Discrete homotopy analysis for optimal trading execution with nonlinear transient market impact

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Abstract

Optimal execution in financial markets is the problem of how to trade a large quantity of shares incrementally in time in order to minimise the expected cost. In this paper we study the problem of the optimal execution in the presence of nonlinear transient market impact. Mathematically such problem is equivalent to solve a strongly nonlinear integral equation, which in our model is a weakly singular Urysohn equation of the first kind. We propose an approach based on homotopy analysis method (HAM), whereby a well behaved initial trading strategy is continuously deformed to lower the expected execution cost. Specifically we propose a discrete version of the HAM, i.e. the DHAM approach, in order to use the method when the integrals to compute have no closed form solution. We find that the optimal solution is front loaded for concave instantaneous impact even when the investor is risk neutral. More important we find that the expected cost of the DHAM strategy is significantly smaller than the cost of conventional strategies.

1 Introduction

The optimization of trading strategies has long been an important goal for investors in financial markets. As demonstrated in the context of a linear equilibrium model by Kyle thirty years ago [33], the optimal strategy for an investor with insider information on the fundamental price of an asset is to trade incrementally through time. This strategy allows the trader to minimize costs whilst also minimizing the revelation of information to the rest of the market. The precise way in which it is optimal to split the large order (herein called *metaorder*) [55] depends on the objective function and on the market impact model, *i.e.* the change in price conditioned on signed trade size. In part due to the increasing tendency toward a full automation of exchanges and in part due to the discovery of new statistical regularities of the microstructure of financial markets, the problem of optimal execution is receiving growing attention from the academic and practitioner communities [6, 29, 25].

As pointed out in Gatheral et al. [26], a first generation of market impact models, studied by Bertsimas, Lo and Almgren [17, 12, 13, 14], distinguishes between two impact components. The first component is temporary and only affects the individual trade that has triggered it. The second component is permanent and affects all current and future trades equally. These models can be either in discrete or in continuous time and can assume either linear or nonlinear market impact for individual trades. The second generation of market impact models focusses on the *transient* nature of market impact [20, 44, 18, 29]. In such models, market impact is typically assumed to factorize into two components: instantaneous market impact and a decay component. The instantaneous component models the reaction of price to traded volume. The decay component describes how the market price relaxes on average after the execution of an order. In such models, each trade affects future price dynamics with an intensity that decays with time.

The problem of optimal execution in the presence of transient impact has been considered in a series of recent studies. In the case of linear instantaneous market impact [26, 9, 21], the problem has been completely solved by showing that the cost minimization problem is equivalent to solving an integral equation. In particular Gatheral et al. [26] proved that optimal strategies can be characterized as measure-valued solutions of a Fredholm integral equation of the first kind. They show that optimal strategies always exist and are nonalternating between buy and sell trades when price impact decays as a convex function of time. This extends the result of Alfonsi et al. [10] on the non existence of transaction triggered price manipulation, *i.e* strategies where the expected execution costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

However, a series of empirical studies [38, 20, 16] has clearly shown that the instantaneous market impact is a strongly concave function of the volume, well approximated by a power law function. The resulting optimal execution problem in the presence of nonlinear and transient impact is mathematically much more complicated than the linear case. Some important results in the nonlinear transient case were established by Gatheral [29] who showed that under certain conditions, the model admits price manipulation, *i.e.* the existence of round trip strategies with positive expected revenues. This money machine should of course be avoided in the modeling of market impact. In particular Gatheral set some necessary conditions for the absence of price

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manipulation (see below for details and Ref. [23]). A step toward the solution of the optimal execution problem under nonlinear transient impact has been made recently by Dang [24]. In his paper, Dang suggests a way to convert the cost minimization problem into a nonlinear integral equation and proposes a numerical fixed point method on a discretization of the trading time interval to solve this equation.

In this paper we propose the Homotopy Analysis Method (HAM) [37, 34, 35] to solve the integral equation proposed by Dang [24]. This method is now widely used in dealing with nonlinear equations. Nonlinear equations are difficult to solve, especially analytically. Perturbation techniques are widely applied in science and engineering, and give a great contribution to help us understand nonlinear phenomena [43]. However, it is well known that perturbation methods are strongly dependent upon small/large physical parameters, and therefore are valid in principle only for weakly nonlinear problems. The so-called non-perturbation techniques, such as the Lyapunov's artificial small parameter method [40], the δ-expansion method [15], Adomian's decomposition method [7], are formally independent of small/large physical parameters. The problem is that all of these traditional non-perturbation methods can not ensure the convergence of the solution series: they are in fact only valid for weakly nonlinear problems too. The homotopy analysis method [37] is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, recently linear and nonlinear integral equations [53, 30] and coupled version of them. Unlike perturbation methods, the HAM is independent of small/large physical parameters, and thus it is valid no matter whether a nonlinear problem contains small/large physical parameters or not. More important, differently from all perturbation and traditional non-perturbation methods, the HAM provides us a simple way to ensure the convergence of solution series, and therefore, the HAM is valid even for strongly nonlinear problems [34, 35]. More and more researchers have successfully applied this method to various nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [52], the KdV-type equations [1], nonlinear heat transfer [2], projectile motion [54], magneto-hydrodynamics [8], Burgers-Huxley equation [42]. The HAM has been successfully applied to solve a few problems in finance [56, 45], e.g. in the case of American Put Option (Ref. [22] and Chapter 13 in [37]). Abbasbandy *et al.* [3] described the usage of HAM for solving the nonlinear Fredholm integral equation of the second kind. In this case the uniqueness of solution was proven and the sufficient condition for convergence of the created series was given.

We apply the HAM to solve a weakly singular Urysohn integral equation of the first kind. The particular form of nonlinearity, described by a nonlinear transient market impact, requires homotopy-derivatives relative to a bi-dimensional system [37]. Moreover, this approach implies the computation of several definite integrals, which cannot be solved analytically. This technical problem requires the use of a discretization of integrals involved in computations and led us to discretize the deformation equations relative to HAM. A similar approach was followed by Allahviranloo et al. [11] to solve nonlinear Fredholm integral equations of the second kind. The method starts from an initial guess for the trading strategy and deforms it continuously in order to find better and better approximations of the solution of the integral equation. In doing this, we are implicitly restricting the space of solutions to continuous nonvanishing functions of the trading rate. We find that the optimal solution is a non time-symmetric U-shape; in the case of concave instantaneous impact, it is optimal to trade more at the beginning of the metaorder in presence of a buy program. A comparative cost analysis shows that our solution outperforms conventional strategies.

The paper is organized as follows. In Section 2, we state the problem and explain why it is difficult to solve. In Section 3, we present the HAM approach to the solution of the cost minimization problem. In Section 4 we show the HAM solutions for three different cases of study. In Section 5, we summarize and conclude. In the Appendix A.1 we report the details on the computation of the homotopy-derivative for a bi-dimensional system.

2 The optimal execution problem

An impact model describes the relation between the trading activity and the price dynamics. Let us call $\dot{x}(t)$ is the rate of trading, *i.e.* number of shares per unit of time, at time t, when the price of the asset is $S(t)$. According to the nonlinear transient market impact model, a trading order starting at time $t = 0$, when the price is $S(0) = S_0$, and lasting until $t = T$ generates a price dynamics

$$S(t) = S_0 + \int_0^t f(\dot{x}(s))\, G(t-s)\, ds + \int_0^t \sigma\, dW(s), \tag{2.1}$$

where$f(\dot{x}(s))$ represents the impact of trading at time s, $G(t-s)$ describes the impact decay, σ is the volatility, and $W(t)$ is a Wiener process. Thus $S(t)$ follows an arithmetic random walk with a drift that depends on the accumulated impacts of previous trades. We refer to $f(\cdot)$ as the *instantaneous market impact function* and to $G(\cdot)$ as the *decay kernel*. In discrete time this is the propagator model originally developed by Bouchaud et al. [20, 19], while the above continuous time formulation (2.1) is due to Gatheral [29]. More recently, Bacry et al.[16] have shown how a market impact model of the form (2.1) may be related to a more fundamental description of order flow using Hawkes processes.

The optimal execution problem consists in finding the trading strategy $\Pi = \{x(t)\}_{t\in[0,T]}$ that minimizes the execution cost for a given total amount X of shares to be traded. The expected cost $C[\Pi]$ associated with a given strategy Π is given by

$$C[\Pi] = \mathbb{E}\left[\int_0^T \dot{x}(t)\,(S(t) - S_0)\, dt\right] = \int_0^T \dot{x}(t) \int_0^t f(\dot{x}(s))\, G(t-s)\, ds\, dt, \tag{2.2}$$

and the constraint that all shares should be traded is

$$\int_0^T \dot{x}(t)\; dt = X. \tag{2.3}$$

Expression (2.2) is the difference between the expected price payed and the price payed in a market without impact, i.e. infinitely liquid, and corresponds to the expected implementation shortfall. We search for a statically optimal strategy. A statically optimal strategy is also dynamically optimal when the cost depends on the unaffected, i.e. without price impact, stock price only through the term $\int_0^T S^u(t)\,\dot{x}(t)\;dt$, with $S^u(t)$ a martingale [47]. This implies that the cost function does not depend by the stochastic price motion. Thus for the model of (2.1), described by an unaffected martingale price process, and the cost function described by (2.2), a statically optimal strategy is also dynamically optimal.

The impact model of equation (2.1) is fully specified by the form of the functions f and G. A large body of empirical evidences points out two empirical facts on the form of these two functions. First, the instantaneous impact function $f(\cdot)$ is strongly concave. For example, based on a large sample of NYSE stocks, Lillo et al. [38] observed a concave function of the transaction volume. The concave function is well fitted by a power law with exponent 0.5 for small volumes and 0.2 for large volumes. Bouchaud et al. [20] analyzed stocks traded at the Paris Bourse and found that a logarithmic form gave the best fit to the data. In addition (in [20] for example), the decay kernel $G(\cdot)$ is found to decay asymptotically as a power law function

$$G(\tau) \sim \frac{1}{\tau^\gamma}. \tag{2.4}$$

The presence of these nonlinearities raises the question of the possible presence of price manipulation. There are different forms of manipulation. Following [28], we define a *price manipulation* as a round trip trade, *i.e.* a strategy with $\int_0^T \dot{x}(t)\,dt = 0$, whose expected cost is negative. An impact model is free from price manipulation if, for any round trip trade the expected cost is non negative. According to Proposition 1 of [27], the model of equation (2.1) admits price manipulation in the nonlinear case unless the decay kernel $G(\tau)$ is singular for $\tau = 0$. For these reasons we will focus our analysis on decay kernels of the form $G(t-s) = (t-s)^{-\gamma}$. Moreover, as shown by Gatheral [29], the requirement of no price manipulation restricts the class of possible joint form for the instantaneous impact and decay kernel. Specifically, for power law impact function, $f(\dot{x}) \propto sign(\dot{x})\,|\dot{x}|^\delta$, and a power-law kernel, $G(t-s) = (t-s)^{-\gamma}$, the following conditions

$$\gamma + \delta \geq 1, \qquad \gamma \geq \gamma^* = 2 - \frac{\log 3}{\log 2} \simeq 0.415, \tag{2.5}$$

are *necessary* for the absence of price manipulations (see Ref. [23] for a discussion on the sufficiency of these conditions). In this paper we always consider parameters δ and γ satisfying the above conditions. However, if these conditions are satisfied, there is no guarantee that the impact model does not admit price manipulation.

2.1 The case of linear market impact

The optimization problem of minimizing the expected cost of equation (2.2) under the constraint of equation (2.3) in the case of linear impact, $f(\dot{x}) \propto \dot{x}$, has been solved and widely studied [26]. In what follows, we use the symbol $v(t)$ to indicate the rate of trading $\dot{x}(t)$. In particular, Proposition

22.9 of Ref. [28] states that if G is positive definite, then $x(t)$ minimizes the expected cost if and only if there is a λ such that $\forall t$, $x(t)$ solves

$$\int_0^T G(|t-s|)dx(s) = \lambda \tag{2.6}$$

As an important example, relevant for this paper, is the case $G(t-s) = (t-s)^{-\gamma}$ where the integral equation (2.6) becomes the Abel equation with solution

$$v(t) = \frac{c}{[t\ (T-t)]^{\frac{1-\gamma}{2}}}, \tag{2.7}$$

where c is uniquely determined by the constraint equation (2.3) as

$$c = X/\left(\sqrt{\pi}\left(\frac{T}{2}\right)^{\gamma}\frac{\Gamma\left(\left(1+\gamma\right)/2\right)}{\Gamma\left(1+\gamma/2\right)}\right), \tag{2.8}$$

where $\Gamma(\cdot)$ is Euler's Gamma function. This solution is U-shaped and symmetric under time reversal, *i.e.* $v(t) = v(T-t)$, $t \in [0, T/2]$. In the following we will refer to this solution as the GSS solution.

2.2 The case of nonlinear market impact

In the general nonlinear case, the problem is mathematically much more complicated. A first step in this direction has been presented very recently by Dang [24] and consists of two contributions.

The first one is the use of calculus of variations for integrals depending on convolution products [50, 48, 49] to transform the cost minimization problem into an integral equation generalizing (2.6). Specifically, given $f \in C^1(\mathbb{R})$ and $G \in L^1[0,T]$, for the class of functions x on $[0,T]$ satisfying

- x is absolutely continuous on $(0,T)$,
- $f \circ v \in L^1[0,T]$,

the following necessary condition for the stationarity of the functional of equation (2.2) holds:

$$\int_0^t f(v(s))\,G(t-s)\,ds + f'(v(t))\int_t^T v(s)\,G(s-t)\,ds = \lambda, \tag{2.9}$$

where again λ is a constant set by the constraint equation (2.3).

In the case of a convex instantaneous impact function, $f(v) \propto sign(v)\,|v|^{\delta}$ with $\delta > 1$, equation (2.9) holds $\forall v \in \mathbb{R}$. In contrast, in the concave case, $\delta < 1$, equation (2.9) is not defined if the trading velocity v vanishes at some time t, because the derivative of f diverges at zero. This observation restricts the class of trajectories that can be considered. The trajectories can describe either pure buy strategies, i.e. positive trading rates $v > 0$, or pure sell strategies, negative trading rates i.e. $v < 0$. This constraint guarantees the smoothness of functions involved in eq. 2.9 when the concave case is considered. Moreover, in the concave case there is no guarantee that the necessary condition (2.9) is also sufficient, because the minimization cost problem could have a large number of extremal points.

Equation (2.9) is a weakly singular[1] Urysohn equation of the first kind [46] taking the form

$$\int_0^T G\left(|t-s|\right)\, F\left(v\left(s\right), t\right)\, ds = \lambda \tag{2.10}$$

where

$$F\left(v\left(s\right), t\right) = \begin{cases} f\left(v\left(s\right)\right), & s \leq t \\ v\left(s\right) f'\left(v\left(t\right)\right), & s > t. \end{cases} \tag{2.11}$$

Note that there are two sources of nonlinearity in the integral equation (2.10): the nonlinear impact function $f(v)$, and the function F. In fact, the nonlinearity depends also on the first derivative of the impact function f', *i.e.* on the response of price to the traded volume per unit time. Moreover, the term involving $f'(v(t))$ entangles the response of price at time t with the future trading rates, *i.e.* $v(s)$ for $s > t$. This represents a coupling between present and future values of the trading rate v. This means that equation (2.10) cannot be classified as a weakly singular nonlinear Fredholm equation, because the function F depends both on t and on s. In the linear impact case, both nonlinearities disappear and one recovers a weakly singular linear Fredholm integral equation of the first kind for the trading rate, where there is no such coupling between present and future times.

It is important to note that F in equation (2.10) is not an invertible function of v, because it depends not only on s but also on t. For this reason, the usual method for solving nonlinear integral equation by setting $u(t) = F(v(t))$ and solving the linear integral equation for u is not applicable here. In the next section we use the Homotopy Analysis Method to solve the integral equation (2.10).

3 The Homotopy Analysis Method

The concept of homotopy can be traced back to Henri Poincaré and describes a *continuous* variation or deformation. In our specific case, let us consider the following general nonlinear equation:

$$\mathcal{N}\left[v\left(t\right)\right] = 0, \tag{3.1}$$

where $\mathcal{N}$ is a nonlinear operator, t denotes the independent variable, and $v(t)$ is the unknown function. Liao [37] constructs the so-called zero-order deformation equation

$$\left(1-p\right)\mathcal{L}\left[\phi\left(t;p\right) - v^0\left(t\right)\right] = p\,\hbar\, H\left(t\right)\mathcal{N}\left[\phi\left(t;p\right)\right], \tag{3.2}$$

[1] A kernel is called singular if it becomes infinite at one or more points in the range of integration, such as in the Abel's equation [53]. A kernel is called weakly singular if its singularity is integrable, *i.e.* the integral of the function on a range that contains the singularity is finite. In our case the weakly singular kernel is given by $G\left(|t-s|\right) = |t-s|^{-\gamma}$.

where $p \in [0,1]$ is called the homotopy parameter, or embedding parameter and $\hbar$ is a non-zero auxiliary parameter which is called the convergence control parameter [4]. $H(t) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $v^0(t)$ is an initial guess of $v(t)$, and $\phi(t;p)$ is an unknown function. There are no particular prescriptions for the choice of auxiliary functions or operators; often the choice depends on the problem to be solved. When $p = 0$ and $p = 1$, we have respectively

$$\phi(t;0) = v^0(t), \qquad \phi(t;1) = v(t). \tag{3.3}$$

Thus, as p increases from 0 to 1, the solution $\phi(t;p)$ varies *continuously* from the initial guess $v^0(t)$ to the sought solution $v(t)$. Expanding $\phi(t;p)$ in a Maclaurin series with respect to p, we have

$$\phi(t;p) = v^0(t) + \sum_{m=1}^{\infty} v^m(t)\, p^m, \tag{3.4}$$

where

$$v^m(t) = \frac{1}{m!} \frac{\partial^m \phi(t;p)}{\partial p^m}\Big|_{p=0}. \tag{3.5}$$

The series representation of ϕ in equation (3.4) is called the homotopy series and $v^m(t)$ in (3.5) is called the mth-order homotopy derivative of ϕ [35]. If the auxiliary linear operator, the initial guess, the convergence control parameter $\hbar$, and the auxiliary function are properly chosen, the homotopy series converges at $p = 1$, giving the sought solution of equation (3.1). Then by using the relationship $\phi(t;1) = v(t)$, one has the so-called homotopy series solution

$$v(t) = v^0(t) + \sum_{m=1}^{\infty} v^m(t), \tag{3.6}$$

which is one of the solutions of the original nonlinear equation, as proved by Liao [36]. Defining the vector

$$\mathbf{v}^m = \left\{ v^0(t), v^1(t), \cdots, v^m(t) \right\}, \tag{3.7}$$

and differentiating the zero-order deformation equation (3.2) m times with respect to the homotopy parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}\left[v^m(t) - \chi^m v^{m-1}(t) \right] = \hbar\, H(t)\, R^m\left(\mathbf{v}^{m-1} \right), \tag{3.8}$$

where

$$R^m\left(\mathbf{v}^{m-1} \right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\phi(t;p)]}{\partial p^{m-1}}\Big|_{p=0} \tag{3.9}$$

and

$$\chi^m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{3.10}$$

The nth-order approximate solution is then given by

$$v^{(n)}(t) = v^0(t) + \sum_{m=1}^{n} v^m(t), \tag{3.11}$$

and the exact solution by $v(t) = \lim_{n\to\infty} v^{(n)}(t)$. The two main difficulties of this approach are the computation of the derivatives of equation (3.9) and the choice of an appropriate value of $\hbar$ in order to guarantee the convergence of the series solution of equation (3.6).

The first problem requires to compute the homotopy derivative of a given nonlinear smooth function f of the homotopy Maclaurin series of equation (3.4), *i.e.* to compute the derivative of $f(\phi)$. We refer the reader to Appendix A.1 for details. The second problem is how to choose an appropriate value of the convergence control parameter $\hbar$ in order to guarantee the convergence of the series of equation (3.6) [39]. To do this, we will adopt in the following, the so-called *optimization* method [37, 8, 5], according to which, we define the squared residual of the governing equation (3.1) as

$$\mathcal{E}^n(\hbar) = \int_0^T \left(\mathcal{N}\left[v^{(n)}(t)\right]\right)^2 dt. \tag{3.12}$$

The optimal value of the convergence control parameter is then obtained by finding the minimum of this squared residual. In fact, if $v^{(n)}(t)$ is the solution of the original problem of equation (3.1), the residual $\mathcal{E}^n(\hbar)$ vanishes.

3.1 Homotopy for nonlinear transient market impact

Now we apply the Homotopy Analysis Method (HAM) to the solution of the nonlinear integral equation (2.10). Thus

$$\mathcal{N}[v(t)] = -\lambda + \int_0^T G(|t-s|)\; F(v(s),t)\; ds. \tag{3.13}$$

As suggested in [30], we choose the linear operator $\mathcal{L}$ to be the identity, that is

$$\mathcal{L}[\phi(t;p)] = \phi(t;p), \tag{3.14}$$

and the auxiliary function to be $H(t) = 1$. The zero-order deformation equation is then

$$(1-p)\left[\phi(t;p) - v^0(t)\right] = \hbar\, p\, \mathcal{N}[\phi(t;p)]. \tag{3.15}$$

Differentiating this zero-order deformation equation m times with respect to p, and finally dividing by $m!$, we obtain the mth-order deformation equation

$$v^m(t) = \chi^m v^{m-1}(t) + \hbar\, R^m\left(\mathbf{v}^{m-1}\right), \tag{3.16}$$

where for $m > 1$

$$R^m\left(\mathbf{v}^{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}\left[\phi\left(t;p\right)\right]}{\partial p^{m-1}}\Big|_{p=0}$$
$$= \int_0^T G\left(|t-s|\right)\left\{\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}F\left(\phi\left(s;p\right),t\right)}{\partial p^{m-1}}\right|_{p=0}\right\} ds\,. \tag{3.17}$$

For example, the first-order deformation equation is

$$v^1\left(t\right) = \hbar\left(-\lambda + \int_0^T G\left(|t-s|\right)F\left(v^0\left(s\right),t\right)\,ds\right). \tag{3.18}$$

To compute the higher-order deformation equations we need to write F as a function of two homotopy-series, defined respectively at the time points s and t

$$F\left(\sum_{i=0}^{\infty} v^i\left(s\right)p^i, \sum_{j=0}^{\infty} v^j\left(t\right)p^j\right) = \begin{cases} f\left(\sum_{i=0}^{\infty} v^i\left(s\right)p^i\right), & s \leq t \\ \left(\sum_{i=0}^{\infty} v^i\left(s\right)p^i\right)\, f'\left(\sum_{j=0}^{\infty} v^j\left(t\right)p^j\right), & s > t. \end{cases} \tag{3.19}$$

and then apply the homotopy-derivative of equation (A.1), relative to a nonlinear one-dimensional system, for $s \leq t$, and the homotopy-derivative of equation (A.2), relative to a nonlinear two-dimensional system, for $s > t$. The expression of the first homotopy-derivative, for $v^0\left(t\right) > 0$, is given by

$$\frac{\partial}{\partial p}F\left(\sum_{i=0}^{\infty} v^i\left(s\right)p^i, \sum_{j=0}^{\infty} v^j\left(t\right)p^j\right)\Bigg|_{p=0} = \begin{cases} \delta\left(v^0\left(s\right)\right)^{\delta-1}v^1\left(s\right), & s \leq t \\ \delta\, v^1\left(s\right)\left(v^0\left(t\right)\right)^{\delta-1} + \delta\left(\delta-1\right)v^0\left(s\right)\left(v^0\left(t\right)\right)^{\delta-2}v^1\left(t\right), & s > t. \end{cases} \tag{3.20}$$

Higher orders may be computed using symbolic computation software such as Mathematica or Maple.

Obviously, for this algorithm to work, the initial guess needs to satisfy the condition $f'(v^0(t)) < \infty$ for all $t \in [0,T]$. Thus, v^0 must have the same sign on the whole interval $[0,T]$. From now on, we consider a buy program. In order to test for possible dependencies on the initial guess, we choose both the VWAP strategy[2], $v^0_{VWAP}\left(t\right) = X/T$, and the GSS solution for the linear case, *i.e.* v^0_{GSS} given by equation (2.7) as initial guesses.

3.2 A Discrete Homotopy Analysis Method

To apply HAM to our problem, we need to compute the definite integrals (3.17), which seem to be analytically intractable. We therefore propose a way to approximate these integrals, and refer to this discretized version of HAM as the DHAM approach.

[2]VWAP stands for "volume-weighted average price "[31, 32], it is a standard trading strategy.

We first discretize equation (2.10) by splitting the time interval $[0, T]$ into N subintervals at the times $t_i = i\,T/N$ where $i \in \{0, 1, \cdots, N\}$. This gives the following nonlinear system of N equations in the variables $v_i = v\,(t_i)$ where $i \in \{1, \cdots, N\}$

$$\sum_{j=1}^{N} G_{ij} F_{ij}\,(v) = \lambda \tag{3.21}$$

where i indicates the time point t_i. The nonlinear function $F\,(.)$ of equation (2.11) becomes a real $N \times N$ matrix

$$F_{ij}\,(v) = \begin{cases} f\,(v_j)\,, & j \leq i \\ v_j f'\,(v_i)\,, & j > i. \end{cases} \tag{3.22}$$

The decay kernel $G\,(|t-s|)$ becomes a Toeplitz real symmetric $N \times N$ matrix given by

$$G_{ij} = \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} G\,(|t-s|)\; ds\; dt. \tag{3.23}$$

If $G\,(\tau) = \tau^{-\gamma}$, for $i > j$ we have

$$G_{ij} = \frac{1}{(1-\gamma)\,(2-\gamma)} \left(\frac{T}{N}\right)^{2-\gamma} \left\{(i-j+1)^{2-\gamma} - 2\,(i-j)^{2-\gamma} + (i-j-1)^{2-\gamma}\right\} \tag{3.24}$$

and the diagonal terms are given by

$$G_{ii} = \frac{2}{(1-\gamma)\,(2-\gamma)} \left(\frac{T}{N}\right)^{2-\gamma}. \tag{3.25}$$

In this scheme the constraint on the traded volume of equation (2.3) is given by

$$\sum_{i=1}^{N} v_i = \frac{NX}{T}. \tag{3.26}$$

We are not able to solve the nonlinear system of equation (3.21) directly. Rather, we search an approximate homotopy series solution

$$v_i = v_i^0 + \sum_{m=1}^{\infty} v_i^m, \;\; i \in \{1, \cdots, N\}\,, \tag{3.27}$$

where the deformation equations (3.16) for $m > 1$ are approximated by

$$v_i^m = v_i^{m-1} + \hbar \sum_{j=1}^{N} G_{ij}\, F_{ij}^{m-1}, \tag{3.28}$$

and where the $(m-1)$-th homotopy derivative is evaluated on the grid points t_i, s_j

$$F_{ij}^{m-1} = \frac{\partial^{m-1}}{\partial p^{m-1}} F\left(\sum_{k=0}^{\infty} v_i^k\, p^k, \sum_{l=0}^{\infty} v_j^l\, p^l\right)\Big|_{p=0}. \tag{3.29}$$

The approximate solution of order n is given by

$$v_i^{(n)} = v_i^0 + \sum_{m=1}^{n} v_i^m, \tag{3.30}$$

and we compute the squared residual error of equation (3.21) as

$$\mathcal{E}^n\left(\hbar\right) = \sum_{i=1}^{N}\left[-\lambda + \sum_{j=1}^{N} G_{ij}\, F_{ij}\left(v^{(n)}\right)\right]^2. \tag{3.31}$$

The value of $\hbar_{min}$ that minimizes this error gives the DHAM solution $v_i^{(n)}\left(\hbar_{min}\right)$ of equation (2.10). This solution can be considered as a piecewise constant approximation of the exact solution corresponding to a sequence of VWAP executions with trading rates $v_i^{(n)}$.

Finally, the expected liquidation cost (2.2) is approximated by

$$C\left[v^{(n)}\right] = \sum_{i=1}^{N}\sum_{j=1}^{N} v_i^{(n)}\, f\left(v_j^{(n)}\right) A_{ij}, \tag{3.32}$$

where the A_{ij} are elements of a Toeplitz matrix that describes the decay kernel $G\left(t-s\right)$

$$\begin{aligned} A_{ij} &= 0;\ j > i, \\ A_{ii} &= G_{ii}/2; \\ A_{ij} &= G_{ij};\ j \leq i. \end{aligned} \tag{3.33}$$

4 Results

We have implemented the DHAM for the Urysohn equation under exam in MATLAB code using the Parallel Computing Toolbox environment [41]. The MATLAB code ran on a system equipped with an Intel(R) Core i7-3930 @ 3.2 GHz, 16 GB 1333 MHz DDR3. The parallelization was performed on the 12 threads allowed by this processor. The parallel part of the code regards the independent computations of the deformation equations for different values of $\hbar$ in order to find the optimal value $\hbar_{min}$ that minimizes the squared residual error. We present the optimal strategies obtained with DHAM in the no-dynamic-arbitrage region given by equation (2.5), analyzing in detail the value $\gamma = 0.5$ in a strong nonlinear regime, i.e. $\delta = 0.5$. We consider the case where the volume to be purchased is $X = 0.1$, which can be interpreted as a metaorder execution where one buys 10% of the available unitary market volume. We consider two initial guesses, namely one corresponding to a VWAP profile and the other to the GSS solution of eq. (2.7)

There are two critical parameters in the numerical computation if the DHAM solution. The first is the discretization step N of the grid and the second is the order of approximation. We choose as a benchmark case $N = 100$ and we compute the solution up to the 7-th order with a tolerance on the constraint on the total quantity executed of 1‰ of X. The choice of these parameters allows a fast computation time. However in order to check for the convergence properties of the homotopy series solution, it is customary to explore the behavior of the residual square error when these parameters are varied. For this reason we have considered two modifications of the benchmark case. In the first one we keep the 7-th order of approximation but we increase the number of intervals to $N = 400$, while in the second we keep $N = 100$ but we increase the approximation up to the 16-th order. Both choices are much more computational expensive than the benchmark case (see below for details).

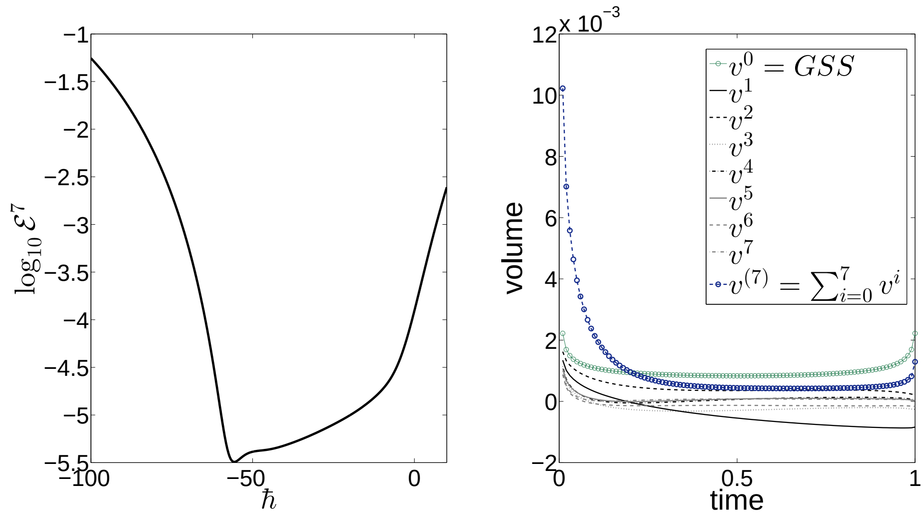

Figure 1: The logarithm of the squared residual $\mathcal{E}^7(\hbar)$ is illustrated on the left panel, the minimum is attained for $\hbar_{min} = -55.7$ where we have $\mathcal{E}^7 = 3.2 \times 10^{-6}$. The GSS initial guess and the DHAM solution are reported on the right panel respectively by a full green line with circles and a dashed blue line with circles, are reported also the results of the seven deformation equations. The nonlinear transient market impact is defined by $\gamma = 0.5$, $\delta = 0.5$.

The results for the benchmark case with GSS initial condition are illustrated in Figure 1. The left panel shows that the minimum of squared residual of the 7-th order iteration is attained at $\hbar = -55.7$. The right panel figures shows the trading profile of the different approximating terms v^i, $(i = 1, ..., 7)$, as well as of the 7-th order approximated solution $v^{(7)}$ obtained by summing the terms. It is worth noticing that higher order terms become smaller and smaller. The DHAM approximated solution is asymmetric with respect to time reversal: it is optimal to trade faster at the beginning of the trading period and slower at the end. Interestingly this asymmetry is similar to the one obtained in a Almgren-Chriss model [13] when the optimization takes into account the variance of the cost (risk averse trader). However the two mechanisms are completely different: an asymmetric trading profile can be due to a risk averse trader using a linear and permanent impact model or to a risk neutral trader using a nonlinear and transient market impact. Finally it is important to notice that different initial conditions (e.g. a VWAP) lead to very similar approximated solutions.

We now discuss the advantage of a DHAM execution with respect to VWAP[3] and GSS executions in terms of expected liquidation costs. The expected liquidation costs of the three strategies are reported in Table 1, where we consider $\gamma = 0.45$ and 0.5 and $\delta \in [1/2, 1]$. As expected, close to the linear case ($\delta = 1$) the difference in cost between the DHAM and the GSS is negligible[4]. Interestingly, the VWAP strategy, which is clearly different from the GSS, has a cost essentially equivalent. Thus in the linear case, the advantage of sophisticated execution strategies relative to a straightforward VWAP is very small. In contrast, as we move toward the strongly nonlinear regime, *i.e.* $\delta \approx 0.5$, the DHAM solution achieves the smallest cost, while the worst strategy is the VWAP. As can be observed from Table 1, the improvement of DHAM with respect to GSS is much larger than the improvement of GSS with respect to VWAP. For example, for $\delta = \gamma = 0.5$, the DHAM has a cost 20% smaller than the GSS, while the latter has a cost which is only 1% smaller than the VWAP. The Table reports also the squared residual error $\mathcal{E}^7(\hbar_{min})$ of the DHAM solution. We observe that the squared residual error is very small, of order 10^{-9}, for $\delta \approx 1$, but increases to an order of 10^{-6} on the strong nonlinear region (as seen in the left panel of Fig. 1). Finally, the computation time required by our computing system to obtain solutions corresponding to the costs given in Table 1 is 3 hours.

We now consider the modification with respect to the benchmark case. In Table 2 we report the cost and the squared residual error when $N = 400$ with 7-th order of approximation and when $N = 100$ with 16-th order of approximation. It is important to notice that the squared residual errors are one order of magnitude smaller than the errors of the benchmark case reported in Table 1. This gives us an indication that homotopy series solution might be convergent. Also the cost reported in Table 2 are up to 4% smaller than those reported for DHAM in Table 1. All these results show that a (slight) improvement from the benchmark case can be obtained by using finer grids and higher order of approximation[5]. However the computation times soar. When $N = 400$ with 7-th order of approximation the time is 185 hours, and for the case $N = 100$ with 16-th order of approximation it is 869 hours. As expected a higher order of approximation seems to guarantee the highest precision, but at a cost of much longer computation times.

[3]The cost of a VWAP execution is $C_{VWAP} = X\, f(X)\, T^{(1-\gamma)} / ((1-\gamma)(2-\gamma))$.

[4]In the linear case, i.e. $\delta = 1$, is possible to find a DHAM strategy with a cost lower than that of GSS strategy. This is only a numerical artifact, because here we implement a discretized version of the GSS strategy defined by (2.7). We compute the cost of a constant piece-wise approximation of the GSS. This approximation is then used as the initial guess of the DHAM solution. In the linear case, when the value of $N \gg 100$ the costs of the approximated GSS and of DHAM strategies become equal.

[5]In both cases the shape of the strategies are very similar to the shape reported in Figure 1 and are not reported.

	VWAP	GSS	DHAM	$\mathcal{E}^7(\hbar_{min})$	VWAP	GSS	DHAM	$\mathcal{E}^7(\hbar_{min})$
δ	$\gamma = 0.45$	$\gamma = 0.45$	$\gamma = 0.45$	$\gamma = 0.45$	$\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 0.5$
1.0	0.0117	**0.0116**	**0.0116**	3.99×10^{-9}	0.0133	0.0132	**0.0131**	3.23×10^{-9}
0.95	0.0132	**0.0130**	**0.0130**	1.15×10^{-8}	0.0150	**0.0148**	**0.0148**	7.96×10^{-9}
0.90	0.0148	0.0146	**0.0143**	3.31×10^{-8}	0.0168	0.0166	**0.0164**	2.43×10^{-8}
0.85	0.0166	0.0164	**0.0162**	7.84×10^{-8}	0.0188	0.0186	**0.0185**	5.63×10^{-8}
0.80	0.0186	0.0184	**0.0179**	1.74×10^{-7}	0.0211	0.0209	**0.0204**	1.26×10^{-7}
0.75	0.0209	0.0206	**0.0198**	3.45×10^{-7}	0.0237	0.0234	**0.0227**	2.52×10^{-7}
0.70	0.0234	0.0231	**0.0218**	6.18×10^{-7}	0.0266	0.0263	**0.0249**	4.60×10^{-7}
0.65	0.0263	0.0260	**0.0235**	8.72×10^{-7}	0.0298	0.0295	**0.0274**	7.43×10^{-7}
0.60	0.0295	0.0291	**0.0251**	8.93×10^{-7}	0.0335	0.0331	**0.0297**	8.47×10^{-7}
0.55	0.0331	0.0327	**0.0275**	2.66×10^{-6}	0.0376	0.0372	**0.0323**	2.25×10^{-6}
0.50					0.0422	0.0417	**0.0347**	3.25×10^{-6}

Table 1: Costs for three different strategies, VWAP, GSS, and DHAM, on the no-dynamic-arbitrage region for $\gamma = 0.45$, 0.5 and $N = 100$. The numbers in boldface indicate strategies achieving the lowest expected cost. The difference between expected costs increases with the degree of non-linearity. In each case we use a GSS initial guess to obtain the DHAM solution. For DHAM strategy we report also the squared residual error $\mathcal{E}^7(\hbar_{min})$ of eq. 3.31 for the 7-th order iteration.

	$C_{N=400}$	$\mathcal{E}^7(\hbar_{min})$	$C_{N=100}$	$\mathcal{E}^{16}(\hbar_{min})$	$C_{N=400}$	$\mathcal{E}^7(\hbar_{min})$	$C_{N=100}$	$\mathcal{E}^{16}(\hbar_{min})$
δ	$\gamma = 0.45$	$\gamma = 0.45$	$\gamma = 0.45$	$\gamma = 0.45$	$\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 0.5$
1.0	0.0115	$\mathbf{4.93 \times 10^{-10}}$	0.0115	5.82×10^{-10}	0.0132	5.36×10^{-10}	0.0130	$\mathbf{3.01 \times 10^{-10}}$
0.95	0.0129	2.03×10^{-9}	0.0129	$\mathbf{1.71 \times 10^{-9}}$	0.0146	1.79×10^{-9}	0.0148	$\mathbf{7.31 \times 10^{-10}}$
0.90	0.0144	7.19×10^{-9}	0.0144	$\mathbf{5.39 \times 10^{-9}}$	0.0165	5.39×10^{-9}	0.0164	$\mathbf{2.66 \times 10^{-9}}$
0.85	0.0160	2.02×10^{-8}	0.0161	$\mathbf{1.41 \times 10^{-8}}$	0.0182	1.53×10^{-8}	0.0183	$\mathbf{7.15 \times 10^{-9}}$
0.80	0.0177	4.72×10^{-8}	0.0179	$\mathbf{2.80 \times 10^{-8}}$	0.0202	3.58×10^{-8}	0.0204	$\mathbf{1.56 \times 10^{-8}}$
0.75	0.0197	9.76×10^{-8}	0.0194	$\mathbf{7.35 \times 10^{-8}}$	0.0228	7.55×10^{-8}	0.0225	$\mathbf{4.03 \times 10^{-8}}$
0.70	0.0216	1.78×10^{-7}	0.0212	$\mathbf{1.01 \times 10^{-7}}$	0.0249	1.41×10^{-7}	0.0247	$\mathbf{7.59 \times 10^{-8}}$
0.65	0.0235	2.64×10^{-7}	0.0233	$\mathbf{2.46 \times 10^{-7}}$	0.0272	2.29×10^{-7}	0.0271	$\mathbf{1.59 \times 10^{-7}}$
0.60	0.0255	$\mathbf{5.05 \times 10^{-7}}$	0.0248	5.21×10^{-7}	0.0297	4.20×10^{-7}	0.0295	$\mathbf{2.32 \times 10^{-7}}$
0.55	0.0271	$\mathbf{7.73 \times 10^{-7}}$	0.0263	8.54×10^{-7}	0.0323	6.97×10^{-7}	0.0317	$\mathbf{5.07 \times 10^{-7}}$
0.50					0.0340	$\mathbf{9.62 \times 10^{-7}}$	0.0332	9.63×10^{-7}

Table 2: Costs and squared residual errors $\mathcal{E}^7(\hbar_{min})$ of DHAM strategies for the cases: $N = 400$ with 7-th order of approximation and $N = 100$ with 16-th order of approximation. The numbers in boldface indicate strategies achieving the lowest squared residual error.

5 Conclusions

In this paper, we have studied the problem of finding optimal execution strategies in the nonlinear transient impact model using a discrete homotopy analysis approach. If the class of admissible strategies is constrained so as to eliminate the possibility of zero trading rates, the cost minimization problem may be cast as an Urysohn integral equation of the first kind. The solutions obtained are by construction continuous deformation of initial guesses, such as VWAP. We have shown that the optimal solution found in this way is not time-reversal symmetric, but front-loaded in the case of concave market impact. Our expected cost analysis shows that such solutions substantially outperform conventional execution strategies such as VWAP and GSS. We have also showed how computationally expensive may be the use of DHAM when we want to achieve a low squared residual error. This can imply the necessity of the use of high performance computing systems in order to achieve tiny values for squared residual errors. It might be interesting to study if the use of better initial conditions for the homotopy analysis, for example one which shares the asymmetric profile observed in this paper, speeds up the computations of high order derivatives. We leave this for a future paper. Finally it is important to stress a much richer phenomenology can be observed by allowing the possibility of vanishing zero rates and arbitrarily high trading speed [23].

A Appendices

A.1 Homotopy derivatives

As explained in Section 3, the so-called homotopy derivative is used to deduce deformation equations of equation (3.8) for any order greater than one, *i.e.* to compute v^m for $m > 1$. Such computations are difficult because they depend on the nonlinear operator $\mathcal{N}$. For our problem, we need to compute the homotopy derivative of a power law function.

A nonlinear power law function with a integer exponent, *i.e.* $f(\phi) = \phi^k,\ k \in \mathtt{N}$, was studied by Molabahrami and Khani [42] to find an approximate solution of the Burgers-Huxley equation. The case of a real power-law index was studied by Wang et al. [52] analyzing the flow of a power-law fluid film on an unsteady stretching surface. Our case (2.11) is more complicated than the case of a simple power-law function. Recent results given by Turkyilmazoglu [51] and Liao [37] show how to compute the homotopy-derivative of any smooth function. The homotopy-derivative $\mathcal{D}^m [f(\phi)] = (1/m!)\, \partial^m f(\phi) / \partial p^m$ is given by the recursive relation

$$\mathcal{D}^m [f(\phi)] = \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}^{m-k} [\phi] \frac{\partial}{\partial \phi} \left\{ \mathcal{D}^k [f(\phi)] \right\}, \tag{A.1}$$

evaluated at $p = 0$. The sum consists of two homotopy-derivative terms. The first gives v^{m-k}, the second term gives polynomial terms of v^i that multiply $f^i(v^0)$, $i = 1, \cdots, m$, derivatives of f evaluated at the initial guess. If the market impact function f is of the form $f(v) \propto v^\delta$ with $\delta < 1$, all such derivatives diverge at $v = 0$. As mentioned in Section 3, we avoid this problem by choosing the initial guess to be a strictly positive (or negative) function of time. Using the $m-1$-th order of equation (A.1) in equation (3.9) we can express the m-th homotopy derivative v^m as a complicated function of the previous $m-1$ derivatives.

To handle the nonlinearity of equation (2.11) in the HAM framework we need a further step. The coupling between past and future values of trading rates implies that we have to consider our problem as a two-dimensional system. This means that we use the homotopy-derivative for systems described by two variables [37], for example u and w, in which we have a nonlinear coupling between them given by $f\left(u,w\right)$

$$\mathcal{D}^m\left[f\left(\phi,\psi\right)\right]=\sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)u^{m-k}\mathcal{D}^k\left[\frac{\partial f\left(\phi,\psi\right)}{\partial\phi}\right]+\left(1-\frac{k}{m}\right)w^{m-k}\mathcal{D}^k\left[\frac{\partial f\left(\phi,\psi\right)}{\partial\psi}\right],\qquad (A.2)$$

where $\phi\,,\psi$ are the Maclaurin series of $u\,,w$ respectively. Thus, given the expression of equation (2.11) as $F\left(v\left(s\right),v\left(t\right)\right)$, we are able to apply the HAM to our optimal execution problem using the equation (A.2) and choosing a function with a given sign as initial guess.

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