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A Hyperintensional Logic of Non-prime Evidence

Pietro Vigiani[1]

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Abstract
We present a logic of evidence that reduces agents' epistemic idealisations by combining classical propositional logic with substructural modal logic for formulas in the scope of epistemic modalities. To this aim, we provide a neighborhood semantics of evidence, which provides a modal extension of Fine's semantics for relevant propositional logic. Possible worlds semantics for classical propositional logic is then obtained by defining the set of possible worlds as a special subset of information states in Fine's semantics. Finally, we prove that evidence is a hyperintensional and non-prime notion in our logic, and provide a sound and complete axiomatisation of our evidence logic.

Keywords Neighborhood semantic · Hyperintensionality · Relevant logic · Substructural logic · Formal epistemology

1 Introduction

Modal logics with epistemic applications famously provide models of reasoning in which epistemic modalities satisfy several closure properties [17, 22]. Given that one of the main purposes of epistemic logic is to formally model the behaviour of agents in everyday inquiry, such closure properties undermine this purpose at its roots, as agents are endowed with great idealisations. In the present paper, we will be interested in addressing such idealisations for modal logics featuring an operator E expressing propositional evidence.

Epistemic idealisations constitute a threat for both normal and non-normal modal logics, since even the minimal modal logic is closed under the following equivalence rule.

$$\varphi \leftrightarrow \psi \Rrightarrow E\varphi \leftrightarrow E\psi \quad \text{(E.E)}$$

✉ Pietro Vigiani
pietro.vigiani@sns.it

1 Department of Philosophy, Scuola Normale Superiore, Pisa, Italy

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Failure of (E.E), motivated by the fact that realistic agents need not discriminate between logically equivalent propositions they have epistemic access to, is the distinctive trait of *hyperintensional* logics (see [33] for an overview).

Neighborhood semantics was recently used in *evidence logic* to provide a philosophically robust formal model of evidence [5, 6]. However, as the minimal modal logic is sound and complete with respect to all neighborhood frames [11], evidence logic is closed under (E.E), thus not being hyperintensional. To appreciate what is odd with the neighborhood analysis of evidence, let us recall that a context is usually said to be hyperintensional if and only if intensionally equivalent propositions cannot be substituted in it salva veritate. Given that standardly intensions are taken to be propositions[1]; that logically equivalent formulas express the same proposition; and that the semantics of $E\varphi$ in neighborhood semantics is given exclusively in terms of the proposition expressed by φ, then (E.E) seems unavoidable.

One way to account for the hyperintensionality of evidential contexts is to make finer-grained distinctions in the concept of proposition. In this paper, we draw from the observation that possible worlds semantics is not adequate to represent the informational content of prospositions (see e.g. [28]) and distinguish between two epistemically salient notions of support, *intensional* and *extensional truth*, with their corresponding classes of semantic objects, *intensional* and *extensional propositions*. The former consists of information states, while the latter of possible worlds.

The use of information states is standard in relevant and substructural logics (see e.g. [1, 31, 32]). In the semantic interpretation of relevant logic, information states constitute a generalisation of possible worlds, insofar as they are able to support both inconsistent and incomplete information. In our framework, information states record what is true according to the agent's information or, alternatively, how the world looks like from the agent's perspective. We label such relation between information states and formulas *intensional truth*, by which we interpret formulas in epistemic contexts (i.e. within the scope of epistemic modalities) according to a logic in the vicinity of relevant logic.

On the other hand, possible worlds can be regarded as maximally consistent information states, and they can be used to describe what is true irrespective of a specific agent's point of view[2]. We then use *extensional truth* to interpret classically, via clauses equivalent to the usual boolean ones, propositional formulas at possible worlds.

The resulting model of evidence is as follows. While agents collect evidence about classical possible worlds, they process the information conveyed by such evidence non-classically[3]. Then, we are able to accomplish hyperintensionality by (i) defining

[1] This is the received view coming from e.g. [40], according to which propositions are taken as sets of states in models (in classical logic, possible worlds). Note however that the concept of proposition depends on a specific theory of sentential content and on the resulting intension-forming operator (see e.g. the two-dimensional approach of [16] and the structural approach to intensions of [12].

[2] Possible worlds can also be conceived of as recording the information of an ideal agent. This intuition exploits the commonly held assumption that classical logic represents an upper bound of deductive reasoning.

[3] We stress that the present strategy does not assign classical logic a privileged status, but is compatible with metaphysical views which are committed to different logics (e.g. paraconsistent) of possible worlds. Likewise, our framework supports many models of reasoning from the family of substructural logics.

possible worlds as a subclass of information states; by (ii) distinguishing between extensional truth and intensional truth; and by (iii) defining validity with respect to extensional truth while (iv) letting extensional epistemic propositions be defined in terms of intensional propositions. This strategy was used in [34], where classical propositional logic and relevant modal logic are combined in a (family of) hyperintensional modal logic(s).

Hyperintensionality has received considerable attention recently, especially as a desideratum in formal epistemology (see e.g. [33]). A related and often neglected desideratum is the invalidity of the following axiom.

$$E(\varphi \vee \psi) \rightarrow E\varphi \vee E\psi \tag{E.DJ}$$

While closure under disjunction is avoided in many modal logics based on relational and neighborhood semantics, its failure does not originate in the standard, lattice-based, semantics of disjunction. However, boolean disjunction has received its share of criticism, as many frameworks have highlighted how it is not adequate in epistemic contexts[4]. We interpret information states as carrying the information *explicitly* supported by an agent. Therefore, it is natural to interpret information states as *non-prime*, in order to model all those contexts where possessing a disjunctive piece of information does not give agents any definite information about either disjunct.

A non-prime account of information states was provided by Fine in [19]. We argue that an epistemic interpretation of Fine's rich semantics can account for the failure of (E.DJ), and we put forward a logic according to which agents having evidential support for a disjunction need not by the same piece of evidence have support for either disjunct, as the information states composing the evidence are not determinate enough (see Example 2 in the next section for a concrete counterexample). Finally, Fine's semantics aligns with the general motivation of the paper: we identify intensional aspects of logical connectives, which arise in epistemic contexts, by treating them as modalities; and we provide logical resources to identify extensional contexts, where such intensional aspects are ineffective (see the discussion in Section 2.3).

The present work can be seen as an application of the framework developed in [34] to a different model of relevant logic. Here, we target specifically evidence's closure properties. The paper additionally provides a modal extension of Fine's relevant logic. The rest of the paper is structured as follows. In Section 2 we provide the necessary epistemological and logical preliminaries. In Section 3 we introduce our semantics, the distinction between extensional and intensional truth, and show that the resulting notion of evidence is not closed under many epistemic principles, including (E.E) and (E.DJ). In Section 4 we provide a sound and complete axiomatisation of our logic of evidence.

[4] To mention a few, in the Dempster-Shafer theory of belief functions, the evidential support received by a disjunction of pairwise disjoint events is greater than the sum of the supports received by the events (see [9] for a recent application in non-classical logic); and in depth-bounded boolean logics (see [14]) the semantics of $\vee$ is formulated non-deterministically, so that agents may accept a disjunction as true while abstaining on both components.

2 Preliminaries

In this section we lay down our target closure principles with a brief discussion on their implausibility; we also review some preliminary results on evidence logic and on propositional relevant logics.

In what follows, we work with the following language $\mathcal{L}$, defined in BNF from a set of propositional atoms At, with $p \in At$.

$$\varphi \in \mathcal{L} ::= \top \mid \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid E\varphi \mid L\varphi$$

We abbreviate $\varphi \leftrightarrow \psi := \varphi \to \psi \wedge \psi \to \varphi$ and use $\forall$, $\exists$, &, $\Rightarrow$ to denote, respectively, universal quantification, existential quantification, conjunction and implication in the meta-language. In this section we consider the sublanguages $\mathfrak{L}_{\top\bot L}$, $\mathfrak{L}_{\top\bot EL}$ restricted to formulas without occurrences of $\top$, $\bot$, $L\varphi$, respectively $\top$, $\bot$, $E\varphi$, $L\varphi$. $L\varphi$ and $E\varphi$ are epistemic formulas reading "the agent has logical information that φ" and "the agent has propositional evidence supporting φ", respectively. Note that we understand E and L as independently motivated modalities. While E constitutes our main object of study, many results rely on the presence of L, whose main technical role is clarified by Lemma 9 (see however Section 3 for an epistemic interpretation). We call the result of removing every occurrence of E from a formula φ the E-free version of φ. Finally, for any binary relation X, we adopt the standard notation $X(x) = \{y \mid Xxy\}$.

2.1 Closure Principles of Interest

Figure 1 shows a non-exhaustive list of closure principles, which we target in the following sections. As a motivation for the study of logics that lack some or all of them, we briefly consider reasons why these principles highlight controversial closure properties of evidence. We refer the interested reader to [22] for a more in depth discussion of the related problem of logical omniscience within the epistemological literature. Here, we just stress that it is highly questionable that logical omniscience can be fully solved by purely semantic means[5], hence the framework we develop just aims at providing a logic of a moderately idealised notion of evidence (see Lemma 9.3 for the distinctive closure principle of evidence in our framework).

Some closure principles highlight agents' possible deductive deficiencies, which come from the fact that agents are resource-bounded, i.e. they may not be in a cognitive position to perform all the deductive steps necessary to derive a conclusion from premises they possess. For a counterexample to (E.M), think of agents not able to deduce all logical consequences of propositions they have evidence for: for example, they may lack the time, computational power, memory, etc. needed to perform the derivation. For a counterexample to (E.C), think of agents who fail to gather, or aggregate, two sources of information into a single piece. This view can be supported by a fragmentation theory of belief storage, according to which the "information utilizable

[5] As remarked in [24], for any two sentences φ and ψ such that they are equivalent in a given logic but syntactically distinct, we can find whatever accidental reason in agents' cognitive architecture so that they have evidence for φ but fail to have evidence for ψ.

Closure principles		Axiom/Rule
(E.E)	Equivalence rule	$\varphi \leftrightarrow \psi \Rrightarrow E\varphi \leftrightarrow E\psi$
(E.M)	Monotonicity rule	$\varphi \rightarrow \psi \Rrightarrow E\varphi \rightarrow E\psi$
(E.N)	Necessitation rule	$\varphi \Rrightarrow E\varphi$
(E.C)	Closure under conjunction	$E\varphi \wedge E\psi \rightarrow E(\varphi \wedge \psi)$
(E.K)	Closure under implication	$E(\varphi \rightarrow \psi) \rightarrow (E\varphi \rightarrow E\psi)$
(E.DJ)	Closure under disjunction	$E(\varphi \vee \psi) \rightarrow (E\varphi \vee E\psi)$
(E.W)	Modal weakening	$E\varphi \rightarrow E(\psi \rightarrow \varphi)$
(E.CO)	Modal contraction	$E(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow E(\varphi \rightarrow \psi)$
(E.MP)	Modal modus ponens	$E(\varphi \wedge (\varphi \rightarrow \psi)) \rightarrow E\psi$
(E.S)	Contradictory source	$E(\varphi \wedge \neg\varphi) \rightarrow E\psi$
(E.DS)	Modal disjuncitve syllogism	$E(\varphi \vee \psi) \wedge E\neg\varphi \rightarrow E\psi$

Fig. 1 Some problematic closure principles for evidence

by cognitive processes is stored in distinct, independently accessible data structures" [4]. For a counterexample to (E.N), think of agents who lack evidence of complex mathematical propositions, which are usually taken to be logically valid. Finally, for a counterexample to (E.K), think of agents who lack the competence to combine information they already have to perform a deduction. We note here that aggregation and combination are often considered distinct epistemic actions: for example, in [39] it is argued that aggregation is just a necessary condition for combination, and that combination (differently from aggregation) is generative and fallible. As hyperintensionality is one of our main desiderata, we will consider in more detail a counterexample to (E.E) adapted from [28].

Example 1 Consider a scenario in which an agent has evidence for some empirical statement φ, but has no information regarding some classical tautology ψ. Since conjunctions of tautologies with empirical statements are true in the same possible worlds as the empirical statements alone, φ and $\varphi \wedge \psi$ are logically equivalent. However, the agent has evidence supporting φ but not supporting $\varphi \wedge \psi$. Note that, were the agent's information maximally consistent, they would have evidence for φ and for $\varphi \wedge \psi$ in exactly the same circumstances.

The remaining principles do not challenge deductive competence, but pertain to specific traits of the information evidential support is based on. They pose a problem to frameworks based on classical modal logic, which are not well positioned to account for e.g. partiality, inconsistency, resource-sensitiveness and non-primeness of information. For a counterexample to (E.W) (a modal version of weakening), it suffices to think about an agent whose non-conclusive evidence towards a proposition is defeated upon considering further information. For a counterexample to (E.CO) (modal contraction), we can adapt an argument from channel theory [29]: think about an agent whose information that 6 is even leads them to having evidence that 10 is even by repeated use of the information conveyed by the rule "if n is even, $n+2$ is even". Such agent has not evidence that 10 is even with a single application of the rule. Counterexamples to (E.MP) (modal modus ponens) involve a similar reference to the failure of idempotency of information combination (see Section 2.3 for further discussion on the properties of information combination). Similar counterexamples can be devised for a modified version of modal modus ponens, $E\varphi \wedge E(\varphi \rightarrow \psi) \rightarrow E\psi$. For a counterexample to (E.S), think about an agent acquiring a contradictory piece of evidence from a source, who does not become thereby altogether irrational. A related problematic principle is $E\varphi \wedge E\neg\varphi \rightarrow E\psi$, where an agent acquires mutually contradictory pieces of evidence from possibly distinct sources. Note that further counterexamples to (E.DS) (as well as to $E\varphi \wedge E(\varphi \rightarrow \psi) \rightarrow E\psi$ and $E\varphi \wedge E\neg\varphi \rightarrow E\psi$) can be devised by appealing to failure of information aggregation, just like in the case of (E.C). Finally, as non-primeness is another of our main desiderata, we will consider in more detail a counterexample to (E.DJ), adapted from [2].

Example 2 Consider a scenario in which an agent fails to access their mail box. While the failed access provides them evidence that either the email address or the password are wrong, they do not have thereby evidence that the email address is wrong, nor that the password is wrong. Note that, were the agent's information sufficiently refined, they would be able to ascertain whether the email address or the password is wrong.

2.2 Classical Evidence Logic

Neighborhood semantics was employed in evidence logic [5, 6] to model epistemic attitudes based on the evidential support that agents gather towards propositions. The main feature of the semantics of evidence logic is that an agent is associated with an evidence base consisting of the distinct pieces of evidence gathered so far. In [5, 6], the rich structure of neighborhood frames has a pivotal role in understanding other interesting epistemic notions, such as (conditional) evidence-based belief and several dynamic actions involving evidence management. The foundational role of evidence in epistemic logic motivates our focus on evidence in isolation, and on its properties in the classical framework.

Definition 1 (**Neighborhood frame**) Let a *neighborhood frame* be a couple $\mathfrak{F}_N = (S, N)$ such that $N \subseteq S \times \mathscr{P}(S)$ and for all $X, Y \in \mathscr{P}(S)$:

$$NsX \;\; \& \;\; X \subseteq Y \Rightarrow NsY \tag{1}$$

Given a neighborhood frame $\mathfrak{F}_N$, let us define the following frame operation for all $X, Y \in \mathscr{P}(S)$.

$$E^{\mathfrak{F}_N} X = \{s \mid NsX\}$$

Definition 2 (**Neighborhood model, validity**) Let a *neighborhood model* be a couple $\mathfrak{M}_N = (\mathfrak{F}_N, V)$ such that $\mathfrak{F}_N$ is a neighborhood frame and $V : At \rightarrow \mathscr{P}(S)$ is extended to a truth relation $\models^N$ on the language $\mathfrak{L}_{\top\bot L}$ by recursion as follows, where $[\![\varphi]\!]_{\mathfrak{M}_N} = \{s \mid \mathfrak{M}_N, s \models^N \varphi\}$.

$$
\begin{aligned}
[\![p]\!]_{\mathfrak{M}_N} &= V(p) \\
[\![\neg\varphi]\!]_{\mathfrak{M}_N} &= \overline{[\![\varphi]\!]}_{\mathfrak{M}_N} \\
[\![\varphi \wedge \psi]\!]_{\mathfrak{M}_N} &= [\![\varphi]\!]_{\mathfrak{M}_N} \cap [\![\psi]\!]_{\mathfrak{M}_N} \\
[\![\varphi \vee \psi]\!]_{\mathfrak{M}_N} &= [\![\varphi]\!]_{\mathfrak{M}_N} \cup [\![\psi]\!]_{\mathfrak{M}_N} \\
[\![\varphi \rightarrow \psi]\!]_{\mathfrak{M}_N} &= \overline{[\![\varphi]\!]}_{\mathfrak{M}_N} \cup [\![\psi]\!]_{\mathfrak{M}_N} \\
[\![E\varphi]\!]_{\mathfrak{M}_N} &= E^{\mathfrak{F}_N} [\![\varphi]\!]_{\mathfrak{M}_N}
\end{aligned}
$$

Let a formula $\varphi \in \mathfrak{L}$ be *N-valid* in a neighborhood model $\mathfrak{M}_N$ (written $\mathfrak{M}_N \models \varphi$) iff $\mathfrak{M}_N, s \models^N \varphi$ for all $s \in S$. A formula φ is *N-valid* (written $\models^N \varphi$) iff φ is valid in all neighborhood models.

In what follows, we will call $N(s) = \{X \mid NsX\}$ the agent's *evidence base* at s, $X \in N(s)$ an *evidence set* in the agent's evidence base, and $x \in X \in N(s)$ an *evidence state* in the agent's evidence base.

Drawing on Levesque's distinction between explicit and implicit beliefs [25], we may distinguish two types of evidence bases. According to the explicit reading, evidence sets in the agent's evidence base constitute the pieces of evidence actually possessed by the agent, while according to the implicit reading they constitute potentially possessed pieces of evidence. That is, according to the implicit reading evidence bases contain all pieces of evidence which include some actually possessed piece of evidence (i.e. the neighborhood relation is monotonic). Likewise, we can distinguish between an explicit and an implicit reading of evidential support. According to the explicit reading of support an agent has evidence that X iff X is in their evidence base, while according to the implicit reading an agent has evidence for X iff there is $Y \in N(w)$ entailing X. Note that, while in [5, 6] evidence is modeled as a monotonic modality by employing the explicit reading of evidence bases and the implicit notion of support in evidence's truth condition, it is technically equivalent to assume the implicit reading of evidence bases and use the explicit notion of support[6].

Note that we do not assume any property on neighborhood frames besides monotonicity, so that the resulting notion of evidence is minimally characterised. Stronger properties can, of course, be obtained by imposing stronger frame conditions, as usual in modal correspondence theory (e.g. see [11, 27]). However, since our purpose is to devise a framework which avoids strong epistemic idealisations, we will be concerned with the weakest modal logic our semantic notion of evidence gives rise to.

Proposition 1 (**Neighborhood closure properties**) (E.E) *and* (E.M) *preserve validity and* (E.W), (E.CO), (E.MP), (E.S) *of Fig. 1 are valid in every neighborhood model* $\mathfrak{M}_N$*.* (E.N) *does not preserve validity and* (E.C), (E.K), (E.DJ), (E.DS) *are not valid in every neighborhood model* $\mathfrak{M}_N$*.*

[6] To complete the picture of evidence logic, [5, 6] include the frame conditions NsS and not $Ns\emptyset$, expressing the minimal requirements that no evidence set is explicitly contradictory per se and that agents "know their space" [5, 6]. We omit them for the sake of simplicity.

Proof The proof is standard. The fact that (E.E) and (E.M) preserve validity is established as in [5, 6], while the validity of the remaining principles follows from the fact that the E-free version of each principle is classically valid. (E.N) does not preserve validity since evidence bases may be empty. (E.DJ) is not valid since e.g. evidence bases are not closed under subset, i.e. $NsX, Y \subseteq X \not\Rightarrow NsY$; while (E.C), (E.K) and (E.DS) are not valid since e.g. evidence bases do not enjoy the finite intersection property, i.e. $X, Y \in N(s) \not\Rightarrow X \cap Y \in N(s)$. □

From the above proposition, it follows that evidence bases are deductively closed, in the specific sense that whenever agents have evidence for φ, they ipso facto have evidence for whatever classically follows from φ. This property is a direct consequence of adopting monotonic neighborhood frames. In Section 3 we introduce a logic according to which evidence bases are not closed under classical consequence, while retaining monotonic frames (and the associated implicit reading of evidence). Note also that, while (E.DJ) is invalid in classical evidence logic, it is so because of the properties of evidence bases rather than of evidence states. However, scenarios such as Example 2 show that evidence states (once they are conceptualised as information states) can themselves support non-prime information. We give logical flesh to this observation in Section 3.

To conclude the presentation of classical evidence logic, we recall the standard axiom system for monotonic non-normal modal logic [11], which is sound and complete with respect to the class of neighborhood models.

Definition 3 (**Classical evidence axiom system**) Let the logic M be the smallest axiom system including the following axioms and rules:

- Classical Propositional Logic (CPC);
- The monotonicity rule (E.M) $\varphi \to \psi \Rrightarrow E\varphi \to E\psi$.

Let provability of a formula φ in M (written $\vdash_{\mathsf{M}} \varphi$) be defined as usual. Provability of formulas in the axiom systems B^-, I and EI to be introduced in the rest of the paper is defined similarly[7].

Theorem 1 (**Evidence determination**) *For all* $\varphi \in \mathfrak{L}_{\top\bot L}$: $\models^N \varphi$ *iff* $\vdash_{\mathsf{M}} \varphi$.

Proof [11, Theorem 9.10]. □

2.3 Relevant Logic with Informational Semantic

We present here an informational interpretation of the semantics of relevant logics (see e.g. [3] for alternative interpretations). While relevant logic was originally developed to capture a notion of primitive entailment and to avoid the fallacies of material implication, it has recently received considerable attention as a formal model of agency (see e.g. [8, 35]).

[7] Note that, since we provide Hilbert-style proof systems for our logics, we will be interested exclusively in theoremhood. We leave the task of a deeper study of consequence relations in our framework for future work. Therefore, our determination results will feature weak completeness theorems.

Models for relevant epistemic logic are based on a partially ordered set of information states $(S, \leq)$, so that $s \leq t$ means that t contains more information than s. In such models information states are understood as carrying the information an agent explicitly possesses, and explicit information can rule out other information or be combined by combining the states carrying such information.

Definition 4 (**Information frame**) Let an *information frame* be a tuple $\mathfrak{F}_I = (S, L, \leq, R, C)$ such that $L \subseteq Up(S) = \{X \subseteq S \mid (s \in X \ \ \& \ \ s \leq t) \Rightarrow t \in X\}$, $\leq \subseteq S^2$, $R \subseteq S^3$ and $C \subseteq S^2$. Moreover, we assume:

$$Cst,\ s' \leq s \ \ \& \ \ t' \leq t \Rightarrow Cs't' \tag{2}$$

$$Rstu,\ s' \leq s,\ t' \leq t \ \ \& \ \ u \leq u' \Rightarrow Rs't'u' \tag{3}$$

$$\forall s \exists t (t \in L \ \ \& \ \ Rtss) \tag{4}$$

$$Rstu \ \ \& \ \ s \in L \Rightarrow t \leq u \tag{5}$$

Definition 5 (**Operations on information frames**) Given an information frame $\mathfrak{F}_I$, let us define the following frame operations for all $X, Y \in \mathscr{P}(S)$.

$$\neg^{\mathfrak{F}_I} X = \{s \mid Cst \Rightarrow t \notin X\}$$

$$X \circ^{\mathfrak{F}_I} Y = \{s \mid \exists t, u (t \in X \ \ \& \ \ u \in Y \ \ \& \ \ Rtus)\}$$

$$X \rightarrow^{\mathfrak{F}_I} Y = \{s \mid \{s\} \circ^{\mathfrak{F}_I} X \subseteq Y\}$$

Note that by (3)-(5), we have that $s \leq t$ iff $\exists l \in L(Rlst)$.

Information frames use C, R and L for the interpretation of negated, implicative and valid formulas, respectively. Negation's accessibility relation, C, is interpreted as expressing compatibility between two information states, so that Cst iff t does not explictly support any information explicitly rejected by s Implication's accessibility relation, R, is the usual ternary relation of relevant logic. Since we do not assume weak commutativty of R ($Rstu \Rightarrow Rtsu$), whenever we have $Rstu$, s can be regarded as containing the contextual information to which, by means of information combination, the information contained in t is applied. Alternatively, s can be thought of as storing programs (wordly regularities) to which input signals can be (non-functionally) applied in order to produce some output target[8]. Finally, L is usually interpreted as containing logical information states. In the semantics of relevant logic, the main motivation for introducing logical information states is a technical one, embodied by the semantic deduction theorem (see Lemma 2.1). This means that the distinction between logical and non-logical states is just a way of individuating relevant deductions (relevantly valid formulas), as per e.g. [26, p.52]. Therefore, logical information states are logical in the minimal sense that they contain valid formulas of relevant logic.

We impose minimal assumptions on R and C in information frames. Again, we note that more or less epistemically motivated stronger frame conditions can be assumed,

[8] This observation makes our picture of information combination reminiscent of how channel theory and situation semantics [15, 23] model the connection between channels, expressing implicative regularities, and situations (see [30] for a comparison with frame semantics for relevant logic).

which correspond to stronger principles concerning $\neg, \rightarrow$. For example, symmetric compatibility relations ($Cst \Rightarrow Cts$), yielding the validity of $\varphi \rightarrow \neg\neg\varphi$ are widely used in frame semantics for relevant logics with epistemic applications [7, 8, 35]. One may further require C to satisfy convergence ($\forall s \exists t(Ctu \Rightarrow u \leq s)$), which together with symmetry and seriality ($\forall s \exists t(Cst)$) yields the validity of $\neg\neg\varphi \rightarrow \varphi$. Moreover, weak commutativity, idempotence ($Rsss$) and associativity ($R(st)uv \Leftrightarrow Rs(tu)v$, where $R(st)uv := \exists x(Rstx \ \& \ Rxuv)$) have all been advocated as characteristic of information combination (see e.g. [13])[9]. In what follows, we set aside these issues, resting on the minimal assumptions that are sufficient to yield an intensional analysis of negation and implication, via the frame operations of Definition 5.

Definition 6 (**Information model, validity**) Let an *information model* $\mathfrak{M}_I$ be a couple $(\mathfrak{F}_I, V)$ such that $\mathfrak{F}_I$ is an information frame and $V : At \rightarrow Up(S)$ is extended to an *intensional truth* relation $\models^I$ on the language $\mathfrak{L}_{\top\bot EL}$ by recursion as follows, where $[\![\varphi]\!]_{\mathfrak{M}_I} = \{s \mid \mathfrak{M}_I, s \models^I \varphi\}$.

$$\begin{aligned}
[\![p]\!]_{\mathfrak{M}_I} &= V(p) \\
[\![\neg\varphi]\!]_{\mathfrak{M}_I} &= \neg^{\mathfrak{F}_I} [\![\varphi]\!]_{\mathfrak{M}_I} \\
[\![\varphi \wedge \psi]\!]_{\mathfrak{M}_I} &= [\![\varphi]\!]_{\mathfrak{M}_I} \cap [\![\psi]\!]_{\mathfrak{M}_I} \\
[\![\varphi \vee \psi]\!]_{\mathfrak{M}_I} &= [\![\varphi]\!]_{\mathfrak{M}_I} \cup [\![\psi]\!]_{\mathfrak{M}_I} \\
[\![\varphi \rightarrow \psi]\!]_{\mathfrak{M}_I} &= [\![\varphi]\!]_{\mathfrak{M}_I} \rightarrow^{\mathfrak{F}_I} [\![\psi]\!]_{\mathfrak{M}_I}
\end{aligned}$$

Let a formula $\varphi \in \mathfrak{L}$ be *I-valid* in an information model $\mathfrak{M}_I$ (written $\mathfrak{M}_I \models \varphi$) iff $\mathfrak{M}_I, l \models^I \varphi$ for all $l \in L$. A formula φ is I-valid (written $\models^I \varphi$) iff φ is valid in all information models.

In information models, situations are not closed under the laws of classical logic. In particular, thanks to the interpretation of negated formulas through C, situations may support classical contradictions $\varphi \wedge \neg\varphi$ and may not support classical tautologies $\varphi \vee \neg\varphi$. Together with the interpretation of implicational sentences through R, this is responsible for the failure of the paradoxes of material implication, such as $\varphi \wedge \neg\varphi \rightarrow \psi$, $\psi \rightarrow (\varphi \vee \neg\varphi)$ or $\varphi \rightarrow (\psi \rightarrow \varphi)$.

Information models are often used to provide formal accounts of inferential patterns that occur in epistemic contexts, in which we reason about agents' evidence, belief, knowledge, etc. For this reason, in many relevant epistemic models, information states have been interpreted as data available to an agent [10], bodies of evidence [35], or even states of explicit knowledge/belief [38].

[9] Note however that each condition comes with some controversy. [37] argues that relevant deductive reasoning should be weakly commutative but not associative and not strongly-commutative ($R(st)uv \Rightarrow R(su)tv$). According to a channel-theoretic interpretation of relevant logic [30], channels' warrant transmission is not idempotent. Finally, it may be argued that combination has a distinctive application order, so that only applying an input signal to a program is felicitous. Since programs constitute different data types than signals and targets, information combination should not be associative nor (weakly or strongly) commutative (see also [41]).

In information models, connectives are analysed as intensional modalties, with the accessibility relations corresponding to each connective making explicit the epistemic labour that agents need to carry out. That is, in epistemic contexts the agent's information depends on their combinatorial abilities or their ability to rule out information.

However, information models' intensional analysis of connectives is partial for two reasons. First, we do not have a way to distinguish in those models epistemic from non-epistemic contexts, in which the truth of a formula does not depends on agents' epistemic actions. In substructural and relevant epistemic models we do not have a way to talk about facts of the world, which obtain irrespective of the information agents possess regarding them[10]. Second, the intensional analysis of connectives in epistemic contexts is limited to implication and negation, and in particular it does not extend to disjunction[11].

In the following section we introduce a semantic model in which it is possible to distinguish epistemic and propositional contexts, and in which the intensional truth of $\rightarrow, \neg, \vee$ depend on agents' epistemic labour (respectively, information combination, ruling out information and information refinement). This is accomplished by introducing a more general and a more specific class of states than information states, non-prime states and possible worlds.

To conclude the presentation of our relevant logic based on informational semantics, we present below the axiom system B^-, the minimal axiom system for relevant propositional logic using relational semantics, which is sound and complete with respect to the class of all information models. As pointed out in Section 4, B^- is closely related to Fine's system B of [19].

Definition 7 (**Relevant axiom system**) Let the logic B^- be the smallest axiom system including the axioms and rules of Fig. 2.

Theorem 2 (**Relevant determination**) *For all* $\varphi \in \mathfrak{L}_{\top\bot EL}$: $\models^I \varphi$ *iff* $\vdash_{\mathsf{B}^-} \varphi$.

Proof Virtually as in [31, Theorems 11.20, 11.37]. □

3 Extensional and Intensional Truth

In this section, we introduce our semantics, which builds on (i) evidence logic [5, 6], (ii) Fine's semantics for relevant logic [19] and (iii) the logic of relevant reasoners in a classical world [34].

In [19], the main novelty compared to the standard semantics of relevant logic is that states are understood as non-prime bodies of information, or theories, so that a state s supporting φ expresses the fact that the theory s commits the agent to the truth of φ. Theories can of course be non-prime, as Example 2 illustrates. Prime

10 Alternatively, if one subscribes to the view of classical logic as an upper bound for deductive reasoning, we lack a way to distinguish standard deductive contexts from ideal deductive contexts.

11 Although conjunction in information models is not treated as a modality, some authors (see e.g. [37]) have suggested that intensional conjunction $\otimes$ should be employed in the analysis of conjunctive statements in epistemic contexts

(A1)	$\varphi \to \varphi$
(A2)	$\varphi \wedge \psi \to \varphi$
(A3)	$\varphi \wedge \psi \to \psi$
(A4)	$(\varphi \to \psi) \wedge (\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)$
(A5)	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
(A6)	$\varphi \to \varphi \vee \psi$
(A7)	$\psi \to \varphi \vee \psi$
(A8)	$\neg\varphi \wedge \neg\psi \to \neg(\varphi \vee \psi)$
(A9)	$\varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \chi)$
(R1)	$\varphi,\ \psi \Rrightarrow \varphi \wedge \psi$
(R2)	$\varphi \to \psi,\ \varphi \Rrightarrow \psi$
(R3)	$\varphi \to \psi,\ \chi \to \xi \Rrightarrow (\psi \to \chi) \to (\varphi \to \xi)$
(R4)	$\varphi \to \psi \Rrightarrow \neg\psi \to \neg\varphi$

Fig. 2 The axiom system B^-

theories correspond in Fine's semantic to theories satisfying further constraints. In our semantics based on *intensional epistemic frames*, we provide a modal extension of [19], in which we are able to encode both the fact that an agent's information supports φ and the stronger fact that an agent has evidence for φ.

On top of that, we build on [34] and understand agents as reasoning according to substructural logic, while gathering information and acquiring evidence within a possible world. We can give a more substantive reading to this claim by considering Mares' *theory of situated inference* [26]. According to such theory, when making an inference "we hypothesise that a proposition is true in a situation and that this situation obtains in a particular world. This world is held fixed for the entire inference"[26, p.42]. The framework we present develops a similar idea, by which the interpretation of implication is relativised to the world where the relevant inference takes place. The inference $\varphi \to \psi$ is situated at a world w in the sense that the information states explicitly supporting $\varphi \to \psi$ are contained in some evidence set contained in the agent's evidence base at w.

As in [34], the semantics based on *epistemic frames* introduces *possible worlds* as a subset of prime information states and distinguish between *extensional truth*, i.e. truth at possible worlds, and *intensional truth*, i.e. truth at information states. Intensional truth represents how the world looks like from the perspective of an agent, while extensional truth represents ground truths. Accordingly, information states represent the information of agents reasoning non-classically about classical possible worlds (maximally consistent information states)[12].

[12] This understanding of the relationship between classical and non-classical logic is not new, see e.g. Grzegorczyk's formal interpretation of intuitionistic logic as the logic of scientific research, against classical logic as the logic of ontological thought [21]. Note also that in our framework we set aside the issue of whether agents can access ground facts or whether they can tell whether they are situated within a world. The distinction between worlds and information states is just a modeler's resource to distinguish whether or not the agent's information corresponds to ground truths, or ontological thought. We thank an anonymous reviewer for pressing us on the issue.

In what follows, we conventionally use s, t, u for general non-prime states and p, q, r for prime states.

Definition 8 (**(Intensional) Epistemic frame**) Let an *intensional epistemic frame* be a tuple $(S, P, L, Prop, \leq, R, C, N, Q)$ where:

- $Prop \subseteq Up(S)$ is a set of admissible propositions;
- $(S, Prop, N)$ is a neighborhood frame such that $N \subseteq S \times Prop$ and for all $X, Y \in Prop$:

$$NsX \ \ \& \ \ X \subseteq Y \Rightarrow NsY \tag{6}$$

$$NsX \ \ \& \ \ s \leq s' \Rightarrow Ns'X \tag{7}$$

- $\leq, L, R$ are as in information frames (i.e. (3)-(5) are satisfied);
- $P \subseteq S$ is a set of *prime states* such that $C, Q \subseteq P^2$, $L \subseteq P$ and:

$$Rstu \ \ \& \ \ u \leq_P p \Rightarrow \exists q \geq_P s(Rqtp) \tag{8}$$

$$Rstu \ \ \& \ \ u \leq_P p \Rightarrow \exists q \geq_P t(Rsqp) \tag{9}$$

$$\forall p \geq_P s(NpX) \Rightarrow NsX \tag{10}$$

$s \leq_P t, s \geq_P t$ abbreviate $s \leq t \ \ \& \ \ t \in P, s \geq t \ \ \& \ \ s \in P$ respectively.
Let an *epistemic frame* be a tuple $\mathfrak{F} = (S, W, P, L, Prop, \leq, R, C, N, Q)$, where:

- $(S, P, L, Prop, \leq, R, C, N, Q)$ is an intensional epistemic frame;
- $W \subseteq P$ is a set of *possible worlds*, such that for all $w \in W$:

$$Cww \tag{11}$$

$$w \leq_P p \ \ \& \ \ Cpq \Rightarrow q \leq w \tag{12}$$

$$Rwww \tag{13}$$

$$Rwst \Rightarrow s \leq w \leq t \tag{14}$$

$$Q(W \uparrow) = L \tag{15}$$

where $Q(W \uparrow) = \{q \mid \exists p \in P, w \in W(p \geq_P w \ \ \& \ \ Qpq)\}$.

Given an (intensional) epistemic frame $\mathfrak{F}$, let us define the following frame operations for all $X, Y \in Prop$.

$$\begin{aligned}
\neg^{\mathfrak{F}} X &= \{s \mid (s \leq_P p \ \ \& \ \ Cpq) \Rightarrow q \notin X\} \\
X \circ^{\mathfrak{F}} Y &= \{s \mid \exists t, u(t \in X \ \ \& \ \ u \in Y \ \ \& \ \ Rtus)\} \\
X \rightarrow^{\mathfrak{F}} Y &= \{s \mid \{s\} \circ^{\mathfrak{F}} X \subseteq Y\} \\
X \vee^{\mathfrak{F}} Y &= \{s \mid s \leq_P t \Rightarrow t \in X \cup Y\} \\
E^{\mathfrak{F}} X &= \{s \mid NsX\} \\
L^{\mathfrak{F}} X &= \{s \mid s \leq_P p \ \ \& \ \ Qpq \Rightarrow q \in X\}
\end{aligned}$$

Moreover, for $\circledast \in \{\neg^{\mathfrak{F}}, \cap, \vee^{\mathfrak{F}}, \rightarrow^{\mathfrak{F}}, L^{\mathfrak{F}}, E^{\mathfrak{F}}\}$ and n arity of $\circledast$:

$$X_1, \ldots X_n \in Prop \Rightarrow \circledast(X_1, \ldots, X_n) \in Prop \tag{16}$$

Note that by Condition (4), L is non-empty, which implies by (15) that W is non-empty, which implies by $W \subseteq P$ that P is non-empty as well. The conditions we impose on (intensional) epistemic frames are the minimal ones that deliver our epistemic desiderata. We set aside the issue about the status of further properties one may wish to impose, e.g. whether $\forall s \in S \exists p \in P(s \leq_P)$ and whether C should yield double negation laws. Note that for technical reasons that have to do with the presence of non-prime states (for which see Footnote 17) we need to restrict the neighborhood relation to a set of admissible propositions, as in general frames for modal logic (see e.g. [27, 36]). Then, Condition (6) is the relativisation of (1) to *Prop*, and Condition (7) provides the monotonicity condition for N, regulating its interaction with information inclusion.

Epistemic frames consist of many components. Let us stop and comment on the role and interpretation of (i) the new modal accessibility relation Q; (ii) the Conditions (8)-(10) on prime states ; and (iii) the Conditions (11)-(15) on possible worlds.

Remark 1 (*Logical information*) A closer scrutiny of Q clarifies the meaning of the modality L. By Qst we mean that the information that can be used in a logical inference at s and its prime extensions is that contained in t (see Section 2.3 for our understanding of the word "logical"). At (prime extensions of) possible worlds, for example, it is reasonable to assume that the information that can be used in a logical inference comes from the relevantly acceptable deductions, i.e. from the set L. This provides an intuition for Condition (15), according to which Q-accessible states from (prime extensions of) worlds are logical states. However, the information that can be used in a logical inference at non-worldly information states may have different properties than logical information. Therefore, in the semantic interpretation of formulas with nested occurrences of L, what counts as logical information varies, i.e. at different information states agents may have different logical resources to carry out deductions.

Remark 2 (*Prime states*) Prime states in (intensional) epistemic models are used to interpret the relational modalties. In particular, in the spirit of [19] we restrict C and Q to prime states, and the frame operations $\neg^{\mathfrak{F}}, \vee^{\mathfrak{F}}$ are relativised accordingly to prime information inclusion $\leq_P$ (see Definition 8)[13]. Prime information inclusion, disjunction's suitable accessiblity relation, can be regarded as a sort of information refinement, so that whenever $s \leq_P p$, p specifies the information contained in s, i.e. p settles all disjunctions $\varphi \vee \psi$ supported by s by explicitly supporting φ or ψ. Then, the conditions on P ensure that $\leq_P$ interact well with respect to each accessibility

[13] As expected, when $\neg^{\mathfrak{F}}, \vee^{\mathfrak{F}}$ and $L^{\mathfrak{F}}$ are applied to sets of prime states, they become equivalent to the corresponding frame operations in information frames. Note that it is not necessary to assume the usual monotonicity Condition 2 on C and Q, as their monotonic behaviour is, so to speak, encoded in the corresponding semantic clauses for $\neg$ and L (see Definition 9). Note also that in our framework, one cannot define the Routley star function $*$ in terms of a symmetric, serial and convergent compatibility relation (as done in e.g. [7]).

relation. Condition (8) ((9)) expresses a density requirement of information combination: whenever it is possible to apply the information contained in t to a context s, then there is a prime refinement q of s (t) such that applying t to q (applying q to s) has the same combinatorial effects. Finally, Condition (10) expresses that the evidence base at prime states refining s is consistent with the evidence base at s.

Remark 3 (*Possible worlds*) The conditions on W account for different classical features of possible worlds. Conditions (11) and (12) imply that worlds are consistent, i.e. $\varphi \wedge \neg\varphi \rightarrow \bot$ is extensionally valid (see Definition 9). Condition (13) flattens information combination to idempotency, so that combining a world with itself results in no new (non-contradictory) information. Similarly, according to Condition (14) worlds are maximal in the sense that it is only possible to apply to them information they already contain and to obtain no less information. Following [34], Conditions (11)-(14) are formulated so as to enforce the extensionality of logical connectives at possible worlds, as shown by Lemma 3. Note that the presence of $\top$ in the language avails us simpler conditions than those used in [34], making it unnecessary to work with bounded models. Finally, Condition (15) has two notable technical consequences. First, they yield a modified version of the standard semantic deduction theorem of relevant logics (see Lemma 2)[14]. Second, they yield Lemma 9, by which the agent's logical information is faithful to the world, in the specific sense that in our system EI, introduced in Section 4, it is provable that agents have logical information that φ only if φ is provable.

Definition 9 (**(Intensional) epistemic model**) Let an *intensional epistemic model* $\mathfrak{M}$ be a tuple $(S, P, L, Prop, \leq, R, C, N, Q, V)$ where $(S, P, L, Prop, \leq, R, C, N, Q)$ is an intensional epistemic frame $\mathfrak{F}$ and $V : At \rightarrow Prop \subseteq Up(S)$ is such that for all $p \in At$:

$$\forall t \geq_P s(t \in V(p)) \Rightarrow s \in V(p) \tag{17}$$

Let an *epistemic model* $\mathfrak{M}$ be a tuple $(\mathfrak{F}, V)$ where $\mathfrak{F}$ is an epistemic frame and V is as in intensional epistemic models. In (intensional) epistemic models, V is extended to the *intensional truth* relation $\models^i$ on the full language $\mathfrak{L}$ by recursion as follows, where $[\![\varphi]\!]^i_{\mathfrak{M}} = \{s \mid \mathfrak{M}, s \models^i \varphi\}$ is the *intensional proposition* expressed by φ.

$$\begin{aligned}
[\![p]\!]^i_{\mathfrak{M}} &= V(p) \\
[\![\top]\!]^i_{\mathfrak{M}} &= S \\
[\![\bot]\!]^i_{\mathfrak{M}} &= \emptyset \\
[\![\neg\varphi]\!]^i_{\mathfrak{M}} &= \neg^{\mathfrak{F}}[\![\varphi]\!]^i_{\mathfrak{M}} \\
[\![\varphi \wedge \psi]\!]^i_{\mathfrak{M}} &= [\![\varphi]\!]^i_{\mathfrak{M}} \cap [\![\psi]\!]^i_{\mathfrak{M}}
\end{aligned}$$

[14] This result is reminiscent of Mares' definition of an entailment from φ to ψ as $\Box_L(\varphi \rightarrow \psi)$, with the aid of a box-like operator $\Box_L$ [26, p.104].

$$\llbracket \varphi \vee \psi \rrbracket^i_{\mathfrak{M}} = \llbracket \varphi \rrbracket^i_{\mathfrak{M}} \vee^{\mathfrak{F}} \llbracket \psi \rrbracket^i_{\mathfrak{M}}$$
$$\llbracket \varphi \rightarrow \psi \rrbracket^i_{\mathfrak{M}} = \llbracket \varphi \rrbracket^i_{\mathfrak{M}} \rightarrow^{\mathfrak{F}} \llbracket \psi \rrbracket^i_{\mathfrak{M}}$$
$$\llbracket E\varphi \rrbracket^i_{\mathfrak{M}} = E^{\mathfrak{F}} \llbracket \varphi \rrbracket^i_{\mathfrak{M}}$$
$$\llbracket L\varphi \rrbracket^i_{\mathfrak{M}} = L^{\mathfrak{F}} \llbracket \varphi \rrbracket^i_{\mathfrak{M}}$$

Let a formula $\varphi \in \mathfrak{L}$ be *intensionally valid* in an intensional epistemic model $\mathfrak{M}$ (written $\mathfrak{M} \models^i \varphi$) iff $\mathfrak{M}, l \models^i \varphi$ for all $l \in L$. Let a formula be *intensionally valid* (written $\models^i \varphi$) iff φ is intensionally valid in all (intensional) epistemic models. Let a formula $\varphi \in \mathfrak{L}$ be extensionally valid in an epistemic model $\mathfrak{M}$ (written $\mathfrak{M} \models^e \varphi$) iff $\mathfrak{M}, w \models^i \varphi$ for all $w \in W$. Let a formula φ be *extensionally valid* (written $\models^e \varphi$) iff φ is extensionally valid in all epistemic models.

In what follows, we often omit reference to $\mathfrak{M}$ when clear from context. We will denote by $\llbracket \varphi \rrbracket^e_{\mathfrak{M}} = \{w \in W \mid \mathfrak{M}, w \models^i \varphi\}$ the *extensional proposition* expressed by φ in an epistemic model $\mathfrak{M}$, and we will write $\mathfrak{M}, w \models^e \varphi$ iff $w \in \llbracket \varphi \rrbracket^e_{\mathfrak{M}}$ for *extensional truth* of φ at w in $\mathfrak{M}$. Note that by definition extensional truth implies intensional truth. At possible worlds also the converse inclusion holds. This is motivated by the fact that possible worlds constitute very special, maximally consistent, bodies of information (see Definition 13) for which substructural truth reduces to classical truth (see Lemma 3). We often use these facts implicitly in the proofs of the following lemmas.

Lemma 1 (**Heredity**) *For all (intensional) epistemic models $\mathfrak{M}$ and $\varphi \in \mathfrak{L}$:*

1. $\mathfrak{M}, s \models^i \varphi$ *iff* $\forall p \geq_P s (\mathfrak{M}, p \models^i \varphi)$;
2. $\mathfrak{M}, s \models^i \varphi \;\;\&\;\; s \leq t$ *only if* $\mathfrak{M}, t \models^i \varphi$.

Proof By induction on the structure of φ as follows, where the base case holds by (17), the induction step for $\varphi := \psi \wedge \chi$ is trivial and that for $\varphi := \psi \vee \chi$ follows easily by $\leq$-reflexivity and transitivity. For $\varphi := \neg\psi$, in one direction assume that $s \not\models^i \neg\psi$. Hence, there are $q, r \in P$ such that $s \leq_P q$ and Cqr such that $r \models^i \psi$. By reflexivity of $\leq$ we have that $q \leq_P q$, by which we conclude that there is $p \geq_P s$, namely q, such that $p \not\models^i \neg\psi$. Conversely, assume there is $p \geq_P s$ such that $p \not\models^i \neg\psi$. Hence, there are $q, r \in P$ such that $p \leq_P q$, Cqr and $r \models^i \psi$. By transitivity of $\leq$ we have that $s \leq_P q$, hence we conclude that $s \not\models^i \neg\psi$. For $\varphi := \psi \rightarrow \chi$, in one direction assume $s \models^i \psi \rightarrow \chi$ and, for some arbitrary $p \in P$, $t, u \in S$ that $p \geq_P s$, $Rptu$ and $t \models^i \psi$. By (3) we have that $Rstu$, hence by $s \models^i \psi \rightarrow \chi$ and $t \models^i \psi$ we have that $u \models^i \psi$, by which we conclude that $p \models^i \psi \rightarrow \chi$. Conversely, assume that $s \not\models^i \psi \rightarrow \chi$. Hence, there are $t, u \in S$ such that $Rstu$, $t \models^i \psi$ and $u \not\models^i \chi$. Hence by IH there is $p \geq_P u$ such that $p \not\models^i \chi$. By (3) we have that $Rstp$ and by (8) there is $q \geq_P s$ such that $Rqtp$, hence we conclude that $q \not\models^i \psi \rightarrow \chi$. For $\varphi := L\psi$, in one direction assume $s \models^i L\psi$ and, for some arbitrary $p, q, r \in P$ that $p \geq_P s$, $q \geq_P p$ and Qqr. By transitivity of $\leq$ we have that $q \geq_P s$. By $s \models L\psi$ and Qqr we have that $r \models \psi$, by which we conclude that $p \models L\psi$. Conversely, assume that $p \models L\psi$ for all $p \geq_p s$ and that Qqr for some arbitrary $q \geq_P s$ and $r \in P$. Hence $q \models L\psi$. By reflexivity of $\leq$ and Qqr we have that $r \models \psi$, by which we conclude that $s \models L\psi$. For $\varphi := E\psi$, in one direction assume $s \models^i E\psi$ and, for some arbitrary $p \in P$, that $p \geq_P s$. Hence, $Ns[\![\psi]\!]^i$, which by $p \geq_P s$ and (7) implies $Np[\![\psi]\!]^i$, by which we conclude that $p \models^i E\psi$. Conversely, assume that for all $p \geq_P s$ we have that $p \models^i E\psi$. Hence, for all $p \geq_P s$ we have that $Np[\![\psi]\!]^i$. By (10) we have that $Ns[\![\psi]\!]^i$, by which we conclude that $s \models^i E\psi$. For Item 2, assume $s \models^i \varphi$, $s \leq t$ and, by contradiction, that $t \not\models^i \varphi$. Hence, by Item 1, there is $q \geq_P t$ such that $q \not\models^i \varphi$, which implies $q \geq_P s$ by transitivity of $\leq$. Hence, again by Item 1 we conclude that $s \not\models^i \varphi$, which is a contradiction. □

Lemma 2 (**Deduction**) *For all $\varphi, \psi \in \mathfrak{L}$ and all (intensional) epistemic models $\mathfrak{M}$:*

1. $\mathfrak{M} \models^i \varphi \to \psi$ *iff* $[\![\varphi]\!]^i_{\mathfrak{M}} \subseteq [\![\psi]\!]^i_{\mathfrak{M}}$*;*
2. $\mathfrak{M} \models^e L(\varphi \to \psi)$ *iff* $[\![\varphi]\!]^i_{\mathfrak{M}} \subseteq [\![\psi]\!]^i_{\mathfrak{M}}$*.*

Proof Item 1 holds by standard arguments in relevant logic (see e.g. [32]), using (4),(5) and Lemma 1.2. Item 2 is established as follows. In one direction, assume by contraposition that $[\![\varphi]\!]^i \not\subseteq [\![\psi]\!]^i$. Hence, there is $s \in S$ such that $s \models^i \varphi$ and $s \not\models^i \psi$. By (4) there is $l \in L$ such that $Rlss$, and by (15) there are $w \in W$ and $p \in P$ such that $w \leq_P p$ and Qpl. We conclude by $L \subseteq P$ that $w \not\models^i L(\varphi \to \psi)$. Conversely, assume that $[\![\varphi]\!]^i \subseteq [\![\psi]\!]^i$ and, for some arbitrary $w \in W$, $p, q \in P$ and $s, t \in S$, that $w \leq_P p$, Qpq, $Rqst$ and $s \models^i \varphi$. By (15) we have that $q \in L$, which implies by (5) and $Rqst$ that $s \leq t$. By $[\![\varphi]\!]^i \subseteq [\![\psi]\!]^i$ and $s \models^i \varphi$ we have that $s \models^i \psi$. We conclude by $s \leq t$ and Lemma 1.2 that $t \models^i \psi$. □

The following Lemma 3 makes clear our remark in Section 1 that propositional formulas are interpreted classically, by means of extensional truth, while formulas in the scope of E are interpreted substructurally, by means of $E^{\mathfrak{F}}$ (which relates worlds to sets of possibly non-wordly information states).

Lemma 3 (**Extensionality**) *For all $\varphi, \psi \in \mathfrak{L}$ and all epistemic models $\mathfrak{M}$:*

1. $[\![\neg\varphi]\!]^e_{\mathfrak{M}} = W \setminus [\![\varphi]\!]^e_{\mathfrak{M}}$*;*
2. $[\![\varphi \vee \psi]\!]^e_{\mathfrak{M}} = [\![\varphi]\!]^e_{\mathfrak{M}} \cup [\![\psi]\!]^e_{\mathfrak{M}}$*;*
3. $[\![\varphi \to \psi]\!]^e_{\mathfrak{M}} = (W \setminus [\![\varphi]\!]^e_{\mathfrak{M}}) \cup [\![\psi]\!]^e_{\mathfrak{M}}$*;*

Proof For Item 1, in one direction assume for some arbitrary $w \in W$ that $w \models^e \varphi$. By reflexivity of $\leq$ we have that $w \leq_P w$ and by (11) we have that Cww. Hence, there are $p, q \in P$, namely $p = q = w$, such that $w \leq_P p$, Cpq and $q \models^e \varphi$. We conclude that $w \not\models^i \neg\varphi$. Conversely, assume for some arbitrary $w \in W$ that there are $p, q \in P$ such that $w \leq_P p$, Cpq and $q \models^i \varphi$. By (12) we have that $q \leq_P w$, by which we conclude by Lemma 1.1 that $w \models^i \varphi$. For Item 2, in one direction assume for some arbitrary $w \in W$ that for all $p \geq_P w$, $p \models^i \varphi$ or $p \models^i \psi$. Then, if $p \models^i \varphi$ ($p \models^i \psi$) by Lemma 1.1 we have that $w \models^i \varphi$ ($w \models^i \psi$), by which we conclude that $w \models^e \varphi$ or $w \models^e \psi$. Conversely, assume for some arbitrary $w \in W$ that $w \models^e \varphi$ or $w \models^e \psi$. If $w \models^e \varphi$ ($w \models^e \psi$), by Lemma 1.1 we have that $p \models^i \varphi$ ($p \models^i \psi$) for all $p \geq_P w$, by which we conclude that $w \models^i \varphi \vee \psi$. For Item 3, in one direction assume for some arbitrary $w \in W$ that $w \models^i \varphi \to \psi$ and that $w \models^e \varphi$. By (13) we have that $Rwww$, hence by $w \models^i \varphi \to \psi$ and $w \models^i \varphi$ we have that $w \models^i \psi$, by which we conclude that $w \models^e \psi$. Conversely, assume for some arbitrary $w \in W$ that there are $s, t \in S$ such that $Rwst$, $s \models^i \varphi$ and $t \not\models^i \psi$. By (14) we have that $s \leq w \leq t$, hence by Lemma 1.2 we conclude that $w \models^e \varphi$ and $w \not\models^e \psi$. □

Proposition 2 below summarises how epistemic models face the challenge posed by the principles of Fig. 1. (E.C), (E.K) and (E.DS) are invalid thanks to the properties of N (since N lacks the finite intersection property). While (E.N) does not preserves validity in neighborhood models, it does once we assume that evidence bases are non-empty, as assumed in [5, 6]. The situation is different in epistemic models, where (E.N) does not preserve validity in general, thanks to the properties of R, C (take e.g. $\varphi \rightarrow \varphi \Rrightarrow E(\varphi \rightarrow \varphi)$). Similarly, (E.DJ) is valid in neighborhood models with the finite intersection property but invalid in general in epistemic models thanks to properties of R, C (take e.g. $E(\neg\varphi \vee \varphi) \wedge E\neg\neg\varphi \rightarrow E\varphi$). Next, axioms (E.W), (E.CO) and (E.MP) are invalid thanks to the properties of R, while (E.S) is invalid thanks to the properties of C (since $\neg$ is paraconsistent). (E.M) and (E.E) do not preserve validity (making E hyperintensional) since $[\![\varphi]\!]^e \subseteq [\![\psi]\!]^e$ does not imply that $[\![\varphi]\!]^i \subseteq [\![\psi]\!]^i$, as shown by Countermodel 1. Finally, (E.DJ) is invalid (making E non-prime) since evidence states need not be prime, as shown by Countermodel 2. Although a deeper study of the logic of L is beyond the scope of the paper, we note that Countermodel 1 can be adapted to show that also L is a hyperintensional modality.

Proposition 2 (**Closure properties**) *None of the axioms (rules) of Fig. 1 are extensionally valid (preserve extensional validity) in every epistemic model* $\mathfrak{M}$.

Proof We show the invalidity of the principles not already covered by (virtually the same argument of) Proposition 1. The invalidity of (E.E) and (E.M) is shown in Countermodel 1. The invalidity of (E.DJ) is shown Countermodel 2. The invalidity of (E.W), (E.CO), (E.MP), (E.S) follow by the fact that the E-free version of each axiom is not valid in all information models. □

Countermodel 1 Consider the model $\mathfrak{M}_1$ represented on the left of Fig. 3, where $S = \{w, v, s, t\}$, $W = \{w, v\}$, $N(w) = \{\{w, s\}\}$, $V(p) = \{w, s\}$ and $V(q) = \{w, t\}$ (the other components are irrelevant and can be specified so that $\mathfrak{M}_1$ is indeed an epistemic model). Clearly, $\mathfrak{M}_1 \models^e p \leftrightarrow q$ as $[\![p]\!]^e_{\mathfrak{M}_1} = [\![q]\!]^e_{\mathfrak{M}_1} = \{w\}$. However, $\mathfrak{M}_1 \not\models^e Ep \leftrightarrow Eq$, as $\mathfrak{M}_1, w \models^e Ep$ (since $Nw[\![p]\!]^i_{\mathfrak{M}_1} = \{w, s\}$) but $\mathfrak{M}_1, w \not\models^e Eq$ (as not $Nw[\![q]\!]^i_{\mathfrak{M}} = \{w, t\}$). Therefore, $\mathfrak{M}_1$ is a countermodel to (E.E).

Countermodel 2 Consider the model $\mathfrak{M}_2$ represented on the right of Fig. 3, where $S = \{w, s, t\}$, $W = \{w\}$, $P = \{w, t\}$, $Nw\{s\}$, $s \leq t$, $V(p) = \{t\}$ and $V(q) = \emptyset$ (the other components are irrelevant and can be specified so that $\mathfrak{M}_2$ is indeed an epistemic model). Clearly, $\mathfrak{M}_2, w \models^e E(p \vee q)$, as $Nw\{s\} = [\![p \vee q]\!]^i_{\mathfrak{M}_2}$. However, $\mathfrak{M}_2, w \not\models Ep \vee Eq$, since $w \models Ep \vee Eq$ iff $w \models Ep$ or $w \models Eq$, $N(w) = \{\{s\}\}$ and neither $\mathfrak{M}_2, s \models^i p$ nor $\mathfrak{M}_2, s \models^i q$. Therefore, $\mathfrak{M}_2$ is a countermodel to (E.DJ).

As a consequence of Proposition 2, it appears prima facie that we are able to get rid of the problem of logical omniscience. This is not quite the case, since whenever agents have evidence for φ and $[\![\varphi]\!]^i \subseteq [\![\psi]\!]^i$, then they also have evidence for ψ.

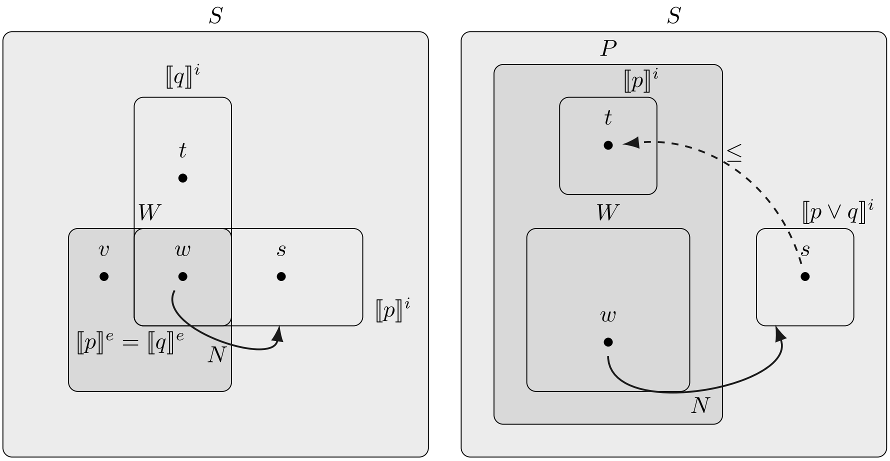

Fig. 3 Representation of the countermodels to (E.E) and (E.DJ)

That is, logical omniscience is limited to substructural, intensional, consequence as opposed to classical, extensional, consequence. Seen as a positive result, we get the following closure property in our system EI, introduced in Section 4.

$$L(\varphi \rightarrow \psi) \Rrightarrow E\varphi \rightarrow E\psi \tag{ER}$$

The above rule (ER) (see Section 4 for a discussion) regiments agents' epistemic reasoning in EI.

4 The Logic of Extensional and Intensional Truth

In this section we define the logic of evidence EI and prove it sound and complete with respect to epistemic models. While the propositional logic of extensional truth is classical logic, the modal logic of intensional truth is a modal extension of Fine's logic B for entailment [19].

Definition 10 (**Intensional axiom system**) Let the logic I be the smallest axiom system including the following axioms and rules:

- The axioms and rules of B^-;
- The axioms and rules for $\top$, $\bot$, L, E of Fig. 4.

Compared to Fine's system, the propositional layer of I does not include $\varphi \vee \neg\varphi$, $\varphi \rightarrow \neg\neg\varphi$, $\neg\neg\varphi \rightarrow \varphi$ and $\neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$[15]. It will also be useful to note that I is closed under uniform substitution and is not hyperintensional, as (E.E) is derivable.

[15] The absence of the De Morgan principle $\neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$ is due to the fact that we employ C, as opposed to the Routley star $*$, in the semantics of $\neg$. This absence can be motivated by the consideration that (weakly) rejecting a conjunction is in general weaker than (weakly) rejecting one of the two conjuncts (see [42] for an analysis of non-classical negation as weak rejection).

($\top$)	$\varphi \to \top$
($\bot$)	$\bot \to \varphi$
(L.C)	$L\varphi \wedge L\psi \to L(\varphi \wedge \psi)$
(L.M)	$\varphi \to \psi \Rrightarrow L\varphi \to L\psi$
(E.M)	$\varphi \to \psi \Rrightarrow E\varphi \to E\psi$

Fig. 4 Axioms and rules for $\top$, $\bot$, L, E in I

Next, let the L-version of an axiom (rule) (X) (labelled L-(X)) be obtained by prefixing the axiom (each of the premises and conclusion of the rule) with L.

Definition 11 (**Axiom system**) Let the logic EI be the smallest axiom system including the following axioms and rules:

- Classical Propositional Calculus (CPC);
- The L-version of each axiom/rule in I;
- The Bridge Rule (BR): $L(\varphi \to \psi) \Rrightarrow \varphi \to \psi$.

Note that by definition all L-free theorems of EI are either instances of classical theorems or obtained by application of (BR). In EI, the L-version of each axiom and rule in I expresses a principle of epistemic reasoning, so that agents' reasoning is regimented by substructural logic. Under this reading, every axiom of the form $L(\varphi \to \psi)$ expresses the fact that ψ can be in principle derived by φ by agents relying only on logical information. By L-($\top$) (L-($\bot$)) , $\top$ ($\bot$) can be taken as expressing some trivial (absurd) information, derivable by agents no matter what their information is (resulting in explosion)[16]. Finally, (BR) says that an agent can have logical information about some entailment relation between propositions only if such entailment expresses a factual relation between the propositions. Note that while (E.M) in I provides us the principle governing agents' epistemic reasoning, in EI such principle is provided by (ER), which is an immediate consequence of L-(E.M) and (BR) and constitutes the only point of interaction between L and E in our axiom system EI.

Our first technical result of the paper is a soundness and completeness theorem for I with respect to intensional epistemic models. This result will pave the way for a soundness and completeness theorem of our main logic EI with respect to epistemic models. In order to prove completeness for I and EI, we present without proof the following standard general syntactic results. Note that our definition of theories is the one of [32]. We do not use the definition of [19], based on a non-standard notion of syntactic consequence, since we are after a weak completeness theorem.

Definition 12 (**Theories, pairs**) For a given logic L, let an ordered pair of sets of formulas (Γ, Δ) be L-*independent* iff for all $\varphi_1 \dots \varphi_n \in \Gamma$, $\psi_1 \dots \psi_m \in \Delta$, we have

[16] Due to the presence of $\top$, I violates the variable sharing principle, a distinctive property of relevant logics, according to which $\varphi \to \psi$ is a theorem only if φ and ψ share a propositional variable. Hence, stricto sensu I is not a relevant logic. Note also that a remedy to this result is to conceive of $\top$, $\bot$ as containing every propositional variable [1]. Finally, note that $\top$ could be omitted from the language (as in [34]), at the expense of more complicated frames.

that $\nvdash_{\mathsf{L}} \varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \psi_1 \vee \cdots \vee \psi_m$. Let an L-*theory* be a set of formulas Γ such that for all $\varphi, \psi \in \mathscr{L}$ (i) $\varphi \in \Gamma$ and $\vdash_{\mathsf{L}} \varphi \rightarrow \psi$ implies $\psi \in \Gamma$ and (ii) $\varphi, \psi \in \Gamma$ implies $\varphi \wedge \psi \in \Gamma$. Moreover, an L-theory Γ is:

- *Consistent* if for all $\varphi \in \mathcal{L}$ Γ does not contain both φ and $\neg\varphi$;
- *Maximal* if Γ is consistent and all its extensions are not consistent;
- *Proper* if Γ does not contain all formulas of $\mathcal{L}$.
- *Regular* if Γ contains all theorems of L;
- *Prime* if for all $\varphi, \psi \in \mathcal{L}$ $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$;
- *Closed under disjunctions* if for all $\varphi, \psi \in \mathcal{L}$ $\varphi, \psi \in \Gamma$ implies $\varphi \vee \psi \in \Gamma$.

Lemma 4 (**Pair extension**) *For every pair of sets of formulas* (Γ, Δ)*:*

1. *If* (Γ, Δ) *is* I*-independent, then there is a prime* I*-theory* Γ' *such that* $\Gamma \subseteq \Gamma'$ *and* $\Gamma' \cap \Delta = \emptyset$*;*
2. *If* Γ *is an* I*-theory,* $\Delta \subseteq \mathcal{L}$ *is closed under disjunction and* $\Gamma \cap \Delta = \emptyset$*, then there is a prime* I*-theory* Γ' *such that* $\Gamma \subseteq \Gamma'$ *and* $\Gamma' \cap \Delta = \emptyset$*;*
3. *If* (Γ, Δ) *is* EI*-independent and* Γ, Δ *are non-empty, then there is a a maximal* EI*-theory* Γ' *such that* $\Gamma \subseteq \Gamma'$ *and* $\Gamma' \cap \Delta = \emptyset$*.*

Proof Item 1-2 are established as in [32, Lemmas 4.3-4.4] and Item 3 is the well known Lindenbaum lemma. □

Theorem 3 (**Intensional soundness**) *For all* $\varphi \in \mathfrak{L}$*:* $\vdash_{\mathsf{I}} \varphi$ *only if* $\models^i \varphi$*.*

Proof By induction on the length of I-proofs. That the axioms of B^- are valid and the rules preserve validity is virtually as in [19], hence we show only the case for (A5), to illustrate where Conditions (9) is used. For (A5), assume by Lemma 2 for some arbitrary $s, t, u \in S$ that $s \models^i \varphi \rightarrow \chi$, $s \models^i \psi \rightarrow \chi$, $Rstu$ and $t \models^i \varphi \vee \psi$ to show $u \models^i \chi$. By $t \models^i \varphi \vee \chi$, we have that $p \models^i \varphi$ or $p \models^i \psi$ for all $p \geq_P t$. By $Rstu$ and (9), we have that for all $q \geq_P u$ there is $r \geq_P t$ such that $Rsrq$. Hence, $r \models^i \varphi$ or $r \models^i \psi$. If the former, by $s \models \varphi \rightarrow \chi$ and $Rsrq$ we have that $q \models^i \chi$. If the latter, by $s \models \psi \rightarrow \chi$ and $Rsrq$ we have that $q \models^i \chi$. Hence, $q \models \chi$ for all $q \geq_u$, by which we conclude by Lemma 1.1 that $u \models^i \chi$. Next, axioms ($\top$), ($\bot$) are trivially valid. For (L.C), assume $s \models^i L\varphi \wedge L\psi$. Hence, for all $p \geq_P s$ and $q \in Q$ such that Qpq, $q \models^i \varphi$ and $q \models^i \psi$, by which $q \models^i \varphi \wedge \psi$, by which we conclude that $s \models L(\varphi \wedge \psi)$. For (L.M), assume $[\![\varphi]\!]^i \subseteq [\![\psi]\!]^i$ and $s \models^i L\varphi$. Hence, for all $p \geq_P s$ and $q \in P$ such that Qpq we have that $q \models^i \varphi$, which implies by $[\![\varphi]\!]^i \subseteq [\![\psi]\!]^i$ that $q \models^i \psi$, by which we conclude that $s \models^i L\psi$. For (E.M), assume $[\![\varphi]\!]^i \subseteq [\![\psi]\!]^i$ and $s \models^i E\varphi$. Hence, we have that $Ns[\![\varphi]\!]^i$, which by (6) implies $Ns[\![\psi]\!]^i$, by which we conclude $s \models^i E\psi$. □

Definition 13 (**Canonical intensional epistemic model**) Let the *canonical intensional epistemic model* be the following tuple $\mathfrak{M}^i_c = (S_c, P_c, L_c, Prop_c, \leq_c, C_c, R_c, N_c, Q_c, V_c)$, where $[\varphi]_c = \{s \in S \mid \varphi \in s\}$:

$$S_c = \{s \mid s \textit{ is a non-empty proper } \mathsf{I}\textit{-theory}\}$$
$$P_c = \{s \mid s \textit{ is a non-empty proper prime } \mathsf{I}\textit{-theory}\}$$

$$L_c = \{s \mid s \textit{ is a non-empty proper prime regular } \mathsf{I}\textit{-theory}\}$$
$$Prop_c = \{[\varphi]_c \mid \varphi \in \mathfrak{L}\}$$
$$\leq_c = \{(s, t) \mid s \subseteq t\}$$
$$C_c = \{(s, t) \in P_c^2 \mid \neg\varphi \in s \Rightarrow \varphi \notin t\}$$
$$R_c = \{(s, t, u) \mid (\varphi \rightarrow \psi \in s, \varphi \in t) \Rightarrow \psi \in u\}$$
$$N_c = \{(s, [\varphi]_c) \mid E\varphi \in s\}$$
$$Q_c = \{(s, t) \in P_c^2 \mid L\varphi \in s \Rightarrow \varphi \in t\}$$
$$V_c(p) = \{s \mid p \in s\}$$

Let the *canonical epistemic model* be the couple $\mathfrak{M}_c = (\mathfrak{M}_c^i, W_c)$, where $\mathfrak{M}_c^i$ is the canonical intensional epistemic model and:

$$W_c = \{s \mid s \textit{ is a maximal } \mathsf{EI}\textit{-theory}\}$$

In what follows, we omit the subscript c from proofs to improve readability.

Lemma 5 (**Theory determination**) *For all $\varphi \in \mathfrak{L}$, $s \in S_c$: $\varphi \in t$ for all $t \geq_{P_c} s$ iff $\varphi \in s$.*

Proof One direction is trivial. For the other direction, assume $\varphi \notin s$ and consider $u = \{\psi \mid \vdash_{\mathsf{I}} \varphi \leftrightarrow \psi\}$. Then, assume $\psi_1, \psi_2 \in u$. Then $\vdash_{\mathsf{I}} \varphi \leftrightarrow \psi_1$ and $\vdash_{\mathsf{I}} \varphi \leftrightarrow \psi_2$. By $\vdash_{\mathsf{I}} \varphi \leftrightarrow \varphi \vee \varphi$ and closure of I under uniform substitution we have that $\vdash_{\mathsf{I}} \varphi \leftrightarrow \psi_1 \vee \psi_2$. By definition of u we have that $\psi_1 \vee \psi_2 \in u$, hence u is closed under disjunctions. By $u \cap s = \emptyset$ and Lemma 4.2 there is $t \in P$ such that $s \leq_P t$, $t \cap u = \emptyset$, by which we conclude that $\varphi \notin t$. □

As a preliminary in showing that $\mathfrak{M}_c^i$ is an intensional epistemic model, we show as a separate result that the frame operations of Definition 8 are well defined.

Lemma 6 (**Canonical frame operations**) *Given the canonical frame $\mathfrak{F}_c$ underlying $\mathfrak{M}_c$, $\circledast \in \{\neg, \vee, \rightarrow, L, E\}$ with n arity of $\circledast$, $\circledast^{\mathfrak{F}_c}([\varphi_1]_c, \ldots [\varphi_n]_c) = [\circledast(\varphi_1, \ldots, \varphi_n)]_c$.*

Proof We show only the cases when $\circledast := \neg$, $\circledast := \vee$ and $\circledast := L$, as the remaining cases are established by standard argument (see e.g. [18, 20]). For $\circledast := \neg$, assume $\neg\varphi \in s$, $p \supseteq s$ and Cpq for some arbitrary $s \in S$ and $p, q \in P$. By $\neg\varphi \in s$ and $p \supseteq s$ we have that $\neg\varphi \in p$. We conclude by Cpq that $\varphi \notin q$. Conversely, assume that $\neg\varphi \notin s$. By Lemma 5 there is $p \in P$ such that $p \supseteq s$ and $\neg\varphi \notin p$. Then, consider the pair $q_0 = (\{\varphi\}, \{\psi \mid \neg\psi \in p\})$. The pair is I-independent, since otherwise we would have for $\neg\psi_1, \ldots, \neg\psi_n \in p$ that $\vdash_{\mathsf{I}} \varphi \rightarrow (\psi_1 \vee \ldots \vee \psi_n)$. The latter would imply that $\vdash_{\mathsf{I}} \neg(\psi_1 \vee \ldots \vee \psi_n) \rightarrow \neg\varphi$. Since $\neg\psi_1, \ldots, \neg\psi_n \in p$, by $\vdash_{\mathsf{I}} (\neg\psi_1 \wedge \ldots \wedge \neg\psi_n) \rightarrow \neg(\psi_1 \vee \ldots \vee \psi_n)$ we would have $\neg(\psi_1 \vee \ldots \vee \psi_n) \in p$, by which we would conclude $\neg\varphi \in p$, which is a contradiction. Hence by Lemma 4.1 there is a prime I-theory $q \supseteq q_0$ such that, by definition of q_0, Cpq and $\varphi \in q$. We conclude that $s \notin \neg^{\mathfrak{F}}[\varphi]$. For $\circledast := \vee$, one direction is trivial. Conversely, assume by contradiction that $s \in [\varphi] \vee^{\mathfrak{F}} [\psi]$ and $\varphi \vee \psi \notin s$. By $\varphi \vee \psi \notin s$ and Lemma 5 there is $q \in P$ such that $q \supseteq s$ and $\varphi \vee \psi \notin q$. Hence, by $s \in [\varphi] \vee^{\mathfrak{F}} [\psi]$ we have that $\varphi \in q$ or $\psi \in q$. If the former (latter), by $\vdash_{\mathsf{I}} \varphi \to \varphi \vee \psi$ ($\vdash_{\mathsf{I}} \psi \to \varphi \vee \psi$) we have that $\varphi \vee \psi \in q$, which is a contradiction. For $\circledast := L$, assume $L\varphi \in s$ and for some arbitrary $p, q \in P$ that $p \supseteq s$ and Qpq. By $p \supseteq s$ and $L\varphi \in s$ we have that $L\varphi \in p$, by which we conclude by Qpq that $\varphi \in q$. Conversely, assume assume $L\varphi \notin s$. By Lemma 5 there is $p \supseteq s$ such that $L\varphi \notin p$. Then, consider the pair $q_0 = (\{\chi \mid L\chi \in p\}, \{\varphi\})$. The pair is I-independent, since otherwise, we would have for $L\chi_1, \ldots, L\chi_n \in p$ that $\vdash_{\mathsf{I}} \chi_1 \wedge \cdots \wedge \chi_n \to \varphi$. The latter would imply by (L.C)-(L.M) that $\vdash_{\mathsf{I}} L\chi_1 \wedge \cdots \wedge L\chi_n \to L\varphi$, by which we would conclude by $L\chi_1, \ldots, L\chi_n \in p$ that $L\varphi \in p$, which is a contradiction. Then, by Lemma 4.1 there is a prime $q \supseteq q_0$ such that $p \supseteq s$, Qpq and $\psi \notin q$. We conclude that $s \notin L^{\mathfrak{F}}[\varphi]$. □

Lemma 7 (**Intensional epistemic model canonicity**) *$\mathfrak{M}^i_c$ is an intensional epistemic model.*

Proof $P \subseteq S$, $L \subseteq P$ and $V : At \to Prop$ follow by inspection of $\mathfrak{M}$. Conditions (3)-(5) and (6)-(7) are established by standard arguments in relevant modal logic (see e.g. [20] and [18], respectively). That the remaining Conditions (8)-(10) and (16)-(17) are satisfied is established as follows. For (8), assume $p \in P$, $Rstu$ and $u \subseteq p$. To show that there is $q \supseteq s$ such that $Rqtp$, we first show that there is x closed under disjunctions and such that $s \cap x = \emptyset$. Take $x = \{\varphi \mid \exists \psi, \chi (\vdash_{\mathsf{I}} \varphi \to (\psi \to \chi), \psi \in t \;\; \& \;\; \chi \notin p)\}$. If $s \cap x \neq \emptyset$, there are φ, ψ, χ such that $\varphi \in s$, $\psi \in t$, $\chi \notin p$ and $\vdash_{\mathsf{I}} \varphi \to (\psi \to \chi)$. By the latter and $\varphi \in s$ we have that that $\psi \to \chi \in s$, hence by $Rstu$ and $\psi \in t$ we conclude that $\chi \in u$. By $u \subseteq p$ we conclude that $\chi \in p$, which is a contradiction. That x is closed under disjunctions is shown as follows. Assume $\varphi_1, \varphi_2 \in x$. Hence, there are $\psi_1, \psi_2, \chi_1, \chi_2$ such that $\vdash_{\mathsf{I}} \varphi_1 \to (\psi_1 \to \chi_1)$, $\vdash_{\mathsf{I}} \varphi_2 \to (\psi_2 \to \chi_2)$, $\psi_1, \psi_2 \in t$ and $\chi_1, \chi_2 \notin p$. By the axioms and rules of I we have that $\vdash_{\mathsf{I}} \varphi_1 \vee \varphi_2 \to (\psi_1 \wedge \psi_2 \to \chi_1 \vee \chi_2)$. By (R1) we have that $\psi_1 \wedge \psi_2 \in t$. By $p \in P$ we have that $\chi_1 \vee \chi_2 \notin p$. Hence, by definition of x we conclude that $\varphi_1 \vee \varphi_2 \in x$. Putting things together, by Lemma 4.2 there is a prime I-theory $q \supseteq s$ such that $q \cap x = \emptyset$. Now, to show $Rqtp$, assume $\alpha \to \beta \in q$ and $\alpha \in t$. By $q \cap x = \emptyset$ we have that $\alpha \to \beta \notin x$, hence for all α_1, β_1 we have that either $\nvdash_{\mathsf{I}} (\alpha \to \beta) \to (\alpha_1 \to \beta_1)$, $\alpha_1 \notin t$ or $\beta_1 \in p$. By $\vdash_{\mathsf{I}} (\alpha \to \beta) \to (\alpha \to \beta)$ and $\alpha \in t$ we conclude that $\beta \in p$. For (9) we proceed in a similar way, showing that there is $q \supseteq t$ such that $Rsqp$ and applying this time Lemma 4.2 to $y = \{\varphi \mid \exists \psi (\varphi \to \psi \in s \;\; \& \;\; \psi \notin p\}$ to obtain q. For (10), assume $Np[\varphi]$ for all $p \in P$ such that $p \supseteq s$. Hence, $E\varphi \in p$ for all $p \supseteq s$, which by Lemma 5 implies that $E\varphi \in s$. We conclude that $Ns[\varphi]$[17]. Finally, (16) follows from Lemma 6 and the definition of *Prop*, while (17) follows from Lemma 5. □

Lemma 8 (**Truth**) *For all $\varphi \in \mathfrak{L}$, $s \in S_c$: $\mathfrak{M}^i_c, s \models^i \varphi$ iff $\varphi \in s$.*

[17] Note that the standard definition of the canonical monotonic neighborhood model ([11, p.257]), according to which $N_c = \{(s, X) \mid [\varphi]_c \subseteq X$ for some $E\varphi \in s\}$, will not work to make (10) satisfied in presence of non-prime states. Indeed, if for all prime $p \supseteq s$ there is φ such that $E\varphi \in p$ and $[\varphi] \subseteq X$, then we have for some $\varphi_1, \ldots, \varphi_n$ that $E(\varphi_1 \vee \ldots \vee \varphi_n) \in s$ but not necessarily that $[\varphi_1 \vee \ldots \ldots \varphi_n] \subseteq X$, since $[\varphi_1] \cup \ldots \cup [\varphi_n] \subseteq X$ does not imply $[\varphi_1 \vee \ldots \varphi_n] \subseteq X$. We thank an anonoymous reviewer for raising the issue.

Proof By induction on the structure of φ, where we show only the cases for $\varphi := \neg\psi$, $\varphi := \psi \vee \chi$ and $\varphi := L\psi$. The remaining cases are established by standard argument in relevant modal logic (see e.g. [20]). For $\varphi := \neg\psi$, in one direction assume for some arbitrary $s \in S$ and $p, q \in P$ that $\neg\psi \in s$, $p \supseteq s$ and Cpq. By $p \supseteq s$ we have $\neg\psi \in p$, which by Cpq implies that $\psi \notin q$. Hence, by the IH $q \not\models \psi$, by which we conclude that $s \models \neg\psi$. Conversely, if $\neg\psi \notin s$, by Lemma 6 we have $s \notin \neg^{\mathfrak{F}}[\psi]$. Hence, there are $p, q \in P$ such that $p \supseteq s$, Cpq and $q \notin [\psi]$. By the IH we conclude that $s \not\models \neg\psi$. For $\varphi := \psi \vee \chi$, in one direction assume for some arbitrary $s \in S$ and $p \in P$ that $\psi \vee \chi \in s$ and $p \supseteq s$. By $p \supseteq s$ we have $\psi \vee \chi \in p$, and by $p \in P$ we have $\psi \in p$ or $\chi \in p$. By the IH we conclude that $p \models \psi$ or $p \models \chi$. Conversely, if $\psi \vee \chi \notin s$, by Lemma 6 we have $s \notin [\psi] \vee^{\mathfrak{F}} [\chi]$. Hence, there is $p \in P$ such that $p \supseteq s$ and $p \notin [\psi]$ or $p \notin [\chi]$. By the IH we conclude that $s \not\models \psi \vee \chi$. For $\varphi := L\psi$, in one direction assume $L\psi \notin s$. By Lemma 6 we have $s \notin L^{\mathfrak{F}}[\psi]$. Hence, there are $p, q \in P$ such that $p \supseteq s$, Qpq and $\psi \notin q$. By IH we have that $q \not\models \psi$, by which we conclude that $s \not\models L\psi$. Conversely, assume that $L\psi \in s$ and for some arbitrary $p, q \in P$ that $p \supseteq s$ and Qpq. By $p \supseteq s$ and $L\psi \in s$ we have $L\psi \in p$, and by Qpq we have that $\psi \in q$. By IH we have that $q \models \psi$, by which we conclude that $s \models L\psi$.

Theorem 4 (**Intensional completeness**) *For all* $\varphi \in \mathfrak{L}$: $\models^i \varphi$ *only if* $\vdash_{\mathsf{I}} \varphi$.

Proof It follows by Lemmas 7 and 8. □

With the completeness of I with respect to intensional epistemic models, we are ready to prove that our main axiom system, the logic of extensional and intensional truth EI, is sound and complete with respect to epistemic models. To this aim, we prove the preliminary Lemma 9, concerning the relations between the substructural logic I and EI. The proof of this result relies on a construction used in [34], simplified by the fact that we do not work here with bounded models.

Lemma 9 is important for technical and conceptual reasons. First, note that Lemma 9.1 is crucial for showing the properties of possible worlds in the canonical epistemic model. Lemma 9.1 shows also that L can be interpreted as internalising I-provability in EI. Second, Lemma 9.3 makes it precise in what sense logical omniscience is limited to intensional consequence. We stress that, while evidence is not closed under many epistemic principles, Lemma 9.3 individuates a logical criterion, also expressed by (ER), for epistemic competence. Finally, note that by 9.1 and 9.2 the factive requirement of logical information expressed by (BR) is extended to arbitrary formulas, i.e $L\varphi \Rrightarrow \varphi$ is an admissible EI-rule.

Theorem 5 (**Soundness**) *For all* $\varphi \in \mathfrak{L}$: $\vdash_{\mathsf{EI}} \varphi$ *only if* $\models^e \varphi$.

Proof By induction on the length of EI-proofs, virtually as in [34]. The axioms and rules of (CPC) are valid, preserve validity respectively, thanks to Lemma 3. To show that the L-versions of the I-axioms are valid, by (15) it suffices to show that $l \models^i \varphi$ for all $l \in L$ and I-axioms φ, which follows by Theorem 3. That the L-versions of the I-rules preserve validity is established similarly. Finally, (BR) preserves validity by Lemmas 2 and 3. □

Lemma 9 (**Bridge**) *For all $\varphi \in \mathfrak{L}$, the following meta-rules hold:*

1. $\vdash_{\mathsf{I}} \varphi \Longleftrightarrow \vdash_{\mathsf{EI}} L\varphi$;
2. $\vdash_{\mathsf{I}} \varphi \Rightarrow \vdash_{\mathsf{EI}} \varphi$;
3. $\vdash_{\mathsf{I}} \varphi \to \psi \Rightarrow \vdash_{\mathsf{EI}} E\varphi \to E\psi$;

Proof Item 1 is established in one direction by induction on the length of $\vdash_{\mathsf{I}}$-proofs, as in [34]. Conversely, assume $\nvdash_{\mathsf{I}} \varphi$. By Theorem 4 there is an intensional epistemic model $\mathfrak{M} = (S, P, L, Prop, \leq, R, C, N, Q, V)$ such that $\mathfrak{M} \not\models^i \varphi$. Then, consider the model $\mathfrak{M}' = (S', W, P', L', Prop', \leq', R', C', N', Q', V)$ such that:

$$\begin{aligned} M' = (&S \cup W, \{w\}, P \cup W, L \cup W, Prop \cup \{X \cup W \mid X \in Prop\}, \leq \cup \\ &\{(w, w)\}, R \cup \{(w, w, w)\}, C \cup \{(w, w)\}, \{(s, X) \mid Ns(X \cap S) \ \ \& \\ &X \in Prop'\}, Q \cup \{(w, s) \mid s \in L \cup W\}, V). \end{aligned}$$

By inspection of the definition, $\mathfrak{M}'$ is an epistemic model. We now claim that for all $s \in S$, $\mathfrak{M}, s \models^i \varphi$ iff $\mathfrak{M}', s \models^i \varphi$. By the claim and $\mathfrak{M} \not\models^i \varphi$, we have that $l \not\models^i \varphi$ for some $l \in L$. Hence, by definition of $\leq'$ and Q' we have that $\mathfrak{M}', w \not\models^e L\varphi$, by which we conclude that $\nvdash_{\mathsf{EI}} L\varphi$ by Theorem 5. The proof of the claim is by induction on the structure of φ, where the base case and the induction step involving $\wedge, \vee$ are trivial. The cases involving $\neg, \to, L$ follow by the fact, easily verifiable by inspection of $\mathfrak{M}'$, that whenever $s \in S$, we have that (Cst iff $C'st$), ($Rstu$ iff $R'stu$) and (Qst iff $Q'st$). For $\varphi := E\psi$, assume $\mathfrak{M}, s \models^i E\psi$. Hence, $Ns[\![\psi]\!]^i_{\mathfrak{M}}$ and by definition of N' we have that $N's([\![\psi]\!]^i_{\mathfrak{M}} \cup \{w\})$ and $N's[\![\psi]\!]^i_{\mathfrak{M}}$. If $w \in [\![\psi]\!]^i_{\mathfrak{M}'}$, by the IH $[\![\psi]\!]^i_{\mathfrak{M}} \cup \{w\} = [\![\psi]\!]^i_{\mathfrak{M}'}$, otherwise $[\![\psi]\!]^i_{\mathfrak{M}} = [\![\psi]\!]^i_{\mathfrak{M}'}$. In both cases $N's[\![\psi]\!]^i_{\mathfrak{M}'}$, by which we conclude that $\mathfrak{M}', s \models^i E\psi$. The other direction is established similarly. Item 2 and 3 are established as in [34].

Lemma 10 (**Epistemic model canonicity**) *$\mathfrak{M}_c$ is an epistemic model.*

Proof By Lemma 7, it suffices to show that $W \subseteq P$ and that W is a set of possible worlds. The former follows by the fact that any maximal EI-theory is non-empty and prime and by the fact that any prime EI-theory is a prime I-theory (by Lemma 9.1 and (BR)). Conditions (11)-(15) are established as follows. For (11), assume $\neg\varphi \in w$. Since w is a maximal EI-theory, we have that $\varphi \notin w$, hence by $W \subseteq P$ we conclude that Cww. For (12), assume $w \subseteq p$, Cpq, $\varphi \in q$ and, by contradiction, $\varphi \notin w$. Since w is a maximal EI-theory, we have that $\neg\varphi \in w$ and by $w \subseteq p$ we have that $\neg\varphi \in p$. By Cpq we conclude that $\varphi \notin q$, contradicting $\varphi \in q$. (13) follows from $\vdash_{\mathsf{EI}} (\varphi \wedge (\varphi \to \psi)) \to \psi$. For (14), assume that $Rwst$. To prove that $w \subseteq t$, assume $\varphi \in w$. Since $\vdash_{\mathsf{EI}} \varphi \to (\top \to \varphi)$, we have that $\top \to \varphi \in w$. Now, note that by ($\top$) and s non-empty we have $\top \in s$ for all s. Hence, we conclude that $\varphi \in t$. To prove that $s \subseteq w$, assume that $\varphi \in s$ and, by contradiction, that $\varphi \notin w$. Since w is a maximal EI-theory, we have $\neg\varphi \in w$, and by $\vdash_{\mathsf{EI}} \neg\varphi \to (\varphi \to \bot)$ we have that $\varphi \to \bot \in w$. Hence, by $Rwst$ and $\varphi \in s$ we have that $\bot \in t$, and by ($\bot$) we have that $\psi \in t$ for all ψ, contradicting t being proper. Hence, we conclude that $\varphi \in w$. (15) is established as follows. In one direction, assume by contradiction that

$q \in Q(W \uparrow)$ and $q \notin L$. Hence, there are $w \in W$ and $p \in P$ such that $p \supseteq w$ and Qpq. By $Q \subseteq P^2$ we have that q is not regular[18], i.e. there is $\varphi \notin q$ such that $\vdash_{\mathsf{I}} \varphi$. By Lemma 9.1 we have that $\vdash_{\mathsf{EI}} L\varphi$, which implies $L\varphi \in w$. Hence, by $p \supseteq w$ we have that $L\varphi \in p$, which together with $\varphi \notin q$ implies that not Qpq, which is a contradiction. Conversely, assume $q \in L$. We have to prove that $q \in Q(W \uparrow)$, i.e. there are $w \in W$ and $p \in P$ such that $p \supseteq w$ and Qpq. To this aim, consider the pair $w_0 = (\{\psi \mid \vdash_{\mathsf{EI}} \psi\}, \{L\varphi \mid \varphi \notin q\})$. The pair is EI-independent, since otherwise we would have for $\varphi_1, \ldots, \varphi_n \notin q$ that $\vdash_{\mathsf{EI}} L\varphi_1 \vee \ldots \vee L\varphi_n$. The latter would imply by $\vdash_{\mathsf{EI}} L\varphi \vee L\psi \rightarrow L(\varphi \vee \psi)$ that $\vdash_{\mathsf{EI}} L(\varphi_1 \vee \ldots \vee \varphi_n)$, which implies by Lemma 9.1 that $\vdash_{\mathsf{I}} \varphi_1 \vee \ldots \vee \varphi_n$. Since $q \in L$, we have that $\varphi_1 \vee \ldots \vee \varphi_n \in q$, which by $L \subseteq P$ implies $\varphi_i \in q$ for some $i \leq n$, which is a contradiction. Then, by Lemma 4.3 there is a maximal EI-theory $w \supseteq w_0$ such that $w \cap \{L\varphi \mid \varphi \notin q\} = \emptyset$, which implies that Qwq. Hence, there is $p \in P$, namely w, such that $p \supseteq w$ and Qpq. □

Theorem 6 (**Completeness**) *For all $\varphi \in \mathfrak{L}$: $\models^e \varphi$ only if $\vdash_{\mathsf{EI}} \varphi$.*

Proof It follows by Lemmas 10 and 8. □

5 Conclusion

In the present paper we provide a modal logic combining classical and substructural logic and prove that it is sound and complete with respect to an epistemically grounded neighborhood semantics. The main features of the semantics are the use of non-prime information states as the main semantic object. From this we defined the set of prime information states, used to give a non-standard modal semantic clause for disjunction; and we defined the set of possible worlds as a special subclass of prime states. Moreover, we used possible worlds to define an extensional notion of support, which corresponds to classical truth, as opposed to intensional truth, which corresponds to truth in substructural models. The resulting notion of evidence is a hyperintensional and non-prime one, in accordance with a philosophically motivated representation of evidence in formal epistemology. Finally, we showed that in our framework evidence does not suffer from many problematic epistemic closure properties, while capturing some logical criterion for epistemic competence.

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[18] Note that the restriction of Q to prime states plays here a crucial role, since otherwise it could be the case that $q \notin L$ because $q \notin P$.

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