

Asymptotic Morse Theory for the Equation

$$\Delta v = 2v_x \wedge v_y$$

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Given a smooth bounded domain $\Omega \subseteq \mathbb{R}^2$, we consider the equation $\Delta v = 2v_x \wedge v_y$ in Ω , where $v : \Omega \rightarrow \mathbb{R}^3$. We prescribe Dirichlet boundary datum, and consider the case in which this datum converges to zero. An asymptotic study of the corresponding Euler functional is performed, analyzing multiple-bubbling phenomena. This allows us to settle a particular case of a question raised by H. Brezis and J.M. Coron in [9].

Key words: H -surfaces, Robin function, non-compactness, nonlinear elliptic systems.

1. Introduction.

Let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded domain. We shall denote by v, \tilde{g} two maps such that $v : \Omega \rightarrow \mathbb{R}^3$ and $\tilde{g} : \partial\Omega \rightarrow \mathbb{R}^3$, with \tilde{g} smooth. Consider the problem

$$\begin{cases} \Delta v = H(\xi, v, \nabla v)v_x \wedge v_y & \text{in } \Omega, \\ v = \tilde{g} & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where H is a smooth scalar function, v_x, v_y are the x and y -derivatives of v , $\xi = (x, y)$ and \wedge denotes the cross-product in \mathbb{R}^3 .

Equation (1) has been the subject of several works, see for example the survey paper [28] by K. Steffen and the recent paper [10]. Existence of solutions of (1) when $\tilde{g} \equiv 0$ strongly depends on the topology of the domain. In fact we show using a Pohozahev-type identity, see Proposition 3.1, that equation (1) has no solution in any simply connected domain when $\tilde{g} = 0$. When $H(\xi, v, \nabla v) \equiv H$, a non-zero constant, such a result was proved by H. Wentz, [16], using reflection techniques and the Kelvin transformation. In the same paper, Wentz also showed that if Ω is an annulus then the study of (1) can be reduced to an ordinary differential equation and (1) does have a non-trivial solution when $v = 0$ on $\partial\Omega$. Thus equation (1) presents features similar to the Yamabe equation on domains with Dirichlet boundary

conditions, studied in particular by A. Bahri and J.M. Coron, [4]. In fact, part of the difficulty in studying (1) is that it is invariant under conformal transformations. This invariance forces the associated variational problem to exhibit non-compactness phenomena, like in the Yamabe problem on domains. We point out that in our case, contrary to the Yamabe problem, simply connected domains always admit only trivial solutions. For the Yamabe problem in dimension greater or equal than three, there are indeed examples of contractible domains which admit non-trivial positive solutions, see [23].

From now on we consider the case of constant H , precisely $H(\xi, v, \nabla v) \equiv 2$. So problem (1) reduces to

$$\begin{cases} \Delta v = 2v_x \wedge v_y, & \text{in } \Omega, \\ v = \tilde{g} & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Under the assumption $\|\tilde{g}\|_\infty < 1$, S. Hildebrandt, [18], constructed a solution of (2) with minimal energy called the *small solution*, while Brezis and Coron, [8], K.Steffen, [27] and M. Struwe, [29], constructed a second solution, referred to as the *large solution*. We remark that the assumption $\|\tilde{g}\|_\infty < 1$ is sharp, see [17].

Results similar to those regarding the Dirichlet problem hold for the *Plateau problem*, in which one looks for solutions of $\Delta u = Hu_x \wedge u_y$ which are conformal and which map the boundary to a given curve (with free parametrization). As a result one obtains surfaces with constant mean curvature.

We mainly focus on the following problem

$$\begin{cases} \Delta v = 2v_x \wedge v_y, & \text{in } \Omega, \\ u = \varepsilon \tilde{g} & \text{on } \partial\Omega. \end{cases} \quad (3)$$

We will study (3) turning it into a variational problem. In view of the non-existence result in [17], it is natural to assume that the boundary datum is small. T. Isobe in particular, [20]-[22], analyzed the behavior of the large solutions of Brezis and Coron in the limit $\varepsilon \rightarrow 0$ (the small solutions converges to the trivial one $v \equiv 0$ as $\varepsilon \rightarrow 0$).

Let g denote the harmonic extension of \tilde{g} in Ω , i.e.

$$\begin{cases} \Delta g = 0 & \text{in } \Omega; \\ g = \tilde{g} & \text{on } \partial\Omega. \end{cases} \quad (4)$$

If v is a solution of (3) and if we set $v = u + \varepsilon g$, the function u solves

$$\begin{cases} \Delta u = \Delta v = 2(u_x + \varepsilon g_x) \wedge (u_y + \varepsilon g_y) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (P_\varepsilon)$$

Problem (P_ε) admits the Euler functional $I_\varepsilon : H_0^1(\Omega; \mathbb{R}^3) \rightarrow \mathbb{R}$, which has the following expression

$$I_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{2}{3} \int_\Omega u \cdot (u_x \wedge u_y) + \varepsilon \int_\Omega u \cdot (u_x \wedge g_y + g_x \wedge u_y) + 2\varepsilon^2 \int_\Omega u \cdot (g_x \wedge g_y). \tag{5}$$

The aim of this paper is to develop a Morse theory for the functional I_ε when ε is small. In order to do this we take advantage of the perturbative approach in [1]. We first recall from [9] that the *fundamental solution (bubble)* of the equation

$$\Delta u = 2u_x \wedge u_y, \quad \text{in } \mathbb{R}^2 \tag{6}$$

is the stereographic projection $\pi : \mathbb{R}^2 \rightarrow S^2 \subseteq \mathbb{R}^3$

$$\pi(x, y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2} \right), \quad (x, y) \in \mathbb{R}^2. \tag{7}$$

Our analysis will use translations, dilations and rotations of the function in (7) and we set

$$R\delta_{a,\lambda}(x, y) \equiv R \circ \pi(\lambda(x - a_1, y - a_2)), \tag{8}$$

for $R \in SO(3)$, $a = (a_1, a_2) \in \mathbb{R}^2$ and $\lambda > 0$. The functions $R\delta_{a,\lambda}$ are mountain-pass critical points of the functional

$$\bar{I}(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{2}{3} \int_{\mathbb{R}^2} u \cdot (u_x \wedge u_y), \quad u \in \mathcal{D}, \tag{9}$$

where \mathcal{D} denotes the functional space

$$\mathcal{D} = \left\{ u \in L_{loc}^2(\mathbb{R}^2; \mathbb{R}^3) : \|u\|_{\mathcal{D}}^2 = \int_{\mathbb{R}^2} |\nabla u|^2 + \int_{\mathbb{R}^2} \frac{|u|^2}{(1+|\xi|^2)^2} < +\infty \right\}.$$

The space \mathcal{D} coincides with $H^1(S^2; \mathbb{R}^3)$ after inverse stereographic projection. We point out that the functionals \bar{I} and I_ε are well defined and smooth on \mathcal{D} and $H_0^1(\Omega, \mathbb{R}^3)$ respectively, see Section 2. It turns out that the manifold constituted by the δ 's is non-degenerate for the functional \bar{I} (modulo constants), as proved in [21] Lemma 5.5, using an isoperimetric inequality. Proving the non-degeneracy condition is equivalent to classify the solutions of

$$\Delta w = 2(w_x \wedge \delta_y + \delta_x \wedge w_y) \quad \text{in } \mathbb{R}^2, \tag{10}$$

which is the linearization of (6) around δ , and to show that the only solutions are the tangent vectors to \bar{Z} at δ , see equation (16). We remark that equation

(6) admits solutions of the form $\pi(z^k)$ (in complex notation) for any integer k , see [9]. We will refer to these solutions as *higher degree bubbles*. For this reason we give in the Appendix an alternative proof of the non-degeneracy, which we believe could adapt naturally to the higher-degree case.

To analyze the problem in Ω , it is convenient to consider the functions $P\delta = \delta - \varphi$, where φ is defined in (21). $P\delta$ is the element of $H_0^1(\Omega; \mathbb{R}^3)$ closest to δ in the Dirichlet norm. We may write

$$u = \sum_{i=1}^k PR_i \delta_{p_i, \lambda_i} + w, \quad (11)$$

where $R_i \in O(3)$, $\lambda_i > 0$, $p_i \in \mathbb{R}^2$ for all i , and w is orthogonal to the manifold $\sum_{i=1}^k PR_i \delta_{p_i, \lambda_i}$. Once we have the non-degeneracy property for \bar{I} , then it is standard to prove that for suitable values of a and λ also the manifold of projected bubbles is non-degenerate for I_ε , and the same holds true for a finite sum of bubbles. This property allows us to solve the equation $I'_\varepsilon(u) = 0$ in w (see Proposition 4.3), and thus our problem is reduced to a finite-dimensional one which involves an auxiliary functional $\tilde{I}_\varepsilon(z)$ (see Section 4) depending only on $\{p_i\}_i, \{\lambda_i\}_i$ and $\{R_i\}_i$. Substituting (11) into I_ε and letting $\varepsilon \rightarrow 0$, we expand $\tilde{I}_\varepsilon(z)$ for large values of λ_i (roughly of order ε^{-1}).

The large solution of Brezis and Coron corresponds to a *one bubble solution* when $\varepsilon \rightarrow 0$, and has been studied in detail by T. Isobe, [20]-[22]. However, from Theorem 0.3 in [9] it is clear that a more complicated configuration may occur. Thus to manufacture this type of solutions we are naturally led to a variational analysis of the functional (5) for multiple bubbles. We point out that from the work of Brezis-Coron the bubbles will not necessarily be all of degree 1. However the variational analysis is more difficult if we allow bubbles of arbitrary degree, and we will return to this point in a subsequent article.

To state our results we need some notation. Given $(a, \xi) = ((a_1, a_2), (x, y)) \in \Omega \times \Omega$, let $h_1, h_2 : \Omega \times \Omega \rightarrow \mathbb{R}$ be the solutions of the problems

$$\begin{cases} \Delta_\xi h_1(a, \xi) = 0 & \text{in } \Omega, \\ h_1(a, \xi) = \frac{\xi_1 - a_1}{|\xi - a|^2} & \text{on } \partial\Omega; \end{cases} \quad \begin{cases} \Delta_\xi h_2(a, \xi) = 0 & \text{in } \Omega, \\ h_2(a, \xi) = \frac{\xi_2 - a_2}{|\xi - a|^2} & \text{on } \partial\Omega, \end{cases} \quad (12)$$

see Remark 5.3 (b). If $G(a, \xi)$ denotes the Green's function of Ω , normalized so that $G(a, \xi) \sim -\log|a - \xi|$ for $a \sim \xi$, and if $H(a, \xi)$ denotes the regular

part of G ($G(a, \xi) = -\log |a - \xi| - H(a, \xi)$), then we have

$$h_1(a, \xi) = \frac{\partial H(a, \xi)}{\partial a_1}; \quad h_2(a, \xi) = \frac{\partial H(a, \xi)}{\partial a_2}. \quad (13)$$

Let also

$$\tilde{H}(a) = \left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y} \right) \Big|_{\xi=a}. \quad (14)$$

It has been proved in [20]-[22] (see also [26]) that the function \tilde{H} plays a crucial role in studying the location of blowing-up solutions of (2), when the boundary datum converges to zero. In fact, the function \tilde{H} appears in the expansion of $I_\varepsilon(u)$, when u is of the form (11) with $k = 1$, as a self-interaction term, see Proposition 5.1. The expansion for $k = 1$ is essentially performed in the works of Isobe, but we derive it in a framework which is convenient to treat the case of $k > 1$, see Section 7. We have the following result, regarding the function \tilde{H} .

Theorem 1.1. (a) *For a simply connected domain Ω there holds*

$$\tilde{H}(a) = 2e^{2H(a,a)}, \quad a \in \Omega,$$

where $H(a, \xi)$ is the regular part of the Green's function of Ω .

(b) *For a multiply connected domain Ω there not exist in general a function F such that $\tilde{H}(a) = F(H(a, a))$. In particular, for some annulus of the form $\{\rho^{-1} < |z| < \rho\}$, $\rho > 1$, the critical points of $H(a, a)$ and of $\tilde{H}(a)$ do not coincide.*

Theorem 1.1 is proved in Section 6. The function $H(a, a)$ is called the Robin function of the domain Ω , see [6]. In dimension 2 it also appears in extremal problems related to the Moser-Trudinger inequality, where the critical points of $H(a, a)$ are shown to be related to the conformal incenter of Ω . Isobe showed that $\tilde{H} > 0$ on any domain, see Remark 5.3, but did not analyze it further. Since \tilde{H} is defined by means of second derivatives of H , we need to use a global argument (the Riemann mapping theorem) to compare the two functions H and \tilde{H} . The new feature of Theorem 1.1 is that the Robin function plays a role in concentration phenomena only for the case of simply connected domains.

The regular part of the Green's function plays an important role in many problems with critical exponent in dimension larger than two, see [2], [5], [7], [15], [24], [25]. The difference here is that the regular part does not appear directly in the expansion, we find the above function \tilde{H} instead, and we

recover the regular part from the Riemann mapping Theorem. The Robin function is also related to the notion of conformal incenter, see [14].

The expansion of $I_\varepsilon(u)$ for multiple bubbles is performed in Section 7, see Proposition 7.4. It turns out that when $\lambda_i \sim \varepsilon^{-1}$ for all i , the mutual interaction among the bubbles is of the same order as the interaction with the boundary (through both the geometry of Ω and the datum g). We observe that the interaction among the bubbles depend on their mutual orientation.

There is a by-product of the expansion in Proposition 7.4. It allows us to settle a particular case of a question raised by Brezis and Coron, see Section 8. In [9] the authors consider a sequence of solutions u_n of (1) and a sequence g_n of boundary data which converge to zero in $H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$. Under these conditions they prove that the sequence u_n *splits* into a finite number of bubbles, and their image converge to a finite and connected union of spheres of radius 1. They ask whether every configuration of spheres can be obtained as a limit of solutions u_n for a suitable sequence of boundary data g_n . We have an affirmative answer if all the spheres pass through the origin.

Theorem 1.2. *Let D denote the unit disk in \mathbb{R}^2 , and let $A = \{S_1 \cup \dots \cup S_k\}$ be any configuration of unit spheres, each passing through the origin of \mathbb{R}^3 . Then there exist a sequence $\tilde{g}_n : S^1 \rightarrow \mathbb{R}^3$ and a sequence of functions u_n solving*

$$\begin{cases} \Delta v = 2v_x \wedge v_y & \text{in } D, \\ v = \tilde{g}_n & \text{on } \partial D, \end{cases}$$

such that the image of the function u_n converge to A in the Hausdorff sense.

The functions u_n in Theorem 1.2 are constructed studying the interactions of the bubbles (of degree 1) with the boundary datum and among themselves. Choosing boundary data with an appropriate *strong concentration* at k points on ∂D , we show that the self interaction among the bubbles becomes negligible. Hence we can find solutions u_n which are highly concentrated at k points close to the boundary of D and with prescribed orientations in \mathbb{R}^3 . We remark that the order of concentration, roughly the parameter λ in (8), turns out to be the same for all the bubbles.

The case of spheres not passing through the origin is not treated here. We believe that it could be possible to achieve such configurations by considering bubbles with higher degree. In fact, in the recent paper by A. Bahri and S. Chanillo, [3], the authors showed that when considering changing-sign solutions of the Yamabe problem, the bubbles can exhibit different orders of concentration. If there is an analogy between higher-degree bubbles

and changing-sign solutions of the Yamabe equation, then one could obtain bubbles with higher and higher concentration and with image not passing through the origin. This will be the object of a future work.

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2. Notation and preliminary facts.

In this Section we introduce some notation and preliminary facts in order to tackle problem (3).

In the following D will denote the unit disk in \mathbb{R}^2

$$D = \{\xi = (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

Let $\bar{I} : \mathcal{D} \rightarrow \mathbb{R}$ be defined by (9). From [12] the last term in $\bar{I}(u)$ is well defined on \mathcal{D} , together with its Frechet derivatives. This makes \bar{I} a smooth functional on \mathcal{D} . The same argument provides regularity of the functional I_ε on $H_0^1(\Omega; \mathbb{R}^3)$.

Using a finite-dimensional reduction, we are going to treat the functional I_ε as a perturbation of \bar{I} . In order to do this, it is essential to consider the critical points of the functional \bar{I} , namely the solutions of

$$\Delta u = 2u_x \wedge u_y \quad \text{in } \mathbb{R}^2, \quad u \in \mathcal{D}. \quad (15)$$

The stereographic projection (7) is indeed a solution of (15), which we call *fundamental solution or bubble*. By invariance its translations, dilations and rotations are also solutions of (15). We set

$$\bar{\mathcal{Z}} = \{R \delta_{a,\lambda}(\cdot) = R \delta_\lambda(\cdot - a) : \lambda > 0, a \in \mathbb{R}^2, R \in SO(3)\}. \quad (16)$$

We remark that, since $SO(3)$ is a three-dimensional manifold, $\bar{\mathcal{Z}}$ is a six-dimensional manifold.

We list now some useful expressions. Note that the function $\delta(\lambda(\xi - a))$ has the explicit form

$$\delta(\lambda(\xi - a)) = \left(\frac{2\lambda(\xi - a)}{1 + \lambda^2|\xi - a|^2}, \frac{\lambda^2|\xi - a|^2 - 1}{\lambda^2|\xi - a|^2 + 1} \right),$$

from which, if a is bounded away from $\partial\Omega$, one can deduce

$$\pi(\lambda(\xi - a)) \sim \left(\frac{2(\xi - a)}{\lambda|\xi - a|^2}, 1 - \frac{2}{\lambda^2|\xi - a|^2} \right) + O(\lambda^{-3}), \quad \text{for } \lambda \text{ large.} \quad (17)$$

Writing for brevity δ instead of $\delta_{a,\lambda}$, we compute some derivatives of δ . We emphasize that throughout the paper, unless explicitly stated, the point a will always be bounded away from $\partial\Omega$, namely we will assume $\text{dist}(a, \partial\Omega) \geq \tau_0$ for some fixed $\tau_0 > 0$. We have

$$\begin{aligned} (\delta_x)_1 &= 2\lambda \frac{1 + \lambda^2(y^2 - x^2)}{(1 + \lambda^2(x^2 + y^2))^2}; & (\delta_x)_2 &= -\frac{4\lambda^3 xy}{(1 + \lambda^2(x^2 + y^2))^2}; \\ (\delta_x)_3 &= \frac{4\lambda^2 x}{(1 + \lambda^2(x^2 + y^2))^2}; \end{aligned} \quad (18)$$

$$\begin{aligned} (\delta_y)_1 &= -\frac{4\lambda^3 xy}{(1 + \lambda^2(x^2 + y^2))^2}; & (\delta_y)_2 &= 2\lambda \frac{1 + \lambda^2(x^2 - y^2)}{(1 + \lambda^2(x^2 + y^2))^2}; \\ (\delta_y)_3 &= \frac{4\lambda^2 y}{(1 + \lambda^2(x^2 + y^2))^2}. \end{aligned}$$

From the last two formulas we deduce

$$(\delta_x \wedge \delta_y)_1 = -\frac{8\lambda^3 x}{(1 + \lambda^2(x^2 + y^2))^3}; \quad (\delta_x \wedge \delta_y)_2 = -\frac{8\lambda^3 y}{(1 + \lambda^2(x^2 + y^2))^3}; \quad (19)$$

$$(\delta_x \wedge \delta_y)_3 = 4\lambda^2 \frac{1 - \lambda^2(x^2 + y^2)}{(1 + \lambda^2(x^2 + y^2))^3}. \quad (20)$$

The functions $R\delta_{a,\lambda}|_\Omega$ do not belong to $H_0^1(\Omega; \mathbb{R}^3)$ since they are non-zero at the boundary. Following [2], [25], it is convenient to project these functions on the space $H_0^1(\Omega; \mathbb{R}^3)$, by subtracting the harmonic function on Ω with the same boundary data. Let $\varphi : \Omega \rightarrow \mathbb{R}^3$ be the unique solution of the problem

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega, \\ \varphi = \delta & \text{on } \partial\Omega, \end{cases} \quad (21)$$

and set $P\delta = \delta - \varphi$. We will often omit the dependence of φ on the parameters a, λ, R , as for δ .

From (18) and some standard computations it is easy to find that, in the case $R = Id$

$$\varphi = \left(\frac{2}{\lambda} h_1(\xi, a) + o(\lambda^{-1}), \frac{2}{\lambda} h_2(\xi, a) + o(\lambda^{-1}), 1 - \frac{2}{\lambda^2} h_3(\xi, a) + o(\lambda^{-2}) \right), \quad (22)$$

and

$$(\delta - \varphi) = \left(\begin{array}{c} \frac{2\lambda(x-a_1)}{1+\lambda^2|\xi-a|^2} - \frac{2}{\lambda} h_1(\xi, a) + o(\lambda^{-1}) \\ \frac{2\lambda(y-a_2)}{1+\lambda^2|\xi-a|^2} - \frac{2}{\lambda} h_2(\xi, a) + o(\lambda^{-1}) \\ -\frac{2}{1+\lambda^2|\xi-a|^2} + \frac{2}{\lambda^2} h_3(\xi, a) + o(\lambda^{-2}) \end{array} \right), \quad (23)$$

where h_1 and h_2 are defined in (12), and where h_3 is the solution of

$$\begin{cases} \Delta_\xi h_3(a, \xi) = 0 & \text{in } \Omega, \\ h_3(a, \xi) = \frac{1}{|\xi-a|^2} & \text{on } \partial\Omega. \end{cases} \quad (24)$$

The quantities $o(\lambda^{-1})$ and $o(\lambda^{-2})$ in formulas (22) and (23) denote functions which $C^k(\bar{\Omega})$ -norm, for any $k \in \mathbb{N}$, is of order $o(\lambda^{-1})$ and $o(\lambda^{-2})$ respectively.

We collect some further estimates, whose proof are trivial, and which we will use later. Given a fixed positive constant $\tau \leq \frac{\tau_0}{2}$, for λ sufficiently large there holds

$$\int_{B_\tau} |P\delta| \leq \frac{C}{\lambda}; \quad \int_{B_\tau} |\nabla\delta| \leq C \frac{\log \lambda}{\lambda}; \quad \int_{B_\tau} |\nabla P\delta| \leq C \frac{\log \lambda}{\lambda}; \quad \int_{B_\tau} |\delta|^2 \leq C \frac{\log \lambda}{\lambda^2}; \quad (25)$$

$$|P\delta|(x) + |\nabla\delta|(x) \leq \frac{C}{\lambda}, \quad \forall x \in \Omega \setminus B_\tau; \quad |\varphi - (0, 0, 1)|(x) + |\nabla\varphi|(x) \leq \frac{C}{\lambda}, \quad \forall x \in \Omega. \quad (26)$$

In the following, for brevity of notation, the constant C will be allowed to vary from formula to formula and from line to line.

For $k \geq 1$ and for $i, j \in \{1, \dots, k\}$, $i \neq j$, we will use the following notation

$$\tilde{e}(\varepsilon, \lambda_1, \dots, \lambda_k) = O(\varepsilon^2) + \sum_{i=1}^k O(\varepsilon \lambda_i^{-1} |\log \lambda_i|) + \sum_{i,j} O\left(\frac{1}{\lambda_i \lambda_j}\right);$$

$$e(\varepsilon, \lambda_i) = o(\varepsilon^2) + o(\varepsilon \lambda_i^{-1});$$

$$e(\lambda_i, \lambda_j) = O\left((\log \lambda_i + \log \lambda_j) \left(\frac{1}{\lambda_i^3} + \frac{1}{\lambda_j^3} + \frac{1}{\lambda_i^2 \lambda_j} + \frac{1}{\lambda_i \lambda_j^2}\right)\right);$$

$$e(\varepsilon, \lambda_1, \dots, \lambda_k) = o(\varepsilon^2) + \sum_{i=1}^k o(\varepsilon \lambda_i^{-1}) + \sum_{i < j} e(\lambda_i, \lambda_j) + \sum_{i < j < k} O\left(\frac{1}{\lambda_i \lambda_j \lambda_k}\right).$$

We will often make use of the identity

$$\int_{\mathbb{R}^2} \frac{1 - |\xi|^2}{(1 + |\xi|^2)^3} = 0, \quad (27)$$

which is immediate to verify (the integrand is indeed the third component of $\Delta\delta$ up to a multiplicative constant).

3. A non-existence result via a Pohozaev-type identity.

In this section we prove a Pohozaev-type identity for the H -surface equation. The proof is elementary and extends a previous result of Wentz, see [31].

Proposition 3.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded and simply-connected domain, and let $v \in C^2(\overline{\Omega}; \mathbb{R}^3)$ be a solution of*

$$\begin{cases} \Delta v = H(\xi, v, \nabla v) v_x \wedge v_y, & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (28)$$

for some continuous function $H(\xi, v, \nabla v)$. Then $v \equiv 0$ in Ω .

PROOF. We assume first that the domain Ω is the unit disk D . In the spirit of the Pohozaev identity, we consider the quantity $\sum_{i=1}^3 (\xi \cdot \nabla v_i) \Delta v_i$ and integrate on D . We claim that

$$\sum_{i=1}^3 (\xi \cdot \nabla v_i) (v_x \wedge v_y)_i = 0, \quad \xi = (x, y). \quad (29)$$

Once (29) is proved, we have

$$\sum_{i=1}^3 (\xi \cdot \nabla v_i) \Delta v_i \equiv 0. \quad (30)$$

Integrating (30) over D and taking into account that the dimension is 2, we find

$$\frac{1}{2} \sum_{i=1}^3 \int_{\partial D} \left(\frac{\partial v_i}{\partial \nu} \right)^2 = \frac{1}{2} \sum_{i=1}^3 \int_{\partial D} (\xi \cdot \nu) \left(\frac{\partial v_i}{\partial \nu} \right)^2 = 0,$$

where ν denoted the exterior unit normal to ∂D . As a consequence we have $\frac{\partial v}{\partial \nu} = 0$ on ∂D . Thus, extending v to zero on the complement of D and also extending H continuously outside D we obtain a C^1 solution of $\Delta v = H(\xi, v, \nabla v)v_x \wedge v_y$ in \mathbb{R}^2 . Hence, applying Theorem 1 in [16] we obtain $v \equiv 0$ in D . Let us now verify (29): using simple computation we find

$$\begin{aligned} \sum_{i=1}^3 (\xi \cdot \nabla v_i)(v_x \wedge v_y)_i &= x [(v_1)_x(v_2)_x(v_3)_y - (v_1)_x(v_3)_x(v_2)_y + (v_2)_x(v_1)_y(v_3)_x \\ &\quad - (v_2)_x(v_1)_x(v_3)_y + (v_3)_x(v_1)_x(v_2)_y - (v_3)_x(v_2)_x(v_1)_y] \\ &\quad + y [(v_1)_x(v_2)_x(v_3)_y - (v_1)_x(v_3)_x(v_2)_y + (v_2)_x(v_1)_y(v_3)_x \\ &\quad - (v_2)_x(v_1)_x(v_3)_y + (v_3)_x(v_1)_x(v_2)_y - (v_3)_x(v_2)_x(v_1)_y] = 0. \end{aligned}$$

This concludes the proof in the case $\Omega = D$. For the general case of a simply-connected domain, it is sufficient to use the Riemann Mapping Theorem and the transformation rule of (28) under conformal mappings. We recall that for Ω smooth, the Riemann map is also smooth up to the boundary, see [30].

■

4. The finite-dimensional reduction.

In this section we show how problem (3) can be reduced to a finite-dimensional one for small values of ε . The starting point is the following Proposition, proven in [21] (Lemma 5.5) using an isoperimetric inequality. We give an alternative proof in the Appendix, using the stereographic projection and shifting the problem from \mathbb{R}^2 to S^2 . We believe that our proof could be naturally extended to the case of higher degree bubbles.

Proposition 4.1. *There exists a constant $C_0 > 0$ such that*

$$\overline{T}''(R\delta_{a,\lambda})[R\delta_{a,\lambda}, R\delta_{a,\lambda}] \leq -C_0 \|R\delta_{a,\lambda}\|^2, \quad \text{for all } R\delta_{a,\lambda} \in \overline{\mathcal{Z}}.$$

and

$$\overline{T}''(R\delta_{a,\lambda})[v, v] \geq C_0 \|\nabla v\|_{L^2(\mathbb{R}^2)}^2, \quad \text{for all } R\delta_{a,\lambda} \in \overline{\mathcal{Z}} \text{ and } v \perp (T_{R\delta_{a,\lambda}} \overline{\mathcal{Z}} \oplus \{tR\delta_{a,\lambda}\}_t).$$

In particular, the equation $\overline{T}''(R\delta_{a,\lambda})[v] = 0$ implies $v - c \in T_{R\delta_{a,\lambda}} \overline{\mathcal{Z}}$ for some $c \in \mathbb{R}^3$.

We are going to consider now problem (15) on the domain Ω . Given $\overline{C} > 0$ we set

$$Z = \left\{ \sum_{i=1}^k P\delta_{p_i, \lambda_i} : \text{dist}(p_i, \partial\Omega) \geq \overline{C}^{-1}, \quad \text{dist}(p_i, p_j) \geq \overline{C}^{-1} \forall i \neq j, \right. \\ \left. \lambda_i \varepsilon \in [\overline{C}^{-1}, \overline{C}], R_i \in SO(3) \right\}. \quad (31)$$

Proposition 4.1 asserts that the manifold \overline{Z} , see (16), is non-degenerate for the functional \overline{I} module translations. As a consequence, it is easy to extremize I_ε in the direction perpendicular to Z . This is stated in the following Proposition 4.3, in the same spirit as [1]. We need first a preliminary Lemma (see also [2], Proposition 3.1).

Lemma 4.2. *Let $k \in \mathbb{N}$, $\overline{C} > 0$, and let Z be as in (31). Then there exists a positive constant C such that*

$$\text{if } v \in H_0^1(\Omega), v \perp T_z Z, v \perp P\delta_i \forall i, \text{ then } I_\varepsilon''(z)[v, v] \geq C^{-1} \|v\|_{H_0^1(\Omega)}^2.$$

PROOF. For $i = 1, \dots, k$, let B_i be the ball of radius $\frac{\overline{C}^{-1}}{2}$ around p_i , and let also \tilde{B}_i be the ball of radius $\frac{\overline{C}^{-1}}{4}$ around p_i . Let us denote by P_i the orthogonal projection of $H_0^1(\Omega)$ onto $H_0^1(B_i)$, and for any $v \in H_0^1(\Omega)$ set $v_1 = v - \sum_{i=1}^k P_i v$. It follows immediately that

$$\|v\|_{H_0^1(\Omega)}^2 = \sum_{i=1}^k \|P_i v\|_{H_0^1(\Omega)}^2 + \|v_1\|_{H_0^1(\Omega)}^2. \quad (32)$$

From standard regularity results, since the function v_1 is harmonic in each B_i , and since it coincides with v on each ∂B_i , there holds

$$\|v_1\|_{C^2(\tilde{B}_i)} \leq C \|v\|_{H_0^1(\Omega)}, \quad \text{for all } i = 1, \dots, k, \quad (33)$$

where C is a constant independent of v . Since v is orthogonal to $P\delta_i$, from (26) we deduce

$$(P_i v, P\delta_i) = \left(v - v_1 - \sum_{j \neq i} P_j v, P\delta_i \right) = -(v_1, P\delta_i) + \sum_{j \neq i} O\left(\frac{1}{\lambda_j}\right) \|v\|_{H_0^1(\Omega)}.$$

To evaluate the scalar product $(v_1, P\delta_i) = \int_{\Omega} \nabla v_1 \cdot \nabla P\delta_i$, we divide the integral in the regions \tilde{B}_i and $\Omega \setminus \tilde{B}_i$. We have clearly $\int_{\Omega \setminus \tilde{B}_i} \nabla v_1 \cdot \nabla P\delta_i = O(\lambda_i^{-1})\|v\|_{H_0^1(\Omega)}$. On the other hand, using (25) and (33) we find

$$\left| \int_{\tilde{B}_i} \nabla v_1 \cdot \nabla P\delta_i \right| = O\left(\int_{\tilde{B}_i} |\nabla P\delta_i| \right) \|v\|_{H_0^1(\Omega)} \leq O\left(\frac{\log \lambda_i}{\lambda_i} \right) \|v\|_{H_0^1(\Omega)}.$$

Using these formulas and (26) we obtain

$$|(P_i v, \delta_i)| = O\left(\frac{\log \lambda_i}{\lambda_i} \right) \|v\|_{H_0^1(\Omega)} + \sum_{j \neq i} O\left(\frac{1}{\lambda_j} \right) \|v\|_{H_0^1(\Omega)} \leq C\varepsilon |\log \varepsilon| \|v\|_{H_0^1(\Omega)}. \quad (34)$$

In the same way as (34), using the explicit expression of the function δ_i and taking the scalar product of v with $\frac{\partial \delta_i}{\partial p_i}$, $\frac{\partial \delta_i}{\partial R_i}$ and $\frac{\partial \delta_i}{\partial \lambda_i}$, one finds $\|\Pi_i P_i v\| \leq C\varepsilon |\log \varepsilon| \|v\|_{H_0^1(\Omega)}$, where Π_i denotes the orthogonal projection onto the space spanned by δ_i , $\frac{\partial \delta_i}{\partial p_i}$, $\frac{\partial \delta_i}{\partial R_i}$ and $\frac{\partial \delta_i}{\partial \lambda_i}$. The functional $I''_{\varepsilon}(z)$ is given by

$$\begin{aligned} I''_{\varepsilon}(z)[v, \tilde{v}] &= \int_{\Omega} \nabla v \cdot \nabla \tilde{v} - 2 \int_{\Omega} z \cdot (v_x \wedge \tilde{v}_y + \tilde{v}_x \wedge v_y) \\ &\quad + 2\varepsilon \int_{\Omega} \tilde{v} \cdot (g_x \wedge v_y + v_x \wedge g_y), \quad v, \tilde{v} \in H_0^1(\Omega). \end{aligned}$$

It follows easily from the expression of I''_{ε} and from Proposition 4.1 that

$$I''_{\varepsilon}(z)[P_i v, P_i v] \geq C_0 \|P_i v\|_{H_0^1(\Omega)}^2 - C\varepsilon |\log \varepsilon| \|v\|_{H_0^1(\Omega)}. \quad (35)$$

For an arbitrary function v there holds

$$I''_{\varepsilon}(z)[v, v] = \sum_{i=1}^k I''_{\varepsilon}(z)[P_i v, P_i v] + I''_{\varepsilon}(z)[v_1, v_1] + 2 \sum_{i=1}^k I''_{\varepsilon}(z)[P_i v, v_1]. \quad (36)$$

From the orthogonality of $P_i v$ and v_1 it follows that

$$\begin{aligned} I''_{\varepsilon}(z)[P_i v, v_1] &= -2 \int_{\Omega} z \cdot ((P_i v)_x \wedge (v_1)_y + (v_1)_x \wedge (P_i v)_y) \\ &\quad + 2\varepsilon \int_{\Omega} P_i v \cdot (g_x \wedge (v_1)_y + (v_1)_x \wedge g_y) \\ &= O\left(\int_{\Omega} |z| |\nabla P_i v| |\nabla v_1| \right) + O(\varepsilon) \|v\|_{H_0^1(\Omega)}^2. \end{aligned}$$

Dividing again the integral into the regions \tilde{B}_i and $\Omega \setminus \tilde{B}_i$ we deduce

$$I''_\varepsilon(z)[P_i v, v_1] = O\left(\frac{\log \lambda_i}{\lambda_i}\right) \|v\|_{H_0^1(\Omega)}^2 + \sum_{j \neq i} O\left(\frac{1}{\lambda_j}\right) \|v\|_{H_0^1(\Omega)}^2 + O(\varepsilon) \|v\|_{H_0^1(\Omega)}^2. \quad (37)$$

Similarly, we obtain

$$\begin{aligned} I''_\varepsilon(z)[v_1, v_1] &= \|v_1\|_{H_0^1(\Omega)}^2 + O\left(\frac{\log \lambda_i}{\lambda_i}\right) \|v\|_{H_0^1(\Omega)}^2 \\ &\quad + \sum_{j \neq i} O\left(\frac{1}{\lambda_j}\right) \|v\|_{H_0^1(\Omega)}^2 + O(\varepsilon) \|v\|_{H_0^1(\Omega)}^2. \end{aligned} \quad (38)$$

From (32), (35), (36), (37) and (38) the Lemma follows. ■

Proposition 4.3. *Let \bar{C} be a fixed positive constant, let $k \in \mathbb{N}$, let $\varepsilon > 0$, and Z be defined as above. Then, if ε is sufficiently small, for every $z \in Z$ there exist a function $w_\varepsilon(z) \in H^1(\Omega; \mathbb{R}^3)$ and $C > 0$ with the following properties*

- i) $w_\varepsilon(z)$ is orthogonal to $T_z Z$, for all $z \in Z$;
- ii) $I'_\varepsilon(z + w_\varepsilon(z)) \in T_z Z$, for all $z \in Z$;
- iii) $\|w_\varepsilon(z)\| \leq C \|I'_\varepsilon(z)\|$, for all $z \in Z$;

By i) and ii), the manifold

$$Z_\varepsilon = \{z + w_\varepsilon(z) : z \in Z\}$$

is a natural constraint for I'_ε . Namely if $u \in Z_\varepsilon$ and $I'_\varepsilon|_{Z_\varepsilon}(u) = 0$, then $I'_\varepsilon(u) = 0$.

PROOF. Given Proposition 4.1, the arguments are quite standard. For convenience, we give a brief sketch in the case $k = 1$. In the proof, we simply write δ for $R\delta_{a,\lambda}$.

Let us define $F_\varepsilon : Z \times H^1(\Omega; \mathbb{R}^3) \times T_z Z \rightarrow H^1(\Omega; \mathbb{R}^3) \times \mathbb{R}$ by setting

$$F_\varepsilon(z, w, q) = \begin{pmatrix} I'_\varepsilon(z + w) - q \\ (w, q) \end{pmatrix}.$$

With this notation, the unknown $(w, q) = (w_\varepsilon, I'_\varepsilon(z + w_\varepsilon(z)))$ can be implicitly defined via the equation $F_\varepsilon(z, w, q) = (0, 0)$. Setting $G_\varepsilon(z, w, q) = F_\varepsilon(z, w, q) - \partial_{(w,q)} F_\varepsilon(z, 0, 0)[(w, q)]$ we have that

$$F_\varepsilon(z, w, q) = 0 \quad \Leftrightarrow \quad \partial_{(w,q)} F_\varepsilon(z, 0, 0)[(w, q)] + G_\varepsilon(z, w, q) = 0.$$

Reasoning as in [1], using Lemma 4.2 one can prove that $\partial_{(w,q)}F_\varepsilon(z, 0, 0)$ is uniformly invertible for $z \in Z$ and ε sufficiently small. Hence we can write

$$\begin{aligned} F_\varepsilon(z, w, q) = 0 &\Leftrightarrow (w, q) = W_\varepsilon(z, w, q) \\ &:= - \left(\partial_{(w,q)}F_\varepsilon(z, 0, 0) \right)^{-1} [F_\varepsilon(z, 0, 0) + Q_\varepsilon(z, w, q)], \end{aligned}$$

where

$$Q_\varepsilon(z, w, q) = F_\varepsilon(z, w, q) - F_\varepsilon(z, 0, 0) - \partial_{(w,q)}F_\varepsilon(z, 0, 0)[(w, q)].$$

It is also standard to prove that $Q_\varepsilon(z, w, q)$ satisfies

$$\begin{cases} \|Q_\varepsilon(z, w, q)\| \leq C\|(w, q)\|^2 \\ \|Q_\varepsilon(z, w, q) - Q_\varepsilon(z, \tilde{w}, \tilde{q})\| \leq C(\|(w, q)\| + \|(\tilde{w}, \tilde{q})\|) \|(w, q) - (\tilde{w}, \tilde{q})\|, \end{cases} \quad (39)$$

where $\|(w, q)\|$ and where $C = C(\Omega, g, \overline{C})$ is a constant depending on Ω , g , \overline{C} , and independent of $z \in Z$ and ε . Using (39) it is possible to prove that the function W_ε is a contraction in a ball of radius $\tilde{C}\|F_\varepsilon(z, 0, 0)\|$ for some positive constant $\tilde{C}(\Omega, g, \overline{C})$. Since $\|F_\varepsilon(z, 0, 0)\| \leq C\|I'_\varepsilon(z)\|$ for some constant C , the conclusion follows. ■

We estimate now the quantity $\|I'_\varepsilon(\sum P\delta_i)\|$ in order to control the norm of $w_\varepsilon(z)$, see *iii*) in Proposition 4.3.

Lemma 4.4. *Let C be a fixed positive constant, let $k \in \mathbb{N}$, let $\varepsilon > 0$, and let Z be as in (31). Then there holds*

$$\|I'_\varepsilon(z)\| \leq \tilde{e}(\varepsilon, \lambda_1, \dots, \lambda_k), \quad \text{for } \varepsilon \text{ sufficiently small and for all } z \in Z,$$

where $\tilde{e}(\varepsilon, \lambda_1, \dots, \lambda_k)$ is defined in Section 2.

PROOF. Let $v \in H_0^1(\Omega; \mathbb{R}^3)$ and $z \in Z$. Using integration by parts we deduce easily

$$I'_\varepsilon(z)[v] = \int_\Omega \nabla z \cdot \nabla v + 2 \int_\Omega v \cdot (z_x \wedge z_y) + 2\varepsilon \int_\Omega z \cdot (v_x \wedge g_y + g_x \wedge v_y) + 2\varepsilon^2 \int_\Omega v \cdot (g_x \wedge g_y). \quad (40)$$

From Hölder's inequality we get

$$\left| \int_\Omega z(v_x \wedge g_y + g_x \wedge v_y) \right| \leq \|g\|_{C^1(\overline{\Omega})} \left(\int_\Omega |z|^2 \right)^{\frac{1}{2}} \|v\|, \quad \text{for all } v \in H^1(\Omega; \mathbb{R}^3).$$

From (25), (26) it is easy to check that $(\int_{\Omega} |z|^2)^{\frac{1}{2}} \leq C \sum_{i=1}^k \frac{1}{\lambda_i} |\log \lambda_i|^{\frac{1}{2}}$, hence we have

$$\left| \varepsilon \int_{\Omega} z \cdot (v_x \wedge g_y + g_x \wedge v_y) \right| \leq C \sum_{i=1}^k \frac{\varepsilon}{\lambda_i} |\log \lambda_i|^{\frac{1}{2}} \|v\|, \text{ for all } v \in H_0^1(\Omega; \mathbb{R}^3).$$

On the other hand, it is also immediate to verify the inequality

$$\varepsilon^2 \left| \int_{\Omega} v \cdot (g_x \wedge g_y) \right| \leq C \varepsilon^2 \|v\|, \quad \text{for all } v \in H_0^1(\Omega) \text{ and all } z \in Z.$$

It remains to estimate the first two terms in (40). Writing for brevity $\delta_i = R_i \delta_{p_i, \lambda_i}$, we have also

$$\int_{\Omega} \nabla z \cdot \nabla v + 2 \int_{\Omega} v \cdot (z_x \wedge z_y) = \int_{\Omega} v \cdot \left[-\Delta \left(\sum P \delta_i \right) + 2 \left(\left(\sum P \delta_i \right)_x \wedge \left(\sum P \delta_i \right)_y \right) \right].$$

Using the equation $\Delta \delta_i = 2((\delta_i)_x \wedge (\delta_i)_y)$ and the fact that $\Delta \delta_i = \Delta(P \delta_i)$, the above quantity becomes

$$2 \int_{\Omega} v \cdot \left[\left(\sum P \delta_i \right)_x \wedge \left(\sum P \delta_i \right)_y - \sum (\delta_i)_x \wedge (\delta_i)_y \right],$$

which can be written as

$$2 \sum_{i \neq j} \int_{\Omega} v \cdot ((P \delta_i)_x \wedge (P \delta_j)_y) + 2 \sum_i \int_{\Omega} v \cdot [(P \delta_i - \delta_i)_x \wedge (\delta_i)_y + (\delta_i)_x \wedge (P \delta_i - \delta_i)_y]. \quad (41)$$

Let us estimate first the term $\int_{\Omega} v \cdot (P \delta_i)_x \wedge (P \delta_j)_y$. Let $\gamma < \frac{1}{2} \bar{C}^{-1}$ (recall the definition of Z) be a fixed positive number and divide the integral in the three regions $B_{\gamma}(p_i)$, $B_{\gamma}(p_j)$ and $\Omega \setminus (B_{\gamma}(p_i) \cup B_{\gamma}(p_j))$. Integrating by parts on the balls $B_{\gamma}(p_i)$ and $B_{\gamma}(p_j)$, the quantity $\int_{\Omega} v \cdot (P \delta_i)_x \wedge (P \delta_j)_y$ becomes

$$\begin{aligned} & - \int_{B_{\gamma}(p_i)} v_x \cdot (P \delta_i \wedge (P \delta_j)_y) - \int_{B_{\gamma}(p_i)} v \cdot (P \delta_i \wedge (P \delta_j)_{xy}) + \int_{\partial B_{\gamma}(p_i)} v \cdot (P \delta_i \wedge (P \delta_j)_y) \nu_x \\ & - \int_{B_{\gamma}(p_j)} v_x \cdot (P \delta_i \wedge (P \delta_j)_y) - \int_{B_{\gamma}(p_j)} v \cdot (P \delta_i \wedge (P \delta_j)_{xy}) + \int_{\partial B_{\gamma}(p_j)} v \cdot (P \delta_i \wedge (P \delta_j)_y) \nu_x \\ & + \int_{\Omega \setminus (B_{\gamma}(p_i) \cup B_{\gamma}(p_j))} v \cdot ((P \delta_i)_x \wedge (P \delta_j)_y). \end{aligned}$$

Hence, since δ_i and its derivatives are of order λ_i^{-1} in $\Omega \setminus B_{\gamma}(p_i)$ one finds

$$\left| \int_{\Omega} v \cdot ((P \delta_i)_x \wedge (P \delta_j)_y) \right| \leq C \frac{1}{\lambda_i \lambda_j} \|v\|.$$

Using similar estimates we find that the whole expression in (41) is of order ε^2 . So we obtain the conclusion. ■

Lemma 4.5. *Let $\bar{C} > 0$ and let Z be as in (31). Then there holds*

$$\left| I''_\varepsilon(z) \left[\frac{\partial z}{\partial p_i} \right] \right| + \left| I''_\varepsilon(z) \left[\frac{\partial z}{\partial R_i} \right] \right| \leq C \left(\varepsilon + \sum_{j=1}^k \lambda_j^{-1} \right);$$

$$\left| I''_\varepsilon(z) \left[\frac{\partial z}{\partial \lambda_i} \right] \right| \leq \frac{1}{\lambda_i} C \left(\varepsilon + \sum_{j=1}^k \lambda_j^{-1} \right),$$

for every $i = 1, \dots, k$.

PROOF. From (40) and some integration by parts it follows that

$$I''_\varepsilon(z)[v, \tilde{v}] = \int_\Omega \nabla v \cdot \nabla \tilde{v} + 2 \int_\Omega \tilde{v} \cdot (z_x \wedge v_y + v_x \wedge z_y) + 2\varepsilon \int_\Omega \tilde{v} \cdot (g_x \wedge v_y + v_x \wedge g_y),$$

where v, \tilde{v} are arbitrary functions in $H_0^1(\Omega, \mathbb{R}^3)$. We choose now $\tilde{v} = \frac{\partial z}{\partial p_i}$, and we let v be an arbitrary test function. We have clearly $\frac{\partial z}{\partial p_i} = \frac{\partial \delta_i}{\partial p_i} - \frac{\partial \varphi_i}{\partial p_i}$, where φ_i is the function in (21) corresponding to δ_i . From the estimates in (26) and from the explicit expression of $\frac{\partial \delta_i}{\partial p_i}$ we find

$$\varepsilon \left| \int \frac{\partial z}{\partial p_i} \cdot (g_x \wedge v_y + v_x \wedge g_y) \right| \leq C \frac{\varepsilon}{\lambda_i} \|v\| + C\varepsilon \left\| \frac{\partial \delta_i}{\partial p_i} \right\|_{L^2(\Omega)} \|v\| \leq C\varepsilon \|v\|. \quad (42)$$

Turning to the remaining two terms, we have

$$\begin{aligned} & \int_\Omega \nabla v \cdot \nabla \frac{\partial z}{\partial p_i} + 2 \int_\Omega \frac{\partial z}{\partial p_i} \cdot (z_x \wedge v_y + v_x \wedge z_y) = \int_\Omega \nabla v \cdot \nabla \frac{\partial \delta_i}{\partial p_i} \\ & + 2 \int_\Omega \frac{\partial \delta_i}{\partial p_i} \cdot ((\delta_i)_x \wedge v_y + v_x \wedge (\delta_i)_y) \\ & + 2 \sum_{j \neq i} \int_\Omega \frac{\partial \delta_i}{\partial p_i} \cdot ((\delta_j)_x \wedge v_y + v_x \wedge (\delta_j)_y) + O(\lambda_i^{-1}) \|v\|. \end{aligned} \quad (43)$$

Integrating by parts and using the fact that $\nabla \frac{\partial \delta_i}{\partial p_i}$ is of order λ_i^{-1} on $\partial\Omega$ we find

$$\begin{aligned} \int_\Omega \nabla v \cdot \nabla \frac{\partial \delta_i}{\partial p_i} + 2 \int_\Omega \frac{\partial \delta_i}{\partial p_i} \cdot ((\delta_i)_x \wedge v_y + v_x \wedge (\delta_i)_y) &= \int_{\partial\Omega} v \cdot \frac{\partial}{\partial \nu} \left(\frac{\partial \delta_i}{\partial p_i} \right) \\ &= O(\lambda_i^{-1}) \|v\|, \end{aligned} \quad (44)$$

since $\frac{\partial \delta_i}{\partial p_i}$ satisfies (10). To estimate $\int_{\Omega} \frac{\partial \delta_i}{\partial p_i} \cdot ((\delta_j)_x \wedge v_y + v_x \wedge (\delta_j)_y)$ for $j \neq i$, we proceed as follows. Let B_i and B_j denote the balls of radius $\frac{1}{2\bar{C}}$ centered at p_i and p_j respectively. By the definition of Z these two balls are disjoint and moreover, by (26) $\nabla \delta_i$ and $\nabla \delta_j$ are of order λ_i^{-1} and λ_j^{-1} respectively outside B_i and B_j . Hence we have

$$\left| \int_{\Omega} \frac{\partial \delta_i}{\partial p_i} \cdot ((\delta_j)_x \wedge v_y + v_x \wedge (\delta_j)_y) \right| \leq C \left(\frac{1}{\lambda_i} + \frac{1}{\lambda_j} + \frac{1}{\lambda_i \lambda_j} \right) \|v\|. \quad (45)$$

Hence (42)-(45) imply $\left| I''_{\varepsilon}(z) \left[\frac{\partial z}{\partial p_i} \right] \right| \leq C \left(\varepsilon + \sum_{j=1}^k \lambda_j^{-1} \right)$. The remaining part of the statement follows from similar arguments. ■

From Proposition 4.3, critical points of I_{ε} restricted to Z_{ε} are true critical points of I_{ε} . We define $\tilde{I}_{\varepsilon} : Z \rightarrow \mathbb{R}$ as $\tilde{I}_{\varepsilon}(z) = I_{\varepsilon}(z + w_{\varepsilon}(z))$. We now analyze the reduced functional \tilde{I}_{ε} .

Proposition 4.6. *Let $\bar{C} > 0$, let Z be as in (31) and let $w_{\varepsilon}(z)$ be as in Proposition 4.3. Then we have*

$$\left| \tilde{I}_{\varepsilon}(z) - I_{\varepsilon}(z) \right| \leq C \tilde{\varepsilon}^2(\varepsilon, \lambda_1, \dots, \lambda_k), \quad \forall z \in Z; \quad (46)$$

Moreover, for all $z = \sum_{i=1}^k R_i P_{p_i, \lambda_i} \in Z$ there holds

$$\left\{ \begin{array}{l} \left| \frac{\partial \tilde{I}_{\varepsilon}(z)}{\partial p_i} - \frac{\partial I_{\varepsilon}(z)}{\partial p_i} \right| + \left| \frac{\partial \tilde{I}_{\varepsilon}(z)}{\partial R_i} - \frac{\partial I_{\varepsilon}(z)}{\partial R_i} \right| \leq \left(\varepsilon + \sum_j \lambda_j^{-1} \right) \tilde{\varepsilon}(\varepsilon, \lambda_1, \dots, \lambda_k); \\ \left| \frac{\partial \tilde{I}_{\varepsilon}(z)}{\partial \lambda_i} - \frac{\partial I_{\varepsilon}(z)}{\partial \lambda_i} \right| \leq \frac{1}{\lambda_i} \left(\varepsilon + \sum_j \lambda_j^{-1} \right) \tilde{\varepsilon}(\varepsilon, \lambda_1, \dots, \lambda_k) \end{array} \right. \quad (47)$$

PROOF. We have

$$\begin{aligned} \tilde{I}_{\varepsilon}(z) - I_{\varepsilon}(z) &= I_{\varepsilon}(z + w) - I_{\varepsilon}(z) = \int_0^1 I'_{\varepsilon}(z + sw)[w] ds \\ &= I'_{\varepsilon}(z)[w] + \int_0^1 (I'_{\varepsilon}(z + sw) - I'_{\varepsilon}(z))[w] ds. \end{aligned}$$

Since the functional I''_{ε} is locally bounded, we have the following estimate

$$\begin{aligned} \left| I'_{\varepsilon}(z + sw) - I'_{\varepsilon}(z) \right| &\leq \left| \int_0^1 I''_{\varepsilon}(z + tsw)[w] dt \right| \\ &\leq \sup_{t, s \in [0, 1]} \|I''_{\varepsilon}(z + tsw)\| \|w\| \leq C \|w\| \end{aligned}$$

for some fixed constant C depending on Ω , $\|g\|_{C^2(\partial\Omega)}$ and the above constant \bar{C} . Using the last three equations, Lemma 4.4 and the property *iii*) in Proposition 4.3 we find

$$\left| \tilde{I}_\varepsilon(z) - I_\varepsilon(z) \right| \leq \|I'_\varepsilon(z)\| \|w_\varepsilon(z)\| + C \|w_\varepsilon(z)\|^2 \leq \tilde{c}^2(\varepsilon, \lambda_1, \dots, \lambda_k).$$

This concludes the proof of (46). We just sketch the proof of (47). Differentiating the equation $F_\varepsilon(z, w, q) = 0$ with respect to p_i we obtain

$$0 = \frac{\partial F_\varepsilon}{\partial z} \frac{\partial z}{\partial p_i} + \frac{\partial F_\varepsilon}{\partial(w, q)} \frac{\partial(w, q)}{\partial p_i} = I''_\varepsilon(z + w_\varepsilon(z)) \frac{\partial z}{\partial p_i} + \frac{\partial F_\varepsilon}{\partial(w, q)} \frac{\partial(w, q)}{\partial p_i}.$$

Similarly as before, one finds that $\partial_{(w, q)} F_\varepsilon$ is uniformly invertible, and hence

$$\left\| \frac{\partial w}{\partial p_i} \right\| \leq C \left\| I''_\varepsilon(z + w_\varepsilon(z)) \frac{\partial z}{\partial p_i} \right\| \leq C \left\| I''_\varepsilon(z) \frac{\partial z}{\partial p_i} \right\| + C \|w_\varepsilon(z)\| \left\| \frac{\partial z}{\partial p_i} \right\|, \quad (48)$$

where we have used the fact that I''_ε is locally Lipschitz. We have

$$\begin{aligned} \frac{\partial \tilde{I}_\varepsilon(z)}{\partial p_i} - \frac{\partial I_\varepsilon(z)}{\partial p_i} &= I''_\varepsilon(z) \left[\frac{\partial z}{\partial p_i}, w_\varepsilon(z) \right] \\ &+ \int_0^1 (I''_\varepsilon(z + sw_\varepsilon(z)) - I''_\varepsilon(z)) \left[\frac{\partial z}{\partial p_i}, w_\varepsilon(z) \right] ds \\ &+ I'_\varepsilon(z) \left[\frac{\partial w}{\partial p_i} \right] + I''_\varepsilon(z) \left[\frac{\partial w}{\partial p_i}, w_\varepsilon(z) \right] \\ &+ \int_0^1 (I''_\varepsilon(z + sw_\varepsilon(z)) - I''_\varepsilon(z)) \left[\frac{\partial w}{\partial p_i}, w_\varepsilon(z) \right] ds. \end{aligned} \quad (49)$$

Equation (49) implies

$$\begin{aligned} \left| \frac{\partial \tilde{I}_\varepsilon(z)}{\partial p_i} - \frac{\partial I_\varepsilon(z)}{\partial p_i} \right| &\leq C \left\| I''_\varepsilon(z) \left[\frac{\partial z}{\partial p_i} \right] \right\| \|w_\varepsilon(z)\| \\ &+ C \|w_\varepsilon(z)\|^2 \left(\left\| \frac{\partial z}{\partial p_i} \right\| + \left\| \frac{\partial w}{\partial p_i} \right\| \right) \\ &+ C \|I'_\varepsilon(z)\| \left\| \frac{\partial w}{\partial p_i} \right\| + C \|w_\varepsilon(z)\| \left\| \frac{\partial w}{\partial p_i} \right\|. \end{aligned}$$

Then the estimate of $\frac{\partial \tilde{I}_\varepsilon(z)}{\partial p_i} - \frac{\partial I_\varepsilon(z)}{\partial p_i}$ in (47) follows from Lemma 4.4, (48) and Lemma 4.5. The remaining part of (47) follows from similar estimates. ■

5. The expansion for one bubble.

In this section we compute the expansion of $I_\varepsilon(z)$, with $z \in Z$, for ε small and in the case $k = 1$. This is essentially performed in [20]-[21], in order to construct blowing-up solutions of (2), and in order to characterize the mountain-pass solutions in the limit $\varepsilon \rightarrow 0$. We derive the expansion here, in a form which is useful for us in the expansion for multiple bubbles in Section 7. Let us first introduce some notation. We recall that $g : \Omega \rightarrow \mathbb{R}^3$ denotes the solution of (4), and letting $R \in SO(3)$, define $d_R g : \Omega \rightarrow \mathbb{R}^3$ by

$$d_R g(a) = \frac{\partial}{\partial x}(R \circ g)_1(a) + \frac{\partial}{\partial y}(R \circ g)_2(a), \quad x \in \Omega.$$

For a fixed boundary datum \tilde{g} , we are interested in expanding the functional $I_\varepsilon(P\delta)$ as a function of the parameters a , λ , R and ε . We have the following Proposition.

Proposition 5.1. *Let $\overline{C} > 0$ be fixed, and let a, λ, R be such that $PR\delta_{a,\lambda} \in Z$. Then, setting*

$$A_0 = \int_{\mathbb{R}^2} \frac{|\xi|^2}{(1+|\xi|^2)^3}; \quad F_{\Omega,g}(\varepsilon, a, \lambda, R) = 8A_0 \left(\frac{1}{\lambda^2} \tilde{H}(a) - \frac{\varepsilon}{\lambda} d_{R^{-1}}g(a) \right),$$

there holds

$$\begin{aligned} I_\varepsilon(PR\delta_{a,\lambda}) &= \frac{8}{9}A_0 + F_{\Omega,g}(a, \lambda, R) + o(\varepsilon^2) + e(\varepsilon, \lambda); \\ \frac{\partial I_\varepsilon(PR\delta_{a,\lambda})}{\partial a} &= \frac{\partial F_{\Omega,g}}{\partial a} + e(\varepsilon, \lambda); \quad \frac{\partial I_\varepsilon(PR\delta_{a,\lambda})}{\partial \lambda} = \frac{\partial F_{\Omega,g}}{\partial \lambda} + \frac{e(\varepsilon, \lambda)}{\lambda}; \\ \frac{\partial I_\varepsilon(PR\delta_{a,\lambda})}{\partial R} &= \frac{\partial F_{\Omega,g}}{\partial R} + e(\varepsilon, \lambda), \end{aligned}$$

where $e(\varepsilon, \lambda)$ is defined in Section 2.

PROOF. We assume that $R = Id$, and we write δ for $\delta_{a,\lambda}$. Let also φ be the solution of (21). We have

$$\begin{aligned} I_\varepsilon(P\delta) &= \frac{1}{2} \int_{\Omega} |\nabla \delta|^2 + \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 - \int_{\Omega} \nabla \varphi \cdot \nabla \delta \\ &\quad + \frac{2}{3} \int_{\Omega} (\delta - \varphi) \cdot ((\delta - \varphi)_x \wedge (\delta - \varphi)_y) \\ &\quad + \varepsilon \int_{\Omega} (\delta - \varphi) \cdot ((\delta - \varphi)_x \wedge g_y + g_x \wedge (\delta - \varphi)_y) \\ &\quad + 2\varepsilon^2 \int_{\Omega} (\delta - \varphi) \cdot (g_x \wedge g_y). \end{aligned} \tag{50}$$

Integrating by parts we can write

$$\frac{1}{2} \int_{\Omega} |\nabla \delta|^2 + \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 - \int_{\Omega} \nabla \varphi \cdot \nabla \delta = \int_{\Omega} (\varphi - \delta) \cdot (\delta_x \wedge \delta_y). \quad (51)$$

We expand first the expression in (51). Let us evaluate the z -component in the scalar product of the integral on the right-hand side in (51). From formulas (18) and (23) we deduce

$$\begin{aligned} & (\varphi - \delta)_3 (\delta_x \wedge \delta_y)_3 \\ &= - \left(1 - \frac{2}{\lambda^2} h_3(a, a) - \frac{\lambda^2 |\xi - a|^2 - 1}{\lambda^2 |\xi - a|^2 + 1} + o(\lambda^{-2}) \right) 4\lambda^2 \frac{1 - \lambda^2 |\xi - a|^2}{(\lambda^2 |\xi - a|^2 + 1)^3} \\ &= -8\lambda^2 \left(\frac{1}{\lambda^2 |\xi - a|^2 + 1} - \frac{1}{\lambda^2} h_3(a, a) + o(\lambda^{-2}) \right) \frac{1 - \lambda^2 |\xi - a|^2}{(\lambda^2 |\xi - a|^2 + 1)^3}. \end{aligned}$$

Integrating on Ω we get

$$\begin{aligned} & \int_{\Omega} (\varphi - \delta)_3 (\delta_x \wedge \delta_y)_3 \quad (52) \\ &= -8\lambda^2 \int_{\Omega} \left(\frac{1}{\lambda^2 |\xi - a|^2 + 1} - \frac{1}{\lambda^2} h_3(a, a) + o(\lambda^{-3}) \right) \frac{1 - \lambda^2 |\xi - a|^2}{(\lambda^2 |\xi - a|^2 + 1)^3}. \end{aligned}$$

Using a change of variable we obtain

$$\begin{aligned} \lambda^2 \int_{\Omega} \frac{1 - \lambda^2 |\xi - a|^2}{(\lambda^2 |\xi - a|^2 + 1)^4} &= \lambda^2 \int_{\mathbb{R}^2} \frac{1 - \lambda^2 |\xi - a|^2}{(\lambda^2 |\xi - a|^2 + 1)^4} - \lambda^2 \int_{\mathbb{R}^2 \setminus \Omega} \frac{1 - \lambda^2 |\xi - a|^2}{(\lambda^2 |\xi - a|^2 + 1)^4} \\ &= \int_{\mathbb{R}^2} \frac{1 - |\xi|^2}{(|\xi|^2 + 1)^4} + O(\lambda^{-4}), \quad (53) \end{aligned}$$

and also

$$\begin{aligned} \int_{\Omega} \frac{1 - \lambda^2 |\xi - a|^2}{(\lambda^2 |\xi - a|^2 + 1)^3} &= \int_{\mathbb{R}^2} \frac{1 - \lambda^2 |\xi - a|^2}{(\lambda^2 |\xi - a|^2 + 1)^3} - \int_{\mathbb{R}^2 \setminus \Omega} \frac{1 - \lambda^2 |\xi - a|^2}{(\lambda^2 |\xi - a|^2 + 1)^3} \\ &= \frac{1}{\lambda^2} \int_{\mathbb{R}^2} \frac{1 - |\xi|^2}{(|\xi|^2 + 1)^3} + O(\lambda^{-4}) = O(\lambda^{-4}). \quad (54) \end{aligned}$$

The last identity follows from (27). In conclusion, from (52), (53) and (54) we get

$$\int_{\Omega} (\varphi - \delta)_3 (\delta_x \wedge \delta_y)_3 = 8 \int_{\mathbb{R}^2} \frac{|\xi|^2 - 1}{(|\xi|^2 + 1)^4} + o(\lambda^{-2}). \quad (55)$$

We consider now the x and y components of the integral on the right-hand side in (51). We have, using (17) and (19)

$$\begin{aligned} & \int_{\Omega} (\varphi - \delta)_1 (\delta_x \wedge \delta_y)_1 \\ &= - \int_{\Omega} \left(\frac{2}{\lambda} h_1(\xi, a) - \frac{2\lambda(x - a_1)}{1 + \lambda^2|\xi - a|^2} + o(\lambda^{-1}) \right) \frac{8\lambda^3(x - a_1)}{(1 + \lambda^2|\xi - a|^2)^3}. \end{aligned}$$

We can write

$$\begin{aligned} \lambda^4 \int_{\Omega} \frac{(x - a_1)^2}{(1 + \lambda^2|\xi - a|^2)^4} &= \lambda^4 \int_{\mathbb{R}^2} \frac{(x - a_1)^2}{(1 + \lambda^2|\xi - a|^2)^4} - \lambda^4 \int_{\mathbb{R}^2 \setminus \Omega} \frac{(x - a_1)^2}{(1 + \lambda^2|\xi - a|^2)^4} \\ &= \int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^4} + O(\lambda^{-4}). \end{aligned}$$

From the smoothness of h_i we have also

$$\begin{aligned} & \left| \int_{\Omega} (h_1(\xi, a) - h_1(a, a) - (\xi - a) \cdot \nabla h_1(a, a)) \frac{x - a_1}{(1 + \lambda^2|\xi - a|^2)^3} \right| \\ & \leq O \left(\int_{\Omega} \frac{|\xi - a|^3}{(1 + \lambda^2|\xi - a|^2)^3} \right). \end{aligned}$$

As a consequence we deduce

$$\begin{aligned} \int_{\Omega} h_1(x, a) \frac{(x - a_1)}{(1 + \lambda^2|\xi - a|^2)^3} &= \int_{\Omega} h_1(a, a) \frac{(x - a_1)}{(1 + \lambda^2|\xi - a|^2)^3} \\ &+ \int_{\Omega} ((\xi - a) \cdot \nabla h_1(a, a)) \frac{(x - a_1)}{(1 + \lambda^2|\xi - a|^2)^3} \\ &+ O \left(\int_{\Omega} \frac{|\xi - a|^3}{(1 + \lambda^2|\xi - a|^2)^3} \right) \\ &= \frac{1}{\lambda^4} \partial_x h_1(a, a) \int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^3} + o(\lambda^{-4}). \end{aligned}$$

From the last equation we deduce

$$\begin{aligned} \int_{\Omega} (\varphi - \delta)_1 (\delta_x \wedge \delta_y)_1 &= 16 \int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^4} \\ &- 16 \frac{1}{\lambda^2} \partial_x h_1(a, a) \int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^3} + o(\lambda^{-2}). \quad (56) \end{aligned}$$

In the same way we obtain

$$\begin{aligned} \int_{\Omega} (\varphi - \delta)_2 (\delta_x \wedge \delta_y)_2 &= 16 \int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^4} \\ &\quad - 16 \frac{1}{\lambda^2} \partial_y h_3(a, a) \int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^3} + o(\lambda^{-2}). \end{aligned} \quad (57)$$

From (55), (56) and (57) it follows that

$$\begin{aligned} \int_{\Omega} (\varphi - \delta) \cdot (\delta_x \wedge \delta_y) &= 8 \int_{\mathbb{R}^2} \frac{|\xi|^2 - 1}{(1 + |\xi|^2)^4} + 32 \int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^4} \\ &\quad - 16 \frac{1}{\lambda^2} \left(\int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^3} \right) \left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y} \right) (a, a) + o(\lambda^{-2}). \end{aligned}$$

It is standard to check that

$$8 \int_{\mathbb{R}^2} \frac{|\xi|^2 - 1}{(1 + |\xi|^2)^4} + 32 \int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^4} = \frac{8}{3} A_0,$$

Hence from the last two equations we find

$$\int_{\Omega} (\varphi - \delta) \cdot (\delta_x \wedge \delta_y) = \frac{8}{3} A_0 - \frac{8}{\lambda^2} A_0 \left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y} \right) (a, a) + o(\lambda^{-2}). \quad (58)$$

We turn now to the fourth term in (50). We have clearly

$$\begin{aligned} \int_{\Omega} (\delta - \varphi) \cdot ((\delta - \varphi)_x \wedge (\delta - \varphi)_y) &= \int_{\Omega} (\delta - \varphi) \cdot (\delta_x \wedge \delta_y) \\ &\quad - \int_{\Omega} (\delta - \varphi) \cdot (\delta_x \wedge \varphi_y + \varphi_x \wedge \delta_y). \end{aligned}$$

Let us consider the term $\delta_x \wedge \varphi_y$. Using the above formulae we deduce

$$\begin{aligned} (\delta_x \wedge \varphi_y)_1 &= \frac{8\lambda(x - a_1)(y - a_2)}{(1 + \lambda^2|\xi - a|^2)^2} \left(\frac{\partial h_3}{\partial y} + O(\lambda^{-1}) \right) \\ &\quad - \frac{8\lambda(x - a_1)}{(1 + \lambda^2|\xi - a|^2)^2} \left(\frac{\partial h_2}{\partial y} + O(\lambda^{-1}) \right); \\ (\delta_x \wedge \varphi_y)_2 &= \frac{8\lambda(x - a_1)}{(1 + \lambda^2|\xi - a|^2)^2} \left(\frac{\partial h_1}{\partial y} + O(\lambda^{-1}) \right) \\ &\quad + \frac{4}{\lambda} \frac{1 + \lambda^2(y^2 - x^2)}{(1 + \lambda^2|\xi - a|^2)^2} \left(\frac{\partial h_3}{\partial y} + O(\lambda^{-1}) \right); \\ (\delta_x \wedge \varphi_y)_3 &= 4 \frac{1 + \lambda^2((y - a_2)^2 - (x - a_1)^2)}{(1 + \lambda^2|\xi - a|^2)^2} \left(\frac{\partial h_2}{\partial y} + O(\lambda^{-1}) \right) \\ &\quad + \frac{8\lambda^2(x - a_1)(y - a_2)}{(1 + \lambda^2|\xi - a|^2)^2} \left(\frac{\partial h_1}{\partial y} + O(\lambda^{-1}) \right). \end{aligned} \quad (59)$$

and also

$$\begin{aligned}
(\varphi_x \wedge \delta_y)_1 &= \frac{8\lambda(y-a_2)}{(1+\lambda^2|\xi-a|^2)^2} \left(\frac{\partial h_2}{\partial x} + O(\lambda^{-1}) \right) \\
&\quad + \frac{4}{\lambda} \frac{1+\lambda^2((x-a_1)^2-(y-a_2)^2)}{(1+\lambda^2|\xi-a|^2)^2} \left(\frac{\partial h_3}{\partial x} + O(\lambda^{-1}) \right); \\
(\varphi_x \wedge \delta_y)_2 &= \frac{8\lambda(x-a_1)(y-a_2)}{(1+\lambda^2|\xi-a|^2)^2} \left(\frac{\partial h_3}{\partial x} + O(\lambda^{-1}) \right) \\
&\quad - \frac{8\lambda(y-a_2)}{(1+\lambda^2|\xi-a|^2)^2} \left(\frac{\partial h_1}{\partial x} + O(\lambda^{-1}) \right); \\
(\varphi_x \wedge \delta_y)_3 &= 4 \frac{1+\lambda^2((x-a_1)^2-(y-a_2)^2)}{(1+\lambda^2|\xi-a|^2)^2} \left(\frac{\partial h_1}{\partial x} + O(\lambda^{-1}) \right) \\
&\quad + \frac{8\lambda^2(x-a_1)(y-a_2)}{(1+\lambda^2|\xi-a|^2)^2} \left(\frac{\partial h_2}{\partial x} + O(\lambda^{-1}) \right).
\end{aligned} \tag{60}$$

The functions $\frac{\partial h_i}{\partial x_j}$ in the last formulas, as before, are evaluated at the point (a, a) . Using (23), (59) and (60) we find

$$\int_{\Omega} (\delta - \varphi) \cdot (\delta_x \wedge \varphi_y) \sim -32 \frac{1}{\lambda^2} \left(\int_{\mathbb{R}^2} \frac{x^2}{(1+|\xi|^2)^3} \right) \frac{\partial h_2}{\partial y}(a, a) + o(\lambda^{-2}), \tag{61}$$

and similarly

$$\int_{\Omega} (\delta - \varphi) \cdot (\varphi_x \wedge \delta_y) \sim -32 \frac{1}{\lambda^2} \left(\int_{\mathbb{R}^2} \frac{x^2}{(1+|\xi|^2)^3} \right) \frac{\partial h_1}{\partial x}(a, a) + o(\lambda^{-2}),$$

Since $|(\varphi_x \wedge \varphi_y)_1|, |(\varphi_x \wedge \varphi_y)_2| \leq O(\lambda^{-3})$ and $|(\varphi_x \wedge \varphi_y)_3| \leq O(\lambda^{-2})$, one can check that

$$\int_{\Omega} (\delta - \varphi) \cdot (\varphi_x \wedge \varphi_y) = o(\lambda^{-2}).$$

Let us now turn to the fifth term in (50). The quantity $(\delta - \varphi) \cdot ((\delta - \varphi)_x \wedge g_y)$ can be estimated as

$$\begin{aligned}
&\left(\frac{2\lambda(x-a_1)}{1+\lambda^2|\xi-a|^2} - \frac{2}{\lambda} h_1 \right) \left[\left(-\frac{4\lambda^3(x-a_1)(y-a_2)}{(1+\lambda^2|\xi-a|^2)^2} - \frac{2}{\lambda} \partial_x h_2 \right) (g_3)_y \right. \\
&\quad \left. - \left(\frac{4\lambda^2(x-a_1)}{(1+\lambda^2|\xi-a|^2)^2} + \frac{2}{\lambda^2} \partial_x h_3 \right) (g_2)_y \right] \\
&\quad + \left(\frac{2\lambda(y-a_2)}{1+\lambda^2|\xi-a|^2} - \frac{2}{\lambda} h_2 \right) \left[\left(\frac{4\lambda^2(x-a_1)}{(1+\lambda^2|\xi-a|^2)^2} + \frac{2}{\lambda^2} \partial_x h_3 \right) (g_1)_y \right.
\end{aligned}$$

$$\begin{aligned}
& - \left(2\lambda \frac{1 + \lambda^2((y - a_2)^2 - (x - a_1)^2)}{(1 + \lambda^2|\xi - a|^2)^2} - \frac{2}{\lambda} \partial_x h_1 \right) (g_3)_y \Big] \\
& + \left(\frac{2}{\lambda^2} h_3 - \frac{2}{1 + \lambda^2|\xi - a|^2} \right) \left[\left(2\lambda \frac{1 + \lambda^2((y - a_2)^2 - (x - a_1)^2)}{(1 + \lambda^2|\xi - a|^2)^2} - \frac{2}{\lambda} \partial_x h_1 \right) (g_2)_y \right. \\
& \left. + \left(\frac{4\lambda^3(x - a_1)(y - a_2)}{(1 + \lambda^2|\xi - a|^2)^2} + \frac{2}{\lambda} \partial_x h_2 \right) (g_1)_y \right].
\end{aligned}$$

Integrating on Ω and reasoning as before we get

$$\int_{\Omega} (\delta - \varphi) \cdot ((\delta - \varphi)_x \wedge g_y) = -\frac{16}{\lambda} \int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^3} (g_2)_y + O(\lambda^{-2}), \quad (62)$$

and similarly

$$\int_{\Omega} (\delta - \varphi) \cdot (g_x \wedge (\delta - \varphi)_y) = -\frac{16}{\lambda} \int_{\mathbb{R}^2} \frac{x^2}{(1 + |\xi|^2)^3} (g_1)_x + O(\lambda^{-2}). \quad (63)$$

Finally, the last term in (50) is easily seen to be of order $o(\varepsilon^2)$. This concludes the proof in the case of $R = Id$. For a generic rotation R it is sufficient, by invariance, to consider the boundary datum $R^{-1}g$ and to substitute $(g_1)_x + (g_2)_y$ with $d_{R^{-1}g}$. ■

Remark 5.2. Propositions 4.6 and 5.1 allow us to find critical points of I_ε extremizing the reduced functional \tilde{I}_ε on Z . Differentiating with respect to R , λ , a we get

$$\frac{\partial}{\partial R} d_{R^{-1}g} = 0; \quad 2\tilde{H}(a) - \varepsilon \lambda d_{R^{-1}g} = 0; \quad \frac{1}{\lambda^2} \nabla \tilde{H}(a) - \frac{\varepsilon}{\lambda} \nabla d_{R^{-1}g} = 0. \quad (64)$$

Using the second and third equations in (64) we deduce

$$\nabla \log \tilde{H}(a) = 2 \nabla \log d_{R^{-1}g}(a).$$

The extremization with respect to R (the first equation in (64)) is performed in [21] Lemma 5.4 and [20] Lemma 3.1.2 and, requiring $d_{R^{-1}g}$ to be positive (from the second equation in (64)) yields

$$\frac{\partial}{\partial R} d_{R^{-1}g} = 0 \quad \Rightarrow \quad d_{R^{-1}g} = (|\nabla g|^2 \pm 2|g_x \wedge g_y|)^{\frac{1}{2}}. \quad (65)$$

Hence, under the conditions $\nabla g \neq 0$, the extremization in (64) becomes

$$\lambda = \frac{2}{\varepsilon} \frac{\tilde{H}(a)}{d_{R^{-1}g}(a)}; \quad \frac{\partial}{\partial R} \operatorname{div} \pi_R g = 0; \quad \nabla \frac{|\nabla g|^2 \pm 2|g_x \wedge g_y|}{\tilde{H}}(a) = 0. \quad (66)$$

In particular the mountain-pass solution of (3) (see [21]) has minimal energy on Z , and one has to chose the + sign in (65)-(66), and the last function in (66) is maximized on Ω .

Remark 5.3. (a) We point out that the expansions in (51) and (58) yield

$$\|P\delta\|_{H^1(\Omega)}^2 = \|\delta\|_D^2 - \frac{16A_0}{\lambda^2}\tilde{H}(a) + o(\lambda^{-2}).$$

Since the norm of $P\delta$, being the projection of δ , is smaller than the norm of δ , the above formula implies $\tilde{H} \geq 0$ on Ω . A little more calculation shows that indeed $\tilde{H} > 0$, as proved in [21].

(b) The functions h_1 , h_2 , and h_3 defined in (12), (24) are related to the boundary values of δ . Since h_1 and h_2 are of order λ^{-1} , while h_3 is of order λ^{-2} , h_3 appears in the expansion only as a lower order term.

6. The role of the Robin function.

In this Section we investigate the relation between the Robin function H and the function \tilde{H} defined in (14). Since \tilde{H} consists of second derivatives of the regular part of the Green's function, while the Robin function involves the regular part itself, we need to use global arguments, based on the Riemann mapping theorem. See [6] for some properties of the Robin function.

6.1. Simply connected domains.

In this subsection we prove the first assertion of Theorem 1.1.

Proposition 6.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a smooth simply connected domain. Then, if $f : \Omega \rightarrow D$ is a Riemann map, there holds*

$$e^{2H(a,a)} = \frac{|f'(a)|^2}{(1 - |f'(a)|^2)^2}; \quad \tilde{H}(a) = 2 \frac{|f'(a)|^2}{(1 - |f'(a)|^2)^2}. \quad (67)$$

PROOF. The first part of the statement is well-known, see e.g. [6], Table 2. Letting $G_D(a, \xi)$ denote the Green's function for D , and letting $\varphi : D \rightarrow \mathbb{R}$ being any smooth function with compact support, we have

$$\varphi(a') = \int_D G_D(a', \xi) \Delta \varphi(\xi) d\xi, \quad a' \in D.$$

Let $\psi : \Omega \rightarrow \mathbb{R}$ be defined by $\psi = \varphi \circ f$. We have $\Delta\psi(\xi) = \frac{1}{|f'(\xi)|^2} \Delta\varphi(\xi)$ and hence, letting $G(a, \xi)$ be the Green's function for Ω and using a change of variables we get

$$\psi(a) = \varphi(f(a)) = \varphi(a') = \int_D G_D(a', \xi) \Delta\varphi(\xi) d\xi = \int_\Omega G_D(f(a), f(\xi)) \Delta\psi(\xi) d\xi,$$

where $a' = f(a)$. Hence, from the explicit expression of G_D it turns out that

$$G(a, z) = G_D(f(a), f(z)) = -\frac{1}{2} \log \frac{|f(z) - f(a)|^2}{|1 - f(z)\overline{f(a)}|^2}; \quad a, z \in \Omega,$$

where we have identified ξ with the point z in the complex plane. It follows that

$$H(a, z) = \frac{1}{2} \left[\log |z - a|^2 - \log \frac{|f(z) - f(a)|^2}{|1 - f(z)\overline{f(a)}|^2} \right]; \quad a, z \in \Omega.$$

The last expression can be rewritten as

$$H(a, z) = \frac{1}{2} \left[\log |1 - f(z)\overline{f(a)}|^2 - \log \frac{|f(z) - f(a)|^2}{|z - a|^2} \right]; \quad a, z \in \Omega.$$

In particular, taking the limit $z \rightarrow a$, we deduce immediately the first equality in (67).

Using complex notation, we have

$$\begin{aligned} \frac{\partial}{\partial a_1} &= \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial \bar{a}} \right); & \frac{\partial}{\partial a_2} &= i \left(\frac{\partial}{\partial a} - \frac{\partial}{\partial \bar{a}} \right); \\ \frac{\partial}{\partial x} &= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right); & \frac{\partial}{\partial y} &= i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right). \end{aligned}$$

It follows that

$$\frac{\partial}{\partial x} \frac{\partial}{\partial a_1} + \frac{\partial}{\partial y} \frac{\partial}{\partial a_2} = 2 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{a}} + 2 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial a} = 4 \operatorname{Re} \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial a} := L. \quad (68)$$

To derive the expression of \tilde{H} , recall (13), we apply L to $H(a, z)$ and evaluate at $z = a$. We have, still in complex notation

$$\frac{\partial}{\partial a} H(a, z) = -\frac{1}{2} \frac{\overline{f(z)} f'(a)}{1 - \overline{f(z)} f(a)} - \frac{1}{2} \frac{\partial}{\partial a} \log \frac{f(z) - f(a)}{z - a}.$$

When we apply the operator $\frac{\partial}{\partial \bar{z}}$ the second term vanishes and we get

$$\frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial a} H(a, z) = -\frac{1}{2} \frac{f'(a) \overline{f'(z)}}{(1 - \overline{f(z)} f(a))^2}.$$

Taking the real part we find

$$4\operatorname{Re} \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial a} H(a, z) = - \left[\frac{f'(a) \overline{f'(z)}}{(1 - \overline{f(z)} f(a))^2} + \frac{\overline{f'(a)} f'(z)}{(1 - \overline{f(a)} f(z))^2} \right].$$

Choosing $z = a$ in the last formula, we obtain the second identity in (67). This concludes the proof. ■

Remark 6.2. (a) The expression of \tilde{H} in (67) does not depend on the choice of the conformal map f of Ω into D .

(b) From the explicit description of \tilde{H} in Proposition 6.1 we obtain $\tilde{H}(a) \rightarrow +\infty$ as $a \rightarrow \partial\Omega$. This is true for any domain, as proved in [21], Lemma 5.7.

(c) In the case of simply connected domains, the function \tilde{H} coincides with the square of the reciprocal of the conformal radius and the hyperbolic radius, see [6], Definitions 1, 7 and Theorem 8. See also Remark 6.4.

(e) Since every convex domain has a single conformal incenter, see [14] Proposition 11, it follows that \tilde{H} possesses a unique critical point in this case. For a general simply connected domain \tilde{H} will have multiple critical points, see [14] page 483. We also point out that, even if a conformal transformation of the domain affects the number of critical points of \tilde{H} , the topology of critical points at infinity (see [2]) at the first level of non-compactness should be an invariant.

6.2. Multiply connected domains.

In this subsection we derive a general formula for \tilde{H} on multiply connected domains. This formula makes use of the covering map and the deck transformation.

Let us recall that the Green's function in the unit disk with pole $z_0 \in D$ is given by

$$G_D(z, z_0) = -\log \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|, \quad z, z_0 \in D.$$

Let us pick a point $w \in \Omega$ and consider the Green's function for Ω with pole at a . From [6], Theorem 4, one has

$$G_\Omega(a, w) = - \sum_k \log \left| \frac{z_k - z}{1 - \bar{z}_k z} \right|, \quad \text{where } f(z) = w, f(z_k) = a \text{ for all } k.$$

Thus the regular part $G_\Omega(a, w)$ is

$$H_\Omega(a, w) = \log |f(z_0) - f(z)| - \sum_k \log |z_k - z| + \sum_k \log |1 - \bar{z}_k z|. \quad (69)$$

Now, as in the previous subsection, it is sufficient to apply the operator L defined in (68). The first two terms vanish when L is applied. To handle the third term, note that f^{-1} is a local diffeomorphism. So f^{-1} is defined from $f^{-1} : \mathcal{U} \rightarrow \mathcal{V}_k$, where \mathcal{U} is a neighborhood of a and \mathcal{V}_k is a neighborhood of z_k . Thus we have

$$\frac{\partial H}{\partial a} = \frac{\partial H}{\partial z_k} \frac{\partial z_k}{\partial a} = \frac{1}{f'(z_k)} \frac{\partial H}{\partial z_k}.$$

Similarly, there holds

$$\frac{\partial^2 H}{\partial \bar{w} \partial a} = \frac{1}{f'(z_k)} \frac{1}{f'(z)} \frac{\partial^2 H}{\partial \bar{z} \partial z_k}. \quad (70)$$

Hence, using (69) and (70) we find

$$4\text{Re} \frac{\partial^2 H}{\partial \bar{w} \partial a} \Big|_{w=a} = -2\text{Re} \sum_k \frac{1}{f'(z_k) \overline{f'(z_0)}} \frac{1}{(1 - z_k \bar{z}_0)^2}.$$

Let T_k be the Mobius (deck) transformation that maps z_0 into z_k . We have

$$f = f \circ T_k \quad \Rightarrow \quad f'(z_0) = f'(z_k) T'_k(z_0).$$

Using the last equation and factoring the term $(1 - |z_0|^2)^2$, we deduce

$$2\text{Re} \frac{\partial^2 H}{\partial \bar{w} \partial a} \Big|_{w=w_1} = -2 \frac{1}{|f'(z_0)|^2 (1 - |z_0|^2)^2} \text{Re} \sum_k \frac{T'_k(z_0) (1 - |z_0|^2)^2}{(1 - z_k \bar{z}_0)^2}. \quad (71)$$

From (71) we obtain immediately the following result.

Proposition 6.3. *Let $\Omega \subseteq \mathbb{R}^2$ be a multiply connected domain, and let $f : D \rightarrow \Omega$ be a conformal covering map. Given $a \in \Omega$, let $\{z_k\}_k$ be the*

pre-image of the point a under the map f , and let $T_k : D \rightarrow D$ denote the deck transformation mapping z_0 into z_k . Then there holds

$$\tilde{H}(a) = \frac{1}{|f'(z_0)|^2(1-|z_0|^2)^2} \sum_k \left(\frac{T_k'(z_0)(1-|z_0|^2)^2}{(1-z_k\bar{z}_0)^2} + \frac{\overline{T_k'(z_0)}(1-|z_0|^2)^2}{(1-\bar{z}_k z_0)^2} \right). \quad (72)$$

We note that when $f^{-1}(a) = \{z_0\}$ we recover the formula for the simply connected domain.

6.3. Some numerical computation.

In this subsection we prove that in general, for a multiply connected domain, the two functions \tilde{H} and $2e^{2H}$ do not coincide (this is the case for simply connected domains, see Proposition 6.1). We consider in particular the case of an annulus of inner radius $\frac{1}{\rho}$ and outer radius ρ , where $\rho > 1$. Our numerical computations show that the critical points of these two functions do not coincide, hence we obtain the statement (b) in Theorem 1.1.

For $\rho > 1$ we set

$$A_\rho = \left\{ (x, y) : \frac{1}{\rho^2} < x^2 + y^2 < \rho^2 \right\}; \quad S_\rho = \{(x, y) : -\log \rho < x < \log \rho\}.$$

It is clear that S_ρ is a covering of A_ρ through the exponential map. We also define $\alpha \in \mathbb{C}$, $h_\rho : S_\rho \rightarrow D$ and $f_\rho : D \rightarrow A_\rho$ by

$$\alpha = -\frac{i\pi}{2 \log \rho}; \quad h_\rho(w) = \frac{e^{\alpha w} - 1}{e^{\alpha w} + 1}, \quad h_\rho^{-1}(z) = \frac{1}{\alpha} \log \left(\frac{1+z}{1-z} \right), \quad f = \exp \circ h_\rho^{-1}.$$

where $w \in S_\rho$ and $z \in D$. Our aim is to compute formula (72) for this particular case. Fixing $z_0 \in D$, the points z_k and the corresponding points $w_k = h_\rho^{-1} z_k$ are given by

$$z_k = \frac{e^{2k\pi i \alpha} \left(\frac{1+z_0}{1-z_0} \right) - 1}{e^{2k\pi i \alpha} \left(\frac{1+z_0}{1-z_0} \right) + 1}; \quad w_k = \frac{1}{\alpha} \log \left(\frac{1+z_0}{1-z_0} \right) + 2k\pi i.$$

Using some elementary computations we obtain

$$z_k = \frac{z_0 + M_k}{z_0 M_k + 1}; \quad \text{where } M_k = \tanh \left(\frac{k\pi^2}{2 \log \rho} \right).$$

If T_k denotes as before the deck transformation, then there holds

$$T_k(z) = \frac{z + M_k}{zM_k + 1}; \quad T'_k(z) = \frac{1 - M_k^2}{(1 + M_k z)^2}.$$

By symmetry, it is sufficient to compute (71) for $f(z_0)$ real and positive. It is convenient to use the following parametrization for the points $a \in A_\rho$ and $z_0 \in D$

$$a = \log x; \quad z_0 = -i \tan\left(\frac{\pi}{4 \log \rho} \log x\right), \quad x \in (-\log \rho, \log \rho).$$

Using this notation, from equation (72) we are left with

$$\tilde{H}(a) = \frac{1}{|f'(z_0)|^2(1 - |z_0|^2)^2} \sum_k \left(\frac{(1 - M_k^2)(1 - |z_0|^2)^2}{(1 + M_k \bar{z}_0)^2(1 - \bar{z}_k z_0)^2} + \frac{(1 - M_k^2)(1 - |z_0|^2)^2}{(1 + M_k z_0)^2(1 - z_k \bar{z}_0)^2} \right).$$

From the above formulit it follows

$$\tilde{H}(a) = \frac{2}{|f'(z_0)|^2(1 - |z_0|^2)^2} \sum_k \frac{(1 - M_k^2)(1 - |z_0|^2)^2 ((1 - |z_0|^2)^2 - 4|z_0|^2 M_k^2)}{((1 - |z_0|^2)^2 + 4|z_0|^2 M_k^2)^2}.$$

From [6] we have

$$e^{2H(a,a)} = \frac{1}{|f'(z_0)|^2(1 - |z_0|^2)^2} \prod_{z_k \neq z_0} \left| \frac{z_k - z_0}{1 - z_k \bar{z}_0} \right|^2.$$

Using elementary computations it turns out that

$$\tilde{H}(a) = \frac{\pi^2}{8(\log \rho)^2} \frac{1}{\cos^2\left(\frac{\pi}{2 \log \rho} \log x\right) |x|^2} \left(1 + 2 \sum_{k=1}^{\infty} W(k, x) \right); \quad (73)$$

$$2e^{2H(a,a)} = \frac{\pi^2}{8(\log \rho)^2} \frac{1}{\cos^2\left(\frac{\pi}{2 \log \rho} \log x\right) |x|^2} \prod_{k=1}^{\infty} Z(k, x)^2, \quad (74)$$

where

$$W(k, x) = \left(1 - \tanh^2\left(\frac{k\pi^2}{2 \log \rho}\right) \right)^2 \left(1 - \tan^2\left(\frac{\pi}{4 \log \rho} \log x\right) \right)^2 \\ \times \frac{\left(1 - \tan^2\left(\frac{\pi}{4 \log \rho} \log x\right) \right)^2 - 4 \tanh^2\left(\frac{k\pi^2}{2 \log \rho}\right) \tan^2\left(\frac{\pi}{4 \log \rho} \log x\right)}{\left(\left(1 - \tan^4\left(\frac{\pi}{4 \log \rho} \log x\right) \right)^2 + 4 \tanh^2\left(\frac{k\pi^2}{2 \log \rho}\right) \tan^2\left(\frac{\pi}{4 \log \rho} \log x\right) \right)^2}, \quad k \geq 1;$$

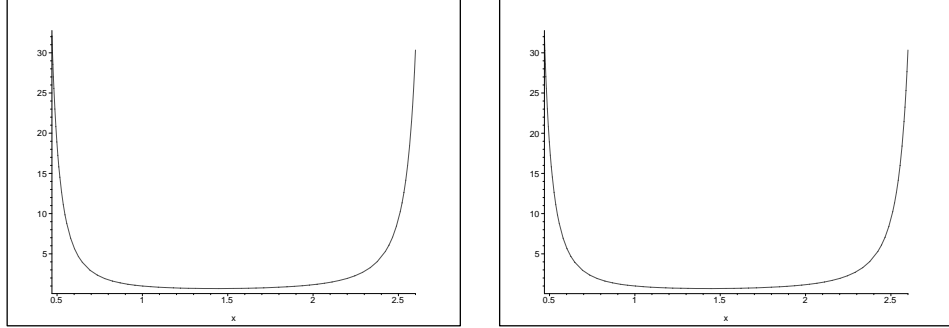


Figure 1: the functions $2e^{2H}$ and \tilde{H} for $\rho = e$, full picture

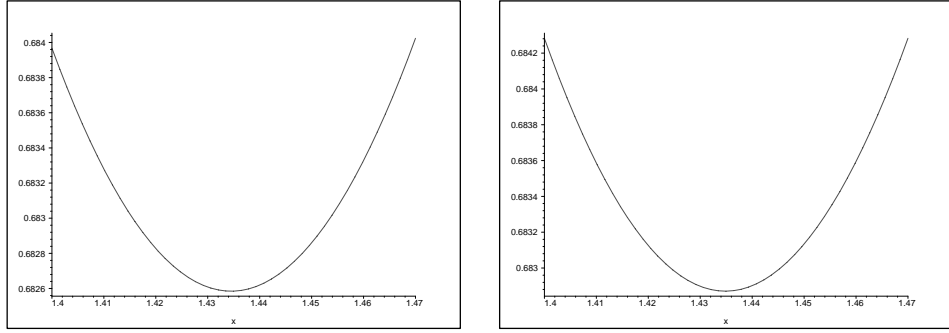


Figure 2: the functions $2e^{2H}$ and \tilde{H} for $\rho = e$, detail

$$Z(k, x) = \frac{\tanh^2\left(\frac{k\pi^2}{2\log \rho}\right) \sec^4\left(\frac{\pi}{4\log \rho} \log x\right)}{\left(1 - \tan^4\left(\frac{\pi}{4\log \rho} \log x\right)\right)^2 + 4 \tanh^2\left(\frac{k\pi^2}{2\log \rho}\right) \tan^2\left(\frac{\pi}{4\log \rho} \log x\right)}, \quad k \geq 1.$$

In Figures 1-3 we plot the functions $2e^{2H}$ and \tilde{H} (modulo the irrelevant factor $\frac{\pi^2}{8(\log \rho)^2}$) for $\rho = e$ and for $\rho = e^{3.5}$. We note that, roughly, $W(k, x) \sim e^{-\frac{2k\pi^2}{\log \rho}}$ and $Z(k, x) \sim 1 - e^{-\frac{k\pi^2}{\log \rho}}$ so for small values of ρ the terms with $k \neq 0$ are almost negligible. This accounts for the fact that for $\rho = e$ the graphs are very similar, see Figure 1, even on a fine scale, see Figure 2. For large values of ρ the difference between the two functions is more pronounced, see Figure 3.

Remark 6.4. We recall that the harmonic and hyperbolic radii are defined by

$$r_{har}(\xi) = e^{-H(\xi, \xi)}; \quad r_{hyp}(f(z)) = |f'(z)|(1 - |z|^2),$$

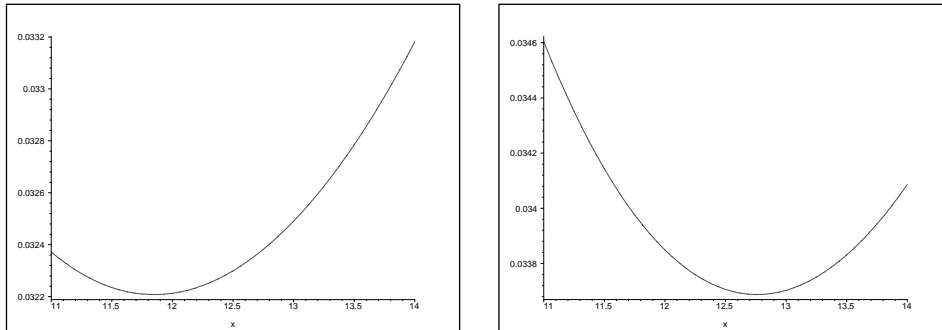


Figure 3: the functions $2e^{2H}$ and \tilde{H} for $\log \rho = 3.5$

where $f : D \rightarrow \Omega$ denotes a conformal covering map. See [6] Definition 1 and page 15. Note that the function \tilde{H} , in the case of general non-simply connected domains, do not even coincide with r_{hyp}^{-2} . However, in the case of small annuli, numerical computation show that r_{har} and r_{hyp} are very close, see [6] Figure 8. We also point out that the harmonic radius is related to the Bergman kernel, see [6] Section 8.4.

7. The expansion for multiple bubbles.

In this section we consider the case of multiple bubbles. We begin considering only two bubbles $R_1\delta_{a,\lambda_1}$ and $R_2\delta_{b,\lambda_2}$, which we denote for simplicity by δ_1 and δ_2 respectively. We assume that $R_1 = Id$, namely the first bubble is not rotated, and we simply write R for R_2 .

For $\bar{C} > 0$, $k = 2$ and $P\delta_1 + P\delta_2 \in Z$, our aim is to expand the functional I_ε (see (P_ε)) on Z in terms of the parameters $a = p_1$, $b = p_2$, λ_1, λ_2 and R . In the following, for brevity, we set (see Section 2)

$$\sigma = b - a, \quad e(\lambda_1, \lambda_2) = O\left((\log \lambda_1 + \log \lambda_2) \left(\frac{1}{\lambda_1^3} + \frac{1}{\lambda_2^3} + \frac{1}{\lambda_1^2 \lambda_2} + \frac{1}{\lambda_1 \lambda_2^2}\right)\right).$$

We recall the explicit form of the functional $I_\varepsilon(u)$, for $u \in H_0^1(\Omega; \mathbb{R}^3)$

$$I_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{2}{3} \int_\Omega u \cdot (u_x \wedge u_y) + \varepsilon \int_\Omega u \cdot (u_x \wedge g_y + g_x \wedge u_y) + 2\varepsilon^2 \int_\Omega u \cdot (g_x \wedge g_y).$$

7.1. Interaction with g .

We consider first the interaction term $\varepsilon \int_\Omega u \cdot (u_x \wedge g_y + g_x \wedge u_y)$ in (P_ε) , with $u = P\delta_1 + P\delta_2$. We recall that throughout this section we assume that

$\overline{C}^{-1}\lambda_i^{-1} \leq \varepsilon \leq \overline{C}\lambda_i^{-1}$ for some fixed constant \overline{C} and for $i = 1, 2$. We have

$$\begin{aligned}
& \int_{\Omega} u \cdot (u_x \wedge g_y + g_x \wedge u_y) \\
&= \int_{\Omega} (P\delta_1 + P\delta_2) \cdot [((P\delta_1)_x + (P\delta_2)_x) \wedge g_y + g_x \wedge ((P\delta_1)_y + (P\delta_2)_y)] \\
&= \int_{\Omega} P\delta_1 \cdot ((P\delta_1)_x \wedge g_y + g_x \wedge (P\delta_1)_y) + \int_{\Omega} P\delta_1 \cdot ((P\delta_2)_x \wedge g_y + g_x \wedge (P\delta_2)_y) \\
& \quad (75) \\
&+ \int_{\Omega} P\delta_2 \cdot ((P\delta_2)_x \wedge g_y + g_x \wedge (P\delta_2)_y) + \int_{\Omega} P\delta_2 \cdot ((P\delta_1)_x \wedge g_y + g_x \wedge (P\delta_1)_y).
\end{aligned}$$

The first term in (75) has been estimated in Section 5, formulas (62)-(63), and gives

$$\varepsilon \int_{\Omega} P\delta_1((P\delta_1)_x \wedge g_y + g_x \wedge (P\delta_1)_y) = -16 \frac{\varepsilon}{\lambda_1} \int_{\Omega} \frac{x^2}{(1+|\xi|^2)^3} d_{Id}g(a) + O(\lambda_1^{-2}). \quad (76)$$

The third term in (75) can be estimated similarly, using the invariance of the problem under rotation, and gives

$$\varepsilon \int_{\Omega} P\delta_2((P\delta_2)_x \wedge g_y + g_x \wedge (P\delta_2)_y) = -16 \frac{\varepsilon}{\lambda_2} \int_{\Omega} \frac{x^2}{(1+|\xi|^2)^3} d_{R^{-1}}g(b) + O(\lambda_2^{-2}). \quad (77)$$

Let us compute now the remaining two terms in (75), starting from the fourth. We write $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Up to an error of order ε^2 , we have

$$\begin{aligned}
& (P\delta_1)_x \wedge g_y = \quad (78) \\
& \left(\begin{array}{l} \left(-\frac{4\lambda_1^3(x-a_1)(y-a_2)}{(1+\lambda_1^2|\xi-a|^2)^2} - \frac{2}{\lambda_1} \partial_x h_2 \right) (g_3)_y - \left(\frac{4\lambda_1^2(x-a_1)}{(1+\lambda_1^2|\xi-a|^2)^2} + \frac{2}{\lambda_1^2} \partial_x h_3 \right) (g_2)_y \\ \left(\frac{4\lambda_1^2(x-a_1)}{(1+\lambda_1^2|\xi-a|^2)^2} + \frac{2}{\lambda_1^2} \partial_x h_3 \right) (g_1)_y - \left(2\lambda_1 \frac{1+\lambda_1^2((y-a_2)^2-(x-a_1)^2)}{(1+\lambda_1^2|\xi-a|^2)^2} - \frac{2}{\lambda_1} \partial_x h_1 \right) (g_3)_y \\ \left(2\lambda_1 \frac{1+\lambda_1^2((y-a_2)^2-(x-a_1)^2)}{(1+\lambda_1^2|\xi-a|^2)^2} - \frac{2}{\lambda_1} \partial_x h_1 \right) (g_2)_y + \left(\frac{4\lambda_1^3(x-a_1)(y-a_2)}{(1+\lambda_1^2|\xi-a|^2)^2} + \frac{2}{\lambda_1} \partial_x h_2 \right) (g_1)_y \end{array} \right)
\end{aligned}$$

and

$$P\delta_2 = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} \frac{2\lambda_2(x-b_1)}{1+\lambda_2^2|\xi-b|^2} - \frac{2}{\lambda_2} h_1(b, \xi) \\ \frac{2\lambda_2(y-b_2)}{1+\lambda_2^2|\xi-b|^2} - \frac{2}{\lambda_2} h_2(b, \xi) \\ \frac{2}{\lambda_2^2} h_3 - \frac{2}{1+\lambda_2^2|\xi-b|^2} \end{pmatrix},$$

where r_{ij} are the entries of the matrix R . We are going to prove that

$$\int_{\Omega} P\delta_2((P\delta_1)_x \wedge g_y + g_x \wedge (P\delta_1)_y) = O(\varepsilon^2 |\log \varepsilon|). \quad (79)$$

If one uses (78) and (79), the integrals involved in the above expression are of the form

$$\lambda_1^3 \lambda_2 \int_{\Omega} \frac{(x - a_1)(y - a_2)(x - b_1)}{(1 + \lambda_1^2 |\xi - a|^2)^2 (1 + \lambda_2^2 |\xi - b|^2)}; \quad (80)$$

$$\lambda_1^3 \lambda_2 \int_{\Omega} \frac{(x - a_1)(y - a_2)(y - b_2)}{(1 + \lambda_1^2 |\xi - a|^2)^2 (1 + \lambda_2^2 |\xi - b|^2)};$$

$$\lambda_1^3 \int_{\Omega} \frac{(x - a_1)(y - a_2)}{(1 + \lambda_1^2 |\xi - a|^2)^2 (1 + \lambda_2^2 |\xi - b|^2)}; \quad \frac{\lambda_1^3}{\lambda_2} \int_{\Omega} \frac{(x - a_1)(y - a_2)}{(1 + \lambda_1^2 |\xi - a|^2)^2}; \quad (81)$$

$$\frac{\lambda_1^3}{\lambda_2^2} \int_{\Omega} \frac{(x - a_1)(y - a_2)}{(1 + \lambda_1^2 |\xi - a|^2)^2}$$

$$\lambda_1^2 \lambda_2 \int_{\Omega} \frac{(x - a_1)(x - b_1)}{(1 + \lambda_1^2 |\xi - a|^2)^2 (1 + \lambda_2^2 |\xi - b|^2)}; \quad (82)$$

$$\lambda_1^2 \lambda_2 \int_{\Omega} \frac{(x - a_1)(y - b_2)}{(1 + \lambda_1^2 |\xi - a|^2)^2 (1 + \lambda_2^2 |\xi - b|^2)};$$

$$\lambda_1^2 \int_{\Omega} \frac{(x - a_1)}{(1 + \lambda_1^2 |\xi - a|^2)^2 (1 + \lambda_2^2 |\xi - b|^2)}; \quad (83)$$

$$\frac{\lambda_1^2}{\lambda_2} \int_{\Omega} \frac{(x - a_1)}{(1 + \lambda_1^2 |\xi - a|^2)^2}; \quad \frac{\lambda_1^2}{\lambda_2^2} \int_{\Omega} \frac{(x - a_1)}{(1 + \lambda_1^2 |\xi - a|^2)^2}$$

$$\lambda_1 \lambda_2 \int_{\Omega} \frac{1 + \lambda_1^2 ((y - a_2)^2 - (x - a_1)^2)(x - b_1)}{(1 + \lambda_1^2 |\xi - a|^2)^2 (1 + \lambda_2^2 |\xi - b|^2)}; \quad (84)$$

$$\lambda_1 \lambda_2 \int_{\Omega} \frac{1 + \lambda_1^2 ((y - a_2)^2 - (x - a_1)^2)(y - b_2)}{(1 + \lambda_1^2 |\xi - a|^2)^2 (1 + \lambda_2^2 |\xi - b|^2)};$$

$$\lambda_1 \int_{\Omega} \frac{1 + \lambda_1^2((y - a_2)^2 - (x - a_1)^2)}{(1 + \lambda_1^2|\xi - a|^2)^2 (1 + \lambda_2^2|\xi - b|^2)}; \quad (85)$$

$$\frac{\lambda_1}{\lambda_2} \int_{\Omega} \frac{1 + \lambda_1^2((y - a_2)^2 - (x - a_1)^2)}{(1 + \lambda_1^2|\xi - a|^2)^2};$$

$$\frac{\lambda_1}{\lambda_2^2} \int_{\Omega} \frac{1 + \lambda_1^2((y - a_2)^2 - (x - a_1)^2)}{(1 + \lambda_1^2|\xi - a|^2)^2}; \quad \frac{\lambda_2}{\lambda_1} \int_{\Omega} \frac{x - b_1}{(1 + \lambda_2^2|\xi - b|^2)}; \quad (86)$$

$$\frac{\lambda_2}{\lambda_1} \int_{\Omega} \frac{y - b_2}{(1 + \lambda_2^2|\xi - b|^2)};$$

$$\frac{1}{\lambda_1} \int_{\Omega} \frac{1}{(1 + \lambda_1^2|\xi - b|^2)}; \int_{\Omega} \frac{1}{\lambda_1 \lambda_2}; \int_{\Omega} \frac{1}{\lambda_1 \lambda_2^2}; \frac{\lambda_2}{\lambda_1^2} \int_{\Omega} \frac{x - b_1}{(1 + \lambda_2^2|\xi - b|^2)}; \quad (87)$$

$$\frac{\lambda_2}{\lambda_1^2} \int_{\Omega} \frac{y - b_2}{(1 + \lambda_2^2|\xi - b|^2)}; \frac{1}{\lambda_1^2} \int_{\Omega} \frac{1}{(1 + \lambda_2^2|\xi - b|^2)}; \int_{\Omega} \frac{1}{\lambda_1^2 \lambda_2}; \int_{\Omega} \frac{1}{\lambda_1^2 \lambda_2^2}. \quad (88)$$

The errors in the expressions of $(P\delta_1)_x \wedge g_y$ and $P\delta_2$ are negligible with respect to the quantities listed in (80)-(88), hence it is sufficient to consider the above expressions.

Estimate of (80). Using the rescaling $\xi \mapsto \lambda_1(\xi - a)$ and setting $\Omega_{\lambda_1, a} = \lambda_1(\Omega - a)$, we get

$$\begin{aligned} & \lambda_1^3 \lambda_2 \int_{\Omega} \frac{(x - a_1)(y - a_2)(x - b_1)}{(1 + \lambda_1^2|\xi - a|^2)^2 (1 + \lambda_2^2|\xi - b|^2)} \\ &= \frac{\lambda_2}{\lambda_1^2} \int_{\Omega_{\lambda_1, a}} \frac{x^2 y}{(1 + |\xi|^2)^2 \left(1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2\right)} \\ &+ \frac{\lambda_2}{\lambda_1} \sigma_1 \int_{\Omega_{\lambda_1, a}} \frac{xy}{(1 + |\xi|^2)^2 \left(1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2\right)}. \end{aligned} \quad (89)$$

Let us consider the first integral. We divide $\Omega_{\lambda_1, a}$ into the regions $|\xi| \leq \frac{\lambda_1 \sigma}{2}$

and $|\xi| \geq \frac{\lambda_1 \sigma}{2}$. If $|\xi| \leq \frac{\lambda_1 \sigma}{2}$, then $|\xi - \lambda_1 \sigma|^2 \geq \frac{\lambda_1^2 |\sigma|^2}{4}$, so we have

$$\begin{aligned} \frac{\lambda_2}{\lambda_1^2} \left| \int_{|\xi| \leq \frac{\lambda_1 \sigma}{2}} \frac{x^2 y}{(1 + |\xi|^2)^2 \left(1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2\right)} \right| &\leq \frac{C \lambda_2}{\lambda_1^2 \lambda_2^2 |\sigma|^2} \int_{|\xi| \leq \frac{\lambda_1 \sigma}{2}} \frac{|\xi|^3}{(1 + |\xi|^2)^2} \\ &\leq \frac{C \lambda_1 \lambda_2}{\lambda_1^2 \lambda_2^2 |\sigma|^2}, \end{aligned}$$

where C is a positive constant depending only on Ω . When $|\xi| \geq \frac{\lambda_1 |\sigma|}{2}$ then for λ_1 large there holds $\frac{|\xi|^3}{(1 + |\xi|^2)^2} \leq \frac{2}{\lambda_1 |\sigma|}$. As a consequence, using also the change of variables $\frac{\lambda_2}{\lambda_1} (\xi - \lambda_1 \sigma) \mapsto \xi$, we deduce

$$\begin{aligned} \frac{\lambda_2}{\lambda_1^2} \left| \int_{|\xi| \geq \frac{\lambda_1 \sigma}{2}} \frac{x^2 y}{(1 + |\xi|^2)^2 \left(1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2\right)} \right| &\leq \frac{C \lambda_2}{\lambda_1^3 |\sigma|} \left(\frac{\lambda_1}{\lambda_2}\right)^2 \int_{\Omega_{\lambda_2, b}} \frac{1}{(1 + |\xi|^2)} \\ &\leq C \frac{\log \lambda_2}{\lambda_1 \lambda_2 |\sigma|}. \end{aligned}$$

Turning to the second integral in the r.h.s. of (89), we again divide the domain into two regions $|\xi| \leq \frac{\lambda_1 \sigma}{2}$ and $|\xi| \geq \frac{\lambda_1 \sigma}{2}$. Reasoning as before we find

$$\begin{aligned} \frac{\lambda_2}{\lambda_1} \sigma_1 \left| \int_{|\xi| \leq \frac{\lambda_1 \sigma}{2}} \frac{xy}{(1 + |\xi|^2)^2 \left(1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2\right)} \right| &\leq C \frac{\lambda_2 \sigma_1}{\lambda_1 \lambda_2^2 |\sigma|^2} \int_{|\xi| \leq \frac{\lambda_1 \sigma}{2}} \frac{|\xi|^2}{(1 + |\xi|^2)^2} \\ &\leq C \frac{|\sigma_1| \log \lambda_1}{|\sigma|^2 \lambda_1 \lambda_2}; \end{aligned}$$

$$\begin{aligned} \frac{\lambda_2}{\lambda_1} \sigma_1 \left| \int_{|\xi| \geq \frac{\lambda_1 \sigma}{2}} \frac{xy}{(1 + |\xi|^2)^2 \left(1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2\right)} \right| &\leq C \frac{\lambda_2 \sigma_1}{\lambda_1^3 |\sigma|^2} \left(\frac{\lambda_1}{\lambda_2}\right)^2 \int_{\Omega_{\lambda_2, b}} \frac{1}{(1 + |\xi|^2)} \\ &\leq C \frac{|\sigma_1| \lambda_2 \log \lambda_2}{|\sigma|^2 \lambda_1^3}. \end{aligned}$$

Since in the definition of Z we assume $\text{dist}(p_i, p_j) \geq \bar{C}^{-1}$, $|\sigma|$ is uniformly bounded from below, the last formulas imply

$$\lambda_1^3 \lambda_2 \left| \int_{\Omega} \frac{(x - a_1)(y - a_2)(x - b_1)}{(1 + \lambda_1^2 |\xi - a|^2)^2 (1 + \lambda_2^2 |\xi - b|^2)} \right| \leq C \varepsilon^2 |\log \varepsilon|.$$

The second expression in (80) can be estimated in the same way.

Estimate of (81)-(88). We just treat some particular cases, since many terms are similar to each-other. First of all, all the terms for which the quantity $(1 + \lambda_1^2|\xi - a|^2)^2 (1 + \lambda_2^2|\xi - b|^2)$ appears in the denominator can be treated as before.

Next, we consider for example the second term in (85) and the last term in (86). Using the changes of variable $\lambda_1(\xi - a) \mapsto \xi$ and $\lambda_2(\xi - b) \mapsto \xi$ we find

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} \int_{\Omega} \frac{1 + \lambda_1^2((y - a_2)^2 - (x - a_1)^2)}{(1 + \lambda_1^2|\xi - a|^2)^2} &= \frac{1}{\lambda_1 \lambda_2} \int_{\Omega_{\lambda_1, a}} \frac{1 + y^2 - x^2}{(1 + |\xi|^2)^2} \leq C \frac{\log \lambda_1}{\lambda_1 \lambda_2} \\ &\leq C \varepsilon^2 |\log \varepsilon|; \end{aligned}$$

$$\frac{\lambda_2}{\lambda_1} \int_{\Omega} \frac{y - b_2}{(1 + \lambda_2^2|\xi - b|^2)} = \frac{1}{\lambda_1 \lambda_2^2} \int_{\Omega_{\lambda_2, b}} \frac{|\xi|}{1 + |\xi|^2} \leq C \frac{1}{\lambda_1 \lambda_2} \leq C \varepsilon^2.$$

Conclusion. Using the estimates of (80) and those of (81)-(88), we obtain (79). In the same way, one can prove that

$$\int_{\Omega} P\delta_1((P\delta_2)_x \wedge g_y + g_x \wedge (P\delta_2)_y) = O(\varepsilon^2 |\log \varepsilon|). \quad (90)$$

Hence, from equations (76), (77), (79) and (90) we deduce

Lemma 7.1. *For $u = P\delta_1 + P\delta_2 \in Z$ there holds*

$$\varepsilon \int_{\Omega} u(u_x \wedge g_y + g_x \wedge u_y) = -8A_0 d_{Id}g(a) - 8A_0 d_{R^{-1}}g(b) + O(\varepsilon^2 |\log \varepsilon|). \quad (91)$$

7.2. Mixed terms in $P\delta_1$ and $P\delta_2$.

For $u = P\delta_1 + P\delta_2$, we consider the first and the second integrals in (5). We are interested in the terms involving both $P\delta_1$ and $P\delta_2$, namely

$$\begin{aligned} &\int_{\Omega} \nabla P\delta_1 \cdot \nabla P\delta_2 + \frac{2}{3} \int_{\Omega} P\delta_1 \cdot ((P\delta_2)_x \wedge (P\delta_2)_y) + \frac{2}{3} \int_{\Omega} P\delta_1 \cdot ((P\delta_2)_x \wedge (P\delta_1)_y) \\ &+ \frac{2}{3} \int_{\Omega} P\delta_1 \cdot ((P\delta_1)_x \wedge (P\delta_2)_y) + \frac{2}{3} \int_{\Omega} P\delta_2 \cdot ((P\delta_2)_x \wedge (P\delta_1)_y) \\ &+ \frac{2}{3} \int_{\Omega} P\delta_2 \cdot ((P\delta_1)_x \wedge (P\delta_2)_y) + \frac{2}{3} \int_{\Omega} P\delta_2 \cdot ((P\delta_1)_x \wedge (P\delta_1)_y). \end{aligned}$$

Integrating by parts it is easy to see that the last expression becomes

$$\int_{\Omega} \nabla P\delta_1 \cdot \nabla P\delta_2 + 2 \int_{\Omega} P\delta_2 \cdot ((P\delta_1)_x \wedge (P\delta_1)_y) + 2 \int_{\Omega} P\delta_1 \cdot ((P\delta_2)_x \wedge (P\delta_2)_y).$$

Integrating by parts the first term we get

$$\int_{\Omega} \nabla P\delta_1 \cdot \nabla P\delta_2 = - \int_{\Omega} \Delta P\delta_2 \cdot P\delta_1 = -2 \int_{\Omega} P\delta_1 \cdot ((\delta_2)_x \wedge (\delta_2)_y),$$

so we are left with

$$2 \int_{\Omega} P\delta_2 \cdot ((P\delta_1)_x \wedge (P\delta_1)_y) + 2 \int_{\Omega} P\delta_1 \cdot ((P\delta_2)_x \wedge (P\delta_2)_y - (\delta_2)_x \wedge (\delta_2)_y).$$

Lemma 7.2. *For $P\delta_1 + P\delta_2 \in Z$, there holds*

$$\begin{aligned} 2 \int_{\Omega} P\delta_2 \cdot ((P\delta_1)_x \wedge (P\delta_1)_y) + 2 \int_{\Omega} P\delta_1 \cdot ((P\delta_2)_x \wedge (P\delta_2)_y - (\delta_2)_x \wedge (\delta_2)_y) \\ = 2 \int_{\Omega} P\delta_2 \cdot ((\delta_1)_x \wedge (\delta_1)_y) + e(\lambda_1, \lambda_2). \end{aligned} \quad (92)$$

PROOF. Since the difference between the l.h.s. and the r.h.s. of (92) is

$$\begin{aligned} 2 \int_{\Omega} P\delta_1 \cdot ((P\delta_2)_x \wedge (P\delta_2)_y - (\delta_2)_x \wedge (\delta_2)_y) \\ + 2 \int_{\Omega} P\delta_2 \cdot ((P\delta_1)_x \wedge (P\delta_1)_y - (\delta_1)_x \wedge (\delta_1)_y), \end{aligned}$$

it is sufficient by symmetry to estimate one of the two terms in the last expression. We have

$$\begin{aligned} \int_{\Omega} P\delta_1 \cdot ((P\delta_2)_x \wedge (P\delta_2)_y - (\delta_2)_x \wedge (\delta_2)_y) \\ = \int_{\Omega} P\delta_1 \cdot [(\varphi_2)_x \wedge (\varphi_2)_y - (\varphi_2)_x \wedge (\delta_2)_y - (\delta_2)_x \wedge (\varphi_2)_y]. \end{aligned}$$

Using equations (25)-(26), choosing $\tau \leq \frac{1}{4}\overline{C}$ and setting $\Omega_{a,b} = \Omega \setminus (B_\tau(a) \cup B_\tau(b))$, we find

$$\begin{aligned} \left| \int_{\Omega} P\delta_1 \cdot ((\varphi_2)_x \wedge (\varphi_2)_y) \right| &\leq \frac{C}{\lambda_1 \lambda_2^2}; \\ \left| \int_{\Omega} P\delta_1 \cdot ((\varphi_2)_x \wedge (\delta_2)_y) \right| &\leq \frac{C}{\lambda_2} \left[\int_{B_\tau(a)} |P\delta_1| |(\delta_2)_y| + \int_{B_\tau(b)} |P\delta_1| |(\delta_2)_y| \right. \\ &\quad \left. + \int_{\Omega_{a,b}} |P\delta_1| |(\delta_2)_y| \right] \\ &\leq \frac{C}{\lambda_2} \left[\frac{C}{\lambda_1 \lambda_2} + C \frac{\log \lambda_2}{\lambda_1 \lambda_2} + \frac{C}{\lambda_1 \lambda_2} \right], \end{aligned}$$

and an analogous estimate for the term $P\delta_1 \cdot ((\delta_2)_x \wedge (\varphi_2)_y)$. This concludes the proof. ■

Lemma 7.3. *For $P\delta_1 + P\delta_2 \in Z$, there holds*

$$\begin{aligned} &2 \int_{\Omega} P\delta_2 \cdot ((\delta_1)_x \wedge (\delta_1)_y) \\ &= \frac{16A_0}{\lambda_1 \lambda_2} r_{11} \left(\frac{\sigma_1^2 - \sigma_2^2}{|\sigma|^4} + \frac{\partial h_1}{\partial x}(a, b) \right) + \frac{16A_0}{\lambda_1 \lambda_2} r_{22} \left(\frac{\sigma_2^2 - \sigma_1^2}{|\sigma|^4} + \frac{\partial h_2}{\partial y}(a, b) \right) \\ &\quad + \frac{16A_0}{\lambda_1 \lambda_2} r_{12} \left(2 \frac{\sigma_1 \sigma_2}{|\sigma|^4} + \frac{\partial h_1}{\partial y}(a, b) \right) + \frac{16A_0}{\lambda_1 \lambda_2} r_{21} \left(2 \frac{\sigma_1 \sigma_2}{|\sigma|^4} + \frac{\partial h_2}{\partial x}(a, b) \right) \quad (93) \\ &\quad + e(\lambda_1, \lambda_2). \end{aligned}$$

PROOF. The left-hand side of (93) is given explicitly by

$$\begin{aligned} 2 \int_{\Omega} P\delta_2 \cdot ((\delta_1)_x \wedge (\delta_1)_y) &= 2 \sum_{i,j=1}^3 r_{ij} \int_{\Omega} (\delta_2)_j \cdot ((\delta_1)_x \wedge (\delta_1)_y)_i \\ &\quad - 2 \sum_{i,j=1}^3 r_{ij} \int_{\Omega} (\varphi_2)_j \cdot ((\delta_1)_x \wedge (\delta_1)_y)_i, \quad (94) \end{aligned}$$

where $\{r_{ij}\}$ are the entries of the matrix R . We are now going to estimate these integrals. We recall that, by (22)

$$\varphi_2(\xi) = \left(\frac{2}{\lambda_2} h_1(\xi, b) + O(\lambda_2^{-2}), \frac{2}{\lambda_2} h_2(\xi, b) + O(\lambda_2^{-2}), 1 - \frac{2}{\lambda_2^2} h_3(\xi, b) + O(\lambda_2^{-3}) \right).$$

Taking this into account, we find that the terms in (94) involving the coefficients r_{11} , r_{12} , r_{13} , r_{31} and r_{33} are given respectively by

$$\begin{aligned} & -32\lambda_1^3\lambda_2 \int_{\Omega} \frac{(x-b_1)(x-a_1)}{(1+\lambda_1^2|\xi-a|^2)^3(1+\lambda_2^2|\xi-b|^2)} \\ & + 32\lambda_1^3 \int_{\Omega} \frac{(\lambda_2^{-1}h_1(\xi,b) + O(\lambda_2^{-2}))(x_1-a_1)}{(1+\lambda_1^2|\xi-a|^2)^3}; \end{aligned} \quad (95)$$

$$\begin{aligned} & -32\lambda_1^3\lambda_2 \int_{\Omega} \frac{(x-b_1)(y-a_2)}{(1+\lambda_1^2|\xi-a|^2)^3(1+\lambda_2^2|\xi-b|^2)} \\ & + 32\lambda_1^3 \int_{\Omega} \frac{(\lambda_2^{-1}h_1(\xi,b) + O(\lambda_2^{-2}))(y-a_2)}{(1+\lambda_1^2|\xi-a|^2)^3}; \end{aligned} \quad (96)$$

$$\begin{aligned} & 16\lambda_1^2\lambda_2 \int_{\Omega} \frac{(x-b_1)(1-\lambda_1^2|\xi-a|^2)}{(1+\lambda_1^2|\xi-a|^2)^3(1+\lambda_2^2|\xi-b|^2)} - \\ & 16\lambda_1^2 \int_{\Omega} \frac{(\lambda_2^{-1}h_1(\xi,b) + O(\lambda_2^{-2}))(1-\lambda_1^2|\xi-a|^2)}{(1+\lambda_1^2|\xi-a|^2)^3}; \end{aligned} \quad (97)$$

$$16\lambda_1^3 \int_{\Omega} \left(\frac{2}{1+\lambda_2^2|\xi-b|^2} - \frac{2}{\lambda_2^2}h_3(\xi,b) + O(\lambda_2^{-3}) \right) \frac{(x-a_1)}{(1+\lambda_1^2|\xi-a|^2)^3}; \quad (98)$$

$$-8\lambda_1^2 \int_{\Omega} \left(\frac{2}{1+\lambda_2^2|\xi-b|^2} - \frac{2}{\lambda_2^2}h_3(\xi,b) + O(\lambda_2^{-3}) \right) \frac{(1-\lambda_1^2|\xi-a|^2)}{(1+\lambda_1^2|\xi-a|^2)^3}. \quad (99)$$

The terms involving the other coefficients of the matrix R can be estimated using the above ones, and will be taken into account later.

Estimate of (95). Using the change of variables $\lambda_1(\xi-a) \mapsto \xi$, equation (95) becomes

$$\begin{aligned} & -32\frac{\lambda_2}{\lambda_1} \int_{\Omega_{\lambda_1,a}} \frac{x(x-\lambda_1\sigma_1)}{(1+|\xi|^2)^3 \left(1 + \frac{\lambda_2^2}{\lambda_1^2}|\xi-\lambda_1\sigma|^2\right)} \\ & + 32 \int_{\Omega_{\lambda_1,a}} \frac{(\lambda_2^{-1}h_1(\lambda_1^{-1}\xi+a,b) + O(\lambda_2^{-2}))x}{(1+|\xi|^2)^3}. \end{aligned} \quad (100)$$

We estimate the first term in (100). Consider the following subsets of the domain of integration

$$\mathcal{B}_1 = \left\{ \xi \in \Omega_{\lambda_1,a} : |\xi| \leq \frac{\lambda_1|\sigma|}{4} \right\}, \quad \mathcal{B}_2 = \left\{ \xi \in \Omega_{\lambda_1,a} : |\xi - \lambda_1\sigma| \leq \frac{\lambda_1|\sigma|}{4} \right\},$$

$$\mathcal{B}_3 = \Omega_{\lambda_1,a} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2).$$

We can write

$$\left(1 + \frac{\lambda_2^2}{\lambda_1^2}|\xi - \lambda_1\sigma|^2\right)^{-1} = (1 + \lambda_2^2|\sigma|^2)^{-1} \left(1 + \frac{\frac{\lambda_2^2}{\lambda_1^2}|\xi|^2 - 2\frac{\lambda_2^2}{\lambda_1}\xi \cdot \sigma}{(1 + \lambda_2^2|\sigma|^2)}\right)^{-1}. \quad (101)$$

On the set \mathcal{B}_1 we have the following inequality

$$\left|\frac{\lambda_2^2}{\lambda_1^2}|\xi|^2 - 2\frac{\lambda_2^2}{\lambda_1}\xi \cdot \sigma\right| \leq \frac{\lambda_2^2|\sigma|^2}{16} + \frac{\lambda_2^2|\sigma|^2}{2} \leq \frac{3}{4}\lambda_2^2|\sigma|^2$$

hence, from a Taylor expansion, we obtain the following uniform estimate

$$\begin{aligned} & \left| \left(1 + \frac{\frac{\lambda_2^2}{\lambda_1^2}|\xi|^2 - 2\frac{\lambda_2^2}{\lambda_1}\xi \cdot \sigma}{(1 + \lambda_2^2|\sigma|^2)}\right)^{-1} - 1 + \frac{\frac{\lambda_2^2}{\lambda_1^2}|\xi|^2 - 2\frac{\lambda_2^2}{\lambda_1}\xi \cdot \sigma}{(1 + \lambda_2^2|\sigma|^2)} \right| \\ & \leq C \left(\frac{|\xi|^4}{\lambda_1^4|\sigma|^4} + \frac{|\xi|^2}{\lambda_1^2|\sigma|^2} \right), \quad \xi \in \mathcal{B}_1. \end{aligned} \quad (102)$$

Using equation (102) and some elementary computations we find

$$\begin{aligned} & \int_{\mathcal{B}_1} \frac{x(x - \lambda_1\sigma_1)}{(1 + |\xi|^2)^3 \left(1 + \frac{\lambda_2^2}{\lambda_1^2}|\xi - \lambda_1\sigma|^2\right)} \\ & = \int_{\mathcal{B}_1} \frac{x(x - \lambda_1\sigma_1)}{(1 + |\xi|^2)^3 (1 + \lambda_2^2|\sigma|^2)} \left(1 + \frac{2\frac{\lambda_2^2}{\lambda_1}\xi \cdot \sigma - \frac{\lambda_2^2}{\lambda_1^2}|\xi|^2}{1 + \lambda_2^2|\sigma|^2}\right) \\ & \quad + e(\lambda_1, \lambda_2) = \frac{A_0}{2} \frac{1}{(1 + \lambda_2^2|\sigma|^2)} - \frac{A_0}{2} \frac{2\sigma_1^2}{(1 + \lambda_2^2|\sigma|^2)|\sigma|^2} + e(\lambda_1, \lambda_2) \quad (103) \\ & = \frac{A_0}{2} \frac{1}{\lambda_2^2|\sigma|^2} \left(1 - 2\frac{\sigma_1^2}{|\sigma|^2}\right) + e(\lambda_1, \lambda_2) = \frac{A_0}{2} \frac{\sigma_2^2 - \sigma_1^2}{\lambda_2^2|\sigma|^4} + e(\lambda_1, \lambda_2). \end{aligned}$$

On the set \mathcal{B}_2 we have

$$|\xi| \geq \lambda_1|\sigma| - |\xi - \lambda_1\sigma| \geq \frac{3}{4}\lambda_1|\sigma|,$$

and hence we deduce easily

$$\begin{aligned} \int_{\mathcal{B}_2} \frac{|\xi||x - \lambda_1\sigma_1|}{(1 + |\xi|^2)^3 \left(1 + \frac{\lambda_2^2}{\lambda_1^2}|\xi - \lambda_1\sigma|^2\right)} & \leq \frac{C}{\lambda_1^4|\sigma|^4} \int_{B_{\frac{\lambda_1|\sigma|}{4}}} \frac{1}{1 + \frac{\lambda_2^2}{\lambda_1^2}|\xi|^2} \\ & \leq C \frac{\log \lambda_1}{\lambda_1^4|\sigma|^4}. \end{aligned} \quad (104)$$

In \mathcal{B}_3 we have $|\xi| \geq \frac{\lambda_1|\sigma|}{4}$ and $|\xi - \lambda_1\sigma| \geq \frac{\lambda_1|\sigma|}{4}$, and hence

$$\int_{\mathcal{B}_3} \frac{|\xi||x - \lambda_1\sigma|}{(1 + |\xi|^2)^3 \left(1 + \frac{\lambda_2^2}{\lambda_1^2}|\xi - \lambda_1\sigma|^2\right)} \leq C|\Omega_{\lambda_1,a}| \frac{1}{\lambda_1^5|\sigma|^5} \frac{1}{\lambda_1|\sigma|} \leq C \frac{1}{\lambda_1^4|\sigma|^6}. \quad (105)$$

Let us now treat the second term in (100). Reasoning as above we find

$$32 \int_{\Omega_{\lambda_1,a}} \frac{(\lambda_2^{-1}h_1(\lambda_1^{-1}\xi + a, b) + O(\lambda_2^{-2}))x}{(1 + |\xi|^2)^3} = \frac{1}{\lambda_1\lambda_2} \frac{A_0}{2} \frac{\partial h_1}{\partial x}(a, b) + e(\lambda_1, \lambda_2). \quad (106)$$

Hence, using formulas (103)-(106), we are able to estimate (95), and we find

$$2 \int_{\Omega} (P\delta_2)_1 ((\delta_1)_x \wedge (\delta_1)_y)_1 = \frac{16A_0}{\lambda_1\lambda_2} \left(\frac{\sigma_1^2 - \sigma_2^2}{|\sigma|^4} + \frac{\partial h_1}{\partial x}(a, b) \right) + e(\lambda_1, \lambda_2). \quad (107)$$

Estimate of (96). The proofs of the estimates of this and the remaining terms will only be sketched, since they are similar to that of (95). Using the usual change of variables, equation (96) becomes

$$\begin{aligned} & -32 \frac{\lambda_2}{\lambda_1} \int_{\Omega_{\lambda_1,a}} \frac{x(y - \lambda_1\sigma_2)}{(1 + |\xi|^2)^3 \left(1 + \frac{\lambda_2^2}{\lambda_1^2}|\xi - \lambda_1\sigma|^2\right)} \\ & + 32 \int_{\Omega_{\lambda_1,a}} \frac{(\lambda_2^{-1}h_1(\lambda_1^{-1}\xi + a, b) + O(\lambda_2^{-2}))y}{(1 + |\xi|^2)^3}. \end{aligned} \quad (108)$$

To treat the first integral in (108) we begin by dividing again $\Omega_{\lambda_1,a}$ into the above sets $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$. Reasoning as before and neglecting the higher-order terms we find

$$\begin{aligned} \frac{\lambda_2}{\lambda_1} \int_{\mathcal{B}_1} \frac{x(y - \lambda_1\sigma_2)}{(1 + |\xi|^2)^3 \left(1 + \frac{\lambda_2^2}{\lambda_1^2}|\xi - \lambda_1\sigma|^2\right)} &= -A_0 \frac{\sigma_1\sigma_2}{\lambda_1\lambda_2|\sigma|^4} + e(\lambda_1, \lambda_2); \\ \int_{\Omega_{\lambda_1,a}} \frac{(\lambda_2^{-1}h_1(\lambda_1^{-1}\xi + a, b) + O(\lambda_2^{-2}))y}{(1 + |\xi|^2)^3} &= \frac{1}{\lambda_1\lambda_2} \frac{A_0}{2} \frac{\partial h_1}{\partial y}(a, b) + e(\lambda_1, \lambda_2). \end{aligned}$$

Hence, using the last two equations we deduce

$$2 \int_{\Omega} (P\delta_2)_1 ((\delta_1)_x \wedge (\delta_1)_y)_2 = \frac{A_0}{\lambda_1\lambda_2} \left(32 \frac{\sigma_1\sigma_2}{|\sigma|^4} + 16 \frac{\partial h_1}{\partial y}(a, b) \right) + e(\lambda_1, \lambda_2). \quad (109)$$

Estimate of (97). We turn now to the term involving the coefficient r_{13} , (97), which can be written as

$$\begin{aligned} & 16 \frac{\lambda_2}{\lambda_1} \int_{\Omega_{\lambda_1, a}} \frac{(1 - |\xi|^2)(x_1 - \lambda_1 \sigma_1)}{(1 + |\xi|^2)^3 \left(1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2\right)} \\ & - 16 \int_{\Omega_{\lambda_1, a}} \frac{(\lambda_2^{-1} h_1(\lambda_1^{-1} \xi + a, b) + O(\lambda_2^{-2}))(1 - |\xi|^2)}{(1 + |\xi|^2)^3 \left(1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2\right)}. \end{aligned} \quad (110)$$

Using (27), (102) and reasoning as in (103) one finds

$$\int_{\mathcal{B}_1} \frac{(1 - |\xi|^2)(x_1 - \lambda_1 \sigma_1)}{(1 + |\xi|^2)^3 \left(1 + \frac{\lambda_2^2}{\lambda_1^2} |x - \lambda_1 \sigma|^2\right)} = e(\lambda_1, \lambda_2).$$

Similar estimates hold if one integrates on the sets \mathcal{B}_2 and \mathcal{B}_3 . Moreover, using (27) and elementary computations one finds

$$\int_{\Omega_{\lambda_1, a}} \frac{(\lambda_2^{-1} h_1(\lambda_1^{-1} \xi + a, b) + O(\lambda_2^{-2}))(1 - |\xi|^2)}{(1 + |\xi|^2)^3 \left(1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2\right)} = e(\lambda_1, \lambda_2).$$

From the last two equations we deduce

$$2 \int_{\Omega} (P\delta_2)_1 ((\delta_1)_x \wedge (\delta_1)_y)_3 = e(\lambda_1, \lambda_2). \quad (111)$$

Estimate of (98). The expression in (98) becomes

$$32 \int_{\Omega_{\lambda_1, a}} \left(\frac{1}{1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2} - \frac{1}{\lambda_2^2} h_3(\lambda_1^{-1} \xi + a, b) + O(\lambda_2^{-3}) \right) \frac{x}{(1 + |\xi|^2)^3}.$$

Reasoning as above, we obtain

$$\int_{\mathcal{B}_1} \left(\frac{2}{1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2} - \frac{2}{\lambda_2^2} h_3(\lambda_1^{-1} \xi + a, b) + O(\lambda_2^{-3}) \right) \frac{x}{(1 + |\xi|^2)^3} = e(\lambda_1, \lambda_2),$$

and that the integrals on the sets \mathcal{B}_2 and \mathcal{B}_3 are also of order $e(\lambda_1, \lambda_2)$. Hence we find

$$2 \int_{\Omega} (P\delta_2)_3 ((\delta_1)_x \wedge (\delta_1)_y)_1 = e(\lambda_1, \lambda_2). \quad (112)$$

Estimate of (99). We turn now to the term involving r_{33} . Using the above change of variables, (99) becomes

$$-16 \int_{\Omega_{\lambda_1, a}} \left(\frac{1}{1 + \frac{\lambda_2^2}{\lambda_1^2} |\xi - \lambda_1 \sigma|^2} - \frac{1}{\lambda_2^2} h_3(\lambda_1^{-1} \xi + a, b) + O(\lambda_2^{-3}) \right) \frac{1 - |\xi|^2}{(1 + |\xi|^2)^3}.$$

Using equation (27) and reasoning as above one finds

$$2 \int_{\Omega} (P\delta_2)_3 ((\delta_1)_x \wedge (\delta_1)_y)_3 = e(\lambda_1, \lambda_2). \quad (113)$$

Other estimates. From the estimates of the terms (95)-(99) one can deduce also those involving the coefficients r_{22} , r_{12} , r_{23} and r_{32} . In fact, it is sufficient to permute the coordinates x and y in a suitable way. Thus one finds

$$2 \int_{\Omega} (P\delta_2)_2 ((\delta_1)_x \wedge (\delta_1)_y)_2 = \frac{16A_0}{\lambda_1 \lambda_2} \left(\frac{\sigma_2^2 - \sigma_1^2}{|\sigma|^4} + \frac{\partial h_2}{\partial y}(a, b) \right) + e(\lambda_1, \lambda_2). \quad (114)$$

$$2 \int_{\Omega} (P\delta_2)_2 ((\delta_1)_x \wedge (\delta_1)_y)_1 = \frac{16A_0}{\lambda_1 \lambda_2} \left(2 \frac{\sigma_1 \sigma_2}{|\sigma|^4} + \frac{\partial h_2}{\partial x}(a, b) \right) + e(\lambda_1, \lambda_2). \quad (115)$$

$$2 \int_{\Omega} (P\delta_2)_3 ((\delta_1)_x \wedge (\delta_1)_y)_2 = e(\lambda_1, \lambda_2). \quad 2 \int_{\Omega} (P\delta_2)_2 ((\delta_1)_x \wedge (\delta_1)_y)_3 = e(\lambda_1, \lambda_2). \quad (116)$$

Hence the conclusion follows from (107), (109), (111), (112), (113), and (114)-(116). ■

7.3. Expansion for k bubbles.

In this subsection we consider the case of k masses. When $k = 2$, from Proposition 5.1, and Lemmas (7.1) 7.2, 7.3 we find

$$\begin{aligned} I_{\varepsilon}(u) &= \frac{16}{9} A_0 + 8A_0 \left(\frac{1}{\lambda_1^2} \tilde{H}(a) + \frac{1}{\lambda_2^2} \tilde{H}(b) - \frac{\varepsilon}{\lambda_1} d_{Idg}(a) - \frac{\varepsilon}{\lambda_2} d_{R^{-1}g}(a) \right) \\ &+ \frac{16A_0}{\lambda_1 \lambda_2} \left[r_{11} \left(\frac{\sigma_1^2 - \sigma_2^2}{|\sigma|^4} + \frac{\partial h_1}{\partial x}(a, b) \right) + r_{22} \left(\frac{\sigma_2^2 - \sigma_1^2}{|\sigma|^4} + \frac{\partial h_2}{\partial y}(a, b) \right) \right. \\ &+ r_{12} \left(2 \frac{\sigma_1 \sigma_2}{|\sigma|^4} + \frac{\partial h_1}{\partial y}(a, b) \right) + r_{21} \left(2 \frac{\sigma_1 \sigma_2}{|\sigma|^4} + \frac{\partial h_2}{\partial x}(a, b) \right) \left. \right] \\ &+ e(\varepsilon, \lambda_1) + e(\varepsilon, \lambda_2) + e(\lambda_1, \lambda_2). \end{aligned} \quad (117)$$

where $u = P\delta_1 + P\delta_2$ and r_{ij} are the entries of the matrix R .

We consider now the more general case of k masses. Given two bubbles $\delta_i = R_i\pi$ and $\delta_k = R_k\pi$ (we recall the definition of the stereographic projection π in Section 2), where $R_i, R_k \in SO(3)$, we denote by R_{ik} the matrix $R_i^{-1} \circ R_k$. By invariance under rotation, it is clear that the interaction between δ_i and δ_k is the same as the interaction between π and $R_i^{-1}\delta_k = R_i^{-1}R_k\pi$.

In the expansion of the Euler functional for k masses, since I_ε is cubic in u , we are going to find mixed terms of the form $\int_\Omega P\delta_i \cdot (P\delta_j \wedge P\delta_k)$, where i, j and k are all different. Since we are assuming that the distance of the points p_i, p_j and p_k is uniformly bounded from below, there holds

$$\int_\Omega P\delta_i \cdot (P\delta_j \wedge P\delta_k) \leq C \frac{1}{\lambda_i \lambda_j \lambda_k}, \quad i \neq j \neq k, i \neq k. \quad (118)$$

It follows that the interaction among three distinct bubbles in Z is negligible with respect to the interactions with g, Ω and the interaction between two bubbles.

We recall the definition of the quantity $e(\varepsilon, \lambda_1, \dots, \lambda_k)$ in Section 2. Using equation (118), and omitting some straightforward but tedious computations we obtain the following Proposition.

Proposition 7.4. *Let $\overline{C} > 0$, let $k \in \mathbb{N}$ and let Z be defined by (31). For $i \neq j$ let us set*

$$\begin{aligned} F_\Omega(p_i, p_j, R_i, R_j) &= 16A_0 \\ &\times \left[(R_{ij})_{11} \left(\frac{(p_j - p_i)_1^2 - (p_j - p_i)_2^2}{|p_j - p_i|^4} + \frac{\partial h_1}{\partial x}(p_i, p_j) \right) \right. \\ &+ (R_{ij})_{22} \left(\frac{(p_j - p_i)_2^2 - (p_j - p_i)_1^2}{|p_j - p_i|^4} + \frac{\partial h_2}{\partial y}(p_i, p_j) \right) \\ &+ (R_{ij})_{12} \left(2 \frac{(p_j - p_i)_1 (p_j - p_i)_2}{|p_j - p_i|^4} + \frac{\partial h_1}{\partial y}(p_i, p_j) \right) \\ &\left. + (R_{ij})_{21} \left(2 \frac{(p_j - p_i)_1 (p_j - p_i)_2}{|p_j - p_i|^4} + \frac{\partial h_2}{\partial x}(p_i, p_j) \right) \right] \end{aligned} \quad (119)$$

and

$$\begin{aligned} \Sigma_{\Omega, g}(\varepsilon, p_1, \dots, p_k, \lambda_1, \dots, \lambda_k, R_1, \dots, R_k) &= \sum_{i=1}^k F_{\Omega, g}(p_i, \lambda_i, R_i) \\ &+ \sum_{i < j} \frac{F_\Omega(\varepsilon, p_i, p_j, R_i, R_j)}{\lambda_i \lambda_j}. \end{aligned}$$

Then there holds

$$I_\varepsilon(u) = \frac{8k}{9}A_0 + \Sigma_{\Omega,g} + e(\varepsilon, \lambda_1, \dots, \lambda_k);$$

and

$$\begin{aligned} \frac{\partial I_\varepsilon(u)}{\partial p_i} &= \frac{\partial \Sigma_{\Omega,g}}{\partial p_i} + e(\varepsilon, \lambda_1, \dots, \lambda_k); & \frac{\partial I_\varepsilon(u)}{\partial \lambda_i} &= \frac{\partial \Sigma_{\Omega,g}}{\partial \lambda_i} + \frac{1}{\lambda_i} e(\varepsilon, \lambda_1, \dots, \lambda_k); \\ \frac{\partial I_\varepsilon(u)}{\partial R_i} &= \frac{\partial \Sigma_{\Omega,g}}{\partial R_i} + e(\varepsilon, \lambda_1, \dots, \lambda_k), \end{aligned}$$

where $u = \sum_{i=1}^k P R_i \delta_{p_i, \lambda_i}$.

7.4. Some remarks.

In this subsection we consider the expansion for 2 masses with zero boundary data. Our goal is to extremize the functional in (117) with respect to $a, b, \frac{\lambda_1}{\lambda_2}$ and R . Letting G denote the Green's function of Ω and setting

$$G_1(a, \xi) = \frac{\partial G}{\partial a_1}(a, \xi); \quad G_2(a, \xi) = \frac{\partial G}{\partial a_2}(a, \xi)$$

there holds

$$\begin{aligned} \left(\frac{\sigma_1^2 - \sigma_2^2}{|\sigma|^4} + \frac{\partial h_1}{\partial x}(a, b) \right) &= \frac{\partial G_1}{\partial x}(a, b); & \left(2 \frac{\sigma_1 \sigma_2}{|\sigma|^4} + \frac{\partial h_1}{\partial y}(a, b) \right) &= \frac{\partial G_1}{\partial y}(a, b); \\ \left(2 \frac{\sigma_1 \sigma_2}{|\sigma|^4} + \frac{\partial h_2}{\partial x}(a, b) \right) &= \frac{\partial G_2}{\partial x}(a, b); & \left(\frac{\sigma_2^2 - \sigma_1^2}{|\sigma|^4} + \frac{\partial h_2}{\partial y}(a, b) \right) &= \frac{\partial G_2}{\partial y}(a, b). \end{aligned}$$

Using these expressions, the expansion of $I_\varepsilon(u)$, with $u = P\delta_1 + RP\delta_2$ becomes

$$\begin{aligned} I_\varepsilon(u) &= \frac{16}{9}A_0 + 8A_0 \left(\frac{1}{\lambda_1^2} \left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y} \right) (a, a) + \frac{1}{\lambda_2^2} \left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y} \right) (b, b) \right) \\ &+ \frac{16A_0}{\lambda_1 \lambda_2} \left[r_{11} \frac{\partial G_1}{\partial x}(a, b) + r_{12} \frac{\partial G_1}{\partial y}(a, b) + r_{21} \frac{\partial G_2}{\partial x}(a, b) + r_{22} \frac{\partial G_2}{\partial y}(a, b) \right] + e(\lambda_1, \lambda_2). \end{aligned}$$

The entries r_{i3} and r_{3i} of the matrix R appear as lower order in the above formula, see Remark 5.3 (b). We can write

$$\begin{aligned} &r_{11} \frac{\partial G_1}{\partial x}(a, b) + r_{12} \frac{\partial G_1}{\partial y}(a, b) + r_{21} \frac{\partial G_2}{\partial x}(a, b) + r_{22} \frac{\partial G_2}{\partial y}(a, b) \\ &= \mathbf{e}_1 \cdot R \nabla G_1(a, b) + \mathbf{e}_2 \cdot R \nabla G_2(a, b). \end{aligned}$$

As in [21], Lemma 5.4, the extremization with respect to R gives

$$\mathbf{e}_1 \cdot R\nabla G_1(a, b) + \mathbf{e}_2 \cdot R\nabla G_2(a, b) = \pm (|\nabla G_1(a, b)|^2 + |\nabla G_2(a, b)|^2 \pm 2|\nabla G_1(a, b) \wedge \nabla G_2(a, b)|)^{\frac{1}{2}}.$$

Hence, setting $\tilde{H} = (h_1)_x + (h_2)_y$, we are left with

$$I_\varepsilon(u) = \frac{16}{9}A_0 + 8A_0 \left(\frac{1}{\lambda_1^2} \tilde{H}(a) + \frac{1}{\lambda_2^2} \tilde{H}(b) \right) \pm \frac{16A_0}{\lambda_1 \lambda_2} \left[\left(\frac{\partial G_1}{\partial x} \pm \frac{\partial G_2}{\partial y} \right)^2 (a, b) + \left(\frac{\partial G_2}{\partial x} \mp \frac{\partial G_1}{\partial y} \right)^2 (a, b) \right]^{\frac{1}{2}} + e(\lambda_1, \lambda_2),$$

where the $+$ and $-$ signs inside the square brackets are opposite (hence there are four different possibilities). $I_\varepsilon(u)$ has the form $c + a_{11}\xi_1^2 + a_{22}\xi_2^2 \pm 2a_{12}\xi_1\xi_2$, with $a_{ij} > 0$. Thus if we consider the case $c + a_{11}\xi_1^2 + a_{22}\xi_2^2 + 2a_{12}\xi_1\xi_2$, we notice that minimizing $\sum a_{ij}\xi_i\xi_j/|\xi|^2$ we necessarily need to select $\xi = (\xi_1, \xi_2)$ with $\xi_1\xi_2 \leq 0$ and so this case does not arise. Thus the only case that remains after extremizing is

$$I_\varepsilon(u) = \frac{16}{9}A_0 + 8A_0 \left(\frac{1}{\lambda_1^2} \tilde{H}(a) + \frac{1}{\lambda_2^2} \tilde{H}(b) \right) - \frac{16A_0}{\lambda_1 \lambda_2} \left[\left(\frac{\partial G_1}{\partial x} \pm \frac{\partial G_2}{\partial y} \right)^2 (a, b) + \left(\frac{\partial G_2}{\partial x} \mp \frac{\partial G_1}{\partial y} \right)^2 (a, b) \right]^{\frac{1}{2}} + e(\lambda_1, \lambda_2). \quad (120)$$

8. Proof of Theorem 1.2.

In this Section we prove Theorem 1.2. We begin with the following Lemma, proved in [26] and which follows from straightforward computations.

Lemma 8.1. *Let $\omega \in (0, 1)$, and let $a_\omega = (\omega, 0) \in D$. Define also $\tilde{g}_\omega : \partial D \rightarrow \mathbb{R}^3$ as*

$$\tilde{g}_\omega(x, y) = \left(\frac{x - \omega}{(x - \omega)^2 + y^2}, \frac{y}{(x - \omega)^2 + y^2}, 0 \right), \quad (x, y) \in \partial D.$$

Then, letting g_ω be the harmonic extension on D of \tilde{g}_ω , there holds

$$g_\omega(x, y) = \left(\frac{x - \omega(x^2 + y^2)}{(1 - \omega x)^2 + (\omega y)^2}, \frac{y}{(1 - \omega x)^2 + (\omega y)^2}, 0 \right), \quad (x, y) \in D, \quad (121)$$

and

$$|\nabla g_\omega|^2 + 2|(g_\omega)_x \wedge (g_\omega)_y| = \frac{4}{((1 - \omega x)^2 + \omega^2 y^2)^2}, \quad (x, y) \in D. \quad (122)$$

By equation (66), the concentration points of blowing-up solutions as $\varepsilon \rightarrow 0$ are critical points of the function $\frac{|\nabla g|^2 \pm 2|g_x \wedge g_y|}{\tilde{H}}$. In the next Lemma we describe the critical points of this function in the case of g_ω .

Lemma 8.2. *Let $\omega \in (0, 1)$, and let g_ω be as in Lemma 8.1. Then one has*

$$W_\omega := \left(\frac{|\nabla g|^2 \pm 2|g_x \wedge g_y|}{\tilde{H}} \right)^{\frac{1}{2}} = \sqrt{2} \frac{(1 - x^2 - y^2)}{(1 - \omega x)^2 + (\omega y)^2}.$$

The point $(\omega, 0)$ is a non-degenerate global maximum for W_ω and

$$W_\omega(\omega, 0) = \frac{\sqrt{2}}{1 - \omega^2}. \tag{123}$$

The Hessian of W_ω at $(\omega, 0)$ is given by

$$D^2W_\omega(\omega, 0) = -2\sqrt{2} \begin{pmatrix} \frac{1}{(1-\omega^2)^3} & 0 \\ 0 & \frac{1}{(1-\omega^2)^3} \end{pmatrix}. \tag{124}$$

Remark 8.3. *From equation (124), the fact that $\nabla g(\omega, 0) \neq 0$, and from Theorem D in [22] it follows that problem (3) admits a solution concentrating at $(\omega, 0)$ as $\varepsilon \rightarrow 0$. The image of these solutions converges to a sphere of radius 1 centered at $(0, 0, -1)$, since $\varphi_{a,\lambda} \rightarrow (0, 0, -1)$ as $\varepsilon \rightarrow 0$, see (22).*

Note that, from (123) and (124), W_ω attains a sharp maximum with highly non-degenerate hessian when ω is close to 1. We will use this fact to glue k single bubbles showing that, for a suitable boundary datum, the interaction of this datum with the bubbles is stronger than the interaction among different bubbles. In the next Lemma we give quantitative estimates of the gradient of F_{D,g_ω} (see Proposition 5.1) in a suitable neighborhood of one of its critical points.

It is classical to represent a rotation $R_0 \in SO(3)$ using the Euler angles in the following way

$$R_0 = \begin{pmatrix} \cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi & \cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi & \sin \psi \sin \theta \\ -\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi & -\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi & \cos \psi \sin \theta \\ \sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta \end{pmatrix},$$

where $\theta \in (0, \pi)$, $\psi, \phi \in (0, 2\pi)$. For us it is convenient to use coordinates different from the Euler angles, in order to have a smooth parametrization near the identity matrix. A rotation R will be parameterized as

$$R^{-1} = \begin{pmatrix} \cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi & \cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi & \sin \psi \sin \theta \\ -\sin \theta \sin \phi & \sin \theta \cos \phi & -\cos \theta \\ -\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi & -\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi & \cos \psi \sin \theta \end{pmatrix}, \tag{125}$$

namely as

$$R^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} R_0.$$

R is the identity matrix for $\theta = \frac{\pi}{2}$, and $\phi = \psi = 0$, and the angles θ, ψ, ϕ are smooth coordinates near the identity. In fact there holds

$$\frac{\partial R^{-1}}{\partial \theta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \frac{\partial R^{-1}}{\partial \psi} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad \frac{\partial R^{-1}}{\partial \phi} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

when $\theta = \frac{\pi}{2}$, $\psi = \phi = 0$. We will show that the identity matrix is critical with respect to the rotations for the quantity $d_{R^{-1}}g_\omega(\omega, 0)$. There holds

$$\begin{aligned} d_{R^{-1}}g_\omega &= (\cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi) \frac{\partial(g_\omega)_1}{\partial x} \\ &\quad + (\cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi) \frac{\partial(g_\omega)_2}{\partial x} \\ &\quad - (\sin \theta \sin \phi) \frac{\partial(g_\omega)_1}{\partial y} + (\sin \theta \cos \phi) \frac{\partial(g_\omega)_2}{\partial y}. \end{aligned}$$

From simple computations one finds

$$\begin{aligned} \frac{\partial(g_\omega)_1}{\partial x} &= \frac{\partial(g_\omega)_2}{\partial y} = \frac{(1 - \omega x)^2 - \omega^2 y^2}{((1 - \omega x)^2 + \omega^2 y^2)^2}, \\ \frac{\partial(g_\omega)_1}{\partial y} &= -\frac{\partial(g_\omega)_2}{\partial x} = -2 \frac{(1 - \omega x)\omega y}{((1 - \omega x)^2 + \omega^2 y^2)^2}, \end{aligned}$$

and hence

$$\begin{aligned} d_{R^{-1}}g_\omega &= (\cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi + \sin \theta \cos \phi) \frac{(1 - \omega x)^2 - \omega^2 y^2}{((1 - \omega x)^2 + \omega^2 y^2)^2} \\ &\quad + 2(\cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi + \sin \theta \sin \phi) \frac{(1 - \omega x)\omega y}{((1 - \omega x)^2 + \omega^2 y^2)^2}. \end{aligned} \quad (126)$$

In the next Lemma we study the critical points of F_{D, g_ω} for $\xi \sim (\omega, 0)$, $R \sim Id$, $\lambda \sim 2\varepsilon^{-1}$ and ε small. We use below the coordinates θ, ψ, ϕ in (125) to parametrize the matrix R .

Lemma 8.4. *Let $\omega \in (0, 1)$ and let g_ω be as above. Then, for fixed ε , the point $x = \omega$, $y = 0$, $\lambda = \frac{2}{\varepsilon}$, $\theta = \frac{\pi}{2}$, $\psi = 0$, $\phi = 0$ is critical for $F_{D,g_\omega}(\varepsilon, \xi, \lambda, \theta, \psi, \phi)$. For $\mu > 0$ define the set*

$$\mathcal{T}_\mu = \left\{ |x - \omega| \leq \mu(1 - \omega^2), |y| \leq \mu(1 - \omega^2), \left| \lambda - \frac{2}{\varepsilon} \right| \leq \frac{\mu}{\varepsilon}, \right. \\ \left. \left| \theta - \frac{\pi}{2} \right| \leq \mu, |\psi| \leq \mu, |\phi| \leq \mu \right\}.$$

Then for μ sufficiently small and ω sufficiently close to 1, there exists a universal constant C_0 independent of ε , μ and ω such that

$$\nabla F_{D,g_\omega}(\varepsilon, \chi) \cdot \chi \geq C_0^{-1} \frac{\varepsilon^2 \mu^2}{(1 - \omega^2)^2} \text{ on } \partial \mathcal{T}_\mu, \text{ and hence } \deg(\nabla F_{D,g_\omega}, \mathcal{T}_\mu, 0) = 1, \quad (127)$$

where χ denotes the set of variables $x, y, \lambda, \theta, \psi, \phi$, and the gradient is taken with respect to χ .

PROOF. We recall that the functional F_{D,g_ω} is defined by

$$F_{D,g_\omega}(\varepsilon, \xi, \lambda, R) = \frac{1}{\lambda^2} \left[\tilde{H}(\xi) - \varepsilon \lambda d_{R-1} g_\omega(\xi) \right], \text{ where } \tilde{H}(\xi) = \frac{2}{(1 - |\xi|^2)^2},$$

and where $d_{R-1} g_\omega(\xi)$ is given by (126). In particular there holds

$$\nabla F_{D,g_\omega} \left(\varepsilon, \omega, 0, \frac{2}{\varepsilon}, \frac{\pi}{2}, 0, 0 \right) = 0;$$

$$\text{Hess} F_{D,g_\omega} \left(\varepsilon, \omega, 0, \frac{2}{\varepsilon}, \frac{\pi}{2}, 0, 0 \right) = 2 \frac{\varepsilon^2}{(1 - \omega^2)^2} A_{\omega, \varepsilon},$$

where

$$A_{\omega, \varepsilon} = \begin{pmatrix} \frac{1}{(1-\omega^2)} + \frac{3\omega^2}{(1-\omega^2)^2} & 0 & -\frac{1}{2} \frac{\varepsilon\omega}{(1-\omega^2)} & 0 & 0 & 0 \\ 0 & \frac{1}{(1-\omega^2)} + \frac{3\omega^2}{(1-\omega^2)^2} & 0 & 0 & 0 & -\frac{\omega}{(1-\omega^2)} \\ -\frac{1}{2} \frac{\varepsilon\omega}{(1-\omega^2)} & 0 & \frac{1}{8} \varepsilon^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & -\frac{\omega}{(1-\omega^2)} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

We point out that the matrix $A_{\omega, \varepsilon}$ is positive-definite and non-degenerate. Using simple but tedious computations, one finds

$$|\nabla F_{D,g_\omega}(\varepsilon, \chi) \cdot \chi - A_{\omega, \varepsilon} \chi| \leq C \frac{\mu^3 \varepsilon^2}{(1 - \omega^2)^2}, \text{ for } x, y, \lambda, \theta, \psi, \phi \in \mathcal{T}_\mu. \quad (128)$$

Then the conclusion follows from the fact that $\omega, 0, \frac{2}{\varepsilon}, \frac{\pi}{2}, 0, 0$ is a critical point of F_{D,g_ω} , from (128), and from the fact that $A_{\omega,\varepsilon}$ is positive definite. ■

Now we are in position to prove Theorem 1.2. For the main idea see Remark 8.3.

PROOF OF THEOREM 1.2 Let $A = \{S_1 \cup \dots \cup S_k\}$ be as in Theorem 1.2, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subseteq \mathbb{R}^3$ denote the centers of S_1, \dots, S_k respectively. Note that, since all the spheres have radius 1 and since they all pass through the origin, one has $|\mathbf{v}_j| = 1$, for all $j = 1, \dots, k$. Let $\mathcal{R}_1, \dots, \mathcal{R}_k \in SO(3)$ satisfy

$$\mathcal{R}_j(0, 0, -1) = \mathbf{v}_j, \quad \text{for all } j = 1, \dots, k, \quad (129)$$

see Remark 8.3. Let $\omega \in (0, 1)$ and define $\tilde{G}_{k,\omega} : \partial D \rightarrow \mathbb{R}^3$ by

$$\tilde{G}_{k,\omega}(x, y) = \sum_{j=1}^k \mathcal{R}_j \tilde{g}_{j,\omega}, \quad (x, y) \in \partial D,$$

where

$$\tilde{g}_{j,\omega}(x, y) = \tilde{g}_\omega \left(\left(\cos \frac{2\pi j}{k} \right) x + \left(\sin \frac{2\pi j}{k} \right) y, - \left(\sin \frac{2\pi j}{k} \right) x + \left(\cos \frac{2\pi j}{k} \right) y \right).$$

It is clear that the harmonic extension $G_{k,\omega}$ of $\tilde{G}_{k,\omega}$ to the interior of D is given by

$$\begin{aligned} G_{k,\omega}(x, y) &= \sum_{j=1}^k \mathcal{R}_j g_{j,\omega}(x, y) \\ &= \sum_{j=1}^k \mathcal{R}_j g_\omega \left(\left(\cos \frac{2\pi j}{k} \right) x + \left(\sin \frac{2\pi j}{k} \right) y, - \left(\sin \frac{2\pi j}{k} \right) x + \left(\cos \frac{2\pi j}{k} \right) y \right), \quad (x, y) \in D. \end{aligned} \quad (130)$$

where g_ω is given by (121). Our goal is now to study the critical points of the functional $\Sigma_{D,G_{k,\omega}}$ defined in Proposition 7.4.

We introduce coordinates θ_j, ψ_j and ϕ_j parameterizing a rotation R_j (note that this is a generic rotation, which differs from the fixed rotation \mathcal{R}_j) in the following way

$$\begin{pmatrix} \cos \psi_j \cos \phi_j - \cos \theta_j \sin \phi_j \sin \psi_j & \cos \psi_j \sin \phi_j + \cos \theta_j \cos \phi_j \sin \psi_j & \sin \psi_j \sin \theta_j \\ -\sin \theta_j \sin \phi_j & \sin \theta_j \cos \phi_j & -\cos \theta_j \\ -\sin \psi_j \cos \phi_j - \cos \theta_j \sin \phi_j \cos \psi_j & -\sin \psi_j \sin \phi_j + \cos \theta_j \cos \phi_j \cos \psi_j & \cos \psi_j \sin \theta_j \end{pmatrix}$$

$$= R_j^{-1} \mathcal{R}_j. \quad (131)$$

The choice of this parametrization will become clear below. Note that for $\theta_j \sim \frac{\pi}{2}$ and ψ_j, ϕ_j close to 0, these angles are a smooth parametrization of $SO(3)$ near \mathcal{R}_j . Define also the set

$$\mathcal{T}_\mu^j = \left\{ \left| (x_j, y_j) - \omega \left(\cos \frac{2\pi j}{k}, \sin \frac{2\pi j}{k} \right) \right| \leq \mu(1 - \omega^2), \left| \lambda_j - \frac{2}{\varepsilon} \right| \leq \frac{\mu}{\varepsilon}, \right. \\ \left. \left| \theta_j - \frac{\pi}{2} \right| \leq \mu, |\psi_j| \leq \mu, |\phi_j| \leq \mu \right\}.$$

We are going to prove that for ε sufficiently small and for ω sufficiently close to 1, the functional $\Sigma_{D, G_{k, \omega}}$ has a critical point with $x_j, y_j, \lambda_j, \theta_j, \psi_j, \phi_j \in \mathcal{T}_\mu^j$ for all $j = 1, \dots, k$.

By Proposition 7.4 and by the definition of $F_{D, G_{k, \omega}}$ we have

$$\begin{aligned} \Sigma_{D, G_{k, \omega}} &= \sum_{j=1}^k F_{D, G_{k, \omega}}(\varepsilon, p_j, \lambda_j, R_j) + \sum_{l < j} \frac{F_D(p_l, p_j, R_l, R_j)}{\lambda_l \lambda_j} \\ &= F_{D, \mathcal{R}_k g_{k, \omega}}(\varepsilon, p_k, \lambda_k, R_k) - 8 \frac{\varepsilon}{\lambda_k} A_0 \sum_{j \neq k} d_{R_k^{-1}} g_{\omega, j}(p_k) \\ &\quad + \sum_{j \neq k} F_{D, G_{k, \omega}}(\varepsilon, p_j, \lambda_j, R_j) + \sum_{l < j} \frac{F_D(p_l, p_j, R_l, R_j)}{\lambda_l \lambda_j}. \end{aligned} \quad (132)$$

By invariance we can write

$$\begin{aligned} F_{D, \mathcal{R}_k g_{k, \omega}}(\varepsilon, p_k, \lambda_k, R_k) &= \frac{1}{\lambda_k^2} \left[\tilde{H}(\xi_k) - \varepsilon \lambda_k d_{R_k^{-1}} \mathcal{R}_k g_\omega(\xi_k) \right] \\ &= \frac{1}{\lambda_k^2} \left[\tilde{H}(\xi_k) - \varepsilon \lambda d_{R_k^{-1}} \mathcal{R}_k g_\omega(\xi) \right] \\ &= F_{D, g_\omega}(\varepsilon, p_k, \lambda_k, \mathcal{R}_k^{-1} R_k) \\ &= F_{D, g_\omega}(\varepsilon, x_k, y_k, \lambda_k, \theta_k, \psi_k, \phi_k). \end{aligned} \quad (133)$$

We remark that the function in (133) is exactly the one studied in Lemma 8.4. This justifies the choice of the coordinates θ_j, ψ_j, ϕ_j in (131).

There holds

$$\begin{aligned} \frac{\partial}{\partial x_k} \Sigma_{D, G_{k, \omega}} &= \frac{\partial}{\partial x_k} F_{D, g_\omega}(\varepsilon, x_k, y_k, \lambda_k, \theta_k, \psi_k, \phi_k) \\ &\quad - 8 \frac{\varepsilon}{\lambda_k} A_0 \sum_{j \neq k} \frac{\partial}{\partial x_k} d_{R_k^{-1}} g_{\omega, j}(p_k) + \sum_{l \neq k} \frac{\partial}{\partial x_k} \frac{F_D(p_l, p_k, R_l, R)}{\lambda_l \lambda_k}. \end{aligned}$$

Since all the λ_i 's are of order ε^{-1} , and since the mutual distance between the points p_j 's is bounded from below, it is easy to check that for μ sufficiently small

$$(p_i, \lambda_i, R_i) \in \mathcal{T}_\mu^i \forall i \quad \Rightarrow \quad \frac{\varepsilon}{\lambda_k} \left| \frac{\partial}{\partial x_k} d_{R_k^{-1}} g_{\omega, j}(p_k) \right| + \left| \frac{\partial}{\partial x_k} \frac{F_D(p_j, p_k, R_j, R_k)}{\lambda_j \lambda_k} \right| \leq C\varepsilon^2 \quad j \neq k,$$

where C is a positive constant independent of ε , ω and μ . Hence the last formula and (132) imply

$$\left| \frac{\partial}{\partial x_k} \Sigma_{D, G_{k, \omega}} - \frac{\partial}{\partial x_k} F_{D, g_\omega}(\varepsilon, x_k, y_k, \lambda_k, \theta_k, \psi_k, \phi_k) \right| \leq C\varepsilon^2$$

if $(p_j, \lambda_j, R_j) \in \mathcal{T}_\mu^j$ for all $j = 1, \dots, k$.

Using similar estimates we find

$$\begin{cases} \left| \nabla_\zeta \Sigma_{D, G_{k, \omega}} - \nabla_\zeta F_{D, g_\omega}(\varepsilon, x_k, y_k, \lambda_k, \theta_k, \psi_k, \phi_k) \right| \leq C\varepsilon^2; \\ \left| \frac{\partial}{\partial \lambda_k} \Sigma_{D, G_{k, \omega}} - \frac{\partial}{\partial \lambda_k} F_{D, g_\omega}(\varepsilon, x_k, y_k, \lambda_k, \theta_k, \psi_k, \phi_k) \right| \leq C\varepsilon^3, \end{cases} \quad (134)$$

provided $(p_j, \lambda_j, R_j) \in \mathcal{T}_\mu^j$ for all $j = 1, \dots, k$. Here ζ denotes the set of variables $x_k, y_k, \theta_k, \psi_k, \phi_k$, and where C is a positive constant independent of ε , ω and μ .

Let us fix μ and ε sufficiently small such that (127) and (134) hold. Then we have

$$\nabla_\chi \Sigma_{D, G_{k, \omega}} \cdot \chi \geq C_0^{-1} \frac{\varepsilon^2 \mu^2}{(1 - \omega^2)^2} - C\varepsilon^2, \quad \chi \in \partial \mathcal{T}_\mu^k; \quad \text{if } (p_j, \lambda_j, R_j) \in \mathcal{T}_\mu^j \text{ for all } j,$$

where C and C_0 are independent of ε , ω and μ . Now, choosing ω sufficiently close to 1, depending on C , C_0 and μ , and reasoning in the same way for the indexes different from k we obtain

$$\nabla_{\chi_j} \Sigma_{D, G_{k, \omega}} \cdot \chi_j \geq \frac{C_0^{-1}}{2} \varepsilon^2, \quad \deg(\nabla_{\chi_j} \Sigma_{D, G_{k, \omega}}, \mathcal{T}_\mu^j, 0) = 1,$$

if $(p_j, \lambda_j, R_j) \in \mathcal{T}_\mu^j$ for all j ,

where χ_j denotes the set of variables $x_j, y_j, \lambda_j, \theta_j, \psi_j, \phi_j$. For the above choices of μ and ω , let $I_{G_{k, \omega(\mu)}, \varepsilon}$ denote the Euler functional I_ε corresponding to the boundary datum $\tilde{G}_{k, \omega(\mu)}$. By Proposition 4.6, for ε sufficiently small we obtain

$$\nabla_{\chi_j} \tilde{I}_{D, G_{k, \omega, \varepsilon}} \cdot \chi_j \geq \frac{C_0^{-1}}{2} \varepsilon^2, \quad \deg(\nabla_{\chi_j} \tilde{I}_{D, G_{k, \omega, \varepsilon}}, \mathcal{T}_\mu^j, 0) = 1,$$

if $(p_j, \lambda_j, R_j) \in \mathcal{T}_\mu^j$ for all j .

Then, by Proposition 4.3 and Lemma 8.5 below, letting $\varepsilon \rightarrow 0$, we find a family of solutions $u_{\varepsilon, \mu}$ of $I'_{G_{k, \omega(\mu)}, \varepsilon} = 0$ satisfying, up to a subsequence

$$u_{\varepsilon, \mu}(D) \rightarrow A_\mu = \{S_{1, \mu}, \dots, S_{k, \mu}\} \text{ in the Hausdorff sense} \quad \text{as } \varepsilon \rightarrow 0,$$

where $S_{1, \mu}, \dots, S_{k, \mu}$ are spheres of radius 1 passing through the origin and lying in a neighborhood of order μ of S_1, \dots, S_k respectively. Now we can choose $\mu(\varepsilon) \rightarrow 0$ sufficiently small as $\varepsilon \rightarrow 0$, and find a corresponding $\omega(\varepsilon) \rightarrow 1$ such that the solution $u_{\varepsilon, \mu(\varepsilon)}$ of $I'_{G_{k, \omega(\mu(\varepsilon)), \varepsilon} = 0$ obtained with the above method satisfies

$$u_{\varepsilon, \mu}(D) \rightarrow A = \{S_1, \dots, S_k\} \text{ in the Hausdorff sense} \quad \text{as } \varepsilon \rightarrow 0.$$

This concludes the proof of the Theorem. ■

Lemma 8.5. *Let $\tilde{g} : \partial\Omega \rightarrow \mathbb{R}^3$ be a smooth function, let $k \in \mathbb{N}$, $\bar{C} > 0$, and let Z be defined as in (31). Let u be a solution of (P_ε) of the form*

$$u = \sum_{i=1}^k PR_i \delta_{p_i, \lambda_i} + w; \text{ with } \sum_{i=1}^k PR_i \delta_{p_i, \lambda_i} \in Z, \text{ and } \|w\|_{H_0^1(\Omega)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Then $\|w\|_{L^\infty(\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

PROOF. In the following we simply write δ_i for $R_i \delta_{p_i, \lambda_i}$, and we let φ_i be the function in (21) corresponding to δ_i . The function w satisfies

$$\begin{cases} \Delta w = 2 \left(\sum_i (\delta_i - \varphi_i) + w + \varepsilon g \right)_x \wedge \left(\sum_j (\delta_j - \varphi_j) + w + \varepsilon g \right)_y \\ \quad - \sum_i (\delta_i)_x \wedge (\delta_i)_y, \\ w = 0 \text{ on } \partial\Omega. \end{cases} \text{ in } \Omega,$$

where g , as before, denotes the harmonic extension of \tilde{g} to Ω . Expanding the wedge produce on the right-hand side we obtain (as before $P\delta_i = \delta_i - \varphi_i$)

$$\begin{aligned} \Delta w &= 2 \sum_{i \neq j} (P\delta_i)_x \wedge (P\delta_j)_y + 2 \sum_i [(P\delta_i)_x \wedge w_y + w_x \wedge (P\delta_i)_y] \\ &\quad + 2\varepsilon \sum_i [(P\delta_i)_x \wedge g_y + g_x \wedge (P\delta_i)_y] - 2 \sum_i [(\delta_i)_x \wedge (\varphi_i)_y + (\varphi_i)_x \wedge (\delta_i)_y] \\ &\quad + \sum_i (\varphi_i)_x \wedge (\varphi_i)_y + 2w_x \wedge w_y + 2\varepsilon(w_x \wedge g_y + g_x \wedge w_y) \\ &\quad + \varepsilon^2 g_x \wedge g_y. \end{aligned} \tag{135}$$

Using (25), (26) and some elementary computations, for any $p > 1$ the first term in the right hand side can be estimated in the following way

$$\begin{aligned} \|(P\delta_i)_x \wedge (P\delta_j)_y\|_{L^p(\Omega)} &\leq C(\bar{C}, p) \left(\frac{1}{\lambda_i \lambda_j} + \frac{\lambda_j^{\frac{p-2}{p}}}{\lambda_i} + \frac{\lambda_i^{\frac{p-2}{p}}}{\lambda_j} \right) \\ &\leq C(\bar{C}, p) \varepsilon^{\frac{2}{p}}, \quad i \neq j. \end{aligned}$$

From standard elliptic estimates it follows that

$$\|(\Delta)^{-1} ((P\delta_i)_x \wedge (P\delta_j)_y)\|_{L^\infty(\Omega)} \leq C(\bar{C}, p) \varepsilon^{\frac{2}{p}}, \quad i \neq j,$$

where $(\Delta)^{-1}$ denotes the Green's operator for Δ in Ω with Dirichlet boundary conditions. Let us focus now on the second term in (135). Writing for brevity $\psi = P\delta_i$, one has

$$\begin{aligned} (\psi_x \wedge w_y + w_x \wedge \psi_y) &= [J(w_2, \psi_3) + J(\psi_2, w_3)]\mathbf{i} + [J(w_1, \psi_3) + J(\psi_1, w_3)]\mathbf{j} \\ &\quad + [J(w_1, \psi_2) + J(\psi_1, w_2)]\mathbf{k}, \end{aligned}$$

where $J(F, G) = F_x G_y - F_y G_x$ is the Jacobian function. By the result in [12] there holds

$$\|(\Delta)^{-1} (\psi_x \wedge w_y + w_x \wedge \psi_y)\|_{L^\infty(\Omega)} \leq C \|P\delta_i\|_{H_0^1(\Omega)} \|w\|_{H_0^1(\Omega)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \tag{136}$$

The remaining terms in (135) can be estimated as in (136). ■

Remark 8.6. *With an easy modification of the above arguments we can obtain the limit configuration $\{S_1, \dots, S_k\}$ with a boundary datum of the form $\varepsilon\tilde{G}$, for some fixed function \tilde{G} on ∂D independent of ε .*

Remark 8.7. *We remark that to obtain L^∞ estimates on the solutions of (P_ε) , we use in a crucial way that these solutions satisfy the H -surface equation with $H \equiv \text{constant}$. Such estimates are not available for general Palais-Smale sequences, as exhibited in [11].*

In Figure 4 we indicate the location of the points p_j in D when ω is close to 1, see the definition of \mathcal{T}_μ^j . We also plot the boundary datum $\varepsilon g_{\omega,j}$, which lies in a plane, and the corresponding bubble (as in Remark 8.3) whose center \mathbf{v}_j is, roughly, perpendicular to the plane of $g_{\omega,j}$. We note that the image of \tilde{g}_ω is a great circle (the Kelvin inversion of ∂D w.r.t. the point $(\omega, 0)$). In Figure 5 we plot the configuration of bubbles generated by the function $\varepsilon G_{k,\omega}$. Each bubble is nearly perpendicular to some g_j, ω (whose sum is $G_{k,\omega}$).

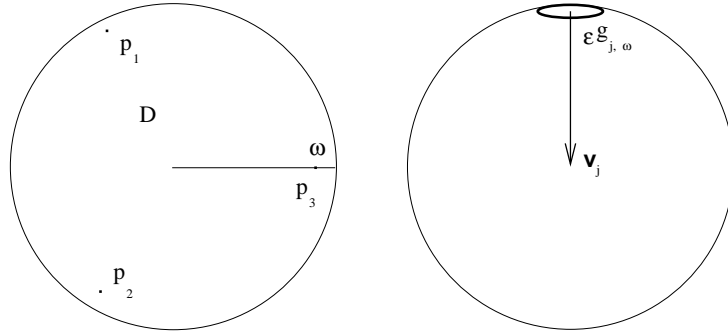


Figure 4: the points p_j on the disk D and the bubble generated by $g_{\omega, j}$

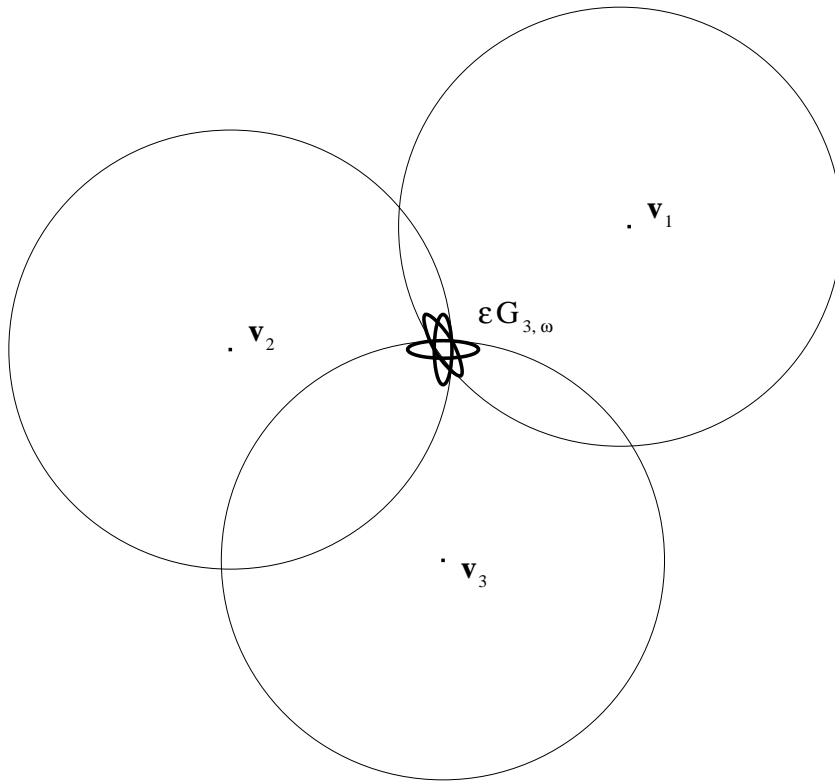


Figure 5: the boundary datum $\varepsilon G_{k, \omega}$ and the corresponding configuration of spheres

9. Appendix.

This Appendix is devoted to the characterization of the solutions of the equation $\bar{T}''(\delta_{a,\lambda})[w] = 0$. We can suppose by invariance that $a = 0$ and $\lambda = 1$, and we set $\delta_{a,\lambda} = \pi$, see Section 2. If a function $w \in \mathcal{D}$ satisfies $\bar{T}''(\delta)[w] = 0$, then it solves the linearization of (15), namely

$$\Delta w = 2(w_x \wedge \delta_y + \delta_x \wedge w_y), \text{ in } \mathbb{R}^2, w \in \mathcal{D}. \quad (137)$$

After inverse stereographic projection, equation (137) can be equivalently viewed on S^2 as follows

$$\Delta_{g_0} w = 2(\sin \varphi)^{-1}(w_\theta \wedge \delta_\varphi + \delta_\theta \wedge w_\varphi), \text{ in } S^2, w \in H^1(S^2; \mathbb{R}^3). \quad (138)$$

where (θ, φ) , $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi$, are spherical coordinates on S^2 and Δ_{g_0} is the Laplacian with respect to standard metric on S^2 .

To analyze (138) we shall use some properties of spherical harmonics that we now recall. Let $P_n(x)$, $x \in (-1, 1)$, denote the n -th Legendre function. We define the associated Legendre function $P_n^k(x)$ by

$$P_n^k(x) = (-1)^k (1-x^2)^{\frac{k}{2}} \frac{d^k}{dx^k} P_n(x), \quad k \geq 0. \quad (139)$$

The spherical harmonics are defined by

$$Y_{n,k}(\theta, \varphi) = c_{n,|k|} P_n^{|k|}(\cos \varphi) e^{ik\theta}, \quad -n \leq k \leq n, \quad (140)$$

where the normalization constant $c_{n,|k|}$, see [13] equation (21), p.171, is given by

$$c_{n,|k|} = \left(\frac{2n+1}{4\pi} \right)^{\frac{1}{2}} \sqrt{\frac{(n-|k|)!}{(n+|k|)!}}. \quad (141)$$

From (139) we easily have, by differentiation

$$P_n^{k+1}(\cos \varphi) = -\sin \varphi (P_n^k)'(\cos \varphi) - m \cot \varphi P_n^k(\cos \varphi), \quad (142)$$

and from equation (41), p.107 in [19],

$$P_n^{k+2}(\cos \varphi) + 2(m+1) \cot \varphi P_n^{k+1}(\cos \varphi) + (n-k)(n+k+1) P_n^k(\cos \varphi) = 0. \quad (143)$$

From (141) we also have, for $0 \leq k \leq n$, $n \geq 2$,

$$d_{n,k} := \frac{c_{n,k}}{c_{n,k+1}} \leq \sqrt{3/2n}, \quad e_{n,k} := (n-k)(n+k+1) \frac{c_{n,k}}{c_{n,k-1}} \leq \sqrt{3/2n}. \quad (144)$$

In the sequel we will often write $P_n^k(\varphi)$ for $P_n^k(\cos \varphi)$.

We define the finite-dimensional subspace \mathcal{H}_n to be the linear span of $Y_{n,k}$, $-n \leq k \leq n$, and \mathcal{H}_n the subspace of $L^2(S^2; \mathbb{R}^3)$ consisting of vectors $w = (f_1, f_2, f_3)$ with $f_i \in H_n$, $i = 1, 2, 3$. Recall that any function $w \in L^2(S^2; \mathbb{R}^3)$ can be decomposed orthonormally as $w = \sum_{n=0}^{\infty} w_n$, with $w_n \in \mathcal{H}_n$. We have

Lemma 9.1. *Let $w : S^2 \rightarrow \mathbb{R}^3$ be a solution of (138), and let $w = \sum_{n=0}^{\infty} w_n$, with $w_n \in \mathcal{H}_n$. Then $w_n = 0$ for $n \geq 4$.*

PROOF. We first claim that for any $F \in \mathcal{H}_n$ there holds

$$\Gamma(F) := \Delta_{g_0} F - \frac{2}{\sin \varphi} (F_\theta \wedge \delta_\varphi + \delta_\theta \wedge F_\varphi) \in \mathcal{H}_n. \quad (145)$$

If our claim is verified, to prove the Lemma it will be enough to pick a solution w to (138) in \mathcal{H}_n and to show that $w_n = 0$ for $n \geq 4$.

We now prove our claim. W.l.o.g. pick $F \in \mathcal{H}_n$ of the form

$$\begin{aligned} F &= \left(\alpha_{k_1} c_{n,k_1} P_n^{k_1}(\varphi) e^{ik_1\theta}, \beta_{k_2} c_{n,k_2} P_n^{k_2}(\varphi) e^{ik_2\theta}, \gamma_{k_3} c_{n,k_3} P_n^{k_3}(\varphi) e^{ik_3\theta} \right) \\ &= (\alpha_{k_1} Y_{n,k_1}, \beta_{k_2} Y_{n,k_2}, \gamma_{k_3} Y_{n,k_3}) \end{aligned}$$

Next we have $\delta_\varphi = (\cos \varphi \cos \theta, \cos \varphi \sin \theta, -\sin \varphi)$ and $\delta_\theta = (-\sin \varphi \sin \theta, \sin \varphi \cos \theta, 0)$. We will show that $\Gamma(F) = -n(n+1)F - 2\mathbf{v}$, where

$$\mathbf{v} = \begin{pmatrix} -ik_2\beta_{k_2}Y_{n,k_2} + 1/2d_{n,k_3}\gamma_{k_3}Y_{n,k_3+1} - 1/2e_{n,k_3}\gamma_{k_3}Y_{n,k_3-1} \\ ik_1\alpha_{k_1}Y_{n,k_1} - i/2d_{n,k_3}\gamma_{k_3}Y_{n,k_3+1} - i/2e_{n,k_3}\gamma_{k_3}Y_{n,k_3-1} \\ i/2d_{n,k_2}\beta_{k_2}Y_{n,k_2+1} - i/2e_{n,k_2}\beta_{k_2}Y_{n,k_2-1} - 1/2d_{n,k_1}\alpha_{k_1}Y_{n,k_1+1} + 1/2e_{n,k_1}\alpha_{k_1}Y_{n,k_1-1} \end{pmatrix}. \quad (146)$$

Since $\mathbf{v} \in \mathcal{H}_n$, our claim follows. It is evident that $\Delta_{g_0} F = -n(n+1)F$, thus it is enough to show that the second expression in (145) is $2\mathbf{v}$. This follows by noting that

$$\begin{aligned} F_\theta \wedge \delta_\varphi &= \begin{pmatrix} -ik_2\beta_{k_2}c_{n,k_2} \sin \varphi P_n^{k_2}(\varphi) e^{ik_2\theta} - ik_3g_{k_3}c_{n,k_3} \cos \varphi \sin \theta P_n^{k_3}(\varphi) e^{ik_3\theta} \\ ik_3g_{k_3}c_{n,k_3} \cos \varphi \sin \theta P_n^{k_3}(\varphi) e^{ik_3\theta} + ik_1\alpha_{k_1}c_{n,k_1} \sin \varphi P_n^{k_1}(\varphi) e^{ik_1\theta} \\ ik_1\alpha_{k_1}c_{n,k_1} \cos \varphi \sin \theta P_n^{k_1}(\varphi) e^{ik_1\theta} - ik_2\beta_{k_2}c_{n,k_2} \sin \varphi P_n^{k_2}(\varphi) e^{ik_2\theta} \end{pmatrix}; \\ \delta_\theta \wedge F_\varphi &= \begin{pmatrix} -g_{k_3}c_{n,k_3} \sin^2 \varphi \cos \theta (P_n^{k_3})'(\varphi) e^{ik_3\theta} \\ -\gamma_{k_3}c_{n,k_3} \sin^2 \varphi \sin \theta (P_n^{k_3})'(\varphi) e^{ik_3\theta} \\ \beta_{k_2}c_{n,k_2} \sin^2 \varphi \sin \theta (P_n^{k_2})'(\varphi) e^{ik_2\theta} + \alpha_{k_1}c_{n,k_1} \sin^2 \varphi \cos \theta (P_n^{k_1})'(\varphi) e^{ik_1\theta} \end{pmatrix}. \end{aligned}$$

Using (142) and (143) it is easily verified that $(\sin \varphi)^{-1} [F_\theta \wedge \delta_\varphi + \delta_\theta \wedge F_\varphi] = \mathbf{v}$. Then from (146) and integrating the equation $\Gamma(w) \cdot \bar{w} = 0$ on S^2 we find

$$-n(n+1) \begin{pmatrix} A^2 \\ B^2 \\ C^2 \end{pmatrix} \tag{147}$$

$$= -2 \sum_k \begin{pmatrix} -ik\beta_k \bar{\alpha}_k + 1/2d_{n,k} \gamma_k \bar{\alpha}_{k+1} - 1/2e_{n,k} \gamma_k \bar{\alpha}_{k-1} \\ ik\alpha_k \bar{\beta}_k - i/2d_{n,k} \gamma_k \bar{\beta}_{k+1} - i/2e_{n,k} \gamma_k \bar{\beta}_{k-1} \\ i/2d_{n,k} \beta_k \bar{\gamma}_{k+1} - i/2e_{n,k} \beta_k \bar{\gamma}_{k-1} - 1/2d_{n,k} \alpha_k \bar{\gamma}_{k+1} + 1/2e_{n,k} \alpha_k \bar{\gamma}_{k-1} \end{pmatrix},$$

where $w = \sum_k (\alpha_k Y_{n,k}, \beta_k Y_{n,k}, \gamma_k Y_{n,k})$ and where $A^2 = \sum |\alpha_k|^2$, $B^2 = \sum |\beta_k|^2$, $C^2 = \sum |\gamma_k|^2$. Using (144), (147) and the Cauchy-Schwartz inequality we find

$$\begin{aligned} & n(n+1) |(A^2, B^2, C^2)| \\ & \leq 2n \left| \left(AB + \sqrt{3/2} AC, AB + \sqrt{3/2} BC \sqrt{3/2} (BC + AC) \right) \right| \\ & \leq 2n \left(\frac{9 + \sqrt{6}}{2} \right)^{\frac{1}{2}} (A^4 + B^4 + C^4)^{1/2}. \end{aligned}$$

Thus $n + 1 \leq (18 + \sqrt{24})^{\frac{1}{2}}$, which implies $n \leq 3$. ■

Lemma 9.2. *The solutions of equation (138) are of the form*

$$w = c + \begin{pmatrix} \alpha x_2 + \beta x_3 \\ -\alpha x_1 + \gamma x_3 \\ -\beta x_1 - \gamma x_2 \end{pmatrix} + (\alpha' x_1 + \beta' x_2 + \gamma' x_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where $c \in \mathbb{R}^3$ and $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{R}$ are arbitrary.

PROOF. We denote by $J(w)$ the r.h.s. of (138). From the proof of Lemma 9.1 it follows that J preserves the degree of spherical harmonic functions. Equivalently, J preserves the degree of polynomial functions in \mathbb{R}^3 restricted to S^2 . By this reason and by Lemma 9.1, we can confine ourselves to study J just on polynomials of order 1, 2 and 3. Since the computations involved in the proof are straightforward, we just give a simple sketch below, omitting some details.

Using simple computations, we obtain

$$J(x_1, 0, 0) = (0, 2x_2, 2x_3); \quad J(x_2, 0, 0) = (0, -2x_1, 0); \quad J(x_3, 0, 0) = (0, 0, -2x_1). \tag{148}$$

With a permutation of coordinates one also finds

$$J(0, x_2, 0) = (2x_1, 0, 2x_3); \quad J(0, x_3, 0) = (0, 0, -2x_2); \quad F(0, x_1, 0) = (-2x_2, 0, 0); \quad (149)$$

$$F(0, 0, x_3) = (2x_1, 2x_2, 0); \quad F(0, 0, x_1) = (-2x_3, 0, 0); \quad F(0, 0, x_2) = (0, -2x_3, 0). \quad (150)$$

Hence, letting $w = (a_1x_1 + a_2x_2 + a_3x_3, b_1x_1 + b_2x_2 + b_3x_3, c_1x_1 + c_2x_2 + c_3x_3)$ we find

$$J(w) = 2 \begin{pmatrix} -b_1x_2 + b_2x_1 - c_1x_3 + c_3x_1 \\ a_1x_2 - a_2x_1 - c_2x_3 + c_3x_2 \\ a_1x_3 - a_3x_1 + b_2x_3 - b_3x_2 \end{pmatrix}; \quad \Delta w = -2 \begin{pmatrix} a_1x_1 + a_2x_2 + a_3x_3 \\ b_1x_1 + b_2x_2 + b_3x_3 \\ c_1x_1 + c_2x_2 + c_3x_3 \end{pmatrix},$$

The system of equations $\Delta w = J(w)$ admits the following solutions;

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}, \quad (151)$$

with α, β, γ arbitrary real numbers.

Let us now consider the homogeneous second order polynomials. We have $\Delta x_i^2 = 2(1 - 3x_i^2)$ and $\Delta(x_ix_j) = -6x_ix_j$. Using the Leibnitz rule and (148)-(150), we can compute $J(w)$ when w has the form;

$$w = \begin{pmatrix} a_1x_1^2 + a_2x_1x_2 + a_3x_2^2 + a_4x_1x_3 + a_5x_2x_3 + a_6x_3^2 \\ b_1x_1^2 + b_2x_1x_2 + b_3x_2^2 + b_4x_1x_3 + b_5x_2x_3 + b_6x_3^2 \\ c_1x_1^2 + c_2x_1x_2 + c_3x_2^2 + c_4x_1x_3 + c_5x_2x_3 + c_6x_3^2 \end{pmatrix}$$

From the relation $\Delta w = J(w)$, and using elementary computations we obtain

$$w = (\alpha x_1 + \beta x_2 + \gamma x_3) + (\delta, \eta, \sigma),$$

where $\alpha, \beta, \gamma, \delta, \eta, \sigma$ are arbitrary real numbers.

Let us now turn to the third order polynomials. We have

$$\Delta x_i^3 = 6x_i(1 - 2x_1^2); \quad \Delta(x_i^2x_j) = 2x_j(1 - 6x_i^2); \quad \Delta(x_ix_jx_k) = -12x_ix_jx_k.$$

Again, the values of F on the third order polynomials can be computed with the Leibnitz rule and (148)-(150).

Letting

$$w = \begin{pmatrix} a_1x_1^3 + a_2x_1^2x_2 + a_3x_1^2x_3 + a_4x_1x_2x_3 + a_5x_1x_2^2 + a_6x_1x_3^2 + a_7x_2^3 + a_8x_3^3 + a_9x_2^2x_3 + a_{10}x_2x_3^2 \\ b_1x_1^3 + b_2x_1^2x_2 + b_3x_1^2x_3 + b_4x_1x_2x_3 + b_5x_1x_2^2 + b_6x_1x_3^2 + b_7x_2^3 + b_8x_3^3 + b_9x_2^2x_3 + b_{10}x_2x_3^2 \\ c_1x_1^3 + c_2x_1^2x_2 + c_3x_1^2x_3 + c_4x_1x_2x_3 + c_5x_1x_2^2 + c_6x_1x_3^2 + c_7x_2^3 + c_8x_3^3 + c_9x_2^2x_3 + c_{10}x_2x_3^2 \end{pmatrix}$$

and equating the coefficients in the expressions for Δw and $J(w)$, we find a system decoupled into four parts. The first part consists of seven equations involving the seven terms $a_2, a_7, a_{10}, b_1, b_5, b_6$ and c_4 . Using simple computations one finds $a_2 = a_7 = a_{10} = \alpha, b_1 = b_5 = b_6 = -\alpha, c_4 = 0$, for some $\alpha \in \mathbb{R}$.

The second part consists of seven equations involving the seven terms a_3, a_8, a_9, b_4 and c_1, c_5, c_6 . Using simple computations one finds $a_3 = a_8 = a_9 = \beta, c_1 = c_5 = c_6 = -\beta, b_4 = 0$, for some $\beta \in \mathbb{R}$.

The third part consists of seven equations involving the seven terms a_4, b_3, b_8, b_9 , and c_2, c_7, c_{10} . Using simple computations one finds $b_3 = b_8 = b_9 = \gamma, c_2 = c_7 = c_{10} = -\gamma, a_4 = 0$, for some $\gamma \in \mathbb{R}$.

The fourth part consists in nine equations involving the terms $a_1, a_5, a_6, b_2, b_7, b_{10}$ and c_3, c_8, c_9 . Using simple computations one finds $a_1 = a_5 = a_6 = b_2 = b_7 = b_{10} = c_3 = c_8 = c_9 = 0$.

The solution obtained in this way represent just the linear functions in (151), taking into account of the identity $x_1^2 + x_2^2 + x_3^2 = 1$ on S^2 . This concludes the proof. ■

PROOF OF PROPOSITION 4.1. Coming back to the space \mathcal{D} , and using some elementary computation, the proof of the last statement follows immediately from Lemma 9.2. The first inequality is immediate to check. The second inequality follows from the proof of Lemma 9.2 when $v \in \oplus_{n \geq 4} \mathcal{H}_n$. When v has some non-zero components in $\oplus_{n \leq 3} \mathcal{H}_n$, then it is sufficient to use straightforward computations, since we have to deal with finite combinations of spherical harmonics. Alternatively note that, since δ is a mountain-pass critical point of \bar{I} , the linearized operator possesses only one negative eigenvalue (with corresponding eigenvector δ), hence if $v \perp \delta$ and $v \perp \text{Ker} \bar{I}''(\delta)$, v must be a combination of positive eigenvectors. ■

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