > \delta\};$$

we call $F_\delta(C)$ the "*delta-complemented set*" since

$$F_\delta(C) \cap C = \varnothing,$$

$F_\delta(C)$ is open, and

$$\bigcup_{\delta>0} F_\delta(C) = X \setminus C.$$

(All this follows from Lemmas 2.16 and 2.19; Figure 2.) Note also that for $0 < s < t$ we have

$$F_s(C) \supseteq F_t(C),$$

while for $C_1 \subseteq C_2$ closed sets we have

$$F_s(C_1) \supseteq F_s(C_2). \tag{2.13}$$

Lemma 2.28. *We assume Hypothesis* 2.24. *Let now* $s, t \in \mathbb{R}$ *with* $0 < s < t$: *there exists an* $\varepsilon > 0$ *such that, for any* $w \in X$, *for any* $C \subseteq X$ *closed, if*

$$B(w, \varepsilon) \cap F_t(C) \neq \varnothing,$$

then

$$B(w, \varepsilon) \subseteq F_s(C).$$

Proof. Let $0 < s < \tilde{s} < t$. By the pseudo reverse triangle inequality (2.3), let $\rho > 0$ be small enough so that

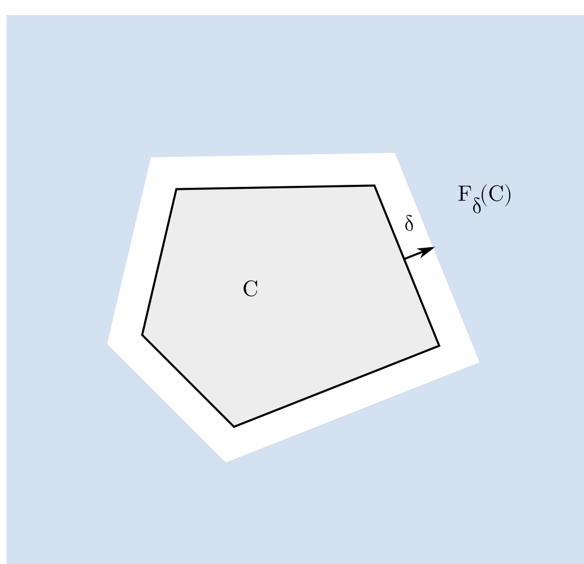


Figure 2: δ complement.

$$d_x(z) \geq t \wedge d_z(y) < \rho \Rightarrow d_x(y) > \tilde{s}.$$

Then choose $\varepsilon > 0$ small enough so that, by Lemma 2.23, we have

$$\forall w \in X, \forall z, y \in B(w, \varepsilon) \Rightarrow d_z(y) < \rho.$$

Let us now suppose that

$$z \in B(w, \varepsilon) \cap F_t(C).$$

For all $x \in C$, we have $d_x(z) > t$. Consider any $y \in B(w, \varepsilon)$, then we have $d_z(y) < \rho$. We conclude that $d_x(y) > \tilde{s}$, hence $\inf_{x \in C} d_x(y) \geq \tilde{s}$. We have so proved that $y \in F_s(A)$. □

2.11 Metrizability

Theorem 2.29. *We assume Hypothesis 2.24. If $\mathcal{A}$ is an open covering of X, then there exists $\mathcal{E}$ an open covering of X that is countably locally finite, and $\mathcal{E}$ is a refinement of $\mathcal{A}$.*

(The proof is in Section A.2 on page 91.)

Theorem 2.30. *If we assume Hypothesis 2.24, then the topological space (X, τ) is metrizable.*

Proof. This follows from the above theorem, Corollary 2.20, and the Nagata-Smirnov metrization theorem (Sections 6-2 and 6-3 in [8]). □

2.12 Example: topological manifolds

As an example, we propose this construction. In this section, we consider a topological manifold (X, τ) that is Hausdorff, second countable, and locally Euclidean with dimension m. (See §36 in [8] for further details.) Then (X, τ) is paracompact and σ-compact. So there exists an atlas of homeomorphisms

$$\varphi_i : V_i \to \mathbb{R}^m$$

satisfying these additional conditions:
- For this atlas we have that the index family is $I = \mathbb{N}$, or I is finite;
- For each $i \in I$ we have that $V_i \subseteq X$ is open with compact closure $\overline{V_i}$;
- Moreover, $(V_i)_{i \in I}$ is a locally finite open cover of X.

Theorem 2.31. *For $x, y \in X$ we can define*

$$d_x(y) = \min\Big\{|\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} : i \in I \wedge x \in V_i \wedge y \in V_i\Big\}. \tag{2.14}$$

Note that the set on the RHS is finite; if it is empty, then we set $d_x(y) = +\infty$. These functions $d_x(y)$ are continuous (jointly in x, y) and are a fundamental family of separation functions that generates the topology τ of X.

(Although the above result seems to be intuitive, the proof is surprisingly long and intricate, so it was moved to Section A.3.)

Interestingly, the modulus ρ_d associated with the family $d_x(y)$ can be defined to satisfy the requirements of Proposition 2.10.

Note that $d_x(y) = d_y(x)$, so this family satisfies Hypothesis 2.18. Consequently, the *set separation function* $d_A(y)$ satisfies Lemmas 2.16 and 2.19: this explicitly proves that the space is T6. So Theorems 2.31 and 2.21 are an alternative way to prove that a manifold (X, τ) is metrizable when it satisfies the hypotheses listed at the beginning of this section.

Example 2.32. In general, the function $d_x(y)$ defined above in (2.14) does not satisfy the triangle inequality. Consider this example of a manifold covered with two charts, where

$$x \in V_1 \setminus V_2, \quad y \in V_1 \cap V_2, \quad z \in V_2 \setminus V_1$$

and

$$d_x(y) < \infty, \quad d_y(z) < \infty, \quad d_x(z) = \infty.$$

Example 2.33. Consider $X = \mathbb{R}$ and cover it with charts having $V_n = (n-1, n+1)$ and

$$\varphi_n(x) = \frac{1}{\max\{1, |n|\}} \psi(x-n), \quad \psi(x) = \frac{x}{(1-x^2)}$$

for $n \in \mathbb{Z}$. Let then

$$x = n, \quad y = n + \frac{1}{2}, \quad z = n + 1,$$

so

$$d_x(y) = d_y(z) = \frac{2}{3n}, \quad d_x(z) = \infty.$$

This explains the importance of the dependence of ρ_a on y.

In some cases, it may happen that we do not know an easy formula for the distance that metrizes the manifold X.

The metrization theorems, such as Nagata-Smirnov metrization theorem and Urysohn's metrization theorem, are usually proven by showing that there is an embedding of X into $\mathbb{R}^{\mathbb{N}}$; to define this embedding, Urysohn's lemma is exploited to define countably many functions $f_n : X \to [0, 1]$; then a distance is defined on $\mathbb{R}^{\mathbb{N}}$ and pulled back on X. While perfectly valid as a proof, it is not an easily manageable definition and it is unsuitable for numerical algorithms.

If X is compact, then X can be embedded in $\mathbb{R}^N$; so this can be used to define a distance on X, by carefully tracking how the embedding is defined (as *e.g.* in §36 in [8]). This plan could be carried on, eventually providing an explicit formula for the distance; in particular, we can assume that the atlas is finite, let $\#I$ be its cardinality, then such proof provides $N = (m+1)\#I$.

We remark that, at the same time, formula (2.14) gives us a very convenient definition of *separation functions* (also when X is not compact): those encode the idea of "nearness" and can be used in further proofs and/or for numerical algorithms.

3 Mild topology

In this section, we propose a novel topology on the space of continuous functions $C^0(M; N)$.

To define the *mild topology* we need that (N, d_N) be a metric space.

We fix a distinguished point $\overline{p} \in N$.

Let $f, g \in C^0 = C^0(M; N)$, we define the "*mild separation*"

$$d_f^{\mathrm{mild},\overline{p}}(g) \stackrel{\mathrm{def}}{=} \sup_{x \in M} \frac{d_N(f(x), g(x))}{1 + d_N(f(x), \overline{p})}. \tag{3.1}$$

For $f \in C^0$ and $\alpha > 0$ we define the "*mild pseudo ball*"

$$B^{\mathrm{mild},\overline{p}}(f, \alpha) \stackrel{\mathrm{def}}{=} \{g \in C^0 : d_f(g) < \alpha\}. \tag{3.2}$$

We omit the superscripts "mild, $\overline{p}$" for ease of notation.

Definition 3.1. The *mild topology* on C^0 is the topology generated by the above sets $B(f, \alpha)$. We will write $C^0_M(M; N)$ to denote this topological space.

The above definitions will be justified in Proposition 3.5. To this end, we prove these two lemmas as follows.

Lemma 3.2. (Pseudo symmetry)

$$d_g(f) \leq \theta(d_f(g))$$

with

$$\theta(\alpha) = \begin{cases} \dfrac{\alpha}{1-\alpha} & \alpha < 1 \\ \infty & \alpha \geq 1. \end{cases}$$

Proof. Suppose $0 < \alpha < 1$ and $d_f(g) \leq \alpha$ then

$$d_N(f(x), g(x)) \leq \alpha(1 + d_N(f(x), \overline{p})) \leq \alpha(1 + d_N(g(x), \overline{p}) + d_N(g(x), f(x))),$$

hence

$$(1-\alpha)d_N(f(x), g(x)) \leq \alpha(1 + d_N(g(x), \overline{p})).$$ □

Lemma 3.3. (Pseudo triangle inequality) *Let* $f, g, h \in C^0$ *and* $\alpha > 0$; *if* $d_f(g) \leq \beta < \alpha$ *and* $d_g(h) \leq \rho_d(\alpha, \beta)$ *with*

$$\rho_d(\alpha, \beta) = \frac{\alpha - \beta}{1 + \beta}, \tag{3.3}$$

then $d_f(h) \leq \alpha$.

Proof. $d_f(g) \leq \beta$ means

$$d_N(f(x), g(x)) \leq \beta(1 + d_N(f(x), \overline{p})),$$

moreover, $d_g(h) \leq \rho$ means

$$d_N(g(x), h(x)) \leq \rho(1 + d_N(g(x), \overline{p}));$$

summing them

$$d_N(f(x), h(x)) \leq d_N(f(x), g(x)) + d_N(g(x), h(x)) \leq (\beta + \rho) + \beta d_N(f(x), \overline{p}) + \rho d_N(g(x), \overline{p}). \tag{3.4}$$

At the same time,

$$d_N(g(x), \overline{p}) \leq d_N(g(x), f(x)) + d_N(f(x), \overline{p}) \leq \beta + (1 + \beta)d_N(f(x), \overline{p}):$$

substituting this in (3.4),

$$d_N(f(x), h(x)) \leq (\beta + \rho(1 + \beta)) + (\beta + \rho(1 + \beta))d_N(f(x), \overline{p}).$$

We just need to find a $\rho > 0$ such that

$$(\beta + \rho(1 + \beta)) \leq \alpha:$$

the value defined in equation (3.3) is such a choice. □

Remark 3.4. Lemma 3.3 proves the *pseudo triangle inequality* in the form in equation (2.5); then the form in equation (2.2) follows from Proposition 2.10. Moreover, the *pseudo reverse triangle inequality* (2.3) holds as well, due to Lemma 3.2 and Proposition 2.26. So the family d_f defined in formula (3.1) is indeed a family of *separation functions*, as defined in Definition 2.1.

Summarizing, we can state the needed result.

Proposition 3.5. *The pseudo balls $B(f,\alpha)$ are a global base for the mild topology. The mild topology is metrizable.*

Proof. The first statement follows from Theorem 2.3. The second statement derives from Theorem 2.30; we need to verify the two Hypothesis 2.24.
- For the first hypothesis, we note that Lemma 3.2 proves that the family of separation functions is pseudo symmetric.
- The second hypothesis requires that the modulus $\rho(y,\alpha,\beta)$ appearing in Definition 2.1 does not depend on y; this is satisfied by the modulus defined in equation (3.3). □

Lemma 3.6. *The mild topology does not depend on the choice of $\overline{p} \in N$.*

Proof. Given $\overline{p}, \tilde{p} \in N$, for any $\beta > 0$, choosing

$$\alpha = \beta \frac{1}{1 + d_N(\overline{p}, \tilde{p})},$$

we have

$$\alpha(1 + d_N(y, \overline{p})) \le \alpha(1 + d_N(y, \tilde{p}) + d_N(\tilde{p}, \overline{p})) \le \beta(1 + d_N(y, \tilde{p}));$$

then we reason as in Remark 3.8. □

Proposition 3.7. *The mild topology is stronger than the weak topology; and it is weaker than the strong topology.*

Proof.
- We show that the mild topology is stronger than the weak topology. Fix $\varepsilon > 0$ and a compact set $K \subseteq M$, let

  $$\beta = \max_{x \in K} d_N(f(x), \overline{p})$$

  and

  $$\rho < \frac{\varepsilon}{1 + \beta},$$

  we know that if

  $$g \in B(f, \rho),$$

  then

  $$\forall x \in K,\ d_N(f(x), g(x)) < \varepsilon.$$
- The fact that the mild topology is weaker than the strong topology follows from Remark 3.8. □

Remark 3.8. We may also define the "*mild neighborhood*"

$$\tilde{B}(f, \alpha) \stackrel{\text{def}}{=} \{g \in C^0 : \forall x \in M,\ d_N(f(x), g(x)) < \alpha(1 + d_N(f(x), \overline{p}))\}. \tag{3.5}$$

Note that, for $0 < \beta < \alpha$

$$\tilde{B}(f, \beta) \subseteq B(f, \alpha) \subseteq \tilde{B}(f, \alpha), \tag{3.6}$$

so "*mild neighborhoods*" can be used to define the *mild topology*; unfortunately, they may fail to be open.

A "*mild neighborhood*" can be built using the same method seen in the *graph topology* (see Definition 1.2): indeed consider open sets of the form

$$U = \{(x, y) \in M \times N : \forall x \in M,\ d_N(f(x), y) < \alpha(1 + d_N(f(x), \overline{p}))\}$$

for $f \in C^0$, $\alpha > 0$, and then

$$\tilde{B}(f, \alpha) = \{g \in C^0 : \operatorname{graph}(g) \in U\}.$$

Consequently, equation (3.6) proves that the *mild topology* is coarser than the *strong topology*.

Remark 3.9. In general, this topology is not separable. For example, when $N = M = \mathbb{R}$, setting $f_s(x) = e^{sx}$, we have

$$d_{f_s}(f_t) = \begin{cases} 1 & s > t \\ \infty & s < t; \end{cases}$$

in these cases the topology does not satisfy the second countability axiom. (This is why we proved Theorem 2.30, that is based on Nagata-Smirnov metrization theorem; we cannot use Urysohn's metrization theorem to prove that the *mild topology* is metrizable.)

3.1 Metrizability

We know by Lemma 3.2 and Theorem 2.30 that the mild topology is metrizable.

At first sight, a reasonable candidate for a distance that generates the mild topology may be

$$d_{\text{mild?}}(f, g) \stackrel{\text{def}}{=} \sup_{x \in M} \frac{d_N(f(x), g(x))}{1 + d_N(f(x), \overline{p}) + d_N(g(x), \overline{p})},$$

where $f, g \in C^0(M; N)$. Note that $0 \le d_{\text{mild?}}(f, g) < 1$.

Lemma 3.10. *Obviously*

$$d_{\text{mild?}}(f, g) \le d_f^{\text{mild},\overline{p}}(g);$$

indeed the formula defining the LHS has one more positive term in the denominator than the formula defining the RHS. Moreover, for $0 < \alpha < 1$

$$\alpha = d_{\text{mild?}}(f, g) \Rightarrow d_f^{\text{mild},\overline{p}}(g) \le \frac{2\alpha}{1 - \alpha}.$$

Proof. If $0 < \alpha < 1$ and $d_{\text{mild?}}(f, g) \le \alpha$, then

$$d_N((fx), g(x)) \le \alpha(1 + d_N(f(x), \overline{p}) + d_N(g(x), \overline{p})) \le \alpha(1 + 2d_N(f(x), \overline{p}) + d_N(g(x), f(x)));$$

hence

$$(1 - \alpha)d_N(f(x), g(x)) \le 2\alpha(1 + d_N(f(x), \overline{p})).$$ □

So if $d_{\text{mild?}}(f, g)$ is a distance, it will generate the mild topology.

But is it a distance? The formula is obviously symmetric, and we have

$$d_{\text{mild?}}(f, g) = 0 \iff f \equiv g;$$

the question is as follows: does it satisfy the triangle inequality?

Consider then this formula

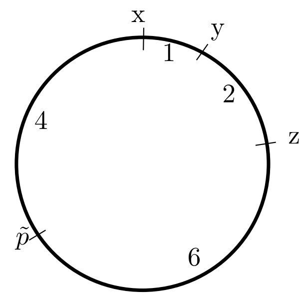
$$d_{\text{N?}}(z, w) \stackrel{\text{def}}{=} \frac{d_N(z, w)}{1 + d_N(z, \overline{p}) + d_N(w, \overline{p})}$$

for $z, w \in N$; so

$$d_{\text{mild?}}(f, g) = \sup_{x \in M} d_{\text{N?}}(f(x), g(x)).$$

We note that $d_{\text{mild?}}(f, g)$ satisfies the triangle inequality if and only if $d_{\text{N?}}(z, w)$ does. (For one implication, consider constant functions; for the other, use standard properties of the supremum.)

Unfortunately, the quantity $d_{\text{N?}}(z, w)$ does not satisfy the triangle inequality for some choices of N; as is seen in this example: let N be a circle of length 13 where the points are posed as in Figure 3.



$$d_N(x, y) = 1$$
$$d_N(y, z) = 2$$
$$d_N(x, z) = 3$$
$$d_N(\overline{p}, x) = 4$$
$$d_N(\overline{p}, y) = 5$$
$$d_N(\overline{p}, z) = 6$$

Figure 3: Points along a circle N of length 13 and distances.

Remark 3.11. At the same time, consider the case when N is a Hilbert space and $\overline{p} = 0$, then

$$d_{\mathrm{N?}}(z, w) \stackrel{\text{def}}{=} \frac{\|z - w\|_N}{1 + \|z\|_N + \|w\|_N}.$$

Note that the formula is invariant for rotations, so it is enough to check the triangle inequality for $N = \mathbb{R}^3$; numerical experiments suggest that it is indeed a distance; to this end, we tested the triangle inequality with randomly sampled points and tried to numerically minimize the difference

$$d_{\mathrm{N?}}(x, y) + d_{\mathrm{N?}}(y, z) - d_{\mathrm{N?}}(x, z)$$

for $x, y, z \in \mathbb{R}^3$. We though could not prove it analytically. See addendum material for more information.

3.2 Properties of proper maps

Lemma 3.12. *Suppose that (N, d_N) is a "proper metric space," i.e., closed balls are compact.*
If $0 < \alpha < 1$ and $f \in B(g, \alpha)$, then g is proper iff f is proper.
Similarly for the "mild neighborhood" $\tilde{B}(f, \alpha)$ defined in Remark 3.8.

Proof. Suppose that $d_g(f) = D < \infty$ and f is proper, we prove that g is proper. Let $K \subseteq N$ be compact, let

$$R = \max_{y \in K} d_N(y, \overline{p}),$$

and let

$$H = \{y \in N : d_N(y, \overline{p}) \leq D + R(D + 1)\};$$

then H is compact. We prove that

$$g^{-1}(K) \subseteq f^{-1}(H),$$

so that $g^{-1}(K)$ is compact. Indeed, if $x \in g^{-1}(K)$, then $g(x) \in K$ so $d_N(g(x), \overline{p}) \leq R$, hence

$$d_N(f(x), \overline{p}) \leq d_N(g(x), \overline{p}) + d_N(g(x), f(x)) \leq D + (D + 1) d_N(g(x), \overline{p}) \leq D + R(D + 1),$$

so $f(x) \in H$. Suppose now that g is proper and $d_g(f) < 1$, then, by pseudo symmetry Lemma 3.2, $d_f(g) < \infty$. So f is proper. □

Corollary 3.13. *The set of proper maps is both open and closed in the mild topology.*

3.3 Properties of affine actions

Theorem 3.14. *Suppose that $N = \mathbb{R}^n$, $d_N(x, y) = |y - x|$ is the usual Euclidean distance; endow $C^0(M; \mathbb{R}^n)$ with the mild topology; then the actions listed in Section 1.5 are (jointly) continuous.*
(*The proof is in page 23 in Section A.4.*)

Remark 3.15. For some specific actions some extra information may be useful.

- Rotation. If we choose $\overline{p} = 0$ for convenience in the definition equation (3.2) (as is made possible by Lemma 3.6), then, given a rotation $R \in O(n)$, the map
$$f \in C^0 \mapsto Rf \in C^0$$
is an "*isometry*": indeed,
$$B(Rf, \alpha) = RB(f, \alpha)$$
because
$$d_{Rf} Rg = d_f g.$$
We also note that for $S, R \in O(n)$,
$$|Rg(x) - Sg(x)| \le \|R - S\| |g(x)|;$$
so $d_{Rg}(Sg) \le \|R - S\|$ where $\|R - S\|$ is a matrix (operator) norm.
- Rescaling. Let $s > 0$, let $m = \min\{1, s\}$, $M = \max\{1, s\}$ then
$$m d_f(g) \le d_{sf}(sg) \le M d_f(g),$$
so
$$f \in C^0 \mapsto sf \in C^0$$
is again a homeomorphism. For the action
$$s \in \mathbb{R} \mapsto sf \in C^0$$
similarly
$$|t - s| m \le d_{sf}(tf) \le |t - s| M.$$

3.4 Caveats

We have then seen many good properties of the *mild topology*; there are some drawbacks though.

- The *mild topology* depends on the choice of distance d_N.
- It is not invariant w.r.t. homeomorphisms as in Proposition 1.4; it is invariant only for right action, *i.e.*, if $\Phi_M : \tilde{M} \to M$ is a homeomorphism, then the map
$$f \mapsto f \circ \Phi_M$$
is a homeomorphism between $C^0(M; N)$ and $C^0(\tilde{M}; N)$, where both spaces are endowed with the *mild topology*.
- The space $C^0(M; N)$ with the mild topology may fail to be connected, since proper maps are open and closed.
- When $N = \mathbb{R}^n$ with Euclidean structure, the space $C^0(M; \mathbb{R}^n)$ with the mild topology is not in general a *topological vector space*; there are many reasons, we list some.
  - For $g \in C^0$ fixed, the map
  $$f \in C^0 \mapsto g + f \in C^0$$
  may fail to be continuous. For example, consider $g(x) = -e^x$ and $C^0 = C^0(\mathbb{R};\mathbb{R})$, $f(x) = e^x$. Then the counter image of the mild pseudo ball $B(0, 1)$ is
  $$\left\{ h(x) + e^x : h \in C^0, \sup_{x \in \mathbb{R}^n} |h(x)| < 1 \right\},$$
  and it does not contain any mild pseudo ball $B(e^x, \varepsilon)$.
  - For $f \in C^0$ fixed, the map
  $$\lambda \in \mathbb{R} \mapsto \lambda f \in C^0$$
  may fail to be continuous at $\lambda = 0$ (adapting the previous example).
  - The space may not be connected.

4 Conclusion

We have discussed a novel method to define topologies, by *separation functions*; we have shown that, even when the topology happens to be metrizable, it may happen that the actual metric is not known and/or that the *separation functions* are more manageable than the metric that metrizes the topology.

We have studied the *mild topology* $C^0(M;\ N)$; it has some good properties: proper maps are a closed and open subset of $C^0(M;\ N)$, as in the case of the *strong topology*; affine actions on $N = \mathbb{R}^n$ are continuous on $C^0(M;\ \mathbb{R}^n)$, as in the case of the *weak topology*.

It is possible to define similar concepts for $C^r(M;\ N)$, the space of r times differentiable maps between two differentiable manifolds M, N; similar properties hold and can be extended to other interesting classes of maps such as immersions, free immersions, submersions, embeddings, diffeomorphisms; this may be argument of a forthcoming paper.

Acknowledgements: The author thank Prof. Laura A. Ba for stylistic and grammatical revision of the text.

Funding information: This project was supported by the RICBASE_2022_MENNUCCI funding.

Conflict of interest: The authors state no conflict of interest.

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Appendix

A Proofs

A.1 Proof of Proposition 2.10

Proof. Fix $y \in X$. In the hypotheses, this form of implicit function theorem holds. Given $0 < \beta < \alpha$, let $r = \rho_d(y, \alpha, \beta)$, then there are small open intervals $I_\alpha, I_\beta \subseteq (0, \infty)$ with $\alpha \in I_\alpha, \beta \in I_\beta$ and a homeomorphic strictly increasing function $R : I_\alpha \to I_\beta$ such that,

$$\forall a \in I_\alpha, \forall b \in I_\beta, \quad 0 < a < b \quad \text{and} \quad (r = \rho_d(y, a, b) \;\Leftrightarrow\; R(a) = b).$$

We now prove the statement.

- Suppose that we want to prove (2.5) namely
$$d_x(y) \le \beta \wedge d_y(z) \le r \Rightarrow d_x(z) \le \alpha$$
knowing that (2.2) holds. If $d_y(z) < r$ then by (2.2) we readily get $d_x(z) < \alpha$; if $d_y(z) = r$ then, for $a > \alpha$ by hypotheses $r < \rho_d(x, \beta, a)$ so by (2.2) $d_x(z) < a$ and by arbitrariness of a this implies $d_x(z) \le \alpha$.
- Suppose that we want to prove (2.2) namely
$$d_x(y) \le \beta \wedge d_y(z) < r \Rightarrow d_x(z) < \alpha$$
knowing that (2.5) holds. Let $a < \alpha$ be near enough so that $d_y(z) < \rho_d(x, \beta, a) < r$ so by (2.5) $d_x(z) \le a < \alpha$.
- Suppose that we want to prove (2.5) namely
$$d_x(y) \le \beta \wedge d_y(z) \le r \Rightarrow d_x(z) \le \alpha,$$
knowing that (2.6) holds. If $d_x(y) < \beta$ then by (2.6) we readily obtain $d_x(z) < \alpha$; if $d_x(y) = \beta$, then we choose $b \in I_\beta$, $b > \beta$ and $a = R^{-1}(b)$ so (2.6) $d_x(z) < a$ then we let $b \to \beta$ and we know that $a \to \alpha$.
- Suppose that we want to prove (2.6) namely
$$d_x(y) < \beta \wedge d_y(z) \le r \Rightarrow d_x(z) < \alpha,$$
knowing that (2.5) holds. We choose $a \in I_\alpha, b \in I_\beta$ such that $a = R^{-1}(b)$ and $d_x(y) < b < \beta$, then by (2.5) $d_x(z) \le a < \alpha$.
- It is easily seen that (2.6) and (2.2) imply (2.7).
- Suppose that we want to prove (2.5) namely
$$d_x(y) \le \beta \wedge d_y(z) \le r \Rightarrow d_x(z) \le \alpha,$$
knowing that (2.7) holds. We choose $a > \alpha$, $a \in I_\alpha$ so $R(a) > \beta$; then
$$b = \frac{\beta + R(a)}{2}$$
so $R(a) > b > \beta$ then $r < \rho_d(x, a, b)$. We can now apply (2.7) in the form
$$d_x(y) < b \wedge d_y(z) < \rho_d(x, a, b) \Rightarrow d_x(z) < a$$
to obtain that $d_x(z) < a$. Eventually, we let $a \to \alpha$ and we know that $b \to \beta$ and $\rho_d(x, a, b) \to r$. ☐

A.2 Proof of Theorem 2.29

Proof. The proof is an adaptation of the analogous proof in Lemma 39.2 in [8]. We will use the *set separation function* to simplify some arguments.

- Given A open we define the *open erosion*
$$S_n(A) = \{x \in X : d_{X \setminus A}(x) > 1/n\}.$$

In relation to Lemma 2.27 we note that

$$S_n(A) = F_{1/n}(X\setminus A),$$

so the results of the lemma imply that $S_n(A)$ is open, $S_n(A) \subseteq S_{n+1}(A)$, and

$$A = \bigcup_{n=1}^{\infty} S_n(A). \tag{A.1}$$

- We also define the *closed erosion*

$$C_n(A) = \{x \in X : d_{X\setminus A}(x) \geq 1/(2n)\}.$$

Obviously, $C_n(A)$ is closed and

$$C_n(A) \subseteq A. \tag{A.2}$$

- By using Zermelo's theorem, we associate with $\mathcal{A}$ a well ordering $\preccurlyeq$; we also define for convenience $x \prec y$ as $x \preccurlyeq y \wedge x \neq y$.
- For each $U \in \mathcal{A}$ we define

$$T_n(U) = S_n(U)\setminus C_n\left(\bigcup_{Z\in\mathcal{A}, Z \prec U} Z\right);$$

it is open.
- Let

$$\mathcal{E}_n = \{T_n(U) : U \in \mathcal{A}\}$$

and let $\mathcal{E} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$: it is clearly a refinement of $\mathcal{A}$ since $T_n(U) \subseteq S_n(U) \subseteq U$.
- We prove that it is an open cover. Fix $x \in X$, let then U be the minimum $U \in \mathcal{A}$ such that $x \in U$, minimum according to the well ordering $\preccurlyeq$ of $\mathcal{A}$. If $V \prec U$, then $x \notin V$ and hence

$$x \notin \bigcup_{V\in\mathcal{A}, V \prec U} V.$$

Then, by equation (A.2),

$$x \notin C_n\left(\bigcup_{Z\in\mathcal{A}, Z \prec U} Z\right);$$

whereas for n large we have $x \in S_n(U)$ (by relation (A.1)). So $x \in T_n(U)$ for n large.
- For any fixed $n \in \mathbb{N}$, we prove that $\mathcal{E}_n$ is locally finite. Indeed, for any fixed $x \in X$, we show that, for $\varepsilon > 0$ small enough, any ball $B(w, \varepsilon)$ intersects at most one set $T_n(U)$, for $U \in \mathcal{A}$.
-

To this end, we choose $\varepsilon > 0$ small enough so that Lemma 2.28 holds for $s = 1/(2n)$, $t = 1/n$. Let then $w \in X$, consider a ball $B(w, \varepsilon)$; suppose that $B(w, \varepsilon)$ intersects any set in $\mathcal{E}_n$; let U be the minimum (in the well ordering $\preccurlyeq$ of $\mathcal{A}$) of all $Z \in \mathcal{A}$ such that $T_n(Z)$ intersects $B(w, \varepsilon)$; we now show that $B(w, \varepsilon)$ does not intersect any $T_n(V)$ for $U \prec V$. Indeed, by Lemma 2.28, we know that

$$B(w, \varepsilon) \subseteq F_{1/2n}(X\setminus U) \subseteq C_n(U) \subseteq C_n\left(\bigcup_{Z\in\mathcal{A}, Z \prec V} Z\right).$$ □

A.3 Proof of Theorem 2.31

Proof.
- We define this useful notation: for any $w \in X$ let $J_w \subseteq I$ be the finite set

$$J_w = \{i \in I : w \in V_i\}.$$

- By the definition we have symmetry $d_x(y) = d_y(x)$; it is also clear that

$$d_x(y) = 0 \iff x = y.$$

- For $i \in I$, we let

$$f_i(x, y) = \begin{cases} |\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} & \text{if } x \in V_i \wedge y \in V_i, \\ +\infty & \text{otherwise;} \end{cases}$$

then each f_i is continuous and we can rewrite the definition equation (2.14) of d_x as

$$d_x(y) = \min_{i \in I} f_i(x, y). \tag{A.3}$$

Using the fact that the cover is locally finite, we know that in a small neighborhood of (x, y) the above RHS in equation (A.3) is the minimum for i in a finite subset of I. Then $d_x(y)$ are jointly continuous in x, y.
- This implies that the "*pseudo balls*"

$$B_d(x, \varepsilon) = \{y \in X : d_x(y) < \varepsilon\}$$

are open; we express them explicitly as

$$B_d(x, \varepsilon) = \bigcup_{i \in J_x} \varphi_i^{-1}(B_{\mathbb{R}^n}(\varphi_i(x), \varepsilon)).$$

where $B_{\mathbb{R}^n}$ are standard balls in $\mathbb{R}^n$. We now show that they generate the topology, showing that for each $x \in X$, $W \subseteq X$ open with $x \in W$, there is $\varepsilon > 0$ such that

$$B_d(x, \varepsilon) \subseteq W.$$

To this end, for each $i \in J_x$ consider

$$\varphi_i(V_i \cap W);$$

this set is open and contains $\varphi_i(x)$, so there is $\varepsilon_i > 0$ such that

$$B_{\mathbb{R}^n}(\varphi_i(x), \varepsilon) \subseteq \varphi_i(V_i \cap W).$$

Now we let

$$\varepsilon = \min_{i \in I_x} \varepsilon_i,$$

so

$$\begin{aligned} d_x(y) < \varepsilon \quad &\Rightarrow \\ \exists i \in I, x \in V_i \wedge y \in V_i \wedge |\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} < \varepsilon \quad &\Rightarrow \\ \exists i \in I, x \in V_i \wedge y \in V_i \wedge \varphi_i(x) \in \varphi_i(V_i \cap W) \quad &\Rightarrow \\ y \in W. \end{aligned}$$

- We want to prove the "pseudo triangle inequality" (2.2) and the "pseudo reverse triangle inequality" (2.3). To this end, we fix $y \in X$, and $0 < \beta < \alpha$, we will show that there exists

$$\rho_d = \rho_d(y, \alpha, \beta)$$

such that, for all $x, z \in X$,

$$d_x(y) \le \beta \wedge d_y(z) < \rho_d \Rightarrow d_x(z) < \alpha,$$

$$d_x(y) \ge \alpha \wedge d_y(z) < \rho_d \Rightarrow d_x(z) > \beta;$$

where these two are respectively the equations (2.2) and (2.3).
- Let

$$A_y = \bigcap_{i \in J_y} V_i = \bigcap_{i \in I, y \in V_i} V_i$$

$$U_y = \bigcup_{i \in J_y} V_i = \bigcup_{i \in I, y \in V_i} V_i$$

that are open.
- Let us fix W open such that $\overline{W}$ is compact,

$$y \in W \subseteq \overline{W} \subseteq A_y,$$

$\overline{W}$ intersects only finitely many V_i; let $\tilde{J} = \{i \in I : V_i \cap \overline{W} \neq \emptyset\}$ be this finite set of indexes; note that $J_y \subseteq \tilde{J}$ since $y \in V_i \cap \overline{W}$ when $i \in J_y$.

- As proven above, there is an $\tilde{\varepsilon} > 0$ such that

$$B_d(y, \tilde{\varepsilon}) \subseteq W,$$

that is

$$\forall z, d_y(z) < \tilde{\varepsilon} \Rightarrow z \in W. \tag{A.4}$$

- Let $z \in W$ be given; for any $i \in J_z$ we have that $z \in W \cap V_i$ and this implies $i \in \tilde{J}$; at the same time, for any $j \in J_y$, we know that $z \in W \subseteq A_y \subseteq V_j$ and this implies $j \in J_z$; summarizing, for any $z \in X$, we have

$$d_y(z) < \tilde{\varepsilon} \Rightarrow \tilde{J} \supseteq J_z \supseteq J_y. \tag{A.5}$$

- Note that $d_x(y) = \infty$ iff $x \notin U_y$; for $\varepsilon > 0$ and $\varepsilon < \tilde{\varepsilon}$ let

$$l(\varepsilon) \stackrel{\text{def}}{=} \inf_{x \notin U_y \wedge d_y(z) < \varepsilon} d_x(z);$$

it is weakly decreasing in ε; we prove that

$$\lim_{\varepsilon \to 0} l(\varepsilon) = +\infty.$$

In the above infimum, we can assume that $x \in V_i$ for an $i \in \tilde{J}$; this follows from equation (A.5), indeed if for all $i \in \tilde{J}$ we have $x \notin V_i$ then, *a fortiori*, for all $i \in J_z$ we have $x \notin V_i$ hence $d_x(z) = +\infty$. Let

$$Z = \bigcup_{i \in \tilde{J}} V_i$$

and we know that $\overline{Z}$ is compact. Suppose by contradiction that the above limit does not hold; there is then a sequence z_n with $d_y(z_n) < 1/n$ and $x_n \notin U_y$ such that $d_{x_n}(z_n)$ is bounded; since

$$x_n \in \overline{Z} \setminus U_y \quad \text{and} \quad z_n \in \overline{W},$$

we can extract converging subsequences, up to renumbering the sequence we can assume that $x_n \to x$, $z_n \to z$ with

$$x \in \overline{Z} \setminus U_y \quad \text{and} \quad z \in \overline{W},$$

but then also $d_y(z_n) \to d_y(z) = 0$ so $y = z$; at the same time $d_{x_n}(z_n) \to_n d_x(y) = +\infty$ since $x \notin U_y$.

- There exists then a continuous decreasing function $\lambda : [0, \infty] \to [0, \tilde{\varepsilon})$ satisfying: if $\varepsilon \leq \lambda(\beta)$ then $l(\varepsilon) > \beta$.
- We prove this other result. Let $\psi_y(z) : W \to [0, \infty]$ be defined by

$$\psi_y(z) = \max_{i \in J_y} |\varphi_i(z) - \varphi_i(y)|_{\mathbb{R}^m};$$

then this function is continuous as well, moreover, $\psi_y(y) = 0$. For $t > 0$ there is then a $\delta(t) > 0$ such

$$(z \in W \wedge d_y(z) < \delta(t)) \Rightarrow \psi_y(z) < t.$$

We can choose $\delta(t)$ to be continuous and increasing, and such that $\delta(t) < \tilde{\varepsilon}$.

- Let eventually

$$\rho_d(y, \alpha, \beta) = \min\{\delta(\alpha - \beta), \lambda(\beta)\, \alpha/(1 - \alpha)\}.$$

Note that this satisfies the requirements of Proposition 2.10. We will show that this solves the problem, in three steps. Note that

$$d_y(z) < \rho_d \Rightarrow \psi_y(z) < \alpha - \beta.$$

- We prove (2.2). Pick now a $x \in X$; if $d_x(y) \leq \beta$, then there is an $i \in I$ such that $x \in V_i \wedge y \in V_i$ and

$$|\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} \leq \beta;$$

if moreover $d_y(z) < \rho_d$, then $z \in W \subseteq V_i$; so the usual triangular inequality tells that

$$|\varphi_i(z) - \varphi_i(x)|_{\mathbb{R}^m} \le |\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} + |\varphi_i(y) - \varphi_i(z)|_{\mathbb{R}^m} \le \beta + \psi_y(z) < \alpha$$

and this proves (2.2).

- Regarding (2.3), assuming $d_y(z) < \tilde{\varepsilon}$, we divide two cases. If $d_x(y) < \infty$, then consider all $i \in I$ such that $x \in V_i \wedge y \in V_i$; then $z \in V_i$ (by the relation (A.4)), so

$$|\varphi_i(z) - \varphi_i(x)|_{\mathbb{R}^m} \ge |\varphi_i(x) - \varphi_i(y)|_{\mathbb{R}^m} - |\varphi_i(y) - \varphi_i(z)|_{\mathbb{R}^m} \ge \alpha - \psi_y(z);$$

but knowing that $d_y(z) < \rho_d \le \delta(\alpha - \beta)$ we obtain $\psi_y(z) < \alpha - \beta$: so

$$|\varphi_i(z) - \varphi_i(x)|_{\mathbb{R}^m} > \beta$$

and passing to the minimum in all such i we obtain $d_x(z) > \beta$.

- Assuming $d_y(z) < \tilde{\varepsilon}$ and $d_x(y) = \infty$, we know that $d_y(z) < \rho_d \le \lambda(\beta)$ implies $l(\varepsilon) > \beta$ so $d_x(z) > \beta$. □

In the above proof, we understand why ρ_d must depend on y.

A.4 Proof of Proposition 3.14

Proof. Let $Fy = Ay + v$ be the affine transformation given by $A \in GL(\mathbb{R}^n)$, $v \in \mathbb{R}^n$; we know that

$$m|x| \le |Ax| \le M|x|$$

(e.g., setting

$$M = \|A\|, \quad m = \frac{1}{\|A^{-1}\|},$$

where $\|A\|$ is an operator norm for the linear transformation A).

We now estimate $d_{Ff}(Fg)$ noting that

$$\frac{|Fy - Fz|}{1 + |Fy|} \ge \frac{|A(y - z)|}{1 + |v| + |Fy|} \ge \frac{m|y - z|}{1 + |v| + M|y|} \ge \frac{|y - z|}{1 + |y|} c_F \textit{ with } c_F = \frac{m}{1 + |v| + M}$$

and using the relation

$$\frac{1}{c + ab} \ge \frac{1}{(b + 1)(a + c)}$$

valid for all $a, b, c > 0$; hence

$$d_{Ff}(Fg) \ge c_F d_f(g)$$

but also

$$d_{Ff}(Fg) \le \frac{1}{c_{F^{-1}}} d_f(g).$$

Let now $F_2 y = A_2 y + v_2$, then

$$d_{Fg}(F_2 g) \le \sup_{y \in \mathbb{R}^n} \frac{|F_2 y - Fy|}{1 + |Fy|} = \sup_{y \in \mathbb{R}^n} \frac{|F_2 F^{-1} y - y|}{1 + |y|} \le c_F(F_2),$$

where

$$c_F(F_2) = \|A_2 - A\| \|A^{-1}\| |v| + |v - v_2|;$$

this $c_F(F_2)$ is a "separation" that generates the topology of the space of affine maps: indeed, having

$$F_2 F^{-1} y - y = A_2 A^{-1}(y - v) + v_2 - y = (A_2 A^{-1} - \mathbb{1})(y - v) + (v_2 - v),$$

(where $\mathbb{I}$ is the identity operator) there follows

$$|F_2F^{-1}y - y| \leq \|A_2A^{-1} - \mathbb{I}\|(|y| + |v|) + |v_2 - v| \leq \|A_2 - A\|\|A^{-1}\|(|y| + |v|) + |v - v_2|.$$

We eventually fix $\alpha > 0$, f and F; then we can choose $\beta < \alpha$ and $\rho = \rho(\alpha, \beta)$ as in Lemma 3.3. If

$$d_f(g) < c_{F^{-1}}\beta, \quad c_F(F_2) < \rho,$$

then

$$d_{Ff}(Fg) < \beta, \quad d_{Fg}(F_2g) < \rho$$

and then $d_{Ff}(F_2g) < \alpha$ by Lemma 3.3. □