SCUOLA
NORMALE
SUPERIORE

Classe di Scienze

Corso di perfezionamento in
Matematica

XXXV ciclo

The Perona-Malik problem: singular perturbation and semi-discrete approximation

Settore Scientifico Disciplinare **MAT/05**

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Anno accademico 2023–2024

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Chapter 1

Introduction

1.1 The Perona-Malik equation

We consider the following initial boundary value problem

$$\begin{cases} u_t(t,x) = \operatorname{div}\left(\dfrac{\nabla u(t,x)}{1+|\nabla u(t,x)|^2}\right) & \forall (t,x) \in (0,T)\times\Omega, \\ \partial_\nu u(t,x) = 0 & \forall (t,x) \in (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \forall x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is an open set and $T > 0$ is a positive number.

This equation was introduced by Perona and Malik in [64] in the context of image processing and, starting from this equation, researchers have developed various numerical schemes which are quite effective in improving the quality of real images.

The heuristic idea that motivates the introduction of this equation is the following. Suppose that $\Omega \subset \mathbb{R}^2$ is a rectangle in the plane and that the function u describes the gray-scale level of an image. Then it was well-known that forward diffusion (like the heat equation) is an efficient tool for removing noise. However, this has an important drawback, namely it blurs the contours of the figures. The idea of Perona and Malik was to reverse the diffusion in regions where $|\nabla u|$ is large, which should correspond to the boundaries of the figures, in order to avoid blurring, and actually enhance these edges. Indeed, equation (1.1) can be rewritten as

$$u_t = \frac{1}{1+|\nabla u|^2}\nabla^2 u \cdot \left(Id - \frac{2\nabla u \otimes \nabla u}{1+|\nabla u|^2}\right).$$

The matrix multiplying the Hessian is positive definite where $|\nabla u| < 1$ (the so-called *subcritical region*) while has a negative eigenvalue in the direction of ∇u where $|\nabla u| > 1$ (the so-called *supercritical region*), and this corresponds to reversing the diffusion from forward to backward in the direction of the gradient. For this reason this model is sometimes called *anisotropic diffusion*.

More generally, the same happens if one considers equations like

$$\begin{cases} u_t(t,x) = \operatorname{div}\left(\varphi'(|\nabla u(t,x)|)\dfrac{\nabla u(t,x)}{|\nabla u(t,x)|}\right) & \forall (t,x) \in (0,T)\times\Omega, \\ \partial_\nu u(t,x) = 0 & \forall (t,x) \in (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \forall x \in \Omega, \end{cases} \tag{1.2}$$

where $\varphi : [0,+\infty) \to [0,+\infty)$ is a non-decreasing C^2 function with derivative vanishing at infinity, that is convex in a neighborhood of the origin and concave elsewhere. In this setting, equation (1.1) corresponds to the choice $\varphi(p) = \log(1+p^2)/2$.

This equation is formally the gradient-flow with respect to the metric of $L^2(\Omega)$ of the energy functional

$$\mathrm{PM}(u) := \int_\Omega \varphi(|\nabla u(x)|)\,dx. \tag{1.3}$$

However, the lack of convexity and the sublinear growth at infinity of the Lagrangian φ imply that the functional is not lower semicontinuous, and actually its relaxation is identically null in any reasonable functional space, so that the interpretation of (1.2) as a gradient-flow can not be made rigorous.

From the PDE viewpoint, (1.2) is a quasilinear forward-backward parabolic equation, and it is well-known that backward parabolic equations are ill-posed, in the sense that a classical solution might not exist even if the initial datum u_0 is smooth. In [54] and [46] it was proved that the situation is the same also with the Perona-Malik equation.

This discrepancy between the good properties exhibited by numerical schemes and the bad behavior of the equation in any rigorous mathematical framework is usually called *Perona-Malik paradox* after [55].

Solving the paradox would mean to find a suitable notion of solution, that exists for a sufficiently large class of initial data, is consistent with the stability of numerical schemes, and to which these converge. At present, however, such a theory seems out of reach, even if many attempts have been made in the last three decades.

1.1.1 Classical solutions

Classical solutions are the most natural notion of solution in every PDE problem, so it is not surprising that many authors considered the problem in this framework.

The first paper in this direction is [54], where many different results were proved, most of which apply also to weak solutions of class C^1 (which are defined as usual via integration by parts, as we explain in the next section). In particular we have a maximum principle [54, Theorem 2.1], a comparison principle for one-dimensional solutions separated by a subcritical function or with disjoint supercritical regions [54, Theorem 4.1], and a uniqueness result under suitable assumptions [54, Theorem 5.1].

Further a priori estimates for classical solutions of (1.2) in the one-dimensional case were proved in [42, Theorem 2.2]. Let us state precisely some of them.

Theorem 1.1.1 (Theorem 2.2 in [42])**.** *Let $\Omega = (a,b)$ be an interval and let $u : [0,T)\times [a,b] \to \mathbb{R}$ be a C^2 solution of (1.2). Then we have the following estimates.*

- *The function* $t \mapsto \max\{u(t,x) : x \in [a,b]\}$ *is non-increasing, while the function* $t \mapsto \min\{u(t,x) : x \in [a,b]\}$ *is non-decreasing.*
- *The function* $t \mapsto \|u_x(t,x)\|_{L^1([a,b])}$ *is non-increasing.*
- *Let us set* $M(t) := \max\{u_x(t,x) : x \in [a,b]\}$. *If* $\varphi''(M(0)) > 0$, *then M(t) is non-increasing, while if* $\varphi''(M(0)) < 0$, *then* $M(t)$ *is non-decreasing.*

In the higher dimensional case the maximum principle continues to hold, while the properties involving derivatives can not be extended (see [42, Theorem 2.17]).

In any case, despite the good properties that they exhibit, at least in the one-dimensional case, it turns out that classical solutions are too rigid to describe the phenomena observed in simulations, as shown by a series of results that we outline now.

First of all, we have the following global existence result for subcritical initial data.

Theorem 1.1.2 (Theorem 6.1 in [54])**.** *Let* Ω *be a convex open set of class* $C^{2,\alpha}$ *for some* $\alpha \in (0,1)$, *and let us assume that* $u_0 \in C^{2,\alpha}(\Omega)$, *that* $\partial_\nu u_0 = 0$ *on* $\partial\Omega$ *and that* $|\nabla u_0(x)| < p_0$, *for some* $p_0 > 0$ *such that* $\varphi''(p) > 0$ *for* $p \in [0, p_0)$. *Then for every* $T > 0$ *there exists a unique classical solution of the problem (1.2).*

Actually, in [54] the assumptions on Ω are not clearly specified, but in some more recent works (see [56, Theorem 1.2] and the subsequent discussion) it was pointed out that convexity is probably necessary.

In any case, this result is not satisfactory because, in the context of images, subcritical initial data correspond to images with no edges (hence with no figures), and the Perona-Malik equation was introduced exactly to deal with the edges.

Unfortunately, outside the subcritical regime, things become much more complicated, because the backward regime becomes relevant and existence of solutions for backward parabolic equation is notoriously problematic. Indeed, in the model case of the backward heat equation, the existence of a classical solution requires a fast decay of the Fourier coefficients of the initial datum.

The following result obtained in [46], which applies also to C^1 solutions, shows that the situation for the Perona-Malik equation is even worse, at least in the one-dimensional case.

Theorem 1.1.3 (Theorem 5.1 in [46])**.** *Let* $\Omega = (a,b)$ *be an interval and let* $u_0 \in C^1([a,b])$ *be such that* $u_0'(a) = u_0'(b) = 0$. *If there exists* $x_0 \in (a,b)$ *such that* $u_0'(x_0) > 1$, *then problem (1.1) has no global (weak) solutions of class* C^1.

On the other hand, the next theorem, which was proved in [43], shows that local classical solutions could exist also for non-subcritical initial data.

Theorem 1.1.4 (Theorem 1.1 in [43])**.** *Let* $\Omega = (a,b)$ *be an interval and let* $\mathcal{R} \subset C^1([a,b])$ *be the set of initial data* u_0 *for which there exist a number* $T > 0$ *and a function* $u \in C^{1,2}([0,T] \times [a,b])$ *satisfying (1.1). Then* $\mathcal{R}$ *is dense in* $C^1([a,b])$.

Finally, it turns out that the situation is even more complicated in higher dimensions, since surprisingly Theorem 1.1.3 can not be extended, as shown by the following theorem (see also [45] for further differences in the evolution of subcritical regions in the one-dimensional case and in the higher dimensional case).

Theorem 1.1.5 (Theorem 1.1 in [44]). *Let $\Omega := \{x \in \mathbb{R}^2 : 1 < |x| < 5\}$ be an annulus. Then there exists a (radially symmetric) function $u_0 \in C^2(\overline{\Omega})$ such that $\{|\nabla u_0(x)| > 1\} = \{x \in \Omega : 2 < |x| < 4\}$ and the problem (1.1) admits a global classical (radially symmetric) solution $u \in C^{1,2}([0,+\infty) \times \overline{\Omega})$.*

Moreover, u becomes subcritical in finite time, namely there exists a positive time $t_0 > 0$ such that $|\nabla u(t,x)| < 1$ for every $(t,x) \in (t_0,+\infty) \times \Omega$.

1.1.2 Weak solutions

Since the notion of classical solutions is too strong to effectively describe the phenomena observed in numerical simulations, the next natural attempt is to require the existence of one less space derivative and to consider the usual notion of weak (distributional) solution for the problem (1.2), namely functions $u : [0,T) \times \Omega \to \mathbb{R}$ that admit one derivative in space and time for which

$$\int_0^T \int_\Omega \left[u_t \psi + \frac{\varphi'(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \psi \right] dxdt = 0 \qquad \forall \psi \in C_0^\infty([0,T] \times \Omega).$$

Of course, one has also to specify in which sense the derivatives of u exist and, if the spatial gradient is not continuous, in which sense the Neumann boundary conditions are satisfied. In fact, it turns out that the behavior of weak solutions depends strongly on their regularity. As we anticipated in the previous section, if one requires that $u \in C^1((0,T) \times \Omega)$, then many properties of smooth solutions, and in particular Theorem 1.1.3, continue to hold, so C^1 weak solutions share the same drawbacks of classical ones.

Surprisingly, as soon as we relax a bit this condition, we end up in a completely different situation. Indeed, it turns out that if one only requires that $u \in W^{1,\infty}((0,T) \times \Omega)$, namely that u is Lipschitz continuous, then for every smooth and non-constant initial datum there exists infinitely many different weak solutions. This result was first established in [71] in the one-dimensional case, and then extended in [56, 57, 58, 59] to the higher dimensional case. The precise statement is the following.

Theorem 1.1.6 (see [71, 56, 57, 58, 59]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$ and $u_0 \in C^{2,\alpha}(\overline{\Omega})$ be a non-constant function with $\partial_\nu u_0 = 0$ on $\partial\Omega$.*

Then there exists infinitely many functions $u \in W^{1,\infty}((0,T) \times \Omega)$ satisfying

$$\int_0^T \int_\Omega \left[u\psi_t - \frac{\varphi'(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \psi \right] dxdt = \int_\Omega \left[u(T,x)\psi(T,x) - u_0(x)\psi(0,x) \right] dx,$$

for every $\psi \in C^\infty([0,T] \times \overline{\Omega})$.

Moreover, if $d = 1$ and $\Omega = (a,b)$, these solutions are classical solution near $\partial\Omega$ and they satisfy the Neumann boundary conditions $u_x(t,a) = u_x(t,b) = 0$ in the classical sense, while if Ω is a ball and u_0 is radially symmetric, there exists infinitely many radial and non-radial solutions.

We point out that this theorem holds also if the initial datum is subcritical, namely in the setting of Theorem 1.1.2, in which the problem is well-posed from the classical point of view.

As a consequence, the notion of weak solutions is too weak to provide a satisfactory explanation of the paradox.

1.1.3 Regularizations

Since neither classical nor weak solutions seem suitable to solve the Perona-Malik paradox, researchers tried to approach the problem in a different way, by introducing various regularized versions of the problem and trying to pass to the limit. We outline below the main approximations that have been introduced in the literature, with a particular focus on two of them, the discrete approximation and the singular perturbation, that have been the object of our research that is illustrated in details in the next chapters.

Semi-discrete approximation

In analogy with the numerical scheme actually proposed by Perona and Malik (which requires a space-time discretization), one can try to approximate the problem (1.2) by discretizing only the space variable.

To be more precise, let us assume for simplicity that $\Omega = (0,1)^d$ is the unit square, and let us fix a positive integer n. Then we restrict the space variable to the lattice $\Omega_n := \mathbb{Z}^d/n \cap \overline{\Omega} = \{0, 1/n, \ldots, 1\}^d$ and we consider functions $u : [0,T) \times \Omega_n \to \mathbb{R}$. We now rewrite the problem (1.2) as

$$u_t = \sum_{i=1}^{d} D_i^- \left(\varphi'(|D_i^+ u|) \frac{D_i^+ u}{|D_i^+ u|} \right),$$

where $D_i^\pm$ denotes the difference quotient in the positive or negative i-th direction.

Since the space variable is restricted to a finite set, the equation can actually be considered as a system of ordinary differential equations, where the variables are the functions $t \mapsto u(t,x)$ with $x \in \Omega_n$.

Hence it is easy to prove that for every initial datum $u_{0n} : \Omega_n \to \mathbb{R}$ there exists a unique global solution of the discretized equation. Therefore, the problem now becomes how to pass to the limit as $n \to +\infty$.

This problem has been studied by many researchers with various different rescalings, both form the point of view of the discretized energy functional in the framework of Gamma-convergence (see [61, 30, 19, 66]) and of the differential equation (see [40, 41, 16, 42, 18, 17, 37, 47]). However, most of the results have been obtained only in the one-dimensional case, in which it is possible to prove the monotonicity of the maximum and the total variation basically with the same arguments that work in the setting of Theorem 1.1.1. In particular, the compactness of the sequence of the solutions to the discretized equations (without any rescaling) is still an open problem in the case $d \geq 2$.

We discuss more details on the state of the art concerning the dynamics of the semi-discrete approximation in Chapter 5, where we describe the content of the paper [47].

Singular perturbation

The regularization by singular perturbation is obtained by adding a term depending on higher order derivatives, multiplied by a small coefficient, to the equation and the functional, so that the non-convexity is confined into lower order terms. In the one-dimensional case $\Omega = (a, b)$, the easiest way to do this is by considering, for $\varepsilon > 0$, the energy functional

$$\mathrm{PM}_\varepsilon(u) = \int_a^b \left[\varepsilon u''(x)^2 + \varphi(u'(x))\right] dx, \tag{1.4}$$

where φ is extended to $\mathbb{R}$ as an even function. The L^2 gradient-flow of PM_ε is the equation

$$u_t = -\varepsilon u_{xxxx} + (\varphi'(u_x))_x,$$

and now the gradient-flow structure is not only formal, because the functional is lower semicontinuous, coercive, and convex with respect to the highest order variable.

This approximation was proposed by De Giorgi in [39], where he conjectured that the solutions of the singularly perturbed problem converge to a limit evolution as $\varepsilon \to 0$. The validity of this conjecture, however, is still an open problem, and very few progresses have been made toward a confirmation or a confutation of it.

The main results that are available for this regularization concern the Gamma-convergence of a suitable rescaling of the energy functionals (see [2, 14, 12]) and the study of their minimizers when a fidelity term is added, which is the content of Chapter 2 (in which we describe the paper [49]), Chapter 3 and Chapter 4 (in which we present further developments that are not yet published).

We quote also [15], which contains also numerical simulations, and [21], where a different higher order perturbation is considered.

Other regularizations

Here we briefly recall other possible regularizations that have been considered in the literature. The first one, which was proposed in [36] (see also [3]), is obtained by regularizing the gradient through convolution with the heat kernel, so that the equation becomes

$$u_t(t,x) = \operatorname{div}\left(\frac{\varphi'(|\nabla(G_\sigma * u)(t,x)|)}{|\nabla(G_\sigma * u)(t,x)|}\nabla u(t,x)\right),$$

where $G_\sigma(x) = (4\pi\sigma)^{-d/2}\exp(-|x|^2/4\sigma)$ and $\sigma > 0$ is a fixed parameter. The equation is now well-posed, but the convergence of solutions as $\sigma \to 0$ has never been investigated.

Another interesting regularization was proposed in [4] and is obtained by a convolution in time, namely

$$u_t(t,x) = \operatorname{div}\left(\frac{\varphi'(|\nabla(\theta * u)(t,x)|)}{|\nabla(\theta * u)(t,x)|}\nabla u(t,x)\right),$$

where $\theta(t)$ is a fixed function that depends only on the time variable like, for example, $\theta(t) = \delta^{-1}\mathbb{1}_{(0,\delta)}(t)$, for some $\delta > 0$. These introduce a time-delay in the equation, namely the gradient is now replaced by its average over an interval of preceding times (instead of a spatial average as in the previous approximation). Also in this case, the equation is well-posed for every $\delta > 0$, but the convergence as $\delta \to 0$ has never been studied.

Other possible regularizations were considered in [52, 50] and involve fractional derivatives, but also in this case the convergence as the regularization parameter vanishes has not been investigated.

Finally, we mention the series of paper [11, 68, 69, 70, 22, 23, 24, 25, 26, 27], where the authors considered the equation

$$v_t = \Delta\varphi'(v), \tag{1.5}$$

and its pseudoparabolic regularization

$$v_t = \Delta\varphi'(v) + \varepsilon\Delta[\psi(v)]_t, \tag{1.6}$$

with various different choices of the function $\psi : \mathbb{R} \to \mathbb{R}$.

This equation is still of forward-backward type and is strongly related to (1.2), with the difference that here the direction of the diffusion is determined by the values of u instead of $|\nabla u|$. This relation is even clearer in the one-dimensional case, since u is a solution of (1.2) if and only if $v = u_x$ is a solution of (1.5). Hence (1.6) provides another possible regularization of the Perona-Malik equation.

1.2 Further problems that I have studied

During the period of my PhD course I also continued to investigate the problems that I had studied for my bachelor and master thesis projects, namely non-local functionals related with Sobolev and BV norms and the phase-field (or Allen-Cahn) approximation of surface energies and flows.

1.2.1 Non-local functionals and constant functions

In this section we briefly describe the content of the papers [8, 9, 10, 48, 65], which concern some characterizations of Sobolev and BV spaces or of constant functions by means of non-local functionals.

The origin of this research is the following result, known as *BBM formula* after [28].

Theorem 1.2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz open set or the whole space and $p \in [1, +\infty)$ be a real number. Let $\rho : [0, +\infty) \to [0, +\infty)$ be an integrable function and for every $\varepsilon > 0$ let us set $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(\varepsilon^{-1}x)$.*

Let us consider the functionals

$$G_{\varepsilon,p}(u) := \iint_{\Omega\times\Omega} \left(\frac{|u(y) - u(x)|}{|y - x|}\right)^p \rho_\varepsilon(|y - x|)\, dx\, dy,$$

and

$$G_{0,p}(u) := \begin{cases} \int_\Omega |\nabla u(x)|^p\, dx & \textit{if } p > 1 \textit{ and } u \in W^{1,p}(\Omega), \\ |Du|(\Omega) & \textit{if } p = 1 \textit{ and } u \in BV(\Omega), \\ +\infty & \textit{otherwise.} \end{cases}$$

Then there exists a constant $C(d,p)>0$ *such that*

$$\Gamma-\lim_{\varepsilon\to0^+}G_{\varepsilon,p}(u)=\lim_{\varepsilon\to0^+}G_{\varepsilon,p}(u)=C(d,p)\|\rho\|_{L^1(0,+\infty)}G_{0,p}(u)\qquad\forall u\in L^p(\Omega).$$

As a corollary of this result, Brezis obtained in [32] the following characterization of constant functions.

Proposition 1.2.2. *Let* $\Omega\subset\mathbb{R}^d$ *be an open set and* $p\in[1,+\infty)$ *be a real number. Let* $u:\Omega\to\mathbb{R}$ *be a measurable function. Then it turns out that*

$$u \textit{ is (essentially) constant} \iff \iint_{\Omega\times\Omega}\left(\frac{|u(y)-u(x)|}{|y-x|}\right)^p\frac{1}{|y-x|^d}\,dx\,dy<+\infty.$$

Several generalizations of this result have been proposed. One of these concerns the functional

$$\mathscr{F}_\omega(u)=\iint_{\Omega\times\Omega}\omega\left(\frac{|u(y)-u(x)|}{|y-x|}\right)\frac{1}{|y-x|^d}\,dx\,dy,$$

where $\omega:[0,+\infty)\to[0,+\infty)$ is a continuous function.

In [53] Ignat investigated the problem of determining necessary and/or sufficient conditions on the functions ω and u so that the following implication holds

$$u \text{ is (essentially) constant} \iff \mathscr{F}_\omega(u)<+\infty.$$

In [48] we considered this problem and we answered, at least partially, to many of the questions raised in [53]. In particular, we found an example that shows that a condition proposed in [53] is not sufficient to ensure the validity of the implication above for every measurable function u, but becomes sufficient if one assumes in addition that u is bounded and approximately differentiable almost everywhere, or if ω satisfies some additional conditions.

Another problem originated from the BBM formula concerns the functionals

$$\Lambda_{\delta,p}(u):=\iint_{\Omega\times\Omega}\phi\left(\frac{|u(y)-u(x)|}{\delta}\right)\frac{\delta^p}{|y-x|^{d+p}}\,dx\,dy. \tag{1.7}$$

In [63] (after many other papers that are quoted therein) the asymptotic behavior of $\Lambda_{\delta,p}$ was studied in the case $\phi(t)=\mathbb{1}_{(1,+\infty)}(t)$, and the following result was proved.

Theorem 1.2.3. *Let* $\Omega\subset\mathbb{R}^d$ *be a bounded Lipschitz open set or the whole space and let us consider the functionals (1.7) with* $\phi(t)=\mathbb{1}_{(1,+\infty)}(t)$.

If $p\in(1,+\infty)$, *then there exist two constants* $C(d,p)>0$ *and* $k(d,p)\in(0,1)$ *such that*

$$\lim_{\delta\to0^+}\Lambda_{\delta,p}(u)=C(d,p)G_{0,p}(u)\qquad\forall u\in L^p(\Omega),$$

and

$$\Gamma-\lim_{\delta\to0^+}\Lambda_{\delta,p}(u)=C(d,p)k(d,p)G_{0,p}(u)\qquad\forall u\in L^p(\Omega).$$

If $p=1$, *then there exist two constants* $C(d,1)>0$ *and* $k(d,1)\in(0,1)$ *such that*

$$\lim_{\delta\to0^+}\Lambda_{\delta,1}(u)=C(d,1)G_{0,1}(u)\qquad\forall u\in C^1_c(\Omega),$$

and

$$\Gamma - \lim_{\delta\to 0^+} \Lambda_{\delta,1}(u) = C(d,1)k(d,1)G_{0,1}(u) \qquad \forall u \in L^1(\Omega),$$

but there exists a function $u \in W^{1,1}(\Omega)$ *such that*

$$\lim_{\delta\to 0^+} \Lambda_{\delta,1}(u) = +\infty.$$

The paper [34] extended this result to more general functions $\phi : [0,+\infty) \to [0,+\infty)$, at least in the case $p = 1$. The result is the following.

Theorem 1.2.4. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz open set or the whole space and let* $\phi : [0,+\infty) \to [0,+\infty)$ *be a bounded, non-decreasing and lower semicontinuous function such that*

$$\phi(t) \le at^2 \qquad \text{and} \qquad \int_0^{+\infty} \frac{\phi(t)}{t^2}\, dt = 1.$$

Then there exist two constants $C(d) > 0$ *and* $k(d,\phi) \in (0,1]$ *such that*

$$\lim_{\delta\to 0^+} \Lambda_{\delta,1}(u) = C(d)G_{0,1}(u) \qquad \forall u \in C^1_c(\Omega),$$

and

$$\Gamma - \lim_{\delta\to 0^+} \Lambda_{\delta,1}(u) = C(d)k(d,\phi)G_{0,1}(u) \qquad \forall u \in L^1(\Omega).$$

In [9] we gave a new proof of the Gamma-convergence result in Theorem 1.2.3, which provides the explicit value of the constant $k(d,p)$, which is independent of d and was only conjectured in [63], since this constant was defined implicitly through a cell problem. In particular, we were able to solve the cell problem, and this made our proof much simpler than the previous one, because we could provide explicit recovery sequences.

Moreover, in [10] we extended our arguments to more general functions ϕ and as a consequence we were able to answer some open questions raised in [34] concerning the dependence of $k(d,\phi)$ on the shape of ϕ.

Our approach consists in reducing the liminf inequality of the Gamma-convergence first to the one-dimensional case, then to functions which take values in a finite set and finally to piecewise constant functions with steps of equal length. At this point we exploit a combinatorial rearrangement inequality to further reduce ourselves to a minimization problem on monotone step functions, which can be easily solved. In this way we obtain the liminf inequality with an explicit value of the constant in front of the limit functional, so the limsup inequality can be proved just by providing an explicit recovery sequence for which we obtain the required explicit value, and the structure of this recovery sequence is suggested by the argument developed for the proof of the liminf inequality (see also [8] for a short summary of our techniques).

More recently, Brezis, Seeger, Van Schaftingen and Yung introduced in [35] the following family of non-local functionals

$$F_{\gamma,\lambda,p}(u) := \iint_{E_{\gamma,\lambda,p}(u)} \lambda^p |y-x|^{d-\gamma}\, dx\, dy, \tag{1.8}$$

where $\gamma \in \mathbb{R}$ and

$$E_{\gamma,\lambda,p}(u) := \left\{(x,y) \in \Omega \times \Omega : |u(y) - u(x)| \geq \lambda |y-x|^{1+\gamma/p}\right\}.$$

We point out that when $\gamma = -p$ (and $\lambda = \delta$) these functionals coincide with the functionals defined in (1.7) with $\phi(t) = \mathbb{1}_{(1,+\infty)}(t)$.

Motivated also by [31], in [65] we considered the case $p = 1$ and we proved some estimates for the pointwise limit as $\lambda \to +\infty$ of $F_{\gamma,\lambda,1}(u)$ in the case in which $u \in BV(\mathbb{R}^d)$ and $\gamma > 0$. This gave a (very) partial answer to some of the questions raised in [35, 31]. However, many interesting problems concerning the functionals (1.8) are still open, and we plan to continue this research in the future.

1.2.2 De Giorgi's approximation of the Willmore functional

In this section we briefly describe the content of the paper [13], in which we have studied the following conjecture posed by De Giorgi in [38, Conjecture 4].

Conjecture 1.2.5. *Let $n \geq 2$ be an integer number and let $E \subset \mathbb{R}^n$ be a set whose boundary $\Sigma := \partial E$ is a hypersurface of class C^2. For any open set $\Omega \subset \mathbb{R}^n$ and any positive number $\lambda > 0$ let us consider the following family of functionals, indexed by the parameter $\varepsilon > 0$,*

$$\mathcal{DG}_\varepsilon(u,\Omega) := \int_\Omega \left[\left(2\varepsilon\Delta u - \frac{\sin u}{\varepsilon}\right)^2 + \lambda\right]\left[\varepsilon|\nabla u|^2 + \frac{1-\cos u}{\varepsilon}\right] dx, \tag{1.9}$$

if $u \in W^{2,1}(\Omega)$, and $\mathcal{DG}_\varepsilon(u) := +\infty$, if $u \in L^1(\Omega) \setminus W^{2,1}(\Omega)$.

Then there exists a constant $k \in \mathbb{R}$ such that

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon \to 0^+} \mathcal{DG}_\varepsilon(2\pi\chi_E, \Omega) = c\lambda\mathcal{H}^{n-1}(\Sigma \cap \Omega) + k\int_{\Sigma\cap\Omega} H^2 d\mathcal{H}^{n-1},$$

where χ_E is the characteristic function of the set E (that is equal to one inside E and null outside), $c = 8\sqrt{2}$, $H(y)$ is the mean curvature of Σ at the point y and $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

We point out that we have added a factor 2 in front of the laplacian so that, if $u \in W^{2,2}(\Omega)$, the squared term is really the L^2-gradient of the Allen-Cahn energy, that is the functional

$$E_\varepsilon(u,\Omega) := \int_\Omega \left[\varepsilon|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right] dx, \tag{1.10}$$

if $u \in W^{1,2}(\Omega)$, and $E_\varepsilon(u) := +\infty$, if $u \in L^1(\Omega) \setminus W^{1,2}(\Omega)$. Here $W\colon \mathbb{R} \to [0,+\infty)$ is a multiple-well potential, like $W(u) = 1 - \cos u$, as in the conjecture of De Giorgi, or the more popular double-well potential $W(u) = (1-u^2)^2$.

The Γ-convergence of the family E_ε is the object of the celebrated Modica-Mortola theorem which in this case says that

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon \to 0^+} E_\varepsilon\left(\chi_E^{a,b}, \Omega\right) = \sigma_W^{a,b}\,\mathrm{Per}(E,\Omega),$$

where now E has finite perimeter $\mathrm{Per}(E,\Omega)$ in Ω, $a<b$ are two consecutive zeros of W, $\chi_E^{a,b}$ is a suitable modification of the characteristic function, defined as

$$\chi_E^{a,b}(x) := \begin{cases} a & \text{if } x \notin E, \\ b & \text{if } x \in E, \end{cases} \tag{1.11}$$

and

$$\sigma_W^{a,b} := 2\int_a^b \sqrt{W(u)}\,du. \tag{1.12}$$

We observe that in the case $W(u) = 1-\cos u$ it turns out that $\sigma_W^{0,2\pi} = c$, so De Giorgi's conjecture is actually saying that the functional

$$G_\varepsilon(u,\Omega) := \int_\Omega \left[2\varepsilon\Delta u - \frac{W'(u)}{\varepsilon}\right]^2 \left[\varepsilon|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right] dx \tag{1.13}$$

is an approximation for a multiple of the Willmore functional

$$\mathcal{W}(\Sigma,\Omega) := \int_{\Sigma\cap\Omega} H^2 d\mathcal{H}^{n-1},$$

provided Σ is of class C^2.

This seems reasonable because the mean curvature is known to represent the first variation of the perimeter and the term $2\varepsilon\Delta u - W'(u)/\varepsilon$ represents the gradient of the functional E_ε. Moreover, if $\{u_\varepsilon\}$ is a family of functions that converges to $\chi_E^{a,b}$ in L^1, then the energy densities, that are the (normalized) measures

$$\mu_\varepsilon := \frac{1}{\sigma_W^{a,b}} \left[\varepsilon|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\right] \mathscr{L}^n, \tag{1.14}$$

where $\mathscr{L}^n$ is the n-dimensional Lebesgue measure, in the limit should be larger than or equal to the measure $\mathcal{H}^{n-1} \llcorner \Sigma$, as a consequence of the Γ-convergence of E_ε.

In the case $W(u) = (1-u^2)^2$, it was proved in [20] that this is what actually happens when one considers the usual recovery sequences for the Γ-limit of E_ε. More specifically, an estimate from above for the Γ-$\limsup$ of $\mathcal{DG}_\varepsilon$ with a positive constant $k>0$ was proved. Moreover, the authors of [20] proposed to investigate the functional

$$\widehat{G}_\varepsilon(u,\Omega) := \int_\Omega \frac{1}{\varepsilon}\left(2\varepsilon\Delta u - \frac{W'(u)}{\varepsilon}\right)^2 dx,$$

in place of G_ε, in order to simplify the problem, and they proved a Γ-$\limsup$ estimate (with a positive constant $k>0$) also for the functionals $E_\varepsilon + \widehat{G}_\varepsilon$.

The modification is motivated by the fact that the second factor in the integrand of G_ε should be proportional to ε^{-1} near Σ, while the contribution of both factors far from this boundary should not be relevant for the Γ-limit.

However, the Γ-$\liminf$ estimate turned out to be much more involved and, after some partial results, the problem has been solved in dimensions 2 and 3 by Röger and Schätzle [67], while it is still open in higher dimensions. More precisely, in the special case $W(u) = (1-u^2)^2$ and $n \in \{2,3\}$, Röger and Schätzle were able to prove that if Σ is of class C^2 then

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0^+} (E_\varepsilon + \widehat{G}_\varepsilon)\left(\chi_E^{-1,1},\Omega\right) = \sigma_W^{-1,1}\mathcal{H}^{n-1}(\Sigma\cap\Omega) + k\int_{\Sigma\cap\Omega} H^2\, d\mathcal{H}^{n-1},$$

for some positive constant $k > 0$. Moreover, they also proved (see Theorem 4.1 and Theorem 5.1 in [67]) that if $\{u_\varepsilon\} \subset W^{2,2}(\Omega)$ is a family of functions for which

$$E_\varepsilon(u_\varepsilon,\Omega) + \widehat{G}_\varepsilon(u_\varepsilon,\Omega) \leq C,$$

then any weak* limit point of the measures $\{\mu_\varepsilon\}$ is an integral $(n-1)$-varifold.

Our main result in [13] is a proof that, surprisingly, De Giorgi's conjecture holds true with $k = 0$. This means that, as opposite to $\widehat{G}_\varepsilon$, the functional G_ε does not contribute to the Γ-limit of $\mathcal{DG}_\varepsilon$ that, instead, turns out to be the same as the one obtained with the functionals λE_ε alone, and this holds with a quite general class of potentials W. This also implies that Conjecture 5 in [38] does not hold, because the perimeter alone, if considered as a function of Ω, is clearly subadditive.

The proof of course consists in finding a family $\{u_\varepsilon\} \subset W^{2,1}_{loc}(\mathbb{R}^n)$ of functions converging in L^1 to $\chi_E^{a,b}$ for which

$$\lim_{\varepsilon\to 0^+} E_\varepsilon(u_\varepsilon,\mathbb{R}^n) = \sigma_W^{a,b}\cdot\mathcal{H}^{n-1}(\Sigma) \quad\text{and}\quad \lim_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon,\mathbb{R}^n) = 0.$$

We construct these functions by perturbing the classical recovery sequences for E_ε in such a way that the two factors in the functional G_ε concentrate in different regions, so that their product becomes small.

We recall that in the functional $\widehat{G}_\varepsilon$ the second factor has been replaced by the constant ε^{-1}, so our strategy, that allows the first factor to be very large in regions where the other one is small, is not effective in decreasing the value of the modified functional, because in this case such regions do not exist (actually $\widehat{G}_\varepsilon(u_\varepsilon,\mathbb{R}^n) \to +\infty$ for our choice of $\{u_\varepsilon\}$).

As a corollary of our main result, we obtain that the limit of the energy densities μ_ε is not necessarily $(n-1)$-rectifiable, even if the functionals are equibounded. In fact, it can also happen that these measures converge to a Dirac mass or, more generally, to a measure that is not absolutely continuous with respect to $\mathcal{H}^{n-1}$.

In the opposite direction, despite this unexpected result, it seems that the boundedness of the family $\{G_\varepsilon(u_\varepsilon,\mathbb{R}^n)\}$ still carries some information on the behavior of the energy densities. Indeed, in the toy model of radial symmetry, we proved that if $\{u_\varepsilon\} \subset W^{2,1}_{loc}(\mathbb{R}^n) \cap W^{1,2}_{loc}(\mathbb{R}^n)$ is a family of functions with

$$E_\varepsilon(u_\varepsilon,\mathbb{R}^n) + G_\varepsilon(u_\varepsilon,\mathbb{R}^n) \leq C,$$

then any weak* limit of μ_ε is an integral $(n-1)$-varifold if restricted to $\mathbb{R}^n \setminus \{0\}$, which of course in this case is simply a union of concentric spheres. The proof of this fact is based on a blow-up argument.

We observe that the radial symmetry and the removal of the origin (where a Dirac mass could appear) automatically imply that the limit measure is absolutely continuous with respect to $\mathcal{H}^{n-1}$, but these assumptions do not prevent a priori that this measure may be supported on sets with larger dimension. In particular, if one only assumes the boundedness of the energies $E_\varepsilon(u_\varepsilon, \mathbb{R}^n)$, without additional assumptions on $G_\varepsilon(u_\varepsilon, \mathbb{R}^n)$, then the limit of the energy densities can be any positive finite radially symmetric measure, so the integrality of the limit measure is not trivial.

We point out that the radial symmetry is not even ruling out the "pathology" that leads to the disappearance of G_ε in the limit, since the recovery sequence for the Γ-limit of $\mathcal{DG}_\varepsilon$ when E is a ball can be made of radially symmetric functions.

1.3 Structure of the thesis

This thesis is organized as follows. In Chapter 2 we describe the paper [49], which deals with minimizers of the singularly perturbed Perona-Malik functional, and we also extend some results to a slightly more general setting. Chapter 3 contains some original (not yet published) results concerning higher order blow-up of the minimizers considered in Chapter 2. In Chapter 4 we present a partial extension of the results in Chapter 2 to the higher dimensional case. Finally, in Chapter 5 we describe the preprint [47], in which we proved some monotonicity results for a general class of evolution curves in dimension one, which includes any limit of the discrete approximation of the Perona-Malik equation and also any weak solution. In section 5.5 and section 5.6 we also extend some of the results in [47] to a more general setting, and we deduce some monotonicity properties of level sets of generalized solutions.

Chapter 2

Singular perturbation: first order blow-up

2.1 Introduction

In this chapter we describe the content of the paper [49], so we consider the minimum problem for the one-dimensional functional

$$\mathrm{PMF}(u) := \int_0^1 \log\left(1 + u'(x)^2\right)\,dx + \beta \int_0^1 (u(x) - f(x))^2\,dx, \qquad (2.1.1)$$

where $\beta > 0$ is a real number, and $f \in L^2((0,1))$ is a given function that we call *forcing term*. The second integral is a sort of *fidelity term*, tuned by the parameter β, that penalizes the distance between u and the forcing term f. The principal part of (2.1.1) is one-dimensional version of the functional (1.3) with the original Perona-Malik lagrangian $\varphi(p) := \log(1 + p^2)$. As a consequence, we know that

$$\inf\left\{\mathrm{PMF}(u) : u \in C^1([0,1])\right\} = 0 \qquad \forall f \in L^2((0,1)).$$

Singular perturbation of the Perona-Malik functional Let us consider the following version of (1.4),

$$\mathrm{PMF}_\varepsilon(u) := \int_0^1 \left\{\varepsilon^{10}|\log\varepsilon|^2 u''(x)^2 + \log\left(1 + u'(x)^2\right) + \beta(u(x) - f(x))^2\right\} dx, \quad (2.1.2)$$

where we have added the fidelity term, and the bizarre form of the ε-dependent coefficient is just aimed at preventing the appearance of decay rates defined in an implicit way in the sequel. For every choice of $\varepsilon \in (0,1)$ and $\beta > 0$ the model is well-posed, in the sense that the minimum problem for (2.1.2) admits at least one minimizer of class C^2 for every choice of the forcing term $f \in L^2((0,1))$. Our goal is to investigate the asymptotic behavior of minima and minimizers as $\varepsilon \to 0^+$. Before describing our results, it is useful to discuss a related problem that has already been studied in the literature.

The Alberti-Müller model Let us consider the functional

$$\mathrm{AM}_\varepsilon(u) := \int_0^1 \left\{\varepsilon^2 u''(x)^2 + (u'(x)^2-1)^2 + \beta(x)u(x)^2\right\} dx, \tag{2.1.3}$$

where $\beta \in L^\infty((0,1))$ is positive for almost every $x \in (0,1)$. The minimizers of (2.1.3) with periodic boundary conditions were studied by G. Alberti and S. Müller in [1] (see also [62]). In this model the forcing term f is identically 0, and the dependence on first order derivatives is described by the double-well potential $\varphi(p) := (p^2-1)^2$. As in (2.1.2) the function $\varphi(p)$ is non-convex, but in this case its convex envelope vanishes just for $|p| \le 1$, while it coincides with $\varphi(p)$ elsewhere, and in particular it is coercive.

From the heuristic point of view, minimizers to (2.1.3) would like to be identically 0, but with constant derivative equal to ± 1. Of course this is not possible if we think of u and u' as functions, but it becomes possible if we consider u as a function whose "derivative" u' is a Young measure. More formally, given a family $\{u_\varepsilon\}$ of minimizers to (2.1.3), one can show that $u_\varepsilon \to 0$ uniformly, $u'_\varepsilon \rightharpoonup 0$ weakly in $L^4((0,1))$, and more precisely u'_ε converges to the Young measure that in every point $x \in (0,1)$ assumes the two values ± 1 with probability $1/2$.

The next step consists in analyzing the asymptotic profile of minimizers. The intuitive idea is that minimizers develop a *microstructure* at some scale $\omega(\varepsilon)$, and this microstructure resembles a triangular wave (sawtooth function). In other words, one expects minimizers to be of the form

$$u_\varepsilon(x) \sim \omega(\varepsilon)\phi\left(\frac{x}{\omega(\varepsilon)} + b(\varepsilon)\right), \tag{2.1.4}$$

where

- the function ϕ that describes the asymptotic profile of minimizers is a triangular wave with slopes ± 1, for example the function defined by $\phi(x) := |x| - 1$ for every $x \in [-2,2]$, and then extended by periodicity to the whole real line,
- $\omega(\varepsilon)$ is a suitable scaling factor that vanishes as $\varepsilon \to 0^+$ and is proportional to the asymptotic "period" of minimizers (which, however, are not necessarily themselves periodic),
- $b(\varepsilon)$ is a sort of phase parameter, that can be assumed to be less than the period of ϕ.

We point out that the limit of u'_ε as a Young measure carries no information concerning the asymptotic behavior of $\omega(\varepsilon)$, and it does not even imply the existence of any form of asymptotic period or asymptotic profile.

The first big challenge is giving a rigorous formal meaning to an asymptotic expansion of the form (2.1.4). In [1] the formalization relies on the notion of Young measure with values in compact metric spaces. In a nutshell, starting from every minimizer u_ε, the authors consider the function that associates to every $x \in (0,1)$ the rescaled function

$$y \mapsto \frac{u_\varepsilon(x+\omega(\varepsilon)y)}{\omega(\varepsilon)},$$

where $\omega(\varepsilon) = \varepsilon^{1/3}$. This new function is interpreted as a Young measure on the interval $(0,1)$ with values in $L^\infty(\mathbb{R})$, which is a *compact* metric space with respect to the distance according to which g_n converges to g_∞ if and only if $\arctan(g_n)$ converges to $\arctan(g_\infty)$ with respect to the weak* convergence in $L^\infty(\mathbb{R})$. The result is that this family of Young measures converges (in the sense of Young measures with values in a compact metric space) to a limit Young measure that in almost every point is concentrated in the translations of the triangular wave. This statement is a rigorous, although rather abstract and technical, formulation of expansion (2.1.4).

From Young measures to varifolds There are some notable differences between our model and (2.1.3). The first one is that in our case the trivial forcing term $f \equiv 0$ would lead to the trivial solution $u_\varepsilon \equiv 0$ for every $\varepsilon \in (0,1)$. Therefore, here a nontrivial forcing term is required if we want nontrivial solutions.

The second difference lies in the growth of the convex envelope of $\varphi(p)$. In the case of (2.1.3) the convex envelope grows at infinity as p^4, and this guarantees a uniform bound in $L^4((0,1))$ for the derivatives of all sequences with bounded energy. In our case the convex envelope vanishes identically, and therefore there is no hope to obtain bounds on derivatives in terms of bounds on the energies.

The third, and more relevant, difference lies in the construction of the convex envelope. In the case of (2.1.3) the convex envelope of φ vanishes in the interval $[-1,1]$ because every p in this interval can be written as a convex combination of ± 1, and $\varphi(1) = \varphi(-1) = 0$. This is the ultimate reason why the derivatives of minimizers tend to stay close to the two values ± 1 when ε is small enough.

In our case the convex envelope of φ vanishes identically on the whole real line, but no real number p can be written as the convex combination of two distinct points where φ vanishes. Roughly speaking, the vanishing of the convex envelope is achieved only in the limit, in some sense by writing every real number p as a convex combination of 0 and $\pm\infty$, depending on the sign of p. This implies that minimizers u_ε tend to assume a staircase-like shape, with regions where they are "almost horizontal" and regions where they are "almost vertical" (as described in the left and central section of Figure 2.1). From the technical point of view, this means that there is no hope that the family $\{u'_\varepsilon\}$ admits a limit in the sense of Young measures.

This is the point in which varifolds come into play, because varifolds allow "functions" whose graph has in every point a mix of horizontal and vertical "tangent" lines.

Our results In our analysis of the asymptotic behavior of minima and minimizers, we restrict ourselves to forcing terms f that are more regular than just L^2, and we prove three main results.

- The first result (Theorem 2.2.2) concerns the asymptotic behavior of minima when $f \in H^1((0,1))$. We prove that the minimum m_ε of (2.1.2) over $H^2((0,1))$ satisfies

  $$m_\varepsilon \sim c_0 \varepsilon^2 |\log \varepsilon| \int_0^1 |f'(x)|^{4/5}\, dx$$

  for a suitable real constant c_0. This is a slight generalization of the result obtained in [49], where we considered only the case $f \in C^1([0,1])$.

- The second result (Theorem 2.2.9) concerns the asymptotic behavior of minimizers u_ε when $f \in C^1([0,1])$. To this end, for every family $x_\varepsilon \to x_0 \in (0,1)$ we consider the families of functions

$$y \mapsto \frac{u_\varepsilon(x_\varepsilon + \omega(\varepsilon)y) - f(x_\varepsilon)}{\omega(\varepsilon)} \qquad \text{and} \qquad y \mapsto \frac{u_\varepsilon(x_\varepsilon + \omega(\varepsilon)y) - u_\varepsilon(x_\varepsilon)}{\omega(\varepsilon)}, \quad (2.1.5)$$

  which correspond to the intuitive idea of zooming the graph of a minimizer u_ε in a neighborhood of $(x_\varepsilon, f(x_\varepsilon))$ and $(x_\varepsilon, u_\varepsilon(x_\varepsilon))$ at scale $\omega(\varepsilon)$. We show that, when $\omega(\varepsilon) = \varepsilon|\log\varepsilon|^{1/2}$, these functions converge (up to subsequences) in a rather strong sense (strict convergence of bounded variation functions, see Definition 2.2.6) to a piecewise constant function, a sort of staircase with steps whose height and length depend on $f'(x_0)$. This result provides a quantitative description of the staircase-like microstructure of minimizers, with a notion of convergence that is much stronger than weak* convergence in $L^\infty(\mathbb{R})$, and without the technical machinery of Young measures with values in metric spaces (see Remark 2.2.12).

- The third result (Theorem 2.2.14) shows that $u_\varepsilon \to f$ first in the sense of uniform convergence, then in the sense of strict convergence of bounded variation functions, and finally in the sense of varifolds, provided that we consider the graph of f as a varifold with a suitable density and a suitable combination of horizontal and vertical tangent lines in every point.

The three results described above are only the first order analysis of what is ultimately a *multi-scale problem*. In chapter 3, we investigate higher-resolution zooms of minimizers (from the center to the right of Figure 2.1), in order to reveal the exact structure of the horizontal and vertical parts of each step of the staircase.

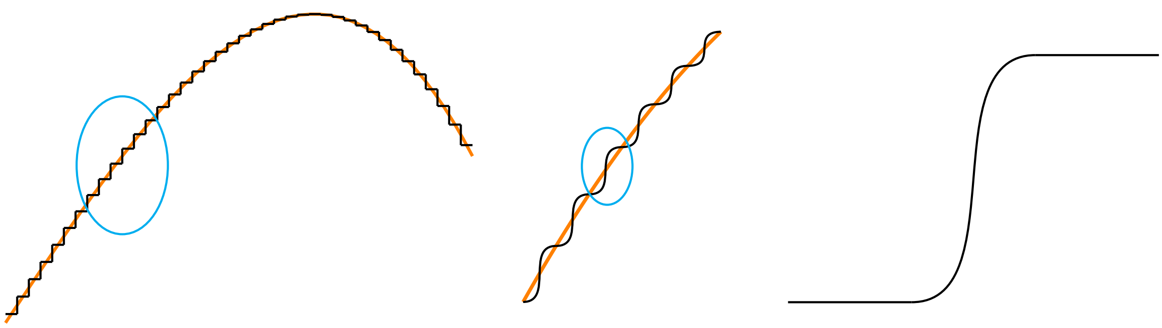

Figure 2.1: description of the multi-scale problem at three levels of resolution. Left: staircasing effect around the forcing term. Center: zoom of the staircase in a region. Right: cubic transition between two consecutive steps.

Overview of the technique Our analysis relies on Gamma-convergence techniques. The easy remark is that minimum values of (2.1.2) tend to 0, and minimizers tend to the forcing term in $L^2((0,1))$. This is because the unstable character of (1.3) comes back again when $\varepsilon \to 0^+$, and forces the Gamma-limit of the family of functionals (2.1.2) to be identically 0.

More delicate is finding the vanishing order of minimum values, and the fine structure of minimizers as $\varepsilon \to 0^+$. The starting observation is that, if $v_\varepsilon(y)$ denotes the blow-up defined in (2.1.5) on the left, with $\omega(\varepsilon) = \varepsilon|\log\varepsilon|^{1/2}$, then $v_\varepsilon(y)$ minimizes a rescaled version of (2.1.2), namely the functional

$$\mathrm{RPMF}_\varepsilon(v) := \int_{I_\varepsilon} \left\{ \varepsilon^6 (v'')^2 + \frac{1}{\varepsilon^2|\log\varepsilon|} \log\left(1+(v')^2\right) + \beta\,(v-g_\varepsilon)^2 \right\} dy, \tag{2.1.6}$$

where the new forcing term $g_\varepsilon(y)$ is a suitable blow-up of $f(x)$, and the new integration interval I_ε depends on the blow-up center x_ε, but in any case its length is equal to $\omega(\varepsilon)^{-1}$, and therefore diverges.

If $f(x)$ is of class C^1, then $g_\varepsilon(y) \to f'(x_0)y$ when $x_\varepsilon \to x_0$. Moreover, the results of [2, 14] suggest that, if we consider the functional (2.1.6) restricted to a *finite fixed interval* (a,b), its Gamma-limit has the form

$$\alpha_0 J_{1/2}(v) + \beta \int_a^b \left(v(y) - f'(x_0)y\right)^2\, dy, \tag{2.1.7}$$

where α_0 is a suitable positive constant, and the functional $J_{1/2}(v)$ is finite only if v is a "pure jump function" (see Definition 2.3.1), and in this class it coincides with the sum of the square roots of the jump heights of v.

At the end of the day, this means that the minimum problem for (2.1.2) can be approximated, at a suitable small scale, by a family of minimum problems for functionals such as (2.1.7), and these minimum problems, due to the simpler form and to the linear forcing term, can be solved almost explicitly.

However, things are not so simple. A first issue is that the integration intervals I_ε in (2.1.6) invade the whole real line. This forces us to work with local minimizers (namely minimizers up to perturbations with compact support) instead of global minimizers. So we have to adapt the classical Gamma-convergence results in order to deal with local minimizers, and we need also to classify all local minimizers to (2.1.7). These local minimizers are characterized in Proposition 2.4.5, and they turn out to be staircases whose steps have length and height that depend on $f'(x_0)$.

The second issue is compactness. We observed before that a bound on $\mathrm{PM}_\varepsilon(u_\varepsilon)$ does not provide compactness of the family $\{u_\varepsilon\}$ in any reasonable space. After rescaling and introducing (2.1.6), on the one hand the good news is that a classical coerciveness result implies that a uniform bound on $\mathrm{RPMF}_\varepsilon(v_\varepsilon)$ is enough to deduce that the family $\{v_\varepsilon\}$ is relatively compact, for example in L^2. On the other hand, the bad news is that an asymptotic estimate of the form $\mathrm{PM}_\varepsilon(u_\varepsilon) \sim c_0\omega(\varepsilon)^2$ yields only a uniform bound on $\omega(\varepsilon)\,\mathrm{RPMF}_\varepsilon(v_\varepsilon)$, which does not exclude that $\mathrm{RPMF}_\varepsilon(v_\varepsilon)$ might diverge as $\varepsilon \to 0^+$.

We overcome this difficulty by showing that a bound of this type in some interval yields a true uniform bound for $\mathrm{RPMF}_\varepsilon(v_\varepsilon)$ in a *smaller interval*, and this is enough to guarantee the compactness of local minimizers. This improvement of the bound (see Proposition 2.6.5) requires a delicate iteration argument in a sequence of nested intervals, which probably represents the technical core of [49].

Possible extensions In order to keep the length of this presentation reasonable, we decided to focus only on the singular perturbation (2.1.2) of the original functional with the logarithm. Nevertheless, many parts of the theory can be extended to more general models. In particular, in [66] we obtained similar results starting from the discrete approximation of the Perona-Malik functional. We discuss further possible generalizations in section 2.7.

Dynamic consequences We hope that our variational analysis could be useful in the investigation of solutions to the evolution equation (1.1). Numerical experiments with different approximating schemes seem to suggest that solutions develop *instantaneously* a staircase-like pattern consistent with the results of this paper. Due to its instantaneous character, this phase of the dynamic is usually referred to as "fast time" (see [15]).

The connection between the dynamic and the variational behavior is hardly surprising if we think of gradient-flows as limits of discrete-time evolutions, as in De Giorgi's theory of minimizing movements. In this context the minimum problem for (2.1.1) with forcing term f equal to the initial datum is just the first step in the construction of the minimizing movement. Transforming this intuition into a rigorous statement concerning the fast-time behavior of solutions to (1.1) is a challenging problem.

Another issue is that the staircasing effect seems to appear only in the so-called supercritical regions of $u_0(x)$, namely where $u_0'(x)$ falls in the concavity region of $\varphi(p)$ (see the simulations in [41, 15, 52, 50, 51]). The variational analysis can not produce this effect, in some sense because the convexification involves a "global procedure", and therefore it is very likely that an explanation should rely also on dynamical arguments.

Structure of the chapter This chapter is organized as follows. In section 2.2 we introduce the notations and we state the main results concerning the asymptotic behavior of minima and minimizers for (2.1.2). In section 2.3 we state the results that we need concerning the rescaled functionals (2.1.6) and their Gamma-limit. In section 2.4 we recall the notion of local minimizers, both for (2.1.2) and for the Gamma-limit, and we state their main properties. In section 2.5 we show that our main results follow from the properties of local minimizers, that we prove later in section 2.6. Finally, in section 2.7 we mention some different models to which our theory can be extended, and in section 2.8 we present some open problems. We also add an appendix with a proof of the results stated in section 2.3, some of which are apparently missing, or present with flawed proofs, in the literature.

2.2 Statements

For every $\varepsilon \in (0,1)$ let us set

$$\omega(\varepsilon) := \varepsilon|\log\varepsilon|^{1/2}. \tag{2.2.1}$$

Let $\beta > 0$ be a real number, let $\Omega \subseteq \mathbb{R}$ be an open set, and let $f \in L^2(\Omega)$ be a function that we call forcing term. In order to emphasize the dependence on all the parameters, we write (2.1.2) in the form

$$\mathbb{PMF}_\varepsilon(\beta, f, \Omega, u) := \int_\Omega \left\{\varepsilon^6\omega(\varepsilon)^4 u''(x)^2 + \log\left(1 + u'(x)^2\right) + \beta(u(x) - f(x))^2\right\} dx. \tag{2.2.2}$$

The first result that we state concerns existence and regularity of minimizers, and their convergence to the fidelity term in $L^2((0,1))$. We omit the proof because it is a standard application of the direct method in the calculus of variations, and of the fact that the convex envelope of the function $p \mapsto \log(1 + p^2)$ is identically 0.

Proposition 2.2.1 (Existence and regularity of minimizers)**.** *Let $\omega(\varepsilon)$ be defined by (2.2.1), and let $\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u)$ be defined by (2.2.2), where $\varepsilon \in (0,1)$ and $\beta > 0$ are two real numbers, and $f \in L^2((0,1))$ is a given function.*

Then the following facts hold true.

(1) (Existence) *There exists*

$$m(\varepsilon, \beta, f) := \min\left\{\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u) : u \in H^2((0,1))\right\}. \tag{2.2.3}$$

(2) (Regularity) *Every minimizer belongs to $H^4((0,1))$, and in particular to $C^2([0,1])$.*

(3) (Minimum value vanishes in the limit) *It turns out that $m(\varepsilon, \beta, f) \to 0$ as $\varepsilon \to 0^+$.*

(4) (Convergence of minimizers to the fidelity term) *If $\{u_\varepsilon\}$ is any family of minimizers for (2.2.3), then $u_\varepsilon \to f$ in $L^2((0,1))$ as $\varepsilon \to 0^+$.*

In the sequel we assume that the forcing term f is more regular. In the case $f \in H^1((0,1))$, our first result concerns the asymptotic behavior of minima.

Theorem 2.2.2 (Asymptotic behavior of minima)**.** *Let $\omega(\varepsilon)$ be defined by (2.2.1), and let $\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u)$ be defined by (2.2.2), where $\varepsilon \in (0,1)$ and $\beta > 0$ are two real numbers, and $f \in H^1([0,1])$ is a given function.*

Then the minimum value $m(\varepsilon, \beta, f)$ defined in (2.2.3) satisfies

$$\lim_{\varepsilon \to 0^+} \frac{m(\varepsilon, \beta, f)}{\omega(\varepsilon)^2} = 10\left(\frac{2\beta}{27}\right)^{1/5} \int_0^1 |f'(x)|^{4/5}\, dx. \tag{2.2.4}$$

The asymptotic behavior of $m(\varepsilon, \beta, f)$ under weaker regularity assumptions on f is a largely open problem. We refer to section 2.8 for further details.

Now we investigate the asymptotic behavior of minimizers. The intuitive idea is that they tend to develop a staircase structure. In order to formalize this idea, we need several definitions. To begin with, we define some classes of “staircase-like functions”.

Definition 2.2.3 (Canonical staircases)**.** Let $S : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$S(x) := 2\left\lfloor \frac{x+1}{2} \right\rfloor \qquad \forall x \in \mathbb{R},$$

where, for every real number α, the symbol $\lfloor \alpha \rfloor$ denotes the greatest integer less than or equal to α. For every pair (H, V) of real numbers, with $H > 0$, we call *canonical (H,V)-staircase* the function $S_{H,V} : \mathbb{R} \to \mathbb{R}$ defined by

$$S_{H,V}(x) := V \cdot S(x/H) \qquad \forall x \in \mathbb{R}. \tag{2.2.5}$$

Roughly speaking, the graph of $S_{H,V}$ is a staircase with steps of horizontal length $2H$ and vertical height $2V$. The origin is the midpoint of the horizontal part of one of the steps. The staircase degenerates to the null function when $V = 0$, independently of the value of H.

Definition 2.2.4 (Translations of the canonical staircase)**.** Let (H, V) be a pair of real numbers, with $H > 0$, and let $S_{H,V}$ be the function defined in (2.2.5). Let $v : \mathbb{R} \to \mathbb{R}$ be a function.

- We say that v is an *oblique translation* of $S_{H,V}$, and we write $v \in \mathrm{Obl}(H, V)$, if there exists a real number $\tau_0 \in [-1, 1]$ such that
$$v(x) = S_{H,V}(x - H\tau_0) + V\tau_0 \qquad \forall x \in \mathbb{R}.$$
- We say that v is a *graph translation of horizontal type* of $S_{H,V}$, and we write $v \in \mathrm{Hor}(H, V)$, if there exists a real number $\tau_0 \in [-1, 1]$ such that
$$v(x) = S_{H,V}(x - H\tau_0) \qquad \forall x \in \mathbb{R}. \tag{2.2.6}$$
- We say that v is a *graph translation of vertical type* of $S_{H,V}$, and we write $v \in \mathrm{Vert}(H, V)$, if there exists a real number $\tau_0 \in [-1, 1]$ such that
$$v(x) = S_{H,V}(x - H) + V(1 - \tau_0) \qquad \forall x \in \mathbb{R}. \tag{2.2.7}$$

Remark 2.2.5. Let us interpret translations of the canonical staircase in terms of graph (see Figure 2.2).

- Oblique translations correspond to taking the graph of the canonical staircase $S_{H,V}(x)$ and moving the origin along the line $Hy = Vx$, namely the line that connects the midpoints of the steps.
- Graph translations of horizontal type correspond to moving the origin to some point in the horizontal part of some step.
- Graph translations of vertical type correspond to moving the origin to some point in the vertical part of some step.

We observe that graph translations of horizontal type with $\tau_0 = \pm 1$ coincide with graph translations of vertical type with the same value of τ_0. In those cases the origin is moved to the "corners" of the graph.

In the sequel $BV((a, b))$ denotes the space of functions with bounded variation in the interval $(a, b) \subseteq \mathbb{R}$. For every function u in this space, Du denotes its distributional derivative, which is a signed measure, and $|Du|((a, b))$ denotes the total variation of u in (a, b). We call jump points of u the points $x \in (a, b)$ where u is not continuous. As usual, $BV_{\mathrm{loc}}(\mathbb{R})$ denotes the set of all functions $u : \mathbb{R} \to \mathbb{R}$ whose restriction to every interval (a, b) belongs to $BV((a, b))$. The staircase-like functions we have introduced above are typical examples of elements of the space $BV_{\mathrm{loc}}(\mathbb{R})$.

Our result for the asymptotic behavior of minimizers involves smooth functions converging to staircases. The strongest sense in which this convergence is possible is the so-called strict converge. We recall here the definitions (see [6, Definition 3.14]).

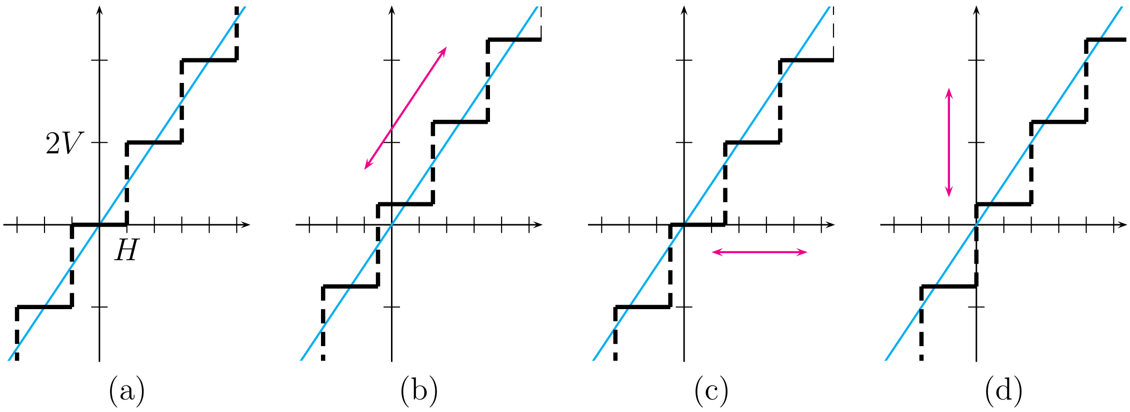

Figure 2.2: (a) Canonical staircase. (b) Oblique translation. (c) Graph translation of horizontal type. (d) Graph translation of vertical type. In all translations the parameter is $\tau_0 = 1/2$.

Definition 2.2.6 (Strict convergence in an interval)**.** Let $(a, b) \subseteq \mathbb{R}$ be an interval. A sequence of functions $\{u_n\} \subseteq BV((a, b))$ converges *strictly* to some $u_\infty \in BV((a, b))$, and we write

$$u_n \rightsquigarrow u_\infty \quad \text{in } BV((a, b)),$$

if

$$u_n \to u_\infty \text{ in } L^1((a, b)) \qquad \text{and} \qquad |Du_n|((a, b)) \to |Du_\infty|((a, b)).$$

Definition 2.2.7 (Locally strict convergence on the whole real line)**.** A sequence of functions $\{u_n\} \subseteq BV_{\text{loc}}(\mathbb{R})$ converges *locally strictly* to some $u_\infty \in BV_{\text{loc}}(\mathbb{R})$, and we write

$$u_n \rightsquigarrow u_\infty \quad \text{in } BV_{\text{loc}}(\mathbb{R}),$$

if $u_n \rightsquigarrow u_\infty$ in $BV((a, b))$ for every interval $(a, b) \subseteq \mathbb{R}$ whose endpoints are not jump points of the limit u_∞.

Both definitions can be extended in the usual way to families depending on real parameters. For example, $u_\varepsilon \rightsquigarrow u_0$ in $BV((a, b))$ as $\varepsilon \to 0^+$ if and only if $u_{\varepsilon_n} \rightsquigarrow u_0$ in $BV((a, b))$ for every sequence $\varepsilon_n \to 0^+$.

In the following remark we recall some consequences of strict convergence.

Remark 2.2.8 (Consequences of strict convergence)**.** Let us assume that $u_n \rightsquigarrow u_\infty$ in $BV((a, b))$. Then the following facts hold true.

(1) It turns out that $\{u_n\}$ is bounded in $L^\infty((a, b))$, and $u_n \to u_\infty$ in $L^p((a, b))$ for every $p \geq 1$ (but not necessarily for $p = +\infty$).

(2) For every $x \in (a, b)$, and every sequence $x_n \to x$, it turns out that

$$\liminf_{y \to x} u_\infty(y) \leq \liminf_{n \to +\infty} u_n(x_n) \leq \limsup_{n \to +\infty} u_n(x_n) \leq \limsup_{y \to x} u_\infty(y),$$

and in particular $u_n(x_n) \to u_\infty(x)$ whenever u_∞ is continuous in x, and the convergence is uniform in (a, b) if the limit u_∞ is continuous in (a, b).

(3) It turns out that $u_n \rightsquigarrow u_\infty$ in $BV((c,d))$ for every interval $(c,d) \subseteq (a,b)$ whose endpoints are not jump points of the limit u_∞.

(4) The positive and negative part of the distributional derivatives converge separately in the *closed* interval (see [6, Proposition 3.15]). More precisely, if D^+u_n and D^-u_n denote, respectively, the positive and negative part of the signed measure Du_n, and similarly for u_∞, then for every continuous test function $\phi : [a,b] \to \mathbb{R}$ it turns out that

$$\lim_{n\to+\infty} \int_{[a,b]} \phi(x)\, dD^+u_n(x) = \int_{[a,b]} \phi(x)\, dD^+u_\infty(x),$$

and similarly with D^-u_n and D^-u_∞.

In our second main result we consider any family $\{u_\varepsilon\}$ of minimizers to (2.2.2) and any family of points $x_\varepsilon \to x_0 \in (0,1)$, and we investigate the asymptotic behavior of the family of fake blow-ups (we call them "fake" because in the numerator we subtract $f(x_\varepsilon)$ instead of $u_\varepsilon(x_\varepsilon)$) defined by

$$w_\varepsilon(y) := \frac{u_\varepsilon(x_\varepsilon + \omega(\varepsilon)y) - f(x_\varepsilon)}{\omega(\varepsilon)} \qquad \forall y \in \left(-\frac{x_\varepsilon}{\omega(\varepsilon)}, \frac{1-x_\varepsilon}{\omega(\varepsilon)}\right), \tag{2.2.8}$$

and the asymptotic behavior of the family of true blow-ups defined by

$$v_\varepsilon(y) := \frac{u_\varepsilon(x_\varepsilon + \omega(\varepsilon)y) - u_\varepsilon(x_\varepsilon)}{\omega(\varepsilon)} \qquad \forall y \in \left(-\frac{x_\varepsilon}{\omega(\varepsilon)}, \frac{1-x_\varepsilon}{\omega(\varepsilon)}\right). \tag{2.2.9}$$

When $f \in C^1([0,1])$, we prove that both families are relatively compact in the sense of locally strict convergence, and all their limit points are suitable staircases.

Theorem 2.2.9 (Blow-up of minimizers at standard resolution). *Let $\omega(\varepsilon)$ be defined by (2.2.1), and let $\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u)$ be defined by (2.2.2), where $\varepsilon \in (0,1)$ and $\beta > 0$ are two real numbers, and $f \in C^1([0,1])$ is a given function.*

Let $\{u_\varepsilon\} \subseteq H^2((0,1))$ be a family of functions with

$$u_\varepsilon \in \operatorname{argmin}\left\{\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u) : u \in H^2((0,1))\right\} \qquad \forall \varepsilon \in (0,1),$$

and let $x_\varepsilon \to x_0 \in (0,1)$ be a family of points. Let us consider the canonical (H,V)-staircase with parameters

$$H := \left(\frac{24}{\beta^2 |f'(x_0)|^3}\right)^{1/5}, \qquad V := f'(x_0)H, \tag{2.2.10}$$

with the agreement that this staircase is identically equal to 0 when $f'(x_0) = 0$.

Then the following statements hold true.

(1) (Compactness of fake blow-ups). *The family $\{w_\varepsilon(y)\}$ defined by (2.2.8) is relatively compact with respect to locally strict convergence, and every limit point is an oblique translation of the canonical (H,V)-staircase.*

More precisely, for every sequence $\{\varepsilon_n\} \subseteq (0,1)$ with $\varepsilon_n \to 0^+$ there exist an increasing sequence $\{n_k\}$ of positive integers and a function $w_\infty \in \mathrm{Obl}(H,V)$ such that

$$w_{\varepsilon_{n_k}}(y) \rightsquigarrow w_\infty(y) \quad \text{in } BV_{loc}(\mathbb{R}).$$

(2) (Compactness of true blow-ups). *The family $\{v_\varepsilon(y)\}$ defined by (2.2.9) is relatively compact with respect to locally strict convergence, and every limit point is a graph translation of the canonical (H,V)-staircase.*

More precisely, for every sequence $\{\varepsilon_n\} \subseteq (0,1)$ with $\varepsilon_n \to 0^+$ there exist an increasing sequence $\{n_k\}$ of positive integers and a function $v_\infty \in \mathrm{Hor}(H,V) \cup \mathrm{Vert}(H,V)$ such that

$$v_{\varepsilon_{n_k}}(y) \rightsquigarrow v_\infty(y) \quad \textit{in } BV_{loc}(\mathbb{R}).$$

(3) (Realization of all possible oblique translations). *Let $w_0 \in \mathrm{Obl}(H,V)$ be any oblique translation of the canonical (H,V)-staircase.*

Then there exists a family $\{x'_\varepsilon\} \subseteq (0,1)$ such that

$$\limsup_{\varepsilon \to 0^+} \frac{|x'_\varepsilon - x_\varepsilon|}{\omega(\varepsilon)} \leq H, \tag{2.2.11}$$

and

$$\frac{u_\varepsilon(x'_\varepsilon + \omega(\varepsilon)y) - f(x'_\varepsilon)}{\omega(\varepsilon)} \rightsquigarrow w_0(y) \quad \textit{in } BV_{loc}(\mathbb{R}). \tag{2.2.12}$$

(4) (Realization of all possible graph translations). *Let $v_0 \in \mathrm{Hor}(H,V) \cup \mathrm{Vert}(H,V)$ be any graph translation of the canonical (H,V)-staircase.*

Then there exists a family $\{x'_\varepsilon\} \subseteq (0,1)$ satisfying (2.2.11) and

$$\frac{u_\varepsilon(x'_\varepsilon + \omega(\varepsilon)y) - u_\varepsilon(x'_\varepsilon)}{\omega(\varepsilon)} \rightsquigarrow v_0(y) \quad \textit{in } BV_{loc}(\mathbb{R}).$$

Let us make some comments about Theorem 2.2.9 above. To begin with, we consider the special case of stationary points, and the special case of blow-ups in boundary points.

Remark 2.2.10 (Stationary points of the forcing term)**.** In the special case where $f'(x_0) = 0$, the canonical (H,V)-staircase is identically equal to 0, and it coincides with all its oblique or graph translations. In this case the whole family of fake blow-ups and the whole family of true blow-ups converge to 0, without any need of subsequences.

Remark 2.2.11 (Internal vs boundary blow-ups)**.** For the sake of shortness, we stated the result in the case where $x_0 \in (0,1)$. The very same conclusions hold true, with exactly the same proof, even if $x_0 \in \{0,1\}$, provided that

$$\lim_{\varepsilon \to 0^+} \frac{\min\{x_\varepsilon, 1 - x_\varepsilon\}}{\omega(\varepsilon)} = +\infty. \tag{2.2.13}$$

When $x_0 \in \{0,1\}$ and (2.2.13) fails, we can again characterize the limits of fake and true blow-ups, more or less with the same techniques. This requires a one-sided variant of the canonical staircases that we discuss later in section 2.4. We refer to Remark 2.5.1 for further details.

In the following remark we present the result from two different points of view.

Remark 2.2.12 (Further interpretations of Theorem 2.2.9)**.** Let us consider any distance in the space $\mathbb{X} := BV_{\mathrm{loc}}(\mathbb{R})$ that induces the locally strict convergence. Given any minimizer u_ε to (2.2.2), we extend it to a continuous function $\widehat{u}_\varepsilon : \mathbb{R} \to \mathbb{R}$ by setting

$$\widehat{u}_\varepsilon(x) := \begin{cases} u_\varepsilon(0) & \text{if } x \leq 0, \\ u_\varepsilon(x) & \text{if } x \in [0,1], \\ u_\varepsilon(1) & \text{if } x \geq 1. \end{cases}$$

Then we consider the function $U_\varepsilon : (0,1) \to \mathbb{X}$ defined by

$$[U_\varepsilon(x)](y) := \frac{\widehat{u}_\varepsilon(x + \omega(\varepsilon)y) - f(x)}{\omega(\varepsilon)} \qquad \forall y \in \mathbb{R},$$

namely the function that associates to every $x \in (0,1)$ the fake blow-up of $\widehat{u}_\varepsilon$ with center in x at scale $\omega(\varepsilon)$.

Finally, for every $x \in (0,1)$ we consider the set $T(x) \subseteq \mathbb{X}$ consisting of all oblique translations of the canonical (H,V)-staircase with parameters given by (2.2.10). We observe that $T(x)$ is homeomorphic to the circle S^1 if $f'(x) \neq 0$, and $T(x)$ is a singleton if $f'(x) = 0$.

Then "$U_\varepsilon(x)$ converges to $T(x)$" in the following senses.

(1) (Hausdorff convergence). For every interval $[a,b] \subseteq (0,1)$ we consider the graph of U_ε over $[a,b]$, namely

$$G_\varepsilon(a,b) := \{(x,w) : x \in [a,b],\ w = U_\varepsilon(x)\} \subseteq [a,b] \times \mathbb{X},$$

and the graph of the multi-function $T(x)$, namely the set

$$G_0(a,b) := \{(x,w) : x \in [a,b],\ w \in T(x)\} \subseteq [a,b] \times \mathbb{X}.$$

Then it turns out that $G_\varepsilon \to G_0$ as $\varepsilon \to 0^+$ with respect to the Hausdorff distance between compact subsets of $(0,1) \times \mathbb{X}$.

This convergence result is a direct consequence of statements (1) and (3) of Theorem 2.2.9. It can also be extended to true blow-ups, just by defining $T(x)$ as the set of graph translations instead of oblique translations.

(2) (Young measure convergence). Let us consider U_ε as a Young measure ν_ε in $(0,1)$ with values in $\mathbb{X}$. Let ν_0 denote the Young measure that associates to every $x \in (0,1)$ the probability measure in $T(x)$ that is invariant by oblique translations. Then it turns out that

$$\nu_\varepsilon \rightharpoonup \nu_0 \qquad \text{as } \varepsilon \to 0^+,$$

where the convergence is in the sense of $\mathbb{X}$-valued Young measures in $(0,1)$. We point out that the strict convergence induced by the distance in our space $\mathbb{X}$ is much stronger than the convergence in [1], where the distance just induces the weak* topology in a ball of L^∞. For this reason, our space $\mathbb{X}$ is not compact, but we could easily recover the compactness by restricting ourselves to the subset consisting of all blow-ups of all minimizers for ε in some interval $(0,\varepsilon_0] \subseteq (0,1)$, together with all their possible limits as $\varepsilon \to 0^+$.

This convergence in the sense of Young measures follows from the Hausdorff convergence and the invariance of ν_0 by oblique translations, which in turn follows from a remake of [1, Proposition 3.1 and Lemma 2.7]. The argument is however analogous to the proof of statement (3) of Theorem 2.2.9. The idea is that any translation of the blow-up point of order $\omega(\varepsilon)$ delivers a proportional oblique translation of the limit.

In the case of true blow-ups we expect the limit Young measure ν_0 to be uniformly concentrated only on graph translations of horizontal type, while graph translations of vertical type should have zero measure because they correspond to a very special choice of the blow-up points.

Theorem 2.2.9 shows that minimizers develop a microstructure at scale $\omega(\varepsilon)$. As a consequence, this microstructure does not appear if we consider blow-ups at a coarser scale, as in the following statement.

Corollary 2.2.13 (Low-resolution blow-ups of minimizers)**.** *Let ε, $\omega(\varepsilon)$, β, f, u_ε, x_0 be as in Theorem 2.2.9. Let $\{x_\varepsilon\} \subseteq (0,1)$ be a family of real numbers such that $x_\varepsilon \to x_0$, and let $\{\alpha_\varepsilon\}$ be a family of positive real numbers such that $\alpha_\varepsilon \to 0$ and $\omega(\varepsilon)/\alpha_\varepsilon \to 0$.*

Then it turns out that

$$\frac{u_\varepsilon(x_\varepsilon + \alpha_\varepsilon y) - u_\varepsilon(x_\varepsilon)}{\alpha_\varepsilon} \rightsquigarrow f'(x_0)y \qquad \text{in } BV_{loc}(\mathbb{R}),$$

and therefore also uniformly on bounded subsets of $\mathbb{R}$.

The second consequence of Theorem 2.2.9 is an improvement of statement (4) in Proposition 2.2.1, at least in the case where the forcing term $f(x)$ is of class C^1. In this case indeed we obtain that minimizers converge to f also in the sense of strict convergence. Moreover, as $u_\varepsilon(x)$ converges to $f(x)$, its derivative $u'_\varepsilon(x)$ converges to a mix of 0 and $\pm\infty$, and this mix is "in the average" equal to $f'(x)$. We state the result using an elementary language, and then we interpret it in the formalism of varifolds.

Theorem 2.2.14 (Convergence of minimizers to the forcing term)**.** *Let ε, β, f, u_ε be as in Theorem 2.2.9.*

Then the family $\{u_\varepsilon\}$ of minimizers converges to f in the following senses.

(1) (Strict convergence). *It turns out that $u_\varepsilon \rightsquigarrow f$ in $BV((0,1))$, and therefore also uniformly in* $[0,1]$.

(2) (Convergence as varifolds). *Let us set*

$$V_0^+ := \{x \in [0,1] : f'(x) > 0\}, \qquad V_0^- := \{x \in [0,1] : f'(x) < 0\}. \qquad (2.2.14)$$

Then for every continuous test function

$$\phi : [0,1] \times \mathbb{R} \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$$

it turns out that

$$\lim_{\varepsilon \to 0^+} \int_0^1 \phi\left(x, u_\varepsilon(x), \arctan(u'_\varepsilon(x))\right) \sqrt{1 + u'_\varepsilon(x)^2}\, dx = \int_0^1 \phi(x, f(x), 0)\, dx$$
$$+ \int_{V_0^-} \phi\left(x, f(x), -\frac{\pi}{2}\right) |f'(x)|\, dx + \int_{V_0^+} \phi\left(x, f(x), \frac{\pi}{2}\right) |f'(x)|\, dx. \quad (2.2.15)$$

The conclusions of Theorem 2.2.14 is weaker than Theorem 2.2.9, because it does not carry so much information about the asymptotic profile of minimizers. Just for comparison, the counterpart of this result in the Alberti-Müller model is the convergence of u'_ε to a Young measure that in every point assumes the two values ± 1 with equal probability. Therefore, we suspect that the same conclusion might be true under weaker assumptions on the forcing term f, and we refer to section 2.8 for further discussion.

Remark 2.2.15 (Varifold interpretation)**.** Let us limit ourselves for a while to test functions such that $\phi(x, s, \pi/2) = \phi(x, s, -\pi/2)$ for all admissible values of x and s. Let us observe that the function $p \mapsto \arctan(p)$ is a homeomorphism between the projective line and the interval $[-\pi/2, \pi/2]$ with the endpoints identified. Under these assumptions we can interpret the two sides of (2.2.15) as the action of two suitable varifolds on the test function ϕ.

In the left-hand side we have the varifold associated to the graph of u_ε in the canonical way, namely with "weight" (projection into $\mathbb{R}^2$) equal to the restriction of the one-dimensional Hausdorff measure to the graph of u_ε, and "tangent component" in the direction of the derivative u'_ε. In the right-hand side we have a varifold with

- "weight" equal to the one-dimensional Hausdorff measure restricted to the graph of f, multiplied by the density

  $$\frac{1 + |f'(x)|}{\sqrt{1 + f'(x)^2}},$$

  which in turn coincides with the push-forward of the Lebesgue measure through the map $x \mapsto (x, f(x))$ multiplied by $1 + |f'(x)|$,

- "tangent component" in the point $(x, f(x))$ equal to

  $$\frac{1}{1 + |f'(x)|}\, \delta_{(1,0)} + \frac{|f'(x)|}{1 + |f'(x)|}\, \delta_{(0,1)},$$

  where $\delta_{(1,0)}$ and $\delta_{(0,1)}$ are the Dirac measures concentrated in the horizontal direction $(1,0)$ and in the vertical direction $(0,1)$, respectively.

It follows that statement (2) of Theorem 2.2.14 above is a reinforced version of varifold convergence. The reinforcement consists in considering the vertical tangent line in the direction $(0,1)$ as different from the vertical tangent line in the direction $(0,-1)$.

Remark 2.2.16 (Minimality is essential)**.** In Theorem 2.2.9 and Theorem 2.2.14 we can not replace the requirement that $\{u_\varepsilon\}$ is a family of minimizers by weaker "almost minimality" conditions such as

$$\lim_{\varepsilon \to 0^+} \frac{\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u_\varepsilon)}{m(\varepsilon, \beta, f)} = 1. \tag{2.2.16}$$

Indeed, one can check that the cost of adding an isolated bump that simulates two opposite jumps in a neighborhood of some point is proportional to $\omega(\varepsilon)^{5/2}$ (see also (2.8.1) below). Since the denominator in (2.2.16) is proportional to $\omega(\varepsilon)^2$, this condition does not even imply a uniform bound on the total variation of u_ε.

2.3 Functional setting and Gamma-convergence

This section deals with the rescaled version of the Perona-Malik functional (2.2.2) and its Gamma-limit. The results are somewhat classical, and rather close to similar results in the literature. On the other hand, in some cases they are not stated in the literature in the form we need, and in some other cases the proofs that we found in the literature do not work. Therefore, for the convenience of the reader we include at least a sketch of the proofs in an appendix at the end of the chapter.

Functional setting Let us consider the functional

$$\mathbb{RPM}_\varepsilon(\Omega, u) := \int_\Omega \left\{ \varepsilon^6 u''(x)^2 + \frac{1}{\omega(\varepsilon)^2} \log\left(1 + u'(x)^2\right) \right\} dx \tag{2.3.1}$$

defined for every real number $\varepsilon \in (0,1)$, every open set $\Omega \subseteq \mathbb{R}$, and every function $u \in H^2(\Omega)$. This functional is a rescaled version of the principal part of (2.2.2). When we add the usual "fidelity term", depending on a real parameter $\beta > 0$ and on a forcing term $f \in L^2(\Omega)$, we obtain the rescaled Perona-Malik functional with fidelity term

$$\mathbb{RPMF}_\varepsilon(\beta, f, \Omega, u) := \mathbb{RPM}_\varepsilon(\Omega, u) + \beta \int_\Omega (u(x) - f(x))^2 \, dx. \tag{2.3.2}$$

The Gamma-limit of (2.3.1) as $\varepsilon \to 0^+$ turns out to be finite only in the space of "pure jump functions", defined as finite or countable linear combination of Heaviside functions. More formally, the notion is the following.

Definition 2.3.1 (Pure jump functions)**.** Let $(a,b) \subseteq \mathbb{R}$ be an interval. A function $u : (a,b) \to \mathbb{R}$ is called a *pure jump function*, and we write $u \in PJ((a,b))$, if there exist a real number c, a finite or countable set $S_u \subseteq (a,b)$, and a function $J : S_u \to \mathbb{R} \setminus \{0\}$ such that

$$\sum_{s \in S_u} |J(s)| < +\infty \tag{2.3.3}$$

and

$$u(x) = c + \sum_{s \in S_u} J(s) \mathbb{1}_{(s,b)}(x) \qquad \forall x \in (a,b), \tag{2.3.4}$$

where $\mathbb{1}_{(s,b)} : \mathbb{R} \to \{0,1\}$ is the indicator function of the interval (s,b), defined as

$$\mathbb{1}_{(s,b)}(x) := \begin{cases} 1 & \text{if } x \in (s,b), \\ 0 & \text{otherwise.} \end{cases}$$

The set S_u is called the *jump set* of u, every element $s \in S_u$ is called a *jump point* of u, and $|J(s)|$ is called the *height of the jump* of u in s.

We call *boundary values* of u the numbers

$$u(a) := \lim_{x \to a^+} u(x) = c \qquad \text{and} \qquad u(b) := \lim_{x \to b^-} u(x) = c + \sum_{s \in S_u} J(s). \tag{2.3.5}$$

Pure jump functions can be defined in an alternative way as those functions in $BV((a,b))$ whose distributional derivative is a finite or countable linear combination of atomic measures. In particular, it can be verified that the representation (2.3.4) is unique, and defines a function $u \in BV((a,b))$ whose total variation is the sum of the series in (2.3.3), and whose distributional derivative Du is the sum of Dirac measures concentrated in the points of the set S_u with weight $J(s)$. Moreover, S_u coincides with the set of discontinuity points of u, and

$$J(s) = \lim_{x \to s^+} u(x) - \lim_{x \to s^-} u(x) \qquad \forall s \in S_u.$$

We can now introduce the functional

$$\mathbb{J}_{1/2}(\Omega, u) := \sum_{s \in S_u \cap \Omega} |J(s)|^{1/2}, \tag{2.3.6}$$

defined for every $u \in PJ((a,b))$ and every open subset $\Omega \subseteq (a,b)$. Of course the convergence of the series in (2.3.3) does not imply the convergence of the series in (2.3.6), and therefore at this level of generality it may happen that $\mathbb{J}_{1/2}(\Omega, u) = +\infty$ for some choices of u and Ω.

Gamma-convergence The following result concerns the convergence of the family $\mathbb{RPM}_\varepsilon$ to a multiple of $\mathbb{J}_{1/2}$. The compactness statement is similar to [14, Theorem 4.1], while the Gamma-convergence statement coincides with [14, Theorem 4.4] in the special case $\phi(p) = \log(1+p^2)$. Unfortunately, even if it applies to a quite general class of Lagrangians, the proof in [14] relies on [14, Lemma 3.1], which is clearly false for this choice of $\phi(p)$. In the appendix at the end of the paper we present a specific proof for this case.

Theorem 2.3.2 (Gamma-convergence, compactness, properties of recovery sequences)**.** *Let $(a,b) \subseteq \mathbb{R}$ be an interval, let us consider the functionals defined in (2.3.1) and (2.3.6), and let us set*

$$\alpha_0 := \frac{16}{\sqrt{3}}. \tag{2.3.7}$$

Then the following statements hold true.

(1) (Gamma convergence) *Let us extend the functionals (2.3.1) and (2.3.6) to the space* $L^2((a,b))$ *by setting them equal to* $+\infty$ *outside their original domains.*

Then with respect to the metric of $L^2((a,b))$ *it turns out that*

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} \mathbb{RPM}_\varepsilon((a,b),u) = \alpha_0\,\mathbb{J}_{1/2}((a,b),u) \qquad \forall u \in L^2((a,b)).$$

(2) (Compactness) *Let* $\{\varepsilon_n\} \subseteq (0,1)$ *be any sequence such that* $\varepsilon_n \to 0^+$*, and let* $\{u_n\} \subseteq H^2((a,b))$ *be any sequence such that*

$$\sup_{n\in\mathbb{N}} \left\{ \mathbb{RPM}_{\varepsilon_n}((a,b),u_n) + \int_a^b u_n(x)^2\,dx \right\} < +\infty. \tag{2.3.8}$$

Then there exist an increasing sequence $\{n_k\}$ *of positive integers, and a function* $u_\infty \in PJ((a,b))$ *such that* $u_{n_k} \to u_\infty$ *in* $L^2((a,b))$ *as* $k \to +\infty$*.*

(3) (Strict convergence of recovery sequences) *Let* $u \in PJ((a,b))$ *be a pure jump function with* $\mathbb{J}_{1/2}((a,b),u) < +\infty$*. Let* $\{\varepsilon_n\} \subseteq (0,1)$ *be any sequence such that* $\varepsilon_n \to 0^+$*, and let* $\{u_n\} \subseteq H^2((a,b))$ *be any sequence such that* $u_n \to u$ *in* $L^2((a,b))$*, and*

$$\lim_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}((a,b),u_n) = \alpha_0\,\mathbb{J}_{1/2}((a,b),u). \tag{2.3.9}$$

Then actually $u_n \rightsquigarrow u$ *in* $BV((a,b))$*, according to Definition 2.2.6.*

(4) (Recovery sequences with given boundary data) *Let* $\{\varepsilon_n\}$ *and* u *be as in the previous statement, and let* $\{A_{0,n}\}$*,* $\{A_{1,n}\}$*,* $\{B_{0,n}\}$*,* $\{B_{1,n}\}$ *be four sequences of real numbers such that*

$$\lim_{n\to+\infty} (A_{0,n}, A_{1,n}, B_{0,n}, B_{1,n}) = (u(a), 0, u(b), 0),$$

where the boundary values of u *are intended as usual in the sense of (2.3.5).*

Then there exists a sequence $\{u_n\} \subseteq H^2((a,b))$ *with boundary data*

$$\big(u_n(a), u_n'(a), u_n(b), u_n'(b)\big) = (A_{0,n}, A_{1,n}, B_{0,n}, B_{1,n}) \qquad \forall n \in \mathbb{N} \tag{2.3.10}$$

such that $u_n \to u$ *in* $L^2((a,b))$ *and (2.3.9) holds true.*

Remark 2.3.3. The choice of the ambient space $L^2((a,b))$ is not essential in Theorem 2.3.2, and actually it can be replaced with $L^p((a,b))$ for any real exponent $p \geq 1$ (but not for $p = +\infty$, at least in statements (2) and (4)).

Convergence of minima and minimizers Since the fidelity term in (2.3.2) is continuous with respect to the metric of $L^2(\Omega)$, and Gamma-convergence is stable with respect to continuous perturbations, we deduce that the Gamma-limit of (2.3.2) is the functional

$$\mathbb{JF}_{1/2}(\alpha,\beta,f,\Omega,u) := \alpha\,\mathbb{J}_{1/2}(\Omega,u) + \beta\int_\Omega (u(x)-f(x))^2\,dx, \tag{2.3.11}$$

with α equal to the constant α_0 defined in (2.3.7). Now we concentrate on the special case where $\Omega = (0, L)$ and the forcing term is the linear function $f(x) = Mx$, for suitable real numbers $L > 0$ and M, and we consider the following minimum values without boundary conditions

$$\mu_\varepsilon(\beta, L, M) := \min_{u \in H^2((0,L))} \mathbb{RPMF}_\varepsilon(\beta, Mx, (0, L), u), \tag{2.3.12}$$

$$\mu_0(\alpha, \beta, L, M) := \min_{u \in PJ((0,L))} \mathbb{JF}_{1/2}(\alpha, \beta, Mx, (0, L), u). \tag{2.3.13}$$

Then we introduce boundary conditions. In the case of (2.3.2) we call $H^2((0, L), M)$ the set of all functions $v \in H^2((0, L))$ such that $v(0) = 0$, $v(L) = ML$, and $v'(0) = v'(L) = 0$. In the case of (2.3.11) we call $PJ((0, L), M)$ the set of all functions $v \in PJ((0, L))$ such that $v(0) = 0$ and $v(L) = ML$, where these boundary values are intended in the sense of (2.3.5). At this point we consider the following minimum values with boundary conditions

$$\mu^*_\varepsilon(\beta, L, M) := \min_{u \in H^2((0,L),M)} \mathbb{RPMF}_\varepsilon(\beta, Mx, (0, L), u), \tag{2.3.14}$$

$$\mu^*_0(\alpha, \beta, L, M) := \min_{u \in PJ((0,L),M)} \mathbb{JF}_{1/2}(\alpha, \beta, Mx, (0, L), u). \tag{2.3.15}$$

The following result contains the properties of these minimum values that we exploit in the sequel (a sketch of the proof is in the appendix).

Proposition 2.3.4 (Asymptotic analysis of minima with linear forcing term)**.** *The minimum values defined in (2.3.12) through (2.3.15) have the following properties.*

(1) (Existence). *The minimum problems (2.3.12) through (2.3.15) admit a solution for every* $(\varepsilon, \alpha, \beta, L, M) \in (0, 1) \times (0, +\infty)^3 \times \mathbb{R}$.

(2) (Symmetry, continuity and monotonicity with respect to M). *For every admissible value of* ε, α, β, L *the four functions*

$$M \mapsto \mu_\varepsilon(\beta, L, M), \qquad M \mapsto \mu^*_\varepsilon(\beta, L, M),$$
$$M \mapsto \mu_0(\alpha, \beta, L, M), \qquad M \mapsto \mu^*_0(\alpha, \beta, L, M),$$

are even, continuous in $\mathbb{R}$, *and nondecreasing in* $[0, +\infty)$.

Moreover, we also have that

$$\mu^*_\varepsilon(\beta, L, M_2) \leq \left(\frac{M_2}{M_1}\right)^2 \mu^*_\varepsilon(\beta, L, M_1), \tag{2.3.16}$$

for every $0 < M_1 \leq M_2$.

(3) (Monotonicity with respect to L). *For every admissible value of* ε, α, β, M, *the three functions*

$$L \mapsto \mu_\varepsilon(\beta, L, M), \qquad L \mapsto \mu_0(\alpha, \beta, L, M), \qquad L \mapsto \mu^*_0(\alpha, \beta, L, M)$$

are nondecreasing with respect to L *in* $(0,+\infty)$. *As for* μ_ε^*, *it turns out that*

$$\mu_\varepsilon^*(\beta, L_2, M) \le \left(\frac{L_2}{L_1}\right)^3 \mu_\varepsilon^*(\beta, L_1, M) \tag{2.3.17}$$

for every $0 < L_1 \le L_2$.

(4) (Pointwise convergence). *For every admissible value of* β, M *and* L *it turns out that*

$$\lim_{\varepsilon\to 0^+} \mu_\varepsilon(\beta, L, M) = \mu_0(\alpha_0, \beta, L, M), \tag{2.3.18}$$

and

$$\lim_{\varepsilon\to 0^+} \mu_\varepsilon^*(\beta, L, M) = \mu_0^*(\alpha_0, \beta, L, M), \tag{2.3.19}$$

where α_0 *is the constant defined in (2.3.7).*

(5) (Uniform convergence). *The limits (2.3.18) and (2.3.19) are uniform for bounded values of* M, *in the sense that for every positive value of* β *and* L *it turns out that*

$$\lim_{\varepsilon\to 0^+} \sup_{|M|\le M_0} |\mu_\varepsilon(\beta, L, M) - \mu_0(\alpha_0, \beta, L, M)| = 0 \qquad \forall M_0 > 0, \tag{2.3.20}$$

and

$$\lim_{\varepsilon\to 0^+} \sup_{|M|\le M_0} |\mu_\varepsilon^*(\beta, L, M) - \mu_0^*(\alpha_0, \beta, L, M)| = 0 \qquad \forall M_0 > 0. \tag{2.3.21}$$

2.4 Local minimizers

In this section we state the key tools for the proof of our main results. The key idea is that also local minimizers for functionals of the form (2.3.2) converge to local minimizers for functionals of the form (2.3.11). This extends the Gamma convergence results of the previous section.

The notion of local minimizers can be introduced in a very general framework by asking minimality with respect to compactly supported perturbations. In many concrete examples this is equivalent to saying that a given function is a minimizer with respect to its own boundary conditions. Of course the number and the form of these boundary conditions depend on the nature of the functional, as we explain below.

Definition 2.4.1 (Local minimizers in intervals)**.** Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let $\mathcal{F}(u)$ be a functional defined in some functional space $\mathcal{S}((a,b))$.

- Let us assume that $\mathcal{S}((a,b)) = H^2((a,b))$. A *local minimizer* is any function $u \in H^2((a,b))$ such that $\mathcal{F}(u) \le \mathcal{F}(v)$ for every function $v \in H^2((a,b))$ such that
$$\big(v(a), v'(a), v(b), v'(b)\big) = \big(u(a), u'(a), u(b), u'(b)\big).$$
- Let us assume that $\mathcal{S}((a,b)) = PJ((a,b))$. A *local minimizer* is any function $u \in PJ((a,b))$ such that $\mathcal{F}(u) \le \mathcal{F}(v)$ for every function $v \in PJ((a,b))$ such that $(v(a), v(b)) = (u(a), u(b))$, where boundary values of pure jump functions are intended in the sense of (2.3.5).

In both cases we write

$$u \in \operatorname{argmin}_{\text{loc}} \{\mathcal{F}(u) : u \in \mathcal{S}((a,b))\}.$$

We observe that in Definition 2.4.1 the two endpoints of the interval play the same role. In the sequel we need also the following notion of one-sided local minimizer, where we focus just on one of the endpoints.

Definition 2.4.2 (One-sided local minimizers in an interval)**.** Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let $\mathcal{F}(u)$ be a functional defined in some functional space $\mathcal{S}((a,b))$.

- Let us assume that $\mathcal{S}((a,b)) = H^2((a,b))$. A *right-hand local minimizer* is any function $u \in H^2((a,b))$ such that $\mathcal{F}(u) \leq \mathcal{F}(v)$ for every function $v \in H^2((a,b))$ such that $(v(b), v'(b)) = (u(b), u'(b))$.
- Let us assume that $\mathcal{S}((a,b)) = PJ((a,b))$. A *right-hand local minimizer* is any function $u \in PJ((a,b))$ such that $\mathcal{F}(u) \leq \mathcal{F}(v)$ for every function $v \in PJ((a,b))$ such that $v(b) = u(b)$.

In both cases we write

$$u \in \operatorname{argmin}_{\text{R-loc}} \{\mathcal{F}(u) : u \in \mathcal{S}((a,b))\}.$$

Left-hand local minimizers are defined in a symmetric way, just focusing on the endpoint a.

Definition 2.4.3 (Entire and semi-entire local minimizers)**.** Let us consider functionals $\mathcal{F}(I,u)$ defined for every interval I and every u in some function space $\mathcal{S}(I)$.

- An *entire local minimizer* is a function $u : \mathbb{R} \to \mathbb{R}$ such that, for every interval $(a,b) \subseteq \mathbb{R}$, the restriction of u to (a,b) is a local minimizer in (a,b).
- A *right-hand semi-entire local minimizer* is a function $u : (0,+\infty) \to \mathbb{R}$ such that, for every real number $L > 0$, the restriction of u to $(0,L)$ is a right-hand local minimizer in $(0,L)$.
- A *left-hand semi-entire local minimizer* is a function $u : (-\infty,0) \to \mathbb{R}$ such that, for every real number $L > 0$, the restriction of u to $(-L,0)$ is a left-hand local minimizer in $(-L,0)$.

The following result is crucial both in the proof of Theorem 2.2.2, and as a preliminary step toward the characterization of entire and semi-entire local minimizers to the limiting functional (2.3.11).

Proposition 2.4.4 (Estimates for minima of the limit problem)**.** *For every choice of the parameters $(\alpha,\beta,L,M) \in (0,+\infty)^3 \times \mathbb{R}$ the minimum values defined in (2.3.13) and (2.3.15) satisfy*

$$c_1|M|^{4/5}L - c_2|M|^{1/5} \leq \mu_0(\alpha,\beta,L,M) \tag{2.4.1}$$

$$\leq \mu_0^*(\alpha,\beta,L,M) \leq c_1|M|^{4/5}L + c_3|M|^{1/5}, \tag{2.4.2}$$

where

$$c_1 := \frac{5}{4}\left(\frac{\alpha^4\beta}{3}\right)^{1/5}, \qquad c_2 := 20\left(\frac{2\alpha^6}{3\beta}\right)^{1/5}, \qquad c_3 := \frac{5}{4}\left(\frac{3\alpha^6}{\beta}\right)^{1/5}. \tag{2.4.3}$$

We are now ready to state the first main result of this section, namely the characterization of all entire and semi-entire local minimizers for the functional (2.3.11).

Proposition 2.4.5 (Classification of entire and semi-entire local minimizers)**.** *For every choice of the real numbers* $(\alpha,\beta,M) \in (0,+\infty)^2 \times \mathbb{R}$ *let us consider the functional* $\mathbb{JF}_{1/2}(\alpha,\beta,M,\mathbb{R},v)$ *defined in (2.3.11). Let us consider the canonical* (H,V)*-staircase* $S_{H,V}$ *with parameters*

$$H := \frac{1}{2}\left(\frac{9\alpha^2}{\beta^2|M|^3}\right)^{1/5}, \qquad V := MH, \tag{2.4.4}$$

and the understanding that $S_{H,V} \equiv 0$ *when* $M = 0$.

Then the following statements hold true.

(1) (Entire local minimizers). *The set of entire local minimizers coincides with the set of the oblique translations of the canonical* (H,V)*-staircase* $S_{H,V}$*, as introduced in Definition 2.2.3 and Definition 2.2.4.*

(2) (Semi-entire local minimizers). *The unique right-hand semi-entire local minimizer is the function* $w : (0,+\infty) \to \mathbb{R}$ *defined by*

$$w(x) := \begin{cases} Mz_0 & \text{if } x \in (0,z_0), \\ S_{H,V}(x - z_0) + Mz_0 & \text{if } x \geq z_0, \end{cases} \tag{2.4.5}$$

where $z_0 := (5/3)^{1/2}H$ *(if* $M = 0$ *the value of* z_0 *is not relevant).*

The unique left-hand semi-entire local minimizer is the function $w(-x)$.

In words, the right-hand semi-entire local minimizer is an oblique translation of the canonical (H,V)-staircase, but with a first step that is longer. Intuitively, this is due to the fact that the "jump at the origin" has no cost in terms of energy.

The second main result of this section is the convergence of local minimizers for (2.3.2) to local minimizers for (2.3.11). Let us start with the symmetric case.

Proposition 2.4.6 (Convergence to entire local minimizers)**.** *Let* M *and* β *be real numbers, with* $\beta > 0$*. For every positive integer* n*, let* $\varepsilon_n \in (0,1)$ *and* $A_n < B_n$ *be real numbers, let* $g_n : (A_n,B_n) \to \mathbb{R}$ *be a continuous function, and let* $w_n \in H^2((A_n,B_n))$.

Let us assume that

(i) as $n \to +\infty$ *it turns out that* $\varepsilon_n \to 0^+$*,* $A_n \to -\infty$*, and* $B_n \to +\infty$*,*

(ii) $g_n(x) \to Mx$ *uniformly on bounded subsets of* $\mathbb{R}$*,*

(iii) for every positive integer n *it turns out that*

$$w_n \in \operatorname{argmin}_{loc}\left\{\mathbb{RPMF}_{\varepsilon_n}(\beta,g_n,(A_n,B_n),w) : w \in H^2((A_n,B_n))\right\},$$

(iv) there exists a positive real number C_0 such that

$$\mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (A_n, B_n), w_n) \leq \frac{C_0}{\varepsilon_n} \qquad \forall n \geq 1. \tag{2.4.6}$$

Then there exists an increasing sequence $\{n_k\}$ of positive integers such that

$$w_{n_k} \rightsquigarrow w_\infty \quad \text{in } BV_{loc}(\mathbb{R}),$$

where w_∞ is an entire local minimizer for the functional (2.3.11) with α given by (2.3.7).

The result for one-sided local minimizers is analogous. We state it in the case of right-hand local minimizers.

Proposition 2.4.7 (Convergence to semi-entire local minimizers)**.** *Let M and β be real numbers, with $\beta > 0$. For every positive integer n, let $\varepsilon_n \in (0, 1)$ and $L_n > 0$ be real numbers, let $g_n : (0, L_n) \to \mathbb{R}$ be a continuous function, and let $w_n \in H^2((0, L_n))$.*

Let us assume that

(i) as $n \to +\infty$ it turns out that $\varepsilon_n \to 0^+$ and $L_n \to +\infty$,

(ii) $g_n(x) \to Mx$ uniformly on bounded subsets of $(0, +\infty)$,

(iii) for every positive integer n it turns out that

$$w_n \in \operatorname{argmin}_{R\text{-}loc} \left\{ \mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (0, L_n), w) : w \in H^2((0, L_n)) \right\},$$

(iv) there exists a positive real number C_0 such that

$$\mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (0, L_n), w_n) \leq \frac{C_0}{\varepsilon_n} \qquad \forall n \geq 1.$$

Let w_∞ denote the unique right-hand semi-entire local minimizer for the functional (2.3.11) with α given by (2.3.7), namely the function defined by (2.4.5).

Then for every $L > 0$ that is not a jump point of w_∞ it turns out that

$$w_n \rightsquigarrow w_\infty \quad \text{in } BV((0, L)).$$

2.5 Proofs of main results

In this section we assume that the results stated in section 2.3 and section 2.4 are valid, and using them we prove all the main results of section 2.2 concerning the behavior of minima and minimizers. We hope that this presentation allows to highlight the main ideas without focusing on the technical details that will be presented in the next section.

2.5.1 Asymptotic behavior of minima (Theorem 2.2.2)

The proof of Theorem 2.2.2 consists of two main parts. In the first part (estimate from below) we consider any family $\{u_\varepsilon\} \subseteq H^2((0,1))$ and we show that

$$\liminf_{\varepsilon\to 0^+} \frac{\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u_\varepsilon)}{\omega(\varepsilon)^2} \geq 10\left(\frac{2\beta}{27}\right)^{1/5} \int_0^1 |f'(x)|^{4/5}\,dx. \tag{2.5.1}$$

In the second part (estimate from above) we construct a family $\{u_\varepsilon\} \subseteq H^2((0,1))$ such that

$$\limsup_{\varepsilon\to 0^+} \frac{\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u_\varepsilon)}{\omega(\varepsilon)^2} \leq 10\left(\frac{2\beta}{27}\right)^{1/5} \int_0^1 |f'(x)|^{4/5}\,dx. \tag{2.5.2}$$

Estimate from below

Interval subdivision and approximation of the forcing term Let us fix two real numbers $L > 0$ and $\eta \in (0,1)$. For every $\varepsilon \in (0,1)$ we set

$$N_{\varepsilon,L} := \left\lfloor \frac{1}{L\omega(\varepsilon)} \right\rfloor \qquad \text{and} \qquad L_\varepsilon := \frac{1}{N_{\varepsilon,L}\,\omega(\varepsilon)}. \tag{2.5.3}$$

We observe that $N_{\varepsilon,L}$ is an integer, and that $L_\varepsilon \to L$ when $\varepsilon \to 0^+$. We observe also that $[0,1]$ is (up to a finite number of points) the disjoint union of the $N_{\varepsilon,L}$ intervals of length $L_\varepsilon\omega(\varepsilon)$ defined by

$$I_{\varepsilon,k} := ((k-1)L_\varepsilon\omega(\varepsilon), kL_\varepsilon\omega(\varepsilon)) \qquad \forall k \in \{1,\ldots,N_{\varepsilon,L}\}, \tag{2.5.4}$$

and we consider the piecewise affine function $f_{\varepsilon,L} : [0,1] \to \mathbb{R}$ that interpolates the values of f at the endpoints of these intervals, namely the function defined by

$$f_{\varepsilon,L}(x) := M_{\varepsilon,L,k}(x - (k-1)L_\varepsilon\omega(\varepsilon)) + f((k-1)L_\varepsilon\omega(\varepsilon)) \qquad \forall x \in I_{\varepsilon,k}, \tag{2.5.5}$$

where

$$M_{\varepsilon,L,k} := \frac{f(kL_\varepsilon\omega(\varepsilon)) - f((k-1)L_\varepsilon\omega(\varepsilon))}{L_\varepsilon\omega(\varepsilon)}.$$

From the H^1 regularity of f we deduce that the family $\{f_{\varepsilon,L}\}$ converges to f in the sense that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\omega(\varepsilon)^2} \int_0^1 (f(x) - f_{\varepsilon,L}(x))^2\,dx = 0. \tag{2.5.6}$$

Moreover, we deduce also that $f'_{\varepsilon,L} \to f'$ in $L^2((0,1))$, and in particular

$$\lim_{\varepsilon\to 0^+} L_\varepsilon\omega(\varepsilon) \sum_{k=1}^{N_{\varepsilon,L}} \phi(M_{\varepsilon,L,k}) = \lim_{\varepsilon\to 0^+} \int_0^1 \phi(f'_{\varepsilon,L}(x))\,dx = \int_0^1 \phi(f'(x))\,dx, \tag{2.5.7}$$

for every continuous function $\phi : \mathbb{R} \to \mathbb{R}$ that grows at most quadratically, namely such that

$$|\phi(p)| \leq C_1 + C_2\,p^2,$$

for some positive constants $C_1, C_2 > 0$.

Finally, from the inequality

$$(a+b)^2 \geq (1-\eta)a^2 + \left(1 - \frac{1}{\eta}\right) b^2 \qquad \forall \eta \in (0,1), \quad \forall (a,b) \in \mathbb{R}^2,$$

we obtain the estimate

$$\int_0^1 (u_\varepsilon - f)^2\, dx \geq (1-\eta) \int_0^1 (u_\varepsilon - f_{\varepsilon,L})^2\, dx + \left(1 - \frac{1}{\eta}\right) \int_0^1 (f - f_{\varepsilon,L})^2\, dx,$$

from which we conclude that

$$\begin{aligned} \mathbb{PMF}_\varepsilon(\beta, f, (0,1), u_\varepsilon) \;&\geq\; (1-\eta)\, \mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, (0,1), u_\varepsilon) \\ &\quad + \left(1 - \frac{1}{\eta}\right) \beta \int_0^1 (f(x) - f_{\varepsilon,L}(x))^2\, dx. \qquad (2.5.8) \end{aligned}$$

Reduction to a common interval We prove that

$$\mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, (0,1), u_\varepsilon) \geq \omega(\varepsilon)^3 \sum_{k=1}^{N_{\varepsilon,L}} \mu_\varepsilon(\beta, L, M_{\varepsilon,L,k}), \qquad (2.5.9)$$

where $\mu_\varepsilon(\beta, L, M_{\varepsilon,L,k})$ is defined by (2.3.12). To this end, we begin by observing that

$$\mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, (0,1), u_\varepsilon) = \sum_{k=1}^{N_{\varepsilon,L}} \mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, I_{\varepsilon,k}, u_\varepsilon). \qquad (2.5.10)$$

Each of the terms of the sum can be reduced to the common interval $(0, L_\varepsilon)$ by introducing the function $v_{\varepsilon,L,k} : (0, L_\varepsilon) \to \mathbb{R}$ defined by

$$v_{\varepsilon,L,k}(y) := \frac{u_\varepsilon((k-1)L_\varepsilon\omega(\varepsilon) + \omega(\varepsilon)y) - f((k-1)L_\varepsilon\omega(\varepsilon))}{\omega(\varepsilon)} \qquad \forall y \in (0, L_\varepsilon). \quad (2.5.11)$$

Indeed, with the change of variable $x = (k-1)L_\varepsilon\omega(\varepsilon) + \omega(\varepsilon)y$, we obtain that

$$\int_{I_{\varepsilon,k}} (u_\varepsilon(x) - f_{\varepsilon,L}(x))^2\, dx = \omega(\varepsilon)^3 \int_0^{L_\varepsilon} (v_{\varepsilon,L,k}(y) - M_{\varepsilon,L,k}\, y)^2\, dy$$

and

$$\int_{I_{\varepsilon,k}} \left\{ \varepsilon^6 \omega(\varepsilon)^4 u_\varepsilon''(x)^2 + \log\left(1 + u_\varepsilon'(x)^2\right) \right\} dx = \omega(\varepsilon)^3\, \mathbb{RPM}_\varepsilon((0, L_\varepsilon), v_{\varepsilon,L,k}),$$

and therefore

$$\begin{aligned} \mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, I_{\varepsilon,k}, u_\varepsilon) \;&=\; \omega(\varepsilon)^3\, \mathbb{RPMF}_\varepsilon(\beta, M_{\varepsilon,L,k}\, x, (0, L_\varepsilon), v_{\varepsilon,L,k}) \\ &\geq\; \omega(\varepsilon)^3 \mu_\varepsilon(\beta, L_\varepsilon, M_{\varepsilon,L,k}) \\ &\geq\; \omega(\varepsilon)^3 \mu_\varepsilon(\beta, L, M_{\varepsilon,L,k}), \end{aligned}$$

where in the last inequality we exploited that $L_\varepsilon \geq L$, and μ_ε is monotone with respect to the length of the interval. Plugging this inequality into (2.5.10) we obtain (2.5.9).

Convergence to minima of the limit problem For every $M_0 > 0$ there exists $\varepsilon_0 \in (0,1)$ such that

$$\mu_\varepsilon(\beta, L, M_{\varepsilon,L,k}) \geq \mu_\varepsilon(\beta, L, \min\{|M_{\varepsilon,L,k}|, M_0\}) \geq \mu_0(\alpha_0, \beta, L, \min\{|M_{\varepsilon,L,k}|, M_0\}) - \eta \tag{2.5.12}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and every $k \in \{1, \ldots, N_{\varepsilon,L}\}$, where the function μ_0 is defined according to (2.3.13), and α_0 is defined by (2.3.7).

Indeed, this estimate follows from Proposition 2.3.4, and in particular from the symmetry and monotonicity of μ_ε with respect to M, and the uniform convergence (2.3.20).

Conclusion Thanks to the estimate from below in (2.4.1) we know that

$$\mu_0(\alpha_0, \beta, L, \min\{|M_{\varepsilon,L,k}|, M_0\}) \geq c_1 \min\{|M_{\varepsilon,L,k}|, M_0\}^{4/5} L - c_2 \min\{|M_{\varepsilon,L,k}|, M_0\}^{1/5},$$

where c_1 and c_2 are given by (2.4.3), and therefore in particular

$$c_1 := \frac{5}{4}\left(\frac{\alpha_0^4 \beta}{3}\right)^{1/5} = 10\left(\frac{2\beta}{27}\right)^{1/5}. \tag{2.5.13}$$

Summing over k, from (2.5.9) and (2.5.12) we obtain that

$$\begin{aligned}
\frac{\mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, (0,1), u_\varepsilon)}{\omega(\varepsilon)^2} &\geq \omega(\varepsilon) \sum_{k=1}^{N_{\varepsilon,L}} \mu_\varepsilon(\beta, L, M_{\varepsilon,L,k}) \\
&\geq \omega(\varepsilon) \sum_{k=1}^{N_{\varepsilon,L}} \mu_0(\alpha_0, \beta, L, \min\{|M_{\varepsilon,L,k}|, M_0\}) - \eta\omega(\varepsilon) N_{\varepsilon,L} \\
&\geq c_1 L \omega(\varepsilon) \sum_{k=1}^{N_{\varepsilon,L}} \min\{|M_{\varepsilon,L,k}|, M_0\}^{4/5} - c_2 \omega(\varepsilon) N_{\varepsilon,L} M_0^{1/5} - \eta \omega(\varepsilon) N_{\varepsilon,L}.
\end{aligned}$$

Finally, plugging this estimate into (2.5.8) we deduce that

$$\begin{aligned}
\frac{\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u_\varepsilon)}{\omega(\varepsilon)^2} &\geq (1-\eta) c_1 \frac{L}{L_\varepsilon} \cdot L_\varepsilon \omega(\varepsilon) \sum_{k=1}^{N_{\varepsilon,L}} \min\{|M_{\varepsilon,L,k}|, M_0\}^{4/5} \\
&\quad - \omega(\varepsilon) N_{\varepsilon,L} \cdot (1-\eta)\left(c_2 M_0^{1/5} + \eta\right) \\
&\quad + \left(1 - \frac{1}{\eta}\right) \frac{\beta}{\omega(\varepsilon)^2} \int_0^1 (f(x) - f_{\varepsilon,L}(x))^2 \, dx.
\end{aligned}$$

Now we let $\varepsilon \to 0^+$, and we exploit (2.5.7) in the first line, the fact that $\omega(\varepsilon) N_{\varepsilon,L} \to 1/L$ in the second line, and (2.5.6) in the third line. We conclude that

$$\liminf_{\varepsilon \to 0^+} \frac{\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u_\varepsilon)}{\omega(\varepsilon)^2} \geq (1-\eta)\left\{c_1 \int_0^1 \min\{|f'(x)|, M_0\}^{4/5} \, dx - \frac{c_2 M_0^{1/5} + \eta}{L}\right\}.$$

Letting $\eta \to 0^+$ and $L \to +\infty$, we deduce that

$$\liminf_{\varepsilon \to 0^+} \frac{\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u_\varepsilon)}{\omega(\varepsilon)^2} \geq c_1 \int_0^1 \min\{|f'(x)|, M_0\}^{4/5}\, dx.$$

Finally, letting also $M_0 \to +\infty$ and recalling that c_1 is given by (2.5.13), we obtain exactly (2.5.1).

Estimate from above

We show the existence of a family $\{u_\varepsilon\} \subseteq H^2((0,1))$ for which (2.5.2) holds true. This amounts to proving the asymptotic optimality of all the steps in the proof of the estimate from below.

Interval subdivision and approximation of the forcing term Let us fix again two real numbers $L > 0$ and $\eta \in (0,1)$, and for every $\varepsilon \in (0,1)$ let us define $N_{\varepsilon,L}$ and L_ε as in (2.5.3), the intervals $I_{\varepsilon,k}$ as in (2.5.4), and the piecewise affine function $f_{\varepsilon,L} : (0,1) \to \mathbb{R}$ as in (2.5.5). Then we exploit the inequality

$$(a+b)^2 \leq (1+\eta)a^2 + \left(1 + \frac{1}{\eta}\right) b^2 \qquad \forall \eta \in (0,1), \quad \forall (a,b) \in \mathbb{R}^2,$$

and for every $u \in H^2((0,1))$ we obtain the estimate

$$\begin{aligned} \mathbb{PMF}_\varepsilon(\beta, f, (0,1), u) \;&\leq\; (1+\eta)\, \mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, (0,1), u) \\ &\quad + \left(1 + \frac{1}{\eta}\right) \beta \int_0^1 (f(x) - f_{\varepsilon,L}(x))^2\, dx. \end{aligned}$$

Reduction to a common interval We claim that there exists $u_\varepsilon \in H^2((0,1))$ such that

$$\begin{aligned} \mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, (0,1), u_\varepsilon) \;&=\; \omega(\varepsilon)^3 \sum_{k=1}^{N_{\varepsilon,L}} \mu_\varepsilon^*(\beta, L_\varepsilon, M_{\varepsilon,L,k}) \\ &\leq\; \left(\frac{L_\varepsilon}{L}\right)^3 \omega(\varepsilon)^3 \sum_{k=1}^{N_{\varepsilon,L}} \mu_\varepsilon^*(\beta, L, M_{\varepsilon,L,k}), \end{aligned}$$

where μ_ε^* is defined by (2.3.14), and the inequality follows from (2.3.17).

To this end, in analogy with the previous case we observe that the equalities

$$\begin{aligned} \mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, (0,1), u_\varepsilon) \;&=\; \sum_{k=1}^{N_{\varepsilon,L}} \mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, I_{\varepsilon,k}, u_\varepsilon) \\ &=\; \omega(\varepsilon)^3 \sum_{k=1}^{N_{\varepsilon,L}} \mathbb{RPMF}_\varepsilon(\beta, M_{\varepsilon,L,k}\, x, (0, L_\varepsilon), v_{\varepsilon,L,k}) \end{aligned}$$

hold true for every $u_\varepsilon \in H^2((0,1))$, provided that $u_\varepsilon(x)$ and $v_{\varepsilon,L,k}(x)$ are related by (2.5.11). At this point it is enough to choose u_ε in such a way that $v_{\varepsilon,L,k}$ coincides with a minimizer in the definition of $\mu_\varepsilon^*(\beta, L_\varepsilon, M_{\varepsilon,L,k})$ for every admissible choice of k.

Due to the boundary conditions in (2.3.14), the resulting function $u_\varepsilon(x)$ coincides with the forcing term $f(x)$ in the nodes of the form $x = kL_\varepsilon\omega(\varepsilon)$, its derivative vanishes in the same points, and the profile in each subinterval is (up to homotheties and translations) a minimizer to (2.3.14). As a consequence, the different pieces glue together in a C^1 way, and thus the resulting function belongs to $H^2((0,1))$.

Convergence to minima of the limit problem As in the case of the estimate from below we rely on Proposition 2.3.4 in order to deduce that for every $M_0 > 0$ there exists $\varepsilon_0 \in (0,1)$ such that

$$\begin{aligned}\left(\frac{L_\varepsilon}{L}\right)^3 \mu_\varepsilon^*(\beta, L, M_{\varepsilon,L,k}) &\leq \left(\frac{|M_{\varepsilon,L,k}|}{\min\{|M_{\varepsilon,L,k}|, M_0\}}\right)^2 \left(\frac{L_\varepsilon}{L}\right)^3 \mu_\varepsilon^*(\beta, L, \min\{|M_{\varepsilon,L,k}|, M_0\}) \\ &\leq \left(\frac{|M_{\varepsilon,L,k}|}{\min\{|M_{\varepsilon,L,k}|, M_0\}}\right)^2 (\mu_0^*(\alpha_0, \beta, L, \min\{|M_{\varepsilon,L,k}|, M_0\}) + \eta)\end{aligned}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and every $k \in \{1, \ldots, N_{\varepsilon,L}\}$. We can absorb the cubic factor into η because $L_\varepsilon \to L$, and $\mu_\varepsilon^*(\beta, L, \min\{|M_{\varepsilon,L,k}|, M_0\})$ is uniformly bounded for ε small because of the continuity of the limit μ_0^* with respect to M.

Conclusion Now we exploit the estimate from above in (2.4.2), and we find that

$$\mu_0^*(\alpha_0, \beta, L, \min\{|M_{\varepsilon,L,k}|, M_0\}) \leq c_1 \min\{|M_{\varepsilon,L,k}|, M_0\}^{4/5} L + c_3 \min\{|M_{\varepsilon,L,k}|, M_0\}^{1/5},$$

where again c_1 is given by (2.5.13), and as in the previous case we conclude that

$$\begin{aligned}\frac{\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u_\varepsilon)}{\omega(\varepsilon)^2} \quad &\leq \quad (1+\eta)c_1 \frac{L}{L_\varepsilon} L_\varepsilon\omega(\varepsilon) \sum_{k=1}^{N_{\varepsilon,L}} \frac{|M_{\varepsilon,L,k}|^2}{\min\{|M_{\varepsilon,L,k}|, M_0\}^{6/5}} \\ &\quad + (1+\eta)\frac{c_3}{L_\varepsilon} L_\varepsilon\omega(\varepsilon) \sum_{k=1}^{N_{\varepsilon,L}} \frac{|M_{\varepsilon,L,k}|^2}{\min\{|M_{\varepsilon,L,k}|, M_0\}^{9/5}} \\ &\quad + (1+\eta)\frac{\eta}{L_\varepsilon} L_\varepsilon\omega(\varepsilon) \sum_{k=1}^{N_{\varepsilon,L}} \frac{|M_{\varepsilon,L,k}|^2}{\min\{|M_{\varepsilon,L,k}|, M_0\}^{2}} \\ &\quad + \left(1 + \frac{1}{\eta}\right) \frac{\beta}{\omega(\varepsilon)^2} \int_0^1 (f(x) - f_{\varepsilon,L}(x))^2\, dx.\end{aligned}$$

Letting $\varepsilon \to 0^+$ and exploiting again (2.5.6) and (2.5.7), we obtain that this family

$\{u_\varepsilon\}$ satisfies

$$\begin{aligned}\limsup_{\varepsilon\to 0^+}\frac{\mathbb{PMF}_\varepsilon(\beta,f,(0,1),u_\varepsilon)}{\omega(\varepsilon)^2} \leq\ & (1+\eta)c_1\int_0^1\frac{|f'(x)|^2}{\min\{|f'(x)|,M_0\}^{6/5}}\,dx\\ &+(1+\eta)\frac{c_3}{L}\int_0^1\frac{|f'(x)|^2}{\min\{|f'(x)|,M_0\}^{9/5}}\,dx\\ &+(1+\eta)\frac{\eta}{L}\int_0^1\frac{|f'(x)|^2}{\min\{|f'(x)|,M_0\}^{2}}\,dx.\end{aligned}$$

Now we observe that the right-hand side tends to the right-hand side of (2.5.2) when $\eta\to 0^+$, $L\to+\infty$ and $M_0\to+\infty$. Therefore, with a standard diagonal procedure we can find a family $\{u_\varepsilon\}\subseteq H^2((0,1))$ for which exactly (2.5.2) holds true. □

2.5.2 Blow-ups at standard resolution (Theorem 2.2.9)

The proof of Theorem 2.2.9 consists of three main steps. In the first two steps we address the compactness of fake and true blow-ups. In the final step we show how to achieve all possible translations of the canonical staircase.

Compactness of fake blow-ups and oblique translations

Let us set for simplicity $x_n := x_{\varepsilon_n}$, and let $w_n(y) := w_{\varepsilon_n}(y)$ denote the corresponding fake blow-ups, defined in the interval (A_n,B_n) with

$$A_n := -\frac{x_n}{\omega(\varepsilon_n)},\qquad B_n := \frac{1-x_n}{\omega(\varepsilon_n)}. \tag{2.5.14}$$

We need to show that the sequence $\{w_n\}$ has a subsequence that converges locally strictly in $BV_{\mathrm{loc}}(\mathbb{R})$ to some oblique translation of the canonical (H,V)-staircase. To this end, we introduce the function $g_n:(A_n,B_n)\to\mathbb{R}$ defined by

$$g_n(y) := \frac{f(x_n+\omega(\varepsilon_n)y)-f(x_n)}{\omega(\varepsilon_n)}\qquad \forall y\in(A_n,B_n). \tag{2.5.15}$$

We are now in a position to apply Proposition 2.4.6. Let us check the assumptions.

- Since $x_n\to x_0\in(0,1)$, passing to the limit in (2.5.14) we see that $A_n\to-\infty$ and $B_n\to+\infty$.
- Since the forcing term f is of class C^1, passing to the limit in (2.5.15) we see that $g_n(y)\to f'(x_0)\cdot y$ uniformly on bounded subsets of $\mathbb{R}$.
- With the change of variable $x=x_n+\omega(\varepsilon_n)y$ we obtain that

$$\mathbb{PMF}_{\varepsilon_n}(\beta,f,(0,1),u_{\varepsilon_n}) = \omega(\varepsilon_n)^3\cdot\mathbb{RPMF}_{\varepsilon_n}(\beta,g_n,(A_n,B_n),w_n). \tag{2.5.16}$$

  Since $u_{\varepsilon_n}(x)$ is a minimizer of the original functional $u\mapsto\mathbb{PMF}_{\varepsilon_n}(\beta,f,(0,1),u)$, it follows that $w_n(y)$ is a minimizer of $w\mapsto\mathbb{RPMF}_{\varepsilon_n}(\beta,g_n,(A_n,B_n),w)$.

- Due to (2.5.16), estimate (2.4.6) follows from Theorem 2.2.2 as soon as $|\log \varepsilon_n| \geq 1$.

At this point, from Proposition 2.4.6 we deduce that the sequence $\{w_n\}$ converges locally strictly in $BV_{\text{loc}}(\mathbb{R})$, at least up to subsequences, to an entire local minimizer of the limiting functional (2.3.11), with α given by (2.3.7). Finally, from Proposition 2.4.5 we know that all these entire local minimizers are oblique translations of the canonical (H, V)-staircase, with parameters given by (2.2.10).

Remark 2.5.1 (Back to Remark 2.2.11)**.** Let consider the case where $x_\varepsilon \to x_0 \in \{0, 1\}$. If (2.2.13) holds true, then again $A_n \to -\infty$ and $B_n \to +\infty$ for every sequence $\varepsilon_n \to 0^+$, and hence the previous proof still works. If (2.2.13) fails, then when $x_0 = 0$ it may happen that $A_n \to A_\infty \in (-\infty, 0]$ and $B_n \to +\infty$ for some sequence $\varepsilon_n \to 0^+$.

In this case it is convenient to introduce the translated functions

$$\widehat{w}_n(y) := w_n(y + A_n) - f'(0)A_n \qquad \text{and} \qquad \widehat{g}_n(y) := g_n(y + A_n) - f'(0)A_n.$$

We observe that these functions are defined in the interval $(0, L_n)$ with $L_n := B_n - A_n$, so that $L_n \to +\infty$. We observe also that

$$\widehat{w}_n \in \operatorname{argmin}_{\text{R-loc}} \left\{ \mathbb{RPMF}_{\varepsilon_n}(\beta, \widehat{g}_n, (0, L_n), v) : v \in H^2((0, L_n)) \right\},$$

and that $\widehat{g}_n(y) \to f'(0) \cdot y$ uniformly on bounded subsets of $(0, +\infty)$.

This means that we are in the framework of Proposition 2.4.7, from which we deduce that the whole sequence $\{\widehat{w}_n\}$ converges to the unique semi-entire local minimizer in $(0, +\infty)$ of the limiting functional (2.3.11), with α given by (2.3.7). This semi-entire local minimizer is given by (2.4.5), and the convergence is strict in $BV((0, L))$ for every $L > 0$ that is not a jump point of the limit. This is a rigorous way of saying that $w_n(y)$ converges to $w(y - A_\infty) + f'(0)A_\infty$, and the latter is the oblique translation of the unique semi-entire local minimizer that "starts in $y = A_\infty$".

The case where $x_0 = 1$, and for some sequence $\varepsilon_n \to 0^+$ it happens that $A_n \to -\infty$ and $B_n \to B_\infty \in [0, +\infty)$, is symmetric.

Compactness of true blow-ups and graph translations

Let us define x_n and $w_n(y)$ as before, and let $v_n(y) := v_{\varepsilon_n}(y)$ denote the corresponding true blow-ups. We observe that true blow-ups are related to the fake blow-ups by the equality

$$v_n(y) = w_n(y) - w_n(0) \qquad \forall y \in (A_n, B_n), \tag{2.5.17}$$

and therefore the asymptotic behavior of the sequence $\{v_n\}$ can be deduced from the asymptotic behavior of the sequence $\{w_n\}$. More precisely, let us assume that

$$w_{n_k}(y) \rightsquigarrow S_{H,V}(y - H\tau_0) + V\tau_0 \quad \text{in } BV_{\text{loc}}(\mathbb{R})$$

for some sequence $n_k \to +\infty$ and some $\tau_0 \in [-1, 1]$. Then we distinguish two cases.

- Let us assume that $|\tau_0| < 1$. In this case $y = 0$ is not a discontinuity point of the limit of fake blow-ups, and hence the strict convergence implies pointwise convergence (see statement (2) in Remark 2.2.8), so that

$$\lim_{k\to+\infty} w_{n_k}(0) = S_{H,V}(-H\tau_0) + V\tau_0 = V\tau_0.$$

Therefore, from (2.5.17) we deduce that $v_{n_k}(y) \rightsquigarrow S_{H,V}(y - H\tau_0)$ in $BV_{\mathrm{loc}}(\mathbb{R})$, and we conclude by observing that the limit is a graph translation of horizontal type of the canonical (H,V)-staircase, as required.

- Let us assume that $\tau_0 = \pm 1$, and hence $\tau_0 = 1$ without loss of generality (because oblique translations corresponding to $\tau_0 = 1$ and $\tau_0 = -1$ coincide). In this case $y = 0$ is a discontinuity point of the limit of fake blow-ups, and hence strict convergence (see statement (2) in Remark 2.2.8) implies only that

$$-V \le \liminf_{k\to+\infty} w_{n_k}(0) \le \limsup_{k\to+\infty} w_{n_k}(0) \le V.$$

As a consequence, up to a further subsequence (not relabeled), $w_{n_k}(0)$ tends to some value in $[-V,V]$ that we can always write in the form $V\tau_1$ for some real number $\tau_1 \in [-1,1]$. Therefore, from (2.5.17) we deduce that, along this further subsequence,

$$v_{n_k}(y) \rightsquigarrow S_{H,V}(y - H) + V - V\tau_1 \quad \text{in } BV_{\mathrm{loc}}(\mathbb{R}),$$

and we conclude by observing that the limit is a graph translation of vertical type of the canonical (H,V)-staircase, as required.

Realization of all possible oblique/horizontal/vertical translations

In the constructions we can assume, without loss of generality, that $f'(x_0) \neq 0$, because otherwise all families of fake or true blow-ups converge to the trivial staircase that is identically 0, in which case there is nothing to prove.

Canonical staircase We show that there exists a family $x'_\varepsilon \to x_0$ satisfying (2.2.11), and (2.2.12) with $w_0(y) := S_{H,V}(y)$. The natural idea is to look for the fake blow-ups that minimize some distance from the desired limit. To this end, for every $\varepsilon \in (0,1)$ small enough we consider the function

$$\psi_\varepsilon(x) := \int_{-H}^{H} \left| \frac{u_\varepsilon(x + \omega(\varepsilon)y) - f(x)}{\omega(\varepsilon)} \right| dy.$$

It is a continuous function of x, and therefore it admits at least one minimum point x'_ε in the interval $[x_\varepsilon - H\omega(\varepsilon), x_\varepsilon + H\omega(\varepsilon)]$. We claim that $\{x'_\varepsilon\}$ is the required family. To begin with, we observe that (2.2.11) is automatic from the definition, and we call

$$w_\varepsilon(y) := \frac{u_\varepsilon(x'_\varepsilon + \omega(\varepsilon)y) - f(x'_\varepsilon)}{\omega(\varepsilon)} \tag{2.5.18}$$

the corresponding fake blow-ups. If we assume by contradiction that $\{w_\varepsilon\}$ does not converge to $S_{H,V}$, then from the compactness result we know that there exists a sequence $\varepsilon_n \to 0^+$ such that w_{ε_n} converges locally strictly in $BV_{\mathrm{loc}}(\mathbb{R})$ to some oblique translation z_0 of $S_{H,V}$, different from $S_{H,V}$ itself, and in particular

$$\lim_{n\to+\infty} \psi_{\varepsilon_n}(x'_{\varepsilon_n}) = \lim_{n\to+\infty} \int_{-H}^{H} |w_{\varepsilon_n}(y)|\, dy = \int_{-H}^{H} |z_0(y)|\, dy > 0.$$

On the other hand, since z_0 is an oblique translation, it can be written in the form

$$z_0(y) = S_{H,V}(y - H\tau_1) + V\tau_1$$

for a suitable $\tau_1 \in [-1,1]$, with $\tau_1 \neq 0$. Now for every positive integer n we set

$$x''_{\varepsilon_n} := x'_{\varepsilon_n} + (2k_n + \tau_1)H\omega(\varepsilon_n),$$

where $k_n \in \{-1,0,1\}$ is chosen in such a way that

$$x_{\varepsilon_n} - H\omega(\varepsilon_n) \le x''_{\varepsilon_n} < x_{\varepsilon_n} + H\omega(\varepsilon_n)$$

(we point out that there is always exactly one possible choice of k_n). We claim that

$$\frac{u_{\varepsilon_n}(x''_{\varepsilon_n} + \omega(\varepsilon_n)y) - f(x''_{\varepsilon_n})}{\omega(\varepsilon_n)} \rightsquigarrow S_{H,V}(y) \qquad \text{in } BV_{\mathrm{loc}}(\mathbb{R}), \tag{2.5.19}$$

and in particular the convergence is also in $L^1((-H,H))$. This implies that $\psi_{\varepsilon_n}(x''_{\varepsilon_n}) \to 0$, and hence $\psi_{\varepsilon_n}(x''_{\varepsilon_n}) < \psi_{\varepsilon_n}(x'_{\varepsilon_n})$ when n is large enough, thus contradicting the minimality of x'_{ε_n}. In order to prove (2.5.19), up to subsequences (not relabeled) we can always assume that k_n is actually a constant k_∞. Now we observe that

$$\frac{u_{\varepsilon_n}(x''_{\varepsilon_n} + \omega(\varepsilon_n)y) - f(x''_{\varepsilon_n})}{\omega(\varepsilon_n)} = w_{\varepsilon_n}(y + (2k_\infty + \tau_1)H) - \frac{f(x''_{\varepsilon_n}) - f(x'_{\varepsilon_n})}{\omega(\varepsilon_n)},$$

so that in particular

$$w_{\varepsilon_n}(y + (2k_\infty + \tau_1)H) \rightsquigarrow z_0(y + (2k_\infty + \tau_1)H) = S_{H,V}(y + 2k_\infty H) + V\tau_1,$$

and

$$\lim_{n\to+\infty} \frac{f(x''_{\varepsilon_n}) - f(x'_{\varepsilon_n})}{\omega(\varepsilon_n)} = (2k_\infty + \tau_1)H \cdot f'(x_0) = (2k_\infty + \tau_1)V.$$

It follows that

$$\frac{u_{\varepsilon_n}(x''_{\varepsilon_n} + \omega(\varepsilon_n)y) - f(x''_{\varepsilon_n})}{\omega(\varepsilon_n)} \rightsquigarrow S_{H,V}(y + 2k_\infty H) - 2k_\infty V,$$

and we conclude by observing that the latter coincides with $S_{H,V}(y)$. This completes the proof of (2.5.19).

All oblique translations Let $x'_\varepsilon \to x_0$ be the family that we found in the previous paragraph, namely a family satisfying (2.2.11), and (2.2.12) with $w_0(y) := S_{H,V}(y)$. If we need to obtain a different oblique translation of the form $w_0(y) = S_{H,V}(y-H\tau_0)+V\tau_0$ for some $\tau_0 \in (-1,1]$, then it is enough to consider the family

$$x''_\varepsilon := x'_\varepsilon - H\tau_0\omega(\varepsilon) + 2k_\varepsilon H\omega(\varepsilon),$$

where $k_\varepsilon \in \{-1,0,1\}$ is chosen in such a way that $x''_\varepsilon \in [x_\varepsilon - H\omega(\varepsilon), x_\varepsilon + H\omega(\varepsilon)]$. Indeed, it is enough to observe that

$$\frac{u_\varepsilon(x''_\varepsilon + \omega(\varepsilon)y) - f(x''_\varepsilon)}{\omega(\varepsilon)} = w_\varepsilon(y + (2k_\varepsilon - \tau_0)H) - \frac{f(x''_\varepsilon) - f(x'_\varepsilon)}{\omega(\varepsilon)}, \tag{2.5.20}$$

where w_ε is the family of fake blow-ups with centers in x'_ε. At this point, if needed we split the family into three subfamilies according to the value of k_ε. In the subfamily where k_ε is equal to some constant k_0 we obtain that

$$w_\varepsilon(y + (2k_\varepsilon - \tau_0)H) \rightsquigarrow S_{H,V}(y + (2k_0 - \tau_0)H),$$

and

$$\frac{f(x''_\varepsilon) - f(x'_\varepsilon)}{\omega(\varepsilon)} \to (2k_0 - \tau_0)H \cdot f'(x_0) = (2k_0 - \tau_0)V.$$

This implies that the left-hand side of (2.5.20) converges locally strictly to

$$S_{H,V}(y + (2k_0 - \tau_0)H) - (2k_0 - \tau_0)V,$$

which is equal to $S_{H,V}(y - H\tau_0) + V\tau_0$, independently of k_0, as required.

Graph translations of horizontal type In this paragraph we show that any graph translation of the form $S_{H,V}(y - H\tau_0)$, with $\tau_0 \in [-1,1]$, can be obtained as the limit of a suitable family of true blow-ups whose centers satisfy (2.2.11).

To begin with, we observe that the set of possible limits is closed with respect to the locally strict convergence in $BV_{\text{loc}}(\mathbb{R})$, and therefore it is enough to obtain all limits with τ_0 in the open interval $(-1,1)$. In this case, we claim that we can take the same family $x'_\varepsilon \to x_0$ whose fake blow-ups converge to $S_{H,V}(y - H\tau_0) - V\tau_0$, with the same value of τ_0. Indeed, we observe again that

$$\frac{u_\varepsilon(x'_\varepsilon + \omega(\varepsilon)y) - u_\varepsilon(x'_\varepsilon)}{\omega(\varepsilon)} = w_\varepsilon(y) - w_\varepsilon(0), \tag{2.5.21}$$

where $w_\varepsilon(y)$ is defined by (2.5.18). Now we know that $w_\varepsilon(y) \rightsquigarrow S_{H,V}(y - H\tau_0) - V\tau_0$. Moreover, if $\tau_0 \in (-1,1)$ the limit function is continuous in $y = 0$, and therefore the strict convergence implies also that $w_\varepsilon(0) \to S_{H,V}(-H\tau_0) - V\tau_0 = -V\tau_0$. Plugging these two results into (2.5.21) we obtain that the left-hand side converges to $S_{H,V}(y - H\tau_0)$, as required.

Graph translations of vertical type In this final paragraph we show that any graph translation of the form $S_{H,V}(y-H)+(1-\tau_0)V$, with $\tau_0\in[-1,1]$, can be obtained as the limit of a suitable family of true blow-ups whose centers satisfy (2.2.11). To begin with, as in the case of graph translations of horizontal type we reduce ourselves to the case where $\tau_0\in(-1,1)$.

In this case we consider the family $x'_\varepsilon\to x_0$ whose fake blow-ups $w_\varepsilon(y)$ defined by (2.5.18) converge to $S_{H,V}(y)$. Since $S_{H,V}(y)$ is continuous in $y=-2H$ and $y=2H$, the strict convergence implies in particular that

$$\lim_{\varepsilon\to0^+} w_\varepsilon(-2H)=-2V \qquad \text{and} \qquad \lim_{\varepsilon\to0^+} w_\varepsilon(2H)=2V.$$

Recalling that $w_\varepsilon(y)$ is continuous in y and vanishes for $y=0$, this means that when $\varepsilon\in(0,1)$ is small enough there exist $a_\varepsilon\in(-2H,0)$ and $b_\varepsilon\in(0,2H)$ such that

$$w_\varepsilon(a_\varepsilon)=(-1+\tau_0)V \qquad \text{and} \qquad w_\varepsilon(b_\varepsilon)=(1+\tau_0)V. \tag{2.5.22}$$

These two conditions imply in particular that

$$\lim_{\varepsilon\to0^+} a_\varepsilon=-H \qquad \text{and} \qquad \lim_{\varepsilon\to0^+} b_\varepsilon=H, \tag{2.5.23}$$

because a limit of blow-ups can be different from an integer multiple of $2V$ only in the jump points of the limit function $S_{H,V}$.

Now we set

$$x''_\varepsilon:=\begin{cases} x'_\varepsilon+\omega(\varepsilon)a_\varepsilon & \text{if } x'_\varepsilon\geq x_\varepsilon,\\ x'_\varepsilon+\omega(\varepsilon)b_\varepsilon & \text{if } x'_\varepsilon< x_\varepsilon,\end{cases}$$

and we claim that this is the required family. Indeed, from the definition it follows that

$$\frac{|x''_\varepsilon-x_\varepsilon|}{\omega(\varepsilon)}\leq\max\left\{\frac{|x'_\varepsilon-x_\varepsilon|}{\omega(\varepsilon)},-a_\varepsilon,b_\varepsilon\right\},$$

and therefore (2.2.11) for x''_ε follows from (2.2.11) for x'_ε and (2.5.23).

In order to compute the limit of the true blow-ups with center in x''_ε, we consider the two subfamilies where x''_ε is defined using a_ε or b_ε. In the first case from (2.5.22) and (2.5.23) we deduce that

$$\begin{aligned}\frac{u_\varepsilon(x''_\varepsilon+\omega(\varepsilon)y)-u_\varepsilon(x''_\varepsilon)}{\omega(\varepsilon)} &= w_\varepsilon(y+a_\varepsilon)-w_\varepsilon(a_\varepsilon),\\ &\rightsquigarrow S_{H,V}(y-H)-(-1+\tau_0)V,\end{aligned}$$

as required. Analogously, in the second case we obtain that

$$\frac{u_\varepsilon(x''_\varepsilon+\omega(\varepsilon)y)-u_\varepsilon(x''_\varepsilon)}{\omega(\varepsilon)}\rightsquigarrow S_{H,V}(y+H)-(1+\tau_0)V,$$

which again coincides with $S_{H,V}(y-H)+(1-\tau_0)V$, as required. □

2.5.3 Convergence of minimizers to the forcing term

Strict convergence (statement (1) of Theorem 2.2.14)

Since the limit f is continuous, we know that uniform convergence in $[0,1]$ follows from strict convergence (see statement (2) in Remark 2.2.8). As for strict convergence, we already know from Proposition 2.2.1 that $u_\varepsilon \to f$ in $L^2((0,1))$. Therefore, it remains to show that (the opposite inequality is trivial)

$$\limsup_{\varepsilon\to 0^+} \int_0^1 |u'_\varepsilon(x)|\,dx \le \int_0^1 |f'(x)|\,dx. \tag{2.5.24}$$

Let us assume by contradiction that this is not the case, and hence there exist a positive real number η_0 and a sequence $\{\varepsilon_n\} \subseteq (0,1)$ such that $\varepsilon_n \to 0^+$ and

$$\int_0^1 |u'_{\varepsilon_n}(x)|\,dx \ge \int_0^1 |f'(x)|\,dx + \eta_0 \qquad \forall n \in \mathbb{N}. \tag{2.5.25}$$

For every fixed positive real number L, in analogy with (2.5.3) we set

$$N_n := \left\lfloor \frac{1}{L\omega(\varepsilon_n)} \right\rfloor \qquad \text{and} \qquad L_n := \frac{1}{N_n\omega(\varepsilon_n)},$$

and we consider the intervals $I_{n,k} := ((k-1)L_n\omega(\varepsilon_n), kL_n\omega(\varepsilon_n))$ with $k \in \{1,\dots,N_n\}$. Since we can rewrite (2.5.25) in the form

$$\sum_{k=1}^{N_n} \int_{I_{n,k}} \left(|u'_{\varepsilon_n}(x)| - |f'(x)|\right)\,dx \ge \eta_0,$$

we deduce that for every $n \in \mathbb{N}$ there exists an integer k_n such that

$$\int_{I_{n,k_n}} \left(|u'_{\varepsilon_n}(x)| - |f'(x)|\right)\,dx \ge \frac{\eta_0}{N_n}. \tag{2.5.26}$$

Now we set $x_n := (k_n-1)\omega(\varepsilon_n)L_n$, and we consider the corresponding fake blow-ups

$$w_n(y) := \frac{u_{\varepsilon_n}(x_n + \omega(\varepsilon_n)y) - f(x_n)}{\omega(\varepsilon_n)}. \tag{2.5.27}$$

With the change of variable $x = x_n + \omega(\varepsilon)y$, we can rewrite (2.5.26) in the form

$$\int_0^{L_n} \left(|w'_n(y)| - |f'(x_n + \omega(\varepsilon_n)y)|\right)\,dy \ge \frac{\eta_0}{N_n\omega(\varepsilon_n)} \ge \eta_0 L. \tag{2.5.28}$$

Up to subsequences (not relabeled) we can always assume that x_n converges to some $x_\infty \in [0,1]$. Let us assume now that $x_\infty \in (0,1)$. From the continuity of f' we deduce that $f'(x_n + \omega(\varepsilon_n)y) \to f'(x_\infty)$ uniformly on bounded subsets of $\mathbb{R}$ and in particular, since $L_n \to L$, we obtain that

$$\lim_{n\to+\infty} \int_0^{L_n} |f'(x_n + \omega(\varepsilon_n)y)|\,dy = |f'(x_\infty)|L. \tag{2.5.29}$$

Moreover, from statement (1) of Theorem 2.2.9 we deduce that, up to a further subsequence (not relabeled), $w_n \rightsquigarrow w_\infty$ in $BV_{\text{loc}}(\mathbb{R})$, where w_∞ is an oblique translation of a canonical staircase with parameters depending on β and $f'(x_\infty)$. As a consequence, from statement (3) of Remark 2.2.8 we obtain that

$$\limsup_{n\to+\infty} \int_0^{L_n} |w_n'(y)|\, dy \le \lim_{n\to+\infty} \int_a^b |w_n'(y)|\, dy = |Dw_\infty|((a,b))$$

for every interval $(a,b) \supseteq [0,L]$ whose endpoints a and b are not jump points of w_∞. If we consider any sequence of such intervals whose intersection is $[0,L]$, we deduce that

$$\limsup_{n\to+\infty} \int_0^{L_n} |w_n'(y)|\, dy \le |Dw_\infty|([0,L]). \tag{2.5.30}$$

From (2.5.28), (2.5.29) and (2.5.30) we conclude that

$$|Dw_\infty|([0,L]) - |f'(x_\infty)|L \ge \eta_0 L. \tag{2.5.31}$$

Now we observe that the left-hand side is the difference between the total variation of w_∞ in $[0,L]$ and the total variation of the line $y \mapsto |f'(x_\infty)|y$ in the same interval. Since $w_\infty(y)$ is a staircase with the property that the midpoints of the vertical parts of the steps lie on the same line, the left-hand side of (2.5.31) is bounded from above by the height of each step of the staircase. Now both w_∞ and x_∞ might depend on L, but in any case the height of the steps depends only on β and $|f'(x_\infty)|$, and the latter is bounded independently on L because f is of class C^1. In conclusion, the left-hand side of (2.5.31) is bounded from above independently of L, and this contradicts (2.5.31) when L is large enough.

Let us consider next the case where $x_\infty = 0$ (the case $x_\infty = 1$ is symmetric). In this case we consider the sequence $\{k_n\}$. If it is unbounded, then up to subsequences we can assume that it diverges to $+\infty$. In this case the intervals where the functions w_n of (2.5.27) are defined invade eventually the whole real line, and therefore the previous argument works without any change (see also Remark 2.2.11).

If the sequence $\{k_n\}$ is bounded, then up to subsequences we can assume that it is equal to some fixed positive integer k_∞. In this case the functions $w_n(y)$ are all defined in the same half-line $y > y_\infty$ with $y_\infty := -(k_\infty - 1)L$, and in this half-line they converge to a limit staircase $w_\infty(y)$ (see Remark 2.5.1). The convergence is strict in every interval of the form (y_∞, b), where b is not a jump point of w_∞, and of course also in all intervals of the form (a,b) where a and b are not jump points of w_∞. Moreover, the function w_∞ is the unique semi-entire right-hand local minimizer of $\mathbb{JF}_{1/2}$ with the appropriate parameters in this half-line, namely the suitable oblique translation of the function defined in (2.4.5), which is again a staircase with the property that the midpoints of the vertical parts of the steps lie on the line $f'(x_\infty)y$.

At the end of the day, we obtain that (2.5.31) holds true also in this case, and as before the right-hand side depends only on the difference between the "values" of $w_\infty(y)$ and of the line $f'(x_\infty)y$ at the two endpoints. This difference is bounded from above by the height of the steps of w_∞. These steps could be either the "ordinary steps" or the "initial step", which is higher, but in any case their height is independent of L.

Varifold convergence (statement (2) of Theorem 2.2.14)

Notations and splitting of the graph In analogy with (2.2.14), for every $\varepsilon \in (0,1)$ we set

$$V_\varepsilon^+ := \{x \in [0,1] : u_\varepsilon'(x) > 0\} \qquad \text{and} \qquad V_\varepsilon^- := \{x \in [0,1] : u_\varepsilon'(x) < 0\}.$$

From statement (4) of Remark 2.2.8 we know that the strict convergence of u_ε to f implies in particular that

$$\lim_{\varepsilon \to 0^+} \int_{V_\varepsilon^+} g(x) u_\varepsilon'(x)\, dx = \int_{V_0^+} g(x) f'(x)\, dx \tag{2.5.32}$$

for every continuous function $g : [0,1] \to \mathbb{R}$, and similarly with V_ε^- and V_0^-.

We observe also that the strict convergence $u_\varepsilon \rightsquigarrow f$ in $BV((0,1))$ implies that the family $\{u_\varepsilon\}$ is bounded in $L^\infty((0,1))$, and therefore there exist real numbers $\varepsilon_0 \in (0,1)$ and $M_0 \geq 0$ such that

$$|\phi(x, u_\varepsilon(x), \arctan p)| \leq M_0 \qquad \forall (x,p) \in [0,1] \times \mathbb{R} \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{2.5.33}$$

Now for every $a \in (0,1)$ we define the three sets

$$I_a := \{(x,s,p) \in [0,1] \times \mathbb{R} \times \mathbb{R} : |s - f(x)| \leq a,\ |p| \leq a\},$$
$$I_a^+ := \{(x,s,p) \in [0,1] \times \mathbb{R} \times \mathbb{R} : |s - f(x)| \leq a,\ p \geq 1/a\},$$
$$I_a^- := \{(x,s,p) \in [0,1] \times \mathbb{R} \times \mathbb{R} : |s - f(x)| \leq a,\ p \leq -1/a\},$$

and the corresponding three constants

$$\Gamma_a := \max\{|\phi(x,s,\arctan p) - \phi(x, f(x), 0)| : (x,s,p) \in I_a\},$$
$$\Gamma_a^+ := \max\left\{|\phi(x,s,\arctan p) - \phi(x, f(x), \pi/2)| : (x,s,p) \in I_a^+\right\},$$
$$\Gamma_a^- := \max\left\{|\phi(x,s,\arctan p) - \phi(x, f(x), -\pi/2)| : (x,s,p) \in I_a^-\right\}.$$

We observe that, due to the boundedness of $f(x)$ and the uniform continuity of ϕ in bounded sets, these constants satisfy

$$\lim_{a \to 0^+} \Gamma_a = \lim_{a \to 0^+} \Gamma_a^+ = \lim_{a \to 0^+} \Gamma_a^- = 0. \tag{2.5.34}$$

Finally, for every $\varepsilon \in (0,1)$ and every $a \in (0,1)$, we write the interval $[0,1]$ as the disjoint union of the four sets

$$H_{a,\varepsilon} := \{x \in [0,1] : |u_\varepsilon'(x)| \leq a\}, \tag{2.5.35}$$

$$V_{a,\varepsilon}^+ := \{x \in [0,1] : u_\varepsilon'(x) \geq 1/a\}, \qquad V_{a,\varepsilon}^- := \{x \in [0,1] : u_\varepsilon'(x) \leq -1/a\}, \tag{2.5.36}$$

$$M_{a,\varepsilon} := \{x \in [0,1] : a < |u_\varepsilon'(x)| < 1/a\}, \tag{2.5.37}$$

and accordingly we write

$$\int_0^1 \phi\left(x, u_\varepsilon(x), \arctan(u'_\varepsilon(x))\right)\sqrt{1+u'_\varepsilon(x)^2}\,dx = I^H_{a,\varepsilon} + I^+_{a,\varepsilon} + I^-_{a,\varepsilon} + I^M_{a,\varepsilon},$$

where the four terms in the right-hand side are the integrals over the four sets defined above. We observe that

$$\begin{aligned}\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u_\varepsilon) &\geq \int_0^1 \log\left(1+u'_\varepsilon(x)^2\right)\,dx \\ &\geq \log\left(1+a^2\right)\left(|V^+_{a,\varepsilon}| + |V^-_{a,\varepsilon}| + |M_{a,\varepsilon}|\right),\end{aligned}$$

and, since the left-hand side tends to 0, we deduce that

$$\lim_{\varepsilon\to 0^+} |V^+_{a,\varepsilon}| = \lim_{\varepsilon\to 0^+} |V^-_{a,\varepsilon}| = \lim_{\varepsilon\to 0^+} |M_{a,\varepsilon}| = 0 \qquad \forall a \in (0,1),$$

and as a consequence

$$\lim_{\varepsilon\to 0^+} |H^+_{a,\varepsilon}| = 1 \qquad \forall a \in (0,1).$$

We claim that for every fixed $a \in (0,1)$ it turns out that

$$\limsup_{\varepsilon\to 0^+} \left| I^H_{a,\varepsilon} - \int_0^1 \phi(x, f(x), 0)\,dx \right| \leq M_0\left(\sqrt{1+a^2} - 1\right) + \Gamma_a, \tag{2.5.38}$$

$$\lim_{\varepsilon\to 0^+} I^M_{a,\varepsilon} = 0, \tag{2.5.39}$$

$$\limsup_{\varepsilon\to 0^+} \left| I^+_{a,\varepsilon} - \int_{V_0^+} \phi(x, f(x), \pi/2)\cdot f'(x)\,dx \right| \leq \Gamma^+_a \int_0^1 |f'(x)|\,dx + M_0 a, \tag{2.5.40}$$

$$\limsup_{\varepsilon\to 0^+} \left| I^-_{a,\varepsilon} - \int_{V_0^-} \phi(x, f(x), -\pi/2)\cdot |f'(x)|\,dx \right| \leq \Gamma^-_a \int_0^1 |f'(x)|\,dx + M_0 a. \tag{2.5.41}$$

If we prove these claims, then we let $a \to 0^+$ and from (2.5.34) we obtain exactly (2.2.15).

In words, this means that the integral in the left-hand side of (2.2.15) splits into the four integrals over the regions (2.5.35), (2.5.36), (2.5.37), which behave as follows.

- The integral over the "intermediate" region $M_{a,\varepsilon}$ disappears in the limit.
- The integral over the "horizontal" region $H_{a,\varepsilon}$ tends to the first integral in the right hand side of (2.2.15), in which the "tangent component" is horizontal.
- The integrals over the two "vertical" regions $V^+_{a,\varepsilon}$ and $V^-_{a,\varepsilon}$ tend to the two integrals over V_0^+ and V_0^- in the right hand side of (2.2.15). In this two integrals the "tangent component" is vertical.

Estimate in the intermediate regime From (2.5.33) we know that

$$|\phi(x, u_\varepsilon(x), \arctan(u'_\varepsilon(x)))| \sqrt{1+u'_\varepsilon(x)^2} \leq M_0\sqrt{1+\frac{1}{a^2}} \qquad \forall x \in M_{a,\varepsilon},$$

and therefore

$$|I^M_{\varepsilon,a}| \leq M_0\sqrt{1+\frac{1}{a^2}} \cdot |M_{a,\varepsilon}|.$$

Since $|M_{a,\varepsilon}| \to 0$ as $\varepsilon \to 0^+$, this proves (2.5.39).

Estimate in the horizontal regime In order to prove (2.5.38), we observe that

$$\begin{aligned} I^H_{a,\varepsilon} - \int_0^1 \phi(x, f(x), 0)\,dx &= \int_{H_{a,\varepsilon}} \phi\left(x, u_\varepsilon(x), \arctan(u'_\varepsilon(x))\right)\left(\sqrt{1+u'_\varepsilon(x)^2}-1\right)\,dx \\ &\quad + \int_{H_{a,\varepsilon}} \left\{\phi\left(x, u_\varepsilon(x), \arctan(u'_\varepsilon(x))\right) - \phi(x, f(x), 0)\right\}\,dx, \\ &\quad + \int_{H_{a,\varepsilon}} \phi(x, f(x), 0)\,dx - \int_0^1 \phi(x, f(x), 0)\,dx. \end{aligned}$$

The absolute value of the first line in the right-hand side is less than or equal to $M_0\left(\sqrt{1+a^2}-1\right)$. The absolute value of the second line is less than or equal to Γ_a provided that

$$|u_\varepsilon(x) - f(x)| \leq a \qquad \forall x \in [0,1], \tag{2.5.42}$$

and this happens whenever ε is small enough. The third line tends to 0 because $|H_{\varepsilon,a}| \to 1$ as $\varepsilon \to 0^+$. This is enough to establish (2.5.38).

Estimate in the vertical regime In order to prove (2.5.40), we observe that

$$\begin{aligned} & I^+_{a,\varepsilon} - \int_{V_0^+} \phi(x, f(x), \pi/2)\cdot f'(x)\,dx = \\ &\quad = \int_{V^+_{a,\varepsilon}} \phi\left(x, u_\varepsilon(x), \arctan(u'_\varepsilon(x))\right)\left(\sqrt{1+u'_\varepsilon(x)^2}-u'_\varepsilon(x)\right)\,dx \\ &\qquad + \int_{V^+_{a,\varepsilon}} \left\{\phi\left(x, u_\varepsilon(x), \arctan(u'_\varepsilon(x))\right) - \phi(x, f(x), \pi/2)\right\} u'_\varepsilon(x)\,dx, \\ &\qquad + \int_{V^+_{a,\varepsilon}} \phi(x, f(x), \pi/2)u'_\varepsilon(x)\,dx - \int_{V^+_\varepsilon} \phi(x, f(x), \pi/2)u'_\varepsilon(x)\,dx \\ &\qquad + \int_{V^+_\varepsilon} \phi(x, f(x), \pi/2)u'_\varepsilon(x)\,dx - \int_{V^+_0} \phi(x, f(x), \pi/2)f'(x)\,dx \\ &\quad =: L_1 + L_2 + L_3 + L_4. \end{aligned}$$

Let us consider the four lines separately. The first line can be estimated as

$$|L_1| \leq M_0 \max\left\{\sqrt{1+p^2}-p : p \geq 1/a\right\} |V^+_{a,\varepsilon}| \leq M_0 \cdot \frac{a}{2} \cdot |V^+_{a,\varepsilon}|,$$

and this tends to 0 when $\varepsilon \to 0^+$. The second line can be estimated as

$$|L_2| \le \Gamma_a^+ \cdot \int_0^1 |u_\varepsilon'(x)|\,dx$$

whenever (2.5.42) holds true, namely when ε is small enough. For the third line we observe that $V_\varepsilon^+ \setminus V_{a,\varepsilon}^+ \subseteq H_{a,\varepsilon} \cup M_{a,\varepsilon}$, and therefore

$$\begin{aligned}
|L_3| &\le \int_{H_{a,\varepsilon}} |\phi(x,f(x),0)|\cdot|u_\varepsilon'(x)|\,dx + \int_{M_{a,\varepsilon}} |\phi(x,f(x),0)|\cdot|u_\varepsilon'(x)|\,dx \\
&\le M_0 a + M_0\cdot\frac{1}{a}\cdot|M_{a,\varepsilon}|.
\end{aligned}$$

Finally, we observe that $L_4 \to 0$ as $\varepsilon \to 0^+$ because of (2.5.32). Recalling (2.5.24) and the fact that $|M_{a,\varepsilon}| \to 0$ as $\varepsilon \to 0^+$, from the previous estimates we conclude that

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} |L_1+L_2+L_3+L_4| &\le \limsup_{\varepsilon\to 0^+} \Gamma_a^+ \cdot \int_0^1 |u_\varepsilon'(x)|\,dx + M_0 a \\
&= \Gamma_a^+ \int_0^1 |f'(x)|\,dx + M_0 a,
\end{aligned}$$

which proves (2.5.40). The proof of (2.5.41) is analogous. □

2.5.4 Low resolution blow-ups (Corollary 2.2.13)

The pointwise convergence for $y = 0$ is trivial, and therefore it is enough to check the convergence of total variations, which in turn reduces to

$$\limsup_{\varepsilon\to 0^+} \int_a^b |u_\varepsilon'(x_\varepsilon + \alpha_\varepsilon y)|\,dy \le |f'(x_0)|(b-a) \tag{2.5.43}$$

for every interval $(a,b) \subseteq \mathbb{R}$. If we assume by contradiction that (2.5.43) fails, than the same argument we exploited in the proof of (2.5.24) shows that there exist a positive real number η_0, a sequence $\{\varepsilon_n\} \subseteq (0,1)$ such that $\varepsilon_n \to 0^+$, and a sequence of positive integers k_n such that

$$\int_0^{L_n} |u_{\varepsilon_n}'(\widehat{x}_n + \omega(\varepsilon_n)y)|\,dy - |f'(x_0)|L_n \ge \eta_0 \cdot \frac{\alpha_{\varepsilon_n}}{\omega(\varepsilon_n)N_n}, \tag{2.5.44}$$

where now

$$N_n := \left\lfloor \frac{(b-a)\alpha_{\varepsilon_n}}{L\omega(\varepsilon_n)} \right\rfloor, \qquad L_n := \frac{(b-a)\alpha_{\varepsilon_n}}{N_n\omega(\varepsilon_n)}, \qquad \widehat{x}_n := x_{\varepsilon_n} + a\alpha_{\varepsilon_n} + L_n\omega(\varepsilon_n)(k_n-1),$$

and $k_n \in \{1,\dots,N_n\}$. The integral in the left-hand side of (2.5.44) coincides with the total variation in the interval $(0,L_n)$ of the fake blow-up of u_{ε_n}, at the standard scale $\omega(\varepsilon_n)$, with center in $\widehat{x}_n$. Since $\widehat{x}_n \to x_0$ (here we exploit again that $k_n \le N_n$ and $\omega(\varepsilon)/\alpha_\varepsilon \to 0$), we know that these fake blow-ups converge strictly (up to subsequences) to some staircase w_∞. Therefore, passing to the limit in (2.5.44) we deduce that

$$|Dw_\infty|([0,L]) - |f'(x_0)|L \ge \frac{\eta_0 L}{b-a},$$

and we conclude exactly as in the proof of (2.5.24). □

2.6 Asymptotic analysis of local minimizers

This section is the technical core of the chapter. Here we prove all the results that we stated in section 2.4.

2.6.1 Preliminary lemmata

Lemma 2.6.1. *Let C_0 and C_1 be two positive real numbers. Let us consider the function $\varphi : (0,1) \to \mathbb{R}$ defined by*

$$\varphi(t) := C_0 \left(\sqrt{t} + \sqrt{1-t}\right) + C_1 \left(t^3 + (1-t)^3\right),$$

and let us assume that there exists $t_0 \in (0,1)$ such that $\varphi(t) \geq \varphi(t_0)$ for every $t \in (0,1)$.

Then it turns out that $t_0 = 1/2$.

Proof. With the variable change $t = \sin^2 \theta$, we can restate the claim as follows. Let us consider the function $g : (0, \pi/2) \to \mathbb{R}$ defined by

$$g(\theta) := C_0 (\cos\theta + \sin\theta) + C_1 \left(\cos^6\theta + \sin^6\theta\right);$$

if there exists $\theta_0 \in (0, \pi/2)$ such that

$$g(\theta) \geq g(\theta_0) \qquad \forall \theta \in (0, \pi/2), \tag{2.6.1}$$

then necessarily $\theta_0 = \pi/4$.

In order to prove this claim, we observe that the derivative of g is

$$g'(\theta) = (\cos\theta - \sin\theta)\left(C_0 - 6C_1 \cos\theta \sin\theta(\cos\theta + \sin\theta)\right). \tag{2.6.2}$$

Let us consider the function $\psi(\theta) := \cos\theta \sin\theta(\cos\theta + \sin\theta)$, whose derivative is

$$\psi'(\theta) = (\cos\theta - \sin\theta)(1 + 3\cos\theta\sin\theta).$$

It follows that ψ is increasing in $[0, \pi/4]$ and decreasing in $[\pi/4, \pi/2]$, and its maximum values is $\psi(\pi/4) = 1/\sqrt{2}$. Now we distinguish two cases.

- If $C_0\sqrt{2} \geq 6C_1$, then the sign of $g'(\theta)$ coincides with the sign of $\cos\theta - \sin\theta$. It follows that $\pi/4$ is the unique stationary point of g in $(0, \pi/4)$, but it is a maximum point, and therefore in this case there is no $\theta_0 \in (0, \pi/2)$ for which (2.6.1) holds true.

- If $C_0\sqrt{2} < 6C_1$, then also the second term in the right-hand side of (2.6.2) changes its sign in two points of the form $\pi/4 \pm \theta_1$ for some $\theta_1 \in (0, \pi/4)$. In this case it turns out that g has three stationary points in $(0, \pi/2)$, namely $\pi/4 \pm \theta_1$ (which are maximum points) and $\pi/4$, which is a minimum point (local or global depending on C_0 and C_1).

In any case, if g has a minimum point in $(0, \pi/2)$, this is necessarily $\pi/4$. □

Lemma 2.6.2. *Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let A_0, A_1, B_0, B_1 be four real numbers. Let us consider the minimum problem*

$$\min\left\{\int_a^b w''(y)^2\,dy : w \in H^2((a,b)),\ \big(w(a),w'(a),w(b),w'(b)\big) = (A_0,A_1,B_0,B_1)\right\}.$$

Then the unique minimum point is the function

$$w_0(y) = P\left(y - \frac{a+b}{2}\right),$$

where $P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ is the polynomial of degree three with coefficients

$$c_0 := \frac{A_0+B_0}{2} - \frac{B_1-A_1}{8}(b-a), \qquad c_1 := \frac{3(B_0-A_0)}{2(b-a)} - \frac{A_1+B_1}{4},$$

$$c_2 := \frac{B_1-A_1}{2(b-a)}, \qquad c_3 := -\frac{2(B_0-A_0)}{(b-a)^3} + \frac{A_1+B_1}{(b-a)^2}.$$

As a consequence, the minimum value is

$$\frac{(B_1-A_1)^2}{b-a} + \frac{12}{(b-a)^3}\left[(B_0-A_0) - \frac{A_1+B_1}{2}(b-a)\right]^2,$$

and the minimum point satisfies the pointwise estimates

$$|w_0(y)| \leq \frac{3(|A_0|+|B_0|)}{2} + \frac{|A_1|+|B_1|}{2}(b-a) \qquad \forall y \in [a,b],$$

and

$$|w_0'(y)| \leq \frac{3|B_0-A_0|}{b-a} + \frac{3(|A_1|+|B_1|)}{2} \qquad \forall y \in [a,b].$$

Proof. From the Euler-Lagrange equation we know that minimizers are polynomials of degree three, and w_0 is the unique such polynomial that fits the boundary conditions. □

Lemma 2.6.3. *Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let D and H be positive real numbers. Let $\varepsilon \in (0,1)$ be a real number such that*

$$2\varepsilon^2\left(\sqrt{H} + \varepsilon^2 D\right) < (b-a) \tag{2.6.3}$$

and

$$\frac{2}{|\log\varepsilon|}\log\left(1 + \frac{45}{2\varepsilon^4}\left(\sqrt{H}+\varepsilon^2 D\right)^2\right) \leq 18. \tag{2.6.4}$$

Then for every $(A_0,B_0) \in [-H,H]^2$ and every $(A_1,B_1) \in [-D,D]^2$ there exists a function $w \in H^2((a,b))$ satisfying the boundary conditions

$$\big(w(a),w'(a),w(b),w'(b)\big) = (A_0,A_1,B_0,B_1), \tag{2.6.5}$$

and the estimates

$$\mathbb{RPM}_\varepsilon((a,b),w) \leq 80\left(\sqrt{H}+\varepsilon^2 D\right), \tag{2.6.6}$$

$$\int_a^b w(x)^2\,dx \leq 10\varepsilon^2\left(\sqrt{H}+\varepsilon^2 D\right)^5. \tag{2.6.7}$$

Proof. For every real number $\eta \in (0,(b-a)/2)$, let us consider the function

$$w(x) := \begin{cases} \varphi_1(x) & \text{if } x \in [a, a+\eta], \\ 0 & \text{if } x \in [a+\eta, b-\eta], \\ \varphi_2(x) & \text{if } x \in [b-\eta, b], \end{cases}$$

where φ_1 is the unique polynomial of degree three such that

$$\varphi_1(a) = A_0, \qquad \varphi_1'(a) = A_1, \qquad \varphi_1(a+\eta) = \varphi_1'(a+\eta) = 0,$$

and φ_2 is the unique polynomial of degree three such that

$$\varphi_2(b) = B_0, \qquad \varphi_2'(b) = B_1, \qquad \varphi_2(b-\eta) = \varphi_2'(b-\eta) = 0.$$

We observe that w belongs to $H^2((a,b))$, and fulfills the boundary conditions (2.6.5). From Lemma 2.6.2 we deduce that w satisfies the integral estimate

$$\int_a^{a+\eta} w''(x)^2\,dx \le \frac{D^2}{\eta} + \frac{12}{\eta^3}\left(H + \frac{D}{2}\eta\right)^2 \le \frac{7D^2}{\eta} + \frac{24H^2}{\eta^3},$$

and the pointwise estimates

$$|w(x)| \le \frac{3H}{2} + \frac{D\eta}{2} \qquad \text{and} \qquad |w'(x)| \le \frac{3H}{\eta} + \frac{3D}{2}$$

for every $x \in [a, a+\eta]$, from which we deduce that

$$\int_a^{a+\eta} w(x)^2\,dx \le \frac{9H^2\eta}{2} + \frac{D^2\eta^3}{2},$$

and

$$\int_a^{a+\eta} \log\left(1 + w'(x)^2\right)\,dx \le \eta \log\left(1 + \frac{18H^2}{\eta^2} + \frac{9D^2}{2}\right).$$

Analogous estimates hold true in the interval $[b-\eta, b]$, while of course there is no contribution from the central interval $[a+\eta, b-\eta]$. It follows that

$$\mathbb{RPM}_\varepsilon((a,b),w) \le \frac{\eta}{\varepsilon^2}\left\{\left(\frac{14D^2}{\eta^2} + \frac{48H^2}{\eta^4}\right)\varepsilon^8 + \frac{2}{|\log\varepsilon|}\log\left(1 + \frac{18H^2}{\eta^2} + \frac{9D^2}{2}\right)\right\}, \tag{2.6.8}$$

and

$$\int_a^b w(x)^2\,dx \le 9H^2\eta + D^2\eta^3. \tag{2.6.9}$$

Now we set $\eta := \varepsilon^2\left(\sqrt{H} + \varepsilon^2 D\right)$. This choice is admissible because $\eta < (b-a)/2$ due to (2.6.3). We observe also that $\eta^4 \ge \varepsilon^8 H^2$ and $\eta^2 \ge \varepsilon^8 D^2$. As a consequence, from (2.6.9) we conclude that

$$\int_a^b w(x)^2\,dx \le 9H^2\eta + D^2\eta^3 \le \frac{10\eta^5}{\varepsilon^8},$$

which proves (2.6.7). Similarly, we obtain that

$$\left(\frac{14D^2}{\eta^2}+\frac{48H^2}{\eta^4}\right)\varepsilon^8\leq 62,$$

and

$$\frac{2}{|\log\varepsilon|}\log\left(1+\frac{18H^2}{\eta^2}+\frac{9D^2}{2}\right)\leq\frac{2}{|\log\varepsilon|}\log\left(1+\frac{45}{2}\frac{\eta^2}{\varepsilon^8}\right)\leq 18,$$

where in the last inequality we exploited (2.6.4). Plugging these estimates into (2.6.8) we obtain (2.6.6). □

2.6.2 Proof of Proposition 2.4.4 and Proposition 2.4.5

In this subsection we prove the two propositions simultaneously. The common idea is that every local minimizer to the functional (2.3.11) is a staircase where all the steps have the same length and the same height, and this staircase intersects the graph of the forcing term Mx in the midpoint of every horizontal step. This structure applies to entire local minimizers, but also to minimizers to (2.3.13), with the possible exception that the length of the two steps at the boundary might be different. Once this structure has been established, we only need to optimize with respect to the length of the steps in both cases.

The proof of the structure result is rather lengthy, because we need first to show that the jump set is discrete, then that the steps are symmetric with respect to the forcing term, and finally that all the steps have the same length.

Since the parameters α, β and M are fixed once and for all, for the sake of shortness in the sequel the functional (2.3.11) is denoted only by $\mathbb{JF}(\Omega, w)$. When needed, we also assume that $M>0$ (the case $M<0$ is symmetric, and the easier case $M=0$ is treated in the last paragraph of the proof).

The jump set of local minimizers is discrete Let us assume that w_0 is a local minimizer for the functional $\mathbb{JF}((a,b),w)$ in some interval $(a,b)\subseteq\mathbb{R}$. We prove that the set of jump points of w_0 in (a,b) is finite.

To this end, let us assume by contradiction that this is not the case. Due to the structure of the elements of the space $PJ((a,b))$, we know that there exist a sequence $\{s_k\}\subseteq(a,b)$ of distinct real numbers, a real number c_0, and a sequence $\{J_k\}$ of real numbers different from zero such that

$$\sum_{k=1}^{\infty}|J_k|<+\infty$$

and

$$w_0(x)=c_0+\sum_{k=1}^{\infty}J_k\mathbb{1}_{(s_k,b)}(x)\qquad\forall x\in(a,b).\tag{2.6.10}$$

For every integer $n\geq 2$ we consider the real number

$$R_n:=\sum_{k=n+1}^{\infty}|J_k|,$$

and the function $w_n : (a, b) \to \mathbb{R}$ defined by

$$w_n(x) := c_0 + \left(J_1 + \sum_{k=n+1}^{\infty} J_k \right) \mathbb{1}_{(s_1,b)}(x) + \sum_{k=2}^{n} J_k \mathbb{1}_{(s_k,b)}(x) \qquad \forall x \in (a, b). \tag{2.6.11}$$

We observe that $R_n \to 0$, and the function w_n has a finite number of jumps located at the arguments $s_1, \ldots, s_n$, and the jump in s_1 has "absorbed" all the heights of the jumps in s_i with $i \geq n+1$ (the jump height in s_1 might also vanish). In this way it turns out that

$$\lim_{x\to a^+} w_n(x) = \lim_{x\to a^+} w_0(x) = c_0 \qquad \text{and} \qquad \lim_{x\to b^-} w_n(x) = \lim_{x\to b^-} w_0(x) = c_0 + \sum_{k=1}^{\infty} J_k,$$

and therefore w_0 and w_n have the same "boundary data". As a consequence, due to the minimality of w_0 this implies that

$$\mathbb{JF}((a, b), w_n) \geq \mathbb{JF}(a, b), w_0) \qquad \forall n \geq 2. \tag{2.6.12}$$

On the other hand, from (2.6.10) and (2.6.11) we obtain that

$$\mathbb{J}_{1/2}((a, b), w_0) - \mathbb{J}_{1/2}((a, b), w_n) = \sum_{k=n+1}^{\infty} |J_k|^{1/2} + |J_1|^{1/2} - \left| J_1 + \sum_{k=n+1}^{\infty} J_k \right|^{1/2}.$$

Due to the subadditivity of the square root, the first term can be estimated as

$$\sum_{k=n+1}^{\infty} |J_k|^{1/2} \geq \left(\sum_{k=n+1}^{\infty} |J_k| \right)^{1/2} = (R_n)^{1/2},$$

while for the second and third term it turns out that

$$|J_1|^{1/2} - \left| J_1 + \sum_{k=n+1}^{\infty} J_k \right|^{1/2} \geq |J_1|^{1/2} - (|J_1| + R_n)^{1/2} \geq -\frac{R_n}{2|J_1|^{1/2}}.$$

From these two inequalities it follows that

$$\mathbb{J}_{1/2}((a, b), w_0) - \mathbb{J}_{1/2}((a, b), w_n) \geq (R_n)^{1/2} - \frac{R_n}{2|J_1|^{1/2}}. \tag{2.6.13}$$

Moreover, from (2.6.11), we obtain also that

$$|w_0(x) - w_n(x)| \leq R_n \qquad \forall x \in (a, b)$$

and

$$|w_n(x) - Mx| \leq |c_0| + \sum_{k=1}^{\infty} |J_k| + M \max\{|a|, |b|\} =: V_\infty \qquad \forall x \in (a, b),$$

and therefore

$$\begin{aligned}\int_a^b (w_0(x)-Mx)^2\,dx &\geq \int_a^b \left[(w_n(x)-Mx)^2+2(w_n(x)-Mx)(w_0(x)-w_n(x))\right]\,dx\\ &\geq \int_a^b (w_n(x)-Mx)^2\,dx-2(b-a)V_\infty R_n \qquad (2.6.14)\end{aligned}$$

for every $n \geq 2$. From (2.6.13) and (2.6.14) we conclude that

$$\mathbb{JF}((a,b),w_0)-\mathbb{JF}((a,b),w_n)\geq \alpha(R_n)^{1/2}-\frac{\alpha R_n}{2|J_1|^{1/2}}-2\beta(b-a)V_\infty R_n$$

When $R_n \to 0^+$ the right-hand side is positive, and this contradicts (2.6.12).

Existence of jump points and intersections Let us assume that $M>0$, and let us set

$$L_0 := \left(\frac{64\alpha^2}{\beta^2 M^3}\right)^{1/5}. \qquad (2.6.15)$$

We claim that, if w_0 is a local minimizer in some interval $(a,b)\subseteq \mathbb{R}$ with length $b-a>L_0$, then w_0 has either at least one jump point in (a,b) or at least one intersection with the line Mx, namely there exists $z_0 \in (a,b)$ such that $w_0(z_0)=Mz_0$.

Indeed, let us assume by contradiction that this is not the case. Then in (a,b) the function w_0 is a constant of the form $Ma-c$ or $Mb+c$ for some real number $c\geq 0$. In both cases it turns out that

$$\mathbb{JF}((a,b),w_0)=\left(\frac{M^2}{3}(b-a)^3+M(b-a)^2c+(b-a)c^2\right)\beta. \qquad (2.6.16)$$

For every real number τ with $0<2\tau<b-a$, let us consider the function $w_\tau : (a,b)\to\mathbb{R}$ defined by

$$w_\tau(x):=\begin{cases}\dfrac{M(a+b)}{2} & \text{if } a+\tau<x<b-\tau,\\ w_0(x) & \text{if } x\in(a,b)\setminus(a+\tau,b-\tau).\end{cases} \qquad (2.6.17)$$

Since w_τ coincides with w_0 in a neighborhood of the boundary, from the minimality of w_0 we deduce that $\mathbb{JF}((a,b),w_\tau)\geq \mathbb{JF}((a,b),w_0)$ for every admissible value of τ, and in particular

$$\lim_{\tau\to 0^+}\mathbb{JF}((a,b),w_\tau)\geq \mathbb{JF}((a,b),w_0). \qquad (2.6.18)$$

The right-hand side is given by (2.6.16). As for the left-hand side, we observe that w_τ has two equal jumps of height $c+M(b-a)/2$, while the integral term can be computed starting from the explicit expression (2.6.17). We obtain that

$$\lim_{\tau\to 0^+}\mathbb{JF}((a,b),w_\tau)=2\alpha\left(c+\frac{M(b-a)}{2}\right)^{1/2}+\frac{\beta M^2}{12}(b-a)^3. \qquad (2.6.19)$$

Plugging (2.6.19) and (2.6.16) into (2.6.18) we conclude that

$$2\alpha\left(c+\frac{M(b-a)}{2}\right)^{1/2} \geq \frac{\beta M^2}{4}(b-a)^3+\beta M(b-a)^2 c+\beta(b-a)c^2. \tag{2.6.20}$$

We claim that this is impossible if $c \geq 0$ and $b-a > L_0$. To this end, we write (2.6.15) in the equivalent form $\beta^2 M^3 L_0^5 = 64\alpha^2$, from which we deduce that

$$\beta^2 M^3 (b-a)^5 > 64\alpha^2 \tag{2.6.21}$$

because $b-a > L_0$. Now we distinguish two cases.

- Let us assume that $c \leq M(b-a)/2$. Multiplying (2.6.21) by $M(b-a)$, and taking the square root, we obtain that

$$\beta M^2(b-a)^3 > 8\alpha[M(b-a)]^{1/2},$$

and therefore

$$2\alpha\left(c+\frac{M(b-a)}{2}\right)^{1/2} \leq 2\alpha[M(b-a)]^{1/2} < \frac{\beta M^2}{4}(b-a)^3.$$

Since the latter is less than or equal to the right-had side of (2.6.20), we have reached a contradiction in this case.

- Let us assume that $c \geq M(b-a)/2$, and, in particular, that c is positive. We observe that this condition can be rewritten as

$$2\sqrt{2}\, c^{3/2} \geq M^{3/2}(b-a)^{3/2},$$

while (2.6.21) can be rewritten in the form

$$\beta(b-a) > \frac{8\alpha}{M^{3/2}(b-a)^{3/2}}.$$

Since $c > 0$, from these inequalities it follows that

$$\beta(b-a)c^2 > \frac{8\alpha}{M^{3/2}(b-a)^{3/2}} \cdot c^2 \geq 2\alpha\sqrt{2c} \geq 2\alpha\left(c+\frac{M(b-a)}{2}\right)^{1/2}.$$

Since the first term is less than or equal to the right-had side of (2.6.20), we have reached a contradiction also in this case.

Symmetry of jumps Let w_0 be a local minimizer in some interval $(a,b) \subseteq \mathbb{R}$, and let $s \in (a,b)$ be a jump point of w_0. From the first step we already know that s is isolated and therefore, up to restricting to a smaller interval, we can assume that $w_0(x)$ is equal to some constant A in (a,s), and to some constant $B \neq A$ in (s,b). We claim that

$$Ms - A = B - Ms \tag{2.6.22}$$

and that, if $M \neq 0$, the two terms have the same sign as M.

To this end, for every $\tau \in (a,b)$ we consider the function $w_\tau : (a,b) \to \mathbb{R}$ that is equal to A in (a,τ), and equal to B in (τ,b), and we set

$$\varphi(\tau) := \mathbb{JF}((a,b), w_\tau) = \alpha\sqrt{B-A} + \beta \int_a^\tau (A - Mx)^2\, dx + \beta \int_\tau^b (B - Mx)^2\, dx.$$

Since w_τ coincides with w_0 in a neighborhood of the boundary of the interval, from the minimality of w_0 we deduce that $\varphi(\tau)$ attains its minimum in (a,b) when $\tau = s$. This implies in particular that

$$0 = \varphi'(s) = \beta\left[(Ms - A)^2 - (B - Ms)^2\right] \tag{2.6.23}$$

and

$$0 \leq \varphi''(s) = 2\beta M[(Ms - A) + (B - Ms)]. \tag{2.6.24}$$

Since $\beta > 0$ and $B \neq A$, equality (2.6.23) implies (2.6.22). If in addition $M \neq 0$, then (2.6.24) implies that the two terms in (2.6.22) have the same sign as M.

Equipartition of intersections Let us assume that $M > 0$, let w_0 be a local minimizer in some interval $(a,b) \subseteq \mathbb{R}$, and let

$$a < z_1 < z_2 < \ldots < z_n < b$$

denote the intersections in (a,b) of $w_0(x)$ with the line Mx, namely the solutions to the equation $w_0(x) = Mx$. We observe that between any two intersections there is necessarily at least one jump point, and therefore from the previous steps we know that their number is finite. We claim that

$$z_2 - z_1 = z_3 - z_2 = \ldots = z_n - z_{n-1}.$$

In order to show the claim it is enough to show that, if $z_1 < z_2 < z_3$ are three consecutive intersections, then $z_2 - z_1 = z_3 - z_2$. To this end, we restrict to the interval (z_1, z_3) and we observe that, due to the previous steps, it turns out that

$$w_0(x) = \begin{cases} Mz_1 & \text{if } x \in (z_1, (z_1+z_2)/2), \\ Mz_2 & \text{if } x \in ((z_1+z_2)/2, (z_2+z_3)/2), \\ Mz_3 & \text{if } x \in ((z_2+z_3)/2, z_3). \end{cases}$$

For every $\tau \in (z_1, z_3)$ we consider the function $w_\tau : (z_1, z_3) \to \mathbb{R}$ defined by

$$w_\tau(x) = \begin{cases} Mz_1 & \text{if } x \in (z_1, (z_1+\tau)/2), \\ M\tau & \text{if } x \in ((z_1+\tau)/2, (\tau+z_3)/2), \\ Mz_3 & \text{if } x \in ((\tau+z_3)/2, z_3). \end{cases}$$

From this explicit expression it follows that

$$\mathbb{JF}((z_1,z_3),w_\tau) = \alpha\sqrt{M}\left(\sqrt{\tau - z_1} + \sqrt{z_3 - \tau}\right) + \frac{\beta M^2}{12}\left\{(\tau - z_1)^3 + (z_3 - \tau)^3\right\}.$$

Since w_τ coincides with w_0 in a neighborhood of the boundary of the interval, from the minimality of w_0 we deduce that this function of τ attains its minimum in (z_1, z_3) when $\tau = z_2$. With the change of variable $\tau = z_1 + t(z_3 - z_1)$ this is equivalent to saying that the function

$$\varphi(t) := C_0\left(\sqrt{t} + \sqrt{1-t}\right) + C_1\left(t^3 + (1-t)^3\right),$$

where

$$C_0 := \alpha\sqrt{M}\sqrt{z_3 - z_1} \qquad \text{and} \qquad C_1 := \frac{\beta M^2}{12}(z_3 - z_1)^3,$$

attains its minimum in $(0,1)$ when $t = (z_2 - z_1)/(z_3 - z_1)$. On the other hand, from Lemma 2.6.1 we know that the only possible minimum point is $t = 1/2$, and this implies that z_2 is the midpoint of (z_1, z_3).

Estimate from below for the minimum We are now ready to prove the estimate from below in (2.4.1). Again we consider the case where $M > 0$.

To begin with, we observe that this estimate is trivial when $L \leq 8L_0$, because in this case the left-hand side is nonpositive. If $L > 8L_0$, then from the previous steps we know that any minimizer $w_0 \in PJ((0,L))$ intersects the line Mx in at least one point $a_0 \in (0, 4L_0)$, and in at least one point $b_0 \in (L - 4L_0, L)$. Indeed, we know that in $(0, 2L_0)$ there exists at least one intersection or jump (because the length of the interval is greater than L_0), and the same in $(2L_0, 4L_0)$, and in any case between any two jumps there exists at least one intersection because of the symmetry of jumps.

Now we know that the interval (a_0, b_0) is divided into $n \geq 1$ intervals of equal length whose endpoints are intersections. Moreover, w_0 has exactly one jump point in the midpoint between any two consecutive intersection. As a consequence, the shape of w_0 in (a_0, b_0) depends only on n, and with an elementary computation we find that

$$\begin{aligned}\mathbb{JF}((0,L),w_0) &\geq \mathbb{JF}((a_0,b_0),w_0)\\ &= n\left\{\alpha\sqrt{\frac{M(b_0-a_0)}{n}} + \frac{\beta M^2}{12}\left(\frac{b_0-a_0}{n}\right)^3\right\}.\end{aligned}$$

Therefore, from the inequality

$$A + B \geq 5\left(\frac{A^4 B}{4^4}\right)^{1/5} \qquad \forall (A,B) \in [0,+\infty)^2$$

we conclude that

$$\mathbb{JF}((0,L),w_0) \geq \frac{5}{4}\left(\frac{\alpha^4\beta M^4}{3}\right)^{1/5}(b_0 - a_0) \geq \frac{5}{4}\left(\frac{\alpha^4\beta M^4}{3}\right)^{1/5}(L - 8L_0).$$

Plugging (2.6.15) into this inequality we obtain the estimate from below in (2.4.1).

Estimate from above for the minimum Let us prove the estimate from above in (2.4.1).

Let $n := \lceil L/(2H) \rceil$ denote the smallest integer greater than or equal to $L/(2H)$, where H is defined by (2.4.4), and let us consider the function $w_0 \in PJ((0, 2nH))$ that has intersections with the line Mx in 0, $2H$, $4H$, ..., $2nH$, and jumps in the midpoints of the intervals between consecutive intersections. Since w_0 is a competitor for the minimum problem (2.3.15) in the interval $(0, 2nH)$, from the monotonicity of μ_0^* with respect to L we deduce that

$$\begin{aligned} \mu_0^*(\alpha, \beta, L, M) &\leq \mu_0^*(\alpha, \beta, 2nH, M) \\ &\leq \mathbb{JF}((0, 2nH), w_0) \\ &= n\left(\alpha\sqrt{2MH} + \frac{2\beta M^2}{3}H^3\right) \\ &\leq \left(\frac{L}{2H} + 1\right)\left(\alpha\sqrt{2MH} + \frac{2\beta M^2}{3}H^3\right), \end{aligned} \tag{2.6.25}$$

and we conclude by remarking that the last term coincides with the right-hand side of (2.4.2) when H is given by (2.4.4).

Structure of entire local minimizers Let w_0 be an entire local minimizer. From the previous steps applied in every interval of the form $(-L, L)$, with $L \to +\infty$, we know that the set of intersection points of $w_0(x)$ with Mx is discrete and divides the line into segments of the same length $2h > 0$, whose midpoints are the unique jump points of w_0. This is enough to conclude that w_0 is an oblique translation of some staircase with steps of horizontal length $2h$ and vertical height $2Mh$. It remains only to show that $h = H$, where H is given by (2.4.4).

Up to an oblique translation, we can always assume that the intersections are the points of the form $2zh$ with $z \in \mathbb{Z}$. Let us consider the interval $(0, 2nh)$, where n is a positive integer. Applying (2.6.25) with $L = 2nh$ we deduce that

$$\begin{aligned} n\left(\alpha\sqrt{2Mh} + \frac{2\beta M^2}{3}h^3\right) &= \mathbb{JF}((0, 2nh), w_0) \\ &= \mu_0^*(\alpha, \beta, 2nh, M) \\ &\leq \left(\frac{2nh}{2H} + 1\right)\left(\alpha\sqrt{2MH} + \frac{2\beta M^2}{3}H^3\right). \end{aligned}$$

Dividing by nh, and letting $n \to +\infty$, we conclude that

$$\alpha\frac{\sqrt{2M}}{\sqrt{h}} + \frac{2\beta M^2}{3}h^2 \leq \alpha\frac{\sqrt{2M}}{\sqrt{H}} + \frac{2\beta M^2}{3}H^2,$$

and this inequality is possible if and only if $h = H$, because H is the unique minimum point of the left-hand side as a function of $h > 0$.

Structure of semi-entire local minimizers Let $w_0 : (0, +\infty) \to \mathbb{R}$ be a right-hand semi-entire local minimizer. Let $z_0 < z_1 < z_2 < \ldots$ denote the intersection points of w_0. Arguing as in the case of entire local minimizers we can show that $z_{k+1} - z_k = 2H$ for every $k \geq 0$. It remains to find the value of z_0. To this end, for every real number τ we consider the function

$$w_\tau(x) := w_0(x) + M\tau \mathbb{1}_{(0,z_0+H)}(x) \qquad \forall x > 0.$$

If we restrict to the interval $(0, z_1) = (0, z_0 + 2H)$, then w_τ and w_0 have the same boundary value in z_1, and therefore by the minimality of w_0 we know that the function $\varphi(\tau) := \mathbb{JF}_{1/2}((0, z_1), w_\tau)$ has a minimum point in $\tau = 0$. On the other hand an easy computation reveals that

$$\varphi(\tau) = \alpha\sqrt{M(2H - \tau)} + \beta \int_0^{z_0+H} M^2(z_0 + \tau - x)^2\, dx + \beta \int_{z_0+H}^{z_1} M^2(z_1 - x)^2\, dx,$$

and therefore

$$0 = \varphi'(0) = -\frac{\alpha\sqrt{M}}{2\sqrt{2H}} + \beta M^2 \left(z_0^2 - H^2\right). \tag{2.6.26}$$

Finally, we observe that the definition of H in (2.4.4) implies that

$$\frac{\alpha\sqrt{M}}{2\sqrt{2H}} = \frac{2\beta M^2}{3} H^2.$$

Plugging this identity into (2.6.26) we obtain that $z_0 = (5/3)^{1/2} H$, as required.

Existence of entire and semi-entire local minimizers Up to this point we have just shown that, if entire or semi-entire local minimizers exist, then they have the prescribed form. It remains to show that all oblique translations of the canonical (H, V)-staircase are actually entire local minimizers, and that the function w defined by (2.4.5) is actually a right-hand semi-entire local minimizer.

The argument is rather standard, and therefore we limit ourselves to sketching the main steps in the case of the canonical (H, V)-staircase $S_{H,V}$ (the case of its oblique translations and of semi-entire minimizers is analogous). It is enough to show that, for every positive integer n, the function $S_{H,V}$ minimizes $\mathbb{JF}_{1/2}((-2nH, 2nH), u)$ among all functions $u \in PJ((-2nH, 2nH))$ that coincide with $S_{H,V}$ at the endpoints. To begin with, we show that the minimum exists. This follows from a standard application of the direct method in the calculus of variations, as in the proof of statement (1) of Proposition 2.3.4. Once we know that the minimum exists, we go back through all the previous steps in order to show that the minimum has only a finite number of equi-spaced intersections points, and their number is the one we expect.

The case $M = 0$ When the forcing term vanishes, the estimates of Proposition 2.4.4 are actually trivial. As for Proposition 2.4.5, we have to show that the function $w_0 \equiv 0$ is the unique entire or semi-entire local minimizer. This can be proved in the following way. Let w_0 be any entire or semi-entire local minimizer.

- We show that the jump set of w_0 is discrete. This can be done as in the general case, since in that paragraph we never used that $M \neq 0$.
- We show the symmetry of jumps (2.6.22) as in the general case, since that equality was proved without using that $M \neq 0$. As a consequence, we deduce that $|w_0|$ is constant.
- We show that w_0 vanishes identically. Indeed, when we consider a long enough interval, any function w_0 with $|w_0|$ constant and different from 0 is worse (due to the overwhelming cost of the fidelity term) than a function with the same boundary values that has two jump points close to the boundary and vanishes elsewhere.

This completes the proof also in this special case. □

Remark 2.6.4. The existence of entire and semi-entire local minimizers follows also as a corollary of Proposition 2.4.6 and Proposition 2.4.7.

2.6.3 Compactness and convergence of local minimizers

In this subsection we prove Proposition 2.4.6 and Proposition 2.4.7. The key point in the argument is the following result, where we show that an estimate of order ε^{-1} for the energy $\mathbb{RPMF}_\varepsilon$ in some interval implies an ε-independent estimate for the same energy in a smaller interval.

Proposition 2.6.5 (Boundedness of the energy in a smaller interval)**.** *Let L, Γ_0, β be positive real numbers.*

Then there exists two real numbers $\varepsilon_0 \in (0,1)$ and $\Gamma_1 > 0$ for which the following statement holds true. Let $f : [-(L+1), L+1] \to \mathbb{R}$ be a continuous function such that

$$|f(x)| \leq \Gamma_0 \qquad \forall x \in [-(L+1), L+1], \tag{2.6.27}$$

let $\varepsilon \in (0, \varepsilon_0)$, and let

$$w \in \operatorname{argmin}_{loc} \left\{ \mathbb{RPMF}_\varepsilon(\beta, f, (-(L+1), L+1), w) : w \in H^2((-(L+1), L+1)) \right\} \tag{2.6.28}$$

be a local minimizer such that

$$\mathbb{RPMF}_\varepsilon(\beta, f, (-(L+1), L+1), w) \leq \frac{\Gamma_0}{\varepsilon}. \tag{2.6.29}$$

Then in the smaller interval $(-L, L)$ the local minimizer w satisfies

$$\mathbb{RPMF}_\varepsilon(\beta, f, (-L, L), w) \leq 4\Gamma_1. \tag{2.6.30}$$

Proof. Let us consider the expression

$$\Gamma_2 := \frac{\Gamma_1^{1/4}}{\beta^{1/4}} + \Gamma_0^{1/2} + 1.$$

We observe that it is possible to choose a real number $\Gamma_1 \geq \Gamma_0$ in such a way that

$$(80+20\beta)\Gamma_2 + 4\beta(L+1)\Gamma_0^2 \leq \Gamma_1, \tag{2.6.31}$$

and it is possible to choose a real number $\varepsilon_0 \in (0,1/4)$ such that the inequalities

$$\Gamma_1\varepsilon^{1/2}|\log\varepsilon| \leq \log 2, \qquad \varepsilon^{3/2}\,\Gamma_2 \leq L, \tag{2.6.32}$$

$$\frac{2}{|\log\varepsilon|}\log\left(1+\frac{45}{2\varepsilon^5}\cdot\Gamma_2^2\right) \leq 18, \qquad \varepsilon^{5/8}\,\Gamma_2^4 \leq 1 \tag{2.6.33}$$

hold true for every $\varepsilon \in (0,\varepsilon_0)$.

In the sequel we show that the statement holds true with these values of ε_0 and Γ_1.

Since $\varepsilon \in (0,\varepsilon_0)$ and $\varepsilon_0 < 1/4$, there exists a unique positive integer n such that

$$\frac{1}{4^{2^n}} = \frac{1}{2^{2^{n+1}}} \leq \varepsilon < \frac{1}{2^{2^n}}. \tag{2.6.34}$$

For every $k \in \{0,1,\ldots,n\}$ we set

$$L_{n,k} := L + 1 - \frac{1}{2^{n-k}},$$

and we observe that for every $k \in \{0,1,\ldots,n-1\}$ it turns out that

$$\begin{aligned} L \leq L_{n,k+1} < L_{n,k} < L+1, \\ L_{n,k} - L_{n,k+1} = \frac{1}{2^{n-k}}, \end{aligned} \tag{2.6.35}$$

and

$$2^{n-k} \leq 2^{2^{n-k-1}} = \left\{\left(2^{2^n}\right)^{1/2}\right\}^{2^{-k}} < \left(\frac{1}{\varepsilon^{1/2}}\right)^{2^{-k}}. \tag{2.6.36}$$

We claim that

$$\mathbb{RPMF}_\varepsilon(\beta,f,(-L_{n,k},L_{n,k}),w) \leq \frac{\Gamma_1}{\varepsilon^{2^{-k}}} \qquad \forall k \in \{0,1,\ldots,n\}. \tag{2.6.37}$$

The case $k=0$ follows from assumption (2.6.29) because the interval $(-L_{n,0},L_{n,0})$ is contained in the interval $(-(L+1),L+1)$ and $\Gamma_1 \geq \Gamma_0$. Since $L_{n,n} = L$, the case $k=n$ implies (2.6.30) because of the estimate from below in (2.6.34).

Now we prove (2.6.37) by finite induction on k. Let us assume that (2.6.37) holds true for some $k \in \{0,1,\ldots,n-1\}$, and let us prove that it holds true also for $k+1$. To begin with, we focus on the interval $(L_{n,k+1},L_{n,k})$, and we observe that

$$\begin{aligned}
\frac{\Gamma_1}{\varepsilon^{2^{-k}}} &\geq \mathbb{RPMF}_\varepsilon(\beta,f,(-L_{n,k},L_{n,k}),w) \\
&\geq \mathbb{RPMF}_\varepsilon(\beta,f,(L_{n,k+1},L_{n,k}),w) \\
&\geq \int_{L_{n,k+1}}^{L_{n,k}} \left\{\frac{1}{\omega(\varepsilon)^2}\log\left(1+w'(y)^2\right)+\beta(w(y)-f(y))^2\right\}dy \\
&\geq (L_{n,k}-L_{n,k+1})\cdot \\
&\quad \cdot\min\left\{\frac{1}{\omega(\varepsilon)^2}\log\left(1+w'(y)^2\right)+\beta(w(y)-f(y))^2 : y \in [L_{n,k+1},L_{n,k}]\right\}.
\end{aligned}$$

If $b_{k,\varepsilon} \in [L_{n,k+1}, L_{n,k}]$ is any minimum point, recalling (2.6.35), (2.6.36), and the first inequality in (2.6.32), this proves that

$$\log\left(1 + w'(b_{k,\varepsilon})^2\right) \leq \frac{\Gamma_1}{\varepsilon^{2^{-k}}} \cdot \omega(\varepsilon)^2 \cdot 2^{n-k} \leq \frac{\Gamma_1}{\left(\varepsilon^{3/2}\right)^{2^{-k}}} \cdot \omega(\varepsilon)^2 \leq \Gamma_1 \varepsilon^{1/2} |\log \varepsilon| \leq \log 2,$$

and

$$\beta(w(b_{k,\varepsilon}) - f(b_{k,\varepsilon}))^2 \leq \frac{\Gamma_1}{\varepsilon^{2^{-k}}} \cdot 2^{n-k} \leq \frac{\Gamma_1}{\left(\varepsilon^{3/2}\right)^{2^{-k}}}.$$

From these two inequalities and (2.6.27) we deduce that $|w'(b_{k,\varepsilon})| \leq 1$ and

$$|w(b_{k,\varepsilon})| \leq \frac{\Gamma_1^{1/2}}{\beta^{1/2}\left(\varepsilon^{3/4}\right)^{2^{-k}}} + \Gamma_0 \leq \frac{(\Gamma_1/\beta)^{1/2} + \Gamma_0}{\left(\varepsilon^{3/4}\right)^{2^{-k}}}.$$

With an analogous argument, we can show that there exists $a_{k,\varepsilon} \in [-L_{n,k}, -L_{n,k+1}]$ such that

$$|w'(a_{k,\varepsilon})| \leq 1 \qquad \text{and} \qquad |w(a_{k,\varepsilon})| \leq \frac{(\Gamma_1/\beta)^{1/2} + \Gamma_0}{\left(\varepsilon^{3/4}\right)^{2^{-k}}}.$$

Now we exploit that w minimizes $\mathbb{RPMF}_\varepsilon$ in the interval $(a_{k,\varepsilon}, b_{k,\varepsilon})$ with respect to its boundary conditions, and we estimate the minimum value by applying Lemma 2.6.3 with

$$(a,b) = (a_{k,\varepsilon}, b_{k,\varepsilon}), \qquad D := 1, \qquad H := \frac{(\Gamma_1/\beta)^{1/2} + \Gamma_0}{\left(\varepsilon^{3/4}\right)^{2^{-k}}}.$$

We observe that

$$\sqrt{H} + \varepsilon^2 D \leq \frac{\left[(\Gamma_1/\beta)^{1/2} + \Gamma_0\right]^{1/2}}{\left(\varepsilon^{3/8}\right)^{2^{-k}}} + 1 \leq \frac{\Gamma_2}{\left(\varepsilon^{3/8}\right)^{2^{-k}}},$$

and in particular from the second inequality in (2.6.32) we obtain that

$$\varepsilon^2\left(\sqrt{H} + \varepsilon^2 D\right) \leq \varepsilon^{3/2}\Gamma_2 \leq L < \frac{b_{k,\varepsilon} - a_{k,\varepsilon}}{2},$$

which shows that assumption (2.6.3) is satisfied, while from the first inequality in (2.6.33) we obtain that

$$\frac{2}{|\log \varepsilon|} \log\left(1 + \frac{45}{2\varepsilon^4}\left(\sqrt{H} + \varepsilon^2 D\right)^2\right) \leq \frac{2}{|\log \varepsilon|} \log\left(1 + \frac{45}{2\varepsilon^4} \cdot \frac{\Gamma_2^2}{\varepsilon}\right) \leq 18.$$

which shows that assumption (2.6.4) is satisfied. Therefore, from Lemma 2.6.3 we deduce the existence of $w_{k,\varepsilon} \in H^2((a_{k,\varepsilon}, b_{k,\varepsilon}))$, with the same boundary values (function and derivative) as w, satisfying

$$\mathbb{RPM}_\varepsilon((a_{k,\varepsilon}, b_{k,\varepsilon}), w_{k,\varepsilon}) \leq 80 \frac{\Gamma_2}{\left(\varepsilon^{3/8}\right)^{2^{-k}}} \leq 80 \frac{\Gamma_2}{\varepsilon^{2^{-k-1}}}$$

and

$$\int_{a_{k,\varepsilon}}^{b_{k,\varepsilon}} w_{k,\varepsilon}(x)^2\,dx \le 10\varepsilon^2 \frac{\Gamma_2^5}{\left(\varepsilon^{15/8}\right)^{2^{-k}}} = 10\frac{\varepsilon^{11/8}}{\left(\varepsilon^{15/8}\right)^{2^{-k}}}\cdot\left(\varepsilon^{5/8}\Gamma_2^4\right)\cdot\Gamma_2 \le 10\frac{\Gamma_2}{\varepsilon^{2^{-k-1}}},$$

where the last inequality follows from the second relation in (2.6.33). From the last two estimates and the minimality of w we conclude that

$$\begin{aligned}
&\mathbb{RPMF}_\varepsilon(\beta, f, (-L_{n,k+1}, L_{n,k+1}), w)\\
&\quad\le\ \mathbb{RPMF}_\varepsilon(\beta, f, (a_{k,\varepsilon}, b_{k,\varepsilon}), w)\\
&\quad\le\ \mathbb{RPMF}_\varepsilon(\beta, f, (a_{k,\varepsilon}, b_{k,\varepsilon}), w_{k,\varepsilon})\\
&\quad\le\ \mathbb{RPM}_\varepsilon((a_{k,\varepsilon}, b_{k,\varepsilon}), w_{k,\varepsilon}) + 2\beta\int_{a_{k,\varepsilon}}^{b_{k,\varepsilon}} w_{k,\varepsilon}(x)^2\,dx + 2\beta\int_{a_{k,\varepsilon}}^{b_{k,\varepsilon}} f(x)^2\,dx\\
&\quad\le\ (80+20\beta)\frac{\Gamma_2}{\varepsilon^{2^{-k-1}}} + 2\beta(2L+2)\Gamma_0^2\\
&\quad\le\ \frac{\Gamma_1}{\varepsilon^{2^{-k-1}}},
\end{aligned}$$

where the last two inequalities we exploited (2.6.27) and (2.6.31), respectively. This completes the inductive step, and hence also the proof. □

Remark 2.6.6. Proposition 2.6.5 can be extended in a straightforward way to one-sided local minimizers. To this end, it is enough to replace in the statement the interval $(-L, L)$ with $(0, L)$, the interval $(-(L+1), L+1)$ with $(0, L+1)$, and "loc" with "R-loc". The proof is analogous and somewhat simpler, because we just need to work on one side of the interval.

Proof of Proposition 2.4.6

Existence of a limit We prove that there exist a function $w_\infty : \mathbb{R} \to \mathbb{R}$ and an increasing sequence $\{n_k\}$ of positive integers such that, for every $L > 0$, the restriction of w_∞ to the interval $(-L, L)$ belongs to $PJ((-L, L))$ and $w_{n_k} \to w_\infty$ in $L^2((-L, L))$.

To this end, it is enough to prove that, for every fixed real number $L > 0$, it holds that

$$\sup\left\{\mathbb{RPM}_{\varepsilon_n}((-L, L), w_n) + \int_{-L}^{L} w_n(x)^2\,dx : n \in \mathbb{N}\right\} < +\infty. \tag{2.6.38}$$

Indeed, once this uniform bound has been established (the supremum might depend on L, of course), the compactness result of statement (2) in Theorem 2.3.2 implies that the sequence $\{w_n\}$ is relatively compact in $L^2((-L, L))$ for this fixed value on L, and any limit function lies in $PJ((-L, L))$. At this point we apply the result to a sequence of intervals $(-L_k, L_k)$ with $L_k \to +\infty$, and with a classical diagonal procedure we obtain the subsequence that converges in all bounded intervals.

In order to prove (2.6.38), we begin by observing that, due to assumption (ii), there exists a constant M_L such that

$$|g_n(x)| \le M_L \qquad \forall x \in [-(L+1), L+1], \quad \forall n \in \mathbb{N}.$$

We now apply Proposition 2.6.5 with

$$w(x) := w_n(x), \qquad f(x) := g_n(x), \qquad \Gamma_0 := \max\{M_L, C_0\}.$$

This is possible because assumptions (2.6.28) and (2.6.29) are satisfied for trivial reasons as soon as $[-(L+1), L+1] \subseteq (A_n, B_n)$ and $|\log \varepsilon_n| \geq 1$. From Proposition 2.6.5 we obtain that there exists a constant Γ_1 such that

$$\mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (-L, L), w_n) \leq 4\Gamma_1 \tag{2.6.39}$$

when n is large enough. This implies (2.6.38) because the left hand-side of (2.6.39) controls the first term in the left-hand side of (2.6.38), while the integral can be estimated as

$$\int_{-L}^{L} w_n(x)^2\,dx \leq 2\int_{-L}^{L}(w_n(x) - g_n(x))^2\,dx + 2\int_{-L}^{L} g_n(x)^2\,dx,$$

where the first integral is controlled again by the left hand-side of (2.6.39), and the second integral is controlled because of the uniform bound on g_n.

Characterization of the limit Let w_∞ be any limit function identified in the first paragraph of the proof. We claim that w_∞ is an entire local minimizer for the functional (2.3.11) with α defined by (2.3.7).

The function w_∞ is by definition the limit in $L^2_{\mathrm{loc}}(\mathbb{R})$ of some sequence w_{n_k}, and from the uniform bounds (2.6.39) we deduce also that $\log(1 + (w'_{n_k})^2) \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. Up to further subsequences (not relabeled) we can assume that in both cases the convergence is also pointwise for almost every $x \in \mathbb{R}$. Now let us consider any interval $(a, b) \subseteq \mathbb{R}$ whose endpoints are not jump points of w_∞, and such that $w_{n_k}(x) \to w_\infty(x)$ and $w'_{n_k}(x) \to 0$ for $x \in \{a, b\}$.

Let $v \in PJ((a, b))$ be any function with the same boundary conditions of w_∞ in the usual sense. From statement (4) of Theorem 2.3.2 applied with the quadruple of boundary data

$$(w_{n_k}(a), w'_{n_k}(a), w_{n_k}(b), w'_{n_k}(b)) \to (w_\infty(a), 0, w_\infty(b), 0) = (v(a), 0, v(b), 0)$$

we obtain a recovery sequence $\{v_k\} \subseteq H^2((a, b))$ for v that has the same boundary conditions as w_{n_k} in a and b (both on the function and on the derivative). From the minimality of w_{n_k} we deduce that

$$\mathbb{RPMF}_{\varepsilon_{n_k}}(\beta, g_{n_k}, (a, b), w_{n_k}) \leq \mathbb{RPMF}_{\varepsilon_{n_k}}(\beta, g_{n_k}, (a, b), v_k)$$

for every positive integer k. Letting $k \to +\infty$, and recalling statement (1) in Theorem 2.3.2, we conclude that

$$\begin{aligned}
\mathbb{JF}_{1/2}(\alpha_0, \beta, M, (a, b), w_\infty) &\leq \liminf_{k\to+\infty} \mathbb{RPMF}_{\varepsilon_{n_k}}(\beta, g_{n_k}, (a, b), w_{n_k}) \\
&\leq \lim_{k\to+\infty} \mathbb{RPMF}_{\varepsilon_{n_k}}(\beta, g_{n_k}, (a, b), v_k) \\
&= \mathbb{JF}_{1/2}(\alpha_0, \beta, M, (a, b), v).
\end{aligned}$$

Since v is arbitrary, this proves that w_∞ is a local minimizer of the limiting functional in the interval (a, b). Since intervals of this type exhaust the real line, this proves that w_∞ is an entire local minimizer for the limiting functional, as required.

Strict convergence In the special case where $v \equiv w_\infty$ in (a,b), the argument of the previous paragraph gives that

$$\lim_{k\to+\infty} \mathbb{RPM}_{\varepsilon_{n_k}}((a,b),w_{n_k}) = \alpha_0\,\mathbb{J}_{1/2}((a,b),w_\infty), \tag{2.6.40}$$

namely $\{w_{n_k}\}$ is a recovery sequence for w_∞ in the interval (a,b). At this point, from statement (3) in Theorem 2.3.2, we conclude that $w_{n_k} \rightsquigarrow w_\infty$ in $BV((a,b))$. Since intervals of this type exhaust the real line, this completes the proof. □

Proof of Proposition 2.4.7

The proof is analogous to the proof of Proposition 2.4.6, and hence we limit ourselves to sketching the argument.

In the first step we show that there exist a function $w_\infty : (0,+\infty) \to \mathbb{R}$ and an increasing sequence $\{n_k\}$ of positive integers such that

- the restriction of w_∞ to the interval $(0,L)$ belongs to $PJ((0,L))$ for every $L>0$,
- $w_{n_k} \to w_\infty$ in $L^2((0,L))$ and $\log(1+(w'_{n_k})^2) \to 0$ in $L^1((0,L))$ for every $L>0$,
- $w_{n_k}(x) \to w_\infty(x)$ and $w'_{n_k}(x) \to 0$ for almost every $x>0$.

The argument relies on the one-sided version of Proposition 2.6.5 (see Remark 2.6.6), and on the compactness result of statement (2) of Theorem 2.3.2.

In the second step we consider intervals of the form $(0,L)$, where L is any positive real number in which we have the pointwise convergence $w_{n_k}(L) \to w_\infty(L)$ and $w'_{n_k}(L) \to 0$, and such that L is not a jump point of w_∞ (both conditions hold true for almost every point in the half-line). Then we consider any function $v \in PJ((0,L))$ such $v(L) = w_\infty(L)$, where boundary values are intended in the usual sense. From statement (4) of Theorem 2.3.2, applied with the quadruple of initial data

$$(v(0),0,w_{n_k}(L),w'_{n_k}(L)) \to (v(0),0,w_\infty(L),0) = (v(0),0,v(L),0),$$

we obtain a recovery sequence $\{v_k\} \subseteq H^2((0,L))$ for v that has the same boundary conditions as w_{n_k} in $x=L$. Thus from the minimality of w_{n_k} in $(0,L)$ we deduce, as in the previous case, that

$$\mathbb{JF}_{1/2}(\alpha_0,\beta,M,(0,L),w_\infty) \leq \mathbb{JF}_{1/2}(\alpha_0,\beta,M,(0,L),v).$$

Since L can be chosen to be arbitrarily large, this is enough to conclude that w_∞ is a right-hand minimizer in $(0,+\infty)$.

Finally, in the third step we conclude as before that the convergence is strict in every interval $(0,L)$ such that L is not a jump point of w_∞. □

2.7 Possible extensions

Our proof of Theorem 2.2.2 relies just on the Gamma-convergence results for the rescaled functionals (2.3.1), and on the estimates of Proposition 2.4.4 for the minima of the limiting functional with linear forcing term. Our proofs of Theorems 2.2.9 and 2.2.14 rely on the characterization of local minima for the limiting functional, and on the compactness result that follows from Proposition 2.6.5. For these reasons, we expect that these results can be extended to more general models by just extending the tools that we exploited here. For example, it is possible to consider more general fidelity terms of the form

$$\int_0^1 \beta(x)|u(x)-f(x)|^p\,dx,$$

for suitable choices of the exponent $p \geq 1$ (but also every $p > 0$ should be fine) and of the coefficient $\beta(x)$, provided that it is continuous and strictly positive.

In the sequel we focus on less trivial generalizations that involve the principal part, and we discuss three possibilities.

Different convex-concave Lagrangians We can replace the function $\varphi(p) := \log(1+p^2)$ with more general functions. This leads to functionals with principal part of the form

$$\mathrm{PM}_\varepsilon(u) := \int_0^1 \left\{\varepsilon^6\omega(\varepsilon)^4 u''(x)^2 + \varphi(u'(x))\right\}\,dx,$$

where now $\omega(\varepsilon) := \varepsilon\varphi(1/\varepsilon^2)^{1/2}$. Under rather general assumptions on φ, the blow-ups of minimizers at scale $\omega(\varepsilon)$ are local minimizers for the rescaled functionals

$$\mathrm{RPM}_\varepsilon(\Omega, v) := \int_\Omega \left\{\varepsilon^6 v''(x)^2 + \frac{1}{\omega(\varepsilon)^2}\varphi(v'(x))\right\}\,dx,$$

and this family Gamma-converges to a suitable multiple of the functional $\mathbb{J}_\sigma(\Omega, v)$, which is the natural generalization of (2.3.6) obtained by replacing $1/2$ with a different exponent $\sigma \in (0,1)$ that depends on the growth at infinity of φ (actually in this case we obtain only exponents in $[1/2, 1)$). All the results of this paper can be easily extended, more or less with the same techniques.

Higher order singular perturbation We can replace second order derivatives with derivatives of higher order. This leads to functionals with principal part of the form

$$\mathrm{PM}_\varepsilon(u) := \int_0^1 \left\{\varepsilon^{4k-2}\omega(\varepsilon)^{2k} u^{(k)}(x)^2 + \log\left(1+u'(x)^2\right)\right\}\,dx,$$

where $u^{(k)}$ denotes the derivative of u of order $k \geq 2$, and $\omega(\varepsilon)$ is defined as in (2.2.1). Also in this case the rescaled functionals

$$\mathrm{RPM}_\varepsilon(\Omega, v) := \int_\Omega \left\{\varepsilon^{4k-2} v^{(k)}(x)^2 + \frac{1}{\omega(\varepsilon)^2}\log\left(1+v'(x)^2\right)\right\}\,dx$$

Gamma-converges to a suitable multiple of $\mathbb{J}_\sigma(\Omega, v)$, now with $\sigma = 1/k$. Therefore, it seems reasonable that the results of this paper can be extended, even if some steps (for example the iteration argument in the compactness result) might require some extra work.

Of course, one can also combine a higher order singular perturbation with a different choice of φ, and/or choose a different exponent for the higher order derivative.

Space discretization In a different direction, it is possible to consider a space discretization of the problem where derivatives are replaced by finite differences. This leads to functionals of the form

$$\mathrm{PM}_n(u) := \int_0^{1-1/n} \log\left(1 + \left(\frac{u(x+1/n) - u(x)}{1/n}\right)^2\right) dx,$$

possibly defined in the space of functions that are piecewise constant with steps of length $1/n$. This is equivalent to considering the original functional (1.3), depending on true derivatives, but restricted to the space of functions that are piecewise affine, again with respect to some grid with size $1/n$. The natural rescaling corresponds to blow-ups at scale ω_n, and leads to the sequence of functionals

$$\mathrm{RPM}_n(\Omega, v) := \frac{1}{\omega_n}\int_\Omega \log\left(1 + \left(\frac{v(x+\delta_n) - v(x)}{\delta_n}\right)^2\right) dx,$$

defined on functions that are piecewise constant with steps of length δ_n, where

$$\omega_n = \left(\frac{\log n}{n}\right)^{1/3} \qquad \text{and} \qquad \delta_n = \frac{1}{n\omega_n} = \frac{1}{n^{2/3}(\log n)^{1/3}}.$$

The Gamma-limit turns out to be a multiple of the functional $\mathbb{J}_0(\Omega, v)$, namely the functional that simply counts the number of jumps of v in Ω, regardless of jump heights. Again it is possible to extend the results of this paper, and some of them can also be proved for more general forcing terms (see [66]).

2.8 Future perspectives and open problems

In this final section we present some questions that remain open, and that may deserve further investigation.

The first one concerns uniqueness of minimizers, which is always a challenging question when the Lagrangian is non-convex. We recall that, for the model (2.1.3), uniqueness is known in some cases (see [62, Theorem 1.1 and subsequent Remark 4]), but, in that case, the forcing term is rather special and there are periodic boundary conditions.

Open problem 1 (Uniqueness of minimizers)**.** *Let us consider the minimum problem (2.2.3), under the same assumptions of Proposition 2.2.1. Determine whether the minimizer is unique, at least when ε is small enough and/or the forcing term f is smooth enough.*

Concerning Theorem 2.2.2, it may be interesting to investigate the asymptotic behavior of minima under weaker regularity assumptions on the forcing term f.

Open problem 2 (Existence of the limit of rescaled minimum values)**.** *Characterize all functions $f \in L^2((0,1))$ such that the limit in (2.2.4) exists, or exists and is a real number, or exists and coincides with the right-hand side, up to defining f' in a suitable way.*

The question is largely open. It is also conceivable that the vanishing order of $m(\varepsilon, \beta, f)$ as $\varepsilon \to 0^+$ depends on the regularity of f in terms of Hölder continuity, Sobolev exponents or even fractional Sobolev spaces, which motivates the following question.

Open problem 3 (Vanishing order of minima vs regularity of the forcing term)**.** *Find any connection between the vanishing order of $m(\varepsilon, \beta, f)$ as $\varepsilon \to 0^+$ and the regularity of the forcing term f.*

Here we present the results that we know at the present time.

- For every $f \in PJ((0,1))$ with a finite number of jumps it turns out that

$$\lim_{\varepsilon \to 0^+} \frac{m(\varepsilon, \beta, f)}{\omega(\varepsilon)^{5/2}} = 4 \left(\frac{2}{3}\right)^{1/2} 5^{3/4} \cdot \mathbb{J}_{1/2}((0,1), f). \tag{2.8.1}$$

  The same should be true when $\mathbb{J}_{1/2}((0,1), f) < +\infty$.

- Heuristically, when minimizing (2.3.2) we can replace the rescaled Perona-Malik functional (2.3.1) by its Gamma-limit (2.3.6). This leads to a minimization problem in the class of pure jump functions, that we can further simplify by restricting to competitors whose jump points are equally spaced at some fixed distance δ, to be optimized with respect to ε. By formalizing this idea we obtain the following two estimates from above.

  - If f is a-Hölder continuous for some $a \in (0,1]$ and some constant H, then it turns out that

$$\limsup_{\varepsilon \to 0^+} \frac{m(\varepsilon, \beta, f)}{\omega(\varepsilon)^{10a/(3a+2)}} \leq c_a H^{4/(3a+2)}.$$

  - If $f \in W^{1,p}((0,1))$ for some $p \in [1,2]$, then it turns out that

$$\limsup_{\varepsilon \to 0^+} \frac{m(\varepsilon, \beta, f)}{\omega(\varepsilon)^{(15p-10)/(7p-4)}} \leq c_p \|f'\|_{L^p((0,1))}^{(5p-2)/(7p-4)}.$$

- The set of forcing terms $f \in L^2((0,1))$ for which the limit in (2.2.4) exists has empty interior, even if we allow the limit to be $+\infty$, and even if we restrict ourselves to a sequence $\varepsilon_n \to 0^+$. Indeed, for every fixed $\varepsilon_n \in (0,1)$, the function $f \to m(\varepsilon_n, \beta, f)$ is continuous in $L^2((0,1))$, and therefore also the function

$$\Psi_n(f) := \arctan\left(\frac{m(\varepsilon_n, \beta, f)}{\omega(\varepsilon_n)^2}\right)$$

is continuous in the same space. Let us assume by contradiction that $\Psi_n(f)$ converges to some $\Psi_\infty(f)$ for every f in some open set $\mathcal{U} \subseteq L^2((0,1))$. Since $\mathcal{U}$ is a Baire space, and Ψ_∞ is the pointwise limit of continuous functions, then necessarily Ψ_∞ is continuous in some G_δ subset $\mathcal{V} \subseteq \mathcal{U}$. Now on the one hand we know from (2.8.1) that $\Psi_\infty(f) = 0$ for every piecewise constant function with a finite number of jumps, and this class is dense in $L^2((0,1))$, and therefore $\Psi_\infty(f) = 0$ for every $f \in \mathcal{V}$. On the other hand, also functions of class C^1 with right-hand side of (2.2.4) greater than 1 are dense in $L^2((0,1))$, which implies that $\Psi_\infty(f) \geq 1$ for every $f \in \mathcal{V}$.

As for the convergence of minimizers, on the one hand we expect that the C^1 regularity of f is required in order to characterize the blow-ups of minimizers with ε-dependent centers as we did in Theorem 2.2.9. On the other hand, the statement of Theorem 2.2.14 seems to require less regularity on f, in contrast with our proof that heavily relies on Theorem 2.2.9 (see [66, Theorem 2.5]).

Open problem 4 (Strict and varifold convergence of minimizers)**.** *Extend the results of Theorem 2.2.14 to less regular forcing terms, and in particular determine whether the results hold true for every $f \in BV((0,1))$ (up to a suitable extension of identity (2.2.15) to bounded variation functions, as in [66, Theorem 2.5]).*

Finally, since we considered the Perona-Malik functional in dimension one, we conclude with the following natural and challenging question.

Open problem 5 (Any space dimension)**.** *Extend the results of this chapter to higher dimensions.*

Some partial results in this directions are described in Chapter 4.

Appendix A

Appendix to Chapter 2

In this final appendix we prove the results stated in section 2.3. To this end, we need three preliminary technical lemmata. The first one is the classical estimate from below for the rescaled Perona-Malik functional in an interval where $|u'(x)|$ is "large" (the argument is analogous to a step in the proof of [2, Proposition 3.3]).

Lemma A.0.1 (Basic estimate from below)**.** *Let $(\alpha, \beta) \subseteq \mathbb{R}$ be an interval, and let $u \in H^2((\alpha, \beta))$. Then the following statements hold true.*

(1) Let us assume that there exists a real number $D > 0$ such that $|u'(x)| \geq D$ for every $x \in (\alpha, \beta)$, and such that either $|u'(\alpha)| = D$ or $|u'(\beta)| = D$.

Then for every $\varepsilon \in (0, 1)$ it holds that

$$\mathbb{RPM}_\varepsilon((\alpha, \beta), u) \geq \frac{M(\varepsilon, D)}{2^{1/2}} \big(|u(\beta) - u(\alpha)| - D(\beta - \alpha)\big)^{1/2}. \tag{A.0.1}$$

where

$$M(\varepsilon, D) := 4 \left(\frac{2}{3}\right)^{1/2} \left(\frac{\log(1 + D^2)}{|\log \varepsilon|}\right)^{3/4}. \tag{A.0.2}$$

(2) Let us assume that there exists a real number $D > 0$ such that $|u'(x)| \geq D$ for every $x \in (\alpha, \beta)$, and such that both $|u'(\alpha)| = D$ and $|u'(\beta)| = D$.

Then for every $\varepsilon \in (0, 1)$ it holds that

$$\mathbb{RPM}_\varepsilon((\alpha, \beta), u) \geq M(\varepsilon, D) \big(|u(\beta) - u(\alpha)| - D(\beta - \alpha)\big)^{1/2}, \tag{A.0.3}$$

where $M(\varepsilon, D)$ is again defined by (A.0.2).

Proof. Let us observe that our assumptions imply that either $u'(x) \geq D$ for every $x \in (\alpha, \beta)$, or $u'(x) \leq -D$ for every $x \in (\alpha, \beta)$. In both cases it turns out that $u'(x)$ has the same sign at the two endpoints of the interval.

Up to a change of sign and a reflection, we can assume that $u'(x) \geq D$ for every $x \in (\alpha, \beta)$ and that $u'(\alpha) = D$, while $C := u'(\beta) \geq D$. As a consequence we have that

$$|u(\beta) - u(\alpha)| = u(\beta) - u(\alpha) \geq D(\beta - \alpha).$$

From Lemma 2.6.2 we obtain that

$$\int_\alpha^\beta u''(x)^2\,dx \geq \frac{(C-D)^2}{\beta-\alpha} + \frac{12}{(\beta-\alpha)^3}\left(u(\beta)-u(\alpha)-\frac{C+D}{2}(\beta-\alpha)\right)^2.$$

We estimate the right-hand side from below by its minimum with respect to C, which is attained when

$$C = \frac{3}{2}\frac{u(\beta)-u(\alpha)}{\beta-\alpha} - \frac{D}{2}.$$

We conclude that

$$\int_\alpha^\beta u''(x)^2\,dx \geq \frac{3}{(\beta-\alpha)^3}\left(u(\beta)-u(\alpha)-D(\beta-\alpha)\right)^2,$$

and therefore

$$\mathbb{RPM}_\varepsilon((\alpha,\beta),u) \geq \frac{3\varepsilon^6}{(\beta-\alpha)^3}\left(u(\beta)-u(\alpha)-D(\beta-\alpha)\right)^2 + \frac{\beta-\alpha}{\omega(\varepsilon)^2}\log\left(1+D^2\right).$$

Applying the classical inequality

$$A+B \geq \frac{4}{3^{3/4}}\left(AB^3\right)^{1/4} \qquad \forall (A,B)\in[0,+\infty)^2,$$

we obtain exactly (A.0.1).

The proof of (A.0.3) is analogous, with the only difference that now $C = D$ by assumption. In this case from Lemma 2.6.2 it follows that

$$\int_\alpha^\beta u''(x)^2\,dx \geq \frac{12}{(\beta-\alpha)^3}\left(u(\beta)-u(\alpha)-D(\beta-\alpha)\right)^2,$$

from which we obtain an additional factor $2^{1/2}$ in the numerator. □

The second lemma shows that for every function $u \in H^2((a,b))$ one can find a function $z \in PJ((a,b))$ that is close to u in terms of the L^p norm and in total variation, and such that the $\mathbb{RPM}_\varepsilon$ energy of u is controlled from below by the $\mathbb{J}_{1/2}$ energy of z. An analogous result is proved in [2, Proposition 3.3].

Lemma A.0.2 (Substitution lemma)**.** *Let $(a,b)\subseteq\mathbb{R}$ be an interval, let $\varepsilon_n \subseteq (0,1)$ be a sequence such that $\varepsilon_n \to 0^+$, and let $\{u_n\}\subseteq H^2((a,b))$ be a sequence of functions such that*

$$\sup\{\mathbb{RPM}_{\varepsilon_n}((a,b),u_n) : n\geq 1\} < +\infty. \tag{A.0.4}$$

Then there exist a sequence of functions $\{z_n\}\subseteq PJ((a,b))$, and a sequence of intervals $(a_n,b_n)\subseteq(a,b)$ with endpoints $a_n\to a^+$ and $b_n\to b^-$, with the following properties.

(1) For every positive integer n the points a_n and b_n are not jump points of the function z_n, and when n is large enough it turns out that

$$\mathbb{RPM}_{\varepsilon_n}((a,b),u_n) \geq M_n\cdot\left\{\mathbb{J}_{1/2}((a_n,b_n),z_n) + \frac{\mathbb{J}_{1/2}((a,a_n)\cup(b_n,b),z_n)}{2^{1/2}}\right\}, \tag{A.0.5}$$

where

$$M_n := 4\left(\frac{2}{3}\right)^{1/2}\left\{\frac{1}{|\log\varepsilon_n|}\log\left(1+\frac{1}{\varepsilon_n^4|\log\varepsilon_n|^8}\right)\right\}^{3/4} \qquad \forall n\geq 1. \tag{A.0.6}$$

(2) The function z_n is asymptotically close to u_n in the sense that

$$\lim_{n\to+\infty}\|u_n-z_n\|_{L^p((a,b))}=0 \qquad \forall p\in[1,+\infty). \tag{A.0.7}$$

(3) The total variation of z_n is asymptotically close to the total variation of u_n in the sense that

$$\lim_{n\to+\infty}\int_a^b|u_n'(x)|\,dx-|Dz_n|((a,b))=0. \tag{A.0.8}$$

Proof. Let us consider the set

$$A_n:=\left\{x\in(a,b):|u_n'(x)|>\frac{1}{\varepsilon_n^2|\log\varepsilon_n|^4}\right\}.$$

Since A_n is an open set, we can write it as a finite or countable union of open disjoint intervals (its connected components), namely in the form

$$A_n=\bigcup_{i\in I_n}(\alpha_{n,i},\beta_{n,i}),$$

where I_n is a suitable index set.

Let $w_n:[a,b]\to\mathbb{R}$ be the function of class C^1 such that $w_n(a)=u_n(a)$, and

$$w_n'(x):=\begin{cases}0 & \text{if } x\in(a,b)\setminus A_n,\\ u_n'(x)-\dfrac{\operatorname{sign}(u_n'(x))}{\varepsilon_n^2|\log\varepsilon_n|^4} & \text{if } x\in A_n.\end{cases}$$

We observe that $w_n'(x)$ is the difference between $u_n'(x)$ and the truncation of $u_n'(x)$ between the two values $\pm\varepsilon_n^{-2}|\log\varepsilon_n|^{-4}$. We deduce that in each of the intervals $(\alpha_{n,i},\beta_{n,i})$ the sign of $w_n'(x)$ is constant and coincides with the sign of $u_n'(x)$ in the same interval, and, in any case, it holds that

$$|w_n(\beta_{n,i})-w_n(\alpha_{n,i})|=|u_n(\beta_{n,i})-u_n(\alpha_{n,i})|-\frac{\beta_{n,i}-\alpha_{n,i}}{\varepsilon_n^2|\log\varepsilon_n|^4}. \tag{A.0.9}$$

Finally, for every $i\in I_n$ we consider the midpoint $\gamma_{n,i}:=(\alpha_{n,i}+\beta_{n,i})/2$ of the interval $(\alpha_{n,i},\beta_{n,i})$, and we introduce the function $z_n\in PJ((a,b))$ whose jump points are located at these midpoints, and have height that amounts to the variation of w_n in the corresponding intervals, and translated vertically so that $z_n(a)=w_n(a)=u_n(a)$. Such a function is given by

$$z_n(x):=u_n(a)+\sum_{i\in I_n}(w_n(\beta_{n,i})-w_n(\alpha_{n,i}))\mathbb{1}_{(\gamma_{n,i},b)}(x) \qquad \forall x\in(a,b).$$

With these definitions, we are now ready to prove the required estimates.

Statement (1) Let us assume for a while that

$$a < \alpha_{n,i} < \beta_{n,i} < b \qquad \forall i \in I_n. \tag{A.0.10}$$

In this case it turns out that necessarily $|u_n'(x)| = D := \varepsilon_n^{-2}|\log \varepsilon_n|^{-4}$ at both the endpoints of each interval $(\alpha_{n,i}, \beta_{n,i})$, and hence from statement (2) of Lemma A.0.1 we deduce that

$$\begin{aligned}
\mathbb{RPM}_{\varepsilon_n}((\alpha_{n,i}, \beta_{n,i}), u_n) &\geq M_n \left(|u_n(\beta_{n,i}) - u_n(\alpha_{n,i})| - \frac{\beta_{n,i} - \alpha_{n,i}}{\varepsilon_n^2 |\log \varepsilon_n|^4} \right)^{1/2} \\
&= M_n \left| w_n(\beta_{n,i}) - w_n(\alpha_{n,i}) \right|^{1/2} \\
&= M_n \left| J_{z_n}(\gamma_{n,i}) \right|^{1/2}
\end{aligned} \tag{A.0.11}$$

for every $i \in I_n$. Summing over all indices we conclude that

$$\begin{aligned}
\mathbb{RPM}_{\varepsilon_n}((a,b), u_n) &\geq \mathbb{RPM}_{\varepsilon_n}(A_n, u_n) \\
&= \sum_{i \in I_n} \mathbb{RPM}_{\varepsilon_n}((\alpha_{n,i}, \beta_{n,i}), u_n) \\
&\geq M_n \sum_{i \in I_n} \left| J_{z_n}(\gamma_{n,i}) \right|^{1/2} \\
&= M_n \, \mathbb{J}_{1/2}((a,b), z_n),
\end{aligned}$$

which proves (A.0.5) with $(a_n, b_n) := (a, b)$.

If (A.0.10) is not true, it means that either $A_n = (a, b)$, or one or two of the connected components of A_n have exactly one endpoint which is either a or b. At the beginning of the proof of the second statement below we show that the measure of A_n tends to 0 as $n \to +\infty$, and this rules out the possibility that $A_n = (a, b)$ when n is large enough.

In the other case we denote the "lateral components" by (a, a_n) and (b_n, b), with the understanding that $a = a_n$ or $b = b_n$ if there is only one such component, and we call $I_n' \subseteq I_n$ the set of indices corresponding to the remaining components. Now the estimate (A.0.11) remains true for every $i \in I_n'$, while in the "lateral components" we can apply statement (1) of Lemma A.0.1 because $|u'|$ is equal to D in at least one of the endpoints. In this way we obtain that

$$\mathbb{RPM}_{\varepsilon_n}\left((a, a_n) \cup (b_n, b), u_n\right) \geq \frac{M_n}{2^{1/2}} \mathbb{J}_{1/2}\left((a, a_n) \cup (b_n, b), z_n\right),$$

while as before

$$\mathbb{RPM}_{\varepsilon_n}\left((a_n, b_n), u_n\right) \geq M_n \, \mathbb{J}_{1/2}((a_n, b_n), z_n).$$

Adding the last two inequalities we obtain (A.0.5) also in this last case.

Statement (2) In order to prove (A.0.7), we show that

$$u_n(x) - w_n(x) \to 0 \qquad \text{uniformly in } [a, b] \tag{A.0.12}$$

and that for every $p \in [1, +\infty)$ it follows that

$$w_n - z_n \to 0 \qquad \text{in } L^p((a,b)). \tag{A.0.13}$$

In order to prove (A.0.12) we introduce the sets

$$B_n := \left\{ x \in (a,b) : \frac{1}{|\log \varepsilon_n|} \le |u_n'(x)| \le \frac{1}{\varepsilon_n^2 |\log \varepsilon_n|^4} \right\}$$

and

$$C_n := \left\{ x \in (a,b) : |u_n'(x)| < \frac{1}{|\log \varepsilon_n|} \right\}.$$

Let us estimate the measure of A_n, B_n, C_n. For C_n we consider the trivial estimate $|C_n| \le b - a$. As for A_n and B_n we consider the term with the logarithm in (2.3.1) and obtain that

$$|A_n| \le \frac{\varepsilon_n^2 |\log \varepsilon_n|}{\log(1 + \varepsilon_n^{-4} |\log \varepsilon_n|^{-8})} \mathbb{RPM}_{\varepsilon_n}((a,b), u_n)$$

and

$$|B_n| \le \frac{\varepsilon_n^2 |\log \varepsilon_n|}{\log(1 + |\log \varepsilon_n|^{-2})} \mathbb{RPM}_{\varepsilon_n}((a,b), u_n).$$

Recalling (A.0.4), these estimates imply that

$$\lim_{n \to +\infty} \frac{|A_n|}{\varepsilon_n^2 |\log \varepsilon_n|^4} = \lim_{n \to +\infty} \frac{|B_n|}{\varepsilon_n^2 |\log \varepsilon_n|^4} = 0.$$

Now let us consider the function $u_n' - w_n'$. In A_n it holds that

$$\int_{A_n} |u_n'(x) - w_n'(x)|\, dx = \int_{A_n} \frac{1}{\varepsilon_n^2 |\log \varepsilon_n|^4}\, dx = \frac{|A_n|}{\varepsilon_n^2 |\log \varepsilon_n|^4}.$$

In B_n and C_n it holds that $w_n'(x) = 0$, and hence

$$\int_{B_n} |u_n'(x) - w_n'(x)|\, dx = \int_{B_n} |u_n'(x)|\, dx \le \frac{|B_n|}{\varepsilon_n^2 |\log \varepsilon_n|^4}$$

and

$$\int_{C_n} |u_n'(x) - w_n'(x)|\, dx = \int_{C_n} |u_n'(x)|\, dx \le \frac{|C_n|}{|\log \varepsilon_n|} \le \frac{b-a}{|\log \varepsilon_n|}.$$

From all these estimates we conclude that

$$\lim_{n \to +\infty} \int_a^b |u_n'(x) - w_n'(x)|\, dx = 0, \tag{A.0.14}$$

which implies (A.0.12) because $u_n(a) = w_n(a) = 0$ for every $n \ge 1$.

In order to prove (A.0.13), we begin by observing that for every $x \in (a,b) \setminus A_n$ it holds that

$$\begin{aligned} w_n(x) &= w_n(a) + \int_a^x w_n'(t)\,dt \\ &= u_n(a) + \sum_{\{i \in I_n : \beta_{n,i} \le x\}} \int_{\alpha_{n,i}}^{\beta_{n,i}} w_n'(t)\,dt \\ &= u_n(a) + \sum_{\{i \in I_n : \beta_{n,i} \le x\}} (w_n(\beta_{n,i}) - w_n(\alpha_{n,i})) \\ &= z_n(x), \end{aligned}$$

which implies that $w_n(x) - z_n(x) = 0$ when $x \notin A_n$. On the other hand, when $x \in A_n$ it holds that

$$|w_n(x) - z_n(x)| \le |w_n(\beta_{n,i}) - w_n(\alpha_{n,i})| = |J_{z_n}(\gamma_{n,i})| \le \left(\mathbb{J}_{1/2}((a,b), z_n)\right)^2,$$

and therefore from (A.0.5) we conclude that

$$\begin{aligned} \int_a^b |w_n(x) - z_n(x)|^p\,dx &= \sum_{i \in I_n} \int_{\alpha_{i,\varepsilon_n}}^{\beta_{i,\varepsilon_n}} |w_n(x) - z_n(x)|^p\,dx \\ &\le \sum_{i \in I_n} (\beta_{i,\varepsilon_n} - \alpha_{i,\varepsilon_n}) \cdot \left(\mathbb{J}_{1/2}((a,b), z_n)\right)^{2p} \\ &= |A_n| \cdot \left(\mathbb{J}_{1/2}((a,b), z_n)\right)^{2p} \\ &\le |A_n| \cdot \left(\frac{2^{1/2}}{M_n} \mathbb{RPM}_{\varepsilon_n}((a,b), u_n)\right)^{2p}, \end{aligned}$$

which implies (A.0.13) because $\mathbb{RPM}_{\varepsilon_n}((a,b), u_n)$ is bounded from above, M_n is bounded from below, and $|A_n| \to 0$.

Statement (3) It remains to prove (A.0.8). To this end, we just observe that

$$|Dz_n|((a,b)) = \sum_{i \in I_n} |w_n(\beta_{n,i}) - w_n(\alpha_{n,i})| = \int_{A_n} |w_n'(x)|\,dx = \int_a^b |w_n'(x)|\,dx,$$

and

$$\left| \int_a^b |u_n'(x)|\,dx - \int_a^b |w_n'(x)|\,dx \right| \le \int_a^b |u_n'(x) - w_n'(x)|\,dx,$$

and conclude thanks to (A.0.14). □

The last preliminary result is the classical lower semicontinuity of $\mathbb{J}_{1/2}$ (see for example [6, Theorems 4.7 and 4.8]). Here we include an elementary proof in the one dimensional case, different from the original proof in [5], because we need to deduce the stronger statement (true in dimension one) that convergence of the energies implies *strict* convergence of the arguments.

Lemma A.0.3 (Lower semicontinuity of $\mathbb{J}_{1/2}$)**.** *Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let $\{z_n\} \subseteq PJ((a,b))$ be a sequence with the following properties:*

(i) there exists a constant M such that

$$\mathbb{J}_{1/2}((a,b),z_n) \leq M \qquad \forall n \geq 1, \tag{A.0.15}$$

(ii) there exists $p \geq 1$ and $z_\infty \in L^p((a,b))$ such that $z_n \to z_\infty$ in $L^p((a,b))$.

Then the following two statements hold true.

(1) (Lower semicontinuity). *It turns out that $z_\infty \in PJ((a,b))$ and*

$$\liminf_{n\to+\infty} \mathbb{J}_{1/2}((a,b),z_n) \geq \mathbb{J}_{1/2}((a,b),z_\infty). \tag{A.0.16}$$

(2) (Strict convergence). *If, in addition, we assume that*

$$\lim_{n\to+\infty} \mathbb{J}_{1/2}((a,b),z_n) = \mathbb{J}_{1/2}((a,b),z_\infty), \tag{A.0.17}$$

then actually $z_n \rightsquigarrow z_\infty$ in $BV((a,b))$.

Proof. For every $n \geq 1$, let us write $z_n(x)$ in the form

$$z_n(x) = c_n + \sum_{i=1}^{\infty} J_n(i)\mathbb{1}_{(s_n(i),+\infty)}(x) \qquad \forall x \in (a,b), \tag{A.0.18}$$

where $c_n := z_n(a)$ (the boundary value is intended in the usual sense), $\{s_n(i)\}_{i\geq 1} \subseteq (a,b)$ is a sequence of distinct points, and $\{J_n(i)\}_{i\geq 1}$ is a sequence of real numbers such that $|J_n(i+1)| \leq |J_n(i)|$ for every $i \geq 1$. We observe that, even when the function z_n has only a finite number of jump points, we can always write it in the form (A.0.18) by introducing infinitely many "jumps" of vanishing height.

From assumptions (i) and (ii) we derive two types of estimates.

- (Uniform bounds). From assumption (i) and the subadditivity of the square root we deduce that

$$\sum_{i=1}^{\infty} |J_n(i)| \leq \left(\sum_{i=1}^{\infty} |J_n(i)|^{1/2}\right)^2 \leq M^2 \qquad \forall n \geq 1. \tag{A.0.19}$$

  Combined with assumption (ii), this implies that there exists a constant M_1 such that

$$|c_n| \leq M_1 \qquad \forall n \geq 1. \tag{A.0.20}$$

  Finally, from (A.0.19) and (A.0.20) we deduce that there exists a constant M_2 such that

$$\|z_n\|_{L^\infty((a,b))} \leq M_2 \qquad \forall n \geq 1. \tag{A.0.21}$$

- (Uniform smallness of the tails). We claim that for every $\varepsilon > 0$ there exists a positive integer i_ε such that

$$\sum_{i=i_\varepsilon}^{\infty} |J_n(i)| \leq M\sqrt{\varepsilon} \qquad \forall n \geq 1. \tag{A.0.22}$$

Indeed, if we define i_ε as the smallest integer greater that $M/\sqrt{\varepsilon}$, then from (A.0.15) it holds that $|J_n(i)| \leq \varepsilon$ for at least one index $i \leq i_\varepsilon$. At this point, from the monotonicity of $|J_n(i)|$ we conclude that

$$|J_n(i)| \leq \varepsilon \qquad \forall n \geq 1, \quad \forall i \geq i_\varepsilon,$$

and therefore

$$\sum_{i=i_\varepsilon}^{\infty} |J_n(i)| = \sum_{i=i_\varepsilon}^{\infty} |J_n(i)|^{1/2} \cdot |J_n(i)|^{1/2} \leq \sqrt{\varepsilon} \cdot \sum_{i=i_\varepsilon}^{\infty} |J_n(i)|^{1/2} \leq \sqrt{\varepsilon} \cdot M,$$

which proves (A.0.22).

From the uniform bounds we obtain that, up to subsequences (not relabeled), the following limits exists as $n \to +\infty$:

$$c_n \to c_\infty, \qquad J_n(i) \to J_\infty(i), \qquad s_n(i) \to s_\infty(i) \in [a, b].$$

From these limits we deduce that

$$\liminf_{n\to+\infty} \mathbb{J}_{1/2}((a,b), z_n) = \liminf_{n\to+\infty} \sum_{i=1}^{\infty} |J_n(i)|^{1/2} \geq \sum_{i=1}^{\infty} |J_\infty(i)|^{1/2}, \tag{A.0.23}$$

and, due to the uniform smallness of the tails,

$$\lim_{n\to+\infty} |Dz_n|((a,b)) = \lim_{n\to+\infty} \sum_{i=1}^{\infty} |J_n(i)| = \sum_{i=1}^{\infty} |J_\infty(i)|. \tag{A.0.24}$$

We can now introduce the function

$$\widehat{z}_\infty(x) := c_\infty + \sum_{i=1}^{\infty} J_\infty(i) \mathbb{1}_{(s_\infty(i),+\infty)}(x) \qquad \forall x \in (a, b). \tag{A.0.25}$$

Exploiting again (A.0.22) we can show that $z_n(x) \to \widehat{z}_\infty(x)$ for every $x \in (a, b)$ that does not appear in the sequence $\{s_\infty(i)\}$. This almost everywhere pointwise convergence, together with the uniform bound (A.0.21), implies that $z_n \to \widehat{z}_\infty$ in $L^p((a, b))$ for every $p \in [1, +\infty)$, and hence in particular that $z_\infty = \widehat{z}_\infty$.

Now we use the representation (A.0.25) in order to compute the total variation of z_∞ and $\mathbb{J}_{1/2}((a, b), z_\infty)$. This is not immediate, because in the representation (A.0.25) the points $s_\infty(i)$ are not necessarily distinct, and some of them might even coincide with the endpoints of the interval (a, b), in which case they do not contribute to the total variation or to $\mathbb{J}_{1/2}$. In any case, the function defined by (A.0.25) belongs to $PJ((a,b))$, and its jump set is contained in the image of the sequence $\{s_\infty(i)\}$ intersected with the open interval (a,b). Moreover, for every s in this set, the jump height of z_∞ in s is given by

$$J_{z_\infty}(s) = \sum_{\{i\geq 1: s_\infty(i)=s\}} J_\infty(i),$$

where of course the sum (or series) might also vanish. In particular, for every jump point s of z_∞ we obtain that

$$|J_{z_\infty}(s)| \leq \sum_{\{i\geq 1: s_\infty(i)=s\}} |J_\infty(i)|, \tag{A.0.26}$$

with equality if and only if all terms in the sum have the same sign. Analogously, we obtain that

$$|J_{z_\infty}(s)|^{1/2} \leq \sum_{\{i\geq 1: s_\infty(i)=s\}} |J_\infty(i)|^{1/2},$$

with equality if and only if at most one term in the sum is different from 0 (here we make use of the fact that the square root is strictly subadditive).

From (A.0.26) it follows that

$$|Dz_\infty|((a,b)) = \sum_{s\in S_{z_\infty}} |J_{z_\infty}(s)| \leq \sum_{i=1}^{\infty} |J_\infty(i)|, \tag{A.0.27}$$

with equality if and only if $s_\infty(i) \in (a,b)$ for every $i \geq 1$ such that $J_\infty(i) \neq 0$, and $J_\infty(i) \cdot J_\infty(j) \geq 0$ for every pair (i,j) of distinct positive integers such that $s_\infty(i) = s_\infty(j) \in (a,b)$.

Analogously, it holds that

$$\mathbb{J}_{1/2}((a,b), z_\infty) = \sum_{s\in S_{z_\infty}} |J_{z_\infty}(s)|^{1/2} \leq \sum_{i=1}^{\infty} |J_\infty(i)|^{1/2}, \tag{A.0.28}$$

with equality if and only if $s_\infty(i) \in (a,b)$ for every $i \geq 1$ such that $J_\infty(i) \neq 0$, and $J_\infty(i) \cdot J_\infty(j) = 0$ for every pair (i,j) of distinct positive integers such that $s_\infty(i) = s_\infty(j) \in (a,b)$. In particular, in all cases where equality occurs in (A.0.28), then equality occurs also in (A.0.27).

At this point we are ready to complete the proof. Indeed, (A.0.16) follows from (A.0.23) and (A.0.28), provided that we start with the subsequence of $\{z_n\}$ that realizes the liminf in (A.0.16). As for the strict convergence, under assumption (A.0.17) we have necessarily equality both in (A.0.23) and in (A.0.28), and hence we have equality also in (A.0.27). At this point from (A.0.24) and (A.0.27) we conclude that $|Dz_n|((a,b)) \to |Dz_\infty|((a,b))$ (to be overly pedantic, what we actually proved is that every subsequence of $\{z_n\}$ has a further subsequence with this property), which is what we need in order to conclude that the convergence is strict. □

Remark A.0.4. The only properties of the square root that we need for Lemma A.0.3 are that it is a nonnegative function that is strictly sub-additive and satisfies $\sqrt{\sigma}/\sigma \to +\infty$ as $\sigma \to 0^+$.

A.1 Proof of Theorem 2.3.2

Statement (1) Let us start with the liminf inequality. We need to prove that

$$\liminf_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}((a,b),u_n) \geq \alpha_0\, \mathbb{J}_{1/2}((a,b),u) \tag{A.1.29}$$

for every sequence $\{u_n\} \subseteq H^2((a,b))$ such that $u_n \to u$ in $L^2((a,b))$, and every sequence $\{\varepsilon_n\} \subseteq (0,1)$ such that $\varepsilon_n \to 0^+$. Up to subsequences (not relabeled), we can assume that the left-hand side is bounded and that the liminf is actually a limit, and in particular that the sequence $\{\mathbb{RPM}_{\varepsilon_n}((a,b),u_n)\}$ is bounded. When this is the case, from Lemma A.0.2 we obtain a sequence $\{z_n\} \subseteq PJ((a,b))$ such that $z_n \to u$ in $L^2((a,b))$ and

$$\mathbb{RPM}_{\varepsilon_n}((a,b),u_n) \geq M_n \cdot \mathbb{J}_{1/2}((a_n,b_n),z_n) \qquad \forall n \geq 1.$$

Now we consider any interval (a',b') whose closure is contained in (a,b). Since $a_n \to a^+$ and $b_n \to b^-$, and since $M_n \to \alpha_0$ as $n \to +\infty$, from the lower semicontinuity of $\mathbb{J}_{1/2}$ (with respect to any L^p convergence) we conclude that

$$\liminf_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}((a,b),u_n) \geq \liminf_{n\to+\infty} M_n \cdot \mathbb{J}_{1/2}((a_n,b_n),z_n) \geq \alpha_0\, \mathbb{J}_{1/2}((a',b'),u).$$

Letting $a' \to a^+$ and $b' \to b^-$ we obtain (A.1.29).

For the limsup inequality, we refer to the proof of [14, Theorem 4.4]. The idea is rather classical. First of all, we reduce ourselves to the case where u has only a finite number of jump points, because this class is dense in $L^2((a,b))$ with respect to the energy $\mathbb{J}_{1/2}$. Given any function $u \in PJ((a,b))$ with a finite number of jumps, we consider the function u_ε that coincides with u outside some small intervals that contain a single jump point, and in each of these small intervals coincides with the cubic polynomial that interpolates the values at the boundary of the interval. From Lemma 2.6.2 we obtain the exact value of the integral of $u_\varepsilon''(x)^2$, and an estimate from above for the integral of $\log(1+u_\varepsilon'(x)^2)$. If we optimize the length of each small interval in terms of ε and of the jump height, the resulting family is the required recovery family.

We stress that, in the case where u has a finite number of jumps, there exists a recovery family that coincides with u in a fixed neighborhood of the boundary points $x = a$ and $x = b$.

Statement (2) Let us apply again Lemma A.0.2. We obtain a sequence $\{z_n\} \subseteq PJ((a,b))$ satisfying (A.0.7) and

$$\mathbb{RPM}_{\varepsilon_n}((a,b),u_n) \geq \frac{M_n}{2^{1/2}} \cdot \mathbb{J}_{1/2}((a,b),z_n)$$

when n is large enough. In particular, since M_n is bounded from below by a positive constant, from (2.3.8) we deduce that this sequence satisfies

$$\sup_{n\in\mathbb{N}} \left\{ \mathbb{J}_{1/2}((a,b),z_n) + \int_a^b z_n(x)^2\,dx \right\} < +\infty.$$

From the classical compactness result for the functional $\mathbb{J}_{1/2}$ (whose proof in dimension one is more or less contained in the proof of Lemma A.0.3 above), it follows that $\{z_n\}$ is relatively compact in $L^p((a,b))$ for every $p \in [1,+\infty)$. Due to (A.0.7), the same is true for $\{u_n\}$.

Statement (3) Let us apply again Lemma A.0.2. The resulting sequence $\{z_n\}$ converges to u. Now let us fix any interval (a',b') whose closure is contained in (a,b). From the lower semicontinuity of $\mathbb{J}_{1/2}$, estimate (A.0.5), and assumption (2.3.9), we deduce that

$$\begin{aligned}
\mathbb{J}_{1/2}((a',b'),u) &\leq \liminf_{n\to+\infty} \mathbb{J}_{1/2}((a',b'),z_n) \\
&\leq \liminf_{n\to+\infty} \mathbb{J}_{1/2}((a_n,b_n),z_n) \\
&\leq \liminf_{n\to+\infty} \left\{ \frac{\mathbb{RPM}_{\varepsilon_n}((a,b),u_n)}{M_n} - \frac{\mathbb{J}_{1/2}\big((a,a_n)\cup(b_n,b),z_n\big)}{2^{1/2}} \right\} \\
&\leq \limsup_{n\to+\infty} \frac{\mathbb{RPM}_{\varepsilon_n}((a,b),u_n)}{M_n} - \limsup_{n\to+\infty} \frac{\mathbb{J}_{1/2}\big((a,a_n)\cup(b_n,b),z_n\big)}{2^{1/2}} \\
&= \frac{1}{\alpha_0}\cdot\alpha_0\,\mathbb{J}_{1/2}((a,b),u) - \frac{1}{2^{1/2}}\limsup_{n\to+\infty}\mathbb{J}_{1/2}\big((a,a_n)\cup(b_n,b),z_n\big),
\end{aligned}$$

which implies that

$$\limsup_{n\to+\infty} \mathbb{J}_{1/2}\big(a,a_n)\cup(b_n,b),z_n\big) \leq 2^{1/2}\left\{\mathbb{J}_{1/2}((a,b),u) - \mathbb{J}_{1/2}((a',b'),u)\right\}.$$

Letting $a' \to a^+$ and $b' \to b^-$ we conclude that

$$\lim_{n\to+\infty} \mathbb{J}_{1/2}\big((a,a_n)\cup(b_n,b),z_n\big) = 0.$$

Now we observe that

$$\begin{aligned}
\mathbb{J}_{1/2}((a,b),u) &\leq \liminf_{n\to+\infty} \mathbb{J}_{1/2}((a,b),z_n) \\
&\leq \limsup_{n\to+\infty} \mathbb{J}_{1/2}((a,b),z_n) \\
&= \limsup_{n\to+\infty} \left\{\mathbb{J}_{1/2}((a_n,b_n),z_n) + \mathbb{J}_{1/2}\big((a,a_n)\cup(b_n,b),z_n\big)\right\} \\
&= \limsup_{n\to+\infty} \mathbb{J}_{1/2}((a_n,b_n),z_n) \\
&\leq \limsup_{n\to+\infty} \frac{1}{M_n}\,\mathbb{RPM}_{\varepsilon_n}((a,b),u_n) \\
&\leq \mathbb{J}_{1/2}((a,b),u),
\end{aligned}$$

and hence $\mathbb{J}_{1/2}((a,b),z_n) \to \mathbb{J}_{1/2}((a,b),u)$. At this point from Lemma A.0.3 we conclude that $z_n \rightsquigarrow u$ in $BV((a,b))$, and therefore also $u_n \rightsquigarrow u$ in $BV((a,b))$ because of (A.0.8).

Statement (4) As in the proof of the limsup inequality for the Gamma-convergence result, we can assume that u is a pure jump function with a finite number of jump points. When this is the case, we already know that there exists a recovery sequence $\widehat{u}_n \to u$ that coincides with u in a neighborhood of the boundary, namely there exists $\eta > 0$ such that for every $n \geq 1$ it holds that $\widehat{u}_n(x) = u(x) = u(a)$ for every $x \in (a, a+\eta)$, and similarly $\widehat{u}_n(x) = u(x) = u(b)$ for every $x \in (b-\eta, b)$.

Now the idea is to modify $\widehat{u}_n$ in the two lateral intervals $(a, a+\eta)$ and $(b-\eta, b)$ in order to fulfill the given boundary conditions (2.3.10). To this end, we set

$$u_n(x) := \begin{cases} u(a) + w_{1,n}(x) & \text{if } x \in (a, a+\eta], \\ \widehat{u}_n(x) & \text{if } x \in [a+\eta, b-\eta], \\ u(b) + w_{2,n}(x) & \text{if } x \in [b-\eta, b). \end{cases}$$

where $w_{1,n}$ is the function given by Lemma 2.6.3 applied to the interval $(a, a+\eta)$ with boundary data

$$\big(w_{1,n}(a), w'_{1,n}(a), w_{1,n}(a+\eta), w'_{1,n}(a+\eta)\big) = (A_{0,n} - u(a), A_{1,n}, 0, 0),$$

and $w_{2,n}$ is the function given by Lemma 2.6.3 applied in the interval $(b-\eta, b)$ with boundary data

$$\big(w_{2,n}(b-\eta), w'_{2,n}(b-\eta), w_{2,n}(b), w'_{2,n}(b)\big) = (0, 0, B_{0,n} - u(b), B_{1,n}).$$

We observe that $u_n \in H^2((a,b))$ and

$$\begin{aligned} \mathbb{RPM}_{\varepsilon_n}((a,b), u_n) \;=\; & \mathbb{RPM}_{\varepsilon_n}((a, a+\eta), w_{1,n}) \\ & + \mathbb{RPM}_{\varepsilon_n}((a+\eta, b-\eta), \widehat{u}_n) \\ & + \mathbb{RPM}_{\varepsilon_n}((b-\eta, b), w_{2,n}). \end{aligned}$$

The second term coincides with $\mathbb{RPM}_{\varepsilon_n}((a,b), \widehat{u}_n)$, and therefore it converges to $\alpha_0 \, \mathbb{J}_{1/2}((a,b), u)$ when $n \to +\infty$. Therefore, it is enough to show that the other two terms vanish in the limit. To this end, we observe that in the interval $(a, a+\eta)$ the assumptions of Lemma 2.6.3 are satisfied with

$$H = H_n := |A_{0,n} - u(a)| \qquad \text{and} \qquad D = D_n := |A_{1,n}|.$$

Since H_n and D_n tend to 0, we conclude that

$$\lim_{n \to +\infty} \mathbb{RPM}_{\varepsilon_n}((a, a+\eta), w_{1,n}) \leq \lim_{n \to +\infty} 80 \left(\sqrt{H_n} + \varepsilon_n^2 D_n \right) = 0.$$

In the same way we obtain that

$$\lim_{n \to +\infty} \mathbb{RPM}_{\varepsilon_n}((b-\eta, b), w_{2,n}) = 0,$$

which completes the proof. □

A.2 Proof of Proposition 2.3.4

Statement (1) In the case of μ_ε, μ_ε^* and μ_0, existence is a standard application of the direct method in the calculus of variations. The case of μ_0^* is less trivial because boundary conditions in $PJ((0,L))$ do not pass to the limit, for example, with respect to L^2 convergence. This issue, however, can be fixed in a rather standard way. To this end, we relax boundary conditions by allowing "jumps at the boundary", namely we minimize

$$\alpha\,\mathbb{J}_{1/2}((0,L),v)+\beta\int_0^L(v(x)-Mx)^2\,dx+\alpha\left(|v(0)|^{1/2}+|v(L)-ML|^{1/2}\right)$$

over $PJ((0,L))$, without boundary conditions. In this case the direct method works, and we claim that any minimizer v satisfies $v(0)=0$ and $v(L)=ML$. Indeed, let v be any minimizer, and let us consider the value in $x=0$ (the argument in $x=L$ is symmetric). Let us assume that $M>0$ (the case $M=0$ is trivial, and the case $M<0$ is symmetric). Arguing as in the beginning of section 2.6.2 we can show that the set of jump points of v is finite, and comparing with a competitor $v_\tau(x)$ which is equal to 0 in $(0,\tau)$, and equal to $v(x)$ elsewhere, we can conclude that $v(0)=0$.

Statement (2) We prove the result in the case of μ_ε, but the argument is analogous in the other three cases.

The symmetry follows from the simple remark that, if v is a minimizer for some M, then $-v$ is a minimizer for $-M$.

Continuity follows from the fact that, if $M_n\to M_\infty$, then the fidelity term in $\mathbb{RPMF}_\varepsilon(\beta,M_nx,(0,L),v)$ converges to the fidelity term in $\mathbb{RPMF}_\varepsilon(\beta,M_\infty x,(0,L),v)$ uniformly on bounded subsets of $L^2((a,b))$.

As for monotonicity, let us consider any pair $0\le M_1<M_2$. Let us choose any minimizer $v_2\in H^2((0,L))$ in the definition of $\mu_\varepsilon(\beta,L,M_2)$, and let us consider the function $v_1(x):=(M_1/M_2)v_2(x)$. Elementary computations show that

$$\mathbb{RPM}_\varepsilon((0,L),v_1)\le\mathbb{RPM}_\varepsilon((0,L),v_2),$$

and

$$\int_0^L(v_1(x)-M_1\,x)^2\,dx=\frac{M_1^2}{M_2^2}\int_0^L(v_2(x)-M_2\,x)^2\,dx\le\int_0^L(v_2(x)-M_2\,x)^2\,dx,$$

and therefore

$$\begin{aligned}\mu_\varepsilon(\beta,L,M_1)&\le\mathbb{RPMF}_\varepsilon(\beta,M_1x,(0,L),v_1)\\&\le\mathbb{RPMF}_\varepsilon(\beta,M_2x,(0,L),v_2)\\&=\mu_\varepsilon(\beta,L,M_2).\end{aligned}$$

Finally, if we reverse the argument, namely we start from a minimizer $v_1\in H^2((0,L))$ in the definition of $\mu_\varepsilon(\beta,L,M_1)$, and we consider the function $v_2(x):=(M_2/M_1)v_1(x)$, we obtain that

$$\int_0^L v_2''(x)^2\,dx=\left(\frac{M_2}{M_1}\right)^2\int_0^L v_1''(x)^2\,dx,$$

$$\int_0^L (v_2(x) - M_1\, x)^2\, dx = \left(\frac{M_2}{M_1}\right)^2 \int_0^L (v_1(x) - M_1\, x)^2\, dx,$$

and

$$\int_0^L \log(1 + v_2'(x)^2)\, dx \leq \left(\frac{M_2}{M_1}\right)^2 \int_0^L \log(1 + v_1'(x)^2)\, dx,$$

because of the inequality

$$\log(1 + \lambda t) \leq \lambda \log(1 + t) \qquad \forall \lambda \geq 1\ \forall t \geq 0.$$

Therefore we have that

$$\begin{aligned}
\mu_\varepsilon(\beta, L, M_2) &\leq \mathbb{RPMF}_\varepsilon(\beta, M_2 x, (0, L), v_2) \\
&\leq \left(\frac{M_2}{M_1}\right)^2 \mathbb{RPMF}_\varepsilon(\beta, M_1 x, (0, L), v_1) \\
&= \left(\frac{M_2}{M_1}\right)^2 \mu_\varepsilon(\beta, L, M_1),
\end{aligned}$$

that is exactly (2.3.16).

Statement (3) Let us consider any pair $0 < L_1 < L_2$, and let us examine separately the behavior of the four functions.

In the case of μ_ε, let v_2 be any minimizer for $\mu_\varepsilon(\beta, L_2, M)$. Then the restriction of v_2 to $(0, L_1)$, which we call v_1, is a competitor in the definition of $\mu_\varepsilon(\beta, L_1, M)$, and therefore as before we conclude that

$$\begin{aligned}
\mu_\varepsilon(\beta, L_1, M) &\leq \mathbb{RPMF}_\varepsilon(\beta, Mx, (0, L_1), v_1) \\
&\leq \mathbb{RPMF}_\varepsilon(\beta, Mx, (0, L_2), v_2) \\
&= \mu_\varepsilon(\beta, L_2, M).
\end{aligned}$$

The same argument works in the case of μ_0.

In the case of μ_0^* we have to take into account boundary conditions, and therefore we define

$$v_1(x) = \frac{L_1}{L_2} v_2\left(\frac{L_2 x}{L_1}\right) \qquad \forall x \in (0, L_1), \tag{A.2.30}$$

and we observe that $\mathbb{J}_{1/2}((0, L_1), v_1) \leq \mathbb{J}_{1/2}((0, L_2), v_2)$ and

$$\int_0^{L_1} (v_1(x) - Mx)^2\, dx = \left(\frac{L_1}{L_2}\right)^3 \int_0^{L_2} (v_2(x) - Mx)^2\, dx,$$

which again implies the conclusion.

Finally, the monotonicity of μ_ε^* with respect to L is in general false (the minimum diverges when $L \to 0^+$ due to the term with second order derivatives). In this case the natural definition (A.2.30), that preserves the boundary conditions (both on the function and on the derivative), reduces the fidelity term and the term with the logarithm, but increases the term with second order derivatives. What we do in this case is the opposite.

We consider a minimizer v_1 in the definition of $\mu_\varepsilon^*(\beta, L_1, M)$, and we define a function v_2 in $(0, L_2)$ in such a way that (A.2.30) holds true. With a simple change of variable we see that

$$\int_0^{L_2} v_2''(x)^2\, dx = \frac{L_1}{L_2} \int_0^{L_1} v_1''(x)^2\, dx,$$

$$\int_0^{L_2} \log\left(1 + v_2'(x)^2\right)\, dx = \frac{L_2}{L_1} \int_0^{L_1} \log\left(1 + v_1'(x)^2\right)\, dx,$$

and

$$\int_0^{L_2} (v_2(x) - Mx)^2\, dx = \left(\frac{L_2}{L_1}\right)^3 \int_0^{L_1} (v_1(x) - Mx)^2\, dx,$$

so that in particular

$$\mathbb{RPMF}_\varepsilon(\beta, (0, L_2), Mx, v_2) \leq \left(\frac{L_2}{L_1}\right)^3 \mathbb{RPMF}_\varepsilon(\beta, (0, L_1), Mx, v_1).$$

Since v_2 is a competitor in the definition of $\mu_\varepsilon^*(\beta, L_2, M)$, this is enough to establish (2.3.17).

Statement (4) Pointwise convergence, namely convergence of minima, is a rather standard consequence of Gamma-convergence and equi-coerciveness. We point out that in the case of (2.3.14) and (2.3.15) the functionals have to take the boundary conditions into account (the usual way is to set the functionals equal to $+\infty$ when the argument does not satisfy the boundary conditions), and in this case the limsup inequality in the Gamma-convergence result is slightly more delicate because it requires the control of boundary conditions for recovery sequences.

Statement (5) The pointwise convergence (2.3.18) is actually uniform with respect to M (on bounded sets) because of the continuity and monotonicity with respect to M of both μ_ε and μ_0. An analogous argument applies in the case of (2.3.19). □

Chapter 3

Singular perturbation: higher resolution blow-up

3.1 Introduction

This chapter is devoted to the study of higher resolution blow-up of minimizers for the functional $\mathbb{PMF}_\varepsilon$ defined in (2.1.1). Our goal is to show that minimizers develop the *multi-scale structure* illustrated in Figure 2.1.

To this end, we consider any sequence $\{u_\varepsilon\}$ of minimizers for the functional (2.1.1) on the interval $(0,1)$, a sequence $x_\varepsilon \to x_0 \in (0,1)$, and the functions v_ε defined in (2.2.9). From Theorem 2.2.9 we know that $\{v_\varepsilon\}$ converges up to subsequences to some graph translations of a suitable staircase $S_{H,V}$, and for each graph translation v_0 of $S_{H,V}$ we can move slightly the points $\{x_\varepsilon\}$ in such a way that $v_\varepsilon \rightsquigarrow v_0$.

The first main result of this chapter states that in this setting, if v_0 is the graph translation of vertical type, then the functions obtained by rescaling horizontally v_ε by a factor ε^2 converge in a very strong sense to a specific function, which is the cubic polynomial that interpolates the constants $v_0(-H)$ and $v_0(H)$ in a prescribed neighborhood of the origin. This result identifies the rightmost function in Figure 2.1, thus revealing the exact form of the transitions between consecutive steps of the staircases.

The second main result of this chapter, instead, improves the convergence of minimizers in the horizontal parts of the staircase. Indeed, we show that if v_0 is a graph translation of horizontal type of the canonical staircase, then the C^1 norm of minimizers is of order $\omega(\varepsilon)^2$ on every closed interval that is contained in the step of the staircase that contains the origin. This means that minimizers are actually flat far from the jumps of the staircase.

As in the previous chapter, we first state precisely our main results in section 3.2. Then we prove them in section 3.3 and section 3.4, exploiting some technical lemmata that we prove later in section 3.5.

3.2 Statements

In order to state precisely our results, let us introduce some notation to describe the cubic transitions arising around the jump points of the staircase.

Definition 3.2.1 (Cubic connection)**.** Let $C : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$C(x) := \begin{cases} -1 & \text{if } x \leq -1, \\ \dfrac{3}{2}x - \dfrac{1}{2}x^3 & \text{if } x \in [-1,1], \\ 1 & \text{if } x \geq 1. \end{cases} \tag{3.2.1}$$

- For every pair (Λ, V) of positive real numbers we call *canonical* (Λ, V)*-cubic connection* the function $C_{\Lambda,V} : \mathbb{R} \to \mathbb{R}$ defined by

$$C_{\Lambda,V}(x) := VC(x/\Lambda) \qquad \forall x \in \mathbb{R}. \tag{3.2.2}$$

- For every $\tau_0 \in (-1, 1)$ we call *graph translation* of the canonical (Λ, V)-cubic connection the function $C_{\Lambda,V,\tau_0} : \mathbb{R} \to \mathbb{R}$ defined by

$$C_{\Lambda,V,\tau_0}(x) := C_{\Lambda,V}(x + x_0) - \tau_0 V \qquad \forall x \in \mathbb{R},$$

  where $x_0 \in (-\Lambda, \Lambda)$ is the unique real number such that $C_{\Lambda,V}(x_0) = \tau_0 V$.

In words, the (Λ, V)-cubic connection is the unique polynomial of degree three that interpolates the constants $-V$ and $+V$ in a C^1 way in the interval $[-\Lambda, \Lambda]$. Every graph translation corresponds to taking the graph of $C_{\Lambda,V}(x)$ and moving the origin to a point of the cubic.

The following theorem contains both the main results of this chapter, concerning higher order blow-ups of minimizers for the functional $\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u)$.

Theorem 3.2.2 (High-resolution blow-ups of minimizers)**.** *Let $\beta > 0$ be a positive number, let $f \in C^1([0,1])$ be a function, and let $x_0 \in (0,1)$ be a point with $f'(x_0) \neq 0$. Let $\varepsilon_n \to 0^+$ be a sequence of positive real numbers, and let $\{x_n\} \subseteq (0,1)$ be a sequence such that $x_n \to x_0 \in (0,1)$. For every positive integer n, let*

$$u_n \in \operatorname{argmin}\left\{\mathbb{PMF}_{\varepsilon_n}(\beta, f, (0,1), u) : u \in H^2((0,1))\right\}, \tag{3.2.3}$$

where $\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u)$ is the family of functionals defined in (2.2.2). Let

$$H := \left(\frac{24}{\beta^2 |f'(x_0)|^3}\right)^{1/5}, \qquad V := f'(x_0)H,$$

be as in (2.2.10), and let us set

$$\Lambda := \frac{\sqrt{3}}{2} \cdot \sqrt{2V}. \tag{3.2.4}$$

Let $\omega(\varepsilon) := \varepsilon|\log\varepsilon|^{1/2}$ *be as in (2.2.1) and let us assume that the sequence*

$$v_n(y) := \frac{u_n(x_n + \omega(\varepsilon_n)y) - u_n(x_n)}{\omega(\varepsilon_n)}$$

of blow-ups at canonical scale converges locally strictly in $BV_{loc}(\mathbb{R})$ *to some* v_∞ *that, due to Theorem 2.2.9, belongs to the family* $\mathrm{Hor}(H,V) \cup \mathrm{Vert}(H,V)$ *of Definition 2.2.4.*

Then the following statements hold true.

(1) (Vertical parts of the steps). *Let us assume that* $v_\infty \in \mathrm{Vert}(H,V) \setminus \mathrm{Hor}(H,V)$*, and more precisely that* $v_\infty(y)$ *is given by (2.2.7) for some* $\tau_0 \in (-1,1)$*, and let* $C_{\Lambda,V,\tau_0}(y)$ *be the corresponding translated cubic connection introduced in Definition 3.2.1. Then it turns out that*

$$v_n(\varepsilon_n^2 y) \to C_{\Lambda,V,\tau_0}(y) \quad \textit{strongly in } H^2_{loc}(\mathbb{R}).$$

(2) (Horizontal parts of the steps). *Let us assume that* $v_\infty \in \mathrm{Hor}(H,V) \setminus \mathrm{Vert}(H,V)$*, and more precisely that* $v_\infty(y)$ *is given by (2.2.6) for some* $\tau_0 \in (-1,1)$*. Then for every closed interval* $[a,b] \subseteq ((\tau_0 - 1)H, (1+\tau_0)H)$ *it turns out that*

$$\limsup_{n\to+\infty} \frac{1}{\omega(\varepsilon_n)^2} \cdot \max\left\{|v_n'(y)| : y \in [a,b]\right\} < +\infty. \tag{3.2.5}$$

As a consequence, for every sequence $\{\alpha_n\} \subset (0,1)$ *such that* $\alpha_n = o(\omega(\varepsilon_n))$ *we deduce that*

$$\frac{u_n(x_n + \alpha_n y) - u_n(x_n)}{\alpha_n} \to 0 \quad \textit{uniformly on bounded subsets of } \mathbb{R}.$$

Remark 3.2.3. We recall that the locally strict convergence of the sequence $\{v_n\}$ to a staircase v_∞ is ensured by Theorem 2.2.9, at least up to subsequences or up to a slight modifications of the centers $\{x_n\}$.

Therefore, there are only two situations in which higher resolution blow-ups are not characterized by Theorem 3.2.2: when $v_\infty \in \mathrm{Hor}(H,V) \cap \mathrm{Vert}(H,V)$, namely when $v_\infty(y) = S_{H,V}(y \pm H)$, and when $f'(x_0) = 0$.

In the first case, it can be seen that the behavior of $\{v_n\}$ could be both "horizontal-like" and "vertical-like", because a sequence $\{x_n\}$ such that $v_\infty \in \mathrm{Hor}(H,V) \cap \mathrm{Vert}(H,V)$ can be obtained with a diagonal procedure both starting from sequences generating staircases in $\mathrm{Hor}(H,V) \setminus \mathrm{Vert}(H,V)$ and in $\mathrm{Vert}(H,V) \setminus \mathrm{Hor}(H,V)$. As a consequence, also some intermediate behavior could arise, and we can not exclude that other non trivial blow-ups might exist at a different scale.

The situation in the second case is similar, at least if f' does not vanish identically in a neighborhood of x_0. Indeed, also in this case $\{x_n\}$ can be obtained with a diagonal procedure starting from sequences such that the corresponding blow-ups exhibit different behaviors.

3.3 Vertical parts (Theorem 3.2.2, statement (1))

In this section we prove Statement (1) in Theorem 3.2.2, which is a direct consequence of the following result about the rescaled functionals $\mathbb{RPM}_\varepsilon$ defined in (2.3.1). We point out that the assumption (3.3.2) follows from the proof of Theorem 2.2.9, and more precisely from (2.6.40) and (2.5.17).

Theorem 3.3.1. *Let $L > 0$ be a positive real number, let $\varepsilon_n \to 0^+$ be a sequence of positive real numbers, and let $\{v_n\} \subseteq H^2((-L,L))$ be a sequence of functions such that $v_n(0) = 0$ for every $n \in \mathbb{N}$. Let us assume that there exist real numbers $V > 0$ and $\tau_0 \in (-1,1)$ such that*

$$v_n(y) \rightsquigarrow V(\operatorname{sign}(y) - \tau_0) \qquad \textit{strictly in } BV((-L,L)), \tag{3.3.1}$$

and

$$\lim_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}((-L,L), v_n) \le \alpha_0\sqrt{2V}, \tag{3.3.2}$$

where α_0 is the constant defined in (2.3.7)

Let us set $w_n(y) := v_n(\varepsilon_n^2 y)$, and let $\Lambda := (\sqrt{3}/2)\sqrt{2V}$ be as in (3.2.4).

Then $w_n(y)$ converges up to order 2 to the cubic connection $C_{\Lambda,V,\tau_0}(y)$ introduced in Definition 3.2.1, in the sense that

$$\lim_{n\to+\infty} \max\left\{|w_n(y) - C_{\Lambda,V,\tau_0}(y)| : |y| \le L/\varepsilon_n^2\right\} = 0,$$

and for every bounded interval $(a,b) \subseteq \mathbb{R}$ it turns out that

$$w_n'(y) \to C_{\Lambda,V,\tau_0}'(y) \quad \textit{uniformly in } (a,b)$$

and

$$w_n''(y) \to C_{\Lambda,V,\tau_0}''(y) \quad \textit{strongly in } L^2((a,b)).$$

Proof. We divide the proof in several steps. First, we introduce some notation, then we show that the transition from $-(1+\tau_0)V$ and $(1-\tau_0)V$ occurs in a single interval, and finally we show that in this interval we have some precise estimates that allow us to characterize the limit.

Notation and definitions We observe that $w_n(y)$ is defined for every $|y| < L/\varepsilon_n^2$, and with a variable change in the integral we obtain that it satisfies

$$\mathbb{RPM}_{\varepsilon_n}((-L,L), v_n) = \mathbb{RPMV}_{\varepsilon_n}((-L/\varepsilon_n^2, L/\varepsilon_n^2), w_n), \tag{3.3.3}$$

where

$$\mathbb{RPMV}_\varepsilon(\Omega, u) := \int_\Omega \left\{ u''(x)^2 + \frac{1}{|\log\varepsilon|}\log\left(1 + \frac{u'(x)^2}{\varepsilon^4}\right)\right\} dx.$$

Since strict convergence implies uniform convergence in compact sets that do not contain jump points of the limit, from (3.3.1) we deduce that for every $a \in (0,L)$ it turns out that

$$\lim_{n\to+\infty} \max\left\{|v_n(y) - V(\operatorname{sign}(y) - \tau_0)| : y \in [-L,L] \setminus (-a,a)\right\} = 0.$$

Since a is arbitrary, with a standard diagonal procedure we can find a sequence $a_n \to 0^+$ such that

$$\lim_{n\to+\infty} \max\left\{|v_n(y) - V(\mathrm{sign}(y) - \tau_0)| : y \in [-L, L] \setminus (-a_n, a_n)\right\} = 0. \tag{3.3.4}$$

Without loss of generality, we can always assume that $a_n \geq \varepsilon_n$. Therefore, if we set $L_n := a_n/\varepsilon_n^2$, then $L_n \to +\infty$ and (3.3.4) is equivalent to

$$\lim_{n\to+\infty} \max\left\{|w_n(y) - V(1 - \tau_0)| : y \in [L_n, L/\varepsilon_n^2]\right\} = \\ = \lim_{n\to+\infty} \max\left\{|w_n(y) + V(1 + \tau_0)| : y \in [-L/\varepsilon_n^2, -L_n]\right\} = 0,$$

so that in particular

$$w_n(-L_n) \to -V(1 + \tau_0) \qquad \text{and} \qquad w_n(L_n) \to V(1 - \tau_0). \tag{3.3.5}$$

Now let us choose a sequence $\{M_n\}$ of real numbers such that

$$M_n \to +\infty, \qquad M_n\varepsilon_n^2 \to 0, \qquad L_nM_n\varepsilon_n^2 \to 0. \tag{3.3.6}$$

We observe that the third condition implies the second one, and a possible choice for this sequence is $M_n := a_n^{-1/2}$. Following [2] (as we did in Lemma A.0.2), for every $n \in \mathbb{N}$ we consider the open set

$$A_n := \left\{y \in (-L_n, L_n) : |w_n'(y)| > \frac{1}{|\log \varepsilon_n|}\right\},$$

and we write it as a union of intervals of the form

$$A_n = \bigcup_{i\in I_n} (\alpha_{n.i}, \beta_{n,i}),$$

where I_n is a finite or countable set of indices. For every $n \in \mathbb{N}$ and every $i \in I_n$, we observe that $w_n'(y)$ has constant sign in $(\alpha_{n,i}, \beta_{n,i})$, and we set

$$\Delta_{n,i} := |w_n(\beta_{n,i}) - w_n(\alpha_{n,i})| = \int_{\alpha_{n,i}}^{\beta_{n,i}} |w_n'(y)|\, dy,$$

and

$$\widehat{\Delta}_{n,i} := \Delta_{n,i} - \frac{\beta_{n,i} - \alpha_{n,i}}{|\log \varepsilon_n|} = \int_{\alpha_{n,i}}^{\beta_{n,i}} \left(|w_n'(y)| - \frac{1}{|\log \varepsilon_n|}\right) dy \geq 0.$$

Identification of the "big jump" We claim that for every $n \in \mathbb{N}$ there exists an index $i(n) \in I_n$ such that

$$\lim_{n\to+\infty} \Delta_{n,i(n)} = \lim_{n\to+\infty} \int_{-L_n}^{L_n} |w_n'(y)|\, dy = 2V. \tag{3.3.7}$$

In words, this means that asymptotically the whole total variation of w_n is realized in a single special interval $(\alpha_{n,i(n)}, \beta_{n,i(n)})$. This is the key point of the proof, and requires seven steps.

- In the first step we show that

$$\limsup_{n\to+\infty} |A_n| < +\infty, \tag{3.3.8}$$

and in particular $|A_n|$ is bounded.

To this end, it is enough to observe that

$$\frac{1}{|\log \varepsilon_n|} \log\left(1 + \frac{1}{|\log \varepsilon_n|\varepsilon_n^4}\right) |A_n| \leq \mathbb{RPMV}_{\varepsilon_n}((-L_n, L_n), w_n),$$

so that from (3.3.2) and (3.3.3) we deduce that

$$\begin{aligned} 4\limsup_{n\to+\infty} |A_n| &\leq \limsup_{n\to+\infty} \mathbb{RPMV}_{\varepsilon_n}((-L_n, L_n), w_n) \\ &\leq \limsup_{n\to+\infty} \mathbb{RPMV}_{\varepsilon_n}((-L, L), v_n) \\ &= \frac{16}{\sqrt{3}}\sqrt{2V}, \end{aligned}$$

which implies (3.3.8).

As a consequence, up to choosing a slightly larger L_n (between the original L_n and $L_n + 4(2V)^{1/2}$), we can assume that none of the intervals $(\alpha_{n,i}, \beta_{n,i})$ lies at the boundary of $(-L_n, L_n)$, so that $|w_n'(\alpha_{n,i})| = |w_n'(\beta_{n,i})| = 1/|\log \varepsilon_n|$ for every $i \in I_n$, at least if n is large enough.

- In the second step we show that

$$\lim_{n\to+\infty} \int_{A_n^c} |w_n'(y)|\, dy = 0. \tag{3.3.9}$$

where A_n^c denote the complement set of A_n in $(-L_n, L_n)$.

To this end, we consider the set

$$B_n := \left\{ y \in A_n^c : |w_n'(y)| \leq M_n \varepsilon_n^2 \right\},$$

and we observe that

$$\limsup_{n\to+\infty} \frac{\log(1 + M_n^2)}{|\log \varepsilon_n|} |A_n^c \setminus B_n| \leq \limsup_{n\to+\infty} \mathbb{RPMV}_{\varepsilon_n}((-L_n, L_n), w_n) < +\infty.$$

Therefore we have that

$$\int_{A_n^c} |w_n'(y)|\, dy \leq M_n\varepsilon_n^2 \cdot |B_n| + \frac{|A_n^c \setminus B_n|}{|\log \varepsilon_n|} \leq 2L_n M_n \varepsilon_n^2 + \frac{|A_n^c \setminus B_n| \log(1 + M_n^2)}{|\log \varepsilon_n| \log(1 + M_n^2)}, \tag{3.3.10}$$

and the conclusion follows from the first and the third condition in (3.3.6).

- In the third step we show that

$$\liminf_{n\to+\infty}\sum_{i\in I_n}\Delta_{n,i}\geq 2V. \tag{3.3.11}$$

To this end, we observe that

$$|w_n(L_n)-w_n(-L_n)|\leq\int_{-L_n}^{L_n}|w_n'(y)|\,dy=\int_{A_n^c}|w_n'(y)|\,dy+\sum_{i\in I_n}\Delta_{n,i},$$

and we conclude by exploiting (3.3.9) and (3.3.5).

- In the fourth step we show that

$$\liminf_{n\to+\infty}\sum_{i\in I_n}\widehat{\Delta}_{n,i}\geq 2V. \tag{3.3.12}$$

Indeed, from the definition we obtain that

$$\begin{aligned}\sum_{i\in I_n}\widehat{\Delta}_{n,i} &= \sum_{i\in I_n}\left(\Delta_{n,i}-\frac{\beta_{n,i}-\alpha_{n,i}}{|\log\varepsilon_n|}\right)\\ &= \sum_{i\in I_n}\Delta_{n,i}-\frac{1}{|\log\varepsilon_n|}\sum_{i\in I_n}(\beta_{n,i}-\alpha_{n,i})\\ &= \sum_{i\in I_n}\Delta_{n,i}-\frac{|A_n|}{|\log\varepsilon_n|},\end{aligned} \tag{3.3.13}$$

and we conclude by exploiting (3.3.11) and (3.3.8).

- In the fifth step we show that

$$\limsup_{n\to+\infty}\sum_{i\in I_n}\left(\widehat{\Delta}_{n,i}\right)^{1/2}\leq\sqrt{2V}. \tag{3.3.14}$$

To this end, for every $i\in I_n$ we apply Lemma 3.5.5 in the interval $(\alpha_{n,i},\beta_{n,i})$ with $M=1/|\log\varepsilon_n|$, so we obtain that

$$\mathbb{RPMV}_{\varepsilon_n}((\alpha_{n,i},\beta_{n,i}),w_n)\geq 4\sqrt{\frac{2}{3}}\left(\frac{1}{|\log\varepsilon_n|}\log\left(1+\frac{1}{\omega(\varepsilon_n)^4}\right)\right)^{3/4}\left(\widehat{\Delta}_{n,i}\right)^{1/2},$$

and therefore

$$\mathbb{RPMV}_{\varepsilon_n}((-L_n,L_n),w_n)\geq 4\sqrt{\frac{2}{3}}\left(\frac{1}{|\log\varepsilon_n|}\log\left(1+\frac{1}{\omega(\varepsilon_n)^4}\right)\right)^{3/4}\sum_{i\in I_n}\left(\widehat{\Delta}_{n,i}\right)^{1/2}.$$

Letting $n\to+\infty$, from (3.3.2) and (3.3.3) we conclude that

$$\frac{16}{\sqrt{3}}\sqrt{2V}=\alpha_0\sqrt{2V}\geq\frac{16}{\sqrt{3}}\limsup_{n\to+\infty}\sum_{i\in I_n}\left(\widehat{\Delta}_{n,i}\right)^{1/2},$$

which is equivalent to (3.3.14).

- In the sixth step we show that

$$\lim_{n\to+\infty}\sum_{i\in I_n}\Delta_{n,i}=\lim_{n\to+\infty}\sum_{i\in I_n}\widehat{\Delta}_{n,i}=2V \tag{3.3.15}$$

and

$$\lim_{n\to+\infty}\sum_{i\in I_n}\left(\widehat{\Delta}_{n,i}\right)^{1/2}=\sqrt{2V}. \tag{3.3.16}$$

To this end, we combine (3.3.12), (3.3.14), and the subadditivity of the square root, and we obtain that

$$\begin{aligned}\sqrt{2V}\geq\limsup_{n\to+\infty}\sum_{i\in I_n}\left(\widehat{\Delta}_{n,i}\right)^{1/2}&\geq\limsup_{n\to+\infty}\left(\sum_{i\in I_n}\widehat{\Delta}_{n,i}\right)^{1/2}\\&\geq\liminf_{n\to+\infty}\left(\sum_{i\in I_n}\widehat{\Delta}_{n,i}\right)^{1/2}=\left(\liminf_{n\to+\infty}\sum_{i\in I_n}\widehat{\Delta}_{n,i}\right)^{1/2}=\sqrt{2V}.\end{aligned}$$

This means that all inequalities are actually equalities, which proves (3.3.16) and the second equality in (3.3.15). At this point, the first inequality in (3.3.15) follows again by letting $n\to+\infty$ in (3.3.13).

- In the last step we apply Lemma 3.5.2 to the index set I_n and the function $f(i):=\widehat{\Delta}_{n,i}$. We deduce that

$$0\leq\left(\sum_{i\in I_n}\widehat{\Delta}_{n,i}-\max_{i\in I_n}\widehat{\Delta}_{n,i}\right)^{1/2}\leq 3\sum_{i\in I_n}\left(\widehat{\Delta}_{n,i}\right)^{1/2}-3\left(\sum_{i\in I_n}\widehat{\Delta}_{n,i}\right)^{1/2}$$

The right-hand side tends to 0 because of (3.3.16) and the second equality in (3.3.15). This is enough to establish the existence of $i(n)\in I_n$ such that

$$\lim_{n\to+\infty}\widehat{\Delta}_{n,i(n)}=2V.$$

Now we know that

$$2V=\lim_{n\to+\infty}\sum_{i\in I_n}\Delta_{n,i}\geq\lim_{n\to+\infty}\Delta_{n,i(n)}\geq\lim_{n\to+\infty}\widehat{\Delta}_{n,i(n)}=2V,$$

and therefore all inequalities are actually equalities. This proves the first part of (3.3.7). The second part follows from the equality

$$\int_{-L_n}^{L_n}|w_n'(y)|\,dy=\Delta_{n,i(n)}+\sum_{i\in I_n\setminus\{i(n)\}}\Delta_{n,i}+\int_{A_n^c}|w_n'(y)|\,dy,$$

since now we know that the last two terms tend to 0.

Uniform estimates in the special interval Let us set for simplicity $\alpha_n := \alpha_{n,i(n)}$ and $\beta_n := \beta_{n,i(n)}$. We claim that

- the sequence $w_n(y)$ tends to $-V(1+\tau_0)$ uniformly in $[-L_n, \alpha_n]$ and to $V(1-\tau_0)$ uniformly in $[\beta_n, L_n]$, in the sense that

$$\lim_{n\to+\infty} \left(\max_{y\in[-L_n,\alpha_n]} |w_n(y) + V(1+\tau_0)| + \max_{y\in[\beta_n,L_n]} |w_n(y) - V(1-\tau_0)| \right) = 0, \tag{3.3.17}$$

  and in particular

$$w_n(\alpha_n) \to -V(1+\tau_0), \qquad w_n(\beta_n) \to V(1-\tau_0), \tag{3.3.18}$$

  while we already know that

$$|w_n'(\alpha_n)| = |w_n'(\beta_n)| = \frac{1}{|\log \varepsilon_n|} \to 0. \tag{3.3.19}$$

- when n is large enough it turns out that

$$-|A_n| < \alpha_n < 0 < \beta_n < |A_n|,$$

  and in particular the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded,

- for every bounded interval $(a,b) \subseteq \mathbb{R}$ it turns out that

$$\{w_n\} \text{ is bounded in } H^2((a,b)), \tag{3.3.20}$$

  of course with a bound that depends on the interval.

Let us prove these claims. To begin with, from (3.3.7) we know that

$$\lim_{n\to+\infty} \int_{-L_n}^{\alpha_n} |w_n'(y)|\, dy + \int_{\beta_n}^{L_n} |w_n'(y)|\, dy = 0.$$

Keeping (3.3.5) into account, this is enough to establish (3.3.17). In particular, from this uniform convergence we deduce that

$$w_n(y) \le -\frac{V(1+\tau_0)}{2} < 0 \qquad \forall y \in [-L_n, \alpha_n]$$

when n is large enough. Since $w_n(0) = 0$ for every $n \in \mathbb{N}$, we conclude that $\alpha_n < 0$, and also

$$\beta_n = \alpha_n + (\beta_n - \alpha_n) \le \alpha_n + |A_n| < |A_n|.$$

In a symmetric way we obtain that $\beta_n > 0$ and $\alpha_n > -|A_n|$.

Finally, the bound on $\mathbb{RPMV}_{\varepsilon_n}((-L_n, L_n), w_n)$ yields immediately a uniform bound on the norm of w_n'' in $L^2((-L_n, L_n))$. Together with the pointwise bounds coming from (3.3.18) and (3.3.19), this is enough to obtain a uniform bound on w_n in $H^2((a,b))$ for every bounded interval $(a,b) \subseteq \mathbb{R}$.

Passing to the limit We are now ready to prove our convergence results. Since the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded, up to subsequences (not relabeled) we can assume that $\alpha_n \to \alpha_\infty$ and $\beta_n \to \beta_\infty$. Moreover, if we fix an interval $(a,b) \supseteq (\alpha_\infty, \beta_\infty)$, then from (3.3.20) we can also assume that there exists $w_\infty \in H^2((a,b))$ such that

$$w_n \to w_\infty \quad \text{and} \quad w_n' \to w_\infty' \quad \text{uniformly in } [a,b],$$

and

$$w_n'' \rightharpoonup w_\infty'' \quad \text{weakly in } L^2((a,b)).$$

We claim that w_∞ is the cubic connection. To prove this, we observe that since $w_n(0) = 0$ we have that $w_\infty(0) = 0$, while from (3.3.18) and (3.3.19) we obtain that

$$w_\infty(\alpha_\infty) = -V(1+\tau_0), \quad w_\infty(\beta_\infty) = V(1-\tau_0), \quad w_\infty'(\alpha_\infty) = w_\infty'(\beta_\infty) = 0. \tag{3.3.21}$$

Therefore, from Lemma 2.6.2 we deduce that

$$\int_{\alpha_\infty}^{\beta_\infty} w_\infty''(y)^2\,dy \geq \frac{12(2V)^2}{(\beta_\infty - \alpha_\infty)^3}.$$

Now we consider the chain of inequalities

$$\begin{aligned}
\frac{16}{\sqrt{3}}\sqrt{2V} &\geq \limsup_{n\to+\infty} \mathbb{RPMV}_{\varepsilon_n}((-L_n, L_n), w_n) \\
&\geq \limsup_{n\to+\infty} \left\{ \int_{\alpha_n}^{\beta_n} \frac{1}{|\log\varepsilon_n|} \log\left(1 + \frac{w_n'(y)^2}{\varepsilon_n^4}\right) dy + \int_a^b w_n''(y)^2\,dy \right\} \\
&\geq \limsup_{n\to+\infty} \left\{ \frac{1}{|\log\varepsilon_n|} \log\left(\frac{1}{\varepsilon_n^4 |\log\varepsilon_n|^2}\right)(\beta_n - \alpha_n) + \int_a^b w_n''(y)^2\,dy \right\} \\
&= 4(\beta_\infty - \alpha_\infty) + \limsup_{n\to+\infty} \int_a^b w_n''(y)^2\,dy \\
&\geq 4(\beta_\infty - \alpha_\infty) + \int_a^b w_\infty''(y)^2\,dy \qquad (3.3.22) \\
&\geq 4(\beta_\infty - \alpha_\infty) + \int_{\alpha_\infty}^{\beta_\infty} w_\infty''(y)^2\,dy \qquad (3.3.23) \\
&\geq 4(\beta_\infty - \alpha_\infty) + 12\frac{(2V)^2}{(\beta_\infty - \alpha_\infty)^3} \qquad (3.3.24) \\
&\geq \frac{16}{\sqrt{3}}\sqrt{2V}. \qquad (3.3.25)
\end{aligned}$$

Since the first and last term coincide, all inequalities are actually equalities, and each of them gives us some piece of information.

- The equality in (3.3.25) implies that $\beta_\infty - \alpha_\infty = 2\Lambda$, where Λ is defined by (3.2.4).
- The equality in (3.3.24) implies that w_∞ is the unique polynomial of degree three that interpolates the boundary data (3.3.21). Combining with the previous point and the fact that $w_\infty(0) = 0$, we conclude that w_∞ coincides with C_{Λ,V,τ_0} in the interval $(\alpha_\infty, \beta_\infty) = (-\Lambda - x_0, \Lambda - x_0)$, where x_0 is as in Definition 3.2.1.

- The equality in (3.3.23) implies that $w''_\infty(y)$ vanishes outside $(\alpha_\infty, \beta_\infty)$. Since $w'_\infty(\alpha_\infty) = w'_\infty(\beta_\infty) = 0$, this is enough to conclude that w_∞ is constant both in (a, α_∞) and in (β_∞, b).
- The equality in (3.3.22) implies that actually $w''_n \to w''_\infty$ strongly in $L^2((a,b))$.

Finally, we observe that the previous steps characterize in a unique way the possible limits of subsequences, and this is enough to conclude the convergence of the whole sequence.

□

3.4 Horizontal parts (Theorem 3.2.2, statement (2))

Let $[a,b] \subseteq ((-1+\tau_0)H, (1+\tau_0)H)$ be a fixed interval. In the first part of the proof we show that

$$\limsup_{n\to+\infty} \frac{1}{\omega(\varepsilon_n)^2} \max\{|v_n(y)| : y \in [a,b]\} < +\infty, \tag{3.4.1}$$

and then in the second part we prove (3.2.5)

3.4.1 Estimate on the functions

Let us set

$$g_n(y) := \frac{f(x_n + \omega(\varepsilon_n)y) - u_n(x_n)}{\omega(\varepsilon_n)} = \frac{f(x_n + \omega(\varepsilon_n)y) - f(x_n)}{\omega(\varepsilon_n)} - \frac{u_n(x_n) - f(x_n)}{\omega(\varepsilon_n)}.$$

We observe that the functions g_n are uniformly bounded on bounded sets because

$$\frac{f(x_n + \omega(\varepsilon_n)y) - f(x_n)}{\omega(\varepsilon_n)} \to f'(x_0)y$$

uniformly on bounded sets, and

$$\frac{u_n(x_n) - f(x_n)}{\omega(\varepsilon_n)}$$

is equal to the value in 0 of the fake blow-up of u_n with center in x_n, which is defined (2.2.8), hence is uniformly bounded because of Theorem 2.2.9.

We also recall that

$$\mathbb{PMF}_{\varepsilon_n}(\beta, f, (0,1), u_n) = \omega(\varepsilon_n)^2\, \mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (-x_n/\omega(\varepsilon_n)), (1-x_n)/\omega(\varepsilon_n)), v_n),$$

and hence (3.2.3) implies that

$$v_n \in \operatorname{argmin}_{\mathrm{loc}}\{\mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (a,b), v) : v \in H^2((a,b))\},$$

for every bounded interval $(a,b) \subset \mathbb{R}$.

Now the proof of (3.4.1) consists of three main steps.

- (Isolation of horizontal and vertical parts). In the first step we define three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ such that

$$a_n \to (-1+\tau_0)H, \qquad v_n(a_n) \to 0, \qquad |v_n'(a_n)| \leq 1, \tag{3.4.2}$$
$$b_n \to (1+\tau_0)H, \qquad v_n(b_n) \to 0, \qquad |v_n'(b_n)| \leq 1, \tag{3.4.3}$$
$$c_n \to (1+\tau_0)H, \qquad v_n(c_n) \to 2V, \qquad |v_n'(c_n)| \leq 1, \tag{3.4.4}$$

and

$$|c_n - b_n| = O(\omega(\varepsilon_n)^2). \tag{3.4.5}$$

Moreover, we set

$$m_n := \max\{|v_n(y)| : y \in [a_n, b_n]\},$$

and we prove that $m_n \to 0$ as $n \to +\infty$.

- (Estimate from below). In the second step we prove that there exists a constant $c_0 > 0$ such that

$$\mathbb{RPM}_{\varepsilon_n}((a_n, b_n), v_n) \geq c_0 \min\left\{\sqrt{m_n}, \frac{m_n^2}{\omega(\varepsilon_n)^2}\right\}. \tag{3.4.6}$$

- (Estimate from above). In the third step we prove that

$$\begin{aligned}\mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (a_n, c_n), v_n) \ \leq\ & \mathbb{RPM}_{\varepsilon_n}((b_n, c_n), v_n) + \beta \int_{a_n}^{b_n} g_n(y)^2\, dy \\ & + O(m_n) + O(\omega(\varepsilon_n)^2),\end{aligned} \tag{3.4.7}$$

Given the three steps, we can conclude as follows. We observe that

$$\int_{a_n}^{b_n} (v_n(y) - g_n(y))^2\, dy = \int_{a_n}^{b_n} g_n(y)^2\, dy + O(m_n)$$

because $|v_n(y)| \leq m_n$ in (a_n, b_n), and that

$$\int_{b_n}^{c_n} (v_n(y) - g_n(y))^2\, dy = O(\omega(\varepsilon_n)^2)$$

because of (3.4.5) and the uniform boundedness of $g_n(y)$ and $v_n(y)$ on every bounded set, and in particular in (b_n, c_n).

As a consequence, by combining (3.4.7) and (3.4.6) we obtain that

$$\begin{aligned}
&\mathbb{RPM}_{\varepsilon_n}((b_n,c_n),v_n)+\beta\int_{a_n}^{b_n} g_n(y)^2\,dy+O(m_n)+O(\omega(\varepsilon_n)^2)\\
&\quad\geq \mathbb{RPMF}_{\varepsilon_n}(\beta,g_n,(a_n,c_n),v_n)\\
&\quad= \mathbb{RPM}_{\varepsilon_n}((a_n,b_n),v_n)+\mathbb{RPM}_{\varepsilon_n}((b_n,c_n),v_n)\\
&\qquad+\beta\int_{a_n}^{b_n}(v_n(y)-g_n(y))^2\,dy+\beta\int_{b_n}^{c_n}(v_n(y)-g_n(y))^2\,dy\\
&\quad\geq \Gamma_0\min\left\{\sqrt{m_n},\frac{m_n^2}{\omega(\varepsilon_n)^2}\right\}+\mathbb{RPM}_{\varepsilon_n}((b_n,c_n),v_n)\\
&\qquad+\beta\int_{a_n}^{b_n} g_n(y)^2\,dy+O(m_n)+O(\omega(\varepsilon_n)^2),
\end{aligned}$$

from which we deduce that

$$c_0\min\left\{\sqrt{m_n},\frac{m_n^2}{\omega(\varepsilon_n)^2}\right\}\leq O(m_n)+O(\omega(\varepsilon_n)^2). \tag{3.4.8}$$

This inequality implies that $m_n = O(\omega(\varepsilon_n)^2)$, which in turn implies (3.4.1). Indeed, if this is not the case, then there exists a subsequence (not relabeled) such that $m_n/\omega(\varepsilon_n)^2 \to +\infty$. Since eventually $m_n \neq 0$ along this subsequence, we can divide (3.4.8) by m_n and obtain that

$$\Gamma_0\min\left\{\frac{1}{\sqrt{m_n}},\frac{m_n}{\omega(\varepsilon_n)^2}\right\}\leq \frac{O(m_n)}{m_n}+\frac{O(\omega(\varepsilon_n)^2)}{\omega(\varepsilon_n)^2}\cdot\frac{\omega(\varepsilon_n)^2}{m_n},$$

which is absurd because the left-hand side tends to $+\infty$ when $n \to +\infty$, while the right-hand side remains bounded.

Isolation of horizontal and vertical parts Let us start by constructing b_n and c_n. The idea is that the limit of $v_n(y)$ is a piecewise constant function that jumps from 0 to $2V$ in the point $H(1+\tau_0)$, and we know from Statement (1) of Theorem 3.2.2 that this jump is entirely achieved in an interval whose length is $O(\varepsilon_n^2)$. In addition, we need a uniform bound on v_n' at the endpoints of this short interval, for which we exploit a bound on the energies to deduce that on every interval whose length is $O(\omega(\varepsilon_n)^2)$ there exists at least a point in which v_n' is bounded.

In order to pursue this path, we begin by observing that $v_n(0)=0$ for every $n\in\mathbb{N}$, and $v_n(2H)\to 2V$. Therefore, there exists $z_n\in(0,2H)$ such that $v_n(z_n)=V$. From the strict convergence we deduce that $z_n\to H(1+\tau_0)$, because the latter is the unique point in $(0,2H)$ where the limit of v_n can be different from an integer multiple of $2V$.

Now we set $y_n := x_n+\omega(\varepsilon_n)z_n$, and we observe that

$$\frac{u_n(y_n+\omega(\varepsilon_n)y)-u_n(y_n)}{\omega(\varepsilon_n)} = v_n(y+z_n)-v_n(z_n) \rightsquigarrow S_{H,V}(y+H)-V.$$

Since the limit function is a graph translation of vertical type of the canonical (H, V)-staircase, from the result for the vertical parts of the steps we know that

$$w_n(y) := \frac{u_n(y_n + \omega(\varepsilon_n)\varepsilon_n^2 y) - u_n(y_n)}{\omega(\varepsilon_n)} \to C_{\Lambda,V}(y),$$

where $C_{\Lambda,V}(y)$ is the canonical (Λ, V)-cubic connection introduced in Definition 3.2.1 with Λ given by (3.2.4), and the convergence is uniform in the sense that

$$\lim_{n\to+\infty} \max\left\{|w_n(y) - C_{\Lambda,V}(y)| : |y| \leq L/\varepsilon_n^2\right\} = 0 \tag{3.4.9}$$

for every $L \in (0, 2H)$.

Now we recall that

$$\mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (-2H, 2H), v_n) \leq E_0, \tag{3.4.10}$$

for some positive constant E_0, because of Proposition 2.6.5. Let us consider the interval

$$I_n := \left[z_n - \Lambda\varepsilon_n^2 - \frac{E_0}{\log 2}\omega(\varepsilon_n)^2, z_n - \Lambda\varepsilon_n^2\right].$$

Since $|I_n| = (E_0/\log 2)\omega(\varepsilon_n)^2$, we deduce that

$$\begin{aligned} E_0 &\geq \mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, I_n, v_n) \\ &\geq \frac{1}{\omega(\varepsilon_n)^2}\int_{I_n} \log(1 + v_n'(y)^2)\, dy \\ &\geq \frac{E_0}{\log 2} \min\left\{\log(1 + v_n'(y)^2) : y \in I_n\right\}. \end{aligned}$$

This implies the existence of $b_n \in I_n$ such that $\log(1 + v_n'(b_n)^2) \leq \log 2$, and therefore $|v_n'(b_n)| \leq 1$. Moreover, $b_n \to H(1 + \tau_0)$ because both endpoints of I_n tend to the same limit of z_n. Finally, we show that $v_n(p_n) \to 0$ for every sequence of points $\{p_n\}$ such that $p_n \in [0, b_n]$, so in particular $v_n(b_n) \to 0$. To this end, we write p_n in the form $z_n + \varepsilon_n^2\beta_n$ for some $\beta_n \in [-z_n/\varepsilon_n^2, -\Lambda]$, and we observe that

$$v_n(p_n) = w_n(\beta_n) + v_n(z_n) = w_n(\beta_n) + V,$$

and we conclude by exploiting (3.4.9) and by remarking that $C_{\Lambda,V}(y) = -V$ when $y \leq -\Lambda$.

In the same way we construct the points c_n satisfying (3.4.4) starting from the interval

$$I_n := \left[z_n + \Lambda\varepsilon_n^2, z_n + \Lambda\varepsilon_n^2 + \frac{E_0}{\log 2}\omega(\varepsilon_n)^2\right].$$

Of course in this case we exploit that c_n is of the form $y_n + \varepsilon_n^2\gamma_n$ with $\gamma_n \geq \Lambda$, so that $w_n(\gamma_n) \to V$.

Finally, we construct a_n in the same way of b_n, but starting with the point $z_n' \in (-2H, 0)$ such that $v_n(z_n') \to -V$ which implies that $z_n' \to (-1 + \tau_0)H$. As before, we can show that $|v_n'(a_n)| \leq 1$ and that $v_n(p_n) \to 0$ for every sequence of points $\{p_n\}$ such that $p_n \in [a_n, 0]$, and in particular $v_n(a_n) \to 0$.

We conclude the proof of the first step by observing that (3.4.5) is an immediate consequence of the definition of b_n and c_n, which implies that

$$0 \leq c_n - b_n \leq \frac{2E_0}{\log 2}\omega(\varepsilon_n)^2,$$

and that $m_n \to 0$ because we have proved that $v_n(p_n) \to 0$ for every sequence $\{p_n\}$ such that $p_n \in [a_n, b_n]$.

Estimate from below We set $M_n := \min\{\sqrt{m_n}, m_n^2/\omega(\varepsilon_n)^2\}$ and we prove that

$$\liminf_{n\to+\infty} \frac{1}{M_n} \mathbb{RPM}_{\varepsilon_n}((a_n, b_n), v_n) > 0, \tag{3.4.11}$$

which in turn implies (3.4.6). To this end, for every $n \in \mathbb{N}$ we define $\widehat{v}_n(y)$ in such a way that $v_n(y) = m_n\widehat{v}_n(y)$, and we observe that

$$\mathbb{RPM}_{\varepsilon_n}((a_n, b_n), v_n) = \int_{a_n}^{b_n} \left\{\varepsilon_n^6 m_n^2 \widehat{v}_n''(y)^2 + \frac{1}{\omega(\varepsilon_n)^2} \log\left(1 + m_n^2 \widehat{v}_n'(y)^2\right)\right\} dy, \tag{3.4.12}$$

and that

$$\widehat{v}_n(0) = 0 \qquad \text{and} \qquad \max\{|\widehat{v}_n(y)| : y \in [a_n, b_n]\} = 1.$$

Now we observe that the sequence of intervals (a_n, b_n) and the sequence of functions $\widehat{v}_n \in H^2((a_n, b_n))$ fit into the framework of Lemma 3.5.6 with $L := 2H$ and $J := 1$, and we claim that the liminf in (3.4.11) is greater than or equal to the constant $c(2H, 1)$ provided by Lemma 3.5.6. It is enough to prove this along the two subsequences where M_n takes either of the values in the minimum, and therefore we distinguish two cases (without relabeling subsequences).

- If $M_n = m_n^2/\omega(\varepsilon_n)^2$, namely $\sqrt{m_n} \geq m_n^2/\omega(\varepsilon_n)^2$, then we have that $\omega(\varepsilon_n)^8 \geq m_n^6$. Therefore, at least if n is large enough, we deduce that

  $$|\log m_n| \geq \frac{4}{3}|\log \omega(\varepsilon_n)| = \frac{4}{3}\left(|\log \varepsilon_n| - \frac{1}{2}\log|\log \varepsilon_n|\right) \geq |\log \varepsilon_n|,$$

  which implies that $\varepsilon_n^6\omega(\varepsilon_n)^2 \geq m_n^6/|\log m_n|^3$. As a consequence, from (3.4.12) we obtain that

  $$\frac{1}{M_n} \mathbb{RPM}_{\varepsilon_n}(a_n, b_n), v_n) \geq \mathbb{RPMH}_{m_n}((a_n, b_n), \widehat{v}_n),$$

  where the functional $\mathbb{RPMH}$ is defined according to (3.5.6).

  Since $m_n \to 0$ and we know that $|\widehat{v}_n'(a_n)| \leq 1/m_n$ and $|\widehat{v}_n'(b_n)| \leq 1/m_n$, the result follows from Lemma 3.5.6.

- If $M_n = \sqrt{m_n}$, namely $\sqrt{m_n} \leq m_n^2/\omega(\varepsilon_n)^2$, then we have that $m_n^2 \geq \omega(\varepsilon_n)^2\sqrt{m_n}$. Thus, from (3.4.12) we obtain that

$$\frac{1}{M_n}\,\mathbb{RPM}_{\varepsilon_n}((a_n,b_n),v_n) \geq \min\left\{1, \frac{|\log(\omega(\varepsilon_n)m_n^{1/4})|^3}{|\log\varepsilon_n|^3}\right\}\mathbb{RPMH}_{\omega(\varepsilon_n)m_n^{1/4}}((a_n,b_n),\widehat{v}_n),$$

and the result follows again from Lemma 3.5.6 because $\omega(\varepsilon_n)m_n^{1/4} \to 0$, the boundary values satisfy

$$|\widehat{v}_n'(a_n)| \leq \frac{1}{m_n} \leq \frac{1}{\omega(\varepsilon_n)m_n^{1/4}} \qquad \text{and} \qquad |\widehat{v}_n'(b_n)| \leq \frac{1}{m_n} \leq \frac{1}{\omega(\varepsilon_n)m_n^{1/4}},$$

and it holds that

$$\liminf_{n\to+\infty}\frac{|\log(\omega(\varepsilon_n)m_n^{1/4})|^3}{|\log\varepsilon_n|^3} \geq \liminf_{n\to+\infty}\frac{|\log(\omega(\varepsilon_n))|^3}{|\log\varepsilon_n|^3} = 1.$$

Estimate from above In this paragraph we prove (3.4.7). Since

$$v_n \in \operatorname{argmin}_{\text{loc}}\left\{\mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (a_n,c_n), v) : v \in H^2((a_n,c_n))\right\},$$

it is enough to exhibit a function $\widetilde{v}_n \in H^2((a_n,c_n))$, with the same boundary conditions (both on function and on the derivative) of v_n, such that $\mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (a_n,c_n), \widetilde{v}_n)$ is bounded from above by the right-hand side of (3.4.7). One would like to choose $\widetilde{v}_n$ as the function identically equal to $v_n(a_n)$ in the horizontal part (a_n,b_n), and equal to a suitable homothety of v_n in the vertical part (b_n,c_n). This choice, however, does not fit the boundary conditions and is not of class H^2. Therefore, we have to smooth out the connections. To this end, we observe that

$$a_n < a_n + \varepsilon_n^3\omega(\varepsilon_n) < b_n - \varepsilon_n^3\omega(\varepsilon_n) < b_n < c_n$$

when n is large enough, so that we can partition $[a_n,c_n]$ into four intervals

$$I_{1,n} := [a_n, a_n+\varepsilon_n^3\omega(\varepsilon_n)], \qquad I_{2,n} := [a_n+\varepsilon_n^3\omega(\varepsilon_n), b_n-\varepsilon_n^3\omega(\varepsilon_n)],$$
$$I_{3,n} := [b_n-\varepsilon_n^3\omega(\varepsilon_n), b_n], \qquad I_{4,n} := [b_n,c_n].$$

Then we consider the constant

$$B_n := \frac{v_n(b_n)-v_n(a_n)}{v_n(c_n)-v_n(b_n)-(c_n-b_n)v_n'(c_n)},$$

and we observe that $B_n = O(m_n)$ because $|v_n(b_n)-v_n(a_n)| \leq 2m_n$ and the denominator tends to $2V$. At this point we define $\widetilde{v}_n$ in a piecewise way as follows.

- In $I_{1,n}$ we define $\widetilde{v}_n(x)$ as the cubic polynomial with boundary conditions

$$\widetilde{v}_n(a_n) = \widetilde{v}_n(a_n+\varepsilon_n^3\omega(\varepsilon_n)) = v_n(a_n), \quad \widetilde{v}_n'(a_n) = v_n'(a_n), \quad \widetilde{v}_n'(a_n+\varepsilon_n^3\omega(\varepsilon_n)) = 0.$$

- In $I_{2,n}$ we define $\widetilde{v}_n(x) \equiv v_n(a_n)$.
- In $I_{3,n}$ we define $\widetilde{v}_n(x)$ as the cubic polynomial such that

$$\widetilde{v}_n(b_n - \varepsilon_n^3\omega(\varepsilon_n)) = \widetilde{v}_n(b_n) = v_n(a_n), \qquad \widetilde{v}_n'(b_n - \varepsilon_n^3\omega(\varepsilon_n)) = 0,$$

  and

$$\widetilde{v}_n'(b_n) = v_n'(b_n) + B_n(v_n'(b_n) - v_n'(c_n)).$$

- In $I_{4,n}$ we set

$$\widetilde{v}_n(y) := v_n(y) + B_n\left(v_n(y) + (c_n - y)v_n'(c_n) - v_n(c_n)\right).$$

This definition guarantees that the connections in the intermediate points are of class C^1, so $\widetilde{v}_n \in H^2((a_n, c_n))$, and that the values of $\widetilde{v}_n(y)$ and $\widetilde{v}_n'(y)$ for $y = a_n$ and $y = c_n$ coincide with the corresponding values of $v_n(y)$ and $v_n'(y)$. We claim that the fidelity term satisfies

$$\int_{a_n}^{c_n} |\widetilde{v}_n(y) - g_n(y)|^2\, dy = \int_{a_n}^{b_n} |g_n(y)|^2\, dy + O(\omega(\varepsilon_n)^2) + O(m_n). \tag{3.4.13}$$

As for the terms of $\mathbb{RPM}_{\varepsilon_n}$ of course in $I_{2,n}$ it holds that $\mathbb{RPM}_{\varepsilon_n}(I_{2,n}, \widetilde{v}_n) = 0$. Moreover, in $I_{1,n}$ and $I_{3,n}$ it turns out that

$$\mathbb{RPM}_{\varepsilon_n}(I_{1,n}, \widetilde{v}_n) = O(\varepsilon_n^3/\omega(\varepsilon_n)) = O(\omega(\varepsilon_n)^2), \tag{3.4.14}$$

and

$$\mathbb{RPM}_{\varepsilon_n}(I_{3,n}, \widetilde{v}_n) = O(\varepsilon_n^3/\omega(\varepsilon_n)) = O(\omega(\varepsilon_n)^2), \tag{3.4.15}$$

while in $I_{4,n}$ it turns out that

$$\int_{b_n}^{c_n} \varepsilon_n^6 \widetilde{v}_n''(y)^2\, dy = \int_{b_n}^{c_n} \varepsilon_n^6 v_n''(y)^2\, dy + O(m_n) \tag{3.4.16}$$

and

$$\int_{b_n}^{c_n} \frac{1}{\omega(\varepsilon_n)^2} \log\left(1 + \widetilde{v}_n'(y)^2\right) = \int_{b_n}^{c_n} \frac{1}{\omega(\varepsilon_n)^2} \log\left(1 + v_n'(y)^2\right) + O(m_n) \tag{3.4.17}$$

These claims, if true, are enough to establish (3.4.7).

Let us start with (3.4.13). We recall that the sequence $\{g_n\}$ is uniformly bounded on bounded sets, hence it is uniformly bounded in $[a_n, c_n]$.

On the other hand, also the sequence $\{\widetilde{v}_n(y)\}$ is uniformly bounded in $[a_n, c_n]$. This is trivial in $I_{2,n}$, while in $I_{1,n}$ and in $I_{3,n}$ it follows from Lemma 2.6.2 and the uniform boundedness of the boundary values, and in $I_{4,n}$ it follows from the uniform bounds on v_n. We point out that in this point it is essential to have an estimate on $v_n'(b_n)$ and

$v_n'(c_n)$. Since the measure of $[a_n, c_n] \setminus I_{2,n}$ is $O(\omega(\varepsilon_n)^2)$ and $|\widetilde{v}_n(y)| = |v_n(a_n)| \leq m_n$ in $I_{2,n}$, it follows that

$$\begin{aligned}\int_{a_n}^{c_n} |\widetilde{v}_n(y) - g_n(y)|^2\, dy &= \int_{I_{2,n}} |\widetilde{v}_n(y) - g_n(y)|^2\, dy + O(\omega(\varepsilon_n)^2) \\ &= \int_{I_{2,n}} |g_n(y)|^2\, dy + O(m_n) + O(\omega(\varepsilon_n)^2) \\ &= \int_{a_n}^{b_n} |g_n(y)|^2\, dy + O(m_n) + O(\omega(\varepsilon_n)^2),\end{aligned}$$

that is exactly (3.4.13).

In order to estimate the terms with second order derivatives, in the intervals $I_{1,n}$ and $I_{3,n}$ we apply again Lemma 2.6.2, and we obtain that

$$\int_{I_{1,n}} \varepsilon_n^6 \widetilde{v}_n''(y)^2\, dy = 4 v_n'(a_n)^2 \frac{\varepsilon_n^3}{\omega(\varepsilon_n)},$$

and

$$\int_{I_{3,n}} \varepsilon_n^6 \widetilde{v}_n''(y)^2\, dy = 4 \left[v_n'(b_n) + B_n(v_n'(b_n) - v_n'(c_n))\right]^2 \frac{\varepsilon_n^3}{\omega(\varepsilon_n)},$$

while for first order derivatives it turns out that

$$\int_{I_{1,n}} \frac{1}{\omega(\varepsilon_n)^2} \log(1 + \widetilde{v}_n'(y)^2)\, dy \leq \frac{1}{\omega(\varepsilon_n)^2} \log\left(1 + \frac{9}{4} v'(a_n)^2\right) |I_{1,n}| = O(\varepsilon_n^3/\omega(\varepsilon_n)),$$

and similarly in $I_{3,n}$. This is enough to establish (3.4.14) and (3.4.15). It remains to consider $I_{4,n} = [b_n, c_n]$, where

$$\widetilde{v}_n'(y) = (1 + B_n) v_n'(y) - B_n v_n'(c_n) \qquad \text{and} \qquad \widetilde{v}_n''(y) = (1 + B_n) v_n''(y).$$

In particular it turns out that

$$\int_{b_n}^{c_n} \varepsilon_n^6 \widetilde{v}_n''(y)^2\, dy = \int_{b_n}^{c_n} \varepsilon_n^6 v_n''(y)^2\, dy + (2B_n + B_n^2) \int_{b_n}^{c_n} \varepsilon_n^6 v_n''(y)^2\, dy,$$

and this implies (3.4.16) because $B_n = O(m_n)$ and the last integral is bounded by E_0 thanks to (3.4.10).

Finally, since $|B_n| \leq 1/2$ if n is large enough, from Lemma 3.5.1 applied with

$$x := v_n'(y), \qquad b := B_n, \qquad d := -B_n v_n'(c_n)$$

we obtain that

$$\int_{b_n}^{c_n} \frac{1}{\omega(\varepsilon_n)^2} \log(1 + \widetilde{v}_n'(y)^2) = \int_{b_n}^{c_n} \frac{1}{\omega(\varepsilon_n)^2} \log(1 + v_n'(y)^2) + R_n,$$

where

$$|R_n| \leq \frac{5}{\omega(\varepsilon_n)^2} \cdot |B_n| \cdot (c_n - b_n).$$

Recalling that $B_n = O(m_n)$ and $c_n - b_n = O(\omega(\varepsilon_n)^2)$, we obtain (3.4.17).

3.4.2 Estimate on the derivatives

In this part of the proof we prove (3.2.5).

Let $[a,b] \subseteq ((-1+\tau_0)H,(1+\tau_0)H)$ be any interval. Let us choose a' and b' with

$$(-1+\tau_0)H < a' < a < b < b' < (1+\tau_0)H.$$

Let us define $w_n(y)$ in such a way that $v_n(y) = \omega(\varepsilon_n)^2 w_n(y)$. From the results of the previous section we know that there exists a constant Γ_0 such that

$$|w_n(y)| \leq \Gamma_0 \qquad \forall y \in [a',b'], \quad \forall n \in \mathbb{N}. \tag{3.4.18}$$

In addition, from the mean value theorem applied in the intervals $[a',a]$ and $[b,b']$, we deduce that there exist $a_n \in (a',a)$ and $b_n \in (b,b')$ such that

$$|w_n'(a_n)| \leq \frac{2\Gamma_0}{a-a'} \qquad \text{and} \qquad |w_n'(b_n)| \leq \frac{2\Gamma_0}{b'-b}. \tag{3.4.19}$$

In the sequel we consider the interval (a_n,b_n), and our claim becomes that

$$\limsup_{n\to+\infty} \max\{|w_n'(y)| : y \in (a_n,b_n)\} < +\infty. \tag{3.4.20}$$

We observe that

$$\begin{aligned}\mathbb{RPMF}_{\varepsilon_n}(\beta,g_n,(a_n,b_n),v_n) = {} & \mathbb{RPM}_{\varepsilon_n}((a_n,b_n),\omega(\varepsilon_n)^2 w_n) \\ & + \int_{a_n}^{b_n} \left(\omega(\varepsilon_n)^4 w_n(y)^2 - 2\omega(\varepsilon_n)^2 g_n(y) w_n(y) + g_n(y)^2\right) dy,\end{aligned}$$

so that we can consider w_n as a minimizer to the right-hand side subject to its own boundary conditions, and we already know from (3.4.18) and (3.4.19) that these boundary conditions satisfy a bound of the form

$$|w_n(a_n)| + |w_n(b_n)| + |w_n'(a_n)| + |w_n'(b_n)| \leq \Gamma_1 \tag{3.4.21}$$

for a suitable real constant Γ_1. Since the integral of $g_n(y)^2$ plays no role in the minimization process, we can neglect it and divide by $\omega(\varepsilon_n)^2$. In this way we obtain that

$$w_n \in \operatorname{argmin}_{\text{loc}} \left\{F_n((a_n,b_n),w) : w \in H^2((a_n,b_n))\right\},$$

where the sequence of functionals F_n is defined by

$$\begin{aligned}F_n(\Omega,w) \;:=\; & \int_\Omega \left(\varepsilon_n^6 \omega(\varepsilon_n)^2 w''(y)^2 + \frac{1}{\omega(\varepsilon_n)^4} \log\left(1+\omega(\varepsilon_n)^4 w'(y)^2\right)\right) dy \\ & + \int_\Omega \left(\omega(\varepsilon_n)^2 w(y)^2 - 2g_n(y)w(y)\right) dy.\end{aligned}$$

One could prove that, if $g_n \to g_\infty$ in L^2, the Gamma-limit of F_n is of the form

$$\int_\Omega \left(w'(y)^2 - 2g_\infty(y)w(y)\right) dy.$$

Therefore, it is reasonable to expect that $w_n(y)$ behaves for large n as a minimizer to the limit problem, which is a standard quadratic functional. This is what actually happens, but the proof is delicate for many reasons, including the lack of convexity of the approximating functionals, defined on intervals that depend also on n, and the boundary layers due to the loss of the boundary conditions on the derivative when the functionals of order two converge to a functional of order one. In particular, we cannot expect the sequence $\{w_n'(y)\}$ to converge uniformly, and we cannot expect all recovery sequences to have bounded derivatives, which forces us to exploit the minimality of $w_n(y)$ to some extent.

The key tool in our analysis is a comparison between $w_n(y)$ and minimizers to a suitable first order functional with a convex Lagrangian, defined as follows. Let us consider the function

$$\varphi_n(\sigma) := \frac{1}{\omega(\varepsilon_n)^4}\log(1+\omega(\varepsilon_n)^4\sigma^2) \qquad \forall\sigma\in\mathbb{R}.$$

An elementary calculation shows that

$$\varphi_n''(\sigma)\geq 1 \qquad \forall\sigma\in\left[-\frac{1}{3\omega(\varepsilon_n)^2},\frac{1}{3\omega(\varepsilon_n)^2}\right].$$

Now we consider the function $\psi_n:\mathbb{R}\to\mathbb{R}$ such that $\psi_n(0)=\psi_n'(0)=0$ and

$$\psi_n''(\sigma) := \max\{\varphi_n''(\sigma),1\} \qquad \forall\sigma\in\mathbb{R}. \tag{3.4.22}$$

This function satisfies

- $\psi_n\in C^2(\mathbb{R})$,
- $\psi''(\sigma)\geq 1$ for every $\sigma\in\mathbb{R}$,
- $\sigma^2/2\leq\psi_n(\sigma)\leq\sigma^2$ for every $\sigma\in\mathbb{R}$,
- $\psi_n(\sigma)=\varphi_n(\sigma)$ for every $\sigma\in[-1/3\omega(\varepsilon_n)^2,1/3\omega(\varepsilon_n)^2]$.

At this point we can consider the functional

$$G_n(\Omega,z) := \int_\Omega \big(\psi_n(z'(y))+\omega(\varepsilon_n)^2 z(y)^2-2g_n(y)z(y)\big)\,dy \tag{3.4.23}$$

and an element

$$z_n\in\operatorname{argmin}\left\{G_n((a_n,b_n),z): z\in H^1((a_n,b_n)),\ z_n(a_n)=w_n(a_n),\ z_n(b_n)=w_n(b_n)\right\}. \tag{3.4.24}$$

We point out that we do not impose any boundary condition on derivatives, because the Lagrangian of G_n is of order one. At this point the proof proceeds as follows.

- (Uniform bounds on z_n). In the first step we show that there exist real numbers Γ_2 and Γ_3 such that

  $$G_n((a_n,b_n),z_n)\leq\Gamma_2, \tag{3.4.25}$$

  and

  $$|z_n(y)|+|z_n'(y)|+|z_n''(y)|\leq\Gamma_3 \qquad \forall y\in(a_n,b_n). \tag{3.4.26}$$

- (Estimates from above). In the second step we show that

$$F_n((a_n, b_n), w_n) \leq G_n((a_n, b_n), z_n) + O(\varepsilon_n^3 \omega(\varepsilon_n)). \tag{3.4.27}$$

- (The contribution of high derivatives is negligible). In the third step we introduce the regions with high derivative

$$A_n := \left\{ y \in (a_n, b_n) : |w_n'(y)| > \frac{1}{4\omega(\varepsilon_n)^2} \right\},$$

and their contribution to the total variation

$$\Delta_n := \int_{A_n} |w_n'(y)| \, dy.$$

We show that they are both negligible, and more precisely that

$$|A_n| = O(\omega(\varepsilon_n)^4) \qquad \text{and} \qquad \Delta_n = O(\varepsilon_n^2 |\log \varepsilon_n|^{5/2}). \tag{3.4.28}$$

- (Estimates from below in regions with "small" derivative). In the fourth step we show that

$$F_n(A_n^c, w_n) \geq G_n((a_n, b_n), z_n) - O(\omega(\varepsilon_n)^4) - O(\Delta_n), \tag{3.4.29}$$

where A_n^c denotes the complement of A_n in (a_n, b_n).

- (Reduction to the convexity zone). In the fifth step we show that

$$|w_n'(y)| \leq \frac{1}{3\omega(\varepsilon_n)^2} \qquad \forall y \in (a_n, b_n) \tag{3.4.30}$$

when n is large enough. This seems to be a weak progress toward (3.4.20), but actually it is a crucial step, because it implies that eventually w_n stays in the region where $\varphi_n(\sigma)$ coincides with $\psi_n(\sigma)$, and hence in the convex regime.

- (Conclusion). Finally, in the sixth step we show that $w_n(y) - z_n(y)$ tends to zero with respect to the C^0 norm, and it is bounded in C^1 norm (but not necessarily vanishing, due to the boundary layers). Since z_n is bounded in the C^1 norm, this is enough to establish (3.4.20).

Uniform bounds on z_n The proof of (3.4.25) and (3.4.26) is a straightforward application of Lemma 3.5.3. We just need to check that the functional G_n defined in (3.4.23) fits into the assumptions, which in turn follow from the properties of ψ_n.

Estimate from above In this paragraph we prove (3.4.27). To this end, we observe that w_n is a solution to a minimum problem, and therefore it is enough to exhibit a competitor for the minimum problem whose energy is bounded by the right-hand side of (3.4.27). A natural choice for this competitor would be $z_n(t)$, which however does not necessarily satisfy the boundary conditions on the derivative. Therefore, we need to modify $z_n(t)$ in a neighborhood of the boundary. To this end, we write $[a_n, b_n]$ as the union of three intervals

$$I_{1,n} := [a_n, a_n+\varepsilon_n^3\omega(\varepsilon_n)], \quad I_{2,n} := [a_n+\varepsilon_n^3\omega(\varepsilon_n), b_n-\varepsilon_n^3\omega(\varepsilon_n)], \quad I_{3,n} := [b_n-\varepsilon_n^3\omega(\varepsilon_n), b_n],$$

and we consider the function $\widehat{w}_n : [a_n, b_n] \to \mathbb{R}$ defined as follows.

- In $I_{1,n}$ we define $\widehat{w}_n$ as the cubic polynomial with boundary conditions

$$\begin{gathered} \widehat{w}_n(a_n) = w_n(a_n) = z_n(a_n), \qquad \widehat{w}_n'(a_n) = w_n'(a_n), \\ \widehat{w}_n(a_n + \varepsilon_n^3\omega(\varepsilon_n)) = z_n(a_n + \varepsilon_n^3\omega(\varepsilon_n)), \\ \widehat{w}_n'(a_n + \varepsilon_n^3\omega(\varepsilon_n)) = z_n'(a_n + \varepsilon_n^3\omega(\varepsilon_n)). \end{gathered}$$

- In $I_{2,n}$ we set $\widehat{w}_n(y) := z_n(y)$.
- In $I_{3,n}$ we define $\widehat{w}_n$ as the cubic polynomial with boundary conditions

$$\begin{gathered} \widehat{w}_n(b_n - \varepsilon_n^3\omega(\varepsilon_n)) = z_n(b_n - \varepsilon_n^3\omega(\varepsilon_n)), \\ \widehat{w}_n'(b_n - \varepsilon_n^3\omega(\varepsilon_n)) = z_n'(b_n - \varepsilon_n^3\omega(\varepsilon_n)), \\ \widehat{w}_n(b_n) = w_n(b_n) = z_n(b_n), \qquad \widehat{w}_n'(b_n) = w_n'(b_n). \end{gathered}$$

This definition guarantees that the connections in the intermediate points are of class C^1, so that the resulting function belongs to $H^2((a_n, b_n))$, and that the values of $\widehat{w}_n(y)$ and $\widehat{w}_n'(y)$ for $y = a_n$ and $y = b_n$ coincide with the corresponding values of $w_n(y)$ and $w_n'(y)$.

Now we show that an estimate of the form (3.4.27) holds true separately in each of the three intervals.

- In the interval $I_{1,n}$ we claim that

$$F_n(I_{1,n}, \widehat{w}_n) = O(\varepsilon_n^3\omega(\varepsilon_n)) \qquad \text{and} \qquad G_n(I_{1,n}, z_n) = O(\varepsilon_n^3\omega(\varepsilon_n)), \tag{3.4.31}$$

  and hence in particular

$$F_n(I_{1,n}, \widehat{w}_n) - G_n(I_{1,n}, z_n) = O(\varepsilon_n^3\omega(\varepsilon_n)).$$

  Regarding $\widehat{w}_n$, we apply Lemma 2.6.2 with

$$\begin{aligned} A_0 &:= w_n(a_n), \quad & B_0 &:= z_n(a_n + \varepsilon_n^3\omega(\varepsilon_n)), \\ A_1 &:= w_n'(a_n), \quad & B_1 &:= z_n'(a_n + \varepsilon_n^3\omega(\varepsilon_n)). \end{aligned}$$

We observe that (3.4.21) and (3.4.26) guarantee that all these values are bounded and that

$$|B_0-A_0| = |z_n(a_n+\varepsilon_n^3\omega(\varepsilon_n))-w_n(a_n)| = |z_n(a_n+\varepsilon_n^3\omega(\varepsilon_n))-z_n(a_n)| = O(\varepsilon_n^3\omega(\varepsilon_n)),$$

Thus from Lemma 2.6.2 we obtain that

$$\int_{I_{1,n}} \varepsilon_n^6\omega(\varepsilon_n)^2\widehat{w}_n''(y)^2\,dy = O(\varepsilon_n^3\omega(\varepsilon_n)),$$

and the existence of a constant Γ_4 such that

$$|\widehat{w}_n(y)| + |\widehat{w}_n'(y)| \le \Gamma_4 \qquad \forall y \in I_{1,n}, \quad \forall n \in \mathbb{N}.$$

Since $|I_{1,n}| = \varepsilon_n^3\omega(\varepsilon_n)$, this implies that

$$\int_{I_{1,n}} \frac{1}{\omega(\varepsilon_n)^4}\log\left(1+\omega(\varepsilon_n)^4\widehat{w}_n'(y)^2\right)\,dy \le \int_{I_{1,n}} \widehat{w}_n'(y)^2\,dy = O(\varepsilon_n^3\omega(\varepsilon_n)),$$

and

$$\int_{I_{1,n}} \left(\omega(\varepsilon_n)^2\widehat{w}_n(y)^2 - 2g_n(y)\widehat{w}_n(y)\right)\,dy = O(\varepsilon_n^3\omega(\varepsilon_n)^3).$$

All these estimates imply the first relation in (3.4.31).

Regarding z_n, it is enough to observe that

$$\int_{I_{1,n}} \psi_n(z_n'(y))\,dy \le \left(\max_{\sigma\in\mathbb{R}}\ \psi_n''(\sigma)\right)\int_{I_{1,n}} \frac{z_n'(y)^2}{2}\,dy \le O(\varepsilon_n^3\omega(\varepsilon_n)),$$

and

$$\int_{I_{1,n}} \left(\omega(\varepsilon_n)^2 z_n(y)^2 - 2g_n(y)z_n(y)\right)dy = O(\varepsilon_n^3\omega(\varepsilon_n)^3),$$

because of the uniform bounds on z_n and g_n.

- In an analogous way, we can show that

$$F_n(I_{3,n},\widehat{w}_n) - G_n(I_{3,n},z_n) = O(\varepsilon_n^3\omega(\varepsilon_n)).$$

- In the interval $I_{2,n}$ we claim that

$$F_n(I_{2,n},\widehat{w}_n) - G_n(I_{2,n},z_n) = O(\varepsilon_n^6\omega(\varepsilon_n)^2).$$

Indeed, the uniform bound on $z_n'(t)$ implies that

$$|z_n'(y)| \le \frac{1}{3\omega(\varepsilon_n)^4} \qquad \forall y \in I_{2,n}$$

when n is large enough. For these values of n it follows that

$$\psi_n(z_n'(y)) = \frac{1}{\omega(\varepsilon_n)^4}\log\left(1+\omega(\varepsilon_n)^4 z_n'(y)^2\right) \qquad \forall y \in I_{2,n},$$

and therefore

$$F_n(I_{2,n},\widehat{w}_n) = F_n(I_{2,n},z_n) = \varepsilon_n^6\omega(\varepsilon_n)^2\int_{I_{2,n}} z_n''(y)^2\,dy + G_n(I_{2,n},z_n),$$

and we conclude by exploiting the uniform bounds on $z_n''(y)$.

The contribution of high derivatives is negligible To begin with, from (3.4.25), (3.4.27) and the uniform bounds on w_n and g_n, we deduce that

$$\int_{A_n}\left(\varepsilon_n^6\omega(\varepsilon_n)^2 w_n''(y)^2+\frac{1}{\omega(\varepsilon_n)^4}\log\left(1+\omega(\varepsilon_n)^4 w_n'(y)^2\right)\right)dy$$
$$\leq F_n((a_n,b_n),w_n)+2\left|\int_{a_n}^{b_n} g_n(y)w_n(y)\,dy\right|\leq\Gamma_5, \quad (3.4.32)$$

for a suitable constant Γ_5. This implies in particular that

$$\Gamma_5\geq\frac{1}{\omega(\varepsilon_n)^4}\int_{A_n}\log\left(1+\omega(\varepsilon_n)^4 w_n'(y)^2\right)dy\geq\frac{1}{\omega(\varepsilon_n)^4}\log\left(\frac{17}{16}\right)|A_n|,$$

which in turn implies that $|A_n|=O(\omega(\varepsilon_n)^4)$. In addition, from Lemma 3.5.5, we deduce that

$$\int_{A_n}\left(\varepsilon_n^6\omega(\varepsilon_n)^2 w_n''(y)^2+\frac{1}{\omega(\varepsilon_n)^4}\log\left(1+\omega(\varepsilon_n)^4 w_n'(y)^2\right)\right)dy$$
$$\geq\frac{4\sqrt{2/3}}{\varepsilon_n|\log\varepsilon_n|^{5/4}}\left(\log\left(\frac{17}{16}\right)\right)^{3/4}\left(\Delta_n-\frac{|A_n|}{4\omega(\varepsilon_n)^2}\right)^{1/2}.$$

Combining this estimate with (3.4.32) we obtain that

$$\frac{\Gamma_5}{4\sqrt{2/3}}\left(\log\left(\frac{17}{16}\right)\right)^{-3/4}\geq\frac{1}{\varepsilon_n|\log\varepsilon_n|^{5/4}}\left(\Delta_n-\frac{|A_n|}{4\omega(\varepsilon_n)^2}\right)^{1/2}\geq\frac{(\Delta_n-O(\omega(\varepsilon_n)^2))^{1/2}}{\varepsilon_n|\log\varepsilon_n|^{5/4}},$$

from which we conclude that $\Delta_n=O(\varepsilon_n^2|\log\varepsilon_n|^{5/2})$.

Estimates from below in regions with "small" derivative In order to prove (3.4.29), we introduce the functions $\widetilde{w}_n\in H^1((a_n,b_n))$ such that $\widetilde{w}_n(a_n)=w_n(a_n)$ and

$$\widetilde{w}_n'(y):=\begin{cases}w_n'(y) & \text{if } x\in A_n^c,\\ 0 & \text{if } x\in A_n.\end{cases}$$

For every $y\in[a_n,b_n]$ it turns out that

$$|w_n(y)-\widetilde{w}_n(y)|\leq\int_{a_n}^{b_n}|w_n'(s)-\widetilde{w}_n'(s)|\,ds=\int_{A_n}|w_n'(s)|\,ds=\Delta_n,$$

and in particular $|\widetilde{w}_n(y)|$ is bounded, independently of y and n.

Then we introduce the function $\widetilde{z}_n$ that minimizes G_n with respect to the same boundary conditions of $\widetilde{w}_n$, namely

$$\widetilde{z}_n\in\operatorname{argmin}\left\{G_n((a_n,b_n),z):z\in H^1((a_n,b_n)),\ z(a_n)=\widetilde{w}(a_n),\ z(b_n)=\widetilde{w}(b_n)\right\}.$$

We claim that

$$F_n(A_n^c,w_n)\geq G_n((a_n,b_n),\widetilde{w}_n)-O(\omega(\varepsilon_n)^4)-O(\Delta_n),$$

and

$$G_n((a_n,b_n),\widetilde{z}_n)=G_n((a_n,b_n),z_n)-O(\Delta_n).$$

Since $G_n((a_n,b_n),\widetilde{w}_n)\geq G_n((a_n,b_n),\widetilde{z}_n)$ due to the minimality of $\widetilde{z}_n$, these two claims, if proved, imply (3.4.29).

In order to prove the first claim, we begin by observing that

$$\begin{aligned}\int_{A_n^c}\frac{1}{\omega(\varepsilon_n)^4}\log\left(1+\omega(\varepsilon_n)^4 w_n'(y)^2\right)\,dy &= \int_{a_n}^{b_n}\frac{1}{\omega(\varepsilon_n)^4}\log\left(1+\omega(\varepsilon_n)^4\widetilde{w}_n'(y)^2\right)\,dy\\ &= \int_{a_n}^{b_n}\psi_n(\widetilde{w}_n'(y))\,dy,\end{aligned}$$

and therefore

$$\begin{aligned}F_n(A_n^c,w_n)\geq{}&\int_{A_n^c}\left[\frac{1}{\omega(\varepsilon_n)^4}\log\left(1+\omega(\varepsilon_n)^4 w_n'(y)^2\right)+\omega(\varepsilon_n)^2 w_n(y)^2-2g_n(y)w_n(y)\right]dy\\ ={}&\int_{a_n}^{b_n}\left(\psi_n(\widetilde{w}_n'(y))+\omega(\varepsilon_n)^2\widetilde{w}_n(y)^2-2g_n(y)\widetilde{w}_n(y)\right)dy\\ &-\int_{A_n}\left(\omega(\varepsilon_n)^2\widetilde{w}_n(y)^2-2g_n(y)\widetilde{w}_n(y)\right)dy\\ &-\int_{A_n^c}\left[\omega(\varepsilon_n)^2\left(\widetilde{w}_n(y)^2-w_n(y)^2\right)-2g_n(y)\left(\widetilde{w}_n(y)-w_n(y)\right)\right]dy.\end{aligned}$$

The first line after the equality sign is exactly $G_n((a_n,b_n),\widetilde{w}_n)$. The second line is $O(|A_n|)$, and therefore $O(\omega(\varepsilon_n)^4)$, due to the uniform bounds on $g_n(y)$ and $\widetilde{w}_n(y)$. For the same reason, the third line is $O(\|\widetilde{w}_n-w_n\|_\infty)$, and hence $O(\Delta_n)$.

The second claim is a straightforward application of Lemma 3.5.4 to the functional G_n. We just need to observe that the difference between the boundary values of z_n and $\widetilde{z}_n$ is $O(\Delta_n)$.

Reduction to the convexity zone Let us show that (3.4.30) holds true whenever n is large enough. Indeed, let us assume by contradiction that this is false. Then along a suitable subsequence (not relabeled) there exists $c_n\in(a_n,b_n)$ such that $|w_n'(c_n)|>(3\omega(\varepsilon_n)^2)^{-1}$. Since $c_n\in A_n$, there exists a point $d_n\in(a_n,b_n)$ such that

$$|w_n'(d_n)|=\frac{1}{4\omega(\varepsilon_n)^2},$$

and that the open interval whose endpoints are c_n and d_n is contained in A_n, so it holds that

$$|d_n-c_n|\leq|A_n|=O(\omega(\varepsilon_n)^4).$$

Let us assume, without loss of generality, that $d_n>c_n$ (the other case is symmetric).

Then it turns out that

$$\begin{aligned}\int_{c_n}^{d_n} \varepsilon_n^6\omega(\varepsilon_n)^2 w_n''(y)^2\,dy &\geq \varepsilon_n^6\omega(\varepsilon_n)^2\frac{1}{d_n-c_n}\left(\int_{c_n}^{d_n} w_n''(y)\,dy\right)^2\\ &= \varepsilon_n^6\omega(\varepsilon_n)^2\frac{1}{d_n-c_n}\left(w_n'(d_n)-w_n'(c_n)\right)^2\\ &\geq \varepsilon_n^6\omega(\varepsilon_n)^2\frac{1}{d_n-c_n}\left(\frac{1}{12\omega(\varepsilon_n)^2}\right)^2.\end{aligned}$$

Recalling the uniform bounds on w_n and g_n, we obtain that

$$\begin{aligned}F_n(A_n,w_n) &\geq \int_{c_n}^{d_n} \varepsilon_n^6\omega(\varepsilon_n)^2 w_n''(y)^2\,dy - 2\int_{A_n}|g_n(y)|\cdot|w_n(y)|\,dy\\ &\geq \frac{1}{144}\frac{\varepsilon_n^6}{(d_n-c_n)\omega(\varepsilon_n)^2} - O(A_n).\\ &= \frac{1}{144}\frac{\varepsilon_n^6}{(d_n-c_n)\omega(\varepsilon_n)^2} - O(\omega(\varepsilon_n)^4).\end{aligned}$$

Combining with (3.4.27) and (3.4.29) we deduce that

$$\begin{aligned}G_n((a_n,b_n),z_n)+O(\varepsilon_n^3\omega(\varepsilon_n)) &\geq F_n((a_n,b_n),w_n)\\ &= F_n(A_n,w_n)+F_n(A_n^c,w_n)\\ &\geq \frac{1}{144}\frac{\varepsilon_n^6}{(d_n-c_n)\omega(\varepsilon_n)^2} - O(\omega(\varepsilon_n)^4)\\ &\quad + G_n((a_n,b_n),z_n) - O(\omega(\varepsilon_n)^4) - O(\Delta_n),\end{aligned}$$

and therefore

$$\frac{\varepsilon_n^6}{O(\omega(\varepsilon_n)^6)} \leq \frac{\varepsilon_n^6}{(d_n-c_n)\omega(\varepsilon_n)^2} \leq O(\Delta_n)+O(\omega(\varepsilon_n)^4).$$

Multiplying both sides by $|\log\varepsilon_n|^3$ this relation leads to a contradiction when $n\to+\infty$, because the left-hand side has a positive liminf, while the right-hand side tends to 0 thanks to the second relation in (3.4.28).

Conclusion In this final paragraph of the proof we define the reminder $r_n(y)$ in such a way that $w_n(y)=z_n(y)+r_n(y)$, and we show that $r_n'(y)$ is uniformly bounded.

To begin with, from (3.4.26) and (3.4.30) we know that both $w_n'(y)$ and $z_n'(y)$ lie in the interval where $\varphi_n(\sigma)$ coincides with the convex function $\psi_n(\sigma)$, at least when n is large enough. Recalling that $\psi''(\sigma)\geq 1$ for every $\sigma\in\mathbb{R}$ we obtain that

$$\varphi_n(w_n'(y)) = \psi_n(z_n'(y)+r_n'(y)) \geq \psi_n(z_n'(y)) + \psi_n'(z_n'(y))\cdot r_n'(y) + \frac{1}{2}r_n'(y)^2.$$

From this inequality we obtain that

$$\begin{aligned} F_n((a_n,b_n),w_n) \;&\geq\; G_n((a_n,b_n),z_n) \\ &+\int_{a_n}^{b_n}\left(\psi_n'(z_n'(y))\cdot r_n'(y)+2\omega(\varepsilon_n)^2 z_n(y)r_n(y)-2g_n(y)r_n(y)\right)dy \\ &+\int_{a_n}^{b_n}\varepsilon_n^6\omega(\varepsilon_n)^2\left(z_n''(y)^2+2z_n''(y)r_n''(y)\right)dy \\ &+\int_{a_n}^{b_n}\left(\varepsilon_n^6\omega(\varepsilon_n)^2 r_n''(y)^2+\frac{1}{2}r_n'(y)^2+\omega(\varepsilon_n)^2 r_n(y)^2\right)dy. \end{aligned}$$

The second line vanishes because it is the first variation of the functional G_n in the minimum point z_n, computed with respect to the variation r_n, which is admissible because $r_n(a_n)=r_n(b_n)=0$. For the third line we exploit the inequality

$$a^2+2ab\geq -\frac{b^2}{2}-a^2 \qquad \forall (a,b)\in\mathbb{R}^2,$$

and, recalling that $z_n''(y)$ is uniformly bounded by (3.4.26), we obtain that

$$\int_{a_n}^{b_n}\varepsilon_n^6\omega(\varepsilon_n)^2\left(z_n''(y)^2+2z_n''(y)r_n''(y)\right)dy\geq -\frac{1}{2}\int_{a_n}^{b_n}\varepsilon_n^6\omega(\varepsilon_n)^2 r_n''(y)^2\,dy - O(\varepsilon_n^6\omega(\varepsilon_n)^2).$$

Combining with (3.4.27) we deduce that

$$\begin{aligned} G_n((a_n,b_n),z_n)+O(\varepsilon_n^3\omega(\varepsilon_n)) &\geq F_n((a_n,b_n),w_n) \\ &\geq G_n((a_n,b_n),z_n)+\frac{1}{2}\int_{a_n}^{b_n}\left(\varepsilon_n^6\omega(\varepsilon_n)^2 r_n''(y)^2+r_n'(y)^2\right)dy - O(\varepsilon_n^6\omega(\varepsilon_n)^2), \end{aligned}$$

and therefore

$$\int_{a_n}^{b_n}\left(\varepsilon_n^6\omega(\varepsilon_n)^2 r_n''(y)^2+r_n'(y)^2\right)dy\leq O(\varepsilon_n^3\omega(\varepsilon_n)).$$

We are now ready to conclude. Since r_n vanishes at the boundary, there exists $y_n\in(a_n,b_n)$ such that $r_n'(y_n)=0$. From the inequality between arithmetic mean ad geometric mean we obtain that

$$\varepsilon_n^3\omega(\varepsilon_n)\frac{d}{ds}[r_n'(s)^2]=2\varepsilon_n^3\omega(\varepsilon_n)r_n'(s)r_n''(s)\leq\varepsilon_n^6\omega(\varepsilon_n)^2 r_n''(y)^2+r_n'(s)^2,$$

and therefore

$$\begin{aligned} \varepsilon_n^3\omega(\varepsilon_n)r_n'(y)^2 &= \varepsilon_n^3\omega(\varepsilon_n)\left(r_n'(y)^2-r_n'(y_n)^2\right) \\ &= \left|\int_{y_n}^{y}\varepsilon_n^3\omega(\varepsilon_n)\frac{d}{ds}[r_n'(s)^2]\,ds\right| \\ &\leq \int_{y_n}^{y}\left(\varepsilon_n^6\omega(\varepsilon_n)^2 r_n''(s)^2+r_n'(s)^2\right)ds \\ &\leq \int_{a_n}^{b_n}\left(\varepsilon_n^6\omega(\varepsilon_n)^2 r_n''(s)^2+r_n'(s)^2\right)ds \\ &\leq O(\varepsilon_n^3\omega(\varepsilon_n)). \end{aligned}$$

Dividing by $\varepsilon_n^3\omega(\varepsilon_n)$ we obtain the uniform bound on $r_n'(y)$.

3.5 Some lemmata

I this section we prove some lemmata that we exploited in the proof of the main results. The first one is an elementary inequality for the logarithm.

Lemma 3.5.1. *It turns out that*

$$\left|\log\left(1+[(1+b)x+d]^2\right)-\log\left(1+x^2\right)\right|\leq 4|b|+|d| \qquad \forall (b,d,x)\in\left[-\frac{1}{2},\frac{1}{2}\right]\times\mathbb{R}^2.$$

Proof. We observe that the function $\psi(\sigma):=\log(1+\sigma^2)$ is Lipschitz continuous, with Lipschitz constant equal to 1, and in particular

$$\left|\log\left(1+[(1+b)x+d]^2\right)-\log\left(1+(1+b)^2x^2\right)\right|\leq |d|,$$

so that it is enough to prove that

$$\left|\log\left(1+(1+b)^2x^2\right)-\log\left(1+x^2\right)\right|\leq 4|b|. \tag{3.5.1}$$

To this end we observe that

$$\left|\log\left(1+(1+b)^2x^2\right)-\log\left(1+x^2\right)\right|=\left|\log\left(1+(2b+b^2)\frac{x^2}{1+x^2}\right)\right|$$

and the right-hand side is a nondecreasing function of x^2, for every fixed $b\in\mathbb{R}$. This implies that

$$\left|\log\left(1+(2b+b^2)\frac{x^2}{1+x^2}\right)\right|\leq\left|\log\left(1+2b+b^2\right)\right| \qquad \forall b\neq -1 \quad \forall x\in\mathbb{R}.$$

Now we exploit the fact that $b\in[-1/2,1/2]$ and we observe that the function $\phi(\sigma):=\log(1+\sigma)$ is Lipschitz continuous on $[-1/2,1/2]$, with Lipschitz constant equal to 2, so we obtain that

$$\left|\log\left(1+2b+b^2\right)\right|=2\left|\log\left(1+b\right)\right|\leq 4|b| \qquad \forall b\in\left[-\frac{1}{2},\frac{1}{2}\right].$$

This completes the proof of (3.5.1). □

The next lemma is a quantitative version of the subadditivity of the square root.

Lemma 3.5.2 (Quantitative subadditivity of the square root)**.** *Let I be a finite or countable nonempty set, and let $f: I\to[0,+\infty)$ be a function. Let us assume that the following three quantities are finite:*

$$S:=\sum_{i\in I} f(i), \qquad R:=\sum_{i\in I}\sqrt{f(i)}, \qquad M:=\max\{f(i): i\in I\}.$$

Then it turns out that

$$\sqrt{S-M}\leq 3\left(R-\sqrt{S}\right). \tag{3.5.2}$$

Proof. We distinguish two cases.

Case 1: $M \geq S/2$ Let $i_0 \in I$ be such that $f(i_0) = M$, and let $I' := I \setminus \{i_0\}$. From the subadditivity of the square root we deduce that

$$R = \sqrt{f(i_0)} + \sum_{i \in I'} \sqrt{f(i)} \geq \sqrt{f(i_0)} + \left(\sum_{i \in I'} f(i) \right)^{1/2} = \sqrt{M} + \sqrt{S - M}.$$

Now we observe that the function $\varphi(x) := 1 + \sqrt{x} - \sqrt{1+x}$ is increasing, and therefore

$$1 + \sqrt{x} - \sqrt{1+x} \geq 2 - \sqrt{2} \geq \frac{1}{3} \qquad \forall x \geq 1.$$

Setting $x := M/(S - M)$ (which is greater than or equal to 1 in this case), from the combination of these two inequalities we conclude that

$$R - \sqrt{S} \geq \sqrt{S - M} + \sqrt{M} - \sqrt{S} \geq \frac{1}{3}\sqrt{S - M},$$

which proves (3.5.2) in this case.

Case 2: $M \leq S/2$ In this case there exists $I_1 \subseteq I$ such that

$$S_1 := \sum_{i \in I_1} f(i) \in \left[\frac{1}{4}S, \frac{3}{4}S \right].$$

Setting $I_2 := I \setminus I_1$, as before from the subadditivity of the square root we obtain that

$$R = \sum_{i \in I_1} \sqrt{f(i)} + \sum_{i \in I_2} \sqrt{f(i)} \geq \left(\sum_{i \in I_1} f(i) \right)^{1/2} + \left(\sum_{i \in I_2} f(i) \right)^{1/2} \geq \sqrt{S_1} + \sqrt{S - S_1}.$$

Now we observe that $\psi(x) := \sqrt{x} + \sqrt{1-x} - 1$ is a concave function with equal values in $1/4$ and $3/4$, and in particular

$$\sqrt{x} + \sqrt{1-x} - 1 \geq \frac{\sqrt{3} - 1}{2} \geq \frac{1}{3} \qquad \forall x \in \left[\frac{1}{4}, \frac{3}{4} \right].$$

Setting $x := S_1/S$ (which lies in the interval $[1/4, 3/4]$), from the combination of these two inequalities we conclude that

$$R - \sqrt{S} \geq \sqrt{S_1} + \sqrt{S - S_1} - \sqrt{S} \geq \frac{1}{3}\sqrt{S} \geq \frac{1}{3}\sqrt{S - M},$$

which proves (3.5.2) in this case. □

Let us consider the minimum problem

$$\min \left\{ \int_a^b L(x, u(x), u'(x))\, dx : u \in H^1((a,b)),\ u(a) = A,\ u(b) = B \right\}, \tag{3.5.3}$$

where $(a,b) \subseteq \mathbb{R}$ is an interval, $L : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function called Lagrangian, and $(A,B) \in \mathbb{R}^2$ is a pair of Dirichlet data.

Let us assume that there exist positive real numbers M_1, ..., M_9 such that

(i) the length of the interval satisfies

$$M_1 \leq b - a \leq M_2,$$

(ii) There exists two functions $\psi \in C^2(\mathbb{R})$ and $f \in C^1([a,b] \times \mathbb{R})$ such that

$$L(x,s,p) = \psi(p) + f(x,s) \qquad \forall (x,s,p) \in [a,b] \times \mathbb{R}^2.$$

(iii) The functions ψ and f satisfy the following conditions

$$M_3 \leq \psi''(p) \leq M_4 \quad \text{and} \quad M_5 p^2 \leq \psi(p) \leq M_6 p^2 \qquad \forall p \in \mathbb{R},$$

$$-M_7(1+|s|) \leq f(x,s) \leq M_7(1+s^2) \quad \text{and} \quad |f_s(x,s)| \leq M_8(1+|s|) \quad \forall (x,s) \in [a,b]\times\mathbb{R}.$$

(iv) the boundary conditions satisfy

$$(A,B) \in [-M_9, M_9]^2.$$

The first result is that minima and minimizers are bounded, and the bound depends only on the constants $M_1, \ldots, M_9$.

Lemma 3.5.3. *Let us consider problem (3.5.3) under assumptions (i)–(iv) described above.*

Let $u(x) \in H^1((a,b))$ be any minimizer. Then actually $u \in C^2([a,b])$, and there exist real numbers C_1 and C_2, depending only on $M_1, \ldots, M_9$, such that

$$\int_a^b L(x,u(x),u'(x))\,dx \leq C_1,$$

and

$$|u(x)| + |u'(x)| + |u''(x)| \leq C_2 \qquad \forall x \in [a,b]. \tag{3.5.4}$$

Proof. The energy estimate follows simply by using the affine function interpolating the boundary conditions as a competitor for the minimum problem.

The regularity and the estimate (3.5.4) follow from the Euler-Lagrange equation solved by minimizers. □

The second result is that minima depend in a Lipschitz way on the Dirichlet boundary conditions.

Lemma 3.5.4. *Let us consider problem (3.5.3) under assumptions (i)–(iv) described above. Let us assume in addition that*

(v) the Lagrangian is locally Lipschitz continuous with respect to the pair (s,p), and more precisely there exists a positive real number M_{10} such that

$$|L(x,s_1,p_1) - L(x,s_2,p_2)| \leq M_{10}(|s_1 - s_2| + |p_1 - p_2|)$$

for every

$$(x,s_1,s_2,p_1,p_2) \in [a,b]\times[-C_2-4M_9, C_2+4M_9]^2\times[-C_2-4M_9/M_1, C_2+4M_9/M_1]^2,$$

where C_2 is the constant in (3.5.4).

Then there exists a positive real number C_3, depending only on $M_1, \ldots, M_{10}$, with the following property. If $u_1(x)$ and $u_2(x)$ are minimizer with boundary data $(A_1, B_1) \in [-M_9, M_9]^2$ and $(A_2, B_2) \in [-M_9, M_9]^2$, respectively, then

$$\left| \int_a^b L(x, u_1(x), u_1'(x))\, dx - \int_a^b L(x, u_2(x), u_2'(x))\, dx \right| \le C_3(|A_1 - A_2| + |B_1 - B_2|).$$

Proof. We observe that the function

$$v(x) = u_2(x) + (A_1 - A_2)\frac{b-x}{b-a} + (B_1 - B_2)\frac{x-a}{b-a}$$

is a competitor for the minimum problem with boundary conditions (A_1, B_1), and that

$$|v(x) - u_2(x)| \le |A_1 - A_2| + |B_1 - B_2|,$$

and

$$|v'(x) - u_2'(x)| \le \frac{|A_1 - A_2|}{b-a} + \frac{|B_1 - B_2|}{b-a}.$$

Therefore from the minimality of u_1 and assumptions (i) and (v) we deduce that

$$\begin{aligned} \int_a^b L(x, u_1(x), u_1'(x))\, dx &\le \int_a^b L(x, v(x), v'(x))\, dx \\ &\le \int_a^b L(x, u_2(x), u_2'(x))\, dx \\ &\quad + M_{10}\left(1 + \frac{1}{M_1}\right)(|A_1 - A_2| + |B_1 - B_2|). \end{aligned}$$

The remaining inequality can be proved by exchanging the role of u_1 and u_2. □

Finally, we prove two estimates from below for rescaled versions of the singularly perturbed Perona-Malik funcitonals.

Lemma 3.5.5. *Let $\alpha, \beta, \gamma, M > 0$ be positive real numbers and let us consider the functional*

$$\mathcal{F}_{\alpha,\beta,\gamma}(\Omega, u) := \int_\Omega \left(\alpha u''(x) + \beta \log(1 + \gamma u'(x)^2)\right)\, dx,$$

defined for every open set $\Omega \subset \mathbb{R}$ and every $u \in H^2(\Omega)$.

Let $(a, b) \subset \mathbb{R}$ be an interval and let $w \in H^2(a, b)$ be a function such that $|w'(a)| \le M$ and $|w'(b)| \le M$. Let us consider the set

$$A_M := \{x \in (a, b) : |w'(x)| > M\},$$

and let us set

$$\Delta_M := \int_{A_M} |w'(x)|\, dx.$$

Then it turns out that

$$\mathcal{F}_{\alpha,\beta,\gamma}(A_M, w) \ge 4\sqrt{\frac{2}{3}}\alpha^{1/4}\beta^{3/4}(\log(1 + \gamma M^2))^{3/4}\sqrt{\Delta_M - M|A_M|} \tag{3.5.5}$$

Proof. Since A_M is an open set, we can write it as a finite or countable union of open disjoint intervals (its connected components), namely in the form

$$A_M = \bigcup_{i \in I} (a_i, b_i),$$

where I is a suitable index set.

We observe that w' has constant sign in each of the intervals (a_i, b_i), and therefore

$$\Delta_M = \sum_{i \in I} \int_{a_i}^{b_i} |w'(x)|\, dx = \sum_{i \in I} |u(b_i) - u(a_i)|$$

Moreover, we have that $|w'(a_i)| = |w'(b_i)| = M$ for every $i \in I$, because w' is continuous and all the points a_i and b_i are internal to (a, b), since $|w'(a)| < M$ and $|w'(b)| < M$.

As a consequence, from Lemma 2.6.2 we deduce that

$$\int_{a_i}^{b_i} w''(x)^2\, dx \geq \frac{12}{(b_i - a_i)^3} (|u(b_i) - u(a_i)| - M(b_i - a_i))^2,$$

and hence

$$\begin{aligned} \mathcal{F}_{\alpha,\beta,\gamma}((a_i, b_i), w) &\geq \frac{12\alpha}{(b_i - a_i)^3} (|u(b_i) - u(a_i)| - M(b_i - a_i))^2 + \beta \log(1 + \gamma M^2)(b_i - a_i) \\ &\geq 4\sqrt{\frac{2}{3}} \alpha^{1/4} \beta^{3/4} (\log(1 + \gamma M^2))^{3/4} \sqrt{|u(b_i) - u(a_i)| - M(b_i - a_i)}, \end{aligned}$$

where the last inequality follows from the inequality

$$a + b \geq \frac{4}{3^{3/4}} (ab^3)^{1/4} \qquad \forall (a, b) \in [0, +\infty)^2.$$

Summing over i and exploiting the subadditivity of the square root we obtain (3.5.5). $\square$

Lemma 3.5.6. *Let (a_n, b_n) be a sequence of intervals such that $b_n - a_n \to L$, for some positive number $L > 0$. Let also $\{\delta_n\} \subset (0, +\infty)$ be a sequence of positive numbers such that $\delta_n \to 0$ and $\{w_n\}$ be a sequence of functions such that $w_n \in H^2((a_n, b_n))$ and*

$$\liminf_{n \to +\infty} \operatorname{osc}(w_n, (a_n, b_n)) \geq J > 0.$$

Let us consider the family of functionals

$$\mathbb{RPMH}_\delta(\Omega, u) := \int_\Omega \left\{ \frac{\delta^6}{|\log \delta|^3} u''(x)^2 + \frac{1}{\delta^2} \log\left(1 + \delta^2 u'(x)^2\right) \right\} dx, \tag{3.5.6}$$

defined for every positive number $\delta > 0$, for every open set $\Omega \subseteq \mathbb{R}$ and every function $u \in H^2(\Omega)$, and let us assume that

$$|w'(a_n)| \leq \frac{1}{\delta_n} \qquad \textit{and} \qquad |w'(b_n)| \leq \frac{1}{\delta_n},$$

for every $n \in \mathbb{N}$.

Then there exists a positive constant $c(L, J)$ *depending only on* L *and* J *such that*

$$\liminf_{n\to+\infty} \mathbb{RPMH}_{\delta_n}((a_n, b_n), w_n) \geq c(L, J).$$

Proof. First of all, up to reducing ourselves to a subsequence, we can assume that

$$\liminf_{n\to+\infty} \mathbb{RPMH}_{\delta_n}((a_n, b_n), w_n) = \lim_{n\to+\infty} \mathbb{RPMH}_{\delta_n}((a_n, b_n), w_n) < +\infty.$$

For every $n \in \mathbb{N}$ let us set

$$A_n := \left\{x \in (a_n, b_n) : |w_n'(x)| > \frac{1}{\delta_n^2 |\log \delta_n|^3}\right\}, \qquad \Delta_n^A := \int_{A_n} |w_n'(x)|\,dx,$$

$$B_n := \left\{x \in (a_n, b_n) : \frac{1}{\delta_n |\log \delta_n|} \leq |w_n'(x)| \leq \frac{1}{\delta_n^2 |\log \delta_n|^3}\right\},$$

$$C_n := \left\{x \in (a_n, b_n) : |w_n'(x)| < \frac{1}{\delta_n |\log \delta_n|}\right\}, \qquad \Delta_n^C := \int_{C_n} |w_n'(x)|\,dx.$$

We observe that

$$|B_n| \leq \frac{\delta_n^2}{\log(1 + |\log \delta_n|^{-2})} \mathbb{RPMH}_{\delta_n}((a_n, b_n), w_n),$$

and therefore

$$\limsup_{n\to+\infty} \int_{B_n} |w_n'(x)|\,dx \leq \frac{|B_n|}{\delta_n^2 |\log \delta_n|^3} \leq \limsup_{n\to+\infty} \frac{\mathbb{RPMH}_{\delta_n}((a_n, b_n), w_n)}{|\log \delta_n|^3 \log(1 + |\log \delta_n|^{-2})} = 0,$$

namely the contribution of B_n to the total variation of w_n is asymptotically negligible.

As a consequence we obtain that

$$\liminf_{n\to+\infty} \Delta_n^A + \Delta_n^C = \liminf_{n\to+\infty} \int_{a_n}^{b_n} |w_n'(x)|\,dx \geq J. \tag{3.5.7}$$

Concerning A_n, we observe that

$$\limsup_{n\to+\infty} \frac{|A_n|}{\delta_n^2 |\log \delta_n|^3} \leq \limsup_{n\to+\infty} \frac{\mathbb{RPMH}_{\delta_n}((a_n, b_n), w_n)}{|\log \delta_n|^3 \log(1 + \delta_n^{-2} |\log \delta_n|^{-6})} = 0,$$

and that Lemma 3.5.5 yields

$$\mathbb{RPMH}_{\delta_n}(A_n, w_n) \geq 4\sqrt{\frac{2}{3}} \left(\frac{1}{|\log \delta_n|} \log\left(1 + \frac{1}{\delta_n^2 |\log \delta_n|^6}\right)\right)^{3/4} \sqrt{\Delta_n^A - \frac{|A_n|}{\delta_n^2 |\log \delta_n|^3}},$$

at least when n is large enough so that $1/\delta_n \leq 1/(\delta_n^2 |\log \delta_n|^3)$.

Hence it holds that

$$\liminf_{n\to+\infty} \mathbb{RPMH}_{\delta_n}(A_n, w_n) \geq 8 \frac{2^{1/4}}{3^{1/2}} \cdot \liminf_{n\to+\infty} \sqrt{\Delta_n^A}. \tag{3.5.8}$$

As for C_n, we observe that

$$\begin{aligned}
\liminf_{n\to+\infty} \mathbb{RPMH}_{\delta_n}(C_n, w_n) &\geq \liminf_{n\to+\infty} \int_{C_n} \frac{\log(1+\delta_n^2 w_n'(x)^2)}{\delta_n^2 w_n'(x)^2} w_n'(x)^2\, dx \\
&\geq \liminf_{n\to+\infty} |\log\delta_n|^2 \log(1+|\log\delta_n|^{-2}) \int_{C_n} w_n'(x)^2\, dx \\
&= \liminf_{n\to+\infty} \int_{C_n} w_n'(x)^2\, dx \\
&\geq \liminf_{n\to+\infty} \frac{(\Delta_n^C)^2}{|C_n|} \\
&= \liminf_{n\to+\infty} \frac{(\Delta_n^C)^2}{L}, \qquad (3.5.9)
\end{aligned}$$

where the second inequality follows form the fact that the function $t \mapsto \log(1+t)/t$ is nonincreasing on $(0, +\infty)$.

Combining (3.5.7), (3.5.8) and (3.5.9) we obtain that

$$\begin{aligned}
\liminf_{n\to+\infty} \mathbb{RPMH}_{\delta_n}((a_n, b_n), w_n) &\geq \liminf_{n\to+\infty} 8\frac{2^{1/4}}{3^{1/2}} \cdot \sqrt{\Delta_n^A} + \frac{(\Delta_n^C)^2}{L} \\
&\geq \min\left\{ 8\frac{2^{1/4}}{3^{1/2}}\sqrt{A} + \frac{C^2}{L} : (A, C) \in [0, +\infty)^2,\ A + C \geq J \right\}.
\end{aligned}$$

We conclude by remarking that

$$c(L, J) := \min\left\{ 8\frac{2^{1/4}}{3^{1/2}}\sqrt{A} + \frac{C^2}{L} : (A, C) \in [0, +\infty)^2,\ A + C \geq J \right\} > 0$$

for every $(L, J) \in (0, +\infty)^2$. □

Chapter 4

Singular perturbation: minimum values in higher dimensions

In this chapter we show how to extend Theorem 2.2.2 to the higher dimensional case, and we describe some difficulties in extending the other results in Chapter 2 (see Remark 4.3.3).

For simplicity and to avoid too many technicalities, we limit ourselves to the case in which $\Omega = (0,1)^d$ and the forcing term is of class C^1. However, it should be clear that the same result can be extended to the case in which Ω is a regular domain and $f \in H^1(\Omega)$.

Before proceeding with the proof, we need to introduce some notation to deal with higher dimensional quantity. So, let $d > 1$ be a positive integer number. Given a matrix $A \in \mathbb{R}^{d\times d}$ we denote with $\|A\|$ its operator norm, namely

$$\|A\| := \sup_{\sigma \in \mathbb{S}^{d-1}} A\sigma \cdot \sigma.$$

We observe that when the matrix is the hessian of a function, then its operator norm is the maximum value of directional second derivatives, namely

$$\|\nabla^2 u(x)\| = \sup_{\sigma \in \mathbb{S}^{d-1}} \partial^2_{\sigma\sigma} u(x).$$

Now, for every real numbers $\varepsilon \in (0,1)$ and $\beta > 0$, every open set $\Omega \subset \mathbb{R}^d$ and every couple of functions $f \in L^2(\Omega)$ and $u \in H^2(\Omega)$ we define the following higher dimensional versions of the functionals (2.2.2), (2.3.1) and (2.3.2)

$$\mathbb{PMF}_\varepsilon(\beta, f, \Omega, u) := \int_\Omega \left[\varepsilon^6 \omega(\varepsilon)^4 \|\nabla^2 u(x)\|^2 + \log(1 + |\nabla u(x)|^2) + \beta(u(x) - f(x))^2\right] dx,$$

$$\mathbb{RPM}_\varepsilon(\Omega, u) := \int_\Omega \left[\varepsilon^6 \|\nabla^2 u(x)\|^2 + \frac{1}{\omega(\varepsilon)^2} \log(1 + |\nabla u(x)|^2)\right] dx,$$

$$\mathbb{RPMF}_\varepsilon(\beta, f, \Omega, u) := \mathbb{RPM}_\varepsilon(u, \Omega) + \beta \int_\Omega (u(x) - f(x))^2 \, dx,$$

where $\omega(\varepsilon)$ is defined as in (2.2.1). Our result is the following.

Theorem 4.0.1. *For every $\beta > 0$ and every $f \in C^1([0,1]^d)$ it turns out that*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\omega(\varepsilon)^2} \min_{u\in H^2([0,1]^d)} \mathbb{PMF}_\varepsilon(\beta, f, [0,1]^d, u) = 10\left(\frac{2\beta}{27}\right)^{1/5} \int_{[0,1]^d} |\nabla f(x)|^{4/5}\, dx.$$

In order to prove this theorem, we need to introduce the higher dimensional analogue of the limit functionals (2.3.6) and (2.3.11), for which we need to define the higher dimensional version of the space PJ, which requires the use of special functions of bounded variation (see [6]). The formal definition is the following

$$PJ(\Omega) := \{u \in GSBV(\Omega) : \nabla u = 0\},$$

where ∇u denotes the approximate gradient of u and

$$GSBV(\Omega) := \{u \in L^1(\Omega) : \min\{\max\{u, -T\}, T\} \in SBV(\Omega) \quad \forall T > 0\},$$

is the space of functions whose truncations are special functions of bounded variation.

In the sequel we also need to consider the larger space

$$GBV(\Omega) := \{u \in L^1(\Omega) : \min\{\max\{u, -T\}, T\} \in BV(\Omega) \quad \forall T > 0\}.$$

We recall that the measure derivative of a PJ function u is supported on a $(d-1)$-rectifiable set S_u, and that for $\mathcal{H}^{d-1}$ almost every point $x \in S_u$ the normal ν_u to S_u and two traces $u^\pm(x)$ of u on the two sides of S_u are well-defined (see [6]).

More precisely, we have that

$$Du = (u^+ - u^-)\nu_u \mathcal{H}^{d-1} \llcorner S_u.$$

Therefore, for $u \in PJ(\Omega)$ we can define the higher dimensional versions of the functionals (2.3.6) and (2.3.11) as

$$\mathbb{J}_{1/2}(\Omega, u) = \int_{\Omega\cap S_u} \sqrt{|u^+(x) - u^-(x)|}\, d\mathcal{H}^{d-1}(x),$$

$$\mathbb{JF}_{1/2}(\alpha, \beta, f, \Omega, u) = \alpha\, \mathbb{J}_{1/2}(\Omega, u) + \beta \int_\Omega (u(x) - f(x))^2\, dx.$$

4.1 Gamma-convergence and compactness

In [12] it is proved that for every bounded open set Ω with Lipschitz boundary it turns out that

$$\Gamma - \lim_{\varepsilon\to 0} \mathbb{RPM}_\varepsilon(\Omega, u) = \alpha_0\, \mathbb{J}_{1/2}(\Omega, u),$$

where α_0 is defined as in (2.3.7).

Here we do not need the full Gamma-convergence result, but only some estimates related with the higher dimensional versions of the minimum problems (2.3.12) and (2.3.13). Unfortunately, we found some issues in the proofs of the equicoerciveness and the liminf inequality in [12]. Indeed, in [12] it is claimed that the domain of the Gamma-limit is contained in $SBV(\Omega)$ and that sequences with equibounded energies (and null average) are relatively compact in $BV(\Omega)$. But the limit functional $\mathbb{J}_{1/2}$ is not lower semicontinous on $L^1(\Omega)$ or $L^2(\Omega)$ if we set it equal to $+\infty$ in $GSBV(\Omega) \setminus SBV(\Omega)$, so it can not be a Gamma-limit.

Furthermore, the proof of the limsup inequality that is contained in [12] (which relies on a density result that is the most original part of that paper) shows itself that the Gamma-limit must be finite on all functions $u \in GSBV(\Omega)$ with $\mathbb{J}_{1/2}(\Omega, u) < +\infty$, and it is not difficult to see that there are actually functions with this property that do not belong to $SBV(\Omega)$ (and this is the reason for which the space $GSBV$ has been introduced).

The problem with the argument in [12] is that it relies on [12, Lemma 3.1], which is said to be proved in [14, Lemma 3.2]. However, looking at the proof of [14, Lemma 3.2], it emerges that the constant $C > 0$ appearing in the statement actually depends on the energy, and this can not be improved, since a functional that is converging to $\mathbb{J}_{1/2}$ can not control the total variation of a function, without additional assumptions on the L^∞ norm of that function.

We also point out that, since the functionals $\mathbb{RPM}_\varepsilon$ involve second derivatives, we can not reduce to the L^∞ case just by truncation, because the truncation of a function of class H^2 is not of class H^2.

For this reason, we include here a different proof of the Gamma-liminf inequality and of the equicoercivess (in GSBV). To this end, we need the following reformulation of [14, Lemma 3.2], that provides explicit values of the constants, and from which it is also clear that truncations are needed if one wants to control the total variation. The proof is an adaptation of the argument of Lemma A.0.2 to the case in which the assumption (A.0.4) is removed.

Lemma 4.1.1. *Let $(a, b) \subseteq \mathbb{R}$ be an interval, let $\varepsilon \in (0, 1)$ be a real number and let $u_\varepsilon \in H^2(\Omega)$ be a function.*

For every positive real number $T > 0$, let us set $u_{\varepsilon,T} := \min\{\max\{u_\varepsilon, -T\}, T\}$. Then it turns out that

$$\int_a^b |u'_{\varepsilon,T}(x)|\, dx \leq \frac{b-a}{|\log \varepsilon|} + \left(\frac{2T^{1/2}}{M(\varepsilon)} + \zeta(\varepsilon)\right) \mathbb{RPM}_\varepsilon((a,b), u_\varepsilon) + 2T, \tag{4.1.1}$$

where

$$M(\varepsilon) := 4\left(\frac{2}{3}\right)^{1/2} \left\{\frac{1}{|\log \varepsilon|} \log\left(1 + \frac{1}{\varepsilon^4 |\log|^8}\right)\right\}^{3/4},$$

and

$$\zeta(\varepsilon) := \frac{1}{|\log \varepsilon|^3} \left(\frac{1}{\log(1 + \varepsilon^{-4}|\log \varepsilon|^{-8})} + \frac{1}{\log(1 + |\log \varepsilon|^{-2})}\right).$$

Proof. We argue as in the proof of Lemma A.0.2, so we consider the sets A_ε, B_ε and C_ε

defined as

$$A_\varepsilon := \left\{ x \in (a,b) : |u_\varepsilon'(x)| > \frac{1}{\varepsilon^2 |\log \varepsilon|^4} \right\},$$

$$B_\varepsilon := \left\{ x \in (a,b) : \frac{1}{|\log \varepsilon|} \le |u_\varepsilon'(x)| \le \frac{1}{\varepsilon^2 |\log \varepsilon|^4} \right\},$$

$$C_\varepsilon := \left\{ x \in (a,b) : |u_\varepsilon'(x)| < \frac{1}{|\log \varepsilon|} \right\},$$

and we observe that

$$|A_\varepsilon| \le \frac{\varepsilon^2 |\log \varepsilon|}{\log(1+\varepsilon^{-4}|\log \varepsilon|^{-8})} \mathbb{RPM}_\varepsilon((a,b),u_\varepsilon), \tag{4.1.2}$$

$$|B_\varepsilon| \le \frac{\varepsilon^2 |\log \varepsilon|}{\log(1+|\log \varepsilon|^{-2})} \mathbb{RPM}_\varepsilon((a,b),u_\varepsilon). \tag{4.1.3}$$

We observe that if $A_\varepsilon = (a,b)$ then u_ε' has constant sign, namely u_ε is monotone, and therefore

$$\int_a^b |u_{\varepsilon,T}'(x)|\, dx = |u_{\varepsilon,T}(b) - u_{\varepsilon,T}(a)| \le 2T,$$

so (4.1.1) holds.

Otherwise, let us write A_ε as the union of its connected components, namely

$$A_\varepsilon = \bigcup_{i\in I} (\alpha_i, \beta_i),$$

and we know that each of the intervals (α_i,β_i) has at least an endpoint internal to (a,b). Therefore $|u_\varepsilon'|$ is equal to $\varepsilon^{-2}|\log\varepsilon|^{-4}$ in that endpoint, so from Lemma A.0.1 we deduce that for every i we have

$$\left(|u_\varepsilon(\beta_i) - u_\varepsilon(\alpha_i)| - \frac{\beta_i - \alpha_i}{\varepsilon^2 |\log\varepsilon|^4} \right)^{1/2} \le \frac{2^{1/2}}{M(\varepsilon)} \mathbb{RPM}_\varepsilon((\alpha_i,\beta_i),u_\varepsilon).$$

Hence we obtain that

$$\begin{aligned}
|u_{\varepsilon,T}(\beta_i) - u_{\varepsilon,T}(\alpha_i)| &= \min\left\{ |u_{\varepsilon,T}(\beta_i) - u_{\varepsilon,T}(\alpha_i)| - \frac{\beta_i - \alpha_i}{\varepsilon^2 |\log\varepsilon|^4} + \frac{\beta_i - \alpha_i}{\varepsilon^2 |\log\varepsilon|^4}, 2T \right\} \\
&\le \min\left\{ |u_{\varepsilon,T}(\beta_i) - u_{\varepsilon,T}(\alpha_i)| - \frac{\beta_i - \alpha_i}{\varepsilon^2 |\log\varepsilon|^4}, 2T \right\} + \frac{\beta_i - \alpha_i}{\varepsilon^2 |\log\varepsilon|^4} \\
&\le (2T)^{1/2} \left(|u_{\varepsilon,T}(\beta_i) - u_{\varepsilon,T}(\alpha_i)| - \frac{\beta_i - \alpha_i}{\varepsilon^2 |\log\varepsilon|^4} \right)^{1/2} + \frac{\beta_i - \alpha_i}{\varepsilon^2 |\log\varepsilon|^4} \\
&\le \frac{2T^{1/2}}{M(\varepsilon)} \mathbb{RPM}_\varepsilon((\alpha_i,\beta_i),u_\varepsilon) + \frac{\beta_i - \alpha_i}{\varepsilon^2 |\log\varepsilon|^4}.
\end{aligned}$$

Summing over i we get that

$$\int_{A_\varepsilon} |u_{\varepsilon,T}'(x)|\, dx = \sum_{i\in I} |u_{\varepsilon,T}(\beta_i) - u_{\varepsilon,T}(\alpha_i)| \le \frac{2T^{1/2}}{M(\varepsilon)} \mathbb{RPM}_\varepsilon(A_\varepsilon,u_\varepsilon) + \frac{|A_\varepsilon|}{\varepsilon^2 |\log\varepsilon|^4}.$$

Therefore, we can estimate the total variation of $u_{\varepsilon,T}$ in the following way

$$
\begin{aligned}
\int_a^b |u'_{\varepsilon,T}(x)|\,dx &= \int_{A_\varepsilon} |u'_{\varepsilon,T}(x)|\,dx + \int_{B_\varepsilon} |u'_{\varepsilon,T}(x)|\,dx + \int_{C_\varepsilon} |u'_{\varepsilon,T}(x)|\,dx \\
&\leq \frac{2T^{1/2}}{M(\varepsilon)}\,\mathbb{RPM}_\varepsilon(A_\varepsilon, u_\varepsilon) + \frac{|A_\varepsilon|}{\varepsilon^2|\log\varepsilon|^4} + \frac{|B_\varepsilon|}{\varepsilon^2|\log\varepsilon|^4} + \frac{|C_\varepsilon|}{|\log\varepsilon|} \\
&\leq \left(\frac{2T^{1/2}}{M(\varepsilon)} + \zeta(\varepsilon)\right)\mathbb{RPM}_\varepsilon((a,b), u_\varepsilon) + \frac{b-a}{|\log\varepsilon|},
\end{aligned}
$$

where in the last line we exploited the estimates (4.1.2) and (4.1.3). This proves (4.1.1) also in this case. □

We point out that as $\varepsilon \to 0^+$ we have that $M(\varepsilon) \to \alpha_0 > 0$, while $\zeta(\varepsilon) \to 0$, so the only possibly unbounded quantity in (4.1.1) is the functional $\mathbb{RPM}_\varepsilon((a,b), u_\varepsilon)$.

In order to treat the higher dimensional case, we need to introduce some notation to deal with one-dimensional sections of higher dimensional functions. So, let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and let us fix $\sigma \in \mathbb{S}^{d-1}$. Let $\pi_\sigma : \mathbb{R}^d \to \sigma^\perp$ be the orthogonal projection and let us set $\Omega_\sigma := \pi_\sigma(\Omega)$. For every $x' \in \Omega_\sigma$ we can consider the one-dimensional section of Ω in direction σ passing through x', namely the set $\Omega_{x',\sigma} := \{y \in \mathbb{R} : x' + \sigma y \in \Omega\}$.

With this notation, for every function $u : \Omega \to \mathbb{R}$, every direction $\sigma \in \mathbb{S}^{d-1}$ and every point $x' \in \Omega_\sigma$, the one-dimensional section of u in direction σ passing through the point x' is the function $u_{x',\sigma}(y) := u(x' + \sigma y)$, which is defined for every y in the set $\Omega_{x',\sigma}$.

We can now state and prove the compactness statement.

Theorem 4.1.2 (Equicoerciveness)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary, $\{\varepsilon_n\} \subset (0,1)$ a sequence such that $\varepsilon_n \to 0^+$ and $\{u_n\} \subset H^2(\Omega)$ be a sequence of functions such that*

$$\sup_{n\in\mathbb{N}} \mathbb{RPM}_{\varepsilon_n}(\Omega, u_n) + \int_\Omega |u_n(x)|\,dx < +\infty. \tag{4.1.4}$$

Then there exists an increasing sequence $\{n_k\}$ of positive integers and a function $u \in PJ(\Omega)$ such that $u_{n_k}(x) \to u(x)$ for almost every $x \in \Omega$ and that

$$\min\{\max\{u_{n_k}, -T\}, T\} \to \min\{\max\{u, -T\}, T\} \quad \forall T > 0,$$

in $L^1(\Omega)$ and in the weak sense of $BV(\Omega)$.

Proof. Let us fix $\sigma \in \mathbb{S}^{d-1}$. We observe that

$$
\begin{aligned}
\mathbb{RPM}_{\varepsilon_n}(\Omega, u_n) &= \int_{\Omega_\sigma} dx' \int_{\Omega_{x',\sigma}} \left[\varepsilon^6 \|\nabla^2 u_n(x' + \sigma y)\|^2 + \frac{1}{\omega(\varepsilon)^2}\log(1 + |\nabla u_n(x' + \sigma y)|^2)\right] dy \\
&\geq \int_{\Omega_\sigma} \mathbb{RPM}_{\varepsilon_n}(\Omega_{x',\sigma}, (u_n)_{x',\sigma})\,dx',
\end{aligned} \tag{4.1.5}
$$

hence for almost every $x' \in \Omega_\sigma$ we have that $\mathbb{RPM}_{\varepsilon_n}(\Omega_{x',\sigma}, (u_n)_{x',\sigma}) < +\infty$ for every $n \in \mathbb{N}$ (but the bound depend on x' and n).

Let us fix $x' \in \Omega_\sigma$ with this property, so that in particular $(u_n)_{x',\sigma} \in H^2(\Omega_{x',\sigma})$ for every $n \in \mathbb{N}$. Let us fix also $T > 0$ and let us set $u_{n,T} = \min\{\max\{u_n, -T\}, T\}$.

Let us assume for a while that $\Omega_{x',\sigma}$ is an open interval for every $x' \in \Omega_\sigma$ (this happens for example if Ω is convex). Then, we can apply Lemma 4.1.1, so we deduce that

$$\begin{aligned}\int_\Omega |\partial_\sigma u_{n,T}(x)|\,dx &= \int_{\Omega_\sigma} dx' \int_{\Omega_{x',\sigma}} |(u_{n,T})'_{x',\sigma}(y)|\,dy \\ &\leq \int_{\Omega_\sigma} \left[\frac{\mathcal{H}^1(\Omega_{x',\sigma})}{|\log\varepsilon_n|} + \left(\frac{2T^{1/2}}{M(\varepsilon_n)} + \zeta(\varepsilon_n)\right)\mathbb{RPM}_{\varepsilon_n}(\Omega_{x',\sigma}, (u_n)_{x',\sigma}) + 2T\right] dx' \\ &\leq \frac{|\Omega|}{|\log\varepsilon_n|} + \left(\frac{2T^{1/2}}{M(\varepsilon_n)} + \zeta(\varepsilon_n)\right)\mathbb{RPM}_{\varepsilon_n}(\Omega, u_n) + 2TK(\Omega),\end{aligned} \tag{4.1.6}$$

where $K(\Omega) := \sup\{\mathcal{H}^{d-1}(\Omega_\sigma) : \sigma \in \mathbb{S}^{d-1}\} < +\infty$.

Now, if Ω is convex, or more generally if it is convex in d different directions, namely if there exist d linearly independent directions $\sigma_1, \ldots, \sigma_d \in \mathbb{S}^{d-1}$ such that for every j the set Ω_{x',σ_j} is an interval for (almost) every $x' \in \Omega_{\sigma_j}$, then we can apply (4.1.6) in every such direction and we deduce that the total variation of $u_{n,T}$ is bounded.

In the general case, we can cover Ω with finitely many open subsets $\Omega_1, \ldots, \Omega_N \subseteq \Omega$ that have this property. Indeed, it is enough to cover $\overline{\Omega}$ with finitely many open cubes $Q_1, \ldots, Q_N$ such that each intersection $\Omega_i := \Omega \cap Q_i$ is either Q_i (namely $Q_i \subseteq \Omega$) or is the subgraph of a Lipshitz function f_i defined on a $(d-1)$-dimensional cube.

In the first case Ω_i is convex. In the second case, if $\sigma_1^i \in \mathbb{S}^{d-1}$ is the direction orthogonal to the domain of f_i, from the Lipschitz continuity of f_i we deduce that there exists a neighborhood $\Sigma^i \subseteq \mathbb{S}^{d-1}$ of σ_1^i such that $(\Omega_i)_{x',\sigma}$ is an interval for every $\sigma \in \Sigma^i$. Therefore we can complete σ_1^i to a base $\sigma_1^i, \ldots, \sigma_d^i$ of $\mathbb{R}^d$ in such a way that $(\Omega_i)_{x',\sigma_j^i}$ is an interval for every $x' \in (\Omega_i)_{\sigma_j^i}$.

Hence, for every $i \in \{1, \ldots, N\}$ there exists a constant C_i (that is equal to 1 when $\Omega_i = Q_i$ and depends only on the base $\sigma_1^i, \ldots, \sigma_d^i$ in the other case) such that

$$\int_{\Omega_i} |\nabla u_{n,T}(x)|\,dx \leq C_i \sum_{j=1}^d \int_{\Omega_i} |\partial_{\sigma_j^i} u_{n,T}(x)|\,dx,$$

so we can apply (4.1.6) and we obtain that

$$\int_{\Omega_i} |\nabla u_{n,T}(x)|\,dx \leq C_i d\left[\frac{|\Omega|}{|\log\varepsilon_n|} + \left(\frac{2T^{1/2}}{M(\varepsilon_n)} + \zeta(\varepsilon_n)\right)\mathbb{RPM}_{\varepsilon_n}(\Omega, u_n) + 2TK(\Omega)\right].$$

Hence, if we set $C := \max_i C_i$, summing over i we get that

$$\begin{aligned}\int_\Omega |\nabla u_{n,T}(x)|\,dx &\leq \sum_{i=1}^N \int_{\Omega_i} |\nabla u_{n,T}(x)|\,dx \\ &\leq NCd\left[\frac{|\Omega|}{|\log\varepsilon_n|} + \left(\frac{2T^{1/2}}{M(\varepsilon_n)} + \zeta(\varepsilon_n)\right)\mathbb{RPM}_{\varepsilon_n}(\Omega, u_n) + 2TK(\Omega)\right].\end{aligned}$$

Hence we deduce that for every $T > 0$ there exists a subsequence $\{n_k\}$ and a function $u_T \in BV(\Omega)$ such that $u_{n_k,T} \to u_T$ weakly in $BV(\Omega)$, strongly in $L^1(\Omega)$ and pointwise almost everywhere. By a diagonal argument we can find a single subsequence such that $u_{n_k,T} \to u_T$ for every $T \in \mathbb{N}$, and hence also for every $T > 0$. It follows that there exists a function $u \in GBV(\Omega)$ for which $u_T = \min\{\max\{u, -T\}, T\}$, and $u_{n_k,T} \to u_T = \min\{\max\{u, -T\}, T\}$ for every $T > 0$. We point out that $u \in L^1(\Omega)$ because

$$\sup_{T>0} \int_\Omega |u_T(x)|\, dx \leq \sup_{T>0} \lim_{k\to+\infty} \int_\Omega |u_{n_k,T}(x)|\, dx \leq \limsup_{k\to+\infty} \int_\Omega |u_{n_k}(x)|\, dx < +\infty,$$

and the left-hand side is equal to the L^1 norm of u.

It remains to prove that $u \in PJ(\Omega)$. To this end, we argue again by sections, so let us fix $\sigma \in \mathbb{S}^{d-1}$. From the pointwise convergence, we deduce that for almost every $x' \in \Omega_\sigma$ we have that

$$(u_{n_k,T})_{x',\sigma} \to \min\{\max\{u_{x',\sigma}, -T\}, T\} \quad \text{in } L^1(\Omega_{x'\sigma}) \quad \forall T > 0. \tag{4.1.7}$$

Moreover, from (4.1.4), (4.1.5) and Fatou's lemma we deduce that

$$\begin{aligned}
&\int_{\Omega_\sigma} \left(\liminf_{k\to+\infty} \mathbb{RPM}_{\varepsilon_{n_k}}(\Omega_{x',\sigma}, (u_{n_k})_{x',\sigma}) + \int_{\Omega_{x',\sigma}} |(u_{n_k})_{x',\sigma}(y)|\, dy \right) dx' \\
&\qquad \leq \liminf_{k\to+\infty} \int_{\Omega_\sigma} \left(\mathbb{RPM}_{\varepsilon_{n_k}}(\Omega_{x',\sigma}, (u_{n_k})_{x',\sigma}) + \int_{\Omega_{x',\sigma}} |(u_{n_k})_{x',\sigma}(y)|\, dy \right) dx' \\
&\qquad \leq \liminf_{k\to+\infty} \mathbb{RPM}_{\varepsilon_{n_k}}(\Omega, u_{n_k}) + \int_\Omega |u_{n_k}(x)|\, dx < +\infty.
\end{aligned}$$

Therefore for almost every $x' \in \Omega_\sigma$ we can find a further subsequence (not relabelled) such that

$$\sup_{k\in\mathbb{N}} \mathbb{RPM}_{\varepsilon_{n_k}}(\Omega_{x',\sigma}, (u_{n_k})_{x',\sigma}) + \int_{\Omega_{x',\sigma}} |(u_{n_k})_{x',\sigma}(y)|\, dy < +\infty. \tag{4.1.8}$$

From the L^1 version of the one-dimensional compactness statement in Theorem 2.3.2 (see Remark 2.3.3) we deduce that there exists a function $z_{x',\sigma} \in PJ(\Omega_{x'\sigma})$ such that, up a further subsequence, $(u_{n_k})_{x',\sigma} \to z_{x',\sigma}$, in $L^1(\Omega_{x'\sigma})$. Therefore, recalling (4.1.7), we deduce that $u_{x',\sigma} = z_{x',\sigma} \in PJ(\Omega_{x',\sigma})$.

We point out that the subsequence for which (4.1.8) holds does depend on x', but anyway it follows that for every $\sigma \in \mathbb{S}^{d-1}$ we have that $u_{x',\sigma} \in PJ(\Omega_{x',\sigma})$ for almost every $x' \in \Omega_\sigma$. This, together with the fact that $u \in GBV(\Omega)$, is enough to conclude that $u \in PJ(\Omega)$, thanks to the relation between the derivatives of one-dimensional sections and the BV differential (see [6, Theorem 3.107 and Theorem 3.108]). □

At this point, the liminf inequality can be proved by standard arguments (see [29]), exploiting the following lemma.

Lemma 4.1.3 (Proposition 1.16 in [29])**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set and let $\mathcal{A}(\Omega)$ denote the family of all open subsets of Ω. Let $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ be a function* *such that* $\mu(A \cup B) \geq \mu(A) + \mu(B)$ *for every* $A, B \in \mathcal{A}(\Omega)$ *such that* $\overline{A} \cap \overline{B} = \emptyset$ *and* $\overline{A}, \overline{B} \subseteq \Omega$. *Let also* λ *be a non-negative Radon measure on* Ω *and let* $\{\psi_i\}$ *be a sequence of non-negative Borel functions defined on* Ω.

Let us assume that $\mu(A) \geq \int_A \psi_i \, d\lambda$ *for every* $i \in \mathbb{N}$ *and every* $A \in \mathcal{A}(\Omega)$ *and let us set* $\psi(x) := \sup_i \psi_i(x)$.

Then it turns out that $\mu(A) \geq \int_A \psi \, d\lambda$ *for every* $A \in \mathcal{A}(\Omega)$.

Theorem 4.1.4 (Liminf inequality)**.** *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded open set with Lipschitz boundary, and let* $\{\varepsilon_n\} \subset (0,1)$ *a sequence such that* $\varepsilon_n \to 0^+$. *Let* $\{u_n\} \subset H^2(\Omega)$ *be a sequence of functions and* $u \in L^1(\Omega)$ *be a function such that* $u_n \to u$ *in* $L^1(\Omega)$. *Then it turns out that*

$$\liminf_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}(\Omega, u_n) \geq \alpha_0 \, \mathbb{J}_{1/2}(\Omega, u),$$

where α_0 *is defined by (2.3.7) and the functional* $\mathbb{J}_{1/2}$ *is set by definition equal to* $+\infty$ *outside* $PJ(\Omega)$.

Proof. Without loss of generality, we can assume that

$$\liminf_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}(\Omega, u_n) = \lim_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}(\Omega, u_n) < +\infty.$$

From Theorem 4.1.2 we immediately deduce that $u \in PJ(\Omega)$.

Now let us fix $\sigma \in \mathbb{S}^{d-1}$. Up to a subsequence, we can assume that $(u_n)_{x',\sigma} \to u_{x',\sigma}$ in $L^1(\Omega_{x',\sigma})$ for almost every $x' \in \Omega_\sigma$. Then by Fatou's lemma and the one-dimensional Gamma-convergence result in Theorem 2.3.2, for every open set $A \subseteq \Omega$ we have that

$$\begin{aligned}
\liminf_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}(A, u_n) &\geq \liminf_{n\to+\infty} \int_{A_\sigma} \mathbb{RPM}_{\varepsilon_n}(A_{x',\sigma}, (u_n)_{x',\sigma}) \, dx' \\
&\geq \int_{A_\sigma} \liminf_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}(A_{x',\sigma}, (u_n)_{x',\sigma}) \, dx' \\
&\geq \int_{A_\sigma} \alpha_0 \, \mathbb{J}_{1/2}(A_{x',\sigma}, u_{x',\sigma}) \, dx' \\
&= \alpha_0 \int_{S_u \cap A} |u^+(x) - u^-(x)|^{1/2} |\nu_u(x) \cdot \sigma| \, d\mathcal{H}^{d-1}(x).
\end{aligned}$$

Now let us fix $T > 0$ and let us set

$$\mu(A) := \liminf_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}(A, u_n), \qquad \lambda_T := \alpha_0 |u_T^+ - u_T^-|^{1/2} \mathcal{H}^{d-1} \llcorner S_u,$$

where $u_T := \min\{\max\{u, -T\}, T\}$. Let also $\{\sigma_i\} \subseteq \mathbb{S}^{d-1}$ be a dense sequence and $\psi_i : \Omega \to \mathbb{R}$ be the functions $\psi_i(x) = |\nu_u(x) \cdot \sigma_i| \mathbb{1}_{S_u}(x)$.

Then $\mu : \mathcal{A}(\Omega) \to \mathbb{R}$ is superadditive, λ_T is a Radon measure and $\mu(A) \geq \int_A \psi_i \, d\lambda_T$ for every i. Hence from Lemma 4.1.3 we deduce that

$$\begin{aligned}
\liminf_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}(\Omega, u_n) &\geq \alpha_0 \int_{S_u} \left(\sup_i |\nu_u(x) \cdot \sigma_i| \right) |u_T^+(x) - u_T^-(x)|^{1/2} \, d\mathcal{H}^{d-1}(x) \\
&= \alpha_0 \, \mathbb{J}_{1/2}(\Omega, u_T),
\end{aligned}$$

and we conclude letting $T \to +\infty$. □

4.2 Estimate from below

Since from now on we always consider cubic domains, for every $L > 0$ let us set $Q_L^d = (0, L)^d$. Similarly to (2.3.12) and (2.3.13), for every $(\alpha, \beta, L, \xi) \in (0, +\infty)^3 \times \mathbb{R}^d$ let us consider the following minimum problems

$$\mu_\varepsilon^d(\beta, L, \xi) := \min_{u \in H^2(Q_L^d)} \mathbb{RPMF}_\varepsilon(\beta, \xi \cdot x, Q_L^d, u), \tag{4.2.9}$$

$$\mu_0^d(\alpha, \beta, L, \xi) := \min_{u \in PJ(Q_L^d)} \mathbb{JF}_{1/2}(\alpha, \beta, \xi \cdot x, Q_L^d, u). \tag{4.2.10}$$

The existence of minimizers for μ_ε^d and μ_0^d is a standard application of the direct method in calculus of variations. Now we prove the properties of μ_ε^d and μ_0^d that we need in the proof of Theorem 4.0.1.

Proposition 4.2.1. *Let us fix $(\beta, L) \in (0, +\infty)^2$, let $\{\varepsilon_n\} \subset (0, 1)$ be a sequence such that $\varepsilon_n \to 0^+$ and let $\{\xi_n\} \subset \mathbb{R}^d$ be a bounded sequence.*

Then it turns out that

$$\liminf_{n \to +\infty} \left(\mu_{\varepsilon_n}^d(\beta, L, \xi_n) - \mu_0^d(\alpha_0, \beta, L, \xi_n)\right) \geq 0.$$

Proof. Without loss of generality, we can assume that $\xi_n \to \xi_\infty \in \mathbb{R}^d$ and that the liminf is a limit.

For every $n \in \mathbb{N}$, let u_n be a minimizer for the minimum problem $\mu_{\varepsilon_n}^d(\beta, L, \xi_n)$. We observe that

$$\mathbb{RPMF}_{\varepsilon_n}(\beta, \xi_n \cdot x, Q_L^d, u_n) \leq \mathbb{RPMF}_{\varepsilon_n}(\beta, \xi_n \cdot x, Q_L^d, 0) = \beta \int_{Q_L^d} (\xi_n \cdot x)^2.$$

As a consequence, we obtain that

$$\sup_{n \in \mathbb{N}} \mathbb{RPM}_{\varepsilon_n}(Q_L^d, u_n) + \int_{Q_L^d} u_n(x)^2 \, dx < +\infty$$

Then, from Theorem 4.1.2, we deduce that there exists a subsequence $n_k \to +\infty$ and $u \in PJ(Q_L^d)$ such that $u_{n_k}(x) \to u(x)$ for almost every $x \in Q_L^d$ and $u_{n_k} \rightharpoonup u$ in $L^2(Q_L^d)$. From [33, Exercise 4.16] it follows that $u_{n_k} \to u$ in $L^1(Q_L^d)$, and hence Theorem 4.1.4 yields

$$\liminf_{k \to +\infty} \mathbb{RPM}_{\varepsilon_{n_k}}(Q_L^d, u_{n_k}) \geq \alpha_0 \, \mathbb{J}_{1/2}(Q_L^d, u).$$

Therefore we have that

$$\begin{aligned}
&\lim_{k\to+\infty} (\mu^d_{\varepsilon_{n_k}}(\beta, L, \xi_{n_k}) - \mu^d_0(\alpha_0, \beta, L, \xi_{n_k})) \\
&= \lim_{k\to+\infty} \mathbb{RPM}_{\varepsilon_{n_k}}(Q^d_L, u_{n_k}) + \beta \int_{Q^d_L} (u_{n_k}(x) - \xi_{n_k}\cdot x)^2\,dx - \mu^d_0(\alpha_0, \beta, L, \xi_{n_k}) \\
&\geq \liminf_{k\to+\infty} \alpha_0\,\mathbb{J}_{1/2}(Q^d_L, u) + \beta \int_{Q^d_L} (u_{n_k}(x) - \xi_{n_k}\cdot x)^2\,dx - \mu^d_0(\alpha_0, \beta, L, \xi_{n_k}) \\
&= \liminf_{k\to+\infty} \mathbb{JF}_{1/2}(\alpha_0, \beta, \xi_{n_k}\cdot x, Q^d_L, u) - \mu^d_0(\alpha_0, \beta, L, \xi_{n_k}) \\
&\quad + \beta \int_{Q^d_L} (u_{n_k}(x) - \xi_{n_k}\cdot x)^2\,dx - \beta \int_{Q^d_L} (u(x) - \xi_{n_k}\cdot x)^2\,dx \\
&\geq \liminf_{k\to+\infty} \beta \int_{Q^d_L} (u_{n_k}(x) - \xi_{n_k}\cdot x)^2\,dx - \beta \int_{Q^d_L} (u(x) - \xi_{n_k}\cdot x)^2\,dx,
\end{aligned}$$

and the last line is non-negative because $u_{n_k} \rightharpoonup u$ and $\xi_{n_k}\cdot x \to \xi_\infty \cdot x$ in $L^2(Q^d_L)$. □

Proposition 4.2.2. *For every $(\alpha, \beta, L, \xi) \in (0, +\infty)^3 \times \mathbb{R}^d$ it turns out that*

$$\mu^d_0(\alpha, \beta, L, \xi) \geq c_1|\xi|^{4/5}L^d - c_2K_d|\xi|^{1/5}L^{d-1}, \tag{4.2.11}$$

where c_1 and c_2 are the constants appearing in Proposition 2.4.4 and

$$K_d := \sup_{\sigma\in\mathbb{S}^{d-1}} \{\mathcal{H}^{d-1}((Q^d_1)_\sigma)\}. \tag{4.2.12}$$

Proof. Let $u \in PJ(Q^d_L)$ be a function and let us set $\sigma = \xi/|\xi|$. Then we have that

$$\int_{\Omega\cap S_u} \sqrt{|u^+(x) - u^-(x)|}\,d\mathcal{H}^{d-1}(x) \geq \int_{(Q^d_L)_\sigma} \mathbb{J}_{1/2}(\Omega_{x',\sigma}, u_{x',\sigma})\,dx',$$

and

$$\int_{Q^d_L} (u(x) - \xi\cdot x)^2\,dx = \int_{(Q^d_L)_\sigma} dx' \int_{(Q^d_L)_{x',\sigma}} (u_{x',\sigma}(y) - |\xi|y)^2\,dy.$$

Hence, recalling the estimate (2.4.1) for the one-dimensional case, we deduce that

$$\begin{aligned}
\mathbb{JF}_{1/2}(\alpha, \beta, \xi\cdot x, Q^d_L, u) &\geq \int_{(Q^d_L)_\sigma} \mathbb{JF}(\alpha, \beta, |\xi|y, (Q^d_L)_{x',\sigma}, u_{x',\sigma})\,dx' \\
&\geq \int_{(Q^d_L)_\sigma} \mu_0(\alpha, \beta, \mathcal{H}^1((Q^d_L)_{x',\sigma}), |\xi|)\,dx' \\
&\geq \int_{(Q^d_L)_\sigma} \left[c_1|\xi|^{4/5}\mathcal{H}^1((Q^d_L)_{x',\sigma}) - c_2|\xi|^{1/5}\right]dx' \\
&= c_1|\xi|^{4/5}L^d - c_2|\xi|^{1/5}\mathcal{H}^{d-1}((Q^d_L)_\sigma) \\
&\geq c_1|\xi|^{4/5}L^d - c_2|\xi|^{1/5}K_dL^{d-1}.
\end{aligned}$$

Taking the infimum of the left-hand side over all $u \in PJ(Q^d_L)$ we obtain (4.2.11). □

We are now ready to prove the estimate from below for the minimum value in Theorem 4.0.1, that is the following proposition.

Proposition 4.2.3. *For every $\beta > 0$, every $f \in C^1(\overline{Q_1^d})$ and every family of functions $\{u_\varepsilon\} \subset H^2(Q_1^d)$ it turns out that*

$$\liminf_{\varepsilon\to 0} \frac{1}{\omega(\varepsilon)^2}\, \mathbb{PMF}_\varepsilon(\beta, f, Q_1^d, u_\varepsilon) \geq 10 \left(\frac{2\beta}{27}\right)^{1/5} \int_{Q_1^d} |\nabla f(x)|^{4/5}\, dx. \tag{4.2.13}$$

Proof. We divide the proof in several steps.

Domain subdivision and approximation of the forcing term Let us fix a positive real number $L > 0$. For every $\varepsilon \in (0,1)$ let us set

$$N_{\varepsilon,L} := \left\lfloor \frac{1}{L\omega(\varepsilon)} \right\rfloor, \qquad L_\varepsilon := \frac{1}{N_{\varepsilon,L}\omega(\varepsilon)},$$

and

$$\mathbb{N}^d_{N_{\varepsilon,L}} := \left\{(z_1, \ldots, z_d) \in \mathbb{N}^d : 1 \leq z_i \leq N_{\varepsilon,L} \quad \forall i \in \{1, \ldots, d\}\right\}.$$

For every $z \in \mathbb{N}^d_{N_{\varepsilon,L}}$ let us define also

$$Q_z := \left\{x \in (0,1)^d : x_i \in ((z_i - 1)L_\varepsilon\omega(\varepsilon), z_i L_\varepsilon \omega(\varepsilon)) \quad \forall i \in \{1, \ldots, d\}\right\},$$

and the functions $f_z : Q_z \to \mathbb{R}$ as

$$f_z(x) = f(zL_\varepsilon\omega(\varepsilon)) + \xi_{\varepsilon,L,z} \cdot (x - zL_\varepsilon\omega(\varepsilon)),$$

where $\xi_{\varepsilon,L,z} = \nabla f(zL_\varepsilon\omega(\varepsilon))$.

Now we can define the function $f_{\varepsilon,L} : Q_1^d \to \mathbb{R}$ such that $f_{\varepsilon,L}(x) = f_z(x)$ for every $x \in Q_z$ and we observe that

$$\begin{aligned}
\int_{Q_1^d} (f - f_{\varepsilon,L})^2 &= \sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} \int_{Q_z} (f - f_z)^2 \\
&\leq \sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} |Q_z| \left(diam(Q_z) \cdot \sup\left\{|\nabla f(x) - \nabla f(zL_\varepsilon\omega(\varepsilon))| : x \in Q_z\right\}\right)^2 \\
&\leq N^d_{\varepsilon,L}(L_\varepsilon\omega(\varepsilon))^{d+2} d \left(\sup\left\{|\nabla f(x) - \nabla f(y)| : |x - y| \leq L_\varepsilon\omega(\varepsilon)\sqrt{d}\right\}\right)^2 \\
&= (L_\varepsilon\omega(\varepsilon))^2 d \left(\sup\left\{|\nabla f(x) - \nabla f(y)| : |x - y| \leq L_\varepsilon\omega(\varepsilon)\sqrt{d}\right\}\right)^2,
\end{aligned}$$

and hence

$$\lim_{\varepsilon\to 0^+} \frac{1}{\omega(\varepsilon)^2} \int_{Q_1^d} (f - f_{\varepsilon,L})^2 = 0, \tag{4.2.14}$$

because $L_\varepsilon \to L$ as $\varepsilon \to 0^+$ and ∇f is uniformly continuous.

Moreover we have that $\nabla f_{\varepsilon,L} \to \nabla f$ uniformly and this implies that

$$\lim_{\varepsilon\to 0^+} \sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} |\xi_{\varepsilon,L,z}|^{4/5}(L_\varepsilon\omega(\varepsilon))^d = \lim_{\varepsilon\to 0^+}\int_{Q_1^d} |\nabla f_{\varepsilon,L}|^{4/5} = \int_{Q_1^d} |\nabla f|^{4/5}. \tag{4.2.15}$$

Finally, for every $\eta \in (0,1)$, we have that

$$\int_{Q_1^d} (u_\varepsilon - f)^2 \geq (1-\eta)\int_{Q_1^d}(u_\varepsilon - f_{\varepsilon,L})^2 + \left(1-\frac{1}{\eta}\right)\int_{Q_1^d}(f-f_{\varepsilon,L})^2,$$

so we conclude that

$$\mathbb{PMF}_\varepsilon(\beta, f, Q_1^d, u_\varepsilon) \geq (1-\eta)\,\mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, Q_1^d, u_\varepsilon) + \left(1-\frac{1}{\eta}\right)\beta\int_{Q_1^d}(f-f_{\varepsilon,L})^2.$$

Recalling (4.2.14), if we divide by $\omega(\varepsilon)^2$ and we take the liminf we obtain that

$$\liminf_{\varepsilon\to 0^+}\frac{1}{\omega(\varepsilon)^2}\,\mathbb{PMF}_\varepsilon(\beta, f, Q_1^d, u_\varepsilon) \geq (1-\eta)\liminf_{\varepsilon\to 0^+}\frac{1}{\omega(\varepsilon)^2}\,\mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, Q_1^d, u_\varepsilon). \tag{4.2.16}$$

Reduction to a common hypercube We prove that

$$\mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, Q_1^d, u_\varepsilon) \geq \omega(\varepsilon)^{d+2}\sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} \mu_\varepsilon^d(\beta, L, \xi_{\varepsilon,L,z}) \tag{4.2.17}$$

where μ_ε^d is defined by (4.2.9)

To this end, let us define the functions $v_{\varepsilon,L,z} : Q^d_{L_\varepsilon} \to \mathbb{R}$ as

$$v_{\varepsilon,L,z}(y) := \frac{f(zL_\varepsilon\omega(\varepsilon)) - u_\varepsilon(zL_\varepsilon\omega(\varepsilon) - \omega(\varepsilon)y)}{\omega(\varepsilon)}.$$

Then, with a change of variable in the integrals we obtain that

$$\begin{aligned}
\mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, Q_1^d, u_\varepsilon) &= \sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} \mathbb{PMF}_\varepsilon(\beta, f_z, Q_z, u_\varepsilon)\\
&= \sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} \omega(\varepsilon)^{d+2}\,\mathbb{RPMF}_\varepsilon(\beta, \xi_{\varepsilon,L,z}\cdot x, Q^d_{L_\varepsilon}, v_{\varepsilon,L,z})\\
&\geq \sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} \omega(\varepsilon)^{d+2}\mu_\varepsilon^d(\beta, L_\varepsilon, \xi_{\varepsilon,L,z})\\
&\geq \sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} \omega(\varepsilon)^{d+2}\mu_\varepsilon^d(\beta, L, \xi_{\varepsilon,L,z}),
\end{aligned}$$

which is exactly (4.2.17). We observe that the last inequality follows from the fact that $L_\varepsilon \geq L$ and the monotonicity of $\mu_\varepsilon^d(\beta, L, \xi)$ with respect to L, which is trivial because the functional $\mathbb{RPMF}_\varepsilon$ decreases if we restrict the domain.

Convergence to minima of the limit problem We prove that there exists $\varepsilon_0 > 0$ such that

$$\mu^d_\varepsilon(\beta, L, \xi_{\varepsilon,L,z}) \geq \mu^d_0(\alpha_0, \beta, L, \xi_{\varepsilon,L,z}) - \eta, \tag{4.2.18}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and every $z \in \mathbb{N}^d_{N_{\varepsilon,L}}$.

Indeed, if (4.2.18) were false, then we could find two sequences $\{\varepsilon_n\} \subset (0,1)$ and $\{\xi_{\varepsilon_n,L,z_n}\} \subseteq \mathbb{R}^d$ such that $\varepsilon_n \to 0^+$ and

$$\mu^d_{\varepsilon_n}(\beta, L, \xi_{\varepsilon_n,L,z_n}) < \mu^d_0(\alpha_0, \beta, L, \xi_{\varepsilon_n,L,z_n}) - \eta,$$

but this contradicts Proposition 4.2.1, because $|\xi_{\varepsilon,L,z}| \leq \|\nabla f\|_\infty$.

Conclusion Combining (4.2.17), (4.2.18) and Proposition 4.2.2 we deduce that

$$\begin{aligned} \mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, Q^d_1, u_\varepsilon) \;\; \geq \;\; & \omega(\varepsilon)^{d+2} \sum_{z \in \mathbb{N}^d_{N_{\varepsilon,L}}} c_1 |\xi_{\varepsilon,L,z}|^{4/5} L^d \\ & - \omega(\varepsilon)^{d+2} N^d_{\varepsilon,L} \left(c_2 K_d \|\nabla f\|^{1/5}_\infty L^{d-1} + \eta \right). \end{aligned}$$

Therefore from (4.2.16) we deduce that

$$\begin{aligned} \liminf_{\varepsilon \to 0^+} \frac{1}{\omega(\varepsilon)^2} \mathbb{PMF}_\varepsilon(\beta, Q^d_1, f, u_\varepsilon) \;\; \geq \;\; & \liminf_{\varepsilon \to 0^+} (1-\eta) c_1 \frac{L^d}{L^d_\varepsilon} \sum_{z \in \mathbb{N}^d_{N_{\varepsilon,L}}} |\xi_{\varepsilon,L,z}|^{4/5} (L_\varepsilon \omega(\varepsilon))^d \\ & - \frac{c_2 K_d \|\nabla f\|^{1/5}_\infty}{L} - \frac{\eta}{L^d}. \end{aligned}$$

Finally, exploiting (4.2.15), we conclude that

$$\liminf_{\varepsilon \to 0^+} \frac{1}{\omega(\varepsilon)^2} \mathbb{PMF}_\varepsilon(\beta, f, Q^d_1, u_\varepsilon) \geq (1-\eta) \left[c_1 \int_{Q^d_1} |\nabla f|^{4/5} - \frac{c_2 K_d \|\nabla f\|^{1/5}_\infty}{L} - \frac{\eta}{L^d} \right]$$

Letting $\eta \to 0^+$ and $L \to +\infty$, and recalling that $c_1 = 10(2\beta/27)^{1/5}$ when $\alpha = \alpha_0$, we get exactly (4.2.13). □

4.3 Estimate from above

In order to estimate the minimum values from above, we have to construct an appropriate family of functions $\{u_\varepsilon\} \subset H^2(Q^d_1)$ showing the asymptotic optimality of all the steps in the estimate from below. To this end, in the one-dimensional case, we introduced the minimum problems with boundary conditions (2.3.14) and (2.3.15), and we proved some estimates based on Gamma-convergence.

Here the situation is more delicate, because in higher dimensions prescribing boundary conditions is more complicated, so we use cutoff functions to ensure that the functions u_ε coincide with the forcing term near the boundary of each of the small cubes in which we divide our domain. In this way, when we glue together the functions defined in each cube we do not lose regularity.

However, this approach requires additional estimates on our approximate minimizers, in order to control the additional terms coming from the cutoff functions, so we now construct explicitly these approximate minimizers. To this hand, let us fix $M \in \mathbb{R}$ and let us consider the canonical (H, V)-staircase with parameters given by (2.4.4), with $\alpha = \alpha_0$.

Let us consider also the canonical (Λ, V)-cubic connection $C_{\Lambda,V}$ defined in (3.2.2) with parameter $\Lambda = (3V/2)^{1/2}$ defined as in (3.2.4).

Then, for every $\varepsilon \in (0,1)$ such that $\Lambda\varepsilon^2 < H/2$, we consider the function $S_{M,\varepsilon} : \mathbb{R} \to \mathbb{R}$ defined by

$$S_{M,\varepsilon}(x) := \begin{cases} (2z+1)V + C_{\Lambda,V}(\varepsilon^{-2}(x-(2z+1)H)) & \text{if } x \in I_{z,M,\varepsilon} \text{ for some } z \in \mathbb{Z}, \\ S_{H,V}(x) & \text{otherwise,} \end{cases} \tag{4.3.19}$$

where $I_{z,M,\varepsilon} = [(2z+1)H - \Lambda\varepsilon^2, (2z+1)H + \Lambda\varepsilon^2]$ for every $z \in \mathbb{Z}$.

We observe that $S_{M,\varepsilon} \in C^1(\mathbb{R}) \cap H^2_{loc}(\mathbb{R})$, and that $S_{M,\varepsilon} \rightsquigarrow S_{H,V}$ locally strictly in $BV_{loc}(\mathbb{R})$ as $\varepsilon \to 0^+$.

Lemma 4.3.1. *Let $\beta, M_0 > 0$ be positive real numbers. Then there exist two real numbers $\varepsilon_0 \in (0,1]$ and $C > 0$ depending only on β and M_0 such that for every $\varepsilon \in (0, \varepsilon_0)$ and every $\xi \in \mathbb{R}^d$ such that $|\xi| \leq M_0$ there exists a function $w_{\xi,\varepsilon} \in H^2_{loc}(\mathbb{R}^d)$ such that*

$$\mathbb{RPMF}_\varepsilon(\beta, \xi \cdot x, Q_L^d, w_{\xi,\varepsilon}) \leq 10 \left(\frac{2\beta}{27}\right)^{1/5} |\xi|^{4/5} L^d + C\left(K_d L^{d-1} + \frac{L^d + K_d L^{d-1}}{|\log \varepsilon|}\right)$$

for every $L > 0$ and

$$|w_{\xi,\varepsilon}(x) - \xi \cdot x| \leq C \qquad |\nabla w_{\xi,\varepsilon}(x)| \leq C/\varepsilon^2 \qquad \|\nabla^2 w_{\xi,\varepsilon}\| \leq C/\varepsilon^4 \qquad \forall x \in \mathbb{R}^d, \tag{4.3.20}$$

where K_d is defined in (4.2.12).

Proof. First of all, we observe that if $\xi = 0$ the function $w_{0,\varepsilon} := 0$ satisfies all the required estimates (with $\varepsilon_0 = 1$ and $C = 0$).

Otherwise, if $\xi \neq 0$ let $H, V, \Lambda \in (0, +\infty)$ be the usual parameters defined in (2.4.4) and (3.2.4), with $\alpha = \alpha_0$ and $M = |\xi|$.

Let us fix $\varepsilon_0 \in (0,1)$ such that $\Lambda\varepsilon_0^2 < H/2$, for every $|\xi| < M_0$. We point out that Λ is increasing and H is decreasing with respect to $|\xi|$, so such ε_0 exists and depends only on β and M_0.

For every $\varepsilon \in (0, \varepsilon_0)$ and every $\xi \in \mathbb{R}^d$ with $|\xi| < M_0$, let $S_{|\xi|,\varepsilon} : \mathbb{R} \to \mathbb{R}$ be the function defined in (4.3.19) with $M = |\xi|$ and let us set also $\sigma = \xi/|\xi|$.

We define the function $w_{\xi,\varepsilon} : \mathbb{R}^d \to \mathbb{R}$ as $w_{\xi,\varepsilon}(x) := S_{|\xi|,\varepsilon}(\sigma \cdot x)$, so we now need to show that this function satisfies the required estimates.

To this end, we observe that the definition of $w_{\xi,\varepsilon}$ implies that

$$w_{\xi,\varepsilon}(x' + \sigma y) = S_{|\xi|,\varepsilon}(y), \quad |\nabla w_{\xi,\varepsilon}(x' + \sigma y)| = |S'_{|\xi|,\varepsilon}(y)|, \quad \|\nabla^2 w_{\xi,\varepsilon}(x' + \sigma y)\| = |S''_{|\xi|,\varepsilon}(y)|,$$

for every $x' \in \sigma^\perp$ and every $y \in \mathbb{R}$, hence we deduce that (4.3.20) holds with

$$C := \sup\left\{\max\{\|C'_{\Lambda,V}\|_\infty, \|C''_{\Lambda,V}\|_\infty\} : M \leq M_0\right\},$$

where the supremum is actually attained at $M = M_0$.

Moreover, using these formulas, we can compute the functional through the one-dimensional section in direction σ, namely

$$\mathbb{RPMF}_\varepsilon(\beta, \xi\cdot x, Q_L^d, w_{\xi,\varepsilon}) = \int_{(Q_L^d)_\sigma} \mathbb{RPMF}_\varepsilon(\beta, |\xi|y, (Q_L^d)_{x',\sigma}, S_{|\xi|,\varepsilon})\, dx'$$

Now we observe that for every $z \in \mathbb{Z}$ we have that

$$\begin{aligned}
&\mathbb{RPMF}_\varepsilon(\beta, |\xi|y, (2zH, 2(z+1)H), S_{|\xi|,\varepsilon}) \\
&\quad\le\ \mathbb{RPM}_\varepsilon((-\Lambda\varepsilon^2, \Lambda\varepsilon^2), C_{\Lambda,V}(\varepsilon^{-2}y)) + \beta\int_0^{2H} (S_{H,V} - |\xi|y)^2\, dy \\
&\quad\le\ \frac{6V^2}{\Lambda^3} + 2\Lambda\frac{\log(1+\varepsilon^{-4}V^2\Lambda^{-2}\|C'_{1,1}\|_\infty^2)}{|\log\varepsilon|} + \frac{2}{3}\beta|\xi|^2H^3 \\
&\quad\le\ \frac{16}{3}6^{1/2}V^{1/2} + \frac{\widehat{C}}{|\log\varepsilon|} + \frac{2}{3}\beta|\xi|^2H^3, \\
&\quad=\ 20\left(\frac{16|\xi|}{9\beta}\right)^{1/5} + \frac{\widehat{C}}{|\log\varepsilon|},
\end{aligned}$$

where $\widehat{C}$ is a constant depending only on β and M_0, and in the last line we have substituted the values of V and H as functions of $|\xi|$.

Thus, if we set $n_{x',\sigma} := \lfloor(\inf{(Q_L^d)_{x',\sigma}})/(2H)\rfloor$ and $N_{x',\sigma} := \lceil(\sup{(Q_L^d)_{x',\sigma}})/(2H)\rceil$, then $(Q_L^d)_{x',\sigma} \subseteq (2n_{x',\sigma}H, 2N_{x',\sigma}H)$, and hence we have that

$$\begin{aligned}
\mathbb{RPMF}_\varepsilon(\beta, \xi\cdot x, Q_L^d, w_{\xi,\varepsilon}) &\le \int_{(Q_L^d)_\sigma}\left[\sum_{z=n_{x',\sigma}}^{N_{x',\sigma}-1} \mathbb{RPMF}_\varepsilon(\beta, |\xi|y, (2zH, 2(z+1)H), S_{|\xi|,\varepsilon})\right] dx' \\
&\le \int_{(Q_L^d)_\sigma} (N_{x',\sigma} - n_{x',\sigma})\left[20\left(\frac{16|\xi|}{9\beta}\right)^{1/5} + \frac{\widehat{C}}{|\log\varepsilon|}\right] dx' \\
&\le \int_{(Q_L^d)_\sigma}\left(\frac{\mathcal{H}^1((Q_L^d)_{x',\sigma})}{2H} + 2\right)\left[20\left(\frac{16|\xi|}{9\beta}\right)^{1/5} + \frac{\widehat{C}}{|\log\varepsilon|}\right] dx' \\
&= \int_{(Q_L^d)_\sigma}\left[10\left(\frac{2\beta}{27}\right)^{1/5}|\xi|^{4/5}\mathcal{H}^1((Q_L^d)_{x',\sigma}) + 40\left(\frac{16|\xi|}{9\beta}\right)^{1/5}\right] dx' \\
&\quad + \frac{\widehat{C}}{|\log\varepsilon|}\int_{(Q_L^d)_\sigma}\left(\frac{\mathcal{H}^1((Q_L^d)_{x',\sigma})}{2}\left(\frac{\beta^2|\xi|^{3/5}}{24}\right)^{1/5} + 2\right) dx' \\
&\le 10\left(\frac{2\beta}{27}\right)^{1/5}|\xi|^{4/5}L^d + \widetilde{C}\left(K_dL^{d-1} + \frac{L^d + K_dL^{d-1}}{|\log\varepsilon|}\right),
\end{aligned}$$

where $\widetilde{C}$ is still a constant depending only on β and M_0. □

We can now prove the asymptotic estimate from above for the minima of $\mathbb{PMF}_\varepsilon$. The following Proposition, together with Proposition 4.2.1 completes the proof of Theorem 4.0.1.

Proposition 4.3.2. *For every $\beta > 0$ and $f \in C^1(\overline{Q_1^d})$ there exists a family of functions $\{u_\varepsilon\} \subset H^2(Q_1^d)$ such that*

$$\limsup_{\varepsilon \to 0^+} \frac{1}{\omega(\varepsilon)^2} \mathbb{PMF}_\varepsilon(\beta, f, Q_1^d, u_\varepsilon) \leq 10 \left(\frac{2\beta}{27}\right)^{1/5} \int_{Q_1^d} |\nabla f(x)|^{4/5} \, dx. \tag{4.3.21}$$

Proof. We divide the proof in several steps. Some of them are similar to those in the proof of the estimate from below, but here we need a more careful approximation of the forcing term, because we need to build a family of H^2 functions.

Regularization of the forcing term Let us extend f to the whole space $\mathbb{R}^d$ in such a way that $f \in C^1(\mathbb{R}^d)$ and ∇f is bounded on $\mathbb{R}^d$.

Let $\rho \in C^\infty(\mathbb{R}^d)$ be a non-negative smooth function supported in the unit ball of $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} \rho = 1$, and let us consider the family of mollifiers

$$\rho_\varepsilon(x) := \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right).$$

Let us set $f_\varepsilon := f * \rho_\varepsilon$. We observe that the following properties hold

$$\int_{Q_1^d} (f(x) - f_\varepsilon(x))^2 \, dx \leq \|\nabla f\|_\infty \left(\int_{\mathbb{R}^d} \rho_\varepsilon(y)|y| \, dy\right)^2 \leq \|\nabla f\|_\infty \varepsilon^2, \tag{4.3.22}$$

$$\|\nabla f_\varepsilon\|_\infty \leq \|\nabla f\|_\infty \qquad \text{and} \qquad \nabla f_\varepsilon \to \nabla f \text{ uniformly,} \tag{4.3.23}$$

$$\|\nabla^2 f_\varepsilon\|_\infty \leq \|\nabla f\|_\infty \|\nabla \rho_\varepsilon\|_{L^1} = \|\nabla f\|_\infty \|\nabla \rho\|_{L^1} \cdot \varepsilon^{-1}. \tag{4.3.24}$$

Domain subdivision and approximation of the forcing term Let us fix a positive real number $L > 0$. For every $\varepsilon \in (0,1)$ let us set

$$N_{\varepsilon,L} := \left\lceil \frac{1}{L\omega(\varepsilon)} \right\rceil, \qquad L_\varepsilon := \frac{1}{N_{\varepsilon,L}\omega(\varepsilon)},$$

and

$$\mathbb{N}^d_{N_{\varepsilon,L}} := \left\{(z_1, \ldots, z_d) \in \mathbb{N}^d : 1 \leq z_i \leq N_{\varepsilon,L} \quad \forall i \in \{1, \ldots, d\}\right\}.$$

For every $z \in \mathbb{N}^d_{N_{\varepsilon,L}}$ let us define also

$$Q_z := \left\{x \in Q_1^d : x_i \in [(z_i - 1)L_\varepsilon\omega(\varepsilon), z_i L_\varepsilon \omega(\varepsilon)) \quad \forall i \in \{1, \ldots, d\}\right\},$$

and the functions $f_z : Q_z \to \mathbb{R}$ as

$$f_z(x) = f_\varepsilon(zL_\varepsilon\omega(\varepsilon)) + \xi_{\varepsilon,L,z} \cdot (x - zL_\varepsilon\omega(\varepsilon)),$$

where $\xi_{\varepsilon,L,z} = \nabla f_\varepsilon(zL_\varepsilon\omega(\varepsilon))$.

Now we can define the function $f_{\varepsilon,L} : Q_1^d \to \mathbb{R}$ such that $f_{\varepsilon,L}(x) = f_z(x)$ for every $x \in Q_z$.

We remark that the definition that we have just introduced are similar to the corresponding definitions in the estimate from below, but now f is replaced by its regularization f_ε and the definition of $N_{\varepsilon,L}$ is slightly different, so that now $L_\varepsilon \leq L$.

Now we observe that, as in the estimate from below, we have that

$$\int_{Q_1^d}(f_\varepsilon - f_{\varepsilon,L})^2 = \sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} \int_{Q_z}(f_\varepsilon - f_z)^2 \leq N^d_{\varepsilon,L}(L_\varepsilon\omega(\varepsilon))^{d+2} d\,\delta_\varepsilon^2,$$

where

$$\delta_\varepsilon := \sup\left\{|\nabla f_\varepsilon(x) - \nabla f_\varepsilon(y)| : |x-y| \leq L_\varepsilon\omega(\varepsilon)\sqrt{d}\right\}, \qquad (4.3.25)$$

and hence

$$\lim_{\varepsilon\to 0^+}\frac{1}{\omega(\varepsilon)^2}\int_{Q_1^d}(f_\varepsilon - f_{\varepsilon,L})^2 = 0,$$

because $\delta_\varepsilon \to 0$ as $\varepsilon \to 0^+$ thanks to (4.3.23).

Therefore, recalling (4.3.22), we deduce that

$$\lim_{\varepsilon\to 0^+}\frac{1}{\omega(\varepsilon)^2}\int_{Q_1^d}(f - f_{\varepsilon,L})^2 \leq \lim_{\varepsilon\to 0^+}\frac{2}{\omega(\varepsilon)^2}\int_{Q_1^d}(f - f_\varepsilon)^2 + (f_\varepsilon - f_{\varepsilon,L})^2 = 0. \qquad (4.3.26)$$

Moreover we have that $\nabla f_{\varepsilon,L} \to \nabla f$ uniformly because of (4.3.23) and the fact that $|\nabla f_\varepsilon - \nabla f_{\varepsilon,L}| \leq \delta_\varepsilon$, and this implies that

$$\lim_{\varepsilon\to 0^+}\sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}}|\xi_{\varepsilon,L,z}|^{4/5}(L_\varepsilon\omega(\varepsilon))^d = \lim_{\varepsilon\to 0^+}\int_{Q_1^d}|\nabla f_{\varepsilon,L}|^{4/5} = \int_{Q_1^d}|\nabla f|^{4/5}. \qquad (4.3.27)$$

Finally, for every $\eta \in (0,1)$ and every function $u \in H^2(Q_1^d)$, we have that

$$\int_{Q_1^d}(u-f)^2 \leq (1+\eta)\int_{Q_1^d}(u - f_{\varepsilon,L})^2 + \left(1+\frac{1}{\eta}\right)\int_{Q_1^d}(f - f_{\varepsilon,L})^2,$$

so we conclude that

$$\mathbb{PMF}_\varepsilon(\beta, f, Q_1^d, u) \leq (1+\eta)\,\mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, Q_1^d, u) + \left(1+\frac{1}{\eta}\right)\beta\int_{Q_1^d}(f - f_{\varepsilon,L})^2.$$

Recalling (4.3.26), if we divide by $\omega(\varepsilon)^2$ and we take the limsup we obtain that

$$\limsup_{\varepsilon\to 0^+}\frac{1}{\omega(\varepsilon)^2}\mathbb{PMF}_\varepsilon(\beta, f, Q_1^d, u_\varepsilon) \leq (1+\eta)\limsup_{\varepsilon\to 0^+}\frac{1}{\omega(\varepsilon)^2}\mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, Q_1^d, u_\varepsilon), \qquad (4.3.28)$$

for every family of functions $\{u_\varepsilon\} \subseteq H^2(Q_1^d)$.

Definition of the family $\{u_\varepsilon\}$ For every $z \in \mathbb{N}^d_{N_{\varepsilon,L}}$ let $w_{z,\varepsilon}$ be the function given by Lemma 4.3.1 with $\xi = \xi_{\varepsilon,L,z}$.

We observe that the numbers $|\xi_{\varepsilon,L,z}|$ are all bounded by $\|\nabla f\|_\infty$, thanks to (4.3.23), so we are in the setting of Lemma 4.3.1 with $M_0 = \|\nabla f\|_\infty$.

Now we would like to define $u_\varepsilon(x)$ so that it coincides with a suitable translation and homothety of $w_{z,\varepsilon}$ in the cube Q_z. However, the function defined in this way does not belong to $H^2(Q^d_1)$, because it could have jump-type discontinuity on the boundaries of the squares Q_z, so we need to be more careful. To this end, let us introduce a family of cut-off functions $\{\vartheta_\varepsilon\} \subseteq C^\infty_c(Q^d_{L_\varepsilon})$ such that

$$\vartheta_\varepsilon(x) = \begin{cases} 1 & \text{if } dist(x, \partial Q^d_{L_\varepsilon}) \geq 2\varepsilon^2, \\ 0 & \text{if } dist(x, \partial Q^d_{L_\varepsilon}) \leq \varepsilon^2, \end{cases}$$

and

$$0 \leq \vartheta_\varepsilon(x) \leq 1, \qquad |\nabla\vartheta_\varepsilon(x)| \leq C/\varepsilon^2, \qquad \|\nabla^2\vartheta_\varepsilon(x)\| \leq C/\varepsilon^4 \qquad \forall x \in Q^d_{L_\varepsilon}, \quad (4.3.29)$$

for some positive constant C that does not depend on ε and L.

For every $z \in \mathbb{N}^d_{N_{\varepsilon,L}}$ and every $\varepsilon \in (0,1)$ let us set

$$\begin{aligned} g_{\varepsilon,L,z}(y) &:= \frac{f_\varepsilon(zL_\varepsilon\omega(\varepsilon)) - f_\varepsilon(zL_\varepsilon\omega(\varepsilon) + \omega(\varepsilon)y)}{\omega_\varepsilon} && \forall y \in Q^d_{L_\varepsilon}, \\ v_{\varepsilon,L,z}(y) &:= \vartheta_\varepsilon(y)w_{z,\varepsilon}(y) + (1-\vartheta_\varepsilon(y))g_{\varepsilon,L,z}(y) && \forall y \in Q^d_{L_\varepsilon}. \end{aligned}$$

Now we can define the functions u_ε by setting

$$u_\varepsilon(x) = f_\varepsilon(zL_\varepsilon\omega(\varepsilon)) - \omega(\varepsilon)v_{\varepsilon,L,z}(zL_\varepsilon - x\omega(\varepsilon)^{-1})$$

whenever $x \in Q_z$ for some $z \in \mathbb{N}^d_{N_{\varepsilon,L}}$.

We observe that the functions $u_\varepsilon \in H^2(Q^d_1)$ because they are of class H^2 in the interior of every cube Q_z and they coincide with the smooth function f_ε near the boundaries of these cubes.

Reduction to a common hypercube With a change of variable we obtain that

$$\begin{aligned} \frac{1}{\omega(\varepsilon)^2}\mathbb{PMF}_\varepsilon(\beta, f_{\varepsilon,L}, Q^d_1, u_\varepsilon) &= \frac{1}{\omega(\varepsilon)^2}\sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} \mathbb{PMF}_\varepsilon(\beta, f_z, Q_z, u_\varepsilon) \\ &= \omega(\varepsilon)^d \sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} \mathbb{RPMF}_\varepsilon(\beta, \xi_{\varepsilon,L,z}\cdot y, Q^d_{L_\varepsilon}, v_{\varepsilon,L,z}) \quad (4.3.30) \end{aligned}$$

Estimates near the boundary Let us set $\Omega_{\varepsilon,L} := (2\varepsilon^2, L_\varepsilon - 2\varepsilon^2)^d$ and $B_{\varepsilon,L} := Q^d_{L_\varepsilon} \setminus \Omega_{\varepsilon,L}$. We prove that there exist a real number $\varepsilon_0 \in (0,1)$ (that does not depend on z) and a positive constant C (that does not depend on z, ε and L) such that

$$\mathbb{RPMF}_\varepsilon(\beta, \xi_{\varepsilon,L,z}\cdot y, B_{\varepsilon,L}, v_{\varepsilon,L,z}) \leq CL^{d-1} \qquad \forall\varepsilon \in (0,\varepsilon_0) \quad \forall z \in \mathbb{N}^d_{N_{\varepsilon,L}}. \qquad (4.3.31)$$

To prove this, let us compute the functional, starting from the second order term.

$$\begin{aligned}\|\nabla^2 v_{\varepsilon,L,z}(y)\| &= \|\nabla^2\vartheta_\varepsilon(w_{z,\varepsilon}-g_{\varepsilon,L,z})+2\nabla\vartheta_\varepsilon\otimes\nabla(w_{z,\varepsilon}-g_{\varepsilon,L,z})\\ &\quad+\vartheta_\varepsilon\nabla^2 w_{z,\varepsilon}+(1-\vartheta_\varepsilon)\nabla^2 g_{\varepsilon,L,z}\|\\ &\le \|\nabla^2\vartheta_\varepsilon\||w_{z,\varepsilon}-g_{\varepsilon,L,z}|+2|\nabla\vartheta_\varepsilon|(|\nabla w_{z,\varepsilon}|+|\nabla g_{\varepsilon,L,z}|)\\ &\quad+\|\nabla^2 w_{z,\varepsilon}\|+\|\nabla^2 g_{\varepsilon,L,z}\|\end{aligned}$$

From (4.3.20) and the mean value theorem we deduce that

$$\begin{aligned}|w_{z,\varepsilon}(y)-g_{\varepsilon,L,z}(y)| &\le |w_{z,\varepsilon}(y)-\xi_{\varepsilon,L,z}\cdot y|+|\xi_{\varepsilon,L,z}\cdot y-g_{\varepsilon,L,z}(y)|\\ &\le C+|\xi_{\varepsilon,L,z}\cdot y-\nabla f_\varepsilon(zL_\varepsilon\omega(\varepsilon)+s_y\omega(\varepsilon)y)\cdot y|,\end{aligned}$$

for some $s_y\in(0,1)$. As a consequence we have that

$$|w_{z,\varepsilon}(y)-g_{\varepsilon,L,z}(y)|\le C+\delta_\varepsilon|y|\le C+\delta_\varepsilon L_\varepsilon\sqrt{d}, \tag{4.3.32}$$

where δ_ε is defined by (4.3.25).

Moreover, we have that

$$\nabla g_{\varepsilon,L,z}(y)=\nabla f_\varepsilon(zL_\varepsilon\omega(\varepsilon)-\omega(\varepsilon)y)\quad\text{and}\quad\nabla^2 g_{\varepsilon,L,z}(y)=-\omega(\varepsilon)\nabla^2 f_\varepsilon(zL_\varepsilon\omega(\varepsilon)-\omega(\varepsilon)y).$$

Therefore, exploiting the estimates (4.3.20), (4.3.23), (4.3.24) and (4.3.29), we obtain that

$$\|\nabla^2 v_{\varepsilon,L,z}\|\le\frac{C}{\varepsilon^4}\left(C+\delta_\varepsilon L_\varepsilon\sqrt{d}\right)+2\frac{C}{\varepsilon^2}\left(\frac{C}{\varepsilon^2}+\|\nabla f\|_\infty\right)+\frac{C}{\varepsilon^4}+\|\nabla f\|_\infty\frac{\|\nabla\rho\|_{L^1}\omega(\varepsilon)}{\varepsilon}\le\frac{C}{\varepsilon^4}$$

for every $\varepsilon\in(0,\varepsilon_0)$. Here and in the sequel of the proof $C>0$ denotes a constant (whose value may vary from an inequality to another) that depends only on the constant β, the space dimension d and the functions f and ρ, but not on ε, L and z, while $\varepsilon_0\in(0,1]$ depends on β, d, f, ρ and L (because we need $\delta_\varepsilon L_\varepsilon$ to be small), but not on z.

Since $|B_{\varepsilon,L}|=L_\varepsilon^d-(L_\varepsilon-4\varepsilon^2)^d\le CL^{d-1}\varepsilon^2$, we deduce that

$$\int_{B_{\varepsilon,L}}\varepsilon^6\|\nabla^2 v_{z,\varepsilon}(y)\|^2\,dy\le CL^{d-1}, \tag{4.3.33}$$

for every $\varepsilon\in(0,\varepsilon_0)$.

Let us now compute the first order term

$$\begin{aligned}|\nabla v_{\varepsilon,L,z}| &= |\nabla\vartheta_\varepsilon(w_{z,\varepsilon}-g_{\varepsilon,L,z})+\vartheta_\varepsilon\nabla w_{z,\varepsilon}+(1-\vartheta_\varepsilon)\nabla g_{\varepsilon,L,z}|\\ &\le |\nabla\vartheta_\varepsilon||w_{z,\varepsilon}-g_{\varepsilon,L,z}|+|\nabla w_{z,\varepsilon}|+|\nabla g_{\varepsilon,L,z}|.\end{aligned}$$

Exploiting again the estimates (4.3.20), (4.3.23), (4.3.29) and (4.3.32) we obtain that

$$|\nabla v_{\varepsilon,L,z}|\le\frac{C}{\varepsilon^2}\left(C+\delta_\varepsilon L_\varepsilon\sqrt{d}\right)+\frac{C}{\varepsilon^2}+\|\nabla f\|_\infty\le\frac{C}{\varepsilon^2},$$

for every $\varepsilon\in(0,\varepsilon_0)$.

Since $|B_{\varepsilon,L}| \leq CL^{d-1}\varepsilon^2$, we deduce that

$$\int_{B_{\varepsilon,L}} \frac{1}{\omega(\varepsilon)^2} \log(1+|\nabla v_{\varepsilon,L,z}(y)|^2)\,dy \leq \frac{CL^{d-1}}{|\log\varepsilon|} \log\left(1+\frac{C}{\varepsilon^4}\right) \leq CL^{d-1}, \qquad (4.3.34)$$

for every $\varepsilon \in (0,\varepsilon_0)$.

Finally, we compute the fidelity term.

$$\begin{aligned} |v_{\varepsilon,L,z}(y) - \xi_{\varepsilon,L,z}\cdot y| &= |\vartheta_\varepsilon(y)(w_{\varepsilon,z}(y) - \xi_{\varepsilon,L,z}\cdot y) + (1-\vartheta_\varepsilon(y))(g_{\varepsilon,L,z}(y) - \xi_{\varepsilon,L,z}\cdot y)| \\ &\leq C + \delta_\varepsilon L_\varepsilon \sqrt{d}. \end{aligned}$$

Since $|B_{\varepsilon,L}| \leq CL^{d-1}\varepsilon^2$, we deduce that the fidelity term actually vanishes as $\varepsilon \to 0^+$, so in particular

$$\beta \int_{B_{\varepsilon,L}} (v_{\varepsilon,L,z}(y) - \xi_{\varepsilon,L,z}\cdot y)^2\,dy \leq CL^{d-1}, \qquad (4.3.35)$$

for every $\varepsilon \in (0,\varepsilon_0)$.

Combining (4.3.33), (4.3.34) and (4.3.35) we obtain (4.3.31).

Internal estimate We observe that $v_{\varepsilon,L,z} = w_{z,\varepsilon}$ in $\Omega_{\varepsilon,L} \subseteq Q^d_{L_\varepsilon} \subseteq Q^d_L$, so Lemma 4.3.1 yields

$$\begin{aligned} &\mathbb{RPMF}_\varepsilon(\beta, \xi_{\varepsilon,L,z}\cdot y, \Omega_{\varepsilon,L}, v_{\varepsilon,L,z}) \\ &\quad\leq \mathbb{RPMF}_\varepsilon(\beta, \xi_{\varepsilon,L,z}\cdot y, Q^d_L, w_{z,\varepsilon}) \\ &\quad\leq 10\left(\frac{2\beta}{27}\right)^{1/5} |\xi_{\varepsilon,L,z}|^{4/5} L^d + C\left(K_d L^{d-1} + \frac{L^d + K_d L^{d-1}}{|\log\varepsilon|}\right). \end{aligned} \qquad (4.3.36)$$

Conclusion Combining (4.3.28), (4.3.30), (4.3.31) and (4.3.36) we obtain that

$$\begin{aligned} &\limsup_{\varepsilon\to 0^+} \frac{1}{\omega(\varepsilon)^2} \mathbb{PMF}_\varepsilon(\beta, f, Q^d_1, u_\varepsilon) \leq \\ &\qquad (1+\eta) \limsup_{\varepsilon\to 0^+} \omega(\varepsilon)^d \sum_{z\in\mathbb{N}^d_{N_{\varepsilon,L}}} \left[10\left(\frac{2\beta}{27}\right)^{1/5} |\xi_{\varepsilon,L,z})|^{4/5} L^d + CL^{d-1}\right], \end{aligned}$$

where again C is a positive constant which depends only on β, the space dimension d and the functions f and ρ, but not on ε, L and z.

Therefore, by (4.3.27) and the fact that $N^d_{\varepsilon,L}\omega(\varepsilon)^d \to L^{-d}$, we deduce that

$$\limsup_{\varepsilon\to 0^+} \frac{1}{\omega(\varepsilon)^2} \mathbb{PMF}_\varepsilon(\beta, f, Q^d_1, u_\varepsilon) \leq (1+\eta) 10\left(\frac{2\beta}{27}\right)^{1/5} \int_{Q^d_1} |\nabla f(x)|^{4/5}\,dx + \frac{C(1+\eta)}{L}.$$

Since $L > 0$ can be chosen arbitrarily large and $\eta > 0$ can be chosen arbitrarily small, we have proved that for every positive real number $\delta > 0$ there exists a family of functions $\{u_\varepsilon\} \subset H^2(Q^d_1)$ such that

$$\limsup_{\varepsilon\to 0^+} \frac{1}{\omega(\varepsilon)^2} \mathbb{PMF}_\varepsilon(\beta, f, Q^d_1, u_\varepsilon) \leq 10\left(\frac{2\beta}{27}\right)^{1/5} \int_{Q^d_1} |\nabla f(x)|^{4/5}\,dx + \delta.$$

Hence, by a diagonal argument, we can also find a family of functions for which (4.3.21) holds true. □

Remark 4.3.3. *In order to extend Theorem 2.2.9 to the higher dimensional case we need two main ingredients: a uniform estimate for the energy of local minimizers as in Proposition 2.6.5, and a characterization of entire local minimizers for the limit functional* $\mathbb{JF}_{1/2}$ *with a linear forcing term.*

The first one can be proved again by an iterative argument, in which Lemma 2.6.3 is replaced by a different estimate that can be obtained using cut-off functions instead of cubic polynomials. This generalization is not immediate, and requires a certain amount of computations, so we do not include it in this thesis.

The characterization of entire local minimizers for $\mathbb{JF}_{1/2}$*, instead, seems to be a challenging problem, that at present we are not able to solve completely. Indeed, even if it can be easily proved that for every* $\xi \in \mathbb{R}^d$ *the functions* $S_\xi(x) = S_{H,V}(\xi \cdot x)$ *are entire local minimizers for* $\mathbb{JF}_{1/2}$ *when the forcing term is* $f(x) = \xi \cdot x$ *and the parameters* H, V *are given by 2.4.4 with* $M = |\xi|$*, it turns out that these are not the only ones.*

On the contrary, at least when the space dimension is $d = 2$*, we know that there exist entire local minimizers that are not "one-dimensional", namely that can not be written as* $S(\xi \cdot x)$ *for some function* $S : \mathbb{R} \to \mathbb{R}$*. However, we are still unable to provide a complete characterization of entire local minimizers, and the situation is probably much more complicated when the space dimension is higher than two.*

Chapter 5

Semi-discrete approximation and monotonicity of the total variation

5.1 Introduction

In this chapter we describe the paper [47] and some extensions of the results contained therein. In particular, we focus on the semi-discrete approximation of the problem (1.2) in the one-dimensional case.

As we briefly mentioned in the introduction, in this case it is known that, for any fixed value of the discretization parameter, both the L^∞ norm and the total variation of the solution are nonincreasing functions of time (see Theorem A). Therefore, if they are bounded for $t = 0$ independently of the discretization parameter, then they remain uniformly bounded for all positive times, and this is enough to show that a limit exists, at least up to subsequences, and it is a bounded variation function for every $t \geq 0$ (see Theorem B). At this point one can consider all possible limits of semi-discrete approximations as generalized solutions to (1.2) (see Definition 5.2.4).

A characterization of all possible limits is still out of reach, apart from some partial results (see Theorem C), because it is not clear how to pass to the limit in the quasi-linear term. In particular, before [47] it was not known whether some good properties of the approximating solutions, such as the maximum principle and the monotonicity of the total variation, remain valid for all possible limits. We observe that these properties are extremely reasonable for what is expected to be a denoising tool.

First main result – A counterexample We already observed that, for every fixed value of the grid size, both the L^∞ norm and the total variation of solutions are nonincreasing functions of time. Therefore, both the maximum principle and the monotonicity of the total variation for generalized solutions would be trivially true if we knew that "the total variation of the limit is the limit of the total variations" or "the maximum of the limit is the limit of the maxima".

Unfortunately, the L^2 convergence provided by the compactness result is not enough to pass such quantities to the limit for all times, even if they pass to the limit at the initial time. Indeed, in Theorem 5.2.6 we provide an explicit example of solutions to the semi-discrete scheme whose maximum and total variation *pass to the limit at the initial time, but do not pass to the limit for every subsequent time.*

In a nutshell, what happens in that example can be described as follows. Assume that we consider equation (1.1) in the interval $(0, \pi)$, with initial datum $u_0(x) = c_0 \sin x$ for some $c_0 \in (0, 1)$, and (for simplicity) Dirichlet boundary conditions. In this case the problem has a unique classical solution $u(t, x)$, which always satisfies $0 \leq u_x(t, x) \leq c_0$, and does not even realize that a backward parabolic regime exists (see Theorem 1.1.2). As expected, the qualitative behavior of $u(t, x)$ is similar to the behavior of the solution of the heat equation with the same initial datum, namely it has a profile that resembles a hill that continues to decrease until it disappears as $t \to +\infty$.

Now let us approximate $u_0(x)$ with a sequence $u_{0n}(x)$ of piecewise constant functions, and let us consider the solution $u_n(t, x)$ of the semi-discrete version of (1.1) with $u_{0n}(x)$ as initial datum. Then we expect that $u_n(t, x)$ mimics $u(t, x)$, but this is true only up to some extent, and it is very sensitive to the choice of the sequence u_{0n}. Indeed, it is enough to modify slightly the values of $u_{0n}(x)$ near the maximum point, and what happens is that those values do not follow the evolution of the rest, but they remain rather close to the initial maximum value for large times.

It is still true that $u_{0n}(x)$ converges uniformly to $u_0(x)$, and also the maximum and the total variation of $u_{0n}(x)$ converge to the corresponding quantities of $u_0(x)$. It is still true that $u_n(t, x)$ converges to $u(t, x)$ in $L^p((0, \pi))$ for every $t \geq 0$ and every finite $p \geq 1$. But the maximum of $u_n(t, x)$ does not tend to the maximum of $u(t, x)$ for every $t > 0$, and the same for the total variation, due to the few anomalous values that remain higher than expected.

One could interpret this pathology by saying that $u_n(t, x)$ does not converge truly to $u(t, x)$, but to some exotic object that coincides with $u(t, x)$ with an isolated anomalous maximum point, and analogously $u_{0n}(x)$ does not converge truly to $u_0(x)$. This approach might be viable, and we plan to explore it in some future research, but it requires the identification of a suitable generalization of the notion of function and derivative, maybe in the spirit of varifolds as in Theorem 2.2.14 and in [66, Theorem 2.5]. Within this approach the dirty job is done by the definition of the exotic objects, and at the end of the day the convergence of maxima/minima and of the total variation are enforced in the definition, and as a consequence the maximum principle and the monotonicity of the total variation for the limit objects are immediate.

Second main result – Monotonicity properties of generalized solutions Here we pursue a different path. We remain in the classical setting of functions with bounded variation, where the compactness result provides the limit, and in Theorem 5.2.7 we show that all possible limits satisfy the expected monotonicity properties, even if we know that the related quantities do not pass to the limit from the semi-discrete to the continuous setting.

In order to prove this result, we exploit a weak characterization of generalized solutions. It is known (see Theorem C) that they satisfy an equation of the form $u_t = v_x$ for a suitable function v. From (1.2) one would expect that

$$v = \varphi'(u_x),$$

but in general this is not true, at least if one understands u_x as the standard derivative of a function with bounded variation. Nevertheless, it remains true that $v \cdot u_x \geq 0$ in the sense of measures (note that for almost every $t \geq 0$ the function v is continuous in x, and u_x is a signed measure).

This leads us to introducing a class of evolution curves that we call uv-evolutions (see Definition 5.2.8), namely solutions to an evolution equation of the form $u_t = v_x$, with suitable regularity requirements and the sign condition $v \cdot u_x \geq 0$. Then we prove that all our generalized solutions of the Perona-Malik equation are actually uv-evolutions (see Proposition 5.2.11), and that all uv-evolutions satisfy the maximum principle and the monotonicity of the total variation (see Proposition 5.2.10).

The class of uv-evolutions contains weak solutions to many parabolic equations in divergence form, including all one-dimesnional weak solutions to the Perona-Malik equation constructed through convex integration techniques (see Theorem 1.1.6). The class of uv-evolutions is stable under weak convergence, and therefore we are confident that it could contain also the limits of trajectories provided by different approximations of the Perona-Malik equation, of course when a compactness result will be available for those models. In all these cases we have now identified a common mechanism that leads to the maximum principle and to the monotonicity of the total variation, and we hope that this could be useful also in different contexts.

A generalization – uvw-evolutions The results that we have just described for uv-evolution actually hold for a more general class of evolution curves, that we call uvw-evolutions, and consists of solutions to equations of the form $u_t = wv_x$, with suitable regularity requirements and a suitable generalized sign condition.

This larger class is also stable under compositions of the function u with monotone functions. This condition allows to prove further monotonicity results for this class, and hence for generalized solutions of the Perona-Malik equations. In particular we obtain a weak form of the monotonicity of the cardinality of level sets, which in some cases generalizes the results for classical solutions of parabolic equations (see [7] and the references quoted therein).

For generalized solutions of the Perona-Malik equation coming from the semi-discrete scheme, a stronger result concerning level sets could also be obtained with a different approach, exploiting the finite dimensional structure of the approximating problems, but we do not include this argument in this thesis.

Finally, we conclude by mentioning a technical point in our proofs that might be interesting in itself. Indeed, the maximum principle and the monotonicity of the total variation seem to be two faces of the same coin. More precisely, in order to show that the total variation of $u(t,x)$ with respect to the space variable $x \in (a,b)$ is a nonincreasing function of time, we consider for every positive integer m the function with $2m$ space variables

$$\sum_{i=1}^{2m} (-1)^i u(t, x_i),$$

defined in the simplex $a < x_1 \leq x_2 \leq \ldots \leq x_{2m} < b$, and we show that its supremum is a nonincreasing function of time. In other words, the monotonicity of the total variation in dimension one can be reduced to a maximum principle in higher dimension.

Structure of the chapter This chapter is organized as follows. In section 5.2 we introduce the notation and we state our main results. In section 5.3 we construct our counterexample to the preservation of strict convergence for positive times. In section 5.4 we develop the theory of *uv*-evolutions that leads to the proof of our monotonicity results. Finally, in section 5.5 and section 5.6 we present some original research concerning *uvw*-evolution and the monotonicity properties of level sets, which is not contained in [47].

5.2 Notation and statements

For the sake of simplicity, here we consider a function $\varphi : \mathbb{R} \to \mathbb{R}$ with the following properties, even if many statements could be proved also under weaker assumptions.

- (Regularity, symmetry, strict convexity in the origin). The function φ is an even function of class C^2 with bounded second derivative, satisfying $\varphi(0) = 0$ and
$$\varphi''(0) > 0. \tag{5.2.1}$$
- (Convex-concavity). There exists a positive real number σ_1 such that
$$\varphi'(\sigma) \text{ is nondecreasing in } [0, \sigma_1] \tag{5.2.2}$$
and
$$\varphi'(\sigma) \text{ is nonincreasing in } [\sigma_1, +\infty). \tag{5.2.3}$$
- (Sublinear growth). It turns out that
$$\lim_{\sigma \to +\infty} \varphi'(\sigma) = 0. \tag{5.2.4}$$

We observe that these properties imply in particular that $\varphi'(0) = 0$ and $\varphi(\sigma) > 0$ for every $\sigma \neq 0$. In addition, σ_1 is a maximum point for $\varphi'(\sigma)$ and

$$\sigma\varphi'(\sigma) \geq 0 \qquad \forall \sigma \in \mathbb{R}. \tag{5.2.5}$$

Now we recall that every function $u \in BV((a, b))$ coincides almost everywhere with a function that is (for example) left-continuous. Identifying u with this left-continuous representative, the *total variation of u in (a, b)* is the non-negative real number

$$TV(u) := \sup \left\{ \sum_{i=1}^{m} |u(x_i) - u(x_{i-1})| : m \geq 1,\ a < x_0 \leq x_1 \leq \ldots \leq x_m < b \right\},$$

and coincides with the total variation of the measure Du. The total variation of u is the sum of the *positive total variation* $TV^+(u)$ of u, defined by

$$TV^+(u) := \sup \left\{ \sum_{i=1}^{2m} (-1)^i u(x_i) : m \geq 1,\ a < x_1 \leq x_2 \leq \ldots \leq x_{2m} < b \right\},$$

and the *negative total variation* $TV^-(u)$ of u, defined as $TV^+(-u)$. The decomposition $TV(u) = TV^+(u) + TV^-(u)$ corresponds to the Hahn decomposition of Du.

5.2.1 The semi-discrete scheme

Let n be a positive integer, and let $u : \{1, \ldots, n\} \to \mathbb{R}$ be any function. Let us extend u by setting

$$u(0) = u(1) \qquad \text{and} \qquad u(n+1) = u(n).$$

We call *forward discrete derivative* of u with step $1/n$ the function $D_n^+ u$ defined as

$$[D_n^+ u](i) := \frac{u(i+1) - u(i)}{1/n} \qquad \forall i \in \{0, 1, \ldots, n\},$$

and we call *backward discrete derivative* of u with step $1/n$ the function $D_n^- u$ defined as

$$[D_n^- u](i) := \frac{u(i-1) - u(i)}{-1/n} \qquad \forall i \in \{1, \ldots, n, n+1\}.$$

We observe that $[D_n^- u](i) = [D_n^+ u](i-1)$ for every admissible value of i. In analogy with the continuous setting, we set

$$\|u\|_\infty := \max\{|u(i)| : i \in \{1, \ldots, n\}\}, \tag{5.2.6}$$

and we call discrete total variation of u the number

$$TV(u) := \frac{1}{n} \sum_{i=1}^{n} |D_n^+ u(i)| = \sum_{i=1}^{n-1} |u(i+1) - u(i)|. \tag{5.2.7}$$

A solution to the *semi-discrete Perona-Malik equation* with step $1/n$ is any function

$$u : [0, +\infty) \times \{1, \ldots, n\} \to \mathbb{R} \tag{5.2.8}$$

that is differentiable with respect to the first variable, and satisfies

$$u'(t, i) = D_n^- \left(\varphi' \left(D_n^+ u(t, i)\right)\right) \qquad \forall t \geq 0, \quad \forall i \in \{1, \ldots, n\}, \tag{5.2.9}$$

with the usual understanding that

$$u(t, 0) = u(t, 1) \qquad \text{and} \qquad u(t, n+1) = u(t, n) \qquad \forall t \geq 0. \tag{5.2.10}$$

Just to avoid any ambiguity, we point out that

$$D_n^- \left(\varphi' \left(D_n^+ u(t, i)\right)\right) = \frac{\varphi' \left(D_n^+ u(t, i)\right) - \varphi' \left(D_n^+ u(t, i-1)\right)}{1/n},$$

so that equation (5.2.9) is the discrete version of (1.2), and (5.2.10) is the discrete version of the Neumann boundary conditions.

Remark 5.2.1 (From discrete to continuum setting)**.** One can always identify a function $u : \{1, \ldots, n\} \to \mathbb{R}$ with the piecewise constant function $\widehat{u} : (0, 1) \to \mathbb{R}$ defined by

$$\widehat{u}(x) := u(\lceil nx \rceil) \qquad \forall x \in (0, 1),$$

where, for every real number α, the symbol $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α. Under this identification, the forward and backward discrete derivatives $D_n^+ u$ and $D_n^- u$ are actually the forward and backward difference quotients of $\widehat{u}$ of step $1/n$, the number $\|u\|_\infty$ defined by (5.2.6) is the usual L^∞ norm of $\widehat{u}$, and the discrete total variation $TV(u)$ defined by (5.2.7) is the total variation of $\widehat{u}$ as a bounded variation function.

In an analogous way, one can always associate to every semi-discrete function u as in (5.2.8) the function $\widehat{u} : [0, +\infty) \times (0,1) \to \mathbb{R}$ defined by

$$\widehat{u}(t,x) := u(t, \lceil nx \rceil) \qquad \forall t \geq 0, \quad \forall x \in (0,1). \tag{5.2.11}$$

In the sequel, with some abuse of notation, we use the same letter u in order to denote the "same function" in three different flavors, namely

- the discrete function with domain and codomain as in (5.2.8),
- the piecewise constant function defined by the right-hand side of (5.2.11),
- the evolution curve from $[0, +\infty)$ to $L^p((0,1))$ that associates to each $t \geq 0$ the function $x \mapsto u(t, \lceil nx \rceil)$, thought as an element of $L^p((0,1))$.

We hope that this abuse of notation could result less confusing than using three different symbols.

Remark 5.2.2 (Discrete Perona-Malik functional)**.** One can define the *discrete Perona-Malik functional* with step $1/n$ as

$$\mathrm{DPM}_n(u) := \frac{1}{n} \sum_{i=1}^{n} \varphi\left(D_n^+ u(i)\right). \tag{5.2.12}$$

Under the identification of Remark 5.2.1, one can interpret it as a functional defined in the space of piecewise constant functions in $(0,1)$ with steps of width $1/n$. In this sense it turns out that the semi-discrete Perona-Malik equation (5.2.9), with the discrete Neumann boundary conditions (5.2.10), is the gradient flow of (5.2.12) with respect to the metric of $L^2((0,1))$. For more details we refer to [42].

We observe that (5.2.9) is actually a system of n ordinary differential equations, and therefore existence and uniqueness of solutions follow from standard theories. In the next result we summarize the properties of solutions that we need in the sequel.

Theorem A (Existence, uniqueness, and monotonicity)**.** *For every positive integer n, and every function $u_0 : \{1, \ldots, n\} \to \mathbb{R}$, the following statements hold true.*

(1) (Existence and uniqueness). *There exists a unique global solution to the semi-discrete Perona-Malik equation with initial datum u_0 and homogeneous Neumann boundary conditions, namely there exists a unique function u that satisfies (5.2.8), (5.2.9), (5.2.10), and the initial condition*

$$u(0,i) = u_0(i) \qquad \forall i \in \{1, \ldots, n\}.$$

(2) (Monotonicity of max/min and total variation). *The three functions*

$$t \mapsto \max_{1\leq i\leq n} u(t,i), \qquad t \mapsto -\min_{1\leq i\leq n} u(t,i), \qquad t \mapsto \frac{1}{n}\sum_{i=1}^{n} |D_n^+ u(t,i)|$$

are nonincreasing.

(3) (L^2 estimate on the time-derivative). *It turns out that*

$$\int_0^{+\infty} \left(\frac{1}{n}\sum_{i=1}^{n} |u'(t,i)|^2\right) dt \leq \mathrm{DPM}_n(u_0), \tag{5.2.13}$$

where DPM_n *is the discrete Perona-Malik functional defined in (5.2.12).*

(4) (Preservation of subcritical regions). *Let* σ_1 *be the threshold that appears in (5.2.2) and (5.2.3). If* $D_n^+ u_0(i) \leq \sigma_1$ *for some index* i*, then* $D_n^+ u(t,i) \leq \sigma_1$ *for every* $t \geq 0$.

(5) (Preservation of monotonicity). *If the initial datum* u_0 *is nondecreasing, then* $u(t,i)$ *is nondecreasing with respect to the second variable for every* $t \geq 0$.

The proof of the first four statements of Theorem A is contained in [42, Theorem 2.5], while statement (5) follows from the monotonicity of the function

$$t \mapsto \frac{1}{n}\sum_{i=1}^{n} \max\left\{-D_n^+ u(t,i), 0\right\},$$

whose proof is analogous to the monotonicity of the total variation.

Solutions to the semi-discrete Perona-Malik equation satisfy the following compactness result.

Theorem B (Compactness for the semi-discrete scheme)**.** *For every positive integer* n*, let* $u_{0n} : \{1,\ldots,n\} \to \mathbb{R}$ *be a function, and let* $u_n : [0,+\infty) \times \{1,\ldots,n\} \to \mathbb{R}$ *denote the solution to the semi-discrete Perona-Malik equation (5.2.9), with discrete Neumann boundary conditions (5.2.10), and initial datum* u_{0n}.

Let us assume that

$$\sup\left\{\|u_{0n}\|_\infty + TV(u_{0n}) : n \geq 1\right\} < +\infty. \tag{5.2.14}$$

Then the sequence $\{u_n\}$ *is relatively compact in* $C^0\left([0,+\infty); L^2((0,1))\right)$ *with respect to the compact-open topology, namely there exist a function* $u : [0,+\infty) \to L^2((0,1))$*, and an increasing sequence* $\{n_k\}$ *of positive integers, such that*

$$\lim_{k\to+\infty} \sup_{t\in[0,T]} \|u_{n_k}(t, \lceil n_k x\rceil) - u(t,x)\|_{L^2((0,1))} = 0 \tag{5.2.15}$$

for every real number $T > 0$.

Remark 5.2.3. The proof of Theorem B, for which we refer to [42, Theorem 2.7] (see also [18]), is a simple application of the classical Arzelà-Ascoli theorem. The two main ingredients are the following.

- For every fixed $t \geq 0$, the sequence $\{u_n(t,x)\}$ is relatively compact in $L^2((0,1))$ due to the maximum principle and the bound on the total variation.
- The sequence u_n is uniformly Hölder continuous with exponent $1/2$ as a function from $[0,+\infty)$ to $L^2((0,1))$. This is due to estimate (5.2.13), and the fact that (5.2.14) implies a uniform bound on $\mathrm{DPM}_n(u_{0n})$ because of the sublinear growth of φ.

We observe also that in the compactness statement the L^2 space can be replaced by any L^p space with finite $p \geq 1$ (but not with $p = +\infty$).

5.2.2 Generalized solutions obtained through the SD scheme

The compactness result of Theorem B motivates the following procedure. Given any function $u_0 \in BV((0,1))$, we approximate it with a sequence $\{u_{0n}\}$ of piecewise constant functions with step $1/n$. For each positive integer n, we consider the solution u_n to the semi-discrete Perona-Malik equation with initial datum u_{0n}. Since u_0 is a bounded variation function, we can choose the approximating sequence in such a way that (5.2.14) holds true, and this guarantees that the sequence $\{u_n\}$ is relatively compact. All possible limits, when also the approximating sequence is allowed to vary, can be considered as some sort of "generalized solutions" to (1.2) with initial datum u_0.

We observe that this procedure, when applied to the heat equation, or to any other forward parabolic equation, delivers the unique classical solution to the equation with initial datum u_0. In the case of the Perona-Malik equation we end up with the following notion.

Definition 5.2.4 (Generalized solutions)**.** A generalized solution to equation (1.2) with homogeneous Neumann boundary conditions in the interval $(0,1)$, obtained through semi-discrete approximation, is any function $u \in C^0\left([0,+\infty); L^2((0,1))\right)$ for which there exist an increasing sequence $\{n_k\}$ of positive integers, and a sequence of functions

$$u_k : [0,+\infty) \times \{1,\ldots,n_k\} \to \mathbb{R},$$

with the following properties.

- (Uniform bounds on initial data). There exists a real number M such that the initial data, defined by $u_{0k}(i) = u_k(0,i)$ for every $i \in \{1,\ldots,n_k\}$, satisfy

$$\|u_{0k}\|_\infty + TV(u_{0k}) \leq M \qquad \forall k \geq 1. \tag{5.2.16}$$

- (Semi-discrete equation and discrete Neumann boundary conditions). For every positive integer k, the function u_k is differentiable with respect to the first variable, and satisfies the semi-discrete Perona-Malik equation

$$u_k'(t,i) = D^-_{n_k}\left(\varphi'\left(D^+_{n_k} u_k(t,i)\right)\right) \qquad \forall t \geq 0, \quad \forall i \in \{1,\ldots,n_k\},$$

  with the usual understanding (discrete Neumann boundary conditions) that

$$u_k(t,0) = u_k(t,1) \qquad \text{and} \qquad u_k(t,n_k+1) = u_k(t,n_k) \qquad \forall t \geq 0. \tag{5.2.17}$$

- (Convergence). As $k \to +\infty$ it turns out that

$$u_k(t, \lceil n_k x \rceil) \to u(t,x) \qquad \text{in } C^0\left([0,+\infty); L^2((0,1))\right)$$

in the sense that (5.2.15) holds true for every real number $T > 0$.

Remark 5.2.5 (Restriction property)**.** The notion of solution of Definition 5.2.4 above is slightly more general than the notion of solution of [42, Definition 2.8]. Indeed, in the latter there was the further requirement that initial data converge strictly in $BV((0,1))$, namely that the total variation of the initial data u_{0k} of the approximating solutions converges to the total variation of the initial datum $u(0,x)$ of the limit solution.

Dropping this extra requirement potentially enlarges the set of generalized solutions, but it has the positive effect that now, if we restrict a solution with initial datum u_0 to some half-line $[T,+\infty)$, what we get is a solution with initial datum $u(T)$.

A partial characterization of generalized solutions is provided by the following result. For a proof, we refer to [42, Theorem 2.9].

Theorem C (Regularity of generalized solution)**.** *Let u be a generalized solution to equation (1.2) with homogeneous Neumann boundary conditions in the interval* $(0,1)$*, obtained through semi-discrete approximation in the sense of Definition 5.2.4 with corresponding sequences $\{n_k\}$ and $\{u_k\}$.*

Then the following statements hold true.

(1) (H^1 regularity in time). *The function u admits a weak derivative u_t with respect to the variable t, and*

$$u_t \in L^2((0,+\infty) \times (0,1)).$$

(2) (BV regularity in space). *For every $t \geq 0$ the function $x \mapsto u(t,x)$ belongs to $BV((0,1))$ and*

$$\|u(t,x)\|_\infty + TV(u(t,x)) \leq M \qquad \forall t \geq 0,$$

where M is the constant that appears in (5.2.16), and both the L^∞ norm and the total variation are intended with respect to the space variable.

(3) (Remnants of the equation). *Let us consider the function v_k defined by*

$$v_k(t,i) := \varphi'\left(D^+_{n_k} u_k(t,i)\right) \qquad \forall t \geq 0, \quad \forall i \in \{1,\ldots,n_k\}. \tag{5.2.18}$$

Then there exists a measurable function $v \in L^\infty((0,+\infty) \times (0,1))$ such that

$$v_k(t, \lceil n_k x \rceil) \rightharpoonup v(t,x) \qquad \text{weakly* in } L^\infty((0,+\infty) \times (0,1)).$$

Moreover, the function v admits a weak derivative v_x with respect to the space variable x. This derivative satisfies

$$D^-_{n_k} v_k(t, \lceil n_k x \rceil) \rightharpoonup v_x(t,x) \qquad \text{weakly in } L^2((0,+\infty) \times (0,1)),$$

and

$$u_t = v_x \qquad \text{as elements of } L^2((0,+\infty) \times (0,1)).$$

Finally, for almost every $t \geq 0$, the function $x \mapsto v(t,x)$ lies in $H^1((0,1))$, and hence it is continuous up to the endpoints and satisfies the boundary conditions $v(t,0) = v(t,1) = 0$.

5.2.3 Main results

The L^∞ norm and the total variation are lower semicontinuous with respect to convergence in L^2, or more generally in L^p with $p < +\infty$. As a consequence, with the notations of Definition 5.2.4, we know that

$$\liminf_{k\to+\infty} \|u_k(t,i)\|_\infty \geq \|u(t,x)\|_\infty \qquad \forall t \geq 0, \tag{5.2.19}$$

and

$$\liminf_{k\to+\infty} TV(u_k(t,i)) \geq TV(u(t,x)) \qquad \forall t \geq 0. \tag{5.2.20}$$

On the other hand, we know from statement (2) of Theorem A that, for every fixed $k \geq 1$, the functions $t \mapsto \|u_k(t,i)\|_\infty$ and $t \mapsto TV(u_k(t,i))$ are nonincreasing.

Unfortunately, the inequalities in (5.2.19) and (5.2.20) are not enough to deduce that the functions $t \mapsto \|u(t,x)\|_\infty$ and $t \mapsto TV(u(t,x))$ are nonincreasing as well. This deduction would be possible if we had equalities instead of inequalities in (5.2.19) and (5.2.20). This would mean that, for every fixed $t \geq 0$, the convergence of $u_k(t, \lceil n_k x \rceil)$ to $u(t,x)$ could be improved from convergence in $L^2((0,1))$ to strict convergence in $BV((0,1))$ in the sense of [6, Definition 3.14].

The first main result of this chapter is a counterexample to strict convergence. We show that it may happen that initial data converge strictly, but solutions do not converge strictly for every positive time, and as a consequence we have strict inequality in both (5.2.19) and (5.2.20) for every $t > 0$.

Theorem 5.2.6 (Potential lack of strict convergence for positive times)**.** *There exists a generalized solution u to the Perona-Malik equation (1.2) with homogeneous Neumann boundary conditions in the interval* $(0,1)$*, obtained through semi-discrete approximation in the sense of Definition 5.2.4 with corresponding sequences $\{n_k\}$ and $\{u_k\}$, which has the following properties.*

(1) (Strict converge of initial data). *For $t = 0$ we have equality in (5.2.19) and (5.2.20).*

(2) (Lack of strict convergence for positive times). *For every $t > 0$ the inequalities in (5.2.19) and (5.2.20) are strict.*

Due to Theorem 5.2.6, it is not possible to deduce the monotonicity of the functions $t \mapsto \|u(t,x)\|_\infty$ and $t \mapsto TV(u(t,x))$ from the corresponding monotonicities at discrete level. Nevertheless, the second main result of this chapter is that those monotonicities hold true anyway.

Theorem 5.2.7 (Monotonicity results for generalized solutions)**.** *Let u be a generalized solution to the Perona-Malik equation (1.2) with homogeneous Neumann boundary conditions in the interval* $(0,1)$*, obtained through semi-discrete approximation in the sense of Definition 5.2.4.*

Then the following monotonicity results hold true.

(1) (Maximum principle). *For every $t \geq 0$, let $M^+(t)$ and $M^-(t)$ denote, respectively, the (essential) supremum and infimum of the function $x \mapsto u(t,x)$.*

Then the function $t \mapsto M^+(t)$ is nonincreasing, while the function $t \mapsto M^-(t)$ is nondecreasing.

(2) (Monotonicity of the total variation). *For every $t \geq 0$, let $TV^{\pm}(t)$ denote the positive/negative total variation of the function $x \mapsto u(t,x)$.*

Then the functions $t \mapsto TV^{\pm}(t)$ are nonincreasing.

5.2.4 uv-evolutions

In the proof of Theorem 5.2.7 we forget that our generalized solutions are limits of solutions to the semi-discrete scheme. We limit ourselves to considering them as functions that satisfy the characterization described in Theorem C, together with a suitable sign condition. This leads to the following notion.

Definition 5.2.8 (uv-evolution with NBC in dimension one)**.** A *uv-evolution with homogeneous Neumann boundary conditions* in an interval $(a,b) \subseteq \mathbb{R}$ is a pair of measurable functions

$$u : (0,+\infty) \times (a,b) \to \mathbb{R} \qquad \text{and} \qquad v : (0,+\infty) \times (a,b) \to \mathbb{R}$$

with the following properties.

- (Time regularity). The function u admits a weak derivative u_t with respect to time, and
$$u_t \in L^1((0,T) \times (a,b)) \qquad \forall T > 0. \tag{5.2.21}$$

- (Space regularity). For almost every $t > 0$ it turns out that
$$\text{the function } x \mapsto u(t,x) \text{ is in } BV((a,b)), \tag{5.2.22}$$
$$\text{the function } x \mapsto v(t,x) \text{ is in } W^{1,1}((a,b)). \tag{5.2.23}$$

- (Evolution equation). The functions u and v satisfy
$$u_t(t,x) = v_x(t,x) \qquad \text{in } (0,+\infty) \times (a,b). \tag{5.2.24}$$

- (Neumann boundary conditions). For almost every $t > 0$ it turns out that
$$v(t,a) = v(t,b) = 0. \tag{5.2.25}$$

- (Sign condition). For almost every $t > 0$ it turns out that
$$v(t,x) \cdot Du(t,x) \geq 0 \qquad \text{as a measure in } (a,b). \tag{5.2.26}$$

Remark 5.2.9. Let us comment on some regularity issues in Definition 5.2.8 above.

- (Evolution equation). The evolution equation (5.2.24) can be seen both as an equality between functions in $L^1((0,T)\times(a,b))$, and as an equality between functions in $L^1((a,b))$ for almost every $t>0$.

- (Time regularity and initial datum). The time regularity assumption (5.2.21) implies that u, as a function from $(0,+\infty)$ to $L^1((a,b))$, is continuous, and actually also absolutely continuous, and hence it can be extended up to $t=0$. In particular, all sections $x\mapsto u(t,x)$ are well defined as functions in $L^1((a,b))$ for every $t\geq 0$, including the initial datum at $t=0$.

- (Neumann boundary conditions). Due to the space regularity assumption (5.2.23), for almost every $t>0$ the function $x\to v(t,x)$ is continuous up to the boundary, and hence the pointwise values in (5.2.25) make sense.

- (Sign condition). For almost every $t>0$, the left-hand side of (5.2.26) is a well defined signed measure. Indeed, due to the space regularity assumptions (5.2.22) and (5.2.23), it is the product of the continuous function $x\to v(t,x)$ and the signed measure Du, which is the derivative of the BV function $x\to u(t,x)$.

The proof of Theorem 5.2.7 follows from the combination of the following two results, where we show that uv-evolutions have some monotonicity properties, and our generalized solutions are uv-evolutions. The formal statements are the following.

Proposition 5.2.10 (Monotonicity properties of uv-evolutions with NBC in dimension one)**.** *Let (u,v) be a uv-evolution with Neumann boundary conditions in an interval $(a,b)\subseteq\mathbb{R}$, in the sense of Definition 5.2.8.*

Then the following monotonicity results hold true (here we always consider the representative of u that is continuous with values in $L^1((a,b))$, see Remark 5.2.9).

(1) (Maximum principle). *For every $t\geq 0$, let $M^+(t)$ and $M^-(t)$ denote, respectively, the (essential) supremum and infimum of the function $x\mapsto u(t,x)$.*

Then the function $t\mapsto M^+(t)$ is nonincreasing, while the function $t\mapsto M^-(t)$ is nondecreasing.

(2) (Monotonicity of the total variation). *For every $t\geq 0$, let $TV^{\pm}(t)$ denote the positive/negative total variation of the function $x\mapsto u(t,x)$.*

Then the functions $t\mapsto TV^{\pm}(t)$ are nonincreasing.

Proposition 5.2.11 (Joining link)**.** *Let u be a generalized solution to equation (1.2) with homogeneous Neumann boundary conditions in the interval $(0,1)$, obtained through semi-discrete approximation in the sense of Definition 5.2.4. Let v be the function defined in statement (3) of Theorem C.*

Then the pair (u,v) is a uv-evolution with homogeneous Neumann boundary conditions in (a,b) in the sense of Definition 5.2.8.

A generalization of Definition 5.2.8 and Proposition 5.2.10 is provided in section 5.5, and some further results concerning the evolution of level sets are deduced in section 5.6.

5.3 A counterexample to strict convergence

In this section we prove Theorem 5.2.6. Before entering into details, we give an outline of the strategy of the proof. Due to assumption (5.2.1), there exist real numbers $0 < \lambda_0 \leq \Lambda_0$ and $\sigma_0 \in (0, \sigma_1)$ such that

$$\lambda_0(\alpha - \beta) \leq \varphi'(\alpha) - \varphi'(\beta) \leq \Lambda_0(\alpha - \beta) \qquad \forall\, 0 \leq \beta \leq \alpha \leq \sigma_0. \tag{5.3.1}$$

Now let us consider the function

$$u_0(x) = \frac{\sigma_0}{2} \sin\left(\frac{\pi}{2}x\right) \qquad \forall x \in [0,1], \tag{5.3.2}$$

and for every positive integer n let us consider the discrete approximation u_{0n} of u_0 defined by

$$u_{0n}(i) := \begin{cases} \dfrac{\sigma_0}{2} \sin\left(\dfrac{\pi}{2}\dfrac{i}{n}\right) & \text{if } i \in \{1, \ldots, m_n\}, \\ \dfrac{\sigma_0}{2} + J_n & \text{if } i \in \{m_n + 1, \ldots, n\}, \end{cases} \tag{5.3.3}$$

where $J_n \to 0^+$ is a suitable sequence of positive real numbers, and $\{m_n\}$ is a suitable sequence of integers such that $0 < m_n < n$ for every $n \geq 2$, and

$$\lim_{n \to +\infty} \frac{m_n}{n} = 1. \tag{5.3.4}$$

Then the following facts hold true.

(1) The sequence $\{u_{0n}\}$ converges to u_0 uniformly in $[0,1]$ (with the usual meaning of Remark 5.2.1), and the total variation of u_{0n} converges to the total variation of u_0.

This is true because of (5.3.4) and the fact that $J_n \to 0$. Roughly speaking, this means that the perturbation for $i > m_n$ is small, both horizontally and vertically.

(2) For every positive integer n, let us consider the solution $u_n(t, i)$ of the semi-discrete equation (5.2.9), with initial datum $u_{0n}(i)$, and the understanding that (the first condition mimics a Dirichlet boundary condition in $x = 0$, the second one a Neumann boundary condition in $x = 1$)

$$u_n(t, 0) = 0 \qquad \text{and} \qquad u_n(t, n+1) = u_n(t, n) \qquad \forall t \geq 0. \tag{5.3.5}$$

The function $u_n(t, i)$ turns out to be increasing with respect to the second variable for every $t \geq 0$, and in particular

$$\|u_n(t, i)\|_\infty = TV(u_n(t, i)) = u_n(t, n) \qquad \forall t \geq 0. \tag{5.3.6}$$

(3) It turns out that

$$\limsup_{n \to +\infty} u_n(t, \lceil nx \rceil) \leq \frac{\sigma_0}{2} \sin\left(\frac{\pi}{2}x\right) \cdot \exp(-\lambda_0 t) \qquad \forall t \geq 0, \quad \forall x \in (0,1), \tag{5.3.7}$$

and

$$\liminf_{n\to+\infty} u_n(t,n) \geq \frac{\sigma_0}{2} \qquad \forall t \geq 0. \tag{5.3.8}$$

These are the two key points of the proof, and they are established in Proposition 5.3.4 by constructing a suitable sub/supersolution.

(4) The sequence $\{u_n\}$ fits into the framework of Theorem B (we point out that both Theorem A and Theorem B remain valid also with discrete Dirichlet/Neumann boundary conditions). Any limit point $u(t,x)$ is nondecreasing with respect to x for every $t \geq 0$. Due to (5.3.7), any limit point satisfies

$$\|u(t,x)\|_\infty = TV(u(t,x)) = u(t,1) \leq \frac{\sigma_0}{2}\exp(-\lambda_0 t) \qquad \forall t \geq 0.$$

On the other hand, from (5.3.6) and (5.3.8) we deduce that

$$\liminf_{n\to+\infty} \|u_n(t,i)\|_\infty = \liminf_{n\to+\infty} TV(u_n(t,i)) \geq \frac{\sigma_0}{2},$$

from which we conclude that there is strict inequality for every $t > 0$ both in (5.2.19) and in (5.2.20). This proves the conclusions of Theorem 5.2.6 for the problem with one Dirichlet and one Neumann boundary condition. If we want an example with Neumann boundary conditions in both endpoints, it is enough to reproduce the phenomenon in $(-1,1)$ by extending $u_n(t,i)$ as an odd function for negative values of i.

Remark 5.3.1 (Characterization of the limit)**.** It is possible to show that the whole sequence u_n converges to the unique classical solution $u(t,x)$ to equation (1.2) with initial datum (5.3.2) and boundary conditions (of Dirichlet type in $x = 0$ and of Neumann type in $x = 1$)

$$u(t,0) = u_x(t,1) = 0.$$

The convergence is in $C^0\left([0,+\infty); L^p((0,1))\right)$ for every finite $p \geq 1$, but of course not for $p = +\infty$, with the usual meaning of Remark 5.2.1.

This fact can be proved either by constructing more refined subsolutions and supersolutions, or by relying on general results concerning the convergence of gradient-flows, as in the proof of [42, Theorem 2.10].

Remark 5.3.2 (Variants of the counterexample)**.** The example described above can be generalized to any interval by translation and/or homothety. Moreover, u_0 and u_{0n} can be extended by periodicity/reflection in order to obtain an example where the anomalous behavior is not at the boundary, but in a neighborhood of some internal maximum/minimum point, as described in the introduction.

The rest of this section is devoted to the proof of (5.3.7) and (5.3.8). The main tool is the following comparison result for solutions to (5.2.9) (see also [17] where a similar idea is exploited).

Lemma 5.3.3 (Comparison principle for discrete sub/supersolutions). *Let $0 < m < n$ be two positive integers, and let T be a positive real number.*

Let $u : [0,T] \times \{1,\ldots,n\} \to \mathbb{R}$ and $v : [0,T] \times \{1,\ldots,n\} \to \mathbb{R}$ be two functions with the following properties, in which σ_1 is the constant that appears in (5.2.2) and (5.2.3).

(i) ("Space" monotonicity). *The functions u and v are nondecreasing with respect to the second variable for every $t \in [0,T]$.*

(ii) (Solution). *The function u is a solution to the semi-discrete equation (5.2.9) in the interval $[0,T]$, where u is extended to the "boundary" according to the discrete Dirichlet/Neumann boundary conditions*

$$u(t,0) = 0 \qquad \text{and} \qquad u(t,n+1) = u(t,n) \qquad \forall t \in [0,T]. \tag{5.3.9}$$

(iii) (Relation between initial data). *The initial data of u and v satisfy*

$$u(0,i) < v(0,i) \qquad \forall i \in \{1,\ldots,m\},$$

and

$$u(0,i) > v(0,i) \qquad \forall i \in \{m+1,\ldots,n\}.$$

(iv) (Subcritical condition except in m). *The functions u and v are subcritical for $i \neq m$, in the sense that for every $t \in [0,T]$ their discrete derivatives satisfy*

$$D_n^+ u(t,i) \leq \sigma_1 \quad \text{and} \quad D_n^+ v(t,i) \leq \sigma_1 \qquad \forall i \in \{1,\ldots,n\} \setminus \{m\} \tag{5.3.10}$$

(v) (Supercritical condition in m). *When $i = m$ it turns out that*

$$D_n^+ v(t,m)) \geq \sigma_1 \qquad \forall t \in [0,T]. \tag{5.3.11}$$

(vi) (Sub/supersolution). *For every $t \in [0,T]$ the function v is a strict supersolution of equation (5.2.12) for $i \leq m$ in the sense that*

$$v'(t,i) > D_n^- \left(\varphi'(D_n^+ v(t,i)\right) \qquad \forall i \in \{1,\ldots,m\},$$

and a strict subsolution for $i > m$ in the sense that

$$v'(t,i) < D_n^- \left(\varphi'(D_n^+ v(t,i)\right) \qquad \forall i \in \{m+1,\ldots,n\}.$$

Like the function u, also the function v is extended to the "boundary" according to the discrete Dirichlet/Neumann boundary conditions

$$v(t,0) = 0 \qquad \text{and} \qquad v(t,n+1) = v(t,n) \qquad \forall t \in [0,T]. \tag{5.3.12}$$

Then for every $t \in [0,T]$ it turns out that

$$u(t,i) < v(t,i) \qquad \forall i \in \{1,\ldots,m\}, \tag{5.3.13}$$

and

$$u(t,i) > v(t,i) \qquad \forall i \in \{m+1,\ldots,n\}. \tag{5.3.14}$$

Proof. Let us set

$$S := \sup\{\tau \in [0,T] : (5.3.13) \text{ and } (5.3.14) \text{ hold true for every } t \in [0,\tau]\}.$$

We observe that (5.3.13) and (5.3.14) hold true when $t = 0$, and therefore a continuity argument implies that $S > 0$. We need to show that $S = T$. So we assume by contradiction that $S \in (0,T)$. Then by the maximality of S there exists $i_0 \in \{1,\ldots,n\}$ such that

$$v(S,i_0) = u(S,i_0). \tag{5.3.15}$$

Moreover, since (5.3.13) and (5.3.14) hold true for every $t \in [0,S)$, passing to the limit we obtain that

$$v(S,i) \geq u(S,i) \qquad \forall i \in \{1,\ldots,m\}, \tag{5.3.16}$$

and

$$v(S,i) \leq u(S,i) \qquad \forall i \in \{m+1,\ldots,n\}. \tag{5.3.17}$$

Due to the discrete boundary conditions (5.3.9) and (5.3.12), inequality (5.3.16) is true also for $i = 0$, and inequality (5.3.17) is true also for $i = n+1$.

Now let us consider the function $w(t,i) := v(t,i) - u(t,i)$, and let us observe that $w(S,i_0) = 0$ because of (5.3.15). Now we distinguish two cases.

Case $i_0 \in \{1,\ldots,m\}$ We observe that $w(t,i_0) > 0$ for every $t \in [0,S)$ because (5.3.13) is true in that interval. It follows that $w'(S,i_0) \leq 0$, and hence

$$v'(S,i_0) \leq u'(S,i_0). \tag{5.3.18}$$

Concerning discrete derivatives, we claim that

$$\varphi'\left(D_n^+ v(S,i_0-1)\right) \leq \varphi'\left(D_n^+ u(S,i_0-1)\right) \tag{5.3.19}$$

and

$$\varphi'\left(D_n^+ v(S,i_0)\right) \geq \varphi'\left(D_n^+ u(S,i_0)\right). \tag{5.3.20}$$

Indeed, from (5.3.15) and from (5.3.16) with $i = i_0 - 1$ we find that

$$v(S,i_0) - v(S,i_0-1) = u(S,i_0) - v(S,i_0-1) \leq u(S,i_0) - u(S,i_0-1),$$

and therefore when we divide by $1/n$ we obtain that

$$D_n^+ v(S,i_0-1) \leq D_n^+ u(S,i_0-1).$$

Now from (5.3.10) we know that these discrete derivatives lie in the interval $[0,\sigma_1]$, and therefore from the monotonicity assumption (5.2.2) we deduce (5.3.19).

As for (5.3.20), in the case where $i_0 \in \{1,\ldots,m-1\}$ we can apply (5.3.15) and (5.3.16) with $i = i_0 + 1$. We find that

$$v(S,i_0+1) - v(S,i_0) = v(S,i_0+1) - u(S,i_0) \geq u(S,i_0+1) - u(S,i_0),$$

and therefore when we divide by $1/n$ we obtain that

$$D_n^+ v(S, i_0) \geq D_n^+ u(S, i_0),$$

from which we deduce (5.3.20) by exploiting again the monotonicity of φ' in $[0, \sigma_1]$.

Finally, in the case where $i_0 = m$ we apply (5.3.15), and (5.3.17) with $i = m + 1$. In the usual way we find that

$$D_n^+ v(S, m) \leq D_n^+ u(S, m).$$

On the other hand, from (5.3.11) we know that these two discrete derivatives lie in the region $[\sigma_1, +\infty)$ where φ' is nonincreasing, and therefore the last inequality implies (5.3.20) in the case $i_0 = m$.

Now from (5.3.20) and (5.3.19) we deduce that

$$\begin{aligned} D_n^- \left(\varphi'(D_n^+ v(S, i_0))\right) &= \frac{\varphi'\left(D_n^+ v(S, i_0)\right) - \varphi'\left(D_n^+ v(S, i_0 - 1)\right)}{1/n} \\ &\geq \frac{\varphi'\left(D_n^+ u(S, i_0)\right) - \varphi'\left(D_n^+ u(S, i_0 - 1)\right)}{1/n} \\ &= D_n^- \left(\varphi'(D_n^+ u(S, i_0))\right). \end{aligned}$$

Since u is a solution, and v is a supersolution for this value of i_0, we conclude that

$$v'(S, i_0) > D_n^- \left(\varphi'(D_n^+ v(S, i_0))\right) \geq D_n^- \left(\varphi'(D_n^+ u(S, i_0))\right) = u'(S, i_0),$$

which contradicts (5.3.18).

Case $i_0 \in \{m+1, \ldots, n\}$ In this case we observe that $w(t, i_0) < 0$ for every $t \in [0, S)$ because (5.3.14) is true in that interval. It follows that $w'(S, i_0) \geq 0$, and hence

$$v'(S, i_0) \geq u'(S, i_0). \tag{5.3.21}$$

Concerning discrete derivatives, in this case it turns out that

$$\varphi'\left(D_n^+ v(S, i_0 - 1)\right) \geq \varphi'\left(D_n^+ u(S, i_0 - 1)\right)$$

and

$$\varphi'\left(D_n^+ v(S, i_0)\right) \leq \varphi'\left(D_n^+ u(S, i_0)\right),$$

and as a consequence

$$D_n^- \left(\varphi'(D_n^+ v(S, i_0))\right) \leq D_n^- \left(\varphi'(D_n^+ u(S, i_0))\right).$$

Since u is a solution, and v is a subsolution for this value of i_0, we conclude that

$$v'(S, i_0) < D_n^- \left(\varphi'(D_n^+ v(S, i_0))\right) \leq D_n^- \left(\varphi'(D_n^+ u(S, i_0))\right) = u'(S, i_0),$$

which contradicts (5.3.21). □

We are now ready to prove (5.3.7) and (5.3.8).

Proposition 5.3.4. *There exist a sequence of positive real numbers $J_n \to 0^+$, and a sequence of positive integers $\{m_n\}$, with $0 < m_n < n$ for every $n \geq 2$, such that the following statement holds true.*

The sequence $\{u_n\}$ of solutions to the semi-discrete equation (5.2.9), with the discrete Dirichlet/Neumann boundary conditions (5.3.5), and initial datum defined by (5.3.3), satisfies (5.3.7) and (5.3.8).

Proof. Before entering into details, let us describe the general strategy. We introduce the function

$$v_n(t,i) := \begin{cases} \dfrac{\sigma_0}{2} \sin\left(\dfrac{\pi}{2}\dfrac{i}{n}\right) \exp(-\lambda_0 t) + A_n \dfrac{i}{n} & \text{if } i \in \{1, \ldots, m_n\}, \\ \dfrac{\sigma_0}{2} + B_n - C_n \left(1 - \dfrac{i}{n}\right)^2 - E_n t & \text{if } i \in \{m_n + 1, \ldots, n\}, \end{cases} \tag{5.3.22}$$

where A_n, B_n, C_n and E_n are four sequences of nonnegative real numbers that tend to 0 as $n \to +\infty$, and m_n is a sequence of integers that satisfies (5.3.4) and $0 < m_n < n$ for n large enough.

At this point we set $J_n := B_n + 1/n$ and we claim that, if the sequences A_n, B_n, C_n, E_n, m_n are chosen properly, then for every real number $T > 0$ there exists a positive integer n_0 such that for every $n \geq n_0$ and for every $t \in [0, T]$ it turns out that

$$u_n(t,i) < v_n(t,i) \qquad \forall i \in \{1, \ldots, m_n\}, \tag{5.3.23}$$

and

$$u_n(t,i) > v_n(t,i) \qquad \forall i \in \{m_n + 1, \ldots, n\}. \tag{5.3.24}$$

These two inequalities are enough to conclude. Indeed, from (5.3.4) we deduce that for every $x \in (0,1)$ it turns out that $\lceil nx \rceil \leq m_n$ when n is large enough (depending on x). At this point from (5.3.22) and (5.3.23) with $i = \lceil nx \rceil$ it follows that

$$u_n(t, \lceil nx \rceil) \leq v_n(t, \lceil nx \rceil) = \frac{\sigma_0}{2} \sin\left(\frac{\pi}{2}\frac{\lceil nx \rceil}{n}\right) \exp(-\lambda_0 t) + A_n \frac{\lceil nx \rceil}{n}$$

when n is large enough. Since $A_n \to 0$, letting $n \to +\infty$ we obtain that (5.3.7) holds true for every $t \in [0, T]$. Since $T > 0$ is arbitrary, that inequality is actually true for every $t \geq 0$.

As for (5.3.8), from (5.3.22) and (5.3.24) with $i = n$ we obtain that for n large enough it turns out that

$$u_n(t,n) \geq v_n(t,n) = \frac{\sigma_0}{2} + B_n - E_n t \qquad \forall t \in [0, T].$$

Since $B_n \to 0$ and $E_n \to 0$, letting $n \to +\infty$ we deduce that the inequality in (5.3.8) is true for every $t \in [0, T]$, and we conclude by the arbitrariness of T.

The two key estimates (5.3.23) and (5.3.24) follow from Lemma 5.3.3, applied to the functions u_n and v_n, provided that we choose the sequences A_n, B_n, C_n, E_n, m_n in such a way that the assumptions of Lemma 5.3.3 are satisfied.

Choice of parameters Let us consider a function $g : [\sigma_1, +\infty) \to [0, \sigma_1]$ such that

$$\varphi'(g(\sigma)) = \varphi'(\sigma) \qquad \forall \sigma \geq \sigma_1, \tag{5.3.25}$$

We observe that such a function g exists because from assumption (5.2.3) we know that $0 \leq \varphi'(\sigma) \leq \varphi'(\sigma_1)$ for every $\sigma \geq \sigma_1$, and from (5.2.3) we know that φ' is nondecreasing and surjective as a function from $[0, \sigma_1]$ to $[0, \varphi'(\sigma_1)]$. We observe also that $g(\sigma) \to 0$ as $\sigma \to +\infty$ due to (5.2.4) and the fact that φ' is strictly increasing in a neighborhood of the origin because of (5.2.2).

Now we consider the two sequences (the key point is that $h_n \to 0$ and $nh_n \to +\infty$)

$$h_n := \frac{1}{\sqrt{n}}, \qquad \mu_n := \left\lceil n\sqrt{g(nh_n)} \right\rceil + 2,$$

and then we set

$$A_n := \varphi'(nh_n) + \frac{\sigma_0 \lambda_0}{2n}, \qquad C_n := \frac{ng(nh_n)}{2\mu_n - 3}, \qquad E_n := 2\Lambda_0 C_n + \frac{1}{n}, \tag{5.3.26}$$

where λ_0 and Λ_0 are the constants that appear in (5.3.1), and finally

$$B_n := A_n + C_n + h_n + \sqrt{E_n}, \qquad m_n := n - \mu_n. \tag{5.3.27}$$

We observe that m_n satisfies (5.3.4) because $\mu_n/n \to 0$, and that the sequences A_n, B_n, C_n, E_n tend to 0. Moreover, the sequences A_n, B_n and E_n are positive, while C_n is just nonnegative (because our assumptions admit that $\varphi'(\sigma)$, and hence also $g(\sigma)$, might vanish when σ is large enough). Therefore, for every $T > 0$ we can choose a positive integer n_0 such that the following four inequalities

$$\frac{\pi}{2}\frac{\sigma_0}{2} + A_n \leq \sigma_0, \qquad 2C_n \leq \sigma_0, \qquad nh_n \geq \sigma_1, \qquad \sqrt{E_n} - E_n T \geq 0 \tag{5.3.28}$$

hold true for every $n \geq n_0$. In the sequel of the proof we check that all the assumptions of Lemma 5.3.3 are satisfied for every $n \geq n_0$.

Solution and space monotonicity The function $u_n(t, i)$ is by definition a solution to the semi-discrete equation (5.2.9) with the discrete Dirichlet/Neumann boundary conditions (5.3.9), and it is nondecreasing with respect to i because of statement (5) of Theorem A.

As for v_n, from the explicit formula (5.3.22) it is immediate that $v_n(t, i+1) > v_n(t, i)$ at least when $i \neq m_n$. In addition, when $i = m_n$ we obtain that

$$v_n(t, m_n + 1) \geq \frac{\sigma_0}{2} + B_n - C_n - E_n T \qquad \text{and} \qquad v_n(t, m_n) \leq \frac{\sigma_0}{2} + A_n.$$

Recalling the definition of B_n in (5.3.27), and the fourth relation in (5.3.28), we conclude that

$$v_n(t, m_n + 1) - v_n(t, m_n) \geq h_n + \left(\sqrt{E_n} - E_n T\right) \geq h_n, \tag{5.3.29}$$

which proves that the difference is positive also in this case.

Relations between initial data We need to check that for $n \geq n_0$ it turns out that

$$u_n(0,i) < v_n(0,i) \qquad \forall i \in \{1, \ldots, m_n\}$$

and

$$u_n(0,i) > v_n(0,i) \qquad \forall i \in \{m_n + 1, \ldots, n\}.$$

Since we set $J_n := B_n + 1/n$, both inequalities are immediate from (5.3.3) and (5.3.22).

Subcritical condition except in m We show that for every $t \in [0,T]$ and every $n \geq n_0$ it turns out that

$$D_n^+ u_n(t,i) \leq \sigma_1 \qquad \text{and} \qquad D_n^+ v_n(t,i) \leq \sigma_0 \leq \sigma_1 \tag{5.3.30}$$

for every $i \in \{1, \ldots, n\} \setminus \{m_n\}$.

As for u_n, the estimate follows from statement (4) of Theorem A, after observing that $|D_n^+ u_{0n}(i)| \leq \sigma_0 \leq \sigma_1$ for all admissible indices $i \neq m_n$.

As for v_n, we distinguish two cases. When $i \in \{1, \ldots, m_n - 1\}$, from the explicit formula (5.3.22) and the Lipschitz continuity of the function $\sin \sigma$ we obtain that

$$D_n^+ v_n(t,i) = n\frac{\sigma_0}{2}\left\{\sin\left(\frac{\pi}{2}\frac{i+1}{n}\right) - \sin\left(\frac{\pi}{2}\frac{i}{n}\right)\right\}\exp(-\lambda_0 t) + A_n \leq n\frac{\sigma_0}{2}\frac{\pi}{2n} + A_n,$$

and the latter is less than or equal to σ_0 because of the first condition in (5.3.28).

In the case where $i \in \{m_n + 1, \ldots, n\}$, from the explicit formula (5.3.22) we obtain that

$$D_n^+ v_n(t,i) = C_n\left(2 - \frac{2i+1}{n}\right) \leq 2C_n \leq \sigma_0,$$

where in the last inequality we exploited the second condition in (5.3.28).

Supercritical condition in m We show that for every $n \geq n_0$ it turns out that

$$D_n^+ v_n(t, m_n) \geq n h_n \geq \sigma_1 \qquad \forall t \in [0,T], \tag{5.3.31}$$

and as a consequence

$$\varphi'(D_n^+ v_n(t, m_n)) \leq \varphi'(n h_n) = \varphi'\left(g(n h_n)\right) \qquad \forall t \in [0,T]. \tag{5.3.32}$$

Indeed, dividing (5.3.29) by $1/n$, and recalling the third condition in (5.3.28), we obtain exactly (5.3.31). At this point, the two relations in (5.3.32) follow from (5.3.31), assumption (5.2.3), and (5.3.25).

Supersolution for $i \in \{1, \ldots, m_n - 1\}$ We need to show that for every $n \geq n_0$ and every $t \in [0, T]$ it turns out that

$$v_n'(t, i) > D_n^- \left(\varphi'(D_n^+ v_n(t, i))\right) \qquad \forall i \in \{1, \ldots, m_n - 1\}. \tag{5.3.33}$$

Computing the time derivative, and rearranging the terms, this inequality can be rewritten as

$$n \left\{\varphi'(D_n^+ v_n(t, i-1)) - \varphi'(D_n^+ v_n(t, i))\right\} > \lambda_0 \frac{\sigma_0}{2} \sin\left(\frac{\pi}{2}\frac{i}{n}\right) \exp(-\lambda_0 t). \tag{5.3.34}$$

From the explicit formula (5.3.22), and the concavity of the function $\sin\sigma$, we find that

$$D_n^+ v_n(t, i-1) \geq D_n^+ v_n(t, i).$$

From (5.3.30) we know that both discrete derivatives lie in the interval $[0, \sigma_0]$, and hence from the estimate from below in (5.3.1) we deduce that

$$\varphi'(D_n^+ v_n(t, i-1)) - \varphi'(D_n^+ v_n(t, i)) \geq \lambda_0 \left\{D_n^+ v_n(t, i-1) - D_n^+ v_n(t, i)\right\}. \tag{5.3.35}$$

Now from the trigonometric identity

$$\sin(a + h) + \sin(a - h) - 2\sin(a) = -4\sin(a)\sin^2\left(\frac{h}{2}\right),$$

applied with $a := (\pi/2)(i/n)$ and $h := (\pi/2)(1/n)$, we find that

$$\begin{aligned} D_n^+ v_n(t, i-1) - D_n^+ v_n(t, i) &= 4\frac{\sigma_0}{2} n \sin^2\left(\frac{\pi}{2}\frac{1}{2n}\right) \sin\left(\frac{\pi}{2}\frac{i}{n}\right) \exp(-\lambda_0 t) \\ &> \frac{\sigma_0}{2n} \sin\left(\frac{\pi}{2}\frac{i}{n}\right) \exp(-\lambda_0 t), \end{aligned}$$

where in the last step we exploited that $\sin(\frac{\pi}{2}x) > x$ for every $x \in (0, 1)$.

Plugging this inequality into (5.3.35) we obtain (5.3.34).

Supersolution for $i = m_n$ We need to show that, for every $n \geq n_0$ and every $t \in [0, T]$, the inequality in (5.3.33) is satisfied also for $i = m_n$.

After computing the derivative and rearranging the terms, the inequality can be rewritten in the form

$$n\varphi'(D_n^+ v_n(t, m_n)) < n\varphi'(D_n^+ v_n(t, m_n - 1)) - \frac{\sigma_0}{2}\lambda_0 \sin\left(\frac{\pi}{2}\frac{m_n}{n}\right) \exp(-\lambda_0 t).$$

From the explicit formula (5.3.22), and the monotonicity of the function $\sin\sigma$, we obtain that $D_n^+ v_n(t, m_n - 1) > A_n$. Therefore, from (5.3.32) and the definition of A_n in (5.3.26) we conclude that

$$\begin{aligned} n\varphi'(D_n^+ v_n(t, m_n)) &\leq n\varphi'(nh_n) = nA_n - \frac{\sigma_0}{2}\lambda_0 \\ &< n\varphi'(D_n^+ v_n(t, m_n - 1)) - \frac{\sigma_0}{2}\lambda_0 \sin\left(\frac{\pi}{2}\frac{m_n}{n}\right) \exp(-\lambda_0 t), \end{aligned}$$

as required.

Subsolution for $i = m_n + 1$ We need to show that for every $n \geq n_0$ and every $t \in [0, T]$ it turns out that

$$v_n'(t, m_n + 1) < \frac{\varphi'(D_n^+ v_n(t, m_n + 1)) - \varphi'(D_n^+ v_n(t, m_n))}{1/n}.$$

The left-hand side is equal to $-E_n$, and hence negative. Therefore, it is enough to show that the right-hand side is nonnegative. To this end, from the explicit formula (5.3.22) we obtain that

$$D_n^+ v_n(t, m_n + 1) = C_n \frac{2(n - m_n) - 3}{n} = C_n \frac{2\mu_n - 3}{n}.$$

At this point, from the definition of C_n in (5.3.26), and estimate (5.3.32), we conclude that

$$\varphi'(D_n^+ v_n(t, m_n + 1)) = \varphi'\left(C_n \frac{2\mu_n - 3}{n}\right) = \varphi'(g(nh_n)) \geq \varphi'(D_n^+ v_n(t, m_n)),$$

as required.

Subsolution for $i \in \{m_n + 2, \ldots, n - 1\}$ We need to show that for every $n \geq n_0$ and every $t \in [0, T]$ it turns out that

$$v_n'(t, i) < D_n^- \left(\varphi'(D_n^+ v_n(t, i))\right) \qquad \forall i \in \{m_n + 2, \ldots, n - 1\}. \tag{5.3.36}$$

From the explicit formula (5.3.22) we obtain that

$$D_n^+ v_n(t, i) = C_n \left(2 - \frac{2i + 1}{n}\right) \quad \text{and} \quad D_n^+ v_n(t, i - 1) = C_n \left(2 - \frac{2i - 1}{n}\right).$$

Exploiting the estimate from above in (5.3.1), and the definition of E_n in (5.3.26), we conclude that

$$\begin{aligned} n\left\{\varphi'(D_n^+ v_n(t, i - 1)) - \varphi'(D_n^+ v_n(t, i))\right\} &\leq n\Lambda_0 \left\{D_n^+ v_n(t, i - 1) - D_n^+ v_n(t, i)\right\} \\ &= 2\Lambda_0 C_n < E_n, \end{aligned}$$

which is equivalent to (5.3.36).

Subsolution for $i = n$ It remains to verify that, for every $n \geq n_0$ and every $t \in [0, T]$, the inequality in (5.3.36) is satisfied also for $i = n$. Due to the discrete Neumann boundary condition, this inequality reduces to

$$-E_n < -n\varphi'\left(D_n^+ v_n(t, n - 1)\right). \tag{5.3.37}$$

From the explicit formula (5.3.22) we deduce that $D_n^+ v_n(t, n-1) = C_n/n$. Therefore, exploiting again the estimate from above in (5.3.1) (now with $\beta = 0$) and the definition of E_n in (5.3.26), we conclude that

$$n\varphi'\left(D_n^+ v_n(t, n - 1)\right) = n\varphi'\left(\frac{C_n}{n}\right) \leq \Lambda_0 C_n < E_n,$$

which proves (5.3.37). □

Remark 5.3.5 (More general setting)**.** We observe that the same proof works if we replace the limiting initial datum $u_0(x)$ defined by (5.3.2) by any smooth function $\widehat{u}_0(x)$ such that $0 \leq \widehat{u}_0(x) \leq u_0(x)$ and $0 \leq \widehat{u}_{0x}(x) \leq \sigma_0$ for every $x \in (0,1)$.

Concerning the nonlinearity φ, the essential hypotheses are (5.3.1) and (5.2.4). The convex-concave assumption can be avoided by modifying a little the definition of v_n and by showing that the discrete derivatives of u_n never enter in the region where $\varphi'(\sigma) > \varphi'(\sigma_0)$. We skip this technical point that only complicates the proof without introducing essentially new ideas.

5.4 Monotonicity results for uv-evolutions

5.4.1 UV-evolutions in any space dimension

In order to prove Proposition 5.2.10 we need to extend the notion of uv-evolution to any space dimension. The extension is almost straightforward, but here we need to consider a combination of Dirichlet and Neumann boundary conditions.

Definition 5.4.1 (UV-evolution with DNBC in any dimension)**.** Let d be a positive integer, and let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set with Lipschitz boundary.

A *UV-evolution with Dirichlet/Neumann boundary conditions* in Ω is a pair of measurable functions

$$U : (0,+\infty) \times \Omega \to \mathbb{R} \qquad \text{and} \qquad V : (0,+\infty) \times \Omega \to \mathbb{R}^d$$

with the following properties.

- (Time regularity). The function U admits a weak derivative U_t with respect to time, and
$$U_t \in L^1((0,T) \times \Omega) \qquad \forall T > 0. \tag{5.4.1}$$

- (Space regularity). For almost every $t > 0$ it turns out that
$$\text{the function } x \mapsto U(t,x) \text{ is in } BV(\Omega), \tag{5.4.2}$$
$$\text{the function } x \mapsto V(t,x) \text{ is in } W^{1,1}(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega). \tag{5.4.3}$$

- (Evolution equation). The functions U and V solve the equation
$$U_t(t,x) = \operatorname{div} V(t,x) \qquad \text{in } (0,+\infty) \times \Omega. \tag{5.4.4}$$

- (Sign condition). For almost every $t > 0$ it turns out that
$$\langle V(t,x), DU(t,x) \rangle \geq 0 \qquad \text{as a measure in } \Omega. \tag{5.4.5}$$

- (Dirichlet/Neumann boundary conditions). There exists a nonincreasing function $D_0 : [0,+\infty) \to \mathbb{R}$ such that for almost every $t > 0$ it turns out that, for almost every $x \in \partial\Omega$ (with respect to the $d-1$ dimensional Hausdorff measure), at least one of the following two conditions
$$U(t,x) \leq D_0(t), \qquad\qquad \langle V(t,x), \nu(x) \rangle = 0 \tag{5.4.6}$$
holds true, where $\nu(x)$ denotes the outer normal to $\partial\Omega$ at point x.

Remark 5.4.2. As we did in dimension one, let us comment on some delicate regularity issues in Definition 5.4.1 above.

- (Evolution equation). As in dimension one, from (5.4.1) and (5.4.3) we know that (5.4.4) can be seen both as an equality between functions in $L^1((0,T)\times\Omega)$, and as an equality between functions in $L^1(\Omega)$ for almost every $t>0$.
- (Time regularity and initial datum). As in dimension one, the time regularity assumption (5.4.1) implies that U is absolutely continuous as a function from $(0,+\infty)$ to $L^1(\Omega)$. In particular, all sections $x\mapsto U(t,x)$ are well defined as functions in $L^1(\Omega)$ for every $t\geq 0$, including the initial datum at $t=0$.
- (Dirichlet/Neumann boundary conditions). From the space regularity assumptions (5.4.2) and (5.4.3) we know that, for almost every $t>0$, the functions $x\mapsto U(t,x)$ and $x\mapsto V(t,x)$ admit a trace on $\partial\Omega$. This implies that the two conditions in (5.4.6) make sense.
- (Sign condition). The left-hand side of (5.4.5) is the sum of d terms that are the product of a bounded function in $W^{1,1}(\Omega)$ and a signed measure. In general this product is not well defined, but in this case the measure is the gradient of a function in $BV(\Omega)$, and therefore it is absolutely continuous with respect to the $d-1$ dimensional Hausdorff measure $\mathcal{H}^{d-1}$, and the precise representative of the Sobolev function V (namely the limit as $r\to 0^+$ of its average in the ball with radius r centered in x) is defined at $\mathcal{H}^{d-1}$ almost every point. Under these assumptions, the product makes sense as a vector measure.

  In any case, the UV-evolutions that we consider here have the additional property that the function $t\mapsto V(t,x)$ is continuous with respect to x for almost every $t>0$, in which case the definition of the product is less delicate.

  We observe that, if we decompose the vector measure Du into its jump part D^Ju and its diffuse part $\widetilde{D}u$ (see [6, Definition 3.91]), then (5.4.5) is equivalent to the sign condition on both components, namely

$$\langle V(t,x),\widetilde{D}U(t,x)\rangle\geq 0 \qquad\text{and}\qquad \langle V(t,x),D^JU(t,x)\rangle\geq 0. \tag{5.4.7}$$

In the next result we show that these UV-evolutions in any dimension satisfy a maximum principle.

Proposition 5.4.3 (Maximum principle for UV-evolutions with DNBC in any space dimension)**.** *Let d be a positive integer, and let $\Omega\subseteq\mathbb{R}^d$ be a bounded open set with Lipschitz boundary.*

Let (U,V) be a UV-evolution with Dirichlet/Neumann boundary conditions in Ω, in the sense of Definition 5.4.1, and let $D_0:[0,+\infty)\to\mathbb{R}$ be the nonincreasing function that appears in (5.4.6).

Then the function defined by

$$M(t):=\max\left\{D_0(t),(\text{ess})\sup\{U(t,x):x\in\Omega\}\right\}\qquad\forall t\geq 0 \tag{5.4.8}$$

is nonincreasing (in the definition of $M(t)$ we consider the representative of U that is continuous with values in $L^1(\Omega)$, see Remark 5.4.2).

Proof. Since we can always restrict to a smaller time interval, it is enough to show that $M(t) \leq M(0)$ for every $t \geq 0$. On the other hand, from the monotonicity of $D_0(t)$ it follows that $D_0(t) \leq D_0(0) \leq M(0)$, and hence we can limit ourselves to showing that for every $t \geq 0$ it turns out that

$$U(x,t) \leq M(0) \qquad \text{for almost every } x \in \Omega. \tag{5.4.9}$$

To this end, we consider a convex function $\psi \in C^2(\mathbb{R})$ that is Lipschitz continuous with

$$\psi(\sigma) = 0 \qquad \text{if and only if} \qquad \sigma \leq M(0), \tag{5.4.10}$$

and then we set

$$E(t) := \int_\Omega \psi(U(t,x))\,dx \qquad \forall t \geq 0. \tag{5.4.11}$$

We observe that (5.4.10) and the convexity of ψ imply that $\psi(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$, and hence

$$E(t) \geq 0 \qquad \forall t \geq 0, \tag{5.4.12}$$

and in addition $\psi(\sigma) > 0$ for every $\sigma > M(0)$, from which we deduce that

$$E(t) = 0 \text{ if and only if (5.4.9) holds true.} \tag{5.4.13}$$

Moreover, the function $E(t)$ is absolutely continuous because of the boundedness of ψ' and the time regularity (5.4.1) of U. We claim that $E'(t) \leq 0$ for almost every $t \geq 0$. Since $E(0) = 0$, this claim, combined with (5.4.12), would imply that $E(t) = 0$ for every $t \geq 0$, and this would complete the proof because of (5.4.13).

Using (5.4.4), we can write the time-derivative of the integral (5.4.11) in the form

$$E'(t) = \int_\Omega \psi'(U(t,x))U_t(t,x)\,dx = \int_\Omega \psi'(U(t,x))\operatorname{div} V(t,x)\,dx.$$

Now the space regularity of U and V is enough to integrate by parts, leading to (with some abuse of notation, because the scalar product is a measure and not a function)

$$E'(t) = -\int_\Omega \langle D[\psi'(U(t,x))], V(t,x)\rangle\,dx. \tag{5.4.14}$$

In the integration by parts we neglected the boundary term

$$\int_{\partial\Omega} \psi'(U(t,x))\langle V(t,x), \nu(x)\rangle\,d\sigma$$

which is equal to 0 for almost every $t \geq 0$ because of (5.4.6). Indeed, for almost every $x \in \partial\Omega$ we know that either the scalar product is equal to 0, or

$$U(t,x) \leq D_0(t) \leq D_0(0) \leq M(0),$$

in which case $\psi'(U(t,x)) = 0$ because of (5.4.10). Now from the chain rule for bounded variation functions (see [6, Theorem 3.96]) we know that

$$D[\psi'(U(t,x))] = \psi''(U(t,x))\widetilde{D}U(t,x) + \frac{\psi'(U^+(t,x)) - \psi'(U^-(t,x))}{U^+(t,x) - U^-(t,x)}D^J U(t,x),$$

where $DU = \widetilde{D}U + D^J U$ is the usual decomposition of the vector measure DU, and U^+ and U^- are the traces of U on the two sides of the jump set.

Plugging this equality into (5.4.14) we deduce that (again with some abuse of notation, because the two scalar products are measures)

$$\begin{aligned} E'(t) &= -\int_\Omega \psi''(U(t,x))\langle \widetilde{D}U(t,x), V(t,v)\rangle\, dx \\ &\quad -\int_\Omega \frac{\psi'(U^+(t,x)) - \psi'(U^-(t,x))}{U^+(t,x) - U^-(t,x)}\langle D^J U(t,x), V(t,v)\rangle\, dx, \end{aligned}$$

and we conclude by observing that both integrals are nonnegative because of the convexity of ψ and the sign conditions (5.4.7). □

Remark 5.4.4 (Case with only Neumann boundary conditions)**.** From the proof it is clear that, when for almost every $t > 0$ the second condition in (5.4.6) is satisfied for almost every $x \in \partial\Omega$, then $D_0(t)$ plays no role. In particular, we do not need to consider the maximum with $D_0(t)$ in (5.4.8), or equivalently we can take $D_0(t) \equiv -\infty$.

5.4.2 Proof of Proposition 5.2.10

Maximum principle We claim that the pair

$$U(t,x) := u(t,x), \qquad V(t,x) := v(t,x)$$

is a UV-evolution with Dirichlet/Neumann (and actually just Neumann in this case) boundary conditions according to Definition 5.4.1 with $d = 1$, $\Omega = (a,b)$, and no need of $D_0(t)$ (see Remark 5.4.4).

Indeed, all the assumption on U and V in Definition 5.4.1 follow immediately from the corresponding assumptions on u and v in Definition 5.2.8.

At this point from Proposition 5.4.3 it follows that the function $M(t)$ defined by (5.4.8) is nonincreasing, but in this case $M(t)$ coincides with the essential supremum $M^+(t)$.

The monotonicity of $M^-(t)$ can be obtained by applying the maximum principle to the pair $(-u,-v)$, which is again a uv-evolution with Neumann boundary conditions.

Monotonicity of the total variation To begin with, we observe that it is enough to prove the monotonicity of the positive total variation, because the negative total variation of u is the positive total variation of $-u$, and we have already observed that $(-u,-v)$ is again a uv-evolution with Neumann boundary conditions.

To this end, for every positive integer m we introduce the positive m-variation

$$TV_m^+(t) := \sup\left\{\sum_{i=1}^{2m}(-1)^i u(t,x_i) : a < x_1 \le x_2 \le \ldots \le x_{2m} < b.\right\}.$$

We observe that

$$TV^+(t) = \sup_{m\ge 1} TV_m^+(t) = \lim_{m\to+\infty} TV_m^+(t)$$

Therefore, if we prove that the function $t \mapsto TV_m^+(t)$ is nonincreasing for every $m \geq 1$, then thesis follows. We prove the monotonicity of $TV_m^+(t)$ by induction on m.

Case $m = 1$ We claim that the pair defined by

$$U(t, x_1, x_2) := u(t, x_2) - u(t, x_1), \qquad V(t, x_1, x_2) := (-v(t, x_1), v(t, x_2))$$

is a UV-evolution with Dirichlet/Neumann boundary conditions in the sense of Definition 5.4.1 with

$$d := 2, \qquad \Omega := \left\{(x_1, x_2) \in (a, b)^2 : a < x_1 < x_2 < b\right\}, \qquad D_0(t) \equiv 0.$$

If the claim is true, then the monotonicity of $TV_1^+(t)$ follows from Proposition 5.4.3, because in this case the function $M(t)$ defined by (5.4.8) coincides with $TV_1^+(t)$.

So let us check that U and V satisfy the properties in Definition 5.4.1. The regularity and the evolution equation follow from the corresponding properties of u and v in Definition 5.2.8. The sign condition (5.4.5) follows from (5.2.26) because

$$\langle V(t, x_1, x_2), DU(t, x_1, x_2)\rangle = v(t, x_1) \cdot Du(t, x_1) + v(t, x_2) \cdot Du(t, x_2)$$

is the sum of two nonnegative measures.

Finally, we observe that Ω is a triangle, and its boundary is contained in the three lines described by the three equalities $a = x_1$, $x_1 = x_2$, and $x_2 = b$.

- In the side with $a = x_1$ the normal vector is $\nu(x_1, x_2) = (-1, 0)$, and hence

$$\langle V(t, x_1, x_2), \nu(x_1, x_2)\rangle = v(t, x_1) = v(t, a) = 0.$$

- In the side with $x_2 = b$ the normal vector is $\nu(x_1, x_2) = (0, 1)$, and hence

$$\langle V(t, x_1, x_2), \nu(x_1, x_2)\rangle = v(t, x_2) = v(t, b) = 0.$$

- In the side with $x_1 = x_2$ it turns out that

$$U(t, x_1, x_2) = 0 \leq D_0(t).$$

Therefore, in all the sides of $\partial\Omega$ the Dirichlet/Neumann boundary conditions (5.4.6) are satisfied, and this completes the proof.

Inductive step We assume that $TV_m^+(t)$ is nonincreasing for some positive integer m, and we prove that also $TV_{m+1}^+(t)$ is nonincreasing.

To this end, we consider the pair defined by

$$U(t, x) := \sum_{i=1}^{2m+2} (-1)^i u(t, x_i), \qquad V(t, x) := \sum_{i=1}^{2m+2} (-1)^i v(t, x_i) e_i,$$

where $x = (x_1, \ldots, x_{2m+2})$, and e_i denotes the i-th vector of the canonical basis of $\mathbb{R}^{2m+2}$. We claim that this pair is a UV-evolution with Dirichlet/Neumann boundary conditions according to Definition 5.4.1 with

$$d := 2m+2, \qquad D_0(t) := TV_m^+(t),$$
$$\Omega := \left\{(x_1, x_2, \ldots, x_{2m+2}) \in (a,b)^{2m+2} : a < x_1 < \ldots < x_{2m+2} < b\right\}.$$

If this is the case, then the monotonicity of $TV_{m+1}^+(t)$ follows from Proposition 5.4.3, because

$$TV_{m+1}^+(t) = \sup\{U(t, x_1, \ldots, x_{2m+2}) : (x_1, \ldots, x_{2m+2}) \in \Omega\},$$

and in particular the function $M(t)$ defined by (5.4.8) in this case is exactly

$$M(t) = \max\{D_0(t), TV_{m+1}^+(t)\} = \max\{TV_m^+(t), TV_{m+1}^+(t)\} = TV_{m+1}^+(t).$$

So let us check that U and V satisfy the assumptions in Definition 5.4.1. As before, the regularity and the evolution equation follow from the corresponding properties of u and v in Definition 5.2.8. The sign condition (5.4.5) follows from (5.2.26) because

$$\langle V(t,x), DU(t,x)\rangle = \sum_{i=1}^{2m+2} v(t, x_i) \cdot Du(t, x_i)$$

is the sum of $2m+2$ nonnegative measures.

Finally, we consider the boundary of Ω, which consists of $2m+3$ "faces" contained in the hyperplanes corresponding to the possible equalities in the definition of Ω.

- In the face with $x_1 = a$ the normal vector is $\nu(x) = -e_1$, and hence

$$\langle V(t,x), \nu(x)\rangle = v(t, x_1) = v(t, a) = 0.$$

- In the face with $x_{2m+2} = b$ the normal vector is $\nu(x) = e_{2m+2}$, and hence

$$\langle V(t,x), \nu(x)\rangle = v(t, x_{2m+2}) = v(t, b) = 0.$$

- Let us finally consider the faces where $x_i = x_{i+1}$ for some index i. In this case two consecutive terms in the definition of U cancel, and what remains is a competitor in the definition of $TV_m^+(t)$. It follows that in all these $2m+1$ faces of $\partial\Omega$ it turns out that

$$U(t, x_1, \ldots, x_{2m+2}) \leq TV_m^+(t) = D_0(t).$$

This proves that the Dirichlet/Neumann boundary conditions (5.4.6) are satisfied, and thus completes the proof. □

5.4.3 Proof of Proposition 5.2.11

The time and space regularity of u and v, as well as the evolution equation that they solve and the boundary conditions, follow exactly from Theorem C. It remains to prove that u and v satisfy the sign condition (5.2.26) for almost every $t \geq 0$.

To this end, let us consider any test function $\phi \in C^1([0,1])$, and its discrete sampling

$$\phi_k(i) := \phi\left(\frac{i}{n_k}\right) \qquad \forall i \in \{0, 1, \ldots, n_k\}.$$

From the space regularity of u and v we know that, for almost every $t \geq 0$, the function $x \mapsto u(t,x)$ lies in $BV((0,1))$, while the function $t \mapsto v(t,x)\phi(x)$ lies in $W^{1,1}((0,1))$ and vanishes at the boundary. Therefore, for any such t it turns out that

$$\int_0^1 Du(t,x)v(t,x)\phi(x)\,dx = -\int_0^1 u(t,x)v_x(t,x)\phi(x)\,dx - \int_0^1 u(t,x)v(t,x)\phi_x(x)\,dx.$$

We point out that, as usual, there is a little abuse of notation in the left-hand side because Du is actually a measure. Integrating with respect to time we deduce that

$$\begin{aligned} \int_{t_1}^{t_2} dt \int_0^1 Du(t,x)v(t,x)\phi(x)\,dx &= -\int_{t_1}^{t_2} dt \int_0^1 u(t,x)v_x(t,x)\phi(x)\,dx \\ &\quad -\int_{t_1}^{t_2} dt \int_0^1 u(t,x)v(t,x)\phi_x(x)\,dx \end{aligned} \tag{5.4.15}$$

for every choice of the times $t_2 \geq t_1 \geq 0$.

Analogously, by a discrete integration by parts (which is actually an algebraic manipulation of the sums, where we exploit also the discrete Neumann boundary conditions (5.2.17)), we obtain that

$$\begin{aligned} &\sum_{i=1}^{n_k} D^+_{n_k} u_k(t,i)\cdot v_k(t,i)\cdot \phi_k(i) \\ &= -\sum_{i=1}^{n_k} u_k(t,i)\cdot D^-_{n_k} v_k(t,i)\cdot \phi_k(i-1) - \sum_{i=1}^{n_k} u_k(t,i)\cdot v_k(t,i)\cdot D^-_{n_k}\phi_k(i) \end{aligned} \tag{5.4.16}$$

for every $t \geq 0$ and every positive integer k. If we integrate with respect to time, and we rewrite the sums as integrals of piecewise constant functions, we deduce that

$$\begin{aligned} &\int_{t_1}^{t_2} dt \int_0^1 D^+_{n_k} u_k(t,\lceil n_k x\rceil)\cdot v_k(t,\lceil n_k x\rceil)\cdot \phi_k(\lceil n_k x\rceil)\,dx \\ &\qquad = -\int_{t_1}^{t_2} dt \int_0^1 u_k(t,\lceil n_k x\rceil)\cdot D^-_{n_k} v_k(t,\lceil n_k x\rceil)\cdot \phi_k(\lceil n_k x\rceil - 1)\,dx \\ &\qquad\qquad -\int_{t_1}^{t_2} dt \int_0^1 u_k(t,\lceil n_k x\rceil)\cdot v_k(t,\lceil n_k x\rceil)\cdot D^-_{n_k}\phi_k(\lceil n_k x\rceil)\,dx \end{aligned}$$

for every positive integer k and every choice of the times $t_2 \geq t_1 \geq 0$.

Now we are allowed to pass to the limit in the two double integrals of the right-hand side, because in each of them the integrand is the product of two terms that converge strongly (the ones with u_k and ϕ_k) and one term that converges weakly in the pair (t,x) (the one with v_k). Since the limits of these two integrals are the two integrals in the right-hand side of (5.4.15), we conclude that

$$\lim_{n\to+\infty}\int_{t_1}^{t_2} dt\int_0^1 D^+_{n_k}u_k(t,\lceil n_k x\rceil)\cdot v_k(t,\lceil n_k x\rceil)\cdot \phi_k(\lceil n_k x\rceil)\,dx$$
$$= \int_{t_1}^{t_2} dt\int_0^1 Du(t,x)\cdot v(t,x)\cdot\phi(x)\,dx. \quad (5.4.17)$$

Now we observe that, if the test function ϕ is nonnegative, then from (5.2.18) and (5.2.5) we deduce that in the left-hand side we have a limit of integrals of nonnegative functions, and hence

$$\int_{t_1}^{t_2} dt\int_0^1 Du(t,x)\cdot v(t,x)\cdot\phi(x)\,dx \geq 0$$

for every choice of the nonnegative test function ϕ and of the times $t_2 \geq t_1 \geq 0$. Since the times are arbitrary, we deduce that, for every nonnegative test function ϕ, there exists a subset $E_\phi \subseteq [0,+\infty)$ with Lebesgue measure equal to 0 such that

$$\int_0^1 Du(t,x)\cdot v(t,x)\cdot\phi(x)\,dx \geq 0 \qquad \forall t\in[0,+\infty)\setminus E_\phi. \quad (5.4.18)$$

The set E_ϕ might depend on ϕ, but we can always take a countable set $\mathcal{D}$ of nonnegative test functions that is dense in the nonnegative functions of $C^1([0,1])$, and a common subset $E\subseteq[0,+\infty)$ with Lebesgue measure equal to 0, such that

$$\int_0^1 Du(t,x)\cdot v(t,x)\cdot\phi(x)\,dx \geq 0 \qquad \forall\phi\in\mathcal{D},\quad \forall t\in[0,+\infty)\setminus E,$$

which guarantees that the sign condition (5.2.26) is satisfied for any such t. □

Remark 5.4.5. We observe that (5.4.17) is equivalent to saying that

$$D^+_{n_k}u_k\cdot v_k \rightharpoonup Du\cdot v \qquad \text{weakly* as measures in } (0,+\infty)\times(0,1).$$

We observe also that the key point in the proof is (5.4.18). One might be tempted to establish this relation by passing to the limit in the right-hand side of (5.4.16) before integrating with respect to time. Indeed, for almost every $t>0$ we have a bound in $L^\infty((0,1))$ on the function $x\mapsto v_k(t,\lceil n_k x\rceil)$, and a bound in $L^2((0,1))$ on its discrete derivative, and therefore these functions admit a weak limit up to subsequences.

The problem with this approach is that in Theorem C the function $v(t,x)$ is defined as the weak limit of v_k in the pair (t,x), and therefore there is no guarantee that the weak limits of the sections of v_k at fixed times have anything to do with the sections of the limit v. This forces us to pass through the double integrals.

5.5 uvw-evolutions

Let us consider the following generalization of Definition 5.2.8.

Definition 5.5.1 (uvw-evolution with NBC in dimension one)**.** A *uvw-evolution with homogeneous Neumann boundary conditions* in an interval $(a, b) \subseteq \mathbb{R}$ is a triple of measurable functions

$$u : (0, +\infty) \times (a, b) \to \mathbb{R}, \qquad v : (0, +\infty) \times (a, b) \to \mathbb{R}, \qquad w : (0, +\infty) \times (a, b) \to \mathbb{R}$$

with the following properties.

- (Boundedness) The function u satisfies

$$u \in L^\infty((0, T) \times (a, b)) \qquad \forall T > 0. \tag{5.5.1}$$

- (Time regularity). The function u admits a weak derivative u_t with respect to time, and

$$u_t \in L^1((0, T) \times (a, b)) \qquad \forall T > 0. \tag{5.5.2}$$

- (Space regularity). For almost every $t > 0$ it turns out that

$$\text{the function } x \mapsto u(t, x) \text{ is in } BV((a, b)),$$

$$\text{the function } x \mapsto v(t, x) \text{ is in } W^{1,1}((a, b)), \tag{5.5.3}$$

$$\text{the function } x \mapsto w(t, x) \text{ is in } BV((a, b)). \tag{5.5.4}$$

- (Evolution equation). The functions u, v and w solve the equation

$$u_t(t, x) = w(t, x) \cdot v_x(t, x) \qquad \text{in } (0, +\infty) \times (a, b). \tag{5.5.5}$$

- (Neumann boundary conditions). For almost every $t > 0$ it turns out that

$$v(t, a) = v(t, b) = 0. \tag{5.5.6}$$

- (Sign condition). There exists a real number $\alpha > 0$ such that for almost every $t > 0$ it turns out that

$$\alpha w^+(t, x)v(t, x)Du(t, x) \geq |v(t, x)Dw(t, x)| \tag{5.5.7}$$

  and

$$\alpha w^-(t, x)v(t, x)Du(t, x) \geq |v(t, x)Dw(t, x)| \tag{5.5.8}$$

  as measures in (a, b).

Remark 5.5.2 (Special case $w \equiv 1$)**.** In the special case where $w(t, x) \equiv 1$, this definition reduces to Definition 5.2.8. In particular, the sign condition (5.5.8) reduces to (5.2.26), without any need of the constant α, and the boundedness assumption (5.5.1) is unnecessary, because it follows from Proposition 5.2.10, at least if the initial datum is bounded (and in our applications to the Perona-Malik equations we always consider initial data with bounded variation).

Remark 5.5.3. Let us comment on some delicate regularity issues in Definition 5.5.1 above.

- (Evolution equation). Due to (5.5.3) and (5.5.4), we know that for almost every $t \in (0,T)$ the right-hand side of (5.5.5) is the product of a bounded function and a function in $L^1((a,b))$, and therefore it is itself in $L^1((a,b))$. Recalling (5.5.2), this means that for almost every $t \in (0,T)$ the evolution equation (5.5.5) is actually an equality between functions in $L^1((a,b))$.

- (Time regularity and initial datum). As for uv-evolutions, the time regularity assumption (5.5.2) implies that u is continuous, and actually also absolutely continuous, as a function from $[0,+\infty)$ to $L^1((a,b))$. In particular the "initial datum" $x \mapsto u(0,x)$ is well defined as a function in $L^1((a,b))$.

- (Neumann boundary conditions). Due to (5.5.3), for almost every $t > 0$ the function $x \to v(t,x)$ is continuous up to the boundary. In particular, the pointwise values in (5.5.6) make sense.

- (Sign conditions). The right-hand sides of (5.5.7) and (5.5.8) are well-defined for almost every $t \in (0,T)$ because they are the total variation of the product of the continuous function $x \to v(t,x)$ and the measure Dw, that is the derivative of the BV function $x \to w(t,x)$. Left-hand sides are more delicate, because they are the product of a signed measure, namely the derivative of the BV function $x \to u(t,x)$, times the continuous function $x \to v(t,x)$, times the functions $x \to w^{\pm}(t,x)$ that in general are not continuous. Nevertheless, since $x \to w(t,x)$ is in $BV((a,b))$, the one-sided limits

$$w^+(t,x) := \lim_{y\to x^+} w(t,y) \qquad \text{and} \qquad w^-(t,x) := \lim_{y\to x^-} w(t,y)$$

  are well-defined for every, and not just almost every, $x \in (a,b)$. This is enough to give a sense to left-hand sides.

  Let us discuss now some implications of the sign conditions. To this end, it is useful to decompose the measures Du and Dw as

$$Du = \widetilde{D}u + D^J u \qquad \text{and} \qquad Du = \widetilde{D}w + D^J w,$$

  where $D^J u$ and $D^J w$ are the atomic jump parts, while $\widetilde{D}u$ and $\widetilde{D}w$ are the sum of the absolutely continuous and the Cantor parts. When we restrict to $\widetilde{D}$, assumptions (5.5.7) and (5.5.8) are equivalent to requiring that

$$\alpha w(t,x)v(t,x)\widetilde{D}u(t,x) \geq |v(t,x)\widetilde{D}w(t,x)| \qquad \text{as a measure in } (a,b), \qquad (5.5.9)$$

  because $w^+(t,x) = w^-(t,x)$ outside the countable set of jump points of $x \mapsto w(t,x)$.

  When restricted to jump parts, assumptions (5.5.7) and (5.5.8) are equivalent to requiring that

$$\alpha w^{\pm}(t,x)v(t,x)\left(u^+(t,x) - u^-(t,x)\right) \geq |v(t,x)| \cdot |w^+(t,x) - w^-(t,x)|$$

for every x in the jump set of the function $x \mapsto w(t,x)$. We observe that this implies in particular that any jump point of w in which $v(t,x) \neq 0$ is necessarily also a jump point of u.

The following definition extends the notion of uvw-evolutions to any dimension.

Definition 5.5.4 (UVW-evolution with DNBC in any dimension)**.** Let d be a positive integer, and let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set with Lipschitz boundary (see Remark 5.5.6 below).

A *UVW-evolution with Dirichlet/Neumann boundary conditions* in Ω is a triple of measurable functions

$$U : (0,+\infty) \times \Omega \to \mathbb{R}, \qquad V : (0,+\infty) \times \Omega \to \mathbb{R}^d, \qquad W : (0,+\infty) \times \Omega \to \mathbb{R}^d$$

with the following properties.

- (Boundedness) The function u satisfies
$$U \in L^\infty((0,T) \times \Omega) \qquad \forall T > 0. \tag{5.5.10}$$

- (Time regularity). The function U admits a weak derivative U_t with respect to time, and
$$U_t \in L^1((0,T) \times \Omega, \mathbb{R}) \qquad \forall T > 0. \tag{5.5.11}$$

- (Space regularity). For almost every $t > 0$ it turns out that
$$\text{the function } x \mapsto U(t,x) \text{ is in } BV(\Omega) \cap L^\infty(\Omega), \tag{5.5.12}$$
$$\text{the function } x \mapsto V_i(t,x) \text{ is in } W^{1,1}(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d), \tag{5.5.13}$$
$$\text{the function } x \mapsto W(t,x) \text{ is in } BV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d). \tag{5.5.14}$$

- (Evolution equation). The functions U, V, and W solve the equation
$$U_t(t,x) = \sum_{i=1}^{d} W_i(t,x) D_{x_i} V(t,x) \qquad \text{in } (0,+\infty) \times \Omega. \tag{5.5.15}$$

- (Dirichlet/Neumann boundary condition). There exists a nonincreasing function $D_0 : [0,+\infty) \to \mathbb{R}$ such that for almost every $t > 0$ and almost every $x \in \partial\Omega$ it turns out that at least one of the following two conditions
$$U(t,x) \leq D_0(t), \qquad\qquad \sum_{i=1}^{d} W_i(t,x) V_i(t,x) \nu_i(x) = 0 \tag{5.5.16}$$
holds true, where $\nu(x)$ denotes the outer normal to $\partial\Omega$ at point x.

- (Sign condition). There exists a real number $\alpha > 0$ such that for almost every $t > 0$ it turns out that

$$\alpha \sum_{i=1}^{d} W_i^+(t,x) V_i(t,x) D_{x_i} U \geq \left| \sum_{i=1}^{d} V_i(t,x) D_{x_i} W_i \right| \tag{5.5.17}$$

and

$$\alpha \sum_{i=1}^{d} W_i^-(t,x) V_i(t,x) D_{x_i} U \geq \left| \sum_{i=1}^{d} V_i(t,x) D_{x_i} W_i \right| \tag{5.5.18}$$

as measures in Ω, where W_i^+ and W_i^- denote the two one-sided traces of W_i on its jump set, and are equal to w elsewhere.

Remark 5.5.5 (Special case $W_i \equiv 1$)**.** In the special case where $W_i(t,x) \equiv 1$ for every $i = 1, \ldots, d$, this definition reduces to Definition 5.4.1. In particular, the sign conditions (5.5.7) and (5.5.8) reduces to (5.4.5), and the second condition in (5.5.16) reduces to the standard Neumann condition $\langle V(t,x), \nu(x) \rangle = 0$ for almost every $x \in \partial\Omega$. Moreover, as in dimension one, the boundedness assumption (5.5.10) becomes unnecessary, because it follows from Proposition 5.4.3, at least if the initial datum is bounded.

Remark 5.5.6. As we did in dimension one, let us comment on some delicate regularity issues in Definition 5.5.4 above.

- (Regularity of the open set). In this sequel we consider UVW-evolutions in bounded open sets that are finite intersections of half-spaces, in which case $\partial\Omega$ is Lipschitz continuous, but not more regular. What we actually need in the proof of Proposition 5.4.3 below is that in Ω the usual integration by parts formula

$$\int_\Omega f(x) \operatorname{div} E(x)\, dx = -\int_\Omega \langle \nabla f(x), E(x) \rangle\, dx + \int_{\partial\Omega} f(x) \langle E(x), \nu(x) \rangle\, d\sigma$$

holds true for every function $f \in W^{1,1}(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d)$ and every vector field $E \in BV(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d)$. We point out that this requires also to define the traces on $\partial\Omega$ of f and E.

- (Evolution equation). As in dimension one, from (5.5.11) we know that the function $x \mapsto U_t(t,x)$ is in $L^1(\Omega, \mathbb{R})$ for almost every $t \in (0,T)$, while from (5.5.13) and (5.5.14) we know that, for almost every $t \in (0,T)$, the function $x \to W_i(t,x) D_{x_i} V_i(t,x)$ is the product of a function in $L^\infty(\Omega)$ and a function in $L^1(\Omega)$. As a consequence, for almost every $t \in (0,T)$ the right-hand side of (5.5.15) is in $L^1(\Omega)$, and the evolution equation is actually an equality between functions in $L^1(\Omega)$.

- (Initial datum). As in dimension one, the time regularity assumption (5.5.11) implies that U is absolutely continuous as a function from $(0,+\infty)$ to $L^1(\Omega)$. In particular the "initial datum" $x \mapsto U(0,x)$ is well defined as a function in $L^1(\Omega)$.

- (Dirichlet/Neumann boundary conditions). From the space regularity assumption (5.5.12) we know that, for almost every $t > 0$, the function $x \mapsto U(t,x)$ has bounded variation, and therefore it admits a trace on $\partial\Omega$. This implies that the first condition in (5.5.16) makes sense.

  Analogously, for almost every $t > 0$, and every index $i = 1, \dots, d$, the function $x \to V_i(t,x)$ and the function $x \to W_i(t,x)$ are bounded and have bounded variation. As a consequence, their product has bounded variation as well (see [6, Example 3.97]), and therefore it has a trace on $\partial\Omega$. For this reason, also the second condition in (5.5.16) makes sense.

- (Sign conditions). The terms in the right-hand sides of (5.5.17) and (5.5.18) are, for almost every $t > 0$, the total variation measure of the scalar product between a vector measure and a bounded vector field in $W^{1,1}(\Omega;\mathbb{R}^d)$. In general this product is not well-defined, but in this case the measure is the gradient of a function in $BV(\Omega)$, and therefore it is absolutely continuous with respect to the $d-1$ dimensional Hausdorff measure $\mathcal{H}^{d-1}$. Under these assumptions, the product makes sense.

  Analogously, also the left-hand sides are the sum of products of the differential of a function in $BV(\Omega)$ times a bounded function (the product $W_i^{\pm}V_i$) whose pointwise values are well-defined for $\mathcal{H}^{d-1}$ almost every point.

  In any case, as for the case of UV-evolutions, here we consider only UVW-evolutions that come from one-dimensional uvw-evolutions, and hence are much more regular. In this case, the definition of the quantities above is much less problematic, and actually almost elementary.

  As in dimension one, when we restrict to the absolutely continuous and Cantor parts, assumptions (5.5.17) and (5.5.18) reduce to the unique requirement that

  $$\alpha \sum_{i=1}^{d} W_i(t,x)V_i(t,x)\widetilde{D}_{x_i}U \geq \left|\sum_{i=1}^{d} V_i(t,x)\widetilde{D}_{x_i}W_i\right| \qquad \text{as measures in } \Omega, \quad (5.5.19)$$

  again because $W_i^+(t,x) = W_i^-(t,x)$ almost everywhere with respect to these measures.

  When we restrict to jump parts, again we deduce that the union of the jump sets of the functions $x \mapsto V_i(t,x)W_i(t,x)$ are contained in the jump set of the function $x \mapsto U(t,x)$, up to a $\mathcal{H}^{d-1}$-negligible set.

In the next results we show that uvw-evolutions in dimension one and UVW-evolutions in any dimension share the same properties of UV-evolutions. The proofs are similar to the ones in the previous section, the main differences being that now the function ψ that we fix at the beginning of the proof depends on the constant α that appears in the sign condition, and in particular we can not choose it in such a way that it is globally Lipschitz continuous. This is the reason for which we need to add the boundedness assumption.

Proposition 5.5.7 (Maximum principle for UVW-evolutions with DNBC in any space dimension)**.** *Let d be a positive integer, and let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Let*

$$U : [0,+\infty) \times \Omega \to \mathbb{R}, \qquad V : [0,+\infty) \times \Omega \to \mathbb{R}^d, \qquad W : [0,+\infty) \times \Omega \to \mathbb{R}^d$$

be a UVW-evolution with DNBC in Ω according to Definition 5.5.4, and let $D_0 : [0,+\infty) \to \mathbb{R}$ be the nonincreasing function that appears in (5.5.16).

Then the function $M(t)$ defined by (5.4.8) is nonincreasing in $[0,+\infty)$.

Proof. Since we can always restrict to a smaller time interval, it is enough to show that $M(t) \leq M(0)$ for every $t \geq 0$.

To this end, we consider a function $\psi_\alpha \in C^2(\mathbb{R})$ such that

$$\psi_\alpha(\sigma) = 0 \qquad \text{if and only if} \qquad \sigma \leq M(0), \tag{5.5.20}$$

and

$$\psi_\alpha''(\sigma) \geq \alpha \psi_\alpha'(\sigma) \geq 0 \qquad \forall \sigma \in \mathbb{R}, \tag{5.5.21}$$

where α is the constant that appears in the sign conditions (5.5.17) and (5.5.18).

A possible choice is, for example, to define $\psi_\alpha(s) = (s - M(0))_+^{2+\alpha}$ for every $s < M(0) + (1+\alpha)/\alpha$ and then extend it to a C^2 function on $\mathbb{R}$ by solving the equation $\psi_\alpha''(s) = \alpha \psi_\alpha'(s)$, whose solution can also be written explicitly.

Then we set

$$E(t) := \int_\Omega \psi_\alpha(U(t,x))\, dx \qquad \forall t \in [0,+\infty). \tag{5.5.22}$$

We observe that (5.5.20) and the convexity of ψ_α imply that $\psi_\alpha(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$, and hence

$$E(t) \geq 0 \qquad \forall t \in [0,+\infty), \tag{5.5.23}$$

and in addition $\psi_\alpha(\sigma) > 0$ for every $\sigma > M(0)$, from which we deduce that

$$E(t) = 0 \qquad \text{if and only if} \qquad M(t) \leq M(0). \tag{5.5.24}$$

We observe that the function $E(t)$ is absolutely continuous because of the boundedness (5.5.10) and the time regularity (5.5.11) of U. We claim that $E'(t) \leq 0$ for almost every $t > 0$. If true, this claim, combined with (5.5.23), would imply that $E(t) = 0$ for every $t > 0$, and this would complete the proof because of (5.5.24).

In order to compute $E'(t)$, we derive the integral (5.5.22) with respect to time, and using (5.5.15) we find that

$$E'(t) = \int_\Omega \psi_\alpha'(U(t,x)) U_t(t,x)\, dx = \sum_{i=1}^d \int_\Omega \psi_\alpha'(U(t,x)) W_i(t,x) D_{x_i} V_i(t,x)\, dx.$$

Now we integrate by parts and we obtain that (with some abuse of notation, because what we integrate is a measure and not a function)

$$E'(t) = -\sum_{i=1}^d \int_\Omega D_{x_i} \left(\psi_\alpha'(U(t,x)) W_i(t,x) \right) \cdot V_i(t,x)\, dx. \tag{5.5.25}$$

In this point we neglected the boundary term

$$\int_{\partial\Omega} \psi'_\alpha(U(t,x)) \left(\sum_{i=1}^{d} W_i(t,x) V_i(t,x) \nu_i(x) \right) d\sigma$$

which is equal to 0 for almost every $t \in [0,T]$ because of (5.5.16). Indeed, for almost every $x \in \partial\Omega$ we know that either the sum is equal to 0, or

$$U(t,x) \leq D_0(t) \leq D_0(0) \leq M(0),$$

in which case $\psi'_\alpha(U(t,x)) = 0$. Now from the chain rule for bounded variation functions we know that (for the sake of shortness we do not write explicitly the dependence on (t,x))

$$D_{x_i}\left(\psi'_\alpha(U)W_i\right) = \psi''_\alpha(U) W_i \widetilde{D}_{x_i} U + \psi'_\alpha(U) \widetilde{D}_{x_i} W_i + \frac{\psi'_\alpha(U^+)W_i^+ - \psi'_\alpha(U^-)W_i^-}{U^+ - U^-} D^J_{x_i} U,$$

at least where $V_i(t,x) \neq 0$, so that the jump set of $W_i(t,x)$ is contained in the jump set of $U(t,x)$. Anyway, since in (5.5.25) this measure is multiplied by $V_i(t,x)$, it is enough to consider the points where $V_i(t,x) \neq 0$.

Plugging this equality into (5.5.25) we obtain that (again with some abuse of notation, because L_1 and L_2 are actually measures)

$$E'(t) = -\int_\Omega L_1(t,x)\,dx - \int_\Omega L_2(t,x)\,dx,$$

where

$$L_1 := \psi''_\alpha(U) \sum_{i=1}^{d} W_i V_i \widetilde{D}_{x_i} U + \psi'_\alpha(U) \sum_{i=1}^{d} V_i \widetilde{D}_{x_i} W_i$$

and

$$L_2 := \frac{\psi'_\alpha(U^+)W_i^+ - \psi'_\alpha(U^-)W_i^-}{U^+ - U^-} V_i D^J_{x_i} U.$$

We claim that L_1 and L_2 are nonnegative measures in Ω. In the case of L_1, from (5.5.21) and the positivity of $W_i V_i \widetilde{D} U$, which follows from (5.5.19), we deduce that

$$L_1 \geq \psi'(U(t,x)) \left\{ \alpha \sum_{i=1}^{d} W_i(t,x) V_i(t,x) \widetilde{D}_{x_i} U - \left| \sum_{i=1}^{d} V_i(t,x) \widetilde{D}_{x_i} W_i \right| \right\},$$

and the latter is nonnegative because of (5.5.19).

In the case of L_2, we observe that it can be rewritten both in the form

$$\begin{aligned} L_2 &= \sum_{i=1}^{d} \psi'(U^+) V_i \frac{W_i^+ - W_i^-}{U^+ - U^-} D^J_{x_i} U + \sum_{i=1}^{d} \frac{\psi'(U^+) - \psi'(U^-)}{U^+ - U^-} W_i^- V_i D^J_{x_i} U \\ &= \psi'(U^+) \sum_{i=1}^{d} V_i D^J_{x_i} W + \frac{\psi'(U^+) - \psi'(U^-)}{U^+ - U^-} \sum_{i=1}^{d} W_i^- V_i D^J_{x_i} U, \end{aligned}$$

and in the form

$$L_2 = \psi'(U^-)\sum_{i=1}^{d} V_i D^J_{x_i} W + \frac{\psi'(U^+) - \psi'(U^-)}{U^+ - U^-}\sum_{i=1}^{d} W_i^+ V_i D^J_{x_i} U.$$

By combining the two expressions we obtain that

$$\begin{aligned} L_2 &= \big(\lambda\psi'(U^-) + (1-\lambda)\psi'(U^+)\big)\sum_{i=1}^{d} V_i D^J_{x_i} W \\ &\quad + \frac{\psi'(U^+) - \psi'(U^-)}{U^+ - U^-}\sum_{i=1}^{d}\left(\lambda W_i^+ + (1-\lambda)W_i^-\right) V_i D^J_{x_i} U \end{aligned}$$

for every Borel function $\lambda : \Omega \to \mathbb{R}$. Now for every x in the jump set of U there exists $\xi(t,x)$ between $U^+(t,x)$ and $U^-(t,x)$ such that

$$\frac{\psi'(U^+(t,x)) - \psi'(U^-(t,x))}{U^+(t,x) - U^-(t,x)} = \psi''(\xi(t,x)).$$

Since ψ' is monotone, there exists $\lambda(t,x) \in [0,1]$ such that

$$\lambda(t,x)\psi'(U^-(t,x)) + (1-\lambda(t,x))\psi'(U^+(t,x)) = \psi'(\xi(t,x)).$$

Recalling (5.5.21), for this choice of the function $x \mapsto \lambda(t,x)$ we deduce that

$$\begin{aligned} L_2 &= \psi'(\xi)\sum_{i=1}^{d} V_i D^J_{x_i} W + \psi''(\xi)\sum_{i=1}^{d}\left(\lambda W_i^+ + (1-\lambda)W_i^-\right) V_i D^J_{x_i} U \\ &\geq \psi'(\xi)\left\{\alpha\sum_{i=1}^{d}\left(\lambda W_i^+ + (1-\lambda)W_i^-\right) V_i D^J_{x_i} U + \sum_{i=1}^{d} V_i D^J_{x_i} W\right\} \\ &\geq \psi'(\xi)\left\{\alpha\sum_{i=1}^{d}\left(\lambda W_i^+ + (1-\lambda)W_i^-\right) V_i D^J_{x_i} U - \left|\sum_{i=1}^{d} V_i D^J_{x_i} W\right|\right\}, \end{aligned}$$

and the latter is a nonnegative measure on Ω because of a convex combination of (5.5.17) and (5.5.18) restricted to jump parts, and the fact that ψ' is a nonnegative function.

This completes the proof. □

Remark 5.5.8 (Case with only Neumann boundary conditions)**.** As for UV-evolutions, from the proof it is clear that, when for almost every $t > 0$ the second condition in (5.5.16) is satisfied for almost every $x \in \partial\Omega$, then $D_0(t)$ plays no role. In particular, we do not need to consider the maximum with $D_0(t)$ in (5.4.8), or equivalently we can take $D_0(t) \equiv -\infty$.

From Proposition 5.5.7 we deduce the following monotonicity properties for *uvw*-evolutions in dimension one, which extend the results of Proposition 5.2.10.

Proposition 5.5.9 (Monotonicity results for UVW-evolutions with NBC in dimension one). *Let (u, v, w) be a uvw-evolution with Neumann boundary conditions in an interval $(a, b) \subseteq \mathbb{R}$ according to Definition 5.5.1.*

Then the following monotonicity results hold true.

(1) (Maximum principle). *For every $t \in [0, +\infty)$ let $M^+(t)$ and $M^-(t)$ denote, respectively, the (essential) supremum and infimum of the function $x \mapsto u(t, x)$.*

Then the function $M^+(t)$ is nonincreasing, while $M^-(t)$ is nondecreasing.

(2) (Monotonicity of the total variation). *For every $t \in [0, +\infty)$ let $TV^{\pm}(t)$ denote the positive/negative variation of the function $x \mapsto u(t, x)$.*

Then the functions $TV^{\pm}(t)$ are nonincreasing.

(3) (Monotonicity of compositions). *For every monotone and Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$ it turns out that the function $g(u(t, x))$ satisfies the maximum principle and the monotonicity of the positive/negative total variation as in the two statements above.*

Proof. As we did for uv-evolutions, we reduce the one-dimensional results to the higher dimensional maximum principle.

Maximum principle We claim that the triple

$$U(t, x) := u(t, x), \qquad V(t, x) := v(t, x), \qquad W(t, x) := w(t, x)$$

is a UVW-evolution with DNBC according to Definition 5.5.4 with $d = 1$, $\Omega = (a, b)$, and no need of D_0 (see Remark 5.5.8).

Indeed, all the assumption on U, V, W in Definition 5.5.4 follow immediately from the corresponding assumptions on u, v, w in Definition 5.5.1.

At this point from Proposition 5.5.7 it follows that the function $M(t)$ defined by (5.4.8) is nonincreasing, but in this case $M(t)$ coincides with the essential supremum $M^+(t)$ because $D_0(t)$ is always less than or equal to the essential supremum.

The monotonicity of $M^-(t)$ can be obtained by applying the maximum principle to the triple $(-u, -v, w)$, which is again a uvw-evolution with Neumann boundary conditions.

Monotonicity of the total variation To begin with, we observe that it is enough to prove the monotonicity of the positive variation, because the negative variation of u is the positive variation of $-u$, and we have already observed that $(-u, -v, w)$ is again a uvw-evolution with Neumann boundary conditions.

As in the proof of Proposition 5.2.10, for every positive integer m, we set

$$TV_m^+(t) := \sup \left\{ \sum_{i=1}^{2m} (-1)^i u(t, x_i) : a \leq x_1 \leq x_2 \leq \ldots \leq x_{2m} \leq b. \right\},$$

and we observe that

$$TV^+(t) = \sup_{m\geq 1} TV_m^+(t) = \lim_{m\to+\infty} TV_m^+(t).$$

Therefore, if we prove that $TV_m^+(t)$ is a nonincreasing function for every $m \geq 1$, then thesis follows. Again, we prove the monotonicity of $TV_m^+(t)$ by induction on m.

Case $m = 1$ We claim that the triple defined by

$$U(t, x_1, x_2) := u(t, x_2) - u(t, x_1),$$
$$V(t, x_1, x_2) := (-v(t, x_1), v(t, x_2)), \qquad W(t, x_1, x_2) := (w(t, x_1), w(t, x_2)).$$

is a UVW-evolution with Dirichlet/Neumann boundary conditions according to Definition 5.5.4 with

$$d := 2, \qquad \Omega := \left\{(x_1, x_2) \in (a, b)^2 : a < x_1 < x_2 < b\right\}, \qquad D_0(t) \equiv 0.$$

If this is the case, then the monotonicity of $TV_1^+(t)$ follows from Proposition 5.5.7, because in this case the function $M(t)$ defined by (5.4.8) coincides with $TV_1^+(t)$.

So let us check that U, V and W satisfy the properties in Definition 5.5.4. The boundedness, the regularity and the evolution equation follow from the corresponding properties of u, v, w in Definition 5.5.1. The sign condition (5.5.17) follows from (5.5.7) because

$$\begin{aligned}
&\left|\sum_{i=1}^{2} V_i(t, x_1, x_2) D_{x_i} W_i(t, x_1, x_2)\right| \\
&\quad = \ |-v(t, x_1)Dw(t, x_1) + v(t, x_2)Dw(t, x_2)| \\
&\quad \leq \ |v(t, x_1)Dw(t, x_1)| + |v(t, x_2)Dw(t, x_2)| \\
&\quad \leq \ \alpha w^+(t, x_1)v(t, x_1)Du(t, x_1) + \alpha w^+(t, x_2)v(t, x_2)Du(t, x_2) \\
&\quad = \ \alpha \sum_{i=1}^{2} W_i^+(t, x_1, x_2)V_i(t, x_1, x_2)D_{x_i}U(t, x_1, x_2).
\end{aligned}$$

Similarly, the sign condition (5.5.18) follows from (5.5.8). Finally, we observe that Ω is a triangle, and its boundary is contained in the three lines described by the three equalities $a = x_1$, $x_1 = x_2$, and $x_2 = b$.

- In the side with $a = x_1$ the normal vector is $\nu(x_1, x_2) = (-1, 0)$, and hence

$$\sum_{i=1}^{2} W_i(t, x_1, x_2)V_i(t, x_1, x_2)\nu_i(x_1, x_2) = w(t, x_1)v(t, x_1) = w(t, a)v(t, a) = 0.$$

- In the side with $x_2 = b$ the normal vector is $\nu(x_1, x_2) = (0, 1)$, and hence

$$\sum_{i=1}^{2} W_i(t, x_1, x_2)V_i(t, x_1, x_2)\nu_i(x_1, x_2) = w(t, x_2)v(t, x_2) = w(t, b)v(t, b) = 0.$$

- In the side with $x_1 = x_2$ it turns out that

$$U(t, x_1, x_2) = 0 \leq D_0(t).$$

Therefore, in all the sides of $\partial\Omega$ the Dirichlet/Neumann boundary conditions (5.5.16) are satisfied, and this completes the proof.

Inductive step We assume that $TV_m^+(t)$ is nonincreasing for some positive integer m and we prove that also $TV_{m+1}^+(t)$ is nonincreasing. To this end, we consider the triple defined by

$$U(t,x) := \sum_{i=1}^{2m+2} (-1)^i u(t, x_i),$$

$$V(t,x) := \sum_{i=1}^{2m+2} (-1)^i v(t, x_i) e_i, \qquad W(t,x) := \sum_{i=1}^{2m+2} w(t, x_i) e_i,$$

where $x = (x_1, \ldots, x_{2m+2})$ and e_i denotes the i-th vector of the canonical basis of $\mathbb{R}^{2m+2}$. We claim that this triple is a UVW-evolution with Dirichlet/Neumann boundary conditions according to Definition 5.5.4 with

$$d := 2m+2, \qquad D_0(t) := TV_m^+(t),$$
$$\Omega := \left\{(x_1, x_2, \ldots, x_{2m+2}) \in (a,b)^{2m+2} : a < x_1 < \ldots < x_{2m+2} < b\right\}.$$

If this is the case, then the monotonicity of $TV_{m+1}^+(t)$ follows from Proposition 5.5.7, because

$$TV_{m+1}^+(t) = \sup\{U(t, x_1, \ldots, x_{2m+2}) : (x_1, \ldots, x_{2m+2}) \in \Omega\}, \tag{5.5.26}$$

and in particular the function $M(t)$ defined by (5.4.8) in this case is exactly

$$M(t) = \max\{D_0(t), TV_{m+1}^+(t)\} = \max\{TV_m^+(t), TV_{m+1}^+(t)\} = TV_{m+1}^+(t). \tag{5.5.27}$$

So let us check that U, V, W satisfy the assumptions in Definition 5.5.4. As before, the boundedness, the regularity and the evolution equation follow from the corresponding properties of u, v, w in Definition 5.5.1. The sign condition (5.5.17) follows from (5.5.7) because

$$\begin{aligned}
\left|\sum_{i=1}^{2m+2} V_i(t,x) D_{x_i} W_i(t,x)\right| &= \left|\sum_{i=1}^{2m+2} (-1)^i v(t,x_i) Dw(t,x_i)\right| \\
&\leq \sum_{i=1}^{2m+2} |v(t,x_i) Dw(t,x_i)| \\
&\leq \alpha \sum_{i=1}^{2m+2} w^+(t,x_i) v(t,x_i) Du(t,x_i) \\
&= \alpha \sum_{i=1}^{2m+2} W_i^+(t,x) V_i(t,x) D_{x_i} U(t,x).
\end{aligned}$$

Similarly, the sign condition (5.5.18) follows from (5.5.8). Finally, we consider the boundary of Ω, which consists of $2m+3$ “faces” contained in the hyperplanes corresponding to the possible equalities in the definition of Ω.

- In the face with $x_1 = a$ the normal vector is $\nu(x) = -e_1$, and hence

$$\sum_{i=1}^{2m+2} W_i(t,x)V_i(t,x)\nu_i(x) = w(t,x_1)v(t,x_1) = w(t,a)v(t,a) = 0.$$

- In the face with $x_2 = b$ the normal vector is $\nu(x) = e_{2m+2}$, and hence

$$\sum_{i=1}^{2m+2} W_i(t,x)V_i(t,x)\nu_i(x) = w(t,x_{2m+2})v(t,x_{2m+2}) = w(t,b)v(t,b) = 0.$$

- Let us finally consider the faces where $x_i = x_{i+1}$ for some index i. In this case two consecutive terms in the definition of U cancel, and what remains is a competitor in the definition of $TV_m^+(t)$. It follows that in all these $2m+1$ faces of $\partial\Omega$ it turns out that

$$U(t,x_1,\ldots,x_{2m+2}) \leq TV_m^+(t) = D_0(t).$$

  This proves that the Dirichlet/Neumann boundary conditions (5.5.16) are satisied.

Monotonicity of compositions Let us assume that the function g is of class C^2, and that there exists three real numbers Γ_1, Γ_2 and η such that

$$0 < \eta \leq |g'(\sigma)| \leq \Gamma_1 \qquad \text{and} \qquad |g''(\sigma)| \leq \Gamma_2 \qquad \forall \sigma \in \mathbb{R}. \tag{5.5.28}$$

We claim that the triple

$$\widehat{u}(x,t) := g(u(x,t)), \qquad \widehat{v}(x,t) := v(x,t), \qquad \widehat{w}(x,t) := g'(u(t,x))w(x,t)$$

is again a *uvw*-evolution with Neumann boundary conditions in the sense of Definition 5.5.1. If this is the case, then the monotonicity results for $g(u(x,t))$ follow from statement (1) and statement (2).

So let us check that $\widehat{u}$, $\widehat{v}$, $\widehat{w}$ satisfy the assumptions in Definition 5.5.1. The boundedness, the regularity, the evolution equation and the Neumann boundary conditions follow from the corresponding properties of u, v, w. We claim that now the sign conditions hold true with

$$\widehat{\alpha} := \frac{\Gamma_2 + \alpha\Gamma_1}{\eta^2}.$$

To begin with, from the chain rule we know that (for the sake of shortness, here we do not write the explicit dependence on (t,x))

$$\begin{aligned} D\widehat{w} &= g''(u)w\widetilde{D}u + g'(u)\widetilde{D}w + \frac{g'(u^+)w^+ - g'(u^-)w^-}{u^+ - u^-}D^J u \\ &= g''(u)w\widetilde{D}u + g'(u)\widetilde{D}w + g'(u^-)D^J w + \frac{g'(u^+) - g'(u^-)}{u^+ - u^-}w^+ D^J u, \end{aligned}$$

at least where $v \neq 0$, so that the jump set of w is contained in the jump set of u.

Therefore, when we multiply by $\widehat{v} = v$ we obtain that

$$
\begin{aligned}
|\widehat{v}D\widehat{w}| \quad &\leq \quad \left|g''(u)vw\widetilde{D}u\right| + \left|g'(u)v\widetilde{D}w\right| + \left|g'(u^-)vD^Jw\right| \\
&\qquad + \left|\frac{g'(u^+) - g'(u^-)}{u^+ - u^-}vw^+D^Ju\right|. \qquad (5.5.29)
\end{aligned}
$$

Let us estimate the four measures in the right-hand side. As for the first measure, from (5.5.28) we deduce that

$$
\left|g''(u)vw\widetilde{D}u\right| \leq \Gamma_2 \left|vw\widetilde{D}u\right| = \Gamma_2 vw\widetilde{D}u = \Gamma_2 vw^+\widetilde{D}u,
$$

where in the two equalities we exploited that the measure $vw\widetilde{D}u$ is nonnegative (this follows from (5.5.9)), and that $w = w^+$ almost everywhere with respect to this measure.

As for the second measure, we exploit (5.5.28) and (5.5.9). We deduce that

$$
\left|g'(u)v\widetilde{D}w\right| \leq \Gamma_1 \left|v\widetilde{D}w\right| \leq \Gamma_1\alpha w^+v\widetilde{D}u.
$$

For the third measure we exploit (5.5.28) and the restriction of (5.5.7) to jump parts, and we obtain that

$$
\left|g'(u^-)vD^Jw\right| \leq \Gamma_1 \left|vD^Jw\right| \leq \Gamma_1\alpha w^+vD^Ju.
$$

Finally, for the fourth measure we estimate the fraction with the second derivative of g, and we obtain that

$$
\left|\frac{g'(u^+) - g'(u^-)}{u^+ - u^-}vw^+D^Ju\right| \leq \Gamma_2 \left|vw^+D^Ju\right| = \Gamma_2 vw^+D^Ju,
$$

where in the last equality we exploited that the measure vw^+D^Ju is nonnegative (this follows from (5.5.7) restricted to the jump part).

Plugging all these estimates into (5.5.29) we conclude that

$$
|\widehat{v}D\widehat{w}| \leq (\Gamma_2 + \alpha\Gamma_1)vw^+Du. \qquad (5.5.30)
$$

Since vw^+Du is a nonnegative measure, we have that

$$
\begin{aligned}
vw^+Du \quad &= \quad vw^+\widetilde{D}u + vw^+D^Ju \\
&= \quad \frac{1}{g'(u)g'(u^+)} \cdot v \cdot g'(u^+)w^+ \cdot g'(u)\widetilde{D}u \\
&\qquad + \frac{1}{g'(u^+)}\frac{u^+ - u^-}{g(u^+) - g(u^-)} \cdot v \cdot g'(u^+)w^+ \cdot \frac{g(u^+) - g(u^-)}{u^+ - u^-}D^Ju \\
&\leq \quad \frac{1}{\eta^2} \cdot v \cdot g'(u^+)w^+ \cdot \left(g'(u)\widetilde{D}u + \frac{g(u^+) - g(u^-)}{u^+ - u^-}D^Ju\right) \\
&= \quad \frac{1}{\eta^2} \cdot \widehat{v} \cdot \widehat{w}^+ \cdot D\widehat{u}.
\end{aligned}
$$

Plugging this estimate into (5.5.30) we finally obtain that

$$|\widehat{v}D\widehat{w}| \leq \frac{\Gamma_2 + \alpha\Gamma_1}{\nu^2} \cdot \widehat{v} \cdot \widehat{w}^+ \cdot D\widehat{u} = \widehat{\alpha} \cdot \widehat{v} \cdot \widehat{w}^+ \cdot D\widehat{u},$$

which proves the sign condition (5.5.7). The proof of the sign condition (5.5.7) is analogous, just starting by rewriting $D\widehat{w}$ in the form

$$D\widehat{w} = g''(u)w\widetilde{D}u + g'(u)\widetilde{D}w + g'(u^+)D^J w + \frac{g'(u^+) - g'(u^-)}{u^+ - u^-} w^- D^J u.$$

Now, if g is just a monotone function of class C^1, it is enough to approximate $g(u)$ with a sequence of functions $\{g_n(u)\}$ with g_n as above and such that $g_n \to g$ and $g_n' \to g'$ uniformly on compact sets. By the chain rule for BV functions, this is enough to deduce that the sequence $\{g_n(u)\}$ converges strictly to $g(u)$ as functions of x for every fixed t, and hence that the total variation of $g(u)$ is a nonincreasing function of t, because it is the pointwise limit of a sequence of nonincreasing functions.

Finally, if g is only Lipschitz continuous, we observe that the argument used in [6, Theorem 3.99] to prove the chain rule actually shows the existence of a sequence $\{g_n\}$ of smooth monotone functions such that $g_n(u)$ converges to $g(u)$ strictly for every $u \in BV(a,b)$. This is enough to deduce the required monotonicity properties also in this case.

□

5.6 Monotonicity properties of level sets

Definition 5.6.1 (Level sets of of BV functions)**.** *Let $(a,b) \subseteq \mathbb{R}$ be an interval and let $f \in BV((a,b))$ be a function.*

For every $y \in \mathbb{R}$ let us consider the following quantities.

$$Z(f,y) := \mathcal{H}^0\left(\left\{x \in (a,b) : \liminf_{x' \to x} f(x') \leq y \leq \limsup_{x' \to x} f(x')\right\}\right), \tag{5.6.1}$$

$$P(f,y) := \mathrm{Per}(\{x \in (a,b) : f(x) < y\}, (a,b)), \tag{5.6.2}$$

$$L(f,y) := \liminf_{n \to +\infty} TV(g_n \circ (f - y)), \tag{5.6.3}$$

where $\mathrm{Per}(E,\Omega)$ denotes the usual weak notion of perimeter of a set E inside Ω (namely the total variation of the characteristic function in $BV(\Omega)$), while

$$g_n(s) := \min\{\max\{ns, -1/2\}, 1/2\}.$$

We recall that for one-dimensional bounded variation functions the left and right limit are well-defined at all points, so the liminf and the limsup at any point are just the minimum and the maximum between the two one-sided limits. Hence, for $f \in BV((a,b))$, it is convenient to set

$$f^*(x) := \limsup_{x' \to x} f(x') = \max\left\{\lim_{x' \to x^-} f(x'), \lim_{x' \to x^+} f(x')\right\},$$

and

$$f_*(x) := \liminf_{x' \to x} f(x') = \min \left\{ \lim_{x' \to x^-} f(x'), \lim_{x' \to x^+} f(x') \right\}.$$

The three quantities defined in (5.6.1), (5.6.2) and (5.6.3) are three weak definition for the number of solutions of the equation $f(x) = y$, in the case in which f is a one-dimensional function of bounded variation. The number $Z(f, y)$ is the more natural definition, since it is basically the classical one, with the difference that we are including also the jump points in which y is between the two values $f_*(x)$ and $f^*(x)$. In particular, if f is continuous, then $Z(f, y)$ coincides with the number of elements of the set $f^{-1}(y)$.

The other two quantities, instead, have some drawbacks. Indeed, $P(f, y)$ can not see local maximum or minimum points, while $L(f, y)$ is even more pathological, because it is possible to find smooth functions f for which $f^{-1}(0)$ is a single point and $L(f, 0)$ is equal to an arbitrary real number larger than 1.

However, the next result shows that this pathologies are in some sense exceptionals, namely they can occur only for a negligible set of y.

Proposition 5.6.2. *Let $(a, b) \subseteq \mathbb{R}$ be an interval and $f \in BV((a, b))$ be a function. Let us consider the three quantities defined in (5.6.1), (5.6.2) and (5.6.3). Then it turns out that*

- *There exists a set $\mathcal{N} \subseteq \mathbb{R}$ that is at most countable such that $Z(f, y) = P(f, y)$ for every $y \in \mathbb{R} \setminus \mathcal{N}$.*
- *It holds that $P(f, y) = L(f, y)$ for almost every $y \in \mathbb{R}$.*

Proof. We divide the proof in several steps.

Step 1 We prove that $P(f, y) \leq Z(f, y)$ for every $y \in \mathbb{R}$.

To this end, we fix $y \in \mathbb{R}$ and we can assume that $Z(f, y) = m \in \mathbb{N}$, so there exist m points $a < x_1 < \cdots < x_m < b$ such that

$$\{x \in (a, b) : f_*(x) \leq y \leq f^*(x)\} = \{x_1, \ldots, x_m\}.$$

Now we observe that the function $f(x) - y$ has constant sign in each of the intervals $(a, x_1), (x_1, x_2), \ldots, (x_m, b)$. As a consequence, we deduce that the set $\{x \in (a, b) : f(x) < y\}$ is the union of some (or none, or all) of those intervals. It follows that $\partial\{x \in (a, b) : f(x) < y\} \subseteq \{x_1, \ldots, x_m\}$, and therefore $P(f, y) \leq m$.

Step 2 We prove that $Z(f, y) \leq P(f, y)$ for every $y \in \mathbb{R} \setminus \mathcal{N}$, where $\mathcal{N}$ is the image of all local maximum and minimum points for f.

To be more precise, a value $y \in \mathbb{R}$ belong to $\mathcal{N}$ if there exist a point $\overline{x} \in (a, b)$ and a positive real number $r > 0$ such that either

$$y = f^*(\overline{x}) \geq f^*(x) \qquad \forall x \in (\overline{x} - r, \overline{x} + r), \tag{5.6.4}$$

or

$$y = f_*(\overline{x}) \leq f_*(x) \qquad \forall x \in (\overline{x} - r, \overline{x} + r).$$

In the first case, we say that $\overline{x}$ is a local maximum point for f, while in the second case we say that $\overline{x}$ is a local minimum point for f.

To prove our claim, we fix $y \in \mathbb{R} \setminus \mathcal{N}$ and we can assume that $P(f,y) < +\infty$. By the characterization of one-dimensional finite perimeter sets (see, for example, [60, Proposition 12.13]), we deduce that $\{x \in (a,b) : f(x) < y\}$ is equivalent to a finite union of intervals with different endpoints, namely there exist a non-negative integer $m \in \mathbb{N}$ and $2m$ points $a \leq x_1 < x_2 < \cdots < x_{2m} \leq b$ such that

$$\{x \in (a,b) : f(x) < y\} = (x_1, x_2) \cup (x_3, x_4) \cup \cdots \cup (x_{2m-1}, x_{2m}),$$

up to a negligible set, and $P(f,y) = \mathcal{H}^0(\{x_1, \ldots, x_{2m}\} \cap (a,b))$.

So it is enough to show that

$$\{x \in (a,b) : f_*(x) \leq y \leq f^*(x)\} \subseteq \{x_1, \ldots, x_{2m}\} \cap (a,b).$$

So let us assume by contradiction that there exists a point $\overline{x}$ belonging to the set in the left-hand side but which is not one of the points $\{x_i\}$. Then there exists an index $i \in \{1, \ldots, 2m-1\}$ and a positive real number $r > 0$ such that $x_i < \overline{x} - r < \overline{x} + r < x_{i+1}$.

It follows that either $f(x) < y$ for almost every $x \in (\overline{x} - r, \overline{x} + r)$ (if i is odd) or $f(x) \geq y$ for almost every $x \in (\overline{x} - r, \overline{x} + r)$ (if i is even).

In the first case, we deduce that

$$f^*(\overline{x}) = y > f(x)$$

for almost every $x \in (\overline{x} - r, \overline{x} + r)$, hence $\overline{x}$ is a local maximum point for f.

In the second case, we deduce that

$$f_*(\overline{x}) = y \leq f(x)$$

for almost every $x \in (\overline{x} - r, \overline{x} + r)$, hence $\overline{x}$ is a local minimum point for f.

In both case, we have that $y \in \mathcal{N}$.

Step 3 We prove that $\mathcal{N}$ is at most countable.

Let us prove that the image of local maximum points is at most countable (the argument for the local minimum points is the same). So, for every positive integer $n \in \mathbb{N}^+$ let $\mathcal{M}_n$ be the set of points $y \in \mathcal{N}$ for which there exists a point $\overline{x} \in (a,b)$ satisfying (5.6.4) with $r = 1/n$.

We claim that $\mathcal{H}^0(\mathcal{M}_n) \leq 1 + (b-a)n$. Indeed, if this were false, we could find two different values $y_1, y_2 \in \mathcal{M}_n$ corresponding to two different maximum points $\overline{x}_1$ and $\overline{x}_2$, which satisfy (5.6.4) with $r = 1/n$ and such that $|\overline{x}_2 - \overline{x}_1| < 1/n$. But this is a contradiction, because we would deduce that $y_1 = y_2$.

Therefore the image of maximum points is the union of countably many finite sets, hence it is at most countable.

Step 4 We prove the second statement, namely that $P(f, y) = L(f, y)$ for almost every $y \in \mathbb{R}$.

From the BV coarea formula (see [6, Theorem 3.40]) we deduce that

$$\begin{aligned}
TV(g_n \circ (f - y)) &= \int_{-\infty}^{+\infty} \mathrm{Per}(\{x \in (a,b) : g_n(f(x) - y) < s\}, (a,b))\, ds \\
&= \int_{-1/2}^{1/2} \mathrm{Per}(\{x \in (a,b) : g_n(f(x) - y) < s\}, (a,b))\, ds \\
&= \int_{-1/2}^{1/2} \mathrm{Per}\left(\left\{x \in (a,b) : f(x) < y + \frac{s}{n}\right\}, (a,b)\right) ds \\
&= n \int_{y-1/2n}^{y+1/2n} \mathrm{Per}(\{x \in (a,b) : f(x) < y'\}, (a,b))\, dy' \\
&= n \int_{y-1/2n}^{y+1/2n} P(f, y')\, dy'
\end{aligned}$$

It follows that $L(f, y) = P(f, y)$ at every Lebesgue point of the function $y \mapsto P(f, y)$, which is measurable and integrable for every $f \in BV((a, b))$ thanks to the BV coarea formula. In particular, the liminf defining $L(f, y)$ is actually a limit for almost every $y \in \mathbb{R}$. □

Now let (u, v, w) be a uvw-evolultion with Neumann boundary conditions in an interval $(a, b) \subseteq \mathbb{R}$. Then from Proposition 5.5.9 we deduce that the total variation (with respect to the variable x) of the function $g_n(u(t, x) - y)$ is nonincreasing with respect to t, and hence also $L(u(t, x), y)$ is nonincreasing. Therefore, if we fix two positive times $0 < t_1 < t_2$ and a point $y \in \mathbb{R}$ such that $L(u, y) = P(u, y) = Z(u, y)$ at both times t_1 and t_2 (and in particular the liminf defining L is a limit), then we know that $Z(u(t_2, x), y) \leq Z(u(t_1, x), y)$, and of course the same inequality holds also with P.

However, this result is not completely satisfactory, because the set of values $y \in \mathbb{R}$ for which the inequality holds depends on the couple of times that we fixed.

On the other hand, we can also say that for almost every $y \in \mathbb{R}$ the functions $t \mapsto Z(u(t, x), y)$ and $t \mapsto P(u(t, x), y)$ coincide for almost everywhere with a nonincreasing function, but we are still allowing many exceptions.

The problem here is that Proposition 5.5.9 lets us control only $L(u(t, x), y)$, but this quantity can actually differ from Z and P even if u is smooth. The only case in which this can not happen is if we assume analiticity, because it turns out that if $f : (a, b) \to \mathbb{R}$ is analytic then $L(f, y) = Z(f, y)$ for every $y \in \mathbb{R}$, with the only exception of the two boundary values.

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