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Modelling Afthairetic Modality

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Abstract
Despite their controversial ontological status, the discussion on arbitrary objects has been reignited in recent years. According to the supporting views, they present interesting and unique qualities. Among those, two define their nature: their assuming of values, and the way in which they present properties. Leon Horsten has advanced a particular view on arbitrary objects which thoroughly describes the earlier, arguing they assume values according to a *sui generis* modality, which he calls *afthairetic*. In this paper, we offer a general method for defining the minimal system of this modality for any given first-order theory, and possible extensions of it that incorporate further aspects of Horsten's account. The minimal system presents an unconventional inference rule, which deals with two different notions of derivability. For this reason and the failure of the *Necessitation* rule, in its full generality, the resulting system is non-normal. Then, we provide conditional soundness and completeness results for the minimal system and its extensions.

Keywords Arbitrary objects · Nonnormal modal logics · Actual world

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1 Introduction

Indefinite descriptions or locutions composed by words such as "arbitrary" – e.g. "a whale," "an arbitrary man" – are commonly used, both in natural language and in mathematical practice. For example, we might say "A star fuses elements together into heavier elements," or "Consider an arbitrary natural number; it has a unique prime factorization," by which we seem to be inferring qualities of objects of a certain class from the consideration of a distinguished object, which is designated by this kind of

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expression. According to many views, any such object may be dispensed in favour of a lighter ontology, and thus there would be no need for a theory of such objects.[1] However, there is a notable minority for which there are, in some sense, objects of that sort: *arbitrary objects*.[2] These new objects stand in opposition to *specific objects* (or, in Fine's terminology, *individual*), the objects of our daily, ordinary life. In a manner familiar to mathematical practice, they may also be called "variables" – when not understood as purely linguistic objects –, or be taken to be the designata of variable expressions. For instance, according to Leon Horsten:

> Arbitrary objects of a given kind F are abstract entities that can be in a state that belongs to the state space associated with F. For instance, an arbitrary person is an abstract entity that can take any value taken from the class of all specific people[3]

Arbitrary objects (henceforth *A-objects*), as have been described in the recent literature, seem to have two distinguished and essential characteristics, according to the two most prominent accounts of them: following Horsten's account, they *assume values* in every state they may be in, and following Kit Fine's account, they present properties according to the way their possible values present them. The relation between the presenting of properties, for Fine, is governed by the *Principle of Generic Attribution* (henceforth *PGA*), according to which an A-object presents a property iff each object they may assume as a value does so. This assuming of values, for Horsten, is captured by a *sui generis* modality, which he calls *afthairetic*. We shall present a framework for working with this modality.

In [8], Horsten and Stanislav Speranski offer an account of the generic ω-sequence, by considering the arbitrary natural numbers – which is later expanded in [7]. They introduce a predicate *Sp* for *being specific*, a modal operator $\Diamond$, which is supposed to behave S5-like, and change the interpretation of $=$, so that $x = y$ means x and y coincide in the state in consideration, in order to describe the structure of natural numbers and the way A-objects behave there. Our framework is supposed to be different in that we not only generalise it to talk about any *non-modal* first-order theory, but in that it is possible to talk about both A-objects and specific objects, and there is no collapse between specific objects and the constant A-objects (those which assume the same value in every possible state).

More concretely, for each theory T, we offer an extension of it which is supposed to be the theory of the afthairetic modality of T, and which therefore is able to describe the A-objects of T. The framework is equipped with an axiomatic system and a semantics, which present not only soundness and completeness – if T does so –, but are also conservative over the theory T. The framework does not reflect Horsten's particular characterisation of A-objects as choice functions, and therefore cannot be claimed, in any way, to be *the* system for Horsten's account. It, nevertheless, reflects the functioning of A-objects as choice functions, and translates naturally Horsten's description

1 See [1, 2, 6, 11, 12], to name a few.

2 Most notably [4] and [7].

3 [7], p. 44.

of the characteristics of afthairetic modality, and of the concepts introduced in the extensions. Later, in Section 4, we attempt to connect Horsten's view with the PGA.

The paper is structured as follows. In Section 2, we introduce the language and the Hilbert-style axiomatic system. In Section 3, we define extended models, with designated worlds, and show the soundness and completeness of the basic axiomatic systems. In Section 4, we show a few extensions of the minimal systems, which reflect Horsten's criterion of identity for arbitrary objects, a comprehension scheme concerning their abundance, how to extend functions from the original theories to the new domain, and a principle to extend predicates from specific objects to A-objects, which allows us to address Fine's PGA. Finally, we conclude with some general remarks in Section 5.

We present here a few notational conventions we shall use throughout the paper. We write n, m, k for natural numbers, i, j for indexes, x, y, z for variables, c for constants, w, u, v for states in a relational model, e, r, s, t for terms in general, f, g for functions, P, Q for predicates, a, b for A-objects, and o for objects in general; we write $\{t_i\}$ for a collection of terms $t_1, t_2, \ldots$, and also t_i for a certain term in that collection; if $t_1, \ldots, t_n$ is a n-ary sequence of terms, we write $\vec{t}$ – for example, instead of $Pt_1 \ldots t_n$, we write $P\vec{t}$; if φ is a formula in which a sequence of terms $t_1, \ldots, t_n$ may or may not occur, we write $\varphi^{r_1,\ldots,r_n}_{t_1,\ldots,t_n}$ for the formula which is just like φ, except each free occurrence of t_i is replaced by an occurrence of r_i, respectively.

2 Language and Hilbert Style System

We wish to describe how A-objects assume states, so we need two distinguished predicates to represent arbitrariness and the fact of being in a state. Thus, we must have a unary predicate A, with the intended meaning of Ax being that *x is arbitrary*, and a binary predicate S, with the intended meaning of Sxy being *y is the value of x* – if there is such a thing –, in the sense that if x is an A-object, then y is the value it assumes, and if x is a specific object, then y is x itself. Our goal would be to see how the A-objects of any (first-order) theory behave, but each theory has its own signature. For now, we restrict ourselves to first-order theories whose signatures are composed by proper predicates, function, and constant symbols – and therefore, as expressed earlier, are non-modal. So, if $\mathcal{L}$ is the language of a certain first-order theory, we add to it the equality symbol $=$ (in case the language is not already equipped with it), the modal operator symbol $\Box$, and the predicates A and S, obtaining $\mathcal{L}^{\Box_{af}}$, the language of its afthairetic modality extension (henceforth *AM-extension*). Let $Atom_{\mathcal{L}}$ be the set of atomic formulas of $\mathcal{L}$, and $Term_{\mathcal{L}}$, of terms. Let $t \in Term_{\mathcal{L}}$. The formulas of $\mathcal{L}^{\Box_{af}}$ are then:

$$\varphi ::= \varphi \in Atom_{\mathcal{L}} \mid t = t \mid At \mid Stt \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\varphi \mid \Box\varphi$$

The operators $\vee, \rightarrow, \leftrightarrow, \Diamond$, and quantifier $\exists x$ may all be defined in the usual manner.

Horsten's account of A-objects introduces a new and peculiar modal profile for A-objects, in which the corresponding modality is brand new: the *afthairetic* modality. In that modality, A-objects *may* always assume a state, but never *actually* do so.

> An arbitrary F might have been in the state of being this specific F and might have been in the state of being that specific F, but isn't actually in any of those. For instance, it makes no sense to ask who the person on (an arbitrary) omnibus actually is. All we can (loosely) say is that it could be this or that specific person; it could be you, and it could be me.[4]

> In general, an arbitrary natural number does not have any specific natural number as its determinate value. There is no determinate matter of fact, for instance, about whether the value of our mathematician's arbitrary number a is 23 - remember: this is loose talk. There can be a determinate fact about whether an arbitrary number x is numerically identical with an arbitrary number y. [...] When an arbitrary natural number does not determinately have some given specific number as its value, there is a sense in which it can be the specific number in question. Thus we say that arbitrary numbers can be in different specific states. There is, however, no actual state in which the arbitrary number is.[5]

The view suggests that in formalizing this modality we need to introduce a distinct actual world: the one in which A-objects never take values. Moreover, since we are dealing with extensions of first-order theories, just as the languages vary, so do the axiomatic systems, but the principles guiding afthairetic modality should be the same in all of their extensions. Therefore, each theory should be supplemented with a similar set of axioms.

From now on, let us call by $\mathrm{T}^{\Box_{af}}$ the target theory of the afthairetic modality of a first-order theory T. Concerning the proper axioms of each first-order theory T, we should have in mind that, if we are introducing new (arbitrary) objects to each theory, the domain of quantification is extended. Therefore, a universally quantified formula which is true (or derivable) in T may no longer be true (or derivable) in $\mathrm{T}^{\Box_{af}}$. However, whatever is true (or derivable) about the objects of T should still be true (or derivable) in $\mathrm{T}^{\Box_{af}}$. We shall accommodate this issue by having the $\mathrm{T}^{\Box_{af}}$ inherit the axioms and inference rules of T after an appropriate translation.

Let $\mathcal{L}$ be the language of T, and $\mathcal{L}^{\Box_{af}}$ that of $\mathrm{T}^{\Box_{af}}$. We first define the following translation $\alpha_{\mathrm{T}} : Form_{\mathcal{L}} \to Form_{\mathcal{L}^{\Box_{af}}}$:

$$\begin{array}{lcl} \alpha_{\mathrm{T}}(Pt_1 ... t_n) & = & Pt_1 ... t_n; \\ \alpha_{\mathrm{T}}(t_1 = t_2) & = & t_1 = t_2; \\ \alpha_{\mathrm{T}}(\neg\varphi) & = & \neg\alpha_{\mathrm{T}}(\varphi); \\ \alpha_{\mathrm{T}}(\varphi \wedge \psi) & = & \alpha_{\mathrm{T}}(\varphi) \wedge \alpha_{\mathrm{T}}(\psi); \\ \alpha_{\mathrm{T}}(\forall x \varphi) & = & \forall x(\neg Ax \to \alpha_{\mathrm{T}}(\varphi)). \end{array}$$

We can now define the syntactic part of afthairetic modality theories. We start by their axioms. Let $\mathbf{A}_i$ be the proper axioms of T. Having the considerations above in mind, $\alpha_{\mathrm{T}}(\mathbf{A}_i)$ are axioms of $\mathrm{T}^{\Box_{af}}$. Furthermore, an AM-extension is still a first-order modal logic theory, so, as such, all of the axioms of first-order modal logic with identity should be axioms of it.[6] However, we cannot assume all terms, in the extended theory, to designate. The reason for that is twofold. First, when moving from

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4 [7], p. 50.

5 [7], p. 62.

6 For an axiomatisation of first-order modal logic with identity, the reader may consult [9] or [5].

a first-order theory to the theory of its A-objects, functions which are originally total may not remain so – for example, it is, in principle, uncertain if the sum of 2 and an arbitrary even is defined at all.[7] Moreover, many theories which are to be extended may already have non-designating terms – such as any with partial functions. Therefore, we shall work with *free* first-order modal logic (henceforth *FFOML*) – that is, with the following modification:

$\mathbf{UI}_f$ $\forall x\varphi \to (\exists y(y = t) \to \varphi[t/x])$,

where $\mathbf{UI}_f$ takes the place of the axiom of universal instantiation. As it shall be important in the completeness proof, we note this variation makes the law of necessity of identity apply to all terms – so that, for example, $\forall x(c = x \to \Box c = x)$ will be a theorem. That ensures constants and functions have constant interpretations throughout states, so that they *designate rigidly*. This consequence is desirable. Consider $c^{\mathcal{M}}$ to be the object designated by a constant c in the actual world and a be the designator of an A-object $a^{\mathcal{M}}$, and suppose we want to represent the fact that, in a possible state, $a^{\mathcal{M}}$ assumes the value $c^{\mathcal{M}}$. From the perspective of the actual world, it should be safe to express that by $\Diamond Sac$. If constants do not designate rigidly, then $\Diamond Sac$ says, instead, that the object designated by a – which, in some possible state, may be distinct from $a^{\mathcal{M}}$ – assumes the value of the object designated by c – which is likewise possibly distinct from $c^{\mathcal{M}}$. If a constant designates non-rigidly, whenever it is inside the scope of a modal operator, the interpretation of a name shifts from the state of evaluation of a sentence to a (in our case, different) possible state, and therefore we loose the ability to make reference across possible states – something that is desired in the case of afthairetic modality. We may extend the argument to functions if we treat function-terms as complex names.

Since the purpose is to model the assuming of values of A-objects, we presume there are A-objects, and since there are such objects, they must *possibly* assume values. In other words, there must be accessible states in which they assume values. We find axiom $\mathbf{D} = \Box\varphi \to \Diamond\varphi$ to adequately capture that. We further adopt axiom $\mathbf{K} = \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ as a basic tenet of modality that extends to the present one.

An important aspect we need to ensure is a constant domain of quantification. In order to see why, we may consider an arbitrary rodent. Anyone would agree that a rodent measures less than 2 meters in length. Indeed, the largest *actual* rodent, which is the capybara, only grows up to around 134 centimeters. If anyone were to object by saying there *could* be a larger species of rodent, or even just a larger individual capybara (not in the sense that actually there is such a species or specimen, unbeknownst to us, but in the sense that such a specimen could exist in some other possible state), we would likely find the objection unsubstantial. As a matter of fact, possible non-actual individuals are not of interest in such an assessment. In a similar way, suppose someone was in a room filled with three leaning chairs, a bowling ball, and nothing else, and they affirmed an arbitrary thing of that room could be a ball. It would be highly strange if a second person immediately maintained that was wrong, on the grounds that the bowling ball could possibly not exist – not only because that person was not there to

[7] Later, in Section 4, we shall see how to extend functions to encompass operations on A-objects.

begin with, but also because there is an *actual* ball in the room, although surely there could have been none. Thus, in the sort of modality concerning arbitrariness, possible objects do not matter in the assuming of states. Therefore, in order to cover the above concerns, we supplement the basic modal system with the usual axiom schemes which came to be known as *Barcan Formula* ($\mathbf{BF} = \forall x \Box \varphi \rightarrow \Box \forall x \varphi$) and *Converse Barcan Formula* ($\mathbf{CBF} = \Box \forall x \varphi \rightarrow \forall x \Box \varphi$), and which are known, together, to ensure a constant domain.[8]

For what concerns the proper axioms, we propose the following:

$\mathbf{S}_{act}$ $\forall x(Ax \leftrightarrow \forall y(\neg Sxy))$
$\mathbf{S}_{pot}$ $\forall x \Box \exists y Sxy$
$\mathbf{S}_{uni}$ $\forall x \forall y(Sxy \rightarrow \forall z(Sxz \rightarrow z = y))$
NSA $\forall x(Sxx \leftrightarrow \neg Ax)$
$\mathbf{NA}_1$ $\forall x(Ax \rightarrow \Box Ax)$
$\mathbf{NA}_2$ $\forall x(\neg Ax \rightarrow \Box \neg Ax)$

$\mathbf{S}_{pot}$ and **NSA** hint the predicate A may be defined in terms of the relation S. In fact, we could have set $Ax ::= \Diamond \exists y(x \neq y \wedge Sxy)$, and that is an equivalence derivable in the system – from axioms $\mathbf{S}_{pot}$, $\mathbf{S}_{uni}$, **NSA**, and rule $Nec_{\mathrm{T}\Box_q}$, introduced below –, so that A is dispensable. We still keep it, from now on, for the sake of simplicity of our presentation. As pointed out earlier, S plays two different roles at the same time, delivering the object itself if it is specific, and the value the object assumes if it is an A-object. Not much should be read into the choice of using such relation. It is merely for aiding the presentation.

We may justify each remaining proper axiom. $\mathbf{S}_{act}$, $\mathbf{S}_{pot}$, $\mathbf{S}_{uni}$, and **NSA** describe how the assuming of values works. An A-object always possibly assume values ($\mathbf{S}_{pot}$), but never actually does so ($\mathbf{S}_{act}$). A value is always unique ($\mathbf{S}_{uni}$), in the sense that at any given state, an object can only assume one value, and only A-objects can *properly* assume values – that is, different from themselves. On the other hand, $\mathbf{NA}_1$ and $\mathbf{NA}_2$ describe the fact that an A-object is necessarily an A-object, for it is of a different nature from a specific (ordinary) object. **NSA** captures the intuition according to which an A-object cannot be in its own scope, if we want to avoid Russell-like paradoxes, and that assuming different values is something exclusive of A-objects.

One may notice that $\mathbf{S}_{act}$, $\mathbf{S}_{pot}$, and **NSA** clearly contradict the reflexivity of an accessibility relation whenever Ax obtains. As such, the normal modal axiom **T** ($\Box \varphi \rightarrow \varphi$) is incompatible with afthairetic modality – since we want there to be A-objects. Furthermore, *Necessitation* is a rule too strong for the system. In order to see that we can argue by contradiction. Assume otherwise, then from $\mathbf{S}_{act}$ we get $\vdash \Box \forall x(Ax \leftrightarrow \forall y \neg Sxy)$, which clearly cannot be valid, since every A-object necessarily assumes a value.

As we said, the afthairetic modality of a first-order theory T must inherit the inference rules of T, in order for it to be a proper extension. This means the resulting theory must have at least *Modus Ponens* (henceforth *MP*) and *Generalisation* (henceforth

[8] As we shall see briefly, the semantics we adopt is slightly different than the usual, but the schemes still aid in the completeness results we offer.

Gen). Consider now $\Gamma \subseteq Form_{\mathcal{L}}$, $\varphi \in Form_{\mathcal{L}}$, and $\alpha_{\mathrm{T}}(\Gamma) = \{\alpha_{\mathrm{T}}(\psi) \mid \psi \in \Gamma\}$. Let $\mathcal{R}$, as described below, be a proper inference rule of T, and $\mathcal{R}_{\Box_{af}}$ be the following:

$\mathcal{R}$ $\vdash^{\mathrm{T}} \Gamma$, then $\vdash^{\mathrm{T}} \varphi$;

$\mathcal{R}_{\Box_{af}}$ $\vdash^{\mathrm{T}^{\Box_{af}}} \alpha_{\mathrm{T}}(\Gamma)$, then $\vdash^{\mathrm{T}^{\Box_{af}}} \alpha_{\mathrm{T}}(\varphi)$.

Then, we admit $\mathcal{R}_{\Box_{af}}$ as an inference rule of $\mathrm{T}^{\Box_{af}}$.

Now, to prove completeness of AM-extensions, we also need to add a new, unconventional rule. It involves two notions of deduction. Let $\alpha^a_{\mathrm{T}}(\mathrm{T}) = \{\alpha_{\mathrm{T}}(\mathbf{A}) \mid \mathbf{A}$ is an axiom of T$\}$, and let FFOML^a be the set of axioms of FFOML. We shall call a formula $\mathrm{T}^{\Box_q}$*-derivable* if it is derivable in the system $\mathrm{T}^{\Box_q}$, so defined:

$$\mathrm{T}^{\Box_q} = \mathrm{FFOML}^a + \alpha^a_{\mathrm{T}}(\mathrm{T}) + \mathit{Necessitation} + \mathit{MP} + \mathit{Gen} + \mathcal{R}_{\Box_{af}} + \{\mathbf{S}_{pot}, \mathbf{S}_{uni}, \mathbf{NSA}, \mathbf{NA}_1, \mathbf{NA}_2, \mathbf{BF}, \mathbf{CBF}, \mathbf{K}\}$$

We exclude **D** from the system $\mathrm{T}^{\Box_q}$ for, otherwise, rule $Nec_{\mathrm{T}^{\Box_q}}$ (introduced below) cannot be valid in the semantics we define, since the system may be valid in models in which, for the actual world w_0, $\mathrm{R}^2(w_0) = \varnothing$.[9]

Likewise, we may call a set of formulas $\mathrm{T}^{\Box_q}$*-consistent* if it is consistent in the theory composed by $\mathrm{T}^{\Box_q}$-derivable formulas. We shall denote the corresponding notion of derivability by $\vdash^{\mathrm{T}^{\Box_q}}$, distinguishing it from derivability in the main theory for afthairetic modality, which we may denote by $\vdash^{\mathrm{T}^{\Box_{af}}}$.[10] We introduce, then, the following rules:

Nec_τ if $\vdash^{\mathrm{T}^{\Box_{af}}} \Box\varphi$, then $\vdash^{\mathrm{T}^{\Box_{af}}} \Box\Box\varphi$;

$Nec_{\mathrm{T}^{\Box_q}}$ if $\vdash^{\mathrm{T}^{\Box_q}} \varphi$, then $\vdash^{\mathrm{T}^{\Box_{af}}} \Box\varphi$.

The theory $\mathrm{T}^{\Box_{af}}$ shall then be defined as:

$$\mathrm{T}^{\Box_{af}} = \mathrm{FFOML}^a + \alpha^a_{\mathrm{T}}(\mathrm{T}) + Nec_\tau + Nec_{\mathrm{T}^{\Box_q}} + \mathit{MP} + \mathit{Gen} + \mathcal{R}_{\Box_{af}} + \{\mathbf{S}_{act}, \mathbf{S}_{pot}, \mathbf{S}_{uni}, \mathbf{NA}_1, \mathbf{NA}_2, \mathbf{NSA}, \mathbf{BF}, \mathbf{CBF}, \mathbf{K}, \mathbf{D}\}$$

The difference between $\mathrm{T}^{\Box_{af}}$ and $\mathrm{T}^{\Box_q}$ is the former encompasses $\mathbf{S}_{act}$, **D**, Nec_τ, and $Nec_{\mathrm{T}^{\Box_q}}$, but lacks the rule *Necessitation*, while the opposite is true for the latter. The purpose of having two theories is to reflect the distinction between how an A-object behaves in the actual world – following $\mathrm{T}^{\Box_{af}}$ –, and how it behaves in possible states – in accordance with $\mathrm{T}^{\Box_q}$.

3 Semantics and Completeness

We start by defining models for the AM-extensions. Let $\langle \mathrm{M}, \mathrm{I}\rangle$ be a first-order model for a first-order theory T (henceforth, T-*model*), whose language is $\mathcal{L}$ – that is, M

[9] We follow the convention of writing $w\mathrm{R}^n_{\mathcal{M}^+}u$ for *there are* $x_1, ..., x_{n-1}$ *such that* $w\mathrm{R}_{\mathcal{M}^+}x_1$ *and* $x_1\mathrm{R}_{\mathcal{M}^+}x_2$ *and ... and* $x_{n-1}\mathrm{R}_{\mathcal{M}^+}u$ for a relational model $\mathcal{M}^+$ with accessibility relation R; and of writing $\mathrm{R}^n_{\mathcal{M}^+}(w)$ for $\{x \mid w\mathrm{R}^n_{\mathcal{M}^+}x\}$. We also use $\mathrm{R}^n(w)$ when the relevant model is clear by context, and $\mathrm{R}(w)$ for when $n = 1$.

[10] In the absence of ambiguity, we may use simply $\vdash$ to mean $\vdash^{\mathrm{T}^{\Box_{af}}}$.

is a set of objects, and for each n-ary predicate P, function f, and constant c of $\mathcal{L}$, $\mathrm{I}(P) \subseteq \mathrm{M}^n$, $\mathrm{I}(f) : \mathrm{M}^n \to \mathrm{M}$, and $\mathrm{I}(c) \in \mathrm{M}$. We may extend it to a model for $\mathrm{T}^{\Box_{af}}$, in a intuitive manner, by adding the relational semantics machinery (a set of states and an accessibility relation), designating a state (the actual world), setting the interpretation of any predicate in the actual world to be that of the model – when restricted to the ordinary objects in the domain, since A-objects are also to be subjects of predication –, and adding adequate interpretations of the distinct predicates A and S. We follow the convention of denoting the domain of a model $\mathcal{M}$ by the standardised M, and by adding the superscript $^+$ to the name of a model – for example, $\mathcal{M}^+$ – to represent an AM-extension of it.[11]

Definition 3.1 ($\mathrm{T}^{\Box_{af}}$-model) Let $\mathcal{M} = \langle \mathrm{M}, \mathrm{I} \rangle$ be a T-model. A model for an AM-extension $\mathcal{M}^+$ of it is a tuple $\langle \mathrm{M}^+, \mathrm{I}^+, \mathrm{W}, w_0, \mathrm{R}, \mathrm{A}, \mathrm{S} \rangle$, such that:

(a) $\mathrm{W} \neq \varnothing$ and $w_0 \in \mathrm{W}$;
(b) $\mathrm{R} \subseteq \mathrm{W} \times (\mathrm{W} \setminus \{w_0\})$ and $\mathrm{R}(w_0) \neq \varnothing$;[12]
(c) for each n-ary predicate P, $\mathrm{I}^+(P, w_0) \restriction_{\mathrm{M}} = \mathrm{I}(P)$;[13] for each constant c, $\mathrm{I}^+(c) = \mathrm{I}(c)$; for each n-ary function symbol f, $\mathrm{I}^+(f)$ is a n-ary function such that $\mathrm{I}^+(f) \restriction_{\mathrm{M}} = \mathrm{I}(f)$;
(d) $\mathrm{M} \subseteq \mathrm{M}^+$ and $\mathrm{M}^+ \setminus \mathrm{M} = \mathrm{A}$;
(e) $\mathrm{S} : \mathrm{M}^+ \times \mathrm{M}^+ \to \mathcal{P}(\mathrm{W})$ is such that:

- $w_0 \notin \bigcup_{i \in \mathrm{A};\, j \in \mathrm{M}^+} \mathrm{S}(i, j)$;[14]
- $o \notin \mathrm{A}$ iff for all $w \in \mathrm{W}$, $w \in \mathrm{S}(o, o)$; and if $a \in \mathrm{A}$, then for all $w \neq w_0$, there is $o \neq a$ such that $w \in \mathrm{S}(a, o)$;
- if $w \in \mathrm{S}(a, b)$ and $w \in \mathrm{S}(a, c)$, then $b = c$.

If $\mathcal{M}^+$ is the AM-extension of a T-model, we call it a $\mathrm{T}^{\Box_{af}}$*-model*.

Definition 3.2 (Assignment and equiadmissible assignment) An *assignment* on an afthairetic modality model $\mathcal{M}^+$ is a function $\eta : Var_{\mathcal{L}} \to \mathrm{M}^+$, where $Var_{\mathcal{L}}$ is the set of variables of $\mathcal{L}$. If η is an assignment such that for all $x \in Var_{\mathcal{L}}$, $\eta(x) \in \mathrm{M}$, then we call η an *equiadmissible assignment*.

Given an assignment η on $\mathcal{M}^+$ (and likewise, μ on $\mathcal{M}$), we may extend it to an assignment $\eta^{\mathcal{M}^+}$ (respectively, $\mu^{\mathcal{M}}$) to any term in the following way: for a variable x, $\eta^{\mathcal{M}^+}(x) = \eta(x)$ (respectively, $\mu^{\mathcal{M}}(x) = \mu(x)$), for a constant c, $\eta^{\mathcal{M}^+}(c) = \mathrm{I}^+(c)$ (respectively, $\mu^{\mathcal{M}}(c) = \mathrm{I}(c)$), and for a n-ary function symbol f, $\eta^{\mathcal{M}^+}(f(t_1, ..., t_n)) = \mathrm{I}^+(f)(\eta^{\mathcal{M}^+}(t_1), ..., \eta^{\mathcal{M}^+}(t_n))$ (respectively,

[11] We use *AM-extension* to refer both to the theory which is an extension of a certain first-order theory, and to the extended model for the afthairetic modality of a certain theory. We trust there to be no ambiguity in each context.

[12] Notice (a) and (b) together imply $\mathrm{W} \setminus \{w_0\} \neq \varnothing$.

[13] The restriction ensures that any new tuple satisfying a predicate in the AM-extension involves an A-object – so that the objects of the first-order theory remain on the same relations between themselves as in the AM-extension.

[14] Intuitively, that means the designated state is not in the set of states in which i assumes the value j, for any $i \in \mathrm{A}$ and $j \in \mathrm{M}^+$.

$\mu^{\mathcal{M}}(f(t_1, ..., t_n)) = \mathrm{I}(f)(\mu^{\mathcal{M}}(t_1), ..., \mu^{\mathcal{M}}(t_n)))$. For $x \in Var_{\mathcal{L}}$, an x-variant of η is an assignment η' on the same model such that for all $y \neq x$, $\eta(y) = \eta'(y)$. If η and η' are x-variant assignments on a model $\mathcal{M}$, we denote it by $\eta \sim_x^{\mathcal{M}} \eta'$. We omit the reference to the model, both when introducing variant assignments and referring to extended assignments, when the context is unambiguous.

Fact 3.3 An assignment is equiadmissible iff it is an assignment on the regular first-order model of which the afthairetic modality model is an AM-extension (or, equivalently, it is an assignment on $\mathcal{M}^+ \restriction_{\mathrm{M}}$). Furthermore, if η is an equiadmissible assignment, then for any $t \in Term_{\mathcal{L}}$, $\eta^{\mathcal{M}^+}(t) = \eta^{\mathcal{M}}(t)$.[15]

Because of the above fact, if η is an equiadmissible assignment, there is no ambiguity in using simply $\eta(t)$ in a context in which there are two relevant models – a first-order model and an AM-extension of it.

Truth, now, may be defined (with respect to a model, a state, and an assignment):

- $\mathcal{M}^+, w \models_\eta t_1 = t_2$ iff $\eta(t_1) = \eta(t_2)$;
- $\mathcal{M}^+, w \models_\eta Pt_1...t_n$ iff $\langle \eta(t_1), ..., \eta(t_n)\rangle \in \mathrm{I}^+(P, w)$.
- $\mathcal{M}^+, w \models_\eta At$ iff $\eta(t) \in \mathrm{A}$;
- $\mathcal{M}^+, w \models_\eta St_1t_2$ iff $w \in \mathrm{S}(\eta(t_1), \eta(t_2))$;
- $\mathcal{M}^+, w \models_\eta \neg\varphi$ iff $\mathcal{M}^+, w \not\models_\eta \varphi$;
- $\mathcal{M}^+, w \models_\eta \varphi \wedge \psi$ iff $\mathcal{M}^+, w \models_\eta \varphi$ and $\mathcal{M}^+, w \models_\eta \psi$;
- $\mathcal{M}^+, w \models_\eta \Box\varphi$ iff for all $u \in \mathrm{R}(w)$, $\mathcal{M}^+, u \models_\eta \varphi$;
- $\mathcal{M}^+, w \models_\eta \forall x\varphi$ iff for any $\eta' \sim_x \eta$, $\mathcal{M}^+, w \models_{\eta'} \varphi$.

Notice the truth of an atomic formula implies the designation of each term occurring in it. Thus, we work with a negative semantics. A formula φ is said to be true in a state w ($\mathcal{M}^+, w \models \varphi$) if it is true in w under any assignment.

Definition 3.4 A formula φ, in the appropriate signature, is said to be true in a model ($\mathcal{M}^+ \models \varphi$) if *it is true in the actual world* w_0. A formula φ is valid ($\models^{\mathrm{T}^{\Box_{af}}} \varphi$) if it is true in any $\mathrm{T}^{\Box_{af}}$-model.[16]

From now on, we shall also use the notion of free occurrences of variables, extending it to terms in the obvious way. We trust the reader to be familiar with the subject.

For simplicity, we use the following conventions: we write μ for assignments on a regular first-order model and η for assignments on AM-extensions; whenever there is a string of related assignments such that $\eta_1 \sim_{x_1} \eta_2 ... \sim_{x_n} \eta_{n+1}$, we may write $\eta_1 \sim_{x_1,...,x_n} \eta_{n+1}$; and we also write $\mathcal{M}^+, w \models_\eta \varphi[o/x]$ to mean φ is true in w under an assignment $\eta' \sim_x \eta$ such that $\eta'(x) = o$. Putting some of those notations together, for example, we have if $\eta_1 \sim_{x_1} \eta_2 ... \sim_{x_n} \eta_{n+1}$, we write $\eta_1 \sim_{\vec{x}} \eta_{n+1}$; if $\forall x_1 ... \forall x_n\varphi$, we write $\forall\vec{x}\varphi$; and if $\mathcal{M}^+, w \models_\eta \forall\vec{x}\varphi$, then for any $\eta' \sim_{\vec{x}} \eta$, $\mathcal{M}^+, w \models_{\eta'} \varphi$.

15 To see that, notice, by Definition 3.2, $\eta^{\mathcal{M}^+}(x) = \eta^{\mathcal{M}}(x)$; by Definition 3.1, for any constant c, $\eta^{\mathcal{M}^+}(c) = \mathrm{I}^+(c) = \mathrm{I}(c) = \eta^{\mathcal{M}}(c)$; and by induction on the complexity of terms, we may get for any n-ary function term f, $\eta^{\mathcal{M}^+}(f(t_1, ..., t_n)) = \mathrm{I}^+(f)(\eta^{\mathcal{M}^+}(t_1), ..., \eta^{\mathcal{M}^+}(t_n)) = \mathrm{I}^+(f)(\eta^{\mathcal{M}}(t_1), ..., \eta^{\mathcal{M}}(t_n)) = \mathrm{I}(f)(\eta^{\mathcal{M}}(t_1), ..., \eta^{\mathcal{M}}(t_n)) = \eta^{\mathcal{M}}(f(t_1, ..., t_n))$.

16 We may note that, under this definition of truth in a model, axiom **D** no longer defines the class of serial models. Instead, it defines the class of models in which $\mathrm{R}(w_0) \neq \varnothing$ – that is, the actual world access at least one other state.

3.1 Soundness of the Hilbert-style System for $T^{\Box_{af}}$

Lemma 3.5 *Let $\mathcal{M}^+$ be an AM-extension of $\mathcal{M}$. Let also $\varphi \in Form_{\mathcal{L}}$ and $\{x_i\}$ be the free variables of φ. If μ is an assignment on $\mathcal{M}$ and η is an assignment on $\mathcal{M}^+$ such that $\eta(x_i) = \mu(x_i)$, then,*

$$\mathcal{M} \models_\mu \varphi \textit{ iff } \mathcal{M}^+, w_0 \models_\eta \alpha_{\mathrm{T}}(\varphi).$$

Theorem 3.6 *Let $\varphi \in Form_{\mathcal{L}^{\Box_{af}}}$ and $\{x_i\}$ be the free variables of φ. If η and η' are assignments on $\mathcal{M}^+$ such that $\eta(x_i) = \eta'(x_i)$, then, for any $w \in \mathrm{W}$,*

$$\mathcal{M}^+, w \models_\eta \varphi \textit{ iff } \mathcal{M}^+, w \models_{\eta'} \varphi.$$

Theorem 3.6 is straightforwardly obtained through the first-order properties of the semantics.

Theorem 3.7 *Let $\mathcal{L}$ be the language of a first-order theory* T *and $\varphi \in Form_{\mathcal{L}}$. Let also $\mathcal{M}$ be a* T*-model, and $\mathcal{M}^+$ be an AM-extension of it. Then,*

$$\mathcal{M} \models \varphi \textit{ iff } \mathcal{M}^+ \models \alpha_{\mathrm{T}}(\varphi).$$

Corollary 3.8 *Let $\mathcal{L}$ be the language of a first-order theory* T *and $\varphi \in Form_{\mathcal{L}}$. Then,*

$$\models^{\mathrm{T}} \varphi \textit{ iff } \models^{\mathrm{T}^{\Box_{af}}} \alpha_{\mathrm{T}}(\varphi).$$

Corollary 3.9 *Let* T *be a first-order theory. Then, $\mathcal{R}_{\Box_{af}}$ preserves validity in $\mathrm{T}^{\Box_{af}}$-models.*

Theorem 3.10 *Let $\mathcal{M} = \langle \mathrm{M}, \mathrm{I} \rangle$ be a* T*-model, $\mathcal{M}^+$ an AM-extension of it, and $\mathcal{L}$ the language of* T*. Then, for any $\varphi \in Form_{\mathcal{L}}$ and equiadmissible assignment η,*

$$\mathcal{M} \models_\eta \varphi \textit{ iff } \mathcal{M}^+ \restriction_{\mathrm{M}}, w_0 \models_\eta \varphi.$$

Corollary 3.8 shows the AM-extension $\mathrm{T}^{\Box_{af}}$ of T is semantically conservative in the sense that, whatever is true of the objects of T *in* T is also true in $\mathrm{T}^{\Box_{af}}$. Indeed, the translation α_{T} only assures us, when moving from a first-order theory to its afthairetic modality, we are still talking about the objects of the theory when using its theorems. Moreover, together with Corollary 3.9, we have that, if T is sound, then the fragment of $\mathrm{T}^{\Box_{af}}$ restricted to the axioms and rules inherited from T is sound with respect to the class of models for $\mathrm{T}^{\Box_{af}}$. Theorem 3.10 is a slightly stronger version of the conservativeness: it shows $\mathcal{M}^+ \restriction_{\mathrm{M}}$ itself is a model for T, as it is to be expected.

Moving on to the remaining axioms and inference rules, Corollaries 3.8 and 3.9 tell us the translated versions of the axioms of predicate logic with identity are valid, and *MP* and *Gen* preserve validity. However, we trust the reader to see there is no substantial modification to the interpretation of regular predicates and identity, so that the regular first-order axioms and inference rules are also valid and preserve validity, respectively, and that **CBF** and **BF** are still valid in monotonous and anti-monotonous models, such as those for $\mathrm{T}^{\Box_{af}}$. Furthermore, for there is always at least one state accessible from the actual world, we may also notice that **D** is valid. We can now concentrate on the proper axioms and rules of $\mathrm{T}^{\Box_{af}}$.

Theorem 3.11 $\mathrm{Nec}_{\mathrm{T}^{\Box q}}$ *and* Nec_τ *preserve validity in the proposed semantics, and* $\models^{\mathrm{T}^{\Box af}}$ $\mathbf{S}_{act}, \mathbf{S}_{pot}, \mathbf{S}_{uni}, \mathbf{NSA}, \mathbf{NA}_1, \mathbf{NA}_2$.

Proof We start by the axioms. We have that $\mathbf{S}_{act}$ holds, since $w_0 \notin \bigcup_{i \in \mathrm{A}; j \in \mathrm{M}^+} \mathrm{S}(i, j)$, and we may see $\mathcal{M}^+, w_0 \models_\eta \neg Sxy$ for any η such that $\eta(x) \in \mathrm{A}$. Thus, $\mathcal{M}^+, w_0 \models \forall x(Ax \leftrightarrow \forall y(\neg Sxy))$.

For $\mathbf{S}_{pot}$, let η be an assignment. Then, $w \in \mathrm{S}(\eta(x), b)$ for some b by the definition of $\mathrm{T}^{\Box af}$-model, so that $\mathcal{M}^+, w \models_\eta Sxy$, for η such that $\eta(x) = a$ and $\eta(y) = b$. Thus, we get $\mathcal{M}^+, w \models_\eta \exists y Sxy$. By the arbitrariness of w, $\mathcal{M}^+, w_0 \models_\eta \Box\exists y Sxy$, and by the arbitrariness of η, we get $\mathbf{S}_{pot}$.

The validity of $\mathbf{S}_{uni}$ and $\mathbf{NSA}$ is clearly implied by Definition 3.1 (e) (by the second and third conditions, respectively).

For $\mathbf{NA}_1$, let η be an assignment and w be any state. Either $\eta(x) \in \mathrm{A}$ or $\eta(x) \notin \mathrm{A}$. If the former, then $\mathcal{M}^+, w \models_\eta Ax$, and since w is any state (particularly, any in $\mathrm{R}(w_0)$), that means $\mathcal{M}^+, w_0 \models_\eta \Box Ax$ – and $\mathcal{M}^+, w_0 \models_\eta Ax \to \Box Ax$. If the latter, then $\mathcal{M}^+, w \not\models_\eta Ax$, which also means $\mathcal{M}^+, w_0 \models_\eta Ax \to \Box Ax$. By the arbitrariness of η, we get $\mathcal{M}^+, w_0 \models \forall x(Ax \to \Box Ax)$. By an analogous argument, the case for $\mathbf{NA}_2$ can be made.

For Nec_τ, suppose $\Box\varphi$ is valid, and $\Box\Box\varphi$ is not. Let $\mathcal{M}^+ \not\models \Box\Box\varphi$. We have $\mathcal{M}^+, w_0 \models \Box\varphi$, and $\mathcal{M}^+, w_0 \not\models \Box\Box\varphi$, which means, for some $u \in \mathrm{R}^2(w_0)$, $\mathcal{M}^+, u \not\models \varphi$. Consider, now, $\mathcal{M}_2^+ = \langle \mathrm{M}^+, \mathrm{I}^+, \mathrm{W}, w_0, \mathrm{R}_2, \mathrm{A}, \mathrm{S}\rangle$ such that $\mathrm{R}_2 = \mathrm{R} \cup \{\langle w_0, u\rangle\}$. It is fairly simple to see that, for any $\psi \in Form_{\mathcal{L}^{\Box af}}$, $\mathcal{M}^+, u \models \psi$ iff $\mathcal{M}_2^+, u \models \psi$, so that $\mathcal{M}_2^+, u \not\models \varphi$, and thus $\mathcal{M}_2^+, w_0 \not\models \Box\varphi$. It is also clear that $\mathcal{M}_2^+$ is a $\mathrm{T}^{\Box af}$-model – since $u \neq w_0$, for w_0 is seen by no state –, so we have a contradiction. Thus, $\Box\Box\varphi$ must also be valid.

For $Nec_{\mathrm{T}^{\Box q}}$, let $\vdash^{\mathrm{T}^{\Box q}} \varphi$. We show, by an induction on the length l of proofs, $\Box\varphi$ is valid. Suppose the length of the proof of φ is $l = 1$. Then, φ is an axiom of $\mathrm{T}^{\Box q}$. We show then its necessitation is valid. The result is trivial for the non-proper axioms of $\mathrm{T}^{\Box q}$. Given that the argument offered above for the truth of the axioms in w_0 works as well for any other state, it is easy to see the boxed version of each proper axiom (of $\mathrm{T}^{\Box q}$) is likewise true in w_0.

Suppose now it works for $l \leq n$, and consider $l = n + 1$. Then, φ is obtained by the application of either *MP*, or *Gen*, or *Necessitation*. Therefore, either (i) there is $\psi \in Form_{\mathcal{L}^{\Box af}}$ such that $\vdash^{\mathrm{T}^{\Box q}} \psi$ and $\vdash^{\mathrm{T}^{\Box q}} \psi \to \varphi$, and the lengths of their proofs is lesser than $n+1$; or (ii) there is $\psi \in Form_{\mathcal{L}^{\Box af}}$ such that $\varphi = \forall x\psi$, $\vdash^{\mathrm{T}^{\Box q}} \psi$, and the length of the proof of ψ is lesser than $n+1$; or (iii) for some $\psi \in Form_{\mathcal{L}^{\Box af}}$, $\varphi = \Box\psi$ and $\vdash^{\mathrm{T}^{\Box q}} \psi$, and the length of its proof is lesser than $n + 1$. If (i), then, by induction hypothesis, $\Box\psi$ and $\Box(\psi \to \varphi)$ are valid. Since $\mathbf{K}$ is valid and *MP* preserves validity, that means $\Box\psi \to \Box\varphi$ is valid, and we have that $\Box\varphi$ is valid. If (ii), by induction hypothesis, $\Box\psi$ is valid. Since *Gen* preserves validity, $\forall x\Box\psi$ is valid, and since $\mathbf{BF}$ is valid and *MP* preserves validity, $\Box\forall x\psi$ is valid. If (iii), then, by induction hypothesis, $\Box\psi$ is valid. Since Nec_τ preserves validity, that means $\Box\Box\psi$ is valid. □

Corollary 3.12 *Let* T *be a first-order theory. Then,* $\mathrm{T}^{\Box_{af}}$ *is sound with respect to the class of AM-extensions of its models (equivalently, the class of all* $\mathrm{T}^{\Box_{af}}$*-models).*

Notice, if the theory T is not consistent, the above Corollaries 3.9 and 3.12 vacuously apply.

3.2 Completeness of the Hilbert-style System for $\mathrm{T}^{\Box_{af}}$

Without loss of generality, we may assume the first-order theory T of which the relevant afthairetic modality theory is an extension is consistent. Therefore, from now on, this is an implicit assumption. We start with some necessary definitions, but first we introduce an important step into the direction of our intended goal.

Theorem 3.13 *Let* $\varphi \in Form_{\mathcal{L}}$*. If* $\vdash^{\mathrm{T}} \varphi$*, then* $\vdash^{\mathrm{T}^{\Box_{af}}} \alpha_{\mathrm{T}}(\varphi)$.

Proof We prove by an induction on the length l of the proof of φ in T. Suppose $l = 1$. Then, φ is an axiom of T, which means $\alpha_{\mathrm{T}}(\varphi)$ is an axiom of $\mathrm{T}^{\Box_{af}}$. Assume the case for $l \leq n$, and consider $l = n + 1$. Then, either φ is obtained by *MP*, or by *Gen*, or by a proper rule $\mathcal{R}$. Therefore, either (i) there is $\psi \in Form_{\mathcal{L}^{\Box_{af}}}$ such that $\vdash^{\mathrm{T}} \psi \to \varphi$ and $\vdash^{\mathrm{T}} \psi$, and the lengths of those proofs are lesser than $n + 1$; or (ii) there is $\psi \in Form_{\mathcal{L}^{\Box_{af}}}$ such that $\varphi = \forall x \psi$ and $\vdash^{\mathrm{T}} \psi$, and the lengths of that proof is lesser than $n + 1$; or (iii) there are ψ_i such that $\vdash^{\mathrm{T}} \psi_i$, and the length of each of those proofs is lesser than $n+1$. If (i), by induction hypothesis, $\vdash^{\mathrm{T}^{\Box_{af}}} \alpha_{\mathrm{T}}(\psi \to \varphi)$ (and $\alpha_{\mathrm{T}}(\psi \to \varphi) = \alpha_{\mathrm{T}}(\psi) \to \alpha_{\mathrm{T}}(\varphi)$) and $\vdash^{\mathrm{T}^{\Box_{af}}} \alpha_{\mathrm{T}}(\psi)$, so by *MP*, $\vdash^{\mathrm{T}^{\Box_{af}}} \alpha_{\mathrm{T}}(\varphi)$. If (ii), by induction hypothesis, $\vdash^{\mathrm{T}^{\Box_{af}}} \alpha_{\mathrm{T}}(\psi)$, so by a few steps, $\vdash^{\mathrm{T}^{\Box_{af}}} \neg Ax \to \alpha_{\mathrm{T}}(\psi)$, and by *Gen*, $\vdash^{\mathrm{T}^{\Box_{af}}} \forall x(\neg Ax \to \alpha_{\mathrm{T}}(\psi))$. If (iii), by induction hypothesis, $\vdash^{\mathrm{T}^{\Box_{af}}} \alpha_{\mathrm{T}}(\psi_i)$, and by $\mathcal{R}_{\Box_{af}}$, $\vdash^{\mathrm{T}^{\Box_{af}}} \alpha_{\mathrm{T}}(\varphi)$. □

Definition 3.14 ($\forall$-property, [9]) Let Γ be a set of formulas. We say Γ has the $\forall$-*property* if for every formula φ and variable x there is a term t such that $\neg\forall x\varphi \to (\neg\varphi^t_x \wedge t = t) \in \Gamma$.

The definition of the $\forall$-property is slightly different from the one usually used. We need that modification to ensure every universally quantified formula which is not verified is falsified by *some* object. Otherwise, since we are using a negative free logic, for example, the fact that φ^t_x is not verified does not mean $\neg\forall x\varphi$, for, as we may see in $\mathbf{UI}_f$, it may be that t does not designate.

Lemma 3.15 *Let* Λ *be a* $\mathrm{T}^{\Box_{af}}$*-consistent set of formulas. If the formulas of* $\mathcal{L}$ *are enumerable, then there is a* $\mathrm{T}^{\Box_{af}}$*-consistent set* Γ *with the* $\forall$*-property such that* $\Lambda \subseteq \Gamma$.

Proof sketch Use the same structure of the proof for first-order modal logic[17] (as $\mathrm{T}^{\Box_{af}}$ also possesses *Gen*), and extend an enumeration of the formulas of $\mathcal{L}$ to an enumeration of the formulas of $\mathcal{L}^{\Box_{af}}$.[18]

[17] For example, see [9], Chapter 14, Theorem 14.1.

[18] Let P^j_i be the j-ary predicates of first-order modal logic, and Q^j_i the j-ary predicates of $\mathcal{L}$. We map A to P^1_1 and Q^1_n to P^1_{n+1}, S to P^2_1 and Q^2_n to P^2_{n+1}, and for all remaining j, Q^j_i to P^j_i.

Lemma 3.16 ([Lindenbaum) *Let* Λ *be a* $\mathrm{T}^{\Box_{af}}$*-consistent set of formulas of* $\mathcal{L}^{\Box_{af}}$*. Then, there is a maximal consistent set of formulas* Γ *with the* $\forall$*-property such that* $\Lambda \subseteq \Gamma$.

Proof sketch The same as that for normal first-order modal logic, using Lemma 3.15 appropriately.

Corollary 3.17 *Let* Λ *be a* $\mathrm{T}^{\Box_q}$*-consistent set of formulas of* $\mathcal{L}^{\Box_{af}}$*. Then, there is a maximal* $\mathrm{T}^{\Box_q}$*-consistent* Γ *with the* $\forall$*-property such that* $\Lambda \subseteq \Gamma$.

We cannot build canonical models, in which each state is a maximal $\mathrm{T}^{\Box_{af}}$-consistent set of formulas with the relevant properties, and for which each formula is true at a state iff the formula is a member of the state. In that sort of construction, it is routine to extend *every* consistent set of formulas to a maximal consistent set of formulas with the relevant properties, and collect them as the states in the canonical model. In the present case, the set $\{\neg Sxy \mid x, y \in Var_{\mathcal{L}^{\Box_{af}}}\}$ can be extended to many different maximal consistent sets. Following the usual method, we would have many non-designated states in which Ax and $\forall y \neg Sxy$ are both true, which is inadequate – in contradiction with $\mathbf{S}_{pot}$. Nevertheless, given that validity in a model is identified with truth in the actual world, for each set of that form, we may build a model that validates all of its formulas, which shall allow us to show the completeness of the system.

Definition 3.18 Let Γ be a set of formulas. Then, $L(\Gamma) = \{\varphi \mid \Box\varphi \in \Gamma\}$.

It is routine, in proving completeness by using canonical models, to show for any maximal consistent set Γ that, for any formula φ such that $\Box\varphi \notin \Gamma$, we may extend $\{\neg\varphi\}$ to a maximal consistent set which contains $L(\Gamma)$. The purpose of that is to make it so there is a maximal consistent set modally compatible with Γ, and so Γ, as a state in the canonical model, may access a state which falsifies φ, therefore reflecting the fact $\Box\varphi \notin \Gamma$. However, in our case, we have two notions of consistency, one for the actual world, and another for any possible state. In the models we have offered, the actual world always accesses possible states, and possible states may access yet other possible states, but the actual world is accessed by no one. Therefore, we need to provide two different lemmas of existence of such sets: one showing that the non-necessary formulas of a $\mathrm{T}^{\Box_{af}}$-consistent set is verified in a $\mathrm{T}^{\Box_q}$-consistent set modally compatible with it – which reflects the actual world accessing a possible state –, and another showing that the non-necessary formulas of a $\mathrm{T}^{\Box_q}$-consistent set is verified in a $\mathrm{T}^{\Box_q}$-consistent set modally compatible with it – which reflects a possible state accessing another.

Lemma 3.19 *Let* Λ *be a maximal* $\mathrm{T}^{\Box_{af}}$*-consistent set of* $\mathrm{T}^{\Box_{af}}$ *formulas of* $\mathcal{L}^{\Box_{af}}$ *with the* $\forall$*-property. Let also* $\varphi \in Form_{\mathcal{L}^{\Box_{af}}}$ *be such that* $\Box\varphi \notin \Lambda$*. Then, there is a maximal* $\mathrm{T}^{\Box_q}$*-consistent set* Γ *with the* $\forall$*-property such that* $L(\Lambda) \cup \{\neg\varphi\} \subseteq \Gamma$.

Lemma 3.20 *Let Λ be a maximal $\mathrm{T}^{\Box_q}$-consistent set of formulas of $\mathcal{L}^{\Box_{af}}$ with the $\forall$-property, and let $\varphi \in Form_{\mathcal{L}^{\Box_{af}}}$ be such that $\Box\varphi \notin \Lambda$. Then, there is a maximal $\mathrm{T}^{\Box_q}$-consistent set of formulas Γ with the $\forall$-property such that $L(\Lambda) \cup \{\neg\varphi\} \subseteq \Gamma$.*[19]

Lemma 3.19 is the reason for the introduction of $Nec_{\mathrm{T}^{\Box_q}}$ into the system, and why we have stated, in Section 2, that the system as it stood then – without the rule – was too weak. It reiterates how that unusual rule is essential to the completeness of the systems with respect to our semantics.

Definition 3.21 (Induced model) Let Γ be a maximal $\mathrm{T}^{\Box_{af}}$-consistent set of formulas of $\mathcal{L}^{\Box_{af}}$ with the $\forall$-property. We define the equivalence classes, for $t \in Term_{\mathcal{L}}$, $[t]_\Gamma = \{r \mid t = r \in \Gamma\}$.[20] The model induced by Γ is $\mathcal{M}^+_\Gamma = \langle \mathrm{M}^+_\Gamma, \mathrm{I}^+, \mathrm{W}, \Gamma, \mathrm{R}, \mathrm{A}, \mathrm{S}\rangle$, such that:

(a) $\mathrm{M}^+_\Gamma = \{[t]_\Gamma \mid t \in Term_{\mathcal{L}} \;\&\; t = t \in \Gamma\}$;
(b) $[\vec{t}]_\Gamma \in \mathrm{I}^+(P, w)$ iff $P\vec{t} \in w$ and $[t_i]_\Gamma \in \mathrm{M}^+_\Gamma$; if $[f(\vec{t})]_\Gamma \in \mathrm{M}^+_\Gamma$, then $\mathrm{I}^+(f)([\vec{t}]_\Gamma) = [f(\vec{t})]_\Gamma$, and otherwise, $\mathrm{I}^+(f)([\vec{t}]_\Gamma)$ is not defined; $\mathrm{I}^+(c) = [c]_\Gamma$;
(c) $\mathrm{W} = \{\Gamma\} \cup \{\Lambda \mid \Lambda$ is a maximal $T^{\Box_q}$-consistent set of formulas with the $\forall$-property such that, for all $t \in Term_{\mathcal{L}}$ such that $At \in \Gamma$, $At \in \Lambda\}$;
(d) $\mathrm{R} = \{\langle w, u\rangle \mid w, u \in \mathrm{W} \;\&\; L(w) \subseteq u\}$;
(e) $\mathrm{A} = \{[t]_\Gamma \mid At \in \Gamma\}$;
(f) $w \in \mathrm{S}([t_1]_\Gamma, [t_2]_\Gamma)$ iff $St_1t_2 \in w$.

Lemma 3.22 *Let η and η' be assignments on a model $\mathcal{M}^+$ (respectively, $\mathcal{M}$), $\varphi \in Form_{\mathcal{L}^{\Box_{af}}}$ (respectively, $\varphi \in Form_{\mathcal{L}}$), and $\{x_i\}$ be the free variables of φ. If $\eta'(x) = \eta(y)$, and for all $x_i \neq x$, $\eta'(x_i) = \eta(x_i)$, then, for any state w,*

$$\mathcal{M}^+, w \models_{\eta'} \varphi \text{ iff } \mathcal{M}^+, w \models_\eta \varphi^y_x$$
$$\text{(respectively, } \mathcal{M} \models_{\eta'} \varphi \text{ iff } \mathcal{M} \models_\eta \varphi^y_x\text{).}$$

Lemma 3.22 for normal first-order modal logic has many proofs in the literature, all of which trivially extend to $\mathrm{T}^{\Box_{af}}$.

Theorem 3.23 *The following are obtained in $\mathrm{T}^{\Box_{af}}$:*

- $\vdash \forall x \forall y(x = y \to \Box x = y)$
- $\vdash \forall x \forall y(x \neq y \to \Box x \neq y)$

The proofs of the above theorems are inherited from FFOML, by the use of $Nec_{\mathrm{T}^{\Box_q}}$.

Corollary 3.24 *Let η and η' be assignments on $\mathcal{M}^+$, $\varphi \in Form_{\mathcal{L}^{\Box_{af}}}$, and $\{t_i\}$ be the free terms of φ. If $\eta'(s) = \eta(t)$, and for all $t_i \neq t$, $\eta'(t_i) = \eta(t_i)$, then, for any state w,*

$$\mathcal{M}^+, w \models_{\eta'} \varphi \text{ iff } \mathcal{M}^+, w \models_\eta \varphi^t_s.$$

[19] To prove this lemma, we only need to make the same argument as that of Lemma 3.19 (which may be found in the Appendix), only using *Necessitation* instead of $Nec_{\mathrm{T}^{\Box_q}}$.

[20] We retain the convention, and write $[\vec{t}]_\Gamma$ for the sequence $[t_1]_\Gamma, ..., [t_n]_\Gamma$.

Lemma 3.25 *Let* Γ *be a maximal* $\mathrm{T}^{\Box_{af}}$*-consistent set of formulas of* $\mathcal{L}^{\Box_{af}}$ *with the* $\forall$*-property, and* $\mathcal{M}^+_\Gamma = \langle \mathrm{M}^+, \mathrm{I}^+, \mathrm{W}, \Gamma, \mathrm{R}, \mathrm{A}, \mathrm{S}\rangle$ *be the model induced by it. Let also* ξ *be the canonical assignment* $\xi(x) = [x]_\Gamma$*. Then,*

$$\mathcal{M}^+_\Gamma, w \models_\xi \varphi \textit{ iff } \varphi \in w.$$

Lemma 3.26 *Let* $\mathcal{M}^+_\Gamma$ *be the model induced by a maximal* $\mathrm{T}^{\Box_{af}}$*-consistent set of formulas* Γ *with the* $\forall$*-property. Let* $\mathcal{M}_\Gamma = \langle \mathrm{M}_\Gamma, \mathrm{I}\rangle$ *be such that:*

- $\mathrm{M}_\Gamma = \mathrm{M}^+_\Gamma \setminus \mathrm{A}$*;*
- $P, f, c \in \mathcal{L}$*,* $\mathrm{I}(P) = \mathrm{I}^+(P, \Gamma) \restriction_{\mathrm{M}_\Gamma}$*;* $\mathrm{I}(f) = \mathrm{I}^+(f) \restriction_{\mathrm{M}_\Gamma}$*; and* $\mathrm{I}(c) = \mathrm{I}^+(c)$*.*

Then, $\mathcal{M}$ *is a* T*-model.*

Theorem 3.27 (Completeness) *Let* T *be a sound theory, and* $\varphi \in Form_{\mathcal{L}^{\Box_{af}}}$*. If* $\models^{\mathrm{T}^{\Box_{af}}} \varphi$*, then* $\vdash^{\mathrm{T}^{\Box_{af}}} \varphi$*.*

Proof As usual, in order to prove the completeness of the system, we only need to show that any model induced by a maximal $\mathrm{T}^{\Box_{af}}$-consistent set of formulas with the $\forall$-property is a $\mathrm{T}^{\Box_{af}}$-model.[21] To do that, consider one such set Γ and its induced model, $\mathcal{M}^+_\Gamma$. We argue it is an AM-extension of $\mathcal{M}_\Gamma$, defined as in Lemma 3.26. We check each condition of Definition 3.1 – skipping (c) and (d), which are covered by Lemma 3.26:

(a) For $\Gamma \in \mathrm{W}$, $\mathrm{W} \neq \varnothing$;

(b) Since $\mathbf{S}_{act}, \mathbf{S}_{pot} \in \Gamma$, by Γ's consistency, there is no state u such that $L(u) \subseteq \Gamma$. Thus, by Definition 3.21 (d), $\mathrm{R} \subseteq \mathrm{W} \times (\mathrm{W} \setminus \Gamma)$. Furthermore, by Lemma 3.19 there is a maximal $\mathrm{T}^{\Box_q}$-consistent Λ with the $\forall$-property such that $L(\Gamma) \subseteq \Lambda$, so by Definition 3.21 (d), $\mathrm{R}(\Gamma) \neq \varnothing$;

(e) For the first point of the criterion, suppose for some $[t_1]_\Gamma, [t_2]_\Gamma \in \mathrm{M}^+_\Gamma$, $\mathcal{M}^+_\Gamma, \Gamma \models_\xi St_1t_2$. By Lemma 3.25, that means $St_1t_2 \in \Gamma$. Then, by Γ's maximal $\mathrm{T}^{\Box_{af}}$-consistency, from $\mathbf{S}_{act}$, $\neg At_1 \in \Gamma$, which means $[t_1]_\Gamma \notin \mathrm{A}$. Therefore, $\Gamma \notin \bigcup_{i \in \mathrm{A}; j \in \mathrm{M}^+} \mathrm{S}(i, j)$. For the second, let $[t]_\Gamma \notin \mathrm{A}$ and $w \in \mathrm{W}$. By Definition 3.21 (e), that is the case iff $At \notin \Gamma$. By Γ's maximal $\mathrm{T}^{\Box_{af}}$-consistency, that happens iff $\neg At \in \Gamma$. By Definition 3.21 (c) and w's maximal $\mathrm{T}^{\Box_q}$-consistency, that is the case iff $\neg At \in w$, and from **NSA**, that is the case iff $Stt \in w$. But, by Definition 3.21 (f), that is so iff $w \in \mathrm{S}([t]_\Gamma, [t]_\Gamma)$. Let now $[t]_\Gamma \in \mathrm{A}$ and $w \in \mathrm{W}$. Then, $At \in \Gamma$. By Γ's maximal $\mathrm{T}^{\Box_{af}}$-consistency, from $\mathbf{S}_{pot}$, $At \to \exists y(t \neq y \wedge Sty) \in L(\Gamma) \subseteq w$. By the previous argument, we also have $At \in w$, and so $\exists y(t \neq y \wedge Sty) \in w$. By Lemma 3.25, that means $\mathcal{M}^+_\Gamma, w \models_\xi \exists y(t \neq y \wedge Sty)$. Therefore, for some $\eta \sim_y \xi$, $\eta(y) \neq [t]_\Gamma$ and $w \in \mathrm{S}([t]_\Gamma, \eta(y))$. Furthermore, by (b) of this proof, $w \neq \Gamma$. For the last point, suppose for some $w \in \mathrm{W}$ and

[21] The reason for that is: suppose $\not\vdash^{\mathrm{T}^{\Box_{af}}} \varphi$. Then, $\{\neg\varphi\}$ is a $\mathrm{T}^{\Box_{af}}$-consistent set of formulas, and thus, by Lemma 3.16, there is a maximal $\mathrm{T}^{\Box_{af}}$-consistent Γ with the $\forall$-property such that $\neg\varphi \in \Gamma$. Let $\mathcal{M}^+_\Gamma$ be the model induced by such set. By Lemma 3.25, that means $\mathcal{M}^+_\Gamma, \Gamma \models_\xi \neg\varphi$, and since Γ is the actual world, $\not\models^{\mathrm{T}^{\Box_{af}}} \varphi$.

$[t]_\Gamma \in$ A, $w \in$ S$([t]_\Gamma, [s]_\Gamma)$ and $w \in$ S$([t]_\Gamma, [s']_\Gamma)$. By Definition 3.21 (f), we have $Sts \in w$ and $Sts' \in w$. By w's maximal T$^{\Box_q}$-consistency, from $\mathbf{S}_{uni}$, we get $s = s' \in w$. By Theorem 3.23, that means $s = s' \in \Gamma$, and therefore, by the definition of $[\cdot]_\Gamma$, $[s]_\Gamma = [s']_\Gamma$.

Therefore, $\mathcal{M}^+_\Gamma$ is a T$^{\Box_{af}}$-model, and we conclude our proof. □

4 Extensions of T$^{\Box_{af}}$ and the PGA

Now that we have the minimal system which models the assuming of values, we may study how to extend it in order to add features that further characterise not only afthairetic modality, but also other characteristics of A-objects, as described by Horsten. We prove a series of semantic characterisations of axioms expressing these features, which show the soundness and completeness of the extensions, conditional on their being consistent. The results are conditional for, with the exception of Theorem 4.5 – which presents a model for the axioms introduced in Subsection 4.2 –, we do not offer model for the extensions. We hope that does not affect extensively our contribution, and leave it open for future work.

The structure of the argument in showing completeness by the characterisation results follows the course usually presented in normal modal logics. When we say that a logic L characterises a class of models $\mathbb{C}$, we mean that any model for L belongs to $\mathbb{C}$ and that any model in $\mathbb{C}$ satisfies every theorem of L. Therefore, any theorem of L is valid over $\mathbb{C}$, and for any non-theorem φ of L there is a model for L $+ \neg\varphi$, which, by the characterisation, will be a model in $\mathbb{C}$. Therefore, the class $\mathbb{C}$ validates all and only the theorems of L – and why, if $\mathbb{C} \neq \varnothing$, that implies the soundness and completeness of the systems.

To make the presentation simpler, we define a few concepts.

Theorem 4.1 *$v(x) = y ::= Sxy$ is a permissible definition (the relation S is a function), and v may be adequately interpreted by* v : M$^+ \times$ W $\to$ M$^+$ *such that* v$(o, w) = o'$ *iff* $w \in$ S(o, o').

Proof The former claim follows from $\mathbf{S}_{uni}$. By the latter, we mean that $\mathcal{M}^+, w \models_\eta v(x) = y$ may be interpreted as v$(\eta(x), w) = \eta(y)$; or analogously, that $\mathcal{M}^+, w \models_\eta Sxy$ iff v$(\eta(x), w) = \eta(y)$. To see that, notice, by Definition 3.1 (e), *the o' such that* $w \in$ S(o, o') is indeed a definite description – that is, there is only one such o'. □

Intuitively, v is the function induced by S that gives the value of an object at a certain state. In fact, we could have as well defined the language and models for AM-extensions with v as a primitive and v as its interpretation, respectively, instead of with the binary predicate S and function S. We avoided doing that for, in that case, v would need to be not defined in the actual world for A-objects, and therefore any term $v(x)$ not occurring inside the scope of a modal operator would fail to have an interpretation, introducing yet more non-designating terms. That would have made the presentation of the current framework needlessly more complex. We trust the reader to see, had we followed that route, we would have obtained a system equivalent to the present one. From now on, we shall use the function v and the relation S interchangeably, depending on which makes the presentation at hand cleaner. The reason for that is we constantly talk about *the* value of some object or another, and we believe that to be better reflected by using a function term.

We extend our convention of abbreviating sequences here as well, so that $v(t_1), ..., v(t_n)$ may be denoted by $v(\vec{t})$, and, as long as we are referring to the values of a collection of objects at a certain state – for example $\mathrm{v}(a_1, w), ..., \mathrm{v}(a_n, w)$ –, we shall also write $\mathrm{v}(\vec{a}, w)$.

The *state space*, or *range*, associated to an A-object a in a model $\mathcal{M}^+$ may be defined as:

$$\mathrm{ran}(a) = \{o \in \mathrm{M}^+ \mid \exists w \in \mathrm{R}(w_0)(o = \mathrm{v}(a, w))\}$$

That is, $\mathrm{ran}(a)$ is the range of the A-object a, the collection of objects a may assume as a value. Though we may not use it, as a curiosity, membership of an object to the range of an A-object may be defined in $\mathcal{L}^{\Box_{af}}$ by:

$$x \in ran(y) ::= Ay \wedge \Diamond v(y) = x$$

The clause requires, for something to have a range, that it be an A-object. That is necessary for v, as mentioned in Section 2, is ambiguous, and when we talk about the range of an object, we are concerned with the sort of assuming of value A-objects perform.

Once the notation is set, we may proceed to the main purpose of the section, which is to extend the minimal system so that it deals with the most common problems faced by an ontology comprising A-objects – and, more specifically, trying to follow Horsten's view.

What is the successor of an arbitrary natural? What about the biological father of an arbitrary man? And the product of an arbitrary even number and 3? Are there such objects, and if so, what is their nature? Would admitting A-objects into our ontology not bring about the endless proliferation of ontological categories? These are a few of the questions Frege posed as criticism of the concept of A-object – or, in the specific context of his work, *indeterminate numbers*. So we find in his writing:

> [Emanuel Czuber] obviously distinguishes two classes of numbers: the determinate and the indeterminate. We may then ask, say, **(1)** to which of these classes the primes belong, or whether maybe some primes are determinate numbers and others indeterminate. [...] **(2)** How many indeterminate numbers are there? **(3)** How are they distinguished from one another? **(4)** Can you add two indeterminate numbers, and if so, how? How do you find the number that is to be regarded as their sum? **(5)** The same questions arise for adding a determinate number to an indeterminate one. To which class does the sum belong? Or maybe it belongs to a third?[22]

Unquestionably, those are problems a theory of A-objects must, at least, attempt to answer, and Horsten's indeed does so:

22 [6], p. 160. The enumerations are ours.

(1) The prime numbers are not arbitrary numbers: they are specific natural numbers. It may be that a concept of prime number can be generalised to the space of arbitrary natural numbers, just as it can be generalised to various algebraic structures, but that is another matter.
(2) [We have] argued that there are 2^ω arbitrary natural numbers. [We] shall see that if you hold a particular structuralist position about the natural number structure, then you will disagree with this argument and instead hold that there are 2^{2^ω} arbitrary natural numbers.
(3) I proposed an identity criterion for arbitrary objects according to which an arbitrary number a is identical with an arbitrary number b if and only if in each situation, a is in the same state as the state that b is in, or, in other words, if and only if a and b take the same specific value in each situation.
(4) A natural notion of sum can be defined for arbitrary numbers in a pointwise manner. If a and b are arbitrary numbers, then their sum $a + b$ is the arbitrary number such that in each state where a is the specific number m and b is the specific number n, $a + b$ is the specific number $n + m$.
(5) Given that a specific number can be seen as a limiting case of an arbitrary number, the sum of a specific natural number and a non-specific natural number is easily seen to be a non-specific arbitrary number.[23]

We shall address each of these points with the extensions. We start with point **(3)**, followed by **(4)** and **(5)**, and then **(2)**. At last, we show how those extensions come together, when using a simple (definable) convention for extending predicates from the original theories to their AM-extensions, to obtain a weak form of the PGA – which allows us to address point **(1)**.

4.1 Identity of A-objects

As in **(3)**, Horsten offers the following identity criterion for A-objects:

For any F, and any arbitrary F's a and b: $a = b$ iff, in every possible situation, the value taken by a is identical to the value taken by b.[24]

That may be formalized as:

Id$_A$ $(Ax \wedge Ay) \rightarrow (x = y \leftrightarrow \Box(v(x) = v(y)))$

That is, two A-objects are the same if they assume the same values on every state – consequently, they have the same range.

Theorem 4.2 $\mathcal{M}^+ \models \mathbf{Id}_A$ *iff for all* $a, b \in \mathrm{A}$, $a = b$ *iff for any* $w \in \mathrm{R}(w_0)$, $\mathrm{v}(a, w) = \mathrm{v}(b, w)$.

Proof sketch One direction of the equivalence in the property is trivial and the other is straightforward from the truth condition of $\mathbf{Id}_A$.

23 [7], p. 67.
24 [7], p. 45.

Theorem 4.2 shows $\mathbf{Id}_A$ adequately encodes Horsten's criterion of identity for A-objects, as the property of the class of frames in which it is valid reflects its intended interpretation.

4.2 Extending Functions

As we have expressed earlier, when going from a first-order theory to its AM-extension, we are extending the domain of quantification, but we are not necessarily extending its functions. AM-extensions in which the functions are interpreted just as in the ordinary first-order models are still $\mathrm{T}^{\Box_{af}}$-models. Therefore, to address issues **(4)** and **(5)** according to Horsten's answer, we add the following three axioms, for each n-ary function symbol f of $\mathcal{L}$:

$\mathbf{FI}_1$ $\forall\vec{x}\forall y(f(x_1, ..., x_n) = y \leftrightarrow \Box f(v(x_1), ..., v(x_n)) = v(y))$
$\mathbf{FI}_2$ $\forall\vec{x}\forall y(f(\vec{x}) = y \rightarrow (\neg Ay \leftrightarrow \bigwedge_{i\leq n} \neg Ax_i))$
$\mathbf{FI}_3$ $\forall\vec{x}(\Box\exists y(f(v(x_1), ..., v(x_n)) = y) \rightarrow \exists y(f(x_1, ..., x_n) = y))$

Theorem 4.3 *Axioms* $\mathbf{FI}_1$, $\mathbf{FI}_2$ *and* $\mathbf{FI}_3$ *define properties* (a), (b) *and* (c) *below, respectively.*

(a) $\mathrm{I}^+(f)(a_1, ..., a_n) = b$ *iff* $\forall w \in \mathrm{R}(w_0)$, $\mathrm{I}^+(f)(\mathrm{v}(a_1, w), ..., \mathrm{v}(a_n, w)) = \mathrm{v}(b, w)$;
(b) $\mathrm{I}^+(f)(o_1, ..., o_n) \in \mathrm{M}$ *iff* $o_1, ..., o_n \in \mathrm{M}$;
(c) *if* $\forall w \in \mathrm{R}(w_0)$, $\mathrm{I}^+(f)(\mathrm{v}(a_1, w), ..., \mathrm{v}(a_n, w))$ *is defined, then so is* $\mathrm{I}^+(f)(a_1, ..., a_n)$.

Proof The equivalence between the validity of $\mathbf{FI}_1$ and $\mathbf{FI}_3$, and properties (a) and (c), respectively, is straightforward. Let $\mathcal{M}^+$ have property (b), and suppose $\mathcal{M}^+, w_0 \models_\eta f(\vec{x}) = y$, so that $\mathrm{I}^+(f)(\eta(\vec{x})) = \eta(y)$, which means these are defined. If $\mathcal{M}^+, w_0 \models_\eta \neg Ay$, then $\eta(y) \in \mathrm{M}$, so by the property, each $\eta(x_i) \in \mathrm{M}$, and therefore $\mathcal{M}^+, w_0 \models_\eta \neg Ax_i$. If now $\mathcal{M}^+, w_0 \models_\eta \neg Ax_i$, then $\eta(x_i) \in \mathrm{M}$, so by the property, $\mathrm{I}^+(f)(\eta(\vec{x})) = \eta(y) \in \mathrm{M}$. Therefore, $\mathcal{M}^+, w_0 \models_\eta \neg Ay$. That gives us $\mathcal{M}^+, w_0 \models_\eta \neg Ay \leftrightarrow \bigwedge \neg Ax_i$, so by the arbitrariness of η, we get $\mathbf{FI}_2$. Let now $\mathcal{M}^+ \models \mathbf{FI}_2$, and $\mathrm{I}^+(f)(o_1, ..., o_n) \in \mathrm{M}$. Let $\eta(x_i) = o_i$ and $\eta(y) = \mathrm{I}^+(f)(o_1, ..., o_n)$, so that $\mathcal{M}^+, w_0 \models_\eta f(\vec{x}) = y$. By the validity of $\mathbf{FI}_2$, $\mathcal{M}^+, w_0 \models_\eta \neg Ay \leftrightarrow \bigwedge \neg Ax_i$, and thus $\mathrm{I}^+(f)(o_1, ..., o_n) \in \mathrm{M}$ iff $o_i \in \mathrm{M}$.[25] □

Theorem 4.4 *Let* $\mathcal{M}^+ \models \mathbf{FI}_2$. *If* $\mathrm{I}(f)$ *is a partial function, then so is* $\mathrm{I}^+(f)$.

Proof Suppose $\mathrm{I}(f)(o_1, ..., o_n)$ is not defined, but $\mathrm{I}^+(f)(o_1, ..., o_n)$ is. Then, by the validity of $\mathbf{FI}_2$, since $o_i \in \mathrm{M}$, $\mathrm{I}^+(f)(o_1, ..., o_n) \in \mathrm{M}$. But that cannot be since, by Definition 3.1, $\mathrm{I}^+ \restriction_{\mathrm{M}} (f) = \mathrm{I}(f)$. □

Therefore, each axiom plays a different role in extending the functions. Axiom $\mathbf{FI}_1$ ensures that, if a function is extended, then it is extended in the appropriate way for

25 Notice throughout the argument we make use of the fact that, for any term t and assignment η, if $\eta(t) \notin \mathrm{A}$ and $\eta(t)$ is defined, then by Definition 3.1 (d), $\eta(t) \in \mathrm{M}$.

A-objects: for example, if a is an arbitrary natural, $a + 2$, if defined, will be an object – not necessarily arbitrary – which assumes as its value the value a assumes plus 2; and if b is another arbitrary natural, then $a + b$, if defined, is an object which always assumes as its value the sum of the values a and b assume. Axiom $\mathbf{FI}_2$ ensures two things: that these objects which are outputs, in these new cases, are indeed A-objects, and that if $\mathrm{I}(f)$ is a partial function, then the tuples of objects of the original theory for which the function are not defined remain so. Axiom $\mathbf{FI}_3$ guarantees objects, like the ones we gave as example of outputs of the function when applied to the extended domain, indeed exist. In other words, axiom $\mathbf{FI}_1$ describes the modal profile the output of the function should possess, axiom $\mathbf{FI}_3$ ensures there is an object with such modal profile, and axiom $\mathbf{FI}_2$ guarantees that object is arbitrary. To see how the last step is necessary, consider two arbitrary numbers a and b such that $\mathrm{ran}(a) = \mathrm{ran}(b) = \{1, 2\}$, and the value b assumes in a state is always that of the value a assumes minus 3. Then, without axiom $\mathbf{FI}_2$, it may be that $a + b = 3$ – as opposed to the product of the sum being an arbitrary number which always assume the value 3 –, which goes against point **(4)**.

For the next result, we make the following observation.

Theorem 4.5 *Let $\mathcal{M}^+$ be a $\mathrm{T}^{\Box_{af}}$-model. Then, for any transfinite cardinal κ, there is a model for $\mathrm{T}^{\Box_{af}} + \{\mathbf{FI}_1, \mathbf{FI}_2, \mathbf{FI}_3\}$ with κ A-objects $\{a_i\}$ such that for some $b \in \mathrm{ran}(a_i)$ $a_i \in \mathrm{ran}(b)$ (that is, the cardinality of the set defined by the condition $\exists y(\Diamond v(x) = y \wedge \Diamond v(y) = x)$ is κ).*

Proof Let $\mathcal{M}^+ = \langle \mathrm{M}^+, \mathrm{I}^+, \mathrm{W}, w_0, \mathrm{R}, \mathrm{A}, \mathrm{S}\rangle$. We show how to construct such model. Let $\mathcal{M}^+_u = \langle \mathrm{M}^+_u, \mathrm{I}^+_u, \mathrm{W}, w_0, \mathrm{R}, \mathrm{A}_u, \mathrm{S}_u\rangle$, such that $\mathrm{M}^+_u = \mathrm{M}^+ \cup (\kappa \times 2)$, $\mathrm{A}_u = \mathrm{A} \cup (\kappa \times 2)$, S_u be in accordance with, for each $\beta \in \kappa$, $\mathrm{ran}(\langle \beta, i\rangle) = \{\langle \beta, |i - 1|\rangle\}$, and I^+_u differ from I^+ only with respect to the interpretation of functions. It is clear neither of the new objects falsify the axioms of $\mathrm{T}^{\Box_{af}}$ – by a quick observation of the proper axioms of $\mathrm{T}^{\Box_{af}}$, and by noticing they are A-objects, and as such consistent with any axiom of the first-order theory T, for the nature of the translation α_{T}. Hence, we show they are consistent with the new axioms. For that, we need to define how the functions work with the new objects. Define $\mathrm{I}^+(f)(o_1, ..., \langle \beta_1, i_1\rangle ..., \langle \beta_k, i_k\rangle, ..., o_n) = \langle \beta_j, i_j\rangle$, for $\beta_j = min\{\beta_l\}_{l \le k}$. We check for the satisfiability of each property described in Theorem 4.3. Clearly, (b) is still satisfied, and so is (c), for the function is defined whenever it takes as input the new objects. For (a), consider $w \in \mathrm{R}(w_0)$. Then,

$$\begin{gathered}\mathrm{I}^+(f)(\mathrm{v}(o_1, w), ..., \mathrm{v}(\langle \beta_1, i_1\rangle, w), ..., \mathrm{v}(\langle \beta_k, i_k\rangle, w), ..., \mathrm{v}(o_n, w)) = \\ \mathrm{I}^+(f)(\mathrm{v}(o_1, w), ..., \langle \beta_1, |1 - i_1|\rangle, ..., \langle \beta_k, |1 - i_k|\rangle, ..., \mathrm{v}(o_n, w)), \\ \text{which by definition, is } \langle \beta_j, |i_j - 1|\rangle \text{ for } \beta_j = min\{\beta_l\}_{l \le k}.\end{gathered}$$

By the definition of our model, $\langle \beta_j, |i_j - 1|\rangle = \mathrm{v}(\langle \beta_j, i_j\rangle, w)$. □

The above theorem shows $\mathrm{T}^{\Box_{af}}$ is insensitive to *ungrounded* A-objects: objects whose existence is not related to the objects of the original theory T. That makes it possible for models with any cardinality of ungrounded objects to equally be $\mathrm{T}^{\Box_{af}}$-models. The existence of such objects is problematic. First, we have no justification for their existence: once again they are (arguably, metaphysically) ungrounded, or in a sense, grounded by themselves – they are essentially *unfounded* with respect to the relation S. Furthermore, in the context of each AM-extended theory, there is no adequate answer to of what they are an A-object. Take for example PA as the original theory. Then, the above result shows there may be non-standard models for its AM-extension, $\mathrm{PA}^{\Box_{af}}$, possessing a transfinite myriad of A-objects which are in not a single way related to natural numbers.

Let us write $Gr_{\mathcal{M}^+}$ for the set of *grounded* A-objects of $\mathcal{M}^+$. We may define it inductively: let $G_0 = \{a \in \mathrm{A} \mid \forall o \in \mathrm{M}^+(o \in \mathrm{ran}(a) \rightarrow o \in \mathrm{M})\}$, $G_{n+1} = G_n \cup \{a \in \mathrm{A} \mid \forall o \in \mathrm{M}^+(o \in \mathrm{ran}(a) \rightarrow o \in G_n \cup \mathrm{M})\}$, and $Gr_{\mathcal{M}^+} = \bigcup_{i<\omega} G_i$. We may then define the set of *ungrounded* A-objects as its complement with respect to A, $Gr^{\mathrm{C}}_{\mathcal{M}^+}$. We shall, from now on, present a series of results we get from the domain without ungrounded A-objects, $\mathrm{M}^+ \setminus Gr^{\mathrm{C}}_{\mathcal{M}^+} = \mathrm{M} \cup Gr_{\mathcal{M}^+}$, so for a cleaner presentation let us name this set M^+_{Gr}.

For the following results, we note the elements of M^+_{Gr} present a natural hierarchy. This hierarchy may be made explicit by defining *ranks of arbitrariness*: for $o \in \mathrm{M}^+_{Gr}$, let $\mathrm{r}_A(o) = 0$ if $o \in \mathrm{M}$, and otherwise, $\mathrm{r}_A(o) = max\{\mathrm{r}_A(a) \mid a \in \mathrm{ran}(o)\} + 1$. We may see, by this definition, given the first iteration of the construction of $Gr_{\mathcal{M}^+}$ in which it appears, the rank of an A-object coincides with the previous level.

Theorem 4.6 *Let $\mathcal{M}^+ \models \mathbf{FI}_1$. Then, if $\mathrm{I}(f)$ is a commutative function, $\mathrm{I}^+(f) \upharpoonright_{\mathrm{M}^+_{Gr}}$ is a commutative function; if $\mathbf{FI}_3$ is also valid and $\mathrm{I}(f)$ is a total function, then $\mathrm{I}^+(f) \upharpoonright_{\mathrm{M}^+_{Gr}}$ is also total; and if $\mathbf{Id}_A$ is also valid and $\mathrm{I}(f)$ is an injective function, then $\mathrm{I}^+(f) \upharpoonright_{\mathrm{M}^+_{Gr}}$ is also injective.*

Theorem 4.6 shows that, in any model for the three axioms with no ungrounded objects, some properties of functions may also be extended. As we shall see, that is due to the possibility of inducing *from the ground up*, on the ranks of M^+_{Gr}. It is the hierarchical nature of the grounded A-objects that allows well behaved extensions, and so there is a further reason for rejecting ungrounded A-objects. In the next subsection, we shall see a possible natural way out of this issue.

4.3 A Note on the Hierarchy of A-objects

Before we continue, let us dig briefly into the naturally rising hierarchy just presented – and more importantly, see what is Horsten's view on the matter:

> A second-order arbitrary number is an entity that can be in a state of being different arbitrary natural numbers, and so on. This gives rise to the following definition of a *hierarchy* of higher-order arbitrary natural numbers:
>
> (1) $\mathrm{A}_0 = \mathbb{N}$;
> (2) $\mathrm{A}_{\beta+1} = \mathrm{A}_\beta^{\mathrm{A}_\beta}$;
> (3) $\mathrm{A}_\lambda = \bigcup_{\beta<\lambda} \mathrm{A}_\beta$ for λ a limit ordinal.[26]

[26] [7], p. 64.

Thus, just as we have defined in the last subsection, the members of M, the original domain, may be seen as “0-th level" objects (those of rank 0), while the A-objects whose range is a subset of it, as “first level" A-objects (those of rank 1).

Define new predicates A_n recursively such that:

- $A_0x ::= \neg Ax$;
- $A_{n+1}x ::= \exists y(\Diamond v(x) = y \wedge A_n y) \wedge \forall y(\Diamond v(x) = y \rightarrow \bigvee_{i \leq n} A_i y)$.

Here, A_0x expresses that x is a specific object. Notice the first conjunct in the definition $A_{n+1}x$ guarantees the rank of x is at least $n+1$, while the second, that its rank is at most $n+1$. Notice, also, our definition is different from what Horsten describes. In the above passage, each level of the hierarchy is formed by the A-objects that may assume as value the objects of the level immediately below. Our definition, however, allows for A-objects of a level to assume as value objects from any of the lower levels – so that our formulation is a little more general. That is no problem, however, as Horsten himself considers such an elaboration.[27]

Lemma 4.7 $\mathcal{M}^+, w_0 \models_\eta A_n x$ *iff* $\mathrm{r}_A(\eta(x)) = n$.

Proof By induction on n, let $\mathcal{M}^+, w_0 \models_\eta A_0 x$. Then, $\eta(x) \notin \mathrm{A}$, $\mathrm{r}_A(\eta(x)) = 0$. Let $n = m+1$. If $\mathcal{M}^+, w_0 \models_\eta \exists y(\Diamond v(x) = y \wedge A_m y) \wedge \forall y(\Diamond v(x) = y \rightarrow \bigvee_{i \leq m} A_i y)$, then, by induction hypothesis, from the first conjunct, for some $a \in \mathrm{ran}(\eta(x))$, $\mathrm{r}_A(a) = m$, and from the second conjunct, for any $a \in \mathrm{ran}(\eta(x))$, $\mathrm{r}_A(a) \leq m$. Therefore, $\mathrm{r}_A(\eta(x)) = m+1$. □

Let $a \in \mathrm{A}$ in some $\mathrm{T}^{\Box_{af}}$-model. Then, $A_n x$ is true of a only if any sequence $o_1, ..., o_k$ such that $o_i \in \mathrm{ran}(o_{i+1})$ and $o_k \in \mathrm{ran}(a)$ is at most of length n, and $o_1 \in \mathrm{M}$. However, with such sort of definition, we may see we cannot define a predicate for A-objects of the level of a limit ordinal λ, since that would require either a second-order quantification on these new predicates – so that $A_\lambda x ::= Ax \wedge \forall \beta < \lambda \exists \alpha < \lambda(\beta < \alpha \wedge \exists y(\Diamond v(x) = y \wedge A_\alpha y))$ –, or an infinitary disjunction – so that $A_\lambda x ::= Ax \wedge \forall y(\Diamond v(x) = y \rightarrow \bigvee_{\beta < \lambda} A_\beta y)$. We may always trivially collect the A-objects of all levels of the hierarchy with the original predicate A, but at the cost of possibly having ungrounded A-objects.

Consider then the following axiom scheme:

$\mathbf{AH}_n$ $\forall x(\bigvee_{i \leq n} A_i x)$.

Then, we have the following result:

Theorem 4.8 $\mathcal{M}^+ \models \mathbf{AH}_n$ *iff* $\mathrm{M}^+_{Gr} = \mathrm{M}^+$ *and* $max\{\mathrm{r}_A(a)\}_{a \in \mathrm{M}^+} \leq n$.

Proof Let $\mathcal{M}^+ \models \mathbf{AH}_n$ and $\eta(x) \in \mathrm{A}$. Then, $\mathcal{M}^+, w_0 \models_\eta A_m x$ for some $m \leq n$. By Lemma 4.7, $\mathrm{r}_A(\eta(x)) = m$. By the arbitrariness of η, that means every object has a defined rank – so that $\mathrm{M}^+_{Gr} = \mathrm{M}^+$ – of, at most, n. Let now the two properties hold. By the properties, $\mathrm{r}_A(\eta(x)) \leq n$ is defined, so by Lemma 4.7, $\mathcal{M}^+, w_0 \models_\eta A_m x$ for some $m \leq n$. Thus, $\mathbf{AH}_n$ holds. □

27 [7], pp. 120–121.

Therefore, finite fragments of the hierarchy may be reflected by the validity of specific axioms in the framework. Furthermore, for any $n < \omega$, we may rule out ungrounded A-objects, at the cost of also ruling out A-objects of rank greater than n. We leave the investigation on how to rule out ungrounded A-objects without this trade off for future work. Still, as we shall briefly see, these predicates reflecting the hierarchy will aid us in the presentation of new axioms and results.

4.4 Arbitrary Comprehension

Concerning the abundance of A-objects, we can divide the investigation in two fronts: we may wonder how many sorts of A-objects there are, and how many A-objects of the same sort there are. That is, we may question what φ's there are that talking about an arbitrary φ makes sense; and given some φ, we may also question how many arbitrary φ's there are. This subsection intends to address the first question, and thus only partially covers **(2)**.[28] In that regard, Horsten expresses the following view:

> For any condition ϕ that holds for every element of a non-empty set A of specific objects of kind K and only of those objects, there is an arbitrary object a that can be in the state of being any element of A and can be in no other state.[29]

This stratified comprehension scheme can be now formalized, by using the ranks of arbitrariness, as follows:

Comp$_A$ $\exists x(A_n x \wedge \varphi(x)) \rightarrow \exists y(Ay \wedge \forall x(\Diamond v(y) = x \leftrightarrow (\bigvee_{i \leq n} A_i x \wedge \varphi(x))))$, for any syntactic condition $\varphi(x)$ and $n < \omega$.

Here, by a *syntactic condition* (which in the absence of ambiguity, we may call simply *condition*), we understand a formula of the language with a single free variable. According to **Comp**$_A$, for any non-empty condition on objects of rank n, there is an A-object which can assume the value of any object, of at most rank n, satisfying that condition, and no other. Notice that implies the rank of the new A-object is $n + 1$.

The formulation of **Comp**$_A$ does not provide for the existence of both an *empty* and a *universal* A-object – that is, one whose range is empty, and another whose range is the whole domain, respectively. Those objects are left out for two reasons. The first is conceptual: as stated in the second section, A-objects are objects which assume values, so trivially, there must be objects in its range. Furthermore, they cannot assume themselves as values, and therefore cannot have the entire domain as their range. The second reason is more formal: as $\mathbf{S}_{pot}$ is an axiom of the system, it would require of the empty A-object that it possibly assumes a value, which might lead to a contradiction. Similarly, as **NSA** is an axiom, the general A-object being in its own range would be contradictory. Still, **Comp**$_A$ implies the existence of a universal *specific* A-object – an A-object whose range is the domain of the original theory – by having $\varphi(x) ::= \neg Ax$ and $n = 0$.

Notice **Comp**$_A$ works as a sort of separation scheme, to avoid Russell-like paradoxes. To see how a paradox may arise from an unrestricted comprehension scheme

[28] We intend to cover the second question in future work.

[29] [7], p. 46.

– like that obtained by just the consequent of the conditional –, consider the A-object a which ranges over the class of objects which cannot assume themselves as a value. Then, a may assume itself as a value iff it does not assume itself as a value. A contradiction. However, in the above formulation, the requirement that the values assumed by the new A-object are of a lesser rank guarantees that condition does not fulfil the necessary requirement to be encompassed by the comprehension scheme.

Definition 4.9 Let $\mathcal{M}^+$ be a $\mathrm{T}^{\Box_{af}}$-model and $X \subseteq \mathrm{M}^+$. We say X is *definable in* $\mathcal{M}^+$ if there is a condition $\varphi(x)$ such that $o \in X$ iff $\mathcal{M}, w_0 \models_\eta \varphi(x)[o/x]$.

The next theorem shows the class of models that allow this comprehension scheme is characterisable. Let us denote the set of objects of rank at most n – that is, $\mathrm{M} \cup \bigcup_{i \leq n} G_i$ – by $G_{i \leq n}$.

Theorem 4.10 $\mathcal{M}^+ \models \mathbf{Comp}_A$ *iff it presents the following property:*

> *For each $n < \omega$, for all definable $X \subset \mathrm{M}^+$ in $\mathcal{M}^+$, if $X \cap G_{i \leq n} \neq \varnothing$ there is $a_X \in G_{n+1}$ such that: for each $o \in X \cap G_{i \leq n}$, there is $w \in \mathrm{R}(w_0)$ such that $\mathrm{v}(a_X, w) = o$; and for any $w \in \mathrm{R}(w_0)$ and $o \in \mathrm{M}^+$, if $\mathrm{v}(a_X, w) = o$, then $o \in X \cap G_{i \leq n}$.*

Proof To see $\mathbf{Comp}_A$ implies the property, let $\mathcal{M}^+ \models \mathbf{Comp}_A$, $X \cap G_{i \leq n} \neq \varnothing$, and let X be defined by $\varphi(x)$. Then,

$$\mathcal{M}^+, w_0 \models_\eta \exists x(A_n x \wedge \varphi(x)) \rightarrow \exists y(Ay \wedge \forall x(\Diamond v(y) = x \leftrightarrow (\bigvee_{i \leq n} A_i x \wedge \varphi(x)))).$$

Clearly, by the non-emptiness of that set, the antecedent is true. Therefore,

$$\mathcal{M}^+, w_0 \models_\eta \exists y(Ay \wedge \forall x(\Diamond v(y) = x \leftrightarrow (\bigvee_{i \leq n} A_i x \wedge \varphi(x)))), \text{ so that}$$
$$\text{for some } \eta' \sim_y \eta, \mathcal{M}^+, w_0 \models_{\eta'} Ay \wedge \forall x(\Diamond v(y) = x \leftrightarrow (\bigvee_{i \leq n} A_i x \wedge \varphi(x))).$$

We get $\eta'(y) \in \mathrm{A}$. Let $o \in X \cap G_{i \leq n}$ and $\eta'' \sim_x \eta'$ be such that $\eta''(x) = o$. Then, we get (i) $\mathcal{M}^+, w_0 \models_{\eta''} \Diamond v(y) = x \leftrightarrow (\bigvee_{i \leq n} A_i x \wedge \varphi(x))$. Since $\eta''(x) \in X \cap G_{i \leq n}$ – which means $\mathcal{M}^+, w_0 \models_{\eta''} \bigvee_{i \leq n} A_i x \wedge \varphi(x)$ –, we get $\mathcal{M}^+, w_0 \models_{\eta''} \Diamond v(y) = x$, and thus there is $w \in \mathrm{R}(w_0)$ such that $\mathrm{v}(\eta''(y), w) = o$. Let now $w \in \mathrm{R}(w_0)$, and $\mathrm{v}(\eta''(y), w) = o$. Since η'' is any x-variant of η', let it be such that $\eta''(x) = o$. Then, $\mathcal{M}^+, w_0 \models_{\eta''} \Diamond v(y) = x$, so that, by (i), $\mathcal{M}^+, w_0 \models_{\eta''} \bigvee_{i \leq n} A_i x \wedge \varphi(x)$, which means $\eta''(x) = o \in X \cap G_{i \leq n}$. Hence, $\eta''(y) = \eta'(y)$ is the object in A making the property true for X.

To see now the property implies $\mathbf{Comp}_A$, suppose it holds, and that $\mathcal{M}^+, w_0 \models_\eta \exists x(A_n x \wedge \varphi(x))$. Let (ii) X be the set defined by $\varphi(x)$. Then, clearly, $X \cap G_{i \leq n} \neq \varnothing$. By the property, there is $a \in \mathrm{A}$ such that (iii) if it assumes the value o in some $w \in \mathrm{R}(w_0)$ then $o \in X \cap G_{i \leq n}$, and (iv) for any $o \in X \cap G_{i \leq n}$ there is w such that a assumes the value o. Let then $\eta' \sim_y \eta$ such that $\eta'(y) = a$, so that (v) $\mathcal{M}^+, w_0 \models_{\eta'} Ay$. Let then $\eta'' \sim_x \eta$. If $\eta''(x) \in X \cap G_{i \leq n}$, then by (ii) $\mathcal{M}^+, w_0 \models_{\eta''} \bigvee_{i \leq n} A_i x \wedge \varphi(x)$. By (iv), there is $w \in \mathrm{R}(w_0)$ such that $\mathrm{v}(\eta''(y), w) = \eta''(x)$, and thus

$\mathcal{M}^+, w_0 \models_{\eta''} \Diamond v(y) = x$. If $\eta''(x) \notin X \cap G_{i \leq n}$, then either $\mathcal{M}^+, w_0 \not\models_{\eta''} \varphi(x)$ or $\mathcal{M}^+, w_0 \not\models_{\eta''} \bigvee_{i \leq n} A_i x$. By (iii), there is no $w \in \mathrm{R}(w_0)$ such that $\mathrm{v}(\eta''(y), w) = \eta''(x)$, so $\mathcal{M}^+, w_0 \not\models_{\eta''} \Diamond v(y) = x$. By the nature of η'' and η', and with (v) the desired conclusion easily follows. □

For a condition φ, let us call an A-object a, in a model $\mathcal{M}^+$, an *arbitrary* φ if it assumes each object satisfying φ in $\mathcal{M}^+$ as a value in some state.

Theorem 4.11 *Let P be a unary predicate, $\mathcal{M}^+$ be a model for* $\mathrm{T}^{\Box_{af}} + \mathbf{Comp}_A$, *and* $a_P \in \mathrm{A}$ *be an arbitrary P. Then,* $a_P \notin \mathrm{I}^+(P, w_0)$.

Proof If $a_P \in \mathrm{I}^+(P, w_0)$, then by Theorem 4.10 there would be $w \in \mathrm{R}(w_0)$ such that $\mathrm{v}(a_P, w) = a_P$, which cannot be the case by Definition 3.1 (e). □

Theorem 4.11 shows a seeming incompatibility between Horsten's theory of A-objects and Kit Fine's [3]. According to the latter, A-objects of a sort P are objects which present all the properties (although within some restrictions) common to all individual P's. This property is best described by the previously mentioned PGA:

> Let $\varphi(x)$ be any condition with free variable x; let a be the name of an arbitrary object a; and let i be a variable that ranges over the individuals in the range of a. (We here follow a general convention whereby a names a.) Then the required formulation of the principle is:
>
> $\varphi(x)(a) \Leftrightarrow \forall i \varphi(i)$ (a φ's iff every individual [in its range] φ's)[30]

Therefore, according to the PGA, an arbitrary P should present P trivially. Theorem 4.11 goes against that. It says an arbitrary P *does not* present, precisely, P. That, however, is in no way a problem for Horsten's account:

> The fundamental question is: is the arbitrary number b a natural number? I say that it is not, and this is (in my view) precisely where Berkeley's argument goes wrong. *The* natural numbers are the *specific* objects 0, 1, 2, 3, ... The domain of arbitrary natural numbers will be seen to, in a sense, contain the natural numbers as limiting cases. The domain of arbitrary numbers can therefore be seen as an extension of the natural numbers. So b is not a natural number. To believe that it is, is to be deceived by language.[31]

So, an arbitrary P may indeed not be a P – not in the same sense an individual P is. We shall see, in the next subsection, how that may be incorporated into the system.

At this point, we note Horsten's work does not intend to focus on this specific aspect of A-objects, nor to engage with the PGA, so that we do not wish to convey him as taking any particular instance on the issue. We do find, however, value in highlighting commitments to the nature of A-objects, and the way they work.

An interesting, but intuitively obvious, consequence of $\mathbf{Comp}_A$ is the following theorem:

[30] [4], p. 59.

[31] [7], p. 48.

Theorem 4.12 *If* $\mathcal{M}^+ \models \textbf{Comp}_A$*, then* $|B_\varphi^{max}| \leq |\mathrm{W}|$*, where* B_φ^{max} *is the largest definable subset of* M^+ *in* $\mathcal{M}^+$.

Proof sketch Just notice $\textbf{Comp}_A$ implies the existence of an arbitrary B_φ^{max}, and therefore there must be at least $|B_\varphi^{max}|$ states in W for it to assume each of its possible values.

Therefore, both the language and the domain of quantification play a role in defining the size of the set of states, which is at least in part in accordance to Horsten's requirement that the state space associated to an arbitrary F is as big as the cardinality of the class of F's. In fact, Horsten argues, depending on the nature of the range of an A-object, there should be *exactly* as many states as the cardinality of its range. Thus, for any A-object whose range is of a cardinality lesser than $|B_\varphi^{max}|$, there will actually be more states than required.[32] If one is to closely follow Horsten's approach, that would be a problem when dealing with *degrees of arbitrariness* of A-objects – which is related to the amount of states in which each different object in the range of an A-object is assumed as a value.[33] The topic, however, is outside the scope of this paper.

Another consequence of $\textbf{Comp}_A$ is that its validity implies the existence of *constant* A-objects, for each constant c – by setting $\varphi(x) ::= x = c$. The further validity of $\textbf{Id}_A$ implies each of these constant A-objects is unique, and therefore nameable. Extending the language with names for each of these new objects allows us to express interesting properties of the extended models.

Define a new constant $c^\uparrow$ by setting, for any $\varphi \in Form_{\mathcal{L}^{\Box_{af}}}$:

$$\varphi(c^\uparrow) ::= Ax \wedge \varphi(x) \wedge \Box(v(x) = c)$$

Having that, we may once again define another new constant, $c^{\uparrow\uparrow}$, which always takes the value of $c^\uparrow$, by setting

$$\varphi(c^{\uparrow\uparrow}) ::= Ax \wedge \varphi(x) \wedge \Box(v(x) = c^\uparrow)$$

Notice, as a consequence, an arbitrary constant, which takes as values the constant of rank n, is of level $n + 1$. That goes on indefinitely – but this is not an issue. By the previous observation, it is clear in the given context there is exactly one object which satisfies any such condition, for each constant. We may call each of these arbitrary constants $c^{n\uparrow}$, referring to its rank – so that the 0th rank is attributed to the original constant c. It is easy to see the interpretation of any model for those axioms may be extend to interpret these new constants, such that $\mathrm{I}^+(c^{n\uparrow})$ is that unique $a \in \mathrm{A}$ such that for any $w \in \mathrm{R}(w_0)$, $\mathrm{v}(a, w) = \mathrm{I}^+(c^{n-1\uparrow})$. Furthermore, notice $\mathrm{I}^+(c^{n\uparrow}) \in \mathrm{M}^+_{Gr}$. Therefore, from now on, we may use $c^{n\uparrow}$ as proper names of such objects, and work as if we have the extended language, and as if any model for $\textbf{Comp}_A$ and $\textbf{Id}_A$ is adequately equipped with an interpretation of $c^{n\uparrow}$.

32 [7], pp. 53–55.

33 [7], pp. 63–64, 178–181.

Now, define a translation on terms, $(\cdot)^{\uparrow}$, as:

$$\begin{array}{lcl}(y)^{\uparrow} & = & y\\ (c^{n\uparrow})^{\uparrow} & = & c^{n+1\uparrow}\\ f(t_1,...,t_n) & = & f(t_1^{\uparrow},...,t_n^{\uparrow})\end{array}$$

That is, $(\cdot)^{\uparrow}$ only replaces each occurrence of a constant with another of a higher rank. Just as $c^{n+1\uparrow}$ names the constant A-object which takes the value of $c^{n\uparrow}$ at every state, we write $t^{n\uparrow}$ for t translated n times, and $\varphi^{n\uparrow}$ for a formula φ with all of its terms translated n times – when we need to make that explicit.

Theorem 4.13 *Let* $\mathcal{M}^+ \models \mathbf{Id}_A, \mathbf{Comp}_A, \mathbf{FI}_1, \mathbf{FI}_2, \mathbf{FI}_3$, *and* $\varphi \in \mathcal{L} \setminus \{P_i\}$ *– that is, the fragment of* $\mathcal{L}$ *with only, in addition to the logical symbols, function symbols and constants. Let* $\{y_i\}$ *be the free variables of* φ*. Then,*

$$\textit{if}\ \forall w \in \mathrm{R}(w_0),\ \mathrm{v}(\eta(x_i), w) = \eta(y_i)\ \textit{implies}\ \mathcal{M}^+ \upharpoonright_{\mathrm{M}^+_{Gr}}, w_0 \models_\eta \varphi,\ \textit{then}$$
$$\mathcal{M}^+ \upharpoonright_{\mathrm{M}^+_{Gr}}, w_0 \models_\eta \varphi^{\uparrow \vec{x}}_{\vec{y}}.$$

For the next result, for each of the predicates A_n previously defined, define:

$$A^s_n x ::= A_n x \wedge \forall y(\Diamond v(x) = y \rightarrow A_{n-1} y)$$

What $A^s_n x$ means is that if an object satisfies it, then not only is it of rank n, but all of the objects in its range are of the rank immediately bellow, $n-1$.

Theorem 4.14 *Let* $\mathcal{M}^+ \models \mathbf{Id}_A, \mathbf{Comp}_A, \mathbf{FI}_1, \mathbf{FI}_2, \mathbf{FI}_3$*. Let also* $\Sigma\varphi(\vec{x}, y) \in \mathcal{L} \setminus \{P_i\}$ *be a condition in prenex normal form, where* Σ *is the prefix and* $\varphi(\vec{x}, y)$ *the matrix. Then, for* $n \geq 1$*,*

$$\textit{if} \models^{\mathrm{T}} \Sigma\varphi(\vec{x}, c),\ \textit{then}\ \mathcal{M}^+ \upharpoonright_{\mathrm{M}^+_{Gr}} \models \Sigma(\textstyle\bigwedge_{i \leq m} A^s_n x_i \rightarrow \varphi^{n\uparrow}(\vec{x}, c^{n\uparrow})).$$

Theorems 4.13 and 4.14 extend the result of Theorem 4.6. For example, if in T there is an inverse for every element with respect to f, then, in $\mathrm{T}^{\Box_{af}}$ with the above axioms, any element of a rank has an inverse of the same rank. Likewise, if there is a neutral element of a certain operation, then in the extended theory there should be a neutral element of that operation for each rank. Thus, a wide variety of properties of functions and constants of the original theory are present in the AM-extended theories.

4.5 Relation Inheritance (or a weak PGA)

As suggested at the beginning of this section, to address **(1)**, we shall define new predicates. Just like we have defined a new constant for each rank – or level of the hierarchy –, so we define new predicates for each rank, counterparts of the original predicates. Thus, for each n-ary predicate P_i, we find new n-ary predicates P_i^{k*}, which we may call *generic predicates* (of k-th level). Intuitively, P^{k*} is supposed to be the (more) generic counterpart to the predicate P^{k-1*}.

Øystein Linnebo offers, in [10], in the context of the abstractionist (or neo-Fregean) program, a proposal for how abstracta introduced by abstraction principles inherit relations from the objects in the classes from which they are generated. Borrowing from his suggestion, we may inductively define:

- $P^{*}\vec{x} ::= \bigwedge_{i\leq m} Ax_i \wedge \Box Pv(x_1)...v(x_m)$;
- $P^{n+1*}\vec{x} ::= \Box P^{n*}v(x_1)...v(x_m)$.

By defining generic predicates in this way, we keep the framework more compatible to the view according to which saying, for example, an arbitrary natural number is even is mere *façon de parler*.

An obvious consequence of this sort of definition is that, for example, if we say an arbitrary natural number is even, then it must clearly be even in a different manner than that of a specific number – accordingly to what we have previously expressed.

Theorem 4.15 *$\mathcal{M}^+, w_0 \models_\eta P^{n*}\vec{x}$ iff $\mathrm{r}_A(\eta(x_i)) = n$ and for any $w \in \mathrm{R}(w_0)$, if $\mathrm{v}(\eta(x_i), w) = \eta(y_i)$, then $\mathcal{M}^+, w_0 \models_\eta P^{n-1*}\vec{y}$.*

Proof sketch By induction on n, the result is easily obtained by the definition of the new predicate.

Corollary 4.16 *If $\mathcal{M}^+, w_0 \models_\eta P^{n*}\vec{x}$, then for any $b \in \mathrm{ran}(\eta(x_i))$, $\mathrm{r}_A(b) = n - 1$.*

The new predicates let us abstract upon the predicates of the original theory. For example, an arbitrary object is *prime** if all the objects in its range are prime numbers; it is *prime*** if the objects in its range are arbitrary primes, and so on. Therefore, by extending the relevant notions, issue **(1)** can adequately be dealt with.

Now, as we intended, we may move towards obtaining a weak form of PGA. To do that, we need to slightly expand our theory. Let

$$\mathrm{T}^{\Box_q+} = \mathrm{T}^{\Box_q} + \{\mathbf{Id}_A, \mathbf{Comp}_A, \mathbf{FI}_1, \mathbf{FI}_2, \mathbf{FI}_3\},$$

and $Nec_{\mathrm{T}^{\Box_q+}}$ be the inference rule which allows us to put a $\Box$ not on every formula derivable in $\mathrm{T}^{\Box_q}$, but on every formula derivable in $\mathrm{T}^{\Box_q+}$. Define then $\mathrm{T}^{\Box_{af}+}$ as $\mathrm{T}^{\Box_{af}}$ with all the new axioms, but with $Nec_{\mathrm{T}^{\Box_q+}}$ instead – that is,

$$\mathrm{T}^{\Box_{af}+} = (\mathrm{T}^{\Box_{af}} - Nec_{\mathrm{T}^{\Box_q}}) + \{\mathbf{Id}_A, \mathbf{Comp}_A, \mathbf{FI}_1, \mathbf{FI}_2, \mathbf{FI}_3\} + Nec_{\mathrm{T}^{\Box_q+}}.$$

For the next theorem, define a translation $(\cdot)^*$ on formulas which replaces each constant and predicate with one of a higher level:

$$
\begin{array}{lcl}
(t_1 = t_2)^* & = & t_1^{\uparrow} = t_2^{\uparrow} \\
(P^{n*}t_1...t_n)^* & = & P^{n+1*}t_1^{\uparrow}...t_n^{\uparrow} \\
(\neg\varphi)^* & = & \neg\varphi^* \\
(\varphi \wedge \psi)^* & = & \varphi^* \wedge \psi^* \\
(\forall x\varphi)^* & = & \forall x\varphi^* \\
(\Box\varphi)^* & = & \Box\varphi^*
\end{array}
$$

Theorem 4.17 *Let $\mathcal{M}^+ \models \mathbf{Id}_A, \mathbf{Comp}_A, \mathbf{FI}_1, \mathbf{FI}_2, \mathbf{FI}_3$. Then, for any $\varphi \in \mathcal{L}^{\Box_{af}} \setminus \{S, A\}$ such that $\neg$ only occurs in expressions of the form $\neg\forall x\neg\psi$,*[34]

$$\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}} \models (\bigwedge_{i\leq n} Ax_i \wedge \Box\varphi(v(x_1), ..., v(x_n))) \leftrightarrow \varphi^*(x_1, ..., x_n).$$

[34] That is, the theorem is not true for negated and disjunctive formulas, but it is for formulas with $\exists x$. The convoluted way in which this definition comes across is for the non-primitiveness of the quantifier.

Proof By induction on the complexity of φ. The base cases are covered by Theorems 4.13 and 4.15. The conjunctive case is trivial. If $\varphi ::= \forall z\psi(\vec{y})$, let $\mathcal{M}^+, w_0 \models_\eta \forall z\psi^*(\vec{x})$. Then, we have for any assignment $\eta' \sim_z \eta$, $\mathcal{M}^+, w_0 \models_{\eta'} \psi^*(\vec{x})$. By induction hypothesis, we get

$$\mathcal{M}^+, w_0 \models_{\eta'} \bigwedge_{i\leq m} Ax_i \wedge \Box\psi(v(\vec{x})).$$

By the arbitrariness of η', we get

$$\mathcal{M}^+, w_0 \models_\eta \forall z(\bigwedge_{i\leq m} Ax_i \wedge \Box\psi(v(\vec{x}))).$$

Since z only (possibly) occurs in $\psi(\vec{y})$, that means

$$\mathcal{M}^+, w_0 \models_\eta \bigwedge_{i\leq m} Ax_i \wedge \Box\forall z\psi(v(\vec{x})).$$

Let now $\mathcal{M}^+, w_0 \not\models_\eta \forall z\psi^*(\vec{x})$, so that for some $\eta' \sim_z \eta$, $\mathcal{M}^+, w_0 \not\models_{\eta'} \psi^*(\vec{x})$. By induction hypothesis, $\mathcal{M}^+, w_0 \not\models_{\eta'} \bigwedge_{i\leq m} Ax_i \wedge \Box\psi(v(\vec{x}))$, so either $\mathcal{M}^+, w_0 \not\models_{\eta'} \bigwedge_{i\leq m} Ax_i$ or $\mathcal{M}^+, w_0 \not\models_{\eta'} \Box\psi(v(\vec{x}))$. If the former, the result is straightforward, so suppose the latter. Then, for some $w \in \mathrm{R}(w_0)$, $\mathcal{M}^+, w \not\models_{\eta'} \psi(v(\vec{x}))$. That means $\mathcal{M}^+, w \not\models_{\eta'} \forall z\psi(v(\vec{x}))$, which gives us $\mathcal{M}^+, w_0 \not\models_{\eta'} \Box\forall z\psi(v(\vec{x}))$.

The case for $\varphi ::= \exists z\psi(v(x))$ is analogous to the universal case. For $\varphi ::= \Box\psi$, just notice although in all the previous arguments we evaluated formulas in w_0, nothing hinges on the state of evaluation being w_0, and so they are all valid for any state accessed by it. Therefore, by the induction hypothesis, the present case can be easily made. $\square$

Once we compare it to Fine's definition, we see Theorem 4.17 is a weak form of the PGA, with a classical reading of a predicate on the left-hand side and a generic reading of it on the right-hand side. Therefore, as we have expressed, Horsten's theory may not be so distant from Fine's PGA.

Notice Theorem 4.17 fails to account for disjunctive and negative relations. Concerning negative relations, the restriction is quite natural. *Being a non-mammal*, or *being neither a chair nor a table* seem less like *actual* (independently obtained) relations, and more like restrictions on other relations. If we consider collections to have similar existence criteria as to sets, then we need collections to exist when they are well-determined sub-collections of pre-existing collections. On the other hand, the power of definition provided by negation can easily encompass this form of determination (e.g. the complement of a pure set is a class). For example, describing something as *the collection of non-aeroplanes* only tells us that, if anything were to belong to that collection, then it would not be an aeroplane. It is only once we supplement the description, by saying *of what* it is a restriction, that the collection may be determined. In that way, there is no loss of generality, as there are no negative relations to be inherited.

Disjunctive relations are of a different matter. The theorem with disjunctive cases cannot be valid (at least with respect to the classes of models we have described), as it would fall prey to Berkeley's argument against A-objects in the semantic framework.

However, it seems adequate to conclude that an arbitrary natural number is either even or odd by seeing that each individual natural number is either even or odd. Therefore, there may be a better way of dealing with additional generic predicates.

5 Conclusion

Concerning the sort of modality that the afthairetic one is, we can easily notice it is distinct from metaphysical necessity. Models are not reflexive, nor serial, nor symmetric, nor transitive. Common to all models, in this regard, is only the fact of having a distinguished world in the center, which accesses every other, but is never accessed. To paint a picture, it is like an octopus, whose tentacles stretch and reach other states, but who can never touch its own head.

Concerning Horsten's view on A-objects, we have shown how to conservatively extend any non-modal first-order theory to incorporate them. More importantly, we have shown how to extend their proof theory, which may allow a more precise study of A-objects in that front. In doing so, we hope we have contributed, even if just in the humblest sense, to a problem Horsten expresses at the end of his book:

Problem 11.1. Articulate and defend a general metaphysical and logical theory of arbitrary entities.[35]

Many questions are left open. First, there is the remaining part of point **(2)** – namely, how many arbitrary φ's there are. Then, there is the problem of finding a way of eliminating ungrounded A-objects, without paying the price of also getting rid of part of the grounded A-objects. On the other hand, it may be argued that some ungrounded A-objects are unique, and therefore there might be interesting applications of models with ungrounded A-objects, for their unfounded nature. Moreover, building models for the AM-extended theories – with all the additional apparatus of Section 4 – may allow us to study the behaviour of A-objects of specific theories – more specifically, PA, so that we may compare our approach to Horsten's model and axiomatisation. We intend to investigate that in future work.

The most pressing question, however, is how to appropriately present a relation inheritance principle, in a way that it adequately rebuts Berkeley's argument. That is not a problem necessarily for Horsten's view, as he is not concerned with this issue. Nevertheless, that same principle suggests a connection to the abstractionist program, and therefore a way of adequately justifying the ontologically committed assertion that there are, in fact, arbitrary objects. That makes the view more promising. Therefore, in future work, we also intend to address that question, with further offering a predicative definition by abstraction of A-objects.

[35] [7], p. 211.

Appendix A Proof of Lemma 3.5

Proof Let x_i be the free variables of φ. We note that the translation works for the purpose of the lemma for it preserves free variables of formulas – that is, φ and $\alpha_{\mathrm{T}}(\varphi)$ indeed share the same free variables.

Now, we proceed by an induction on the complexity of φ. We skip the Boolean cases. First, by an induction on the complexity of terms, we show that for any free term t occurring in φ, $\mu(t) = \eta(t)$. The base cases are trivial. The case of $t ::= f(\vec{s})$ is easily obtained by the induction hypothesis and Definition 3.21 (c).

Now, suppose $\mathcal{M} \models_\mu t_1 = t_2$. That is the case iff $\mu(t_1) = \mu(t_2)$. Since t_1 and t_2 occur freely, $\mu(t_1) = \eta(t_1)$ and $\mu(t_2) = \eta(t_2)$, so $\eta(t_1) = \eta(t_2)$, and therefore $\mathcal{M}^+, w_0 \models_\eta t_1 = t_2$. The other direction goes by an analogous argument. Suppose now $\mathcal{M} \models_\mu P\vec{t}$. Then $\mu(\vec{t}) \in \mathrm{I}(P)$. By Definition 3.1 (c), $\mathrm{I}(P) \subseteq \mathrm{I}^+(P, w_0)$. Since the terms occur freely, $\mu(\vec{t}) = \eta(\vec{t})$, and therefore $\eta(\vec{t}) \in \mathrm{I}^+(P, w_0)$, so that $\mathcal{M}^+, w_0 \models_\eta P\vec{t}$. Once again, we get the other direction by an analogous argument.

Suppose now $\mathcal{M} \models_\mu \forall x\varphi$. If $\eta(x) \notin \mathrm{M}$, then $\mathcal{M}^+, w_0 \not\models_\eta \neg Ax$, and therefore $\mathcal{M}^+, w_0 \models_\eta \neg Ax \to \alpha_{\mathrm{T}}(\varphi)$. If on the other hand $\eta(x) \in \mathrm{M}$, then $\mathcal{M}^+, w_0 \models_\eta \neg Ax$. Let μ' be an assignment on $\mathcal{M}$ such that, for $y = x$ or $y = x_i$, $\mu'(y) = \eta(y)$, and $\mu'(y) = \mu(y)$ otherwise. Then, $\mu' \sim_x^{\mathcal{M}} \mu$, so that $\mathcal{M} \models_{\mu'} \varphi$, and also, by induction hypothesis, $\mathcal{M}^+, w_0 \models_\eta \alpha_{\mathrm{T}}(\varphi)$. We conclude $\mathcal{M}^+, w_0 \models_\eta \forall x(\neg Ax \to \alpha_{\mathrm{T}}(\varphi))$. Suppose now $\mathcal{M} \not\models_\mu \forall x\varphi$. Then, for some $\mu' \sim_x^{\mathcal{M}} \mu$, $\mathcal{M} \not\models_{\mu'} \varphi$. Let η be an assignment on $\mathcal{M}^+$ such that $\eta(x_i) = \mu'(x_i)$ for the free variables x_i of $\forall x\varphi$, and $\eta(x) = \mu(x)$. Then, by induction hypothesis, $\mathcal{M}^+, w_0 \not\models_\eta \alpha_{\mathrm{T}}(\varphi)$. Furthermore, $\eta(x) = \mu(x) \in \mathrm{M}$, so that $\mathcal{M}^+, w_0 \models_\eta \neg Ax$. Therefore, $\mathcal{M}^+, w_0 \not\models_\eta \neg Ax \to \alpha_{\mathrm{T}}(\varphi)$, and we conclude $\mathcal{M}^+, w_0 \not\models_\eta \forall x(\neg Ax \to \alpha_{\mathrm{T}}(\varphi))$. □

Appendix B Proof of Theorem 3.7

Proof The proof is an induction on the complexity of φ. Skipping the trivial cases, let $\varphi ::= \forall x\psi$ and $\mathcal{M} \models \forall x\psi$. By the definition of validity of open formulas, that is the case iff $\mathcal{M} \models \psi$, which by induction hypothesis, means $\mathcal{M}^+ \models \alpha_{\mathrm{T}}(\psi)$. From that, one gets $\mathcal{M}^+ \models \neg Ax \to \alpha_{\mathrm{T}}(\psi)$, and then the desired conclusion. For the other direction, suppose $\mathcal{M}^+ \models \alpha_{\mathrm{T}}(\forall x\psi)$, which means $\mathcal{M}^+ \models \forall x(\neg Ax \to \alpha_{\mathrm{T}}(\psi))$. Let μ be an assignment on $\mathcal{M}$. We get $\mathcal{M}^+, w_0 \models_\mu \neg Ax \to \alpha_{\mathrm{T}}(\psi)$. Since μ is an equiadmissable assignment, $\mu(x) \notin \mathrm{A}$, so $\mathcal{M}^+, w_0 \models_\mu \alpha_{\mathrm{T}}(\psi)$. By Lemma 3.5, we get $\mathcal{M} \models_\mu \psi$, and by the arbitrariness of μ, we have $\mathcal{M} \models \forall x\psi$. □

Appendix C Proof of Theorem 3.10

Proof We prove it by an induction on the complexity of φ. We skip the cases of identity (in which the all terms designate, since the functions are total in M) and the Boolean operators. Suppose $\mathcal{M} \models_\eta P\vec{t}$. That is the case iff $\eta(\vec{t}) \in \mathrm{I}(P)$, which by Definition 3.1 (c), is the case iff $\eta(\vec{t}) \in \mathrm{I}^+(P, w_0) \upharpoonright_{\mathrm{M}}$ iff $\mathcal{M}^+ \upharpoonright_{\mathrm{M}}, w_0 \models_\eta P\vec{t}$. Suppose now $\mathcal{M} \models_\eta \forall x\varphi$. Then, for any $\eta' \sim_x^{\mathcal{M}} \eta$, $\mathcal{M} \models_{\eta'} \varphi$. By induction hypothesis, $\mathcal{M}^+ \restriction_{\mathrm{M}}, w_0 \models_{\eta'} \varphi$. Notice both $\mathcal{M}^+ \restriction_{\mathrm{M}}$ and $\mathcal{M}$ possess the same domain, and thus any $\eta'' \sim_x^{\mathcal{M}^+ \restriction_{\mathrm{M}}} \eta$ is an equiadmissible assignment. Therefore, η' is also any x-variant of η in $\mathcal{M}^+ \restriction_{\mathrm{M}}$, so that $\mathcal{M}^+ \restriction_{\mathrm{M}}, w_0 \models_\eta \forall x \varphi$. The converse case goes by an analogous argument. □

Appendix D Proof of Lemma 3.19

Proof Let Λ be such a set of formulas. Suppose $L(\Lambda) \cup \{\neg\varphi\}$ is not $\mathrm{T}^{\Box_q}$-consistent. Then, either (i) $L(\Lambda)$ is inconsistent, so that for some $\psi_1, ..., \psi_n, \chi \in L(\Lambda)$,

$$\vdash^{\mathrm{T}^{\Box_q}} (\psi_1 \wedge ... \wedge \psi_n) \to \neg\chi,$$

or (ii) $L(\Lambda)$ is consistent, so that for some $\gamma_1, ..., \gamma_m \in L(\Lambda)$,

$$\vdash^{\mathrm{T}^{\Box_q}} (\gamma_1 \wedge ... \wedge \gamma_m) \to \neg\neg\varphi.$$

Suppose (i). Then, by $Nec_{\mathrm{T}^{\Box_q}}$ and **K**,

$$\vdash^{\mathrm{T}^{\Box_{af}}} \Box(\psi_1 \wedge ... \wedge \psi_n) \to \Box\neg\chi.$$

Since $\psi_i \in L(\Lambda)$, $\Box\psi_i \in \Lambda$, and since it is maximal consistent, $(\Box\psi_1 \wedge ... \wedge \Box\psi_n) \in \Lambda$, which means $\Box(\psi_1 \wedge ... \wedge \psi_n) \in \Lambda$. That gives us $\Box\neg\chi \in \Lambda$. But $\chi \in L(\Lambda)$, so $\Box\chi \in \Lambda$. By an argument analogous to the last one, we have $\Box(\chi \wedge \neg\chi) \in \Lambda$. Now,

$\vdash^{\mathrm{T}^{\Box_q}} (\chi \wedge \neg\chi) \to \varphi$, which by $Nec_{\mathrm{T}^{\Box_q}}$ and **K**, gives us
$\vdash^{\mathrm{T}^{\Box_{af}}} \Box(\chi \wedge \neg\chi) \to \Box\varphi$.

However, since Λ is maximal and $\mathrm{T}^{\Box_{af}}$-consistent, that would mean $\Box\varphi \in \Lambda$, contrary to our assumption. Let us suppose now (ii). Then, we get

$\vdash^{\mathrm{T}^{\Box_q}} (\gamma_1 \wedge ... \wedge \gamma_m) \to \varphi$, and by $Nec_{\mathrm{T}^{\Box_q}}$ and **K**, we get
$\vdash^{\mathrm{T}^{\Box_{af}}} \Box(\gamma_1 \wedge ... \wedge \gamma_m) \to \Box\varphi$.

Since $\gamma_i \in L(\Lambda)$, $\Box\gamma_i \in \Lambda$, which means $\Box(\gamma_1 \wedge ... \wedge \gamma_m) \in \Lambda$. But Λ is maximal and $\mathrm{T}^{\Box_{af}}$-consistent, which would mean $\Box\varphi \in \Lambda$, contrary to our supposition. Therefore, $L(\Lambda) \cup \{\neg\varphi\}$ is $\mathrm{T}^{\Box_q}$-consistent. By Corollary 3.17, there is a maximal $\mathrm{T}^{\Box_q}$-consistent Γ with the $\forall$-property such that $L(\Lambda) \cup \{\neg\varphi\} \subseteq \Gamma$. □

Appendix E Proof of Lemma 3.25

Proof First, notice for any $t \in Term_{\mathcal{L}}$, if $\xi(t)$ is defined – or, equivalently, $\xi(t) \in \mathrm{M}_\Gamma^+$ –, then $\xi(t) = [t]_\Gamma$.[36] Now, we proceed by an induction on the complexity of formulas

[36] By the definition of the equivalence classes $[\cdot]_\Gamma$ and the canonical assignment, the base cases are trivial. If t is $f(\vec{s})$, let $\xi(f(\vec{s})) = \mathrm{I}^+(f)(\xi(\vec{s})) \in \mathrm{M}_\Gamma^+$. By Definition 3.21 (b), $\mathrm{I}^+(f)(\xi(\vec{s})) = [f(\vec{s})]_\Gamma$.

of $\mathcal{L}^{\Box_{af}}$. Suppose $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} P\vec{t}$. That is the case iff $[\vec{t}] \in \mathrm{I}^{+}(P, w)$ iff $P\vec{t} \in w$, by Definition 3.21 (b).

If $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} t_1 = t_2$, then the $\xi(t_i)$ are defined, so $[t_1]_{\Gamma} = [t_2]_{\Gamma}$, which means $t_1 = t_2 \in \Gamma$, so if w is Γ, we are done. Otherwise, since Γ is maximal $\mathrm{T}^{\Box_{af}}$-consistent, by Theorem 3.23, $\Box(t_1 = t_2) \in \Gamma$, so by Definition 3.21 (c) and (d), $t_1 = t_2 \in L(\Gamma) \subseteq w$. Let now $\mathcal{M}_{\Gamma}^{+}, w \not\models_{\xi} t_1 = t_2$. Then, either the $\xi(t_i)$ are defined or not. If they are, then by the definition of $[\cdot]_{\Gamma}$, $t_1 = t_2 \notin \Gamma$, so if w is Γ, we are done. Otherwise, by the maximality of Γ, $t_1 \neq t_2 \in \Gamma$, so by Theorem 3.23, $\Box t_1 \neq t_2 \in \Gamma$, and thus $t_1 \neq t_2 \in L(\Gamma) \subseteq w$. If either of the $\xi(t_i)$ are not defined, by Definition 3.21, $t_1 = t_2 \notin \Gamma$, so by maximality, $t_1 \neq t_2 \in \Gamma$, and the previous argument follows.

If $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} At$, that is the case iff $[t]_{\Gamma} \in \mathrm{A}$, iff $At \in \Gamma$, by Definition 3.21 (e). By Definition 3.21 (c) and (d), that is the case iff $At \in w$.

If $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} St_1t_2$, that is the case iff $w \in \mathrm{S}([t_1]_{\Gamma}, [t_2]_{\Gamma})$ iff $St_1t_2 \in w$, by Definition 3.21 (f).

If $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} \neg\varphi$, that is the case iff $\mathcal{M}_{\Gamma}^{+}, w \not\models_{\xi} \varphi$, which, by induction hypothesis, happens iff $\varphi \notin w$, iff $\neg\varphi \in w$, since w is maximal.

If $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} \varphi \wedge \psi$, that is the case iff $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} \varphi$ and $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} \psi$, which, by induction hypothesis, happens iff $\varphi \in w$ and $\psi \in w$ iff $\varphi \wedge \psi \in w$, since w is maximal and consistent.

If $\forall x\varphi \in w$, since w is maximal and consistent (it does not matter if $\mathrm{T}^{\Box_{af}}$ or $\mathrm{T}^{\Box_{q}}$-consistent), by $\mathbf{UI}_f$, $\exists y(y = t) \to \varphi_x^t \in w$ for any $t \in Term_{\mathcal{L}}$. Let now $\eta \sim_x^{\mathcal{M}_{\Gamma}^{+}} \xi$. Then, by Definition 3.21 (a), for some $s \in Term_{\mathcal{L}}$, $\eta(x) = [s]_{\Gamma} = \xi(s)$. By the maximal consistency of w, $\varphi_x^s \in w$, so by induction hypothesis, we get $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} \varphi_x^s$. By Corollary 3.24, that means $\mathcal{M}_{\Gamma}^{+}, w \models_{\eta} \varphi$. By the arbitrariness of η, we have $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} \forall x\varphi$. Now, suppose $\forall x\varphi \notin w$. Since w has the $\forall$-property, for some t, $\neg\forall x\varphi \to (\neg\varphi_x^t \wedge t = t)$. Since w is maximal and consistent, we get $\neg\varphi_x^t \wedge t = t \in w$. By induction hypothesis, we have $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} \neg\varphi_x^t$ and $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} t = t$. Therefore, for $\eta \sim_x \xi$ such that $\eta(x) = \xi(t)$, $\mathcal{M}_{\Gamma}^{+}, w \models_{\eta} \neg\varphi$.[37] We conclude $\mathcal{M}_{\Gamma}^{+}, w \not\models_{\xi} \forall x\varphi$.

If now $\Box\varphi \in w$, consider some $u \in \mathrm{R}(w)$. Then, $\varphi \in L(w)$, which means $\varphi \in u$, by Definition 3.21 (d). By induction hypothesis, $\mathcal{M}_{\Gamma}^{+}, u \models_{\eta} \varphi$. Since u is any such relevant state, $\mathcal{M}_{\Gamma}^{+}, w \models_{\xi} \Box\varphi$. Suppose now $\Box\varphi \notin w$. Since w is maximal and consistent, $\neg\Box\varphi \in w$, which means $\Diamond\neg\varphi \in w$. If w is Γ, by Lemma 3.19, there is $u \in \mathrm{R}(\Gamma)$ and $\neg\varphi \in u$. Otherwise, by Lemma 3.20, there is a $u \in \mathrm{R}(w)$ and $\neg\varphi \in u$. In either case, by induction hypothesis, $\mathcal{M}_{\Gamma}^{+}, u \models_{\xi} \neg\varphi$, which means $\mathcal{M}_{\Gamma}^{+}, w \not\models_{\xi} \Box\varphi$. □

Appendix F Proof of Lemma 3.26

Proof First, we note, for μ an assignment on $\mathcal{M}_{\Gamma}$, we have $\mu^{\mathcal{M}_{\Gamma}}(t) = \mu^{\mathcal{M}_{\Gamma}^{+}}(t)$.[38] Now, we show, by an induction on the complexity of formulas of $\mathcal{L}$, that if $\vdash^{\mathrm{T}} \varphi$,

[37] Notice the modification in Definition 3.14 plays an important role here, as otherwise we could not guarantee the existence of such x-variant assignment, for $\xi(t)$ would not be defined.

[38] To see that, we proceed by an induction on the complexity of terms, and skip the base cases, so consider the case of $f(\vec{t})$. Then, $\mu^{\mathcal{M}_{\Gamma}}(f(\vec{t})) = \mathrm{I}(f)(\mu^{\mathcal{M}_{\Gamma}}(\vec{t})) = \mathrm{I}^{+}(f)(\mu^{\mathcal{M}_{\Gamma}}(\vec{t}))$. Since, by induction hypothesis, $\mu^{\mathcal{M}_{\Gamma}}(\vec{t}) = \mu^{\mathcal{M}_{\Gamma}^{+}}(\vec{t})$, we get $\mathrm{I}^{+}(f)(\mu^{\mathcal{M}_{\Gamma}}(\vec{t})) = \mathrm{I}^{+}(f)(\mu^{\mathcal{M}_{\Gamma}^{+}}(\vec{t})) = \mu^{\mathcal{M}_{\Gamma}^{+}}(f(\vec{t}))$.

then $\mathcal{M}_\Gamma \models \varphi$. We skip the cases of the Boolean operators. Suppose $\vdash^{\mathrm{T}} t = s$, which means $\vdash^{\mathrm{T}} \forall\vec{x}(t = s)$, where $\vec{x}$ are the free variables of t and s. Then, by Theorem 3.13, $\vdash^{\mathrm{T}^{\Box af}} \forall\vec{x}(\bigwedge\neg Ax_i \rightarrow t = s)$. Since Γ is $\mathrm{T}^{\Box af}$-consistent and maximal, $\forall\vec{x}(\bigwedge\neg Ax_i \rightarrow t = s) \in \Gamma$, and by Lemma 3.25, $\mathcal{M}^+_\Gamma, \Gamma \models_\xi \forall\vec{x}(\bigwedge\neg Ax_i \rightarrow t = s)$. Let μ be any assignment on $\mathcal{M}_\Gamma$. Then, μ is an equiadmissible assignment (on $\mathcal{M}^+_\Gamma$). Let η be an assignment on $\mathcal{M}^+_\Gamma$ such that, $\eta(x_i) = \mu(x_i)$, and $\eta(y) = \xi(y)$ for $y \neq x_i$. Then, $\eta \sim^{\mathcal{M}^+_\Gamma}_{\vec{x}} \xi$, so $\mathcal{M}^+_\Gamma, \Gamma \models_\eta \bigwedge\neg Ax_i \rightarrow t = s$, and since x_i are the free variables of that formula, $\mathcal{M}^+_\Gamma, \Gamma \models_\mu \bigwedge\neg Ax_i \rightarrow t = s$, by Theorem 3.6. Since μ is an equiadmissible assignment, $\mu(x_i) \notin \mathrm{A}$, and therefore $\mathcal{M}^+_\Gamma, \Gamma \models_\mu t = s$, so that $\mu^{\mathcal{M}^+_\Gamma}(t) = \mu^{\mathcal{M}^+_\Gamma}(s)$. Here we recall our remark from the beginning, which means we get $\mu^{\mathcal{M}_\Gamma}(t) = \mu^{\mathcal{M}_\Gamma}(s)$, and therefore $\mathcal{M}_\Gamma \models_\mu t = s$. Since μ is any assignment, we conclude $\mathcal{M}_\Gamma \models t = s$.

Suppose $\vdash^{\mathrm{T}} P\vec{t}$, which means $\vdash^{\mathrm{T}} \forall\vec{x}P\vec{t}$, for $\vec{x}$ the free variables of $\vec{t}$. Then, by Theorem 3.13, $\vdash^{\mathrm{T}^{\Box af}} \forall\vec{x}(\bigwedge\neg Ax_i \rightarrow P\vec{t})$, so that $\forall\vec{x}(\bigwedge\neg Ax_i \rightarrow P\vec{t}) \in \Gamma$, and thus by Lemma 3.25, $\mathcal{M}^+_\Gamma, \Gamma \models_\xi \forall\vec{x}(\bigwedge\neg Ax_i \rightarrow P\vec{t})$. Once again, let μ be any assignment on $\mathcal{M}_\Gamma$ (so, an equiadmissible assignment), and consider η on $\mathcal{M}^+_\Gamma$ such that $\eta(y) = \mu(x_i)$ for $y = x_i$ for each x_i in $\vec{x}$, and $\eta(y) = \xi(y)$ otherwise. Then, $\mathcal{M}^+_\Gamma, \Gamma \models_\eta \bigwedge\neg Ax_i \rightarrow P\vec{t}$, and since $\eta(x_i) = \mu(x_i) \notin \mathrm{A}$, $\mathcal{M}^+_\Gamma, \Gamma \models_\eta P\vec{t}$. By Theorem 3.6, that means $\mathcal{M}^+_\Gamma, \Gamma \models_\mu P\vec{t}$, so that $\mu^{\mathcal{M}^+_\Gamma}(\vec{t}) \in \mathrm{I}^+(P, \Gamma)$. Since, for the t_i, $\mu^{\mathcal{M}^+_\Gamma}(t_i) = \mu^{\mathcal{M}_\Gamma}(t_i) \in \mathrm{M}_\Gamma$, that means $\mu^{\mathcal{M}^+_\Gamma}(\vec{t}) \in \mathrm{I}^+(P, \Gamma) \restriction_{\mathrm{M}_\Gamma} = \mathrm{I}(P)$. Therefore, $\mathcal{M}_\Gamma \models_\mu P\vec{t}$, so by the arbitrariness of μ, $\mathcal{M}_\Gamma \models P\vec{t}$.

At last, suppose $\vdash^{\mathrm{T}} \forall x\varphi$. By universal instantiation, $\vdash^{\mathrm{T}} \varphi$, so by induction hypothesis, $\mathcal{M}_\Gamma \models \varphi$, which is the case iff $\mathcal{M}_\Gamma \models \forall x\varphi$. □

Appendix G Proof of Theorem 4.6

Proof (Totality) Let $\mathcal{M}^+ \models \mathbf{FI}_1, \mathbf{FI}_3$ and $\mathrm{I}(f)$ be a total function. We show, by an induction on $max\{\mathrm{r}_A(o_j)\}_{j\leq n}$, that if $o_i \in \mathrm{M}^+_{Gr}$, then $\mathrm{I}^+(f)(o_1, ..., o_n)$ is defined. The case in which that is 0 is straightforward, by assumption. Let it work then for a max rank of m, and suppose $max\{\mathrm{r}_A(o_j)\}_{j\leq n} = m + 1$. If $o_i \in \mathrm{M}$, then for any $w \in \mathrm{R}(w_0)$, $\mathrm{r}_A(o_i) = \mathrm{r}_A(\mathrm{v}(o_i, w)) = 0 \leq \mathrm{r}_A(o_k)$, for any k. Otherwise, $o_i \in Gr_{\mathcal{M}^+}$, and thus by definition, $\mathrm{r}_A(\mathrm{v}(o_i, w)) < \mathrm{r}_A(o_i) \leq m + 1$ for any $w \in \mathrm{R}(w_0)$. Thus, by induction hypothesis, we get $\mathrm{I}^+(f)(\mathrm{v}(o_1, w), ..., \mathrm{v}(o_1, w))$ is defined. Let $\eta(x_i) = o_i$, and $\eta(y) = \mathrm{I}^+(f)(\mathrm{v}(o_1, w), ..., \mathrm{v}(o_1, w))$, so $\mathcal{M}^+, w \models_\eta f(v(x_1), ..., v(x_n)) = y$. By the arbitrariness of w, we get $\mathcal{M}^+, w_0 \models_\eta \Box f(v(x_1), ..., v(x_n)) = y$. From $\mathbf{FI}_3$, that gives us $\mathcal{M}^+, w_0 \models_\eta \exists y(f(x_1, ..., x_n) = y)$, and so, for some $\eta' \sim_y \eta$, $\mathrm{I}^+(f)(\eta(\vec{x})) = \mathrm{I}^+(f)(o_1, ..., o_n) = \eta'(y)$, and so that is defined. Let $\eta'(y) = b$ and $w \in \mathrm{R}(w_0)$. By $\mathbf{FI}_1$, $\mathrm{v}(b, w) = \mathrm{I}^+(f)(\mathrm{v}(o_1, w), ..., \mathrm{v}(o_n, w))$. Once again, by induction on $max\{\mathrm{r}_A(o_j)\}_{j\leq n}$, we show $\mathrm{v}(b, w) \in \mathrm{M}^+_{Gr}$. If that max rank is 0, then $o_i \in \mathrm{M}$, and since $\mathrm{I}(f)$ is a total function, we get $\mathrm{I}^+(f)(\mathrm{v}(o_1, w), ..., \mathrm{v}(o_1, w)) = \mathrm{I}^+(f)(o_1, ..., o_n) = \mathrm{I}(f)(o_1, ..., o_n) \in \mathrm{M} \subseteq \mathrm{M}^+_{Gr}$.[39] Let it work for a max rank of m,

[39] For then $o_i \in \mathrm{M}$, and thus $\mathrm{v}(o_i, w) = o_i$ for any $w \in \mathrm{W}$.

and suppose that max rank is $m+1$. As we have seen, $\mathrm{r}_A(\mathrm{v}(o_i, w)) < \mathrm{r}_A(o_i) \leq m+1$, so that by induction hypothesis, $\mathrm{I}^+(f)(\mathrm{v}(o_1, w), ..., \mathrm{v}(o_n, w)) = \mathrm{v}(b, w) \in \mathrm{M}^+_{Gr}$. We conclude $\mathrm{I}^+(f)$ is a total function on M^+_{Gr}.

(Commutativity) Let $a_1, a_2 \in \mathrm{M}^+_{Gr}$, such that $\mathbf{I}^+(f)(a_1, a_2)$ is defined and b is its output. By the validity of $\mathbf{FI}_1$, for any $w \in \mathrm{R}(w_0)$, $\mathbf{I}^+(f)(\mathrm{v}(a_1, w), \mathrm{v}(a_2, w)) = \mathrm{v}(b, w)$. For the sake of clarity, let $\mathrm{v}(a_i, w) = o_i$ and $\mathrm{v}(b, w) = b'$. By an induction on $max\{\mathrm{r}_A(o_j)\}_{j\leq 2}$, we show $\mathbf{I}^+(f)(o_1, o_2) = \mathbf{I}^+(f)(o_2, o_1)$. If that max rank is 0, then $o_i \in \mathrm{M}$, so since $\mathrm{I}(f)$ is commutative, we are done. Let it work for a max rank of m, and suppose now $max\{\mathrm{r}_A(o_j)\}_{j\leq 2} = m+1$. By $\mathbf{FI}_1$, we have $\mathbf{I}^+(f)(o_1, o_2) = b'$ iff for any $u \in \mathrm{R}(w_0)$, $\mathbf{I}^+(f)(\mathrm{v}(o_1, u), \mathrm{v}(o_2, u)) = \mathrm{v}(b', u)$. As per the last paragraph, $\mathrm{r}(\mathrm{v}(o_i, u)) < \mathrm{r}(o_i) \leq m+1$, so that by induction hypothesis, $\mathbf{I}^+(f)(\mathrm{v}(o_1, u), \mathrm{v}(o_2, u)) = \mathbf{I}^+(f)(\mathrm{v}(o_2, u), \mathrm{v}(o_1, u)) = \mathrm{v}(b', u)$. Therefore, by $\mathbf{FI}_1$, $\mathbf{I}^+(f)(o_2, o_1) = \mathbf{I}^+(f)(\mathrm{v}(a_2, w), \mathrm{v}(a_1, w)) = b' = \mathrm{v}(b, w)$. By the arbitrariness of w and $\mathbf{FI}_1$, we get $\mathbf{I}^+(f)(a_2, a_1) = b = \mathbf{I}^+(f)(a_1, a_2)$.

(Injectivity) Let $a_1, a_2 \in \mathrm{M}^+_{Gr}$ such that $a_1 \neq a_2$. Then, by the validity of $\mathbf{Id}_A$, $\mathrm{v}(a_1, u) \neq \mathrm{v}(a_2, u)$ for some $u \in \mathrm{R}(w_0)$. Let $\mathrm{I}^+(f)(a_1)$ be defined and b_1 be its output. By the validity of $\mathbf{FI}_1$, that is the case iff for any $w \in \mathrm{R}(w_0)$, $\mathrm{I}^+(f)(\mathrm{v}(a_1, w)) = \mathrm{v}(b_1, w)$. If $\mathrm{I}^+(f)(a_2)$ is not defined, we are done, so suppose it is and its output is b_2. That is the case iff for any $w \in \mathrm{R}(w_0)$, $\mathrm{I}^+(f)(\mathrm{v}(a_2, w)) = \mathrm{v}(b_2, w)$. One more time, we proceed by an induction on $max\{\mathrm{r}_A(\mathrm{v}(a_j, u)\}_{j\leq 2}$ to show that means $\mathrm{I}^+(f)(\mathrm{v}(a_1, u)) \neq \mathrm{I}^+(f)(\mathrm{v}(a_2, u))$. If that is 0, then $\mathrm{v}(a_i, u) \in \mathrm{M}$, so since $\mathrm{I}(f)$ is injective, we are done. Let it work for a max rank of m, and suppose $max\{\mathrm{r}_A(\mathrm{v}(a_j, u)\}_{j\leq 2} = m+1$. By $\mathbf{FI}_1$, $\mathrm{I}^+(f)(\mathrm{v}(a_i, u)) = \mathrm{v}(b_i, u)$ iff for any $u' \in \mathrm{R}(w_0)$, $\mathrm{I}^+(f)(\mathrm{v}(\mathrm{v}(a_i, u), u')) = \mathrm{v}(\mathrm{v}(b_i, u), u')$. Since $\mathrm{r}_A(\mathrm{v}(\mathrm{v}(a_i, u), u')) < \mathrm{r}_A(\mathrm{v}(a_i, u)) \leq m+1$, by induction hypothesis, $\mathrm{I}^+(f)(\mathrm{v}(\mathrm{v}(a_1, u), u')) \neq \mathrm{I}^+(f)(\mathrm{v}(\mathrm{v}(a_2, u), u'))$, and therefore $\mathrm{v}(\mathrm{v}(b_1, u), u') \neq \mathrm{v}(\mathrm{v}(b_2, u), u')$. That means $\mathrm{v}(b_1, u) \neq \mathrm{v}(b_2, u)$, and therefore $\mathrm{I}^+(f)(\mathrm{v}(a_1, u)) \neq \mathrm{I}^+(f)(\mathrm{v}(a_2, u))$. Thus, $b_1 \neq b_2$, and we conclude $\mathrm{I}^+(f)(a_1) \neq \mathrm{I}^+(f)(a_2)$. □

Appendix H Proof of Theorem 4.13

Before we give the full proof, we need to show a few lemmas.

Lemma H.1 *Let $\mathcal{M}^+ \models \mathbf{Id}_A, \mathbf{Comp}_A, \mathbf{FI}_1, \mathbf{FI}_2$. Let also $t^{\uparrow\vec{x}}_{\vec{y}}$ be the translation $t^{\uparrow}$ of a term t with each y_i replaced by x_i not occurring previously, and η be an assignment such that $\eta(x_i) \in \mathrm{M}^+_{Gr}$. If $\eta(t^{\uparrow\vec{x}}_{\vec{y}})$ is defined, then $\eta(t^{\uparrow\vec{x}}_{\vec{y}}) \in \mathrm{M}^+_{Gr}$.*

Proof The argument for that is a simple induction on the complexity of t. If $t ::= y$, then $t^{\uparrow x}_y$ is x, so by the assumption $\eta(x) \in \mathrm{M}^+_{Gr}$; if $t ::= c^{n\uparrow}$, then $t^{\uparrow}$ is $c^{n+1\uparrow}$ and $\eta(t^{\uparrow}) = \mathrm{I}^+(c^{n+1\uparrow}) \in \mathrm{M}^+_{Gr}$; and if $t ::= f(s_1, ..., s_n)$, then $t^{\uparrow\vec{x}}_{\vec{y}} = f(s^{\uparrow}_1, ..., s^{\uparrow}_n)^{\vec{x}}_{\vec{y}} = f(s^{\uparrow\vec{x}}_{1\vec{y}}, ..., s^{\uparrow\vec{x}}_{n\vec{y}})$. For a cleaner presentation, let us call each $s^{\uparrow\vec{x}}_{n\vec{y}}$ by r_i. By induction hypothesis, $\eta(r_i) \in \mathrm{M}^+_{Gr}$. We proceed by an induction on $max\{\mathrm{r}_A(\eta(r_i))\}_{i\leq n}$ to show $\eta(t^{\uparrow\vec{x}}_{\vec{y}}) \in \mathrm{M}^+_{Gr}$. If that max rank is 0, then $\eta(r_i) \in \mathrm{M}$, so by $\mathbf{FI}_2$, $\eta(t^{\uparrow\vec{x}}_{\vec{y}}) = \mathrm{I}^+(f)(\eta(r_1), ..., \eta(r_n)) = \eta(t^{\uparrow\vec{x}}_{\vec{y}}) \in \mathrm{M} \subseteq \mathrm{M}^+_{Gr}$. Supposing it works for a max rank of m, let it now be $m+1$. By $\mathbf{FI}_1$, that is the case iff $\mathrm{I}^+(f)(\mathrm{v}(\eta(r_n), w), ..., \mathrm{v}(\eta(r_n), w)) = \mathrm{v}(\eta(t^{\uparrow\vec{x}}_{\vec{y}}), w)$ for any $w \in \mathrm{R}(w_0)$. As we have seen, $\mathrm{r}_A(\mathrm{v}(\eta(r_i), w)) < \mathrm{r}_A(\eta(r_i)) \leq m+1$, so by induction hypothesis, $\mathrm{v}(\eta(t^{\uparrow\vec{x}}_{\vec{y}}), w)) \in \mathrm{M}^+_{Gr}$. By the arbitrariness of w, we conclude $\eta(t^{\uparrow\vec{x}}_{\vec{y}}) = \mathrm{I}^+(f)(\eta(r_1), ..., \eta(r_n)) \in \mathrm{M}^+_{Gr}$. □

Lemma H.2 *Let* $\mathcal{M}^+ \models \mathbf{Id}_A, \mathbf{Comp}_A, \mathbf{FI}_1, \mathbf{FI}_2$, $w \in \mathrm{R}(w_0)$, *and* η *be an assignment such that* $\eta(t^{\uparrow\vec{x}}_{\vec{y}})$ *is defined. If* $\eta(x_i) \in \mathrm{M}^+_{Gr}$ *and* $\mathrm{v}(\eta(x_i), w) = \eta(y_i)$, *then* $\mathrm{v}(\eta(t^{\uparrow\vec{x}}_{\vec{y}}), w) = \eta(t)$.

Proof We prove by an induction on the complexity of t. If $t ::= y$, the result is straightforward. If $t ::= c^{n\uparrow}$, then $(t)^\uparrow = c^{n+1\uparrow}$, so we have $\eta(c^{n\uparrow}) = \mathrm{I}^+(c^{n\uparrow})$, and $\eta(c^{n+1\uparrow}) = \mathrm{I}^+(c^{n+1\uparrow})$, so by our definitin of $c^{n+1\uparrow}$ we are done. If $t ::= f(s_1, ..., s_n)$, suppose $\eta(t^{\uparrow\vec{x}}_{\vec{y}}) = \eta(f(s_1^\uparrow, ..., s_n^\uparrow)^{\vec{x}}_{\vec{y}})$ is defined. Then, each $\eta(s^{\uparrow\vec{x}}_{i\vec{y}})$ must be defined. Since $\eta(x_i) \in \mathrm{M}^+_{Gr}$, by Lemma H.1, $\eta(s^{\uparrow\vec{x}}_{i\vec{y}}) \in \mathrm{M}^+_{Gr}$, and so $\eta(t^{\uparrow\vec{x}}_{\vec{y}}) = \mathrm{I}^+(f^\uparrow)(\eta(s^{\uparrow\vec{x}}_{1\vec{y}}), ..., \eta(s^{\uparrow\vec{x}}_{n\vec{y}})) \in \mathrm{M}^+_{Gr}$. By $\mathbf{FI}_2$, for any $w \in \mathrm{R}(w_0)$, $\mathrm{v}(\eta(t^{\uparrow\vec{x}}_{\vec{y}}), w) = \mathrm{I}^+(f)(\mathrm{v}(\eta(s^{\uparrow\vec{x}}_{1\vec{y}})), w), ..., \mathrm{v}(\eta(s^{\uparrow\vec{x}}_{n\vec{y}})), w))$. By induction hypothesis, $\mathrm{v}(\eta(s^{\uparrow\vec{x}}_{i\vec{y}}), w) = \eta(s_i)$, and so $\mathrm{v}(\eta(t^{\uparrow\vec{x}}_{\vec{y}}), w) = \mathrm{I}^+(f)(\eta(s_1), ..., \eta(s_n)) = \eta(t)$. □

Lemma H.3 *Let* $\mathcal{M}^+ \models \mathbf{Id}_A, \mathbf{Comp}_A, \mathbf{FI}_1, \mathbf{FI}_2$, t *be a term, and* $\{x_i\}$ *be its free variables. If* $\eta(x_i) \in \mathrm{M}^+_{Gr}$ *and* $\eta(t)$ *is defined, then* $\eta(t) \in \mathrm{M}^+_{Gr}$.

Proof Once again, by induction on t. The base cases are trivial, so suppose $t ::= f(s_1, ..., s_n)$. Then, $\eta(t) = \mathrm{I}^+(f)(\eta(s_1), ..., \eta(s_n))$. By induction hypothesis, $\eta(s_i) \in \mathrm{M}^+_{Gr}$. We make an induction on $max\{\mathrm{r}_A(\eta(s_i))\}_{i\leq n}$ to show $\eta(t) \in \mathrm{M}^+_{Gr}$. If that is 0, then $\eta(s_i) \in \mathrm{M}$, so by $\mathbf{FI}_2$, $\mathrm{I}^+(f)(\eta(s_1), ..., \eta(s_n)) \in \mathrm{M} \subseteq \mathrm{M}^+_{Gr}$. Let it work for a rank of m, and suppose that max rank is $m+1$. By $\mathbf{FI}_1$, $\eta(t) = \mathrm{I}^+(f)(\eta(s_1), ..., \eta(s_n))$ iff $\mathrm{v}(\eta(t), w) = \mathrm{I}^+(f)(\mathrm{v}(\eta(s_1), w), ..., \mathrm{v}(\eta(s_1), w))$ for any $w \in \mathrm{R}(w_0)$. As we have argued before, $\mathrm{r}_A(\mathrm{v}(\eta(s_i), w) < \mathrm{r}_A(\eta(s_i)) \leq m+1$, and therefore, by induction hypothesis, $\mathrm{v}(\eta(t), w) = \mathrm{I}^+(f)(\mathrm{v}(\eta(s_1), w), ..., \mathrm{v}(\eta(s_1), w)) \in \mathrm{M}^+_{Gr}$. By the arbitrariness of w, $\eta(t) \in \mathrm{M}^+_{Gr}$. □

Lemma H.4 *Let* $\mathcal{M}^+ \models \mathbf{Id}_A, \mathbf{Comp}_A, \mathbf{FI}_1, \mathbf{FI}_2, \mathbf{FI}_3$, $t \in Term_{\mathcal{L}}$, *and* η *be an assignment such that* $\eta(x_i) \in \mathrm{M}^+_{Gr}$ *and* $\mathrm{v}(\eta(x_i), w) = \eta(y_i)$. *If for any* $w \in \mathrm{R}(w_0)$, $\eta(t)$ *is defined, then* $\eta(t^{\uparrow\vec{x}}_{\vec{y}})$ *is defined.*

Proof By induction on the complexity of t. We skip the base cases, so suppose $t ::= f(s_1, ..., s_n)$. We have (i) $\eta(t) = \mathrm{I}^+(f)(\eta(s_1), ..., \eta(s_n))$, and so the $\eta(s_i)$ are defined. By Lemma H.3, $\eta(s_i) \in \mathrm{M}^+_{Gr}$, so by induction hypothesis, the $\eta(s^{\uparrow\vec{x}}_{i\vec{y}})$ are defined. By Lemma H.2, $\eta(s_i) = \mathrm{v}(\eta(s^{\uparrow\vec{x}}_{i\vec{y}}), w)$, so (i) means $\mathrm{I}^+(f)(\mathrm{v}(\eta(s_{1\vec{y}}^{\uparrow\vec{x}}), w), ..., \mathrm{v}(\eta(s_{n\vec{y}}^{\uparrow\vec{x}}), w))$ is always defined. By $\mathbf{FI}_3$, that means $\mathrm{I}^+(f)(\eta(s_{1\vec{y}}^{\uparrow\vec{x}}), ..., \eta(s_{n\vec{y}}^{\uparrow\vec{x}})) = \eta(t_{\vec{y}}^{\uparrow\vec{x}})$ is defined. □

We are now ready to offer the proof of Theorem 4.13:

Proof By an induction on the complexity of formulas. Let $\mathcal{M}^+ \upharpoonright_{\mathrm{M}_{Gr}^+}, w_0 \models_\eta t_1 = t_2$. We proceed by an induction on the complexity of terms. First, let $t_1 ::= y_1$, and suppose $t_2 ::= c^{n\uparrow}$. Then, we have $\mathrm{v}(\eta(x_1), w) = \mathrm{I}^+(c^{n\uparrow})$, which by $\mathbf{Id}_A$ means $\eta(x_1) = \mathrm{I}^+(c^{n+1\uparrow})$. Therefore, $\mathcal{M}^+ \upharpoonright_{\mathrm{M}_{Gr}^+}, w_0 \models_\eta x_1 = c^{n+1\uparrow}$. If $t_2 ::= x_2$, then $\mathrm{v}(\eta(x_1), w) = \mathrm{v}(\eta(x_2), w)$, so by $\mathbf{Id}_A$, $\mathcal{M}^+ \upharpoonright_{\mathrm{M}_{Gr}^+}, w_0 \models_\eta x_1 = x_2$. Let now $t_2 ::= f(s_1, ..., s_n)$. Then, we have $\mathrm{v}(\eta(x_1), w) = \mathrm{I}^+(f)(\eta(s_1), ..., \eta(s_n))$. Since this is defined, $\eta(s_i) \in \mathrm{M}_{Gr}^+$, so by Lemma H.4, $\mathrm{I}^+(f)(\eta(s_{1\vec{y}}^{\uparrow\vec{x}}), ..., \eta(s_{n\vec{y}}^{\uparrow\vec{x}})) = \eta(t_{2\vec{y}}^{\uparrow\vec{x}})$ is defined. By Lemma H.2, $\mathrm{v}(\eta(t_{2\vec{y}}^{\uparrow\vec{x}}), w) = \mathrm{I}^+(f)(\eta(s_1), ..., \eta(s_n)) = \mathrm{v}(\eta(x_1), w)$. By the arbitrariness of w, and the validity of $\mathbf{Id}_A$, that means $\eta(x_1) = \eta(t_{2\vec{y}}^{\uparrow\vec{x}})$. Thus, $\mathcal{M}^+ \upharpoonright_{\mathrm{M}_{Gr}^+}, w_0 \models_\eta x_1 = t_{2\vec{y}}^{\uparrow\vec{x}}$. The induction cases for t_1 are analogous, so we conclude our induction on the complexity of terms.

We skip the cases of negation and conjunction, so suppose now $\mathcal{M}^+ \upharpoonright_{\mathrm{M}_{Gr}^+}, w_0 \models_\eta \forall z\psi$. Then, for any $\eta' \sim_z \eta$, $\mathcal{M}^+ \upharpoonright_{\mathrm{M}_{Gr}^+}, w_0 \models_{\eta'} \psi$. By induction hypothesis, $\mathcal{M}^+ \upharpoonright_{\mathrm{M}_{Gr}^+}, w_0 \models_{\eta'} \psi_{\vec{y}}^{\uparrow\vec{x}}$, so by the arbitrariness of η', $\mathcal{M}^+ \upharpoonright_{\mathrm{M}_{Gr}^+}, w_0 \models_\eta \forall z\psi_{\vec{y}}^{\uparrow\vec{x}}$. □

Appendix I Proof of Theorem 4.14

Proof We prove by induction on the complexity of $\Sigma\varphi(\vec{x}, c)$. Let $\models^{\mathrm{T}} t_1 = t_2$. By Theorem 3.7, we have $\eta, \mathcal{M}^+, w_0 \models_\eta t_1 = t_2$. We proceed by an induction on the complexity of terms. Let $t_1 ::= c$, and suppose $t_2 ::= c_2$. Then, we have $\mathcal{M}^+, w_0 \models_\eta c = c_2$, which means $\mathrm{I}^+(c) = \mathrm{I}^+(c_2)$. By our definition of $c^{n\uparrow}$ and $c_2^{n\uparrow}$, we may see that is the case iff $\mathrm{I}^+(c^{n\uparrow}) = \mathrm{I}^+(c_2^{n\uparrow})$, and therefore $\mathcal{M}^+, w_0 \models_\eta c^{n\uparrow} = c_2^{n\uparrow}$. Let now $t_2 ::= f(s_1, ..., s_m)$. Then, we have $\mathcal{M}^+, w_0 \models_\eta c = f(s_1, ..., s_m)$, so that $\mathrm{I}^+(c) = \mathrm{I}^+(f)(\eta(s_1), ..., \eta(s_m))$. By Lemma H.2, $\mathrm{v}(\mathrm{I}^+(f)(\eta(s_1^{n\uparrow}), ..., \eta(s_m^{n\uparrow})), w) = \mathrm{I}^+(f)(\eta(s_1), ..., \eta(s_m)) = \mathrm{I}^+(c^{n\uparrow})$ for any $w \in \mathrm{R}(w_0)$. Thus, by the definition of $c^{n\uparrow}$ and the validity of $\mathbf{Id}_A$, $\mathrm{I}^+(c^{n\uparrow}) = \mathrm{I}^+(f)(\eta(s_1^{n\uparrow}), ..., \eta(s_m^{n\uparrow}))$, so that $\mathcal{M}^+, w_0 \models_\eta c^{n\uparrow} = f(s_1^{n\uparrow}, ..., s_m^{n\uparrow})$. The induction cases for t_1 are analogous.

Skipping the cases of negation and conjunction, suppose now $\models^{\mathrm{T}} \forall y\varphi(y, c)$. By Theorem 3.7, for any η,

$$(i)\mathcal{M}^+ \upharpoonright_{\mathrm{M}_{Gr}^+}, w_0 \models_\eta \forall y(\neg Ay \to \varphi(y, c)). \quad (1)$$

Let then $\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}}, w_0 \models_\eta A^s_n x$. We show, by an induction on n, that $\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}}, w_0 \models_\eta \varphi^{n\uparrow}(x, c^{n\uparrow})$.[40] Let $n = 1$, $w \in \mathrm{R}(w_0)$, and $\eta' \sim_y \eta$ be such that $\eta'(y) = \mathrm{v}(\eta(x), w)$. Then, $\eta(y) \notin \mathrm{A}$. Therefore, from (i), we have $\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}}, w_0 \models_{\eta'} \varphi(y, c)$. By the arbitrariness of w, and Theorem 4.13, we get $\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}}, w_0 \models_{\eta'} \varphi^{\uparrow}(x, c^{\uparrow})$, which means $\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}}, w_0 \models_\eta \varphi^{\uparrow}(x, c^{\uparrow})$, since $\eta'(x) = \eta(x)$. Let that work for $n \leq k$. If $\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}}, w_0 \models_\eta A^s_{k+1} x$, then by Lemma 4.7, $\mathrm{r}_A(\eta(x)) = k + 1$. Let once again $w \in \mathrm{R}(w_0)$, and $\eta'(y) = \mathrm{v}(\eta(x), w)$. Then, $\mathrm{r}_A(\eta'(y)) = k$, which means $\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}}, w_0 \models_{\eta'} A_k y$. By induction hypothesis, we get $\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}}, w_0 \models_\eta \varphi^{k\uparrow}(y, c^{k\uparrow})$. By the arbitrariness of w and Theorem 4.13, we get $\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}}, w_0 \models_{\eta'} \varphi^{k+1\uparrow}(x, c^{k+1\uparrow})$. Since $\eta'(x) = \eta(x)$, that is also the case for η, and we conclude our induction on β. Therefore, $\mathcal{M}^+ \restriction_{\mathrm{M}^+_{Gr}}, w_0 \models_\eta \forall x(A_n x \rightarrow \varphi^{n\uparrow}(x, c^{n\uparrow}))$. □

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[40] Notice the induction being performed is actually on the rank r_A, just as in the previous results.

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