Non-compactness and multiplicity results for the Yamabe problem on S^n

Massimiliano Berti - Andrea Malchiodi

Scuola Internazionale Superiore di Studi Avanzati
via Beirut, 2-4 – 34014 Trieste, Italy

Abstract

The aim of this paper is to show the existence of metrics $\overline{g}_\varepsilon$ on S^n, where $\overline{g}_\varepsilon$ is a perturbation of the standard metric $\overline{g}_0$, for which the Yamabe problem possesses a sequence of solutions unbounded in $L^\infty(S^n)$. The metrics $\overline{g}_\varepsilon$ that we find are of class C^k on S^n with ($k \leq \frac{n-3}{4}$). We also prove some new multiplicity results.

1 Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 3$ with scalar curvature R_g. The conformal deformation $g' = u^{\frac{4}{n-2}} g$ of g, where $u : M \to \mathbb{R}$ is a smooth positive function, has scalar curvature $R_{g'}$ related to R_g by

$$-2c_n \Delta_g u + R_g u = R_{g'} u^{\frac{n+2}{n-2}}; \qquad c_n = 2\frac{(n-1)}{(n-2)},$$

where Δ_g is the Laplace-Beltrami operator on (M, g), see [5]. The Yamabe problem consists in finding some metric g' in the conformal class $[g]$ of g such that its scalar curvature $R_{g'}$ is a constant function. Choosing $R_{g'} \equiv 1$ then the problem is equivalent to finding a solution to the equation on M

$$(1) \qquad -2c_n \Delta_g u + R_g u = u^{\frac{n+2}{n-2}}, \quad u > 0.$$

A positive answer to the Yamabe problem has been given by Th. Aubin, see [4, 5], who proved that if (M^n, g), $n \geq 6$, is not locally conformally flat, then the Yamabe problem has at least one solution. The locally conformally flat case and dimensions $n = 3, 4, 5$ have been handled by R. Schoen [18], see also [20]. For a detailed treatment of this topic see for example the review [12]. See also [6] and [7] for different proofs.

In [19], R. Schoen announced the following compactness Theorem, giving a detailed proof for the locally conformally flat case.

Theorem 1.1 *Let (M, g) be a compact C^∞ manifold not conformally equivalent to the standard sphere. Then the set of solutions of problem (1) is compact in $C^{2,\alpha}(M)$.*

It is a natural question to see if Theorem 1.1 can be extended to C^k metrics on manifolds of arbitrary dimension. The main purpose of our paper is to show that this is not the case. Let $\overline{g}_0$ denote the standard metric on S^n. Our main result is the following.

Theorem 1.2 *Let $k \geq 2$ and $n \geq 4k+3$. Then there exists a family of C^k metrics $\overline{g}_\varepsilon$ on S^n, with $\|\overline{g}_\varepsilon - \overline{g}_0\|_{C^k(S^n)} \to 0$ as $\varepsilon \to 0$, which possess the following property. For every ε small enough, problem (1) on $(S^n, \overline{g}_\varepsilon)$ has a sequence of solutions v_ε^i with $\|v_\varepsilon^i\|_{L^\infty(S^n)} \to +\infty$ as $i \to \infty$.*

Remark 1.1 *It is an open problem to find the sharpest condition on n and k for which the above non-compactness result is true.*

The proof of Theorem 1.2 is based on a sharpening of a construction introduced in [3]; since this paper is the starting point of our work we discuss it in more detail. There the authors consider on S^n a suitable class of metrics $\overline{g}_\varepsilon = \overline{g}_0 + \varepsilon \overline{h}$, perturbations of the standard one, and prove the existence of two solutions of the Yamabe problem.

Using stereographic coordinates problem (1) for $(M, g) = (S^n, \overline{g}_\varepsilon)$ can be reduced to study

$$-2c_n \Delta_g u + R_g u = u^{(n+2)/(n-2)} \quad \text{in } \mathbb{R}^n, \qquad u > 0. \tag{2}$$

Here $g = g_\varepsilon$ is the metric with components $g_{ij} = z_0^{-\frac{4}{n-2}} \overline{g}_{ij}$, where $z_0 : \mathbb{R}^n \to \mathbb{R}$ is given by

$$z_0(x) = \kappa_n \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}}, \qquad \kappa_n = (4n(n-1))^{\frac{n-2}{4}}.$$

Taking $\overline{g}_\varepsilon = \overline{g}_0 + \varepsilon \overline{h}$, it turns out that

$$g_{ij} = \delta_{ij} + \varepsilon h_{ij}, \tag{3}$$

for some symmetric matrix h_{ij}. The Weyl tensor W_g of the metric g in (3) can be expanded in powers of ε as $W_g = \varepsilon \overline{W}_h + o(\varepsilon)$, where $\overline{W}_h$ depends only on h. The main result of [3] is the following.

Theorem 1.3 *Let $n \geq 6$, and let h be of the form*

$$h(x) = \tau(x) + \omega(x - x_0), \tag{4}$$

where τ, ω are of class C^∞, with compact support, and with $\overline{W}_\tau, \overline{W}_\omega \not\equiv 0$. Then there exists $\overline{L} > 0$ such that for $|x_0| \geq \overline{L}$ there exists $\widetilde{\varepsilon} > 0$ for which, for $|\varepsilon| \leq \widetilde{\varepsilon}$, there exist at least two different solutions $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ of problem (2).

Coming back to the original problem on S^n, Theorem 1.3 implies the existence of at least two solutions for problem (1) on $(S^n, \overline{g}_\varepsilon)$.

Solutions of (2) can be found as critical points of the functional $f_\varepsilon : E = \mathcal{D}^{1,2}(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$f_\varepsilon(u) = \int_{\mathbb{R}^n} \left(c_n |\nabla_g u|^2 + \frac{1}{2} R_g u^2 - \frac{1}{2^*} |u|^{2^*} \right) dV_g, \qquad u \in E, \tag{5}$$

where $2^* = \frac{2n}{n-2}$. The positive solutions of $f_0' = 0$ constitute an $(n+1)$-dimensional manifold Z given by

$$Z = \left\{ z_{\mu,\xi} = \mu^{-\frac{n-2}{2}} z_0 \left(\frac{x-\xi}{\mu} \right) \;\middle|\; \mu > 0,\ \xi \in \mathbb{R}^n \right\} \simeq \mathbb{R}_+ \times \mathbb{R}^n.$$

Using the Implicit Function Theorem it is shown, see [1], [2], that there exists a manifold Z_ε, perturbation of Z, which is a natural constraint for f_ε, namely if $f_\varepsilon'|_{Z_\varepsilon}(u) = 0$ for some $u \in Z_\varepsilon$, then also $f_\varepsilon'(u) = 0$. In the case of (5) it turns out that

$$f_\varepsilon(z_\varepsilon) = b_0 + \varepsilon^2 \Gamma(z_\varepsilon) + o(\varepsilon^2); \qquad b_0 = f_0(z_0),$$

for some $\Gamma : Z \to \mathbb{R}$. Hence, roughly, critical points of Γ give rise, for ε small, to solutions of (2). If $\overline{W} \not\equiv 0$, then Γ admits some minima and, when $|x_0|$ is large, Γ inherits a double well structure: this guarantes the existence of at least two solutions $u_{1,\varepsilon}, u_{2,\varepsilon}$ of (2).

In this paper, the above result is extended by showing the existence of metrics on S^n, perturbations of the standard one, for which problem (1) possesses infinitely many distinct solutions, which are not bounded in $L^\infty(S^n)$. This is done by considering on $\mathbb{R}^n$ a metric $g = g_\varepsilon = \delta + \varepsilon h$ with

$$h(x) = \sum_{i \in \mathbb{N}} \sigma_i \, \tau(x - x_i), \tag{6}$$

where $\tau : \mathbb{R}^n \to \mathbb{M}^{n \times n}$ is a C^∞ matrix-valued function with compact support, $\overline{W}_\tau \not\equiv 0$, $\sigma_i \in \mathbb{R}$, and $|x_i| \to +\infty$ as $i \to \infty$. Using the fact that the metric g possesses infinitely many "bumps", we prove that the function $f_\varepsilon|_{Z_\varepsilon}$ inherits infinitely many local minima provided the points x_i are sufficiently far away one from each other. The last step of the proof of Theorem 1.2 consists in proving that:

i) the metric g_ε gives rise to a C^k metric $\overline{g}_\varepsilon$ on S^n;

ii) for ε small, problem (1) for $(S^n, \overline{g}_\varepsilon)$ has a sequence of solutions whose L^∞ norm blows up.

The method we use can be extended to prove some new multiplicity results. Let us recall that the existence of multiple solutions for the Yamabe problem has been studied in [10], [19] and [17]. In [10] multiplicity is obtained under symmetry assumptions while in [19] the author considers the specific case of $S^1(T) \times S^n$, where $S^1(T)$ is the one dimensional circle of radius T. He proves that when $T \to +\infty$, problem (1) possesses an increasing number of solutions. In [17] the author proves that, given any manifold of dimension greater or equal than 3 and with positive scalar curvature, then, for some suitable C^0 perturbation of the metric, the solutions of (1) have a multibump structure.

Our multiplicity results are of two types:

1) we improve Therorem 1.3 by showing the existence of a non-minimal third solution, see Theorem 5.1;

2) in the specific case of the sphere S^n, we improve the result in [17], by proving the same result for C^k perturbations of the standard metric, provided $n \geq 4k + 3$, see Theorem 5.2.

The paper is organized as follows. Section 2 contains some preliminaries. Section 3 deals with the construction of the natural constraint Z_ε for f_ε. In Section 4 Theorem 1.2 is proved, and in Section 5 some related results are treated. The Appendix contains some technical proofs.

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Notations

We denote by $E = \mathcal{D}^{1,2}(\mathbb{R}^n)$ the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the Dirichlet norm $\|u\|^2 = \int_{\mathbb{R}^n} |\nabla u|^2 dx$. (u, v) is the standard scalar product $\int_{\mathbb{R}^n} \nabla u \, \nabla v \, dx$, for $u, v \in E$. Given $u \in E$, the function $u^* \in E$ is defined as

$$u^*(x) = \frac{1}{|x|^{n-2}} u \left(\frac{x}{|x|^2} \right), \qquad x \in \mathbb{R}^n.$$

If $f \in C^1(E)$, we denote by f' or ∇f its gradient. We set $Crit(f) = \{x \in E : \; f'(x) = 0\}$. If $f \in C^2(E)$, $f''(x) : E \to E$ is the linear operator defined by duality in the following way

$$(f''(x)v, w) = D^2 f(x)[v, w], \qquad \forall v, w \in E.$$

If $x \in Crit(f)$, we denote by $m(x,f)$ the Morse index of f at x, namely the maximal dimension of a subspace of E on which f'' is negative definite. We also denote by $m^*(x,f)$ the extended Morse index, the maximal dimension of a subspace of E on which f'' is non-positive definite. For all $u \in E$, $\mu \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ we set $u_{\mu,\xi} = \mu^{-\frac{n-2}{2}} u\left(\frac{x-\xi}{\mu}\right)$. The map π denotes the stereographic projection $\pi : S^n = \left\{x \in \mathbb{R}^{n+1} : \, |x| = 1\right\} \to \mathbb{R}^n$ through the north pole P_N of S^n, $P_N = (0,\dots,0,1)$, where we identify $\mathbb{R}^n$ with $\left\{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\right\}$. The map $\mathcal{R} : S^n \to S^n$ is the reflection through the hyperplane $\{x_{n+1} = 0\}$, i.e. for $(x', x_{n+1}) \in S^n$, it is $\mathcal{R}(x', x_{n+1}) = (x', -x_{n+1})$. Given a function $v : \mathbb{R}^n \to \mathbb{R}$, we define $v^\sharp : \mathbb{R}^n \to \mathbb{R}$ in the following way

$$v^\sharp(x) = v\left(\frac{x}{|x|^2}\right), \qquad x \in \mathbb{R}^n.$$

We set $\mathcal{S}_n = \{h : \mathbb{R}^n \to M(n \times n) : h_{ij} = h_{ji}, \forall i,j\}$. In the following, for brevity, the positive constant C will assume possibly different values from line to line.

2 Preliminaries

In this paper we consider metrics on $\mathbb{R}^n$ possessing "infinitely many bumps". In order to describe precisely such metrics we introduce some notations.

Let $\tau : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function with compact support with $\overline{W}_\tau \not\equiv 0$, see formula (13). For $A > 0$, let $\mathcal{H}_A \subseteq \mathcal{S}_n$ be defined by

$$\mathcal{H}_A = \left\{ h \; : \; h(x) = \sum_{i \in \mathbb{N}} \sigma_i \, \tau(x - x_i), \, |x_i - x_j| \geq 4 \; \text{diam}\; (supp\, \tau), \, i \neq j, \, \sum_i |\sigma_i|^{\frac{n}{2}} \leq A \right\}. \tag{7}$$

We will consider the following class of metrics on $\mathbb{R}^n$

$$g_{ij} = (g_\varepsilon)_{ij} = \delta_{ij} + \varepsilon h_{ij}, \tag{8}$$

where ε is a small parameter and $h = h_{ij} \in \mathcal{H}_A$.

Geometric preliminaries and expansion of f_ε

We recall some formulas given in [3] which will be useful for our computations. It will always be understood that the expansions in ε below are uniform for $h \in \mathcal{H}_A$. We denote with $g_{ij} = \delta_{ij} + \varepsilon h_{ij}$ the coefficients of the metric g and with g^{ij} the elements of the inverse matrix $(g^{-1})_{ij}$. The volume element dV_g of the metric g is

$$dV_g = |g|^{\frac{1}{2}}\, dx = \left(1 + \varepsilon \, \frac{1}{2} tr \; h + \varepsilon^2 \left(\frac{1}{8}(trh)^2 - \frac{1}{4} tr(h^2)\right) + o(\varepsilon^2)\right) dx. \tag{9}$$

The Christoffel symbols are given by $\Gamma^l_{ij} = \frac{1}{2}[D_i g_{kj} + D_j g_{ki} - D_k g_{ij}] g^{kl}$. The components of the Riemann tensor, the Ricci tensor and the scalar curvature are given respectively by

$$R^l_{kij} = D_i \Gamma^l_{jk} - D_j \Gamma^l_{ik} + \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^l_{jm} \Gamma^m_{ik}; \qquad R_{kj} = R^l_{klj}; \qquad R = R_{kj} g^{kj}. \tag{10}$$

The Weyl tensor W_{ijkl} is defined by

$$W_{ijkl} \;\; = \;\; R_{ijkl} - \frac{1}{n-2}\left(R_{ik} g_{jl} - R_{il} g_{jk} + R_{jl} g_{ik} - R_{jk} g_{il}\right) + \frac{R}{(n-1)(n-2)}(g_{jl} g_{ik} - g_{jk} g_{il}).$$

For a smooth function u the components of $\nabla_g u$ are $(\nabla_g u)^i = g^{ij} D_j u$, so we have

$$(\nabla_g u)^i = \nabla u \, (1 + O(\varepsilon)), \tag{11}$$

and moreover

$$|\nabla_g u|^2 = |\nabla u|^2 - \varepsilon \sum_{i,j} h_{ij} D_i u D_j u + \varepsilon^2 \sum_{i,j,l} h_{il} h_{lj} D_i u D_j u + o(\varepsilon^2). \tag{12}$$

Let R_ε be the scalar curvature of g. There holds, see [3],

$$R_\varepsilon(x) = \varepsilon \, R_1(x) + \varepsilon^2 \, R_2(x) + o(\varepsilon^2),$$

where

$$R_1 = \sum_{i,j} D^2_{ij} h_{ij} - \Delta \; tr \, h,$$

and

$$\begin{aligned} R_2 \; = \; & -2 \sum_{k,j,l} h_{kj} D^2_{lk} h_{lj} + \sum_{k,j,l} h_{kj} D^2_{ll} h_{kj} + \sum_{k,j,l} h_{kj} D^2_{jk} h_{ll} + \frac{3}{4} \sum_{k,j,l} D_k h_{jl} D_k h_{jl} \\ & - \sum_{k,j,l} D_l h_{jl} D_k h_{jk} + \sum_{k,j,l} D_l h_{jl} D_j h_{kk} - \frac{1}{4} \sum_{k,j,l} D_j h_{ll} D_j h_{kk} - \frac{1}{2} \sum_{k,j,l} D_j h_{lk} D_l h_{jk}. \end{aligned}$$

Similarly we define the tensor $\overline{W}_{ijkl}$ by

$$W_{ijkl} = \varepsilon \, \overline{W}_{ijkl} + o(\varepsilon). \tag{13}$$

By formulas (9) and (11) the functionals $u \to \int |\nabla_g u|^2 dV_g$, $u \to \int |u|^{2^*} dV_g$ are well defined for $u \in E$ and $h \in \mathcal{H}_A$. Moreover, for $h \in \mathcal{H}_A$, the supports of the functions $\tau(\cdot - x_i)$ are all disjoint, so there holds $R_{g_\varepsilon} \leq |\varepsilon| \, R_h$, with $R_h \in L^{\frac{n}{2}}(\mathbb{R}^n)$, and $\|R_h\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}$ uniformly bounded, by the condition $\sum_i |\sigma_i|^{\frac{n}{2}} < A$. Hence also the map $u \to \int \; R_g \, u^2 dV_g$ is well defined. In conclusion the Euler functional $f_\varepsilon : E \to \mathbb{R}$

$$f_\varepsilon(u) = \int \left(c_n |\nabla_g u|^2 + \frac{1}{2} R_g u^2 - \frac{1}{2^*} |u|^{2^*} \right) dV_g, \qquad g = \delta + \varepsilon \, h, \tag{14}$$

is well defined, provided $h \in \mathcal{H}_A$ and ε is sufficiently small. The functional f_ε in (14) admits the following expansion

$$\forall u \in E, \qquad f_\varepsilon(u) = f_0(u) + \varepsilon \, G_1(u) + \varepsilon^2 \, G_2(u) + o(\varepsilon^2),$$

where

$$f_0(u) = \int \left(c_n |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} \right) dx;$$

$$G_1(u) = \int \left(-c_n \sum_{i,j} h_{ij} D_i u D_j u + \frac{1}{2} R_1 u^2 + \left(c_n |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} \right) \frac{1}{2} trh \right) dx;$$

$$\begin{aligned} G_2(u) \; = \; & \int \Bigg[c_n \sum_{i,j,l} h_{il} h_{lj} D_i u D_j u + \frac{1}{2} R_2 u^2 + \left(c_n |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} \right) \left(\frac{1}{8} (tr \; h)^2 - \frac{1}{4} tr(h^2) \right) \\ & + \frac{1}{2} trh \left(\frac{1}{2} R_1 u^2 - c_n \sum_{i,j} h_{ij} D_i u D_j u \right) \Bigg] \, dx. \end{aligned}$$

We now describe in some detail how problem (1) on S^n can be reduced to problem (2) on $\mathbb{R}^n$, and viceversa. The stereographic projection $\pi : S^n \to \mathbb{R}^n$ induces an isomorphism $\iota : H^1(S^n) \to E$ defined by

$$(\iota u)(x) = z_0(x)\, u(\pi^{-1}(x)), \qquad u \in H^1(S^n), \quad x \in \mathbb{R}^n. \tag{15}$$

In particular the following relations hold for all $u, v \in H^1(S^n)$

$$2c_n \int_{\mathbb{R}^n} \nabla \iota u \cdot \nabla \iota v = \int_{S^n} (2c_n \nabla_{g_0} u \cdot \nabla_{g_0} v + uv)\, dV_{g_0}, \quad \int_{\mathbb{R}^n} (\iota u)^{2^*-1} \iota v = \int_{S^n} u^{2^*-1} v. \tag{16}$$

If $\overline{g}$ is a Riemannian metric on S^n, the Euler functional $J : H^1(S^n) \to \mathbb{R}$ associated to problem (1) is

$$J(v) = \int_{S^n} \left(c_n |\nabla_{\overline{g}} v|^2 + \frac{1}{2} R_{\overline{g}} v^2 - \frac{1}{2^*} |v|^{2^*} \right) dV_{\overline{g}}, \qquad v \in H^1(S^n).$$

Using stereographic coordinates on S^n, we define the metric g on $\mathbb{R}^n$ to be

$$g_{ij}(x) = z_0^{-\frac{4}{n-2}}(x)\, \overline{g}_{ij}(x) \tag{17}$$

and, associated to g, the functional $f : E \to \mathbb{R}$

$$f(u) = \int_{\mathbb{R}^n} \left(c_n |\nabla_g u|^2 + \frac{1}{2} R_g u^2 - \frac{1}{2^*} |u|^{2^*} \right) dV_g, \qquad u \in E.$$

The functional J is related to f from the equation

$$J(u) = f(\iota(u)), \qquad u \in H^1(S^n). \tag{18}$$

From equality (18) one deduces immediately that the functions $\{\iota^{-1} z_{\mu,\xi}\}_{\mu,\xi}$ are positive solutions of $J_0' = 0$.

Let $\overline{g}_{\mathcal{R}}$ be the pull back of $\overline{g}$ through $\mathcal{R}$, see Notations. Then $\overline{g}_{\mathcal{R}}$ gives rise to the metric

$$g^\sharp_{ij}(x) := z_0^{-\frac{4}{n-2}}(x) (\overline{g}_{\mathcal{R}})_{ij}(x), \qquad x \in \mathbb{R}^n. \tag{19}$$

It turns out, using straightforward computations, that

$$\sum_{ij} g^\sharp_{ij}(x)\, dx_i\, dx_j = \delta_{ij} + \sum_{ij} \left(g_{ij}\left(\frac{1}{x}\right) - \delta_{ij} \right) \left(dx_i - \frac{2\, x_i \sum_k x_k\, dx_k}{|x|^2} \right) \left(dx_j - \frac{2\, x_j \sum_l x_l\, dx_l}{|x|^2} \right). \tag{20}$$

Denoting by $f^\sharp$ the functional on E associated to the metric $g^\sharp$, there holds

$$f(u) = f^\sharp(u^*), \qquad u \in E. \tag{21}$$

It is a simple calculation to check that

$$(z_{\mu,\xi})^* = z_{\overline{\mu},\overline{\xi}}, \qquad \text{with} \quad \overline{\mu} = \frac{\mu}{\mu^2 + \xi^2}, \, \overline{\xi} = \frac{\xi}{\mu^2 + \xi^2}. \tag{22}$$

Technical Lemmas

We now collect some technical Lemmas, proved in the Appendix, which will be useful in the sequel.

Lemma 2.1 *Let $n \geq 3$ and $p > 0$. There exists $C > 0$, depending on p, such that for all $a, b \in \mathbb{R}$*

$$|a+b|^p \leq C\,(|a|^p + |b|^p); \tag{23}$$

$$\left| |a+b|^{2^*} - |a|^{2^*} - |b|^{2^*} \right| \leq C\,\left(|a|^{2^*-1}\,|b| + |a|\,|b|^{2^*-1} \right); \tag{24}$$

$$\left| |a+b|^{2^*-2}(a+b) - |a|^{2^*-2}a - |b|^{2^*-2}b \right| \leq C\,\left(|a|^q\,|b|^r + |a|^r\,|b|^q \right), \tag{25}$$

where $q = \frac{(n+2)^2}{2n(n-2)}$, and $r = \frac{(n+2)}{2n}$. Note that $r + q = 2^* - 1$. Moreover, for $n \geq 6$

$$\left| |a+b|^{2^*-2} - |a|^{2^*-2} \right| \leq |b|^{2^*-2}, \qquad \forall\, a, b \in \mathbb{R}. \tag{26}$$

Lemma 2.2 *Let $n \geq 3$. There exists $C > 0$ such that for all $h \in \mathcal{H}_A$ and for all $|\varepsilon|$ sufficiently small there holds*

$$\forall u \in E, \qquad f_\varepsilon(u) - f_0(u) - \varepsilon\,G_1(u) - \varepsilon^2\,G_2(u) = o(\varepsilon^2)\,\left(\|u\|^2 + \|u\|^{2^*} \right); \tag{27}$$

$$\forall u \in E, \qquad \|f'_\varepsilon(u) - f'_0(u) - \varepsilon\,G'_1(u)\| \leq C\,\varepsilon^2\,\left(\|u\| + \|u\|^{\frac{n+2}{n-2}} \right); \tag{28}$$

$$\forall z \in Z, \qquad \|f'_\varepsilon(z)\| \leq C\,|\varepsilon|; \tag{29}$$

$$\forall u \in E, \qquad \|f''_\varepsilon(u) - f''_0(u)\| \leq C\,|\varepsilon|\,\left(1 + \|u\|^{\frac{4}{n-2}} \right); \tag{30}$$

$$\forall u, w \in E, \qquad |f_\varepsilon(u+w) - f_\varepsilon(u)| \leq C\,\|w\|\,\left(1 + \|u\|^{\frac{n+2}{n-2}} + \|w\|^{\frac{n+2}{n-2}} \right); \tag{31}$$

$$\forall u, w \in E, \qquad \|f'_\varepsilon(u+w) - f'_\varepsilon(u)\| \leq C\,\|w\|\,\left(1 + \|u\|^{\frac{4}{n-2}} + \|w\|^{\frac{4}{n-2}} \right); \tag{32}$$

$$\forall u, w \in E, \qquad \|G'_1(u+w) - G'_1(u)\| \leq C\,\|w\|\,\left(1 + \|u\|^{\frac{4}{n-2}} + \|w\|^{\frac{4}{n-2}} \right); \tag{33}$$

For $n = 3, 4, 5$ we have

$$\forall u, w \in E, \qquad \|f''_\varepsilon(u+w) - f''_\varepsilon(u)\| \leq C\,\|w\|\,\left(\|u\|^{\frac{6-n}{n-2}} + \|w\|^{\frac{6-n}{n-2}} \right). \tag{34}$$

For $n \geq 6$, the last expression becomes

$$\forall\, u, w \in E, \qquad \|f''_\varepsilon(u+w) - f''_\varepsilon(u)\| \leq C\,\|w\|^{\frac{4}{n-2}}. \tag{35}$$

3 Reduction of the functional

The aim of this section is to construct the natural constraint Z_ε for f_ε. This will provide the existence of solutions to (2) close to solutions of the unperturbed problem (36) below. The advantage of our construction respect to [1] and [2] is that it works uniformly for all $h \in \mathcal{H}_A$ and for ε sufficiently small.

The natural constraint

Our starting point is the following Proposition, see [2, 16].

Proposition 3.1 *The unperturbed functional f_0 possesses an $(n+1)$-dimensional manifold Z of critical points, diffeomorphic to $\mathbb{R}_+ \times \mathbb{R}^n$, given by*

$$Z = \left\{ z_{\mu,\xi} := \mu^{-\frac{n-2}{2}} z_0\left(\frac{x-\xi}{\mu}\right) \;\middle|\; \mu > 0,\ \xi \in \mathbb{R}^n \right\} \simeq \mathbb{R}_+ \times \mathbb{R}^n,$$

namely every element $z_{\mu,\xi} \in Z$ is a solution of

$$\begin{cases} -2c_n \Delta u = u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n; \\ u > 0, u \in E. \end{cases} \tag{36}$$

Moreover f_0 satisfies the following properties

(i) $f_0''(z) = I - \mathcal{K}$, where $\mathcal{K}$ is a compact operator for every $z \in Z$;

(ii) $T_z Z = Ker f_0''(z)$ for all $z \in Z$.

From (i)-(ii) it follows that the restriction of f_0'' to $(T_z Z)^\perp$ is invertible. Moreover, denoting by L_z its inverse, there exists $C > 0$ such that

$$\|L_z\| \leq C \quad \text{for all} \quad z \in Z. \tag{37}$$

Through a Lyapunov-Schmidt reduction, using Proposition 3.1, we can reduce problem (2) to a finite dimensional one.
For brevity, we denote by $\dot{z} \in E^{n+1}$ an orthonormal $(n+1)$-tuple in $T_z Z = span\{D_\mu z, D_{\xi_1} z, \ldots, D_{\xi_n} z\}$.

Proposition 3.2 *Let $n \geq 3$. Given $A > 0$, there exist $\varepsilon_0, C > 0$, such that for every $h \in \mathcal{H}_A$ there exists a C^1 function*

$$(w_\varepsilon, \alpha_\varepsilon) = (w(\varepsilon, z), \alpha(\varepsilon, z)) : (-\varepsilon_0, \varepsilon_0) \times Z \to (E, \mathbb{R}^{n+1})$$

which satisfies

(i) $w(\varepsilon, z)$ is orthogonal to $T_z Z \quad \forall z \in Z$, i.e. $(w, \dot{z}) = 0$;

(ii) $f_\varepsilon'(z + w(\varepsilon, z)) = \alpha(\varepsilon, z)\dot{z} \quad \forall z \in Z$;

(iii) $\|w(\varepsilon, z)\| \leq C\,|\varepsilon| \quad \forall z \in Z$.

From (i)-(ii) it follows that

(iv) the manifold $Z_\varepsilon = \{z + w(\varepsilon, z) \mid z \in Z\}$ is a natural constraint for f_ε.

Proof. The unknown (w,α) satisfying (i) and (ii) can be implicitly defined by means of the function $H : Z \times E \times \mathbb{R}^{n+1} \times \mathbb{R} \to E \times \mathbb{R}^{n+1}$

$$H(z,w,\alpha,\varepsilon) = \begin{pmatrix} f'_\varepsilon(z+w) - \alpha \dot z \\ (w, \dot z) \end{pmatrix}.$$

Since every $z \in Z$ solves $f'_0(z) = 0$, it is $H(z,0,0,0) = 0$ and we can write

$$H(z,w,\alpha,\varepsilon) = 0 \quad \Leftrightarrow \quad \frac{\partial H}{\partial (w,\alpha)}|_{(z,0,0,0)}[w,\alpha] + R(z,w,\alpha,\varepsilon) = 0,$$

where we have set $R(z,w,\alpha,\varepsilon) = H(z,w,\alpha,\varepsilon) - \frac{\partial H}{\partial (w,\alpha)}|_{(z,0,0,0)}[w,\alpha]$. From (37) it is easy to check, see [1], that $\frac{\partial H}{\partial (w,\alpha)}|_{(z,0,0,0)}$ is invertible and there holds

$$\left\| \left(\frac{\partial H}{\partial (w,\alpha)}|_{(z,0,0,0)} \right)^{-1} \right\| \leq C, \qquad \forall z \in Z. \tag{38}$$

Hence we can write

$$H(z,w,\alpha,\varepsilon) = 0 \quad \Leftrightarrow \quad (w,\alpha) = -\left(\frac{\partial H}{\partial (w,\alpha)}(z,0,0,0) \right)^{-1} R(z,w,\alpha,\varepsilon) := F_{z,\varepsilon}(w,\alpha).$$

We will prove that, for ρ and ε sufficiently small, the map $F_{z,\varepsilon}(\ ,\ )$ is a contraction in some $B_\rho = \{(w,\alpha) \in E \times \mathbb{R}^{n+1} \ : \ \|w\| + |\alpha| \leq \rho\}$. First we show that there exists $C > 0$ such that for all $\|(w,\alpha)\|, \|(w',\alpha')\| \leq \rho$ small enough

$$\begin{cases} \|F_{z,\varepsilon}(w,\alpha)\| \leq C \left(|\varepsilon| + \rho^{\min\{2, \frac{n+2}{n-2}\}} \right), \\ \|F_{z,\varepsilon}(w',\alpha') - F_{z,\varepsilon}(w,\alpha)\| \leq C \left(|\varepsilon| + \rho^{\min\{1, \frac{4}{n-2}\}} \right) \|(w,\alpha) - (w',\alpha')\|. \end{cases} \tag{39}$$

By (38), condition (39) is equivalent to the following two inequalities

$$\|f'_\varepsilon(z+w) - f''_0(z)[w]\| \leq C \left(|\varepsilon| + \rho^{\min\{2, \frac{n+2}{n-2}\}} \right); \tag{40}$$

$$\|(f'_\varepsilon(z+w) - f''_0(z)[w]) - (f'_\varepsilon(z+w') - f''_0(z)[w'])\| \leq C \left(|\varepsilon| + \rho^{\min\{1, \frac{4}{n-2}\}} \right) \|(w,\alpha) - (w',\alpha')\|. \tag{41}$$

We now prove (40). Using formulas (29) and (30) we have, since $||z||$ is bounded

$$\begin{aligned} f'_\varepsilon(z+w) - f''_0(z)[w] &= \Big(f'_\varepsilon(z+w) - f'_\varepsilon(z) - f''_\varepsilon(z)[w] \Big) + f'_\varepsilon(z) + \Big(f''_\varepsilon(z) - f''_0(z) \Big)[w] \\ &= \int_0^1 \Big(f''_\varepsilon(z+sw) - f''_\varepsilon(z) \Big)[w] ds + O(\varepsilon) + O(\varepsilon)\|w\|. \end{aligned}$$

Hence, using (34) and (35), since $\|z\|$ and $\|w\|$ are bounded, we deduce that

$$\|f'_\varepsilon(z+w) - f''_0(z)[w]\| \leq C \left(|\varepsilon| + \|w\|^{\min\{2, \frac{n+2}{n-2}\}} + |\varepsilon| \, ||w|| \right) \leq C \left(|\varepsilon| + \rho^{\min\{2, \frac{n+2}{n-2}\}} \right),$$

and (40) is proved. We turn now to (41). There holds

$$\begin{aligned} \|f'_\varepsilon(z+w) - f'_\varepsilon(z+w') - f''_0(z)[w-w']\| &= \left| \int_0^1 \Big(f''_\varepsilon(z+w+s(w'-w)) - f''_0(z) \Big)[w'-w] ds \right| \\ &\leq \sup_{s \in [0,1]} \|f''_\varepsilon(z+w+s(w'-w)) - f''_0(z)\| \, \|w'-w\|. \end{aligned}$$

Using again formulas (30), (34) and (35) we have that

$$||f''_\varepsilon(z+w'+s(w-w'))-f''_0(z)||\le C\,\left(|\varepsilon|+\rho^{\min\{2,\frac{n+2}{n-2}\}}\right),$$

hence also (41) holds. Now that (39) is proved, if $C\,\left(|\varepsilon|+\rho^{\min\{2,\frac{n+2}{n-2}\}}\right)<\rho$ and if $C\,(|\varepsilon|+\rho^{\min\{1,\frac{4}{n-2}\}})<1$, then $F_{z,\varepsilon}(w,\alpha)$ is a contraction in B_ρ. These inequalities are solved, for example, choosing $\rho=2\,C\,|\varepsilon|$, for $|\varepsilon|\le\varepsilon_0$ with ε_0 sufficiently small. Hence we find a unique solution $(w_\varepsilon,\alpha_\varepsilon)$ satisfying $\|(w_\varepsilon,\alpha_\varepsilon)\|\le 2C\,|\varepsilon|$. The fact that the map (w,α) is of class C^1 is standard and follows from the Implicit Function Theorem. ■

Expansion of $f_{\varepsilon|Z_\varepsilon}$

By Proposition 3.2-(iv) problem (2) is solved if one finds critical points of $f_\varepsilon|_{Z_\varepsilon}$. This is done by expanding $f_\varepsilon|_{Z_\varepsilon}$ in powers of ε as stated in Proposition 3.3 below. We recall that G_1 and G_2 denote the coefficients of the expansion in ε of $f_\varepsilon(u)$, see Section 2.

In [3] the following Lemma is estabilished.

Lemma 3.1 *For all $z\in Z$ it is $G_1(z)=0$. Hence $G'_1(z)\perp T_zZ$ for all $z\in Z$.*

The function $w_\varepsilon(z)$ is estimated in terms of $G'_1(z)$ in the following Lemma.

Lemma 3.2 *Let $n\ge 6$. The following expansion holds*

$$w(\varepsilon,z)=-\varepsilon\,L_zG'_1(z)+O\left(|\varepsilon|^{\frac{(n+2)}{(n-2)}}\right). \tag{42}$$

PROOF. We can write $f'_\varepsilon(z+w_\varepsilon)=\beta_1+\beta_2+\beta_3+(f''_0(z)[w_\varepsilon]+\varepsilon\,G'_1(z))$, where

$$\begin{aligned}\beta_1&=f'_\varepsilon(z+w_\varepsilon)-f'_0(z+w_\varepsilon)-\varepsilon\,G'_1(z+w_\varepsilon);\quad \beta_2=f'_0(z+w_\varepsilon)-f''_0(z)[w_\varepsilon];\\ \beta_3&=\varepsilon\,G'_1(z+w_\varepsilon)-\varepsilon\,G'_1(z).\end{aligned}$$

From (28), since $\|z+w_\varepsilon\|$ is uniformly bounded, we have that $\|\beta_1\|=O(\varepsilon^2)$. There holds

$$\beta_2=\int_0^1(f''_0(z+sw_\varepsilon)-f''_0(z))[w_\varepsilon]ds,$$

so (35) and (iii) in Proposition 3.2 imply $\|\beta_2\|=O\left(|\varepsilon|^{\frac{(n+2)}{(n-2)}}\right)$. Then, from (33) it follows also that $\|\beta_3\|=O(\varepsilon^2)$. Hence we deduce that $\beta_1+\beta_2+\beta_3=O\left(|\varepsilon|^{\frac{(n+2)}{(n-2)}}\right)$. Thus the relation $f'_\varepsilon(z+w_\varepsilon)=\alpha_\varepsilon\dot z$ can be written as $f''_0(z)[w_\varepsilon]+\varepsilon\,G'_1(z)+O\left(|\varepsilon|^{\frac{(n+2)}{(n-2)}}\right)=\alpha_\varepsilon\dot z$. Projecting this equation on $(T_zZ)^\perp$ and applying the operator L_z, we obtain (42). ■

We finally furnish the expansion of $f_\varepsilon|_{Z_\varepsilon}$.

Proposition 3.3 *Let $n\ge 6$. Given $A>0$, the following expansion holds, uniformly in $z\in Z$ and in $h\in\mathcal{H}_A$*

$$f_\varepsilon(z_{\mu,\xi}+w_\varepsilon(z_{\mu,\xi}))=b_0+\varepsilon^2\,\Gamma(\mu,\xi)+o(\varepsilon^2), \tag{43}$$

where $\Gamma:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}$ is defined by

$$\Gamma(\mu,\xi)=G_2(z_{\mu,\xi})-\frac{1}{2}\Big(L_{z_{\mu,\xi}}G'_1(z_{\mu,\xi}),G'_1(z_{\mu,\xi})\Big). \tag{44}$$

Proof. We can write $f_\varepsilon(z+w_\varepsilon)=\gamma_1+\gamma_2+\gamma_3$, where

$$\gamma_1=f_\varepsilon(z), \qquad \gamma_2=f'_\varepsilon(z)[w_\varepsilon], \qquad \gamma_3=f_\varepsilon(w_\varepsilon+z)-f_\varepsilon(z)-f'_\varepsilon(z)[w_\varepsilon].$$

By (27), since $G_1|_Z\equiv 0$, we deduce that

$$\gamma_1=f_0(z)+\varepsilon\, G_1(z)+\varepsilon^2\, G_2(z)+o(\varepsilon^2)=b_0+\varepsilon^2\, G_2(z)+o(\varepsilon^2).$$

Turning to γ_2, from (28), (42) and from $f'_0(z)=0$ we obtain

$$\gamma_2=(f'_0(z),w_\varepsilon)+\varepsilon\,(G'_1(z),w_\varepsilon)+o(\varepsilon^2)=-\varepsilon^2\left(L_zG'_1(z),G'_1(z)\right)+o(\varepsilon^2).$$

We now estimate γ_3. We can write

$$\gamma_3=\int_0^1(f'_\varepsilon(z+sw_\varepsilon)-f'_\varepsilon(z),w_\varepsilon)ds.$$

Using (28) we have

$$\gamma_3=\int_0^1\Big((f'_0(z+sw_\varepsilon)-f'_0(z))+\varepsilon\,(G'_1(z+sw_\varepsilon)-G'_1(z))\,,w_\varepsilon\Big)ds+o(\varepsilon^2).$$

Using (33), (35) and $\|w_\varepsilon\|\le C\,|\varepsilon|$, it follows that

$$\begin{aligned}
\gamma_3 &= \int_0^1(f'_0(z+sw_\varepsilon)-f'_0(z),w_\varepsilon)\,ds+o(\varepsilon^2)\\
&= \int_0^1\left(\int_0^1(f''_0(z+tsw_\varepsilon)-f''_0(z))[sw_\varepsilon]dt\right)[w_\varepsilon]ds+\int_0^1\left(\int_0^1 f''_0(z)[sw_\varepsilon]dt\right)[w_\varepsilon]ds+o(\varepsilon^2)\\
&= \frac{1}{2}f''_0(z)[w_\varepsilon,w_\varepsilon]+o(\varepsilon^2).
\end{aligned}$$

From the above estimates for γ_1,γ_2 and γ_3, we deduce the Proposition. ∎

Study of the function Γ

We report here the main properties of the function Γ, which are obtained in [3].

Proposition 3.4 *The function Γ can be extended to the hyperplane $\{\mu=0\}$ by setting*

$$\Gamma(0,\xi)=0, \tag{45}$$

and there holds

$$\Gamma(\mu,\xi)\to 0, \qquad \textit{as } \mu+|\xi|\to+\infty. \tag{46}$$

If $n\ge 6$, then

$$\frac{\partial\Gamma}{\partial\mu}(0,\xi)=0, \quad \frac{\partial^2\Gamma}{\partial\mu^2}(0,\xi)=0, \quad \frac{\partial^3\Gamma}{\partial\mu^3}(0,\xi)=0, \qquad \forall\xi\in\mathbb{R}^n, \tag{47}$$

moreover

$$\begin{cases} \lim_{\mu\to 0}\mu^{-4}\Gamma(\mu,\xi)=-\infty & \textit{if } \overline{W}_h(\xi)\neq 0, \quad \textit{for } n=6;\\ \frac{\partial^4\Gamma}{\partial\mu^4}(0,\xi)<0 & \textit{if } \overline{W}_h(\xi)\neq 0, \quad \textit{for } n>6. \end{cases} \tag{48}$$

4 Infinitely many solutions

In this Section we prove our main result Theorem 1.2. We consider on $\mathbb{R}^n$ metrics g of the form (8). Since these metrics possess infinitely many "bumps", we expect that the function $f_\varepsilon|_{Z_\varepsilon}$ inherits infinitely many local minima when the points x_i are sufficiently far away one from each other.

Let f_ε^i be the Euler functional corresponding to the metric $g^i(x) = g_\varepsilon^i(x) = \delta + \varepsilon\,\sigma_i\,\tau(x - x_i)$. Since $\sigma_i\,\tau(\cdot - x_i) \in \mathcal{H}_A$, the construction of Proposition 3.2 can be performed also for f_ε^i. We denote by $Z^i = \{z + w_\varepsilon^i \mid z \in Z\}$ the corresponding natural constraint. We will often set for brevity

$$A_i := supp\,\tau(\cdot - x_i); \qquad z_\varepsilon^i := z + w_\varepsilon^i.$$

Let Γ^τ denote the function as in (44) associated to the metric $\delta(x) + \varepsilon\,\tau(x)$. By Proposition 3.4, the function Γ^τ possesses some negative minimum and tends to zero at the boundary of $\mathbb{R}_+ \times \mathbb{R}^n$. Hence we can find a compact set K of $\mathbb{R}_+ \times \mathbb{R}^n$ such that

$$\left\{ y \in \mathbb{R}_+ \times \mathbb{R}^n : \Gamma^\tau(y) \leq \frac{1}{2} \min \Gamma^\tau \right\} \subseteq K.$$

In the following this compact set K will be kept fixed.

If $(\mu,\xi) \in K + (0, x_i)$, then the functions $z_{\mu,\xi} + w_\varepsilon^i$ satisfies an uniform decay estimate. This is stated precisely in the following Lemma.

Lemma 4.1 *Let $|\varepsilon| \leq \varepsilon_0$. There exist $C > 0, R > 1$ such that for every i and for every $(\mu,\xi) \in K + (0, x_i)$ there holds*

$$(49) \qquad |z_{\mu,\xi} + w_\varepsilon^i|\,(x) \leq \frac{C}{|x - x_i|^{n-2}}, \qquad |\nabla(z_{\mu,\xi} + w_\varepsilon^i)|\,(x) \leq \frac{C}{|x - x_i|^{n-1}}; \qquad |x - x_i| \geq R.$$

PROOF. We can suppose without loss of generality that $x_i = 0$ and the support of τ is contained in $B_1 = \{x \in \mathbb{R}^n \ : \ |x| \leq 1\}$.

The function z_ε^i, satisfies $\nabla f_\varepsilon^i(z_\varepsilon^i) = \alpha_\varepsilon^i \dot{z}$, hence it solves the equation

$$-2c_n \Delta(z_\varepsilon^i) - |z_\varepsilon^i|^{2^*-2} z_\varepsilon^i = -\alpha_\varepsilon^i \Delta \dot{z}, \qquad \text{in } \mathbb{R}^n \setminus B_1.$$

Performing the transformation (see the Notations for the definition of the map $u \to u^*$)

$$z_\varepsilon^i(x) \to u_\varepsilon^i(x) := \mu^{\frac{n-2}{2}} (z_\varepsilon^i)^*(\mu x),$$

one easily verifies that the function u_ε^i solves

$$(50) \qquad -\Delta u_\varepsilon^i(x) = |u_\varepsilon^i|^{2^*-2}(x) u_\varepsilon^i(x) + \mu^{\frac{n+2}{2}} q_z(\mu x), \qquad \text{in } B_1,$$

where $q_z = -\alpha_\varepsilon^i(z)\Delta(\dot{z}^*)$. Since (μ_1,ξ_1) belongs to the fixed compact set K, the norm

$$(51) \qquad \|q_z\|_{C^3(B_1)} \ \text{ is uniformly bounded for } (\mu_1,\xi_1) \in K.$$

Moreover, since w_ε^i is a continuous function of z, it turns out that

$$(52) \qquad \zeta_\mu = \sup_{(\mu,\xi)\in K} \int_{B_1} |\nabla u_\varepsilon^i|^2 \to 0, \quad \eta_\mu = \sup_{(\mu,\xi)\in K} \int_{B_1} |u_\varepsilon^i|^{2^*} \to 0, \qquad \text{as } \mu \to 0.$$

Under conditions (50), (51) and (52), the arguments in the proof of Proposition 1.1 in [13] imply that for some $\mu = \mu_0$ sufficiently small it is $\|u_\varepsilon^i\|_{C^1(B_{\frac{1}{2}})} \leq C$ uniformly in for $(\mu_1,\xi_1) \in K$. From this one can easily deduce that

$$z_\varepsilon^i(x) \leq \frac{C}{\mu_0^{\frac{n-2}{2}}} \frac{1}{|x|^{n-2}}, \qquad \text{for } |x| \geq \frac{2}{\mu_0}; \qquad (\mu_1,\xi_1) \in K,$$

which is the first inequality in (49). The second inequality follows in the same way from the boundedness of $\|u^i_\varepsilon\|_{C^1(B_{\frac{1}{2}})}$. ■

Lemma 4.2 *There exist $C > 0$, $\varepsilon_1 > 0$ such that for $|\varepsilon| \leq \varepsilon_1$ there holds*

$$||w_\varepsilon - w^i_\varepsilon|| \leq C\, ||\nabla f_\varepsilon(z + w^i_\varepsilon) - \nabla f^i_\varepsilon(z + w^i_\varepsilon)||. \tag{53}$$

Proof. Let us consider the function

$$\overline{H} \,:\, Z \times E \times \mathbb{R}^{n+1} \to E \times \mathbb{R}^{n+1} \times \mathbb{R}$$

with components $\overline{H}_1 \in E$ and $\overline{H}_2 \in \mathbb{R}^{n+1}$ given by

$$\begin{array}{rcl} \overline{H}_1(z,w,\alpha,\varepsilon) & = & \nabla f_\varepsilon(z + w^i_\varepsilon + w) - (\alpha^i_\varepsilon + \alpha)\, \dot{z}, \\ \overline{H}_2(z,w,\alpha,\varepsilon) & = & (w, \dot{z}). \end{array}$$

We have

$$\overline{H}(z,w,\alpha,\varepsilon) = 0 \quad \Leftrightarrow \quad \overline{H}(z,0,0,\varepsilon) + \frac{\partial \overline{H}}{\partial(w,\alpha)}|_{(z,0,0,\varepsilon)}[w,\alpha] + \overline{R}(z,w,\alpha,\varepsilon) = 0,$$

where $\overline{R}(z,w,\alpha,\varepsilon) = \overline{H}(z,w,\alpha,\varepsilon) - \overline{H}(z,0,0,\varepsilon) - \frac{\partial \overline{H}}{\partial(w,\alpha)}|_{(z,0,0,\varepsilon)}[w,\alpha]$.

It is easy to see that for $|\varepsilon|$ small enough there holds

$$\left| \left(\frac{\partial \overline{H}}{\partial(w,\alpha)}|_{(z,0,0,\varepsilon)} \right)^{-1} \right| \leq C \qquad \forall z \in Z.$$

Moreover we have

$$\overline{H}(z,w,\alpha,\varepsilon) = 0 \qquad \Leftrightarrow \qquad (w,\alpha) = \overline{F}_{\varepsilon,z}(w,\alpha),$$

where

$$\overline{F}_{\varepsilon,z}(w,\alpha) := -\left(\frac{\partial \overline{H}}{\partial(w,\alpha)}|_{(z,0,0,\varepsilon)} \right)^{-1} \left(\overline{H}(z,0,0,\varepsilon) + \overline{R}(z,w,\alpha,\varepsilon) \right).$$

We claim that the following two estimates hold. For all $\|(w,\alpha)\|, \|(w',\alpha')\| \leq \rho$ small enough

$$\|\overline{F}_{\varepsilon,z}(w,\alpha)\| \leq C\, \|\nabla f_\varepsilon(z + w^i_\varepsilon) - \nabla f^i_\varepsilon(z + w^i_\varepsilon)\| + C\, \rho^{\frac{n+2}{n-2}}, \tag{54}$$

$$\|\overline{F}_{\varepsilon,z}(w,\alpha) - \overline{F}_{\varepsilon,z}(w',\alpha')\| \leq C\, \rho^{\frac{4}{n-2}}\, \|w' - w\|. \tag{55}$$

Let us prove (54). For all $(w,\alpha) \in B_\rho$

$$\|\overline{F}_{\varepsilon,z}(w,\alpha)\| \leq C\, \|\overline{H}(z,0,0,\varepsilon)\| + C\, \|\overline{R}(z,w,\alpha,\varepsilon)\| \tag{56}$$

We have, using the same arguments of Proposition 3.2

$$\begin{array}{rcl} \|\overline{R}(\varepsilon,z,w,\alpha)\| & = & \left\| \overline{H}(z,w,\alpha,\varepsilon) - \overline{H}(z,0,0,\varepsilon) - \dfrac{\partial \overline{H}}{\partial(w,\alpha)}|_{(z,0,0,\varepsilon)}[w,\alpha] \right\| \\ & = & \|\nabla f_\varepsilon(z + w^i_\varepsilon + w) - \nabla f_\varepsilon(z + w^i_\varepsilon) - D^2 f_\varepsilon(z + w^i_\varepsilon)[w]\| \leq C\, \|w\|^{\frac{n+2}{n-2}}. \end{array}$$

Since $\overline{H}(z,0,0,\varepsilon)=\nabla f_\varepsilon(z+w_\varepsilon^i)-\nabla f_\varepsilon^i(z+w_\varepsilon^i)$, (54) follows from (56). Let us turn to (55). For all $(w,\alpha),(w',\alpha')\in B_\rho$ it is

$$
\begin{aligned}
\|\overline{F}_{\varepsilon,z}(w,\alpha)-\overline{F}_{\varepsilon,z}(w',\alpha')\| &= \left\|\left(\frac{\partial \overline{H}}{\partial(w,\alpha)}|_{(z,0,0,\varepsilon)}\right)^{-1}\left(\overline{R}(z,w,\alpha,\varepsilon)-\overline{R}(z,w',\alpha',\varepsilon)\right)\right\| \\
&\le C\left\|\int_0^1 (f_\varepsilon)''(z+w_\varepsilon^i+w'+s(w-w'))-(f_\varepsilon)''(z+w_\varepsilon^i)ds\right\| \\
&\times \|w'-w\|\le C\,\rho^{2^*-2}\,\|w'-w\|,
\end{aligned}
$$

so (55) holds true. Now, arguing as in Proposition 3.2, we deduce that there exists a unique $(w_\varepsilon^D,\alpha_\varepsilon^D)$ such that

- *(i)* $(w_\varepsilon^D,\dot z)=0$;
- *(ii)* $\nabla f_\varepsilon(z+w_\varepsilon^i+w_\varepsilon^D)=(\alpha_\varepsilon^i+\alpha_\varepsilon^D)\dot z$;
- *(iii)* $\|w_\varepsilon^D\|\le C\,\|\nabla f_\varepsilon(z+w_\varepsilon^i)-\nabla f_\varepsilon^i(z+w_\varepsilon^i)\|$ for ε sufficiently small.

The couple $\left(w_\varepsilon^i+w_\varepsilon^D,\alpha_\varepsilon^i+\alpha_\varepsilon^D\right)$ satisfies $(i)-(iv)$ in Proposition 3.2, hence by uniqueness it must be $w_\varepsilon=w_\varepsilon^i+w_\varepsilon^D$; by *(iii)*, inequality (53) follows. ■

In the next Lemma we estimate the quantity $||\nabla f_\varepsilon(z_\varepsilon^{i_0})-\nabla f_\varepsilon^{i_0}(z_\varepsilon^{i_0})||$ with respect to ε, $\{\sigma_i\}_i$, and $\{x_i\}_i$.

Lemma 4.3 *There exist $C>0$, $L_1>0$ such that, if $|x_{i_0}-x_i|\ge L_1$ for all $i\ne i_0$, then*

$$
||\nabla f_\varepsilon(z_{\mu,\xi}+w_\varepsilon^{i_0})-\nabla f_\varepsilon^{i_0}(z_{\mu,\xi}+w_\varepsilon^{i_0})||\le C\,|\varepsilon|\sum_{i\ne i_0}\frac{\sigma_i}{|x_i-x_{i_0}|^{n-2}}, \tag{57}
$$

for all $(\mu,\xi)\in(0,x_{i_0})+K$.

Proof. Since the metric $g_\varepsilon^{i_0}$ is flat on A_i for $i\ne i_0$, for $v\in E$ there holds

$$
\begin{aligned}
|(\nabla f_\varepsilon(z_\varepsilon^{i_0})-\nabla f_\varepsilon^{i_0}(z_\varepsilon^{i_0}),v)| &= \Bigg|\sum_{i\ne i_0}\int_{A_i}2c_n\nabla_g z_\varepsilon^{i_0}\cdot\nabla_g v+R_g z_\varepsilon^{i_0}v-|z_\varepsilon^{i_0}|^{2^*-2}z_\varepsilon^{i_0}v\;dV_g \\
&- \sum_{i\ne i_0}\int_{A_i}2c_n\nabla z_\varepsilon^{i_0}\cdot\nabla v-|z_\varepsilon^{i_0}|^{2^*-2}z_\varepsilon^{i_0}v\;dx\Bigg|.
\end{aligned}
$$

Using the Hölder inequality on each A_i we get

$$
|(\nabla f_\varepsilon(z_\varepsilon^{i_0})-\nabla f_\varepsilon^{i_0}(z_\varepsilon^{i_0}),v)|\le C\,|\varepsilon|\sum_{i\ne i_0}\sigma_i\int_{A_i}|\nabla z_\varepsilon^{i_0}|\,|\nabla v|+|z_\varepsilon^{i_0}|\,|v|+|z_\varepsilon^{i_0}|^{2^*-1}|v|\;dx.
$$

By Lemma 4.1 we know that for $(\mu,\xi)\in(0,x_{i_0})+K$

$$
|z_\varepsilon^{i_0}(x)|\le\frac{C}{|x-x_{i_0}|^{n-2}},\qquad |\nabla z_\varepsilon^{i_0}(x)|\le\frac{C}{|x-x_{i_0}|^{n-1}}\qquad\text{for}\qquad |x-x_{i_0}|\ge R.
$$

Hence we deduce, using the Hölder and the Sobolev inequalities, if $|x_{i_0}-x_i|\ge L_1$, $i\ne i_0$, with $L_1\ge R$, there holds

$$
|(\nabla f_\varepsilon(z_\varepsilon^{i_0})-\nabla f_\varepsilon^{i_0}(z_\varepsilon^{i_0}),v)|\le C\,|\varepsilon|\,\|v\|\sum_{i\ne i_0}\sigma_i\left(\frac{1}{|x_i-x_{i_0}|^{n-1}}+\frac{1}{|x_i-x_{i_0}|^{n-2}}+\frac{1}{|x_i-x_{i_0}|^{n+2}}\right).
$$

This concludes the proof. ■

In the next Proposition we compare $f_\varepsilon|_{Z_\varepsilon}$ with the reduced functional $f_\varepsilon^{i_0}|_{Z_{i_0}}$ corresponding to one-bump metrics.

Proposition 4.1 *Set*

$$Q_{i_0} = f_\varepsilon(z_{\mu,\xi} + w_\varepsilon) - f_\varepsilon^{i_0}(z_{\mu,\xi} + w_\varepsilon^{i_0}).$$

Then, if $|x_{i_0} - x_i| \geq L_1$ *for all* $i \neq i_0$, *for all* $(\mu, \xi) \in (0, x_{i_0}) + K$ *and for all* $|\varepsilon| < \varepsilon_1$ *there holds*

$$|Q_{i_0}| \leq C\,|\varepsilon| \Big(\sum_{i\neq i_0} \frac{1}{|x_i - x_{i_0}|^n} \Big)^{\frac{n-2}{n}}. \tag{58}$$

Proof. We have by (31), (53) and (57)

$$\begin{aligned} |Q_{i_0}| &= |f_\varepsilon(z+w_\varepsilon) - f_\varepsilon^{i_0}(z_\varepsilon^{i_0})| \leq |f_\varepsilon(z+w_\varepsilon) - f_\varepsilon(z+w_\varepsilon^{i_0})| + |f_\varepsilon(z_\varepsilon^{i_0}) - f_\varepsilon^{i_0}(z_\varepsilon^{i_0})| \\ &\leq C\,||w_\varepsilon - w_\varepsilon^{i_0}|| + |f_\varepsilon(z_\varepsilon^{i_0}) - f_\varepsilon^{i_0}(z_\varepsilon^{i_0})| \leq C\,||\nabla f_\varepsilon(z_\varepsilon^{i_0}) - \nabla f_\varepsilon^{i_0}(z_\varepsilon^{i_0})|| + |f_\varepsilon(z_\varepsilon^{i_0}) - f_\varepsilon^{i_0}(z_\varepsilon^{i_0})| \\ &\leq C\,|\varepsilon| \sum_{i\neq i_0} \frac{\sigma_i}{|x_i - x_{i_0}|^{n-2}} + |f_\varepsilon(z_\varepsilon^{i_0}) - f_\varepsilon^{i_0}(z_\varepsilon^{i_0})|. \end{aligned} \tag{59}$$

Arguing as in Lemma 4.3 we deduce

$$\begin{aligned} |f_\varepsilon(z_\varepsilon^{i_0}) - f_\varepsilon^{i_0}(z_\varepsilon^{i_0})| &= \sum_{i\neq i_0} \int_{A_i} c_n |\nabla_g (z_\varepsilon^{i_0})|^2 + R_g (z_\varepsilon^{i_0})^2 - \frac{1}{2^*} |z_\varepsilon^{i_0}|^{2^*}\, dV_g \\ &- \sum_{i\neq i_0} \int_{A_i} c_n |\nabla (z_\varepsilon^{i_0})|^2 - \frac{1}{2^*} |z_\varepsilon^{i_0}|^{2^*}\, dx \\ &\leq C\,|\varepsilon| \sum_{i\neq i_0} \sigma_i \int_{A_i} |\nabla (z_\varepsilon^{i_0})|^2 + |z_\varepsilon^{i_0}|^2 + |z_\varepsilon^{i_0}|^{2^*}\, dx. \end{aligned}$$

Then, using the fact that $|x_i - x_{i_0}| \geq L_1$

$$|f_\varepsilon(z_\varepsilon^{i_0}) - f_\varepsilon^{i_0}(z_\varepsilon^{i_0})| \leq C\,|\varepsilon| \sum_{i\neq i_0} \sigma_i \left(\frac{1}{|x_i - x_{i_0}|^{2(n-1)}} + \frac{1}{|x_i - x_{i_0}|^{2(n-2)}} + \frac{1}{|x_i - x_{i_0}|^{2n}} \right).$$

The last inequlity and (59) imply that $|Q_{i_0}| \leq C\,|\varepsilon| \sum_{i\neq i_0} \frac{\sigma_i}{|x_i - x_{i_0}|^{n-2}}$. Applying the Hölder inequality and taking into account that $\sum_i |\sigma_i|^{\frac{n}{2}} < A$, (58) follows. ■

Lemma 4.4 *Let* $\alpha > 1$, $\gamma > 1$. *There exists a constant* $C > 0$ *depending only on* α *and* γ, *such that*

$$\sum_{i\neq i_0} \frac{1}{|i^\alpha - i_0^\alpha|^\gamma} \sim C\, \frac{1}{i_0^{(\alpha-1)\gamma}}, \qquad i_0 \to +\infty.$$

Proof. For i_0 large enough there holds

$$\sum_{i<i_0} \frac{1}{|i^\alpha - i_0^\alpha|^\gamma} \sim \int_0^{(i_0-1)} \frac{dx}{(i_0^\alpha - x^\alpha)^\gamma}, \qquad \sum_{i>i_0} \frac{1}{|i^\alpha - i_0^\alpha|^\gamma} \sim \int_{(i_0+1)}^{\infty} \frac{dx}{(x^\alpha - i_0^\alpha)^\gamma}.$$

Hence, we are reduced to estimate the above two integrals. Let us start with the first one: using the change of variables $i_0 y = x$, we deduce that

$$\int_0^{(i_0-1)} \frac{dx}{(i_0^\alpha - x^\alpha)^\gamma} = i_0 \int_0^{1-\frac{1}{i_0}} \frac{dy}{i_0^{\alpha\gamma}\,(1-y^\alpha)^\gamma} = \frac{1}{i_0^{\alpha\gamma-1}} \int_0^{1-\frac{1}{i_0}} \frac{dy}{(1-y^\alpha)^\gamma}.$$

Since $(1-y^\alpha)^\gamma \sim C\,(1-y)^\gamma$, for y close to 1 it follows that $\int_0^{1-\frac{1}{i_0}} \frac{dy}{(1-y^\alpha)^\gamma} \sim C\, i_0^{\gamma-1}$. Hence it turns out that $\int_0^{(i_0-1)} \frac{dx}{(i_0^\alpha - x^\alpha)^\gamma} \sim C\, \frac{1}{i_0^{(\alpha-1)\gamma}}$. An analogous estimate holds for the other integral $\int_{(i_0+1)}^\infty \frac{dx}{(x^\alpha - i_0^\alpha)^\gamma}$. This concludes the proof. ■

4.1 Proof of Theorem 1.2

Existence of infinitely many solutions

Fix $\mathbf{a} \in \mathbb{R}^n$ with $|\mathbf{a}| = 1$, and let h be of the form (6) with $\sigma_i = i^{-\beta}$ and $x_i = D\, i^\alpha\, \mathbf{a}$. We choose

$$D = \frac{C_0}{|\varepsilon|^{1/(n-2)}}; \qquad \alpha > 4k+1; \qquad 2\alpha k < \beta < 2\alpha k + \frac{\alpha - (4k+1)}{2}, \tag{60}$$

where C_0 is a constant to be fixed later. With the above choice of σ_i there holds $\sum_{i+1}^{+\infty} |\sigma_i|^{n/2} < +\infty$, since $\beta > 1 > \frac{2}{n}$. Since also $\alpha > 1$, we have $\inf_{i \neq j} |x_i - x_j| > 4\, \mathrm{diam}(supp\, \tau)$ for i, j large enough. Hence, if we take $\sigma_i = 0$ for i sufficiently small, then h belongs to $\mathcal{H}_A$.

From the expansion in (43) we know that

$$f_\varepsilon^{i_0}(z_\varepsilon^{i_0}) = b_0 + \varepsilon^2 \sigma_i^2\, \Gamma^{\tau(\cdot - x_{i_0})}(\mu, \xi) + o(\varepsilon^2 \sigma_i^2), \qquad z_\varepsilon^{i_0} = z_{\mu,\xi} + w_\varepsilon^{i_0},$$

and so $f_\varepsilon^{i_0}|_{Z^{i_0}}$ attains absolute minimum in a point $\widetilde{z}_\varepsilon^{i_0} = z_{\widetilde{\mu},\widetilde{\xi}} + w_\varepsilon^{i_0}$ with $(\widetilde{\mu}, \widetilde{\xi}) \in (0, x_{i_0}) + K$. Moreover there exists a smooth open set $U \subseteq K$ such that for σ_{i_0} sufficiently small

$$\min_{(\mu,\xi) \in \partial U} f_\varepsilon^{i_0}(z_{\mu,\xi} + w_\varepsilon^{i_0}) - f_\varepsilon^{i_0}(\widetilde{z}_\varepsilon^{i_0}) \geq \frac{1}{4} d_\tau \sigma_{i_0}^2 \varepsilon^2; \qquad d_\tau = |\min \Gamma^\tau|. \tag{61}$$

We assume i_0 to be so large that $\min_{i \neq i_0} |x_{i_0} - x_i| \geq L_1$, so (58) holds. Hence we have that

$$|Q_{i_0}| \leq \frac{C\, |\varepsilon|}{D^{(n-2)}} \Big(\sum_{i \neq i_0} \frac{1}{|i^\alpha - {i_0}^\alpha|^n} \Big)^{\frac{n-2}{n}}.$$

So, by Lemma 4.4, for i_0 sufficiently large there holds

$$|Q_{i_0}| \leq \frac{C\, |\varepsilon|}{D^{(n-2)}}\, \frac{1}{i_0^{(\alpha-1)(n-2)}}. \tag{62}$$

By our choice of σ_i and by (61), in order to find for ε small a minimum of $f_\varepsilon|_{Z_\varepsilon}$ near $\widetilde{z}_\varepsilon^{i_0}$, it is sufficient that

$$|Q_{i_0}| \leq \frac{1}{8} d_\tau\, {i_0}^{-2\beta}\, |\varepsilon|^2. \tag{63}$$

Taking into account (62), inequality (63) is satisfied, for i_0 large enough, when $D = \frac{C_0}{|\varepsilon|^{1/(n-2)}}$, C_0 is sufficiently large, and

$$(\alpha - 1)(n-2) \geq 2\beta. \tag{64}$$

We have then proved that if (64) holds, then for all i_0 large enough and ε small enough $f_\varepsilon(z_{\mu,\xi} + w_\varepsilon)$ attains a minimum $(\widetilde{\mu}_{i_0}, \widetilde{\xi}_{i_0}) \in (0, x_{i_0}) + K$. Hence there are infinitely many distinct solutions v_ε^i of (1) on $(S^n, \overline{g}_\varepsilon)$.

Regularity of the metrics

Now we want to determine the regularity of $\overline{g}_\varepsilon$ on S^n. Clearly $\overline{g}_\varepsilon$ is of class C^∞ on $S^n \setminus P_N$. Moreover, the regularity of $\overline{g}_\varepsilon$ at P_N is the same as that of $(\overline{g}_\varepsilon)_{\mathcal{R}}$ at the south pole P_S and so, recalling formula (19), it is the same of $g_\varepsilon^\sharp$ in 0. From equation (20), it follows that the functions $g_{ij}^\sharp(x)$ are of the form

$$g_{ij}^\sharp(x) = \delta_{ij} + \sum_{kj} \Lambda_{ijkl}\left(\frac{x}{|x|}\right)\left(g_{kl}\left(\frac{1}{x}\right) - \delta_{kl}\right), \tag{65}$$

where Λ_{ijkl} are smooth angular functions. Set $N_\varepsilon^i = \|(g_\varepsilon^i)^\sharp - \delta\|_{C^k}$. Since $(g_\varepsilon^i)^\sharp - \delta$ has support in $A^i := \left\{x \in \mathbb{R}^n : \frac{x}{|x|^2} \in A_i\right\}$, and since $diam(A^i) \sim |x_i|^{-2}$, one can easily check from (65) that N_ε^i can be estimated as

$$N_\varepsilon^i \leq C\,|\varepsilon|\,|\sigma_i|\,|x_i|^{2k} \leq C\,|\varepsilon|^{1-\frac{2k}{n-2}}\, i^{2\alpha k-\beta}.$$

Let $g_{\varepsilon,j}^\sharp$ be the metric constituted by the first j bumps of $g_\varepsilon^\sharp$. Hence, since all the bumps of $g_\varepsilon^\sharp$ have disjoint support, there holds

$$\|g_{\varepsilon,j}^\sharp - g_{\varepsilon,l}^\sharp\|_{C^k(\mathbb{R}^n)} \leq \sup_{i=j+1,\dots,l} N_\varepsilon^i \leq C\,|\varepsilon|^{1-\frac{2k}{n-2}} \sup_{i=j+1,\dots,l} i^{2\alpha k-\beta}; \qquad j < l.$$

So, if $2\alpha k - \beta < 0$, the sequence $g_{\varepsilon,j}^\sharp$ is Cauchy in $C^k(B_1)$, and hence $\overline{g}_\varepsilon$ is also of class C^k. If moreover there holds $1 - \frac{2k}{n-2} > 0$, then $\|\overline{g}_\varepsilon - \overline{g}_0\|_{C^k} \to 0$ when $\varepsilon \to 0$. The three inequalities we are requiring, namely (64) and

$$\beta > 2\alpha k; \qquad n-2 > 2k$$

are satisfied with $n \geq 4k+3$ and our choices in (60). We have proved that $\overline{g}_\varepsilon$ are of class C^k and that $\|\overline{g}_\varepsilon - \overline{g}_0\|_{C^k(S^n)}$ tends to 0 as ε tends to 0.

Since the solutions u_ε^i of (2) are close in E to some $z_{\widetilde{\mu}_i,\widetilde{\xi}_i}$ with $(\widetilde{\mu}_i,\widetilde{\xi}_i) \in (0,x_i)+K$, the solutions $v_\varepsilon^i = \iota^{-1}u_\varepsilon^i$ of (1) on S^n are close in $H^1(S^n)$ to $\iota^{-1}z_{\widetilde{\mu}_i,\widetilde{\xi}_i}$. From the fact that the functions $\iota^{-1}z_{\widetilde{\mu}_i,\widetilde{\xi}_i}$ blow-up at P_N as $i \to +\infty$, one can deduce that $\|v_\varepsilon^i\|_{L^\infty(S^n)} \to +\infty$ as $i \to +\infty$. Standard regularity arguments, see [9], imply that the weak solutions v_ε^i are indeed of class C^k on S^n. From the fact that $\|v_\varepsilon^i - \iota^{-1}z_{\widetilde{\mu}_i,\widetilde{\xi}_i}\|_{H^1(S^n)}$ is small and from the maximum principle, it is also easy to check that the solutions we find are positive. This concludes the proof.

5 Further results

In this section we prove some multiplicity results, which are consequences of the method applied above.

We consider on S^n a smooth bilinear and symmetric form $\overline{h}$, and the metric $\overline{g} = \overline{g}_\varepsilon$ given by

$$\overline{g}_\varepsilon = \overline{g}_0 + \varepsilon\,\overline{h}. \tag{66}$$

Let g be the metric on $\mathbb{R}^n$ associated to $\overline{g}$ by formula (17). Using the isometry ι, it is possible to prove that the Euler functional $f_\varepsilon : E \to \mathbb{R}$ corresponding to g is well defined, and one can repeat all the arguments of Section 3. Let again $Z_\varepsilon = \{z + w_\varepsilon\}$ denote the natural constraint for f_ε: to study $f_\varepsilon|_{Z_\varepsilon}$, for brevity we define $\varphi_\varepsilon(\mu,\xi) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ as

$$\varphi_\varepsilon(\mu,\xi) = f_\varepsilon(z_{\mu,\xi} + w_\varepsilon(z_{\mu,\xi})).$$

We have the following Proposition, proved in the Appendix.

Proposition 5.1 *Suppose $n \geq 3$. Let $\overline{h}$ be a smooth bilinear and symmetric form on S^n, and, for ε small, let $\overline{g}_\varepsilon$ be given by (66). Then φ_ε can be extended by continuity to $\{\mu = 0\}$ by setting*

$$\varphi_\varepsilon(0, \xi) = b_0, \qquad \xi \in \mathbb{R}^n. \tag{67}$$

Moreover there holds

$$\lim_{\mu + |\xi| \to +\infty} \varphi_\varepsilon(\mu, \xi) = b_0. \tag{68}$$

As a first application of Proposition 5.1 we improve Theorem 1.3.

Theorem 5.1 *Under the same assumptions of Theorem 1.3 there exist $\overline{L}, \hat{\varepsilon} > 0$ such that, for $|x_0| \geq \overline{L}$ and for $|\varepsilon| \leq \hat{\varepsilon}$, problem (2) admits a third solution $u_{3,\varepsilon}$. In the non-degenerate case this solution has Morse index $m(u_{3,\varepsilon}, f_\varepsilon) \geq 2$, or in general extended Morse index $m^*(u_{3,\varepsilon}, f_\varepsilon) \geq 2$.*

Proof. In [3] it is proved that for $|x_0| \geq \overline{L}$ large enough and for $|\varepsilon| \leq \hat{\varepsilon}$ small enough, φ_ε possesses two points e_0, e_1 of local minimum with $\varphi_\varepsilon(e_0), \varphi_\varepsilon(e_1) < b_0$. These minima give rise to two solutions $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ of problem (2). Now three cases can occur. The first one is that $\sup_{\mathbb{R}_+ \times \mathbb{R}^n} \varphi_\varepsilon > b_0$, the second is that $\varphi_\varepsilon \leq b_0$ and $\varphi_\varepsilon(\mu, \xi) = b_0$ for some $(\mu, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n$, and the third case is that $\varphi_\varepsilon(\mu, \xi) < b_0$ for all $(\mu, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n$. In the first two cases φ_ε possesses an interior maximum, while in the third case, by the mountain pass Theorem, there exists a critical level $c^\varepsilon > \max\{\varphi_\varepsilon(e_0), \varphi_\varepsilon(e_1)\}, c^\varepsilon < b_0$. In each case there is a third solution $u_{3,\varepsilon}$ to problem (2). In the non-degenerate case we show that $m(u_{3,\varepsilon}, f_\varepsilon) \geq 2$.

The operator $f''_\varepsilon(u_{3,\varepsilon})$ is negative definite on the one-dimensional subspace $\{t u_{3,\varepsilon}, t \in \mathbb{R}\}$, so there it is $m(u_{3,\varepsilon}, f_\varepsilon) \geq 1$. Suppose by contradiction that $m(u_{3,\varepsilon}, f_\varepsilon) = 1$. Then, since we are in the non-degenerate case, $f''_\varepsilon(u_{3,\varepsilon})$ would be positive definite on the finite dimensional space $T_{u_{3,\varepsilon}} Z_\varepsilon$, and $u_{3,\varepsilon}$ would be a strict minimum for $f_\varepsilon|_{Z_\varepsilon}$. Clearly this is a contradiction when $u_{3,\varepsilon}$ is an interior maximum. When $u_{3,\varepsilon}$ is a mountain pass critical point, the result follows from [11]. In the degenerate case, the same argument shows that $m^*(u_{3,\varepsilon}, f_\varepsilon) \geq 2$. ■

Remark 5.1 *As a byproduct of Proposition 5.1, we can immediately deduce that φ_ε possesses a critical point, and hence problem (1) admits a solution for $g = \overline{g}_\varepsilon$. We point out that, in the present very specific situation, we do not need to distinguish between different dimensions and between the locally conformally flat or non-locally conformally flat case.*

Our last result deals with thte existence of multibump solutions as in [17]. Given an integer $\ell > 0$, an *ℓ-bump* solution of (1) is a function u satisfying (1) and such that $u \sim \sum_{i=1}^{\ell} z_{\mu_i, \xi_i}$.

Theorem 5.2 *For all integers $\ell > 0$, there exists $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon < \varepsilon_0$, there exists a metric $\overline{g}_\varepsilon$ on S^n for which problem (1) possesses ℓ-bump solutions. If $k \geq 2$ and $n \geq 4k + 3$ then $\overline{g}_\varepsilon$ can be chosen in such a way that $\|\overline{g}_\varepsilon - \overline{g}_0\|_{C^k(S^n)} \to 0$ as $\varepsilon \to 0$.*

For the sake of brevity we will only outline the main steps of the arguments, referring to [15] for more details and complete proofs.

Step 1 We fix $\ell \in \mathbb{N}$ and we take $x_1, \dots, x_\ell \in \mathbb{R}^n$ and g_ε of the form

$$g_\varepsilon(x) = \delta_{ij} + \varepsilon \sum_{i=1}^{\ell} \tau(x - x_i), \qquad \text{in } \mathbb{R}^n.$$

The multibump solution is found near the following set of functions

$$Z^\ell = \{z_1 + \cdots + z_\ell : z_i \in Z\},$$

obtained "gluing" together ℓ elements of Z. We show that

$$\|f'_\varepsilon(z)\| = O\left(\max_{i\neq j}|x_i - x_j|^{-\frac{(n+2)(n-2)}{2n}} + \varepsilon^2\right), \qquad z \in Z^\ell.$$

Step 2 Following the arguments of [7], we use the last estimate to prove the existence of a manifold

$$Z^\ell_\varepsilon = \{z + w : z \in Z^\ell\}, \qquad \|w\| = O(\|f'_\varepsilon(z)\|),$$

which is a natural constraint for f_ε. Moreover, it turns out that

$$f_e(z+w) = \ell\, b_0 + \varepsilon^2 \sum_{i=1}^{\ell} \Gamma(z_i) + R,$$

where

$$|R| = O\left(\varepsilon \max_{i\neq j}|x_i - x_j|^{-\frac{(n+2)(n-2)}{2n}} + \varepsilon^2\right). \tag{69}$$

Step 3 Each of the functions $\Gamma(z_i)$ attains a minimum at $z_i = z_{\mu_i,\xi_i}$ with μ_i bounded above and below, and with ξ_i close to x_i. By means of equation (69), we prove that, if we choose $\max_{i\neq j}|x_i - x_j|^{-(n-2)} \sim \varepsilon^2$, these minima persist, and we find a critical point of f_ε on Z^ℓ_ε. Furthermore, the metric g_ε gives rise to a metric $\overline{g}_\varepsilon$ on S^n with $\overline{g}_\varepsilon \to \overline{g}_0$ in C^k.

6 Appendix

Proof of technical Lemmas

Proof of Lemma 2.1. Equation (23) is a trivial consequence of the subadditivity of the function $t \to |t|^p$ for $0 < p \leq 1$, and of the convexity of $t \to |t|^p$ for $p > 1$. When $n \geq 6$, then the number $2^* - 2 = \frac{4}{(n-2)}$ is greater than 0 and smaller or equal to 1, so equation (26) is also a consequence of the subadditivity of $t \to |t|^p$, with $0 < p \leq 1$. Turning to (25) it is sufficient, by homogeneity, to prove that for every $t \in \mathbb{R}$ there holds

$$\left||1+t|^{p-1}(1+t) - |t|^{p-1}t - 1\right| \leq C\,\left(|t|^r + |t|^q\right). \tag{70}$$

Equation (70) is satisfied near $t = 0$ for every $C > 0$, since $0 < r < 1$. At infinity, the left-hand side goes to $+\infty$ as $|t|^{p-1}$, while the right hand side goes to $+\infty$ as $|t|^q$, since $q > r$. Moreover $p - 1 < q$, so (70) holds for C sufficiently large and for all t. Inequality (24) can be obtained reasoning in the same way. ■

Proof of Lemma 2.2 We start proving (35). Given two functions $v_1, v_2 \in E$, there holds

$$\begin{aligned} |(f''_\varepsilon(u+w) - f''_\varepsilon(u))[v_1,v_2]| &= (2^*-1)\left|\int (|u+w|^{2^*-2} - |u|^{2^*-2})v_1 v_2 dV_g\right| \\ &\leq (2^*-1)\,(1+O(\varepsilon))\left|\int \left||u+w|^{2^*-2} - |u|^{2^*-2}\right| |v_1||v_2| dx\right|. \end{aligned}$$

Using the Hölder and the Sobolev inequalities we deduce that

$$\int \left||u+w|^{2^*-2} - |u|^{2^*-2}\right| |v_1||v_2| dx \leq C\left(\int \left||u+w|^{2^*-2} - |u|^{2^*-2}\right|^{\frac{n}{2}}\right)^{\frac{2}{n}} \|v_1\|\,\|v_2\|.$$

For $n \geq 6$, using inequality (26) with $a = u(x)$, $b = w(x)$, we deduce that $\left||u+w|^{2^*-2} - |u|^{2^*-2}\right|^{\frac{n}{2}} \leq C\,|w|^{2^*}$, so (35) holds.

We now prove (30). Taking into account formulas (9) and (11), we have that

$$f_\varepsilon''(u)[v_1, v_2] = \int \Big(\nabla v_1 \cdot \nabla v_2 \,(1 + O(\varepsilon)) + R_g v_1 v_2 - (2^* - 1)|u|^{2^*-2} v_1 v_2\Big)\, dx\,(1 + O(\varepsilon)).$$

From the Hölder and the Sobolev inequalities, and using the fact that the support of R_g is compact, it follows that

$$(f_\varepsilon''(u) - f_0''(u))[v_1, v_2] = O(\varepsilon)\,\Big(1 + O(\varepsilon) + \|u\|^{\frac{4}{n-2}}\Big)\,\|v_1\|\,\|v_2\|,$$

and (30) is proved.

Let us turn to (32). For every $v \in E$ there holds

$$(f_\varepsilon'(u+w) - f_\varepsilon'(u), v) = \int \Big(2c_n \nabla_g w \cdot \nabla_g v + R_g w v + |u+w|^{2^*-2}(u+w)v - |u|^{2^*-2} u v\Big)\, dV_g. \tag{71}$$

This implies that

$$\|f_\varepsilon'(u+w) - f_\varepsilon'(u)\| \leq O(1)\,\|w\|\,(1 + O(\varepsilon)) + \left(\int \left||u+w|^{2^*-2}(u+w) - |u|^{2^*-2}u\right|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} (1 + O(\varepsilon)).$$

Since

$$|u+w|^{2^*-2}(u+w) - |u|^{2^*-2}u = (2^* - 1)\int_0^1 |u+sw|^{2^*-2} w\, ds,$$

setting $y(x) = (2^* - 1)\int_0^1 |u+sw|^{2^*-2} ds$, we have $|u+w|^{2^*-2}(u+w) - |u|^{2^*-2}u = y(x)w(x)$. Hence there holds

$$\left(\int \left||u+w|^{2^*-2}(u+w) - |u|^{2^*-2}u\right|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} \leq C\,\|w\|\,\left(\int |y|^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

Using again the Hölder inequality, we have that $|y| \leq \left(\int_0^1 |u+sw|^{2^*} ds\right)^{\frac{2}{n}}$. So from the Fubini Theorem

$$\int |y|^{\frac{n}{2}}\, dx \leq \int \left|\int_0^1 |u+sw|^{2^*} ds\right| dx = \int_0^1 \left(\int |u+sw|^{2^*} dx\right) ds \leq \sup_{s\in[0,1]} \|u+sw\|_{2^*}^{2^*}.$$

Taking into account the Sobolev inequality, it turns out that, by (23)

$$\left(\int |y|^{\frac{n}{2}}\right)^{\frac{2}{n}} \leq \sup_{s\in[0,1]} \|u+sw\|^{\frac{4}{(n-2)}} \leq C\,(\|u\|^{\frac{4}{(n-2)}} + \|w\|^{\frac{4}{(n-2)}}).$$

In conclusion we obtain (32).

We now prove (28). Given $v \in E$, we have

$$(f_\varepsilon'(u), v) = \int \Big(2c_n \nabla_g u \cdot \nabla_g v + R_g u v - |u|^{2^*-2} u v\Big)\, dV_g.$$

Taking into account formulas (9) and (11), we deduce

$$\begin{aligned}(f_\varepsilon'(u), v) \;&=\; \int \left(2c_n \nabla u \cdot \nabla v - \varepsilon \sum_{ij} h_{ij} D_i u D_j v + O(\varepsilon^2)|\nabla u||\nabla v| + \varepsilon R_1 u v + O(\varepsilon^2)|u||v| - |u|^{2^*-2} u v\right) \\ &\quad\times\; \left(1 + \frac{1}{2}\varepsilon tr\, h + O(\varepsilon^2)\right) dx.\end{aligned}$$

Expanding the last expression in ε, and $O(\varepsilon^2)$, and using again the Hölder and the Sobolev inequality, we obtain (28). Formulas (27), (29), (31), (33) and (34) can be obtained with similar computations. ■

Proof of Proposition 5.1

Let $f_\varepsilon^\delta : E \to \mathbb{R}$ be the Euler functional (5) corresponding to the metric $g^\delta(x) = g(\delta x)$, $\delta > 0$. For all $u \in E$ there holds

$$f_\varepsilon^\delta(u) = f_\varepsilon\left(\delta^{-\frac{n-2}{2}} u(\delta^{-1}x)\right) = f_\varepsilon(u_{\delta,0}) \tag{72}$$

and inversely

$$f_\varepsilon(u) = f_\varepsilon^\delta(\delta^{\frac{n-2}{2}} u(\delta x)).$$

The map $T_\delta : E \to E$ defined by $T_\delta(u) := u_{\delta,0}$ is a linear isometry and by (72) f_ε^δ is nothing but $f_\varepsilon^\delta(u) = f_\varepsilon \circ T_\delta$. In particular for all $u \in E$ it is

$$\nabla f_\varepsilon(u) = T_\delta \nabla f_\varepsilon^\delta(T_\delta^{-1} u). \tag{73}$$

Since f_ε^δ is related to f_ε by the isometry T_δ, one can apply without changes the construction of Section 3 to f_ε^δ. Hence there exists $w_\varepsilon^\delta \in (T_{z_0} Z)^\perp$ such that

$$\nabla f_\varepsilon^\delta(z_0 + w_\varepsilon^\delta) \in T_{z_0} Z.$$

Since $\nabla f_\varepsilon(z_{\delta,0} + w_\varepsilon(z_{\delta,0})) \in T_{z_{\delta,0}} Z$, by uniqueness and by (73) it turns out that

$$w_\varepsilon^\delta(x) = \delta^{\frac{n-2}{2}} w_\varepsilon(z_{\delta,0})(\delta x). \tag{74}$$

We consider also the functional

$$f_\varepsilon^0(u) = \int_{\mathbb{R}^n} \left(c_n \sum_{i,j} g^{ij}(0) D_i u D_j u - \frac{1}{2^*}|u|^{2^*} \right) dV_{g(0)},$$

which corresponds to the metric in $\mathbb{R}^n$ which is identically equal to $g(0)$. With respect to some orthonormal system of coordinates the symmetric matrix $g^{ij}(0)$ has the diagonal form $(\lambda_1, \ldots, \lambda_n)$, where for brevity we have omitted the dependence of λ_i from ε. We note that the numbers λ_i are positive since $g^{ij}(0)$ is close to the identity matrix.

Since f_ε^0 is a perturbation of f_0, reasoning as above we find an unique $w_\varepsilon^0 \in (T_{z_0} Z)^\perp$ satisfying $\nabla f_\varepsilon^0(z_0 + w_\varepsilon^0) \in T_{z_0} Z$. We note that, by symmetry reasons, w_ε^0 must be an even function in $\mathbb{R}^n$. In the next Lemma we prove some further properties of w_ε^0. Define

$$\widetilde{z}_0(x) = z_0\left(\frac{x_1}{\sqrt{\lambda_1}}, \ldots, \frac{x_n}{\sqrt{\lambda_n}}\right).$$

Lemma 6.1 *The function w_ε^0 satisfies $\nabla f_\varepsilon^0(z_0 + w_\varepsilon^0) = 0$. Moreover there holds*

$$w_\varepsilon^0 = T_\mu \widetilde{z}_0 - z_0, \qquad \textit{for some } \mu > 0, \textit{ and } \quad f_\varepsilon^0(z_0 + w_\varepsilon^0) = b_0.$$

PROOF. The functional f_ε^0 is invariant under the transformations $u \to u_{\mu,\xi}$, for all $\mu > 0$ and $\xi \in \mathbb{R}^n$. From this fact one can deduce that $f_\varepsilon^0(z_{\mu,\xi} + w_\varepsilon^0(z_{\mu,\xi}))$ is independent of μ, ξ. Hence, by Proposition 3.2 (iv), the points $z_{\mu,\xi} + w_\varepsilon^0(z_{\mu,\xi})$ are all critical for f_ε^0, and in particular it is $\nabla f_\varepsilon^0(z_0 + w_\varepsilon^0) = 0$.

The positive solutions u of $\nabla f_\varepsilon^0(u) = 0$ can be completely classified. In fact, using the coordinates introduced above, a critical point u of f_ε^0 is a solution of the problem

$$-2c_n \sum_i \lambda_i D_{ii}^2 u = u^{2^*-1}, \qquad u \in E.$$

Using the change of variables $x_i = \lambda_i y_i$, and taking into account that the only solutions of $-\Delta\, u = u^{2^*-1}$ are of the form $z_{\mu,\xi}$, one can deduce that $z_0 + w_\varepsilon^0 = T_\mu \widetilde{z}_0$, for some $\mu > 0$ (here we have used the fact that w_ε^0 must be an even function).

Now we prove that $f_\varepsilon^0(z_0 + w_\varepsilon^0) = b_0$: in fact there holds

$$f_\varepsilon^0(T_\mu \widetilde{z}_0) = f_\varepsilon^0(\widetilde{z}_0) = \int \left(c_n \sum_i \lambda_i \frac{1}{\lambda_i} |D_i z_0|^2 - \frac{1}{2^*} |z_0|^{2^*} \right) \left(\frac{x_1}{\sqrt{\lambda_1}}, \dots, \frac{x_n}{\sqrt{\lambda_n}} \right) |\Pi_i \lambda_i|^{\frac{1}{2}} \; dx.$$

Using again the change of variables $x_i = \lambda_i y_i$, we obtain the result. The proof of the Lemma is complete. ■

Proof of Proposition 5.1. For all $u \in E$ there holds

$$\lim_{\delta \to 0} ||\nabla f_\varepsilon^\delta(u) - \nabla f_\varepsilon^0(u)|| = 0; \tag{75}$$

$$\lim_{\delta \to 0} f_\varepsilon^\delta(u) = f_\varepsilon^0(u). \tag{76}$$

Equations (75) and (76) are easy to verify, for example starting with $u \in C_c^\infty(\mathbb{R}^n)$ and proceeding by density. Furthermore, arguing as in Lemma 4.2, one can deduce that for some $C > 0$ it is $\|w_\varepsilon^\delta - w_\varepsilon^0\| \leq C \|\nabla f_\varepsilon^\delta(\widetilde{z}_0) - \nabla f_\varepsilon^0(\widetilde{z}_0)\| = C \|\nabla f_\varepsilon^\delta(\widetilde{z}_0)\|$. Hence by (75), applied with $u = T_\mu \widetilde{z}_0$, and by Lemma 6.1, it turns out that

$$w_\varepsilon^\delta \to w_\varepsilon^0 = \widetilde{z}_0 - z_0 \quad \text{as} \quad \delta \to 0. \tag{77}$$

Using (72) and (73) we deduce that

$$\varphi_\varepsilon(\delta, 0) = f_\varepsilon(z_{\delta,0} + w_\varepsilon(z_{\delta,0})) = f_\varepsilon^\delta(z_0 + w_\varepsilon^\delta).$$

We can write

$$f_\varepsilon^\delta(z_0 + w_\varepsilon^\delta) - f_\varepsilon^0(z_0 + w_\varepsilon^0) = \left(f_\varepsilon^\delta(z_0 + w_\varepsilon^\delta) - f_\varepsilon^\delta(z_0 + w_\varepsilon^0) \right) + \left(f_\varepsilon^\delta(z_0 + w_\varepsilon^0) - f_\varepsilon^0(z_0 + w_\varepsilon^0) \right).$$

There holds

$$f_\varepsilon^\delta(z_0 + w_\varepsilon^\delta) - f_\varepsilon^\delta(z_0 + w_\varepsilon^0) = f_\varepsilon(z_{\delta,0} + T_\delta w_\varepsilon^\delta) - f_\varepsilon(z_{\delta,0} + T_\delta w_\varepsilon^0),$$

and from (31) it follows that

$$|f_\varepsilon(z_{\delta,0} + T_\delta w_\varepsilon^\delta) - f_\varepsilon(z_{\delta,0} + T_\delta w_\varepsilon^0)| \leq C \|T_\delta w_\varepsilon^\delta - T_\delta w_\varepsilon^0\|.$$

By (77), and since T_δ is an isometry, it is $f_\varepsilon^\delta(z_0 + w_\varepsilon^\delta) - f_\varepsilon^\delta(z_0 + w_\varepsilon^0) \to 0$ as $\delta \to 0$. From (76) we deduce that also $f_\varepsilon^\delta(z_0 + w_\varepsilon^0) - f_\varepsilon^0(z_0 + w_\varepsilon^0) \to 0$ as $\delta \to 0$. Hence $f_\varepsilon^\delta(z_0 + w_\varepsilon^\delta) - f_\varepsilon^0(z_0 + w_\varepsilon^0) \to 0$ as $\delta \to 0$. By means of the last computations we have proved that

$$\lim_{\delta \to 0} \varphi_\varepsilon(\delta, \xi) = b_0, \qquad \xi = 0. \tag{78}$$

Actually the above reasoning can be performed uniformly if ξ varies in a fixed compact set of $\mathbb{R}^n$; this implies (67). Equation (68) can be proved using the Kelvin transform. In fact, since the same computations can be repeated in the same way for $f_\varepsilon^\sharp$, one has, by formula (22)

$$\lim_{\mu + |\xi| \to +\infty} \varphi_\varepsilon(\mu, \xi) = \lim_{(\overline{\mu}, \overline{\xi}) \to 0} \varphi_\varepsilon^\sharp(\overline{\mu}, \overline{\xi}) = 0.$$

This concludes the proof. ■

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