

Small-scale turbulence limit of Fokker–Planck equation for polymers in turbulent flow

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We study the singular limit of Fokker–Planck equation of polymers density as the dominant time scale of small-scale component of turbulent flow goes to zero. Here, we complete the study of Flandoli–Tahraoui [Stretching of polymers and turbulence: Fokker Planck equation, special stochastic scaling limit and stationary law, *J. Differential Equations* **452** (2026) 113789] about scaling limit as the space scale of small-scale component of turbulent flow goes to zero by using stochastic modeling of turbulence. Depending on certain parameters related to turbulence modeling, the limit density has a generalized Cauchy distribution for the end-to-end vector. We also discuss the limit when we don't have a probability density limit. Our approach is based on the derivation of appropriate estimates on L^2 with appropriate weight and investigation of the convergence.

Keywords: Fokker–Planck equation; turbulence; singular limit; polymer flow; Cauchy distribution.

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1. Introduction

Polymers are complex molecular systems that can be vaguely thought of as a chain of springs. Understanding polymer dynamics is important, both theoretically and in practice. One application is that the presence of low concentrations of polymers can lead to a significant change in hydrodynamics, and one of the most important effects is the drag reduction in turbulence. In a turbulent fluid, polymers are usually found in two states called coil and stretched. The coil state is like a spherical or ellipsoidal rolled chain, which may be more or less elongated, but still in the roll position. The stretched state is when the chain is elongated, more similar to a straight line than a sphere. Strongly stretching turbulence may lead to the stretched state; when the polymer passes from one state to the other, we speak of coil-stretch transition.

One of the aims of this contribution is to rigorously justify the power-law distribution of the end-to-end vector of polymers. Consider in dimension 2 or 3 the following *Hookean* model:

$$\begin{cases} dR_t = \nabla u(X_t, t)R_t dt - \frac{1}{\beta}R_t dt + \sqrt{2}\sigma d\mathcal{W}_t, \\ dX_t = u(X_t, t)dt, \end{cases} \quad (1.1)$$

where X_t is the polymer position (the center of mass) and R_t is the end-to-end vector, representing the orientation and elongation of the chain, see e.g. Ref. 7. The polymer is embedded into a fluid having velocity $u(t, x)$, which stretches R_t by $\nabla u(x, t)$. The equation for R_t contains also a damping (restoring) term with relaxation time β and Brownian fluctuations $\sqrt{2}\sigma d\mathcal{W}_t$ where, to simplify the notation, we have denoted by σ^2 the product $\frac{kT}{\beta}$, k being the Boltzmann constant and T being the temperature.

We consider a dilute (non-interacting) family of polymers subject to equations (1.1), thus described by the kinetic equation for the density $f^{N,\tau} := f^{N,\tau}(x, r, t)$ of polymers with position x and length r at time t ,

$$\begin{cases} \partial_t f^{N,\tau} + \operatorname{div}_x(u^{N,\tau} f^{N,\tau}) + \operatorname{div}_r\left(\left(\nabla u^{N,\tau} r - \frac{1}{\beta}r\right) f^{N,\tau}\right) = \sigma^2 \Delta_r f^{N,\tau}, \\ u^{N,\tau} := u^{N,\tau}(x, t) = u_L(x, t) + u_s(x, t) = u_L(x, t) + \circ \sum_{k \in K} \sigma_k^{N,\tau}(x) \partial_t W_t^k, \\ f^{N,\tau}|_{t=0} = f_0^\tau. \end{cases} \quad (1.2)$$

Here $u^{N,\tau}(x, t)$ is the fluid velocity and it is assumed that $u^{N,\tau}(x, t)$ is made of two components, a deterministic large-scale one $u_L(x, t)$ and a stochastic one, modeling small-scale turbulence, of the form $\sum_{k \in K} \sigma_k^{N,\tau}(x) \partial_t W_t^k$, acting in Stratonovich form. We stress the fact that the coefficients $\sigma_k^{N,\tau}(x)$ of the turbulent part depend on a parameter N so that, when N increases, they represent smaller and smaller space scales, precisely Fourier frequencies $N \leq |k| \leq 2N$, providing a separation of scale regime. The noise acts on $f^{N,\tau}(x, r, t)$ in transport form, but incorporating also the stretching action by the term $\nabla u^{N,\tau}(x, t)r$. The coefficients $(\sigma_k^{N,\tau})_k$ depend as well on the dominant time scale τ of u_s , see Sec. 2.3.

In Ref. 22, under suitable intensity assumption, such that the stretching term has a finite limit covariance, we proved that $f^{N,\tau}$ weakly converges to the solution $f_\tau := f_\tau(x, r, t)$ of a deterministic equation, with a new diffusion term in the r -variable, of the form

$$\begin{cases} \partial_t f_\tau + \operatorname{div}_x(u_L f_\tau) + \operatorname{div}_r(\nabla u_L r f_\tau) \\ \quad = \operatorname{div}_r\left(\frac{1}{\beta}r f_\tau\right) + \sigma^2 \Delta_r f_\tau + \frac{k_T}{2} \operatorname{div}_r(A_b(r) \nabla_r f_\tau), \\ f_\tau|_{t=0} = f_0^\tau, \end{cases} \quad (1.3)$$

where $A_b(r) = ((b+1)|r|^2 I - br \otimes r)$ ($b = 2$ in 2D and $b = 1$ in 3D), $k_T = C_d a_\tau^2$ ($C_d = \frac{\pi \log(2)}{8}$ in 2D and $C_d = \frac{8\pi \log(2)}{15}$ in 3D) and a_τ is an intensity parameter of the noise depending on τ . We refer to Sec. 2, where we recall the assumptions and main results from Ref. 22 about the rigorous derivation of (1.3), as scaling limit of (1.2).

In the regime where the relaxation time of polymers β and the dominant time scale τ satisfy $\beta = \zeta\tau$, $\zeta > 0$, (1.3) becomes

$$\begin{cases} \partial_t f_\tau + \operatorname{div}_x(u_L f_\tau) + \operatorname{div}_r(\nabla u_L r f_\tau) = \frac{1}{\zeta \alpha \tau} \operatorname{div}_r \left(\alpha r f_\tau + \nabla_r f_\tau + \frac{1}{2} A_b(r) \nabla_r f_\tau \right), \\ f_\tau|_{t=0} = f_0^\tau, \quad (x, r) \in \mathbb{T}^d \times \mathbb{R}^d, d = 2, 3, \end{cases} \quad (1.4)$$

where $\alpha > 0$ and is related to the intensity of turbulence, see Sec. 2.3 how to get (1.4) from (1.3) and (2.7) for the interpretation of α .

The aim of this contribution is to consider the singular limit as $\tau \rightarrow 0$ of (1.4). This completes the study done in Ref. 22, and rigorously justifies the physical predictions about the power tail of the probability density function of polymer end-to-end vector R (see e.g. Ref. 2). In Ref. 22, we proved a first scaling limit (in space) by using a stochastic modeling of small-scale turbulent flow. In other words, we proved $\lim_{N \rightarrow +\infty} f^{N, \tau} = f^\tau$ exists and f^τ satisfies (1.3). Here, we prove that $\lim_{\tau \rightarrow 0} f^\tau = \lim_{\tau \rightarrow 0} \lim_{N \rightarrow +\infty} f^{N, \tau}$ exists in an appropriate sense and we identify the limit. To the best of the author's knowledge, this is the first time that scaling and singular limits,^a in space and time, have been considered using stochastic modeling of small scales of turbulent flow in the presence of stretching and transport terms. The main novelty of our contribution lies in the rigorous analysis based on analytical tools derived from the stochastic diffusion limit, homogenization theory and kinetic equations.

Sketch of the main results

For the convenience of the reader, let us explain heuristically the main result of this work; then precise results (see Theorems 3.1 and 3.2) and rigorous proofs are provided in the main body of the paper. We distinguish three cases:

- The case $\alpha > \frac{d}{2}$: there exists $f_0 \in L^2(\mathbb{T}^d \times \mathbb{R}^d; (1 + \frac{|r|^2}{2})^\alpha dr dx)$ such that

$$\lim_{\tau \rightarrow 0} \int_{\mathbb{T}^d \times \mathbb{R}^d} |f_0^\tau - f_0|^2 \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx = 0,$$

^aWe use *scaling limit* to denote the limit as $N \rightarrow +\infty$, which corresponds to a spatial scaling limit. On the other hand, the term *singular limit* denotes the limit $\tau \rightarrow 0$, which corresponds to the temporal singular parameter τ . We use the word “singular” because the limit problem (equilibrium) does not have the same nature as the initial one (1.4) (kinetic).

then f_τ converges to $\rho \otimes \frac{1}{Z}(1 + \frac{|r|^2}{2})^{-\alpha}$ in $L^2(0, T; L^2(\mathbb{T}^d \times \mathbb{R}^d; (1 + \frac{|r|^2}{2})^\alpha dr dx))$, where $Z = \int_{\mathbb{R}^d} (1 + \frac{|r|^2}{2})^{-\alpha} dr$ and ρ is the unique L^2 -valued solution of

$$\begin{cases} \partial_t \rho + u_L \cdot \nabla_x \rho = 0, \\ \rho(x)|_{t=0} = \rho_0 = \int_{\mathbb{R}^d} f_0(x, r) dr. \end{cases} \quad (1.5)$$

- The case $0 < \alpha \leq \frac{d}{2}$ and $(\int_{\mathbb{T}^d \times \mathbb{R}^d} |f_0^\tau|^2 (1 + \frac{|r|^2}{2})^\alpha dr dx)_\tau$ is bounded: there exists a subsequence of $(f_\tau)_\tau$ denoted by $(f_{\tau_k})_{\tau_k}$ such that

$$f_{\tau_k} \text{ converges weakly-}^* \text{ to } 0 \text{ in } L^\infty\left(0, T; L^2\left(\mathbb{T}^d \times \mathbb{R}^d; \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx\right)\right).$$

- The case $0 < \alpha \leq \frac{d}{2}$ and $(\int_{\mathbb{T}^d \times \mathbb{R}^d} |f_0^\tau| dr dx)_\tau$ is bounded: there exists a subsequence of $(f_\tau)_\tau$ denoted by $(f_{\tau_k})_{\tau_k}$ such that f_{τ_k} converges weakly- * to ν in $L^\infty(0, T; \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d))$, where $\mathcal{M}$ denotes the space of Radon measures and ν satisfies in weak sense the following:

$$\operatorname{div}_r \left(\alpha r \nu + \nabla_r \nu + \frac{1}{2} A_b(r) \nabla_r \nu \right) = 0. \quad (1.6)$$

The solution to (1.6) is $\nu = g \otimes \mu_\alpha$, where g is an element of $L^\infty(0, T; \mathcal{M}(\mathbb{T}^d))$ and $\mu_\alpha \in \mathcal{M}(\mathbb{R}^d)$ defined as follows:

$$\langle \mu_\alpha, \varphi \rangle_{\mathcal{M}, C_c} := \int_{\mathbb{R}^d} \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} \varphi(r) dr, \quad \forall \varphi \in C_c(\mathbb{R}^d).$$

We refer to Sec. 3 for a discussion about the physical interpretation of the above results.

Stochastic diffusion limit results are related to homogenization theory in very broad sense. Regarding this latter and without trying to be exhaustive, we refer to Ref. 3 for deterministic techniques, Ref. 34 for PDE homogenization and stochastic modeling (see also Refs. 31 and 29). However, it is important to notice that the strategies in stochastic diffusion limits are different from classical homogenization and especially they are very efficient for generalizations to nonlinear problems.

The origin of stochastic diffusion limits, based on the Itô–Stratonovich corrector, is Ref. 23 that threw a new light on passive scalars subject to turbulent models. The result of Ref. 23 has been generalized in several directions, showing in particular its strength for nonlinear scalar problems (see Ref. 21 for a review). In particular, we refer to Refs. 17 and 18 for the results in this direction about the 2D Euler and 2D Navier–Stokes equations. Moreover, this approach leads to a delayed blow-up and dissipation enhancement results, see e.g. Refs. 16 and 30. We refer also to Ref. 22 for extended literature review on the stochastic diffusion limits and transport noise. It is worth underlying that the mentioned results concern scalar problems with transport noise. However, when the turbulent fluid acts on vector fields, in addition to transport there is *stretching*: namely the differential $D\phi_t$ of the Lagrangian flow

ϕ_t of a fluid produces a modification of the length of vectors v , which leads to an increase of length for many possible directions v . The result of this work can be classified in this direction.

More difficult has been adapting the ideas to advected vector fields, because of lack of control on the stochastic stretching when the diffusive scaling limit is performed. Nevertheless, positive results for particular models have been obtained, adapting the classical scaling of Ref. 23, see e.g. Refs. 8, 19, 20 and 33. Recently, after considering a new scaling such as the noise covariance goes to zero but a suitable covariance built on derivatives of the noise converges to a non-zero limit, new results have been obtained in Refs. 9 and 22 in the case of stochastic transport-stretching. In Ref. 9, we introduced a background stochastic Vlasov equation behind stochastic transport and advection equations which gives additional information on the fluctuations and oscillations of solutions based on Young measures. We first developed the theory for the stochastic transport of a passive scalar. Then we developed the theory for a passive vector field, where stretching also acts in addition to transport. In the case of a passive vector field the background Vlasov equation adds completely new statistical information to the stochastic advection equation and the theory developed may help to recognize the existence of large values of the length of the vectors, where the physical phenomenon of magnetic dynamo is an example. In Ref. 22, we considered the new scaling by considering the Fokker–Planck equation (1.2) of polymers density and obtained the result described in the beginning of this introduction, namely, (1.3) as a limit of (1.2).

On the other hand, kinetic equations in general, which correspond to the mesoscopic scale, appear in the study of many problems in physics, biology, and other fields. Different types of these equations have been studied extensively, both in terms of the existence of solutions and their properties, as well as the asymptotic behavior with respect to the parameters appearing in the equations. In general, the main interest when studying asymptotic is to derive equations of macroscopic quantities such as density and probability distribution with respect to mesoscopic variable such as particle velocity and the polymers end-to-end vector in our case. Without trying to be exhaustive, the author studied in Ref. 36 the asymptotic of electron distribution function in semiconductor kinetic theory where the equation is given by Boltzmann transport equation for high electric fields. He proved that the limit distribution function is given by the tensor product of the density (satisfying a linear transport equation) and a probability distribution function of the velocity variable satisfying an appropriate PDE. In plasma, people are interested in studying the asymptotic of the Vlasov–Poisson–Fokker–Planck System for example. There are two important scalings, the low field limit and the high field limit. This leads to different scalings of the kinetic equations and requires a separate analysis, we refer for example to Refs. 13 and 32. In Refs. 13 and 37, the limit distribution function is given by the tensor product of the density (satisfying a drift-diffusion equation) and a normalized Maxwellian with zero mean for the microscopic velocity. In Ref. 15, the authors investigate a zero inertia limit of some kinetic equation

where the limit is given by the tensor product of the density (satisfying a nonlinear transport equation) and monokinetic distribution of the velocity variable. We refer for example to Refs. 25–28 and 32 for more results about the asymptotics of the kinetic equations.

Finally, it is worth mentioning that due to the stretching term in (1.1) and the appropriate stochastic scaling in Ref. 22, the matrix A_b is generated after the first scaling limit. This matrix A_b is a fundamental key giving rise to the Cauchy distribution for the end-to-end vector.

Structure of the paper

The paper is organized as follows: in Sec. 2, we present some preliminaries and formulate the problem. Then, we collect the main results and the physical interpretation of our work in Sec. 3. Section 4 is devoted to the proof of a uniform estimate with respect to τ and some results about the continuity equation associated with (1.4). In Sec. 5, we prove the convergence results, while discussion and outlook are provided in Sec. 6. We conclude the paper with Appendix A containing a proof of Lemma 4.1, Appendix B talks about the weighted Poincaré inequality and commutator estimate lemma in Appendix C.

2. Preliminaries and Formulation of the Problem

The aim of this section is to recall the main ingredients of the rigorous derivation of (1.4) as the scaling limit of (1.2). We consider a dilute family of polymers described by the kinetic equation (1.2) for the density $f^{N,\tau}$. In order to make the paper self-contained and for the convenience of the reader, let us recall the structure of $(\sigma_k^{N,\tau})_k$. For more details we refer to Ref. 22.

2.1. Structure of the small scales

In order to clarify the stochastic modeling of small scales, τ denotes the dominant time scale of u_s and $\ell \sim N^{-1}$ its space scale.

The 2D case

Consider $\mathbb{Z}_0^2 := \mathbb{Z}^2 - \{(0, 0)\}$ divided into its four quadrants (write $k = (k_1, k_2)$)

$$K_{++} = \{k \in \mathbb{Z}_0^2 : k_1 \geq 0, k_2 > 0\}; \quad K_{-+} = \{k \in \mathbb{Z}_0^2 : k_1 < 0, k_2 \geq 0\},$$

$$K_{--} = \{k \in \mathbb{Z}_0^2 : k_1 \leq 0, k_2 < 0\}; \quad K_{+-} = \{k \in \mathbb{Z}_0^2 : k_1 > 0, k_2 \leq 0\}$$

and set

$$K_+ = K_{++} \cup K_{+-}; \quad K_- = K_{-+} \cup K_{--} \quad \text{and} \quad K = K_+ \cup K_-.$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ be a complete filtered probability space. Define^b

$$\sigma_k^{N,\tau}(x) = \theta_k^{N,\tau} \frac{k^\perp}{|k|} \cos k \cdot x, \quad k \in K_+, \quad \sigma_k^{N,\tau}(x) = \theta_k^{N,\tau} \frac{k^\perp}{|k|} \sin k \cdot x, \quad k \in K_-,$$

where

$$\theta_k^{N,\tau} = \frac{a_\tau}{|k|^2}, \quad N \leq |k| \leq 2N, \quad N \in \mathbb{N}^*; \quad \theta_k^{N,\tau} = 0 \quad \text{elsewhere}, \quad (2.1)$$

where a_τ is a positive constant measuring the intensity, see Sec. 2.3. Let us also consider a family $(W_t^k)_{t \in \mathbb{Z}_0^2}^{k \in \mathbb{Z}_0^2}$ of independent Brownian motions on the probability space $(\Omega, \mathcal{F}, P)$. Then, set $u_s(x, t) = \sum_{k \in K} \sigma_k^{N,\tau}(x) \partial_t W_t^k$.

The 3D case

We introduce the partition $\mathbb{Z}_0^3 = \Gamma_{3,+} \cup \Gamma_{3,-}$ ^c such that $\Gamma_{3,+} = -\Gamma_{3,-}$ and we consider a family of real-valued independent Brownian motions $(B_t^{k,j})_{t \in \mathbb{Z}_0^3}^{k \in \mathbb{Z}_0^3, j \in \{1,2\}}$ defined on the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$, that is $\mathbb{E}(B_t^{k,j} B_s^{l,m}) = \min(t, s) \delta_{k,l} \delta_{j,m}$. Then, we introduce a sequence of complex-valued Brownian motions adapted to $(\mathcal{F}_t)_t$ defined as follows:

$$W_t^{k,j} = \begin{cases} B_t^{k,j} + iB_t^{-k,j} & \text{if } k \in \Gamma_{3,+}, \\ B_t^{-k,j} - iB_t^{k,j} & \text{if } k \in \Gamma_{3,-}. \end{cases}$$

Let $N \in \mathbb{N}^*$, define $\theta_{k,j}^{N,\tau} = \frac{a_\tau}{|k|^{5/2}} \mathbf{1}_{\{N \leq |k| \leq 2N\}}$, for a positive constant a . Then, for each $k \in \mathbb{Z}_0^3$, $j \in \{1,2\}$ we denote $\sigma_{k,j}^{N,\tau}(x) = \theta_{k,j}^{N,\tau} a_{k,j} e^{ik \cdot x}$, where $\{\frac{k}{|k|}, a_{k,1}, a_{k,2}\}$ is an orthonormal system of $\mathbb{R}^3$ for $k \in \Gamma_{3,+}$ and $a_{k,j} = a_{-k,j}$ if $k \in \Gamma_{3,-}$. Set

$$\mathbf{W}^{N,\tau}(t, x) = \sum_{k \in \mathbb{Z}_0^3, j \in \{1,2\}} \theta_{k,j}^{N,\tau} a_{k,j} e^{ik \cdot x} W^{k,j}(t), \quad (2.2)$$

and consider $u_s = \partial_t \mathbf{W}^{N,\tau}$ in the 3D case.

Seeking not to overload the notation, we simply recall the result of the scaling limit and refer to the main theorems in Ref. 22 for the precise mathematical framework of well-posedness and the various estimates necessary for the analysis. Thus, the following subsection concerns the rigorous derivation of (1.3) as a scaling limit of (1.2).

2.2. Scaling limit results

Let us present the following scaling limit result based on Ref. 22.

^bFor $y = (y_1, y_2) \in \mathbb{R}^2$, $y^\perp$ stands for $(-y_2, y_1)$.

^c $\mathbb{Z}_0^3 = \mathbb{Z}^3 - \{(0, 0, 0)\}$.

2.2.1. Notation and functional setting

Let $d \in \{2, 3\}$, we consider the periodic boundary conditions with respect to the spacial variable x , namely x belongs to the d -dimensional torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$. On the other hand, the end-to-end vector variable r belongs to $\mathbb{R}^d$. Let $m \in \mathbb{N}^*$ and we introduce the following Lebesgue and Sobolev spaces with polynomial weight, namely

$$L^2_{r,m}(\mathbb{T}^d \times \mathbb{R}^d) := \left\{ f : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R} : \int_{\mathbb{T}^d \times \mathbb{R}^d} |f(x, r)|^2 (1 + |r|^2)^{\frac{m}{2}} dx dr := \|f\|_{L^2_{r,m}}^2 < \infty \right\}.$$

Note that $L^2_{r,0}(\mathbb{T}^d \times \mathbb{R}^d)$ coincides with the Lebesgue space $L^2(\mathbb{T}^d \times \mathbb{R}^d)$. We will use the following notation $L^2_{r,2}(\mathbb{T}^d \times \mathbb{R}^d) := H$. We recall the definition of inner products defined on the spaces H and $L^2(\mathbb{T}^d \times \mathbb{R}^d)$.

$$(h, g)_H := \int_{\mathbb{T}^d \times \mathbb{R}^d} h(x, r) g(x, r) (1 + |r|^2) dx dr, \quad \forall g, h \in H;$$

$$(h, g) := \int_{\mathbb{T}^d \times \mathbb{R}^d} h(x, r) g(x, r) dx dr, \quad \forall g, h \in L^2(\mathbb{T}^d \times \mathbb{R}^d).$$

2.2.2. Scaling limit as $N \rightarrow +\infty$

Let $T > 0$ and $u_L = u_L(x, t)$ be a given large-scale component such that $u_L \in C([0, T], C^2(\mathbb{T}^d; \mathbb{R}^d))$ and $\operatorname{div}_x(u_L) = 0$.

Since $L^\infty(0, T; H)$ is not separable, it is convenient to recall the following space:

$$L^2_{w-*}(\Omega; L^\infty(0, T; H)) = \{u : \Omega \rightarrow L^\infty(0, T; H) \text{ is weakly-}^* \text{ measurable and } \mathbb{E}\|u\|_{L^\infty(0, T; H)}^2 < \infty\},$$

where weakly- * measurable stands for the measurability when $L^\infty(0, T; H)$ is endowed with the σ -algebra generated by the Borel sets of weak- * topology, we recall that (see Ref. 12)

$$L^2_{w-*}(\Omega; L^\infty([0, T]; H)) \simeq (L^2(\Omega; L^1([0, T]; H')))'.$$

Assume that $f_0^\tau \in H$ and $f^{N,\tau}$ be the unique quasi-regular weak solution of (1.2) (see Ref. 22). Let us recall the main ideas to get (1.3) from (1.2).

By using the stochastic modeling of small scale, we get a stochastic Fokker–Planck equation in Stratonovich form:

$$\begin{cases} \partial_t f^{N,\tau} + \operatorname{div}_x(u_L^{N,\tau} f^{N,\tau}) + \operatorname{div}_r \left(\left(\nabla u_L^{N,\tau} r - \frac{1}{\beta} r \right) f^{N,\tau} \right) = \sigma^2 \Delta_r f^{N,\tau} \\ \quad - \sum_{k \in K} \sigma_k^N \cdot \nabla_x f^{N,\tau} \circ \partial_t W^k - \sum_{k \in K} (\nabla \sigma_k^N r) \cdot \nabla_r f^{N,\tau} \circ \partial_t W^k, \\ f^{N,\tau}|_{t=0} = f_0^\tau. \end{cases} \quad (2.3)$$

We write (2.3) in Itô form, where we need to compute Itô–Stratonovich corrector, see Ref. 22. By using the space-homogeneity and mirror symmetry of the covariance operator $Q(x, y) := \sum_{k \in K} \sigma_k^N(x) \otimes \sigma_k^N(y)$, we get the following Itô form of the stochastic Fokker–Planck equation:

$$\left\{ \begin{aligned} & df^{N,\tau} + \operatorname{div}_x(u_L^{N,\tau} f^{N,\tau}) dt + \operatorname{div}_r \left(\left(\nabla u_L^{N,\tau} r - \frac{1}{\beta} r \right) f^{N,\tau} \right) dt \\ &= \sigma^2 \Delta_r f^{N,\tau} dt - \sum_{k \in K} \sigma_k^{N,\tau} \cdot \nabla_x f^{N,\tau} dW^k - \sum_{k \in K} (\nabla \sigma_k^{N,\tau} r) \cdot \nabla_r f^{N,\tau} dW^k \\ &+ \alpha_N \Delta_x f^{N,\tau} dt + \frac{1}{2} \operatorname{div}_r \left(\sum_{k \in K} ((\nabla \sigma_k^{N,\tau} r) \otimes (\nabla \sigma_k^{N,\tau} r)) \nabla_r f^{N,\tau} \right) dt, \\ &f^{N,\tau}|_{t=0} = f_0^\tau, \end{aligned} \right. \quad (2.4)$$

where

$$A_N(r) = \sum_{k \in K} ((\nabla \sigma_k^N r) \otimes (\nabla \sigma_k^N r)) = k_T A_b(r) + O\left(\frac{1}{N}\right) P(r),$$

P is a polynomial of second degree.

By using the scaling (2.1), we see that $A_N(r) \rightarrow k_T A_b(r)$ and $\alpha_N \rightarrow 0$. Thus, we can interpret the emergence of the part including $A_b(r)$ as a consequence of the stretching effect and the choice of scaling (2.1). More precisely, we get (see Ref. 22) the following result.

Theorem 2.1. *Under the assumption that the small scales have the given structure in Sec. 2.1. There exists a new probability space, denoted by the same way (for simplicity) $(\Omega, \mathcal{F}, P)$, $f_\tau \in L^2_{w-*}(\Omega; L^\infty([0, T]; H))$, $\nabla_r f_\tau \in L^2(\Omega; L^2([0, T]; H))$ such that the following convergence holds (up to a subsequence)*

$$\begin{aligned} f^{N,\tau} &\rightharpoonup f_\tau \text{ in } L^2_{w-*}(\Omega; L^\infty([0, T]; H)), \\ \nabla_r f^{N,\tau} &\rightharpoonup \nabla_r f_\tau \text{ in } L^2(\Omega; L^2([0, T]; H)). \end{aligned}$$

Moreover f_τ is the unique solution of the following problem: P -a.s. for any $t \in [0, T]$:

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(x, r, t) \phi(x) \psi(r) dr dx - \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_0^\tau(x, r) \phi(x) \psi(r) dr dx \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(x, r, s) u_L(x, s) \cdot \nabla_x \phi(x) \psi(r) dr dx ds \\ &+ \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(x, r, s) \left(\nabla u_L(s, x) r - \frac{1}{\beta} r \right) \cdot \nabla_r \psi(r) \phi(x) dr dx ds \\ &- \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \sigma^2 \nabla_r f_\tau(x, r, s) \cdot \nabla_r \psi(r) \phi(x) dr dx ds \\ &+ \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{k_T}{2} A_b(r) \nabla_r f_\tau(x, r, s) \cdot \nabla_r \psi(r) \phi(x) dr dx ds, \end{aligned}$$

for any $\phi \in C^\infty(\mathbb{T}^d)$ and $\psi \in C_c^\infty(\mathbb{R}^d)$, $A_b(r) = ((b+1)|r|^2 I - br \otimes r)$ ($b = 2$ in 2D and $b = 1$ in 3D), $k_T = C_d a_\tau^2$ ($C_d = \frac{\pi \log(2)}{8}$ in 2D and $C_d = \frac{8\pi \log(2)}{15}$ in 3D) and a_τ is an intensity parameter of the noise.

2.3. The derivation of (1.4)

We recall that σ^2 is the product $\frac{k_T}{\beta}$, k being Boltzmann constant and T being the temperature. Hence $\sigma^2 = \frac{C_2}{\beta}$, where $C_2 = kT > 0$. In order to formulate the main equation, let us present the following.

2.3.1. Noise specification and the parameter a_τ

We focus on the 2D case and the 3D case follows in the same way. Denote by u_s the small scale of the turbulent velocity, its vortex structures have a dominant time scale τ and space scale ℓ , and the associated turbulent kinetic energy k_T (of order $\frac{\ell^2}{\tau^2}$, in other words, $k_T = \kappa \frac{\ell^2}{\tau^2}$ for a positive constant κ). In a Gaussian approximation of the model, a reasonable choice is

$$u_s(x, t) := \frac{\sqrt{k_T}}{\ell} \sum_{N \leq |k| \leq 2N} \frac{1}{|k|^2} \frac{k^\perp}{|k|} e_k(x) Z_t^k,$$

where the functions $e_k(x)$, sine and cosine, $N \sim \ell^{-1}$, and Z_t^k are stationary Ornstein–Uhlenbeck processes solution of

$$dZ_t^k = -\frac{1}{\tau} Z_t^k dt + \sqrt{\frac{2}{\tau}} dW_t^k,$$

where $(W_t^k)_k$ are independent Brownian motions. The average kinetic energy is given by

$$\mathbb{E}[|u_s(x, t)|^2] = \frac{k_T}{\ell^2} \sum_{N \leq |k| \leq 2N} \frac{1}{|k|^4} e_k^2(x),$$

and $\frac{1}{\ell^2} \sum_{N \leq |k| \leq 2N} \frac{1}{|k|^4} e_k^2(x)$ has a unitary variance size. The intensity of the turbulent flow is given by $\sqrt{k_T}$. Therefore, after using the approximation

$$\lim_{\tau \rightarrow 0} \mathbb{E} \left[\left(\frac{1}{\sqrt{\tau}} \int_0^t Z_s^k ds - W_t^k \right)^2 \right] = 0,$$

we write

$$u_s(x, t) dt := \sqrt{\tau k_T} \sum_{N \leq |k| \leq 2N} \frac{1}{\ell} \frac{1}{|k|^2} \frac{k^\perp}{|k|} e_k(x) dW_t^k.$$

Thus, a_τ is of the order $\frac{\ell}{\sqrt{\tau}}C(\ell)$ with $C(\ell) = \frac{1}{\ell}$. More precisely, $a_\tau = \sqrt{\kappa}\frac{\ell}{\sqrt{\tau}}C(\ell)$. Thus we infer that $a_\tau = \frac{C_3}{\sqrt{\tau}}$ with $C_3 = \sqrt{\kappa} > 0$. Let us formulate the main equation. It is clear that we can write the limit equation given by Theorem 2.1 as follows:

$$\begin{cases} \partial_t f_\tau + \operatorname{div}_x(u_L f_\tau) + \operatorname{div}_r(\nabla u_L r f_\tau) \\ \quad = \operatorname{div}_r\left(\frac{1}{\beta} r f_\tau\right) + \sigma^2 \Delta_r f_\tau + \frac{C_d a_\tau^2}{2} \operatorname{div}_r(A_b(r) \nabla_r f_\tau), \\ f_\tau|_{t=0} = f_0^\tau. \end{cases} \quad (2.5)$$

In the regime where the relaxation time of polymers and the dominant time scale of the small-scale turbulent flow satisfies $\beta = \zeta\tau, \zeta > 0$, we obtain

$$\begin{cases} \partial_t f_\tau + \operatorname{div}_x(u_L f_\tau) + \operatorname{div}_r((\nabla u_L r) f_\tau) \\ \quad = \operatorname{div}_r\left(\frac{1}{\zeta\tau} r f_\tau + \frac{C_2}{\zeta\tau} \nabla_r f_\tau + \frac{C_d C_3^2}{2\tau} A_b(r) \nabla_r f_\tau\right), \\ f_\tau|_{t=0} = f_0^\tau. \end{cases} \quad (2.6)$$

By setting

$$\alpha = \frac{1}{\zeta C_d C_3^2} > 0 \quad \text{and} \quad \gamma = \frac{C_2}{\zeta C_d C_3^2} > 0, \quad (2.7)$$

it is worth making some comments on the parameter α . Note that from (2.7), α becomes smaller either when ζ becomes larger (i.e. the relaxation time of polymer β becomes larger than the dominant time scale τ) or C_3 becomes larger (i.e. the intensity of the turbulent flow becomes larger). In both cases, this corresponds to a stronger turbulent flow for the polymers.

Therefore, we obtain the following equation for $f_\tau = f_\tau(x, r, t)$:

$$\begin{cases} \partial_t f_\tau + \operatorname{div}_x(u_L f_\tau) + \operatorname{div}_r(\nabla u_L r f_\tau) = \frac{1}{\zeta\alpha\tau} \operatorname{div}_r\left(\alpha r f_\tau + \gamma \nabla_r f_\tau + \frac{1}{2} A_b(r) \nabla_r f_\tau\right), \\ f_\tau|_{t=0} = f_0^\tau. \end{cases}$$

Without loss of generality set $\gamma = 1$. Our aim is to investigate the limit behavior of $(f_\tau)_\tau$ as $\tau \rightarrow 0$ in (2.8).

$$\begin{cases} \partial_t f_\tau + \operatorname{div}_x(u_L f_\tau) + \operatorname{div}_r(\nabla u_L r f_\tau) = \frac{1}{\zeta\alpha\tau} \operatorname{div}_r\left(\alpha r f_\tau + \nabla_r f_\tau + \frac{1}{2} A_b(r) \nabla_r f_\tau\right), \\ f_\tau|_{t=0} = f_0^\tau. \end{cases} \quad (2.8)$$

For clarity of the presentation, it is worth summarizing the various key parameters (see Table 1) of the paper and those related to it from Ref. 22. First, we recall that $C_3 > 0$ is a dimensionless constant measuring the intensity of the turbulent flow, $C_d = \frac{\pi \log(2)}{8}$ in 2D and $C_d = \frac{8\pi \log(2)}{15}$ in 3D occurring after taking the scaling limit and depending on the choice of coefficients $(\sigma_k^{N,\tau})_N$.

Table 1. Key parameters.

Parameter	Scaling limit ²² ($N \uparrow +\infty$)	Singular limit ($\tau \downarrow 0$)
Boltzmann constant $\times$ Temperature	kT	kT
Polymer relaxation time	β	β
Dominant space scale of small-scale turbulence	$\ell \sim N^{-1}$	$\ell \sim N^{-1}$
Dominant time scale of small-scale turbulence	$/$	τ
Intensity parameter (turbulence)	a	$a_\tau = \frac{C_3}{\sqrt{\tau}}$
Constant $\zeta > 0$	$/$	$\beta = \zeta \tau$
Exponent α (see also (2.7))	$/$	$\alpha = \frac{1}{\zeta C_d C_3^2}$

3. Main Results

Let us introduce the following space

$$H_\alpha = \left\{ g : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R} : \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g^2(x, r) \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx < \infty \right\}, \quad \alpha > 0.$$

Remark 3.1. Note that $H_\alpha \hookrightarrow H \cap L^1(\mathbb{T}^d \times \mathbb{R}^d)$ if $\alpha > \frac{d}{2}$.

In the following, we always consider $f_0^\tau \in H$ for any $\tau > 0$.

3.1. Statement of the main result

Consider the following assumptions:

(H₁) There exists $\Lambda > 0$ independent of τ such that $\sup_{\tau > 0} \|f_0^\tau\|_{H_\alpha}^2 \leq \Lambda$.

(H₂) There exists $f_0 \in H_\alpha$ such that $\lim_{\tau \rightarrow 0} \|f_0^\tau - f_0\|_{H_\alpha} = 0$.

The first result concerns the case $\alpha > \frac{d}{2}$.

Theorem 3.1. *Let $\tau > 0$ and $\alpha > \frac{d}{2}$, assume that assumption H_1 is satisfied. Let f_τ be the unique quasi-regular weak solution to (4.2), then*

$$\begin{aligned} \sup_{t \in [0, T]} \|f_\tau(t)\|_{H_\alpha}^2 + \frac{1}{\zeta \alpha \tau} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha \right) \right|^2 \\ \times \left(1 + \frac{|r|^2}{2} \right)^{-\alpha+1} dr dx ds \leq e^{\mathbf{K}T} \Lambda, \end{aligned}$$

where $\mathbf{K} = 2\alpha \|\nabla_x u_L\|_\infty$. If moreover H_2 is satisfied, the following convergence holds as $\tau \rightarrow 0$

$$f_\tau \rightarrow_{\tau} \rho \otimes \frac{1}{Z} \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \quad \text{in } L^2(0, T; H_\alpha), \quad (3.1)$$

where ρ is the unique solution to (4.12) and $Z := \int_{\mathbb{R}^d} (1 + \frac{|r|^2}{2})^{-\alpha} dr = C(d, \alpha)$ is a normalizing factor. In addition, the following rate of convergence holds

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_\tau - \rho \otimes p_\alpha)^2(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx ds \leq \mathbf{C}_\alpha \left(\tau + \frac{T}{Z} \|f_0^\tau - f_0\|_{H_\alpha}^2 \right),$$

where $\mathbf{C}_\alpha$ is a positive constant.

Proof. It is a consequence of Propositions 4.1, 5.1 and Lemma 4.2. $\square$

Note that (3.1), in Theorem 3.1, asserts convergence in the space $L^2(0, T; H_\alpha)$ with rate $O(\tau)$ if $\|f_0^\tau - f_0\|_{H_\alpha}^2 = O(\tau)$. This convergence is the consequence of the combination of the inequality (5.5) and the weighted Poincaré inequality (B.3). Additionally, from (5.5) it follows

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(t) - \rho(t) \otimes p_\alpha|^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \\ & \leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_0^\tau - \rho_0 \otimes p_\alpha|^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx + \tilde{C}\sqrt{\tau}, \end{aligned}$$

which asserts convergence in the space $L^\infty(0, T; H_\alpha)$ with rate $O(\sqrt{\tau})$ if $\|f_0^\tau - \rho_0 \otimes p_\alpha\|_{H_\alpha}^2 = O(\sqrt{\tau})$. However, it is worth noticing that the convergence in $L^\infty(0, T; H_\alpha)$ requires a stronger assumption of convergence of the initial data, namely it requires f_0 to have the form $\rho_0 \otimes p_\alpha$.

Remark 3.2. It is worth mentioning that the same conclusion of Theorem 3.1 holds if H_2 is replaced by $f_0 \in L^2(\mathbb{T}^d; L^1(\mathbb{R}^d))$ such that $\lim_{\tau \rightarrow 0} \|f_0^\tau - f_0\|_{L_x^2 L_r^1} = 0$.

In the following, $\mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)$ denotes the space of Radon measures. Concerning the case $0 < \alpha \leq \frac{d}{2}$, we have the following theorem.

Theorem 3.2. *Let $\tau > 0$ and $0 < \alpha \leq \frac{d}{2}$. We distinguish two cases:*

- (i) *Assume that H_1 is satisfied and let f_τ be the unique quasi-regular weak solution to (4.2). Then*

$$\begin{aligned} & \sup_{t \in [0, T]} \|f_\tau(t)\|_{H_\alpha}^2 + \frac{1}{\zeta \alpha \tau} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha \right) \right|^2 \\ & \times \left(1 + \frac{|r|^2}{2}\right)^{-\alpha+1} dr dx ds \leq e^{\mathbf{K}T} \Lambda. \end{aligned}$$

Moreover, there exists a subsequence $(f_{\eta_k})_k$ of $(f_\tau)_\tau$ such that the following convergence holds:

$$f_{\eta_k} \xrightarrow{*} 0 \quad \text{in } L^\infty(0, T; H_\alpha) \quad \text{as } k \rightarrow +\infty. \quad (3.2)$$

- (ii) *Let $f_0^\tau \in L^1(\mathbb{T}^d \times \mathbb{R}^d) \cap H$ and f_τ be the unique quasi-regular weak solution to (4.2). Assume the existence of $\Lambda_2 > 0$ independent of τ such that $\sup_{\tau > 0} \|f_0^\tau\|_{L^1} \leq \Lambda_2$. Then*

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(t)| dr dx \leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_0^\tau| dr dx \leq \Lambda_2. \quad (3.3)$$

Moreover, there exists a subsequence $(f_{\tau_k})_k$ of $(f_\tau)_\tau$ and $\nu \in L^\infty(0, T; \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d))$ such that

$$f_{\tau_k} \xrightarrow{*} \nu \quad \text{in } L^\infty(0, T; \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)) \quad \text{as } k \rightarrow +\infty, \quad (3.4)$$

and ν solves, in the sense of distributions,^d the following:

$$\operatorname{div}_r \left(\alpha r \nu + \nabla_r \nu + \frac{1}{2} A_b(r) \nabla_r \nu \right) = 0 \quad \text{in } \mathcal{D}'. \quad (3.5)$$

Proof. It is a consequence of Propositions 4.1, 5.2 and 5.3. $\square$

Remark 3.3. Since the differential operator acts only with respect to r -variable, a particular solution is of the form $\nu = \mathbf{g}_x(\cdot) \otimes \mu_r$, where $\mathbf{g}_x$ is an element of $L^\infty(0, T; \mathcal{M}(\mathbb{T}^d))$ and $\mu_r \in \mathcal{M}(\mathbb{R}^d)$ solving

$$\operatorname{div}_r \left(\alpha r \mu_r + \nabla_r \mu_r + \frac{1}{2} A_b(r) \nabla_r \mu_r \right) = 0 \quad \text{in } (C_c^\infty(\mathbb{R}^d))'. \quad (3.6)$$

On the other hand, $P_\alpha(r) = (1 + \frac{|r|^2}{2})^{-\alpha} \in C_b^\infty(\mathbb{R}^d)$ and $\alpha r P_\alpha(r) + \nabla_r P_\alpha(r) + \frac{1}{2} A_b(r) P_\alpha(r) = 0$. Define the following Radon measure μ_α :

$$\langle \mu_\alpha, \varphi \rangle_{\mathcal{M}, C_c} := \int_{\mathbb{R}^d} P_\alpha(r) \varphi(r) dr, \quad \forall \varphi \in C_c(\mathbb{R}^d). \quad (3.7)$$

Indeed, let $\varphi \in C_c(\mathbb{R}^d)$, one gets

$$|\langle P_\alpha, \varphi \rangle_{\mathcal{M}, C_c}| = \left| \int_{\mathbb{R}^d} P_\alpha(r) \varphi(r) dr \right| \leq \|\varphi\|_{C_c} \int_{\operatorname{supp}(\varphi)} P_\alpha(r) dr \leq C \|\varphi\|_{C_c}$$

for some positive constant C . Consequently, $\mathbf{g}_x(\cdot) \otimes \mu_\alpha \in L^\infty(0, T; \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d))$ solves (3.5).

3.1.1. Non-uniqueness of measure-valued solutions and selection principle ($0 < \alpha \leq \frac{d}{2}$)

Since the differential operator in (3.5) acts only with respect to r -variable, the solution has the form

$$\nu = \mathbf{g}(t, x) \otimes \varrho_r, \quad (3.8)$$

where $\mathbf{g}$ is any element of $L^\infty(0, T; \mathcal{M}(\mathbb{T}^d))$ and $\varrho_r \in \mathcal{M}(\mathbb{R}^d)$ is the solution of

$$\operatorname{div}_r \left(\alpha r \varrho_r + \nabla_r \varrho_r + \frac{1}{2} A_b(r) \nabla_r \varrho_r \right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \quad (3.9)$$

• **(Non-uniqueness of ν)** Note that at this level, the uniqueness fails by choosing $\mathbf{g}_1, \mathbf{g}_2 \in L^\infty(0, T; \mathcal{M}(\mathbb{T}^d))$ such that $\mathbf{g}_1 \neq \mathbf{g}_2$ and note from Remark 3.3 that $\mathbf{g}_1(t, x) \otimes \mu_r$ and $\mathbf{g}_2(t, x) \otimes \mu_r$ are two distinct solutions to (3.5).

• **(The uniqueness of ϱ_r)** Let us discuss the uniqueness for the solution to (3.9). Namely, ϱ_r satisfies (see (5.20))

$$\left\langle \varrho_r, -\alpha r \cdot \nabla_r \varphi + \Delta_r \varphi + \frac{1}{2} \operatorname{div}_r (A_b(r) \nabla_r \varphi) \right\rangle_{\mathcal{M}, C_c} ds = 0 \quad \text{for any } \varphi \in C_c^2(\mathbb{R}^d). \quad (3.10)$$

^dIt is the dual space of $C_c^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R}^d)$, denoted by $\mathcal{D}'$.

By using Lebesgue decomposition theorem of Radon measures (cf. Ref. 39), there exists a unique pair $(\varrho_{ac}^r, \varrho_s^r)$ of measures such that $\varrho_r = \varrho_s^r + \varrho_{ac}^r$ where

- $\varrho_{ac}^r \ll \lambda_d$ (absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$). Moreover, $\varrho_{ac} = h(r)dr$ with $h \in L^1_{\text{loc}}(\mathbb{R}^d)$.
- $\varrho_s^r \perp \lambda_d$ (singular with respect to Lebesgue measure) and may include Dirac measures. In other words, concentrations are authorized in the space $\mathcal{M}(\mathbb{R}^d)$.

As we already indicated in Remark 3.3 that $\mu_r(dr) = (1 + \frac{|r|^2}{2})^{-\alpha} dr \in \mathcal{M}(\mathbb{R}^d)$ solves (3.6). In accordance with Lebesgue decomposition theorem, μ_r belongs to ϱ_{ac}^r , i.e.

$$\varrho_{ac}^r = \mu_r + C\mu_{r,2}, C \geq 0 \text{ where } \mu_{r,2} \text{ is another absolutely continuous component.} \quad (3.11)$$

In general, it is difficult to explicitly compute all the solutions to (3.9) but if uniqueness holds, then it is necessary to get $\varrho_r = \mu_r$, since (3.9) is a linear equation. This is the goal of the next proposition.

Proposition 3.1. *There exists a unique Radon measure solution to (3.9), in the sense (3.10).*

Proof. Recall that (3.10) is a linear PDE. Hence the difference of two solutions, still denoted by ϱ_r , satisfies (3.10). In order to prove uniqueness, we use the standard duality argument and Riesz representation theorem (see e.g. Ref. 39) for Radon measures. Namely, the uniqueness follows if we show

$$\langle \varrho_r, \psi \rangle_{\mathcal{M}, C_c^\infty} = 0 \quad \text{for any } \psi \in C_c^\infty(\mathbb{R}^d). \quad (3.12)$$

Let us see if (3.12) holds. To abbreviate the notation, define

$$\mathcal{L}^* \varphi = -\alpha r \cdot \nabla_r \varphi + \Delta_r \varphi + \frac{1}{2} \operatorname{div}_r (A_b(r) \nabla_r \varphi).$$

From (3.10), we have $\langle \varrho_r, \mathcal{L}^* \varphi \rangle = 0$ for any $\varphi \in C_c^2(\mathbb{R}^d)$. Let us analyze the adjoint equation. Namely, let $\psi \in C_c^\infty(\mathbb{R}^d)$ and denote by B_M the ball with radius M such that $\operatorname{supp}(\psi) \subset B_M$. Then, consider the following problem in B_M with Dirichlet boundary condition:

$$\begin{cases} \mathcal{L}^* \varphi = \psi & \text{in } B_M, \\ \varphi = 0 & \text{on } \partial B_M. \end{cases} \quad (3.13)$$

The existence of solution in $H_0^1(B_M)$ to (3.13) follows by using “Fredholm alternative for linear PDEs” (see e.g. Ref. 14). Since ∂B_M is C^∞ -boundary, the classical elliptic theory (see e.g. Ref. 14) ensures the existence of $\varphi \in C^\infty(\overline{B_M})$. Then, extend φ by zero outside B_M to get $\varphi \in C_c^2(\mathbb{R}^d)$. Therefore, we obtain

$$\langle \varrho_r, \psi \rangle_{\mathcal{M}, C_c^\infty} = \langle \varrho_r, \mathcal{L}^* \varphi \rangle_{\mathcal{M}, C_c} = 0. \quad (3.14)$$

The claim (3.12) follows from the arbitrariness of ψ . □

Finally, let us discuss a selection principle for $\mathbf{g}$.

• **(Selection principle for $\mathbf{g}$)** Since $\sup_{\tau>0} \|f_0^\tau\|_{L^1} \leq \Lambda_2$, by using (3.3) we can set $g_\tau = \int_{\mathbb{R}^d} f_\tau dr$ and integrate (2.8) with respect to r , therefore we get

$$\begin{cases} \partial_t g_\tau + \operatorname{div}_x(u_L g_\tau) = 0, & \tau > 0, \\ g_\tau|_{t=0} = g_0^\tau = \int_{\mathbb{R}^d} f_0^\tau dr \in L^1(\mathbb{T}^d). \end{cases} \quad (3.15)$$

Note that (3.15) is linear transport equation studied in Ref. 11. Therefore, by writing Ref. 11 in the case of torus, we have the following proposition.

Proposition 3.2. *Let $u_L \in C([0, T], C^2(\mathbb{T}^d; \mathbb{R}^d))$, $\operatorname{div}_x(u_L) = 0$ and $g_0^\tau \in L^1(\mathbb{T}^d)$. There exists a unique weak solution $g_\tau \in L^\infty([0, T]; L^1(\mathbb{T}^d))$ to (3.15) in the following sense:*

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} (g_\tau(t, x) \partial_t \varphi(t, x) + g_\tau(t, x) u_L(t, x) \cdot \nabla \varphi(t, x)) \, dx \, dt \\ & + \int_{\mathbb{R}^d} g_0^\tau(x) \varphi(0, x) \, dx = 0 \end{aligned}$$

for every test function $\varphi \in C_c^\infty([0, T] \times \mathbb{T}^d)$. Moreover, the solution is renormalized:

$$\partial_t \beta(g_\tau) + u_L \cdot \nabla \beta(g_\tau) = 0$$

for all $\beta \in C^1(\mathbb{R})$ with bounded derivative and vanishing near 0.

A priori, at the limit, we will be interested in (3.15) with initial Radon measure data (a weak-* limit, as $\tau \rightarrow 0$, of $(f_0^\tau)_\tau$ in $\mathcal{M}(\mathbb{T}^d)$), which requires more involved analysis so we restrict ourself to the case L^1 -data at the limit, in order to explain the idea of selection.

Proposition 3.3. *Let $g_0 \in L^1(\mathbb{T}^d)$ and assume moreover that $(g_0^\tau)_\tau$ in (3.15) converges to g_0 in $L^1(\mathbb{T}^d)$ as $\tau \rightarrow 0$. Then g_τ converges to $\tilde{g}$ in $L^\infty([0, T]; L^1(\mathbb{T}^d))$ as $\tau \rightarrow 0$, where $\tilde{g} \in L^\infty([0, T]; L^1(\mathbb{T}^d))$ is the unique weak solution to*

$$\begin{cases} \partial_t \tilde{g} + \operatorname{div}_x(u_L \tilde{g}) = 0, \\ \tilde{g}|_{t=0} = g_0. \end{cases} \quad (3.16)$$

Proof. The existence and uniqueness to (3.16) follow as above, namely by using Ref. 11. On the other hand, using the linearity of (3.15) and (3.16), the difference $g_\tau - \tilde{g}$ is the unique weak solution to

$$\begin{cases} \partial_t (g_\tau - \tilde{g}) + \operatorname{div}_x(u_L (g_\tau - \tilde{g})) = 0, & \tau > 0, \\ g_\tau - \tilde{g}|_{t=0} = g_0^\tau - g_0 \in L^1(\mathbb{T}^d). \end{cases} \quad (3.17)$$

Recall that the solution is renormalized. By choosing appropriate sequence $\beta_M \in C^1(\mathbb{R})$ with bounded derivative, vanishing near 0 and converging to the absolute

value function (note that we can also use similar computation as in Proposition 5.3 as well), we get

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^d} |g_\tau - \tilde{g}| dx \leq \int_{\mathbb{T}^d} |g_0^\tau - g_0| dx \rightarrow 0 \quad \text{as } \tau \rightarrow 0. \quad (3.18)$$

□

Consequently, as a natural selection to avoid the trivial non-uniqueness, we can select $\mathbf{g}$ in (3.8) to be $\tilde{g}$, the unique *weak solution* of (3.16).

Remark 3.4. A natural assumption guarantees that $(g_0^\tau)_\tau$ converges toward g_0 in $L^1(\mathbb{T}^d)$ and to assume f_0^τ converges to some $f_0 \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$, instead assuming only the boundedness in $L^1(\mathbb{T}^d \times \mathbb{R}^d)$ as in Theorem 3.2(ii).

3.2. On the role of the unbounded potential

Let us show that the unbounded nature of the restoring force leads to the emergence of a measure valued limit if $0 < \alpha \leq \frac{d}{2}$. Note that the unbounded nature of the force allows the polymer end-to-end vector to take values in $\mathbb{R}^d$, which plays a role in the lack of L^1 -compactness in this regime.

Proposition 3.4. *Under the assumptions of Theorem 3.2(ii), let $f_\tau \in L^\infty(0, T; L^1(\mathbb{T}^d \times \mathbb{R}^d))$ be the unique solution to (2.8) satisfying*

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(t)| dr dx \leq \Lambda_2, \quad \forall \tau > 0.$$

*Then $(f_\tau)_\tau$ does not contain any weakly- * convergent subsequence in $L^\infty(0, T; L^1(\mathbb{T}^d \times \mathbb{R}^d))$.*

Proof. We reason by contradiction. Assume that $(f_\tau)_\tau$ has a subsequence, denoted by $(f_{\tau_k})_{\tau_k}$, $k \in \mathbb{N}$, which converges weakly- * to some $\bar{f}$ in $L^\infty(0, T; L^1(\mathbb{T}^d \times \mathbb{R}^d))$. Namely, the following holds:

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_{\tau_k}(s, x, r) \xi(s) \psi(x) \phi(r) ds dx dr \\ & \rightarrow \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \bar{f}(s, x, r) \xi(s) \psi(x) \phi(r) ds dx dr, \end{aligned} \quad (3.19)$$

for any $\xi \in L^1([0, T])$, $\psi \in L^\infty(\mathbb{T}^d)$, $\phi \in L^\infty(\mathbb{R}^d)$. On the other hand, we know that f_{τ_k} is the unique solution to (2.8) for any $k \in \mathbb{N}$ and it is bounded in $L^\infty(0, T; L^1(\mathbb{T}^d \times \mathbb{R}^d))$. Therefore, it converges, with respect to the weak- * topology, in $L^\infty(0, T; \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d))$ (up to subsequence) to some $\tilde{\nu}$ such that

$$\operatorname{div}_r \left(\alpha r \tilde{\nu} + \nabla_r \tilde{\nu} + \frac{1}{2} A_b(r) \nabla_r \tilde{\nu} \right) = 0 \text{ in } \mathcal{D}',$$

$$\text{and } \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_{\tau_k}(s, x, r) \xi(s) \psi(x) \phi(r) ds dx dr \rightarrow \int_0^T \langle \tilde{\nu}, \psi \phi \rangle \xi(s) ds, \quad (3.20)$$

for any $\xi \in L^1([0, T])$, $\psi \in C_c(\mathbb{T}^d)$ and $\phi \in C_c(\mathbb{R}^d)$. We proved in Sec. 3.1.1 that $\tilde{\nu}$ has necessarily the form $h \otimes (1 + \frac{|r|^2}{2})^{-\alpha} dr$ for some $h \in L^\infty(0, T; \mathcal{M}(\mathbb{T}^d))$. Now, set $\psi \in C_c(\mathbb{T}^d)$ and $\phi \in C_c(\mathbb{R}^d)$ in (3.19). Using the uniqueness of the limit, we obtain from (3.20)

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \bar{f}(s, x, r) \psi(x) \phi(r) \xi(s) dx dr ds = \int_0^T \langle h \otimes \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} dr, \psi \phi \rangle \xi(s) ds.$$

The arbitrariness of ψ, ϕ and ξ gives $\bar{f} = h \otimes (1 + \frac{|r|^2}{2})^{-\alpha} dr$. Finally, recall that

$$\int_{\mathbb{R}^d} \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} dr = +\infty \quad \text{if } 0 < \alpha \leq \frac{d}{2},$$

which contradicts the assumption $\bar{f} \in L^\infty(0, T; L^1(\mathbb{T} \times \mathbb{R}^d))$. $\square$

Remark 3.5. We can use a similar argument and Dunford–Pettis theorem (see e.g. Ref. 6) to show that the sequence $(f_\tau)_\tau$ is not uniformly integrable.

3.3. *Physical interpretation*

From a physical point of view, the statistics of the polymer length R have been investigated by several authors in the physical literature, see for instance Refs. 2, 24 and 35. In the coil state, the distribution of polymer end-to-end vector R is found to be power law

$$f(R) \sim R^{-1-\theta} \quad \text{for relatively large } R \tag{3.21}$$

with the exponent θ positive (so that f is normalizable). The exponent θ depends on the stretching properties of the turbulent flow: the highest is the stretching intensity; the lowest is θ . At $\theta = 0$ one has the coil-stretch transition.

To make links with the physical literature on the *coil-stretch transition*, it is worth recalling that the tail exponent θ is related to the Lyapunov exponents of the turbulent flow and polymer relaxation time. More precisely, if $\phi_t(x)$ denotes the Lagrangian flow associated to the turbulent flow, and we set

$$\mathbf{L}(q) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[|D\phi_t(x)|^q]$$

then θ satisfies $\frac{\theta}{2\beta} = \mathbf{L}(\theta)$. It is important to mention that the computation of $\mathbf{L}(\theta)$ is not easy in general and we would like to predict the exponent θ based on turbulence features. In our case, we obtain the following probability density

$$p(r) = \frac{1}{Z} \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} \sim |r|^{-2\alpha} \quad \text{for relatively large } |r|,$$

where $|r|$ denotes the length of the end-to-end vector of the polymers and α depends on some interpretable constants, see (2.7). Taking into account the space dimension, we get

$$p(|r|) \sim |r|^{-1+d-2\alpha}$$

in accordance with the prediction of Ref. 2. Now, comparing to (3.21), we get

$$\theta = 2\alpha - d = \frac{2}{\zeta C_d C_3^2} - d \quad \text{where} \quad \zeta = \frac{\beta}{\tau} = \frac{\text{relaxation time of polymer}}{\text{dominant time scale of small-scale turbulence}}. \quad (3.22)$$

Let us discuss the results of Theorems 3.1 and 3.2. We distinguish two main cases.

(i) If $\alpha > \frac{d}{2}$,^e then we have two subcases:

- The case $\alpha > \frac{d+1}{2}$: the average (mean) associated with the probability density in Theorem 3.1 is finite and the most of polymers have an equilibrium size. One can consider that polymers are in coil state in this regime.
- The case $\frac{d}{2} < \alpha \leq \frac{d+1}{2}$: The mean is not finite and larger values of polymer end-to-end vector are more probable, which means that the most of polymers have large size and the polymers are in stretched state. Moreover, at $\alpha = \frac{d+1}{2}$ the average (mean) associated with the probability density in Theorem 3.1 is not finite anymore, this can be interpreted as the criterion for the coil-stretch transition. Note also that $\frac{d}{2}$ corresponds to $\theta = 0$ from (3.22), which is the coil-stretch transition threshold following.²

(ii) The case $0 < \alpha \leq \frac{d}{2}$: In this case, we no longer have a probability density in Theorem 3.2. Moreover, we don't have *a priori* uniqueness of the limit as $\tau \rightarrow 0$. Depending on the regularity of the initial data, we obtain different limits, the first one is the trivial limit. More importantly, the second limit ν , Radon measure valued one in Theorem 3.2. The latter seems to be more natural because it corresponds to the L^1 -framework and we are dealing with Fokker–Planck-type equation (see Ref. 22). As noticed in Remark 3.3 and Sec. 3.1.1, we can define a family of explicit solutions given by

$$g(t, x) \otimes \mu_\alpha, \quad g \in L^\infty(0, T; \mathcal{M}(\mathbb{T}^d)) \text{ and } \mu_\alpha \text{ given by (3.7).}$$

Note that μ_α shows a power-law decay with the exponent α in the generalized sense of Radon measure. From (2.7), as α decreases, the turbulent flow becomes stronger. Thus, we can say that the polymers are strongly stretched in comparison with the second subcase above, where the pdf must be interpreted as a less regular object, namely as a Radon measure absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$, with density (Radon–Nikodym derivative) showing the power-law decay behavior. In addition, in the stretched state, the precise mathematics depends on the idealizations of the model. If we had introduced a

^eNote that $\int_{\mathbb{R}^d} (1 + \frac{|r|^2}{2})^{-\alpha} dr = |S^{d-1}| \int_0^{+\infty} (1 + v^2)^{-\alpha} v^{d-1} dv < \infty$ if and only if $\alpha > \frac{d}{2}$ ($|S^{d-1}|$ denotes the surface area of the unit sphere in $\mathbb{R}^d$).

^fNote that first absolute moment $\int_{\mathbb{R}^d} |r| (1 + \frac{|r|^2}{2})^{-\alpha} dr = |S^{d-1}| \int_0^{+\infty} (1 + v^2)^{-\alpha} v^d dv < \infty$ if and only if $\alpha > \frac{d+1}{2}$.

superlinear damping instead of the linear damping $-\frac{1}{\beta}R_t$, this would produce a sort of cut-off at very high lengths (e.g. Finitely Extensible Nonlinear Elastic (FENE) model, see Sec. 6), so that the behavior (3.21) would be true only in a range

$$R_{\min} \ll R \ll R_{\max}$$

and globally the function f would still be a pdf. We refer to Sec. 6 for further discussion on FENE model. In our idealization of linear damping, the stretching may overcome the damping and lead to infinite length in the asymptotic regime, which is the idealized signature of stretch state. More precisely, note that the mathematical idealization in the Hookean model allows the infinite extensibility mechanism. Recall that the term including $A_b(r)$ (see the operator $\mathcal{L}$ in (3.24)) is a consequence of stretching effect, resulting after taking the first scaling limit²² and it is responsible for the loss of compactness in L^1 if $0 < \alpha \leq \frac{d}{2}$. Indeed, in the absence of the term $A_b(r)$ we see that $\mathcal{L}$ is the Fokker–Planck operator associated with Ornstein–Uhlenbeck process and we expect a Gaussian distribution as $\tau \rightarrow 0$, even if the finite extensibility constraint is missing. On the other hand, since $A_b(r)r = |r|^2r$ and $|r|^2$ is the smallest eigenvalue of the matrix $A_b(r)$, the effective diffusion increases with $|r|$ and enhances the outward motion for large values of $|r|$. Consequently, it allows the mass to escape to infinity and, mathematically speaking, it indicates a failure of tightness of the associated probability measures and the loss of compactness in L^1 -space, see Sec. 3.2. In other words, the term including $A_b(r)$ overcomes the drift term containing α (which confines the mass) if $0 < \alpha \leq \frac{d}{2}$. To conclude, the notion of “infinite length” reflects the possibility of polymers reaching infinite length and this corresponds to the loss of compactness from a mathematical point of view. Summarizing, this phenomenon is a consequence of a combination of the stretching effect of the turbulence and the infinite extensibility possibility of the Hookean model.

In all cases, our main results justified rigorously match the physical prediction (see e.g. Ref. 2) concerning the power-law distribution of polymers embedded in turbulent flow.

3.4. The coil-stretch operator

A first step in the study of the limit as $\tau \rightarrow 0$ in (2.8) requires understanding the singular term $\frac{1}{\zeta\alpha\tau}\text{div}_r(\alpha r f_\tau + \nabla_r f_\tau + \frac{1}{2}A_b(r)\nabla_r f_\tau)$. This will be the objective of this subsection. Note that $P_\alpha(r) = (1 + \frac{|r|^2}{2})^{-\alpha g}$ satisfies

$$\alpha r P_\alpha(r) + \nabla_r P_\alpha(r) + \frac{1}{2}A_b(r)\nabla_r P_\alpha(r) = 0, \quad (3.23)$$

²²In the following, we use p_α for the renormalized Cauchy-type function and P_α for the non-renormalized one.

where we denoted $A_b(r) = (b+1)|r|^2I - br \otimes r$. Thus

$$\begin{aligned} & \alpha r g + \nabla_r g + \frac{1}{2} A_b(r) \nabla_r g \\ &= \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} \left[\nabla_r \left\{ g \left(1 + \frac{|r|^2}{2}\right)^\alpha \right\} + \frac{1}{2} A_b(r) \nabla_r \left\{ g \left(1 + \frac{|r|^2}{2}\right)^\alpha \right\} \right] \\ &= \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} \left[\nabla_r g_\alpha + \frac{1}{2} A_b(r) \nabla_r g_\alpha \right], \end{aligned}$$

where (to simplify the notation) we used $g_\alpha = g(1 + \frac{|r|^2}{2})^\alpha$. Thus, we can guess formally that the penalization $\tau \rightarrow 0$ leads to

$$f_\tau(t, x, r) \simeq \rho(t, x) p_\alpha(r),$$

where $p_\alpha(r) = \frac{1}{Z}(1 + \frac{|r|^2}{2})^{-\alpha}$ (Z is a normalizing factor) is generalized Cauchy density with parameter $\alpha > \frac{d}{2}$. The aim of this subsection is to study some spectral properties of the following operator:

$$\mathcal{L}(f) = \operatorname{div}_r \left(\alpha r f + \nabla_r f + \frac{1}{2} ((b+1)|r|^2I - br \otimes r) \nabla_r f \right), \quad b \in \{1, 2\}. \quad (3.24)$$

Let $d \in \{2, 3\}$ and consider the following space:

$$\mathbf{X}_\alpha = \left\{ g : \mathbb{R}^d \rightarrow \mathbb{R} : \int_{\mathbb{R}^d} g^2(r) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr < \infty \right\}, \quad \alpha > 0.$$

Note that $\mathbf{X}_\alpha$ is a Hilbert space endowed with its natural inner product, namely

$$(f, g)_{\mathbf{X}_\alpha} = \int_{\mathbb{R}^d} f(r) g(r) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr.$$

We introduce the following unbounded operator $\mathcal{L}$ acting on $\mathbf{X}_\alpha$ as follows:

$$\mathcal{L} : \mathbf{X}_\alpha \rightarrow \mathbf{X}_\alpha,$$

$$g \mapsto \mathcal{L}(g) = \operatorname{div}_r \left(\left(1 + \frac{|r|^2}{2}\right)^{-\alpha} \left[\nabla_r g_\alpha + \frac{1}{2} A_b(r) \nabla_r g_\alpha \right] \right),$$

with domain

$$\mathcal{D}(\mathcal{L}) = \left\{ g \in \mathbf{X}_\alpha, \nabla_r \cdot \left(\left(1 + \frac{|r|^2}{2}\right)^{-\alpha} \left[\left(I + \frac{1}{2} A_b(r) \right) \nabla_r g_\alpha \right] \right) \in \mathbf{X}_\alpha \right\}.$$

It is clear that $A_b(r)^* = A_b(r)$. Let us introduce the following space $\mathbf{Z}_\alpha$, which serves to prove the closedness of $-\mathcal{L}$.

$$\mathbf{Z}_\alpha = \left\{ g \in \mathbf{X}_\alpha : \int_{\mathbb{R}^d} \left(1 + \frac{|r|^2}{2}\right)^{-\alpha+1} \left| \nabla_r f \left(1 + \frac{|r|^2}{2}\right)^\alpha \right|^2 dr < +\infty \right\}.$$

It is not difficult to check that $\mathbf{Z}_\alpha$ is a Hilbert space equipped by its natural inner product. Let us present the main spectral properties of the operator $\mathcal{L}$.

Proposition 3.5. Let $\alpha > \frac{d}{2}$, the operator $-\mathcal{L}$ is self-adjoint on $\mathbf{X}_\alpha$ and satisfies:

- (1) $-\mathcal{L}$ is positive and $\text{Ker}(\mathcal{L}) = \{cP_\alpha, \quad c \in \mathbb{R}\}$,
- (2) $R(\mathcal{L}) = \{g \in \mathbf{X}_\alpha : \int_{\mathbb{R}^d} g(r)dr = 0\}$,
- (3) For all $g \in R(\mathcal{L})$ there exists $f \in \mathcal{D}(\mathcal{L})$ such that $\mathcal{L}(f) = g$. The solution is unique under the solvability condition $\int_{\mathbb{R}^d} f(r)dr = 0$.

Proof. $-\mathcal{L}$ is symmetric on $\mathbf{X}_\alpha$. Indeed, let $f, g \in \mathcal{D}(\mathcal{L})$. Since $A_b(r)^t = A_b(r)$, we get

$$\begin{aligned} (\mathcal{L}(f), g)_{\mathbf{X}_\alpha} &= - \int_{\mathbb{R}^d} \left(\left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \left[\nabla_r f_\alpha + \frac{1}{2} A_b(r) \nabla_r f_\alpha \right] \right) \cdot \nabla_r g_\alpha dr \\ &= - \int_{\mathbb{R}^d} \nabla_r f_\alpha \cdot \left(\left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \left[\nabla_r g_\alpha + \frac{1}{2} A_b(r) \nabla_r g_\alpha \right] \right) dr \\ &= (f, \mathcal{L}(g))_{\mathbf{X}_\alpha}. \end{aligned}$$

In particular, since $A_b(r) \nabla_r f_\alpha \cdot \nabla_r f_\alpha \geq |r|^2 |\nabla_r f_\alpha|^2$, we obtain

$$\begin{aligned} (-\mathcal{L}(f), f)_{\mathbf{X}_\alpha} &= \int_{\mathbb{R}^d} \left(\left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \left[\nabla_r f_\alpha + \frac{1}{2} A_b(r) \nabla_r f_\alpha \right] \right) \cdot \nabla_r f_\alpha dr \\ &= \int_{\mathbb{R}^d} \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \left[|\nabla_r f_\alpha|^2 + \frac{1}{2} A_b(r) \nabla_r f_\alpha \cdot \nabla_r f_\alpha \right] dr \\ &\geq \int_{\mathbb{R}^d} \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \left[\left(1 + \frac{|r|^2}{2} \right) |\nabla_r f_\alpha|^2 \right] dr \\ &= \int_{\mathbb{R}^d} \left(1 + \frac{|r|^2}{2} \right)^{-\alpha+1} |\nabla_r f_\alpha|^2 dr \geq 0. \end{aligned} \quad (3.25)$$

Thus $-\mathcal{L}$ is symmetric and positive. On the other hand, note that $\mathcal{D}(\mathcal{L})$ is dense in $\mathbf{X}_\alpha$ since $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{L})$. By using Lax–Milgram theorem, we obtain the existence and uniqueness of $h \in \mathbf{Z}_\alpha$ for any $g \in \mathbf{X}_\alpha$ such that

$$\begin{aligned} \int_{\mathbb{R}^d} \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \left[\nabla_r h_\alpha + \frac{1}{2} A_b(r) \nabla_r h_\alpha \right] \cdot \nabla_r \phi_\alpha dr + \int_{\mathbb{R}^d} h_\alpha \phi \left(1 + \frac{|r|^2}{2} \right)^\alpha dr \\ = \int_{\mathbb{R}^d} g_\alpha \phi \left(1 + \frac{|r|^2}{2} \right)^\alpha dr, \quad \forall \phi \in \mathbf{Z}_\alpha. \end{aligned}$$

In other words, $(I - \mathcal{L})h = g$. Let us show that $I - \mathcal{L}$ is closed, let $(h_n, (I - \mathcal{L})h_n)$ converges to (h, g) in $\mathbf{X}_\alpha \times \mathbf{X}_\alpha$. We need to prove that $h \in \mathcal{D}(\mathcal{L})$ and $(I - \mathcal{L})h = g$. By using Lax–Milgram theorem, there exists a unique $\psi \in \mathbf{Z}_\alpha$ such that $(I - \mathcal{L})\psi = g$ hence $\psi \in \mathcal{D}(\mathcal{L})$. Therefore $(I - \mathcal{L})(\psi - h_n)$ converges to 0 in $\mathbf{X}_\alpha$ which ensures that $\psi - h_n$ goes to 0 in $\mathbf{X}_\alpha$ and $\psi = h \in \mathbf{X}_\alpha$. We deduce that $-\mathcal{L}$ is also closed and -1 belongs to the resolvent of $-\mathcal{L}$.

- (1) From (3.25), we get $\text{Ker}(\mathcal{L}) = \{cP_\alpha, c \in \mathbb{R}\}$ since $f_\alpha = f(1 + \frac{|r|^2}{2})^\alpha$.

(2) Since $-\mathcal{L}$ is closed symmetric and -1 belongs to the resolvent of $-\mathcal{L}$ then it is self-adjoint thanks to Ref. 38. By using Ref. 6, we have

$$R(\mathcal{L}) = \text{Ker}(\mathcal{L})^\perp := \{g \in \mathbf{X}_\alpha : (g, P_\alpha)\mathbf{x}_\alpha = 0\} = \left\{g \in \mathbf{X}_\alpha : \int_{\mathbb{R}^d} g(r)dr = 0\right\}.$$

The point (3) holds obviously. $\square$

Remark 3.6. Note that in the case $0 < \alpha \leq d/2$, we still have that $-\mathcal{L}$ is positive. On the other hand, $P_\alpha(r) = (1 + \frac{|r|^2}{2})^{-\alpha} \notin \mathbf{X}_\alpha$ which implies that $\text{Ker}(\mathcal{L}) = \{0\}$.

4. Uniform Estimates and Continuity Equation

4.1. Uniform estimates with respect to τ

Let $\tau > 0$ and $\alpha > 0$, the aim of this section is to prove some uniform estimates with respect to τ . First, we improve the regularity with respect to the r -variable comparing to the one proved in Ref. 22 with a bound depending on τ (see Lemma 4.1). Then, we combine Lemma 4.1 and (3.23) to obtain uniform estimates with respect to τ .

To simplify the notation, let's introduce the following spaces:

$$\mathcal{Y} = \{\varphi \in H : \nabla_r \varphi \in H, \nabla_x \varphi \in L^2(\mathbb{T}^d \times \mathbb{R}^d)\},$$

$$H_\alpha = \left\{g : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R} : \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g^2(x, r) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx < \infty\right\}, \quad \alpha > 0.$$

By approximation arguments and Theorem 2.1, we can deduce that (2.8) is satisfied in $\mathcal{Y}$ -sense, namely there exists a unique quasi-regular weak solution f_τ such that

$$f_\tau \in L^2_{w-*}(\Omega; L^\infty([0, T]; H)), \quad \nabla_r f_\tau \in L^2(\Omega; L^2([0, T]; H)) \quad (4.1)$$

and $f_\tau(t) := f_\tau(t, x, r)$ satisfies P-a.s. for any $t \in [0, T]$:

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t) \phi dr dx - \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_0^\tau \phi dr dx \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(s) (u_L(s) \cdot \nabla_x \phi + (\nabla u_L(s)r) \cdot \nabla_r \phi) dr dx ds \\ & \quad - \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha f_\tau(s) r \cdot \nabla_r \phi + \nabla_r f_\tau(s) \cdot \nabla_r \phi \\ & \quad + \frac{1}{2} A_b(r) \nabla_r f_\tau(s) \cdot \nabla_r \phi dr dx ds, \end{aligned} \quad (4.2)$$

for any $\phi \in \mathcal{Y}$. In order to derive some necessary estimates, we introduce a regularization kernel. More precisely, let $\delta > 0$ and Θ be a smooth radially symmetric density of a probability measure on $\mathbb{R}^d$, compactly supported in $B(0, 1)$ and define the approximation of identity for the convolution on $\mathbb{R}^d$ as $\Theta_\delta(y) = \frac{1}{\delta^d} \Theta(\frac{y}{\delta})$. Since we are working on $\mathbb{T}^d \times \mathbb{R}^d$, we recall that for any integrable function h on $\mathbb{T}^d, h$

can be extended periodically to a locally integrable function on the whole $\mathbb{R}^d$ and convolution $\Theta_\delta * h$ is meaningful and $[h]_\delta := \Theta_\delta * h$ is still a C^∞ -periodic function.

Now, we present the following lemma which serves for the proof of Proposition 4.1.

Lemma 4.1. *There exists $\mathbf{C} = \frac{d/2+10\alpha}{\zeta\tau} + 2\alpha\|\nabla_x u_L\|_\infty$ such that*

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \\ & \quad + \frac{1}{2\zeta\tau\alpha} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r f_\tau(s)|^2 \left(1 + \frac{|r|^2}{2}\right)^{\alpha+1} dr dx ds \\ & \leq e^{\mathbf{C}T} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_0^\tau)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \leq \Lambda e^{\mathbf{C}T}. \end{aligned} \quad (4.3)$$

Although the proof of Lemma 4.1 follows by classical arguments, we give a proof in Appendix A for the convenience of the reader.

Remark 4.1. (L^2 -estimate) It is worth mentioning that we can prove the L^2 estimates using arguments similar to the one used in Appendix A but the bound depends on $\frac{1}{\tau}$. For the convenience of the reader, let us give just the formal argument. The estimate is based on the analysis of the following:

$$\begin{aligned} & \frac{1}{\zeta\alpha\tau} \left\langle \operatorname{div}_r \left(\alpha r f_\tau + \nabla_r f_\tau + \frac{1}{2} A_b(r) \nabla_r f_\tau \right), f_\tau \right\rangle \\ & = -\frac{1}{\zeta\alpha\tau} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\alpha r f_\tau + \nabla_r f_\tau + \frac{1}{2} A_b(r) \nabla_r f_\tau \right) \cdot \nabla_r f_\tau dr dx \\ & = -\frac{1}{\zeta\tau} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} r f_\tau \cdot \nabla_r f_\tau dr dx \\ & \quad - \frac{1}{\zeta\alpha\tau} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \overbrace{\left(|\nabla_r f_\tau|^2 + \frac{1}{2} A_b(r) \nabla_r f_\tau \cdot \nabla_r f_\tau \right)}^{\geq 0} dr dx \\ & = \frac{d}{2\zeta\tau} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2 dr dx - \frac{1}{\zeta\alpha\tau} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \overbrace{\left(|\nabla_r f_\tau|^2 + \frac{1}{2} A_b(r) \nabla_r f_\tau \cdot \nabla_r f_\tau \right)}^{\geq 0} dr dx. \end{aligned}$$

This is why we consider the space H_α to obtain uniform bound with respect to τ .

Proposition 4.1. *Let $\alpha > 0$ and assume that (H_1) holds. There exists $\mathbf{K} = 2\alpha\|\nabla_x u_L\|_\infty$ such that*

$$\begin{aligned} & \sup_{t \in [0, T]} \|f_\tau(t)\|_{H_\alpha}^2 + \frac{1}{\zeta\alpha\tau} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r (f_\tau(s) P_{-\alpha})|^2 \left(1 + \frac{|r|^2}{2}\right)^{1-\alpha} dr dx ds \\ & \leq e^{\mathbf{K}T} \Lambda := \overline{\mathbf{K}}. \end{aligned} \quad (4.4)$$

Proof. Let $t \in [0, T]$ and $\phi \in \mathcal{Y}$, if we denote $X = (x, r) \in \mathbb{T}^d \times \mathbb{R}^d$ and $\Theta_\delta(X) = \Theta_\delta(x)$, then $\phi_\delta := \Theta_\delta * \phi$ is an appropriate test function in (4.2), a classical computation yields

$$\begin{aligned}
 & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [f_\tau(t)]_\delta \phi dr dx - \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [f_0^\tau]_\delta \phi dr dx \\
 &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [u_L(s) \cdot \nabla_x f_\tau(s)]_\delta \phi + [(\nabla u_L(s)r) \cdot \nabla_r f_\tau(s)]_\delta \phi dr dx ds \\
 &+ \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha [\operatorname{div}_r(f_\tau(s)r)]_\delta \phi dr dx ds \\
 &+ \frac{1}{\zeta \alpha \tau} \int_0^t \langle [\Delta_r f_\tau(s)]_\delta, \phi \rangle + \frac{1}{2} \langle [\operatorname{div}_r(A_b(r) \nabla_r f_\tau(s))]_\delta, \phi \rangle ds, \quad \forall \phi \in \mathcal{Y},
 \end{aligned} \tag{4.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{Y}'$ and $\mathcal{Y}$. Let us introduce a sequence of cut-off function $\mathcal{T}_M(s) = \max(-M, \min(s, M))$, $M \in \mathbb{N}$ and define

$$T_M(r) := \mathcal{T}_M \left(\left(1 + \frac{|r|^2}{2} \right)^\alpha \right), \quad r \in \mathbb{R}^d, \quad d = 2, 3.$$

Recall that $|T_M(r)| \leq (1 + \frac{|r|^2}{2})^\alpha$ and $T_M(r) \uparrow (1 + \frac{|r|^2}{2})^\alpha$ as $M \uparrow +\infty$. Let $\delta > 0$, $M \in \mathbb{N}$, notice that $\phi_{\delta, M}(x, r) = T_M(r)[f_\tau(\cdot)]_\delta(x, r)$ is an appropriate test function in (4.5), therefore we obtain

$$\begin{aligned}
 & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [f_\tau(t)]_\delta^2 T_M dr dx - \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [f_0^\tau]_\delta^2 T_M dr dx \\
 &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [u_L(s) \cdot \nabla_x f_\tau(s)]_\delta T_M [f_\tau(s)]_\delta dr dx ds \\
 &- \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [(\nabla u_L(s)r) \cdot \nabla_r f_\tau(s)]_\delta T_M [f_\tau(s)]_\delta dr dx ds \\
 &+ \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha [\operatorname{div}_r(f_\tau(s)r)]_\delta T_M [f_\tau(s)]_\delta dr dx ds \\
 &+ \frac{1}{\zeta \alpha \tau} \left(\int_0^t \langle [\Delta_r f_\tau(s)]_\delta, T_M [f_\tau(s)]_\delta \rangle ds \right. \\
 &\left. + \int_0^t \left\langle \frac{1}{2} [\operatorname{div}_r(A_b(r) \nabla_r f_\tau(s))]_\delta, T_M [f_\tau(s)]_\delta \right\rangle ds \right).
 \end{aligned} \tag{4.6}$$

Now, we use (4.3) to prove some uniform estimates independent of τ . For the analysis of the term independent of τ , see the first part of the proof of Lemma 4.1 in Appendix A. We will discuss only the τ dependent terms.

4.1.1. *Uniform estimates with respect to τ*

Note that

$$\begin{aligned}
 & - \left(\int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha [\operatorname{div}_r(f_\tau(s)r)]_\delta T_M[f_\tau(s)]_\delta dr dx ds + \int_0^t \langle [\Delta_r f_\tau(s)]_\delta, T_M[f_\tau(s)]_\delta \rangle ds \right) \\
 & \quad - \int_0^t \left\langle \frac{1}{2} [\operatorname{div}_r(A_b(r)\nabla_r f_\tau(s))]_\delta, T_M[f_\tau(s)]_\delta \right\rangle ds \\
 & = \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\alpha r[f_\tau(s)]_\delta + \nabla_r[f_\tau(s)]_\delta + \frac{1}{2} A_b(r)\nabla_r[f_\tau(s)]_\delta \right) \\
 & \quad \times \nabla_r(T_M[f_\tau(s)]_\delta) dr dx ds.
 \end{aligned}$$

By using (3.23), we obtain

$$\begin{aligned}
 & \alpha r[f_\tau(s)]_\delta + \nabla_r[f_\tau(s)]_\delta + \frac{1}{2} A_b(r)\nabla_r[f_\tau(s)]_\delta \\
 & = \nabla_r \left([f_\tau(s)]_\delta \left(1 + \frac{|r|^2}{2} \right)^\alpha \right) \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \\
 & \quad + \frac{1}{2} A_b(r)\nabla_r \left([f_\tau(s)]_\delta \left(1 + \frac{|r|^2}{2} \right)^\alpha \right) \left(1 + \frac{|r|^2}{2} \right)^{-\alpha}.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 & - \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\alpha r[f_\tau(s)]_\delta + \nabla_r[f_\tau(s)]_\delta + \frac{1}{2} A_b(r)\nabla_r[f_\tau(s)]_\delta \right) \\
 & \quad \times \nabla_r(T_M[f_\tau(s)]_\delta) dr dx ds \\
 & = - \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \nabla_r \left([f_\tau(s)]_\delta \left(1 + \frac{|r|^2}{2} \right)^\alpha \right) \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \\
 & \quad \times \nabla_r(T_M[f_\tau(s)]_\delta) dr dx ds \\
 & \quad - \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{1}{2} A_b(r)\nabla_r \left([f_\tau(s)]_\delta \left(1 + \frac{|r|^2}{2} \right)^\alpha \right) \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \\
 & \quad \times \nabla_r(T_M[f_\tau(s)]_\delta) dr dx ds \\
 & := I_\delta^M.
 \end{aligned}$$

By using (4.1), it is possible to let $\delta \rightarrow 0$ in the last equality. Moreover, note that

$$\begin{aligned}
 \nabla_r(T_M f_\tau) & = \nabla_r \left(\left(1 + \frac{|r|^2}{2} \right)^\alpha f_\tau \right) T_M \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \\
 & \quad + \nabla_r \left(T_M \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \right) \left(1 + \frac{|r|^2}{2} \right)^\alpha f_\tau
 \end{aligned}$$

thus

$$\begin{aligned}
 \lim_{\delta \rightarrow 0} I_{\delta}^M &= -\frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(f_{\tau}(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha} \right) \right|^2 \\
 &\quad \times \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} T_M \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} dr dx ds - \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(I + \frac{1}{2} A_b(r) \right) \\
 &\quad \nabla_r \left(f_{\tau}(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha} \right) \cdot \nabla_r \left(T_M \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \right) f_{\tau}(s) dr dx ds \\
 &\quad - \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\frac{1}{2} A_b(r) \nabla_r \left(f_{\tau}(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha} \right) \right) \\
 &\quad \cdot \left(\nabla_r \left(f_{\tau}(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha} \right) T_M \left(1 + \frac{|r|^2}{2} \right)^{-2\alpha} \right) dr dx ds.
 \end{aligned}$$

Concerning the first and the third terms, we have

$$\begin{aligned}
 J^M &:= \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(f_{\tau}(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha} \right) \right|^2 \\
 &\quad \times \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} T_M \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} dr dx ds \\
 &\quad + \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\frac{1}{2} A_b(r) \nabla_r \left(f_{\tau}(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha} \right) \right) \\
 &\quad \times \left(\nabla_r \left(f_{\tau}(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha} \right) T_M \left(1 + \frac{|r|^2}{2} \right)^{-2\alpha} \right) dr dx ds \\
 &\geq \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(f_{\tau}(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha} \right) \right|^2 \\
 &\quad \times \left(1 + \frac{|r|^2}{2} \right)^{1-\alpha} T_M \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} dr dx ds,
 \end{aligned}$$

hence, by using Fatou's lemma (see e.g. Ref. 6) we get

$$\liminf_{M \rightarrow +\infty} J^M \geq \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(f_{\tau}(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha} \right) \right|^2 \left(1 + \frac{|r|^2}{2} \right)^{1-\alpha} dr dx ds.$$

Finally, we use (4.3) to show that $\lim_{M \rightarrow +\infty} |R^M| = 0$ where

$$\begin{aligned}
 R^M &= -\frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\left(I + \frac{1}{2} A_b(r) \right) \nabla_r \left(f_{\tau}(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha} \right) \right) \\
 &\quad \times \left(\nabla_r \left(T_M \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \right) f_{\tau}(s) \right) dr dx ds.
 \end{aligned}$$

Note that $T_M(r)(1 + \frac{|r|^2}{2})^{-\alpha} = \tilde{T}_M(r) \circ (1 + \frac{|r|^2}{2})^\alpha$, where $\tilde{T}_M(s) = \frac{T_M(s)}{s}$, $s > 0$ thus

$$\nabla_r \left(T_M(r) \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} \right) = -\alpha r M \left(1 + \frac{|r|^2}{2} \right)^{-\alpha-1} 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) \quad (4.7)$$

in a weak sense. On the other hand, we have

$$\begin{aligned} & \left| -\alpha r M \left(1 + \frac{|r|^2}{2} \right)^{-\alpha-1} 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) \right| \\ & \leq 2\alpha M \left(1 + \frac{|r|^2}{2} \right)^{-\alpha-\frac{1}{2}} 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) \\ & \leq 2\alpha \left(1 + \frac{|r|^2}{2} \right)^{-\frac{1}{2}} 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) \end{aligned}$$

and consequently

$$\begin{aligned} |R^M| & \leq \frac{2}{\zeta\tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \left(I + \frac{1}{2} A_b(r) \right) \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha \right) \right| \\ & \quad \times f_\tau(s) \left(1 + \frac{|r|^2}{2} \right)^{-\frac{1}{2}} 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) dr dx ds. \end{aligned}$$

Since

$$\begin{aligned} \lim_{M \rightarrow +\infty} & \left| \left(I + \frac{1}{2} A_b(r) \right) \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha \right) \right| f_\tau(s) \left(1 + \frac{|r|^2}{2} \right)^{-\frac{1}{2}} 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) \\ & = 0 \quad \text{a.e. } (x, r, s) \in \mathbb{T}^d \times \mathbb{R}^d \times (0, T). \end{aligned}$$

By using (4.3), we obtain the following dominated function:

$$\begin{aligned} \mathbf{y} & = \left| \left(I + \frac{1}{2} A_b(r) \right) \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha \right) \right| f_\tau(s) \left(1 + \frac{|r|^2}{2} \right)^{-\frac{1}{2}} \\ & \in L^1(\mathbb{R}^d \times \mathbb{T}^d \times (0, T)), \end{aligned}$$

Lebesgue dominated convergence theorem (see e.g. Ref. 6) ensures that

$$\lim_{M \rightarrow +\infty} |R^M| = 0.$$

Thus

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t)^2 \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx \\ & \quad + \frac{1}{\zeta\alpha\tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha \right) \right|^2 \left(1 + \frac{|r|^2}{2} \right)^{1-\alpha} dr dx ds \end{aligned}$$

$$\begin{aligned} &\leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_0^\tau)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \\ &\quad + 2\alpha \|\nabla_x u_L\|_\infty \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds. \end{aligned}$$

Set $\mathbf{K} = 2\alpha \|\nabla_x u_L\|_\infty$ and apply Gronwall inequality to deduce

$$\begin{aligned} &\sup_{t \in [0, T]} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \\ &\quad + \frac{1}{\zeta \alpha \tau} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha \right) \right|^2 \left(1 + \frac{|r|^2}{2}\right)^{1-\alpha} dr dx ds \\ &\leq e^{\mathbf{K}T} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_0^\tau)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx. \end{aligned} \quad \square$$

4.2. The continuity equation

Let $\alpha > \frac{d}{2}$ and $\tau > 0$, after integration “formally” with respect to the variable r in (2.8), set

$$\rho_\tau = \rho_\tau(t, x) := \int_{\mathbb{R}^d} f_\tau(t, x, r) dr$$

and consider the following continuity equation:

$$\begin{cases} \partial_t \rho_\tau + u_L \cdot \nabla_x \rho_\tau = 0, \\ \rho_\tau(x)|_{t=0} = \rho_0^\tau = \int_{\mathbb{R}^d} f_0^\tau(x, r) dr. \end{cases} \quad (4.8)$$

Under the assumption H_1 , we have $(\rho_0^\tau)_\tau$ is bounded in $L^2(\mathbb{T}^d)$.

Proposition 4.2. *Assume that H_1 holds, there exists a unique solution*

$$\rho_\tau \in C([0, T]; L^2(\mathbb{T}^d))$$

to (4.8) in the following sense:

$$\int_{\mathbb{T}^d} \rho_\tau(t) \varphi dx - \int_0^t \int_{\mathbb{T}^d} \rho_\tau(s) u_L \cdot \nabla \varphi dx ds = \int_{\mathbb{T}^d} \rho_0^\tau \varphi dx, \quad \forall \varphi \in H^1(\mathbb{T}^d). \quad (4.9)$$

Moreover, $(\rho_\tau)_\tau$ is bounded in $L^\infty(0, T; L^2(\mathbb{T}^d))$.

Proof. The existence follows by a classical arguments, for example, use the Galerkin approximation and prove that the approximation sequence is a Cauchy sequence in $C([0, T]; L^2(\mathbb{T}^d))$ (see also Ref. 11 for a different proof). The uniqueness and the boundedness are consequence of Lemma 4.2. $\square$

Lemma 4.2. (Stability) *Let ρ_1 and ρ_2 be two solutions to (4.8) in the sense (4.9), with initial conditions ρ_0^1 and ρ_0^2 , respectively. Then*

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^d} |(\rho_1 - \rho_2)(t)|^2 dx \leq \int_{\mathbb{T}^d} |\rho_0^1 - \rho_0^2|^2 dx. \quad (4.10)$$

Proof. Let $\varphi \in H^1(\mathbb{T}^d)$ and $t \in [0, T]$. Note that $\rho_1 - \rho_2$ satisfies

$$\int_{\mathbb{T}^d} (\rho_1 - \rho_2)(t) \varphi dx - \int_0^t \int_{\mathbb{T}^d} (\rho_1 - \rho_2)(s) u_L \cdot \nabla \varphi dx ds = \int_{\mathbb{T}^d} (\rho_0^1 - \rho_0^2) \varphi dx.$$

Recall that $\varphi_\delta := \Theta_\delta * \varphi^h$ is an appropriate test function in last equation, hence

$$\int_{\mathbb{T}^d} [(\rho_1 - \rho_2)]_\delta(t) \varphi dx + \int_0^t \int_{\mathbb{T}^d} [u_L \cdot \nabla (\rho_1 - \rho_2)(s)]_\delta \varphi dx ds = \int_{\mathbb{T}^d} [(\rho_0^1 - \rho_0^2)]_\delta \varphi dx.$$

Set $\varphi := \Theta_\delta * (\rho_1 - \rho_2) = [\rho_1 - \rho_2]_\delta$, which is an appropriate test function in the last equality. We get

$$\begin{aligned} & \int_{\mathbb{T}^d} [(\rho_1 - \rho_2)]_\delta^2(t) dx + \int_0^t \int_{\mathbb{T}^d} [u_L \cdot \nabla (\rho_1 - \rho_2)(s)]_\delta [\rho_1 - \rho_2]_\delta dx ds \\ &= \int_{\mathbb{T}^d} [(\rho_0^1 - \rho_0^2)]_\delta^2 dx. \end{aligned} \quad (4.11)$$

Now, we are in position to use commutator estimates (Lemma C.1). Namely, notice that

$$\begin{aligned} [u_L \cdot \nabla (\rho_1 - \rho_2)]_\delta [\rho_1 - \rho_2]_\delta &= ([u_L \cdot \nabla (\rho_1 - \rho_2)]_\delta - u_L \cdot \nabla [(\rho_1 - \rho_2)]_\delta) [(\rho_1 - \rho_2)(s)]_\delta \\ &\quad + u_L \cdot \nabla [(\rho_1 - \rho_2)]_\delta [\rho_1 - \rho_2]_\delta. \end{aligned}$$

Recall that $\rho_1 - \rho_2 \in C([0, T]; L^2(\mathbb{T}^d))$ and by applying Lemma C.1, we get

$$\begin{aligned} ([u_L \cdot \nabla (\rho_1 - \rho_2)]_\delta - u_L \cdot \nabla [(\rho_1 - \rho_2)]_\delta) [\rho_1 - \rho_2]_\delta &\rightarrow 0 \quad \text{in } L^2(\mathbb{T}^d \times (0, T)) \\ &\text{as } \delta \rightarrow 0. \end{aligned}$$

On the other hand, we have

$$\int_0^t \int_{\mathbb{T}^d} u_L \cdot \nabla [(\rho_1 - \rho_2)]_\delta [\rho_1 - \rho_2]_\delta dx ds = \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \operatorname{div}(u_L [(\rho_1 - \rho_2)]_\delta^2) dx ds = 0.$$

Now, we pass to the limit in (4.11) as $\delta \rightarrow 0$ to conclude the proof. $\square$

Lemma 4.3. *Assume that H_1 and H_2 hold. Then, there exists a unique solution ρ in the sense of (4.9) to*

$$\begin{cases} \partial_t \rho + u_L \cdot \nabla_x \rho = 0, \\ \rho(x)|_{t=0} = \rho_0 = \int_{\mathbb{R}^d} f_0(x, r) dr, \end{cases} \quad (4.12)$$

and the sequence $(\rho_\tau)_\tau$ converges to ρ in $C([0, T], L^2(\mathbb{T}^d))$ as $\tau \rightarrow 0$.

^hRecall that Θ_δ is a symmetric regularization kernel.

Proof. Recall that $\rho_0 = \int_{\mathbb{R}^d} f_0(x, r) dr \in L^2(\mathbb{T}^d)$, the existence and uniqueness follow like Proposition 4.2. On the other hand, we get $\|\rho_0^\tau - \rho_0\|_{L^2} \rightarrow 0$. Indeed, notice that

$$\begin{aligned} \|\rho_0^\tau - \rho_0\|_{L^2}^2 &= \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}^d} (f_0^\tau(x, r) - f_0(x, r)) dr \right)^2 dx \\ &\leq \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}^d} |f_0^\tau(x, r) - f_0(x, r)| dr \right)^2 dx \\ &\leq \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}^d} |f_0^\tau(x, r) - f_0(x, r)| dr \right)^2 dx \\ &\leq \int_{\mathbb{R}^d} \left(1 + \frac{|r|^2}{2} \right)^{-\alpha} dr \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_0^\tau(x, r) - f_0(x, r)|^2 \\ &\quad \times \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx \rightarrow 0 \quad \text{as } \delta \downarrow 0, \end{aligned}$$

thanks to H_2 and $\int_{\mathbb{R}^d} (1 + \frac{|r|^2}{2})^{-\alpha} dr < +\infty$ since $\alpha > \frac{d}{2}$. Thus, the convergence is a consequence of Lemma 4.2 by setting $\rho_1 = \rho_\tau$ and $\rho_2 = \rho$. $\square$

5. Convergence Result as $\tau \rightarrow 0$

5.1. The case $\alpha > \frac{d}{2}$

Let $\alpha > \frac{d}{2}$, we prove the following result.

Proposition 5.1. *Let $(f_\tau)_\tau$ be the unique quasi-regular weak solution to (4.2) for any $\tau > 0$. Assume that assumptions H_1 and H_2 are satisfied then f_τ converges to $\rho \otimes p_\alpha$ in $L^2(0, T; H_\alpha)$ as $\tau \rightarrow 0$. More precisely, there exists $\mathbf{C}_\alpha > 0$ (independent of τ) such that the following inequality holds:*

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_\tau - \rho \otimes p_\alpha)^2(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx ds \leq \mathbf{C}_\alpha \tau + \frac{1}{Z} \int_0^T \int_{\mathbb{T}^d} |\rho_\tau - \rho|^2 dx ds,$$

where ρ is the unique solution to (4.12) and $p_\alpha(r) = \frac{1}{Z} (1 + \frac{|r|^2}{2})^{-\alpha}$ is the Cauchy probability density on $\mathbb{R}^d$.

Proof. Let $t \in [0, T]$ and set $f := f(t, x, r) = \rho(t, x) \otimes p_\alpha(r)$, define $g_\tau = f_\tau - f$ and $g_0^\tau = f_0^\tau - \rho_0 \otimes p_\alpha$. By using (3.23) and (4.12) we get for any $\phi \in \mathcal{Y}$:

$$\begin{aligned} &\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g_\tau(t) \phi dr dx - \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g_0^\tau \phi dr dx \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g_\tau(s) u_L(s) \cdot \nabla_x \phi dr dx ds + \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(s) (\nabla u_L(s) r) \cdot \nabla_r \phi dr dx ds \\ &\quad - \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha g_\tau(s) r \cdot \nabla_r \phi + \nabla_r g_\tau(s) \cdot \nabla_r \phi + \frac{1}{2} A_b(r) \nabla_r g_\tau(s) \\ &\quad \times \nabla_r \phi dr dx ds. \end{aligned} \tag{5.1}$$

Let $\phi \in \mathcal{Y}$ and $\delta > 0$, the arguments already detailed ensure that

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [g_\tau(t)]_\delta \phi dr dx - \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [g_0^\tau]_\delta \phi dr dx \\ &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [u_L(s) \cdot \nabla_x g_\tau(s)]_\delta \phi + [(\nabla u_L(s)r) \cdot \nabla_r f_\tau(s)]_\delta \phi dr dx ds \\ & \quad + \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha [\operatorname{div}_r(g_\tau(s)r)]_\delta \phi dr dx ds \\ & \quad + \frac{1}{\zeta \alpha \tau} \int_0^t \langle [\Delta_r g_\tau(s)]_\delta, \phi \rangle + \frac{1}{2} \langle [\operatorname{div}_r(A_b(r) \nabla_r g_\tau(s))]_\delta, \phi \rangle ds, \end{aligned} \quad (5.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{Y}'$ and $\mathcal{Y}$. Consider

$$\phi_{\delta, M}(x, r) = T_M(r)[g_\tau(\cdot)]_\delta(x, r) \in \mathcal{Y},$$

which is an appropriate test function in (5.2). Thus, we obtain

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [g_\tau(t)]_\delta^2 T_M dr dx - \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [g_0^\tau]_\delta^2 T_M dr dx \\ &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [u_L(s) \cdot \nabla_x g_\tau(s)]_\delta T_M[g_\tau(s)]_\delta dr dx ds \\ & \quad - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [(\nabla u_L(s)r) \cdot \nabla_r f_\tau(s)]_\delta T_M[g_\tau(s)]_\delta dr dx ds \\ & \quad + \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha [\operatorname{div}_r(g_\tau(s)r)]_\delta T_M[g_\tau(s)]_\delta dr dx ds \\ & \quad + \frac{1}{\zeta \alpha \tau} \int_0^t \langle [\Delta_r g_\tau(s)]_\delta, T_M[g_\tau(s)]_\delta \rangle ds \\ & \quad + \frac{1}{\zeta \alpha \tau} \int_0^t \left\langle \frac{1}{2} [\operatorname{div}_r(A_b(r) \nabla_r g_\tau(s))]_\delta, T_M[g_\tau(s)]_\delta \right\rangle ds. \end{aligned}$$

Now, let us pass to the limit as $\delta \rightarrow 0$ then $M \rightarrow +\infty$ in the last equality.

- A standard argument (see e.g. the proof of Lemma 4.1) ensures

$$\liminf_{M \rightarrow +\infty} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g_\tau(t)^2 T_M dr dx \leq \liminf_{M \rightarrow +\infty} \liminf_{\delta \rightarrow 0} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [g_\tau(t)]_\delta^2 T_M dr dx,$$

and by monotone convergence theorem (see e.g. Ref. 6), we deduce

$$\begin{aligned} \liminf_{M \rightarrow +\infty} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g_\tau(t)^2 T_M dr dx &= \lim_{M \rightarrow +\infty} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g_\tau(t)^2 T_M dr dx \\ &= \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g_\tau(t)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx. \end{aligned}$$

- Since $g_0^\tau \in H_\alpha$, we get $\lim_{\delta \rightarrow 0} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [g_0^\tau]_\delta^2 T_M dr dx = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (g_0^\tau)^2 T_M dr dx$, again monotone convergence theorem ensures

$$\lim_{M \rightarrow +\infty} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (g_0^\tau)^2 T_M dr dx = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (g_0^\tau)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx.$$

- $\lim_{\delta \rightarrow 0} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [u_L(s) \cdot \nabla_x g_\tau(s)]_\delta T_M [g_\tau(s)]_\delta dr dx ds = 0$. Indeed, note that

$$\begin{aligned} & [u_L(s) \cdot \nabla_x g_\tau(s)]_\delta T_M [g_\tau(s)]_\delta \\ &= ([u_L(s) \cdot \nabla_x g_\tau(s)]_\delta - u_L(s) \cdot \nabla_x [g_\tau(s)]_\delta) T_M [g_\tau(s)]_\delta \\ &+ u_L(s) \cdot \nabla_x [g_\tau(s)]_\delta T_M [g_\tau(s)]_\delta. \end{aligned}$$

On the one hand, since T_M depends only on r , we have

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} u_L(s) \cdot \nabla_x [g_\tau(s)]_\delta T_M [g_\tau(s)]_\delta dr dx ds \\ &= \int_0^t \int_{\mathbb{R}^d} T_M \int_{\mathbb{T}^d} u_L(s) \cdot \nabla_x [g_\tau(s)]_\delta [g_\tau(s)]_\delta dx dr ds \\ &= \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} T_M \int_{\mathbb{T}^d} \operatorname{div}[u_L(s) [g_\tau(s)]_\delta^2] dx dr ds = 0. \end{aligned}$$

On the other hand, since $|T_M| \leq M$ and by using Cauchy–Schwarz inequality we have

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} ([u_L(s) \cdot \nabla_x g_\tau(s)]_\delta - u_L(s) \cdot \nabla_x [g_\tau(s)]_\delta) T_M [g_\tau(s)]_\delta dr dx ds \right| \\ &\leq M \| [u_L(s) \cdot \nabla_x g_\tau(s)]_\delta - u_L(s) \cdot \nabla_x [g_\tau(s)]_\delta \|_{L^2(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))} \\ &\quad \times \| [g_\tau(s)]_\delta \|_{L^2(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))} \\ &\leq M \| [u_L \cdot \nabla_x g_\tau]_\delta - u_L \cdot \nabla_x [g_\tau]_\delta \|_{L^2(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))} \| g_\tau \|_{L^2(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))} \rightarrow 0, \end{aligned}$$

by using that $g_\tau \in L^2(0, T; H)$ and the commutator estimate, see Lemma C.1.

- The term $\int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [(\nabla u_L(s)r) \cdot \nabla_r f_\tau(s)]_\delta T_M [g_\tau(s)]_\delta dr dx ds$ requires special attention, by using (4.1) and the properties of convolution one gets

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [(\nabla u_L(s)r) \cdot \nabla_r f_\tau(s)]_\delta T_M [g_\tau(s)]_\delta dr dx ds \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) T_M g_\tau(s) dr dx ds. \end{aligned}$$

Note that

$$\int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) T_M g_\tau(s) dr dx ds$$

$$\begin{aligned}
&= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) T_M f_\tau(s) dr dx ds \\
&\quad - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) T_M \rho(s) \otimes p_\alpha dr dx ds.
\end{aligned}$$

Concerning the second term, we have

$$|(\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) T_M \rho(s) \otimes p_\alpha| \leq \frac{1}{Z} \|\nabla u_L\|_\infty |r| |\nabla_r f_\tau(s)| |\rho(s)|. \quad (5.3)$$

Thanks to (4.3), we get $\|\nabla u_L\|_\infty |r| |\nabla_r f_\tau(s)| |\rho(s)| \in L^1(\mathbb{R}^d \times \mathbb{T}^d \times (0, T))$ and Lebesgue's dominated convergence theorem (see e.g. Ref. 6) ensures

$$\begin{aligned}
&\lim_{M \rightarrow +\infty} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) T_M \rho(s) \otimes p_\alpha dr dx ds \\
&= \frac{1}{Z} \int_0^t \int_{\mathbb{T}^d} \rho(s) \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) dr dx ds \\
&= \int_0^t \int_{\mathbb{T}^d} \rho(s) \int_{\mathbb{R}^d} \operatorname{div}_r (\nabla u_L(s)r f_\tau(s)) dr dx ds = 0.
\end{aligned}$$

On the other hand, concerning the first term we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) T_M f_\tau(s) dr dx ds \\
&= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) T_M \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds \\
&= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) (\nabla u_L(s)r) \cdot \nabla_r \left(T_M \left(1 + \frac{|r|^2}{2}\right)^{-\alpha}\right) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds \\
&\quad - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(s) T_M \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} \\
&\quad \times \left(\nabla u_L(s)r) \cdot \nabla_r (f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha)\right) dr dx ds \\
&= A_1^M + A_2^M.
\end{aligned}$$

We have $\lim_{M \rightarrow +\infty} |A_1^M| = 0$. Indeed, by using (4.7) we get

$$|A_1^M| \leq \alpha \|\nabla u_L\|_\infty \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(f_\tau^2(s) |r|^2 M \left(1 + \frac{|r|^2}{2}\right)^{-\alpha-1}\right) dr dx ds$$

$$\begin{aligned} & \times 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds \\ & \leq \alpha \|\nabla u_L\|_\infty \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) dr dx ds. \end{aligned}$$

(i) Since $0 \leq 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) \leq 1$ for any $r \in \mathbb{R}^d$, we have

$$\begin{aligned} |\mathbf{f}_M(s, r)| &:= f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) \\ &\leq f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha := \mathbf{f}(s, r), \quad \forall M \in \mathbb{N}, \end{aligned}$$

for $(s, r) \in [0, T] \times \mathbb{R}^d$. Thanks to Proposition 4.1, we have

$$\mathbf{f} \in L^1(\mathbb{T}^d \times \mathbb{R}^d \times (0, T)),$$

namely

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \mathbf{f} dr dx ds = \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds < +\infty.$$

(ii) Since $\lim_{M \rightarrow +\infty} 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) = 0$ for every $r \in \mathbb{R}^d$ and $f_\tau^2(s)(1 + \frac{|r|^2}{2})^\alpha$ is independent of M almost everywhere $(x, r, s) \in \mathbb{T}^d \times \mathbb{R}^d \times (0, T)$, it follows that

$$\lim_{M \rightarrow +\infty} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) dr dx ds = 0$$

a.e. $(x, r, s) \in \mathbb{T}^d \times \mathbb{R}^d \times (0, T)$.

Therefore, Lebesgue's dominated convergence theorem (see e.g. Ref. 6) ensures

$$\lim_{M \rightarrow +\infty} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha 1_{\{(1+\frac{|r|^2}{2})^\alpha > M\}}(r) dr dx ds = 0$$

and $\lim_{M \rightarrow +\infty} |A_1^M| = 0$. Concerning A_2^M , since $|T_M(1 + \frac{|r|^2}{2})^{-\alpha}| \leq 1$ we have

$$\begin{aligned} G_M &:= \left| f_\tau(s) T_M \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} (\nabla u_L(s)r) \cdot \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha \right) \right| \\ &\leq \left| f_\tau(s) (\nabla u_L(s)r) \cdot \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha \right) \right| = G. \end{aligned}$$

By using (4.4) and Young's inequality, we have $G \in L^1(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))$. Indeed,

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |G| dr dx ds \\ & \leq \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds + 2\|\nabla u_L\|_\infty^2 \\ & \quad \times \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha \right) \right|^2 \left(1 + \frac{|r|^2}{2}\right)^{1-\alpha} dr dx ds < +\infty. \end{aligned}$$

On the other hand, note that

$$\begin{aligned} \lim_{M \rightarrow +\infty} f_\tau(s) T_M \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} (\nabla u_L(s)r) \cdot \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha\right) \\ = f_\tau(s) (\nabla u_L(s)r) \cdot \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha\right) \quad \text{a.e. } (x, r, s) \in \mathbb{T}^d \times \mathbb{R}^d \times (0, T). \end{aligned}$$

Lebesgue's dominated convergence theorem (see e.g. Ref. 6) gives

$$\begin{aligned} \lim_{M \rightarrow +\infty} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(s) T_M \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} \\ \times (\nabla u_L(s)r) \cdot \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha\right) dr dx ds \\ = \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(s) (\nabla u_L(s)r) \cdot \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha\right) dr dx ds. \end{aligned}$$

By using (4.4), we obtain

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(s) (\nabla u_L(s)r) \cdot \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha\right) dr dx ds \right| \\ \leq \frac{\sqrt{\tau}}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds + \frac{1}{\sqrt{\tau}} \|\nabla u_L\|_\infty^2 \\ \times \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |r|^2 \left| \nabla_r \left(f_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha\right) \right|^2 \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} dr dx ds \\ \leq \bar{\mathbf{K}} \left(\frac{T}{2} + \|\nabla u_L\|_\infty^2 \zeta \alpha \right) \sqrt{\tau} := \tilde{\mathbf{C}} \sqrt{\tau}. \end{aligned} \tag{5.4}$$

- Concerning the other terms, we have

$$\begin{aligned} J_\delta^M &:= \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha [\operatorname{div}_r(g_\tau(s)r)]_\delta T_M[g_\tau(s)]_\delta dr dx ds \\ &\quad + \frac{1}{\zeta \alpha \tau} \int_0^t \langle [\Delta_r g_\tau(s)]_\delta, T_M[g_\tau(s)]_\delta \rangle ds \\ &\quad + \frac{1}{\zeta \alpha \tau} \int_0^t \left\langle \frac{1}{2} [\operatorname{div}_r(A_b(r) \nabla_r g_\tau(s))]_\delta, T_M[g_\tau(s)]_\delta \right\rangle ds \\ &= -\frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\alpha r [g_\tau(s)]_\delta + \nabla_r [g_\tau(s)]_\delta + \frac{1}{2} A_b(r) \nabla_r [g_\tau(s)]_\delta \right) \\ &\quad \cdot \nabla_r (T_M[g_\tau(s)]_\delta) dr dx ds. \end{aligned}$$

Note that the smallest eigenvalue of $A_b(r)$ is $|r|^2$ and by repeating the arguments presented in Sec. 4.1.1, we infer

$$\liminf_{M \rightarrow +\infty} J_\delta^M \geq \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(g_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha\right) \right|^2 \left(1 + \frac{|r|^2}{2}\right)^{1-\alpha} dr dx ds.$$

Summarizing, we get

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g_\tau(t)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \\ & \quad + \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(g_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha \right) \right|^2 \left(1 + \frac{|r|^2}{2}\right)^{1-\alpha} dr dx ds \\ & \leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (g_0^\tau)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx + \tilde{\mathbf{C}} \sqrt{\tau}. \end{aligned}$$

Hence

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g_\tau(t)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \\ & \quad + \frac{1}{\zeta \alpha \tau} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left(g_\tau(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha \right) \right|^2 \left(1 + \frac{|r|^2}{2}\right)^{1-\alpha} dr dx ds \\ & \leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (g_0^\tau)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx + \tilde{\mathbf{C}} \sqrt{\tau}. \end{aligned}$$

In other words, we obtain

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(t) - \rho(t) \otimes p_\alpha|^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \\ & \quad + \frac{1}{\zeta \alpha \tau} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left((f_\tau - \rho \otimes p_\alpha)(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha \right) \right|^2 \\ & \quad \times \left(1 + \frac{|r|^2}{2}\right)^{-\alpha+1} dr dx ds \\ & \leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_0^\tau - \rho_0 \otimes p_\alpha|^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx + \tilde{\mathbf{C}} \sqrt{\tau}. \end{aligned} \quad (5.5)$$

Let us use (5.5) to deduce the convergence results. Indeed, (5.5) implies

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left| \nabla_r \left((f_\tau - \rho \otimes p_\alpha)(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha \right) \right|^2 \left(1 + \frac{|r|^2}{2}\right)^{-\alpha+1} dr dx ds \\ & \leq \zeta \alpha \tau \left(\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_0^\tau - \rho_0 \otimes p_\alpha|^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx + \tilde{\mathbf{C}} \sqrt{\tau} \right) \\ & \leq \overline{\mathbf{C}} \tau \rightarrow 0 \quad \text{as } \tau \rightarrow 0. \end{aligned} \quad (5.6)$$

Recall that for $\alpha > \frac{d}{2}$, $\nu_\alpha(dr) = \frac{1}{Z} (1 + \frac{|r|^2}{2})^{-\alpha} dr$ is a Borel probability measure on $\mathbb{R}^d$, set

$$h_\alpha = (f_\tau - \rho \otimes p_\alpha) \left(1 + \frac{|r|^2}{2}\right)^\alpha,$$

by using the weighted Poincaré inequality (see Proposition B.1), there exists $C_\alpha > 0$ such that

$$\int_{\mathbb{R}^d} h_\alpha^2 d\nu_\alpha - \left(\int_{\mathbb{R}^d} h_\alpha d\nu_\alpha \right)^2 \leq C_\alpha \int_{\mathbb{R}^d} |\nabla h_\alpha|^2 \left(1 + \frac{|r|^2}{2} \right) d\nu_\alpha.$$

On the other hand, note that

$$\int_{\mathbb{R}^d} h_\alpha d\nu_\alpha = \int_{\mathbb{R}^d} (f_\tau - \rho \otimes p_\alpha) \left(1 + \frac{|r|^2}{2} \right)^\alpha d\nu_\alpha = \frac{1}{Z} \left(\int_{\mathbb{R}^d} f_\tau dr - \rho \right) = \frac{1}{Z} (\rho_\tau - \rho).$$

Gathering the last inequalities to get

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_\tau - \rho \otimes p_\alpha)^2(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx ds \leq C_\alpha \overline{C} \tau + \frac{1}{Z} \int_0^T \int_{\mathbb{T}^d} |\rho_\tau - \rho|^2 dx ds.$$

By using Lemma 4.3, we have $\rho_\tau \rightarrow \rho$ in $L^2(\mathbb{T}^d \times (0, T))$ as $\tau \rightarrow 0$ and the result holds. $\square$

Remark 5.1. Since $\alpha > \frac{d}{2}$, the last inequality implies the convergence in $L^1(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))$.

5.2. The limit as $\tau \rightarrow 0$ if $0 < \alpha \leq \frac{d}{2}$

We will discuss the following two cases.

- (i) $f_0^\tau \in H_\alpha \cap H$ and there exists $\Lambda > 0$ independent of τ such that $\sup_{\tau > 0} \|f_0^\tau\|_{H_\alpha}^2 \leq \Lambda$.
- (ii) $f_0^\tau \in L^1(\mathbb{T}^d \times \mathbb{R}^d) \cap H$ and there exists $\Lambda > 0$ independent of τ such that $\sup_{\tau > 0} \|f_0^\tau\|_{L^1_{x,r}} \leq \Lambda$.

5.2.1. The case $(f_0^\tau)_\tau$ is bounded in H_α

Following Proposition 4.1, there exists $\mathbf{K} = 2\alpha \|\nabla_x u_L\|_\infty$ such that

$$\begin{aligned} & \sup_{t \in [0, T]} \|f_\tau(t)\|_{H_\alpha}^2 + \frac{1}{\zeta \alpha \tau} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r (f_\tau(s) P_{-\alpha})|^2 \\ & \quad \times \left(1 + \frac{|r|^2}{2} \right)^{1-\alpha} dr dx ds \leq e^{\mathbf{K}T} \Lambda := \overline{\mathbf{K}}. \end{aligned} \quad (5.7)$$

Proposition 5.2. Let $0 < \alpha \leq \frac{d}{2}$ and assume that $(f_0^\tau)_\tau$ is bounded in H_α . Then there exists a subsequence of $(f_\tau)_\tau$ denoted by $(f_{\tau_k})_{\tau_k}$ such that

$$f_{\tau_k} \xrightarrow{*} 0 \quad \text{in } L^\infty(0, T; H_\alpha) \quad \text{as } \tau_k \rightarrow 0. \quad (5.8)$$

Proof. By using (5.7) and Banach–Alaoglu–Bourbaki theorem (see e.g. Ref. 6), one gets the existence of subsequence of $(f_\tau)_\tau$ (denoted by $(f_{\tau_k})_{\tau_k}$) and $\bar{f} \in L^\infty(0, T; H_\alpha)$ such that

$$f_{\tau_k} \xrightarrow{*} \bar{f} \quad \text{in } L^\infty(0, T; H_\alpha) \quad \text{as } \tau_k \rightarrow 0, \quad (5.9)$$

and

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r(f_{\tau_k}(s)P_{-\alpha})|^2 \left(1 + \frac{|r|^2}{2}\right)^{-\alpha+1} dr dx ds \leq \bar{\mathbf{K}} \zeta_\alpha \tau_k. \quad (5.10)$$

From (5.10), we obtain

$$|\nabla_r(f_{\tau_k}(s)P_{-\alpha})|^2 \left(1 + \frac{|r|^2}{2}\right)^{1-\alpha} \rightarrow 0 \quad \text{in } L^1([0, T] \times \mathbb{T}^d \times \mathbb{R}^d) \text{ as } \tau_k \rightarrow 0.$$

Therefore, up to subsequence if necessary, we get

$$|\nabla_r(f_{\tau_k}(s)P_{-\alpha})|^2 \rightarrow 0 \quad \text{a.e. in } [0, T] \times \mathbb{T}^d \times \mathbb{R}^d \text{ as } \tau_k \rightarrow 0, \quad (5.11)$$

since $(1 + \frac{|r|^2}{2})^{1-\alpha} > 0$. On the other hand, thanks to (5.9) and (5.11) we get the following convergence in the sense of distributions

$$\langle \nabla_r(f_{\tau_k}(s)P_{-\alpha}), \Phi \rangle_{\mathcal{D}', \mathcal{D}} \rightarrow \langle \nabla_r(\bar{f}(s)P_{-\alpha}), \Phi \rangle_{\mathcal{D}', \mathcal{D}} = 0, \quad \forall \Phi \in \mathcal{D},$$

where $\mathcal{D} = C_c^\infty([0, T]) \otimes C^\infty(\mathbb{T}^d) \otimes C_c^\infty(\mathbb{R}^d)$. Hence, we deduce the existence of distribution $h_{t,x}$ such that

$$\bar{f}P_{-\alpha} = h_{t,x} \quad \text{in } \mathcal{D}' \Leftrightarrow \bar{f} = h_{t,x} \otimes P_\alpha(r) = h_{t,x} \otimes \left(1 + \frac{|r|^2}{2}\right)^{-\alpha} \quad \text{in } \mathcal{D}'.$$

We recall that $\int_{\mathbb{R}^d} (1 + \frac{|r|^2}{2})^{-\alpha} dr$ diverges if $0 < \alpha \leq \frac{d}{2}$. Since $\bar{f} \in L^\infty(0, T; H_\alpha)$ from (5.9), we deduce that $h_{t,x} \equiv 0$. $\square$

5.2.2. The L^1 -setting

We prove the following result.

Proposition 5.3. *Assume that $f_0^\tau \in L^1(\mathbb{T}^d \times \mathbb{R}^d) \cap H$ and there exists $\Lambda > 0$ (independent of τ) such that $\sup_{\tau>0} \|f_0^\tau\|_{L^1} \leq \Lambda$. Then*

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(t)| dr dx \leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_0^\tau| dr dx \leq \Lambda. \quad (5.12)$$

Moreover, there exists a subsequence of $(f_\tau)_\tau$ (denoted by $(f_{\tau_k})_k$) and $\nu \in L^\infty(0, T; \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d))$ such that

$$f_{\tau_k} \xrightarrow{*} \nu \quad \text{in } L^\infty(0, T; \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)) \text{ as } k \rightarrow +\infty, \quad (5.13)$$

and ν solves, in the sense of distributions, the following equation:

$$\operatorname{div}_r \left(\alpha r \nu + \nabla_r \nu + \frac{1}{2} A_b(r) \nabla_r \nu \right) = 0 \quad \text{in } \mathcal{D}'. \quad (5.14)$$

Proof. Let $\epsilon > 0$, we define η_ϵ the odd continuous function on $\mathbb{R}$ as follows:

$$\eta_\epsilon(s) = \begin{cases} 1 & \text{if } \epsilon < s, \\ \frac{1}{2} \left[1 + \sin \left(\frac{\pi}{2\epsilon} (2s - \epsilon) \right) \right] & \text{if } 0 \leq s \leq \epsilon. \end{cases}$$

Note that $\eta_\epsilon \in C^1(\mathbb{R})$, $\eta''_\epsilon \in L^\infty(\mathbb{R})$ and satisfies:

- $|\eta_\epsilon(s)| \leq 1 \wedge \frac{\pi s}{\epsilon}, \forall s \in \mathbb{R}$;
- $\eta'_\epsilon \geq 0$, $\text{supp}(\eta'_\epsilon) \subset [-\epsilon, \epsilon]$ and $|\eta'_\epsilon(s)| \leq \frac{\pi}{2\epsilon}, \forall s \in \mathbb{R}$;
- $\eta_\epsilon(s)$ converges pointwise to $\text{sign}(s)$ as $\epsilon \rightarrow 0$.

Let $\delta, \epsilon > 0$, note that $\eta_\epsilon([f_\tau(\cdot)]_\delta) \in \mathcal{Y}$. By using (4.5), we get for any $t \in [0, T]$

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [f_\tau(t)]_\delta \eta_\epsilon([f_\tau(t)]_\delta) dr dx - \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [f_0^\tau]_\delta \eta_\epsilon([f_0^\tau]_\delta) dr dx \\ &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [u_L(s) \cdot \nabla_x f_\tau(s)]_\delta \eta_\epsilon([f_\tau(s)]_\delta) dr dx ds \\ & \quad - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [(\nabla u_L(s)r) \cdot \nabla_r f_\tau(s)]_\delta \eta_\epsilon([f_\tau(s)]_\delta) dr dx ds \\ & \quad + \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha [\text{div}_r(f_\tau(s)r)]_\delta \eta_\epsilon([f_\tau(s)]_\delta) dr dx ds \\ & \quad + \frac{1}{\zeta \alpha \tau} \int_0^t \langle [\Delta_r f_\tau(s)]_\delta, \eta_\epsilon([f_\tau(s)]_\delta) \rangle \\ & \quad + \frac{1}{2} \langle [\text{div}_r(A_b(r) \nabla_r f_\tau(s))]_\delta, \eta_\epsilon([f_\tau(s)]_\delta) \rangle ds. \end{aligned}$$

First, note that

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [u_L(s) \cdot \nabla_x f_\tau(s)]_\delta \eta_\epsilon([f_\tau(s)]_\delta) dr dx ds \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} ([u_L(s) \cdot \nabla_x f_\tau(s)]_\delta - u_L(s) \cdot \nabla_x [f_\tau(s)]_\delta) \eta_\epsilon([f_\tau(s)]_\delta) dr dx ds \\ & \quad + \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} u_L(s) \cdot \nabla_x [f_\tau(s)]_\delta \eta_\epsilon([f_\tau(s)]_\delta) dr dx ds \\ &= A_\delta + B_\delta \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \end{aligned}$$

Concerning the term A_δ , recall that $|\eta_\epsilon([f_\tau(s)]_\delta)| \leq \frac{\pi}{\epsilon} |[f_\tau(s)]_\delta|$, which ensures that $(\eta_\epsilon([f_\tau(s)]_\delta))_\delta$ is bounded in $L^2(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))$. Therefore, we can use the commutator estimate (see Lemma C.1) to get $\lim_{\delta \rightarrow 0} A_\delta = 0$. On the other hand, denote by N_ϵ the primitive function of η_ϵ , then by using “divergence theorem” we get

$$\begin{aligned} B_\delta &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} u_L(s) \cdot \nabla_x [f_\tau(s)]_\delta \eta_\epsilon([f_\tau(s)]_\delta) dr dx ds \\ &= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \text{div}_x(u_L(s) N_\epsilon([f_\tau(s)]_\delta)) dx dr ds = 0. \end{aligned}$$

Second, we have $\lim_{\delta} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [\operatorname{div}_r(\nabla u_L(s)r) f_{\tau}(s)]_{\delta} \eta_{\epsilon}([f_{\tau}(s)]_{\delta}) dr dx ds = 0$. Indeed, note that

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [\operatorname{div}_r(\nabla u_L(s)r f_{\tau}(s))]_{\delta} \eta_{\epsilon}([f_{\tau}(s)]_{\delta}) dr dx ds \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} ([\operatorname{div}_r(\nabla u_L(s)r f_{\tau}(s))]_{\delta} - \operatorname{div}_r(\nabla u_L(s)r[f_{\tau}(s)]_{\delta})) \eta_{\epsilon}([f_{\tau}(s)]_{\delta}) dr dx ds \\ &+ \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \operatorname{div}_r(\nabla u_L(s)r[f_{\tau}(s)]_{\delta}) \eta_{\epsilon}([f_{\tau}(s)]_{\delta}) dr dx ds = C_{\delta} + D_{\delta}. \end{aligned}$$

Recall that $\operatorname{div}_r(\nabla u_L(s)r) = 0$, by using “divergence theorem” we get

$$\begin{aligned} D_{\delta} &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \operatorname{div}_r(\nabla u_L(s)r[f_{\tau}(s)]_{\delta}) \eta_{\epsilon}([f_{\tau}(s)]_{\delta}) dr dx ds \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \operatorname{div}_r(\nabla u_L(s)r N_{\epsilon}([f_{\tau}(s)]_{\delta})) dr dx ds = 0. \end{aligned}$$

Concerning C_{δ} , note that

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} ([\operatorname{div}_r(\nabla u_L(s)r f_{\tau}(s))]_{\delta} - \operatorname{div}_r(\nabla u_L(s)r[f_{\tau}(s)]_{\delta})) \eta_{\epsilon}([f_{\tau}(s)]_{\delta}) dr dx ds \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\operatorname{div}_r(\Theta_{\delta} * (\nabla u_L(s)r f_{\tau}(s))) - \operatorname{div}_r(\nabla u_L(s)r \Theta_{\delta} * f_{\tau}(s))) \\ &\quad \times \eta_{\epsilon}(\Theta_{\delta} * f_{\tau}(s)) dr dx ds \\ &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\Theta_{\delta} * (\nabla u_L(s)r f_{\tau}(s)) - (\nabla u_L(s)r \Theta_{\delta} * f_{\tau}(s))) \\ &\quad \times \Theta_{\delta} * \nabla_r f_{\tau}(s) \eta'_{\epsilon}(\Theta_{\delta} * f_{\tau}(s)) dr dx ds. \end{aligned}$$

On the other hand, for any $(s, x, r) \in [0, T] \times \mathbb{T}^d \times \mathbb{R}^d$ note that

$$\begin{aligned} & (\Theta_{\delta} * (\nabla u_L(\cdot)r f_{\tau})) (s, x, r) - \nabla u_L(s, x)r(\Theta_{\delta} * f_{\tau})(s, x, r) \\ &= \int_{\mathbb{R}^d} [\nabla u_L(x - y, s) - \nabla u_L(x, s)]r \Theta_{\delta}(y) f_{\tau}(s, x - y, r) dy. \end{aligned}$$

By mean-value theorem we get $|\nabla u_L(x - y, s) - \nabla u_L(x, s)| \leq |y| \|u_L\|_{C^2}$ and

$$\begin{aligned} & |(\Theta_{\delta} * (\nabla u_L(\cdot)r f_{\tau})) (s, x, r) - \nabla u_L(s, x)r(\Theta_{\delta} * f_{\tau})(s, x, r)| \\ &\leq \|u_L\|_{C^2} \int_{\mathbb{R}^d} |y| |r| \Theta_{\delta}(y) |f_{\tau}(s, x - y, r)| dy \\ &\leq \delta \|u_L\|_{C^2} \int_{\mathbb{R}^d} \Theta_{\delta}(y) |r| |f_{\tau}(s, x - y, r)| dy, \end{aligned}$$

since $\text{supp } \Theta \subset B(0, 1)$. Therefore, we get

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |(\Theta_\delta * (\nabla u_L(s) r f_\tau(s)) \\
 & \quad - (\nabla u_L(s) r \Theta_\delta * f_\tau(s))) \Theta_\delta * \nabla_r f_\tau(s) \eta'_\epsilon(\Theta_\delta * f_\tau(s))| dr dx ds \\
 & \leq \frac{\pi}{2\epsilon} \delta \|u_L\|_{C^2} \int_0^t \left\| \int_{\mathbb{R}^2} \Theta_\delta(y) |r| |f_\tau(s, x - y, r)| dy \right\|_{L^2_{x,r}} \|\Theta_\delta * \nabla_r f_\tau(s)\|_{L^2_{x,r}} ds \\
 & \leq \frac{\pi}{2\epsilon} \delta \|u_L\|_{C^2} \int_0^t \|\Theta_\delta * |r| f_\tau(s)\|_{L^2_{x,r}} \|\Theta_\delta * \nabla_r f_\tau(s)\|_{L^2_{x,r}} ds \\
 & \leq \frac{\pi}{2\epsilon} \delta \|u_L\|_{C^2} \int_0^t \|f_\tau(s)\|_H \|\nabla_r f_\tau(s)\|_{L^2_{x,r}} ds \rightarrow 0 \quad \text{as } \delta \rightarrow 0.
 \end{aligned}$$

Thanks to the properties of the matrix A_b and $\eta'_\epsilon \geq 0$, we deduce

$$\begin{aligned}
 & \frac{1}{\zeta \alpha \tau} \int_0^t \langle [\Delta_r f_\tau(s)]_\delta, \eta_\epsilon([f_\tau(s)]_\delta) \rangle + \frac{1}{2} \langle [\text{div}_r(A_b(r) \nabla_r f_\tau(s))]_\delta, \eta_\epsilon([f_\tau(s)]_\delta) \rangle ds \\
 & = -\frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(|[\nabla_r f_\tau(s)]_\delta|^2 + \frac{1}{2} A_b(r) \nabla_r [f_\tau(s)]_\delta \cdot \nabla_r [f_\tau(s)]_\delta \right) \\
 & \quad \times \eta'_\epsilon([f_\tau(s)]_\delta) dr dx ds \\
 & \leq -\frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(1 + \frac{|r|^2}{2} \right) |[\nabla_r f_\tau(s)]_\delta|^2 \eta'_\epsilon([f_\tau(s)]_\delta) dr dx ds. \quad (5.15)
 \end{aligned}$$

Finally, we have

$$\begin{aligned}
 & \frac{1}{\tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [\text{div}_r(f_\tau(s) r)]_\delta \eta_\epsilon([f_\tau(s)]_\delta) dr dx ds \\
 & = -\frac{1}{\zeta \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [f_\tau(s)]_\delta r \cdot \nabla_r [f_\tau(s)]_\delta \eta'_\epsilon([f_\tau(s)]_\delta) dr dx ds \\
 & \leq \frac{1}{\zeta \alpha \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{|r|^2}{2} |[\nabla_r f_\tau(s)]_\delta|^2 \eta'_\epsilon([f_\tau(s)]_\delta) dr dx ds \\
 & \quad + \frac{\alpha}{2\zeta \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [f_\tau(s)]_\delta^2 \eta'_\epsilon([f_\tau(s)]_\delta) dr dx ds.
 \end{aligned}$$

By using (5.15), then letting $\delta \rightarrow 0$ we obtain

$$\begin{aligned}
 & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t) \eta_\epsilon(f_\tau(t)) dr dx \\
 & \leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_0^\tau \eta_\epsilon(f_0^\tau) dr dx + \frac{\alpha}{2\zeta \tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \eta'_\epsilon(f_\tau(s)) dr dx ds
 \end{aligned}$$

$$\begin{aligned} &\leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_0^\tau| dx + \frac{\alpha}{2\zeta\tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{\pi}{2\epsilon} f_\tau^2(s) 1_{\{|f_\tau| \leq \epsilon\}} dx ds \\ &\leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_0^\tau| dx + \frac{\alpha\pi}{4\zeta\tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(s)| 1_{\{|f_\tau| \leq \epsilon\}} dx ds. \end{aligned}$$

By using Fatou's lemma (see e.g. Ref. 6), we get

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(t)| dx \leq \liminf_{\epsilon \rightarrow 0} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t) \eta_\epsilon(f_\tau(t)) dx.$$

In conclusion, we get

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(t)| dx \leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_0^\tau| dx + \frac{\alpha\pi}{4\zeta\tau} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(s)| 1_{\{|f_\tau| \leq \epsilon\}} dx ds. \quad (5.16)$$

By using Gronwall inequality, we deduce

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(t)| dx \leq \exp\left(\frac{\alpha\pi}{4\zeta\tau} T\right) \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_0^\tau| dx \leq \exp\left(\frac{\alpha\pi}{4\zeta\tau} T\right) \Lambda. \quad (5.17)$$

Since $|f_\tau| 1_{\{|f_\tau| \leq \epsilon\}} \rightarrow 0$ a.e. as $\epsilon \rightarrow 0$ and $f_\tau \in L^1([0, T] \times \mathbb{T}^d \times \mathbb{R}^d)$ thanks to (5.17), dominated convergence theorem (see e.g. Ref. 6) ensures

$$\lim_{\epsilon \rightarrow 0} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |f_\tau(s)| 1_{\{|f_\tau| \leq \epsilon\}} dx ds = 0.$$

Therefore, after passing to the limit as $\delta \rightarrow 0$ then $\epsilon \rightarrow 0$ in (5.16), we obtain (5.12).

The limit as $\tau \rightarrow 0$

We recall that Riesz–Markov–Kakutani representation theorem ensures $\mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d) \simeq (C_c(\mathbb{T}^d \times \mathbb{R}^d))'$, see Ref. 39. Since $L^1(\mathbb{T}^d \times \mathbb{R}^d) \hookrightarrow \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)$. By using (5.12) and Banach–Alaoglu–Bourbaki theorem (see e.g. Ref. 6), we can extract a subsequence of $(f_\tau)_\tau$ (denoted by $(f_{\tau_k})_{\tau_k}$) and $\nu \in L^\infty(0, T; \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d))$ such that

$$f_{\tau_k} \xrightarrow{*} \nu \quad \text{in } L^\infty(0, T; \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)) \quad \text{as } k \rightarrow 0. \quad (5.18)$$

By using (4.2), we have

$$\begin{aligned} \left\langle \frac{df_\tau(s)}{ds}, \phi \right\rangle &= \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} G(f_\tau(s), \phi) dx \\ &:= \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(s) (u_L(s) \cdot \nabla_x \phi + (\nabla u_L(s)r) \cdot \nabla_r \phi) dx \\ &\quad - \frac{1}{\zeta\alpha\tau} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha f_\tau(s) r \cdot \nabla_r \phi + \nabla_r f_\tau(s) \cdot \nabla_r \phi \\ &\quad + \frac{1}{2} A_b(r) \nabla_r f_\tau(s) \cdot \nabla_r \phi dx, \quad \forall \phi \in \mathcal{Y}. \end{aligned} \quad (5.19)$$

Let $\xi \in C_c^\infty([0, T])$, multiplying the last equality by ξ and integrating by parts with respect to s , we obtain

$$-\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(s) \phi \partial_s \xi ds dr dx = \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} G(f_\tau(s), \phi) \xi dr dx ds.$$

Now, set $\phi = \vartheta \otimes \varphi \in C^\infty(\mathbb{T}^d) \otimes C_c^\infty(\mathbb{R}^d)$ and integrate by parts with respect to r in the right-hand side, we infer

$$\begin{aligned} & -\zeta \alpha \tau_k \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_{\tau_k}(s) \partial_t \xi \otimes \vartheta \otimes \varphi \\ & \quad + f_{\tau_k}(s) (u_L(s) \cdot \xi \otimes \nabla_x \vartheta \otimes \varphi + (\nabla u_L(s)r) \cdot \xi \otimes \vartheta \otimes \nabla_r \varphi)) dr dx ds \\ & = \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(-\alpha f_{\tau_k}(s) r \cdot \xi \otimes \vartheta \otimes \nabla_r \varphi + f_{\tau_k}(s) \xi \otimes \vartheta \otimes \Delta_r \varphi \right. \\ & \quad \left. + \frac{1}{2} f_{\tau_k}(s) \operatorname{div}_r (A_b(r) \xi \otimes \vartheta \otimes \nabla_r \varphi) \right) dr dx ds \\ & = \int_0^T \left\langle f_{\tau_k}(s), -\alpha r \cdot \xi \otimes \vartheta \otimes \nabla_r \varphi + \xi \otimes \vartheta \otimes \Delta_r \varphi \right. \\ & \quad \left. + \frac{1}{2} \operatorname{div}_r (A_b(r) \xi \otimes \vartheta \otimes \nabla_r \varphi) \right\rangle_{\mathcal{M}, C_c} ds. \end{aligned}$$

By using (5.18) and (5.12), we deduce as $\tau \rightarrow 0$

$$\begin{aligned} & \int_0^T \left\langle \nu(s), -\alpha r \cdot \xi \otimes \vartheta \otimes \nabla_r \varphi + \xi \otimes \vartheta \otimes \Delta_r \varphi \right. \\ & \quad \left. + \frac{1}{2} \operatorname{div}_r (A_b(r) \xi \otimes \vartheta \otimes \nabla_r \varphi) \right\rangle_{\mathcal{M}, C_c} ds = 0. \end{aligned} \quad (5.20)$$

In particular, ν is weak solution to the following equation:

$$\operatorname{div}_r \left(\alpha r \nu + \nabla_r \nu + \frac{1}{2} A_b(r) \nabla_r \nu \right) = 0 \quad \text{in } \mathcal{D}'. \quad (5.21)$$

□

6. Discussion and Outlook

6.1. Summary

We study the distribution of polymers in turbulent flow by using stochastic modeling of the small scales and the Fokker–Planck equation in the Hookean case. We rigorously prove the emergence of the heavy-tail distribution of polymer elongation. More precisely, the fluid velocity is decomposed into a smooth large-scale component and a rapidly fluctuating small-scale turbulent field. We send both the spatial and temporal scales of these turbulent fluctuations to zero, by using scaling and singular limits techniques. This allows to fill a gap between formal theory,

numerical simulations in polymer-turbulence literature and rigorous mathematical analysis. We distinguish two main regimes for polymers advected and stretched by small-scale turbulence. Namely, we have

- *The probability density regime* ($\alpha > d/2$): The limit of the Fokker–Planck density factorizes as $\rho(t, x) \otimes (1 + |r|^2/2)^{-\alpha}$, which is a power-law distribution in the end-to-end vector r . α becomes smaller in stronger turbulent flow and encodes the competition between polymer relaxation and turbulent strain fluctuations. This corresponds to a well-defined probability density, representing coil and moderate stretching states. Physically, polymer extension statistics remain bounded and close to equilibrium as $\alpha > d + \frac{1}{2}$, and polymers are in “coil-state”.
- *The Radon measure regime and coil-stretch transition* ($0 < \alpha \leq d/2$): $\alpha = \frac{d}{2}$ is the *coil-stretch transition* threshold and the limit object should be understood as Radon measure $h_x(t) \otimes (1 + |r|^2/2)^{-\alpha} dr$ exhibiting power-law behavior in the end-to-end vector. This reflects the *extreme stretching* of polymers induced by turbulence and the possibility of polymers reaching infinite length, which indicates that small-scale turbulence dominates. This corresponds also to the loss of L^1 -compactness from a mathematical point of view.

6.2. Outlook

The first natural extension includes the analysis of polymers in the FENE case, see Sec. 6.3. Next is the study of polymer feedback on the flow and links with drag reduction, by coupling Fokker–Planck and Navier–Stokes equations or any other possible alternatives. Another future project is to consider turbulence with non-Gaussian statistics and study the effect on the limit distribution.

6.3. FENE model

It is worth making some comment on the FENE model. While the Hookean model assumes linear elasticity and infinite extensibility, the FENE model introduces non-linearity and a finite maximum extension. Namely, in the FENE model we replace $\frac{1}{\beta} R_t$ in (1.1) by $\frac{1}{\beta} \frac{R_t}{1 - |R_t|^2/b}$, where $\sqrt{b}$ is the upper bound of the length extension of the polymer chains, i.e. the chains cannot stretch beyond the length $\sqrt{b}$. From mathematical point of view, the analysis must be considered in a bounded domain with respect to r -variable with the appropriate boundary conditions. The rigorous justification of the scaling and singular limits in the case of FENE model is the subject of the ongoing work.¹⁰ However, for the convenience of the reader, let us write what do we expect for the limit density. In the case of FENE model, we expect to get on the right-hand side of (1.4) (after the scaling limit)

$$\frac{1}{\tau} \operatorname{div}_r \left(\frac{br}{b - |r|^2} f_\tau + \nabla_r f_\tau + \frac{C_d}{2} A_b(r) \nabla_r f_\tau \right), \quad |r|^2 < b.$$

Concerning the singular limit, i.e. as $\tau \rightarrow 0$, we expect $f_\tau \rightarrow \mathfrak{f}$ where

$$\operatorname{div}_r \left(\frac{br}{b - |r|^2} \mathfrak{f} + \nabla_r \mathfrak{f} + \frac{C_d}{2} A_b(r) \nabla_r \mathfrak{f} \right) = 0, \quad |r|^2 < b,$$

with the interpretation that C_d becomes larger as the turbulence becomes stronger. Therefore, let us look for radial solution to the following equation:

$$\frac{br}{b - |r|^2} \mathfrak{f} + \nabla_r \mathfrak{f} + \frac{C_d}{2} A_b(r) \nabla_r \mathfrak{f} = 0.$$

Set $t = |r|$, we get the following ODE in t : $\frac{b}{C_d} \frac{1}{b-t^2} \mathfrak{f} + \left(\frac{2+C_d t^2}{2C_d t} \right) \mathfrak{f}' = 0$. By solving the ODE, we get

$$\mathfrak{f}(r) = \frac{1}{Z} \left(\frac{b - |r|^2}{b} \right)^\gamma \left(\frac{2 + C_d |r|^2}{2} \right)^{-\gamma}, \quad \gamma = \frac{b}{2 + C_d b} > 0 \text{ and} \\ Z \text{ is a renormalized constant.} \quad (6.1)$$

Note that the last formula (6.1) exhibits power-law behavior, with the term reflecting the fact that the domain is bounded. It is also similar to the stationary probability density derived in Ref. 1. Now, let us make some remarks on the FENE model in comparison with the Hookean one. First, note that $\mathfrak{f}$ is always a probability density function, unlike the Hookean case. Furthermore, the stretching saturates at $\sqrt{b}$ whereas in the Hookean model, we could obtain an infinite length. Finally, for intermediate extensions, $R_{\min} \ll |r| \ll \sqrt{b}$, we have

$$\mathfrak{f} \sim |r|^{d-1} \left(1 + C_d \frac{|r|^2}{2} \right)^{\frac{-b}{2+C_d b}} \sim |r|^{-1+d-\frac{2b}{2+C_d b}}.$$

In accordance with Ref. 2, the coil-stretch holds at $(2 + C_d b)d = 2b$. A more in-depth discussion of this transition and its link to the Hookean case is part of the ongoing work,¹⁰ see also Ref. 35 for some recent numerical simulation and discussions.

Appendix A. Proof of Lemma 4.1

Let us pass to the limit as $\delta \rightarrow 0$ then $M \rightarrow +\infty$ in (4.6).

- Recall that $f \in C_w([0, T], H)$, by using the properties of convolution and the lower semi-continuity of weak convergence, we obtain

$$\liminf_{M \rightarrow +\infty} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t)^2 T_M dr dx \leq \liminf_{M \rightarrow +\infty} \liminf_{\delta \rightarrow 0} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [f_\tau(t)]_\delta^2 T_M dr dx,$$

by Fatou's lemma (see e.g. Ref. 6), we deduce

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t)^2 \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx \leq \liminf_{M \rightarrow +\infty} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t)^2 T_M dr dx.$$

- Since $f_0^\tau \in H_\alpha$, we have $\lim_{\delta \rightarrow 0} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [f_0^\tau]_\delta^2 T_M dr dx = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_0^\tau)^2 T_M dr dx$ and monotone convergence theorem ensures

$$\lim_{M \rightarrow +\infty} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_0^\tau)^2 T_M dr dx = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_0^\tau)^2 \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx \leq \Lambda.$$

- $\lim_{\delta \rightarrow 0} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [u_L(s) \cdot \nabla_x f_\tau(s)]_\delta T_M[f_\tau(s)]_\delta dr dx ds = 0$. Indeed, it holds by using commutators estimate, see Lemma C.1 (see also e.g. Ref. 22 for a similar arguments).
- The term $\int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [(\nabla u_L(s)r) \cdot \nabla_r f_\tau(s)]_\delta T_M[f_\tau(s)]_\delta dr dx ds$. By using (4.1) and the properties of convolution, one gets

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [(\nabla u_L(s)r) \cdot \nabla_r f_\tau(s)]_\delta T_M[f_\tau(s)]_\delta dr dx ds \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) T_M f_\tau(s) dr dx ds. \end{aligned}$$

Now, by using that $|T'_M| \leq 1$ we get

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot \nabla_r f_\tau(s) T_M f_\tau(s) dr dx ds \right| \\ & \leq \alpha \left| \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla u_L(s)r) \cdot r f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^{\alpha-1} T'_M dr dx ds \right| \\ & \leq 2\alpha \|\nabla_x u_L\|_\infty \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{|r|^2}{2} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^{\alpha-1} dr dx ds \\ & \leq 2\alpha \|\nabla_x u_L\|_\infty \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds. \end{aligned}$$

- Concerning the term $\frac{1}{\zeta_{\alpha\tau}} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha [\operatorname{div}_r(f_\tau(s)r)]_\delta T_M[f_\tau(s)]_\delta dr dx ds$, we have

$$[\operatorname{div}_r(f_\tau(s)r)]_\delta T_M[f_\tau(s)]_\delta = d[f_\tau(s)]_\delta^2 T_M + r \cdot \nabla_r [f_\tau(s)]_\delta [f_\tau(s)]_\delta T_M,$$

by using (4.1) and the properties of convolution, it is possible to let $\delta \rightarrow 0$ and obtain

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} [\operatorname{div}_r(f_\tau(s)r)]_\delta T_M[f_\tau(s)]_\delta dr dx ds \\ &= d \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) T_M + r \cdot \nabla_r f_\tau(s) f_\tau(s) T_M dr dx ds \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \operatorname{div}_r(f_\tau(s)r) T_M f_\tau(s) dr dx ds \\ &= \frac{d}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) T_M dr dx ds - \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) r \cdot \nabla_r T_M dr dx ds \\ &= \frac{d}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) T_M dr dx ds - \frac{\alpha}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) |r|^2 \\ & \quad \times \left(1 + \frac{|r|^2}{2}\right)^{\alpha-1} T'_M dr dx ds \\ & \leq \frac{d}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds. \end{aligned}$$

- The term $\frac{1}{\zeta\alpha\tau} \int_0^t \langle [\Delta_r f_\tau(s)]_\delta, T_M[f_\tau(s)]_\delta \rangle ds$. Note that

$$\begin{aligned} & \int_0^t \langle [\Delta_r f_\tau(s)]_\delta, T_M[f_\tau(s)]_\delta \rangle ds \\ &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |[\nabla_r f_\tau(s)]_\delta|^2 T_M + [\nabla_r f_\tau(s)]_\delta \cdot \nabla_r T_M[f_\tau(s)]_\delta dr dx ds, \end{aligned}$$

by using (4.1) we can pass to the limit as $\delta \rightarrow 0$ and get

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_0^t \langle [\Delta_r f_\tau(s)]_\delta, T_M[f_\tau(s)]_\delta \rangle ds \\ &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r f_\tau(s)|^2 T_M + \nabla_r f_\tau(s) \cdot \nabla_r T_M f_\tau(s) dr dx ds \\ &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r f_\tau(s)|^2 T_M + \alpha \nabla_r f_\tau(s) \cdot r \left(1 + \frac{|r|^2}{2}\right)^{\alpha-1} T'_M f_\tau(s) dr dx ds. \end{aligned}$$

By using Young inequality we get

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \alpha \nabla_r f_\tau(s) \cdot r \left(1 + \frac{|r|^2}{2}\right)^{\alpha-1} T'_M f_\tau(s) dr dx ds \right| \\ & \leq \alpha^2 \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds \\ & \quad + \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r f_\tau(s)|^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & \limsup_{M \rightarrow +\infty} \lim_{\delta \rightarrow 0} \int_0^t \langle [\Delta_r f_\tau(s)]_\delta, T_M[f_\tau(s)]_\delta \rangle ds \\ & \leq - \liminf_{M \rightarrow +\infty} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r f_\tau(s)|^2 T_M dr dx ds \\ & \quad + \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r f_\tau(s)|^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds \\ & \quad + \alpha^2 \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds. \end{aligned}$$

- The term $\frac{1}{\zeta\alpha\tau} \int_0^t \langle \frac{1}{2} [\operatorname{div}_r(A_b(r)\nabla_r f_\tau(s))]_\delta, T_M[f_\tau(s)]_\delta \rangle ds$. We have

$$\begin{aligned} & \int_0^t \langle [\operatorname{div}_r(A_b(r)\nabla_r f_\tau(s))]_\delta, T_M[f_\tau(s)]_\delta \rangle ds \\ &= - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left((A_b(r)\nabla_r[f_\tau(s)]_\delta) \cdot \nabla_r[f_\tau(s)]_\delta T_M \right. \\ & \quad \left. + \alpha(A_b(r)\nabla_r[f_\tau(s)]_\delta) \cdot r \left(1 + \frac{|r|^2}{2}\right)^{\alpha-1} T'_M[f_\tau(s)]_\delta \right) dr dx ds. \end{aligned}$$

By using (4.1), we can let $\delta \rightarrow 0$ in the last equality and obtain

$$\begin{aligned}
 & \lim_{\delta \rightarrow 0} - \int_0^t \langle [\operatorname{div}_r(A_b(r)\nabla_r f_\tau(s))]_\delta, T_M[f_\tau(s)]_\delta \rangle ds \\
 &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (A_b(r)\nabla_r f_\tau(s)) \cdot \nabla_r f_\tau(s) T_M + (\alpha(A_b(r)\nabla_r f_\tau(s))) \\
 & \quad \cdot \left(r \left(1 + \frac{|r|^2}{2} \right)^{\alpha-1} T'_M f_\tau(s) \right) dr dx ds \\
 &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (b+1)|r|^2 |\nabla_r f_\tau(s)|^2 T_M - b|r \cdot \nabla_r f_\tau(s)|^2 T_M dr dx ds \\
 & \quad + \alpha \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (A_b(r)\nabla_r f_\tau(s)) \cdot r \left(1 + \frac{|r|^2}{2} \right)^{\alpha-1} T'_M f_\tau(s) dr dx ds \\
 &\geq \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |r|^2 |\nabla_r f_\tau(s)|^2 T_M dr dx ds \\
 & \quad + \alpha \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (A_b(r)\nabla_r f_\tau(s)) \cdot r \left(1 + \frac{|r|^2}{2} \right)^{\alpha-1} T'_M f_\tau(s) dr dx ds.
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 & \left| (A_b(r)\nabla_r f_\tau(s)) \cdot r \left(1 + \frac{|r|^2}{2} \right)^{\alpha-1} T'_M f_\tau(s) \right| \\
 & \leq 3 \left| |r|^2 \nabla_r f_\tau(s) \cdot r T'_M f_\tau(s) \left(1 + \frac{|r|^2}{2} \right)^{\alpha-1} \right| \leq 6 |\nabla_r f_\tau(s)| |r| f_\tau(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha \\
 & \leq \frac{1}{2\alpha} |\nabla_r f_\tau(s)|^2 |r|^2 \left(1 + \frac{|r|^2}{2} \right)^\alpha + 18\alpha f_\tau^2(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha,
 \end{aligned}$$

therefore, we get

$$\begin{aligned}
 & \lim_{\delta \rightarrow 0} - \int_0^t \langle [\operatorname{div}_r(A_b(r)\nabla_r f_\tau(s))]_\delta, T_M[f_\tau(s)]_\delta \rangle ds \\
 & \geq \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |r|^2 |\nabla_r f_\tau(s)|^2 T_M dr dx ds \\
 & \quad - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{1}{2} |\nabla_r f_\tau(s)|^2 |r|^2 \left(1 + \frac{|r|^2}{2} \right)^\alpha + 18\alpha^2 f_\tau^2(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx ds.
 \end{aligned}$$

Hence by using monotone convergence theorem, we get

$$\begin{aligned}
 & \liminf_{M \rightarrow +\infty} \lim_{\delta \rightarrow 0} - \int_0^t \langle [\operatorname{div}_r(A_b(r)\nabla_r f_\tau(s))]_\delta, T_M[f_\tau(s)]_\delta \rangle ds \\
 & \geq \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{|r|^2}{2} |\nabla_r f_\tau(s)|^2 \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx ds \\
 & \quad - 18\alpha^2 \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2} \right)^\alpha dr dx ds.
 \end{aligned}$$

By gathering the previous estimates we get

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx + \frac{1}{2\zeta\tau\alpha} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r f_\tau(s)|^2 \left(1 + \frac{|r|^2}{2}\right)^{\alpha+1} dr dx ds \\ & \leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_0^\tau)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \\ & \quad + \left(\frac{d/2 + 10\alpha}{\zeta\tau} + 2\alpha \|\nabla_x u_L\|_\infty\right) \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau^2(s) \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx ds. \end{aligned}$$

Gronwall's inequality ensures the existence of $\mathbf{C} = \frac{d/2+10\alpha}{\zeta\tau} + 2\alpha \|\nabla_x u_L\|_\infty$ such that

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_\tau(t)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \\ & \quad + \frac{1}{2\zeta\tau\alpha} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\nabla_r f_\tau(s)|^2 \left(1 + \frac{|r|^2}{2}\right)^{\alpha+1} dr dx ds \\ & \leq e^{\mathbf{C}T} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f_0^\tau)^2 \left(1 + \frac{|r|^2}{2}\right)^\alpha dr dx \leq \Lambda e^{\mathbf{C}T}. \end{aligned}$$

Appendix B. Weighted Poincaré Inequality

Let ϑ_σ be a family of the generalized Cauchy distributions on $\mathbb{R}^d$, which have densities

$$\frac{d\vartheta_\sigma(x)}{dx} = \frac{1}{A} (1 + |x|^2)^{-\sigma}; \quad \sigma > \frac{d}{2} \text{ and a normalizing constant } A.$$

Recall that $\text{Var}_{\vartheta_\sigma}(g) = \int_{\mathbb{R}^d} g^2 d\vartheta_\sigma - (\int_{\mathbb{R}^d} g d\vartheta_\sigma)^2$. We have the following result.

Theorem B.1. (Theorem 3.1 from Ref. 4) *The generalized Cauchy distribution ϑ_σ with $\sigma \geq d$ satisfies the weighted Poincaré-type inequality*

$$\text{Var}_{\vartheta_\sigma}(g) \leq \mathbf{C}_\sigma \int_{\mathbb{R}^d} |\nabla g(x)|^2 (1 + |x|^2) d\vartheta_\sigma(x) \quad (\text{B.1})$$

for all bounded smooth functions g on $\mathbb{R}^d$ with $\mathbf{C}_\sigma = \frac{1}{2(\sigma-1)} (\sqrt{1 + \frac{2}{\sigma-1}} + \sqrt{\frac{2}{\sigma-1}})^2$.

The authors in Ref. 4 state that Theorem B.1 can involve the values $\frac{d}{2} < \sigma \leq d$ with constant depending on the dimension d (see Ref. 4). However, general results on Riemannian manifold,⁵ based on the spectral gap estimate, guarantee weighted Poincaré inequality with explicit constants. Let us recall them (see Ref. 5). Following Ref. 5, the constant C_σ in (B.1) is given by

$$\mathbf{C}_\sigma = \frac{1}{(\sigma - \frac{d}{2})^2} \quad \text{if } \frac{d}{2} < \sigma \leq d; \quad d = 2, 3. \quad (\text{B.2})$$

On the other hand, we recall that Poincaré inequality is understood in the following sense: if the right-hand side is finite, then the function is square-integrable, and the

inequality holds true. Therefore, based on Refs. 4 and 5 briefly recalled above, we present the following result, which serves in our analysis. Namely, we consider ν_σ , a family of the generalized Cauchy distributions on $\mathbb{R}^d$, which have densities

$$\frac{d\nu_\sigma(y)}{dy} = \frac{1}{Z} \left(1 + \frac{|y|^2}{2}\right)^{-\sigma}; \quad \sigma > \frac{d}{2} \text{ and a normalizing constant } Z.$$

Proposition B.1. *Let $d \in \{2, 3\}$, $\sigma > \frac{d}{2}$ and $g : \mathbb{R}^d \rightarrow \mathbb{R}$. Assume that g satisfies*

$$\int_{\mathbb{R}^d} |\nabla g(y)|^2 \left(1 + \frac{|y|^2}{2}\right) d\nu_\sigma(y) < +\infty.$$

Then, the weighted Poincaré-type inequality holds

$$\text{Var}_{\nu_\sigma}(g) \leq C_\sigma \int_{\mathbb{R}^d} |\nabla g(y)|^2 \left(1 + \frac{|y|^2}{2}\right) d\nu_\sigma(y), \quad (\text{B.3})$$

where

$$C_\sigma = \begin{cases} \frac{1}{4(\sigma-1)} \left(\sqrt{1 + \frac{2}{\sigma-1}} + \sqrt{\frac{2}{\sigma-1}} \right)^2 & \text{if } d \leq \sigma, \\ \frac{1}{2(\sigma - \frac{d}{2})^2} & \text{if } \frac{d}{2} < \sigma \leq d. \end{cases}$$

Proof. By using (B.1), (B.2), and the change of variable $x = \frac{y}{\sqrt{2}}$ and noticing that

$$Z = \int_{\mathbb{R}^d} \left(1 + \frac{|y|^2}{2}\right)^{-\sigma} dy = 2^{\frac{d}{2}} \int_{\mathbb{R}^d} (1 + |x|^2)^{-\sigma} dx = 2^{\frac{d}{2}} A,$$

we obtain (B.3). $\square$

Appendix C. Commutator Estimate

Lemma C.1. *Let $p \in [1, +\infty[$ and $h \in L^p(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))$. Then, we have*

$$\lim_{\delta \rightarrow 0} ([u_L(s) \cdot \nabla_x h(s)]_\delta - u_L(s) \cdot \nabla_x [h(s)]_\delta) = 0 \quad \text{in } L^p(\mathbb{T}^d \times \mathbb{R}^d \times (0, T)).$$

Similarly, if $h \in L^p(\mathbb{T}^d \times (0, T))$ only, then we have

$$\lim_{\delta \rightarrow 0} ([u_L(s) \cdot \nabla_x h(s)]_\delta - u_L(s) \cdot \nabla_x [h(s)]_\delta) = 0 \quad \text{in } L^p(\mathbb{T}^d \times (0, T)).$$

Proof. Set $r_\delta(s) := [u_L(s) \cdot \nabla_x h(s)]_\delta - u_L(s) \cdot \nabla_x [h(s)]_\delta$, we have, a.e. $s \in [0, T]$,

$$\|r_\delta(s)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} \leq C \|\nabla u_L(s)\|_\infty \|h(s)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)}, \quad (\text{C.1})$$

$$\lim_{\delta} r_\delta = 0 \quad \text{in } L^p(\mathbb{T}^d \times \mathbb{R}^d \times (0, T)), \quad (\text{C.2})$$

where $C > 0$ independent of δ . Indeed, let us show (C.1), note that

$$r_\delta(s, x, r) = - \int_{\mathbb{R}^d} (u_L(x, s) - u_L(y, s)) \cdot \nabla_x \rho_\delta(x - y) h(s, y, r) dy.$$

Consider the following change of variables $z = \frac{x-y}{\delta}$ to get

$$\begin{aligned} r_\delta(s, x, r) &= - \int_{\mathbb{R}^2} \frac{u_L(x, s) - u_L(x - \delta z, s)}{\delta} \cdot \nabla \rho(z) h(s, x - \delta z, r) dz \\ &= - \int_{\mathbb{R}^d} \int_0^1 \nabla u_L(x - \gamma \delta z) z d\gamma \cdot \nabla \rho(z) h(s, x - \delta z, r) dz, \end{aligned}$$

where we used mean value theorem, i.e.

$$u_L(x, s) - u_L(x - \delta z, s) = \delta \int_0^1 \nabla u_L(x - \gamma \delta z) z d\gamma.$$

Recall that $\text{supp } \rho \subset B(0, 1)$, thus

$$\begin{aligned} |r_\delta(s, x, r)| &\leq \|\nabla u_L(s)\|_\infty \int_{\text{supp } \rho} |z| |\nabla \rho(z)| |h(s, x - \gamma \delta z, r)| dz \\ &\leq \|\nabla u_L(s)\|_\infty \|\nabla \rho\|_{C_b^1} \int_{\text{supp } \rho} |h(s, x - \gamma \delta z, r)| dz \end{aligned}$$

and by Hölder's inequality and Fubini's theorem, we get

$$\begin{aligned} \|r_\delta(s)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)}^p &\leq \|\nabla u_L(s)\|_\infty^p \|\rho\|_{C_b^1}^p \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \left(\int_{\text{supp } \rho} |h(s, x - \gamma \delta z, r)| dz \right)^p dx dr \\ &\leq \|\nabla u_L(s)\|_\infty^p \|\rho\|_{C_b^1}^p \int_{\text{supp } \rho} \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} |h(s, x - \gamma \delta z, r)|^p dx dr dz \\ &\leq \|\nabla u_L(s)\|_\infty^p \|\rho\|_{C_b^1}^p \|h(s)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)}^p, \end{aligned}$$

by using the periodicity with respect to the x -variable. Concerning (C.2), we have

$$L^p(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))\text{-}\lim_{\delta \rightarrow 0} r_\delta(s) = -h(s, \cdot) \left(\int_{\mathbb{R}^2} \nabla u_L(s, \cdot) z \cdot \nabla \rho(z) dz \right).$$

Indeed, by using that $z \in \text{supp } \rho := \mathbf{F} \subset B(0, 1)$, we get

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \left| \int_{\mathbb{R}^d} \int_0^1 \nabla u_L(x - \gamma \delta z) z d\gamma \cdot \nabla \rho(z) h(s, x - \delta z, r) dz \right. \\ &\quad \left. - h(s, x, r) \left(\int_{\mathbb{R}^d} \nabla u_L(s, x) z \cdot \nabla \rho(z) dz \right) \right|^p dx dr ds \\ &\leq \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \int_{\mathbf{F}} \int_0^1 |\nabla u_L(x - \gamma \delta z) z \cdot \nabla \rho(z) h(s, x - \delta z, r)| \\ &\quad - h(s, x, r) \nabla u_L(s, x) z \cdot \nabla \rho(z)|^p d\gamma dz dx dr ds \\ &\leq \int_0^T \int_{\mathbf{F}} \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} |[h(s, x - \delta z, r) \nabla u_L(x - \gamma \delta z) \\ &\quad - h(s, x, r) \nabla u_L(s, x)] z \cdot \nabla \rho(z)|^p dx dr d\gamma dz ds \end{aligned}$$

$$\begin{aligned}
 &\leq \|\rho\|_{C_b^1}^p \int_0^T \int_{\mathbf{F}} \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} |h(s, x - \delta z, r) \nabla u_L(x - \gamma \delta z) \\
 &\quad - h(s, x, r) \nabla u_L(s, x)|^p dx dr d\gamma dz ds \\
 &\leq 2^{p-1} \|\rho\|_{C_b^1}^p \left(\int_0^T \int_{\mathbf{F}} \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} |h(s, x - \delta z, r) \right. \\
 &\quad \times [\nabla u_L(x - \gamma \delta z) - \nabla u_L(x - \delta z)]|^p dx dr d\gamma dz ds \\
 &\quad + \int_0^T \int_{\mathbf{F}} \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} |h(s, x - \delta z, r) \nabla u_L(x - \delta z) - h(s, x, r) \\
 &\quad \times \nabla u_L(s, x)|^p dx dr d\gamma dz ds \Big) \\
 &:= I_\delta^1 + I_\delta^2.
 \end{aligned}$$

Using the continuity of translations in $L^2(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))$ for the function $h \nabla u_L$, we get $\limsup_{\delta \rightarrow 0} I_\delta^2 = 0$. Concerning I_δ^1 , by mean-value theorem we get

$$\begin{aligned}
 &2^{p-1} \|\rho\|_{C_b^1}^p \int_0^T \int_{\mathbf{F}} \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} |h(s, x - \delta z, r) \\
 &\quad \times [\nabla u_L(x - \gamma \delta z) - \nabla u_L(x - \delta z)]|^p dx dr d\gamma dz ds \\
 &\leq 2^{p-1} \|\rho\|_{C_b^1}^p \delta^p \|u_L\|_{C_b^2}^p \int_0^T \int_{\mathbf{F}} \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} |h(s, x - \delta z, r)|^p dx dr dz ds \\
 &\leq 2^{p-1} \|\rho\|_{C_b^1}^p \delta^p \|u_L\|_{C_b^2}^p \|h(s)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))}^p \rightarrow 0 \quad \text{as } \delta \rightarrow 0.
 \end{aligned}$$

Finally, since ρ is radially symmetric, we get $\int_{\mathbb{R}^d} z_i \partial_j \rho(z) dz = -\delta_{ij}$ and so

$$\int_{\mathbb{R}^d} \nabla u_L(s, \cdot) z \cdot \nabla \rho(z) dz = -\operatorname{div} u_L = 0,$$

which gives (C.2). Then, the first claim of Lemma C.1 follows by integrating from 0 to T the above estimates. For the second part, it is clear that it follows by neglecting integration with respect to the variable r in the above calculation. $\square$

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